Question: How to maximize the revenue earned by the auctioneer?

To answer this, we need to benchmark mechanisms w.r.t. their expected revenue, where expectation is taken w.r.t. the prior distribution of the types. ~~Acc~~ Accordingly, the notions of incentive compatibility and individual rationality have to change.

Bayesian Incentive Compatibility:

$T_i = [0, b_i]$, Common prior G over $T = \bigtimes_{i=1}^{n} T_i$

G is a common prior. — g denotes the density.

$G_{-i}(\cdot \mid s_i)$ is the conditional distribution over other types s_{-i} given agent i's type is s_i

$g_{-i}(\cdot \mid s_i)$ denote the density of the distribution

$g_{-i}(s_{-i} \mid s_i)$ is derived from g using Bayes rule.

Every mechanism $(f, p_1, \ldots, p_n)$ induces an expected allocation and payment rule $(\alpha, \underline{\pi})$

$$\alpha_i(s_i \mid t_i) = \int_{s_{-i} \in T_{-i}} f_i(s_i, s_{-i})\, g_{-i}(s_{-i} \mid t_i)\, ds_{-i}$$

Expectation: Two levels

1. w.r.t. the types of other agents — coming from the common prior.
2. w.r.t. the randomization of the mechanism — this we have used before.

Expected payment

$$\pi_i(s_i|t_i) = \int_{s_{-i} \in T_{-i}} p_i(s_i, s_{-i})\, g_{-i}(s_{-i}|t_i)\, ds_{-i}$$

Expected utility of agent i is

$$t_i \alpha_i(t_i|t_i) - \pi_i(t_i|t_i)$$

Defn: A mechanism $(f, \underline{p})$ is Bayesian incentive compatible (BIC) if $\forall i \in N \quad \forall s_i, t_i \in T_i$

$$t_i \alpha_i(t_i|t_i) - \pi_i(t_i|t_i) \geq t_i \alpha_i(s_i|t_i) - \pi_i(s_i|t_i)$$

Similarly, f is Bayesian implementable if $\exists \underline{p}$ s.t. $(f, \underline{p})$ is BIC.

Independence and Characterization of BIC mechanisms

Assume that the priors are independent, i.e., every agents value/type is drawn independently from a distribution G_i (with density g_i).

$$G(s_1, \ldots, s_n) = \prod_{i=1}^{n} G_i(s_i)$$

$$G_{-i}(s_{-i}|t_i) = \prod_{j \neq i} G_j(s_j)$$

Defn: An allocation rule is non-decreasing in expectation (NDE) if $\forall i \in N \quad \forall s_i, t_i \in T_i$ s.t. $s_i < t_i$ we have $\alpha_i(s_i) \leq \alpha_i(t_i)$.

Compare: allocation rules that are non-decreasing [previously defined] are always NDE. But there can be more allocation rules.

Characterization of BIC rules

Theorem: A mechanism $(f, \underline{p})$ in the independent prior setting is BIC iff

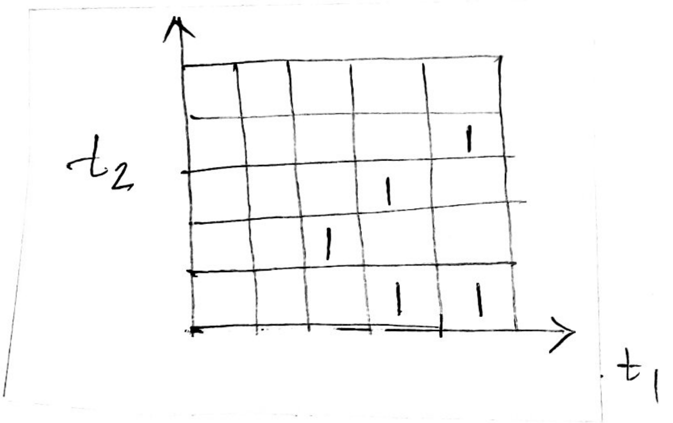
1. f is NDE and
2. p_i satisfies

$$\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(s_i)\, ds_i$$

$\forall t_i \in T_i \ \forall i \in N.$

Proof: In same lines as before [exercise].

A BIC rule may not be DSIC. In particular f can be NDE but not non-decreasing.



f is NDE but not non-decreasing.

As we are in the Bayesian setting now, we can define an analog of individual rationality.

Defn: A mechanism $(f, \underline{p})$ is interim individually rational (IIR) if for every bidder $i \in N$ we have

$$\alpha_i(t_i) \cdot t_i - \pi_i(t_i) \geq 0 \quad \forall t_i \in T_i.$$

Lemma: A mechanism (f, p) is BIC and IIR iff

1. f is NDE
2. For all $i \in N$
$$\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(s_i)\, ds_i \,, \forall t_i \in T_i$$
3. $\forall i \in N$, $\pi_i(0) \leq 0$.

Sketch: (1) and (2) uniquely ~~determine~~ identify a BIC mechanism so the proof requires to show that IIR along with (1) and (2) are equivalent to (3).

$(\Rightarrow)$ ~~Apply~~ IIR at $t_i = 0$ on (2) and get $\pi_i(0) \leq 0$.

$(\Leftarrow)$ $t_i \alpha_i(t_i) - \pi_i(t_i) \geq 0$ given $\pi_i(0) \leq 0$

Single Agent Problem

We'll spend some more time understanding the optimal mechanism design problem for a single agent.

Type set $T = [0, \beta]$. Mechanism (f, p)

$$f : [0, \beta] \to [0, 1] \,, \quad p : [0, \beta] \to \mathbb{R}.$$

- Incentive Compatibility [BIC and DSIC are equivalent]
$$t f(t) - p(t) \geq t f(s) - p(s) \quad \forall s, t \in T.$$
- Individual Rationality [IR and IIR are same].
$$t f(t) - p(t) \geq 0 \quad \forall t \in T.$$

The revenue earned by a mechanism M is given by

$$\pi^M := \int_0^\beta p(t)\, g(t)\, dt$$

Let $\mathcal{M}$ be the class of all IC and IR mechanisms

M^* is optimal if $\pi^{M^*} \geq \pi^M \quad \forall M \in \mathcal{M}$

Structure of an optimal (revenue) mechanism?

Consider a IC and IR mechanism $(f, p) \equiv M$

By the characterization theorem

$$p(t) = p(0) + t f(t) - \int_0^t f(x)\,dx \quad [\text{IC}]$$

$$p(0) \leq 0 \qquad [\text{IR}]$$

Since we want to maximize revenue, $p(0) = 0$

- $p(t) = t f(t) - \int_0^t f(x)\,dx \quad \cdots (1)$

completely given by the allocation rule.

Since the type is distributed as G (density g)

The expected revenue

$$\pi^f = \int_0^\beta p(t)\, g(t)\,dt$$

$$= \int_0^\beta \left[t f(t) - \int_0^t f(x)\,dx\right] g(t)\,dt \quad \cdots (2)$$

If f^* is revenue optimal, then

- $\pi^{f^*} \geq \pi^f \quad \forall f$
- $\forall t \in T, \quad p(t) = t f(t) - \int_0^t f(x)\,dx.$

Lemma: [~~Optimal Mechanism~~]

For any implementable allocation rule f, we have

$$\pi^f = \int_0^\beta w(t) f(t) g(t)\,dt$$

Where $w(t) = \left(t - \frac{1 - G(t)}{g(t)}\right).$