Lec 9

Part 2: Uniqueness.

Define a game called the ~~as~~ carrier game

A coalition is winning if it contains a distinguished set.

Intuition: say T is an influential coalition - any ~~co~~ coalition containing it can pass a bill/do a change.

Defn: Let $T \subseteq N$ be a non-empty coalition. The carrier game over T is the game (N, u_T) s.t. for each coalition $S \subseteq N$

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{ow.} \end{cases}$$

Theorem 1: Every game (N, v) is a linear combination of carrier games.

To define any TU game, we need to define the valuations over all non-empty subsets. Hence it has 2^n-1 degrees of freedom. Hence, every game (N, v) is a point in $\mathbb{R}^{2^n-1}$. Want to show that carrier games span this space. Find carrier games that are linearly independent and forms a basis.

Suppose, carrier games are linearly dependent (for contradiction). $\exists$ real numbers $(\alpha_T)_{\{T \subseteq N, T \neq \phi\}}$ not all zero, s.t.

$$\sum_{\{T \subseteq N, T \neq \phi\}} \alpha_T u_T(S) = 0, \quad \forall S \subseteq N.$$

Let $\mathcal{T} = \{T \subseteq N : T \neq \phi, \alpha_T \neq 0\}$ collections of non-empty coalitions with non-zero coefficients in the above equation. Since $\{\alpha_T\}_{\{T \subseteq N, T \neq \phi\}}$ are not all zero, $\exists$ a minimal coalition in $\mathcal{T}$, i.e. coalition with smallest cardinality. Say $S_0 \in \mathcal{T}$ is one such coalition. $\nexists$ any subset of S_0 with positive coefficients. Consider

$$\sum_{\{T \subseteq N, T \neq \phi\}} \alpha_T u_T(S_0) = \underbrace{\sum_{\{T \subset S_0, T \neq \phi\}} \alpha_T u_T(S_0)}_{=0} + \alpha_{S_0} u_{S_0}(S_0) + \underbrace{\sum_{T \not\subseteq S_0} \alpha_T u_T(S_0)}_{=0}$$

$$= \alpha_{S_0} \neq 0$$

$\square$

Theorem 2: Let T be a non-empty coalition, and $\alpha \in \mathbb{R}$. Define a game $(N, u_{T,\alpha})$ as follows

$$u_{T,\alpha}(S) = \begin{cases} \alpha & \text{if } T \subseteq S \\ 0 & \text{ow} \end{cases}$$

If ϕ is a solution concept that satisfies efficiency, symmetry, null player property, Then

$$\phi_i(N, u_{T,\alpha}) = \begin{cases} \frac{\alpha}{|T|} & \text{if } i \in T \\ 0 & \text{ow} \end{cases}$$

Obs 1: If $i \notin T$ $\quad u_{T,\alpha}(S \cup \{i\}) = u_{T,\alpha}(S) \quad \forall S \subseteq N$

i is a null player

obs 2: If $i, j \in T$, they are symmetric.

$$u_{T,\alpha}(S \cup \{i\}) = u_{T,\alpha}(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i,j\}$$

~~From~~

From the fact that ϕ is efficient, symmetric, null-player compliant solution concept, the result follows.

Finishing the proof of Part 2: Uniqueness

Shapley value satisfies the four properties.

Need to show: any ϕ satisfying these four properties is identical to Sh.

Thm 1 says that for any game (N, v), we can write v as sum of u_{T,α_T}'s

$\exists (\alpha_T)_{\{T \subseteq N,\, T \neq \phi\}}$ s.t.

$$v(S) = \sum_{\{T \subseteq N,\, T \neq \phi\}} u_{T,\alpha_T}(S)$$

Thm 2 says, since both ϕ and Sh satisfies efficiency, symmetry, and null player property

$$\phi(N, u_{T,\alpha_T}) = Sh(N, u_{T,\alpha_T}), \quad \forall T \subseteq N, T \neq \phi.$$

Since both ϕ and Sh satisfy additivity

$$\phi(N,v) = \sum_{\{T \subseteq N, T\neq \phi\}} \phi(N, u_{T,\alpha_T}) = \sum_{\{T \subseteq N, T \neq \phi\}} Sh(N, u_{T,\alpha_T}) = Sh(N,v).$$

we started with an arbitrary game (N,v), hence this holds for all such games. $\square$

Examples

① Two player bargaining: (N, v)

$v(1) = v(2) = 0, \quad v(1,2) = 1$

symmetric players: 1 & 2, Shapley value is efficient

$Sh(N,v) = \left(\frac{1}{2}, \frac{1}{2}\right)$

② Majority game

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq \frac{n}{2} \\ 1 & \text{if } |S| > \frac{n}{2} \end{cases}$$

all players are symmetric, hence Shapley values are same, together with efficiency

$Sh(N,v) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$

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(3) Gloves game

$$v(1) = v(2) = v(3) = v(1,2) = 0$$
$$v(1,3) = v(2,3) = v(1,2,3) = 1$$

Perm	Player 1	Player 2	Player 3
1, 2, 3	$v(1) - v(\phi) = 0$	$v(1,2) - v(1) = 0$	1
1, 3, 2	0	$v(123) - v(13) = 0$	1
2, 1, 3	$v(1,2) - v(2) = 0$	0	1
2, 3, 1	$v(1,2,3) - v(2,3) = 0$	0	1
3, 1, 2	$v(1,3) - v(3) = 1$	0	0
3, 2, 1	0	$v(2,3) - v(3) = 1$	0
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$

$Sh(N, v) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$

emphasizes that ~~the~~ player 3 is the most powerful player.
but players 1 and 2 do not get zero in the allocation.

Core : $(0, 0, 1)$ is the singleton.

Hence Shapley value is not in core.

Application: Shapley - Shubik power index

Defn : (Simple, monotone games)

Simple games : The value of any coalition can either be 0 or 1.

Monotone games : If any coalition has value 1, every superset of that coalition also has value 1.

Motivation: to model legislations/ decisions based on committees.

Defn: The Shapley-Shubik power index is a function associating each simple monotonic game with its Shapley value. The i^{th} co-ordinate denotes the "power" of player i in this game.

$$Sh_i(N,v) = \sum_{\substack{\{S \subseteq N\setminus\{i\}: \\ S\cup\{i\} \text{ is winning} \\ S \text{ is losing}\}}} \frac{|S|!\,(n-|S|-1)!}{n!}$$

counting all such scenarios where ~~step~~ player i is pivotal.

Case study: UN Security council

UN: Body of international political system, established in 1945 (after WW II)

till 1965: five permanent members, six nonpermanent members

Resolution adopted if it receives at least 7 votes but all permanent members have to be unanimous - all of them have veto powers.

Debated about unequal distribution of powers in the security council.

after 1965: five permanent members, 10 nonpermanent members resolution needed 9 votes but veto power remains with the permanent members

This is a simple, monotonic game. Compute the Shapley-Shubik power index.

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P: permanent members , NP: non permanent members

pre-1965:

$$v(S) = \begin{cases} 1 & \text{if } S \supseteq P \text{ and } |S| \geq 7 \\ 0 & \text{ow} \end{cases}$$

non permanent i

$$Sh_i(N,v) = \binom{5}{1} \frac{6!\,4!}{11!} = \frac{1}{462}$$

all non-permanent members are symmetric
all permanent members are symmetric
Shapley value is efficient, hence for a permanent j

$$Sh_j(N,v) = \frac{1}{5}\left(1 - \frac{6}{462}\right) = \frac{91\cdot 2}{462}$$

Power ratio of nonpermanent to permanent = $1 : 91\cdot 2$.

post-1965:

$$v(S) = \begin{cases} 1 & \text{if } S \supseteq P \text{ and } |S| \geq 9 \\ 0 & \text{ow} \end{cases}$$

non-permanent i:

$$Sh_i(N,v) = \binom{9}{3} \frac{8!\,6!}{15!} = \frac{4}{2145}$$

permanent j:

$$Sh_j(N,v) = \frac{1}{5}\left(1 - 10 \times \frac{4}{2145}\right) = \frac{421}{2145}$$

ratio = $1 : 105\cdot 25$

~~Restructi~~ Restructuring actually increased the power of the permanent members.

Convex games

$$\forall S, T \subseteq N$$

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

Thm: If (N, v) is a convex game, the SV is in the core.

Proof: For any permutation $\pi \in \Pi(N)$, consider the imputation w^{π}

$$w_i^{\pi} = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)).$$

We have shown that this imputation is in core for every $\pi \in \Pi(N)$. Since core is a convex set, any ~~the~~ convex combination of these points will be in core. In particular, the Shapley value

$$Sh(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} w^{\pi}$$

will also be in the core.

Consistency of the SV.

Defn: Let ϕ be a single valued solution concept, let (N, v) be a coalitional game and $S \subseteq N$ and $S \neq \phi$. The Hart-Mas-colell reduced game over S relative to ϕ is the game $(S, \tilde{v}_{S,\phi})$ s.t.

$$\tilde{v}_{S,\phi}(R) = \begin{cases} v(R \cup S^c) - \sum_{i \in S^c} \phi_i(R \cup S^c, v), & \forall R \subseteq S, R \neq \phi \\ 0 & \text{if } R = \phi \end{cases}$$

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Difference with Davis-Maschler reduced game

- solution concept for single-valued solutions
- DM selects the most beneficial coalition of S^c

but HM considers the whole of S^c.

Defn: A solution concept ϕ is consistent wrt the HM reduced game if for every game (N, v), every nonempty coalition S, and for every $i \in S$,

$$\phi_i(N, v) = \phi_i(S, \tilde{v}_{S,\phi})$$

Theorem: SV is HM reduced game consistent.