CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 4: Mixed Strategy Nash Equilibrium

Lecturer: Swaprava Nath *Scribe(s): Sachin K Salim*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

4.1 Mixed Strategy Nash Equilibrium

For a finite set A, $\Delta(A)$ is defined as the set of all probability distributions over A, $\Delta(A) = \{p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1\}$. Then $\sigma_i \in \Delta(S_i)$ is a **mixed strategy** of player i, where S_i is their finite strategy set. Mixed strategy is a distribution σ_i over the strategies in S_i , i.e., $\sigma_i : S_i \mapsto [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

Utility of a mixed strategy profile (σ_i, σ_{-i}) is $u_i(\sigma_i, \sigma_{-i}) = \sum\limits_{s \in S} \prod\limits_{i \in N} \sigma_i(s_i)\ u_i(s_i, s_{-i})$

For example, consider the following game as given in Table 4.1. Note that there is no Pure Strategy Nash equilibrium in this game. Now suppose Player 1 plays the mixed strategy H with probability p and Player 2 plays H with probability q.

		P_2: H	P_2: T
P_1	H	$+1, -1$	$-1, +1$
	T	$-1, +1$	$+1, -1$

Table 4.1: Matching Coins Game

Then the utility u_1 of the player 1 is $u_1((p, 1-p), (q, 1-q))$
$= pq\ u_1(H,H) + p(1-q)\ u_1(H,T) + (1-p)q\ u_1(T,H) + (1-p)(1-q)\ u_1(T,T)$

For a mixed strategy profile $\sigma' = ((1,0), (\frac{1}{2}, \frac{1}{2}))$, $u_1(\sigma') = 1.\frac{1}{2}(+1) + 1.\frac{1}{2}(-1) = 0$

When player i plays pure strategy while all others play mixed strategy, the utility function is given by
$u_i(s_i, \sigma_{-i}) = \sum\limits_{s_{-i} \in S_{-i}} \prod\limits_{j \neq i} \sigma_j(s_j)\ u_i(s_i, s_{-i})$

Definition 4.1 (Mixed Strategy Nash Equilibrium) *MSNE is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ s.t. $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)\ \forall \sigma_i' \in \Delta(S_i),\ \forall i \in N$*

Definition 4.2 (Support of a Mixed Strategy) *Set of strategies/subset of the strategy space on which the mixed strategy σ_i has positive mass*
$\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$

Theorem 4.3 (Characterization of a MSNE) *A mixed strategy profile (σ_i, σ_{-i}) is a MSNE iff $\forall i \in N$*

1. *$u_i(s_i, \sigma_{-i})$ is the same for all $s_i \in \delta(\sigma_i^*)$ and*
2. *$u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}),\ \forall\ s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$*