Group Contraction Lemma

Let $G \subseteq N$, $G \neq \phi$ be decisive. Then $\exists\, G' \subsetneq G$ which is also decisive

Proof: $|G| = 1$ nothing to prove

$|G| \geq 2$.

let $G_1 \subset G$ $\quad G_2 = G \setminus G_1$

Construct R

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

$a P_i b \;\; \forall i \in G$ and G decisive

$\Rightarrow a \hat{F}(R) b$ -- (1)

Case (A): $a \hat{F}(R) c$

but consider G_1

$a P_i c \;\; \forall i \in G_1 \quad c P_j a \;\; \forall j \notin G_1$

$\forall R'$ where this is the case, $\Rightarrow a \hat{F}(R') c$

hence $\bar{D}_{G_1}(a, c) \xRightarrow{FEL} G_1$ is decisive.

Case (B): $\neg\, a \hat{F}(R) c \Rightarrow c F(R) a$

also from (1) $a \hat{F}(R) b \overset{\text{transitivity}}{\Rightarrow} c \hat{F}(R) b$

but consider G_2

$c P_i b \;\; \forall i \in G_2$ and $b P_j c \;\; \forall j \notin G_2$

using IIA $\bar{D}_{G_2}(b, c) \xRightarrow{FEL} G_2$ is decisive $\square$

By WP, N is decisive

By GCL $\exists\, i \in N$ $\{i\}$ is decisive

i is the dictator

8-2

Arrovian Social Welfare setup asks for a lot
- Social ordering is unachievable in a democratic way.
- ~~Res~~ One way to mitigate is to allow relaxing the rationality ordering OR looking at restricted preferences - e.g. single peaked preferences
- There are some positive results

However, we'll approach it from a "social ordering" perspective to "social choice" perspective transition.
Instead of a collective order, aim for a collective alternative

$$f : P^n \rightarrow A$$ [Assumption: only strict preferences]

The most representative setting : Voting

Examples of voting protocols

1. Plurality : single candidate to vote, candidate with highest number of votes wins.
   Britain, US, India, Canada
2. Plurality with runoff : two stages : the top 2 most voted candidates advance to second stage where the highest voted candidate wins
   - French presidential election
3. Approval : voters submit only approved candidates
4. Scoring rule : $(s_1, \dots, s_m)$ for each candidate
   - highest scored candidate wins
5. Maximin : Candidate with largest margin wins
6. Copeland : Candidate with maximum pairwise wins wins.

Why so many voting rules? Democracy — Proportional representation / Desirable axioms