Topics in Game Theory and Collective Choice

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1 Introduction

Game Theory is an interesting field, and you will often encounter counter-intuitive results while analyzing problems. First, the stage is set with the discussion of one such problem- The Neighboring Kingdoms dilemma. Then we discuss discuss preference relations, and the requirements for the preference relation to have a utility function.

2 The Neighboring Kingdoms dilemma

Suppose there are two kingdoms, each having limited resources. The kingdoms have been hostile for some time, and thus need to boost their defense. Meanwhile, they are also starving, and need to improve their agriculture. Due to limited resources, they can only do one of the two things- boost defense or agriculture.

Let's call the kingdoms **A** and **B**. Kings of A and B need to make a call between the two options. They don't know what the other king will choose, and they cannot coordinate.

Let's look at some numbers-

A\B	Agri	Def
Agri	(5,5)	(0,6)
Def	(6,0)	(1,1)

Let's say the values represent the happiness quotient of the two kingdoms. If both A and B choose agriculture, they both are happy. Thus the high values! If A chooses agriculture and B defense, then B will attack A and loot their resources (A can't attack because of B's defense). So B get 6, and A 0. Same happens the other way around. When both choose defense, then although they don't have anything to eat, but are still content that nobody can loot them! Thus (1,1). What do you think will they choose?

Let's analyze this problem from A's perspective. If B chooses agri, then A is better of choosing Def, since the quotient he gets is 6 (compared to 5 if he chooses agri).
If B chooses Def, then A is better off choosing Def, since the quotient in this case is 1 (compared to 0 if he chooses Agri).

So whether B chooses Agri or Def, A is better of choosing Def. Same happens other way around, since the problem is similar from B's perspective.
Thus both end up choosing Def, and get a quotient of 1 each. If they were allowed to cooperate, they could have chosen Agri each, leading to a better quotient of 5 each.

A similar version of this problem is the Prisoner's Dilemma

3 Notation

Set of Players $N = 1...n$
Set of actions a_i for player i, specific action a_i in A_i
Utility/payoff of agent i : $u_i = A_1 * A_2 * ... * A_n - > R$

Now that we have understood the basic notations, it is important to note that not every preference relation is expressible using utility. Let's look at an example.

3.1 Preference relation without utility function

Consider a family of three people- child, father, mother. Three dishes a, b, c.
Preference ordering:
child : $a > b > c$
father : $b > c > a$
mother : $c > a > b$
Let's we consider the majority preference order - ie. the order of dishes by majority voting.
If we compare a and b, child and father prefer a, while only mother prefers b. So a is preferred over b.
If you do this for pairs (b, c), (c, d), and (c, a), you will arrive at $(a > b > c > a)$. Clearly, this preference relation does not have a utility representation, as you cannot assign real values that satisfy these inequalities.

Next we discuss the conditions sufficient for a preference relation to have a utility representation.

4 Von Neumann–Morgenstern utility theorem

Axioms for utility representation:

1. **Completeness**: $\forall a_1, a_2 \in A$, either $a_1 \succ a_2$, or $a_2 \succ a_1$ or $a_1 \sim a_2$
2. **Transitivity**: If $a_1 \succ a_2$ and $a_2 \succ a_3$ then $a_1 \succ a_3$
3. **Substitutability**: If $a_1 \sim a_2$, then for every sequence $a_3...a_k$, and preference masses $p, p_3...p_k < 1$, st. $\sum_{i=2}^{n} p_i + p = 1$, the following holds: $[p : a_1, p_3 : a_3...p_k : a_k] \sim [p : a_2, p_3 : a_3...p_k : a_k]$
4. **Decomposability**: For lotteries l_1, l_2, if $P_{l1}(a_i) = P_{l2}(a_i)$, $\forall a_i \in A$, then $l_1 \sim l_2$
example:

$l_1 = [0.5 : [0.4 : a_1, 0.6 : a_2], 0.5 : a_3]$
$l_2 = [0.2 : a_1, 0.3 : a_2, 0.5 : a_3]$
here $l_1 \sim l_2$

5. **Monotonicity**: If $a_1 \succ a_2$, and $1 \succ p \succ q \succ 0$, then
   $[p : a_1, 1 - p : a_2] \succ [q : a_1, 1 - q : a_2]$

6. **Continuity**: If $a_1 \succ a_2$ and $a_2 \succ a_3$, then
   $\exists p \in [0, 1]$ st $a_2 \sim l_2 = [p : a_1, 1 - p : a_3]$

4.1 Theorem

The Von Neumann–Morgenstern (1944) states if a preference relation $\succeq$ satisfies axioms 1 to 6, then $\exists u : A- > [0, 1]$ st

1. $u(a_1) \geqslant u(a_2) \iff a_1 \succeq a_2$
2. $u([p_1 : a_1, p_2 : a_2, ...p_k : a_k]) = \sum_{i=1}^{k} (p_i * u(a_i))$

5 Conclusion

We studied 6 axioms that ensure the existence of a utility function for a preference relation. These axioms are quite intuitive, and the proof of the theorem has not been covered in class. In many real life situations, these axioms are satisfies, and henceforth, we shall assume the existence of a utility function without proving checking the axioms again. However, the reader is encouraged to read about preference relations that do not satisfy these axioms.