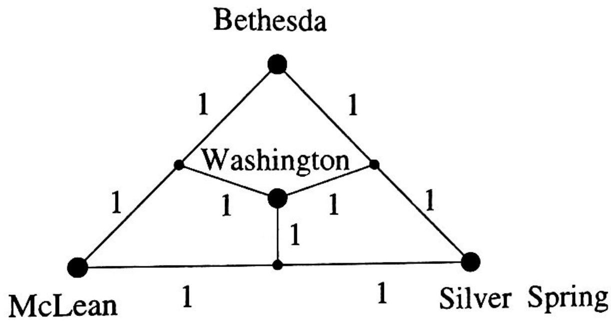
16.5 The board of directors of a certain company contains four members (including the chairman of the board). A motion is passed by the board only if the chairman approves it, and it is supported by a majority of the board (i.e., gets at least three votes). Write down the coalitional function of the corresponding game.

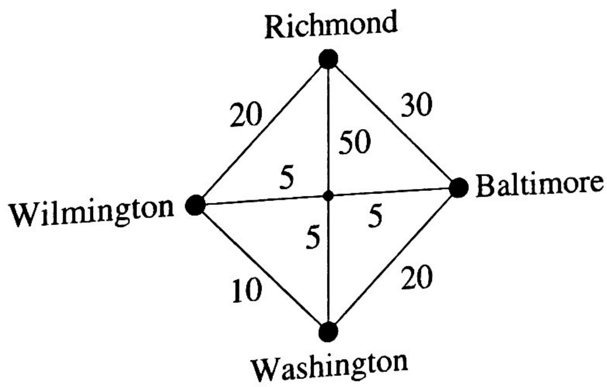
16.6 **Spanning tree games** The following figure depicts a network of roads connecting the capital city, Washington, with three nearby towns, Bethesda, Silver Spring, and McLean. The towns are responsible for maintaining the roads between themselves and the capital. The maintenance cost of every segment of road is listed as a unit.

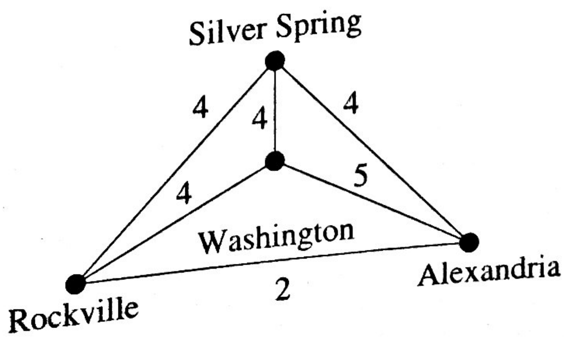
In the figure, a large dot indicates a vertex at which a town is located, and a small dot indicates a vertex at which no town is located.

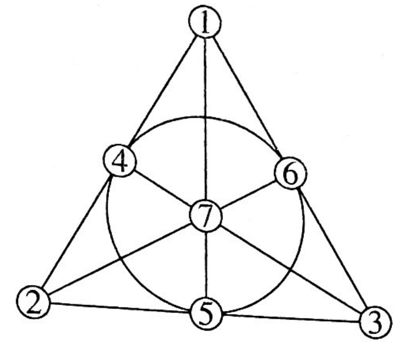
Define $c(S)$ as the minimal cost required for all the towns in coalition S to be connected with the capital. Write down the coalitional function.



16.7 Repeat Exercise 16.6 for each of the following networks. The maintenance cost of each road segment is indicated next to it.







Prove that this game cannot be represented as a weighted majority game.

16.11 A player i in a simple game $(N; v)$ is called a *veto player* if $v(S) = 0$ for every coalition S that does not contain i. The player is called a *dictator* if $v(S) = 1$ if and only if $i \in S$.

(a) Prove that in a simple game satisfying the property that $v(S) + v(N \setminus S) = 1$ for every coalition $S \subseteq N$, there exists at most one veto player, and that player is a dictator.

(b) Find a simple three-player game satisfying $v(S) + v(N \setminus S) = 1$ for every coalition $S \subseteq N$ that has no veto player.

16.12 **Market game** A set of merchants $N_1 = \{1, 2\}$ sell their wares in the market. Each merchant has an "initial endowment" $(0, \frac{1}{2})$: the first number represents the amount of gin the merchant has and the second number the amount of tonic the merchant has. A second set of merchants $N_2 = \{3, 4, 5\}$ also sells in the same market, and each member of this set has an initial endowment of $(1, 0)$. The total bundle available to a coalition $S \subseteq N_1 \cup N_2$ is

$$\left(|S \cap N_2|, \tfrac{1}{2}|S \cap N_1|\right),$$

where $|R|$ denotes the number of members of coalition R.

Consumers will only buy cocktails containing equal parts gin and tonic. The net profit from selling α units of cocktail is α dollars. Describe this situation as a coalitional game, and write down in detail the coalitional function.

16.13 Repeat Exercise 16.12, but assume that the initial endowment of the merchants in N_1 is $(0, \frac{2}{3})$.

16.14 Are the following three-player games strategically equivalent? Justify your answer.

$$v(1) = 6, \quad v(2) = 5, \quad v(3) = 8, v(1, 2) = 10, \quad v(1, 3) = 20,$$
$$v(2, 3) = 50, \quad v(1, 2, 3) = 80,$$
$$w(1) = 13, \quad w(2) = 10, \quad w(3) = 19, w(1, 2) = 25, \quad w(1, 3) = 55,$$
$$w(2, 3) = 140, \quad w(1, 2, 3) = 235.$$

16.15 Let $(N; v)$ be the coalitional game with $N = \{1, 2, 3\}$ and the following coalitional function:

$$v(1) = 3, \quad v(2) = 6, \quad v(3) = 8, \quad v(1,2) = 12, \quad v(1,3) = 15,$$
$$v(2,3) = 18, \quad v(1,2,3) = 80.$$

Write down a $0-1$ normalized coalitional game $(N; w)$ that is strategically equivalent to $(N; v)$.

16.16 What is the coalitional function of the game derived from

$$v(1) = 20, \quad v(2) = 30, \quad v(3) = 50, \quad v(1,2) = 10, \quad v(1,3) = 15,$$
$$v(2,3) = 40, \quad v(1,2,3) = 5,$$

if each player is given an initial sum of \$1,000?

16.17 Prove Theorem 16.7 (page 670): let $(N; v)$ be a coalitional game. Then

(a) $(N; v)$ is strategically equivalent to a $0-1$ normalized game if and only if $v(N) > \sum_{i \in N} v(i)$.
(b) $(N; v)$ is strategically equivalent to a $0-0$ normalized game if and only if $v(N) = \sum_{i \in N} v(i)$.
(c) $(N; v)$ is strategically equivalent to a $0-(-1)$ normalized game if and only if $v(N) < \sum_{i \in N} v(i)$.

16.18 Describe the family of all superadditive games in which the set of players is $N = \{1, 2\}$.

16.19 Let $(N; v)$ be a coalitional game with a set of players $N = \{1, 2, 3\}$ and coalitional function

$$v(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ 1 & \text{if } S = \{1\}, \{2\}, \\ 2 & \text{if } S = \{3\}, \\ 4 & \text{if } |S| = 2, \\ 5 & \text{if } |S| = 3. \end{cases}$$

(a) Is $(N; v)$ a superadditive game?
(b) What is the set of imputations of this game?

16.20 Prove that the convex combination of superadditive games is also superadditive. In other words, if $(N; v)$ and $(N; w)$ are superadditive games, and if $0 \leq \lambda \leq 1$, then the game $(N, \lambda v + (1-\lambda)w)$ defined by

$$(\lambda v + (1-\lambda)w)(S) := \lambda v(S) + (1-\lambda)w(S) \tag{16.29}$$

is also superadditive.

16.21 Give an example of a monotonic game that is not superadditive, and an example of a superadditive game that is not monotonic.

16.22 Prove that every game that is strategically equivalent to a superadditive game is itself superadditive.

16.23 Prove that a convex combination of monotonic games is also monotonic. In other words, if $(N; v)$ and $(N; w)$ are monotonic games, and if $0 \leq \lambda \leq 1$, then the game $(N, \lambda v + (1-\lambda)w)$ defined by

$$(\lambda v + (1-\lambda)w)(S) := \lambda v(S) + (1-\lambda)w(S) \tag{16.30}$$

is also monotonic.

16.24 (a) Is the three-player game $(N; v)$ in which v is given by

$$v(1) = 3, \quad v(2) = 13, \quad v(3) = 4, \quad v(1,2) = 12, \quad v(1,3) = 15,$$
$$v(2,3) = 1, \quad v(1,2,3) = 10$$

monotonic? Justify your answer.

(b) Find a monotonic game that is strategically equivalent to $(N; v)$.

(c) Prove that every game is strategically equivalent to a monotonic game. It follows that the property of monotonicity is not invariant under strategic equivalence.

16.25 Let $(N; v)$ be a nonnegative coalitional game, i.e., $v(S) \geq 0$ for every coalition $S \subseteq N$.

(a) Prove that if $(N; v)$ is superadditive then $(N; v)$ is monotonic.

(b) Show by example that the converse does not hold: it is possible for $(N; v)$ to be monotonic but not superadditive.

16.26 The 0-*normalization* of a coalitional game $(N; v)$ is a coalitional game $(N; w)$ that is strategically equivalent to $(N; v)$ and satisfies $w(i) = 0$ for every player $i \in N$. A coalitional game is called 0-*monotonic* if its 0-normalization is a monotonic game.

(a) Which of the following monotonic games with set of players $N = \{1, 2, 3\}$ is 0-monotonic?

(i) $v(1) = 5$, $v(2) = 8$, $v(3) = 15$, $v(1,2) = 10$, $v(1,3) = 30$, $v(2,3) = 50$, $v(1,2,3) = 80$.

(ii) $v(1) = 5$, $v(2) = -2$, $v(3) = 7$, $v(1,2) = 9$, $v(1,3) = 30$, $v(2,3) = 17$, $v(1,2,3) = 30$.

(b) Give an example of a coalitional game that is not monotonic, but is 0-monotonic.

16.27 Prove that a coalitional game $(N; v)$ is 0-monotonic if and only if $v(S \cup \{i\}) \geq v(S) + v(i)$ for every coalition S and every player $i \notin S$.

16.28 Let $(N; v)$ be a coalitional game. The *superadditive cover* of $(N; v)$ is the coalitional game $(N; w)$ satisfying the properties:

- $(N; w)$ is a superadditive game.
- $w(S) \geq v(S)$ for every coalition S.
- Every game $(N; u)$ satisfying the previous two properties also satisfies $u(S) \geq w(S)$ for every coalition S.