CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

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Problem 1

P_1	P_2	$\hat{P}_1$	$\hat{P}_2$	P'_1	P'_2
a	c	b	c	b	a
b	b	a	a	a	b
c	a	c	b	c	c

Part a

Given: $f(P_1, P_2) = a$
To Prove: $f(P'_1, P'_2) = b$
Proof:
Consider the preference profile $(\hat{P}_1, \hat{P}_2)$
Since $\forall(\bar{P}_1, \bar{P}_2), f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$, therefore

$$f(\hat{P}_1, \hat{P}_2) \in \{b, c\}$$

Assume $f(\hat{P}_1, \hat{P}_2) = c$.
Now we see that since dominating set of c in (P_1, P_2) is the same (and hence subsets) as dominating set of c in $(\hat{P}_1, \hat{P}_2)$ for each player, $f(P_1, P_2) = c$. (By Monotonicity implied from Strategy Proofness)
However, this contradicts with the given statement that $f(P_1, P_2) = a$.
Hence, $f(\hat{P}_1, \hat{P}_2) = b$

Now we see that since dominating set of b in (P'_1, P'_2) is the subset of dominating set of b in $(\hat{P}_1, \hat{P}_2)$ for each player, $f(P'_1, P'_2) = b$. (By Monotonicity implied from Strategy Proofness)
Hence, $f(P'_1, P'_2) = b$.
Hence Proved

Part b

Since we are using Singly Peaked Preferences, we must be choosing the leftmost(least) peak with ordering as $a < b < c$.
This implies that $f(\bar{P}_1, \bar{P}_2) = min(\bar{P}_1(1), \bar{P}_2(1))$
By this rule, we see that $f(P'_1, P'_2) = a$. **Therefore the earlier conclusion does not hold**.

The previous proof does not go through because we have used an ordering $\hat{P}_2 = c > a > b$, which is not allowed in singly peaked preferences with given ordering.

The rule $f(\bar{P}_1, \bar{P}_2) = min(\bar{P}_1(1), \bar{P}_2(1))$ gives $f(P_1', P_2') = a$.

Problem 2

We know that for Gibbard-Satterthwaite result applies only when the domain is unrestricted. This means that all preference profiles must have some chance of coming out as the outcome of the process.

Consider some subsets A and B of X such that $A \subset B$.

According to the given preference rules, any person will either prefer B over A or will be indifferent between them. (If the person's top profile is in A, it definitely has to be in B, but not vice versa. Therefore, every player weakly prefers B).

But this implies that all preference profiles which strictly prefer A over B are impossible to come out as outcome. **Hence the domain is restricted**
Therefore, Gibbard-Satterthwaite result will not apply here.

Problem 3

Suppose there exists a subset of K such agents (say K-set) who can change the outcome of the social choice function. The SCF is that out of singly peaked preferences, the middle one (median) would be selected. Consider the following exhaustive cases:

Case 1: Some median voter belongs to K set: Let us assume that some median voter M belongs to the group. Now, if the group is able to shift the outcome, clearly the new outcome would be strictly less preferred by the M (Since his preference peaked at the previous median). Therefore, M will not participate in such a group. Hence, this case is closed.

Case 2: All of the K set lie to the left of the median: If all of the K set agents lie to the left of the median, they can only have the median shifted to the right by moving their peaks to the right. However, this would result the outcome to be farther from the actual preference peak of all of these agents, which will strictly reduce the utility of all of the K set agents. Hence, $f(P_K, P_{-K})P_i f(P_K', P_{-K})$. Hence this case is closed.

Case 3: All of the K set lie to the right of the median: If all of the K set agents lie to the right of the median, they can only have the median shifted to the left by moving their peaks to the left. However, this would result the outcome to be farther from the actual preference peak of all of these agents, which will strictly reduce the utility of all of the K set agents. Hence, $f(P_K, P_{-K})P_i f(P_K', P_{-K})$. Hence this case is closed.

Case 4: Some agents out of K lie to the left of the median, rest lie to the right of the median: Let us assume that set A out of the K-set lie to the left of the median, while set B lie to the right of the median. Now, assume that this K-set makes the median shift in either direction (WLOG, say right). However, this would make the utilities of all agents of set A strictly reduce (since the outcome is now farther from their actual preference peak). Hence, such a group cannot form. Hence this case is also closed.

Since no distribution of the K-set exists such that they can shift the median in their preferred direction, such a K-set cannot exist. Therefore, **Median voting rule is Group Strategy Proof**.