Mechanism Design

N = Set of agents

X = set of outcomes

Θ_i = set of types of i

u_i = utility of i $\quad u_i : X \times \Theta_i \to \mathbb{R}$

The social planner wants to have a specific outcome for a given type profile.

This is encapsulated in Social Choice Function

$$f : \Theta \to X.$$

Example: public project choice, every agent has different values for a public project.

$X = \{\text{bridge, road}, \ldots\}$

$\theta_i \equiv v_i : X \to \mathbb{R}$ type is the value for a specific project

$$f(v) = \underset{a \in X}{\arg\max}\ \sum_{i=1}^{n} v_i(a) \quad \text{(with tie breaking)} \to \text{utilitarian SCF}$$

$$= \underset{a \in X}{\arg\max}\ \min_{i \in N} v_i(a) \to \text{egalitarian SCF}$$

Social planner wants to ~~take~~ choose a desirable outcome but the θ_i's are unknown to him.

Need for mechanisms

Defn: A mechanism is a collection of message spaces and a decision rule, $M = \langle M_1, M_2, \ldots, M_n, g \rangle$

- M_i is the message space of agent i.
- $g : M_1 \times M_2 \times \cdots \times M_n \to X$

direct mechanism when $M_i = \Theta_i$, $g = f$.

(15-2) Defn: In a mechanism $\langle M, g \rangle$
A message m_i is a weakly dominant strategy for agent i at θ_i if

$$u_i(g(m_i, \underline{m}_{-i}), \theta_i) \geq u_i(g(m_i', \underline{m}_{-i}), \theta_i) \quad \forall m_i' \in M_i$$
$$\forall \underline{m}_{-i} \in \underline{M}_{-i}$$

Defn: An SCF $f : \Theta \to X$ is implemented in dominant strategies by $\langle M, g \rangle$ if

(1) $\exists$ message mappings $m_i : \Theta_i \to M_i$ s.t. $m_i(\theta_i)$ is a dominant strategy for agent i at θ_i, $\forall \theta_i \in \Theta_i, \forall i \in N$.

(2) $g(m_i(\theta_i), \underline{m}_{-i}(\underline{\theta}_{-i})) = f(\theta_i, \underline{\theta}_{-i}), \quad \forall \theta \in \Theta.$

This is an indirect implementation, f is DSI by $\langle M, g \rangle$.

Defn: A direct mechanism is strategyproof or dominant strategy incentive compatible (DSIC) if

$$u_i(f(\theta_i, \underline{\theta}_{-i}), \theta_i) \geq u_i(f(\theta_i', \underline{\theta}_{-i}), \theta_i) \quad \forall \theta_i \in \Theta_i$$
$$\forall \theta_i' \in \Theta_i$$
$$\forall \underline{\theta}_{-i} \in \underline{\Theta}_{-i}, \forall i \in N.$$

To find if an SCF f is dominant strategy implementable, we need to search for all possible indirect mechanisms $\langle M, g \rangle$, but there is a result that reduces this search space.

Revelation principle (for DSI ~~mechanisms~~ SCFs)

If there exists an indirect mechanism that implements f in dominant strategies, $\exists$ a direct mechanism to implement f, i.e., f is DSIC.

▷ Implication: One can focus on DSIC SCFs WLOG.

Proof: Let f is implemented by $\langle M, g \rangle$, hence $\exists\, m_i : \Theta_i \to M_i$

s.t. $\forall i, \forall m_{-i}, \forall m_i', \forall \theta_i \in \Theta_i$:

$$u_i(g(m_i(\theta_i), m_{-i}), \theta_i) \geq u_i(g(m_i', m_{-i}), \theta_i) \quad \text{--- (1)}$$

and $$g(m_i(\theta_i), m_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}), \ \forall \theta \in \Theta.$$

From (1), we see that this holds even for

$$m_i' = m_i(\theta_i') \text{ and } m_{-i} = m_{-i}(\theta_{-i})$$

Then $$u_i(\underbrace{g(m_i(\theta_i), m_{-i}(\theta_{-i}))}_{= f(\theta_i, \theta_{-i})}, \theta_i) \geq u_i(\underbrace{g(m_i(\theta_i'), m_{-i}(\theta_{-i}))}_{f(\theta_i', \theta_{-i})}, \theta_i)$$

$$\Rightarrow u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\theta_i', \theta_{-i}), \theta_i)$$

$\Rightarrow$ f is DSIC.