Recap: Stability at a profile where no pair of men and women can block an allocation/matching.

— Deferred acceptance algorithm ensures stability (both versions)

Question: comparison between stable matches?

Defn: A matching μ is men-optimal stable matching if μ is stable and for every other stable matching μ' we have $\mu(m) \, P_m \, \mu'(m)$ or $\mu(m) = \mu'(m)$ $\forall \, m \in M$.

Similar definition for women-optimal stable matching.

Remark: If there exists two ~~men~~ men-optimal stable matching, then they must differ for at least one man (in fact for two men), and since preferences are strict this man must be worse off in one of these stable matches - and hence the men-optimal stable matching is unique.

Theorem: The men-proposing version of the Deferred Acceptance algorithm terminates at the unique men-optimal stable matching.

[similarly, the women-proposing version terminates at the women-optimal stable matching].

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Proof: Define a woman w is "possible" for a man m if (m, w) is matched in some stable matching.

In a stable matching, for every man who is matched to a woman, there exists at least the same number of men as the number of women above the matched woman ~~on~~ (acc to his preference) in the preferences of those women.

Claim: A woman who is possible for a man never rejects him in this algorithm.

Proof via induction: At stage 1, ~~no one rejects~~ clearly this holds, if someone is rejected, must be impossible hence the claim holds. Suppose this is true till stage n.

Suppose at round n+1, woman w rejects m in favor of m'.

- this implies that m' approached w in round ~~n~~ n+1 ~~at~~ and all women that m' prefers to w (whom he made prior approaches but were rejected) must be impossible for him - according to the induction hypothesis.

Then w must be impossible for m.

- Suppose not, there exists some stable match involving (m, w), then m' must be matched to someone else.
  - cannot put m' with women above w (they are impossible for him)
  - cannot put with women below w since then (m', w) makes a blocking pair.

Hence the claim is proved.

~~Aco~~ The men-proposing DA algorithm gives every man their ~~be~~ most preferred "possible" woman. - hence men-optimal.

Let us denote the men-optimal stable match as μ^m and women-optimal stable match as μ^W

Question: Can both sides be happy? i.e., does there exist a matching that is both sides' optimal?

The general answer is NO. We saw example of mens and women optimal solutions to be different. But something more is true. Let us explore the structure of the stable matchings a bit more.

Theorem: Let μ and μ' be a pair of stable matchings.

Then $\quad \mu(m) \; P_m \; \mu'(m) \;$ or $\; \mu(m) = \mu'(m) \quad \forall m \in M$

iff $\quad \mu'^{-1}(w) \; P_w \; \mu^{-1}(w) \;$ or $\; \mu'^{-1}(w) = \mu^{-1}(w) \;$ for all $w \in W$.

Proof: ($\Rightarrow$) [The other direction is very similar]

Let μ and μ' be s.t.

$$\mu(m) \; P_m \; \mu'(m) \text{ or } \mu(m) = \mu'(m) \quad \forall m \in M.$$

Suppose for contradiction

$$\mu^{-1}(w) \; P_w \; \mu'^{-1}(w) \quad \text{for some } w \in W.$$

Let $\mu^{-1}(w) = m$ and from above $\mu'^{-1}(w) \neq m \Rightarrow \mu'(m) \neq w$

hence $\quad \underbrace{\mu(m)}_{=w} \; P_m \; \mu'(m)$

Then (m, w) forms a blocking pair of μ'.

Contradiction to the fact that μ' is stable $\quad \square$

The previous two theorems say that the men-optimal stable matching is the worst stable matching for women and vice-versa.

We can define a binary relation between stable matchings.

Defn: We say $\mu \triangleright \mu'$ if for every $m \in M$, either $\mu(m) P_m \mu'(m)$ or $\mu(m) = \mu'(m)$.

[Equivalently, $\mu'^{-1}(w) P_w \mu^{-1}(w)$ or $\mu'^{-1}(w) = \mu^{-1}(w)$ $\forall w \in W$]

Note: $\triangleright$ is not a complete relation, cannot compare all stable matchings. But an immediate corollary of the previous two theorems.

Corollary: For any stable matching μ,

$$\mu^m \triangleright \mu \triangleright \mu^w .$$

There are more structures of stable matchings.

For any pair of stable matchings μ, μ', we can construct another matching $\mu'' \equiv (\mu \vee^m \mu')$ as follows: for every $m \in M$

$$\mu''(m) = \begin{cases} \mu(m) & \text{if } \mu(m) P_m \mu'(m) \text{ or } \mu(m) = \mu'(m), \text{ and} \\ \mu'(m) & \text{if } \mu'(m) P_m \mu(m). \end{cases}$$

Hence

$$(\mu \vee^m \mu')(m) = \max_{P_m} \left(\mu(m), \mu'(m)\right)$$

Similarly we can define such a matching using the womens' preferences

$$(\mu \vee^{w} \mu')^{-1}(w) = \max_{P_w} \{ \mu^{-1}(w), \mu'^{-1}(w) \}.$$

Not clear if μ'' is a matching. But the next result ~~shows~~ answers that.

<u>Theorem</u>: For every pair of stable matchings μ and μ' both $(\mu \vee^{m} \mu')$ and $(\mu \vee^{w} \mu')$ are stable matchings.

<u>Proof</u>: <u>Part 1: μ'' is a matching</u>

Say for contradiction μ'' is not a matching

$\exists$ some $m, m' \in M$ s.t.

$\mu''(m) = \mu''(m')$

	μ''	μ	μ'
m	w	w	
m'	w		w

Then it must be the case that for one of μ or μ' m is assigned w and for the other m' is assigned w, WLOG assume

$\mu(m) = w$ and $\mu'(m') = w$

also $w \, P_m \, \mu'(m)$ and <u>$w \, P_{m'} \, \mu(m')$</u>.

Now μ' is a stable matching

<u>$m' \, P_w \, m$</u>

[Else, $m \, P_w \, m'$ and (m, w) is a blocking pair of μ']

But now, (m', w) forms a blocking pair to μ.

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Part 2: μ'' is a stable matching

Assume for contradiction (m, w) is a blocking pair of μ''. Hence

	μ''	μ	μ'
m	w_1	w_1	w_2
m_1	w	?	?

$\mu''(m) = w_1$ but $m \, P_w \, m_1$

$\mu''(m_1) = w$ $\quad w \, P_m \, w_1$

By definition of μ'', either μ or μ' matches m to w_1, WLOG assume

$w_1 = \mu(m)$ and $w_2 = \mu'(m)$, where ~~w_2 can~~ w_1 can be same as w_2

By definition of μ'', $w_1 \, P_m \, w_2$ or $w_1 = w_2$

in both cases,

$w \, P_m \, w_2$ and we already have $w \, P_m \, w_1$.

Now for m_1, ~~either~~ w must be matched to him either in μ or μ', we show neither is possible which is a contradiction.

a) if $w = \mu(m_1)$, $m \, P_w \, m_1$ and $w \, P_m \, w_1 \Rightarrow w \, P_m \, \mu(m)$

so (m, w) blocks μ.

b) if $w = \mu'(m_1)$, $m \, P_w \, m_1$ and $w \, P_m \, w_2 \Rightarrow w \, P_m \, \mu'(m)$

$\Rightarrow (m, w)$ blocks μ'.

A similar proof for $(\mu \vee^w \mu')$ $\square$

Using this result, given any pair of stable matchings, one can move towards men or women optimal.

Exercise: a similar definition using min $(\mu \wedge^m \mu')$, $(\mu \wedge^w \mu')$