Game Theory Assignment 2

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1 Question 1

No. of agents, $N = 2$
Set of alternatives, $A = \{a, b, c\}$
We are also given that the SCF f is *onto*.
The given two preference profiles are:

P_1	P_2	P_1'	P_2'
a	c	b	a
b	b	a	b
c	a	c	c

$f(P_1, P_2) = a$

Part (a):
Given: The domain of preference profiles is of unrestricted strict preferences. f is strategyproof.
To prove: $f(P_1', P_2') = b$.

Proof: Consider the preference order $P_2'' = (c > a > b)$. We will use it to prove the result.
Note that since f is strategyproof, it is also monotonous.
Let $P = (P_1, P_2), P' = (P_1', P_2'), P'' = (P_1, P_2''), P''' = (P_1', P_2'')$
Now $D(a, P_1) \subseteq D(a, P_1)$, $D(a, P_2) \subseteq D(a, P_2'')$ and $f(P) = a$.
$\therefore f(P) = f(P'') = f(P_1, P_2'') = a$ from monotonicity of f.
Now, let if possible, $f(P''') = c$.
But then $aP_1'c$, that is, $f(P_1, P_2'')P_1'f(P_1', P_2'')$, which contradicts f is strategyproof.
But we also know that $f(P_1', P_2'') \in \{P_1'(1), P_2''(1)\}$, that is, $f(P_1', P_2'') \in \{b, c\}$.
$\therefore f(P''') = f(P_1', P_2'') = b$.
But now $D(b, P_1') \subseteq D(b, P_1')$, $D(b, P_2'') \subseteq D(b, P_2')$ and $f(P''') = b$.

$\therefore$ From monotonicity of f we have $f(P''') = f(P') = f(P_1', P_2') = b$.
Hence proved.

Part (b):
Now the domain of preferences is restricted to the single-peaked preferences domain.
Intrinsic ordering of preferences is $a < b < c$.
The earlier conclusion does not hold in this case. This is because the earlier proof made use of the preference order P_2'' of agent 2 in the preference profiles P'' and P'''. But this preference order is not a single-peaked preference, hence the earlier proof does not go through.

Consider the following SCF;
$f(P_1^*, P_2^*) = min\{P_1^*(1), P_2^*(1)\}$, for some preference profile (P_1^*, P_2^*)
First it is easy to note that this SCF is unanimous. Hence it is also onto.
Now $f(P_1, P_2) = min\{a, c\} = a$ and $f(P_1', P_2') = min\{b, a\} = a$.
Therefore, this provided SCF is onto with $f(P_1, P_2) = a$ and $f(P_1', P_2') = a$.

2 Question 2

set of projects $= X$
$\therefore$ set of alternatives $= A = \{Y|Y \subseteq X \ and Y \neq \phi\}$
Note that, since $|X| \geq 2$ we have $|A| \geq 3$.

Now consider $V, W \in A$ such that $V \subset W$.

Therefore, for any given agent i and preference P_i;
Case 1: If the highest ranked project in W is in V, agent i is indifferent between alternatives W and V at profile P_i.
Case 2: else, if the highest ranked project in W is in $W - V$, then agent i prefers W over V.

Therefore it can never happen that alternative V is preferred over alternative W, for all such pairs of alternatives V, W defined above.
Therefore this is a setting where the preference profiles are restricted. All possible orderings of the alternatives are not possible.

Hence we may not be able to apply the Gibbard-Satterthwaite theorem here.

3 Question 3

Yes, the median voter SCF is group strategyproof under the single-peaked domain model. This is proved below.

Given: The preferences are single-peaked.
To prove: Median voter SCF is group strategyproof.

Proof: Let, if possible, there exists a group $K \subseteq N$ which can manipulate the median voting rule. Lets call the median of the true preferences, m. And m itself is part of the left side of m. (without loss of generality).

Case 1:
The actual peaks of all the members of K are on the same side of m.
Without loss of generality, let all members of K have their peaks to the left side of m. Then;

i) if the reported peaks of all agents $i \in K$ are to the left of m, the output of the SCF doesn't change. Hence, such a manipulation is useless.
ii) if any of the agents in K report their peaks to be to the right of m, the median (hence the output of the SCF) will shift to the right; farther away from the actual peaks of all the agents in K. Hence, such a manipulation is useless as well.
Therefore, there are no beneficial manipulations in this case.
The argument for the case when the actual peaks of all the agents in K are to the right of m,
is symmetrical.

Case 2:
$K = L \cup R$, where the actual peaks of all members of L are to the left of m, and that of R are to the right of m. $L, R \neq \phi$.
Now any manipulation that causes the output of SCF to shift to the left of m, benefits members of L, but members of R are worse off. Similarly, any manipulation that causes the output of the SCF to shift to the right of m, benefits members of R, but then members of L are worse off.
Therefore there doesn't exist any manipulation (at any preference profile), that benefits all the members of such a group K. Therefore no such group K satisfies the definition of group-manipulable given in the question.

Combining cases 1 and 2 exhausts all possible types of K. Therefore the assumption that there exists a group K which can manipulate the median voter SCF, is false. Therefore the median voter SCF is group strategy-proof.