Midsem Answer Key

CS711: Game Theory and Mechanism Design

October 30, 2020

1. False.
   Counter Example: The Neighbouring Kingdoms's Dilemma.

	A	D
A	5,5	0,6
D	6,0	1,1

   The only PSNE is (D,D) but the best utility achievable by both players is 6 which is not achievable in the PSNE.

2. (a) For $\alpha = 1$, the strategic form of the altruistic version becomes,

	A	D
A	10,10	6,6
D	6,6	2,2

   This game has (A,A) as the SDSE, which is also the strategy profile that maximizes the utility for both. The Neighbouring Kingdoms' Dilemma game has a suboptimal equilibrium and hence the players were in a dilemma. But here there isn't any dilemma and therefore this game is no longer the classical Neighbouring Kingdoms' Dilemma game.
   Note: The change of SDSE from (D,D) to (A,A) is not a change in the conclusion. The change is because of the fact that a player was in dilemma whether the other player will play A or not, in which case, the utility would be better than the SDSE, but here the maxima is at the SDSE and there is no dilemma.

   (b) i. Classical Neighbouring Kingdoms' Dilemma game requires (D,D) as the SDSE, which implies that,
   For Player 1,
$$6 > 5 + 5\alpha \quad \text{and} \quad 1 + 1\alpha > 6\alpha \quad \Rightarrow \alpha < \frac{1}{5}$$
   For Player 2,
$$6 > 5 + 5\alpha \quad \text{and} \quad 1 + 1\alpha > 6\alpha \quad \Rightarrow \alpha < \frac{1}{5}$$
   Hence, for $\alpha < \frac{1}{5}$, the game is a classical Neighbouring Kingdoms' Dilemma.

   ii. The game is not Neighbouring Kingdoms' Dilemma for $\alpha \geq \frac{1}{5}$. We use the Characterization Theorem and find that for $\alpha = \frac{1}{5}$, all possible mixed strategy profiles are MSNEs. In particular, when one player picks a pure strategy and the other mixes or both picks a pure strategy.
   For $\alpha > \frac{1}{5}$, applying the same Characterization Theorem, we find the only Nash Equilibrium to be (A,A) which is PSNE.

3. (a) (Turn, Don't Turn) and (Don't Turn, Turn) are the two PSNEs.

   (b) (Turn, Don't Turn) is the only PSNE.

(c) Applying the Characterization Theorem, we get the MSNE to be $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{1+T}, \frac{T}{1+T}))$.
Payoff of Player 1 $= -(\frac{T}{1+T})$,
Payoff of Player 2 $= -(\frac{1}{2})$.

(d) At $T = 2$, Player 1 is more likely to turn, probability $= \frac{1}{2}$ as opposed to $\frac{1}{3}$ for Player 2. Payoff of Player 1 is $-\frac{2}{3}$ and that of Player 2 is $-\frac{1}{2}$. Hence, expected payoff of Player 2 is higher at $T = 2$. Player 2's equilibrium mixed strategy depends on T.

(e) Note that, T affects the utility of Player 1 alone and in a positive way, i.e. payoff increases as T increases. However, the expected payoff of Player 1 decreases as T increases. That seems quite paradoxical. The paradox, as we know from the analysis above is because of the rather fast drop in the probability of Player 2 playing turn $(\frac{1}{1+T})$.

4. (a) $(\frac{2}{5}, \frac{3}{5}, 0)$
   (b) $(\frac{22}{25}, \frac{2}{25}, \frac{1}{25})$
   (c) We need to show that the mixed strategy profile given by the answers of 4.(a) and 4.(b) is a MSNE. This can be shown directly or via the characterization theorem of MSNE, i.e., the expected utility of player 2 is same across all pure strategies of *player 2* when player 1 picks the mixed strategy $(\frac{2}{5}, \frac{3}{5}, 0)$. [similar for the converse].
   Note: This is different from the equalizing strategy. Equalizing strategy gives equal payoffs to *player 2* when it mixes strategies and player 1 picks pure strategies. It is equivalent only in the case of matrix games since the payoffs of the players are just negative of each other.
   (d) If both players have an equalizing strategy, using the fact that it is a matrix game, the equalizing strategy of player 1 is mixed strategy profile that satisfies the characterization theorem's requirement for player 2 (and vice-versa). [utilities are negative of each other, hence the equality holds for the other players too.]
   (e) 4.(d) assumes that each player has an equalizing strategy, only then it is an optimal strategy. One can construct a game where one player has an equalizing strategy while the other doesn't and the equalizing strategy is not optimal.

	A	B
A	1	-1
B	2	3

   Player 1 has an equalizing such that $(\frac{1}{3}, \frac{2}{3})$ but Player 2 doesn't. For Player 1, playing B is clearly better than this mixed strategy.

5. The game is $G = \langle N, (A_i), (u_i) \rangle$ with $N = \{1, 2\}$, $A_1 = A_2 = \{T_1, T_2, T_3\}$ and u_1 and u_2 are defined by the following matrix:

	T_1	T_2	T_3
T_1	$0, 0$	$v1, -v1$	$v1, -v1$
T_2	$v1, -v2$	$0, 0$	$v2, -v2$
T_3	$v3, -v3$	$v3, -v3$	$0, 0$

Since, utility of player 2 is the negative of utility of player 1, for any strategy, the game is strictly competitive.
It is a zero sum NFG with maxmin value$(= 0) \neq$ minmax value$(= v_2)$, there is *no PSNE.*

A mixed strategy equilibrium must be of the form, $\alpha^* = (\alpha_1^*, \alpha_2^*) = ((a_1, a_2, a_3), (b_1, b_2, b_3))$, with $0 \leq a_i, b_j \leq 1$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1$

We first want to calculate the support of α_1^* and α_2^*;. Note that for all Nash Equilibria, $\alpha_1^* \supseteq \alpha_2^*$. Only targets that are actually attacked should be defended. To sec this, note that if $b_1 > 0$

and $a_1 = 0$, then $(0, b_2 + \frac{1}{2}b_1, b_3 + \frac{1}{2}b_1)$ would be a better response to α_1^* than (b_1, b_2, b_3). Similarly, if $b_2 > 0$ and $a_2 = 0$, $(b_1 + \frac{1}{2}b_2, 0, b_3 + \frac{1}{2}b_2)$ would be better, and if $b_3 > 0$ and $a3 = 0$, $(b_1 + \frac{1}{2}b_3, b_2 + \frac{1}{2}b_3, 0)$ would be better. We now make a case distinction.

Case 1: $|supp(\alpha_2^*)| = 1$
Because $\alpha_1^* \subseteq \alpha_2^*$. this means that the same target is attacked and defended all the time, leading to payoffs of 0 to both players. Then α_1^* is not a best response to α_2^*,: Attacking another target with positive probability would improve the payoff. Hence α^* cannot be a Nash Equilibrium.
Case 2: $|supp(\alpha_2^*)| = 2$
In this case, player 2 defends two different targets. Because $\alpha_1^* \subseteq \alpha_2^*$, these targets are under attack with non zero probability. It must be the case that $supp(\alpha_2^*) = (T_1, T_2)$: If T_3 were one of the defended (and hence attacked) targets, and either T_1 or T_2 were undefended, then player 1 could increase his payoff by never attacking the third target and instead increasing the probability of attacking the undefended target by a_3.
This means that there are two possibilities:

$$supp(\alpha_1^*) = \{T_1, T_2\} \text{ } and \text{ } supp(\alpha_2^*) = \{T_1, T_2\} \tag{1}$$

$$supp(\alpha_1^*) = \{T_1, T_2, T_3\} \text{ } and \text{ } supp(\alpha_2^*) = \{T_1, T_2\} \tag{2}$$

Case 3: $|supp(\alpha_2^*)| = 3$
This leads to the third and last possibility:

$$supp(\alpha_1^*) = \{T_1, T_2, T_3\} \text{ } and \text{ } supp(\alpha_2^*) = \{T_1, T_2, T_3\} \tag{3}$$

We will now discuss these *three possibilities* in order. Each possibility corresponds to different requirements on payoffs. In general, the payoff of a strategy profile $\alpha = ((a1, a2, a3), (b1, b2, b3))$ is as follows:

$$\begin{aligned} U_1(\alpha) =& a_1 b_1 \cdot 0 + a_1 b_2 \cdot v_1 + a_1 b_3 \cdot v_1 + \\ & a_1 b_2 \cdot v_2 + a_1 b_2 \cdot 0 + a_1 b_3 \cdot v_2 + \\ & a_1 b_3 \cdot v_3 + a_1 b_2 \cdot v_3 + a_1 b_3 \cdot 0 \\ =& a_1 (1 - b_1) v_1 + a_2 (1 - b_2) v_2 + a_3 (1 - b_3) v_3 \end{aligned}$$

The equation is true because $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1$. For the payoff of the second player, we have $U_2(\alpha) = -U_1(\alpha)$

Possibility 1: $supp(\alpha_1^*) = \{T_1, T_2\}$ and $supp(\alpha_2^*) = \{T_1, T_2\}$
In this case, we have $a1, a2, b1, b2 > 0$ and $a3 = b3 = 0$. T_1 and T_2 must be best responses for both players. given the mixed strategy of the other player. Therefore these two actions must yield the same payoff. and the payoff of T_3 may not be higher than that. Thus, for α^* to be a Nash Equilibrium, the following conditions must be met:

$$U_1((1,0,0),(b_1,b_2,0)) = U_1((0,1,0),(b_1,b_2,0)) \tag{4}$$

$$U_1((1,0,0),(b_1,b_2,0)) \geq U_1((0,0,1),(b_1,b_2,0)) \tag{5}$$

$$U_2((a_1,a_2,0),(1,0,0)) = U_2((a_1,a_2,0),(0,1,0)) \tag{6}$$

$$U_2((a_1,a_2,0),(1,0,0)) \geq U_2((a_1,a_2,0),(0,0,1)) \tag{7}$$

Using the definition of U_1 and U_2, condition (4) reads $(1 - b_1) v_1 - (1 - b_2) v_2$ and condition (6) reads $-a_2 v_2 = -a_1 v_1$. Additionally, we know that $a_1 + a_2 = 1$ and $b_1 + b_2 = 1$. These linear equations in a_i and b_i have the unique solution

$$a_1 = b_2 = \frac{v_2}{v_1 + v_2}, a_2 = b_1 = \frac{u_1}{v_1 + v_2}$$

We must check if these solutions also satisfy the other conditions. For condition (5) we get $(1-b_1)\,v_1 \geq v_3$. Substituting the value of b_1, we get the condition $v_3 \leq \frac{v_1 v_2}{v_1+v_2}$. α^* can only be a Nash Equilibrium if the values of the targets satisfy this requirement.

Condition (7) translates to $-a_2v_2 \geq -(a_1v_1 + a_2v_2)$. This is always true if $a_1, v_1 > 0$. We must also verify that all a_i and b_i are in the interval $(0,1]$, which is obviously the case.

Possibility 2: $\operatorname{supp}(\alpha_1) = \{T_1, T_2, T_3\}$, $\operatorname{supp}(\alpha_2) = \{T_1, T_2\}$
In this case, we have $a_1, a_2, a_3, b_1, b_2 > 0$ and $b_3 = 0$. All actions of player 1 must be best responses, given the mixed strategy of the other player. Therefore T_1, T_2 and T_3 must yield the same payoff for player 1. For player 2, T_1 and T_2 must be best responses and hence their payoff must be equal to one another and at least as big as the payoff of T_3. Thus, for α^* to be a Nash Equilibrium, the following conditions must be met:

$$\begin{aligned}
U_1((1,0,0),(b_1,b_2,0)) &= U_1((0,1,0),(b_1,b_2,0))\\
U_1((1,0,0),(b_1,b_2,0)) &= U_1((0,0,1),(b_1,b_2,0))\\
U_2((a_1,a_2,a_3),(1,0,0)) &= U_2((a_1,a_2,a_3),(0,1,0))\\
U_2((a_1,a_2,a_3),(1,0,0)) &\geq U_2((a_1,a_2,a_3),(0,0,1))
\end{aligned}$$

Using the definition of U_1 and U_2, this translates to:

$$\begin{aligned}
(1-b_1)\,v_1 &= (1-b_2)\,v_2\\
(1-b_1)\,v_1 &= v_3\\
-(a_2v_2 + a_3v_3) &= -(a_1v_1 + a_3v_3)\\
-(a_2v_2 + a_3v_3) &\geq -(a_1v_1 + a_2v_2)
\end{aligned}$$

Additionally, $a_1+a_2+a_3 = 1$ and $b_1+b_2 = 0$. We have three linear equations in b_1 and b_2. Solving these, we obtain a unique solution if $v_3 - \frac{v_1 v_2}{v_1+v_2}$ (there is no solution if this is not true). This solution is $b_1 = \frac{v_1}{v_1+v_2}, b_2 = \frac{\mathrm{v2}}{v1+v_2}$ Solving the linear equations and inequation in a_i and making use of the condition $v_3 = \frac{v_1v_2}{\mathrm{v1+v2}}$, we observe that there is no solution if $a_3 > \frac{1}{2}$ and exactly one solution for each $a_3 \in \left(0, \frac{1}{2}\right]$:

$$a_1 = \frac{v_2}{v_1+v_2}(1-a_3)\,, a_2 = \frac{v_1}{v_1+v_2}(1-a_3)$$

Again we must verify that all a_i and b_i are in the interval $(0,1]$, which is obviously the case for $a_3 \in \left(0, \frac{1}{2}\right]$.

Possibility 3: $\operatorname{supp}(\alpha_1) = \{T_1, T_2, T_3\}$, $\operatorname{supp}(\alpha_2) = \{T_1, T_2, T_3\}$
In this case, we have $a_1, a_2, a_3, b_1, b_2, b_3 > 0$. All actions must be best responses for both players, given the mixed strategy of the other player. Therefore for each player, all three actions must yield the same payoff given the other player's strategy. Thus, for α^* to be a Nash Equilibrium, the following conditions must be met:

$$\begin{aligned}
U_1((1,0,0),(b_1,b_2,b_3)) &= U_1((0,1,0),(b_1,b_2,b_3))\\
U_1((1,0,0),(b_1,b_2,b_3)) &= U_1((0,0,1),(b_1,b_2,b_3))\\
U_2((a_1,a_2,a_3),(1,0,0)) &= U_2((a_1,a_2,a_3),(0,1,0))\\
U_2((a_1,a_2,a_3),(1,0,0)) &= U_2((a_1,a_2,a_3),(0,0,1))
\end{aligned}$$

Using the definition of U_1 and U_2, this translates to:

$$\begin{aligned}
(1-b_1)\,v_1 &= (1-b_2)\,v_2\\
(1-b_1)\,v_1 &= (1-b_3)\,v_3\\
-(a_2v_2 + a_3v_3) &= -(a_1v_1 + a_3v_3)\\
-(a_2v_2 + a_3v_3) &= -(a_1v_1 + a_2v_2)
\end{aligned}$$

Also using the fact that $a_1 + a_2 + a_3 = 1$ and $b_1 + b_2 + b_3 = 1$, we obtain a set of six linear equations in six variables with the following unique solution:

$$a_1 = \frac{v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, a_2 = \frac{v_1 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, a_3 = \frac{v_1 v_2}{v_1 w_2 + v_1 v_3 + v_2 v_3}$$

$$b_1 = \frac{v_1 v_2 + v_1 v_3 - v_2 v_3}{v_1 v_2 + v_1 v_3 + w_2 v_3}, b_2 = \frac{v_1 v_2 - v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, b_3 = \frac{-v_1 v_2 + v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}$$

We must also verify that all a_i and b_i are in the interval $(0, 1]$. For the a_i, this. is obviously the case. Because $v_1 v_2 > v_2 v_3$ and $v_1 v_2 > v_1 v_3$, this is also the case for b_1 and b_2. For b_3, the numerator $-v_1 v_2 + v_1 v_3 + v_2 v_3$ must be greater than zero. This translates to the condition $v_3 > \frac{v_1 v_2}{v_1 + v_2}$.

Summarizing the results, we obtain the following Nash Equilibria:
If $v_3 \leq \frac{v_1 v_2}{v_1 + v_2}$

$$\alpha^* = \left(\left(\frac{v_2}{v_1 + v_2}, \frac{v_1}{v_1 + v_2}, 0 \right), \left(\frac{v_1}{v_1 + v_2}, \frac{v_2}{v_1 + v_2}, 0 \right) \right)$$

If $v_3 = \frac{v_1 v_2}{v_1 + v_2}$, for each $a_3 \in \left(0, \frac{1}{2}\right]$:

$$\alpha^* = \left(\left(\frac{v_2}{v_1 + v_2} (1 - a_3), \frac{v_1}{v_1 + v_2} (1 - a_3), a_3 \right), \left(\frac{v_1}{v_1 + v_2}, \frac{v_2}{v_1 + v_2}, 0 \right) \right)$$

If $v_3 > \frac{v_1 v_2}{v_1 + v_2}$

$$\alpha^* = \left(\left(\frac{v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \frac{v_1 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \frac{v_1 v_2}{v_1 v_2 + v_1 v_3 + v_2 v_3} \right), \right.$$
$$\left. \left(\frac{v_1 v_2 + v_1 v_3 - v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \frac{v_1 v_2 - v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \frac{-v_1 v_2 + v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3} \right) \right)$$

6. (a) As a normal form game, the situation is:
$G = (N, (S_i)_{i \epsilon N}, (u_i)_{i \epsilon N})$ where, $N = \{1, 2\}$
$S_i = \{x_i | x_i \epsilon [0, 1]\}$
$u_i = \frac{1}{2}(f(x_i, x_i) - c(x_i))$

(b) i. $f(x_1, x_2) = 3x_1 x_2, c(x_i) = x_i^2$

For player $1(P_1)$: $u_1 = \frac{3}{2} x_1 x_2 - x_1^2$
P_1's best reply to P_2's strategy x_2 is the value of x_1 maximising $u_1(x_1, x_2)$ and that is at the point where derivative of the function u_i is zero.

$$\frac{\partial u_1}{\partial x_1}(x_1, x_2) = 0 \implies \frac{3}{2} x_2 - 2x_1 = 0 \implies x_1 = \frac{3}{4} x_2 \tag{1}$$

similarly, P_2's best reply is:

$$x_2 = \frac{3}{4} x_1 \tag{2}$$

from the above two equations (1) and (2), the equilibrium is when $x_1 = x_2 = 0$ and payoffs are $u_1(0, 0) = u_2(0, 0) = 0$

ii. For P_1: $u_1(x_1, x_2) = \frac{1}{2}(4x_1 x_2) - x_1$
The derivative wrt x_1 is $2x_2 - 1$, therefore, if this derivative is positive, if $(x_2 > \frac{1}{2})$, P_1 will maximise x_1 at $x_1 = 1$. If it is negative if $(x_2 < \frac{1}{2})$, P_1 will minimise x_1 at $x_1 = 0$. If it is 0, then any choice of x_1 gives the same payoff, so $x_1 \epsilon [0, 1]$ is the best response. Similarly for P_2.
NE's are: $x_1 = x_2 = 0, x_1 = x_2 = \frac{1}{2}, x_1 = x_2 = 1$

(c) For (i), $x_1 = x_2 = \frac{1}{2}$ yields higher payoffs ($u_1 = u_2 = \frac{1}{2}$)
For (ii) $x_1 = x_2 = 1$ yields highest payoffs and is an NE

7. If (s_1, s_2) is PSNE then,

$$\begin{aligned} & u_1(s_1, s_2) \geq u_1(s_1^{'}, s_2) \qquad \forall s_1^{'} \in S_1 \\ \text{and,} \quad & u_2(s_1, s_2) \geq u_2(s_1, s_2^{'}) \qquad \forall s_2^{'} \in S_2 \end{aligned}$$

as the game is symmetric and $S_1 = S_2$, the above inequalities imply that,

$$\begin{aligned} & u_2(s_2, s_1) \geq u_2(s_2, s_1^{'}) \qquad \forall s_1^{'} \in S_2 \\ \text{and,} \quad & u_1(s_2, s_1) \geq u_1(s_2^{'}, s_1) \qquad \forall s_2^{'} \in S_1 \end{aligned}$$

$$\Rightarrow \quad (s_2, s_1) \quad \text{is PSNE.} \tag{1}$$

8. (i) Start with some strategy profile.
   (ii) While exists a player who can switch to a better response, do so:
   Let the profile before switch is $(s_1, ..., s_i, ..., s_n)$ and after switch is $(s_1, ..., s_i^{'}, ..., s_n)$. This implies that, $u_1(s_i^{'}, s_{-i}^{'}) > u_i(s_i, s_{-i})$ and therefore, $\phi(s_t^{'}, s_{-i}) > \phi(s_i, s_{-i})$.

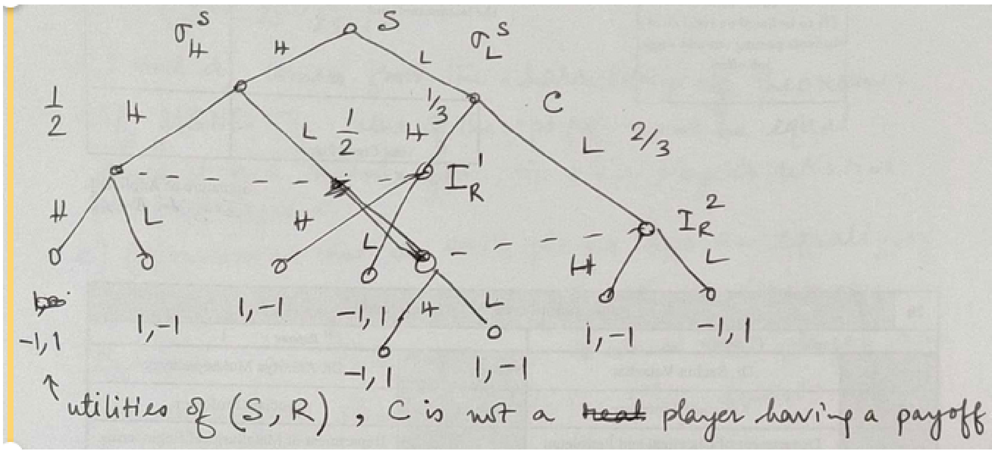
   Step (ii) will terminate if there does not exists any player who can improve his utility (keeping the strategies of other players same), we reach a local maxima of ϕ. The strategy profile at the end of step (ii) is a PSNE. Step (ii) will converge as the set of strategies is finite.

9. Since all of the money is gone before period 6 begins, the game is till period 5 (say ($T = 5$)). So, SPNE can be found by backward induction. Let offer (x,y) represents x fraction of a dollar to P1 and y fraction of a dollar to P2. In backward,

   - At $T = 5$, there is 0.20\$ left. P1 offers $(0.20 - 0, 0) = (0.20\$, 0)$ to P2 and P2 accepts.
   - At $T = 4$, there is 0.40\$ left. P2 offers $(0.20\$, 0.40\$ - 0.20\$) = (0.20\$, 0.20\$)$ and P1 accepts.
   - At $T = 3$, there is 0.60\$ left. P1 offers $(0.60\$ - 0.20\$, 0.20\$) = (0.40\$, 0.20\$)$ and P2 accepts.
   - At $T = 2$, there is 0.80\$ left. P2 offers $(0.40\$, 0.80\$ - 0.40\$) = (0.40\$, 0.40\$)$ and P1 accepts.
   - At $T = 1$, there is 1\$ left. P1 offers $(1\$ - 0.40\$, 0.40\$) = (0.60\$, 0.40\$)$ and P2 accepts.

   So, SPNE outcome is P1 offers $(0.60\$, 0.40\$)$ in period 1 and offer is accepted.

10. (a) Player S, C, R move in sequence as shown in the game tree below: The game is a game with



perfect recall, hence WLOG, we can assume behavioral strategies for every player. (σ_H^S, σ_L^S) for player S and (σ_H^R, σ_L^R) for player R.

(b) To find PBE, compare the expected payoff from H and L at information set I_R^1 for R. (I_R^1 is when the player R receives the signal H).
Expected payoff from strategy H at I_R^1 is,

$$EU_1^{I_R^1}(H) = (1) * Pr(\sigma_H^S | I_R^1) + (-1) * Pr(\sigma_L^S | I_R^1)$$
$$= (1) * \frac{\frac{1}{2}\sigma_H^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S} + (-1) * \frac{\frac{1}{3}\sigma_L^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S}$$

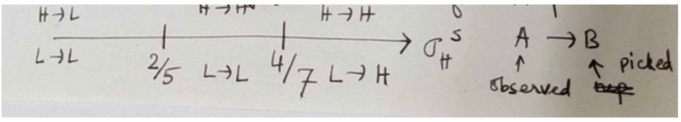
Expected payoff from strategy L at I_R^1 is,

$$EU_1^{I_R^1}(L) = (-1) * Pr(\sigma_H^S | I_R^1) + (1) * Pr(\sigma_L^S | I_R^1)$$
$$= (-1) * \frac{\frac{1}{2}\sigma_H^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S} + (1) * \frac{\frac{1}{3}\sigma_L^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S}$$

comparing the expected payoff from H and L at information set I_R^1 for R,

$$\frac{\frac{1}{2}\sigma_H^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S} - \frac{\frac{1}{3}\sigma_L^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S} \geq - \frac{\frac{1}{2}\sigma_H^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S} + \frac{\frac{1}{3}\sigma_L^S}{\frac{1}{2}\sigma_H^S + \frac{1}{3}\sigma_L^S}$$

The above inequality is true only if $\sigma_H^S \geq \frac{2}{5} \implies$ If $\sigma_H^S \geq \frac{2}{5}$, the optimal strategy for R is to picks H if it sees H.
Similarly, by comparing the expected payoff from H and L at information set I_R^1 for R, $\implies$ If $\sigma_H^S \geq \frac{4}{7}$, the optimal strategy for R is to picks H if it sees L.
Hence, the optimal policy for R is in the following figure snap:



We look at the three cases for the strategy of player R wrt σ_H^S:
Case 1 $\sigma_H^S \leq \frac{2}{5}$: Player R picks L if it sees H or L. The expected utility of player S is,

$$\begin{aligned} EU_S &= \sigma_H * \frac{1}{2} * (1) + \sigma_H * \frac{1}{2} * (1) + \sigma_L * \frac{1}{3} * (-1) + \sigma_L * \frac{2}{3} * (-1) \\ &= \sigma_H - \sigma_L \\ &= \sigma_H - (1 - \sigma_H) \\ &\leq -\frac{1}{5} \end{aligned}$$

Player S gets maximum expected utility when it picks H with maximum possible probability to pick H (in this case, $= \frac{2}{5}$).
Case 2 $\frac{2}{5} \leq \sigma_H^S \leq \frac{4}{7}$: Player R picks L if it sees L and picks H if it sees H. The expected utility of player S is,

$$\begin{aligned} EU_S &= \sigma_H * \frac{1}{2} * (-1) + \sigma_H * \frac{1}{2} * (1) + \sigma_L * \frac{1}{3} * (1) + \sigma_L * \frac{2}{3} * (-1) \\ &= -\frac{\sigma_L}{3} \\ &\leq -\frac{1}{7} \end{aligned}$$

Player S gets maximum expected utility when it picks L with minimum possible probability to pick L that means, player S gets maximum expected utility when it picks H with maximum possible

probability to pick H (in this case, $= \frac{4}{7}$).
Case 3 $\sigma_H^S \geq \frac{4}{7}$: Player R picks L if it sees H or L. The expected utility of player S is,

$$\begin{aligned} EU_S &= \sigma_H * \frac{1}{2} * (-1) + \sigma_H * \frac{1}{2} * (-1) + \sigma_L * \frac{1}{3} * (1) + \sigma_L * \frac{2}{3} * (1) \\ &= 1 - 2\sigma_H \\ &\leq -\frac{1}{7} \end{aligned}$$

Player S gets maximum expected utility when it picks H with minimum possible probability to pick H (in this case, $= \frac{4}{7}$).

Concluding from all the three cases, the best strategy for S given the strategy of player R (as shown in the above snap) is to pick H with probability $\frac{4}{7}$ and L with probability $\frac{3}{7}$.

Hence the PBE of the game is: S picks H with probability $= \frac{4}{7}$ and R picks H for both information sets,

$$\left(\sigma_H^S = \frac{4}{7}\ ,\ \sigma_L^S = \frac{3}{7}\right)$$

$$\sigma_H^R(I_R^1) = \begin{cases} 1, & \text{if } \sigma_H^S > \frac{2}{5} \\ [0,1], & \text{if } \sigma_H^S = \frac{2}{5} \\ 0, & \text{otherwise} \end{cases}$$

$$\sigma_H^R(I_R^2) = \begin{cases} 1, & \text{if } \sigma_H^S > \frac{4}{7} \\ [0,1], & \text{if } \sigma_H^S = \frac{4}{7} \\ 0, & \text{otherwise} \end{cases}$$