Develop the notation and definitions.

Example of a two-player game

Prisoner's dilemma

1 \ 2	Cooperate	Defect
Cooperate	−5, −5	0, −10
Defect	−10, 0	−1, −1

Neighboring Kingdom's dilemma

1 \ 2	A	D
A	5, 5	0, 6
D	6, 0	2, 2

- Representation : Normal form / Strategic Form Game
  - Simultaneous move, one shot games.

- Notation : Set of players $N = \{1, 2, \ldots, n\}$
  Set of actions A_i for player i, specific action $a_i \in A_i$

utility / payoff of agent i, $u_i : \underbrace{A_1 \times A_2 \times \cdots \times A_n}_{\times_{i \in N} A_i = A} \to \mathbb{R}$

Warning : Not every preference may have utility representation.
Special case that the preferences are expressible using functions as described.

Example : (Preferences without utility representation)
student − F − M − relative cooks a, b, c (any two items)

$a \succ b \succ c$
$b \succ c \succ a$
$c \succ a \succ b$

The preference of the majority of this family does not have an utility representation.

Axioms for utility representation (Concept #1)
Use outcomes rather than alternatives

① Completeness : $\forall\, a_1, a_2$, either $a_1 \succ a_2$ or $a_2 \succ a_1$ or $a_1 \sim a_2$.

② Transitivity : If $a_1 \succeq a_2$ and $a_2 \succeq a_3 \Rightarrow a_1 \succeq a_3$

③ Substitutability : If $a_1 \sim a_2$, then $\forall$ sequences of outcomes $a_3, \ldots, a_k$ and sets of probabilities $p, p_3, \ldots, p_k$ s.t. $p + \sum_{i=3}^{k} p_i = 1$, $p, p_i \geq 0$

$$[p : a_1, p_3 : a_3, \ldots, p_k : a_k] \sim [p : a_2, p_3 : a_3, \ldots, p_k : a_k]$$

4. Decomposability: If $\forall a_i \in A,\ P_{\ell_1}(a_i) = P_{\ell_2}(a_i) \Rightarrow \ell_1 \sim \ell_2$

5. Monotonicity: If $a_1 \succ a_2$ and $1 > p > q > 0$

$$[p : a_1, 1-p : a_2] \succ [q : a_1, 1-q : a_2]$$

6. Continuity: If $a_1 \succ a_2$ and $a_2 \succ a_3$ then $\exists\, p \in [0,1]$ s.t. $a_2 \sim [p : a_1, 1-p : a_3]$

(von-Neumann, Morgenstern 1944)

Theorem: If a preference relation $\succeq$ satisfies axioms 1 - 6

then $\exists\, u : A \to [0,1]$ s.t.

1. $u(a_1) \geq u(a_2) \Leftrightarrow a_1 \succeq a_2$
2. $u([p_1 : a_1, p_2 : a_2, \ldots, p_k : a_k]) = \sum_{i=1}^{k} p_i u(a_i)$

Proof: Indifferent - Part 1 is obvious $u(a_i) = 0\ \forall a_i$ - all permutation $u(\ldots)$ everything in simplex is same

2 follows from ~~substitutability~~ decomposability define $u(a_i) = $ constant.

Case 2: $\exists$ ~~one~~ at least two alternatives that are preferred over one another

$\bar{a}$ most preferred, $\underline{a}$ least preferred.

for any alternative a_i, define $u(a_i)$ as p_i s.t.

$a_i \sim [p_i : \bar{a}, 1-p_i : \underline{a}]$ by continuity.

Part 1: $u(a_1) \geq u(a_2) \Leftrightarrow a_1 \succeq a_2$

($\Rightarrow$) define $a_1 \sim \ell_1 = [u(a_1) : \bar{a}, 1-u(a_1) : \underline{a}]$ → applies to lotteries too

$a_2 \sim \ell_2 = [u(a_2) : \bar{a}, 1-u(a_2) : \underline{a}]$

(a) $u(a_1) > u(a_2)$ & by monotonicity,

$a_1 \sim \ell_1 \succ \ell_2 \sim a_2 \Rightarrow a_1 \succ a_2$ (transitivity)

(b) $u(a_1) = u(a_2)$ $a_1 \sim \ell_1 \equiv \ell_2 \sim a_2 \Leftrightarrow a_1 \sim a_2$

($\Leftarrow$) $u(a_1) \geq u(a_2) \Leftarrow a_1 \succeq a_2$

$\Leftrightarrow u(a_1) < u(a_2) \Rightarrow a_1 \prec a_2$ already shown.

completeness.

Part 2 : $u^* = u([p_1 : a_1, p_2 : a_2, \ldots, p_k : a_k]) = \sum_{i=1}^{k} p_i u(a_i)$

$$a_1 \sim l_1 = [u(a_1) : \bar{a},\ 1-u(a_1) : \underline{a}]$$

✓ Substitutability:

$$u^* = u([p_1 : l_1, \ldots, p_k : l_k]) \qquad \sum_{i=1}^{k} p_i u(a_i)$$

✓ decomposability

$$= u\Big(\Big[p_1 u(a_1) + p_2 u(a_2) + \cdots + p_k u(a_k) : \bar{a}$$

$$1 - \sum_{i=1}^{k} p_i u(a_i) : \underline{a}\Big]\Big)$$

$$u^* = u(l^*) = u\Big(\Big[\underbrace{\sum_{i=1}^{k} p_i u(a_i)}_{} : \bar{a},\ 1-p : \underline{a}\Big]\Big)$$

$$= u(l^*) = u^*$$

Concept # 2 : Rationality and Intelligence

- Agents pick actions to maximize their utility
- Knows the rule of the game and pick action to maximize utility.

Concept # 3 : Common Knowledge behavioral model

blue/black eyed people

everybody knows the rules, everybody knows that everybody knows it, ... and so on.