CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

Nikita Awasthi

150453

Question 1

Part a

Given: A two agent model with alternatives {a, b, c} and two preference profiles P and P'. f is an onto SCF such that $f(P_1, P_2) = a$.
To show: If f is strategyproof then $f(P_1^{'}, P_2^{'}) = b$
Proof: We construct two new preference profiles $P^{''}$ and $P^{'''}$ such that $P^{''} = (P_1, P_2^{''})$ and $P^{'''} = (P_1^{'}, P_2^{''})$ such that $P_2^{''} = \{c, a, b\}$ in that preference order as given in the table. Given that f is strategyproof implies that f is MONO. Given that $f(P) = a$ and by construction $D(a, P_i) \subseteq D(a, P_i^{''}) \forall i \in$ N $\implies$ that $f(P^{''}) = a$ (using the fact that f is MONO).
Let us now consider $P^{'''}$. We know that if f is onto and strategyproof then for every preference profile P, $f(P) \in \{P_1(1), P_2(1)\}$ considering that there are two agents. Since f is both onto and strategyproof, therefore $f(P^{'''}) \in \{b, c\}$. Let us assume that $f(P^{'''}) = c$. Then agent 1 can manipulate at $P^{'''}$ to $P^{''}$ such that $f(P^{''}) = a$ and $f(P^{''}) P_1 f(P^{'''})$ leading to a contradiction since f is strategyproof. Therefore, $f(P^{'''}) = b$.
Given that f is MONO, $f(P^{'''}) = b$ and $D(b, P_1^{'}) = \{a, c\}$, $D(b, P_2^{''}) = \emptyset$, $D(b, P_2^{'}) = \{c\}$, therefore $D(b, P_i^{''}) \subseteq D(b, P_i^{'}) \forall i \in$ N $\implies$ that $f(P_1^{'}, P_2^{'}) = b$ using the property that f is MONO.

Preference Profiles							
P		$P^{'}$		$P^{''}$		$P^{'''}$	
P_1	P_2	$P_1^{'}$	$P_2^{'}$	P_1	$P_2^{''}$	$P_1^{'}$	$P_2^{''}$
a	c	b	a	a	c	b	c
b	b	a	b	b	a	a	a
c	a	c	c	c	b	c	b

■

Part b

No, the earlier conclusion does not hold in this case.
The earlier proof cannot go through in this case because single-peaked preference domain is a restricted domain with an intrinsic ordering of alternatives being $a < b < c$. As a result, we cannot construct the preference profile $P_2^{''} = \{c, a, b\}$ as we cannot have a single peak in this type of preference ordering. In this case if the peak is at c, then there would be a dip at b and again a rise at a which does not indicate a single peaked preference.
For this part, we need to construct a mechanism that can have $f(P_1^{'}, P_2^{'}) = a$ given that $f(P_1, P_2) = a$. The peaks for P_1 and P_2 are a and c respectively and the peaks for $P_1^{'}$ and $P_2{'}$ are b and a respectively. Therefore,

if we define SCF f : $S^n \mapsto A$ where S is the set of single-peaked preferences w.r.t common order $<$ for $P \in S^n$

$$f(P) = \min_{i \in N}\{P_i(1)\}$$

Therefore f will pick the left-most peak among the peak of the agents depending on the intrinsic ordering $<$. This is the required mechanism which has been proven to be strategy-proof in class.

Question 2

Gibbard-Satterthwaite(GS) result was proved under the assumptions that all preference profiles are possible i.e the domain is not restricted. We cannot apply the GS result in cases where the domain is restricted as it may not hold. Also, indifference among various alternatives may also be one of the reasons when GS may not hold.
Given that $|X| \geq 2 \implies |A| \geq 3$ where the set of alternatives(A) is the set of all non-empty subsets of objects i.e $\{A : A \subseteq X\}$. The preference profiles of agents over the various alternatives depends on the highest ranked projects in the alternative(which is a subset of X). The ranking of projects depends on a linear ordering over the projects. Let us consider two alternatives A and B such that $A \subsetneq B$. Let us consider the highest ranked projects in A and B which are p_A and p_B respectively. Since $A \subsetneq B$, p_A might be the same as p_B or p_B might be greater in ranking than p_A i.e p_B does not belong to subset A. Therefore, there cannot exist a preference profile in which $A \subsetneq B$ and A is more preferred than B. This is a restriction of the domain and therefore we cannot apply GS rule here.

Question 3

Yes, median voter SCF is group strategy-proof. Here we discuss the proof that median voter SCF is group strategy-proof
Proof: The proof is along the same lines as the proof of median voter SCF being strategy-proof as discussed in class. We need to consider the peak preferences of each agent and for ease will denote the preference profiles of agents using their peak preference only. If we denote the preference of agents by their peaks such that $P = (P_1(1) \ldots P_n(1))$ and $f(P) = a \in A$ be such that a is the median of the preference peaks and the phantom peaks. Let us now consider a group $K \subseteq N$ where is N is the set of all agents.

- Considering the case that if $\forall i \in K$, if $P_i(1) = a$, then there is no reason for the group K to deviate
- Considering the case where $P_i(1) \neq a \forall i \in K$. Suppose the size of the group K is k. Consider that of this group of k agents, $b < k$ have $P_i(1) \neq a$ where b cannot be 0. The following scenarios can arise:
  - Suppose S and R such that $|S| > \frac{b}{2}$ have $P_i(1) < a$ and $|R| = b - |S|$ such that $P_i(1) > a$. If those $|S|$ agents shift their preference to further left of a and the remaining shift their preferences to further right of a i.e $(P_k, P_{-k}) \mapsto (P_k^{'}, P_{-k})$. The new peak $a^{'} < a$. For $r \in R$ $a^{'} < a < P_r(1) < P_r^{'}(1)$ and this is a non-profitable deviation. As a result, they will not deviate to the right in this case. If only the agents in S, deviate to the left of a, it will not affect the median. However, if all agents in R and S deviate to the left of a, then the median will shift to the left of a which would be even less profitable to agents in R as it would be farther left to their peak.
    Let us now consider the scenario that agents in S deviate to the right of a and agents in R deviate to the left of a. This would lead to a resultant shift in the median towards right of a. Considering agents in S, $\forall s \in S$ $P_s(1) < a < P_s^{'}(1)$ and since P_s is a single peaked preference therefore $f(P_k, P_{-k}) P_s f(P_k^{'}, P_{-k})$. Therefore, agents in S will not deviate. Now if only the agents

in R deviate to the left of a, $P_r^{'}(1) < a < P_r^{(}1)$ and by definition of single peaked preferences $f(P_k, P_{-k})P_r f(P_k^{'}, P_{-k})$ and thus R will also not deviate.

– Suppose S and R such that $|S| < \frac{b}{2}$ have $P_i(1) < a$ and $|R| = b - |S|$ such that $P_i(1) > a$. By symmetrical arguments, for this case as well, group K has no profitable manipulation.

– For the case when S and R are equal in numbers, and agents in S deviate to the left of a and those in R deviate to the right of a, the median does not change and same is the case for converse.

– Let us now consider the case when $|S| = |R|$ and agents in both the subsets of group K deviate to the left. The median would shift to the left which would be non profitable for agents in group R as discussed above. Similar case hold when both S and R deviate to the right.

Therefore, there is no profitable manipulation for any group $K \subseteq N$. This proves that median voter SCF is group strategy-proof. ■