Lecture 35: November 7, 2017

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35.1 Revenue Maximization

We need to benchmark mechanisms w.r.t. their expected revenue. The benchmark is typically done w.r.t. a given prior distribution of the types. Therefore, we move from a prior-free environment to a weaker setup due to two major reasons: (1) for the prior-free setup, we can ask the worst-case revenue earned, which can be arbitrarily bad, hence we cannot achieve any useful result, and (2) in majority of real-world applications, the agents interact repeatedly with the system and it is possible to build a model for their valuation distribution.

Since we move to a Bayesian setting, all the notions of truthfulness and participation constraints need to be redefined.

35.2 Bayesian Incentive Compatibility

Model: Let the type set of agent i be $T_i = [0, b_i], \forall\, i \in N$. The common prior G is defined over the set of type profiles $T := \bigtimes_{i \in N} T_i$, and g denotes the density of the joint distribution.

The conditional distribution over the types of agents except i when the type of agent i is t_i is denoted by $G_{-i}(\cdot \mid t_i)$. Let $g_{-i}(\cdot \mid t_i)$ denote the density of this distribution. The conditional density $g_{-i}(t_{-i} \mid t_i)$ is derived from g using Bayes' rule.

Every mechanism $(f, p_1, \dots, p_n)$ induces an expected allocation and payment rule (α, π) given by

$$\alpha_i(s_i \mid t_i) = \int_{s_{-i} \in T_{-i}} f_i(s_i, s_{-i}) g_{-i}(s_{-i} \mid t_i) ds_{-i},$$
$$\pi_i(s_i \mid t_i) = \int_{s_{-i} \in T_{-i}} p_i(s_i, s_{-i}) g_{-i}(s_{-i} \mid t_i) ds_{-i}.$$

Where the notation $(s_i \mid t_i)$ denote that s_i is reported when the true type is t_i. Therefore, the expected utility of agent i is

$$t_i \cdot \alpha_i(s_i \mid t_i) - \pi_i(s_i \mid t_i).$$

Note: since the randomization in a Bayesian setting has two following levels, the meaning of 'expected' has to be understood appropriately.

- The first level is w.r.t. the types of other agents, which comes from the common prior in this setting.
- The second level is w.r.t. the randomization of the mechanism (allocation rule) – we have used this in our discussions earlier.

In the context of Bayesian common prior setting, the term ‘expected’ will refer to the expectation w.r.t. **both** the randomization effects. We define the notion of truthfulness in this setting.

Definition 35.1 (Bayesian Incentive Compatibility) *A mechanism (f, p) is* Bayesian Incentive compatible *(BIC) if $\forall s_i, t_i \in T_i, i \in N$*

$$t_i \cdot \alpha_i(t_i \mid t_i) - \pi_i(t_i \mid t_i) \geqslant t_i \cdot \alpha_i(s_i \mid t_i) - \pi_i(s_i \mid t_i).$$

Consequently, an allocation rule f is Bayesian implementable if $\exists p$ such that (f, p) is BIC. We will use the shorthand $\alpha_i(t_i)$ and $\pi_i(t_i)$ to denote that the true and reported types are both t_i.

35.2.1 Independence of types

We assume that the types are independent, i.e., every agent's type is drawn independently from a distribution G_i (with density g_i). This assumption is well motivated in the settings where agents cannot communicate with each other.

$$G(s_1, \ldots, s_n) = \prod_{i \in N} G_i(s_i)$$
$$G_{-i}(s_{-i} \mid t_i) = \prod_{j \neq i} G_j(s_j)$$

Now we define a property of the allocation rule that will help us characterize the BIC rules.

Definition 35.2 (Non-decreasing in Expectation) *An allocation rule is* non-decreasing in expectation *(NDE) if $\forall s_i, t_i \in T_i$ such that $s_i < t_i$, we have*

$$\alpha_i(s_i) \leqslant \alpha_i(t_i), \ \forall i \in N.$$

Non-decreasing allocation rules are always NDE. Therefore, we expanded the space of mechanisms in the Bayesian setting.

Theorem 35.3 (Characterization of BIC rules) *A mechanism (f, p) in the independent prior setting is BIC iff*

1. *f is NDE, and*
2. *p_i satisfies*

$$\pi_i(t_i) := \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(s_i) ds_i, \forall t_i \in T_i, i \in N.$$

Proof is left as an exercise.

Note: A BIC mechanism may not be DSIC. We can see this through the following example where the allocation rule is NDE but **not** non-decreasing. Figure 35.1 shows that the types of two agents t_1 and t_2 are uniformly distributed over an unit square. The ‘1’s represent that the object is allocated to agent 1 when

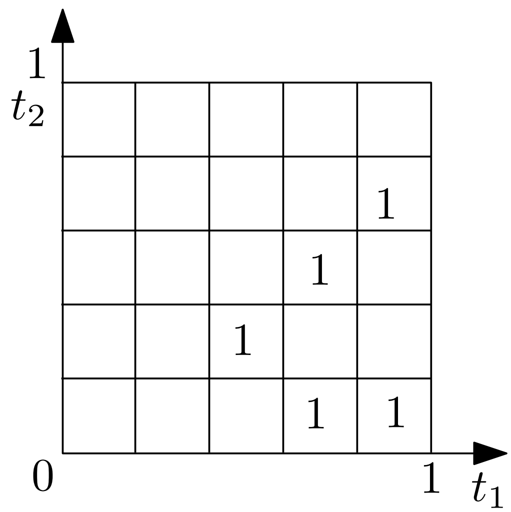

Figure 35.1: An allocation rule that is NDE but not ND.

the types of the two agents fall within that tile. In all the other tiles the object is allocated to agent 2. We see that the expected allocation of agent 1

$$\alpha_1(t_1) = \begin{cases} 0 & \text{for } 0 \leqslant t_1 < 2/5 \\ 1/5 & \text{for } 2/5 \leqslant t_1 < 3/5 \\ 2/5 & \text{for } 3/5 \leqslant t_1 < 4/5 \\ 2/5 & \text{for } 4/5 \leqslant t_1 \leqslant 1 \end{cases} \tag{35.1}$$

Hence the allocation of agent 1 is NDE, similarly it can be shown that allocation of agent 2 is NDE too. However, it is clear that the allocation is not ND, since for $t_2 \in [1/5, 2/5)$, agent 1's allocation decreases from 1 to 0.

35.2.2 Interim Individual Rationality

In the Bayesian setup, we need to modify the definition of individual rationality as follows.

Definition 35.4 (Interim Individual Rationality) *A mechanism (f, p) is* interim individually rational *(IIR) if for every agent $i \in N$, we have*

$$t_i \cdot \alpha_i(t_i) - \pi_i(t_i) \geqslant 0, \forall t_i \in T_i.$$

The following result summarizes the requirements of a BIC and IIR mechanism.

Lemma 35.5 *A mechanism (f, p) is BIC and IIR iff*

1. *f is NDE,*
2. *p_i satisfies*

$$\pi_i(t_i) := \pi_i(0) + t_i\alpha_i(t_i) - \int_0^{t_i} \alpha_i(s_i)ds_i, \forall t_i \in T_i, i \in N,$$

3. $\forall i \in N,\ \pi_i(0) \leqslant 0.$

Proof: Conditions (1) and (2) characterize a BIC mechanism (Theorem 35.3). Hence, it remains to be shown that IIR together with (1) and (2) is equivalent to (3).

(IIR + 1 + 2 $\Rightarrow$ 3): apply IIR at $t_i = 0$ in (2) to get $\pi_i(0) \leqslant 0$.

(IIR + 1 + 2 $\Leftarrow$ 3): $t_i \cdot \alpha_i(t_i) - \pi_i(t_i) = -\pi_i(0) + \int_0^{t_i} \alpha_i(s_i)ds_i \geqslant 0$ if $\pi_i(0) \leqslant 0$. ∎

35.3 Single Agent Problem

TO understand the dynamics of an optimal auction, we focus on the problem when there is exactly one buyer. The setting here is $T = [0, \beta]$, the mechanism is $(f, p) =: M$ where $f : [0, \beta] \mapsto [0, 1]$ and $p : [0, \beta] \mapsto \mathbb{R}$. The desirable properties are

- Incentive compatibility [DSIC and BIC are equivalent]

$$tf(t) - p(t) \geqslant tf(s) - p(s), \forall s, t \in T.$$

- Individual rationality [IR and IIR are equivalent]

$$tf(t) - p(t) \geqslant 0, \forall t \in T.$$

The revenue earned by the mechanism M is given by

$$\Pi^M := \int_0^\beta p(t)g(t)dt.$$

An **optimal mechanism** is a mechanism M^* such that it is IC and IR and satisfies $\Pi^{M^*} \geqslant \Pi^M, \forall M$.

From IC characterization, we know that

$$p(t) = p(0) + tf(t) - \int_0^t f(s)ds.$$

But from IR requirement, we have $p(0) \leqslant 0$. Since our goal is to maximize revenue, clearly $p(0) = 0$. Hence the payment reduces to

$$p(t) = tf(t) - \int_0^t f(s)ds.$$

This is completely given by the allocation rule f. Since the prior is G (with density g), the expected revenue is given by

$$\begin{aligned}\Pi^f &= \int_0^\beta p(t)g(t)dt \\ &= \int_0^\beta \left(tf(t) - \int_0^t f(s)ds\right) g(t)dt.\end{aligned}$$

Theorem 35.6 *For any implementable allocation rule f, the revenue earned is given by*

$$\Pi^f = \int_0^\beta w(t)f(t)g(t)dt$$

where $w(t) = \left(t - \frac{1-G(t)}{g(t)}\right)$, which is known as the virtual valuation of the agent.