CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2018**

Project: Assignment 1

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Solution 1: Given: $f(P_1, P_2) = a$

Table 1.1: Two preference profiles

P_1	P_2	P'_1	P'_2
a	c	b	a
b	b	a	b
c	a	c	c

(a) Consider $f(P'_1, P_2)$:

P'_1	P_2
b	c
a	b
c	a

$\therefore f(P'_1, P_2) \in \{b, c\}$
Suppose $f(P'_1, P_2) = c$
$\implies f(P_1, P_2) = c$ (by MONO) $\Rightarrow \Leftarrow$
$\therefore f(P'_1, P_2) = b$

Consider:

P'_1	P''_2
b	c
a	a
c	b

$\therefore f(P'_1, P''_2) \in b, c$
Again, $f(P'_1, P''_2) = b \quad (\because f(P'_1, P''_2) = c \implies f(P_1, P_2) = c \; by \; Monotonicity \Rightarrow \Leftarrow)$

$\therefore$ **By Monotonicity:**
$f(P'_1, P'_2) = b$
Hence Proved

(b) No, the earlier conclusion does NOT hold in this scenario.
That is so because we used $P_2'' : (c, a, b)$ which is not single-peaked (i.e. $P_2'' \notin \mathcal{S}$) to prove $f(P_1, P_2) = b$.
The mechanism which would allow $f(P_1', P_2') = a$ is as follows:
$f(P) = \min_{i \in N} \{P_i(1)\}$
We already know that the above mechanism is strategy proof and ONTO.
Verification:
$f(P_1', P_2') = \min_{i \in \{1,2\}} \{P_i'(1)\}$
$\therefore f(P_1', P_2') = \min\{b, a\}$
$\therefore f(P_1', P_2') = a$

Solution 2: The set of alternatives is the set of all subsets of objects: $\{S : S \subseteq X\}$. If there are at least 2 projects then, the set of alternatives is at least 3 (the cardinality of set of alternatives being $2^{|X|} - 1$). Now, consider two alternatives S and T such that $S \subset T$. By definition of the preference ordering, any agent is either indifferent between S and T or prefers T over S. Hence, the preference ordering where S is ranked higher than T can never arise. This is a, clearly, a restriction of the domain and hence, we cannot apply the Gibbard-Satterthwaite result here.

Solution 3: Let P be a preference profile. Since the given SCF, f, is median-voter SCF, we need to consider only the peak preferences of all agents, i.e., $P = \{P_1(1), P_2(1), \ldots, P_n(1)\}$.
Let $f(P) = a \in A$, (the set of alternatives)
Let $K \subseteq N$ and $R = \{P_i(1) \in P : i \in K\}$.
So the following situations are possible:

- $a \in R$ and $P_i(1) = a$, for some $i \in K$. Then, $\nexists b \in A$ s.t. bP_ia. So there is no possibility of manipulation in this case.
- $b < a, \forall\, b \in R$. In this case, if any agent(s) shift their preferences anywhere to the left of a, the median will not change. If any of them shift their peak preferences to the right of a, then the median will shift right. Let $f(P') = c \in A$. But as all $P_i's$ are single-peaked preference and $a > P_i(1), \forall i \in K$, aP_ic, $\forall\, i \in K$. Thus, no manipulation is profitable in this case.
- Symmetrically, when $b > a$, $\forall\, i \in K$, there's no profitable manipulation either.
- Now suppose $b < a$, for some $b \in R$ and $c > a$, for some $c \in R$. Let $i, j \in K$ be such that $P_i(1) < a$ and $P_j(1) > a$. Let $p = P_i(1)$ and $q = p_j(1)$.Now, after some manipulation, suppose the median shifts somewhat to the right and the new median is c, the new peak preferences of all agents is P. Then as P_i and P_j are single-peaked preferences and $p < a < c$, $(a = f(P))P_i(f(P') = c)$. So, the manipulation was not profitable in this case. Suppose, after some manipulation, the median, instead, shifts left, the new median becomes c and the new peak preferences of all agents becomes P'. In that case too, since P_i and P_j are single-peaked preferences and $c < a < q$, $(a = f(P))P_j(f(P') = c)$. So, the manipulations is not profitable again.

So, by the above arguments, there does not exist any profitable manipulation. Hence, f, the median-voter SCF, is group-strategy proof.