Priyesh Zalani 160522 **CS711**

Project: Assignment 2

1.1 Problem 1

1.1.1 A.

We are given that f is onto SCF with $f(P_1, P_2) = a$ such that all possible linear orderings of alternatives are possible as preference profiles.
We know that when$f$ is *onto* and *strategyproof*, then for every preference profile (Q_1, Q_2), $f(Q_1, Q_2) \;\epsilon \{\, Q_1(1), Q_2(1)\}$ and **SP** $\Longleftrightarrow$ **MONO**. By considering some transition preference profiles, we will prove that $f(P_1', P_2') \;=\; b$.

1. Consider the profile:

P_1	Q_2
a	c
b	a
c	b

Here, the position of a improves. By Monotonocity: $f(P_1, Q_2) = a$

2. Consider the profile:

$Q_1 = P_1'$	Q_2
b	c
a	a
c	b

Here, $f(P_1', Q_2) \;\epsilon\; \{b, c\}$.

Suppose, $f(P_1', Q_2) = c$, then $f(P_1, Q_2) = c$. But, $f(P_1, Q_2) = a$. By contradiction, $f(P_1', Q_2) = b$

Hence, if the SCF is strategyproof then $f(P_1', Q_2) \;=\; \mathbf{b}$.

3. Consider the profile:

P_1'	P_2'
b	a
a	b
c	c

Here, the position of b improves. By Monotonocity: $f(P_1', P_2') = b$

1.1.2 B.

If the preferences are generated from a single-peaked preference domain with the given intrinsic ordering of alternatives, we can not say anything about $f(P_1', P_2')$. Thus, the earlier conclusion can not hold in this case since preferences Q_2 and Q_3 are not allowed.

Q_2	Q_3
c	a
a	c
b	b

We have to propose a new mechanism such that: $f(P_1', P_2') = a$. Preferences generated from a single-peaked preference domain are ($|\mathcal{S}| = 4$):

P'_1	P'_2	X	Y
b	a	c	b
a	b	b	c
c	c	a	a

We take a median vector SCF. Every median vector SCF is SP. *Onto*ness has to be verified.
Phantom voter whose peak is at a:

- $f(P'_2, P'_2) = \text{median}(a, a, a) = a$.
- $f(P'_1, P'_1) = \text{median}(c, c, a) = b$.
- $f(X, X) = \text{median}(b, b, a) = c$

As all outcomes are possible, SCF is ONTO. phantom voter whose peak is at a.

- $f(P_1, P_2) = \text{median}(a, a, c) = a$
- Similarly, $f(P'_1, P'_2) = \text{median}(a, b, a) = a$

1.2 Problem 2

Let us first discuss about the domain of social choice function f: $\mathcal{P}^n \mapsto A$. (A $\geqslant 3$)

$\mathcal{P}^n$ is set of all possible ordering of preferences for alternatives by all agents n. The GS theorem requires unrestricted preferences to hold. All possible linear orderings $|\mathcal{P}| = |A|!$ should be available to be chosen by agents.In a setting where preferences are restricted, **Gibbard-Satterthwaite** theorem may not hold.

For the given problem, the set of alternatives is given by all possible subsets of X.As $|X| \geqslant 2$, possible alternatives is $\geqslant 3$. We will show that domain is restricted. Take 2 subsets of projects $S, T \subseteq X$ S is preferred to T if the highest ranked project in S (according to P_i) is better than the highest ranked project in T - if these two projects are the same, then S and T are indifferent.
Suppose $S \subsetneq T$. Clearly, highest ranked project in T $\succeq$ highest ranked project in S. The preference ordering in which set S is ranked higher than set T can never arise. Therefore, domain is restricted and **GS result can not be applied here**.

1.3 Problem 3

. **Claim:** In a single peaked domain model, median vector SCF is group strategy proof.

Proof: The proof is similar to the proof that shows that a median vector SCF is SP **(Theorem 21.9)**. We need to consider only the peak preferences of all the agents.So let us denote the preferences denoted only by their peaks, *i.e*, $P = (P_1(1), .., P_i(1), .., P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and the phantom peaks. Consider

- If $P_i(1) = a \; \forall i \in K$, then there is no reason for group to manipulate and f is group strategy proof.

- $\exists$ some $i \in K$ such that $P_i \neq a$. A group of agents can shift a median if they can shift their peak to the other side of the median, and this will shift the outcome to the other side, which this agent will not like. If the peak preferences of all the agents stay on the same side of the median as they were before, the median cannot change.

- Consider WLOG, If $P_i(1) < a$, then if the agent shifts her preference to further left of a, the median will not change. If she manipulates to report her peak to further right of a, *i.e.* $(P_i, P_{-i}) \mapsto (P_i', P_i)$ $s.t.$ $a < P_i'(1)$, this will imply that $P_i(1) < a < P_i'(1)$, and since P_i' is a single peaked preference, $a = (P_i, P_{-i})P_i(P_i'., P_{-i})$ Thus, $(\ P_K, P_{-K})P_K(P_K', P_{-K})$ for agent i. Therefore, i has no profitable manipulation.

Hence, f is group strategy proof. ■