CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

1.1 Solution to Question 1:

(Part a)Given preference profile P:

P_1	P_2
a	c
b	b
c	a

and f(P_1,P_2)=a.
For preference profile P':

$P_1{}'$	$P_2{}'$
b	a
a	b
c	c

From the result given in question f($P_1{}'$,$P_2{}'$) $\in \{a, b\}$. ...(1)
Let a preference profile Q given by:

$P_1{}'$	P_2
b	c
a	b
c	a

Here by the result given in question f($P_1{}'$, P_2) $\in \{b, c\}$.
Assume f($P_1{}'$, P_2)=c.
$P_1{}'$ can manipulate the result by changing its preference order to P_1 then f($P_1{}'$, P_2) will reduce to f(P_1,P_2)=a but as f is strategy-proof it contradicts the fact as strategy proof function cannot be manipulated.Thus it is a contradiction to fact f($P_1{}'$, P_2)=c and thus f($P_1{}'$, P_2)=b.
Let another preference profile R such that where P_3 is a agent who has priority as b<a<c:

$P_1{}'$	P_3
b	c
a	a
c	b

Here by the result given in question f($P_1{}'$, P_3) $\in \{b, c\}$.
Preference profile R follows monotonicity with preference profile Q as c is least preferred by $P_1{}'$ in both preference profile and the same alternative is most preferred by its other agent in the preference profile.
Thus f($P_1{}'$,P_3)=f($P_1{}'$, P_2)=b

From (1) we know that f($P_1{}'$, $P_2{}'$) $\in \{a, b\}$.
Assume f($P_1{}'$, $P_2{}'$)=a.
$P_2{}'$ can manipulate the result by changing its preference order to P_3 then f($P_1{}'$, P_2) will reduce to f(P_1, P_2)=b but as f is strategy-proof it contradicts the fact as strategy proof function cannot be manipulated.Thus it is a contradiction to fact f($P_1{}'$, $P_2{}'$)=a and thus f($P_1{}'$, $P_2{}'$)=b.

(Part b)-Let these preferences are generated from a single-peaked preference domain. The earlier conclusion does not hold in this case as we assumed a preference order P_3 in part (a) which assumes a ordering b<a<c which is not possible in a single peaked preference domain and thus earlier proof donot go through in this case.
For the part of finding the mechanism such that f($P_1{}'$, $P_2{}'$)=a Lets use the median voting SCF with a phantom peak just on the left of a.In that case f($P_1{}'$, $P_2{}'$)= a as we have peaks at left of a(phantom peak),a and b(greater than a) and thus a is the median peak and this mechanism also satisfies preference profile P.

1.2 Answer to Question 2:

Let us take a case:
Let S and T be two subsets of X and S $\subset$ T then the outcomes are achieved by cases:
1)S indifferent to T- This will be possible when S $\cap$ T contains the highest priority element of S $\cup$ T.
2)T P_i S- This will be possible when S doesnot contains highest priority element of S $\cup$ T and T contains the highest priority element of S $\cup$ T.
3)S P_i T- This is impossible case as it implies that the highest priority element of S $\cup$ T is in S and not T which is not possible as S $\subset$ T.
So, here the preferences are restricted to only two outcomes out of three outcome which are possible.
As we know that in a setting where preferences are restricted,Gibbard-Satterthwaite theorem may not hold and thus Gibbard-Satterthwaite result cannot be applied here.

1.3 Answer to Question 3:

Aim: To prove median voter SCF group strategy-proof.
As we have assumed single-peaked domain model we need to consider only the peak preferences of all agents.So let us denote the preferences denoted only by their peaks that is P=(p_1,p_2,p_3,...,p_i,....,p_n) and let f(P)=a $\in$ A is the median of these peaks and phantom peaks.Let us take a subset of voters represented as P_K then the preference profile can be represented as (P_K,P_{-K}).
For the question we have to prove that the social choice function f is group strategy-proof that is it cannot be manipulated by group of agents.Mathematically,
A social choice function f is manipulable by a group of agents K $\subseteq$ N if for some preference profile (P_K, P_K) there exists some preference profile $P_K{}'$ of agents in K such that f($P_K{}'$, P_K) P_i f(P_K, P_K) for all $\in$ K.
Let us take a set of voters out of which some are peaked at point on left of median voter, few are peaked at point on right of median voter and rest peaked at median point itself.Let us see individual situation for them:
1)Ones which are peaked at median position-These will not try to manipulate as they prefer median and no reason to deviate from median.
2)Ones which are peaked to left of median voter-If they try to manipulate to left side then also median voter doesnot changes.When they try to shift to right of median voter then only median will deviate.This deviation will not happen because this voter preference peak is to the left of median voter and manipulating its choice to right side just decreases its priority and no one wants to select less priority in order to shift median.Thus it will not be able to manipulate the median voting.

3)Ones which are peaked to right of median voter-Similar case as the peaked to left side of median and thus will not try to manipulate.
Thus median voter SCF is group strategy proof as no one tries to modify its preference order and thus no group of any size tries to manipulate the scf as any manipulation may not be beneficial to anyone in the group.