let S be the triangle whose vertices are $(0,0)$, $(a,0)$, and $(0,b)$. Prove that $\varphi(S,(0,0)) = (\frac{a}{2}, \frac{b}{2})$.

15.10 Let φ be a solution concept (for $\mathcal{F}$) satisfying symmetry, efficiency, and independence of the units of measurement. Let $a, b > 0$ be positive numbers, and let $x = (x_1, x_2)$ be a point on the ray emanating from $(0,0)$ and passing through (a,b), satisfying $x_1 > \frac{a}{2}$. Let S be the quadrangle whose vertices are $(0,0)$, $(a,0)$, $(0,b)$, and x. Prove that $\varphi(S,(0,0)) = x$.

15.11 Prove that under a positive affine transformation of the plane, an equilateral triangle whose base is the x_1-axis is transformed into an equilateral triangle whose base is parallel to the x_1-axis.

15.12 Let φ be a solution concept (for $\mathcal{F}$) satisfying the properties of efficiency and individual rationality. Let (S,d) be a bargaining game satisfying the following property: there is an alternative $x \in S$ satisfying $x \geq d$, but there is no $x \in S$ satisfying $x \gg d$. What is $\varphi(S,d)$?

15.13 A *set solution concept* for a family of bargaining games $\widetilde{\mathcal{F}}$ is a function φ associating every bargaining game (S,d) in $\widetilde{\mathcal{F}}$ with a subset of S (which may contain more than a single point). Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a function. Define a set solution concept φ as follows:

$$\varphi(S,d) = \mathrm{argmax}_{x \in S} f(d,x). \tag{15.60}$$

(a) Give an example of a function f for which $\varphi(S,d)$ is not always a single point.
(b) Prove that φ satisfies independence of irrelevant alternatives. In other words, if $T \supseteq S$ and $x \in \varphi(T,d) \cap S$, then $x \in \varphi(S,d)$.

15.14 Let φ_1 and φ_2 be two solution concepts for the family of bargaining games $\mathcal{F}$. Define another solution concept φ for the family of bargaining games $\mathcal{F}$ as follows:

$$\varphi(S,d) = \tfrac{1}{2}\varphi_1(S,d) + \tfrac{1}{2}\varphi_2(S,d). \tag{15.61}$$

For each of the following properties, prove or disprove the following claim: if φ_1 and φ_2 satisfy the property, then the solution concept φ also satisfies the same property.

(a) Symmetry.
(b) Efficiency.
(c) Independence of irrelevant alternatives.
(d) Covariance under positive affine transformations.

15.15 Two players are to divide \$2,000 between them. The utility functions of the players are the amounts of money they receive: $u_1(x) = u_2(x) = x$. If they cannot come to agreement, neither of them receives anything. For the following cases, describe the bargaining game derived from the given situation, in utility units, and find its Nash solution of the game.

(a) Given any division which the players agree upon, the first player receives his full share under the agreed division, and the second player pays a tax of 40%.
(b) Repeat the situation in the previous item, but assume that the second player pays a tax of 60%.
(c) The first player pays a tax of 20%, and the second pays a tax of 30%.

15.16 Two players are to divide \$2,000 between them. The utility function of the first player is $u_1(x) = x$. The utility function of the second player is $u_2(x) = \sqrt{x}$. For each of the following two situations, describe the bargaining game derived from the situation, in utility units, and find its Nash solution.

(a) If the two players cannot come to an agreement, neither of them receives any payoff.
(b) If the two players cannot come to an agreement, the first one receives \$16, and the second receives \$49 (note that in this case the disagreement point in the utility space is (16, 7)).

15.17 Find the Nash solution for the bargaining game in which

$$S = \left\{ x \in \mathbb{R}^2 : \frac{x_1^2}{16^2} + \frac{x_2^2}{20^2} \leq 1 \right\}, \tag{15.62}$$

and

(a) The disagreement point is (0, 0).
(b) The disagreement point is (10, 0).

15.18 Prove Theorem 15.22 on page 638: for every bargaining game (S, d), the only alternative that constitutes a solution concept according to Definition 15.21 (page 638) is the Nash solution $\mathcal{N}(S, d)$. Moreover, the constant c equals minus the slope of the supporting line of S at the point $\mathcal{N}(S, d)$.

15.19 Let $(S, d) \in \mathcal{F}$ be a bargaining game.

(a) Prove that there exists a unique efficient alternative in S minimizing the absolute value $|(x_1 - d_1) - (x_2 - d_2)|$. Denote this alternative by x^*.

Let Y be the collection of efficient alternatives y in S satisfying the property that the sum of their coordinates $y_1 + y_2$ is maximal.

(b) Show that the Nash solution $\mathcal{N}(S, d)$ is on the efficient boundary between x^* and the point in Y that is closest to x. In particular, if $x^* \in Y$ then $x^* = \mathcal{N}(S, d)$.

15.20 Let $(S, d) \in \mathcal{F}$ be a bargaining game. Denote by $x_2 = g(x_1)$ the equation defining the north-east boundary of S. Prove that if g is strictly concave and twice differentiable, then the point $x^* = \mathcal{N}(S, d)$ is the only efficient point x in S satisfying $-g'(x_1)(x_1 - d_1) = (x_2 - d_2)$.

15.21 Suppose two players have utility functions for money given by

$$u_1(x) = x^{\alpha_1}, \quad u_2(x) = x^{\alpha_2}, \tag{15.63}$$

where $0 < \alpha_1 < \alpha_2 < 1$. The Arrow–Pratt risk aversion index of player i is $r_{u_i}(x) := -\frac{u_i''(x)}{u_i'(x)}$ (see Exercise 2.28 on page 37 for an explanation of this index).

(a) Are the players risk-seeking or risk-averse? In other words, are their utility functions convex or concave?
(b) Player i is more risk-averse than player j if $r_{u_i}(x) \geq r_{u_j}(x)$ for every x. Which of the two players is more risk-averse?
(c) The players are to divide between them a potential profit of A dollars, but this profit can only be realized if the players can come to agreement on how to divide it. What is the Nash solution of this bargaining game, when the outcomes are in units of utility? Which of the players receives the greater payoff?
(d) What is the effect of risk aversion on the Nash outcome of a bargaining game in this example?

15.22 Two bargaining games, $(S, (0, 0))$ and $(T, (0, 0))$, are given by

$$S = \{x \in \mathbb{R}^2_+ : 2x_1 + x_2 \leq 100\}, \tag{15.64}$$
$$T = \{x \in \mathbb{R}^2_+ : x_1 + 2x_2 \leq 100\}. \tag{15.65}$$

David and Jonathan face the following situation. With probability $\frac{1}{2}$, they will negotiate tomorrow over the bargaining game $(S, (0, 0))$, and with probability $\frac{1}{2}$, they will negotiate over the bargaining game $(T, (0, 0))$.

David and Jonathan believe in the Nash solution, but they do not know which bargaining game they will play. Jonathan proposes that they apply the Nash solution in each of the two bargaining games (when it is reached), and therefore their expected utility is

$$\tfrac{1}{2}\mathcal{N}(S, (0, 0)) + \tfrac{1}{2}\mathcal{N}(T, (0, 0)). \tag{15.66}$$

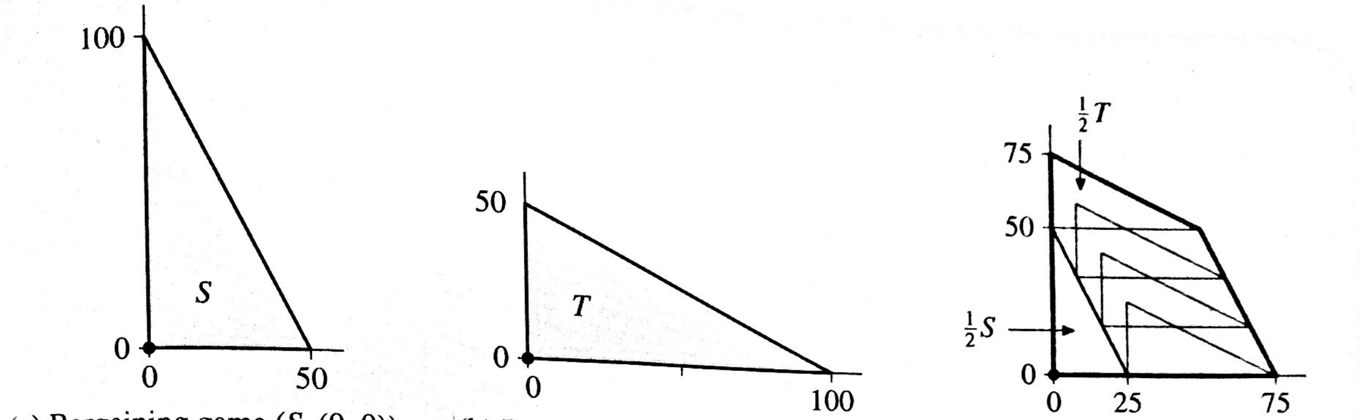
David counterproposes as follows: since the probability that they will negotiate over each of the bargaining games is $\frac{1}{2}$, the players are actually facing the bargaining game $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$, where the set $\frac{1}{2}S + \frac{1}{2}T$ is defined by

$$\tfrac{1}{2}S + \tfrac{1}{2}T = \left\{\tfrac{1}{2}x + \tfrac{1}{2}y : x \in S, y \in T\right\}. \tag{15.67}$$

They should therefore implement the Nash solution over the game $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$, which then should be

$$N\left(\tfrac{1}{2}S + \tfrac{1}{2}T, (0, 0)\right). \tag{15.68}$$

To compute the set $\frac{1}{2}S + \frac{1}{2}T$, draw the set $\frac{1}{2}S$, and "slide" the set $\frac{1}{2}T$ along its efficient points (see Figure 15.26(c)).



(a) Bargaining game $(S, (0, 0))$ (b) Bargaining game $(T, (0, 0))$ (c) Bargaining game $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$

Figure 15.26 The bargaining games $(S, (0, 0))$, $(T, (0, 0))$ and $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$

Compute $\frac{1}{2}\mathcal{N}(S, (0, 0)) + \frac{1}{2}\mathcal{N}(T, (0, 0))$, and $\mathcal{N}\left(\frac{1}{2}S + \frac{1}{2}T, (0, 0)\right)$. Did you get the same result in both cases?

15.23 Repeat Exercise 15.22 for the following bargaining games $(S, (0, 0))$ and $(T, (0, 0))$:

$$S = \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 4\}, \tag{15.69}$$

$$T = \left\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 5, \tfrac{3}{4}x_1 + x_2 \leq 4\right\}. \tag{15.70}$$

15.24 Prove that for every $0° < \alpha < 90°$, the solution λ_α defined on page 640 satisfies weak efficiency, homogeneity, strict individual rationality, and full monotonicity. In addition, if $\alpha = 45°$, the solution λ_α satisfies symmetry.

15.25 Prove the minimality of the set of properties characterizing the solution concept λ_α, as listed in Theorem 15.31 (page 645). In other words, prove (by examples) that for every three properties out of the four properties mentioned in the statement of Theorem 15.31 there exists a solution concept satisfying all three properties, but not the fourth.

15.26 Solve Exercises 15.15 and 15.16, assuming the players accept the Kalai–Smorodinsky solution, not the Nash solution. To convert the bargaining game (S, d) in $\mathcal{F}$ to a bargaining game in $\mathcal{F}_0$ (over which the Kalai–Smorodinsky solution is defined), apply the positive affine transformation $x \mapsto x - d$ and remove all the points y that do not satisfy $y \geq (0, 0)$.

15.27 Solve Exercises 15.22 and 15.23, assuming the players accept the Kalai–Smorodinsky solution, rather than the Nash solution. To convert the bargaining game (S, d) in $\mathcal{F}$ to a bargaining game in $\mathcal{F}_0$ (over which the Kalai–Smorodinsky solution is defined), apply the positive affine transformation $x \mapsto x - d$ and remove all the points y that do not satisfy $y \geq (0, 0)$.

15.28 Prove the minimality of the set of properties defining the Kalai–Smorodinsky solution. In other words, prove (by examples) that for any set of three of the four properties characterizing the solution concept, there exists a solution concept over $\mathcal{F}_0$ that satisfies those three properties, but not the fourth property.

15.29 Prove Theorem 15.35 (page 651), which characterizes the Nash solution for bargaining games with any number of players.
Hint: Look at the function $\ln(\prod_{i=1}^{n}(x_i - d_i))$.