Core limitation 1: many solutions - set-valued

Shapley Value

- Single-valued solution concept.
- Based on axioms (known as Shapley axioms) similar to Nash bargaining.

Notation: ϕ be a single-valued solution concept

$\phi_i(N, v)$ is called the allocation of player $i \in N$.

Axioms:

① Efficiency: A solution concept ϕ satisfies efficiency if for every TU game (N, v)

$$\sum_{i \in N} \phi_i(N, v) = v(N). \quad \text{[no-wastage property]}$$

② Symmetry:

Defn: Players i and j are symmetric players if for every coalition $S \subseteq N \setminus \{i, j\}$,

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

symmetric players give same marginal contribution to every coalition.

Defn: A solution concept ϕ satisfies symmetry if for every coalitional game (N, v) and every pair of symmetric players i and j in the game

$$\phi_i(N, v) = \phi_j(N, v).$$

[equal treatment for equals]

(3) Null player property :

Defn: A player i is called a null player in (N, v) if for every $S \subseteq N$, $v(S) = v(S \cup \{i\})$

– clearly $v(i) = 0$.

Defn: A solution concept ϕ satisfies null player property if for every coalitional game (N, v) and for every null player i, $\phi_i(N, v) = 0$.

(4) Additivity: A solution concept ϕ satisfies additivity if for every pair of coalitional games (N, v) and (N, w)

$$\phi(N, v+w) = \phi(N, v) + \phi(N, w).$$

" To what extent a single game is equivalent to playing two games individually? "

This property says independence – the share/allocation from a game with added valuation is exactly the same as ~~the~~ playing the games independently and collecting the rewards.

Examples

(1) $\Psi_i(N, v) = v(i)$

additivity – $\Psi_i(N, v+w) = (v+w)(i) = v(i) + w(i)$

$= \Psi_i(N, v) + \Psi_i(N, w)$

Symmetry – If $\forall S \subseteq N \setminus \{i, j\}$ $v(S \cup \{i\}) = v(S \cup \{j\})$

then apply $S = \phi \Rightarrow v(i) = v(j) \Rightarrow \Psi_i(N, v) = \Psi_j(N, v)$.

null player – for every null player $v(i)=0$, $\Psi_i(N,v)=0$.

efficiency – not necessarily. $\sum v(i) \neq v(N)$.

(2) A player i is called a <u>dummy player</u> if

$v(S \cup \{i\}) = v(S) + v(i) \quad \forall S \subseteq N \setminus \{i\}$

Every null player is a dummy player.

Let $d(v)$ be the number of dummy players in (N,v)

Consider a solution concept.

$$\Psi_i(N,v) = \begin{cases} v(i) + \dfrac{v(N) - \sum_{j \in N} v(j)}{n - d(v)} & i \text{ is not a dummy player} \\ v(i) & i \text{ is a dummy player.} \end{cases}$$

efficiency – yes, Null – yes, null is dummy – gets zero.

symmetry – clearly if both the players are dummy

Then this is true, both players are non-dummy

then also true.

What about i non-dummy and j dummy

Can they be symmetric?

– ~~let D be the~~ dummy set, clearly ~~$v(D) = \sum_{j \in D} v(j)$~~

– if $i \in ND$ and $j \in D$ and they are symmetric

Then we have $v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i,j\}$

clearly $v(i) = v(j)$ $\quad = v(S) + v(j)$

$S = \phi$ $\quad = v(S) + v(i)$

$\Rightarrow v(S \cup \{i\}) + v(j) = v(S) + v(j) + v(i)$

$\Rightarrow v(S \cup \{j\} \cup \{i\}) = v(S \cup \{j\}) + v(i)$

$\Rightarrow v(\bar{S} \cup \{i\}) = v(\bar{S}) + v(i) \quad \forall \bar{S} \subseteq N \setminus \{i\}$

$\Rightarrow i$ is a dummy player $\rightarrow\leftarrow$

(8-4) does not satisfy additivity.

$v(1) = v(2) = v(3) = v(1,2) = v(1,3) = 0$

$v(2,3) = v(1,2,3) = 1$ ------ (1)

and $u(1) = u(2) = u(3) = u(1,3) = 0$, $u(1,2) = u(2,3) =$

$u(1,2,3) = 1$

$(N, v) \to$ player 1 is the dummy $\quad (0, \frac{1}{2}, \frac{1}{2})$

(N, u) no dummy $\quad (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Consider $(N, u+v)$

$(u+v)(1) = (u+v)(2) = (u+v)(3) = (u+v)(1,3) = 0$

$(u+v)(1,2) = 1$, $(u+v)(2,3) = 2 = (u+v)(1,2,3)$

no-dummy $\quad \psi(N, u+v) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\psi(N,u) + \psi(N,v) = \left(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right)$.

(3) $\psi_i(N, v) = \max\limits_{\{S : i \notin S\}} \left(v(S \cup \{i\}) - v(S)\right)$

symmetry, null player ✓

efficiency, additivity ×.

(4) $\psi_i(N, v) = v(1, 2, \ldots, i) - v(1, 2, \ldots, i-1)$

efficiency – ✓

additivity – ✓

null player – ✓

not symmetry – consider the first game of example (2)

eqn. (1) $\quad \psi(N,v) = (0, 0, 1)$

but 2, 3 are symmetric.

The solution concept can be defined for any order of the players (not just the identity order)

Say $\Pi(N)$ denote the set of all possible orders.

~~Defn~~ Call $\pi \in \Pi(N)$ to be one ordering/permutation of the players.

Call the predecessor of player i in the permutation π as

$$P_i(\pi) = \{j \in N : \pi(j) < \pi(i)\}$$

$$P_i(\pi) = \phi \text{ if } \pi(i) = 1.$$

$$P_i(\pi) \cup \{i\} = P_k(\pi) \Leftrightarrow \pi(k) = \pi(i) + 1.$$

Now, we can define the solution concept

$$\varphi_i^{\pi}(N, v) = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)).$$

As we saw in ~~ex~~ example 4 before, this solution concept satisfies efficiency, null-player property, additivity, not symmetry.

Shapley value

Q. Is there a solution concept that satisfies all ~~4~~ four properties?

A. Yes, and it is unique.

Defn (Shapley 1953) The Shapley value is the solution concept Sh defined as,

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \left[v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) \right] \quad \forall i \in N.$$

A simple average over all ψ_i^{π}'s

Hence $$Sh_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \psi^{\pi}(N,v)$$

Theorem: The Shapley value is the only single-valued solution concept satisfying efficiency, additivity, null player, and symmetry.

An equivalent formula for Shapley value

$$\frac{1}{n!} \sum_{\pi \in \Pi(N)} \psi_i^{\pi}(N,v) = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \sum_{\substack{\pi \in \Pi(N) \\ P_i(\pi) = S}} \left(v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))\right)$$

$$= \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \sum_{\substack{\pi \in \Pi(N) \\ P_i(\pi) = S}} \left(v(S \cup \{i\}) - v(S)\right)$$

$$= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right]$$

Interpretation: "Average marginal contribution to all other coalitions"

Proof of Shapley theorem:

Part 1: Shapley value satisfies the four axioms.

Each of ψ^{π} satisfies efficiency, additivity, null player, so does their average [Exercise]

Symmetry: Let i and j be two symmetric players. Given a permutation π, define the following permutation $f(\pi)$, s.t.

$$f : \Pi(N) \to \Pi(N)$$

$f(\pi)$ just swaps the positions of i and j

$$(f(\pi))(k) = \begin{cases} \pi(j) & \text{if } k = i \\ \pi(i) & \text{if } k = j \\ \pi(k) & \text{if } k \neq i, j \end{cases}$$

Clearly f is a bijective mapping.

Claim: $\psi_i^{\pi}(N, v) = \psi_j^{f(\pi)}(N, v)$

$$\Leftrightarrow v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) = v(P_j(f(\pi)) \cup \{j\}) - v(P_j(f(\pi))) \quad \text{--- (2)}$$

Case 1: Player i appears before j in π, i.e., $j \notin P_i(\pi)$

clearly $P_j(f(\pi)) = P_i(\pi) \Rightarrow v(P_j(f(\pi))) = v(P_i(\pi))$

and since i and j are symmetric players,

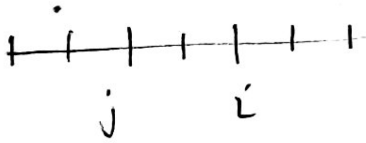
$$v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$$

equ (2) is holds.

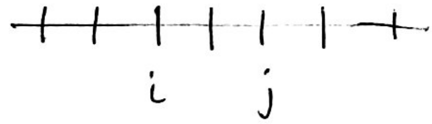
Case 2: Player i appears after j in π, i.e., $j \in P_i(\pi)$

here

π



$f(\pi)$



$$P_i(\pi) \cup \{i\} = P_j(f(\pi)) \cup \{j\}$$

$$\Rightarrow \quad v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$$

also $P_i(\pi) \setminus \{j\} = P_j(f(\pi)) \setminus \{i\}$ both $\not\ni i, j$

since i, j are symmetric

$$v(P_i(\pi)) = v(P_j(f(\pi)))$$

$\square$

Eq. (2) holds