CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2018**

Assignment 2

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Question 1

Part (a)

We are given that f is strategyproof and $f(P_1, P_2) = a$.

Also, for any preference (P_a, P_b) we have:

$$f(P_a, P_b) \in \{P_a(1), P_b(1)\} \qquad (1)$$

We need to prove that $f(P_1', P_2') = b$

Consider the following strategy profiles:

P_1	P_2	P_1'	P_2
a	c	b	c
b	b	a	b
c	a	c	a

Let's assume (for contradiction) that $f(P_1', P_2) = c$

Since $aP_1'c$, we have $f(P_1, P_2)\ P_1'\ f(P_1', P_2)$. Hence, f is manipulable, which is a contradiction.

So from (1), we have $f(P_1', P_2) = b$ (2)

Consider the following strategies:

P_1'	P_2	P_1'	$\hat{P}_2$
b	c	b	c
a	b	a	a
c	a	c	b

Let's assume (for contradiction) that $f(P_1', \hat{P}_2) = c$

Since cP_2b and from (2), we have $f(P_1', \hat{P}_2)\ P_2\ f(P_1', P_2)$. Hence, f is manipulable, which is contradiction.

So from (1), we have $f(P_1', \hat{P}_2) = b$ (3)

Consider the following strategies:

P_1'	$\hat{P}_2$	P_1'	P_2'
b	c	b	a
a	a	a	b
c	b	c	c

Let's assume (for contradiction) that $f(P_1', P_2') = a$

Since $a\hat{P}_2 b$ and from (3), we have $f(P_1', P_2')\ \hat{P}_2\ f(P_1, \hat{P}_2)$. Hence, f is manipulable, which is a contradiction.

Hence from (1), we have $f(P_1', P_2') = b$.

Part b

No, the earlier conclusion does not hold in the given single-peaked domain.

The proof won't go through because in one of the steps we have considered a transition from (P_1', P_2) to $(P_1', \hat{P}_2)$. But in the given single-peaked domain with intrinsic ordering of the alternatives as $a < b < c$, we can't have a preference ordering $\hat{P}_2 : c \succ a \succ b$ (this has two peaks). Since this type of transition in preference profiles is essential for the proof, it won't go through in this case.

Consider the SCF $f : S^n \to A$ that picks the left-most peak among the peaks of the agents, defined as -

$$f(P) = \underset{i \in S}{min}\{P_i(1)\}$$

Hence, we have -

$$f(P_1, P_2) = min(a, c) = a$$
$$f(P_1', P_2') = min(b, a) = a$$

The above SCF is strategyproof and non-dictatorial(proved in class), and it satisfies the given conditions $f(P_1, P_2) = a$ and $f(P_1', P_2') = a$. Hence, we have a mechanism that doesn't follow the conclusion in part-(a).

Question 2

Let Y be the set of all non-empty subsets of projects in X. Let all preference orderings among these subsets be Q.

Consider the social choice function F that given preference orderings of non-empty subsets of projects in X i.e, $\mathcal{Q}^n$ chooses a non-empty subset.

Let's assume (for the sake of contradiction) that we have an unrestricted domain over the preference orderings.

Since, we have unrestricted domain, consider a scenario where for some "agent i" and projects X_1, X_2, the preference ordering of "agent i" satisfies $\{X_1\}\mathcal{Q}_i\{X_2\}$ and $\{X2\}\mathcal{Q}_i\{X_1, X_2\}$.

From $\{X_1\}\mathcal{Q}_i\{X2\}$, we can infer that in the linear order corresponding to "agent i" i.e, P_i, we have $X_1 P_i X_2$.

But from $X_1 P_i X_2$, we have a restriction on the preference ordering that $\{X_1, X_2\}\mathcal{Q}_i\{X_2\}$. Hence, we have a contradiction.

So, we have restricted domain and since Gibbard-Satterthwaite theorem needs unrestricted preferences, we can't apply Gibbard-Satterthwaite in this case.

Question 3

Let's consider the median voter SCF f that is defined in single-peaked domain.

Consider any group of agents $K \subseteq N$ and let's denote the preferences of all agents denoted only by their peaks i.e, $P = (P_1(1), P_2(1)...P_n(1))$ and $f(P) = f(P_K, P_{-K}) = a$.

Case 1: $\forall i \in K$**,** $P_i(1) = a$

Then, there is no reason for any agent in K to manipulate.

Case 2: $\exists i \in K$**,** $P_i(1) = a$ **and** $\exists j \in K$**,** $P_j(1) \neq a$

So, consider any preference profile P'_K such that $f(P'_K, P_{-K}) = b, b \neq a$. Then, for some agent i (with $P_i(1) = a$), we have $f(P_K, P_{-K}) P_i f(P'_K, P_{-K})$. So, we can't manipulate using the agents in K for this case.

Case 3: $\forall i \in K$, $P_i(1) \neq a$

Consider the case where $f(P'_K, P_{-K}) = b$ and $a < b$ in the linear order.

For this to happen at least one agent who reported the peak less than a should change it to greater than a. So, there exists an agent i, such that -

$$P_i(1) < a < P'_i(1)$$

But since P_i is a single-peaked preference, the alternative a must be preferred over b (because in linear order $P_i(1) < a < b$) i.e, aP_ib. So, we have $f(P_K, P_{-K}) P_i f(P'_K, P_{-K})$. Hence, the group of agents can't manipulate in this case.

The other case where $f(P'_K, P_{-K}) = b$ and $b < a$ in the linear order can be analyzed similarly.

Hence, any median voter SCF is a group strategy-proof.