CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 37: Revenue Optimization in Single Object Allocation

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37.1 Recap

Continuing our discussion on revenue optimization in single object allocation from the previous lecture, in which we had devised an optimization problem and characterized that in order to ensure the mechanism to be BIC, f had to be non-decreasing in expectation (NDE). The problem was formulated as:

$$\begin{aligned} \text{OPT1} : \max_f \int_T (\sum_{i \in N} w_i(t) \cdot f_i(t)) g(t) dt \\ \text{subj to} : f \quad \text{is NDE} \end{aligned} \tag{37.1}$$

where $w_i(t)$ is the virtual valuation function, $f_i(t)$ is the allocation function for the type profile t and $g(t)$ is the prior on the type profile of agents. We also proved that if we impose the "regularity" condition on the virtual valuation function, the unrestricted optimization problem (above), is maximized by an allocation function f^* that is NDE. The virtual valuation w_i was defined as regular if it was an increasing function. We shall now prove the lemma that stated the above argument formally and present a proof of the same.

37.2 Solving the Optimization Problem

Lemma 37.1 *Suppose every agent's virtual valuations are regular. The solution of the constrained problem is same as the unconstrained problem.*

Proof: To obtain a solution for the unconstrained optimization problem, we observe that,

$$\sum_{i \in N} w_i(t) \cdot f_i(t) \tag{37.2}$$

is a convex combination of the virtual valuations weighted by the allocation function. This implies that the optimal solution would place all the weight on the $\{i \in N : w_i(t) \geq w_j(t) \forall j\}$, with ties broken arbitrarily. We have to consider the fact that the virtual valuations can be negative and hence if

$$\begin{aligned} w_i(t) < 0 \quad \forall i \in N \\ \Rightarrow f_i(t) = 0 \quad \forall i \in N \\ \text{o/w} \quad f_i(t) = \begin{cases} 1 & w_i(t_i) \geq w_j(t_j) \quad \forall j \\ 0 & o/w \end{cases} \end{aligned} \tag{37.3}$$

Now, for a given t_{-i}, if

$$f_i(t_i, t_{-i}) \geq f_i(s_i, t_{-i}) \quad t_i > s_i \tag{37.4}$$

regularity of w_i will ensure that t_i gets the allocation for $t_i > s_i$. This proves that the given allocation function is non-decreasing. It is important to note that we had set out to prove that f was NDE, which was a weaker requirement. This concludes the proof. ■

Remark 37.2 *f is DSIC since it is non-decreasing.*

37.3 Payment Rule

Let us now look at the payment rule in this allocation, since we want the allocation to optimal w.r.t. revenue. Let us define

$$\kappa_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i} = 1)\} \tag{37.5}$$

The above function lists down the minimum value type/bid of agent i, that ensures allocation to her. It is important to note that it includes the condition that the virtual valuation function of the agent be positive, for the allocation to take place. More formally,

$$\begin{aligned}\kappa_i^*(t_{-i}) &\geq w_i^{-1}(0) \\ p_i &= t_i f_i(t) - \int_0^{t_i} f_i(x_i, t_{-i}) dx_i \\ p_i &= \kappa_i^*(t_{-i}) f_i(t)\end{aligned} \tag{37.6}$$

Theorem 37.3 *If Virtual valuations are regular. For every type t, if*

$$\begin{aligned} w_i(t) &< 0 \quad \forall i \in N \\ \Rightarrow f_i(t) &= 0 \quad \forall i \in N \\ o/w \quad f_i(t) &= \begin{cases} 1 & w_i(t_i) \geq w_j(t_j) \quad \forall j \\ 0 & o/w \end{cases}\end{aligned} \tag{37.7}$$

And the payment is given by, $p_i(t) = \kappa_i^*(t_{-i}) f_i(t)$. Then (f, p) is an revenue optimal mechanism.

So, while we wanted to obtain a mechanism that is BIC, IIR and randomized, the one we just proved was BSIC, IR and deterministic, which is a subset of what we desired. These mechanisms need priors, which different from the Vickrey Auction we had seen before that was a prior-free mechanism.

37.4 Some Examples

Let us now look at some examples through which the payment rule and the entire mechanism will become clearer.

1. Consider an auction with two agents $(\mathcal{A}_1, \mathcal{A}_2)$, with uniform priors given by $\mathcal{T}_1 = [0, 12]$, $\mathcal{T}_1 = [0, 18]$. The virtual valuations are given by,

$$\begin{aligned} w_1(t_1) &= t_1 - \frac{1 - G_1(t_1)}{g(t_1)} \\ &= t_1 - \frac{1 - \frac{t_1}{2}}{\frac{1}{12}} \\ &= 2t_1 - 12 \\ w_2(t_2) &= 2t_2 - 18 \end{aligned} \tag{37.8}$$

where we $G_1(t_1)$ is the cumulative distribution function of the uniform distribution prior of $\mathcal{A}_1$. Now, we shall take some values of the bids t_1, t_2 and look at the allocation function and the payment made by the agent in each case.

t_1	t_2	Allocated to	p_1	p_2	$w_1(t_1)$	$w_2(t_2)$
4	8	unsold	0	0	-4	-2
2	12	$\mathcal{A}_2$	0	9	-8	6
6	6	$\mathcal{A}_1$	6	0	0	-6
9	9	$\mathcal{A}_1$	6	0	6	0
8	15	$\mathcal{A}_2$	0	11	4	12

2. **Symmetric Bidders** : As visible from the previous example, if the bidders were symmetric, that is they had the same prior distributions and hence the same cdf, their virtual valuation functions should be the same.

$$\begin{aligned} T_i &= T \quad \forall i \in N \\ G_i &= G \quad \forall i \in N \\ \Rightarrow w_i &= w \end{aligned} \tag{37.9}$$

Now since, $w_i(t_i) > w_j(t_j) \iff t_i > t_j$. The payment

$$p_i = \max\{w_i^{-1}(0), \max_{j \neq i} t_j\} \tag{37.10}$$

This looks a lot like the second price auction with a reserve price.

3. **Efficiency and Optimality**: We moved from optimizing the Social welfare function to optimizing the revenue function, let us now look at the impact revenue optimization that we have discussed in this lecture has on the efficiency of the allocation function. Recall that, efficiency says that the agent with the highest bid should be allocated the object. Consider an auction with two agents, $\mathcal{A}_1$, $\mathcal{A}_2$ with uniform priors given as, $T_1 = [0, 10]$ and $T_2 = [0, 6]$. The virtual valuation function evaluates to,

$$\begin{aligned} w_1(t_1) &= 2t_1 - 10 \\ w_2(t_2) &= 2t_2 - 6 \\ 2t_1 - 10 &= 2t_2 - 6 \qquad \text{Decision boundary for the winning agent} \\ t_2 &= t_1 - 2 \end{aligned} \tag{37.11}$$

Now, in the figure presented, there is a region (marked 1) where the product is not allocated to anyone, even though the type/bids of the agents is positive. This is not efficient.
Efficiency would require the allocation decision be made around the line $t_1 = t_2$, however, the payment rule here does it around the line $t_2 = t_1 - 2$, making the allocation in the shaded area inefficient.

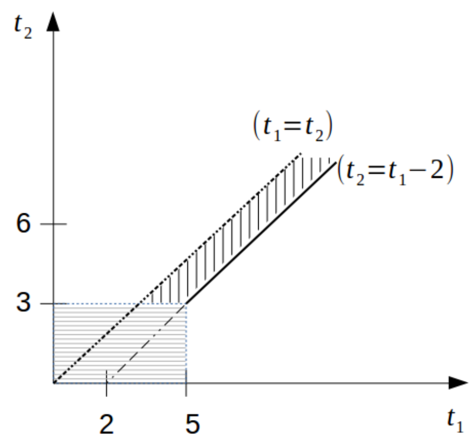

Figure 37.1: A plot of the bid and the allocation decision boundary of the given mechanism