Games : Strategic interaction between decision making agents that are rational and intelligent.

Game Theory : Study of such interactions

Example : Prisoner's dilemma

1 \ 2	~~Confess~~ Cooperate	Defect
C	−2, −2	−10, 0
D	0, −10	−5, −5

Agent perspective

Predictive guarantees

Mechanism Design : Designer's perspective

Prescriptive guarantees

Example : Two women claiming to be the mother of a child - comes to the king to decide.

The women perfectly know the truth - the king does not. Can he design a "game" such that its outcome is the desired one - true mother gets the child.

Solution given in the mythology : King orders to cut the child and give to each of them.

Real mother withdraws her claim.

King awards her the child.

That is an example of a mechanism.

Flaw : The mechanism does not commit to its declared principle of operation.

The mechanism will not work a second time.

1-2

A mechanism must be committed to its rules and must be reproducible.

Proposal 2: If every woman was given an option to pay for the baby.

Rule: Highest bidder wins and pays the losing bid.

Assuming that the real mother's value for the child is much higher than the value of the false mother

- the child goes to its rightful mother and at a small payment.
- But this is reproducible and committed to its rules.

Mechanism: Rules of a game such that desired outcomes result in the equilibrium of the game.

Formal model:

$N = \{1, 2, \dots, n\}$ set of agents/players

X = set of outcomes

θ_i : private information of agent i, also called type.

Θ_i : set of all possible types of i.

$u_i : X \times \Theta_i \to \mathbb{R}$ utility of agent i

Examples:

① Voters satisfaction with candidates denoted by θ_i. ~~Say~~ for agent 1, value of 1 when a is elected is $u_1(a, \theta_1)$ which is more than $u_1(b, \theta_1)$ and $u_1(c, \theta_1)$, $u_1(a, \theta_1) > u_1(b, \theta_1) > u_1(c, \theta_1)$

If type changes, the utility changes too

$u_1(c, \theta_1') > u_1(b, \theta_1') > u_1(a, \theta_1')$

② Single object allocation (indivisible)

outcome $x \in X$ is a tuple (a, p)

a is the allocation and p is the payment vector.

θ_i denotes agent i's value for the object.

$$a = (a_1, \ldots, a_n), \quad p = (p_1, \ldots, p_n)$$

$$a_i \in \{0,1\}, \quad \sum_{i=1}^{n} a_i \leq 1$$

$$u_i((a,p), \theta_i) = a_i \theta_i - p_i$$

The objective of a mechanism is captured via a social choice function (SCF).

$$f : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \rightarrow X$$

The planner/social decision maker wants to take this decision – given a type profile take a certain outcome – but he does not know the true types – private to the agents. Therefore a mechanism is needed to uncover the types truthfully – we call that a mechanism <u>implements</u> a social choice function.

Back to the King-Baby example. The types are the true mother's identity and the goal is to <u>assign the baby to the true mother</u>, but (θ_1, θ_2) are unknown to the king – hence he can ~~take ad~~ use the mechanism second price auction to implement this SCF.

(1-4)

Mechanism

Defn: A *mechanism* is a collection of message spaces and a decision rule, $\langle M_1, M_2, \ldots, M_n, g\rangle$

- M_i is the message space for agent i.
- $g : M_1 \times M_2 \times \cdots \times M_n \to X$

A mechanism is called *direct* when $M_i = \Theta_i$, $g = f$.

Defn: In a mechanism $\langle M, g\rangle$, a message m_i is *weakly dominant* for agent i at θ_i if

$$u_i(g(m_i, \underline{m}_i), \theta_i) \geq u_i(g(m_i', \underline{m}_i), \theta_i) \quad \forall m_i' \in M_i$$
$$\forall \underline{m}_i \in M_{-i}$$

Defn: A SCF $f : \Theta \to X$ is *implemented in dominant strategies* by $\langle M, g\rangle$ if

(1) $\exists$ message mappings $m_i : \Theta_i \to M_i$ s.t. $m_i(\theta_i)$ is a dominant strategy for agent i at θ_i, $\forall \theta_i \in \Theta_i$ $\forall i \in N$

(2) $g(m_i(\theta_i), \underline{m}_i(\underline{\theta}_i)) = f(\theta_i, \underline{\theta}_i)$, $\forall \theta \in \Theta$.

f is dominant strategy implementable by $\langle M, g\rangle$.

Defn: A direct mechanism is strategyproof (or dominant strategy incentive compatible, DSIC) if

$$u_i(f(\theta_i, \underline{\theta}_i), \theta_i) \geq u_i(f(\theta_i', \underline{\theta}_i), \theta_i) \quad \forall \theta_i, \theta_i' \in \Theta_i$$
$$\forall \underline{\theta}_i \in \underline{\Theta}_i$$
$$\forall i \in N.$$

Revelation principle: If f is DSI, then f is DSIC.

Cooperative Game Theory

A game paradigm where agents can communicate and jointly fix strategies. However, there is a threat to the jointly fixed strategies, whether it is self-enforcing. Will they not gain by breaking the agreement?

E.g. Prisoner's dilemma, both prisoners can jointly decide to play "cooperate", but one agent can deviate and make her better off. The self-enforcing criteria in cooperative games come from the idea of correlated equilibrium.

Correlated equilibrium concept was introduced by Robert Aumann in 1974.

Ex1: Traffic signal.

1 \ 2	Drive	Stop
D	−10, −10	5, 0
S	0, 5	0, 0

Two pure Nash equilibrium
- good prediction if the players are non-cooperative.
- but how should an individual driver know which equilibrium to play? need for a mediator.
- traffic police / lights do this role of a mediator.
- given the trusted mediator gives a specific "suggestion" to each of the players, is it best for them to follow this?
- depends on how the mediator picks the suggestions
  - the distribution over the strategy profiles
  - e.g. if the traffic lights pick (S,D) and (D,S) w.p. $\frac{1}{2}$ each, perhaps it is best for the drivers to follow this.

Ex 2: Game Selection Problem

1 \ 2	C	F
C	2,1	0,0
F	0,0	1,2

This game has two pure strategy and one mixed strategy Nash equilibrium.

$\left(\frac{2}{3},\frac{1}{3}\right) \to 1 \quad \left(\frac{1}{3},\frac{2}{3}\right) \to 2$

Expected utility of every player $= \frac{2}{3}$

Correlated Strategy

Defn: A correlated strategy is a mapping $\pi : S \to [0,1]$ s.t. $\sum_{s \in S} \pi(s) = 1$, where $S = S_1 \times S_2 \times \cdots \times S_n$, S_i denotes the strategy of player i.

A correlated strategy π is a joint probability distribution over the strategy profiles.

A correlated strategy become correlated equilibrium if it becomes 'self enforcing', i.e., no player 'gains' by deviating from it.

π is a common knowledge.

Correlated equilibrium (Aumann '74)

Defn: A correlated equilibrium is a correlated strategy π s.t. $\forall s_i \in S_i$ and $\forall i \in N$

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i})\, u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i.$$

Interpretations

1. Player i does not gain any advantage (in expected utilities) if she deviated from the suggested action, when others listen to the suggestion of the trusted mediator.

2. A correlated equilibrium is a randomization device (e.g. a dice or coin) which gives a random outcome of a strategy profile, but a player only observes the strategy corresponding to her. Given that observation, she computes her expected utility - if no other strategy gives her a strict better utility, and if this is true for every agent, the randomization device is a correlated equilibrium.

Illustration with the examples

1. Game selection problem.

Consider the correlated strategy

$$\pi(C,C) = \frac{1}{2} = \pi(F,F), \quad \pi(C,F) = \pi(F,C) = 0.$$

Suppose player 1 is suggested to play F.

$$\sum_{s_{-1} \in S_{-1}} \pi(F, s_{-1})\, u_1(F, s_{-1}) = \pi(F,F)\, u_1(F,F) + \pi(F,C)\, u_1(F,C)$$

$$= \frac{1}{2} \cdot 1 + 0 \cdot 0 = \frac{1}{2} \quad \text{[utility from following]}$$

$$\sum_{s_{-1} \in S_{-1}} \pi(F, s_{-1})\, u_1(C, s_{-1}) = \pi(F,F)\, u_1(C,F) + \pi(F,C)\, u_1(C,C)$$

$$= 0 \quad \text{[utility from violating the suggestion]}$$

similar case for ~~player~~ the player when C is suggested and for player 2.

(2-4)

Expected utility of every player $\frac{1}{2}(2+1) = 3/2$
as opposed to $\frac{2}{3}$ in the MSNE.

(2) Traffic signal

1 \ 2	S	G
S	0,0	1,2
G	2,1	-10,-10

Consider correlated strategy

$\pi(SS) = \pi(SG) = \pi(GS) = \frac{1}{3}$

Player 1 suggested to stop,

$$\sum_{s_{-1} \in S_{-1}} \pi(S, s_{-1})\, u_1(S, s_{-1}) = \pi(SS)\,u_1(SS) + \pi(SG)\,u_1(SG)$$
$$= \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 = \frac{1}{3}$$

by violating

$$\sum_{s_{-1} \in S_{-1}} \pi(S, s_{-1})\, u_1(G, s_{-1}) = \pi(SS)\,u_1(GS) + \pi(SG)\,u_1(GG)$$
$$= \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot(-10) = -\frac{8}{3}$$

Similar conclusions when suggested to go and for player 2.

Interpretation from a best response viewpoint.

The best response set (of i for s_i) in a correlated strategy π is

$$B_i(\pi, s_i) = \arg\max_{\tilde{s}_i \in S_i} \left[\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i})\, u_i(\tilde{s}_i, s_{-i}) \right]$$

π is a CE if $\forall s_i \in S_i \; \forall i \in S_i$

$$s_i \in B_i(\pi, s_i).$$

Computation of Correlated Equilibrium.

Feasibility LP

Variables : $\pi(s)$, $\forall s \in S$.

$$\sum_{s_{-i} \in S_{-i}} \pi(s)\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s)\, u_i(s_i', s_{-i}) \quad \forall s_i, s_i' \quad \forall i \in N$$

-- $O(nm^2)$ inequalities

$$\pi(s) \geq 0 \quad \forall s \in S \qquad \text{-- } O(m^n)$$

$$\sum_{s \in S} \pi(s) = 1 \qquad \text{--- } 1$$

Compare this with MSNE computation which needed all supports of every player to be enumerated hence complexity $O(2^{mn})$ – exponentially larger than CE computation.

- CE guarantees a cooperative self-enforcing decision using a trusted mediator
- Much easier to compute
- Quite natural in several game settings than a non-cooperative solution.

Axiomatic Bargaining

- Earliest results in cooperative game theory
- Axioms represents the goal of a designer when a solution is proposed,
  e.g., how to divide a joint profit ?

(2-6)

Axiomatic bargaining was introduced by Nash 1950

Bargaining refers to

- two individuals have the possibility of concluding to a mutually beneficial agreement
- there is a conflict of interest on which agreement to conclude
- no agreement may be imposed without every player's approval.

The payoff of agents ~~depend on~~ has two components

- payoffs in case the negotiation fails
- payoffs jointly feasible by their negotiation/arbitration.

Examples:

1. Management - Labor arbitration
2. International relations
3. Property settlement.

The Bargaining Problem

Two person bargaining problem consists of a pair (F, v), where F is the feasible set and v is the 'disagreement point'.

- F is a closed, convex subset of $\mathbb{R}^2$
- $v = (v_1, v_2) \in \mathbb{R}^2$ represents the disagreement payoff allocation for the two players, default point.
- $F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1, x_2 \geq v_2\} \neq \phi$, bounded.

Discussions on the assumptions:

- Convexity due to the natural reason of correlated equilibrium. A convex combination of two potential actions that gives that utility will also be a correlated equilibrium.
- Closedness - sequence of bargaining converving to an infeasible solution.
- $\exists$ feasible solution points above the disagreement and not unbounded.

Axiomatic Bargaining

- Two agents
- negotiating on a mutually beneficial agreement which is self enforcing.
- Desirable properties : given by axioms

Axiom 1: Strong Efficiency

Problem setup : $\langle F, v \rangle$ - bargaining problem

F : feasible set , v : disagreement point.

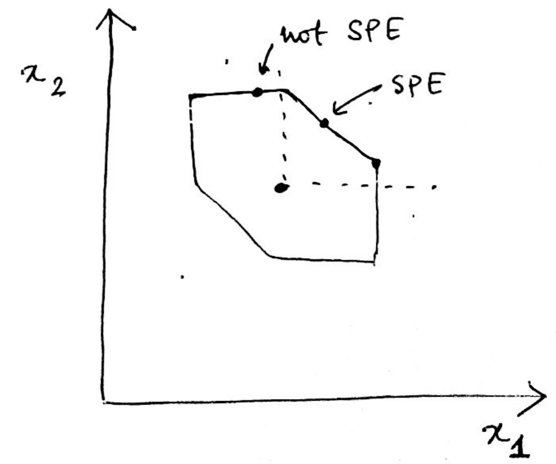
$f(F, v)$: bargaining solution

$$f(F,v) = (f_1(F,v), f_2(F,v)) \quad , \quad f_i(F,v) \in \mathbb{R} \ , i = 1,2.$$

Given a feasible set F, we say an allocation $x = (x_1, x_2) \in F$ is strongly (Pareto) efficient if $\nexists$ another $y = (y_1, y_2) \in F$ s.t. $y_1 \geq x_1$ and $y_2 \geq x_2$ with the inequality being strict for at least one agent.

- an allocation $x = (x_1, x_2) \in F$ is weakly (Pareto) efficient if $\nexists$ another $y \in F$ s.t. $y_1 > x_1$ and $y_2 > x_2$.

We want the bargaining solution to be strongly efficient. Implies that there does not exist another allocation which will make both the players better off and at least one of them strictly.



Axiom 2 : Individual Rationality

$$f(F, v) \geq v$$

$$\Rightarrow f_i(F,v) \geq v_i \ , \forall i = 1,2.$$

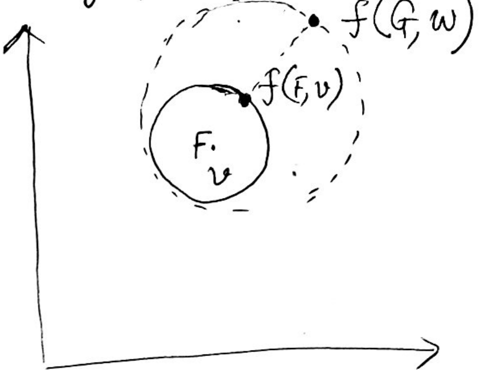
Axiom 3: Scale Covariance

Consider an affine transformation of the feasible space F, i.e., let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1, \lambda_2 > 0$ and $G := \{ (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F \}$

and $w := (\lambda_1 v_1 + \mu_1, \lambda_2 v_2 + \mu_2)$

[scaling and translating the feasible space and the disagreement point]

Then $(\lambda_1 f_1(F, v) + \mu_1, \lambda_2 f_2(F, v) + \mu_2)$ must be a solution of the scaled bargaining problem (G, w).

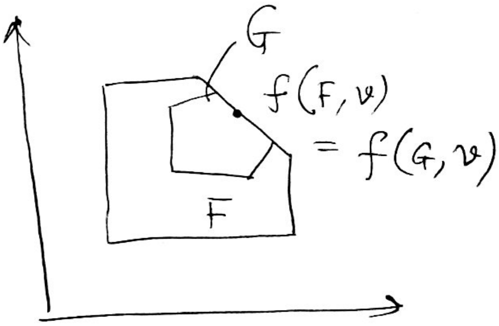


Axiom 4: Independence of Irrelevant Alternatives.

For any closed, convex set G

$G \subseteq F$ and $f(F, v) \in G$

$\Rightarrow f(G, v) = f(F, v)$.



Axiom 5: Symmetry

If positions of players 1 and 2 are symmetric, the solution should treat them symmetrically

$v_1 = v_2$ and $\{ (x_2, x_1) : (x_1, x_2) \in F \} \subseteq F \Rightarrow f_1(F, v) = f_2(F, v)$

The Nash Bargaining Solution

<u>Thm</u>: Given a two person bargaining problem (F, v), there exists a unique solution function f that satisfies axioms 1–5, and is given by

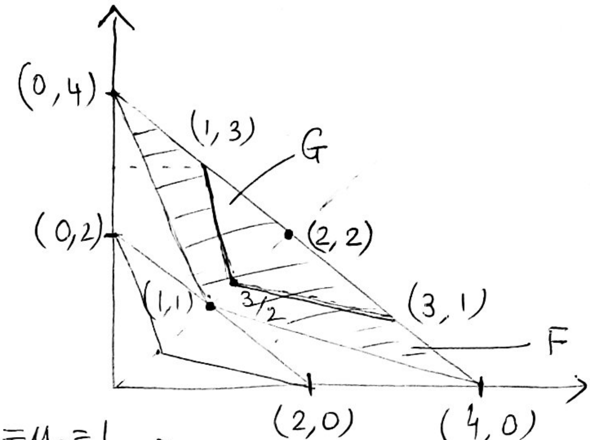
$$f(F,v) \in \underset{\substack{(x_1,x_2)\in F \\ x_1 \geq v_1,\ x_2 \geq v_2}}{\operatorname{argmax}} \left((x_1 - v_1)(x_2 - v_2) \right)$$

Illustrative example

F : convex hull of
$(4,0)$, $(1,1)$, $(0,4)$

$v = (1,1)$

$f(F,v) = (2,2)$ PE, IR, Sym.



Obtain G by $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\mu_1 = \mu_2 = 1$ → remains $(2,2)$ scale covariance

" H by $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\mu_1 = \mu_2 = 0$
→ $(1,1)$

Proof of Bargaining Theorem

We will consider a special (but almost general) subclass where $\exists$ at least one $y \in F$ s.t. $y_1 > v_1$ and $y_2 > v_2$.
We call such bargaining problem as "<u>essential</u>" bargaining problem.

<u>Defn</u>: A function defined over a non-empty convex set $f : S \to \mathbb{R}$, S is convex and non-empty, is <u>quasi-concave</u>

if $\quad f(\lambda x + (1-\lambda) y) \geq \min \{ f(x), f(y) \} \ \forall x, y \in S, \forall \lambda \in [0,1]$

f is strictly convex if

$$f(\lambda x + (1-\lambda) y) > \min \{ f(x), f(y) \} \ \forall x, y \in S,\ x \neq y \quad \forall \lambda \in (0,1)$$

(3-4)

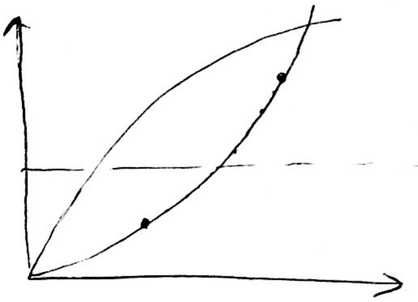
Alternative definition

upper contour set of f.

$U_a = \{x \in S : f(x) \geq a\}$

is (strictly) convex for every a, if f is (strictly) quasi-concave.

Examples of QC functions



Observation:

$N(x) = (x_1 - v_1)(x_2 - v_2)$ is strictly quasi-concave

for essential bargaining games in the region $x_1 \geq v_1, x_2 \geq v_2$.

- easier to see from the alternative definition.

The Nash Bargaining Theorem states that the five axioms are satisfied for the unique bargaining solution

$$f^N(F, v) = \underset{(x_1,x_2) \in F \cap \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq v_1, x_2 \geq v_2\}}{\arg\max} N(x) \quad \text{------} \quad (1)$$

$N(x)$ is called the Nash product.

Fact: A strict quasi-concave function has unique maxima.

Proof: (Part 1) Solution $x^* = (x_1^*, x_2^*)$ of (1) satisfies the 5 axioms

(1) Strong efficiency:

$$\text{Given } (x_1^*, x_2^*) = \underset{\substack{(x_1,x_2) \in F \\ x_1 \geq v_1, x_2 \geq v_2}}{\arg\max} N(x)$$

Suppose $\exists$ $(\hat{x}_1, \hat{x}_2)$ s.t. $\hat{x}_1 \geq x_1^*$ and $\hat{x}_2 \geq x_2^*$ at least one of them is strict.

Since we consider essential bargaining problem
$N(x^*) > 0$, but by assumption

$$N(\hat{x}_1, \hat{x}_2) > N(x^*) > 0$$

which is a contradiction to the definition of x^*.

(2) Individual rationality is obvious from the definition of x^*.

(3) Scale covariance:
Consider $\lambda_1, \lambda_2 > 0$, μ_1, μ_2 and define

$$G = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F\}$$

The Nash product problem in G

$$\max_{\substack{(y_1, y_2) \in G \\ y_1 \geq w_1, y_2 \geq w_2}} (y_1 - w_1)(y_2 - w_2) \quad \text{where } \begin{array}{l} w_1 = \lambda_1 v_1 + \mu_1 \\ w_2 = \lambda_2 v_2 + \mu_2 \end{array}$$

$$\Rightarrow \max_{\substack{(x_1, x_2) \in F \\ x_1 \geq v_1, x_2 \geq v_2}} \lambda_1 \lambda_2 (x_1 - v_1)(x_2 - v_2)$$

The maximum is attained at (x_1^*, x_2^*)
therefore the above problem attains maxima at $(\lambda_1 x_1^* + \mu_1, \lambda_2 x_2^* + \mu_2)$
unique as this is quasi-concave

(4) IIA: $G \subseteq F$ is convex and closed.
(x_1^*, x_2^*) is optimal to (F, v) and let (y_1^*, y_2^*) be optimal to (G, v), also $(x_1^*, x_2^*) \in G$.

Since $G \subseteq F$ $\quad N(x_1^*, x_2^*) \geq N(y_1^*, y_2^*)$

but y^* is optimal in G, $\Rightarrow$ $N(y_1^*, y_2^*) \geq N(x_1^*, x_2^*)$

$\Rightarrow$ $N(x^*) = N(y^*)$

but the optima is unique $\Rightarrow$ $x^* = y^*$.

3-6

(5) Symmetry: Suppose F is symmetric, i.e.

$\{(x_2, x_1) : (x_1, x_2) \in F\} = F$ and $v_1 = v_2 = v$

by definition (x_1^*, x_2^*) maximizes $(x_1^* - v)(x_2^* - v) = N(x_1^*, x_2^*)$

which is same as $N(x_2^*, x_1^*)$. Since optima is unique $x_1^* = x_2^*$.

(Part 2) Given: $f(F, v)$ is a bargaining solution that satisfies all the five axioms

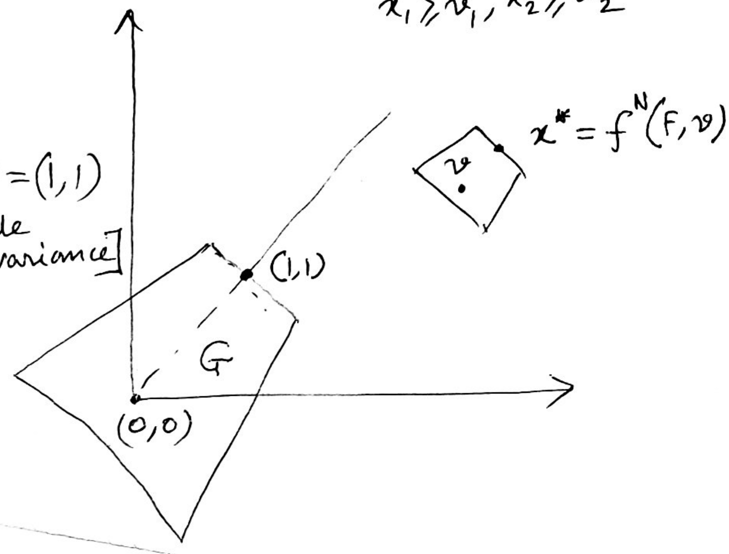
TST: $f(F, v) = f^N(F, v)$; $f^N(F, v) = \underset{\substack{(x_1, x_2) \in F \\ x_1 \geq v_1, x_2 \geq v_2}}{\arg\max} (x_1 - v_1)(x_2 - v_2)$

Plan: $f(F, v) = f^N(F, v)$

$\Leftrightarrow f(G, (0,0)) = f^N(G, (0,0)) = (1,1)$ [both satisfy scale covariance]

finally, need to show

$f(G, (0,0)) = (1,1)$



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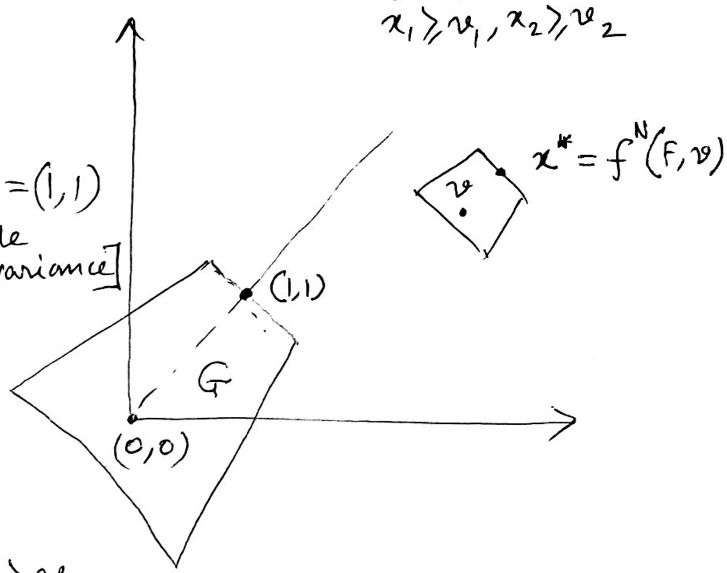


Since this is essential bargaining, i.e., $x_1^* > v_1$, $x_2^* > v_2$

$$L(x_1, x_2) = (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2)$$

$$\lambda_1 = \frac{1}{x_1^* - v_1}, \quad \lambda_2 = \frac{1}{x_2^* - v_2}, \quad \mu_1 = \frac{-v_1}{x_1^* - v_1}, \quad \mu_2 = \frac{-v_2}{x_2^* - v_2}$$

$$L(x_1, x_2) = \left(\frac{x_1 - v_1}{x_1^* - v_1}, \frac{x_2 - v_2}{x_2^* - v_2} \right)$$

$$L(v) = (0,0), \quad L(x^*) = (1,1)$$

$$G = \{ L(x) : x \in F \}$$

$$L(x^*) = f^N(G, (0,0)) = (1,1)$$

Claim : $y_1 + y_2 \leq 2$, $\forall (y_1, y_2) \in G$

Suppose not, then $\exists (y_1, y_2)$ s.t. $y_1 + y_2 > 2$

$f^N(G,(0,0))$ maximizer $y_1 y_2$

G is convex, as F was convex

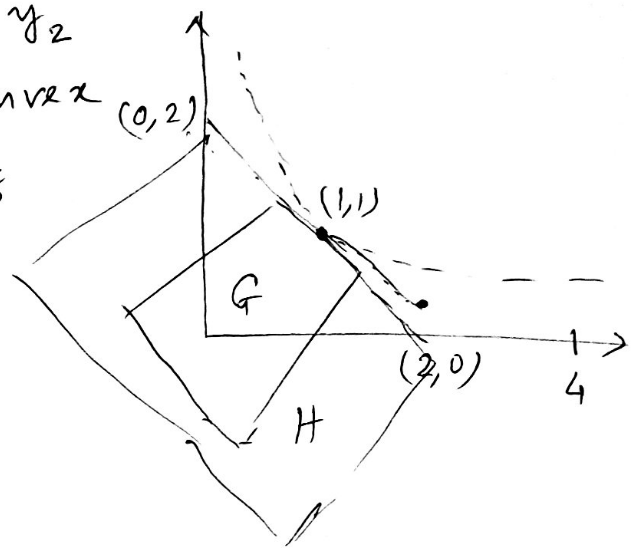
construct $\lambda y + (1-\lambda)(1,1) = z$

pick λ sufficiently small

s.t. the product $z_1 z_2 > 1$

which is a contradiction

to $f^N(G,(0,0))$ being 1.



Enclose G with H s.t. H is symmetric around $x_1 = x_2$ and $G \subseteq H$, with $(1,1)$ on the ~~Pareto~~ boundary of H.

- Strong Pareto Efficiency and symmetry $\Rightarrow$

$$f(H,(0,0)) = (1,1)$$

- IIA $\Rightarrow$ $f(G,(0,0)) = (1,1)$ done!

Exercise: Extend the proof for inessential bargaining problem.

Exercise: Find at least one other solution for any combination of the three properties among SPE, Symmetry, Scale covariance, IIA.

Multi-person cooperative games

$(F, (v_1, \ldots, v_n))$ defines the game in this setting

- Nash bargaining solution in this context
  does it reasonably capture the coalitional characteristics?

Examples:

(1) Divide the dollar — version 1:

$N = \{1, 2, 3\}$, want to divide a total wealth of 300.

Each player can propose a division so that the sum ≤ 300.

$F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 300\}$
feasible set

In this version of the game a division is implemented only if all agents agree to that division. Gets zero otherwise.

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{o/w} \end{cases}$$

Every player has equal power in this game.
The disagreement value is zero for every agent.
Hence the Nash bargaining solution is $(100, 100, 100)$
which looks perfectly reasonable — no agent or group can have a profitable deviation.

(2) Version 2:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ 0 & \end{cases}$$

disagreement point $v = (0, 0, 0)$ still.
Hence Nash bargaining solution remains $(100, 100, 100)$
But this does not look reasonable - group $\{1, 2\}$ can profitably deviate from this allocation.

Effective negotiation:

The members of a coalition of players can negotiate effectively (and form an effective coalition) if the players, on realizing that there is a feasible change in their strategies that can benefit all of them, actually makes such a change.

An n-person Nash bargaining solution would be relevant if the only coalition that can negotiate effectively is the grand coalition N.

(3) Version 3 (DTD):

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = x \\ & \text{or } s_1 = s_3 = x \\ 0 & \text{ow} \end{cases}$$

again the disagreement point is the same, however the Nash solution is far from reasonable, both $\{1,2\}$ and $\{1,3\}$ has a profitable deviation. Player 1 seems to have a lot more power in the decision making - therefore he can make very biased offers and other players have no choice but to accept it.

(4) Version 4 - majority voting game:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_j = s_k = (x_1, x_2, x_3) \text{ for some } j \neq k \\ 0 & \text{ow.} \end{cases}$$

Now again the negotiation may continue forever, since for every proposal among a pair of players, the third player has a better proposal for at least one of the players.

Need a better model for coalitional games with 3 or more players and better solution concepts.

Transferable Utility Games (TU Games)

Introduction of a fluid commodity that can transfer utility - this is called money. With this transfer being possible, we can define cooperative game by a characteristic function

$$v : 2^N \to \mathbb{R} \quad , \quad N: \text{set of players}$$

$v(S)$: value of the coalition $S \subseteq N$

$v(\phi) = 0$.

Defn. A Transferable Utility (TU) game is given by the tuple (N, v) where N is the set of players and v is the characteristic function.

Examples: (1) DTD : Ver 1 :

$$v(\{1,2,3\}) = 300 \; , \; v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = v\{1,3\} = v(\{2,3\}) = 0$$

Ver 2 : $v(\{1,2\}) = v(\{1,2,3\}) = 300$, all others $= 0$

Ver 3 : $v(\{1,2\}) = v(\{1,3\}) = v(\{1\,2\,3\}) = 300$, others $= 0$

Ver 4 : $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 300$

(2) Minimum cost spaning tree game.

$v(1) = 10 - 5 = 5$, benefit-cost.

$v(2) = 10 - 1 = 9$

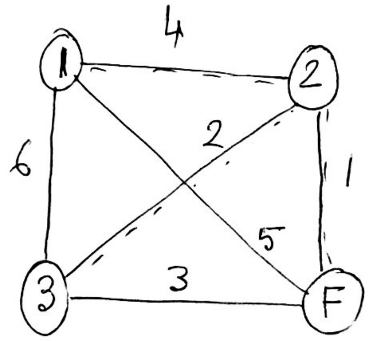
$v(3) = 10 - 3 = 7$

$v(1,2) = 20 - 5 = 15$

$v(2,3) = 20 - 3 = 17$

$v(1,3) = 20 - 8 = 12$

$v(1,2,3) = 30 - 7 = 23$



③ Bankruptcy game (E, c) : $E \geq 0$ is the market value of an estate/company that went bankrupt. The vector c denotes the claim vector of different stakeholders of the estate, $c \in \mathbb{R}^n_{\geq 0}$

Value of a coalition $S \subseteq N$ is

$$v(S) = \left[E - \sum_{i \in N \setminus S} c_i\right]^+ \qquad x^+ := \max\{0, x\}$$

Say $N = \{1,2,3\}$, $c = (10, 50, 70)$, $E = 100$

$v(1) = 0$, $v(2) = 20$, $v(3) = 40$

$v(1,2) = 30$, $v(2,3) = 90$, $v(1,3) = 50$

$v(1,2,3) = 100$

Special classes of TU games

① Monotonic game : A TU game (N, v) is called monotonic if $v(C) \leq v(D) \quad \forall\, C \subseteq D \subseteq N$

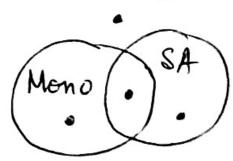
— almost all reasonable games

② Superadditive game : A TU game (N, v) is superadditive if

$$v(C \cup D) \geq v(C) + v(D) \quad , \forall\, C, D \subseteq N \text{ s.t. } C \cap D = \phi .$$

Monotonic and superadditive are independent features

Exercise: construct examples of all possibilities of monotonic and superadditive games.



③ Convex games : A TU game is convex if

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D) \qquad \forall\, C, D \subseteq N$$

A convex game is always superadditive, but the converse is not true.

Proposition: (N, v) is convex iff $v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D) \quad \forall\, C \subseteq D \subseteq N \setminus \{i\}$ $\forall\, i \in N$.

Solution concepts – Questions for cooperative games

1. What coalitions will form? If a coalition S is formed, how does it divide the worth $v(S)$ among its members
2. What would a trusted mediator/arbitrator recommend to the players?

– Answer to 1 is hard. We will assume that the grand coalition forms (and find conditions when it is likely to form), and then ask how agents will divide the grand worth among themselves in a rational way.

– On the other hand, recommendation by an arbitrator is a point solution.

Imputation: (Share of the valuation among the players)

Defn: An imputation $x \in \mathbb{R}^n$ is a share of the players that satisfies

1. $x_i \geq v(\{i\}) \quad \forall i \in N$ ... individually rational
2. $\sum_{i \in N} x_i = v(N)$. – grand coalitionally rational.

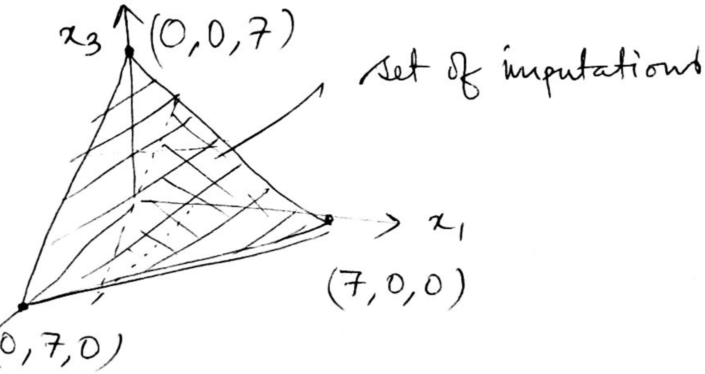
imputations are guaranteed?

Example: $N = \{1, 2, 3\}$

$v(1) = v(2) = v(3) = 0$

$v(1,2) = 2 \quad v(1,3) = 3, \; v(2,3) = 4, \; v(1,2,3) = 7$

if $v(1) = 2$



Defn:

An allocation $x \in \mathbb{R}^n$ is coalitionally rational if $\sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N$.

Note: this implies individual rationality.

Core:

Defn: An imputation is in the core if it is coalitionally rational, i.e.

(1) $\sum_{i \in S} x_i \geq v(S) \quad \forall S \subsetneq N$

(2) $\sum_{i \in N} x_i = v(N)$.

For the previous example

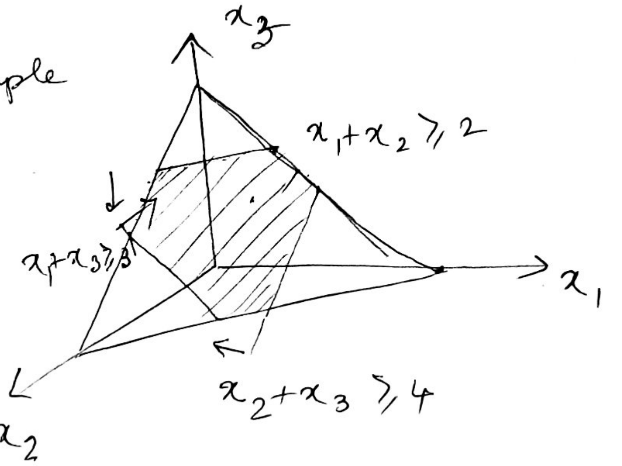
$x_1 + x_2 + x_3 = 7$

$x_1 + x_2 \geq 2$

$x_2 + x_3 \geq 4$

$x_1 + x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$



Core is a polytope.

Core for the previous examples

(1) DTD - ver 1:

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 300, \ x_1, x_2, x_3 \geq 0\}$$

ver 2:

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 300, \ x_1 \geq 0, x_2 \geq 0, \ x_3 = 0\}.$$

ver 3:

$$\left.\begin{array}{l} x_1 + x_2 + x_3 = 300 \\ x_1 + x_2 \geq 300 \\ x_1 + x_3 \geq 300 \\ x_1, x_2, x_3 \geq 0 \end{array}\right\} \Rightarrow x_2 = x_3 = 0$$

$(300, 0, 0)$ is the only point in the core.

ver 4:

$$\left.\begin{array}{ll} x_1 + x_2 + x_3 = 300 & \\ x_1 + x_2 \geq 300 & x_2 + x_3 \geq 300 \\ x_1 + x_3 \geq 300 & x_1, x_2, x_3 \geq 0 \end{array}\right\} \ C(N, v) = \phi.$$

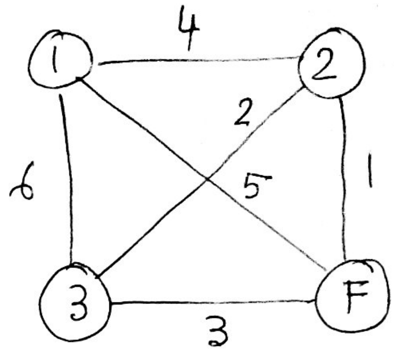
② Core for MST game

$v(1) = 5$ | $v(12) = 15$

$v(2) = 9$ | $v(13) = 12$

$v(3) = 7$ | $v(23) = 17$

$v(123) = 23.$

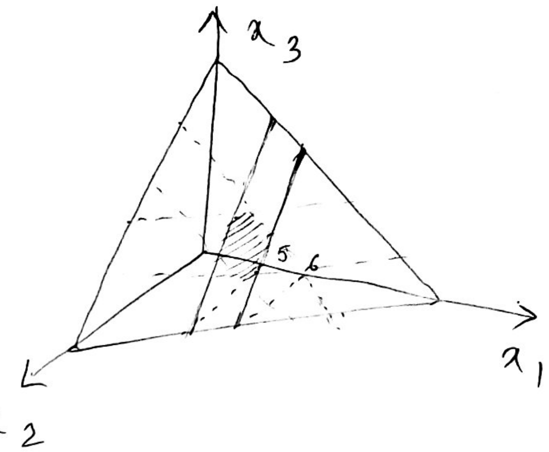


$x_1 \geq 5$, $x_2 \geq 9$, $x_3 \geq 7$ $\qquad x_1 + x_2 + x_3 = 23$

$x_1 + x_2 \geq 15 \Rightarrow x_3 \leq 8$

$x_2 + x_3 \geq 17 \Rightarrow x_1 \leq 6$

$x_1 + x_3 \geq 12 \Rightarrow x_2 \leq 11$



uncountably many points in the core.

Exercise: core of the bankruptcy game.

For all such TU games, grand coalition is rational only if core is non-empty.

Balanced collection of coalitions

Consider $N = \{1, 2, 3\}$, an imputation x is in core iff the following holds:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= v(123) \\
x_1 + x_2 &\geq v(1,2) \\
x_1 + x_3 &\geq v(13) \\
x_2 + x_3 &\geq v(23) \\
x_1 &\geq v(1) \\
x_2 &\geq v(2) \\
x_3 &\geq v(3)
\end{aligned}
$$

necessary condition means that we must have a solution for this system of inequalities, accordingly there will be restrictions on v.

(5-4)

Combining :

$$v(123) \geq v(1) + v(2) + v(3) \quad - - - (1)$$

$$v(123) \geq v(12) + v(3) \quad - - (2)$$

$$v(123) \geq v(13) + v(2) \quad - - (3)$$

$$v(123) \geq v(23) + v(1) \quad - - - (4)$$

$$v(123) \geq \frac{1}{2} v(12) + \frac{1}{2} v(13) + \frac{1}{2} v(23) - - (5)$$

<u>necessary</u>

It can ~~a~~ also be proved (needs work) that these are sufficient conditions for ~~the~~ a 3-player game to have a non-empty core. (exercise).

Goal : generalize this for any number of players.

Collection of coalitions	coefficients on the RHS
$\{\{1\},\{2\},\{3\}\}$	1, 1, 1
$\{1,2\},\{3\}$	1, 1
$\{1,3\},\{2\}$	1, 1
$\{2,3\},\{1\}$	1, 1
$\{1,2\},\{2,3\},\{1,3\}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

consider 1 – 4, close to superadditivity but not quite, since this holds only for the grand coalition.

Incidence matrix = I   coefficients = c

$$\begin{array}{c} \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \\ 1 \end{array}$$

matrix rows are coalitions, columns are players

$$\begin{array}{c} \{1,2\} \\ \{3\} \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \end{array}$$

$$\begin{array}{c} \{1,3\} \\ \{2\} \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \end{array}$$

$$\begin{array}{c} \{2,3\} \\ \{1\} \end{array} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \end{array}$$

$$\begin{array}{l} \{1,2\} \\ \{1,3\} \\ \{2,3\} \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{array}{c} 1/2 \\ 1/2 \\ 1/2 \end{array}$$

Observe that $C^T I = (1,1,1)$ in all cases.
A collection of coalitions that has a vector of positive coefficients satisfying this property is called a <u>balanced collection</u>. The coefficients are called balanced coefficients.

<u>Defn</u>: A collection of coalitions $\mathcal{D}$ is a balanced collection if $\exists$ a vector of positive numbers $(\delta_S)_{S\in\mathcal{D}}$ s.t.

$$\sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S = 1, \quad \forall i \in N.$$

$(\delta_S)_{S\in\mathcal{D}}$ is a vector of balancing weights for that collection of coalitions. If positive is replaced with nonnegative - weakly balanced collection/weights.

Balancing weights are like soft partition of every individual.

From the example, say a collection $\mathcal{D}_1 = \{\{1\},\{2\},\{3\}\}$ is balanced $\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = 1$

& $\mathcal{D}_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$ is balanced

$$\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = 1/2$$

Consider $\mathcal{D}_1 \cup \mathcal{D}_2 = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$

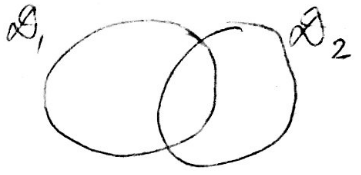
$$\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = \lambda \qquad \delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \tfrac{1}{2}(1-\lambda)$$

$\lambda \in [0,1]$ for balanced.

Claim: If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two balanced collections, then their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is also a balanced collection.

(5-6)

$$\sum_{S\in\mathcal{D}_1\cup\mathcal{D}_2 : i\in S} \delta_S \overset{?}{=} 1 \text{ , find } \delta_S$$



Given $\exists\, \delta_S^{(1)}\ \forall S\in\mathcal{D}_1$ s.t. $\sum_{S\in\mathcal{D}_1 : i\in S} \delta_S^{(1)} = 1$

and $\exists\, \delta_S^{(2)}\ \forall S\in\mathcal{D}_2$ s.t. $\sum_{S\in\mathcal{D}_2 : i\in S} \delta_S^{(2)} = 1$

$$\sum_{S\in\mathcal{D}_1\cup\mathcal{D}_2 : i\in S} \delta_S = \underbrace{\sum_{S\in\mathcal{D}_1\setminus\mathcal{D}_2 : i\in S} \delta_S}_{\lambda\delta_S^{(1)}} + \underbrace{\sum_{S\in\mathcal{D}_2\setminus\mathcal{D}_1 : i\in S} \delta_S}_{(1-\lambda)\delta_S^{(2)}} + \underbrace{\sum_{S\in\mathcal{D}_1\cap\mathcal{D}_2 : i\in S} \delta_S}_{\lambda\delta_S^{(1)} + (1-\lambda)\delta_S^{(2)}}$$

$$= \lambda \sum_{S\in\mathcal{D}_1 : i\in S} \delta_S^{(1)} + (1-\lambda)\sum_{S\in\mathcal{D}_2 : i\in S} \delta_S^{(2)} = 1.$$

Consider the collection $\mathcal{D} = \{\{1,2\},\{1,3\}\}$

$\delta_{\{1,2\}}, \delta_{\{1,3\}}$ for 1 : $\delta_{\{1,2\}} + \delta_{\{1,3\}} = 1$

for 2 : $\delta_{\{1,2\}} = 1$, for 3 : $\delta_{\{1,3\}} = 1$

infeasible, i.e., $\mathcal{D}$ is not a balanced collection.

$\mathcal{D} = \{\{1,3\},\{2,3\},\{1\}\}$ is weakly balanced but not ~~the~~ balanced collection.

$\delta_{\{1,3\}} + \delta_{\{1\}} = 1$

$\left.\begin{array}{l}\delta_{\{1,3\}} + \delta_{\{2,3\}} = 1 \\ \delta_{\{2,3\}} = 1\end{array}\right\}$ $\delta_{\{1,3\}} = 0$, $\delta_{\{1\}} = \delta_{\{2,3\}} = 1$

Relationship with the core

Theorem [Bondareva'63, Shapley'67] The necessary and sufficient condition for a coalitional game (N, v) to have a non-empty core is that for every balanced collection $\mathcal{D}$ of coalitions, and every ~~ea~~ vector of balancing weights $(\delta_S)_{S\in\mathcal{D}}$

$$v(N) \geq \sum_{S\in\mathcal{D}} \delta_S v(S)$$

The condition is also called balanced condition - and the game as balanced game.
An alternative statement says ~~the~~ that a ~~game~~ game has non-empty core iff the game is balanced.

Remark: 1 The theorem holds even when balancedness condition is relaxed to weakly balanced, because the inequality holds for every balanced collection iff it holds for every weakly balanced collection.

2 BS theorem is useful to find a counterexample of non-~~empty~~ empty core, also for guaranteeing non-empty core for a class of games.
- it is not very useful to find the core of a game.

To prove we will need <u>LP duality</u>.
A special set of balanced weights where
$\mathcal{D}^* = 2^N$ all subsets of N.
and call that set of balanced weights λ^*, balanced* weights, i.e., $\sum_{\{S \subseteq N : i \in S\}} \lambda^*(S) = 1 \quad \forall i \in N$.

EX: show that $\mathcal{D}^*$ is a balanced collection.

<u>Thm:</u> (BS, second formulation)
A ~~coalitional~~ TU game (N, v) has a non-empty core iff for all balanced* weights λ, we have

$$v(N) \geq \sum_{S \subseteq N} \lambda(S)\, v(S).$$

Check: DTD - ver 4 (majority)
$v(i) = 0 \ \forall i = 1,2,3$, $v(\{i,j\}) = 300 \quad i \neq j,\ i,j \in N$, $v(123) = 300$
$\lambda(12) = \lambda(23) = \lambda(13) = 1/2 \rightarrow$ counterexample.

 Proof of Bondareva - Shapley Theorem (second formulation)

Proof: Consider the following linear program to check the feasibility of the core

$$\text{minimize} \quad \sum_{i \in N} x_i \qquad \cdots (1)$$

$$\text{s.t.} \quad \sum_{i \in S} x_i \geq v(S) \qquad \forall S \subseteq N.$$

The value of this OPT problem is at least $v(N)$.

Claim: If there is a non-empty core then the OPT $= v(N)$.

if OPT $> v(N) \iff$ core empty

Consider the dual of (1)

$$\text{maximize} \sum_{S \subseteq N} \lambda(S)\, v(S)$$

$$\text{s.t.} \quad \sum_{S \subseteq N:\, i \in S} \lambda(S) = 1 \qquad \forall i \in N$$

$$\lambda(S) \geq 0 \quad \forall S \subseteq N.$$

The constraints are that of a balanced* weights.

Weak duality

$$\sum_{S \subseteq N} \lambda(S)\, v(S) \leq \underbrace{\sum_{i \in N} x_i^*}_{\text{Primal solution}} = \underbrace{v(N)}_{\text{core is non-empty}}$$

for all balanced* weights

□

A coalitional game satisfying B-S condition is called a balanced game, i.e., $\forall$ balanced weights λ

$$v(N) \geq \sum_{S \subseteq N} \lambda(S)\, v(S).$$

Market Games

Concentrate on coalitional games that arises naturally in practice — and apply the B-S theorem to prove non-empty cores.

- Producers $N = \{1, 2, \dots, n\}$ trade L commodities
- Set of commodities, $\mathcal{C} = \{1, 2, \dots, L\}$
  e.g., different kinds of raw materials
  wood, metal, human resources, expert consultation hours etc.
- A commodity vector is denotes by $x \in \mathbb{R}^L_{\geq 0}$
  x_j, $j = 1, \dots, L$ denote the amount/quantity of commodity j.
  assuming these are fluid items.

  Refer this as a "bundle". Bundle of producer i is denoted by $x_i \in \mathbb{R}^L_{\geq 0}$, x_{ij} is the quantity of commodity j player i gets.

- Production/Utility function of producer i

$$u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$$

  $u_i(x_i)$: The amount of money producer i can generate from the bundle x_i. [Ex. $p^T x_i$]

- Initial endowment of producer i is $a_i \in \mathbb{R}^L_{\geq 0}$

Coalitional strategy:

If a coalition S forms, the members trade commodities among themselves/pool them. The goal is to maximize the total money generated.

Total endowment of S, $\quad a(S) = \sum_{i \in S} a_i \in \mathbb{R}^L_{\geq 0}$

The coalition can only redistribute these items among its members, $x_i \in \mathbb{R}^L_{\geq 0}$ with

$$x(S) = \sum_{i \in S} x_i = a(S)$$

Hence, by redistributing the items They can generate a collective wealth of $\sum_{i \in S} u_i(x_i)$

Defn: A market is given by a vector $(N, C, (a_i, u_i)_{i \in N})$ where

- $N = \{1, 2, \ldots, n\}$ is the set of producers
- $C = \{1, 2, \ldots, L\}$ is the set of commodities
- $\forall i \in N$, $a_i \in \mathbb{R}^L_{\geq 0}$ is the initial endowment of producer i.
- $\forall i \in N$, $u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ is the production function of i.

Set of allocations for coalition S

$$X^S := \{(x_i)_{i \in S} : x_i \in \mathbb{R}^L_{\geq 0} \ \forall i \in S, \ x(S) = a(S)\}$$

Result: For every coalition S, X^S is compact [Closed and Bounded]

Assume all production functions are continuous.

Worth of ~~each~~ coalition S

$$v(S) = \max\left\{ \sum_{i \in S} u_i(x_i) : x = (x_i)_{i \in S} \in X^S \right\} \quad \text{--- } (1)$$

Since u_i's are continuous and X^S is compact the maxima is attained within X^S.

Example: $N = \{1, 2, 3\}$, $C = \{1, 2\}$

- $a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = (2, 2)$
- $u_1(x_1) = x_{11} + x_{12}$, $u_2(x_2) = x_{21} + 2x_{22}$

$u_3(x_3) = \sqrt{x_{31}} + \sqrt{x_{32}}$

$v(1) = 1$, $v(2) = 2$, $v(3) = 2\sqrt{2}$

Compute $v(123)$, leave $v(12), v(13), v(23)$ as exercise

Consider $\sum_{i=1}^{3} u_i(x_i)$ every unit of 1 contributes equally for producers 1 and 2, and for that of unit 2, producer 2's share contributes twice as that of 1. Hence in a maximum utility, $x_1 = (0,0)$ and the whole share of 1 can be transferred to 2.

$$v(123) = \max\{x_{21} + 2x_{22} + \sqrt{3 - x_{21}} + \sqrt{3 - x_{22}} : 0 \leq x_{21} \leq 3,\ 0 \leq x_{22} \leq 3\}$$

$$x_2 = \left(\frac{11}{4}, \frac{47}{16}\right), \quad x_3 = \left(\frac{1}{4}, \frac{1}{4}\right).$$

<u>Defn</u>: A coalitional game (N, v) is a <u>market game</u> if $\exists\ L > 0$, $\forall i \in N$ $\exists\ a_i \in \mathbb{R}^L_{\geq 0}$, and $u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ continuous and concave $\forall i \in N$ s.t. eq. (1) is satisfied for all non-empty $S \subseteq N$.

Theorem (Shapley and Shubik (1969))
The core of a market game is non-empty.

<u>Proof</u>: We'll use B-S theorem to prove this result.
To prove: every market game is a balanced game.
Consider a market game $(N, \mathcal{C}, (a_i, u_i)_{i \in N})$.
Fix an arbitrary coalition S
let $x^S = (x_i^S)_{i \in S}$ be the allocation that maximizes $\sum_{i \in N} u_i(x_i^S)$ - by definition of u_i, $x^S \in X^S$.

We have

- $x_i^S \in \mathbb{R}^L_{\geq 0}$
- $x^S(S) = \sum_{i \in S} x_i^S = a(S)$
- $\sum_{i \in S} u_i(x_i^S) = v(S)$

(6-5) let $\delta = (\delta_S)_{S \subseteq N}$ be a balanced weight vector (arbitrary)

$$v(N) \geq \sum_{S \subseteq N} \delta_S v(S) ,$$

Define, $$z_i := \sum_{\{S \subseteq N : i \in S\}} \delta_S x_i^S \in \mathbb{R}^L_{\geq 0}$$

Claim: z_i is a feasible bundle, i.e. $\sum_{i \in N} z_i = a(N)$

Pf: $$z(N) = \sum_{i \in N} z_i = \sum_{i \in N} \sum_{S \subseteq N : i \in S} \delta_S x_i^S$$

$$= \cancel{\sum_{S \subseteq N : i \in S} \delta_S \sum_{i \in N} x_i^S} = \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \delta_S x_i^S$$

$$= \sum_{S \subseteq N} \sum_{i \in N} I\{i \in S\} \delta_S x_i^S$$

$$= \sum_{S \subseteq N} \sum_{i \in S} \delta_S x_i^S = \sum_{S \subseteq N} \delta_S \sum_{i \in S} x_i^S$$

$$= \sum_{S \subseteq N} \delta_S x^S(S) = \sum_{S \subseteq N} \delta_S a(S)$$

by definition of x^S

$$= \sum_{S \subseteq N} \delta_S \sum_{i \in S} a_i$$

$$= \sum_{i \in N} \sum_{S \subseteq N : i \in S} \delta_S a_i$$

$$= \sum_{i \in N} a_i \underbrace{\left(\sum_{S \subseteq N : i \in S} \delta_S \right)}_{=1} = a(N).$$

By definition of v

$$
\begin{aligned}
v(N) &\geq \sum_{i\in N} u_i(z_i) \\
&= \sum_{i\in N} u_i\Big(\sum_{S\subseteq N:\, i\in S} \delta_S x_i^S\Big) \\
&\geq \sum_{i\in N}\sum_{S\subseteq N:\, i\in S} \delta_S\, u_i(x_i^S) \\
&= \sum_{S\subseteq N}\sum_{i\in S} \delta_S\, u_i(x_i^S) \\
&= \sum_{S\subseteq N} \delta_S \sum_{i\in S} u_i(x_i^S) \\
&= \sum_{S\subseteq N} \delta_S\, v(S) \quad \text{[Balanced condition]}. \ \square
\end{aligned}
$$

If the producers leave from market, leading to $(N, C, (a_i, u_i)_{i\in N})$ being reduced to $(S, C, (a_i, u_i)_{i\in S})$

- We can define a restriction of v in (N, v) to the v restricted to S which is same as $v(T)$ $\forall\, T\subseteq S$. Hence we can consider subgame (S, v) of the market game (N, v)

Let $(S, \tilde{v})$ be the reduced game, $\forall\, T\subseteq S$

$$
\begin{aligned}
\tilde{v}(T) &= \max\Big\{\sum_{i\in T} u_i(x_i) : x_i \in \mathbb{R}^L_{\geq 0}\ \forall i\in T, \sum_{i\in T} x_i = \sum_{i\in N} a_i\Big\} \\
&= v(T)
\end{aligned}
$$

Corollary [of Shapley - Shubik theorem]

If (N, v) is a market game, every subgame (S, v) of it is a market game, and in particular is balanced.

Such games are called totally balanced.

A coalitional game is totally balanced if every subgame of it has non-empty core.

Restatement of Shapley-Shubik:

Every market game is totally balanced.

The converse of this result is also true.

Theorem: Every totally balanced game is a market game.

Core is a set-solution concept.

$v(\{i\}) = 0 \quad \forall i = 1,2,3$, $v(1,2) = v(2,3) = 1$ $\quad v(1,3) = 2$

$v(1,2,3) = 3$.

$(2, 0.5, 0.5)$ is in the core.

If player 3 leave with the core share 0.5.
Does the other two players do a better division of the remaining money on table? E.g. $(1,2)$ may divide 2.5 as 1.25 each, which is not in core.

Davis - Maschler reduced game property

Defn: Let (N, v) be a TU game, let S be a non-empty coalition, let x be an efficient vector, i.e., $x(N) = v(N)$
The Davis - Maschler reduced game to S relative to x, denoted by (S, w_S^x) is the coalitional game with the set of players S and a coalition function.

$$w_S^x(R) = \begin{cases} \max\limits_{Q \subseteq N\setminus S} \{v(R \cup Q) - x(Q)\}, & \text{if } \phi \neq R \subset S \\ x(S) & , \text{if } R = S \\ 0 & \text{if } R = \phi \end{cases}$$

Idea: If players outside S accept the ~~~~ offer x then any $Q \subseteq N \setminus S$ will be happy with $x(Q)$.
The rest can be the worth of R, and R picks that Q which maximizes the leftover worth.
Of course, they can at most have $x(S)$ which they can divide in case $R = S$.

7-2 Consistency of the core

Defn: A set-solution concept ϕ satisfies the Davis-Maschler reduced game property if for every TU game (N,v), for every non-empty coalition $S \subseteq N$, and for every vector $x \in \phi(N,v)$, it holds that

$$(x_i)_{i \in S} \in \phi(S, w_S^x).$$

Remark: The reduced game property is a consistency property: if ~~those~~ the players believe in x, they will refrain from redistributing $x(S)$ since that is already a solution in the subgame reduced to S.

Theorem: The core satisfies the Davis-Maschler reduced game property.

Proof: Let x be a point in the core of (N,v) and let S be a non-empty coalition. We will show that $(x_i)_{i \in S}$ is in the core of (S, w_S^x).

Need to show: 1) $x(R) \geq w_S^x(R) \quad \forall R \subset S, R \neq \emptyset$

2) $x(S) = w_S^x(S)$

2) is satisfied by definition of w_S^x.

To show 1) consider $R \subset S$. By definition of w_S^x $\exists$ coalition $Q \subseteq N \setminus S$ s.t.

$$w_S^x(R) = v(R \cup Q) - x(Q)$$
$$= v(R \cup Q) - \left(x(R \cup Q) - x(R)\right)$$

since x is in core, $v(R \cup Q) \leq x(R \cup Q)$, hence

$$x(R) \geq w_S^x(R).$$

$\square$

Convex games revisited

Def: A coalitional game (N, v) is convex if for every pair of coalitions S and T,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \quad \cdots ①$$

Fact (Proof: exercise) If (N, v) is a convex game, then for every coalition $S \subseteq N$, the subgame (S, v) restricted to the players in S and v restricted to the power set of S, is also a convex game.

Convex games are characterized by the property that the marginal contribution of a player is larger in a larger coalition. Formally it is stated as:

Theorem: For any TU game (N, v) the following statements are equivalent

① (N, v) is a convex game

② For every $S \subseteq T \subseteq N$ and for every $R \subseteq N \setminus T$

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T).$$

③ For every $S \subseteq T \subseteq N$ and $\forall i \in N \setminus T$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Remark: Any of these conditions can be used as definition.

Proof: We'll prove ① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ①

(7-2)

① ⇒ ②: Given (N, v) is a convex game

Suppose S, T be s.t. $S \subseteq T \subseteq N$ and $R \subseteq N \setminus T$

use convexity for $S \cup R$ and T

$$v(S \cup R) + v(T) \leq v(\underbrace{(S \cup R) \cup T}_{= S \cup T \cup R = T \cup R}) + v(\underbrace{(S \cup R) \cap T}_{= (S \cap T) \cup (R \cap T) = S \cup \phi = S})$$

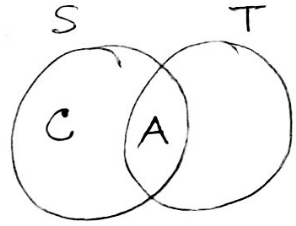
$$\Rightarrow v(S \cup R) - v(S) \leq v(T \cup R) - v(T).$$

② ⇒ ③ Obvious, put $R = \{i\}$.

③ ⇒ ① If $S \subseteq T$, then eqn. ① holds with equality.

Consider S and T are not contained in one another.

Define, $A := S \cap T$, $C = S \setminus T \neq \phi$



Let $C = \{i_1, i_2, \dots, i_k\}$

Since $A \subseteq T$

$$A \cup \{i_1, \dots, i_\ell\} \subseteq T \cup \{i_1, \dots, i_\ell\} \quad \forall \ell = 0, 1, \dots, k-1$$

Also $i_{\ell+1} \notin T \cup \{i_1, \dots, i_\ell\}$

③ gives

$$v(A \cup \{i_1, \dots, i_\ell, i_{\ell+1}\}) - v(A \cup \{i_1, \dots, i_\ell\}) \leq v(T \cup \{i_1, \dots, i_{\ell+1}\}) - v(T \cup \{i_1, \dots, i_\ell\})$$

writing these inequalities for $\ell = 0, \dots, k-1$ and summin

$$v(A \cup \{i_1\}) - v(A) \leq v(T \cup \{i_1\}) - v(T)$$

$$v(A \cup \{i_1, i_2\}) - v(A \cup \{i_1\}) \leq v(T \cup \{i_1, i_2\}) - v(T \cup \{i_1\})$$

$$\vdots$$

$$v(A \cup C) - v(A \cup \{i_1, \dots, i_{k-1}\}) \leq v(T \cup C) - v(T \cup \{i_1, \dots, i_{k-1}\})$$

$$\Rightarrow v(\underbrace{A \cup C}_{= S}) - v(\underbrace{A}_{= T \cap S}) \leq v(\underbrace{T \cup C}_{= T \cup S}) - v(T)$$

Convex games have non-empty core.

<u>Theorem</u>: Let (N, v) be a convex game, let x be the imputation

$$x_1 = v(1)$$
$$x_2 = v(1,2) - v(1)$$
$$\cdots$$
$$x_n = v(1,2,\ldots,n) - v(1,2,\ldots,n-1)$$

Then x is in the core of (N, v).

<u>Proof</u>: x is efficient

$$\sum_{i \in N} x_i = v(N).$$

Need to show $x(S) \geq v(S)$, $\forall S \subseteq N$.

Let $S = \{i_1, \ldots, i_k\}$ be an arbitrary coalition.

WLOG $i_1 < i_2 < \cdots < i_k$

$$\underbrace{\{i_1, i_2, \ldots, i_{j-1}\}}_{S} \subseteq \underbrace{\{1, 2, \ldots, i_j - 1\}}_{T} \quad \forall j = 1, \ldots, k \qquad i_j \in N \setminus T$$

Implication (3) of previous theorem gives

$$v(1,2,\ldots,i_j) - v(1,2,\ldots,i_j - 1) \geq v(i_1,\ldots,i_j) - v(i_1,\ldots,i_{j-1})$$

Hence

$$\begin{aligned} x(S) &= \sum_{j=1}^{k} x_{i_j} \\ &= \sum_{j=1}^{k} \left[v(1,2,\ldots,i_j) - v(1,2,\ldots,i_j - 1)\right] \\ &\geq \sum_{j=1}^{k} \left[v(i_1,\ldots,i_j) - v(i_1,\ldots,i_{j-1})\right] \\ &= v(i_1,\ldots,i_k) = v(S). \end{aligned}$$

$\square$

7-6

The theorem shows if the agents are ordered lexicographically and correspondingly the x_i's are defined

$$x_i = v(1,2,\ldots,i) - v(1,2,\ldots,i-1)$$

Then x is in the core. But clearly the same construction holds for any permutation of the players, say $\pi = (i_1, i_2, \ldots, i_n)$

$$w^{\pi} := (v(i_1), v(i_1,i_2) - v(i_1), \ldots, v(N) - v(N\setminus\{i_n\}))$$

Description: Players enter a room in the order π and everyone is paid his/her marginal contribution. The imputation derived from this is w^{π}, and is in the core.

Other game classes having non-empty core.

- Spanning tree games
- Flow games.

Set-solution concept to point (single-value) solution concept.

(1) – Core has many solutions – what to expect

(2) – Core may not exist – some solution concept that is guaranteed to exist.

Shapley properties.

Core limitation 1: many solutions - set-valued

Shapley Value

- Single-valued solution concept.
- Based on axioms (known as Shapley axioms) similar to Nash bargaining.

Notation: ϕ be a single-valued solution concept

$\phi_i(N, v)$ is called the allocation of player $i \in N$.

Axioms:

① Efficiency: A solution concept ϕ satisfies efficiency if for every TU game (N, v)

$$\sum_{i \in N} \phi_i(N, v) = v(N).$$ [no-wastage property]

② Symmetry:

Defn: Players i and j are symmetric players if for ~~any~~ every coalition $S \subseteq N \setminus \{i, j\}$,

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

symmetric players give same marginal contribution to every coalition.

Defn: A solution concept ϕ satisfies symmetry if for every coalitional game (N, v) and every pair of symmetric players i and j in the game

$$\phi_i(N, v) = \phi_j(N, v).$$

[equal treatment for equals]

(3) Null player property :

Defn: A player i is called a null player in (N, v) if for every $S \subseteq N$, $v(S) = v(S \cup \{i\})$

– clearly $v(i) = 0$.

Defn: A solution concept ϕ satisfies null player property if for every coalitional game (N, v) and for every null player i, $\phi_i(N, v) = 0$.

(4) Additivity: A solution concept ϕ satisfies additivity if for every pair of coalitional games (N, v) and (N, w)

$$\phi(N, v+w) = \phi(N, v) + \phi(N, w).$$

"To what extent a single game is equivalent to playing two games individually?"

This property says independence – the share/allocation from a game with added valuation is exactly the same as ~~the~~ playing the games independently and collecting the rewards.

Examples

(1) $\Psi_i(N, v) = v(i)$

additivity – $\Psi_i(N, v+w) = (v+w)(i) = v(i) + w(i)$

$= \Psi_i(N, v) + \Psi_i(N, w)$

Symmetry – If $\forall S \subseteq N \setminus \{i, j\}$ $v(S \cup \{i\}) = v(S \cup \{j\})$

then apply $S = \phi \Rightarrow v(i) = v(j) \Rightarrow \Psi_i(N, v) = \Psi_j(N, v)$.

null player – for every null player $v(i)=0$, $\psi_i(N,v)=0$.

efficiency – not necessarily. $\sum v(i) \neq v(N)$.

(2) A player i is called a dummy player if

$$v(S\cup\{i\}) = v(S) + v(i) \quad \forall S \subseteq N\setminus\{i\}$$

Every null player is a dummy player.

Let $d(v)$ be the number of dummy players in (N,v)

Consider a solution concept.

$$\psi_i(N,v) = \begin{cases} v(i) + \dfrac{v(N) - \sum_{j\in N} v(j)}{n-d(v)} & i \text{ is not a dummy player} \\ v(i) & i \text{ is a dummy player.} \end{cases}$$

efficiency – yes, Null – yes, null is dummy – gets zero.

symmetry – clearly if both the players are dummy Then this is true, both players are non-dummy then also true.

What about i non-dummy and j dummy

Can they be symmetric?

– ~~let D be the dummy set~~, ~~clearly $v(D) = \sum_{j\in D} v(j)$~~

– if $i \in ND$ and $j \in D$ and they are symmetric

Then we have $v(S\cup\{i\}) = v(S\cup\{j\}) \quad \forall S \subseteq N\setminus\{i,j\}$

clearly $v(i) = v(j)$ $\quad = v(S) + v(j)$

$S=\phi$ $\quad = v(S) + v(i)$

$\Rightarrow v(S\cup\{i\}) + v(j) = \underbrace{v(S) + v(j)} + v(i)$

$\Rightarrow v(S\cup\{j\}\cup\{i\}) = v(S\cup\{j\}) + v(i)$

$\Rightarrow v(\bar{S}\cup\{i\}) = v(\bar{S}) + v(i) \quad \forall \bar{S} \subseteq N\setminus\{i\}$

$\Rightarrow i$ is a dummy player $\rightarrow\leftarrow$

(8-4) does not satisfy additivity.

$v(1) = v(2) = v(3) = v(1,2) = v(1,3) = 0$

$v(2,3) = v(1,2,3) = 1$ - - - - - - (1)

and $u(1) = u(2) = u(3) = u(1,3) = 0$, $u(1,2) = u(2,3) =$

$u(1,2,3) = 1$

$(N, v) \to$ player 1 is the dummy $\left(0, \frac{1}{2}, \frac{1}{2}\right)$

(N, u) no dummy $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Consider $(N, u+v)$

$(u+v)(1) = (u+v)(2) = (u+v)(3) = (u+v)(1,3) = 0$

$(u+v)(1,2) = 1$, $(u+v)(2,3) = 2 = (u+v)(1,2,3)$

no-dummy $\psi(N, u+v) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\psi(N,u) + \psi(N,v) = \left(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right)$.

(3) $\psi_i(N, v) = \max\limits_{\{S : i \notin S\}} \left(v(S \cup \{i\}) - v(S)\right\}$

symmetry, null player ✓

efficiency, additivity ×.

(4) $\psi_i(N, v) = v(1, 2, \ldots, i) - v(1, 2, \ldots, i-1)$

efficiency – ✓

additivity – ✓

null player – ✓

not symmetry – consider the first game of example (2)

eqn. (1) $\psi(N, v) = (0, 0, 1)$

but 2, 3 are symmetric.

The solution concept can be defined for any order of the players (not just the identity order)
Say $\Pi(N)$ denote the set of all possible orders.
~~Defn~~: Call $\pi \in \Pi(N)$ to be one ordering/permutation of the players.
Call the predecessor of player i in the permutation π as

$$P_i(\pi) = \{ j \in N : \pi(j) < \pi(i) \}$$

$$P_i(\pi) = \phi \text{ if } \pi(i) = 1.$$

$$P_i(\pi) \cup \{i\} = P_k(\pi) \iff \pi(k) = \pi(i) + 1.$$

Now, we can define the solution concept

$$\varphi_i^{\pi}(N, v) = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)).$$

As we saw in ~~ex~~ example (4) before, this solution concept satisfies efficiency, null-player property, additivity, not symmetry.

Shapley value

Q. Is there a solution concept that satisfies all ~~4~~ four properties?

A. Yes, and it is unique.

Defn (Shapley 1953) The Shapley value is the solution concept Sh defined as,

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \left[v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) \right] \quad \forall i \in N.$$

A simple average over all ψ_i^{π}'s

Hence
$$Sh_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \psi^{\pi}(N,v)$$

Theorem: The Shapley value is the only single-valued solution concept satisfying efficiency, additivity, null player, and symmetry.

An equivalent formula for Shapley value

$$\frac{1}{n!} \sum_{\pi \in \Pi(N)} \psi_i^{\pi}(N,v) = \frac{1}{n!} \sum_{S \subseteq N\setminus\{i\}} \sum_{\substack{\pi \in \Pi(N) \\ P_i(\pi) = S}} \left(v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))\right)$$

$$= \frac{1}{n!} \sum_{S \subseteq N\setminus\{i\}} \sum_{\substack{\pi \in \Pi(N) \\ P_i(\pi) = S}} \left(v(S \cup \{i\}) - v(S)\right)$$

$$= \sum_{S \subseteq N\setminus\{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \left[v(S\cup\{i\}) - v(S)\right]$$

Interpretation: "Average marginal contribution to all other coalitions"

Proof of Shapley theorem:

Part 1: Shapley value satisfies the four axioms.

Each of ψ^{π} satisfies efficiency, additivity, null player, so does their average [Exercise]

Symmetry: Let i and j be two symmetric players. Given a permutation π, define the following permutation $f(\pi)$, s.t.

$$f : \Pi(N) \to \Pi(N)$$

$f(\pi)$ just swaps the positions of i and j

$$(f(\pi))(k) = \begin{cases} \pi(j) & \text{if } k = i \\ \pi(i) & \text{if } k = j \\ \pi(k) & \text{if } k \neq i, j \end{cases}$$

Clearly f is a bijective mapping.

Claim: $\psi_i^{\pi}(N, v) = \psi_j^{f(\pi)}(N, v)$

$$\Leftrightarrow \; v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) = v(P_j(f(\pi)) \cup \{j\}) - v(P_j(f(\pi))) \quad \cdots \; (2)$$

Case 1: Player i appears before j in π, i.e., $j \notin P_i(\pi)$

clearly $P_j(f(\pi)) = P_i(\pi) \Rightarrow v(P_j(f(\pi))) = v(P_i(\pi))$

and since i and j are symmetric players,

$$v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$$

eqn (2) is holds.

(8-8)

Case 2: Player i appears after j in π, i.e., $j \in P_i(\pi)$

here

$$P_i(\pi) \cup \{i\} = P_j(f(\pi)) \cup \{j\}$$

$$\Rightarrow \quad v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$$

also $\quad P_i(\pi) \setminus \{j\} = P_j(f(\pi)) \setminus \{i\}$ $\quad$ both $\not\ni i, j$

since i, j are symmetric

$$v(P_i(\pi)) = v(P_j(f(\pi))$$

$\square$

Eq. (2) holds

Lec 9

Part 2: Uniqueness.

Define a game called the ~~as~~ carrier game

A coalition is winning if it contains a distinguished set.

Intuition: say T is an influential coalition - any ~~co~~ coalition containing it can pass a bill/do a change.

Defn: Let $T \subseteq N$ be a non-empty coalition. The carrier game over T is the game (N, u_T) s.t. for each coalition $S \subseteq N$

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{ow.} \end{cases}$$

Theorem 1: Every game (N, v) is a linear combination of carrier games.

To define any TU game, we need to define the valuations over all non-empty subsets. Hence it has $2^n - 1$ degrees of freedom. Hence, every game (N, v) is a point in $\mathbb{R}^{2^n - 1}$. Want to show that carrier games span this space. Find carrier games that are linearly independent and forms a basis.

Suppose, carrier games are linearly dependent (for contradiction). $\exists$ real numbers $(\alpha_T)_{\{T \subseteq N, T \neq \phi\}}$ not all zero, s.t.

$$\sum_{\{T \subseteq N, T \neq \phi\}} \alpha_T u_T(S) = 0, \quad \forall S \subseteq N.$$

Let $\mathcal{T} = \{T \subseteq N : T \neq \phi, \alpha_T \neq 0\}$ collections of non-empty coalitions with non-zero coefficients in the above equation. Since $\{\alpha_T\}_{\{T \subseteq N, T \neq \phi\}}$ are not all zero, $\exists$ a minimal coalition in $\mathcal{T}$, i.e. coalition with smallest cardinality. Say $S_0 \in \mathcal{T}$ is ~~the~~ one such coalition. $\nexists$ any subset of S_0 with positive coefficients. Consider

$$\sum_{\{T \subseteq N, T \neq \phi\}} \alpha_T u_T(S_0) = \underbrace{\sum_{\{T \subset S_0, T \neq \phi\}} \alpha_T u_T(S_0)}_{=0} + \alpha_{S_0} u_{S_0}(S_0) + \underbrace{\sum_{T \not\subseteq S_0} \alpha_T u_T(S_0)}_{=0}$$

$$= \alpha_{S_0} \neq 0$$

□

(9-3)

Theorem 2: Let T be a non-empty coalition, and $\alpha \in \mathbb{R}$. Define a game $(N, u_{T,\alpha})$ as follows

$$u_{T,\alpha}(S) = \begin{cases} \alpha & \text{if } T \subseteq S \\ 0 & \text{ow} \end{cases}$$

If ϕ is a solution concept that satisfies efficiency, symmetry, null player property, Then

$$\phi_i(N, u_{T,\alpha}) = \begin{cases} \frac{\alpha}{|T|} & \text{if } i \in T \\ 0 & \text{ow} \end{cases}$$

Obs 1: If $i \notin T$ $\quad u_{T,\alpha}(S \cup \{i\}) = u_{T,\alpha}(S) \quad \forall S \subseteq N$

i is a null player

Obs 2: If $i, j \in T$, they are symmetric.

$$u_{T,\alpha}(S \cup \{i\}) = u_{T,\alpha}(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i,j\}$$

~~From~~

From the fact that ϕ is efficient, symmetric, null-player compliant solution concept, the result follows.

Finishing the proof of Part 2: Uniqueness

Shapley value satisfies the four properties.

Need to show: any ϕ satisfying these four properties is identical to Sh.

Thm 1 says that for any game (N, v), we can write v as sum of u_{T,α_T}'s

$\exists (\alpha_T)_{\{T \subseteq N, T \neq \phi\}}$ s.t.

$$v(S) = \sum_{\{T \subseteq N, T \neq \phi\}} u_{T,\alpha_T}(S)$$

Thm 2 says, since both ϕ and Sh satisfies efficiency, symmetry, and null player property

$$\phi(N, u_{T,\alpha_T}) = Sh(N, u_{T,\alpha_T}), \quad \forall T \subseteq N, T \neq \phi.$$

Since both ϕ and Sh satisfy additivity

$$\phi(N,v) = \sum_{\{T \subseteq N, T \neq \phi\}} \phi(N, u_{T,\alpha_T}) = \sum_{\{T \subseteq N, T \neq \phi\}} Sh(N, u_{T,\alpha_T}) = Sh(N,v).$$

we started with an arbitrary game (N, v), hence this holds for all such games. $\square$

Examples

① Two player bargaining: (N, v)

$v(1) = v(2) = 0, \quad v(1,2) = 1$

symmetric players : 1 & 2, Shapley value is efficient

$Sh(N,v) = \left(\frac{1}{2}, \frac{1}{2}\right)$

② Majority game

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq \frac{n}{2} \\ 1 & \text{if } |S| > \frac{n}{2} \end{cases}$$

all players are symmetric, hence Shapley values are same, together with efficiency

$Sh(N,v) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$

(9-5)

3. Gloves game

$$v(1) = v(2) = v(3) = v(1,2) = 0$$
$$v(1,3) = v(2,3) = v(1,2,3) = 1$$

Perm	Player 1	Player 2	Player 3
1, 2, 3	$v(1) - v(\phi) = 0$	$v(1,2) - v(1) = 0$	1
1, 3, 2	0	$v(123) - v(13) = 0$	1
2, 1, 3	$v(1,2) - v(2) = 0$	0	1
2, 3, 1	$v(1,2,3) - v(2,3) = 0$	0	1
3, 1, 2	$v(1,3) - v(3) = 1$	0	0
3, 2, 1	0	$v(2,3) - v(3) = 1$	0
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$

$Sh(N, v) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$

emphasizes that ~~the~~ player 3 is the most powerful player. but players 1 and 2 do not get zero in the allocation.

Core : $(0, 0, 1)$ is the singleton.

Hence Shapley value is not in core.

Application: Shapley - Shubik power index

Defn : (Simple, monotone games)

Simple games : The value of any coalition can either be 0 or 1.

Monotone games : If any coalition has value 1, every superset of that coalition also has value 1.

Motivation: to model legislations/ decisions based on committees.

Defn: The Shapley-Shubik power index is a function associating each simple monotonic game with its Shapley value. The i^{th} co-ordinate denotes the "power" of player i in this game.

$$Sh_i(N,v) = \sum_{\substack{\{S \subseteq N\setminus\{i\}: \\ S\cup\{i\} \text{ is winning} \\ S \text{ is losing}\}}} \frac{|S|!\,(n-|S|-1)!}{n!}$$

counting all such scenarios where player i is pivotal.

Case study: UN Security council

UN: Body of international political system, established in 1945 (after WW II)

till 1965: five permanent members, six nonpermanent members

Resolution adopted if it receives at least 7 votes but all permanent members have to be unanimous - all of them have veto powers.

Debated about unequal distribution of powers in the security council.

after 1965: five permanent members, 10 nonpermanent members

resolution needed 9 votes but veto power remains with the permanent members

This is a simple, monotonic game. Compute the Shapley-Shubik power index.

(9-7)

P: permanent members , NP: non permanent members

pre-1965:

$$v(S) = \begin{cases} 1 & \text{if } S \supseteq P \text{ and } |S| \geq 7 \\ 0 & \text{ow} \end{cases}$$

non permanent i

$$Sh_i(N,v) = \binom{5}{1} \frac{6!\,4!}{11!} = \frac{1}{462}$$

all non-permanent members are symmetric
all permanent members are symmetric
Shapley value is efficient, hence for a permanent j

$$Sh_j(N,v) = \frac{1}{5}\left(1 - \frac{6}{462}\right) = \frac{91 \cdot 2}{462}$$

Power ratio of nonpermanent to permanent $= 1 : 91 \cdot 2$.

post-1965:

$$v(S) = \begin{cases} 1 & \text{if } S \supseteq P \text{ and } |S| \geq 9 \\ 0 & \text{ow} \end{cases}$$

non-permanent i:

$$Sh_i(N,v) = \binom{9}{3} \frac{8!\,6!}{15!} = \frac{4}{2145}$$

permanent j:

$$Sh_j(N,v) = \frac{1}{5}\left(1 - 10 \times \frac{4}{2145}\right) = \frac{421}{2145}$$

ratio $= 1 : 105 \cdot 25$

~~Restructi~~ Restructuring actually increased the power of the permanent members.

Convex games

$$\forall S, T \subseteq N$$

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

Thm: If (N, v) is a convex game, the SV is in the core.

Proof: For any permutation $\pi \in \Pi(N)$, consider the imputation w^π

$$w_i^\pi = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)).$$

We have shown that this imputation is in core for every $\pi \in \Pi(N)$. Since core is a convex set, any ~~the~~ convex combination of these points will be in core. In particular, the Shapley value

$$Sh(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} w^\pi$$

will also be in the core.

Consistency of the SV.

Defn: Let ϕ be a single valued solution concept, let (N, v) be a coalitional game and $S \subseteq N$ and $S \neq \phi$. The Hart-Mas-colell reduced game over S relative to ϕ is the game $(S, \tilde{v}_{S,\phi})$ s.t.

$$\tilde{v}_{S,\phi}(R) = \begin{cases} v(R \cup S^c) - \sum_{i \in S^c} \phi_i(R \cup S^c, v), & \forall R \subseteq S, R \neq \phi \\ 0 & \text{if } R = \phi \end{cases}$$

(9-9)

Difference with Davis-Maschler reduced game

- solution concept for single-valued solutions
- DM selects the most beneficial coalition of S^c but HM considers the whole of S^c.

Defn: A solution concept ϕ is consistent wrt the HM reduced game if for every game (N, v), every nonempty coalition S, and for every $i \in S$,

$$\phi_i(N, v) = \phi_i(S, \tilde{v}_{S,\phi}).$$

Theorem: SV is HM reduced game consistent.

Core limitation 2: non-existence

For various game settings, core may be empty.
What can be expected in such games?
- Stable sub coalitions may form
- Or a weaker notion of equilibrium may be proposed

Refinements of the core

ϵ-core: idea similar in spirit to ϵ-Nash equilibrium.

Defn: A payoff vector x is in the ϵ-core of a coalitional game (N, v) if

$$\sum_{i \in S} x_i \geq v(S) - \epsilon, \quad \forall S \subseteq N, S \neq \phi.$$

One interpretation/motivation: There is a cost to move from the grand coalition, which is denoted by ϵ. If the value is not decreasing more than that cost, it is possibly not meaningful to deviate from the grand coalition.

Mathematically, no reason why $\epsilon \geq 0$. If $\epsilon < 0$, the condition of ϵ-core is giving some "bonus" for forming a coalition. The allocation not only is coalitionally rational, it awards something that is strictly better by a constant margin.
The ϵ-core allocation is more stable than a core allocation.

But for a given ϵ, the ϵ-core may still be empty. We can continue relaxing the ϵ-core until some non-empty ϵ-core is found.

Least core

Defn: A payoff vector x is in the least core of (N, v) if x is a solution to the following linear program

$$\text{LP-1} \quad \left[\begin{array}{ll} \min \ \epsilon & \\ \text{s.t.} \quad \sum_{i \in S} x_i \geq v(S) - \epsilon & \forall S \subset N. \\ x(N) = v(N) & \end{array} \right.$$

- objective is non positive iff the core of the game is non empty
- For sufficiently large ϵ, the constraints can be always satisfied
- When the core is non-empty, least core does not contain all core allocations - rather gives the least opportunity for every coalition to deviate - is a core refinement.

Still is a set-solution.

The solution of the LP may return a ~~solutions~~ vector x and ϵ s.t. not all inequalities are tight.

Exercise: construct examples of multiple solutions of least core.

Strengthening the least core

Idea: make the slack inequalities tight

Formally, say ϵ_1 is the optimal value of the LP-1. Now we optimize
[Also let S_1 be the set of coalitions for which the inequalities are tight]

$$\text{minimize } \quad \epsilon$$

$$\text{s.t.} \quad \sum_{i \in S} x_i = v(S) - \epsilon_1 \qquad \forall S \in S_1$$

$$\sum_{i \in S} x_i \geq v(S) - \epsilon \qquad \forall S \in 2^N \setminus S_1$$

$$x(N) = v(N)$$

This makes few more inequalities tight.
There could still be some slack inequalities.
We sequentially repeat this procedure until all inequalities become tight.
Since there are finite number of inequalities, this always converge to a unique payoff vector, known as "nucleolus."

Defn: An allocation x is in the nucleolus of a coalitional game (N, v) if it is the solution of a series of LPs

$$(LP_1) \qquad \min \epsilon$$

$$\text{s.t.} \quad \sum_{i \in S} x_i \geq v(S) - \epsilon \qquad \forall S \subseteq N.$$

$$x(N) = v(N)$$

$$(LP_2) \qquad \min \epsilon.$$

$$\text{s.t.} \quad \sum_{i \in S} x_i = v(S) - \epsilon_1 \qquad \forall S \in S_1$$

$$\sum_{i \in S} x_i \geq v(S) - \epsilon \qquad \forall S \in 2^N \setminus S_1$$

$$x(N) = v(N)$$

$$(LP_k) \quad \begin{aligned} \min \ & \epsilon \\ \text{s.t.} \ & \sum_{i \in S} x_i = v(S) - \epsilon_1 \quad \forall S \in \mathcal{S}_1 \\ & \vdots \\ & \sum_{i \in S} x_i = v(S) - \epsilon_{k-1} \quad \forall S \in \mathcal{S}_{k-1} \setminus \mathcal{S}_{k-2} \\ & \sum_{i \in S} x_i \geq v(S) - \epsilon \quad \forall S \in 2^N \setminus \mathcal{S}_{k-1} \\ & x(N) = v(N) \end{aligned}$$

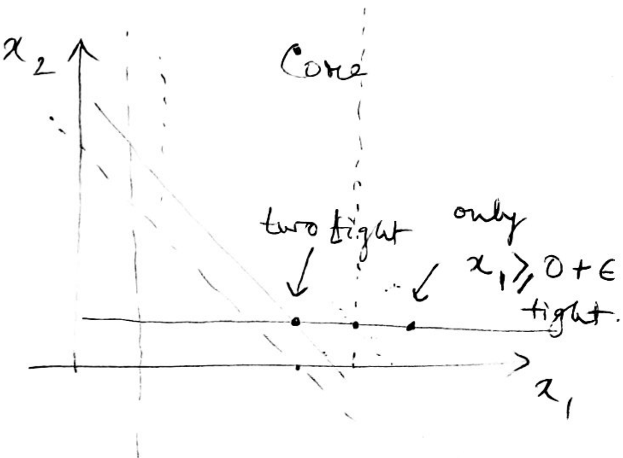
Needs at most n iterations (argue over the dimensions of the variables space. [Exercise]

Intuition: $v(1) = 0$, $v(2) = 1$, $v(1,2) = 3$

core is non-empty

$\epsilon < 0$, more negative is better

if $v(2) = 4$, core empty.



Theorem: For any game (N, v), nucleolus exists and is unique.

Proof: existence: The series of LPs can be solved and reach some assignment of x's s.t. all inequalities are met with equalities. In every round, at least one inequality will become tight, and it will converge (property of LP) and therefore a solution always exists.

Uniqueness: Earlier LPs influence the latter LPs only via the values. Therefore, the set of LPs will always lead to the same set of solutions $(\epsilon_1, \epsilon_2, \ldots)$ [In particular $\epsilon_1 > \epsilon_2 > \ldots$]. After all the iterations are over, we are left with 2^n equations over n variables. It has rank of ~~at most~~ n - if a solution exists, it must be unique.

$$\begin{array}{c|c}
 & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\
\begin{array}{c} S_1 \\ \\ \text{coalitions} \\ \\ S_{2^n} \end{array} & \left[\begin{array}{c} \\ \\ \text{Association} \\ \text{matrix} \\ \\ \end{array} \right] x = \begin{bmatrix} v(S_1) - \epsilon_j \\ \vdots \\ \vdots \\ v(S_{2^n}) - \epsilon_k \end{bmatrix}
\end{array}$$

players

columns are linearly indept.

we have made the RHS live in the space spanned by the columns of association matrix. Therefore, there exists a unique linear combination of those columns to yield the RHS [fact from linear algebra]. ☐

An alternate definition

Nucleolus is also defined wrt excesses

Defn: The excess of a coalition S in (N, v) wrt payoff vector x is denoted as

$$e(S, x, v) = v(S) - \sum_{i \in S} x_i$$

Obs: if core is non-empty, then $\exists\, x$ s.t. all excesses are non-positive.

(10-6)

Given a coalitional game (N, v) and a payoff vector x compute all excesses except coalitions N and ϕ

This $2^n - 2$ dimensional vector is the "raw excess" vector.

The vector is sorted in decreasing order - sorted excess vector denoted $\theta(x, v)$

Given two payoff vectors x and y, we say excesses due to x are <u>lexicographically smaller</u> than those due to y, written $x \preceq_L y$ if for the smallest index i where $\theta(x,v)$ and $\theta(y,v)$ differ, $\theta_i(x,v) < \theta_i(y,v)$.

This is a valid binary relation, which is reflexive, transitive, complete, but not symmetric.

<u>Defn</u>: (Nucleolus, alternative defn)

Given a coalitional game (N, v), the nucleolus is the payoff vector x such that for all other payoff vectors y, $y \succeq_L x$, i.e. x lexicographically minimizes the excesses of all coalitions.

Compact representation of coalitional games

- Have seen several solution concepts – how to compute them in practice.
- The representation of the game is important in answering that question – straightforward representation will take enormous space.
  - gives a feeling that even brute force methods are also "good". Hence a compact representation is important.

Example: <u>Weighted Graph Games</u>

Game is defined by an undirected graph with edge weights and the value of a collection of nodes is the sum of the edge weights that run between the nodes.

Examples: Cities that are connected via high-speed toll highways – how to share the revenue among them.

<u>important</u>: easy to represent the values just $\binom{n}{2}$ numbers for the edge weights.

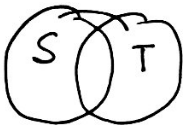
Defn: (WGG) Let (V, W) denote an undirected weighted graph, V = set of vertices, $W \in \mathbb{R}^{|V| \times |V|}$ is the set of edge weights, W symmetric, weight between i and j denoted by $w(i,j)$. The coalitional game WGG is a game (N, v) s.t.

1. $N = V$
2. $v(S) = \sum_{i,j \in S} w(i,j)$. $\forall S \subseteq N$.

(10-8)

Prop : If all weights are non-negative, WGG is convex.

$$v(S) + v(T) \overset{?}{\leq} v(S \cup T) + v(S \cap T)$$



$\downarrow$
edges from $S \setminus T$ and $T \setminus S$ are counted as extra.

Shapley value :

Thm: The SV of the game induced by a WGG (V, W) is

$$Sh_i(N, v) = \frac{1}{2} \sum_{j \neq i} w(i,j).$$

Proof:
$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} w_i^{\pi}$$

$$w_i^{\pi} = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))$$

$$v(P_i(\pi)) = \sum_{j,k \in P_i(\pi)} w(j,k)$$

$$v(P_i(\pi) \cup \{i\}) = \sum_{j,k \in P_i(\pi)} w(j,k) + \sum_{k \in P_i(\pi)} w(i,k)$$

$$w_i^{\pi} = \sum_{k \in P_i(\pi)} w(i,k)$$

sum over all possible permutations, how many times should a specific k appear before i ? $\frac{n!}{2}$

$$\sum_{\pi \in \Pi(N)} w_i^{\pi} = \frac{n!}{2} \sum_{j \neq i} w(i,j) \qquad \square$$

Obs: We can compute Shapley value in $O(n^2)$ time

Answering questions regarding the core of WGG is more complex.

Cut: is a set of edges that divide the nodes of a graph into two parts

$(S, V\setminus S)$ — edges between them form a cut.

Weight of a cut is the sum of its weights

$$\sum_{i\in S,\, j\in N\setminus S} w(i,j)$$

Thm: The Shapley value is in the core of a WGG iff there is no negative cut in the weighted graph.

Proof: From the previous result, if $\exists$ a cut $S, N\setminus S$ s.t. the weight is negative

then
$$\sum_{i\in S} Sh_i = \frac{1}{2}\sum_{i\in S}\sum_{j\neq i} w(i,j)$$

$$= \frac{1}{2}\sum_{i\in S}\left[\sum_{j\neq i, j\in S} w(i,j) + \sum_{j\in N\setminus S} w(i,j)\right]$$

$$= \frac{1}{2}\cdot 2\sum_{(i,j)\in S^2} w(i,j) + \frac{1}{2}\sum_{i\in S,\, j\in N\setminus S} w(i,j)$$

$$= v(S) + \frac{1}{2}\text{ weight of cut}$$

$$< v(S) \qquad \text{iff weight of cut is negative.}$$

<u>Thm</u>: The core is non-empty iff there is no negative cut in the weighted graph.

<u>Pf</u>: ($\Leftarrow$) is obvious from previous claim.

($\Rightarrow$) Suppose $\exists$ a negative cut, then we show core is empty. Say $(S, N\setminus S)$ has a negative cut

$$\sum_{i\in S} Sh_i - v(S) = \frac{1}{2}\sum_{i\in S, j\in N\setminus S} w(i,j) \quad \text{--- (1)}$$

$$\sum_{i\in N\setminus S} Sh_i - v(N\setminus S) = \frac{1}{2}\sum_{i\in N\setminus S\ j\in S} w(i,j) \quad \text{--- (2)}$$

same

Pick any efficient payoff vector x.

$$v(N) = \sum_{i\in S} x_i + \sum_{i\in N\setminus S} x_i = \sum_{i\in S} Sh_i + \sum_{i\in N\setminus S} Sh_i \quad \text{--- (3)}$$

$$\left(\sum_{i\in S} x_i - v(S)\right) + \left(\sum_{i\in N\setminus S} x_i - v(N\setminus S)\right) = \sum_{i\in S, j\in N\setminus S} w(i,j) < 0$$

at least one of them is negative, hence core empty $\square$

However, general WGG are harder to analyze.

<u>Thm</u>: Testing nonemptiness of core for general WGGs is NP-complete.

uses reduction from MAXCUT.

One sided matching - object allocation mechanisms

Quick recap of social choice setting

- Agents have types $\theta_i \in \Theta_i$: private to agents
- Set of outcomes X
- Social choice function maps a type profile to an outcome

$$f : \Theta \to X$$

example: types are preferences over candidates (voting)

$$\theta_i = a \succ b \succ c \succ d .$$

The voting rule collects the preferences and selects a candidate. $X = A$: set of candidates.
In a voting setting, any preference order over the candidates is plausible, and therefore difficult to design truthful mechanisms.

Thm (Gibbard - Sattenthwaite)
If $|A| \geq 3$ and ~~vote~~ preferences are unrestricted total orders, every onto and truthful social choice function must be dictatorial.

Restricted preferences and positive results
- Single-peaked preferences, mechanisms with transfers etc
- Another setting without money: object allocation mechanisms - one sided matching.

(11-2)

Setting of one sided matching

- $M = \{a_1, \dots, a_m\}$ finite set of objects
- $N = \{1, \dots, n\}$ set of agents, $m \geq n$.
- objects are houses, jobs, projects, positions,
- Each agent has a linear order over the objects
  - linear order: R_i of agent i

    complete: $\forall a, b \in M$ either $a R_i b$ or $b R_i a$

    transitive: if $a R_i b$ and $b R_i c \Rightarrow a R_i c$.

    anti-symmetric: if $a R_i b$ and $b R_i a \Rightarrow a = b$.

we will denote such linear orders with P_i

$(P_1, P_2, \dots, P_n)$ is a preference profile

- $\mathcal{M}$: set of all possible linear ~~ob~~ orders over M.
- $P_i(k, S)$ is the k-th top ~~alternati~~ object that belongs to $S \subseteq M$.

Departure from the classic G-S setting:

The preferences are over objects and not over alternatives.

Alternative in this setting is a matching / assignment of the objects to the agents.

A feasible matching is a mapping

$a : N \to M$ injective : distinct objects are allocated to distinct agents

set of alternatives A : collection of such mappings

$a(i) = j \in M$: object j assigned/matched to i.

Why is this a restricted domain?

- There cannot exist any preference profile where certain alternatives can have both permutations

  e.g. let a and b be two alternatives/matchings where player i gets the same object. He is indifferent between the alternatives, hence $\nexists P_i$ s.t. $a P_i b$ or $b P_i a$.

- Good news: G-S theorem does not hold anymore
- expect to have non-trivial truthful mechanisms.

Example: An SCF $f : \mathcal{M}^n \to A$. Define a fixed priority (serial dictatorship) mechanism.

A priority is a bijective mapping

$$\sigma : N \to N$$

The mechanism: every agent in the order σ picks her favorite object from the leftover list.

$$a(\sigma(i)) = P_{\sigma(i)}\left(1, N \setminus \{a(\sigma(1)), \ldots, a(\sigma(i-1))\}\right) \quad i = 1, \ldots, n$$

$$a(\sigma(0)) = \phi.$$

$$f^{\sigma}(P) = a.$$

Remarks:

- A generalization of dictatorship
- Easy to see that this is strategyproof

Defn: An SCF $f : \mathcal{M}^n \to A$ is <u>strategyproof</u> (house alloc. model)

if $\quad f(P_i, \underline{P}_{-i})(i) \; P_i \; f(P_i', \underline{P}_{-i})(i)$

$$\forall P_i \in \mathcal{M} \;\; \forall \underline{P}_{-i} \in \mathcal{M}^{n-1} \;\; \forall i \in N.$$

– also, this is efficient in the following sense.

Defn: An SCF is <u>efficient</u> (house allocation model) if for all preference profiles P and all matchings a, if there exists another matching $a' \neq a$ s.t. either $\quad a'(i) \; P_i \; a(i)$ or $a'(i) = a(i) \;\; \forall i \in N$,

then $\quad f(P) \neq a$.

Proposition: Every fixed priority SCF is strategyproof and efficient.

Pf: Let σ be any fixed priority. f^σ is the SCF

Fix ~~an~~ agent i, $\quad \sigma^+(i) = \{ j \in N : \sigma(j) < \sigma(i) \}$

set of ~~play~~ agents having higher priority than i.

Being truthful agent i gets $\quad P_i(1, M \setminus M^{\sigma^+(i)})$

$M^{\sigma^+(i)}$ – set of objects assigned to $\sigma^+(i)$.

By deviating, agent i cannot get any better since $M^{\sigma^+(i)}$ remains fixed.

<u>Efficiency</u>: Suppose f^σ is not efficient.

$\exists P$ s.t. $f^\sigma(P) = a$. $\exists a' \neq a$ s.t.

$a'(i) \; P_i \; a(i)$ or $a'(i) = a(i) \;\; \forall i \in N$. Consider the first j s.t. $a'(j) \; P_j \; a(j)$. Since all before this were same $a'(j)$ was available to j – contradiction $\square$

However there are strategyproof SCFs that are not necessarily serial dictators.

Ex: $N = \{1, 2, 3\}$ $M = \{a_1, a_2, a_3\}$

The preference order changes depending on the top choice of a specific player.

$$\sigma = \begin{cases} (1, 2, 3) & \text{if } P_1(1) = a_1 \\ (2, 1, 3) & \text{if } P_1(1) \neq a_1 \end{cases}$$

Truthful: For a fixed priority, this is strategyproof, now 2 and 3 cannot change the priority, therefore it is strategyproof for them. Player 1 can,

if $P_1(1) = a_1$: she gets a_1

if $P_1(1) \neq a_1 \Rightarrow P_1(1) \in \{a_2, a_3\}$, the only way she can change the priority is by reporting a_1 as $P_1(1)$, but then she gets a_1 - which is at most her second choice. If she reported truthfully, she could get either top or second choice - hence this is strategyproof for agent 1 too.

Efficiency: similar argument as before.

For a given priority, the outcome is always efficient.

Top-trading cycle with fixed endowments

A different mechanism (actually a class of mechanisms) that is truthful and has other nice properties.

Assume $m = n$ for simplicity.

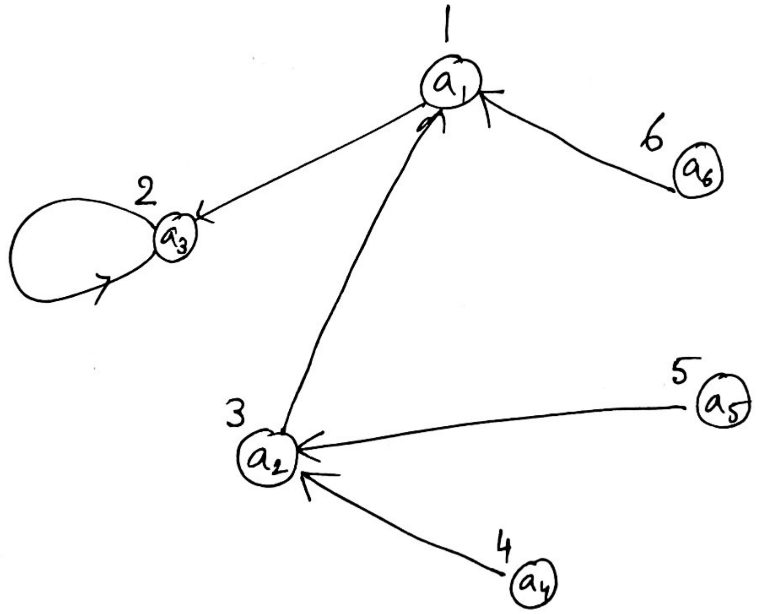
Initialization: Each agent is endowed with a house

Suppose there are 6 agents and are endowed with

a^* : $a^*(1) = a_1$, $a^*(2) = a_3$, $a^*(3) = a_2$, $a^*(4) = a_4$, $a^*(5) = 5$

$a^*(6) = a_6$

P_1	P_2	P_3	P_4	P_5	P_6
a_3	a_3	a_1	a_2	a_2	a_1
a_1	a_2	a_4	a_1	a_1	a_3
a_2	a_1	a_3	a_5	a_6	a_2
a_4	a_5	a_2	a_4	a_4	a_4
a_5	a_4	a_6	a_3	a_5	a_6
a_6	a_6	a_5	a_6	a_3	a_5



Initial graph of the TTC

Algorithm: Top-trading cycle

Step 1: Set $M' = M$, $N' = N$ construct a directed graph G' with nodes N'.

- There is a directed edge from $i \in N'$ to $j \in N'$ iffs $P_i(1, M') = a^*(j)$.
- Allocate houses along every cycle of graph G'. i.e. if $(i', i^2, \dots, i^p, i')$ is a directed cycle in G', set $a(i') = a^*(i^2), \dots, a(i^p) = a^*(i')$.

let $\hat{N}'$ be the set of agents allocated houses $\hat{M}'$

$N^2 = N' \setminus \hat{N}'$, $M^2 = M' \setminus \hat{M}'$

Step k: Continue to get G^k with nodes N^k edge ~~between~~ from $i \in N^k$ ~~and~~ to $j \in N^k$ if

$$P_i(1, M^k) = a^*(j)$$

- - - -

$$N^{k+1} = N^k \setminus \hat{N}^k, \quad M^{k+1} = M^k \setminus \hat{M}^k.$$

Stop: if $N^{k+1} = \phi$. ELSE: REPEAT.

Theorem: TTC with fixed endowment mechanism is strategyproof and efficient.

Proof: Agent i is truthful, then gets room in round k (say)

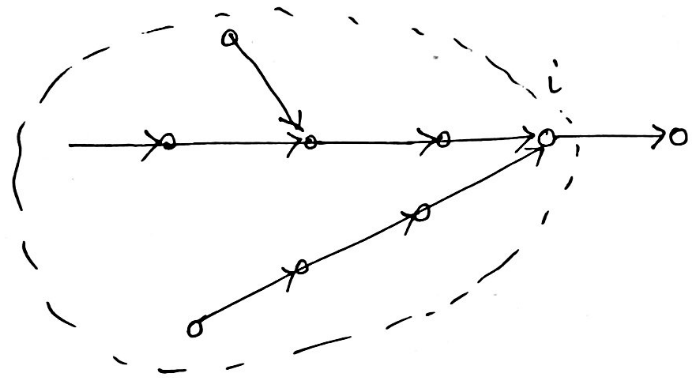
$H^k \rightarrow$ rooms allocated till round k (including that round)

i gets his best choice from $M \setminus H^{k-1}$.

How can agent i deviate:

1. i's deviation gives her a room on or after round k. Gets from $M \setminus H^{k-1}$ rooms – since the rounds till H^{k-1} are unaffected by the misreport – but truthful gives the best in this case, so no reason to misreport.

2. i's deviation gives her a room in round $r < k$.



$\Pi_i = \{$nodes that lead a path to $i\}$

The only way i can change the allocation is by pointing to some node in Π_i

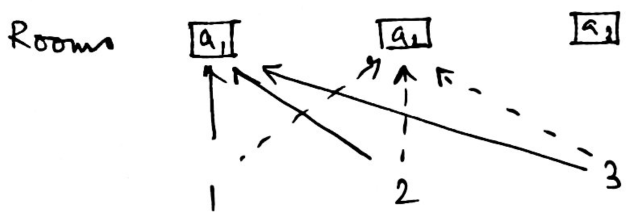
– note: i can't change the cycles by pointing to $M \setminus \Pi_i$

– Why is this important? Since the room i is currently pointing to may not be available till round k, and we do not know what is her next best choice.

Claim: i's room is available till k, then all rooms of agents in Π_i are also available till k.

Hence the choice that i has at any round $r < k$ stays until making the choice in round k, which she'll do in truthful manner

TTC is not serial dictatorship



case 1 : $1 \to a_1$

case 2 : $2 \to a_2$ but not 1.

Efficiency : If this is not efficient, then there must exist some a' which is strictly better for some agent and gives the same house for others. Consider the first stage where in the TTC algorithm ~~gives a~~ differs from a', and $a'(i)\ P_i\ a(i)$ [by assumption]. Hence, $a(i)$ cannot be given to i – contradiction.

Stable house allocation with initial endowments

Stability ensures that when the agents have their initial endowments, and an allocation is suggested, there is ~~not~~ no better group deviation.

Previous example :
initial endowments : $a^*(1) = a_1$, $a^*(2) = a_3$, $a^*(3) = a_2$, $a^*(4) = a_4$, $a^*(5) = a_5$, $a^*(6) = a_6$.

P_3	P_4
a_1	a_2
a_4	
a_3	

If an allocation is proposed $a(i) = i$.

$a(3) = a_3$ and $a(4) = a_4$

$\{3,4\}$ can deviate and allocate a better choice.
3 gets a_4 and 4 gets a_2. ~~and~~ $a_4\ P_3\ a_3$, $a_2\ P_4\ a_4$

Such an allocation is not stable since the group $\{3,4\}$ ~~block~~ block such an allocation.

(12-4) Let a^* denote an initial endowment of agents.

a^S denote the allocation of $S \subseteq N$. — denotes the matchings of players in S with the ~~hoo~~ houses available to players in S.

- A coalition $S \subseteq N$ can block a matching a at a preference profile P if $\exists$ a matching a^S s.t.

$$a^S(i)\ P_i\ a(i) \text{ or } a^S(i) = a(i) \text{ for all } i \in N$$

and the strict preference occurs for at least one $j \in N$.

- A matching is in the core at a profile P if no coalition can block a at P.

- An SCF f is stable if $\forall P$, $f(P)$ is in the core at P.

Note: Stability implies efficiency.

Efficiency only requires that the grand coalition cannot block an allocation

Ex: Efficient but not stable.

$$a^*(1) = a_1, \quad a^*(2) = a_2, \quad a^*(3) = a_3$$

Agent 1, 2 : $a_1 \succ a_2 \succ a_3$

Agent 3 : $a_2 \succ a_1 \succ a_3$

$$a(1) = a_3, \quad a(2) = a_1, \quad a(3) = a_2$$

efficient as 2 and 3 gets their top choices.

not stable, since 1 can deviate and retain his house.

<u>Theorem</u>: The TTC mechanism is stable. Moreover, there is a unique core matching for every preference profile.

<u>Proof</u>: Suppose TTC is not stable. $\exists P$ s.t. matching produced by TTC is not in the core. Let coalition S blocks it. $\exists a^S$ s.t. $a^S(i) P_i a(i)$ or $a^S(i) = a(i)$ $\forall i \in S$ with at least one strict ~~pre~~ preference.

Let $T = \{i \in S : a^S(i) P_i a(i)\}$ The set of all strict improvement individuals.

By assumption $T \neq \phi$.

~~Remind~~ Remember $\hat{N}^k$: people assigned houses in round k

$\hat{M}^k$: houses allocated in round k.

We will look at ~~the~~ how these people in S appear in these sets, i.e., the people $S \cap \hat{N}^k$.

Clearly, $S \cap \hat{N}^1$ are getting their top ranked houses, so they must not be in T, i.e., $S \cap \hat{N}^1 \subseteq S \setminus T$

We will use induction. Let $S \cap \hat{N}^k =: S^k$

~~Let $S^k \subseteq$~~

<u>Claim</u>: If $(S^1 \cup S^2 \cdots \cup S^{k-1}) \subseteq S \setminus T$, we show $S^k \subseteq S \setminus T$

all S^1 to S^{k-1}'s are subsets of $\hat{N}^1 \ldots \hat{N}^{k-1}$ respectively they got houses from $\hat{M}^1, \ldots, \hat{M}^{k-1}$. Hence $S \cap \hat{N}^k$ gets houses from $M \setminus (\hat{M}^1 \cup \hat{M}^2 \cdots \cup \hat{M}^{k-1})$

a gives the best available houses from $M \setminus (\hat{M}^1 \cup \hat{M}^2 \cdots \cup \hat{M}^{k-1})$

hence a^S cannot give them any better houses. Hence $S \cap \hat{N}^k \subseteq S \setminus T$. Hence $S = \bigcup_{k=1}^{K} S^k \subseteq S \setminus T \Rightarrow T = \phi$ □

(12-6)

Uniqueness: Suppose TTC returns a and $\exists a' \neq a$ which is also in core.

Note: In $\hat{N}^1$ every agent gets their top choice.

Hence $a(i) = a'(i) \ \forall i \in \hat{N}^1$, because if not, the agents in $\hat{N}^1$ will block a'.

Now, induction is used.

Suppose $a(i) = a'(i) \ \forall \ i \in \hat{N}^1 \cup \hat{N}^2 \cup \dots \cup \hat{N}^{k-1}$

if $a(i) \neq a'(i)$ in $\hat{N}^k$, we see that every agent $i \in \hat{N}^k$ gets their top remaining houses. $M \setminus (\hat{M}^1 \cup \dots \cup \hat{M}^{k-1})$

~~Then $\hat{N}^k$ will block~~

Since all agents in $\hat{N}^1 \cup \hat{N}^2 \dots \cup \hat{N}^{k-1}$ get same houses in a and a', if there is any difference in a and a' in $\hat{N}^k$, then $\hat{N}^k$ blocks a'.

it must be $a(i) \ P_i \ a'(i)$

This contradicts that a' is a core matching $\square$

We can weaken the notion of stability to an individual level.

Defn: f is individually rational if at every profile P, the matching $f(P) \equiv a$ satisfies $a(i) \ P_i \ a^*(i)$ or $a(i) = a^*(i)$.

Clearly, stability implies individual rationality, since we want the single agent coalitions to be non-blocking.

Hence TTC satisfies individual rationality too. However, ~~the~~ this weaker condition along with the other two properties characterize TTC.

<u>Theorem</u>: A one sided matching mechanism is strategyproof, efficient, and individually rational iff it is a TTC mechanism.

<u>Generalized TTC mechanism</u>

It mixes the fixed priority and TTC in a convenient way.

It defines a priority order for every house - how the initial endowment will be transferred.

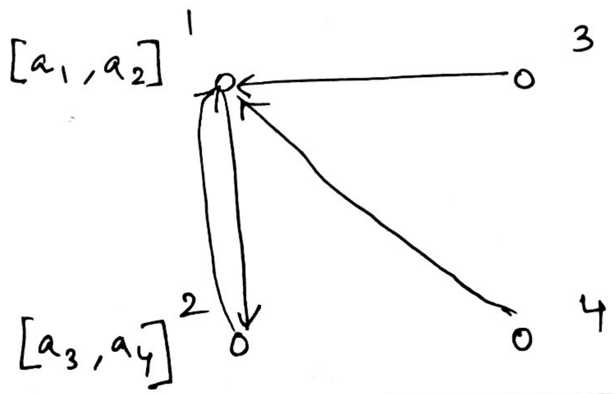
~~Every~~ $\sigma_j : N \to N$ for every $j \in M$

- one agent may endow more than one house

P_1	P_2	P_3	P_4
a_3	a_2	a_2	a_1
a_2	a_3	a_4	a_4
a_1	a_4	a_3	a_3
a_4	a_1	a_1	a_2

$\sigma_1 = \sigma_2 = (1, 2, 3, 4)$ for rooms a_1 and a_2

$\sigma_3 = \sigma_4 = (2, 1, 4, 3)$ for rooms $a_3 a_4$.



$1 \to a_3$, $2 \to a_2$

$4 \to a_4$

$3 \to a_1$

Theorem: GTTC is strategyproof and efficient.

Lec 13

Two sided matching

- More widely used because of its application domain.

Examples:

- Marriage and ~~d~~ dating markets
- Medical residencies
- University - Student matching (IITJEE seat allocation)
- Job market : Employers and Candidates

We will refer to this setting as marriage problem

M : set of men

W : set of women

For simplicity, we assume $|M| = |W|$, but this is not necessary. The results extend to more general settings too.

Every $m \in M$ has a strict preference P_m over W.

Similarly $w \in M$ has a strict preference P_w over M.

$x \, P_m \, y \iff m$ strictly prefers x over y, $x, y \in W$.

A matching is a bijective mapping $\mu : M \to W$.

$\mu(m)$: woman matched to m

$\mu^{-1}(w)$: man matched to w.

Stable marriage problem

Stability in this context is slightly different from the one-sided matching model.

Example: $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$

Table 1:

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

A candidate matching:

$$\mu : \mu(m_1) = w_1 \, , \; \mu(m_2) = w_2 \, , \; \mu(m_3) = w_3$$

but, $w_2 \, P_{m_3} \, \mu(m_3)$ and $m_3 \, P_{w_2} \, \mu^{-1}(w_2)$

(m_3, w_2) can move out and block this matching.

Defn (Stability): A matching μ is pairwise unstable at a preference profile P if $\exists$ m, m' such that ⓐ $\mu(m') \, P_m \, \mu(m)$ and ⓑ $m \, P_{\mu(m')} \, m'$.

- The pair $(m, \mu(m'))$ is called a blocking pair of μ at P.
- If a matching μ has no blocking pairs at ~~any~~ a preference profile P, then it is called a pairwise stable matching at P.

Stable

$m_1 - w_1$
$m_2 - w_3$
$m_3 - w_2$

Questions:

1. Existence of pairwise stable matching?
2. Why pairwise? Contrast with the one-sided matching – a group of people can redistribute their initial endowments and be strictly better off. Group blocking.

Group blocking:

A coalition $S \subseteq (M \cup W)$ blocks a matching μ at a profile P if $\exists$ another matching μ' s.t.

(i) for all $m \in M \cap S$, $\mu'(m) \in W \cap S$, and for all $w \in W \cap S$, $\mu'^{-1}(w) \in M \cap S$, and

(ii) for all $m \in M \cap S$, $\mu'(m)\ P_m\ \mu(m)$ and for all $w \in W \cap S$, $\mu'^{-1}(w)\ P_w\ \mu^{-1}(w)$.

A matching μ is in the core of the induced coalitional game at a profile P if no coalition can block μ at P.

The following result shows that this condition is equivalent to pairwise blocking.

Theorem: A matching is pairwise stable at a profile iff it belongs to the core at that profile.

Pf: ($\Leftarrow$) direction is trivial. If no coalition of arbitrary size can block the matching, clearly a coalition of size 2 cannot block it – hence pairwise stable.

($\Rightarrow$)

(13-4)

($\Rightarrow$) Let μ be pairwise stable at P. For contradiction, assume μ is not in the core at P.

Then $\exists$ $S \subseteq (M \cup W)$ and a matching $\hat{\mu}$ such that for all $m \in M \cap S$ and $w \in W \cap S$ with $\hat{\mu}(m), \hat{\mu}^{-1}(w) \in S$ we have $\hat{\mu}(m) \; P_m \; \mu(m)$ and $\hat{\mu}^{-1}(w) \; P_w \; \mu^{-1}(w)$.

This means that $\exists$ $m \in M \cap S$ s.t. $\hat{\mu}(m) \in W \cap S$

Call $\hat{\mu}(m) = w$, hence

$$w \; P_m \; \mu(m) \quad \text{and} \quad m \; P_w \; \mu^{-1}(w)$$

hence (m, w) is a blocking pair of μ at P. This is a contradiction to μ being pairwise stable at P. $\square$

Stable from now on will refer to pairwise stability.

Answering the other question of existence.

Deferred Acceptance Algorithm (Gale - Shapley)

A stable matching always exist in a marriage market. This is proved via exhibiting an algorithm to find such a matching.

2 versions : men - proposing and women - proposing

One - side of the market proposes the other side, and the proposed agent may accept or reject the offer.

Men-proposing Deferred Acceptance Algorithm

Step 1 : Every man proposes their top-ranked woman

Step 2 : Every woman who got at least one proposal *tentatively* keeps the top man among the received proposals and rejects the rest.

Step 3 : Every man who was rejected in the last round, proposed to the top woman who has not rejected him in earlier rounds.

Step 4 : Every woman who gets at least one proposal, including the tentative accepted proposal tentatively keeps the top man and rejects the rest.

The process is repeated from step 3 till each woman gets at least one proposal, at this point the tentative accepts become final accepts.

Example : Construct the men and women proposing versions in table 1

Illustration : see the app www.facebook.com/adstudmatch/

Remarks :

- Since each woman is allowed to keep only one proposal, no woman gets more than one man
- Similarly, if a man's proposal is tentatively accepted he is not allowed to propose more, that ensures one woman is assigned to one man.
- The algorithm terminates in finite steps since the set of woman a man proposes does not increase and strictly decreases for at least one man.
- This also shows that the algorithm terminates in a matching.

13-6

Stability and Optimality of the DA algorithm

Thm: At every preference profile, the DA algorithm terminates at a stable matching for that profile.

Proof: Consider men-proposing DA algorithm (similar proof for women-proposing) for a preference profile P. Let μ be the matching of the DA algorithm. Assume for contradiction, μ is not stable. Hence $\exists$ $m \in M$ and $w \in W$ s.t. (m, w) is a blocking pair. By assumption, ~~we~~ $w \neq \mu(m)$ ~~hence~~ and $w \, P_m \, \mu(m)$. Then in this algorithm m must have proposed w at some round and ~~was~~ have been rejected before being matched to $\mu(m)$. But w rejected m since she got a better proposal. Therefore $\mu^{-1}(w) \, P_w \, m$. This contradicts the fact that (m, w) is a blocking pair. $\square$

Questions:

1. Men-proposing and Women-proposing versions of DA ~~to~~ may lead to different stable matches. Is there a reason to prefer one?
2. How should we define a desirable criterion for selecting one stable matching?

Recap: Stability at a profile where no pair of men and women can block an allocation/matching.

— Deferred acceptance algorithm ensures stability (both versions)

Question: comparison between stable matches?

Defn: A matching μ is men-optimal stable matching if μ is stable and for every other stable matching μ' we have $\mu(m)\ P_m\ \mu'(m)$ or $\mu(m) = \mu'(m)$ $\forall\, m \in M$.

Similar definition for women-optimal stable matching.

Remark: If there exists two ~~men~~ men-optimal stable matching, then they must differ for at least one man (in fact for two men), and since preferences are strict this man must be worse off in one of these stable matches — and hence the men-optimal stable matching is unique.

Theorem: The men-proposing version of the Deferred Acceptance algorithm terminates at the unique men-optimal stable matching.

[similarly, the women-proposing version terminates at the women-optimal stable matching].

14-2

Proof: Define a woman w is "possible" for a man m if (m, w) is matched in some stable matching.

In a stable matching, for every man who is matched to a woman, there exists at least the same number of men as the number of women above the matched woman ~~on~~ (acc to his preference) in the preferences of those women.

Claim: A woman who is possible for a man never rejects him in this algorithm.

Proof via induction: At stage 1, ~~no one rejects~~ clearly this holds, if someone is rejected, must be impossible hence the claim holds. Suppose this is true till stage n.

Suppose at round n+1, woman w rejects m in favor of m'.

- this implies that m' approached w in round ~~n~~ n+1 ~~at~~ and all women that m' prefers to w (whom he made prior approaches but were rejected) must be impossible for him - according to the induction hypothesis.

Then w must be impossible for m.

- Suppose not, there exists some stable match involving (m, w), then m' must be matched to someone else.
  - cannot put m' with women above w (they are impossible for him)
  - cannot put with women below w since then (m', w) makes a blocking pair.

Hence the claim is proved.

~~Aso~~ The men-proposing DA algorithm gives every man their ~~be~~ most preferred "possible" woman. - hence men-optimal.

Let us denote the men-optimal stable match as μ^m and women-optimal stable match as μ^w

<u>Question</u>: Can both sides be happy? i.e., does there exist a matching that is both sides' optimal?

The general answer is NO. We saw example of men and women optimal solutions to be different. But something more is true. Let us explore the structure of the stable matchings a bit more.

<u>Theorem</u>: Let μ and μ' be a pair of stable matchings. Then

$$\mu(m) \; P_m \; \mu'(m) \quad \text{or} \quad \mu(m) = \mu'(m) \quad \forall m \in M$$

iff

$$\mu'^{-1}(w) \; P_w \; \mu^{-1}(w) \quad \text{or} \quad \mu'^{-1}(w) = \mu^{-1}(w) \quad \text{for all } w \in W.$$

Proof: ($\Rightarrow$) [The other direction is very similar]

Let μ and μ' be s.t.

$$\mu(m) \; P_m \; \mu'(m) \quad \text{or} \quad \mu(m) = \mu'(m) \quad \forall m \in M.$$

Suppose for contradiction

$$\mu^{-1}(w) \; P_w \; \mu'^{-1}(w) \quad \text{for some } w \in W.$$

Let $\mu^{-1}(w) = m$ and from above $\mu'^{-1}(w) \neq m \Rightarrow \mu'(m) \neq w$

hence $\underbrace{\mu(m)}_{=w} \; P_m \; \mu'(m)$

Then (m, w) forms a blocking pair of μ'.

Contradiction to the fact that μ' is stable $\square$

The previous two theorems say that the men-optimal stable matching is the worst stable matching for women and vice-versa.

We can define a binary relation between stable matchings.

Defn: We say $\mu \triangleright \mu'$ if for every $m \in M$, either $\mu(m)\, P_m\, \mu'(m)$ or $\mu(m) = \mu'(m)$.

[Equivalently, $\mu'^{-1}(w)\, P_w\, \mu^{-1}(w)$ or $\mu'^{-1}(w) = \mu^{-1}(w)$ $\forall w \in W$]

Note: $\triangleright$ is not a complete relation, cannot compare all stable matchings. But an immediate corollary of the previous two theorems.

Corollary: For any stable matching μ,

$$\mu^m \triangleright \mu \triangleright \mu^w .$$

There are more structures of stable matchings. For any pair of stable matchings μ, μ', we can construct another matching $\mu'' \equiv (\mu \vee^m \mu')$ as follows: for every $m \in M$

$$\mu''(m) = \begin{cases} \mu(m) & \text{if } \mu(m)\, P_m\, \mu'(m) \text{ or } \mu(m) = \mu'(m), \text{ and} \\ \mu'(m) & \text{if } \mu'(m)\, P_m\, \mu(m) . \end{cases}$$

Hence

$$(\mu \vee^m \mu')(m) = \max_{P_m} \left(\mu(m), \mu'(m)\right)$$

Similarly we can define such a matching using the womens' preferences

$$(\mu \vee^{W} \mu')^{-1}(w) = \max_{P_w} \{\mu^{-1}(w), \mu'^{-1}(w)\}.$$

Not clear if μ'' is a matching. But the next result ~~shows~~ answers that.

<u>Theorem</u>: For every pair of stable matchings μ and μ' both $(\mu \vee^{m} \mu')$ and $(\mu \vee^{W} \mu')$ are stable matchings.

<u>Proof</u>: <u>Part 1: μ'' is a matching</u>

Say for contradiction μ'' is not a matching

$\exists$ some $m, m' \in M$ s.t.

$\mu''(m) = \mu''(m')$

	μ''	μ	μ'
m	w	w	
m'	w		w

Then it must be the case that for one of μ or μ' m is assigned w and for the other m' is assigned w, WLOG assume

$\mu(m) = w$ and $\mu'(m') = w$

also $w \, P_m \, \mu'(m)$ and <u>$w \, P_{m'} \, \mu(m')$</u>.

Now μ' is a stable matching

<u>$m' \, P_w \, m$</u>

[Else, $m \, P_w \, m'$ and (m, w) is a blocking pair of μ']

But now, (m', w) forms a blocking pair to μ.

(14-6)

Part 2: μ'' is a stable matching

Assume for contradiction (m, w) is a blocking pair of μ''. Hence

	μ''	μ	μ'
m	w_1	w_1	w_2
m_1	w	?	?

$\mu''(m) = w_1$ but $m \, P_w \, m_1$

$\mu''(m_1) = w$ $\quad w \, P_m \, w_1$

By definition of μ'', either μ or μ' matches m to w_1, WLOG assume

$w_1 = \mu(m)$ and $w_2 = \mu'(m)$, where ~~w_2~~ w_1 can be same as w_2

By definition of μ'', $w_1 \, P_m \, w_2$ or $w_1 = w_2$

in both cases,

$w \, P_m \, w_2$ and we already have $w \, P_m \, w_1$.

Now for m_1, ~~either~~ w must be matched to him either in μ or μ', we show neither is possible which is a contradiction.

a) if $w = \mu(m_1)$, $m \, P_w \, m_1$ and $w \, P_m \, w_1 \Rightarrow w \, P_m \, \mu(m)$

so (m, w) blocks μ.

b) if $w = \mu'(m_1)$, $m \, P_w \, m_1$ and $w \, P_m \, w_2 \Rightarrow w \, P_m \, \mu'(m)$

$\Rightarrow (m, w)$ blocks μ'.

A similar proof for $(\mu \vee^w \mu')$ $\square$

Using this result, given any pair of stable matchings, one can move towards men or women optimal.

Exercise: a similar definition using min $(\mu \wedge^m \mu')$, $(\mu \wedge^w \mu')$

Strategic Issues in DA algorithm

Consider the men-proposing version of this algorithm

The earlier example of a preference profile

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3		
			m_2		

~~m_2~~	m_1
~~m_1~~	m_2
~~m_3~~	m_3

Truthful:

$m_1 \to w_2$ $\quad m_1 - w_2$

$m_2 \to w_1$ $\quad m_3 - w_1$

$m_3 \nearrow$

$m_2 \to w_3$ $\quad m_2 - w_3$

Can anyone improve by a misreport of the preference

let w_1 report ~~$m_2 \succ m_1 \succ m_3$~~ $m_1 \succ m_2 \succ m_3$

$m_1 \to w_2$ $\quad m_1 - w_2$

$m_2 \to w_1$ $\quad m_2 - w_1$

$m_3 \nearrow$

$m_3 \to w_2$ $\quad w_2$ $\times$ m_1 $\quad w_2 - m_3$

$\checkmark$ m_3

$m_1 \to w_1$ $\quad w_1 - m_1$

Theorem: The men (women)-proposing DA algorithm is strategyproof for men (women).

Q: Can there be a mechanism that is truthful for both?

A: No.

Theorem: No stable matching algorithm can be strategyproof for both men and women.

Some open research directions in matching

- Fairness considerations — is there a stable match that is more egalitarian for both men and women
- Feasibility / multiple attributes — ~~lke~~ the kidney exchange problem - it is a house allocation but not everyone can receive any kidney.
  - Similarly, if preferences are multidimensional, one preference for each attribute —
    - students to universities
    - advertisers to viewers
- Monetary transfers — classical quasi-linear setting
  - questions of revenue can be asked.

Strategic Network Formation

Networks are formed via connections between individuals. We ask for the incentives for individuals to form links - and reason for which sort of networks may result due to their strategic choices.

Game Theoretic model of network formation

- costs and benefits for agents associated with networks
- agents are the nodes, and they choose links
  - countries with trade relations
  - people choosing friends
  - researchers with research ~~co~~ collaborations
  - employees with companies
- Contrast individual and social choices.

Modeling choices for adding/forming / or deleting links

- consensus needed (undirected / directed)
- coordinate changes (network structures, influence)
- dynamic or static
- sophisticated agents - can compute the values
- can they compensate each other to form links?
- links adjustable in intensity.

Questions:

1. Which networks are likely to form?
2. Stable against perturbations?
3. Efficient from a global perspective? - ~~Government subsidies~~
4. How inefficient they are if not efficient?
5. Can intervention help improve efficiency?

Jackson-Wolinsky (1996) model of network formation

- $u_i(g)$: payoff to agent i if the network is g.

<u>Connections model</u> (JW 1996)

- $0 \leq \delta_{ij} \leq 1$ ... a benefit parameter for connection between i and j
- $0 \leq c_{ij}$ cost to i to maintain a link with j
- $\ell(i,j)$ - length of the shortest path between i and j

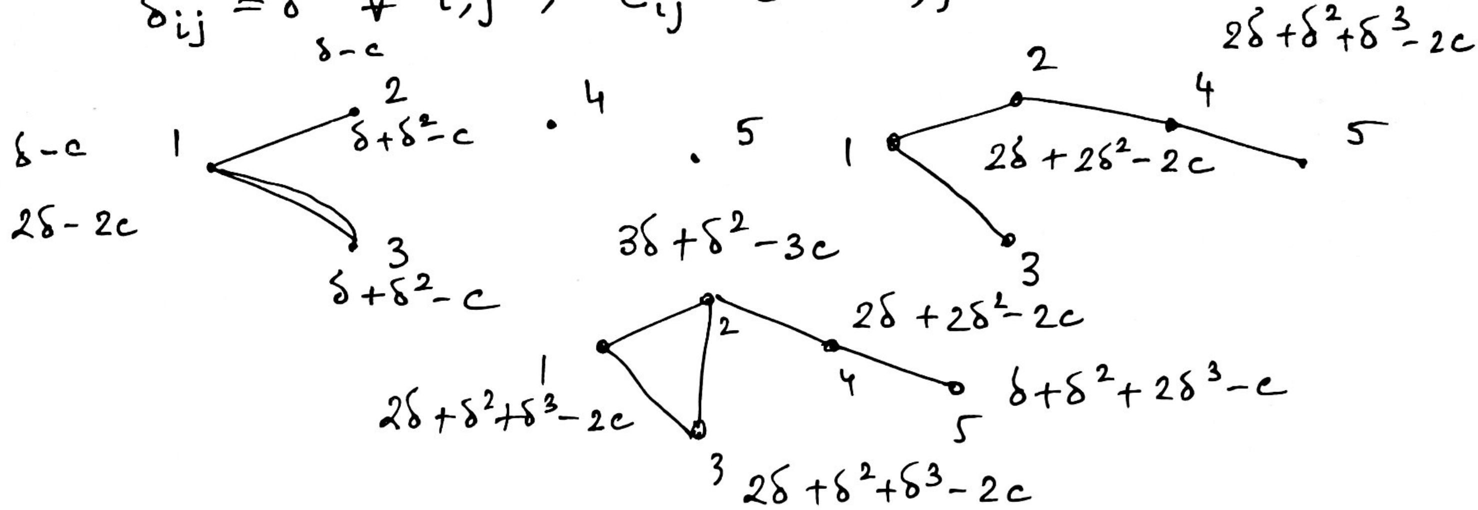
Utility model:

$$u_i(g) = \sum_{j \in N \setminus \{i\}} \delta_{ij}^{\ell(i,j)} - \sum_{k \in N_i(g)} c_{ik}$$

(15-4)

Symmetric version

$\delta_{ij} = \delta \ \forall \ i,j \ , \quad c_{ij} = c \ \forall \ i,j$



Shortest path is considered as the contribution flow.

- What network will form?
- Is that optimal from the global perspective?

Modeling Incentives / Equilibrium

consensus is needed to form a link.

- Every agent announces her agents of choice and link forms if mutual agreements are received.
- Nash equilibrium: no agent can gain from unilateral deviation.

Not a good notion:

Both are NE but unsatisfactory since it says anything can happen

Any reasonable model should allow to form the link.

Other equilibrium notions have similar trouble

- the off-the-shelf concepts from non-cooperative game theory may not work.

Pairwise Stability

~~Current links~~

No single agent gain by deleting a link

No pair of agent gains by adding a link

Defn: A graph g is pairwise stable if

a) $\forall (ij) \in g \quad u_i(g) \geq u_i(g \setminus (ij))$ and

$$u_j(g) \geq u_j(g \setminus (ij))$$

b) $\forall (ij) \notin g$ if $u_i(g + (ij)) > u_i(g)$

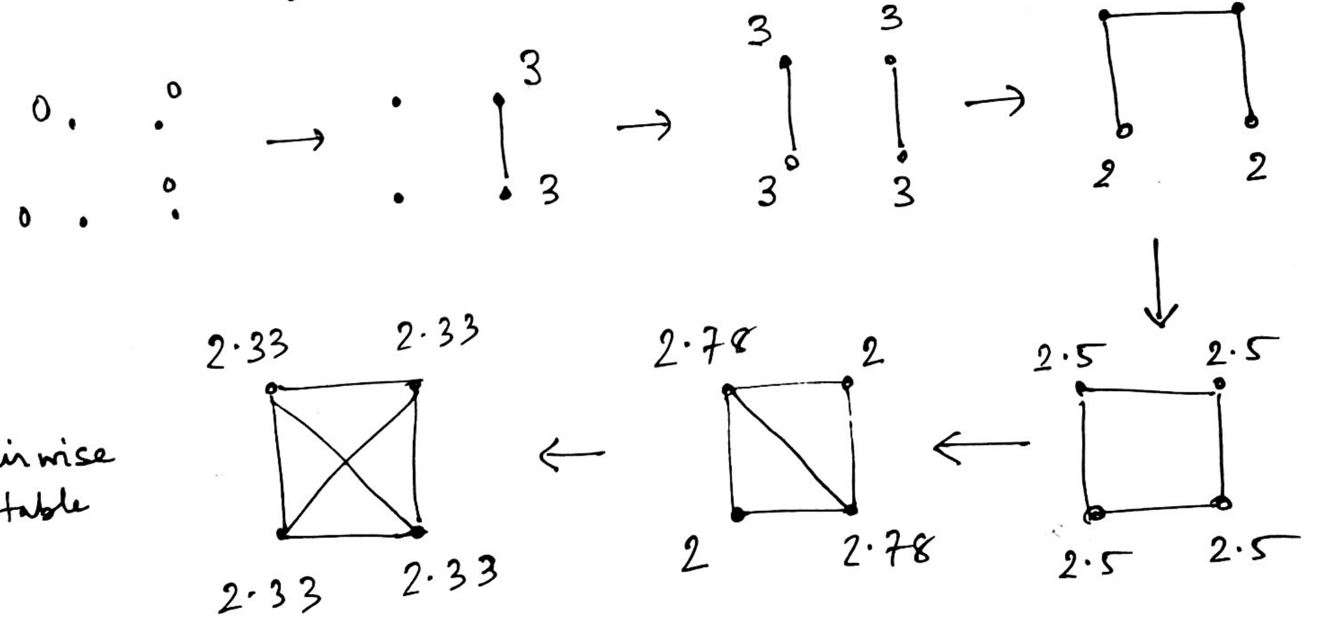
then $u_j(g + (ij)) < u_j(g)$.

This is weak assumption – minimal to work with

Example: (1)

pairwise stable.

(2)



pairwise stable

Note: The pairwise stable network gives worse payoff than ~~the~~ some unstable ones.

– Individual incentives drag the network from that optimal network.

(15-6)

Pareto efficiency

A network g is PE if $\nexists\, g'$ s.t.

$u_i(g') \geq u_i(g) \;\forall i \in N$ and strict for some $j \in N$.

Efficiency : $\quad g \in \underset{g \in \mathcal{G}}{\operatorname{argmax}} \sum_{i \in N} u_i(g)$.

utilitarian

Efficient $\Rightarrow$ PE. $\quad$!PE $\Rightarrow$!EFF.

Explanation using the previous example.

$\longrightarrow$ Back to connections model (δ, c) - symmetric version

~~Theorem~~: Consider efficient networks

Theorem: • When $c < \delta - \delta^2$ [low cost]

Classical market — complete network is uniquely efficient.

• when $\delta - \delta^2 < c < \delta + (n-2)\delta^2/2$ [medium cost]

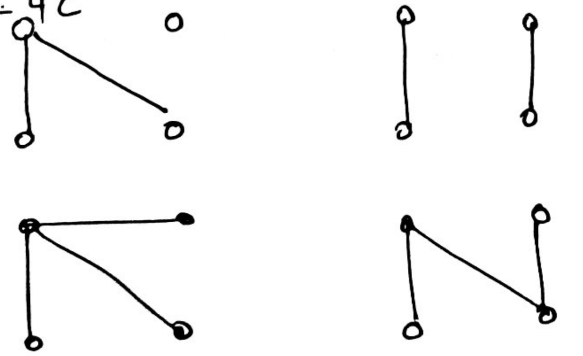
Amazon/ Flipkart — star networks with all agents are uniquely efficient.

• when $\delta + (n-2)\delta^2/2 < c$ [high cost]

RIP trade platforms — empty network is uniquely efficient.

Why Stars?

$4\delta + 2\delta^2 - 4c$ $\quad\quad$ $4\delta - 4c$

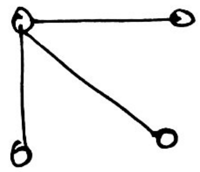


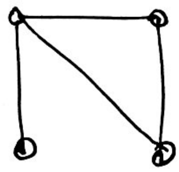
$6\delta + 6\delta^2 - 6c$ $\quad\quad$ $6\delta + 4\delta^2 + 2\delta^3 - 6c$

indirect connections are longer

With given number of links, stars are most efficient ways to connect individuals.

Star vs Complete





$6\delta + 6\delta^2 - 6c$ $\qquad$ $8\delta + 4\delta^2 - \underline{8c}$

made some indirect connections direct - thereby more benefit ~~so~~ but increased the cost

When is the gain more than the cost

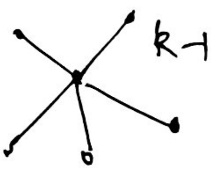
if $\delta - \delta^2 > c$, adding direct link is beneficial.

Proof: Part 1: If $i\,j$ are not directly connected then the benefit is at most δ^2. If they are connected, $\delta - c$ is the ~~cost~~ benefit but $\delta - c > \delta^2$ ~~hence~~ in the low cost region. Hence adding the edge is always beneficial. Others are weakly better.

Part 2: $c > \delta - \delta^2$: first show that the value of a component is maximum when the component is a star

- value of a star with k players is

$$2(k-1)(\delta - c) + (k-1)(k-2)\delta^2$$



- value of a network with k players and m links $(m \geq k-1)$ is at most

$$2m(\delta - c) + \left(2 \cdot \binom{k}{2} - 2m\right)\delta^2$$

$\binom{k}{2} - m$ indirect links counted twice for each player

The difference between value of star and value of any other network is at least

$$2\left(m - (k-1)\right)\left[\delta^2 - (\delta - c)\right] > 0 \quad \text{if } m > k-1$$

in a region $\delta^2 > \delta - c$

(5-8)

If $m = k-1$ and not a star, then some pair of nodes is at a distance of more than 2, so less value than a star:

- star is better.

Can two stars be better than one star?

Exercise: show that two stars with k and k' nodes give less welfare than one consolidated star.

~~Part 3~~ Finally, whether to keep a star or empty

look at a star of size n — if the total utility is > 0 then star is optimal

$$2(n-1)(\delta - c) + (n-1)(n-2)\delta^2 > 0$$

$$\Rightarrow \quad c < \delta + (n-2)\delta^2/2 .$$

else empty is better [gives cases 2 and 3] $\square$.

Pairwise stability

Low cost : $c < \delta - \delta^2$ - complete network is pairwise stable

medium/low cost : $\delta - \delta^2 < c < \delta$

- star is pairwise stable
- others are too

medium/high cost : $\delta < c < \delta + (n-2)\delta^2/2$

- star is not pairwise stable

high cost : $c > \delta + (n-2)\delta^2/2$

- empty is pairwise stable.