Introduction to Game Theory
and Mechanism Design

Objectives of the course

Engineering approach to Economic Theory

Typical Engineering approach
- Analysis
- Synthesis

Examples:

Circuit – analyze with different resistors, capacitors
find out voltage, current

– synthesize a circuit with desired voltage, current

Algorithms – analyze to find complexities, then design
according to a desired complexity

In Game Theory, the setup is:

Multiple agents with possibly conflicting objectives – GAME

Given a game - find most probable outcomes or responses of the agents/players : Game Theory.

- analysis part, predictive approach

Given a "reasonable" outcome - find/build the game that yields that as a probable outcome : Mechanism Design

- synthesis part, prescriptive approach.

Example of Game Theory : Neighboring Kingdom's Dilemma

Kingdoms A and B have limited options to invest wealth

1. Agriculture : save people from starvation
2. Warfare : ransack other kingdom and have their wealth

Outcome is dependent on the joint action of both

e.g., if A chooses Agri and B chooses War, then B gets all the agricultural produce of A since A has not developed techniques to protect itself.

A \ B	Agri	War
Agri	5, 5	0, 6
War	6, 0	1, 1

Question: What is a "reasonable" outcome of the above game?

Little more formally,

A game is a formal representation of the strategic interaction between multiple agents called players.

The choices available to the players are called actions.

The mapping from the state of the game $\rightarrow$ set of actions
strategy

Depending on the context, games can be represented in many ways:

Normal form, Extensive form, Repeated, Stochastic, ...

Game theory is the formal study of strategic interaction between players that are rational and intelligent.

A player is rational if she picks actions to achieve her most desired outcome, e.g., maximize her payoff.

A player is intelligent if she knows the rules of the game perfectly and picks action considering that there are other rational and intelligent players in the game.

Intelligence implies that the player has sufficient computational ability to find the "optimal" action.

against other players

Objectives of game theory:

provide predictions of an outcome

A different example

Goal: divide the cake such that each kid is happy with his/her portion.

Kid_1 thinks that he got at least half of the cake.
Kid_2 thinks she got at least half of the cake.

in their view

Notion of at least half is subjective.

A third party, e.g., the mother, may not know what is at least half.

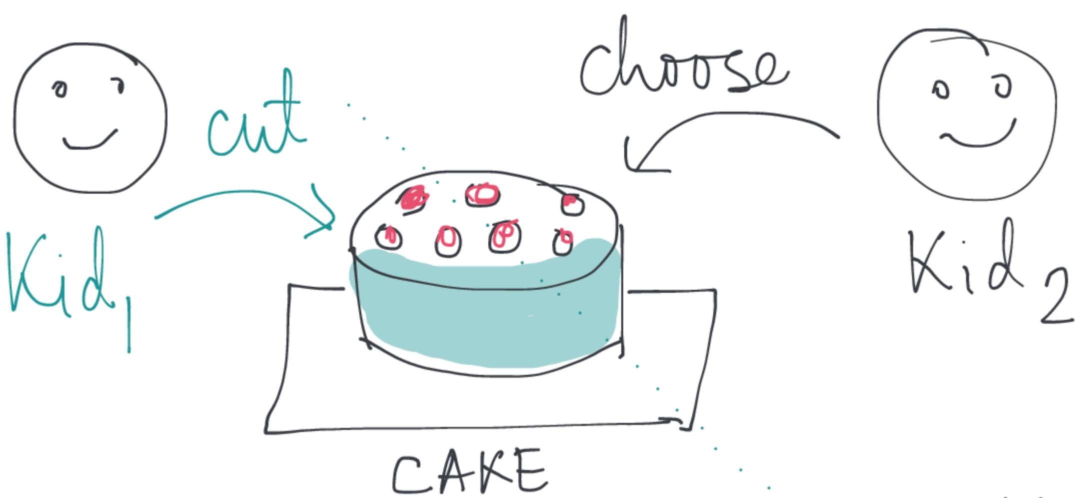
What if one kid complains after the mother made a division?

Challenge:

- The mother wants to achieve a "fair" division
- Does not have enough information to do it (doesn't see the cake through kids' views)
- Does not know what is a fair division

Question:

Can she design a mechanism with that incomplete knowledge to achieve a fair division?



I CUT YOU CHOOSE MECHANISM

Why does this work?

Kid_1 will divide it equally in his view

- because, if not, Kid_2 may pick the bigger one.
- hence he is indifferent between the two pieces.

Kid_2 will pick the bigger one in her view.

Mechanism Design: Inverse Game Theory

- Start with an objective
- Design a game, such that the "reasonable" outcome of that game satisfies that objective.
- Provides a prescription

Why should we design a game?

Sports tournaments generally have groups

Round robin in every group - top 2 qualifies

Is this a good tournament design?

World Cup Football (Soccer) 1982, Group II

Teams: Austria, Algeria, West Germany, Chile

Game 1: Algeria beat West Germany 2-1 : shock

Game 2: Austria beat Algeria 2-0

Game 3: Algeria beat Chile 3-2

Algeria was almost going to be the first African team to qualify for knockout

But W. Germany and Austria made contract - Austria lost - Disgrace of Gijon.

Course outline

Non-cooperative game theory

Mechanism design

Applications (interspersed)

Takeaways

- Apply principles of Economic Theory and computation to understand incentives in social systems and on the internet.
- Build a taste for mathematical description of social problems
- Make deployable AI system that does it automatically.

Self contained course materials

- Game Theory : Maschler, Solan, Zamir
- Multiagent Systems : Shoham and Leyton-Brown

Quick recap: Game Theory

- Analytical approach for predicting reasonable outcome
- Building blocks: players, strategies, utilities
- Difference between action and strategy
- Key assumptions: rationality and intelligence

Example to illustrate: Game of Chess (von Neumann and Morgenstern, 1944)

Formal description

- Two player game: White and Black — 16 pieces each.
- Every piece has some legal moves — ACTIONS
- Starts with W, players take turns
- Ends: W win, if W captures B king
  B win, if B captures W king
  Draw, if nobody has legal moves but kings are not in check, both players agree to a draw, board position is such that nobody can win, many more ...

Natural questions from a theorist's perspective

- Does W have a winning strategy, i.e., a plan of moves s.t. it wins IRRESPECTIVE of the moves of B?
- Does B have a winning strategy?
- Or at least guarantee a draw?
- Neither may be possible — not synonymous with end of game.

What is a strategy?

In the context of chess,

board position if different from Game Situation

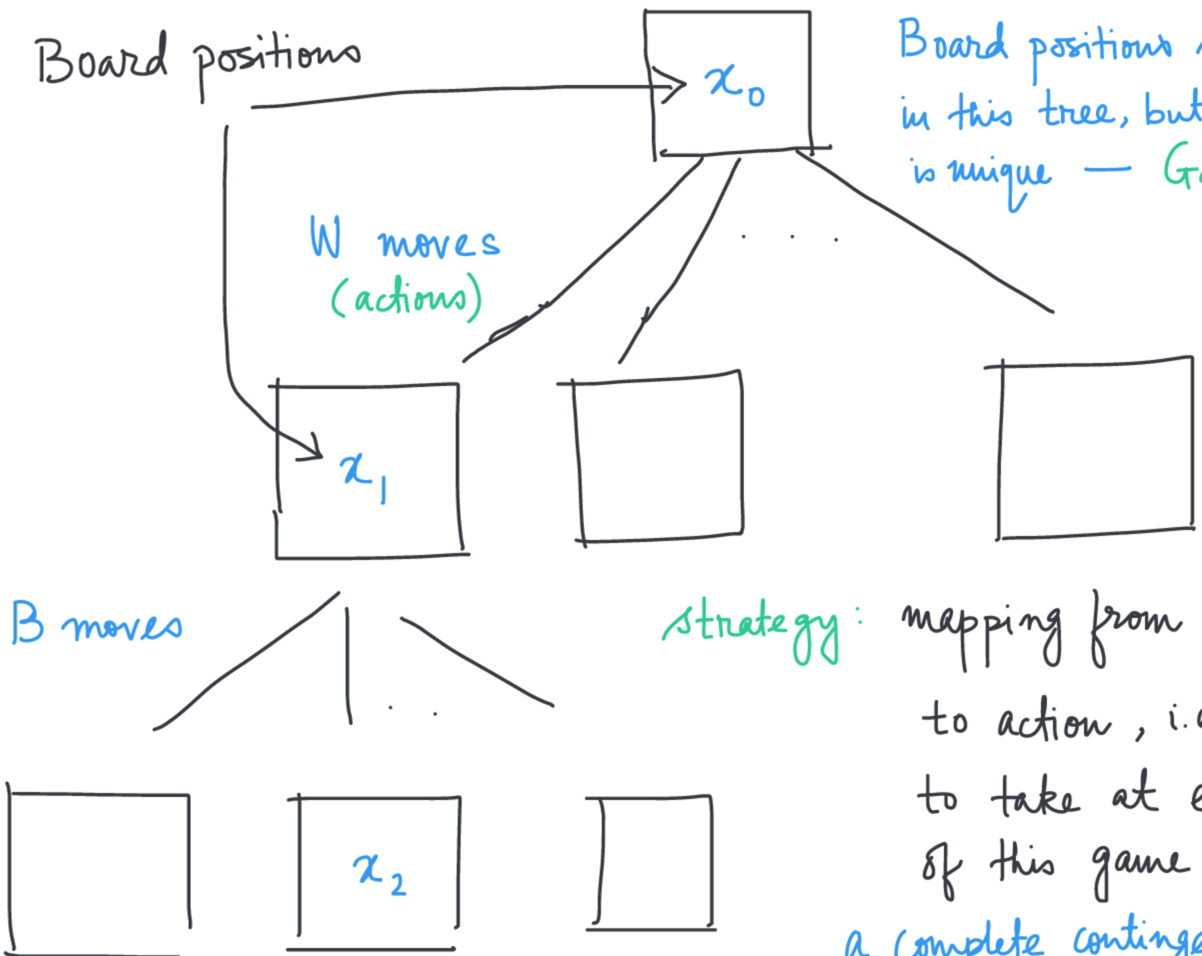
more than one sequence of moves can bring to the same board position.

denote a board position by x_k

Game Situation is a finite sequence $(x_0, x_1, x_2, \dots, x_K)$ of board positions s.t.

- x_0 is the opening board position
- $x_k \to x_{k+1}$, k even – created by a single action of W

  k odd – created by a single action of B



Board positions may repeat in this tree, but a vertex is unique — Game Situation

strategy: mapping from game situation to action, i.e. what action to take at every vertex of this game tree.

a complete contingency plan

A strategy for W is a function s_W that associates every game situation $(x_0, x_1, \ldots, x_K) \in H$ (set of all game situations), K even, with a board position x_{K+1} such that the move $x_K \to x_{K+1}$ is a single valid move of W.

Similar definition of s_B for B.

Note:

- strategy pair (s_W, s_B) determines $\boxed{\text{an outcome}}$ also called one play of the game. - a path through the game tree

Questions: (1) this is a finite game - where does it end?

(2) can a player guarantee an outcome?

The game ends: (a) W wins OR (b) B wins OR (c) Draw.

A winning strategy for W is a strategy s_W^* s.t. for every s_B (s_W^*, s_B) ends in a win for W.

A strategy guaranteeing at least a draw for W is s_W' s.t. for every s_B, (s_W', s_B) either ends in a draw or win for W.

analogous definitions of s_B^* and s_B'

Not obvious if such strategies exist

An early result of Game Theory (Von Neumann, 1928)

In chess, one and only one of the following statements is true

1. W has a winning strategy
2. B has a winning strategy
3. Each player has a strategy guaranteeing a draw

- there were other possibilities, e.g., nothing can be guaranteed
- it does not say what is that strategy

  actually it is not known: which one is true and what is that strategy

Chess would have been a boring game if any of these answers were known.

An early result of Game Theory (von Neumann, 1928)

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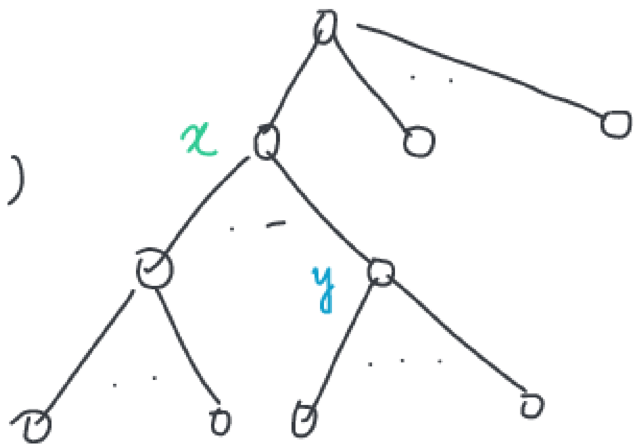


Proof: Each vertex is a game situation

$\Gamma(x)$: subtree rooted at x (includes itself)

n_x : number of vertices in $\Gamma(x)$

y is a vertex in $\Gamma(x)$, $y \neq x$

$\Gamma(y)$ is a subtree of $\Gamma(x)$, $n_y < n_x$

$n_x = 1 \Rightarrow x$ is a terminal vertex

The proof is via induction on n_x

The theorem holds for $n_x = 1$, why?

if W king is removed, B wins

if B king is removed, W wins

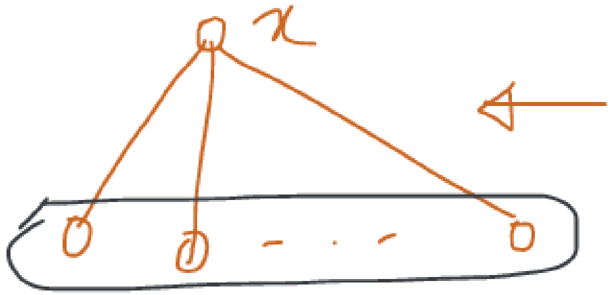
if both kings present, but game ends — draw

Suppose x is a vertex with $n_x > 1$

Induction hypothesis: for all vertices $y \in \Gamma(y)$, s.t. $n_y < n_x$, the statement holds

in particular, $\boxed{\Gamma(y) \text{ is a subgame of } \Gamma(x)}$

WLOG W moves



$C(x)$ = Vertices reachable from x in one step.

Case (i) if $\exists y_0 \in C(x)$, s.t. (1) is true in $\Gamma(y_0)$, Then (1) is true in $\Gamma(x)$
W just picks that

Case (ii) if $\forall y \in C(x)$, (2) is true, Then (2) is true in $\Gamma(x)$
B sees that action and picks The appropriate action to win.

Case (iii)

- (i) does not hold, W does not have a winning strategy in any $y \in C(x)$
  Since induction hypothesis holds for every $y \in C(x)$, either B has a winning strategy or both have draw-guaranteeing strategy
- (ii) doesn't hold, $\exists y' \in C(x)$ where B doesn't have a winning strategy
  since (i) doesn't hold either, W can't guarantee a win in y'
  - hence they both have strategies guaranteeing a draw.

    W picks The action to reach y'.

    B picks action that guarantees a draw or win.

This concludes The proof.

Exercise: prove this when The length of game is infinite, (ex 1.3 MSZ)

Normal Form Games

It is a representation technique for games

$N = \{1, 2, \dots, n\}$ set of players

S_i : set of strategies of player i, $s_i \in S_i$

Set of strategy profiles $S = \times_{i \in N} S_i$

A strategy profile $s = (s_1, s_2, \dots, s_n) \in S$

strategy profile without i

$$\underline{s}_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$$

$u_i : S \to \mathbb{R}$ utility function of player i

NFG representation is the tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

If S_i is finite $\forall i \in N$, this is called a finite game.

Example: Penalty Kick Game

Shooter \ Goalkeeper	L	C	R
L	-1, 1	1, -1	1, -1
C	1, -1	-1, 1	1, -1
R	1, -1	1, -1	-1, 1

$N = \{1, 2\}$

$S_1 = S_2 = \{L, C, R\}$

$u_1(L, L) = -1$, $u_1(L, C) = 1$,

$u_1(L, R) = 1$

$u_2(L, L) = 1$, $u_2(L, C) = -1$

$u_2(L, R) = -1$

Rationality: A player is rational if she picks actions to maximize her utility

Intelligence: A player is intelligent if she knows the rules of the game perfectly and picks action considering that there are other rational and intelligent players.

Common Knowledge:

A fact is common knowledge if

1. all players know the fact
2. all players know that all players know the fact
3. all players know that all other players know that all other players know the fact

... ad infinitum

Implication:

- Isolated island: three blue-eyed people (eye color can be blue or black) no reflecting medium on the island, nobody talks about eye color
- One day a sage comes to the island and says

  Blue-eyed people are bad for the island and must leave. There is at least one blue-eyed person in this island.

sage cannot be disputed - if someone realizes that his/her eye color is blue he/she leaves at the end of the day.

How does common knowledge percolate?

If there were only one blue-eyed person, he would see the other two person has black eyes. Sage is always correct, hence he must be the only blue-eyed person – leaves at end of day 1.

If there were two, each of them would see one blue, one black. Watch the other blue-eyed person's move till day 2 (since the other blue-eyed person also knows that fact). When the other person doesn't leave by day 1, both are certain about their eye-color and leaves at the end of day 2. The black-eyed person watches this till day 3 and does not leave.

Since there are 3 people with blue eyes, all of them leaves on day 3.

Assumption: The fact that all players are rational and intelligent is a common knowledge.

Domination in NFGs

strategies of player 2

Players 2 / 1	L	M	R
U	1,0	1,3	3,2
D	-1,6	0,5	3,3

Strategies of Player 1

Will a rational player ever play R?

Dominated Strategy

A strategy $s_i' \in S_i$ of player i is strictly dominated if there exists another strategy s_i of i such that for every strategy profile $\underline{s}_{-i} \in \underline{S}_{-i}$ of the other players $u_i(s_i, \underline{s}_{-i}) > u_i(s_i', \underline{s}_{-i})$.

A strategy $s_i' \in S_i$ of player i is weakly dominated if there exists another strategy s_i of i such that for every strategy profile $\underline{s}_{-i} \in \underline{S}_{-i}$ of the other players $u_i(s_i, \underline{s}_{-i}) \geq u_i(s_i', \underline{s}_{-i})$, and

there exists some $\underline{\tilde{s}}_{-i} \in \underline{S}_{-i}$ such that

$$u_i(s_i, \underline{\tilde{s}}_{-i}) > u_i(s_i', \underline{\tilde{s}}_{-i}).$$

Example: R is strictly dominated, D is weakly dominated.

Dominant Strategy

A strategy s_i is strictly (weakly) dominant strategy for player i if s_i strictly (weakly) dominates all other $s_i' \in S_i \setminus \{s_i\}$

Examples: ① Neighboring Kingdoms' dilemma

	Agri	Defence
Agri	5,5	0,6
Defence	6,0	1,1

Dominant strategy?

Which kind?

② One indivisible item for sale

Two players having values v_1 and v_2 respectively

Each player can choose a number in $[0, M]$, $(M \gg v_1, v_2)$

Player quoting the largest number "wins" the object (tie broken in favor of 1), and "pays" the losing player's chosen number

utility of winning player = her value − her payment

utility of losing player = 0

NFG representation: $N = \{1, 2\}$, $S_1 = S_2 = [0, M]$

$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2, & \text{if } s_1 \geqslant s_2 \\ 0 & \text{ow} \end{cases} \quad \Bigg| \quad u_2(s_1, s_2) = \begin{cases} v_2 - s_1, & \text{if } s_1 < s_2 \\ 0 & \text{ow} \end{cases}$$

Dominant strategy? Which kind?

Dominant Strategy Equilibrium

A strategy profile $(s_1^*, s_2^*, \cdots, s_n^*)$ is a strictly (weakly) dominant strategy equilibrium (SDSE/WDSE) if s_i^* is a strictly (weakly) dominant strategy for i, $\forall i \in N$.

Question: What kind of equilibrium in this game?

	D	E
A	5, 5	0, 5
B	5, 0	1, 1
C	4, 0	1, 1

Rationality and Dominated Strategies

Rational players do not play dominated strategies

To obtain rational outcomes of a game – eliminate dominated strategies

For strictly dominated strategies, the order of elimination does NOT matter

It matters for the weakly dominated strategies – some reasonable outcomes are also eliminated

	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,1	0,0	1,0

Order: T, R, B, C → (M,L): 2,2

Order: B, L, C, T → (M,R): 3,2

Existence of dominant strategies (and DSE)

Not guaranteed!

Coordination game

	L	R
L	1,1	0,0
R	0,0	1,1

Football or Cricket?

	F	C
F	2,1	0,0
C	0,0	1,2

If dominance cannot explain reasonable outcome – Refine the equilibrium concept

Nash Equilibrium (Nash 1951)

"No player gains by a unilateral deviation"

A strategy profile (s_i^*, s_{-i}^*) is a pure strategy Nash equilibrium (PSNE) if $\forall i \in N$ and $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Football or Cricket?

	F	C
F	2,1	0,0
C	0,0	1,2

A best response view:

A best response of player i against the strategy profile s_{-i} of the other players is a strategy that gives the maximum utility, i.e.,

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \ \forall s_i' \in S_i\}$$

PSNE is a strategy profile (s_i^*, s_{-i}^*) s.t.

$$s_i^* \in B_i(s_{-i}^*), \ \forall i \in N.$$

PSNE gives stability - once there, no rational player unilaterally deviates

Question: Relationship between SDSE, WDSE and PSNE?

Risk aversion of Players

Risk : if the other player does not pick the equilibrium action

Less risky for player 1 : T.

1 \ 2	L	R
T	2,1	1,-20
M	3,0	-10,1
B	-100,2	3,3

Another type of rationality : players making pessimistic estimates of others

This worst case optimal choice is max-min strategy

$$s_i^{maxmin} \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

maxmin value

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^{maxmin}, t_{-i}) \geqslant \underline{v}_i \, , \quad \forall t_{-i} \in S_{-i}$$

Max-min and dominant strategies

Theorem: If s_i^* is a dominant strategy for player i, then it is a maxmin strategy for i.

Proof outline [for strictly dominant strategies]

Let s_i^* be the strictly dominant strategy of player i

(a) $\quad u_i(s_i^*, \underline{s}_{-i}) > u_i(s_i', \underline{s}_{-i}), \ \forall \underline{s}_{-i} \in \underline{S}_{-i}, \forall s_i' \in S_i \setminus \{s_i^*\}$

let $\underline{s}_{-i}^{\min}(s_i') \in \arg\min_{s_{-i} \in \underline{S}_{-i}} u_i(s_i', \underline{s}_{-i})$ - worst choice of the other players for i.

but (a) holds for all $\underline{s}_{-i}$

$$u_i(s_i^*, \underline{s}_{-i}^{\min}(s_i')) > u_i(s_i', \underline{s}_{-i}^{\min}(s_i')), \ \forall s_i' \in S_i \setminus \{s_i^*\}$$

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{\underline{s}_{-i} \in \underline{S}_{-i}} u_i(s_i, \underline{s}_{-i}).$$

Weak dominance : Homework.

Relationship with PSNE

Every PSNE $s^* = (s_1^*, \dots, s_n^*)$ of an NFG satisfies

$$u_i(s^*) \geq \underline{v}_i, \ \forall i \in N.$$

$1 \backslash 2$	L	R
T	2,1	1,-20
M	3,0	-10,1
B	-100,2	3,3

Proof:

$$u_i(s_i, \underline{s}_{-i}^*) \geq \min_{\underline{s}_{-i} \in \underline{S}_{-i}} u_i(s_i, \underline{s}_{-i}) \quad \text{[by defn. of min]}$$

$$u_i(s_i^*, \underline{s}_{-i}^*) \geq u_i(s_i, \underline{s}_{-i}^*), \ \forall s_i \in S_i \quad \text{[by defn of PSNE]}$$

$$u_i(s_i^*, \underline{s}_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \underline{s}_{-i}^*) \geq \max_{s_i \in S_i} \min_{\underline{s}_{-i} \in \underline{S}_{-i}} u_i(s_i, \underline{s}_{-i}) = \underline{v}_i$$

Recap: (1) dominance cannot explain all reasonable outcomes

(2) PSNE - unilateral deviation [STABILITY]

(3) Maxmin - rationality for risk-aversion [SECURITY]

What happens to stability and security when some strategies are eliminated?

Iterated elimination of dominated strategies

	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,0	0,0	1,0

Order: T, R, B, C → (M, L): 2,2

Order: B, L, C, T → (M, R): 3, 2

Does it change the maxmin value?

		Pl 1	Pl 2
Consider the example above:	maxmin	2	0
B is eliminated (dominated for 1)	maxmin	2	2

Maxmin value is not affected for the player whose dominated strategy is removed

Theorem: Consider NFG $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$, let $\hat{s}_j \in S_j$ be a dominated strategy. Let $\hat{G}$ be the residual game after removing $\hat{s}_j$. The maxmin value of j in $\hat{G}$ is equal to her maxmin value in G.

Intuition: maxmin is the max of the mins - elimination affects one min but that doesn't affect the max since the strategy was dominated.

Proof: maxmin value of j in G, $\underline{v}_j = \max\limits_{s_j \in S_j} \min\limits_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(s_j, \underline{s}_{-j})$

maxmin value of j in $\hat{G}$, $\hat{\underline{v}}_j = \max\limits_{s_j \in S_j \setminus \{\hat{s}_j\}} \min\limits_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(s_j, \underline{s}_{-j})$

let t_j dominates $\hat{s}_j$ in G, $t_j \in S_j \setminus \{\hat{s}_j\}$

$$u_j(t_j, \underline{s}_{-j}) \geqslant u_j(\hat{s}_j, \underline{s}_{-j}), \quad \forall \underline{s}_{-j} \in \underline{S}_{-j}$$

Therefore,

$$\min_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(t_j, \underline{s}_{-j}) = u_j(t_j, \tilde{\underline{s}}_{-j}) \geqslant u_j(\hat{s}_j, \tilde{\underline{s}}_{-j}) \geqslant \min_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(\hat{s}_j, \underline{s}_{-j})$$

$$\Rightarrow \max_{s_j \in S_j \setminus \{\hat{s}_j\}} \min_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(s_j, \underline{s}_{-j}) \geqslant \min_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(t_j, \underline{s}_{-j}) \geqslant \min_{\underline{s}_{-j} \in \underline{S}_{-j}} u_j(\hat{s}_j, \underline{s}_{-j})$$

$\underline{v}_j$ [max min value in G for j]

$$= \max_{s_j \in S_j} \min_{s_{-j} \in \underline{S}_{-j}} u_j(s_j, s_{-j})$$

$S_j \setminus \{\hat{s}_j\}$ $\quad$ $\hat{s}_j$

$$= \max \left\{ \max_{s_j \in S_j \setminus \{\hat{s}_j\}} \min_{s_{-j} \in \underline{S}_{-j}} \cdots , \min_{s_{-j} \in \underline{S}_{-j}} u_j(\hat{s}_j, s_{-j}) \right\}$$

$\geq$

$$= \max_{s_j \in S_j \setminus \{\hat{s}_j\}} \min_{s_{-j} \in \underline{S}_{-j}} u_j(s_j, s_{-j}) = \hat{\underline{v}}_j \text{ [max min of } j \text{ in } \hat{G}]$$

What happens to equilibrium after iterative elimination?

Theorem: Consider G and $\hat{G}$ are games before and after elimination of a strategy [not necessarily dominated]. If s^* is a PSNE in G and survives in $\hat{G}$, the s^* is a PSNE in $\hat{G}$ too.

Intuition: PSNE strategy was the maxima, removing others will continue keeping this as maxima. Proof: exercise.

Can new equilibrium be generated?

Theorem: Consider NFG G. Let $\hat{s}_j$ be a weakly dominated strategy of j. If $\hat{G}$ is obtained from G eliminating $\hat{s}_j$, every PSNE of $\hat{G}$ is a PSNE of G

No new PSNE if the eliminated strategy is dominated.

Proof: $\quad \hat{G}: \hat{S}_j = S_j \setminus \{\hat{s}_j\}, \quad \hat{S}_i = S_i, \forall i \neq j.$

TST: if $s^* = (s_j^*, s_{-j}^*)$ is a PSNE in $\hat{G}$, it is a PSNE in G

$$u_i(s^*) \geqslant u_i(s_i, s_{-i}^*), \quad \forall i \neq j, \forall s_i \in \hat{S}_i = S_i$$

$$u_j(s^*) \geqslant u_j(s_j, s_{-j}^*), \quad \forall s_j \in \hat{S}_j$$ - this has one less

need to show that there is no profitable deviation for any player in G

for $i \neq j$, this is immediate — no strategies are removed

for j, this is true for all strategies except $\hat{s}_j$

Since $\hat{s}_j$ is dominated, $\exists\, t_j \in \hat{S}_j = S_j \setminus \{\hat{s}_j\}$

s.t. $u_j(t_j, \underline{s}_{-j}) \geq u_j(\hat{s}_j, \underline{s}_{-j}),\ \forall \underline{s}_{-j} \in \underline{S}_{-j}$

so, in particular, $u_j(t_j, \underline{s}^*_{-j}) \geq u_j(\hat{s}_j, \underline{s}^*_{-j})$

since s^* is a PSNE in $\hat{G}$ and $t_j \in \hat{S}_j$,

$$u_j(s^*_j, \underline{s}^*_{-j}) \geq u_j(t_j, \underline{s}^*_{-j}) \geq u_j(\hat{s}_j, \underline{s}^*_{-j})$$

Summary:

- Elimination of strictly dominated strategies have no effect on PSNE.
- Elimination of weakly dominated strategies may reduce the set of PSNEs, but never adds new.
- The maxmin value is unaffected by the elimination of strictly or weakly dominated strategies

Matrix games (Two player zero sum games)

A special class with certain nice properties of the stability and security notions

$\langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$, with $N = \{1,2\}$, $u_1 + u_2 \equiv 0$

Example: Penalty shootout game

G	L	R
S: L	−1, 1	1, −1
R	1, −1	−1, 1

An arbitrary game

	L	C	R
T	3, −3	−5, 5	−2, 2
M	1, −1	4, −4	1, −1
B	6, −6	−3, 3	−5, 5

Possible to represent the game with one matrix u, considering the utilities of only player 1

Player 2's utilities are negative of the matrix

Player 2's maxmin strategies are the minmax of this matrix (security criterion)

u	L	R	maxmin
L	−1	1	−1
R	1	−1	−1
minmax	1	1	

u	L	C	R	maxmin
T	3	−5	−2	−5
M	1	4	1	1
B	6	−3	−5	−5
minmax	6	4	1	

What are the PSNEs of these games?

Saddle point (of a matrix): The value is maximum for player 1, minimum for player 2.

Rephrase: what are the saddle point of the two games?

Theorem: In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point if and only if it is a PSNE.

Proof: (s_1^*, s_2^*) is a saddle point $\Leftrightarrow$

$$u(s_1^*, s_2^*) \geq u(s_1, s_2^*), \forall s_1 \in S_1, \text{ and } u(s_1^*, s_2^*) \leq u(s_1^*, s_2) \quad \forall s_2 \in S_2$$

$\Leftrightarrow$ it is a PSNE, since $u_1 \equiv u$, $u_2 = -u$.

Consider the maxmin and minmax values

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$

How are they related?

Lemma: For matrix games $\overline{v} \geq \underline{v}$.

Proof: $$u(s_1, s_2) \geq \min_{t_2 \in S_2} u(s_1, t_2)$$

$$\max_{t_1 \in S_1} u(t_1, s_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \quad \forall s_2$$

$$\min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \qquad \square$$

Earlier matrix game examples

u	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
minmax	1	1	

$\overline{v} = 1 > -1 = \underline{v}$

PSNE doesn't exist

u	L	C	R	maxmin
T	3	-5	2	-5
M	1	4	1	1
B	6	-3	-5	-5
minmax	6	4	1	

$\overline{v} = 1 = \underline{v}$

PSNE exists

Define $s_1^* \in \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$: maxmin strategy of 1

$s_2^* \in \arg\min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$: minmax strategy of 2

Theorem: A matrix game u has a PSNE (saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively. In particular, (s_1^*, s_2^*) is a PSNE.

Proof: $(\Rightarrow)$ i.e., PSNE $\Rightarrow \overline{v} = \underline{v} = u(s_1^*, s_2^*)$

Say the PSNE is (s_1^*, s_2^*), i.e., $u(s_1^*, s_2^*) \geq u(s_1, s_2^*), \forall s_1 \in S_1$

$$\Rightarrow u(s_1^*, s_2^*) \geq \max_{t_1 \in S_1} u(t_1, s_2^*)$$

$$\geq \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2)$$, since s_2^* is a specific strategy

$$= \overline{v}$$

Similarly, using the same argument for player 2, we get

$$\underline{v} \geqslant u(s_1^*, s_2^*), \text{ for player 2 utility } u_2 = -u$$

But $\bar{v} \geqslant \underline{v}$ [from previous lemma]

Hence, $u(s_1^*, s_2^*) \geqslant \bar{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*)$

$\Rightarrow u(s_1^*, s_2^*) = \bar{v} = \underline{v}$, also implies that the maxmin for 1 and minmax for 2 are s_1^* and s_2^* resp.

($\Leftarrow$) given $u(s_1^*, s_2^*) = \bar{v} = \underline{v}$ $= v$ (say), s_1^*, s_2^* are maxmin and minmax resp. for 1 and 2.

$$u(s_1^*, s_2) \geqslant \min_{t_2 \in S_2} u(s_1^*, t_2) \quad : \text{by defn of min}$$

$$\forall s_2 \in S_2 \qquad = \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \quad : \text{since } s_1^* \text{ is the maxmin strategy for 1.}$$

$$= v \text{ (given)}$$

Similarly show, $u(s_1, s_2^*) \leqslant v \quad \forall s_1 \in S_1$

but $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE

Recap: ① iterated elimination of dominated strategies
② Preservation of equilibrium
③ stability & security coincide for matrix games
④ limited to pure strategies – PSNE may not exist

	L	R
L	-1,1	1,-1
R	1,-1	-1,1

Mixed strategies

probability distribution over the set of strategies

		$4/5$	$1/5$
		L	R
$2/3$	L	-1,1	1,-1
$1/3$	R	1,-1	-1,1

Consider a finite set A

define $\Delta A = \{ p \in [0,1]^{|A|} : \sum_{a \in A} p_a = 1 \}$

set of all probability distributions over A.

σ_i is a mixed strategy of player i

$\sigma_i \in \Delta(S_i)$, i.e., $\sigma_i : S_i \to [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

We are discussing non-cooperative games, the players choose their strategies independently

The joint probability of 1 picking s_1 and 2 picking $s_2 = \sigma_1(s_1)\,\sigma_2(s_2)$

utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1)\,\sigma_2(s_2) \cdots \sigma_n(s_n)\; u_i(s_1, s_2, \ldots, s_n)$$

we are overloading u_i to denote the utility at pure and mixed strategies.

Utility at a mixed strategy is the expectation of the utilities at pure strategies.

So, all the rules of expectation holds, e.g., linearity.

Example:

		4/5	1/5
		L	R
2/3	L	-1,1	1,-1
1/3	R	1,-1	-1,1

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot 1 + \frac{1}{3} \cdot \frac{4}{5} \cdot 1 + \frac{1}{3} \cdot \frac{1}{5} \cdot (-1)$$

mixture of mixed strategies

$$u_i(\lambda \sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda u_i(\sigma_i, \sigma_{-i}) + (1-\lambda)\, u_i(\sigma_i', \sigma_{-i}).$$

Mixed strategy Nash equilibrium (MSNE)

<u>Defn</u>: MSNE is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \quad \forall \sigma_i' \in \Delta(S_i), \forall i \in N$$

Relation between PSNE and MSNE?

An alternative definition

<u>Theorem</u>: A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE if and only if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*), \quad \forall s_i \in S_i, \forall i \in N$$

<u>Proof</u>: $(\Rightarrow)$ s_i is a special case of the mixed strategy

The mixed strategy with s_i having prob. 1. Inequality holds by definition of MSNE.

$(\Leftarrow)$ Pick an arbitrary mixed strategy σ_i of player i

$$u_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i(s_i) \; u_i(s_i, \sigma_{-i}^*)$$

(Given)

$$\leq \sum_{s_i \in S_i} \sigma_i(s_i) \; u_i(\sigma_i^*, \sigma_{-i}^*)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*) \sum_{s_i \in S_i} \sigma_i(s_i) = u_i(\sigma_i^*, \sigma_{-i}^*)$$

Example of MSNE

Is the mixed strategy profile an MSNE?

		$4/5$	$1/5$
		L	R
$2/3$	L	$-1,1$	$1,-1$
$1/3$	R	$1,-1$	$-1,1$

to prove this, need to show there does not exist any better mixed strategy for the player.

expected utility of player 2 from $L = \frac{2}{3}\cdot 1 + \frac{1}{3}(-1) = \frac{1}{3}$,

from $R = -\frac{1}{3}$

expected utility will increase if some probability is transferred from R to L $\Rightarrow$ the current profile is not an MSNE.

Some balance in the utilities is needed

Redo the calculations

		$1/2$	$1/2$
		L	R
$1/2$	L	$-1,1$	$1,-1$
$1/2$	R	$1,-1$	$-1,1$

does there exist any improving mixed strategy?

How to find an MSNE?

Support of mixed strategy (prob. distribution)

For mixed strategy σ_i, the subset of strategy space of i on which σ_i has positive mass is the support of σ_i, i.e.,

$$\delta(\sigma_i) = \{ s_i \in S_i : \sigma_i(s_i) > 0 \}$$

using the definition of support, here is a characterization of MSNE

Theorem: A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff $\forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in \delta(\sigma_i^*)$
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$

Implication: consider penalty shootout game

	L	R
L	-1,1	1,-1
R	1,-1	-1,1

Case 1: Supports $(\{L\}, \{L\})$

for player 1, $s_1' = R$ violates 2

Case 2: $(\{L, R\}, \{L\})$ - symmetric for the other case

for player 1, the expected utility has to be same for L and R - not possible - violates 1

Case 3: $(\{L, R\}, \{L, R\})$

(2) is vacuously satisfied

for (1), player 1 chooses L w.p. p and player 2 chooses L w.p. q

(1) for player 1

$$u_1(L, (q, 1-q)) = u_1(R, (q, 1-q))$$

$$(-1)\, q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q)$$

$$\Rightarrow \quad q = 1/2$$

	L	R
L	-1,1	1,-1
R	1,-1	-1,1

(1) for player 2

$$u_2((p, 1-p), L) = u_2((p, 1-p), R) \quad \Rightarrow \quad p = 1/2$$

$$\text{MSNE} = \left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)$$

Exercises:

	F	C
F	2,1	0,0
C	0,0	1,2

	F	C	D
F	2,1	0,0	1,1
C	0,0	1,2	2,0

MSNE characterization theorem

<u>Theorem</u>: A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff $\forall i \in N$

(1) $u_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in \delta(\sigma_i^*)$

(2) $u_i(s_i, \sigma_{-i}^*) \geqslant u_i(s_i', \sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$

Observations:

(1) $$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

maximizing w.r.t. a distribution $\equiv$ whole probability mass at max

(2) $$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$

$(\sigma_i^*, \sigma_{-i}^*)$ MSNE

the maximizer must lie in $\delta(\sigma_i^*)$ – if no maximizer in $\delta(\sigma_i^*)$ then put all probability mass on that $s_i' \notin \delta(\sigma_i^*)$ that has the maximum value of the utility – $(\sigma_i^*, \sigma_{-i}^*)$ is not a MSNE.

<u>Proof</u>: $(\Rightarrow)$ given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \quad \text{----} \quad (1) \end{aligned}$$

by definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*)$$

$$= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*) \quad \text{- - - - - } (2)$$

(1) and (2) are equal - max is equal to positive weighted average

- can happen only when all values are same. proves condition 1

for condition 2: suppose for contradiction

$$\exists\ s_i \in \delta(\sigma_i^*) \text{ and } s_i' \notin \delta(\sigma_i^*)$$

$$\text{s.t. } u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$$

shift the probability mass $\sigma_i^*(s_i)$ to s_i', this new mixed strategy gives a strict better utility - contradiction to MSNE.

($\Leftarrow$) Given the two conditions of the theorem hold

let $u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*)$ - condition 1

note $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$ - condition 2

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*) \quad \text{- by defn. of } \delta(\sigma_i^*)$$

$$= m_i(\sigma_{-i}^*) \quad \text{- previous conclusion}$$

$$= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \quad \text{- previous conclusion}$$

from the observation

$$= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \geqslant u_i(\sigma_i, \sigma_{-i}^*),\ \forall \sigma_i \in \Delta(S_i)$$

algorithmic way to find MSNE

MSNE characterization Theorem to algorithm

NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

All possible supports of $S_1 \times S_2 \times \cdots \times S_n$

number $= K = (2^{|S_1|}-1) \times (2^{|S_2|}-1) \times \cdots \times (2^{|S_n|}-1)$

for every support profile $X_1 \times X_2 \cdots \times X_n$, where $X_i \subseteq S_i$

solve the following feasibility program

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \quad \forall s_i \in X_i, \forall i \in N \quad \text{— cond } 1$$

$$w_i \geqslant \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N \quad \text{— cond } 2$$

$$\sigma_j(s_j) \geqslant 0, \forall s_j \in S_j, \forall j \in N, \text{ and } \sum_{s_j \in S_j} \sigma_j(s_j) = 1, \forall j \in N.$$

feasibility program with variables w_i, $i \in N$, $\sigma_j(s_j)$, $s_j \in S_j$, $j \in N$.

Remarks : this is not a linear program unless $n=2$

For general games, there is no poly-time algorithm

Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs]

Daskalakis, Goldberg, Papadimitriou "The complexity of computing a Nash equilibrium" 2009.

MSNE and dominance

The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

Dominated strategy in this game?

	L	R
T	4, 1	2, 5
M	1, 3	6, 2
B	2, 2	3, 3

domination can also be via mixed strategy

weak dominated strategy removal can remove equilibrium

for strictly dominated strategies

<u>Theorem</u>: If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$, then in every MSNE of the game, s_i is chosen with probability zero.

So, can remove without loss of equilibrium.

<u>Existence</u> of MSNE

Finite game: number of players and the strategies are finite

Theorem (Nash 1951)

Every finite game has a (mixed) Nash equilibrium.

Proof requires a few tools and a result from real analysis

- A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda) x \in S$
- A set $S \subseteq \mathbb{R}^n$ is closed if it contains all its limit point (points whose every neighborhood contains a point in S – a set not closed $[0,1)$ – every ball of radius $\epsilon > 0$ around 1 has a member of $[0,1)$, but 1 is not in the set $[0,1)$)
- A set $S \subseteq \mathbb{R}^n$ is bounded if $\exists\, x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$ s.t.

  $\forall x \in S, \ \|x - x_0\|_2 < R$
- A set $S \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

A result from real analysis (without proof)

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is convex and compact and $T : S \to S$, is continuous

Then T has a fixed point, i.e., $\exists\, x^* \in S$ s.t. $T(x^*) = x^*$.

Nash theorem and its proof

Theorem 1 (Nash (1951)) *Every finite game has a (mixed) Nash equilibrium.*

Proof: Define simplex to be

$$\Delta_k = \{x \in \mathbb{R}^{k+1}_{\geq 0} : \Sigma_{i=1}^{k+1} x_i = 1\}.$$

Clearly, this is a convex and compact set in $\mathbb{R}^{k+1}$. Consider two players (the case with n players is an extension of this idea). Say, player 1 has m strategies labeled $1, \ldots, m$ and player 2 has n strategies labeled $1, \ldots, n$. So, player 1's mixed strategy is a point in Δ_{m-1} and player 2's mixed strategy is a point in Δ_{n-1}. The set of mixed strategy profiles is a point in $\Delta_{m-1} \times \Delta_{n-1}$. Since we are in a two player game, the utilities can be expressed in terms of two matrices A and B, both in $\mathbb{R}^{m \times n}$, denoting the utilities of players 1 and 2 respectively at the pure strategy profiles given by the rows and columns of the matrices. For mixed strategies $p \in \Delta_{m-1}$ and $q \in \Delta_{n-1}$ for players 1 and 2 respectively

$$u_1(p, q) = p^\top Aq, u_2(p, q) = p^\top Bq.$$

Define the following quantities,

$$c_i(p, q) = \max\{A_i q - p^\top Aq, 0\}, \text{ where } A_i \text{ is the } i^{th} \text{ row of } A,\ i \in \{1, \ldots, m\}.$$
$$d_j(p, q) = \max\{p^\top B_j - p^\top Bq, 0\}, \text{where } B_j \text{ is the } j^{th} \text{ column of } B,\ j \in \{1, \ldots, n\}.$$

Clearly, both quantities are non-negative for all i, j.

Now, we define two functions P and Q as follows

$$P_i(p, q) = \frac{p_i + c_i(p, q)}{1 + \sum_{k=1}^m c_k(p, q)},\ i \in \{1, \ldots, m\};$$
$$Q_j(p, q) = \frac{q_j + d_j(p, q)}{1 + \sum_{k=1}^n d_k(p, q)},\ j \in \{1, \ldots, n\}.$$

Clearly, $P_i(p, q) \geq 0, \forall i$ and $\sum_{i=1}^m P_i(p, q) = 1$. Hence $P(p, q) \in \Delta_{m-1}$ and similarly we see that $Q(p, q) \in \Delta_{n-1}$. Define the transformation function

$$T(p, q) = (P(p, q), Q(p, q)).$$

We see that, $T : \Delta_{m-1} \times \Delta_{n-1} \mapsto \Delta_{m-1} \times \Delta_{n-1}$, and maps a convex and compact set onto itself. From the definitions it is clear that c_i and d_j's are continuous in (p, q), therefore, P_i's and Q_j's are also continuous which implies that T is continuous. Hence, by Brouwer's fixed point theorem,

$$\exists\ (p^*, q^*) \text{ s.t. } T(p^*, q^*) = (p^*, q^*).$$

Claim 2

$$\sum_{k=1}^{m} c_k(p^*, q^*) = 0; \qquad \sum_{k=1}^{n} d_k(p^*, q^*) = 0.$$

Proof:[of Claim] Suppose the claim is false, i.e., $\sum_{k=1}^{m} c_k(p^*, q^*) > 0$. Since (p^*, q^*) is a fixed point of T

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \sum_{k=1}^{m} c_k(p^*, q^*)} \Rightarrow p_i^* \left(\sum_{k=1}^{m} c_k(p^*, q^*) \right) = c_i(p^*, q^*). \tag{1}$$

Define a subset of indices as $I = \{i : p_i^* > 0\}$. We see that

$$I = \{i : p_i^* > 0\} = \{i : c_i(p^*, q^*) > 0\} = \{i : A_i q^* > p^{*\top} A q^*\}. \tag{2}$$

The first equality follows from eq. (1) and our assumption that $\sum_{k=1}^{m} c_k(p, q) > 0$. The second equality come from the definition of c_i. Define $u_i^* := p^{*\top} A q^*$.

Now we see

$$u_1^* = \sum_{i=1}^{m} p_i^* A_i q^* = \sum_{i \in I} p_i^* (A_i q^*) > \left(\sum_{i \in I} p_i^* \right) u_1^* = u_1^*.$$

The first equality is by definition, the second inequality holds since p_i^* is positive only on I (by definition), the inequality holds from eq. (2), and the last equality holds since u_i^* is a scalar and comes out of the summation. The inequality above is a contradiction. Similarly we can prove the claim for $\sum_k d_k$. Hence our claim is proved. ■

By this claim, $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$. Since $c_k(p^*, q^*) \geq 0, \forall k = 1, \ldots, m$, it implies that $c_k(p^*, q^*) = 0 \forall k = 1, \ldots, m$. By definition of c_i's, we then have

$$\begin{aligned} A_i q* &\leq p^{*\top} A q^* \\ \Rightarrow \sum_{i=1}^{m} p_i' A_i q^* &\leq p^{*\top} A q^*. \end{aligned}$$

The implication holds for any arbitrary mixed strategy p' of player 1. Similarly we can show that q^* is a best response for player 2 against the mixed strategy p^* played by player 1. Therefore (p^*, q^*) is a MSNE. ■

Correlated strategy and equilibrium

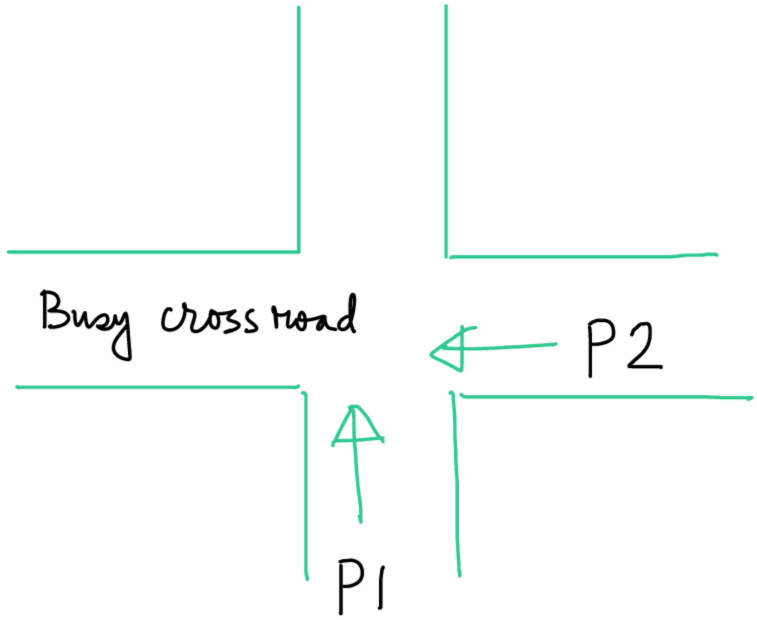
<u>Recap</u>: MSNE → weakest notion of equilibrium so far
existence is guaranteed for finite games
computationally expensive

Alternative approach – entry of a mediating agent/device

Why? (1) Alternative explanation of player rationality
(2) Utility for all players may get better
(3) Computational tractability

<u>Example</u>

P1 \ P2	Wait	Go
Wait	0,0	1,2
Go	2,1	-10,-10



Nash solutions are (1) one waits other goes or (2) large probability on waiting

In practice something else happens

A traffic light guides the players – and the players agree to this plan – Why?

The trusted third party is called the mediator

Role: randomize over the strategy profiles and suggest the corresponding strategies to the players

If the strategies are enforceable then it is an equilibrium (correlated)

Definition: A correlated strategy is a mapping $\pi : S \to [0,1]$ s.t. $\sum_{s \in S} \pi(s) = 1$

example: $\pi(W,W) = 0$, $\pi(W,G) = \pi(G,W) = \frac{1}{2}$, $\pi(G,G) = 0$.

A correlated strategy is a correlated equilibrium when no player "gains" from deviating while others are following the suggested strategies

The correlated strategy π is a common knowledge

Definition: A correlated equilibrium is a correlated strategy π s.t.

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i}), \forall s_i, s_i' \in S_i, \forall i \in N$$

Discussions: the mediator suggests the actions after running its randomization device π, every agent's best response is to follow it if others are also following it.

Ex. 1

	F	C
F	2,1	0,0
C	0,0	1,2

MSNE : $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$

Q: Is $\pi(C,C) = \frac{1}{2} = \pi(F,F)$ a CE ?

Expected utility : MSNE $= \frac{2}{3}$, CE $= \frac{3}{2}$

Ex. 2

	Wait	Go
Wait	0,0	1,2
Go	2,1	-10,-10

Consider $\pi(W,G) = \pi(W,W) = \pi(G,W) = \frac{1}{3}$

Is this a CE?

Are there other CEs?

Computing Correlated Equilibrium

CE finding is to solve a set of linear equations

Two sets of constraints

1. $$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i})\, u_i(s_i, s_{-i}) \geqslant \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i})\, u_i(s_i', s_{-i}), \ \forall s_i, s_i' \in S_i, \forall i \in N$$

Total number of inequalities $= O(nm^2)$, assuming $|S_i| = m$, $\forall i \in N$.

2. $\pi(s) \geqslant 0, \ \forall s \in S$ $\quad$ m^n inequalities

$$\sum_{s \in S} \pi(s) = 1$$

The inequalities together represent a feasibility LP that is easier to compute than MSNE.

MSNE : total number of support profiles $O(2^{mn})$

CE : number of inequalities $O(m^n)$ – exponentially smaller than the above

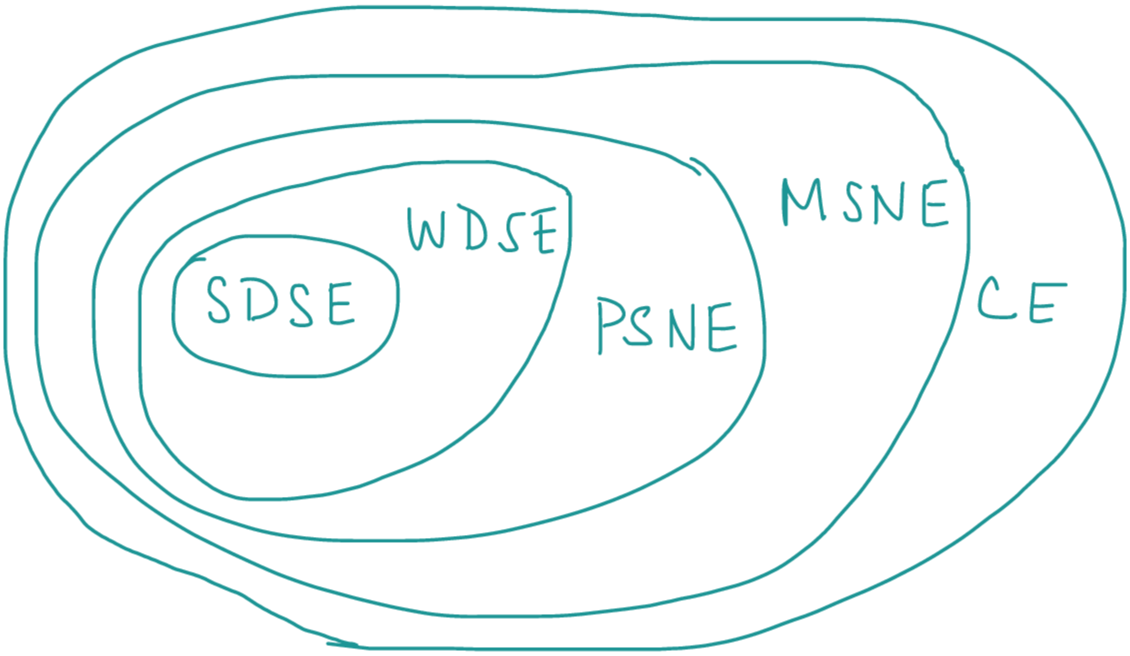
[take log of both quantities to understand this point]

Moreover, this can also be used to optimize some objective function, e.g., maximize utilities of the players

Comparison with the previous equilibrium notions

Theorem : For every MSNE σ^*, there exists a CE π^*.

Proof hint : Use $\pi^*(s_1, \dots, s_n) = \prod_{i=1}^{n} \sigma_i^*(s_i)$ and the MSNE characterization theorem. [Homework]



Summary so far

- Normal form games
- rationality, intelligence, common knowledge
- strategy and action
- dominance – strict and weak – equilibria : SDSE, WDSE
- unilateral deviation – PSNE, generalization : MSNE, existence (Nash)
- characterization of MSNE – computing, hardness
- trusted mediator – correlated strategies – equilibrium

Richer representation of games

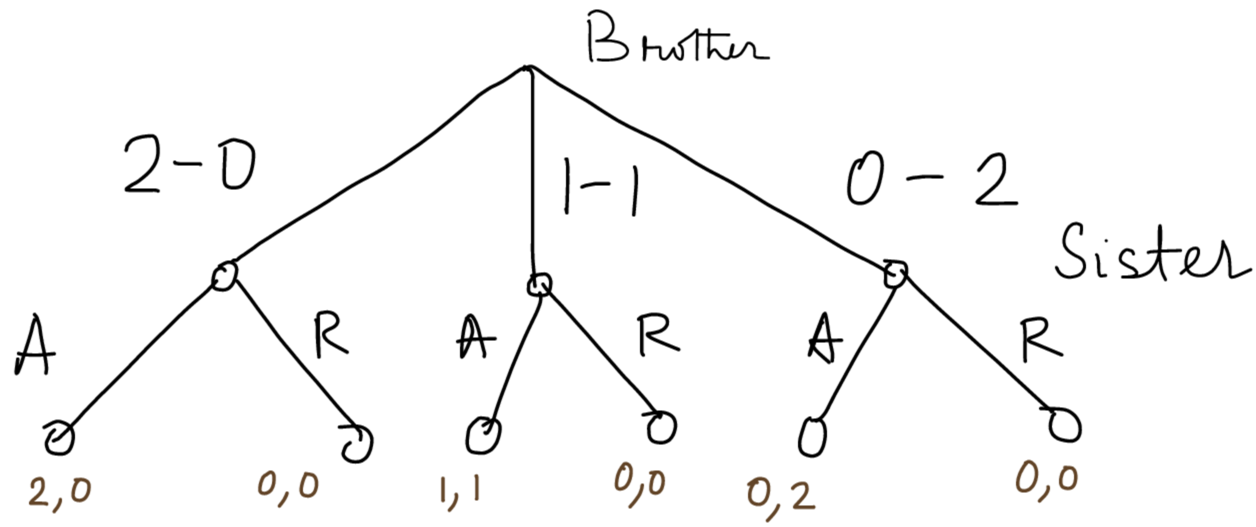
More appropriate for multi-stage games, e.g., chess

Players interact in a sequence – the sequence of actions is the history of the game

Perfect Information Extensive Form Games (PIEFG)

Warning: more notation :)

Ex. Brother-Sister chocolate division



Disagreement → both chocolates taken away

Formal capture: PIEFG $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N} \rangle$

- N : set of players
- A : set of all possible actions (of all players)
- $\mathcal{H}$: set of all sequence of actions (histories) satisfying
  - empty $\emptyset \in \mathcal{H}$
  - if $h \in \mathcal{H}$, any sub-sequence h' of h starting at the root must be in $\mathcal{H}$
  - $h = (a^{(0)}, a^{(1)}, \ldots, a^{(T-1)})$ is terminal if $\nexists\ a^{(T)} \in A$ s.t. $(a^{(0)}, a^{(1)}, \ldots, a^{(T)}) \in \mathcal{H}$

- $Z \subseteq \mathcal{H}$: set of all terminal histories
- $\mathcal{X} : \mathcal{H} \setminus Z \to 2^A$: action set selection function
- $P : \mathcal{H} \setminus Z \to N$: player function
- $u_i : Z \to \mathbb{R}$: utility of i

The strategy of a player in an EFG is a tuple of actions at every history where the player plays.

$$S_i = \bigtimes_{\{h \in \mathcal{H} : P(h) = i\}} \mathcal{X}(h)$$

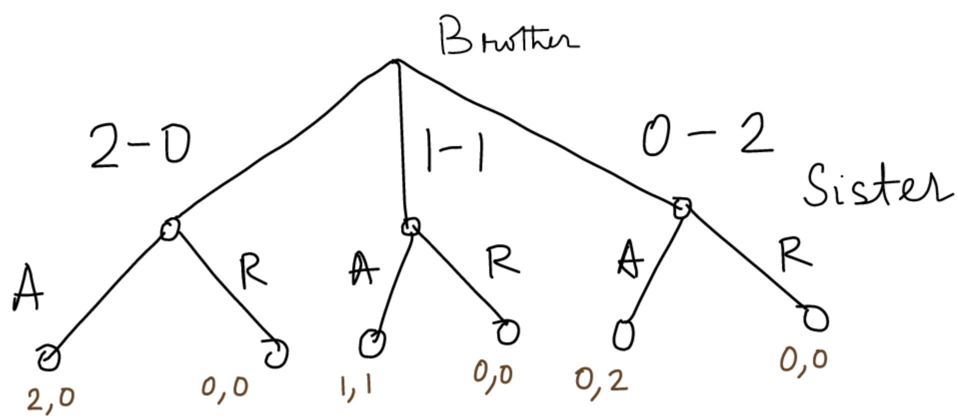
Remember: strategy is a complete contingency plan of the player.

It enumerates potential actions a player can take at every node where she can play, even though some combination of actions may never be executed together.

$N = \{1\,(B), 2\,(S)\}$

$A = \{2\text{-}0, 1\text{-}1, 0\text{-}2, A, R\}$

$\mathcal{H} = \{\phi, (2\text{-}0), (1\text{-}1), (0\text{-}2), (2\text{-}0, A), (2\text{-}0, R), (1\text{-}1, A), (1\text{-}1, R), (0\text{-}2, A), (0\text{-}2, R)\}$



$Z = \{(2\text{-}0, A), (2\text{-}0, R), (1\text{-}1, A), (1\text{-}1, R), (0\text{-}2, A), (0\text{-}2, R)\}$

$\mathcal{X}(\phi) = \{(2\text{-}0), (1\text{-}1), (0\text{-}2)\}$, $\mathcal{X}(2\text{-}0) = \mathcal{X}(1\text{-}1) = \mathcal{X}(0\text{-}2) = \{A, R\}$

$P(\phi) = 1$, $P(2\text{-}0) = P(1\text{-}1) = P(0\text{-}2) = 2$

$u_1(2\text{-}0, A) = 2$, $u_1(1\text{-}1, A) = 1$, $u_2(1\text{-}1, A) = 1$, $u_2(0\text{-}2, A) = 2$ [utilities are zero at other terminal histories]

$S_1 = \{2\text{-}0, 1\text{-}1, 0\text{-}2\}$

$S_2 = \{A,R\} \times \{A,R\} \times \{A,R\} = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR\}$

Transforming PIEFG into NFG

Once we have the S_1 and S_2, the game can be represented as an NFG

	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0
1-1	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
0-2	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

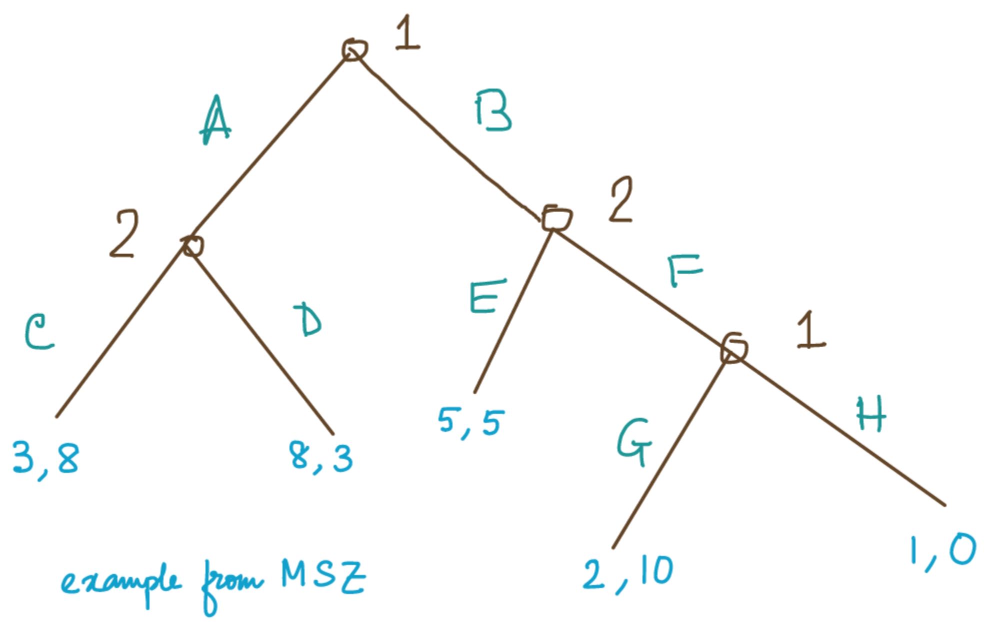
Nash equilibrium like {2-0, RRA} not quite reasonable -

why R at 1-1? {2-0, RRR} is not a credible threat

hence this equilibrium concept is not good enough for predicting outcomes in PIEFGs.

Also, the representation has huge redundancy. EFG is succinct.

PIEFG to NFG : Equilibrium guarantees are weak in PIEFG



example from MSZ

Strategies of player 1 : AG, AH, BG, BH

Strategies of player 2 : CE, CF, DE, DF

PSNEs : (AG, CF), (AH, CF), (BH, CE)

non-credible threat

Better notion of rational outcome will be that which considers a history and ensures utility maximization

Subgame : game rooted at an intermediate vertex

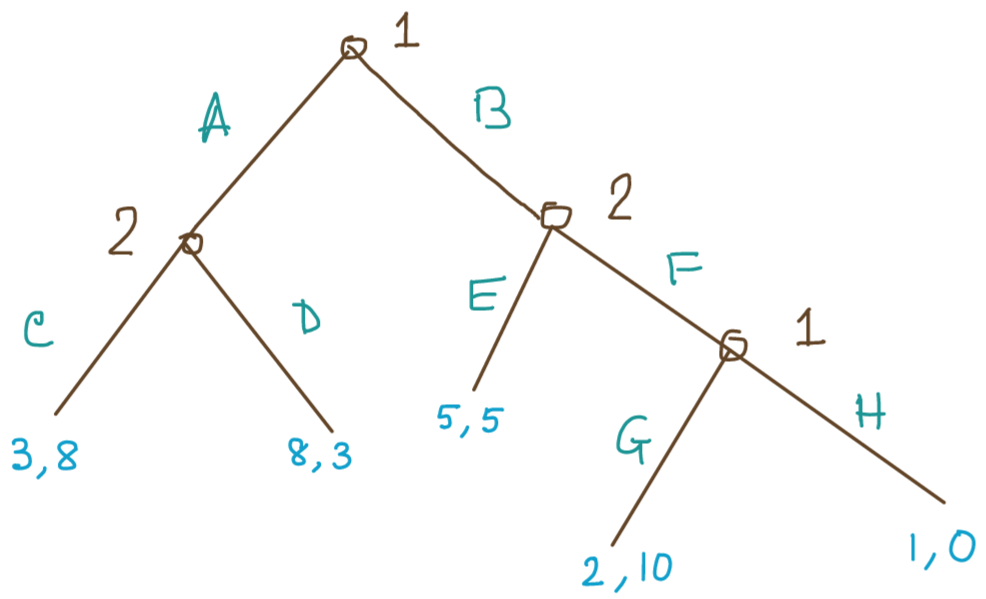
The subgame of a PIEFG G rooted at a history h is the restriction of G to the descendants of h.

The set of subgames of G is the collection of all subgames at some history of G.

Subgame perfection : best response at every subgame

Definition: The subgame perfect Nash equilibrium (SPNE) of an PIEFG G are all strategy profiles $s \in S$ s.t. for any subgame G' of G, the restriction of s to G' is a PSNE of G'.

Example



PSNEs: (AH, CF), (BH, CE), (AG, CF)

Are they all SPNEs? How to compute them?

Algorithm: Backward Induction

function BACK_IND(history h) ...... returns utility and the action

if $h \in Z$ then

return $u(h), \emptyset$

$\text{best\_util}_{P(h)} \leftarrow -\infty$

forall $a \in \mathcal{X}(h)$ do

$\text{util\_at\_child}_{P(h)} \leftarrow \text{BACK\_IND}((h, a))$

if $\text{util\_at\_child}_{P(h)} > \text{best\_util}_{P(h)}$ then

$\text{best\_util}_{P(h)} \leftarrow \text{util\_at\_child}_{P(h)}$, $\text{best\_action}_{P(h)} \leftarrow a$

return $\text{best\_util}_{P(h)}$, $\text{best\_action}_{P(h)}$

Computational cost of SPNE

function BACK_IND (history h)

  if $h \in Z$ then

    return $u(h), \emptyset$

  $best\_util_{P(h)} \leftarrow -\infty$

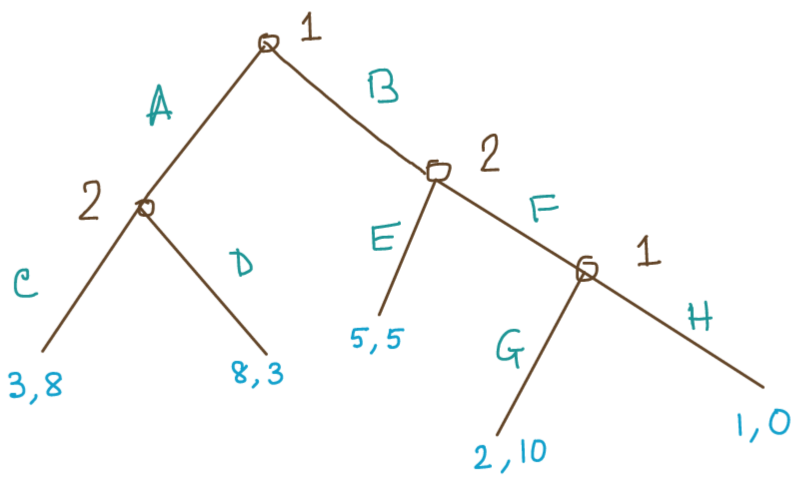
  forall $a \in \mathcal{X}(h)$ do

    $util\_at\_child_{P(h)} \leftarrow$ BACK_IND$((h,a))$

    if $util\_at\_child_{P(h)} > best\_util_{P(h)}$ then

      $best\_util_{P(h)} \leftarrow util\_at\_child_{P(h)}$, $best\_action_{P(h)} \leftarrow a$

  return $best\_util_{P(h)}$, $best\_action_{P(h)}$



The idea of subgame perfection inherently is based on backward induction.

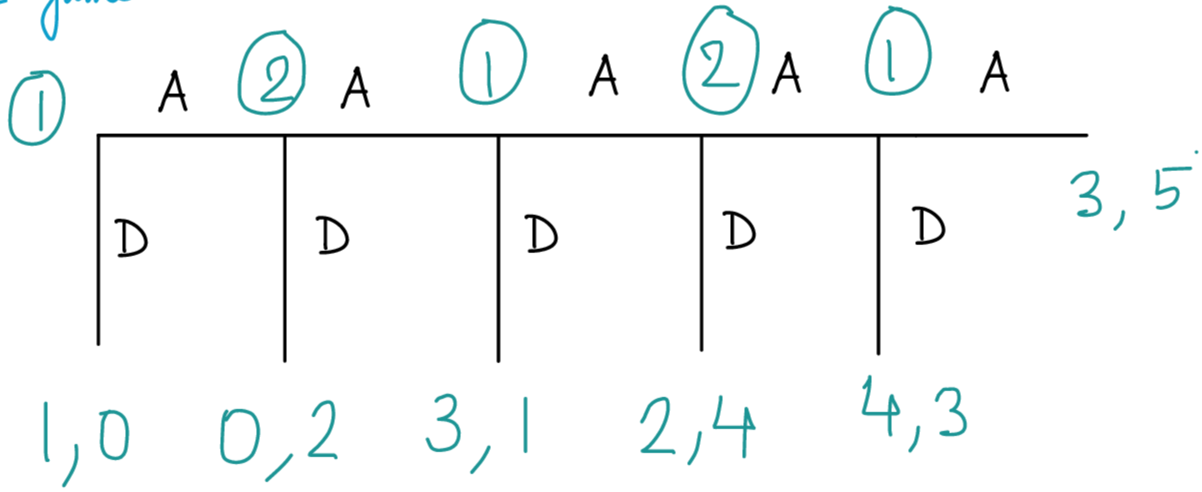
Advantages:

1. SPNE is guaranteed to exist in finite PIEFGs (requires proof)
2. An SPNE is a PSNE ... found a class of games where PSNE is guaranteed to exist.
3. The algorithm to find SPNE is quite simple.

Disadvantage: The whole tree has to be parsed to find the SPNE - which can be computationally expensive (or maybe impossible)
e.g., chess has $\sim 10^{150}$ vertices

Other criticism: about the cognitive limit (of real players)

Centipede game



What is/are the SPNE(s) of this game ?

What is the problem with that prediction ?

This game has been experimented with various populations
- random participants, university students, grandmasters

Most of the subjects (except grandmasters) continue till a few rounds

Reasons claimed : altruism, limited computational capacity of individuals, incentive difference

Criticism of the principle of SPNE

It talks about "what action if the game reached this history"
but the equilibrium in some stage above can show that it "cannot reach that history".

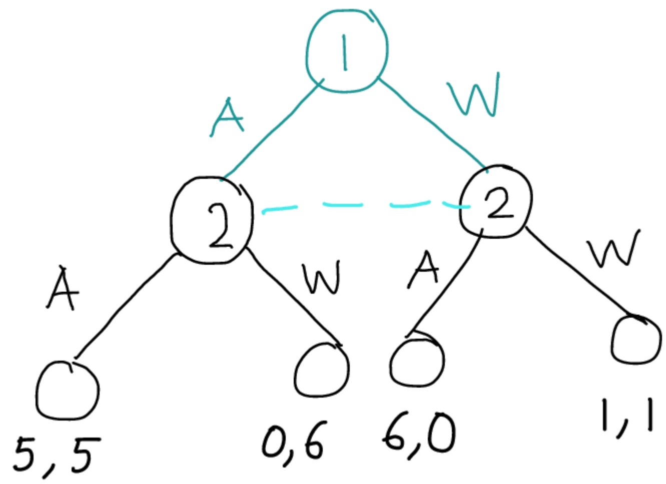
Extension using the idea of player beliefs.

Games with imperfect information

- Games discussed so far (EFGs) are of perfect information
- Every player has perfect knowledge about all the developments in the game until that round
- Limited practical use – several games have states that are unknown to certain agents – e.g., card games
- not possible to represent simultaneous move games using EFGs

	Agri	War
Agri	5, 5	0, 6
War	6, 0	1, 1

Neighboring Kingdom's Dilemma

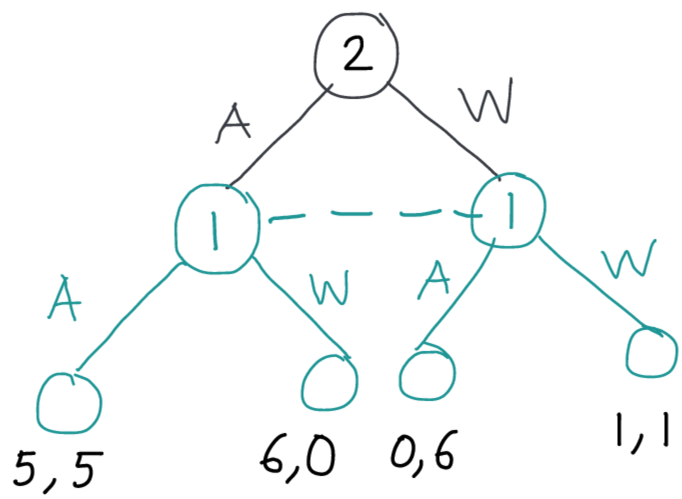


Imperfect information EFGs, indistinguishable nodes are connected via a dotted line

Player 2 does NOT know which node/history the game is in

- these indistinguishable histories form an information set for player 2.
- more general representation than PIEFGs, since information sets can be singleton

- IIEFGs are not unique for a given simultaneous move game
- The kingdom's dilemma can also be represented this way with (non singleton) information set for 1.



Imperfect Information Extensive Form Games

An IIEFG is a tuple $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N}, (I_i)_{i\in N} \rangle$

where $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N} \rangle$ is a PIEFG

for every $i \in N$, $I_i := (I_i^1, I_i^2, \dots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that $\mathcal{X}(h) = \mathcal{X}(h')$ and $P(h) = P(h') = i$, whenever $\exists j$ s.t. $h, h' \in I_i^j$

I_i^j's are called the information sets of player i, I_i is the collection of information sets of i.

At an information set, player and her available actions are same

That player is uncertain about which history in the information set is reached

Some differences with the PIEFG

- Since actions at an information set are identical, $\mathcal{X}$ can be defined over I_i^js, i.e., $\mathcal{X}(h) = \mathcal{X}(h') = \mathcal{X}(I_i^j)$, $\forall h, h' \in I_i^j$.

- Strategies can also be defined over information sets

Strategy set of player $i \in N$ is defined as the cartesian product of the actions available to i at her information sets, i.e.,

$$S_i = \bigtimes_{\tilde{I} \in I_i} \mathcal{X}(\tilde{I}) = \bigtimes_{j=1}^{k(i)} \mathcal{X}(I_i^j)$$

With IIEFGs, NFGs can be represented using EFGs, although not very succinct. Representations are appropriate for certain kind of games.

However, IIEFG is a richer representation than both NFG and PIEFG.

Strategies in IIEFGs

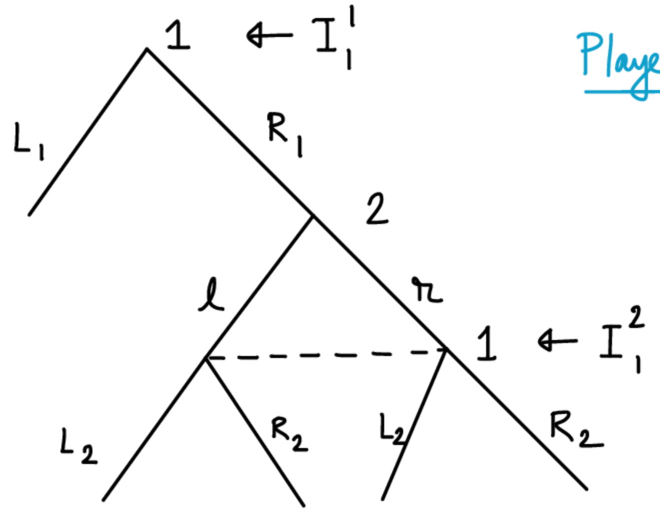
Strategy set of i : $S_i = \bigtimes_{j=1}^{k(i)} X(I_i^j)$

Randomized strategies in IIEFG

In NFGs, mixed strategies randomize over pure strategies

In EFGs, randomization can happen in different ways

- randomize over the strategies defined at the beginning of the game
- randomize over the action at an information set - behavioral strategy



Player 1

Pure strategies at the beginning

$(L_1 L_2), (L_1 R_2), (R_1 L_2), (R_1 R_2)$

mixed strategy σ_1

$\sigma_1(L_1 L_2), \sigma_1(L_1 R_2), \sigma_1(R_1 L_2), \sigma_1(R_1 R_2)$

---

actions at I_1^1 : L_1, R_1 ;

at I_1^2 : L_2, R_2

behavioral strategies

$b_1(I_1^1) \in \Delta(L_1, R_1)$

$b_1(I_1^2) \in \Delta(L_2, R_2)$

Definition: Behavioral Strategy

A behavioral strategy of a player in an IIEFG is a function that maps each of her information sets to a probability distribution over the set of possible actions at that information set.

Question: What is the relation between mixed and behavioral strategies?

In this example: MSs live in $\mathbb{R}^4$, BSs live in two $\mathbb{R}^2$ spaces mixed strategies look a "richer" or "larger" concept

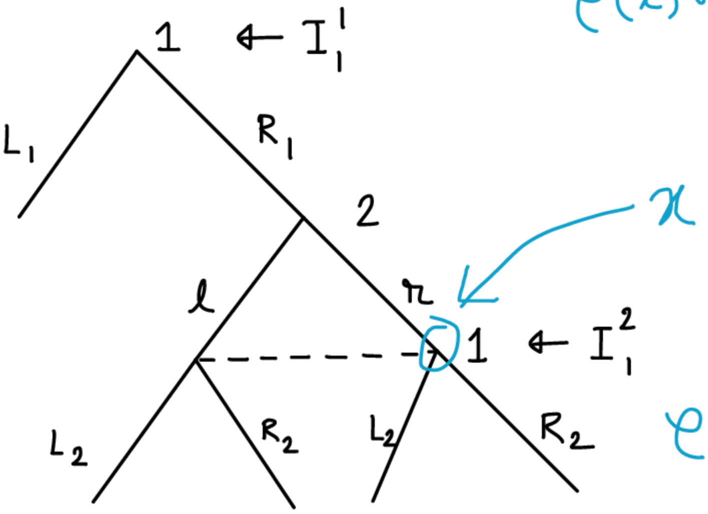
Can a player attain higher payoff in one strategy than the other?

Question: Can we have an equivalence?

Equivalence in terms of the probability of reaching a vertex/history x

Say $\varphi(x;\sigma)$ is the probability of reaching node x under mixed strategy profile σ.

Similarly, $\varphi(x;b)$ is the same for behavioral strategy profile b.



$$\varphi(x;\sigma) = \sigma_1(R_1)\,\sigma_2(r)$$
$$= \big(\sigma_1(R_1L_2) + \sigma_1(R_1R_2)\big)\cdot \sigma_2(r)$$

$$\varphi(x;b) = b_1(I_1^1)(R_1)\cdot b_2(I_2^1)(r)$$

Important: different players can choose different kind of strategies

e.g., if 1 chooses σ_1 above and 2 chooses b_2 then

$$\varphi(x;\sigma_1,b_2) = \big(\sigma_1(R_1L_2) + \sigma_1(R_1R_2)\big)\cdot b_2(I_2^1)(r)$$

Definition: equivalence

A mixed strategy σ_i and a behavioral strategy b_i of a player i in an IIEFG are equivalent if every mixed/behavioral strategy vector ξ_{-i} of the other players and every vertex x in the game tree

$$\varphi(x;\sigma_i,\xi_{-i}) = \varphi(x;b_i,\xi_{-i})$$

Example: in the game above

$$b_1(I_1^1)(L_1) = \sigma_1(L_1L_2) + \sigma_1(L_1R_2)$$

$$b_1(I_1^1)(R_1) = \sigma_1(R_1L_2) + \sigma_1(R_1R_2)$$

b_1 and σ_1 are equivalent

$$b_1(I_2^1)(L_2) = \sigma_1(L_2 \mid R_1)$$

$$b_1(I_2^1)(R_2) = \sigma_1(R_2 \mid R_1)$$

equivalent strategies induce same probability of reaching a vertex

More on equivalent strategies

The equivalence, by definition, holds at the leaf nodes too

Claim: it is enough to check the equivalence only at the leaf nodes

Reason: pick an arbitrary non-leaf node, the probability of reaching that node is equal to the sum of the probabilities of reaching the leaf nodes in its subtree.

This argument can be extended further

Theorem (Utility equivalence)

If σ_i and b_i are equivalent, then for every mixed/behavioral strategy vector of the other players ξ_{-i}, the following holds

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}), \ \forall j \in N.$$

Repeat the argument for any equivalent mixed and behavioral str profiles

Corollary: Let σ and b are equivalent, i.e., σ_i and b_i are equivalent $\forall i \in N$.

Then $u_i(\sigma) = u_i(b)$.

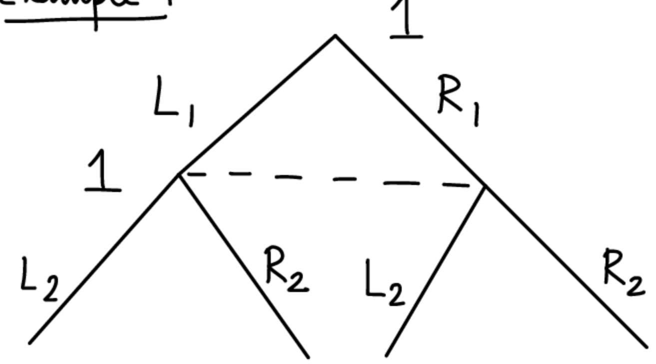
Why behavioral strategies are desirable ?

① More natural in large IIEFGs
- players plan at a stage (information set) rather than a master plan

② Smaller number of variables to deal with
- consider a player having 4 information sets with 2 actions each
- needs (2^4-1) variables to represent mixed strategies
- needs 4 variables for behavioral strategies

Question: can one construct one from the other ? OR does equivalence always hold ?

<u>Example 1</u>:



mixed strategy

$\sigma_1(L_1, L_2)$, $\sigma_1(L_1, R_2)$, $\sigma_1(R_1 L_2)$,

$\sigma_1(R_1, R_2)$

behavioral strategy

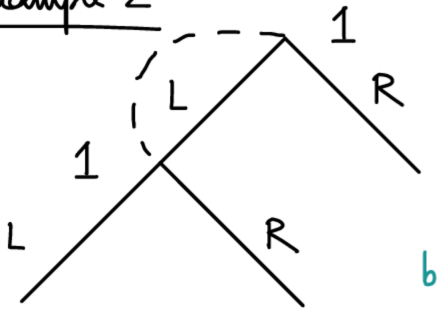
$b_1(L_1)$, $b_1(L_2)$

mixed strategy has more control over the profiles, e.g., $\sigma_1(L_1 R_2) = \sigma_1(R_1 L_2) = 0$

but not possible in behavioral strategies

mixed strategy with no equivalent behavioral strategies

<u>Example 2</u>



A behavioral strategy can have positive mass on LR, but mixed strategy cannot

behavioral strategy with no equivalent mixed strategy

<u>Ex 1</u>: player remembers that it made a move but forgets what move.

<u>Ex 2</u>: player forgets whether it made a move or not

The equivalence does not hold if players are forgetful.

When does behavioral strategy has no equivalent mixed strategy?

Let x be a non-root node

action at x_1 leading to x : The unique edge emanating from x_1 that is on the path from root to x.

In ex 2, there is a node which has a path from root to itself that crosses the same information set twice

if the path from root to x passes through vertices x_1 and $\hat{x}_1$ that are in the same information set of player i, and

the action leading to x at x_1 and $\hat{x}_1$ are different, then no pure strategy can ever lead to x

mixed strategy is randomization over pure strategies, every mixed strategy will put zero mass on x.

but behavioral strategy randomizes on every vertex independently, hence x can be reached in behavioral strategies with positive probability

The above observation can be stated as a lemma

Lemma: If there exists a path from the root to some vertex x that passes through the same information set at least twice, and if the action leading to x is not the same at each of those vertices, then the player of the information set has a behavioral strategy that has no equivalent mixed strategy.

The lemma helps us in proving the following characterization result of equivalence

Theorem (6.11 of MSZ)

Consider an IIEFG s.t. every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy iff each information set of a player intersects every path emanating from the root at most once.

Proof: reading exercise from MSZ.

Mixed strategy equivalent of behavioral strategy

<u>Theorem</u> (6.11 of MSZ)
Consider an IIEFG s.t. every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy iff each information set of a player intersects every path emanating from the root at most once.

Behavioral strategy equivalent of mixed strategy

To formalize (i.e., set the conditions when the equivalence holds), we need to formalize the forgetfulness of the player

- saw few examples of players' forgetfulness
- our conditions need to ensure that none of those forgetfulness happens

Definition (Choice of same action at an information set)

Let $X = (x^0, x^1, \ldots, x^K)$ and $\hat{X} = (x^0, \hat{x}^1, \ldots, \hat{x}^L)$ be two paths in the game tree
Let I_i^j be an information set of player i that intersects these two paths only at one vertex, say x^k and $\hat{x}^l$ respectively.
These two paths choose the same action at information set I_i^j if

- $k < K$ and $l < L$
- actions x^k leading to x^{k+1} and $\hat{x}^l$ leading to $\hat{x}^{l+1}$ are identical denoted by $a_i(x^k \to x^{k+1}) = a_i(\hat{x}^l \to \hat{x}^{l+1})$

"leading to" may not be a relation between parent and child nodes it can be any descendant of the former since the path is unique in a tree.

Games with Perfect Recall

Definition

Player i has perfect recall if the following conditions are satisfied

1. Every information set of player i intersects every path from the root to a leaf at most once.

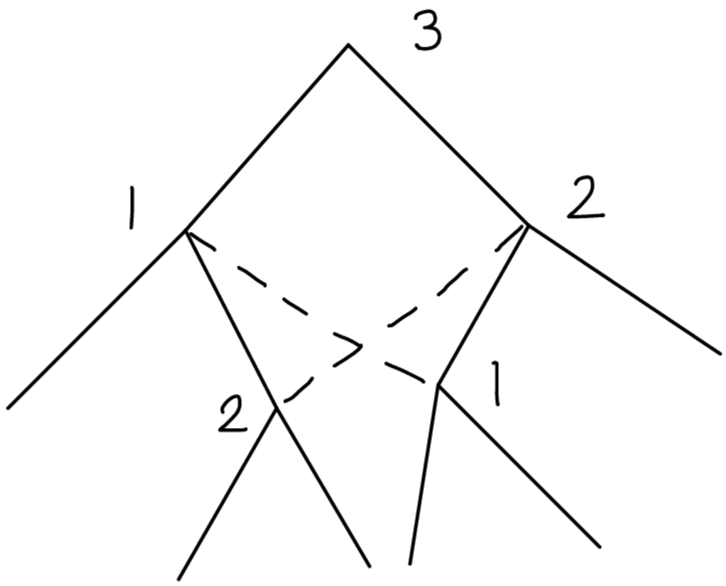
2. Every two paths that end in the same information set of player i pass through the same information sets of i in the same order and in every such information set the two paths choose the same action.

Rephrasing: for every I_i^j of i and every pair of vertices x and $x' \in I_i^j$
if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$, and $x_i'^1, x_i'^2, \ldots, x_i'^{L'} = x'$ respectively for the two paths from root to x and x'
then
1. $L = L'$
2. $x_i^\ell, x_i'^\ell \in I_i^k$ for some k, and
3. $a_i(x_i^\ell \to x_i^{\ell+1}) = a_i(x_i'^\ell \to x_i'^{\ell+1})$, $\forall \ell = 1, 2, \ldots, L-1$.

A game is of perfect recall if every player has perfect recall.

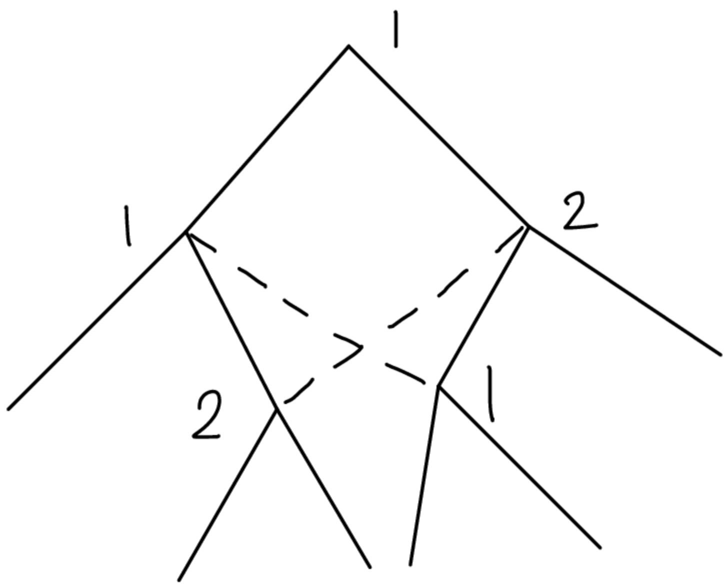
Note: definition of perfect recall subsumes the condition the theorem where every behavioral strategy has equivalent mixed strategy (point 1)

Examples



This example satisfies the conditions of the definition

game with perfect recall



Player 1 takes two different actions at the first information set to reach two different vertices of the second information set

game with imperfect recall

Implications of perfect recall

Let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses actions leading to x — i.e., intersections of members of S_i with the path from root to x.

Theorem: If i is a player with perfect recall and x and x' are two vertices in the same information set of i. Then $S_i^*(x) = S_i^*(x')$.

The above conclusion comes from the same sequence of information sets and same actions. The next implication gives the equivalence of mixed and behavioral strategies.

Theorem (Kuhn 1957)

In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of i, there exists a behavioral strategy

The converse is already true (beh has equiv mixed) since the sufficient condition for that is already subsumed in the definition of perfect recall.

Proof: reading exercise (MSZ Theorem 6.15)

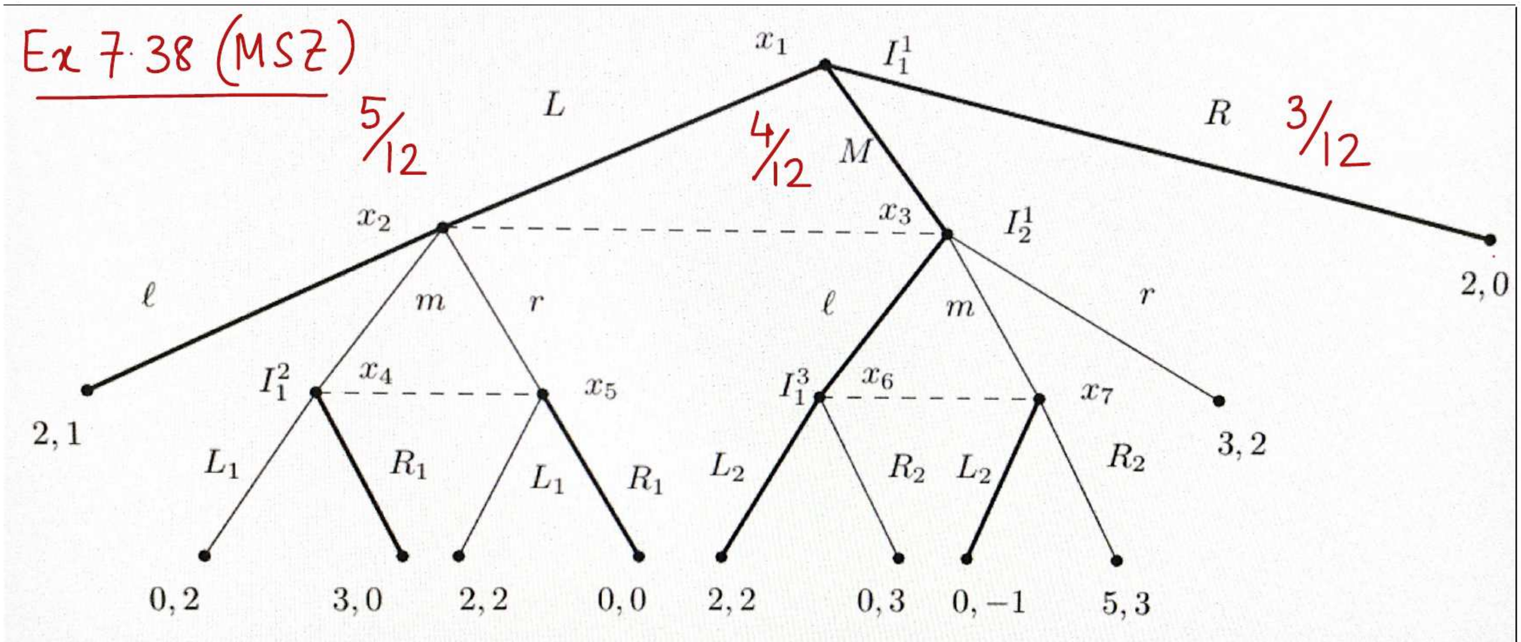
Remarks: the proof is constructive. It starts with a mixed strategy and constructs the behavioral strategies s.t. the probabilities of reaching a leaf are same. The arguments show that such a construction is always possible because of perfect recall.

Equilibrium notions in IIEFGs

- Can extend the subgame perfection of PIEFG, but since the nodes/histories are uncertain, we need to extend to mixed strategies
- Because of the information sets, best response cannot be defined without the belief of each player

Belief is a conditional probability distribution over the histories in an information set – conditioned on reaching the information set.

Example: an IIEFG with perfect recall, i.e., mixed and behavioral strategies are equiv.



Consider the behavioral strategy profile: σ_1, at I_1^1 $\left(\frac{5}{12} L, \frac{4}{12} M, \frac{3}{12} R\right)$

at I_1^2, choose R_1, at I_1^3, choose L_2

σ_2 : choose ℓ

Q: Is this an equilibrium? Which implies

- are the Bayesian beliefs consistent with P_σ – that visits vertex x w.p. $P_\sigma(x)$
- The actions and beliefs are consistent for every player, i.e., maximizes their expected utility

- player 1, at I_1^3, believes that x_6 is reached w.p. 1

  if the belief was $> \frac{2}{7}$ in favor of x_7, should have chosen R_2

Choose an action maximizing expected utility at each information set

— sequential rationality

The strategy vector σ induces the following probabilities to the vertices

$$P_\sigma(x_2) = \frac{5}{12}, \quad P_\sigma(x_3) = \frac{4}{12}, \quad P_\sigma(x_4) = P_\sigma(x_5) = P_\sigma(x_7) = 0, \quad P_\sigma(x_6) = \frac{4}{12}$$

- player 2, at I_2^1, believes that x_3 is reached w.p.

$$P_\sigma(x_3 \mid I_2^1) = \frac{P_\sigma(x_3)}{P_\sigma(x_2) + P_\sigma(x_3)} = \frac{4/12}{4/12 + 5/12} = 4/9$$

Similarly, $P_\sigma(x_2 \mid I_2^1) = 5/9$

Is the action of player 2 sequentially rational wrt her belief?

by picking l, her expected utility $= \frac{5}{9} \times 1 + \frac{4}{9} \times 2 = \frac{13}{9}$, this is larger than any other choice of actions.

- Given this, what will be the sequentially rational strategy of player 1 at I_1^1?

- L, M, R all gives the same expected utility for 1 (utility $= 2$)

mixed/behavioral strategy profile σ is sequentially rational for all players

---

Formal definitions

① **Belief**: Let the information sets of player i be $I_i = \{I_i^1, \dots, I_i^{k(i)}\}$

The belief of player i is a mapping $\mu_i^j : I_i^j \to [0,1]$, s.t.

$$\sum_{x \in I_i^j} \mu_i^j(x) = 1 .$$

② **Bayesian belief**: A belief $\mu_i = (\mu_i^j, j = 1, \dots, k(i))$ of player i is Bayesian wrt the behavioral strategy σ, if it is derived from σ using Bayes rule, i.e.,

$$\mu_i^j(x) = \frac{P_\sigma(x)}{\sum_{y \in I_i^j} P_\sigma(y)} , \quad \forall x \in I_i^j , \ \forall j = 1, 2, \dots, k(i) .$$

③ **Sequential rationality**:

A strategy σ_i of player i at an information set I_i^j is sequentially rational given $\underline{\sigma}_{-i}$ and partial belief μ_i^j if

$$\sum_{x \in I_i^j} \mu_i^j(x)\, u_i(\sigma_i, \underline{\sigma}_{-i} | x) \geqslant \sum_{x \in I_i^j} \mu_i^j(x)\, u_i(\sigma_i', \underline{\sigma}_{-i} | x) .$$

The tuple (σ, μ) is sequentially rational if it is sequentially rational for every player at every information set.

The tuple (σ, μ) is also called an assessment.

Sequential rationality is a refinement of Nash equilibrium

The notion coincides with SPNE when applied to PIEFGs

Theorem: In a PIEFG, a behavioral strategy profile σ is an SPNE iff the tuple $(\sigma, \hat{\mu})$ is sequentially rational.

[In PIEFG, every information set is a singleton; $\hat{\mu}$ is the degenerate distribution at that singleton]

Equilibrium with sequential rationality

Perfect Bayesian equilibrium

An assessment (σ, μ) is a perfect Bayesian equilibrium (PBE) if for every player $i \in N$

1. μ_i is Bayesian wrt σ, and
2. σ_i is sequentially rational given σ_{-i} and μ_i

Often represented only with σ, since μ is obtained from σ.

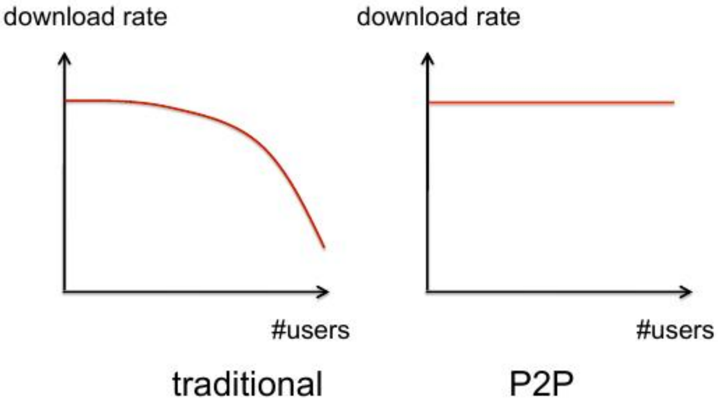
Self-enforcing (like the SPNE) in a Bayesian way.

CS711: Introduction to Game Theory and Mechanism Design

Teacher: Swaprava Nath

P2P file sharing
slides adapted from CS186 Harvard

Peer to Peer



Desired Properties and Terminology

- Scalability
- Failure resilience

Terminology:

- **Protocol:** messages that can be sent, actions that can be taken over the network
- **Client:** a particular process for sending messages, taking actions
- **Reference client:** particular implementation
- **Peer**

Early P2P Technologies

Napster (1999 - 2001)

- Centralized database
- Users download music from each other

Early P2P Technologies

Napster (1999 - 2001)

- Centralized database
- Users download music from each other

Gnutella (2000 -)

- Get list of IP addresses of peers from set of known peers (no server)
- To get a file: Query message broadcast by peer A to known peers
- Query response: sent by B if B has the desired file (routed back to requestor)
- A can then download directly from B

The File Sharing Game

		Player 2	
		Share	Free Ride
Player 1	Share	$2,\ 2$	$-1,\ 3$
	Free Ride	$3,\ -1$	$0,\ 0$

The File Sharing Game (Contd.)

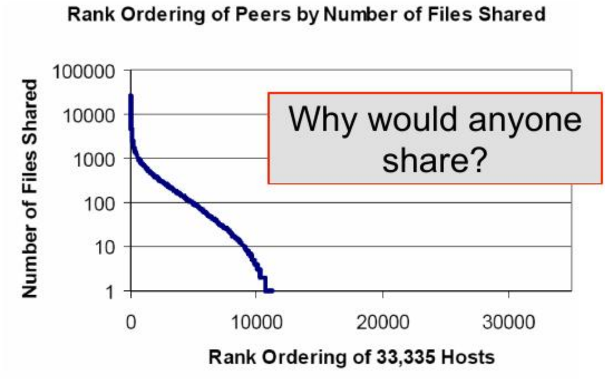


Image courtesy: Adar and Huberman (2000)

Incentives for Client Developers

- Client developers can ensure file sharing
- But competition among the developers

Incentives for Client Developers

- Client developers can ensure file sharing
- But competition among the developers

- 85% peers free-riding by 2005; Gnutella less than 1% of ww P2P traffic by 2013
- Few other P2P systems met the same fate

New Protocol

BitTorrent (2001 -)

- Approx 85% of P2P traffic in US
- File sharing
- Also used for S/W distribution (e.g., Linux)

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Key innovations

- Break file into pieces: A repeated game!
- "If you let me download, I'll reciprocate."

BitTorrent Schematic

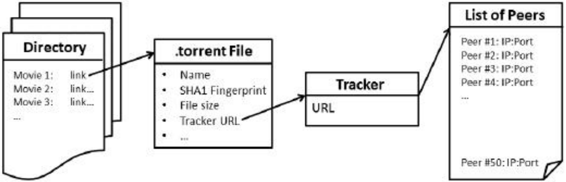

Figure 5.4.: Starting a download process in the BitTorrent protocol: 1) A user goes to a searchable directory to find a link to a .torrent file corresponding to the desired content; 2) the .torrent file contains metadata about the content, in particular the URL of a tracker; 3) the tracker provides a list of peers participating in the swarm for the content (i.e., their IP address and port); 4) the user's BitTorrent client can now contact all these peers and download content.

Image courtesy: Parkes and Seuken (2017)

BitTorrent Optimistic Unchoking Algorithm

Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

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- If a peer j uploaded to i at a rate $< r$, choke j in the next period
- Every three time periods, optimistically unchoke a random peer from the neighborhood who is currently choked, and leave that peer unchoked for three time periods.

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Forcing a repeated game by fragmenting the files

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Forcing a repeated game by fragmenting the files
The leecher-seeder game is a repeated Prisoners' Dilemma

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Forcing a repeated game by fragmenting the files
The leecher-seeder game is a repeated Prisoners' Dilemma
Strategy of the seeder is tit-for-tat

Illustration

Illustration

Strategic Behaviors

- How often to contact tracker?
- Which pieces to reveal?
- How many upload slots, which peers to unchoke, at what speed?
- What data to allow others to download?
- Possible goals: min upload, max download speed, some balance

Attacks on BitTorrent

- BitThief
- Strategic piece revealer
- BitTyrant

BitThief

- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!

BitThief

- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!
- Fix: modify the tracker (block same IP address within 30 minutes).

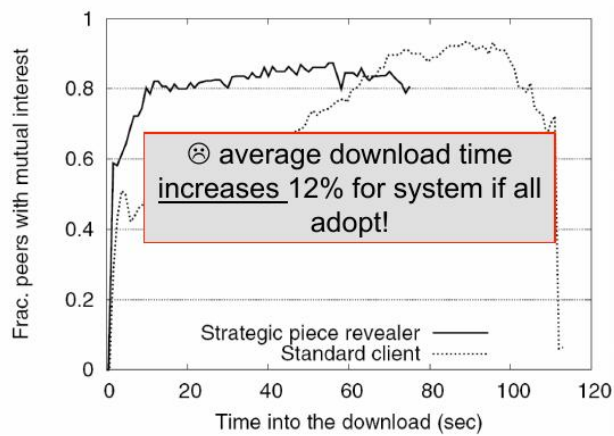
Ref: Locher et al., "Free Riding in BitTorrent is Cheap", HotNets 2006

Strategic Piece Revealer

- Reference client: tell neighbors about new pieces, use "rarest-first" to request
- Manipulator strategy: reveal most common piece that reciprocating peer does not have!
- Try to protect a monopoly, keep others interested

Ref: Levin et al., "BitTorrent is an Auction: Analyzing and Improving BitTorrents Incentives", SIGCOMM 2008

Strategic Piece Revealer



Summary

- P2P demonstrates importance of game-theory in computer systems
- Early systems were easily manipulated
- BitTorrent's innovation was to break files into pieces, enabling TitForTat.
- Still some vulnerabilities, but generally very successful example of incentive-based protocol design.

Games
- Cooperative (players form coalitions, utilities are defined over coalitions)
- Non-cooperative
  - Incomplete information (players do not deterministically know which game they're playing)
  - Complete information
    - Normal form representation, appropriate for simultaneous move
      Equilibrium notions: SDSE, WDSE, PSNE, MSNE, Correlated
    - Extensive form representation, appropriate for multistage games
      Equilibrium notions: SPNE (PIEFG), mixed and behavioral strategies (IIEFG), PBE

Other types of games: repeated, stochastic, etc.

Games with complete information -

Players deterministically know the game they are playing

- there can be some chance moves but those probabilities are known

Games with incomplete information

Players do not deterministically know which game they are playing

- players receive private signals / types
- To discuss: special subclass: games with incomplete information with common priors (Harsanyi 1967) - Bayesian Games

Example: Soccer game - two competing teams

Each can choose a gameplan: aim to WIN or aim to DRAW

We will call the gameplan as their type

these are private signals to them, often caused by external factors, e.g., weather condition, player injury, ground condition etc.

- there are four possible type profiles in this example WW, WD, DW, DD. The payoff matrices differ.

WW

	Attack	Defence
Attack	1, 1	2, 0
Defence	0, 2	0, 0

WD [DW is symmetrically opposite]

	Attack	Defence
Attack	2, 0	2, 1
Defence	0, 1	1, 0

DD

	Attack	Defence
Attack	0, 0	1, 0
Defence	0, 1	-1, -1

Assumptions: (1) The probabilities of choosing the different games (or the type profiles) come from a common prior distribution.

(2) The common prior is a common knowledge

Definition: A Bayesian game is represented by

$$\langle N, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta \in \times_{i\in N} \Theta_i} \rangle,$$

where N: set of players, Θ_i: set of types of player i, P: common prior distribution over $\Theta = \times_{i\in N} \Theta_i$, with the restriction that

$\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_i, \theta_{-i}) > 0, \forall \theta_i \in \Theta_i, \forall i \in N$, i.e., marginals for every type is positive (otherwise we can prune the type set)

Γ_θ: NFG for the type profile $\theta \in \Theta$, i.e.,

$$\Gamma_\theta = \langle N, (A_i(\theta))_{i\in N}, (u_i(\theta))_{i\in N} \rangle$$ [we assume $A_i(\theta) = A_i, \forall \theta$]

$u_i : A \times \Theta \to \mathbb{R}$, $A = \times_{i\in N} A_i$.

Stages of a Bayesian game

1. $\theta = (\theta_i, \theta_{-i})$ is chosen randomly according to P
2. Each player observes her own type θ_i
3. Player i picks action $a_i \in A_i$, $\forall i \in N$.
4. Player i's payoff realizes $u_i(a_i, a_{-i}; \theta_i, \theta_{-i})$.

Bayesian games

$$\langle N, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta \in \times_{i\in N} \Theta_i} \rangle$$

$$\Gamma_\theta = \langle N, (A_i(\theta))_{i\in N}, (u_i(\theta))_{i\in N} \rangle$$ [we assume $A_i(\theta) = A_i, \forall \theta$]

Strategy : a plan to map type to action

Pure : $s_i : \Theta_i \to A_i$, Mixed : $\sigma_i : \Theta_i \to \Delta A_i$

The player can experience its utility in two stages (depending on the realization of θ_i).

Ex-ante utility : expected utility before observing own type

$$U_i(\sigma) = \sum_{\theta\in\Theta} P(\theta)\, u_i(\sigma(\theta); \theta)$$

$$= \sum_{\theta\in\Theta} P(\theta) \sum_{(a_1,\ldots,a_n)\in A} \prod_{j\in N} \sigma_j(\theta_j)[a_j]\, u_i(a_1,\ldots,a_n; \theta_1,\cdots,\theta_n)$$

The belief of player i over others' types changes after observing her own type θ_i according to Bayes rule on P

$$P(\underline{\theta}_{-i} \mid \theta_i) = \frac{P(\theta_i, \underline{\theta}_{-i})}{\sum_{\tilde{\underline{\theta}}_{-i}\in \Theta_{-i}} P(\theta_i, \tilde{\underline{\theta}}_{-i})}$$; positive marginals assumption is crucial

Ex-interim utility : expected utility after observing one's own type

$$U_i(\sigma \mid \theta_i) = \sum_{\underline{\theta}_{-i}\in \Theta_{-i}} P(\underline{\theta}_{-i} \mid \theta_i)\, u_i(\sigma(\theta); \theta)$$

Special case : for independent types, observing θ_i does not give any information on $\underline{\theta}_{-i}$. Both utilities are same.

Relationship between these two utilities

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i)\, u_i(\sigma \mid \theta_i)$$

<u>Example</u> 1 : Two player bargaining game

Player 1 : seller, type : price at which he is willing to sell

Player 2 : buyer, type : price at which he is willing to buy

$\Theta_1 = \Theta_2 = \{1, 2, \dots, 100\}$, $A_1 = A_2 = \{1, 2, \dots, 100\}$, bids

If the bid of the seller is smaller or equal to that of the buyer, trade happens at a price average of the two bids. Else, trade does not happen.

Suppose type generation is independent and uniform over Θ_1, Θ_2 resp.

$$P(\theta_2 \mid \theta_1) = P(\theta_2) = \frac{1}{100} \quad \forall \theta_1, \theta_2.$$

$$P(\theta_1 \mid \theta_2) = P(\theta_1) = \frac{1}{100}, \quad \forall \theta_1, \theta_2.$$

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1 + a_2}{2} - \theta_1 & \text{if } a_2 \geqslant a_1 \\ 0 & \text{ow} \end{cases}$$

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1 + a_2}{2} & \text{if } a_2 \geqslant a_1 \\ 0 & \text{ow} \end{cases}$$

Common prior $P(\theta_1, \theta_2) = \frac{1}{10000}, \ \forall \theta_1, \theta_2$

Example 2: Sealed bid auction

Two players, both willing to buy an object. Their values and bids lie in $[0,1]$

allocation function: $O_1(b_1,b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{ow} \end{cases}$ | $O_2(b_1,b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$

beliefs: $f(\theta_2 | \theta_1) = 1, \ \forall \theta_1, \theta_2$ $\quad f(\theta_1, \theta_2) = 1, \ \forall (\theta_1, \theta_2) \in [0,1]^2$

$f(\theta_1 | \theta_2) = 1, \ \forall \theta_1, \theta_2$

$u_i(b_1, b_2 ; \theta_1, \theta_2) = O_i(b_1, b_2)(\theta_i - b_i)$ [winner pays his bid]

Equilibrium concepts in Bayesian games

Ex-ante: before observing own type

Nash equilibrium (σ^*, P): $u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i', \sigma_{-i}^*)$, $\forall \sigma_i'$, $\forall i \in N$

Ex-interim: after observing own type

Bayesian equilibrium (σ^*, P)

$$u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i) \geqslant u_i(\sigma_i'(\theta_i), \sigma_{-i}^* \mid \theta_i), \quad \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i$$

The RHS of the definition can be replaced by a pure strategy a_i, $\forall a_i \in A_i$

The reason is exactly same as that of MSNE (these definitions are equivalent)

NE notion takes expectation over $P(\theta)$, BE notion takes expectation over $P(\theta_{-i} \mid \theta_i)$

Equivalence of the two equilibrium concepts

Theorem: In finite Bayesian games, a strategy profile is a Bayesian equilibrium iff it is a Nash equilibrium.

Proof: $(\Rightarrow)$ Suppose (σ^*, P) is a BE, consider

$$u_i(\sigma_i', \sigma_{-i}^*) = \sum_{\theta_i \in \Theta_i} P(\theta_i)\, u_i(\sigma_i'(\theta_i), \sigma_{-i}^* \mid \theta_i)$$

$$\overset{\text{BE}}{\leqslant} \sum_{\theta_i \in \Theta_i} P(\theta_i)\, u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i) = u_i(\sigma_i^*, \sigma_{-i}^*).$$

$(\Leftarrow)$ Proof by contradiction. Suppose (σ^*, P) is not a BE, i.e.,

there exists some $i \in N$, some $\theta_i \in \Theta_i$, and some $a_i \in A_i$, s.t.

$$u_i(a_i, \sigma_{-i}^* \mid \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i)$$

Construct the strategy $\hat{\sigma}_i$, $\quad \hat{\sigma}_i(\theta_i') = \sigma_i^*(\theta_i') \;\; \forall \theta_i' \in \Theta_i \setminus \{\theta_i\}$

$$\hat{\sigma}_i(\theta_i)[a_i] = 1 \,, \quad \hat{\sigma}_i(\theta_i)[b_i] = 0 \quad \forall\, b_i \in A_i \setminus \{a_i\}$$

Then, $$u_i(\hat{\sigma}_i, \underline{\sigma}_{-i}^*) = \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i(\tilde{\theta}_i), \underline{\sigma}_{-i}^* \mid \tilde{\theta}_i)$$

$$= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i(\tilde{\theta}_i), \underline{\sigma}_{-i}^* \mid \tilde{\theta}_i) + P(\theta_i)\, \underbrace{u_i(\hat{\sigma}_i(\theta_i), \underline{\sigma}_{-i}^* \mid \theta_i)}_{> u_i(\sigma_i^*(\theta_i), \underline{\sigma}_{-i}^* \mid \theta_i)}$$

$$> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i(\tilde{\theta}_i), \underline{\sigma}_{-i}^* \mid \tilde{\theta}_i) + P(\theta_i)\, u_i(\sigma_i^*(\theta_i), \underline{\sigma}_{-i}^* \mid \theta_i)$$

$$= u_i(\sigma_i^*, \underline{\sigma}_{-i}^*)$$

Hence $(\sigma_i^*, \underline{\sigma}_{-i}^*)$ is not a Nash equilibrium.

Existence of Bayesian equilibrium

Theorem: Every finite Bayesian game has a Bayesian equilibrium

[finite Bayesian game: set of players, action set, type set are finite]

proof idea: transform the Bayesian game into a complete information game treating each type a player, and invoke Nash Theorem for existence of equilibrium - which is a BE in the original game. [see addendum for details]

Addendum to transform Bayesian game to complete information NFG.

$$\underset{\substack{\uparrow \\ \text{new player set}}}{\bar{N}} = \bigcup_{i \in N} \Theta_i = \{\theta_1^1, \theta_1^2, \dots, \theta_1^{|\Theta_1|}, \dots \}$$

Consider two players, type sets $\Theta_1 = \{\theta_1^1, \theta_1^2\}$, $\Theta_2 = \{\theta_2^1, \theta_2^2\}$

utility of player θ_1^1 — original payoffs of Bayesian game

$$\bar{u}_{\theta_1^1}(a_{\theta_1^1}, a_{\theta_1^2}, a_{\theta_2^1}, a_{\theta_2^2}) = P(\theta_2^1 | \theta_1^1)\, u_1(a_{\theta_1^1}, a_{\theta_2^1}, \theta_1^1, \theta_2^1) + P(\theta_2^2 | \theta_1^1)\, u_1(a_{\theta_1^1}, a_{\theta_2^2}, \theta_1^1, \theta_2^2)$$

[defining $a_{\theta_1^1} = a_1(\theta_1^1)$, $a_{\theta_1^2} = a_1(\theta_1^2)$ etc.] --- (1)

consider a mixed strategy $(\sigma_{\theta_1^1}, \sigma_{\theta_1^2}, \sigma_{\theta_2^1}, \sigma_{\theta_2^2})$ in this new game

$$\bar{u}_{\theta_1^1}(\sigma_{\theta_1^1}, \sigma_{\theta_1^2}, \sigma_{\theta_2^1}, \sigma_{\theta_2^2}) =$$

$$\sum_{a_{\theta_2^2} \in A_2} \sum_{a_{\theta_2^1} \in A_2} \sum_{a_{\theta_1^2} \in A_1} \sum_{a_{\theta_1^1} \in A_1} \sigma_{\theta_1^1}(a_{\theta_1^1})\, \sigma_{\theta_1^2}(a_{\theta_1^2})\, \sigma_{\theta_2^1}(a_{\theta_2^1})\, \sigma_{\theta_2^2}(a_{\theta_2^2}) \times \underbrace{\bar{u}_{\theta_1^1}(a_{\theta_1^1}, a_{\theta_1^2}, a_{\theta_2^1}, a_{\theta_2^2})}_{\text{now plug this in from (1), irrelevant } a_{\theta_i} \text{ terms will sum to 1}}$$

$$= \sum_{a_{\theta_2^1} \in A_2} P(\theta_2^1 | \theta_1^1)\, \underbrace{\sigma_{\theta_2^1}(a_{\theta_2^1})}_{=:\ \sigma_2(\theta_2^1, a_{\theta_2^1})}\, u_1(a_{\theta_1^1}, a_{\theta_2^1}, \theta_1^1, \theta_2^1) + \sum_{a_{\theta_2^2} \in A_2} P(\theta_2^2 | \theta_1^1)\, \underbrace{\sigma_{\theta_2^2}(a_{\theta_2^2})}_{\text{define as } \sigma_2(\theta_2^2, a_{\theta_2^2})}\, u_1(a_{\theta_1^1}, a_{\theta_2^2}, \theta_1^1, \theta_2^2)$$

$$= \sum_{\theta_2 \in \Theta_2} P(\theta_2 | \theta_1^1)\, U_1(\sigma_1, \sigma_2 | \theta_1^1)$$

Hence a mixed strategy $(\sigma_{\theta_1^1}, \sigma_{\theta_1^2}, \sigma_{\theta_2^1}, \sigma_{\theta_2^2})$ in the complete information game is a mixed strategy (σ_1, σ_2) in the Bayesian game.

It follows that the MSNE in that game will be a BE in this game.

Bayesian equilibria in Bayesian games

Sealed bid auction

Two players, both willing to buy an object. Their values and bids lie in $[0,1]$

allocation function: $O_1(b_1,b_2) = I\{b_1 \geq b_2\}$; $O_2(b_1,b_2) = I\{b_2 > b_1\}$

beliefs: $f(\theta_2|\theta_1)=1$, $\forall \theta_1, \theta_2$ $\quad f(\theta_1,\theta_2) = 1$, $\forall (\theta_1,\theta_2) \in [0,1]^2$

$f(\theta_1|\theta_2)=1$, $\forall \theta_1, \theta_2$

① First price auction: if $b_1 \geq b_2$, player 1 wins and pays her bid
ow, player 2 wins and pays her bid

$$u_1(b_1,b_2,\theta_1,\theta_2) = (\theta_1 - b_1)\, I\{b_1 \geq b_2\}$$

$$u_2(b_1,b_2,\theta_1,\theta_2) = (\theta_2 - b_2)\, I\{b_1 < b_2\}$$

$b_1 = s_1(\theta_1)$, $b_2 = s_2(\theta_2)$, assume $s_i(\theta_i) = \alpha_i \theta_i$, $\alpha_i > 0$, $i=1,2$.

To find the BE, we need to find the s_i^* (or α_i^*) that maximizes the ex-interim utility of player i

$$\max_{\sigma_i} \; u_i(\sigma_i, \underline{\sigma}_i^* \,|\, \theta_i),$$

For player 1, this reduces to:

$$\max_{b_1 \in [0,\alpha_2]} \int_0^1 f(\theta_2|\theta_1)(\theta_1 - b_1)\, I\{b_1 \geq \alpha_2\theta_2\}\, d\theta_2$$

(since $\theta_2 \in [0,1]$ b_1 never needs to be larger than α_2)

$$= \max_{b_1 \in [0,\alpha_2]} (\theta_1 - b_1)\frac{b_1}{\alpha_2} \Rightarrow b_1^* = \begin{cases} \frac{\theta_1}{2} & \text{if } \alpha_2 > \frac{\theta_1}{2} \\ \alpha_2 & \text{ow} \end{cases}$$

$$s_1^*(\theta_1) = \min\left\{\frac{\theta_1}{2}, \alpha_2\right\}, \quad s_2^*(\theta_2) = \min\left\{\frac{\theta_2}{2}, \alpha_1\right\}$$

if $\alpha_1 = \alpha_2 = 1/2$, Then $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ is a BE

In The Bayesian game induced by uniform prior on first price auction, bidding half The true value is a Bayesian equilibrium.

② Second price auction: highest bidder wins but pays The second highest bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)\, I\{b_1 \geqslant b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)\, I\{b_1 < b_2\}$$

Player 1's bidding problem is to maximize

$$\int_0^1 f(\theta_2|\theta_1)\,(\theta_1 - s_2(\theta_2))\, I(b_1 \geqslant s_2(\theta_2))\, d\theta_2$$

$$= \int_0^1 1 \cdot (\theta_1 - \alpha_2\theta_2)\, I\left(\theta_2 \leqslant \frac{b_1}{\alpha_2}\right) d\theta_2$$

$$= \frac{1}{\alpha_2}\left(b_1\theta_1 - \frac{\theta_1^2}{2}\right) \Rightarrow \text{maximized when } b_1 = \theta_1$$

similarly for $b_2 = \theta_2$.

If The distributions of θ_1 and θ_2 were arbitrary but independent The maximization problem would have been

$$\int_0^{b_1/\alpha_2} f(\theta_2)\,(\theta_1 - \alpha_2\theta_2)\, d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{b_1/\alpha_2} \theta_2 f(\theta_2)\, d\theta_2$$

differentiating wrt b_1, we get

$$\theta_1 \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \frac{b_1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) \cdot \frac{1}{\alpha_2} = 0$$

$$\Rightarrow \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right)(b_1 - \theta_1) = 0 \Rightarrow b_1 = \theta_1 \text{ (similar for 2), if } f\left(\frac{b_1}{\alpha_2}\right) > 0$$

For any independent, positive prior, bidding true type is a BE of The induced Bayesian game in second price auction.

Mechanism Design (Inverse Game Theory)

The objectives / desired outcomes are set - task is to set the rules of the game

E.g., Election, license scarce resource (spectrum, cloud), matching students to universities

General model:

N : set of players

X : set of outcomes, e.g., winner in an election, which resource allocated to whom etc.

Θ_i : set of private information of agent i (type). A type $\theta_i \in \Theta_i$

The type may manifest in the preferences over the outcomes in different ways

1. Ordinal: θ_i defines an ordering over the outcomes
2. Cardinal: an utility function u_i maps an (outcome, type) pair to real numbers, $u_i : X \times \Theta_i \to \mathbb{R}$ (private value model)
   or $u_i : X \times \Theta \to \mathbb{R}$ (interdependent value model)

Examples: Voting: X is the set of candidates

θ_i is a ranking over the candidates, e.g., $\theta_i = (a, b, c)$, i.e., a is more preferred than b which in turn is more preferred than c.

Single object allocation: an outcome is $x = (\underline{a}, \underline{p}) \in X$

$\underline{a} = (a_1, a_2, \cdots, a_n)$, $a_i \in \{0,1\}$, $\sum_{i \in N} a_i \leq 1$, allocations

$\underline{p} = (p_1, p_2, \ldots, p_n)$, p_i is the payment charged to i

θ_i : value of i for the object

$u_i(x, \theta_i) = a_i \theta_i - p_i$.

But the designer has an objective

This is captured through a Social Choice Function (SCF)

$$f : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \to X$$

E.g., in voting, if there is a candidate who beats everyone else in pairwise contests must be chosen as a winner.

in public project choice, where $\theta_i : X \to \mathbb{R}$, value for each project pick $f(\theta) \in \underset{a \in X}{\operatorname{argmax}} \sum_{i \in N} \theta_i(a)$.

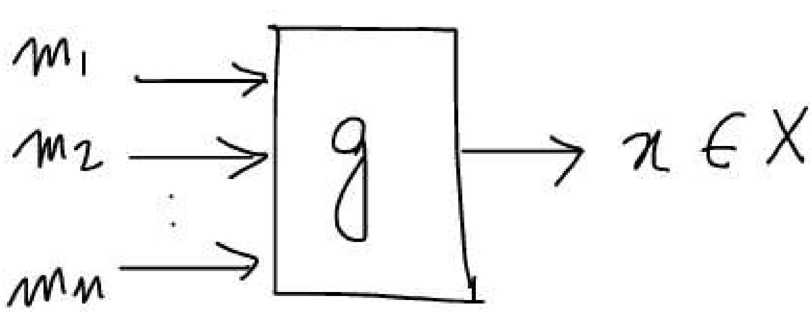
Q: How can we create a game where $f(\theta)$ emerges as an outcome of an equilibrium?

A: we need mechanisms.

<u>Defn</u>. An (indirect) mechanism is a collection of message spaces and a decision rule $\langle M_1, M_2, \ldots, M_n, g \rangle$

- M_i is the message space of agent i
- $g : M_1 \times M_2 \times \cdots \times M_n \to X$

A direct mechanism is same as above with $M_i = \Theta_i$, $\forall i \in N$, $g \equiv f$.



The message space is similar to equipping every agent with a card deck and asking to pick some

Q: Why these are not so commonplace?

A: due to a result that will follow.

Defn. In a mechanism $\langle M_1, \dots, M_n, g \rangle$, a message m_i is weakly dominant for player i at θ_i if

$$u_i(g(m_i, \tilde{\underline{m}}_{-i}), \theta_i) \geqslant u_i(g(m_i', \tilde{\underline{m}}_{-i}), \theta_i), \forall \tilde{\underline{m}}_{-i}, \forall m_i'$$

[all subsequent definitions assume cardinal preferences, however they can be replaced with ordinal, e.g., the above one could be defined as

$$g(m_i, \tilde{\underline{m}}_{-i}) \ \theta_i \ g(m_i', \tilde{\underline{m}}_{-i}) \ \forall m_i', \forall \tilde{\underline{m}}_{-i} \quad]$$

this outcome is preferred at least as much as the latter

Defn. An SCF $f: \Theta \to X$ is implemented in dominant strategies by $\langle M_1, \dots, M_n, g \rangle$ if

1. $\exists$ message mappings $s_i : \Theta_i \to M_i$, s.t., $s_i(\theta_i)$ is a dominant strategy for agent i at θ_i, $\forall \theta_i \in \Theta_i$, $\forall i \in N$, and
2. $g(s_1(\theta_1), \dots, s_n(\theta_n)) = f(\theta)$, $\forall \theta \in \Theta$.

We call this an indirect implementation, i.e., SCF f is dominant strategy implementable (DSI) by $\langle M_1, \dots, M_n, g \rangle$.

Defn. A direct mechanism $\langle \Theta_1, \dots, \Theta_n, f \rangle$ is dominant strategy incentive compatible (DSIC) if

$$u_i(f(\theta_i, \tilde{\underline{\theta}}_{-i}), \theta_i) \geqslant u_i(f(\theta_i', \tilde{\underline{\theta}}_{-i}), \theta_i), \forall \theta_i, \theta_i', \tilde{\underline{\theta}}_{-i}$$
$$\forall i \in N.$$

To find if an SCF f is dominant strategy implementable, we need to search over all possible indirect mechanisms $\langle M_1, \dots, M_n, g \rangle$

But luckily, there is a result that reduces the search space.

Relationship between DSI and DSIC

Revelation principle (for DSI SCFs): If there exists an indirect mechanism that implements f in dominant strategies, then f is DSIC.

Implication: can focus on DSIC mechanisms WLOG.

Proof: Let f is implemented by $\langle M_1, \dots, M_n, g \rangle$, hence $\exists\, s_i : \Theta_i \to M_i$ s.t. $\forall i \in N, \forall\, \tilde{\underline{m}}_{-i}, m_i', \theta_i$,

$$u_i\left(g(s_i(\theta_i), \tilde{\underline{m}}_{-i}), \theta_i\right) \geq u_i\left(g(m_i', \tilde{\underline{m}}_{-i}), \theta_i\right) \quad \text{—— (1)}$$

and

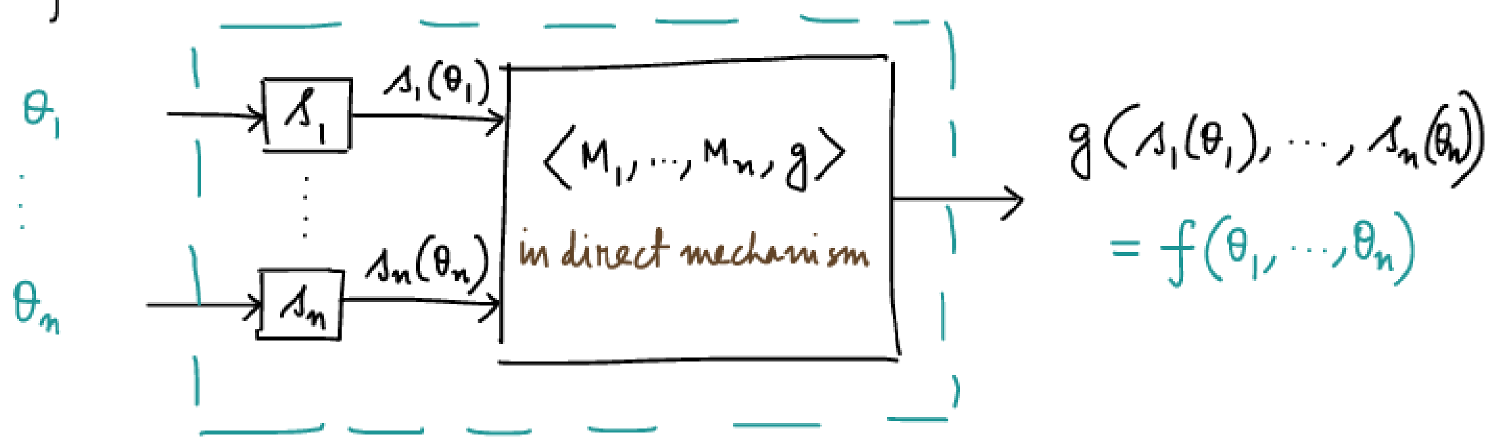
$$g\left(s_i(\theta_i), \underline{s}_{-i}(\underline{\theta}_{-i})\right) = f(\theta_i, \underline{\theta}_{-i}) \quad \text{—— (2)}$$

Eqn. (1) holds for all $m_i', \tilde{\underline{m}}_{-i}$, in particular, $m_i' = s_i(\theta_i')$, $\tilde{\underline{m}}_{-i} = \underline{s}_{-i}(\tilde{\underline{\theta}}_{-i})$ where θ_i' and $\tilde{\underline{\theta}}_{-i}$ are arbitrary. Hence

$$u_i\big(\underbrace{g(s_i(\theta_i), \underline{s}_{-i}(\tilde{\underline{\theta}}_{-i}))}_{= f(\theta_i, \tilde{\underline{\theta}}_{-i})}, \theta_i\big) \geq u_i\big(\underbrace{g(s_i(\theta_i'), \underline{s}_{-i}(\tilde{\underline{\theta}}_{-i}))}_{= f(\theta_i', \tilde{\underline{\theta}}_{-i})}, \theta_i\big) \quad [\text{By } (2)]$$

$$\Rightarrow u_i\left(f(\theta_i, \tilde{\underline{\theta}}_{-i}), \theta_i\right) \geq u_i\left(f(\theta_i', \tilde{\underline{\theta}}_{-i}), \theta_i\right)$$

$\Rightarrow$ f is DSIC.



$\langle \Theta_1, \dots, \Theta_n, f \rangle$, direct mechanism

Bayesian extension (agents may have probabilistic information about others' types)

Types are generated from a common prior (common knowledge) and are revealed only to the respective agents.

Recall: Bayesian games

$$\langle N, (M_i)_{i\in N}, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta\in\Theta} \rangle$$

taking the place of actions

$$s_i : \Theta_i \to M_i \text{, message mapping}$$

Defn. An (indirect) mechanism $\langle M_1, \dots, M_n, g\rangle$ implements an SCF f in Bayesian equilibrium if

① $\exists$ a message mapping profile $(s_1, \dots, s_n)$, s.t., $s_i(\theta_i)$ maximizes the ex-interim utility of agent i, $\forall \theta_i, \forall i \in N$, i.e.,

$$\mathbb{E}_{\underline{\theta}_{-i}|\theta_i}[u_i(g(s_i(\theta_i), \underline{s}_{-i}(\underline{\theta}_{-i})), \theta_i)] \geq \mathbb{E}_{\underline{\theta}_{-i}|\theta_i}[u_i(g(m_i', \underline{s}_{-i}(\underline{\theta}_{-i})), \theta_i)]$$

$$\forall m_i', \forall \theta_i, \forall i \in N, \quad \text{and}$$

② $g(s_i(\theta_i), \underline{s}_{-i}(\underline{\theta}_{-i})) = f(\theta_i, \underline{\theta}_{-i}), \forall \theta$.

We call f is Bayesian implementable via $\langle M_1, \dots, M_n, g\rangle$ under the prior P.

Lemma: If an SCF f is dominant strategy implementable, then it is Bayesian implementable.

Proof: homework.

A direct mechanism $\langle \Theta_1, \dots, \Theta_n, f \rangle$ is Bayesian Incentive Compatible (BIC) if $\forall \theta_i, \theta_i', \forall i \in N$

$$\mathbb{E}_{\underline{\theta}_{-i} | \theta_i} \left[u_i\left(f(\theta_i, \underline{\theta}_{-i}), \theta_i \right) \right] \geqslant \mathbb{E}_{\underline{\theta}_{-i} | \theta_i} \left[u_i\left(f(\theta_i', \underline{\theta}_{-i}), \theta_i \right) \right].$$

Revelation principle (for BI SCFs)

If an SCF f is implementable in Bayesian equilibrium, then f is BIC.

Proof idea is similar to the DSI, with expected utilities at appropriate places.

For truthfulness of these two kinds, we will only consider incentive compatibility.

These results hold even for ordinal preferences and mechanisms.

Aggregating opinions (not worrying about truthful revelation)

Can we create social preference orders from individual preferences?

Arrow's social welfare function setup

Finite set of alternatives, $A = \{a_1, a_2, \dots, a_m\}$

Finite set of players, $N = \{1, \dots, n\}$

Each player i has a preference order R_i over A [a binary relation over A]. $a R_i b \Rightarrow a$ is at least as good as b.

Properties of R_i:

1. Completeness: for every pair of alternatives $a, b \in A$, either $a R_i b$ or $b R_i a$ or both
2. Reflexivity: $\forall a \in A$, $a R_i a$
3. Transitivity: if $a R_i b$ and $b R_i c$, then $a R_i c$, $\forall a, b, c \in A$ and $i \in N$.

Set of all preference ordering is $\mathcal{R}$

An ordering is linear if for every $a, b \in A$ s.t. $a R_i b$ and $b R_i a$, it holds that $a = b$. [indifferences are not allowed]

Set of all linear orderings is $\mathbb{P}$

Hence any arbitrary ordering R_i can be decomposed into (a) asymmetric part P_i, and (b) symmetric part I_i

E.g., $R_i = \begin{bmatrix} a \\ b, c \\ d \end{bmatrix} = \{(a,b), (a,c), (b,c), (c,b), (b,d), (c,d)\}$

$$\Rightarrow P_i = \begin{bmatrix} a & a \\ b & c \\ d & d \end{bmatrix} = \{(a,b), (a,c), (b,d), (c,d)\}, \quad I_i = \{(b,c), (c,b)\}$$

Arrovian Social Welfare Function (ASWF)

$F : \mathcal{R}^n \to \mathcal{R}$, domain and range both are rankings

motivation: The collective ordering of the society - if the most preferred is not feasible, the society can move to the next and so on.

$F(R) = F(R_1, R_2, \cdots, R_n)$ is an ordering

$\hat{F}(R)$ is the asymmetric part of $F(R)$

$\bar{F}(R)$ is the symmetric part of $F(R)$

Defn: Weak Pareto

An ASWF F satisfies weak Pareto if $\forall a, b \in A$

$$[a P_i b, \forall i \in N] \Rightarrow [a \hat{F}(R) b]$$

This notation is read as "whenever <the condition inside> holds, the implication follows"

$\forall R \in \mathcal{R}^n$, if $a P_i b, \forall i \in N$, then $a \hat{F}(R) b$.

There could be R's where the if condition doesn't hold, there the implication is vacuously true.

Defn. Strong Pareto

An ASWF F satisfies strong Pareto if $\forall a, b \in A$

$$[a R_i b, \forall i \in N, \text{and } a P_j b, \exists j] \Rightarrow [a \hat{F}(R) b]$$

Q: Which property implies the other?

We say $R_i, R_i' \in \mathcal{R}$ agree on $\{a, b\}$ if for agent i

$$a P_i b \Leftrightarrow a P_i' b, \quad b P_i a \Leftrightarrow b P_i' a, \quad a I_i b \Leftrightarrow a I_i' b$$

We use the shorthand $R_i|_{a,b} = R_i'|_{a,b}$ to denote this

If this holds for every agent, $R|_{a,b} = R'|_{a,b}$.

Defn. An ASWF F satisfies IIA if $\forall a,b \in A$,

$$\left[R|_{a,b} = R'|_{a,b}\right] \Rightarrow \left[F(R)|_{a,b} = F(R')|_{a,b}\right]$$

If the relative positions of two alternatives are same in two different preference profiles, then the aggregate must also retain the same relative positions.

Example:

R

a	a	c	d
b	c	b	c
c	b	a	b
d	d	d	a

R′

d	c	b	b
a	a	c	a
b	b	a	d
c	d	d	c

Consider scoring rules $(s_1, s_2, \ldots, s_m)$, $s_i \geqslant s_{i+1}$, $i=1,\ldots,m-1$

one special rule: plurality $s_1 = 1$, $s_2 = \cdots = s_m = 0$.

Does plurality satisfy IIA?

check $a\, F^{Pl}(R)\, b$, but $b\, F^{Pl}(R')\, a$, $R|_{a,b} = R'|_{a,b}$

Does dictatorship satisfy IIA?

Theorem (Arrow 1951)

Assume $|A| \geqslant 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial.

Arrow's impossibility result

Theorem (Arrow 1951)

Assume $|A| \geq 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial.

For the proof, we need the notions of decisiveness.

Defn. Let $F: \mathcal{R}^n \to \mathcal{R}$ be given, $G \subseteq N$, $G \neq \phi$

① G is almost decisive over $\{a, b\}$ if

$$[\, a P_i b, \ \forall i \in G, \text{ and } \ b P_j a \ \ \forall j \notin G \,] \Rightarrow [\, a \hat{F}(R) b \,]$$

We write this with the shorthand $\bar{D}_G(a,b)$: G is almost decisive over $\{a,b\}$ w.r.t. F

② G is decisive over $\{a, b\}$ if

$$[\, a P_i b \, , \forall i \in G \,] \Rightarrow [\, a \hat{F}(R) b \,]$$

Shorthand $D_G(a,b)$: G is decisive over $\{a,b\}$ w.r.t. F

Clearly, $D_G(a,b) \Rightarrow \bar{D}_G(a,b)$

The proof of the theorem proceeds in two parts

Part 1: Field expansion lemma

if a group is decisive over a pair of alternatives, it is decisive over all pairs of alternatives.

Part 2 : Group contraction lemma

if a group is decisive, then a strict subset of that group is also decisive.

Note that, these two lemmas immediately proves the theorem.

Part I: Field expansion lemma

Let F satisfy WP and IIA, then $\forall\, a,b,x,y$, $G \subseteq N$, $G \neq \phi$, $a \neq b$, $x \neq y$

$$\bar{D}_G(a,b) \Rightarrow D_G(x,y).$$

Remark: under WP and IIA, the two notions of decisiveness are identical.

Proof: Cases to consider

1. $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$, i.e., $x=a$, $y \neq a,b$
2. $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$, i.e., $x \neq a,b$, $y=b$
3. $\bar{D}_G(a,b) \Rightarrow D_G(x,y)$, i.e., $x \neq a,b$, $y \neq a,b$
4. $\bar{D}_G(a,b) \Rightarrow D_G(x,a)$, i.e., $x \neq a,b$, $y=a$
5. $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$, i.e., $x=b$, $y \neq a,b$
6. $\bar{D}_G(a,b) \Rightarrow D_G(a,b)$
7. $\bar{D}_G(a,b) \Rightarrow D_G(b,a)$

Case 1: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$, i.e., pick arbitrary $R \in \mathcal{R}^n$ s.t.

$a\, P_i\, y \;\; \forall i \in G$, need to show that $a\, \hat{F}(R)\, y$.

Construct R'

G	$N \setminus G$	
a	b	
b	a	y
y		

ensure $R_i'|_{a,y} = R_i|_{a,y}$, $\forall i \in N$

$\bar{D}_G(a,b) \Rightarrow a\, \hat{F}(R')\, b$

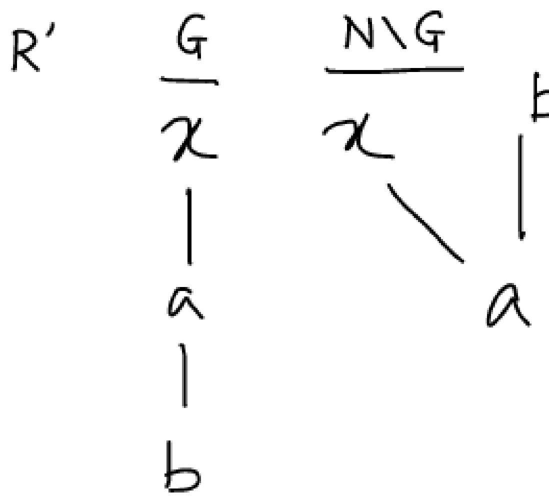
WP over $b,y \Rightarrow b\, \hat{F}(R')\, y$

transitivity $\Rightarrow a\, \hat{F}(R')\, y$

$\underset{\text{IIA}}{\Rightarrow} a\, \hat{F}(R)\, y$. Hence $D_G(a,y)$

Case 2: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$

Pick arbitrary R s.t. $x P_i b$, $\forall i \in G$. Need to show $x \hat{F}(R) b$.



Ensure $R_i'|_{x,b} = R_i|_{x,b}$ $\forall i \in N$.

$\bar{D}_G(a,b) \Rightarrow a \hat{F}(R') b$

WP on $x, a \Rightarrow x \hat{F}(R') a$

transitivity $\Rightarrow x \hat{F}(R') b \overset{IIA}{\Rightarrow} x \hat{F}(R) b$.

Case 3: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ [case 1]

$\Rightarrow \bar{D}_G(a,y)$ [definition]

$\Rightarrow D_G(x,y)$ [case 2]

Case 4: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ [case 2] $\quad x \neq a, b$

$\Rightarrow \bar{D}_G(x,b)$ [definition]

$\Rightarrow D_G(x,a)$ [case 1]

Case 5: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ [case 1] $\quad y \neq a, b$

$\Rightarrow \bar{D}_G(a,y)$ [definition]

$\Rightarrow D_G(b,y)$ [case 2]

Case 6: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ [case 2] $\quad x \neq a, b$

$\Rightarrow \bar{D}_G(x,b)$ [definition]

$\Rightarrow D_G(a,b)$ [case 2]

Case 7: $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$ [case 5] $\quad y \neq a, b$

$\Rightarrow \bar{D}_G(b,y)$ [definition]

$\Rightarrow D_G(b,a)$ [case 1]

Part 2: Group contraction lemma

Let F satisfy WP and IIA. Let $G \subseteq N$, $G \neq \phi$, $|G| \geqslant 2$, be decisive. Then $\exists\, G' \subset G$, $G' \neq \phi$ which is also decisive.

Proof: If $|G| = 1$, nothing to prove. WLOG assume $|G| \geqslant 2$

Let $G_1 \subset G$, $G_2 = G \setminus G_1$, construct R

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

$a P_i b \quad \forall i \in G$

and G decisive

$\Rightarrow \quad a \hat{F}(R) b \quad — (1)$

Case 1: $a \hat{F}(R) c$, now consider G_1

$$a P_i c \ \forall i \in G_1, \ c P_i a \ \forall i \notin G_1$$

Consider all R', where this holds, by IIA $a \hat{F}(R') c$

hence $\bar{D}_G(a,c) \overset{FEL}{\Longrightarrow} G_1$ is decisive

Case 2: $\neg(a \hat{F}(R) c) \Rightarrow c F(R) a$

from (1) we get $a \hat{F}(R) b \underset{\text{trans}}{\Rightarrow} c \hat{F}(R) b$

Consider G_2,

$$c P_i b \ \forall i \in G_2, \text{ and } b P_i c \ \forall i \notin G_2$$

using IIA as before $\bar{D}_{G_2}(b,c) \overset{FEL}{\Longrightarrow} G_2$ is decisive

This concludes the proof.

Arrovian social welfare setup is too demanding

It says that achieving a "social ordering" in a democratic way is impossible

Steps to mitigate:

1. Consider a social choice setting – instead of an ordering, select an alternative.
2. Put restrictions on agents' preferences

Social Choice Function

$$f : \mathbb{P}^n \to A$$, assuming only strict preferences

most representative example: voting

Various voting rules

1. Scoring rule: $(s_1, s_2, \cdots, s_m)$ common score vector. Every voter's k^{th} preferred alternative is given a score of s_k. Scores are summed for each candidate – highest score wins.
   - special cases: plurality – $(1, 0, \cdots, 0)$
     - veto – $(1, 1, \cdots, 1, 0)$
     - Borda – $(m-1, m-2, \cdots, 0)$
     - harmonic – $(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{m})$
     - k-approval – $(\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0)$
2. Plurality with runoff: two phases – first, top 2 highest scored candidates remain and the voters vote again [French presidential]
3. Maximin: candidate with largest margin of victory wins
4. Copeland: based on score = # of wins in pairwise elections

A Condorcet winner is a candidate that beats every other candidate in pairwise election. It is not guaranteed to exist.

a	b	c
b	c	a
c	a	b

If a Condorcet winner exists, the voting rules that returns it as the winner are called "Condorcet consistent"

Easy to check that Copeland is Condorcet consistent (by design)

But plurality is not

30%	30%	40%
a	b	c
b	a	a
c	c	b

pairwise election
a beats b 70-30
a beats c 60-40
a is the Condorcet winner

Actually, no scoring rule is Condorcet consistent.

Back to Social Choice Functions: $f : \mathbb{P}^n \to A$

Pareto domination: An alternative a is Pareto dominated by b if $\forall i \in N \quad b P_i a$. [it is Pareto dominated if some such b exists]

Pareto Efficiency: An SCF f is PE if for every preference profile P and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.

Unanimity: An SCF f is UN if for every preference profile P having $P_1(1) = P_2(1) = \cdots = P_n(1) = a$ [where $P_i(k)$ is the k^{th} preferred alternative of i], $f(P) = a$.

Clearly PE $\subset$ UN, when the top candidate is the same a for all agents, all other alternatives are Pareto dominated by a

Hence a PE SCF can choose nothing but a.

Why strict? consider a profile where the top alternative is not the same, a UN SCF can pick a dominated alternative

Ontoness : An SCF is ONTO if $\forall a \in A$, $\exists P^{(a)} \in \mathcal{P}^n$ s.t.

$$f(P^{(a)}) = a.$$

Claim : UN $\subset$ ONTO

Manipulability : An SCF f is manipulable if $\exists\, i \in N$ and a profile P s.t. $f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i})$ for some P_i'.

Ex. Plurality (tie breaking in favor of $a > b > c$)

a	b	c
b	a	b
c	c	a
4	4	1

$\leftarrow$ true votes

last voter should vote for b

Copeland (tie breaking in favor of $a > b > c$)

a	b	c
b	c	a
c	a	b
1	1	1

each candidate has Copeland score $= 1$, a is the winner.

a is least preferred

if reports $\begin{matrix} c \\ b \\ a \end{matrix}$ then c is the Copeland winner.

An SCF is strategyproof if it is not manipulable by any agent at any profile.

Implications of strategyproofness

Defn : Dominated set of a at preference P_i

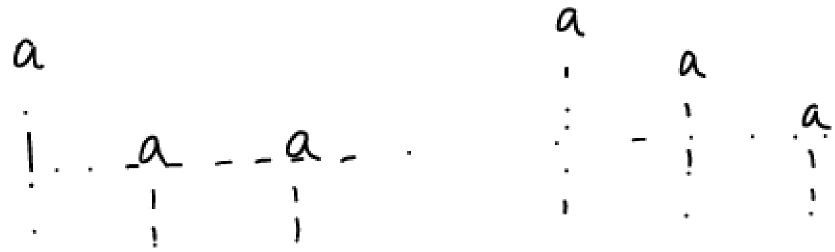
$$D(a, P_i) = \{ b \in A : a\, P_i\, b \}$$

The set of alternatives below a in that preference

e.g., $P_i = \begin{matrix} b \\ d \\ a \\ c \end{matrix} \Rightarrow D(d, P_i) = \{a, c\}$

<u>Monotonicity</u>: An SCF f is monotone if for any two profiles P and P' with $f(P) = a$ and
$$D(a, P_i) \subseteq D(a, P_i'), \forall i \in N, \text{ must imply}$$
$$f(P') = a.$$

The relative position of a has weakly improved from R to R'. This property says if a was the outcome in P, then it must continue to be the outcome in P'



P if $f(P) = a$, then P'

$f(P') = a$

<u>Theorem</u>: An SCF f is strategyproof (SP) iff it is monotone (MONO).

Theorem: An SCF f is strategyproof (SP) iff it is monotone (MONO).

Note: The proof technique, will be used later as well.

Proof: SP $\Rightarrow$ MONO, consider the "if" condition of MONO

P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i') \;\forall i \in N$

Break the transition from P to P' into n stages

$$\underset{P = P^{(0)}}{(P_1\, P_2 \cdots P_n)} \to \underset{P^{(1)}}{(P_1'\, P_2 \cdots P_n)} \to \underset{P^{(2)}}{(P_1'\, P_2' \cdots P_n)} \to \underset{P^{(k)}}{(P_1' \cdots P_k'\, P_{k+1} \cdots P_n)} \to \cdots \underset{P^{(n)} = P'}{(P_1' \cdots P_n')}$$

Claim: $f(P^{(k)}) = a$, $\forall k = 1, \dots, n$

Suppose not, i.e., $\exists\, P^{(k-1)}, P^{(k)}$ s.t. $f(P^{(k-1)}) = a$, $f(P^{(k)}) = b \neq a$

$P_1' \cdots P_{k-1}'\, P_k \cdots P_n$ $\qquad$ $P_1' \cdots P_{k-1}'\, P_k' \cdots P_n$

a $\qquad$ $a \leftarrow$ position has weakly bettered

outcome a $\qquad$ outcome b

there can be three cases:

$a P_k b$ and $a P_k' b$ $\to$ voter k misreports $P_k' \to P_k$

$b P_k a$ and $b P_k' a$ $\to$ voter k misreports $P_k \to P_k'$

$b P_k a$ and $a P_k' b$ $\to$ voter k misreports in both

contradiction to f SP.

SP $\Leftarrow$ MONO, we will prove $\neg$SP $\Rightarrow$ $\neg$MONO

suppose not, i.e., f is $\neg$SP but MONO

$\neg$SP implies that $\exists\, i, P_i, P_i', P_{-i}$ s.t. $\underbrace{f(P_i', P_{-i})}_{=:\, b}\; P_i\; \underbrace{f(P_i, P_{-i})}_{=:\, a}$

hence $b \, P_i \, a$. construct P'' s.t. $P''_{-i} = P_{-i}$.

$P_i''(1) = b$, $P_i''(2) = a$

P_i''	P_{-i}
b	$\vdots$
a	$\vdots$
$\vdots$	

Consider two transitions

① $(P_i, P_{-i}) \rightarrow (P_i'', P_{-i})$

$$D(a, P_i) \subseteq D(a, P_i'') \underset{\text{MONO}}{\Rightarrow} f(P_i'', P_{-i}) = a$$

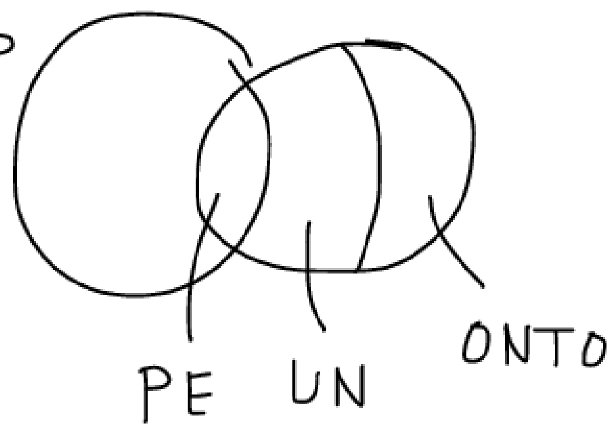
② $(P_i', P_{-i}) \rightarrow (P_i'', P_{-i})$

$$D(b, P_i') \subseteq D(b, P_i'') \underset{\text{MONO}}{\Rightarrow} f(P_i'', P_{-i}) = b \quad \text{(contradiction)}$$

This concludes the proof.

Lemma: If an SCF f is MONO and ONTO, then f is PE.

MONO $\Leftrightarrow$ SP



Proof: Suppose not, i.e., f is MONO and ONTO but not PE

then $\exists \, a, b, P$ s.t., $b \, P_i \, a \;\; \forall i \in N$ but $f(P) = a$.

ONTO: $\exists \, P'$ s.t. $f(P') = b$.

Construct P'' s.t. $P_i''(1) = b$, $P_i''(2) = a$, $\forall \, i \in N$

P''			
b	b	$\cdots$	b
a	a		a
$\vdots$	$\vdots$		$\vdots$

Clearly $D(b, P_i') \subseteq D(b, P_i'') \;\; \forall i \in N$

$$\underset{\text{MONO}}{\Rightarrow} f(P'') = b$$

Also $D(a, P_i) \subseteq D(a, P_i'') \ \forall i \in N$

$\underset{\text{MONO}}{\Rightarrow} \quad f(P'') = a$ (contradiction). Hence proved.

<u>Corollary</u>: f is SP + PE $\Leftrightarrow$ f is SP + UN $\Leftrightarrow$ f is SP + ONTO

Gibbard - Satterthwaite Theorem (G73, S75)

Suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial.

The statements with f is PE/UN and SP are equivalent.

Corollary: f is SP+PE $\Leftrightarrow$ f is SP+UN $\Leftrightarrow$ f is SP+ONTO

Gibbard-Satterthwaite Theorem (G73, S75)

Suppose $|A| \geqslant 3$, f is ONTO and SP iff f is dictatorial.

The statements with f is PE/UN and SP are equivalent.

Few points to note:

1. $|A|=2$: GS theorem does not hold. Plurality with a fixed tie breaking rule is SP, ONTO, and non-dictatorial.
2. The domain is $\mathcal{P}$: all permutations of the alternatives are feasible. Intuitively, every voter has many options to misreport. If the domain was limited, then GS may not hold.
3. Indifference in preferences: in general, GS theorem does not hold. In the proof, we use some specific constructions. If they are possible, then GS theorem holds.
4. Cardinalization: GS theorem will hold as long as all possible ordinal ranks are feasible in the cardinal preferences.

For the proof, we will follow a direct approach (Sen 2001)

First prove for $n=2$ and then apply induction on the number of agents.

Lemma: Suppose $|A| \geq 3$, $N = \{1,2\}$, f is ONTO and SP, then for every preference profile P, $f(P) \in \{P_1(1), P_2(1)\}$.

Proof: If $P_1(1) = P_2(1)$, then unanimity implies $f(P) = P_1(1)$ (Corollary above)

Say $P_1(1) = a \neq b = P_2(1)$. For contradiction assume $f(P) = c \neq a, b$ (need 3 alt)

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
⋮	⋮	⋮	a	b	a	b	⋮
			⋮	⋮	⋮	⋮	

outcome here is c

Now $f(P_1 P_2') \in \{a, b\}$ [because all other alternatives except b is Pareto dominated by a]

But if $f(P_1 P_2') = b$, then player 2 manipulates from P_2 to P_2'. Hence, $f(P_1 P_2') = a$.

By a similar argument, $f(P_1' P_2) = b$

But now MONO will lead to a contradiction

$P_1' P_2 \to P_1' P_2'$, outcome should be b

$P_1 P_2' \to P_1' P_2'$, outcome should be a $\square$

Lemma: Suppose $|A| \geq 3$, $N = \{1,2\}$, f is ONTO and SP.

Let P: $P_1(1) = a \neq b = P_2(1)$, P': $P_1'(1) = c$, $P_2'(1) = d$.

If $f(P) = a$, then $f(P') = c$

If $f(P) = b$, then $f(P') = d$.

This proves dictatorship for two players.

Proof: If $c = d$, unanimity proved the lemma. Hence consider $c \neq d$.

Cases	c	d
1	a	b
2	$\neq a, b$	b
3	$\neq a, b$	$\neq b$
4	a	$\neq a, b$
5	b	$\neq a, b$
6	b	a

These cases are exhaustive

Enough to consider the case

If $f(P) = a \Rightarrow f(P') = c$

The other case is symmetric

Case 1: $c = a$, $d = b$,

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	a	b	a	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	b	a
				$\vdots$	$\vdots$

We know (by previous lemma)

$f(P') \in \{a, b\}$

say for contradiction $f(P') = b$

$P_1 P_2 \xrightarrow{\text{MONO}} \hat{P}_1 \hat{P}_2$

$a \qquad\qquad a$

$P_1' P_2' \xrightarrow{\text{MONO}} \hat{P}_1 \hat{P}_2$

$b \qquad\qquad b$

Case 2: $c \neq a, b$, $d = b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	P_2
a	b	c	b	c	b
⋮	⋮	⋮	⋮	a	⋮
				⋮	

$f(P') \in \{c, b\}$

assume $f(P') = b$ (for contradiction)

$P_1' \; P_2' \rightarrow \hat{P}_1 \; P_2$ (apply case 1)

b b

agent 1 misreports $\hat{P}_1 \rightarrow P_1$ as a $\hat{P}_1$ b.

Case 3: $c \neq a, b$, $d \neq b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	c	d	c	b
⋮	⋮	⋮	⋮	⋮	⋮

Say $f(P') = d$

$P' \rightarrow \hat{P}$ $\quad f(\hat{P}) = b$ (case 2)

$P \rightarrow \hat{P}$ $\quad f(\hat{P}) = c$ (case 2)

Case 4: $c = a$, $d \neq b, a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	c=a	d	a	b
⋮	⋮	⋮	⋮	⋮	⋮

Say $f(P') = d$

$P' \rightarrow \hat{P}$ $\quad f(\hat{P}) = b$ (case 2)

$P \rightarrow \hat{P}$ $\quad f(\hat{P}) = a$ (case 1)

Case 5: $c = b$, $d \neq b, a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	c=b	d	c	d
⋮	⋮	⋮	⋮	⋮	⋮

Say $f(P') = d$

$P' \rightarrow \hat{P}$ $\quad f(\hat{P}) = d$ (case 4)

$P \rightarrow \hat{P}$ $\quad f(\hat{P}) = a$ (case 4)

Case 6: $c = b$, $d = a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	P_2'	$\tilde{P}_1$	P_2'
a	b	$c=b$	$d=a$	b	a	x	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	x	$\vdots$		
				$\vdots$			

$f(P') = a$

$x \neq a, b$

$P' \to (\hat{P}_1\ P_2')$, $f(\hat{P}_1\ P_2') = a$ (case 1)

$P \to (\tilde{P}_1\ P_2')$, $f(\tilde{P}_1\ P_2') = x$ (case 3)

Player 1 manipulates from $\hat{P}_1\ P_1' \to \tilde{P}_1\ P_2'$

since $x\ \hat{P}_1\ a$

More than 2 agents → induction on the number of agents. See Sen (2001): "A direct proof of GS theorem".

GS Theorem holds for unrestricted preferences

$$f : \mathcal{P}^n \longrightarrow A$$

all preferences admissible

One reason for a restrictive result like GS theorem is that the domain of the SCF is large – a potential manipulator has many options to manipulate.

Strategyproofness (defined alternatively)

$$f(P_i, \underline{P}_{-i}) \; P_i \; f(P_i', \underline{P}_{-i}) \;, \forall P_i, P_i' \in \mathcal{P} \;, \forall i \in N$$

$$\text{OR} \quad f(P_i, \underline{P}_{-i}) = f(P_i', \underline{P}_{-i}). \qquad \forall \underline{P}_{-i} \in \mathcal{P}^{n-1}$$

If we reduce the set of feasible preferences from $\mathcal{P}$ to $\mathcal{S} \subset \mathcal{P}$ The SCF f strategyproof on $\mathcal{P}$ continues to be strategyproof over $\mathcal{S}$, but there can potentially be more f's that can be strategyproof, i.e., satisfy the condition above.

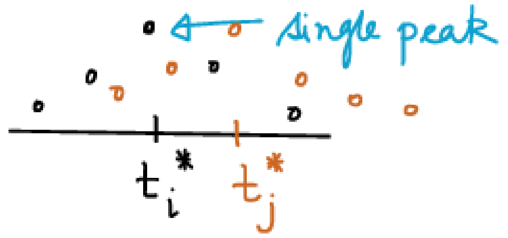
Domain restrictions

1. Single peaked preferences
2. Divisible goods allocation
3. Quasi-linear preferences

Each of these domains have interesting non-dictatorial SCFs that are strategyproof.

Single peaked preferences

Ex. temperature of a room – for every agent, most comfortable temperature t_i^* – anything above or below are monotonically less preferred.



One common order over the alternatives

Agent preferences are single peaked wrt that common order

Other examples:

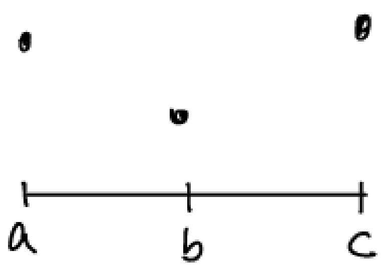
1. Facility location: School / Hospital / Post office
2. Political ideology: Left, Center, Right

The natural ordering (or common ordering) of the alternatives is denoted via $<$ [as in real numbers]

in general, any relation over the alternatives that is transitive and antisymmetric. In this course, we will assume
1. alternatives live on a real line
2. consider only one-dimensional single-peakedness

How is it a domain restriction?

Consider $a < b < c$



all possible preferences

a	b	b	c	a	c
b	a	c	b	c	a
c	c	a	a	b	b

Defn. A preference ordering P_i (linear over A) of agent i is single-peaked wrt the common order $<$ of the alternatives if
1. $\forall\, b, c \in A$ with $b < c \leq P_i(1)$, $c \, P_i \, b$
2. $\forall\, b, c \in A$ with $P_i(1) \leq b < c$, $b \, P_i \, c$.

Let $\mathcal{S}$ be the set of single peaked preferences

The SCF : $f : \mathcal{S}^n \to A$.

How does it circumvent GS theorem?

Each player's preference has a peak. Suppose, f picks the leftmost peak. For the agent having the leftmost peak, no reason to misreport. For any other agent, the only way she can change the outcome is by reporting her peak to be left of the leftmost – but that is strictly worse than the current outcome.

Repeat this argument for any fixed k-th peak from left. Even the rightmost peak choosing SCF is also strategyproof, so is median $\left(k = \left\lfloor \frac{n}{2} \right\rfloor\right)$

Median voter SCF:

An SCF $f : \mathcal{S}^n \to A$ is a median voter SCF if there exists $B = \{y_1, y_2, \ldots, y_{n-1}\}$ s.t. $f(P) = \text{median}(B, \text{peaks}(P))$ for all preference profiles $P \in \mathcal{S}$. [median wrt $<$]

The points in B are called the peaks of "phantom voters".

Note: B is fixed for f and does not change with P.

Why phantom voters?

$f^{\text{leftmost}} \equiv (B_{\text{Left}}, \text{peaks}(P))$; $B_{\text{Left}} = \{y_L, \ldots, y_L\}$

if all phantom peaks are on the left, it corresponds to leftmost peak SCF. Similarly, $f^{\text{rightmost}}(\cdot)$ can be found in a similar way.

phantom voters give a complete description of the SCFs.

Theorem (Moulin 1980): Every median voter SCF is strategyproof.

Proof sketch: argue that if $f(P) = a$ and a player has a peak $P_i(1)$ to the left of a, it has no benefit by misreporting the peak to be on the right of a, which is the only way of changing the outcome of f. Similar for $P_i(1)$ on the right of a.

Note: mean does not have this property.

Claim: Let $p_{\min}$ and $p_{\max}$ are the leftmost and rightmost peaks of P according to $<$, then f is PE iff $f(P) \in [p_{\min}, p_{\max}]$

Proof: $\Rightarrow$ Suppose $f(P) \notin [p_{min}, p_{max}]$. WLOG, $f(P) < p_{min}$.
Then every agent prefers p_{min} over $f(P)$, i.e., $f(P)$ is dominated.
Hence $f(P)$ is not PE.

$\Leftarrow$ If $f(P) \in [p_{min}, p_{max}]$, then the condition $b \, P_i \, f(P) \; \forall i \in N$ never occurs. In other words, there does not exist an alternative b that Pareto dominates $f(P)$. Hence $f(P)$ is PE.

Consider monotonicity (MONO). The results similar to unrestricted preferences hold here too, but the proofs differ since we cannot construct preferences as freely as before.

Thm: f is SP $\Rightarrow$ f is MONO.

This proof is similar to the previous one. To prove the reverse implication one needs to argue why the construction is valid in the single peaked domain. (Or provide counterexample)

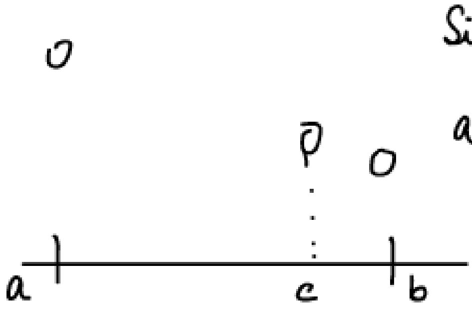
Thm: Let $f: \mathcal{S}^n \to A$ is a SP SCF. Then,

$$f \text{ is ONTO} \Leftrightarrow f \text{ is UN} \Leftrightarrow f \text{ is PE}$$

Proof: We know PE $\Rightarrow$ UN $\Rightarrow$ ONTO. To prove the above implication, we need to show that ONTO $\Rightarrow$ PE when f is SP.

Suppose, for contradiction, f is SP and ONTO, but not PE.

Then $\exists \, a, b \in A$ s.t. $a \, P_i \, b \; \forall i \in N$ but $f(P) = b$.



Since preferences are single peaked, $\exists$ another alternative $c \in A$, which is a neighbor of b s.t. $c \, P_i \, b \; \forall i \in N$. c can be a itself.

ONTO $\Rightarrow \exists\ P'$ s.t. $f(P') = c$.

Construct P'' s.t. $P_i''(1) = c$, $P_i''(2) = b$, $\forall i \in N$.

$P \to P''$, MONO $\Rightarrow f(P'') = b$, $P' \to P''$ MONO $\Rightarrow f(P'') = c$.

We are interested in non-dictatorial SCFs.

Anonymity: (outcome insensitive to agent identities)

Permutation of agents $\sigma : N \to N$.

We apply a permutation σ to a profile P to construct another profile as: The preference of i goes to agent $\sigma(i)$ in the new profile. Denote this new profile as P^σ.

Example: $N = \{1, 2, 3\}$, σ: $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$

P_1	P_2	P_3
a	b	b
b	a	c
c	c	a

P_1^σ	P_2^σ	P_3^σ
b	a	b
c	b	a
a	c	c

The social outcome should not alter due to agent renaming.

Defn: An SCF $f : \mathcal{P}^n \to A$ is anonymous (ANON) if for every profile P and for every permutation of the agents σ,

$$f(P^\sigma) = f(P).$$

Example of a non-anonymous SCF?

Equivalence of SP, ONTO, ANON and median voting rule in single peaked domain

Theorem (Moulin 1980)

A strategyproof SCF f is onto and anonymous iff it is a median voter SCF.

Proof: $\Leftarrow$ median voter SCF is SP (previous theorem).

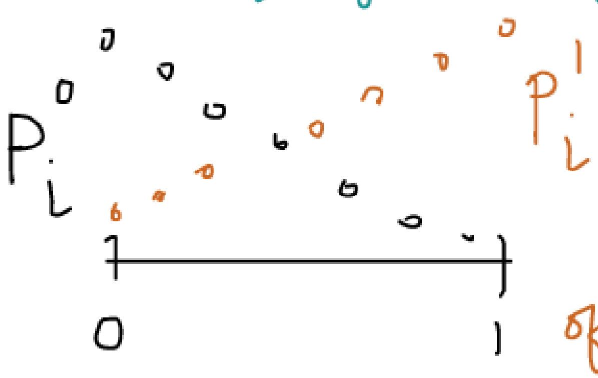
It is anonymous, if we permute the agents with peaks unchanged the outcome doesnot change.

It is onto, pick any arbitrary alternative a, put peaks of all players at a. The outcome will be a irrespective of the positions of the phantom peaks – since there are $(n-1)$ phantom peaks and n agent peaks.

$\Rightarrow$ Given, $f : \mathcal{S}^n \to A$ is SP, ANON, and ONTO.

define, P_i^0 : agent i's preference with peak at leftmost wrt $<$

P_i^1 : agent i's preference with peak at rightmost wrt $<$



The proof is constructive, we will construct the median voting rule (which needs the phantom peaks s.t. the outcome of an arbitrary f matches the outcome of the median SCF.

First, construct phantom peaks

$$y_j = f(\underbrace{P_1^0, P_2^0, \ldots, P_{n-j}^0}_{(n-j)\text{ peaks leftmost}}, \underbrace{P_{n-j+1}^1, \ldots, P_n^1}_{j\text{ peaks rightmost}}), \quad j=1,\ldots,n-1$$

Which agents have which peaks does not matter because of anonymity.

Claim: $y_j \leq y_{j+1}$, $j=1,\ldots,n-2$, i.e., peaks are non-decreasing.

Proof: $y_{j+1} = f(P_1^0, P_2^0, \ldots, P_{n-j}^1, P_{n-j+1}^1, \ldots, P_n^1)$

Due to SP, $y_j \, P_{n-j}^0 \, y_{j+1}$ — or they are same, but P_{n-j}^0 is single peaked with peak at 0, hence $y_j \leq y_{j+1}$. $\square$

Consider an arbitrary profile, $P = (P_1, P_2, \ldots, P_n)$, $P_i(1) = p_i$ (the peaks).

Claim: Suppose f satisfies SP, ONTO, ANON, then

$$f(P) = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1}).$$

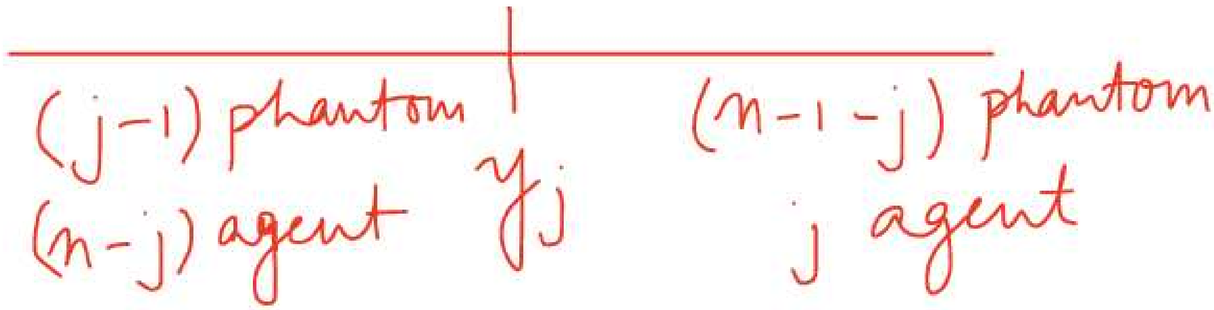
WLOG, can assume $p_1 \leq p_2 \leq \cdots \leq p_n$ due to ANON.

also say, $a = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1})$

Case 1: a is a phantom peak

Say $a = y_j$, for some $j \in \{1, 2, \cdots, n-1\}$.

this is a median of $2n-1$ points, of which $(j-1)$ phantom peaks lie on the left (see the claim before). Rest $(n-j)$ points are agent peaks.



Hence, $p_1 \leqslant \cdots \leqslant p_{n-j} \leqslant y_j = a \leqslant p_{n-j+1} \leqslant \cdots \leqslant p_n$.

Use a similar transformation as we used earlier

$$f(P_1^0, P_2^0, \cdots, P_{n-j}^0, P_{n-j+1}^1, \cdots, P_n^1) = y_j \quad \text{(definition)}$$

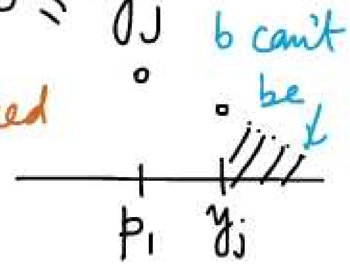
$$f(P_1, P_2^0, \cdots, P_{n-j}^0, P_{n-j+1}^1, \cdots, P_n^1) = b \quad \text{(say)}$$

By SP, $y_j \, P_1^0 \, b \Rightarrow y_j \leqslant b$

again by SP, $b \, P_1 \, y_j$, but $p_1 \leqslant y_j \Rightarrow b \leqslant y_j$

single peaked

hence $b = y_j$



repeat this argument for first $(n-j)$ agents to get

$$f(P_1, P_2, \ldots, P_{n-j}, P'_{n-j+1}, \ldots, P'_n) = y_j$$

now consider

$$f(P_1, P_2, \ldots, P_{n-j}, P'_{n-j+1}, \ldots, P_n) = b \text{ (say)}$$

apply very similar argument

$$\left.\begin{array}{l} y_j \, P'_n \, b \Rightarrow b \leq y_j \\ \\ b \, P_n \, y_j \text{ and } y_j \leq P_n \Rightarrow y_j \leq b \end{array}\right\} \quad b = y_j$$

Hence $f(P_1 \ldots, P_n) = y_j$.

Claim: Suppose f satisfies SP, ONTO, ANON, then

$$f(P) = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1}).$$

Case 1: a is a phantom peak - proved

Case 2: a is an agent peak

We will prove this for 2 players. The general case repeats this argument.

Claim: $N = \{1,2\}$, let P and P' be such that

$P_i(1) = P_i'(1)$, $\forall i \in N$. Then $f(P) = f(P')$.

Proof: Let $a = P_1(1) = P_1'(1)$, and $P_2(1) = P_2'(1) = b$.

$f(P) = x$ and $f(P_1', P_2) = y$

Since f is SP, $x P_1 y$ and $y P_1' x$

Since peaks of P_1 and P_1' are the same, if x, y are on the same side of the peak, they must be the same, as the domain is single peaked.

The only other possibility is that x and y fall on different sides of the peak. We show that this is impossible.

WLOG $x < a < y$ and $a < b$

f is SP+ONTO $\Leftrightarrow$ f is SP+PE

PE requires $f(P) \in [a, b]$, but $f(P) = x < a$ $\rightarrow\leftarrow$

now repeat this argument for $(P_1', P_2) \rightarrow (P_1', P_2')$ $\square$

Profile : $(P_1, P_2) = P$, $P_1(1) = a$, $P_2(1) = b$

y_1 is the phantom peak.

by assumption, median (a, b, y_1) is an agent peak

WLOG assume the median is a.

Assume for contradiction $f(P) = c \neq a$.

By PE, c must be within a and b. We have two cases to consider : $b < a < y_1$ and $y_1 < a < b$.

Case 2.1: $b < a < y_1$, by PE $c < a$

construct P_1' s.t. $P_1'(1) = a = P_1(1)$

and $y_1 P_1' c$ (possible since they are on different sides of a)

by the earlier claim, $f(P) = c \Rightarrow f(P_1', P_2) = c$.

now consider the profile (P_1', P_2)

peak at the rightmost

$P_2(1) = b < y_1 \leq P_1'(1)$, hence the median of $\{b, y_1, P_1'(1)\}$ is y_1 (which is a phantom peak, hence case 1 applies).

$$f(P_1', P_2) = y_1.$$

But $y_1 P_1' c$ (by construction) and $f(P_1', P_2) = c$

agent 1 manipulates $P_1' \to P_1'$, contradiction to f being SP.

Case 2.2: $y_1 < a < b$, PE $\Rightarrow a < c$

construct P_1' s.t. $P_1'(1) = a = P_1(1)$ and $y_1 P_1' c$

$f(P_1', P_2) = c$ (by claim)

Consider (P_1^0, P_2), $P_1^0(1) \leq y_1 < b \Rightarrow f(P_1^0, P_2) = y_1$

but $y_1 P_1' c$, hence manipulable by agent 1.

This completes the proof for two agents (case 2). For the generalization to n players, see Moulin (1980) "On strategyproofness and single peakedness".

Task allocation domain (related but different than single-peaked)

Unit amount of task to be shared among n agents

Agent i gets a share $s_i \in [0,1]$ of the job, $\sum_{i \in N} s_i = 1$.

Agent payoff: every agent has a most preferred share of work.

Example: The task has rewards – wages per unit time $= w$

if agent i works for t_i time then gets $w t_i$

the task also has costs, e.g., physical tiredness / less free time etc.

let the cost is quadratic $= c_i t_i^2$

net payoff $= w t_i - k_i t_i^2 \Rightarrow$ maximized at $t_i^* = \frac{w}{2c_i}$

and monotone decreasing on both sides.

This is single-peaked over the share of the task and not over the alternatives. Suppose, two alternatives are $(0.2, 0.4, 0.4)$ and $(0.2, 0.6, 0.2)$ – player 1 likes both of them equally.

There can't be a single common order over the alternatives s.t. the preferences are single-peaked for all.

Denote this domain of task allocation with T (single peaked over task share)

SCF: $f : T^n \to A$,

Let $P \in T^n$, $f(P) = (f_1(P), f_2(P), \cdots, f_n(P))$

$$f_i(P) \in [0,1], \forall i \in N; \quad \sum_{i \in N} f_i(P) = 1$$

Player i has a peak p_i over the share of task.

Pareto Efficiency: An SCF f is PE if there does not exist another share of task that is weakly preferred by all agents and strictly preferred by at least one, i.e.,

$$\nexists a \in A \text{ s.t. } a R_i f(P), \forall i \in N \text{ and } \exists j \text{ s.t. } a P_j f(P)$$

Implications:

① $\sum_{i \in N} p_i = 1$, allocate tasks according to the peaks of the agents. This is the unique PE.

② $\sum_{i \in N} p_i > 1$, $\exists k \in N$ s.t. $f_k(P) < p_k$.

Q: Can there be an agent j s.t. $f_j(P) > p_j$ if f is PE?

If so, increasing k's share of task and reducing j's makes both players strictly better off. Therefore

$$\forall j \in N, \; f_j(P) \leq p_j.$$

③ If $\sum_{i \in N} p_i < 1$, similarly $\forall j \in N, f_j(P) \geq p_j$.

Anonymity: (if agent preferences are permuted, the shares will also get permuted accordingly.

$$f_{\sigma(j)}(P^{\sigma}) = f_j(P)$$

$N = \{1,2,3\}$, $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$

$P = (0.7, 0.4, 0.3) \Rightarrow P^{\sigma} = (0.3, 0.7, 0.4)$

$f_1(0.7, 0.4, 0.3) = f_2(0.3, 0.7, 0.4)$

Candidate SCFs:

<u>Serial dictatorship</u>: A predetermined sequence of the agents is fixed. Each agent is given either his peak share or a leftover share. If $\sum p_i < 1$, then the last agent is given the leftover share.

Properties: PE, SP, but not ANON. Also quite unfair for the last agent.

<u>Proportional</u>: Every player is assigned a share that is c times their peaks, s.t. $c \sum_{i \in N} p_i = 1$

overload if $\sum p_i < 1$, underload if $\sum p_i > 1$.

Q: Is it ANON, PE, SP?

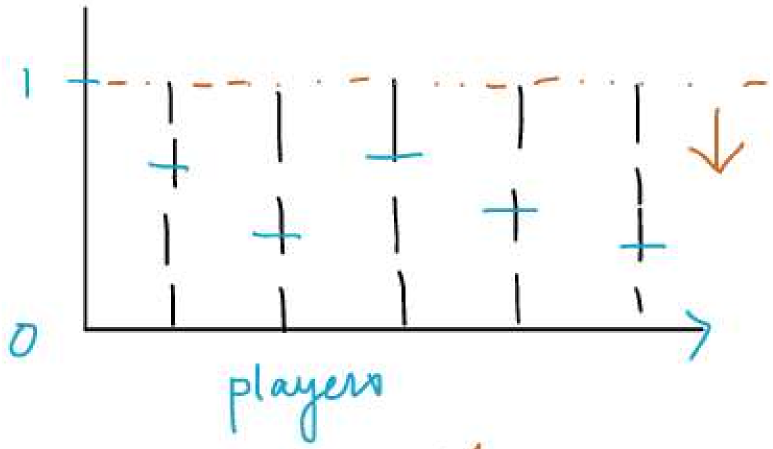
Suppose peaks are 0.2, 0.3, 0.1 for 3 players, $c = \frac{1}{0.6}$

player 1 gets $\frac{1}{3}$ (more than 0.2)

if the report is 0.1, 0.3, 0.1, $c = \frac{1}{0.5}$, player 1 gets 0.2.

How to ensure PE, ANON, and SP in task allocation domain?

Uniform rule (Sprumont 1991)



Suppose, $\sum_{i \in N} p_i < 1$

Begin with everyone's allocation being 1. Keep reducing until $\sum f_i = 1$

Whenever some agent's peak is reached, set the allocation for that agent to be its peak

Definition:

① $f_i^u(P) = p_i$ if $\sum_{i \in N} p_i = 1$

② $f_i^u(P) = \max\{p_i, \mu(P)\}$ if $\sum_{i \in N} p_i < 1$

where $\mu(P)$ solves $\sum_{i \in N} \max\{p_i, \mu\} = 1$.

③ $f_i^u(P) = \min\{p_i, \lambda(P)\}$ if $\sum_{i \in N} p_i > 1$

where $\lambda(P)$ solves $\sum_{i \in N} \min\{p_i, \lambda\} = 1$.

Q: Is this ANON, PE, and SP?

Theorem (Sprumont 1991)

The uniform rule SCF is ANON, PE, and SP.

Proof: ANON is obvious - only the peaks matter and not their owners.

PE: the allocation is s.t.

$$f_i^u(P) = p_i, \forall i \in N, \text{ if } \sum p_i = 1$$

$$f_i^u(P) \geq p_i, \forall i \in N, \text{ if } \sum p_i < 1$$

$$f_i^u(P) \leq p_i, \forall i \in N, \text{ if } \sum p_i > 1$$

for some players the peaks are allocated, and for others the allocation is the same. This is PE, since any other allocation can only improve the allocation of a player at the cost of another player's allocation.

Strategyproofness.

for case $\sum p_i = 1$, every agent gets their peak - no reason to deviate.

Case $\sum p_i < 1$, then $f_i^u(P) \geq p_i \ \forall i \in N$.

only possible manipulation for agents that have $f_i^u(P) > p_i$

$\Rightarrow \mu(P) > p_i$, i.e., the allocation stopped before reaching p_i. The only way i can change the allocation is by reporting $p_i' > \mu(P) > p_i$ — but this is a worse allocation for i than $\mu(P)$.

Similar argument for case $\sum p_i > 1$. This completes the proof.

The converse is also true. We skip the proof.

<u>Thm</u>: An SCF is SP, PE, and ANON iff it is the uniform rule.

<u>Ref</u>: Sprumont (1991) : Division problem with single peaked preferences.

Mechanism Design with Transfers

Social Choice Function $F : \Theta \rightarrow X$

X : space of all outcomes

In this domain, an outcome x has two components allocation a and payment vector $\pi = (\pi_1, \dots, \pi_n)$, $\pi_i \in \mathbb{R}$

Examples of allocations

(1) A public decision of building a bridge, park, or museum.

$a \in A = \{\text{park}, \text{bridge}, \cdots\}$

(2) Allocation of a divisible good, e.g., a shared spectrum

$a = (a_1, a_2, \dots, a_n)$, $a_i \in [0,1]$, $\sum_{i \in N} a_i = 1$

a_i : fraction of the resource i gets.

(3) Single indivisible object allocation

$a = (a_1, \dots, a_n)$, $a_i \in \{0,1\}$, $\sum_{i \in N} a_i \leqslant 1$

(4) Partition of indivisible objects.

S = set of objects

$A = \{(A_1, \dots, A_n) : A_i \subseteq S \ \forall i \in N, A_i \cap A_j = \phi \ \forall i \neq j\}$

Type of an agent i is $\theta_i \in \Theta_i$, this is a private information of i.

Agents' benefit from an allocation is defined via the valuation function

Valuation depends on the allocation and type of the player

$v_i : A \times \Theta_i \rightarrow \mathbb{R}$ [independent private values]

E.g., if i has a type "environment saver" θ_i^{env}

and $a \in \{$ Bridge, Park $\}$, $v_i(B, \theta_i^{env}) < v_i(P, \theta_i^{env})$

the value can change if the type changes to "business friendly" θ_i^{bus}

$$v_i(B, \theta_i^{bus}) > v_i(P, \theta_i^{bus})$$

Payments $\pi_i \in \mathbb{R}$, $\forall i \in N$

Payment vector $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$

Utility of player i, when its type is θ_i and the outcome is $(a, \underline{\pi})$ is given by $u_i((a, \underline{\pi}), \theta_i) = v_i(a, \theta_i) - \pi_i$

possibly non-linear

linear in payment

Quasi-linear payoffs

Q: Why is this a domain restriction?

Consider two alternatives (a, π) and (a, π')

Suppose $\pi_i' < \pi_i$, there cannot be any preference profile in the quasi-linear domain where (a, π) is more preferred than (a, π') for agent i.

The utilities are $v_i(a, \theta_i) - \pi_i' > v_i(a, \theta_i) - \pi_i$

In the complete domain both orders would have been feasible.

This simple restriction opens up the opportunity for a lot of SCFs to satisfy interesting properties

Quasi linear preferences

The SCF is decomposed into two components

Allocation rule component

$$f : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \to A$$

When the types are θ_i, $i \in N$, $f(\theta_1, \ldots, \theta_n) = a \in A$

Payment function

$$p_i : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \to \mathbb{R}, \ \forall i \in N$$

When the types are θ_i, $i \in N$, $p_i(\theta_1, \ldots, \theta_n) = \pi_i \in \mathbb{R}$

Examples of allocation rules

① Constant rule, $f^c(\theta) = a \quad \forall \theta \in \Theta$

② Dictatorial rule, $f^D(\theta) \in \underset{a \in A}{\arg\max}\ v_d(a, \theta_d)$ for some $d \in N$ $\forall \theta \in \Theta$.

③ Allocatively efficient rule/ utilitarian rule

$$f^{AE}(\theta) \in \underset{a \in A}{\arg\max} \sum_{i \in N} v_i(a, \theta_i)$$

Note: this is different from Pareto efficiency (PE is a property defined for the outcome which also considers the payment)

④ Affine maximizer rule:

$$f^{AM}(\theta) \in \underset{a \in A}{\arg\max} \left(\sum_{i \in N} \lambda_i v_i(a, \theta_i) + K(a) \right), \ \lambda_i \geq 0, \text{ not all zero.}$$

⑤ Max-min / egalitarian

$$f^{MM}(\theta) \in \underset{a \in A}{\arg\max} \ \min_{i \in N} \ v_i(a, \theta_i)$$

Examples of payment rules

① No deficit: $\sum_{i \in N} p_i(\theta) \geq 0 \quad \forall \theta \in \Theta$.

② No subsidy: $p_i(\theta) \geq 0, \ \forall \theta \in \Theta, \ \forall i \in N$.

③ Budget balanced: $\sum_{i \in N} p_i(\theta) = 0, \quad \forall \theta \in \Theta$.

Recall: Incentive Compatibility

A mechanism is the tuple of the allocation and payment rule $(f, \underline{p})$

A mechanism $(f, \underline{p})$ is dominant strategy incentive compatible (DSIC) if $\forall i \in N$

$$v_i(f(\theta_i, \tilde{\underline{\theta}}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\underline{\theta}}_{-i}) \geq v_i(f(\theta_i', \tilde{\underline{\theta}}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\underline{\theta}}_{-i}),$$

$$\forall \underline{\theta}_{-i} \in \underline{\Theta}_{-i}, \ \forall \theta_i, \theta_i' \in \Theta_i.$$

DSIC means truthtelling is a weakly DSE.

We say that the payment rule $\underline{p}$ implements f in dominant strategies OR f is implementable in dominant strategies (by a payment rule)

In QL domain, we are often more interested in the allocation rule than the whole SCF (includes payment).

What needs to be satisfied for a DSIC mechanism $(f, \underline{p})$?

$N = \{1,2\}$, $\Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$, $f : \Theta_1 \times \Theta_2 \to A$

the following conditions must hold

$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geqslant v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2), \ \forall \theta_2$$

$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geqslant v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2), \ \forall \theta_2$$

for player 2 :

$$v_2(f(\theta_1, \theta^H), \theta^H) - p_2(\theta_1, \theta^H) \geqslant v_2(f(\theta_1, \theta^L), \theta^H) - p_2(\theta_1, \theta^L), \ \forall \theta_1$$

$$v_2(f(\theta_1, \theta^L), \theta^L) - p_2(\theta_1, \theta^L) \geqslant v_2(f(\theta_1, \theta^H), \theta^L) - p_2(\theta_1, \theta^H), \ \forall \theta_1$$

Properties of the payment that implements an allocation rule

① Say $(f, \underline{p})$ is incentive compatible. Consider another payment

$$q_i(\theta_i, \underline{\theta}_{-i}) = p_i(\theta_i, \underline{\theta}_{-i}) + h_i(\underline{\theta}_{-i}) \quad \forall \theta, \forall i \in N.$$

Q: is $(f, \underline{q})$ DSIC ?

A: Yes.

$$v_i(f(\theta_i, \underline{\tilde{\theta}}_{-i}), \theta_i) - p_i(\theta_i, \underline{\tilde{\theta}}_{-i}) - h_i(\underline{\tilde{\theta}}_{-i})$$
$$\geqslant v_i(f(\theta_i', \underline{\tilde{\theta}}_{-i}), \theta_i) - p_i(\theta_i', \underline{\tilde{\theta}}_{-i}) - h_i(\underline{\tilde{\theta}}_{-i})$$
$$\forall \theta_i, \theta_i', \underline{\tilde{\theta}}_{-i}, \ \forall i \in N.$$

if we can find a payment that implements an allocation rule, there exists uncountably many payments that can implement it.

The converse question: when do the payments that implement f differ only by a factor $h_i(\underline{\theta}_{-i})$?

② Implication of incentive compatibility on payment

suppose the allocation is same in two type profiles θ and $\tilde{\theta} = (\tilde{\theta}_i, \underline{\theta}_{-i})$

$f(\theta) = f(\tilde{\theta}) = a$, then if $\underline{p}$ implements f, then

$$p_i(\theta) = p_i(\tilde{\theta}) .$$ [exercise]

Pareto optimality in Quasi-linear domain

Defn: A mechanism $(f, (p_1, \dots, p_n))$ is Pareto Optimal if at every type profile $\theta \in \Theta$, there does not exist an allocation $b \neq f(\theta)$ and payments $(\pi_1, \dots, \pi_n)$ with $\sum_{i \in N} \pi_i \geq \sum_{i \in N} p_i(\theta)$ s.t.

$$v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - p_i(\theta), \quad \forall i \in N.$$

with the inequality being strict for some $i \in N$.

Pareto optimality is meaningless if there is no restriction on the payment. One can always put excessive subsidy to every agent to make everyone better off. So, the condition requires to spend at least the same budget.

Theorem: A mechanism $(f, (p_1, \dots, p_n))$ is Pareto optimal iff it is allocatively efficient.

Proof: $(\Rightarrow)$ we'll prove $!AE \Rightarrow\, !PO$

$$!AE \Rightarrow \exists\, b \neq f(\theta) \text{ s.t. } \sum_{i \in N} v_i(b, \theta_i) > \sum_{i \in N} v_i(f(\theta), \theta_i)$$

for some θ

$$\text{let } \delta = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > 0.$$

Consider payment $\pi_i = v_i(b, \theta_i) - v_i(f(\theta), \theta_i) + p_i(\theta) - \delta/n$

$$\text{hence, } [v_i(b, \theta_i) - \pi_i] - [v_i(f(\theta), \theta_i) - p_i(\theta)] = \delta/n > 0 \;\; \forall i \in N$$

also $\sum_{i \in N} \pi_i = \sum_{i \in N} p_i(\theta)$. Hence f is not PO.

$\Leftarrow$ again we prove $!PO \Rightarrow !AE$

$!PO$, $\exists\, b, \underline{\pi}$ s.t. $\sum_{i\in N} \pi_i \geqslant \sum_{i\in N} p_i(\theta)$

$$v_i(b,\theta_i) - \pi_i \geqslant v_i(f(\theta),\theta_i) - p_i(\theta) \quad \forall i \in N$$

strict for some $j \in N$.

summing over the second inequality,

$$\sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N}\pi_i > \sum_{i\in N} v_i(f(\theta),\theta_i) - \sum_{i\in N} p_i(\theta)$$

$$\Rightarrow \sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i) > \sum_{i\in N}\pi_i - \sum_{i\in N} p_i(\theta) \geqslant 0$$

$\Rightarrow$ f is $!AE$.

Allocative efficient rule is implementable

$$f^{eff}(\theta) \in \operatorname*{argmax}_{a\in A} \sum_{i\in N} v_i(a,\theta_i)$$

Consider the following payment:

$$p_i^G(\theta_i,\underline{\theta}_{-i}) = h_i(\underline{\theta}_{-i}) - \sum_{j\neq i} v_j(f^{eff}(\theta_i,\underline{\theta}_{-i}),\theta_j).$$

[Groves payment]

where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function.

<u>Example</u>: Single indivisible item allocation. $N=\{1,2,3,4\}$

$\theta_1 = 10$, $\theta_2 = 8$, $\theta_3 = 6$, $\theta_4 = 4$, when they get the object, zero

Otherwise. Let $h_i(\underline{\theta}_{-i}) = \min \underline{\theta}_{-i}$

if everyone reports their true type, The values of h_i are

$h_1 = 4, \; h_2 = 4, \; h_3 = 4, \; h_4 = 6$

The efficient allocation gives the item to agent 1.

$p_1 = 4 - 0 = 4, \; p_2 = 4 - 10 = -6, \; p_3 = 4 - 10 = -6$

$p_4 = 6 - 10 = -4$, i.e., only player 1 pays, others get paid.

Surprisingly, this is a truthful mechanism.

Theorem: Groves mechanisms are DSIC.

Proof: Consider player i. Let $f^{eff}(\theta_i, \underline{\tilde{\theta}}_{-i}) = a$, and

$f^{eff}(\theta_i', \underline{\tilde{\theta}}_{-i}) = b$

by definition, $v_i(a, \theta_i) + \sum_{j \neq i} v_j(a, \tilde{\theta}_j)$

$$\geqslant v_i(b, \theta_i) + \sum_{j \neq i} v_j(b, \tilde{\theta}_j) \quad \cdots\cdots (1)$$

utility of player i when he reports θ_i

$$v_i(f^{eff}(\theta_i, \underline{\tilde{\theta}}_{-i}), \theta_i) - p_i(\theta_i, \underline{\tilde{\theta}}_{-i})$$

$$= v_i(\underbrace{f^{eff}(\theta_i, \underline{\tilde{\theta}}_{-i})}_{a}, \theta_i) - h_i(\underline{\tilde{\theta}}_{-i}) + \sum_{j \neq i} v_j(\underbrace{f^{eff}(\theta_i, \underline{\tilde{\theta}}_{-i})}_{a}, \tilde{\theta}_j)$$

$$\geqslant v_i(\underbrace{f^{eff}(\theta_i', \underline{\tilde{\theta}}_{-i})}_{b}, \theta_i) - \underbrace{h_i(\underline{\tilde{\theta}}_{-i}) + \sum_{j \neq i} v_j(\underbrace{f^{eff}(\theta_i', \underline{\tilde{\theta}}_{-i})}_{b}, \tilde{\theta}_j)}_{= p_i(\theta_i', \underline{\tilde{\theta}}_{-i})}$$

$$= v_i\left(f^{\text{eff}}(\theta_i', \tilde{\underline{\theta}}_{-i}), \theta_i\right) - p_i(\theta_i', \tilde{\underline{\theta}}_{-i}) .$$

Since player i was arbitrary, this holds for all $i \in N$.

Hence the claim.

The Vickrey-Clarke-Groves Mechanism (VCG)

The most popular mechanism in the Groves class

Also known as the pivotal mechanism (V'61, C'71, G'73)

Given by a unique $h_i(\underline{\theta}_{-i})$ function

$$h_i(\underline{\theta}_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

The payment is modified to

$$p_i^{VCG}(\theta_i, \underline{\theta}_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{eff}(\theta_i, \underline{\theta}_{-i}), \theta_j)$$

Note: $p_i^{VCG}(\theta) \geqslant 0 \quad \forall \theta \in \Theta, \forall i \in N$ [no subsidy ⇒ no deficit]

another interpretation of the payment:

sum value of others (in absence of i − in presence of i)

interpretation of the utility under VCG mechanism

$$v_i(f^{eff}(\theta_i, \underline{\theta}_{-i}), \theta_i) - p_i^{VCG}(\theta_i, \underline{\theta}_{-i})$$

$$= \underbrace{\sum_{j \in N} v_j(f^{eff}(\theta_i, \underline{\theta}_{-i}), \theta_j)}_{\text{max social welfare in presence of } i} - \underbrace{\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)}_{\text{max social welfare in absence of } i}$$

= marginal contribution of i in the social welfare

Examples:

① Single object allocation. Type = value for the object if allocated, the agent gets this value and zero otherwise.

$$p_i^{VCG}(\theta_i, \underline{\theta}_i) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{eff}(\theta_i, \underline{\theta}_i), \theta_j)$$

efficient allocation would give the object to the individual whose reported type is highest.

Consider 4 players, types: $\{10, 8, 9, 5\} \Rightarrow \{9, 0, 0, 0\}$

② What is pivotal in the VCG payment?

3 players having the following valuations

	Football	Library	Museum
A	0	70	50
B	95	10	50
C	10	50	50

VCG allocation : M (maximizes SW)

A pays = 105 − 100 = 5

B pays = 120 − 100 = 20

C pays = 100 − 100 = 0 ← non pivotal agent

The agent whose presence changes the outcome is charged money

they are the pivotal players.

③ Combinatorial allocation : sale of multiple objects

	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
θ_1	0	8	6	12
θ_2	0	9	4	14

value is the type itself

$v_i(a, \theta_i) = \theta_i(a)$

Efficient allocation : $\{1\} \to 2$ and $\{2\} \to 1$: call this a^*

$$p_1^{VCG}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq 1} \theta_j(a) - \sum_{j \neq 1} \theta_j(a^*)$$

$$= 14 - 9 = 5 \quad ; \text{ payoff} = 6 - 5 = 1$$

$$p_2^{VCG}(\theta_1, \theta_2) = 12 - 6 = 6 \quad ; \text{ payoff} = 9 - 6 = 3$$

VCG mechanism in combinatorial auctions

$M = \{1, \dots, m\}$: set of objects

$\Omega = 2^M := \{S : S \subseteq M\}$: set of bundles

$\theta_i : \Omega \to \mathbb{R}$: type/value of agent i

we assume $\theta_i(S) \geq 0 \;\; \forall S \in \Omega$, objects are "goods"

An allocation in this case is a partition of the objects

$a = \{a_0, a_1, a_2, \dots, a_n\}$, $a_i \in \Omega$, $a_i \cap a_j = \phi$ if $i \neq j$

$\bigcup_{i=0}^{n} a_i = M$. Let A be the set of all such allocations.

a_0 : set of unallocated objects.

Assume $\theta_i(\phi) = 0$

Also assume selfish valuations, i.e., $\theta_i(a) = \theta_i(a_i)$

agent i's valuation does NOT depend on the allocations to others.

Claim: In the allocation of goods, the VCG payment for agent, that gets no object in the efficient allocation, is zero.

Proof sketch: $a^* \in \underset{a \in A}{\text{argmax}} \sum_{j \in N} \theta_j(a)$, $a_i^* = \phi$

$$a_{-i}^* \in \underset{a \in A}{\text{argmax}} \sum_{j \in N \setminus \{i\}} \theta_j(a)$$

we know, $p_i^{VCG}(\theta) \geq 0$, also $p_i^{VCG}(\theta) = \sum_{j \neq i} \theta_j(a_{-i}^*) - \sum_{j \neq i} \theta_j(a^*)$

[add $\theta_i(a^*_{-i}) = 0$ and subtract $\theta_i(a^*) = \theta_i(a_i^*) = 0$]

$$= \sum_{j\in N} \theta_j(a^*_{-i}) - \sum_{j\in N} \theta_j(a^*) \leq 0$$ [a^* maximizes this sum by definition]

Hence $p_i^{VCG}(\theta) = 0$.

Defn: (Individual Rationality)

A mechanism $(f, \underline{p})$ is individually rational if

$$v_i(f(\theta), \theta_i) - p_i(\theta) \geq 0, \; \forall \theta \in \Theta, \; \forall i \in N.$$

Claim: In the allocation of goods, VCG mechanism is individually rational.

Proof sketch: $\theta_i(a^*) - p_i^{VCG}(\theta)$

$$= \theta_i(a^*) - \left(\sum_{j\neq i} \theta_j(a^*_{-i}) - \sum_{j\neq i} \theta_j(a^*)\right)$$

$$= \sum_{j\in N} \theta_j(a^*) - \sum_{j\neq i} \theta_j(a^*_{-i}) - \theta_i(a^*_{-i}) + \theta_i(a^*_{-i})$$

$$= \underbrace{\sum_{j\in N} \theta_j(a^*) - \sum_{j\in N} \theta_j(a^*_{-i})}_{\geq 0, \text{ by defn. of } a^*} + \underbrace{\theta_i(a^*_{-i})}_{\geq 0} \quad \geq 0$$

Application domain: Internet advertising

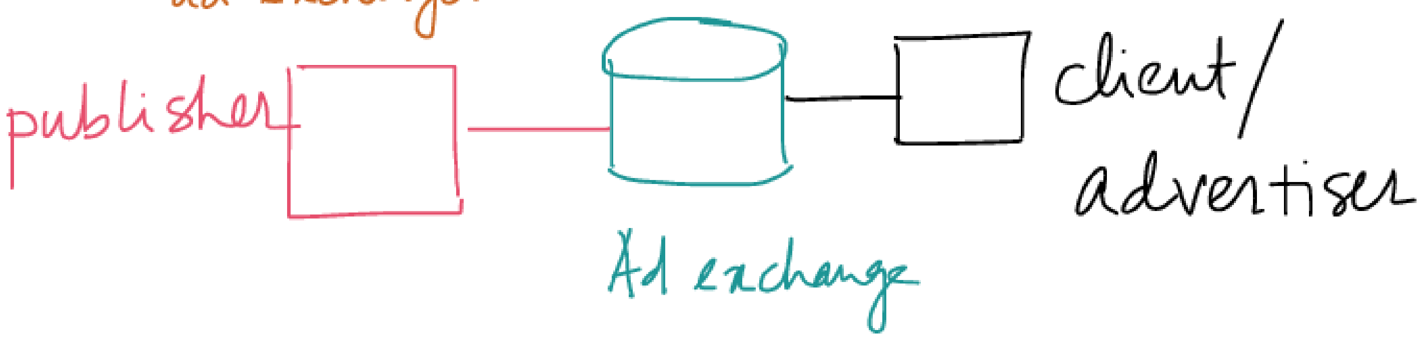
The success of internet advertising

1. User data: advertiser can gather a lot of data of the user to design targeted products.
2. Measurable actions: can classify buyers into categories and measure the interest and take appropriate actions
3. Low latency: real time bidding, automated bidding, decisions on the fly possible.

Types of ads on the internet

1. Sponsored search ad: advertisers bid on the keywords entered by the users during search.
2. Contextual ads: depending on the content of the page, post or email message
3. Display ads: traditional modes of advertising, e.g., banner ads in newspapers.

Ads are complex - modern internet advertising is handled via ad exchanges



Small businesses can customize their ads via exchanges.

Position Auctions: auctions to sell multiple ad positions on a page.

Let $N = \{1, 2, \dots, n\}$: set of advertisers

$M = \{1, 2, \dots, m\}$: set of slots

assume : $m \geq n$ – every ad is shown

1 : best position, m : worst position.

Evolution of position auctions

① Early position auctions ordered the ads via bid-per-impression

- just for showing the ad.
- newspaper ads e.g.
- all risk on the advertiser

② Bids on clicks – pay-per-click model

- risk is shared by the publisher
- ranked by bid-per-click
- shown ads are not clicked, publisher earns nothing

③ Today's approach : rank advertisers based on the product of probability of click and bid value.

- probability of click is called click through rate (CTR)
- rank by expected revenue

Advertiser value

Assumptions: ① clicks generate value to the advertisers

② all clicks are valued equally – no matter what position the ad is displayed. The position only affects the chance of getting the click.

these assumptions help decouple the value effect and position effect

Agent i's expected value when her ad is shown at position $j \in M$: $v_{ij} = CTR_{ij} \cdot v_i$ ← value of a click

↑ click through rate

$CTR_{ij} \in [0,1]$: probability of getting a click on i's ad at j.

quality component e_i position component p_j

$$CTR_{ij} = e_i \cdot p_j \quad ; \text{ user effect, position effect}$$

hence the expected value: $v_{ij} = p_j (e_i v_i)$

position effect is assumed to be decreasing with position

$$p_1 = 1, \; p_j > p_{j+1} \; ; \; j = 1, \dots, m-1.$$

v_i is the only private information of the advertiser.

p_j and e_i are measurable

search engines estimate the e_i : say $\hat{e}_i$

bidders bid b_i, ads are ranked in decreasing order of $\hat{e}_i b_i$

Allocation of slots in position auctions

Value of an agent i = $P_{a_i}(\hat{e}_i \cdot \theta_i) = v_i(a, \theta_i)$

where $a = (a_1, \ldots, a_n)$ is the allocation, a_i is the slot allocated to i.

Pick allocations $a^* \in \underset{a \in A}{\arg\max} \sum_{i \in N} v_i(a, \theta_i)$ efficient allocation

Claim: An allocation of slots is efficient iff it is rank-by-expected revenue mechanism.

Proof sketch: maximizing the weighted sum problem. Sum is maximized when maximum weight is put on maximum value.

The slot allocation problem is a sorting problem - hence computationally tractable.

Allocation decision is done, need payments to make it DSIC.

natural candidate: VCG [used in Facebook]

Note: actual implementation in practice might be different. Here we discuss only an abstract notion of how it can be done.

VCG in position auction

Given bids $(b_1, \ldots, b_n)$ [note, $\hat{\theta}_i$: reported type and b_i are same]

WLOG ordered such that $\hat{e}_1 b_1 \geqslant \hat{e}_2 b_2 \geqslant \cdots \geqslant \hat{e}_n b_n$

- allocation a^* is s.t. $a_i^* = i$.

- define $\underline{a}_{-i}^* \in \arg\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$

note: allocations of the agents after i, i.e., $i+1$ to n get one slot better.

hence $p_i^{VCG}(b) = \sum_{j \neq i} v_j(\underline{a}_{-i}^*, \theta_j) - \sum_{j \neq i} v_j(a^*, \theta_j)$

$$= \sum_{j=i}^{n-1} p_j (\hat{e}_{j+1} b_{j+1}) - \sum_{j=i}^{n-1} p_{j+1} (\hat{e}_{j+1} b_{j+1})$$

$$= \sum_{j=i}^{n-1} (p_j - p_{j+1})(\hat{e}_{j+1} b_{j+1}) \quad , \forall\, i = 1, \dots, n-1$$

$p_n^{VCG}(b) = 0$.

This is the total expected payment. To convert this to the pay-per-click: $\frac{1}{p_i \hat{e}_i} p_i^{VCG}(b)$.

Pros and cons of VCG mechanism

1. DSIC – hence very low cognitive load on the bidders
2. No deficit (and subsidy) if items are goods
3. Never charges a losing agent
4. Individually rational to participate – nobody loses money.

Criticisms of VCG:

1. Privacy and transparency:

(a) it reveals true valuations/types. two competing companies would not like to make the private information public.

(b) a malicious auctioneer may introduce fake bidders to extract more payment from the bidders.

2. Susceptibility to collusion:

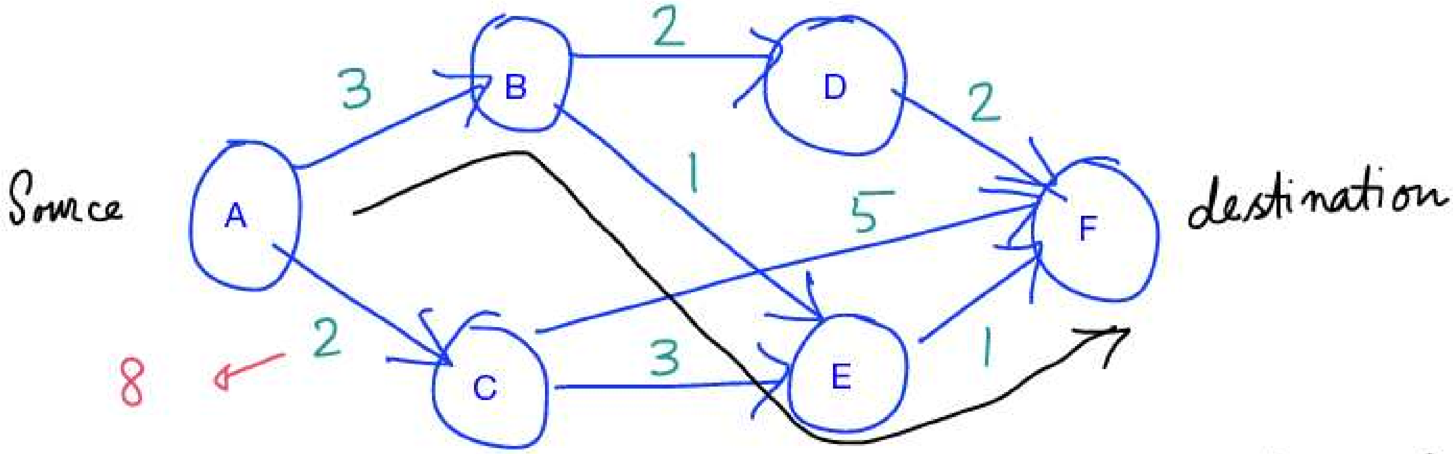
example: public goods

Players	A	B	payment
1	(250) 200	0	150 (100)
2	(150) 100	0	50 (0)
3	0	250	0 (0)

If 1 and 2 collude and bid higher, both of them reduces their payments $\Rightarrow$ utility increases.

3. Not frugal: payment could be very large

VCG is guaranteed to be no deficit, but can charge payment much larger than the cost.



Example: item delivery network (e.g., Amazon)
This is a cost setup, hence the values can be considered to be negative. Each edge is a player.

Efficient allocation: $A \rightarrow B \rightarrow E \rightarrow F$

$$p_{AB} = (-2-3-1) - (-1-1) = -4$$

$$p_{AB} = (-8-3-1) - (-1-1) = -10$$

effect of the other players' costs.

④ Revenue monotonicity violated

revenue monotonicity: revenue weakly increases with number of players.

	F	M	payment
1	0	90	0 → 0
2	100	0	90 → 0
3	100	0	→ 0

nobody's pivotal

⑤ Not budget balanced

This is a no deficit mechanism, but it almost always keeps surplus – which can be large.

Problem: this money cannot be redistributed among the same players, since that will change their payoffs and the resulting mechanism can be not DSIC.

If the players are partitioned into two groups and the surplus of one group is redistributed over the other group – then it is budget balanced, but the overall efficiency is compromised.

This surplus has to be taken away or destroyed –

money burning

To understand this tradeoff better, see

Nath and Sandholm (2019): Efficiency and budget balance in general quasi-linear domains, Games and Econ Behavior.

Remark: these are certain limitations, VCG still is quite elegant and widely used in various settings. Good to know the limitations for effective use.

Generalization of VCG mechanism

Need: incorporate a larger class of DSIC mechanisms in the quasi-linear domain.

Affine Maximizer Allocation Rule

$$f^{AM}(\theta) \in \underset{a \in A}{\arg\max} \left(\sum_{i \in N} w_i \theta_i(a) + K(a) \right)$$

where $w_i \geqslant 0 \; \forall i \in N$, not all zero – different weights for players

$K : A \to \mathbb{R}$ is any arbitrary function – translation

Special cases: $K \equiv 0$ and (i) $w_i = 1 \; \forall i \in N$ (Efficient)

(ii) $w_d = 1$ and $w_i = 0 \; \forall i \neq d$ (Dictatorial)

w_i's are different $\Rightarrow$ not ANON

K is a non-constant function $\Rightarrow$ different importance is given to different allocations

- AM is a superclass of VCG/efficient allocations. Hence it can satisfy more properties.
- We can ask a characterization question (like GS theorem) in the quasi-linear setting with public goods.

Defn: An AM rule f^{AM} with weights w_i, $i \in N$ and the function K satisfies independence of non-influential agents (INA) if for all $i \in N$ with $w_i = 0$ we have $f^{AM}(\theta_i, \underline{\theta}_{-i}) = f^{AM}(\theta_i', \underline{\theta}_{-i})$, $\forall \theta_i, \theta_i', \underline{\theta}_{-i}$.

Remark: this is a tie-breaking requirement. The weight zero agent does not influence the allocation decision, hence it should not break any tie either.

If INA was not satisfied, then the AM can be manipulated.
E.g., suppose there is a tie when $w_i = 0$ for some valuation profile, but the allocation is the less preferred one for agent i.

Theorem: An AM rule satisfying INA is implementable in dominant strategies.

Proof sketch: we need to construct a payment function to make (f^{AM}, p^{AM}) DSIC. Consider

$$p_i^{AM}(\theta_i, \underline{\theta}_{-i}) = \begin{cases} \frac{1}{w_i}\left[h_i(\underline{\theta}_{-i}) - \left(\sum_{j \neq i} w_j \theta_j (f^{AM}(\theta)) + K(f^{AM}(\theta)) \right) \right] & \forall i : w_i > 0 \\ 0, & \forall i : w_i = 0. \end{cases}$$

Payoff of i if $w_i > 0$

$$\theta_i(f^{AM}(\theta_i, \underline{\theta}_{-i})) - p_i^{AM}(\theta_i, \underline{\theta}_{-i})$$

$$= \frac{1}{w_i}\left[\left(\sum_{j \in N} w_j \theta_j (f^{AM}(\theta_i, \underline{\theta}_{-i})) + K(f^{AM}(\theta_i, \underline{\theta}_{-i})) \right) - h_i(\underline{\theta}_{-i}) \right]$$

$$\geqslant \frac{1}{w_i}\left[\left(\sum_{j \in N} w_j \theta_j (f^{AM}(\theta_i', \underline{\theta}_{-i})) + K(f^{AM}(\theta_i', \underline{\theta}_{-i})) \right) - h_i(\underline{\theta}_{-i}) \right]$$

$$= \theta_i(f^{AM}(\theta_i', \underline{\theta}_{-i})) - p_i^{AM}(\theta_i', \underline{\theta}_{-i})$$

for i where $w_i = 0$, the payments are zero and

$$f^{AM}(\theta_i, \underline{\theta}_{-i}) = f^{AM}(\theta_i', \underline{\theta}_{-i}) \quad \forall \theta_i, \theta_i', \underline{\theta}_{-i}$$

hence the payoffs are identical for all types of i.

Similar to GS theorem, we ask what if the valuations are unrestricted.

$\theta_i : A \to \mathbb{R}$; Θ_i contains all such valuation functions, no restriction on the functions is imposed.

With this unrestricted space of valuations, we can characterize the class of DSIC mechanisms in the quasi-linear domain.

Theorem (Roberts 1979)

Let A be finite with $|A| \geqslant 3$. If the type space is unrestricted, then every ONTO and dominant strategy implementable allocation rule must be an affine maximizer.

Similarity with GS theorem: there it is restricting the class to dictatorships here to affine maximizers.

Restricted domains are open research domains.

Proof reference: Lavi, Mualem, Nisan (2009): Two Simplified Proofs of Roberts' theorem.

Mechanism design for selling a single indivisible object

Motivation: simplest yet elegant results

Setup: type set of agent i : $T_i \subseteq \mathbb{R}$

$t_i \in T_i$ denotes the value of agent i if she wins the object

An allocation a is a vector of length n that represents the probability of winning the object by the respective agent. Hence,

Set of allocations: $\Delta A = \{ a \in [0,1]^n : \sum_{i=1}^{n} a_i = 1 \}$

Allocation rule: $f : T_1 \times T_2 \times \cdots \times T_n \to \Delta A$

valuation: $v_i(a, t_i) = a_i \cdot t_i$ (product form) – expected valuation

Hence, $f_i(t_i, t_{-i})$ is the probability of winning the object for agent i when the type profile is (t_i, t_{-i}).

Recall: Vickrey / Second-price auction: type is v_i.

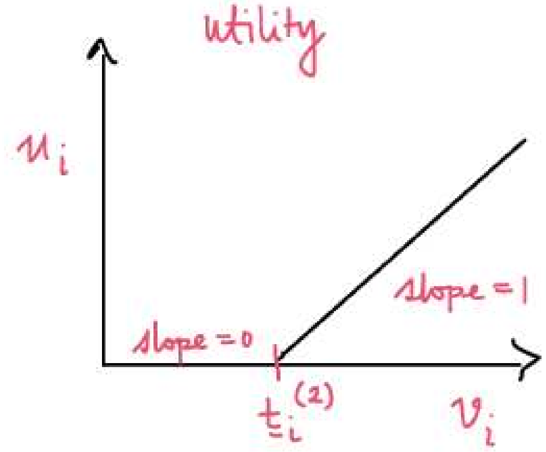
define $t_{-i}^{(2)} = \max_{j \neq i} \{ v_j \}$

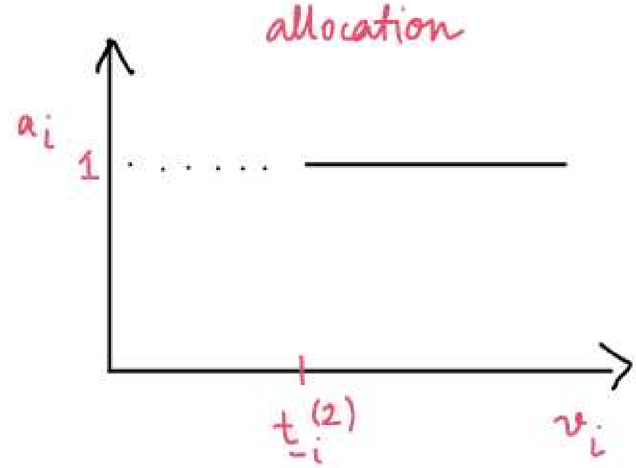
agent i wins if $v_i > t_{-i}^{(2)}$, loses if $v_i < t_{-i}^{(2)}$

a tie-breaking rule decides if equality.

Since, payment is $t_{-i}^{(2)}$ if i is the winner, the utility is zero in case of a tie.

$$u_i = \begin{cases} 0 & \text{if } v_i \leq t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & \text{if } v_i > t_{-i}^{(2)} \end{cases}$$





Observations:

1. utility is convex, derivative is zero if $v_i < \underline{t}_{-i}^{(2)}$ and 1 if $v_i > \underline{t}_{-i}^{(2)}$ — not differentiable at $v_i = \underline{t}_{-i}^{(2)}$.
2. Whenever differentiable, it coincides with the allocation probability.

Known facts from convex analysis (see, e.g., Rockafeller (1980))

Fact 1: Convex functions are continuous in the interior of its domain. Jumps can occur only at the boundaries.

Fact 2: Convex functions are differentiable "almost everywhere".

The points where the function is not differentiable form a countable set (see the example before) — has measure zero.

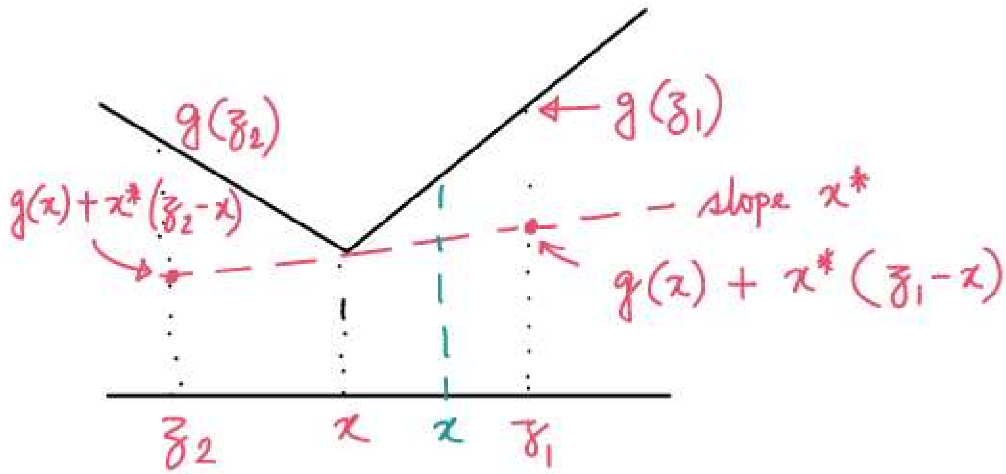
Recall: A function $g : I \to \mathbb{R}$ (where I is an interval) is convex if for every $x, y \in I$ and $\lambda \in [0,1]$

$$\lambda g(x) + (1-\lambda) g(y) \geq g(\lambda x + (1-\lambda) y).$$

If g is differentiable at $x \in I$, we denote the derivative by $g'(x)$. The following definition extends the idea of gradient

Defn: For any $x \in I$, x^* is a subgradient of g at x if

$$g(z) \geq g(x) + x^*(z - x) \quad \forall z \in I.$$



many possible x^* for x.

only one x^* possible for x

Few standard results (proofs: any standard text on convex analysis)

Lemma 1: Let $g: I \to \mathbb{R}$ be a convex function. Suppose x is in the interior of I and g is differentiable at x. Then $g'(x)$ is the unique subgradient of g.

Lemma 2: Let $g: I \to \mathbb{R}$ be a convex function. Then for every $x \in I$ a subgradient of g at x exists.

Fact 3: Let $I' \subseteq I$ be the set of points where g is differentiable. The set $I \setminus I'$ is of measure zero. The set of subgradients at a point forms a convex set.

Define
$$g'_+(x) = \lim_{\substack{z \to x \\ z \in I',\, z > x}} g'(z), \qquad g'_-(x) = \lim_{\substack{z \to x \\ z \in I',\, z < x}} g'(z)$$

Fact 4: The set of subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$

We will denote the set of subgradients of g at $x \in I$ as $\partial g(x)$

Lemma 1 says $\partial g(x) = \{g'(x)\}$ $\forall x \in I'$.

Lemma 2 says that $\partial g(x) \neq \emptyset$ $\forall x \in I$.

Lemma 3: Let $g: I \to \mathbb{R}$ be a convex function. Let $\phi: I \to \mathbb{R}$ be a subgradient function, i.e., $\phi(z) \in \delta g(z)$ $\forall z \in I$.
Then for all $x, y \in I$ s.t. $x > y$, we have $\phi(x) \geq \phi(y)$.

$\phi(z)$ picks one value at every z (even if subgradients can be many)

This result says that subgradient functions are monotone non-decreasing.

Lemma 4: Let $g: I \to \mathbb{R}$ be a convex function. Then for any $x, y \in I$,

$$g(x) = g(y) + \int_y^x \phi(z)\, dz,$$

where $\phi: I \to \mathbb{R}$ is s.t. $\phi(z) \in \delta g(z)$, $\forall z \in I$.

Monotonicity and Myerson's lemma

<u>Defn</u>: An allocation rule is non-decreasing if for every agent $i \in N$ and $\underline{t}_{-i} \in \underline{T}_{-i}$ we have $f_i(t_i, \underline{t}_{-i}) \geqslant f_i(s_i, \underline{t}_{-i})$, $\forall s_i, t_i \in T_i$, $t_i > s_i$

Holding the types of other agents fixed, the probability of allocation never decreases with valuation.

Theorem (Myerson 1981)

Suppose $T_i = [0, b_i]$ $\forall i \in N$, and the valuations are in the product form. An allocation rule $f : T \to \Delta A$ and a payment rule $(p_1, \ldots, p_n)$ are DSIC iff

1. f is non-decreasing, and
2. payments are given by

$$p_i(t_i, \underline{t}_{-i}) = p_i(0, \underline{t}_{-i}) + t_i\, f_i(t_i, \underline{t}_{-i}) - \int_0^{t_i} f_i(x, \underline{t}_{-i})\, dx .$$

$$\forall t_i \in T_i, \forall \underline{t}_{-i} \in \underline{T}_{-i}, \forall i \in N.$$

Remark: difference with the Roberts' theorem: Roberts' result gives a functional form, while Myerson's result is a more implicit property. Sometimes function forms help answering questions in a more direct manner.

<u>Proof</u>: ($\Rightarrow$) given that $(f, \underline{p})$ is DSIC.

Utility of agent i

$$u_i(t_i, \underline{t}_{-i}) = t_i\, f_i(t_i, \underline{t}_{-i}) - p_i(t_i, \underline{t}_{-i}), \text{ and}$$

$$u_i(s_i, \underline{t}_{-i}) = s_i\, f_i(s_i, \underline{t}_{-i}) - p_i(s_i, \underline{t}_{-i}).$$

Since $(f, \underline{p})$ is DSIC,

$$u_i(t_i, \underline{t}_i) = t_i f(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)$$
$$\geq t_i f(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)$$
$$= s_i f_i(s_i, \underline{t}_i) + f_i(s_i, \underline{t}_i)(t_i - s_i) - p_i(s_i, \underline{t}_i)$$
$$= u_i(s_i, \underline{t}_i) + f_i(s_i, \underline{t}_i)(t_i - s_i) \quad \cdots\cdots (1)$$

fixing $\underline{t}_i$, define $g(t_i) = u_i(t_i, \underline{t}_i), \phi(t_i) = f_i(t_i, \underline{t}_i)$.

Hence, Eq (1) can be written as

$$g(t_i) \geq g(s_i) + \phi(s_i)(t_i - s_i)$$

$\Rightarrow$ $\phi(s_i)$ is a subgradient of g at s_i. $\cdots\cdots$ (2)

Next, need to show: g is convex.

pick $x_i, z_i \in T_i$, define $y_i = \lambda x_i + (1-\lambda) z_i$, $\lambda \in [0,1]$.

DSIC implies

$$g(x_i) \geq g(y_i) + \phi(y_i)(x_i - y_i), \quad \text{and}$$
$$g(z_i) \geq g(y_i) + \phi(y_i)(z_i - y_i)$$
$$\Rightarrow \lambda g(x_i) + (1-\lambda) g(z_i) \geq g(y_i) + \phi(y_i)[\underbrace{\lambda x_i + (1-\lambda) z_i - y_i}_{=0}]$$
$$= g(\lambda x_i + (1-\lambda) z_i)$$

$\Rightarrow$ g is convex. $\cdots\cdots$ (3)

Apply lemmas 3 and 4

Lemma 3 $\Rightarrow$ ϕ is non-decreasing, i.e., $f_i(\cdot, \underline{t}_i)$ is non-decreasing

$\Rightarrow$ Part (1) is proved.

Lemma 4 $\Rightarrow$ $g(t_i) = g(0) + \int_0^{t_i} \phi(x)\,dx$

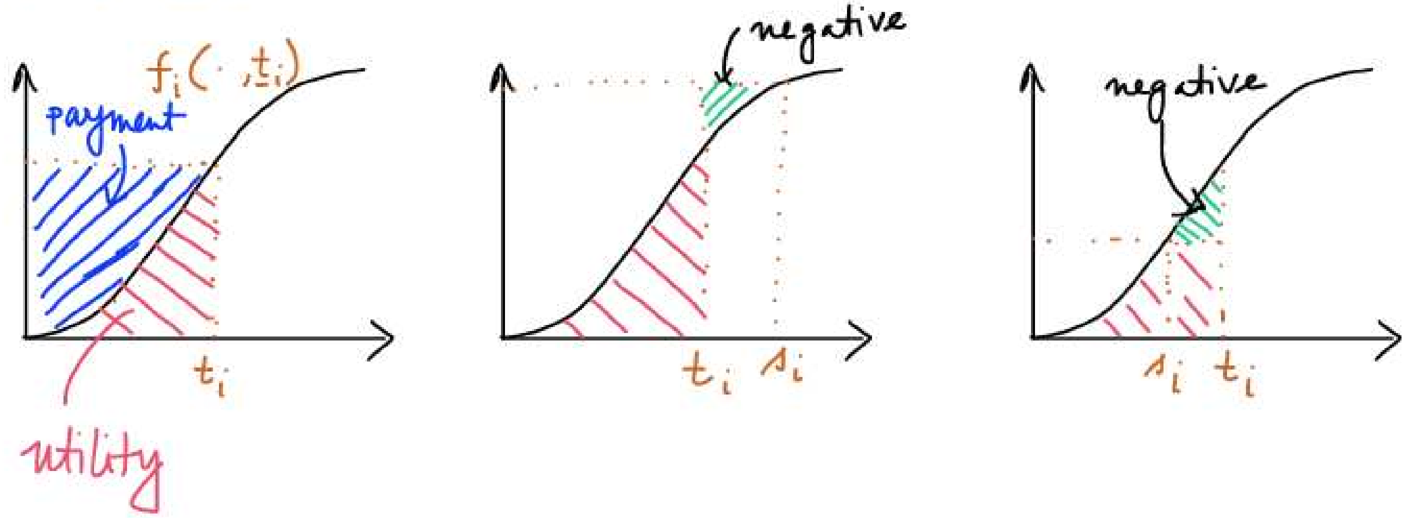
$$\Rightarrow u_i(t_i, \underline{t}_i) = u_i(0, \underline{t}_i) + \int_0^{t_i} f_i(x, \underline{t}_i)\, dx$$

$$\Rightarrow t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i) = -p_i(0, \underline{t}_i) + \int_0^{t_i} f_i(x, \underline{t}_i)\, dx$$

$$\Rightarrow p_i(t_i, \underline{t}_i) = p_i(0, \underline{t}_i) + t_i f_i(t_i, \underline{t}_i) - \int_0^{t_i} f_i(x, \underline{t}_i)\, dx.$$

($\Leftarrow$) Given: f is non-decreasing and payment formula.

proof by pictures – assume $p_i(0, \underline{t}_i) = 0$



$$[t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)] - [t_i f_i(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)]$$

$$= (s_i - t_i) f_i(s_i, \underline{t}_i) + \int_{s_i}^{t_i} f_i(x, \underline{t}_i)\, dx \geqslant 0$$

Corollary: An allocation rule in single object allocation setting is implementable in dominant strategies if it is non-decreasing.

Examples of some single object allocation mechanisms

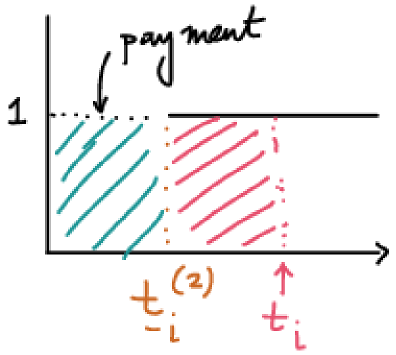
① Constant allocation rule – non-decreasing, payment = constant (e.g., 0)

② Dictatorial – give the object only to the dictator – non-decreasing, payment = constant / zero.

③ Second price auction

$$\underbrace{p_i(0, t_{-i})}_{=0} + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})\, dx .$$

step function



④ Efficient allocation with a reserve price is also non decreasing.

If the highest value is below a reserve price r, nobody gets the object. Otherwise, the item goes to the highest bidder.

allocated to i if $v_i > \max\{t_{-i}^{(2)}, r\}$. payment $= \max\{t_{-i}^{(2)}, r\}$

⑤ Not so common allocation rule : $N = \{1, 2\}$, $A = \{a_0, a_1, a_2\}$

unsold given to 1

Given a type profile $t = (t_1, t_2)$, the seller computes

$U(t) = \max\{2, t_1^2, t_2^3\}$ – select a_0, a_1, a_2 depending on which of the three expressions is the maxima – break ties in favor of $0 > 1 > 2$.

Player 1 gets the object if $t_2 > \sqrt{\max\{2, t_2^3\}}$

Player 2 gets the object if $t_3 > \sqrt[3]{\max\{2, t_1^2\}}$

both monotone.

Individual Rationality

Defn: A mechanism $(f, \underline{p})$ is *ex post individually rational* if

$$t_i f_i(t_i, \underline{t}_{-i}) - p_i(t_i, \underline{t}_{-i}) \geqslant 0, \ \forall t_i \in T_i, \forall \underline{t}_{-i} \in \underline{T}_{-i}, \forall i \in N.$$

Ex-post: even after all agents have revealed their types, participating is weakly preferred.

Lemma: In the single object allocation setting, consider a DSIC mechanism $(f, \underline{p})$.

1. It is IR iff $\forall i \in N$ and $\forall \underline{t}_{-i} \in \underline{T}_{-i}$, $p_i(0, \underline{t}_{-i}) \leqslant 0$.

2. It is IR and satisfies no subsidy, i.e., $p_i(t_i, \underline{t}_{-i}) \geqslant 0$, $\forall t_i, \underline{t}_{-i}$ $\forall i \in N$ iff $\forall i \in N$, $\underline{t}_{-i} \in \underline{T}_{-i}$, $p_i(0, \underline{t}_{-i}) = 0$.

Proof: (Part 1) Suppose $(f, \underline{p})$ is IR, then $0 - p_i(0, \underline{t}_{-i}) \geqslant 0$

hence $p_i(0, \underline{t}_{-i}) \leqslant 0$.

Conversely, if $p_i(0, \underline{t}_{-i}) \leqslant 0$, then the payoff of i is

$$t_i f_i(t_i, \underline{t}_{-i}) - p_i(t_i, \underline{t}_{-i})$$

$$= t_i f_i(t_i, \underline{t}_{-i}) \underbrace{- p_i(0, \underline{t}_{-i})}_{\geqslant 0} - t_i f_i(t_i, \underline{t}_{-i}) + \int_0^{t_i} f_i(x, \underline{t}_{-i})\, dx \geqslant 0$$

(Part 2): IR $\Rightarrow p_i(0, \underline{t}_{-i}) \leqslant 0$, if $p_i(t_i, \underline{t}_{-i}) \geqslant 0 \ \forall t_i \Rightarrow$

$p_i(0, \underline{t}_{-i}) = 0$.

Clearly if $p_i(0, \underline{t}_{-i}) = 0 \Rightarrow (f, \underline{p})$ is IR and no-subsidy.

Some non-Vickrey auctions – focus: budget balance

① The object goes to the highest bidder, but the payment is such that everyone is compensated some amount.

– highest and second highest bidders are compensated $\frac{1}{n}$ of the third highest bid. $p_1(0,\underline{t}_{-1}) = p_2(0,\underline{t}_{-2}) = -\frac{1}{n}t_3$

– everyone else receives $\frac{1}{n}$ of the second highest bid

$$p_i(0,\underline{t}_{-i}) = -\frac{1}{n} \text{ second highest in } \{t_j, j\neq i\}$$

WLOG $t_1 > t_2 > \cdots > t_n$

Agent 1 pays $= -\frac{1}{n}t_3 + t_1 - \int_0^{t_1} f_1(x,\underline{t}_{-1})\,dx = -\frac{1}{n}t_3 + t_2$

2 pays $= -\frac{1}{n}t_3$, all others $= -\frac{1}{n}t_2$

total payments $= -\frac{1}{n}t_3 + t_2 - \frac{1}{n}t_3 - \frac{n-2}{n}t_2 = \frac{2}{n}(t_2 - t_3)$

tends to 0 for large n.

deterministic mechanism that redistributes the money.

② Allocate the object w.p. $\left(1-\frac{1}{n}\right)$ to the highest bidder

w.p. $\frac{1}{n}$ to the second highest bidder

$$p_i(0,\underline{t}_{-i}) = -\frac{1}{n} \text{ second highest bid in } \{t_j, j\neq i\}$$

1 pays $= -\frac{1}{n}t_3 + \left(1-\frac{1}{n}\right)t_1 - \frac{1}{n}(t_2 - t_3) - \left(1-\frac{1}{n}\right)(t_1 - t_2)$

$= \left(1-\frac{2}{n}\right)t_2$

2 pays $= -\frac{1}{n}t_3 + \frac{1}{n}t_2 - \frac{1}{n}(t_2 - t_3) = 0$

all others $= -\frac{1}{n}t_2$. Together $= 0$.

How to maximize the revenue earned by the auctioneer?

maximize w.r.t. what knowledge of the auctioneer? — The common prior distribution over the types.

Accordingly, the notions of incentive compatibility and individual rationality have to change.

Bayesian Incentive Compatibility

$T_i = [0, b_i]$, common prior G over $T = \overset{n}{\underset{i=1}{\times}} T_i$ — g denotes the density

$G_{-i}(\underline{s}_{-i} | s_i)$ is the conditional distribution over s_{-i}, given i's type is s_i.

similarly $g_{-i}(\underline{s}_{-i} | s_i)$ is derived via Bayes rule from g.

Every mechanism $(f, p_1, \dots, p_n)$ induces an expected allocation and payment rule $(\alpha, \underline{\pi})$

$$\alpha_i(\underset{\text{reported}}{s_i} | \underset{\text{true}}{t_i}) = \int_{\underline{s}_{-i} \in \underline{T}_{-i}} \underbrace{f_i(s_i, \underline{s}_{-i})}_{\substack{\text{probabilistic} \\ \text{allocation}}} \underbrace{g_{-i}(\underline{s}_{-i} | t_i)}_{\substack{\text{types of other agents} \\ \text{common prior}}} d\underline{s}_{-i}$$

two levels of expectation

expected payment

$$\pi_i(s_i | t_i) = \int_{\underline{s}_{-i} \in \underline{T}_{-i}} p_i(s_i, \underline{s}_{-i}) \, g_{-i}(\underline{s}_{-i} | t_i) \, d\underline{s}_{-i}$$

Expected utility of agent i

$$t_i \alpha_i(t_i | t_i) - \pi_i(t_i | t_i)$$

Defn: A mechanism $(f, \underline{p})$ is Bayesian incentive compatible (BIC) if $\forall i \in N$, $\forall s_i, t_i \in T_i$

$$t_i \alpha_i(t_i | t_i) - \pi_i(t_i | t_i) \geqslant t_i \alpha_i(s_i | t_i) - \pi_i(s_i | t_i).$$

Similarly, f is Bayesian implementable if $\exists\ \underline{p}$ s.t. $(f, \underline{p})$ is BIC

Independence and Characterization of BIC mechanisms

Assume that the priors are independent, i.e., agent i's value is drawn from a distribution G_i (density g_i) independently from other agents.

$$G(s_1, s_2, \dots, s_n) = \prod_{i \in N} G_i(s_i)$$

$$G(\underline{s}_{-i} \mid t_i) = \prod_{j \neq i} G_j(s_j)$$

We will use the shorthand $\alpha(t_i) = \alpha(t_i \mid t_i)$

<u>Defn</u>: An allocation rule is non-decreasing in expectation (NDE) if $\forall i \in N$, $\forall s_i, t_i \in T_i$ with $s_i < t_i$ we have $\alpha_i(s_i) \leq \alpha_i(t_i)$.

Note: The rules that are non-decreasing (defined before) are always NDE. But there can be more rules that are NDE.

Characterization of BIC rules

<u>Theorem</u>: A mechanism $(f, \underline{p})$ in the independent prior setting is BIC iff ① f is NDE, and

② p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx$

$\forall t_i \in T_i,\ \forall i \in N$.

Remark: Bayesian version of the earlier theorem

Proof: in similar lines as before [exercise]

An allocation rule may be NDE but not non-decreasing.

t_2

				1
			1	
			1	1
		1		1

t_1

all 5 types are equally likely

$\alpha_1(t_1)$ and $\alpha_2(t_2)$ are monotone but $f(t_1, t_2)$ is not.

As we are in the Bayesian setting now, we can define an analog of individual rationality

Defn: A mechanism $(f, \underline{p})$ is interim individually rational (IIR) if for every bidder $i \in N$, we have

$$t_i \alpha_i(t_i) - \pi_i(t_i) \geqslant 0 \quad \forall t_i \in T_i .$$

Lemma: A mechanism $(f, \underline{p})$ is BIC and IIR iff

① f is NDE,

② p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx$

$\forall t_i \in T_i, \forall i \in N.$

③ $\forall i \in N, \ \pi_i(0) \leqslant 0.$

Proof sketch: ① and ② uniquely identify a BIC mechanism. So, the proof requires to show that IIR along with ① and ② are equivalent to ③

($\Rightarrow$) apply IIR at $t_i = 0$ on ② and get $\pi_i(0) \leqslant 0$

($\Leftarrow$) $t_i \alpha_i(t_i) - \pi_i(t_i) = -\pi_i(0) + \int_0^{t_i} \alpha_i(s_i)\, ds_i \geqslant 0$ if $\pi_i(0) \leqslant 0$

Optimal mechanism design for a single agent

Motivation: analyze a simpler problem to understand the problem of revenue maximization. Will generalize later to multiple agents.

Setup: Type set $T=[0,\beta]$. Mechanism (f,p)

$$f:[0,\beta]\to[0,1], \quad p:[0,\beta]\to\mathbb{R}$$

- Incentive compatibility [BIC and DSIC equivalent]

$$t f(t)-p(t)\geqslant t f(s)-p(s), \quad \forall t,s\in T.$$

- Individual rationality [IR and IIR equivalent]

$$t f(t)-p(t)\geqslant 0, \quad \forall t\in T.$$

The expected revenue earned by a mechanism M is given by

$$\pi^M := \int_0^\beta p(t)\,g(t)\,dt$$

We need to find a mechanism M^* in the class of all IC and IR mechanisms s.t. $\pi^{M^*}\geqslant \pi^M, \ \forall M$.

We will call M^* the optimal mechanism.

Q: What is the structure of an optimal mechanism?

Consider an IC and IR mechanism $(f,p)\equiv M$

By the characterization theorems and lemmas, we know

$$p(t)=p(0)+tf(t)-\int_0^t f(x)\,dx \quad [\text{IC}]$$

$$p(0)\leqslant 0 \quad [\text{IR}]$$

since we want to maximize revenue, $p(0)=0$.

Hence, the payment formula is

$$p(t) = tf(t) - \int_0^t f(x)\,dx$$

Note: in optimal mechanism, payment is completely given once the allocation is fixed. Hence, we need to optimize only over one variable.

Expected revenue: $\pi^f = \int_0^\beta p(t)\,g(t)\,dt$

$$= \int_0^\beta \left(tf(t) - \int_0^t f(x)\,dx\right) g(t)\,dt$$

Need to maximize this wrt f.

Lemma: For any implementable allocation rule f, we have

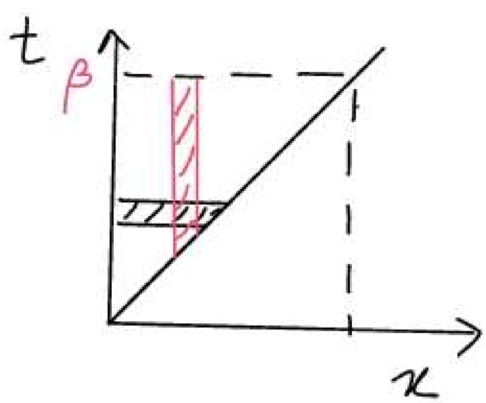
$$\pi^f = \int_0^\beta \left(t - \frac{1-G(t)}{g(t)}\right) g(t)\,dt\,.$$

Proof: $\pi^f = \int_0^\beta \left(tf(t) - \int_0^t f(x)\,dx\right) g(t)\,dt$

$$= \int_0^\beta tf(t)\,g(t)\,dt - \int_0^\beta \int_0^t f(x)\,dx\; g(t)\,dt$$

$$= \int_0^\beta tf(t)\,g(t)\,dt - \int_0^\beta \int_x^\beta g(t)\,dt\; f(x)\,dx$$

[standard limit switching]



$$= \int_0^\beta tf(t)\,g(t)\,dt - \int_0^\beta \int_t^\beta g(x)\,dx\; f(t)\,dt$$

$$= \int_0^\beta \left[t f(t) g(t) - (1 - G(t)) f(t) \right] dt$$

$$= \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t)\, dt . \qquad \square$$

Hence the optimal mechanism finding problem reduces to

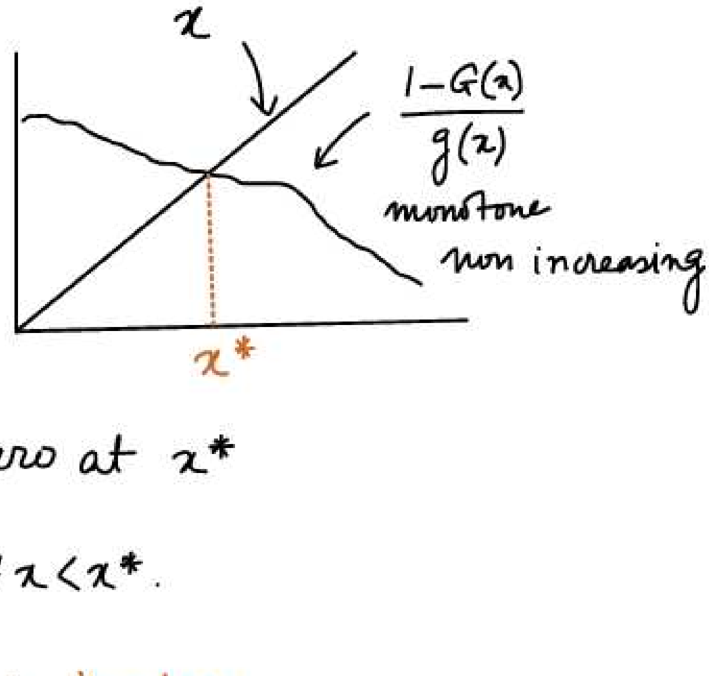
OPT1: $$\max_{f : f \text{ is nondecreasing}} \int_0^\beta \left(t - \frac{1-G(t)}{g(t)} \right) g(t) f(t)\, dt$$

Assumption: G satisfies the monotone hazard rate condition (MHR), i.e., $\frac{g(x)}{1-G(x)}$ is non decreasing in x.

Standard distributions like uniform and exponential satisfy MHR condition.

Fact: If G satisfies MHR condition, there is a unique solution to

$$x = \frac{1 - G(x)}{g(x)} .$$



Intuition:

Let x^* be the unique solution of this equation

Hence, $w(x) = x - \frac{1 - G(x)}{g(x)}$ is zero at x^*

$w(x) > 0 \ \forall x > x^*$ and $< 0 \ \forall x < x^*$.

The unrestricted solution to OPT1 is therefore

$$f(t) = \begin{cases} 0 & \text{if } t < x^* \\ 1 & \text{if } t > x^* \\ \alpha & \text{if } t = x^*, \ \alpha = [0,1] \end{cases} \qquad (1)$$

But this f is non-decreasing, therefore it is the optimal solution of OPT1.

Theorem: A mechanism (f, p) under the MHR condition is optimal iff (1) f is given by eqn. (1) where x^* is the unique solution of $x = \frac{1-G(x)}{g(x)}$, and

(2) For all $t \in T$, $p(t) = \begin{cases} x^* & \text{if } t \geq x^* \\ 0 & \text{ow} \end{cases}$

Optimal mechanism design for multiple agents

In this context, we will call a mechanism optimal if it is BIC and IIR and maximizes revenue.

By previous results, this reduces to

① f_i's are NDE, $\forall i \in N$,

② $\pi_i(t_i)$ has a specific formula and $\pi_i(0)=0$.

The expected payment made by agent i is

$$\int_{T_i} \pi_i(t_i)\, g_i(t_i)\, dt_i \quad ; \quad T_i = [0, b_i]$$

in a way similar to the earlier exercise, simplify to the following

$$\int_0^{b_i} w_i(t_i)\, g_i(t_i)\, \underbrace{\alpha_i(t_i)}_{= \int_{T_{-i}} f_i(t_i, t_{-i})\, g_{-i}(t_{-i})\, dt_{-i}}\, dt_i \qquad w_i(t_i) = t_i - \frac{1 - G_i(t_i)}{g_i(t_i)}$$

also called virtual valuation of player i

$$= \int_T w_i(t_i)\, f_i(t)\, g(t)\, dt$$

Hence, the total revenue generated by all players is

$$\sum_{i \in N} \int_T w_i(t_i)\, f_i(t)\, g(t)\, dt$$

$$= \int_T \underbrace{\left(\sum_{i \in N} w_i(t_i)\, f_i(t) \right)}_{\text{expected total virtual valuation}} g(t)\, dt$$

Hence the optimal mechanism design problem reduces to

$$\max \int_T \left(\sum_{i \in N} w_i(t_i)\, f_i(t) \right) g(t)\, dt \text{ , s.t. } f \text{ is NDE.}$$

As before, we attempt to solve the unconstrained optimization problem.

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geq w_j(t_j) \ \forall j \quad \text{(sold)} \\ 0 & \text{ow} \end{cases} \qquad \text{①}$$

$$f_i(t) = 0, \forall i \in N, \text{ if } w_i(t_i) < 0 \ \forall i \in N \quad \text{(unsold)}$$

But it can lead to a situation where f is not NDE (for an example, see Myerson (1981): "Optimal Auction Design" — The example is such that the following condition is violated)

Defn: A virtual valuation w_i is regular if $\forall s_i, t_i \in T_i$ with $s_i < t_i$, it holds that $w_i(s_i) < w_i(t_i)$.

This condition is weaker than the MHR condition as MHR implies regularity.

Lemma: Suppose every agent's valuations are regular. The allocation rule of the optimal mechanism is same as the solution of the unconstrained problem.

Proof sketch: The solution is as given in eqn. ①.

Regularity ensures that $w_i(t_i) > w_i(s_i) \ \forall s_i < t_i$

Then the optimal allocation rule also satisfies

$$f_i(t_i, \underline{t}_{-i}) \geq f_i(s_i, \underline{t}_{-i}) \ \forall \underline{t}_{-i} \in \underline{T}_{-i}, \forall s_i < t_i.$$

i.e., f_i is non-decreasing (hence NDE).

Examples of optimal mechanisms

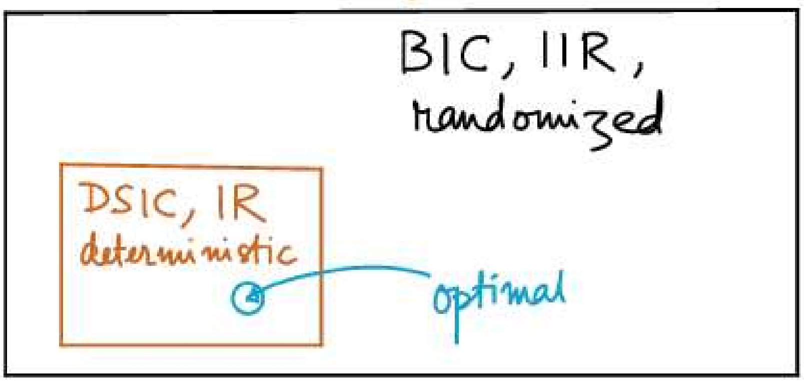
Optimal mechanism design problem:

$$\max \int_T \Big(\sum_{i\in N} w_i(t_i)\, f_i(t)\Big) g(t)\, dt \;,\; \text{s.t. } f \text{ is NDE.}$$

Solution for regular w_i's:

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geq w_j(t_j) \ \forall j \\ 0 & \text{ow} \end{cases}$$

We wanted to find an allocation that is NDE, but found an f that is non-decreasing. Also, it is deterministic.



Space of regular virtual valuations

Theorem: Suppose every agent's valuation is regular. Then, for every type profile t,

if $w_i(t_i) < 0 \ \forall i \in N$, $f_i(t) = 0 \ \forall i \in N$.

otherwise, $f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geq w_j(t_j) \ \forall j \in N \\ 0 & \text{ow} \end{cases}$

ties are broken arbitrarily. Payments are given by

$$p_i(t) = \begin{cases} 0 & \text{if } f_i(t) = 0 \\ \max\{ w_i^{-1}(0), K_i^*(\underline{t}_{-i}) \} & \text{if } f_i(t) = 1 \end{cases}$$

Then $(f, \underline{p})$ is an optimal mechanism.

$w_i^{-1}(0)$: The value of t_i where $w_i(t_i) = 0$.

$K_i^*(t_{-i}) = \inf \{ t_i : f_i(t_i, t_{-i}) = 1 \}$

The minimum value of t_i where i begins to be the winner

Example 1: Two buyers : $T_1 = [0, 12]$, $T_2 = [0, 18]$

Uniform, independent prior.

$$w_1(t_1) = t_1 - \frac{1 - G_1(t_1)}{g_1(t_1)} = t_1 - \frac{1 - \frac{t_1}{12}}{\frac{1}{12}} = 2t_1 - 12$$

$$w_2(t_2) = 2t_2 - 18$$

t_1	t_2	action	p_1	p_2
4	8	unsold	0	0
2	12	sold to 2	0	9
6	6	sold to 1	6	0
9	9	sold to 1	6	0
8	15	sold to 2	0	11

Example 2: Symmetric bidders : The valuations are drawn from the same distribution, $g_i = g$, $T_i = T$, $\forall i \in N$

virtual valuation : $w_i = w$.

$$w(t_i) > w(t_j) \text{ iff } t_i > t_j$$

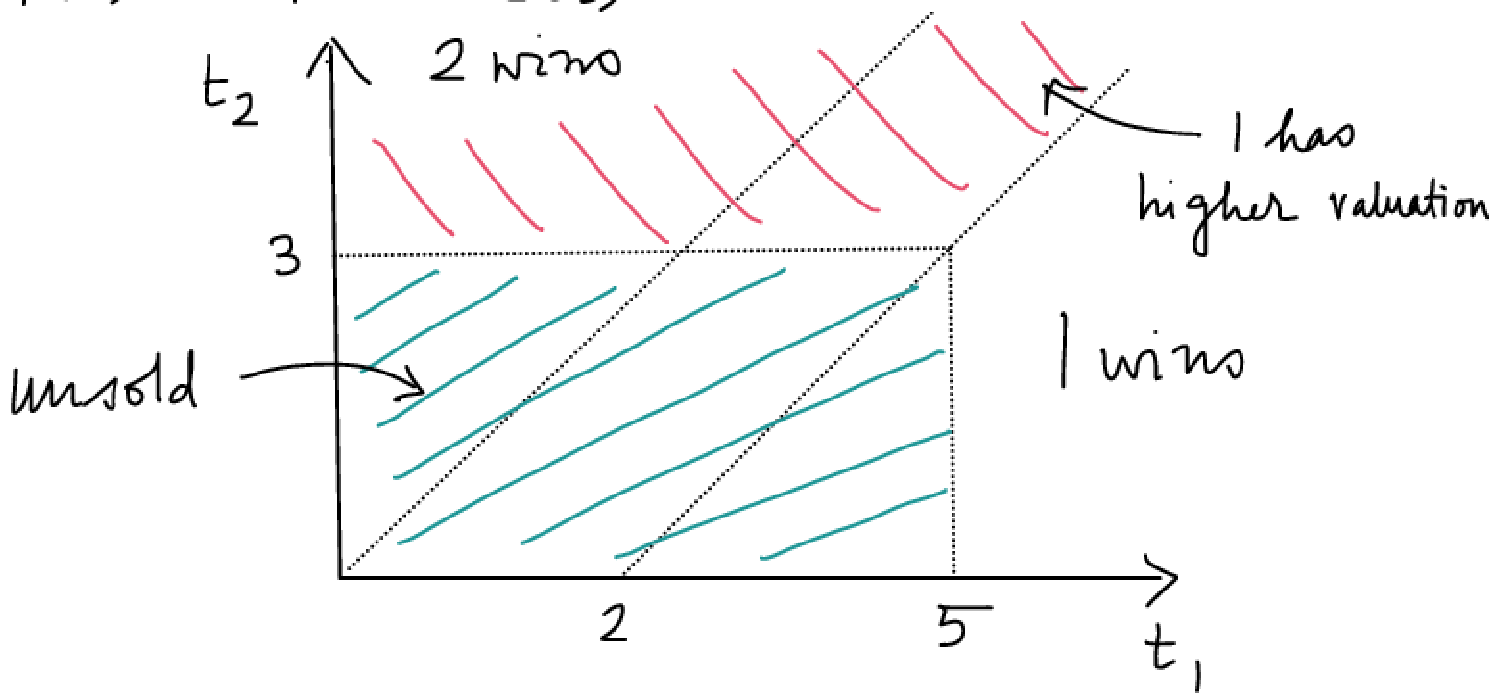
the object goes to the highest bidder. Not sold if $w^{-1}(0) > t_i$ $\forall i \in N$. $\text{Payment}_i = \max \{ w^{-1}(0), \max_{j \neq i} t_j \}$

Second price auction with a reserve price, and is efficient when the object is sold.

Example 3: Efficiency and Optimality

$T_1 = [0, 10]$, $T_2 = [0, 6]$, uniform, independent prior

$w_1(t_1) = 2t_1 - 10$, $w_2(t_2) = 2t_2 - 6$



Unsold is inefficient, also in the region of the plane.

Efficiency and Budget Balance

Uniqueness of Groves for Efficiency

$$f^{eff}(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a)$$

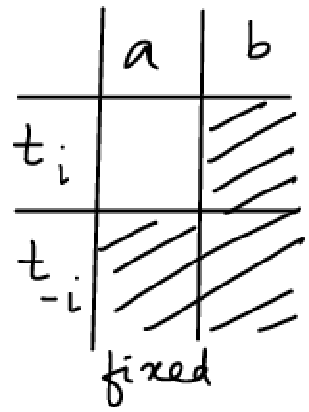
Theorem (Green and Laffont (1979), Holmström (1979))

If the type space is 'sufficiently' rich, every efficient and DSIC mechanism is a Groves mechanism.

Proof sketch: two alternatives $A = \{a, b\}$

welfares $\sum t_i(a)$ and $\sum t_i(b)$

$\sum t_i(a) \geq \sum t_i(b)$ then a is chosen

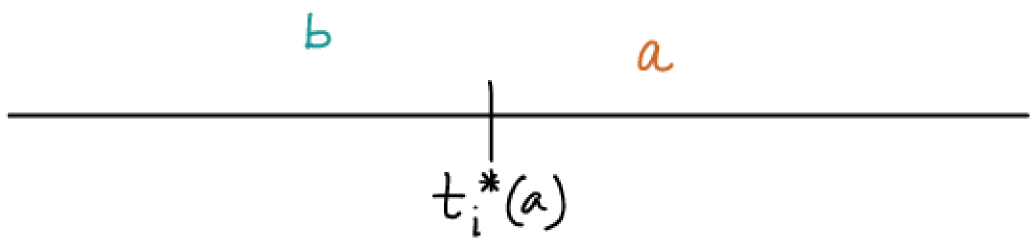


- fix the valuations of the other agents to t_{-i}
- fix value of i at alternative b at $t_i(b)$

∃ some threshold $t_i^*(a)$ s.t.

∀ $t_i(a) \geq t_i^*(a)$ a is the outcome

$t_i(a) < t_i^*(a)$ b is the outcome



Use DSIC for $t_i^*(a) + \epsilon = t_i(a)$, $\epsilon > 0$

$t_i^*(a) + \epsilon - p_{ia} \geq t_i(b) - p_{ib}$

note: payment for a player has to be same for an allocation.

similarly, $t_i'(a) = t_i^*(a) - \delta$, $\delta > 0$

$t_i(b) - p_{ib} \geq t_i^*(a) - \delta - p_{ia}$

since ϵ, δ are arbitrary, then

$$t_i^*(a) - p_{ia} = t_i(b) - p_{ib} \quad \text{①}$$

But $t_i^*(a)$ is the threshold of the efficient outcome

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b) \quad \text{②}$$

from ① and ②

$$p_{ia} - p_{ib} = \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a)$$

hence the payment has to be of the form $p_{ix} = h_i(t_{-i}) - \sum_{j \neq i} t_j(x)$

Theorem (Green and Laffont (1979))

No Groves mechanism is budget balanced, i.e., $\nexists p_i^G$ s.t.,

$$\sum_{i \in N} p_i^G(t) = 0 \quad \forall t \in T.$$

Proof sketch: Two alternatives $\{0,1\}$ 0: a project is undertaken
1: project is not undertaken

in outcome 0, every agent has zero value.

Suppose, $\exists h_i$ s.t. $\sum_{i \in N} p_i(t) = 0$

Consider two types w_1^+, w_1^- for player 1, and one type w_2 for player 2, such that

$w_1^+ + w_2 > 0$ hence project is built

$w_1^- + w_2 < 0$ and project is not built

Budget balance at type profile (w_1^+, w_2)

$$h_1(w_2) - w_2 + h_2(w_1^+) - w_1^+ = 0$$

at (w_1^-, w_2), $h_1(w_2) + h_2(w_1^-) = 0$

eliminating $h_1(w_2)$, $w_2 = h_2(w_1^+) - h_2(w_1^-) - w_1^+$

The RHS depends only on w_1, hence it is possible to alter w_2 slightly to retain the inequalities, but then the above equality cannot hold.

Corollary: If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and BB.

Weakening DSIC for positive results

Allocation is still the efficient one.

Payment in this setting is also defined via a prior

$$\delta_i(t_i) = \mathbb{E}_{\underline{t}_i | t_i} \sum_{j \neq i} t_j(a^*(t))$$

allocation, $a^*(t) \in \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} t_i(a)$

payment, $p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i)$

(named after d'Aspremont, Gerard-Varet (1979), Arrow (1979))

This payment implements the efficient allocation rule in Bayes Nash equilibrium.

$$\mathbb{E}_{\underline{t}_i | t_i} \left[t_i(a^*(t)) - p_i^{dAGVA}(t) \right]$$

$$= \mathbb{E}_{\underline{t}_i | t_i} \sum_{j \in N} t_j(a^*(t)) - \mathbb{E}_{\underline{t}_i | t_i} \left[\frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) \right]$$

$$\geq \mathbb{E}_{\underline{t}_{-i}|t_i} \sum_{j\in N} t_j\left(a^*(t_i', \underline{t}_{-i})\right) - \mathbb{E}_{\underline{t}_{-i}|t_i}\Big[\frac{1}{n-1}\underbrace{\sum_{j\neq i}\delta_j(t_j)}_{\text{not a function of } t_i}\Big]$$

$$= \mathbb{E}_{\underline{t}_{-i}|t_i}\left[t_i\left(a^*(t_i', \underline{t}_{-i})\right) - p_i^{dAGVA}(t_i', \underline{t}_{-i})\right]$$

To show budget balance, consider

$$\sum_{i\in N} p_i^{dAGVA}(t) = \frac{1}{n-1}\sum_{i\in N}\sum_{j\neq i}\delta_j(t_j) - \sum_{i\in N}\delta_i(t_i)$$

$$= \frac{n-1}{n-1}\sum_{j\in N}\delta_j(t_j) - \sum_{i\in N}\delta_i(t_i) = 0$$

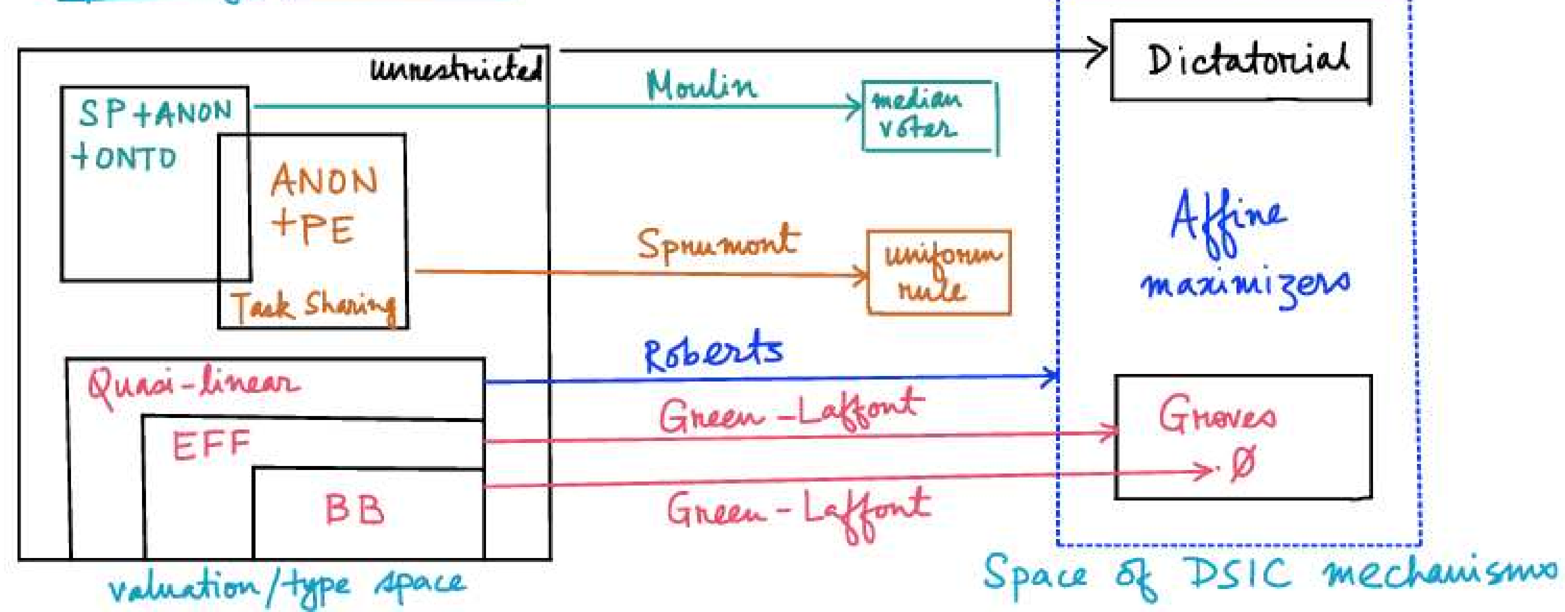
Theorem: The dAGVA mechanism is efficient, BIC, and budget balanced.

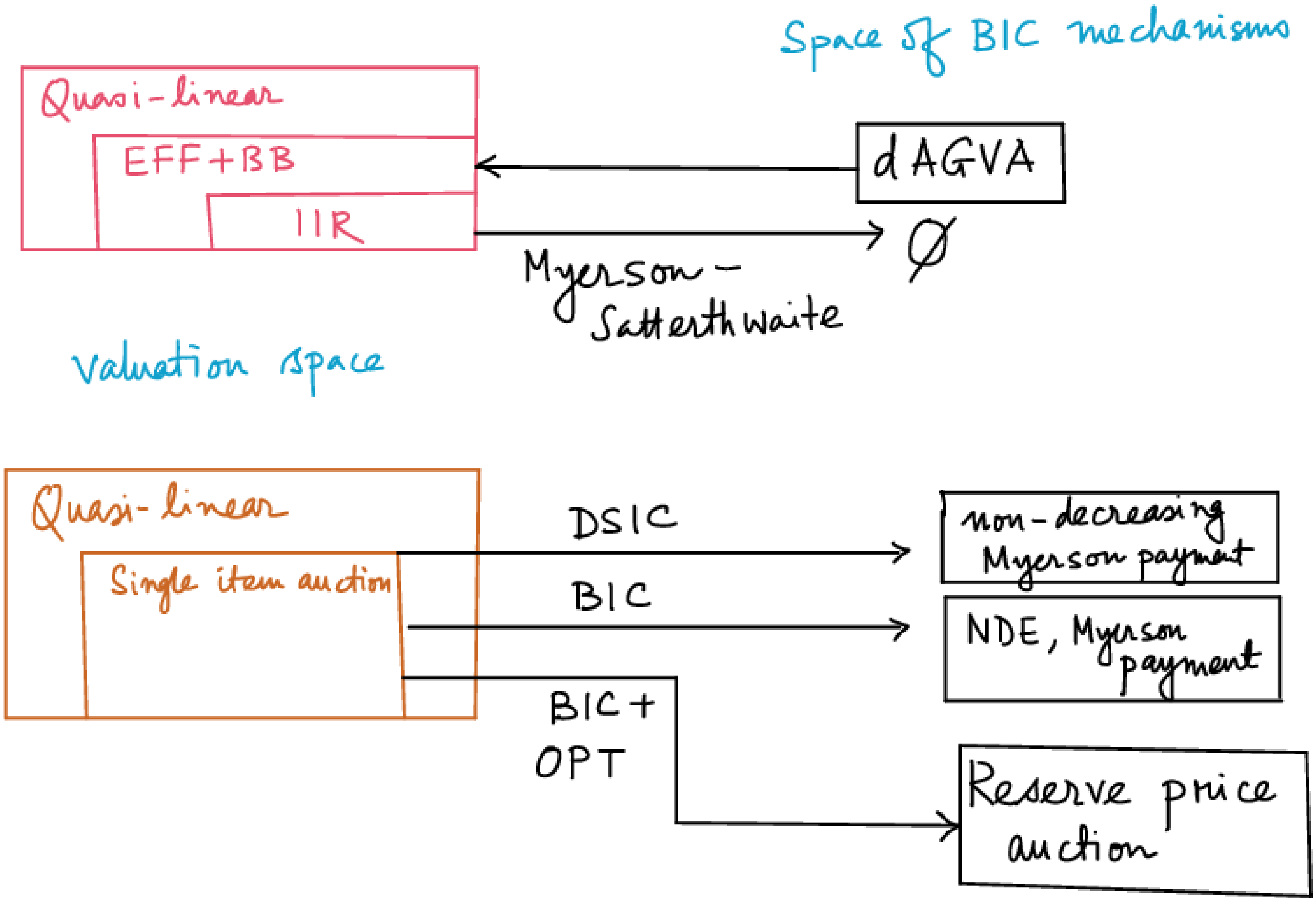
Q: participation guarantee?

A: dAGVA is not IIR.

Theorem (Myerson, Satterthwaite (1983))
In a bilateral trade (that involves two types of agents: seller and buyer) no mechanism can be simultaneously BIC, efficient, IIR, and budget balanced.

Space of mechanisms




Space of BIC mechanisms
Quasi-linear
EFF+BB
IIR
dAGVA
Ø
Myerson–Satterthwaite
Valuation space
Quasi-linear
Single item auction
DSIC
non-decreasing Myerson payment
BIC
NDE, Myerson payment
BIC+OPT
Reserve price auction