CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 17: Game Theory – An Introduction

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

17.1 Recap

In the last class we had defined the Arrovian Social Welfare Function. ASWF is a mapping from the set of all preference profiles of n agents to a single preference profile to estimate preference ranking. It satisfies two properties:

Definition 17.1 *A Social Welfare Function F satisfies* ***Weak Pareto*** *if*
$\forall R \in \mathscr{R}^n$, $\forall a, b \in A$ $[aP_i b, \forall i \in N] \implies [a\hat{F}(R)b]$

Definition 17.2 *A Social Welfare Function F satisfies* ***Strong Pareto*** *if*
$\forall R \in \mathscr{R}^n$, $\forall a, b \in A$ $[aR_i b,\ \forall i \in N,\ \exists j,\ aP_j b] \implies [a\hat{F}(R)b]$

Note: *In last class an extra notation "$\forall R \in \mathscr{R}^n$" was used, which is not required since it is taken care of by this condition itself.* If all individuals strictly prefer a over b, this immediately implies that the social ranking should place a strictly above b. Whenever you have a R that satisfies this condition, it automatically satisfies the social ranking.

17.2 Independence of Irrelevant Alternatives(IIA)

The Second property that is the crux of Arrow's Impossibility result is known as Independence of Irrelevant Alternatives. It essentially connects two different preference profiles.
Two preference profiles, say R_i and $R_i' \in \mathscr{R}$ are said to agree over $\{a, b\}$
if for agent i

- $aP_i b \Leftrightarrow aP_i' b$
- $bP_i a \Leftrightarrow bP_i' a$
- $aI_i b \Leftrightarrow aI_i' b$

R, R' agree if for every i, $R_i\Big|_{a,b} = R_i'\Big|_{a,b}$ which implies,

$$R\Big|_{a,b} = R'\Big|_{a,b}$$

17.2.1 Scoring Rules

If an agent i has a preference profile, he can assign scores to each alternatives, say

$$(s_1, s_2, s_3, \ldots, s_m) \quad s_i \geq s_{i+1}, i = 1, 2, \ldots, m-1, s_i \geq 0, \forall i$$

This is one special class of ASWF. There are many special class of Scoring rules, as described below:

- **Plurality**: In this case we assing top score, i.e 1 to s_1 and 0 to all others, So
  $s_1 = 1$,and $s_2 = s_3 = \ldots = s_m = 0$
  Let us consider two preference profiles R and R'.
  The preference rules are:

R	R'
a a c d	d c b b
b c b c	a a c a
c b a b	b b a d
d d d a	c d d c

  In this setting of preference profiles, we see that for each agent the ordering of a and b are same in R and R', but overall Social Welfare Ranking is not consistent.
  That is,
  $$a\hat{F}^{Pl}(R)b \text{ and } b\hat{F}^{Pl}(R')a$$
  which are not consistent. Thus we conclude that Plurality does not satisfy *IIA*.

- **Borda**: The scoring rule in this case is,
  $s_1 = m-1, s_2 = m-2, \ldots, s_{m-1} = 1, s_m = 0$
  We can check by suitable example that this kind of voting rule too does not satisfies —emphIIA

- **Veto**: The scoring rule is,
  $s_1 = s_2 = \ldots = s_{m-1} = 1, s_m = 0$

- **Dictatorial**:A voting rule is dictatorial if and only if it always selects the most-liked candidate of the dictator among the possible outcomes; in particular, it does not depend on the other voters' ballots. Thus it is trivial that this kind of voting rule satisfies *IIA*.

Theorem 17.3 ***(Arrow 1951)**:If* $\mid A \mid \geq 3$*, if an ASWF F satisfies* Weak Pareto *and* IIA *then it must be dictatorial.*

Proof: The proof of the following two Lemmas will lead us to eventually prove Arrow's Impossibility Theorem.

1. <u>Field Expansion Lemma</u>
   if a group G is decisive over a, b ,then it is decisive over all pairs of alternatives.
   Note: G⊆N *and* G$\neq \phi$

2. <u>Group Contraction Lemma</u>
   if a group G is decisive, there exists a strict subset of G, that is also decisive.

Definition 17.4 *Given* $F : \mathscr{R}^n \to \mathscr{R}$ *and* $G \subseteq N$ *and* $G \neq \phi$

1. *G is* <u>*almost decisive*</u> *over a, b if*
   $[aP_ib \; \forall i \in G \text{ and } bP_ja \; \forall j \notin G] \implies [a\hat{F}(R)b]$

2. *G is called* <u>*decisive*</u> *over a, b if*
   $[aP_ib \; \forall i \in G] \implies [a\hat{F}(R)b]$

17.2.2 Notation

- $\overline{D}_G(a,b)$ means, G is almost decisive over a, b
- $D_G(a,b)$ means, G is decisive over a, b

We can clearly see, that $D_G(a,b) \implies \overline{D}_G(a,b)$

Lemma 17.5 ***Field Expansion Lemma***
Let F satisfy Weak Pareto *and* IIA *then $\forall a, b, x, y, a \neq b, x \neq y$, we have,*

$$\overline{D}_G(a,b) \implies D_G(x,y)$$

Cases:

1. $\overline{D}_G(a,b) \implies D_G(a,y)$ where $y \neq a, b$
2. $\overline{D}_G(a,b) \implies D_G(x,b)$ where $x \neq a, b$
3. $\overline{D}_G(a,b) \implies D_G(x,y)$ where $x \neq a, b$ and $y \neq a, b$
4. $\overline{D}_G(a,b) \implies D_G(x,a)$ where $x \neq a, b$
5. $\overline{D}_G(a,b) \implies D_G(b,y)$ where $y \neq a, b$
6. $\overline{D}_G(a,b) \implies D_G(b,a)$
7. $\overline{D}_G(a,b) \implies D_G(a,b)$

Proof: <u>**Case 1**</u>

Given: $\overline{D}_G(a,b)$ we need to show $D_G(a,y)$
Pick arbitrary R such that,

$$aP_iy \; \forall i \in G, \; \underline{\text{to show}} \; a\hat{F}(R)y$$

Construct R' in this fashion,

G	$N \setminus G$
$a \succ b \succ y$	$b \succ a$ and $b \succ y$

Where $a \succ b$ denotes a is more preferred than b.

$$R_i\Big|_{a,y} = R'_i\Big|_{a,y} \quad \forall i \in N$$

Now,

$$\overline{D}_G(a,b) \implies a\hat{F}(R')b$$

Since a is preferred over b by all agents in the group G and the opposite way for all agents not in G
Weak Pareto over $b, y \implies b\hat{F}(R')y$.
Using transitivity of $F(R')$,we have,

$$a\hat{F}(R')y$$

IIA,

$$\implies a\hat{F}(R)y$$

■

Proof: <u>Case2</u>

Given :$\overline{D}_G(a,b)$ we need to show $D_G(x,b)$
Given R such that $xP_ib\ \forall i \in G$, need to show, $x\hat{F}(R)b$
Construct R' in this fashion,

G	$N \setminus G$
$x \succ a \succ b$	$x \succ a$ and $b \succ a$

$R_i\Big|_{x,b} = R'_i\Big|_{x,b}$ $\forall i \in N$ Now, $\overline{D}_G(a,b) \implies a\hat{F}(R')b$
Weak Pareto over $x, a \implies x\hat{F}(R')a$
Using transitivity of $F(R')$, we have,

$$x\hat{F}(R')b$$

IIA,

$$\implies x\hat{F}(R)b$$

■

Similarly we can prove for all other cases by minor modifications. ■

In the next class we will prove the *Group Contraction Lemma* to complete our proof of the *Arrow's Impossibility Theorem.*