CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

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Solution 1

(a)

Consider following preference profiles:

P_1	P_2	P'_1	P'_2	P_1	$\hat{P}_2$	P'_1	$\hat{P}_2$
a	c	b	a	a	c	b	c
b	b	a	b	b	a	a	a
c	a	c	c	c	b	c	b

Given, f is strategyproof (SP) and hence monotone (MONO)

Consider transition from (P_1, P_2) to $(P_1, \hat{P_2})$. The relative position of alternative a is getting weakly better and also $f(P_1, P_2) = a$. Hence using monotonicity of f,

$$\boxed{f(P_1, \hat{P}_2) = a}$$

Also,

$$\boxed{f(P'_1, \hat{P}_2) = b}$$

because if $f(P'_1, \hat{P}_2) = c$, then at profile $(P'_1, \hat{P}_2)$, agent 1 can report P_1 instead of P'_1 and get outcome $f(P_1, \hat{P}_2) = a$ which it prefers over current outcome $f(P'_1, \hat{P}_2) = c$. This contradicts SP of f. So, since $f(P'_1, \hat{P}_2) \in \{b, c\}$ and cannot be c, it has to be b.

Now consider transition from $(P'_1, \hat{P}_2)$ to (P'_1, P'_2). The relative position of alternative b gets relatively better and $f(P'_1, \hat{P}_2) = b$. Using monotonicity of f,

$$\boxed{f(P'_1, P'_2) = b}$$

Hence Proved.

(b)

No, the conclusion will not hold in this restricted domain.

The earlier proof doesn't go through because preference $\hat{P}_2$ cannot exist in this restricted domain which we have used in the proof.

Consider a SCF, that picks the *left-most* peak among the peak of all the agents. i.e

$$f(P) = \min_{i \in N}\{P_i(1)\}$$

where minimum is taken w.r.t order $a < b < c$.

According to this SCF:

$$f(P_1, P_2) = a$$

$$f(P'_1, P'_2) = a$$

Thus, a mechanism that uses above described SCF will have $f(P'_1, P'_2) = a$.

Solution 2

Let

$$S = \{s \in X | s \neq \phi\}$$

i.e. , S is the power set of X, minus empty set.

Also, let R_i be the extended linear ordering of P_i, over S.

Claim: *R_i is in restricted domain*
Proof: Let $a \in X$.

Let

$$\begin{aligned} A &= \{a\} \in S \\ B &= X - \{a\} \in S \end{aligned}$$

NOTE: $B \neq \phi$ because $|X| \geq 2$

Now consider an R_i which satisfies:

$$\begin{aligned} A \quad R_i \quad B \qquad\qquad & \ldots 1 \\ B \quad R_i \quad X \qquad\qquad & \ldots 2 \end{aligned}$$

If $aP_ib \quad \forall b \in X - \{a\}$ (a is most preferred by agent i), then condition 2 above can never be satisfied because $a \in X$, $a \notin B$ and a is most preferred project, always resulting in $X \quad R_i \quad B$.

Otherwise, condition 1 can never be satisfied because $\exists b \in B$ s.t. bP_ia, always resulting in $B \quad R_i \quad A$.

So, such an R_i cannot exist which satisfies both the conditions.

Hence, R_i is in restricted domain.

Now, for *Gibbard-Satterthwaite* rule to apply on SCF $f : R^n \to S$, R must be a unrestricted domain, which it is not. Hence GS rule doesn't apply here.

Solution 3

Consider a preference profile $P = (P_K, P_{-K})$ where say *non-empty* $K \subseteq N$ is trying to manipulate f, a median voter SCF.

Let, $f(P_k, P_{-k}) = a$ be the median of peaks in P and the phantom peaks.

NOTE: We can rule out the agent with peak a being in manipulating group K because it cannot get better off with shifting of outcome either way.

Consider another preference profile $P' = (P'_K, P_{-K})$

Let, $M_1 = \{i \in K \quad | \quad P_i(1) < a < P'_i(1)\}$ be the subset of K who moved their peak from left of a to its right (according to natural ordering $<$ of outcomes)

Similarly, $M_2 = \{i \in K \quad | \quad P'_i(1) < a < P_i(1)\}$ be the subset of K who moved their peak from right of a to its left.

Case 1: $|M_1| - |M_2| = 0$
The median of peaks doesn't change.

$$f(P') = f(P) = a$$

Case 2: $|M_1| - |M_2| > 0$
The new median is to the right of a. i.e.

$$f(P) = a < f(P') \quad \ldots 1$$

Consider some agent $j \in M_1$. By definition of M_1,

$$P_j(1) < a = f(P) \quad \ldots 2$$

So using 1 and 2,

$$P_j(1) < a = f(P) < f(P')$$

By definition of peak,

$$f(P) P_j f(P')$$

So, $j \in K$ doesn't benefit from this manipulation.

Case 3: $|M_1| - |M_2| < 0$
Symmetric argument can be given as in case 2.

Thus for any $P = (P_k, P_{-K})$, $\nexists P'_K$ s.t.

$$f(P'_K, P_{-K}) P_i f(P) \quad \forall i \in K$$

Hence, median voter SCF is *group strategy-proof.*