Cooperative Game Theory

A game paradigm where agents can communicate and jointly fix strategies. However, there is a threat to the jointly fixed strategies, whether it is self-enforcing. Will they not gain by breaking the agreement?

E.g. Prisoner's dilemma, both prisoners can jointly decide to play "cooperate", but one agent can deviate and make her better off. The self-enforcing criteria in cooperative games come from the idea of correlated equilibrium.

Correlated equilibrium concept was introduced by Robert Aumann in 1974.

Ex1: Traffic signal.

1 \ 2	Drive	Stop
D	−10, −10	5, 0
S	0, 5	0, 0

Two pure Nash equilibrium

- good prediction if the players are non-cooperative.
- but how should an individual driver know which equilibrium to play? need for a mediator.
- traffic police / lights do this role of a mediator.
- given the trusted mediator gives a specific "suggestion" to each of the players, is it best for them to follow this?
- depends on how the mediator picks the suggestions
  - the distribution over the strategy profiles
  - e.g. if the traffic lights pick (S,D) and (D,S) w.p. $\frac{1}{2}$ each, perhaps it is best for the drivers to follow this.

Ex 2: Game Selection Problem

1 \ 2	C	F
C	2,1	0,0
F	0,0	1,2

This game has two pure strategy and one mixed strategy Nash equilibrium.

$\left(\frac{2}{3}, \frac{1}{3}\right) \rightarrow 1 \quad \left(\frac{1}{3}, \frac{2}{3}\right) \rightarrow 2$

Expected utility of every player $= \frac{2}{3}$

Correlated Strategy

Defn: A correlated strategy is a mapping $\pi : S \rightarrow [0,1]$ s.t. $\sum_{s \in S} \pi(s) = 1$, where $S = S_1 \times S_2 \times \cdots \times S_n$, S_i denotes the strategy of player i.

A correlated strategy π is a joint probability distribution over the strategy profiles.

A correlated strategy become correlated equilibrium if it becomes 'self enforcing', i.e., no player 'gains' by deviating from it.

π is a common knowledge.

Correlated equilibrium (Aumann '74)

Defn: A correlated equilibrium is a correlated strategy π s.t. $\forall s_i \in S_i$ and $\forall i \in N$

$$\sum_{\underline{s}_i \in \underline{S}_i} \pi(s_i, \underline{s}_i)\, u_i(s_i, \underline{s}_i) \geq \sum_{\underline{s}_i \in \underline{S}_i} \pi(s_i, \underline{s}_i)\, u_i(s_i', \underline{s}_i) \quad \forall s_i' \in S_i.$$

Interpretations

(1) Player i does not gain any advantage (in expected utilities) if she deviated from the suggested action, when others listen to the suggestion of the trusted mediator.

(2) A correlated equilibrium is a randomization device (e.g. a dice or coin) which gives a random outcome of a strategy profile, but a player only observes the strategy corresponding to her. Given that observation, she computes ~~the~~ her expected utility - if no other strategy gives her a strict better utility, and if this is true for every agent, the randomization device is a correlated equilibrium.

Illustration with the examples

(1) Game selection problem.

Consider the correlated strategy

$\pi(C,C) = \frac{1}{2} = \pi(F,F)$, $\pi(C,F) = \pi(F,C) = 0$.

Suppose player 1 is suggested to play F.

$$\sum_{s_{-1} \in S_{-1}} \pi(F, s_{-1})\, u_1(F, s_{-1}) = \pi(F,F)\, u_1(F,F) + \pi(F,C)\, u_1(F,C)$$

$$= \frac{1}{2} \cdot 1 + 0 \cdot 0 = \frac{1}{2}$$ [utility from following]

$$\sum_{s_{-1} \in S_{-1}} \pi(F, s_{-1})\, u_1(C, s_{-1}) = \pi(F,F)\, u_1(C,F) + \pi(F,C)\, u_1(C,C)$$

$$= 0$$ [utility from violating the suggestion]

similar case for ~~player~~ the player when C is suggested and for player 2.

(2-4)

Expected utility of every player $\frac{1}{2}(2+1) = 3/2$
as opposed to $\frac{2}{3}$ in the MSNE.

(2) Traffic signal

Consider correlated strategy

$\pi(SS) = \pi(SG) = \pi(GS) = \frac{1}{3}$

1 \ 2	S	G
S	0,0	1,2
G	2,1	-10,-10

Player 1 suggested to stop,

$$\sum_{s_{-1} \in S_{-1}} \pi(S, s_{-1})\, u_1(S, s_{-1}) = \pi(SS)\,u_1(SS) + \pi(SG)\,u_1(SG)$$
$$= \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 = \frac{1}{3}$$

by violating

$$\sum_{s_{-1} \in S_{-1}} \pi(S, s_{-1})\, u_1(G, s_{-1}) = \pi(SS)\,u_1(GS) + \pi(SG)\,u_1(GG)$$
$$= \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot(-10) = -\frac{8}{3}$$

Similar conclusions when suggested to go and for player 2.

Interpretation from a best response viewpoint.

The best response set of i for s_i in a correlated strategy π is

$$B_i(\pi, s_i) = \arg\max_{\tilde{s}_i \in S_i} \left[\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i})\, u_i(\tilde{s}_i, s_{-i}) \right]$$

π is a CE if $\forall s_i \in S_i \; \forall i \in S_i$

$$s_i \in B_i(\pi, s_i).$$

Computation of Correlated Equilibrium.

Feasibility LP

Variables : $\pi(s)$, $\forall s \in S$.

$$\sum_{s_{-i} \in S_{-i}} \pi(s)\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s)\, u_i(s_i', s_{-i}) \quad \forall s_i, s_i' \;\; \forall i \in N$$

-- $O(nm^2)$ inequalities

$$\pi(s) \geq 0 \quad \forall s \in S \qquad \text{-- } O(m^n)$$

$$\sum_{s \in S} \pi(s) = 1 \qquad \text{--- } 1$$

Compare this with MSNE computation which needed all supports of every player to be enumerated hence complexity $O(2^{mn})$ – exponentially larger than CE computation.

- CE guarantees a cooperative self-enforcing decision using a trusted mediator
- Much easier to compute
- Quite natural in several game settings than a non-cooperative solution.

Axiomatic Bargaining

– Earliest results in cooperative game theory

– Axioms represents the goal of a designer when a solution is proposed,
e.g., how to divide a joint profit ?

(2-6)

Axiomatic bargaining was introduced by Nash 1950

Bargaining refers to

- two individuals have the possibility of concluding to a mutually beneficial agreement
- there is a conflict of interest on which agreement to conclude
- no agreement may be imposed without every player's approval.

The payoff of agents ~~depend on~~ has two components

- payoffs ~~a~~ in case the negotiation fails
- payoffs jointly feasible by their negotiation/arbitration.

Examples:

(1) Management - Labor arbitration

(2) International relations

(3) Property settlement.

The Bargaining Problem

Two person bargaining problem consists of a pair (F, v), where F is the feasible set and v is the 'disagreement point'.

- F is a closed, convex subset of $\mathbb{R}^2$
- $v = (v_1, v_2) \in \mathbb{R}^2$ represents the disagreement payoff allocation for the two players, default point.
- $F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1, x_2 \geq v_2\} \neq \phi$, bounded.

Discussions on the assumptions:

- Convexity due to the natural reason of correlated equilibrium. A convex combination of two potential actions that gives that utility will also be a correlated equilibrium.
- Closedness - sequence of bargaining converving to an infeasible solution.
- $\exists$ feasible solution points above the disagreement and not unbounded.