CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 33: November 1, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

33.1 Recap

We have studied some basic properties and results dealing with convex functions. We will use lemma 3 and lemma 4 explained in previous lecture to prove Myserson's theorem.

33.2 Monotonicity

Definition 33.1 ***Monotonicity****: An allocation rule is non-decreasing if for every agent $\forall\ i \in N$ and $\forall\ t_{-i} \in T_{-i}$ we have $f_i\left(t_i, t_{-i}\right) \geq f_i\left(s_i, t_{-i}\right)\ \forall\ t_i\ ,\ s_i \in T_i$ with $t_i > s_i$.*

In words if the types of other agents are held fixed then the probability of allocation of the object to the agent weakly increases with the increase in its valuation.

33.3 Myerson's Theorem

Theorem 33.2 *Suppose $T_i = [0, b_i]\ \forall\ i \in N$ and the valuations are in product form. An allocation rule $f : T \to \Delta A$ and a payment rule $(p_1, p_2, ..., p_n)$ is DSIC iff*

1. *f is non-decreasing.*
2. *Payment is given by*

$$p_i\left(t_i, t_{-i}\right) = p_i\left(0, t_{-i}\right) + t_i f_i\left(t_i, t_{-i}\right) - \int_0^{t_i} f_i\left(x_i, t_{-i}\right)\ dx_i$$

Proof: Consider the utility of agent i

$$u_i\left(t_i, t_{-i}\right) = t_i f_i\left(t_i, t_{-i}\right) - p_i\left(t_i, t_{-i}\right)$$

$$u_i\left(s_i, t_{-i}\right) = s_i f_i\left(s_i, t_{-i}\right) - p_i\left(s_i, t_{-i}\right)$$

now since (f, p) is DSIC $\forall s_i, t_i, t_{-i}$

$$u_i\left(t_i, t_{-i}\right) = t_i f_i\left(t_i, t_{-i}\right) - p_i\left(t_i, t_{-i}\right)$$

$$\geq t_i f_i\left(s_i, t_{-i}\right) - p_i\left(s_i, t_{-i}\right)$$

$$= s_i f_i(s_i, t_{-i}) + f_i(s_i, t_{-i})(t_i - s_i) - p_i(s_i, t_{-i})$$
$$= u_i(s_i, t_{-i}) + f_i(s_i, t_{-i})(t_i - s_i)$$

define g $(t_i) = u_i(t_i, t_{-i})$ $\phi(t_i) = f_i(t_i, t_{-i})$

$$g(t_i) \geq g(s_i) + \phi(s_i)(t_i - s_i)$$

$\phi(s_i)$ is a subgradient of g at s_i

Convexity of g

pick x_i ,$z_i \in T_i$ define $y_i = \lambda x_i + (1-\lambda) z_i$ where $\lambda \in (0,1)$

DSIC implies

$$g(x_i) \geq g(y_i) + \phi(y_i)(x_i - y_i)$$
$$g(z_i) \geq g(y_i) + \phi(y_i)(z_i - y_i)$$

Multiply the eqn 1 with λ and eqn 2 with $(1-\lambda)$ and add both eqns

$$\lambda g(x_i) + (1-\lambda) g(z_i) \geq g(y_i) + \phi(y_i)[\lambda x_i + (1-\lambda) z_i - y_i$$

$$\Rightarrow \lambda g(x_i) + (1-\lambda) g(z_i) \geq g(y_i)]$$

$\Rightarrow$ g is convex

Now since g is convex we can apply leema 3 discussed in last lecture By leema 3 since the subgradient of a convex function is non- decreasing. As $\phi \equiv f_i(., t_{-i})$. Therefore $f_i(., t_{-i})$ is non-decreasing.
By leema 4

$$g(t_i) = g(0) + \int_0^{t_i} \phi(x)\ dx$$

$$\Rightarrow u_i(t_i, t_{-i}) = u_i(0, t_{-i}) + \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

$$\Rightarrow t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = -p_i(0, t_{-i}) + \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

$$\Rightarrow p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

$(\Leftarrow)$ For the reverse direction

It is given that allocation rule is monotone and payment rule is given by

$$\Rightarrow p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

We need to prove that f is DSIC

Let t_i be the true type of player and s_i be the misleading reported type by the player

$$t_i f_i\left(t_i, t_{-i}\right) - p_i\left(t_i, t_{-i}\right) = t_i f_i\left(t_i, t_{-i}\right) - p_i\left(0, t_{-i}\right) - t_i f_i\left(t_i, t_{-i}\right) + \int\limits_0^{t_i} f_i\left(x_i, t_{-i}\right)\ dx_i$$

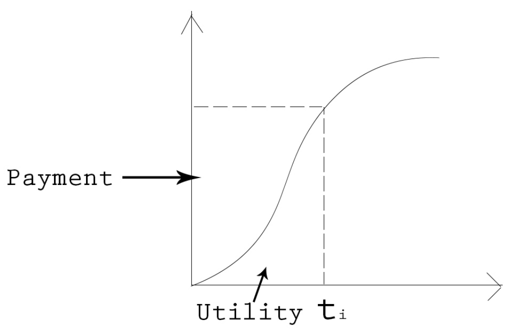
$$t_i f_i\left(s_i, t_{-i}\right) - p_i\left(s_i, t_{-i}\right) = t_i f_i\left(s_i, t_{-i}\right) - p_i\left(0, t_{-i}\right) - s_i f_i\left(s_i, t_{-i}\right) + \int\limits_0^{s_i} f_i\left(x_i, t_{-i}\right)\ dx_i$$

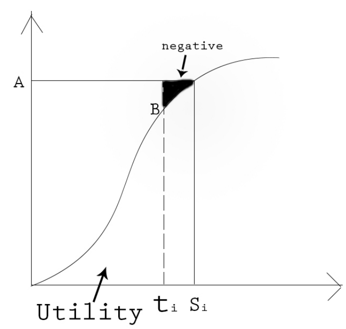
$$t_i f_i\left(t_i, t_{-i}\right) - p_i\left(t_i, t_{-i}\right) - \left[t_i f_i\left(s_i, t_{-i}\right) - p_i\left(s_i, t_{-i}\right)\right] = (s_i - t_i) f_i(s_i - t_i) + \int\limits_{s_i}^{t_i} f_i\left(x_i, t_{-i}\right)\ dx_i \geq 0$$

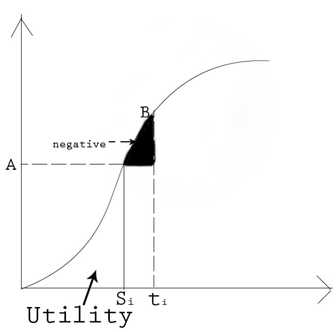
From Graphs we can see that the quantity on RHS of the above equality is always greater than or equal to zero.

We can see that when reported type is greater than the true type then the quantity on the RHS of the equality turns out to be the shaded region in the graph since $(s_i - t_i) f_i(s_i - t_i)$ is the rectangle and the integral term is area under the curve from s_i to t_i but since $s_i > t_i$ the integral turns out to be negative and the total sum is equivalent to the shaded region of the graph which is positive. The above explanation is true since f is monotone.

Similar observation for the case when $s_i < t_i$ leads to the conclusion that f is DSIC.







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