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36.1 Recap

In the last lecture, we started discussing the optimal auction for a single object allocation. To build our intuition, we started with a single agent, i.e., there is only one potential buyer for the object and the auctioneer wants to maximize the revenue earned (maximize the payment of that agent subject to some conditions). We mentioned a result which gives a structure for the revenue function. In this lecture, we prove the result, and discuss the solution of *optimal revenue problem* for the single-agent case. Finally, we will extend the result to the multi-agent scenario.

36.2 Optimal Auction for a Single Agent

36.2.1 Struction of the revenue function

Theorem 36.1 *For any implementable allocation rule f, the revenue earned is given by*

$$\Pi^f = \int_0^\beta w(t)f(t)g(t)dt$$

where $w(t) = \left(t - \frac{1-G(t)}{g(t)}\right)$, which is known as the virtual valuation of the agent.

Proof: Consider the expected payment of the agent

$$\begin{aligned}
\Pi^f &= \int_0^\beta p(t)g(t)dt \\
&= \int_0^\beta \left[tf(t) - \int_0^t f(x)dx\right] g(t)dt \\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \left(\int_0^t f(x)dx\right) g(t)dt \\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \left(\int_x^\beta f(x)g(t)dt\right) dx && \text{(changing the order of integration of the second term)} \\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \left(\int_x^\beta g(x)dx\right) f(t)dt && \text{(exchanging the variables } t \text{ and } x\text{)} \\
&= \int_0^\beta \left[tg(t) - \int_t^\beta g(x)dx\right] f(t)dt
\end{aligned}$$

$$= \int_0^{\beta} \left[tg(t) - (G(\beta) - G(t))\right] f(t)dt$$
$$= \int_0^{\beta} \left[tg(t) - (1 - G(t)\right] f(t)dt$$
$$= \int_0^{\beta} \left(t - \frac{1 - G(t)}{g(t)}\right) g(t)f(t)dt$$
$$= \int_0^{\beta} w(t)g(t)f(t)dt \qquad \text{(where } w(t) = t - \tfrac{1-G(t)}{g(t)}\text{)}$$

The second equality follows because IC implies that the payment is given by Myerson's characterization result, and IR together with revenue maximization implies $p(0) = 0$. ■

36.2.2 Optimal revenue problem

Now we want to find the allocation function which maximizes the revenue of the auctioneer. So we need to solve the following optimization problem :

$$\max_{f\,:\,f \text{ is non-decreasing}} \Pi^f \tag{OPT1}$$

Solving the above optimization problem is difficult, while the unconstrained version (given below) of the above problem is easier.

$$\max_{f} \Pi^f \tag{OPT2}$$

Under the assumption of *monotone hazard rate*, the solutions of $OPT1$ and $OPT2$ are identical.

Assumption 36.2 *A prior distribution G satisfies the* Monotone Hazard Rate *(MHR) condition if $\frac{g(x)}{1-G(x)}$ is non-decreasing in x.*

Standard distributions like uniform and exponential satisfy MHR condition. We state a fact concerning MHR condition without proof.

Fact 36.3 *If G satisfies MHR, then $\exists$ a unique x^* s.t.*

$$x^* = \frac{1 - G(x^*)}{g(x^*)} \implies w(x^*) = 0$$

Observe that if G satisfies MHR condition, w is strictly increasing. Therefore for $x < x^*$, w is negative and for $x > x^*$, w is positive. This observation gives us an easy way to solve the unconstrained optimization problem OPT2. As the goal is to maximize the total expected revenue, and for $t < x^*$, the virtual valuation is negative, we want not to allocate the object. Similarly, we want the sell the object for $t > x^*$. When $t = x^*$, the virtual valuation is zero, hence we can allocate the object with any probability and it does not affect the revenue. Therefore the allocation function is given as:

$$f(t) = \begin{cases} 0 & \text{for } t < x^* \\ \alpha \in [0,1] & \text{for } t = x^* \\ 1 & \text{for } t > x^* \end{cases} \tag{36.1}$$

Note that the above allocation function is non-decreasing as well. Hence it is a valid solution to constrained optimization problem OPT1. Therefore, f defined as above is implementable in DSIC. We now state a theorem to summarize the discussions we have had so far.

Theorem 36.4 *A mechanism (f, p) under MHR condition is optimal iff the following two conditions hold:*

1. *allocation f is given by Equation 36.1, and*
2. *payment is given by $p(t) = f(t)x^*, \ \forall t \in T$.*

36.3 Optimal Auction Mechanism for Multiple Agents

36.3.1 Struction of the revenue function

The optimal auction mechanism $M \equiv (f, p)$ is defined as one that is BIC, IIR and satisfies $\Pi^M \geqslant \Pi^{M'} \ \forall M'$. We know that

- BIC $\implies$ f_i's are non-decreasing in expectation (NDE) and expected payment $\Pi_i(t_i)$ has a specefic formula.
- IIR and revenue optimality $\implies$ $\Pi_i(0) = 0$.

Expected payment made by agent i is given by (assuming $T_i = [0, b_i]$):

$$\begin{aligned}
\text{payment}_i &= \int_0^{b_i} \Pi_i(t_i) g(t_i) dt_i \\
&= \int_0^{b_i} \left(t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x_i) dx_i \right) g(t_i) dt_i \\
&= \int_0^{b_i} \left(t_i - \frac{1 - G_i(t_i)}{g_i(t_i)} \right) g_i(t_i) \alpha_i(t_i) dt_i && \text{(using the proof from the previous section)} \\
&= \int_0^{b_i} w_i(t_i) \alpha_i(t_i) g_i(t_i) dt_i \\
&= \int_0^{b_i} w_i(t_i) \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i}) dt_{-i} g_i(t_i) dt_i && \text{(substituting the expression of } \alpha_i(t_i)) \\
&= \int_T w_i(t_i) f_i(t) g(t) dt && (\text{where } T = \textstyle\bigtimes_{i=1}^n T_i)
\end{aligned}$$

Now, we have total revenue earned by the auctioneer as:

$$\Pi^M = \sum_{i \in N} \text{payment}_i = \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t) dt \tag{36.2}$$

Since the expected revenue is completely determined by the allocation rule f, we can replace Π^M with Π^f.

36.3.2 Optimal Revenue Problem

To find the allocation function which maximizes the revenue of the auctioneer, we need to solve the following constrained optimization problem

$$\max_{f \,:\, f \text{ is NDE}} \Pi^f \tag{OPT3}$$

Again we can see solving the above optimization problem is difficult. But solving the unconstrained version (given below) of the above problem is easier.

$$\max_f \Pi^f \tag{OPT4}$$

Observe that Π^f involves taking convex combination of the virtual valuations of every agent (Eq. 36.2). From that expression of Π^f, it is clear that to maximize the integration we need to maximize a convex combination. We know that any convex combination $\sum_i \alpha_i x_i$ where $\sum_i \alpha_i = 1$ and $\alpha_i \in [0,1]$ is maximized (where α_is are variables) only when we set $\alpha_j = 1$ for $x_j : x_j \geqslant x_i$, $\forall i$ and set $\alpha_i = 0$ for $\forall i \neq j$. However, in the allocation problem we also have the option of not assigning the object to anyone (unsold), which will be optimal if the virtual valuations are negative for every agent. So following this scheme, we have the allocation of any agent i for the unconstrained problem to be the following when $\exists\, i \in N, w_i(t_i) \geqslant 0$.

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j), \forall j \in N \\ 0 & \text{otherwise} \end{cases} \tag{36.3}$$

With an arbitrary tie-breaking rule. The allocation is $f_i(t) = 0, \forall\, i \in N$ if $w_i(t_i) < 0, \forall\, i \in N$.

Clearly the allocation rule is **not** non-decreasing and therefore **not** non-decreasing even in expectation (because some of w_is can be decreasing w.r.t. t_is which might cause the corresponding agent to have its allocation reduced from 1 to 0 as result of having her type increased).

Similar to single-agent case, we want the solution of OPT3 to be the same as OPT4. The reason why the two solutions do not match is because we do not have any guarantee for virtual valuation of agents (w_is). Like in the single-agent case, we make the following assumption on the virtual valuation of agents.

Definition 36.5 (Regular Virtual Valuation) *A virtual valuation w_i is regular if $\forall s_i, t_i \in T_i$ with $s_i < t_i$, then $w_i(s_i) < w_i(t_i)$.*

This assumption regarding virtual valuations is weaker than the MHR condition.

Lemma 36.6 *If the hazard rate $\frac{g_i(t_i)}{1-G_i(t_i)}$ is non-decreasing, then w_i is regular.*

Assumption 36.7 *The virtual valuations of all the agents are regular, i.e, w_i is regular $\forall i \in N$.*

This assumption addresses the issue with virtual valuations mentioned earlier. The implication of this assumption is summarized in the following lemma.

Lemma 36.8 *If every agent's virtual valuation is **regular**, then the solution of the constrained optimization problem is the same as the unconstrained problem.*