CS711 Assignment-2
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Solution 1

(a) We are given that $f(P_1, P_2) = a$. Now, consider

$P_1^{'}$	P_2
b	c
a	b
c	a

We see that $D(c, P_1^{'}, P_2) \subseteq D(c, P_1, P_2)$. So, since SP $\Rightarrow$ MONO, $f(P_1^{'}, P_2)$ cannot be c, otherwise it would mean $f(P_1, P_2) = c$, which is false. But $f(P_1^{'}, P_2) \in \{b, c\}$, so $f(P_1^{'}, P_2) = b$. Now, consider

$P_1^{'}$	$P_2^{''}$
b	c
a	a
c	b

We see that $D(c, P_1^{'}, P_2^{''}) \subseteq D(c, P_1^{'}, P_2)$. Again, $f(P_1^{'}, P_2^{''}) \in \{b, c\}$, and $f(P_1^{'}, P_2^{''})$ cannot be c, otherwise it would mean $f(P_1^{'}, P_2) = c$, which is false. So, $f(P_1^{'}, P_2^{''}) = b$. But then, $D(b, P_1^{'}, P_2^{''}) \subseteq D(b, P_1^{'}, P_2^{'})$, so by MONO, we get $f(P_1^{'}, P_2^{'}) = b$.

(b) No, the earlier conclusion does not hold because now, the domain consists only of single-peaked preferences, whereas we used the preference

$P_2^{''}$
c
a
b

which is not single-peaked. The mechanism $f(P) = min_{i \in N}\{P_i(1)\}$ is ONTO and SP when the domain consists of single-peaked preferences. Then, $f(P_1^{'}, P_2^{'}) = min_{i \in N}\{b, a\} = a$, since $a < b$.

Solution 2 In this problem, a subset X of set of projects A is placed in the preference order of an agent by seeing only the alternative with highest (individual) preference in the subset. Note that the entire set of alternatives A always has rank 1 in any ordering, since it always consists of the alternative having individual rank 1. This means that the domain of preference orders is restricted, since an ordering wherein the entire set A has rank other than 1 is not allowed. Gibbard-Satterthwaite theorem demands that the set of preferences be unrestricted, i.e. all possible arrangements of order be the domain. Restriction would lead to inability in constructing the preferences for arguing dictatorship in the proof. So we cannot apply the Gibbard-Satterthwaite result in this case.

Solution 3 Let f be a median voter SCF, $P = (P_1, ..., P_n)$ be a single-peaked profile such that $f(P) = a$, and $B = (y_1, .., y_{n-1})$ be a phantom vector used in calculating the median. Let $K \subseteq N$. For $K = \{i\}$, we do not have $f(P^{'}_K, P_{-K}) P i f(P_K, P_{-K})$ since a median voter SCF is strategy-proof(Moulin). Let $i, j \in K$ such that $i \neq j$. Consider the following cases:

- If either $P_i(1) = a$ or $P_j(1) = a$, that agent would get no benifit in misreporting her preferential order. So the situation is tantamount to manipulation of preference order of solely the other candidate, which does not bring any benifit as we remarked above.

- If both $P_i(1) < a$ and $P_j(1) < a$,

  If any of the agents (say i) chooses profile $P^{'}_i$ such that $P^{'}_i(1) < P_i(1)$, then the median of $(B, P^{'})$ (where $P^{'} = (P_1, ..P^{'}_i, .., P^{'}_j, ..P_n)$) is going to remain the same as that of $(B, (P^{'}_j, P^{'}_{-j}))(\because P^{'}_i(1) < P_i(1) < median(B, P))$. So $f(P^{'}_K, P_{-K}) = f(P^{'}_j, P_{-j})$, which we saw can never benifit j. Same would be the outcome when both $P^{'}_i(1) < P_i(1)$ and $P^{'}_j(1) < P_j(1)$.

  If both $P^{'}_i(1) > P_i(1)$ and $P^{'}_j(1) > P_j(1)$, the new median of $(B, P^{'})$ can either remain a or increase in terms of the prescribed order of alternatives. If it remains a, they did not benifit and if it increased, they are further away from their (true) preference, since $P_i(1) < a < f(P^{'}_K, P_{-K})$ now.

- If both $P_i(1) > a$ and $P_j(1) > a$, the argument is similar to above.

- If $P_i(1) < a$ and $P_j(1) > a$,

  We can see that $P^{'}_i(1) < P_i(1)$ or $P^{'}_j(1) > P_j(1)$ would do no good since it would not alter the median that was without doing this(as we argued earlier as well).

  Suppose $P^{'}_i(1) > P_i(1)$ and $P^{'}_j(1) < P_j(1)$, then the median of $(B, (P^{'}_K, P_{-K}))$ could possibly change in either direction. If it does not change, it does not benifit any agent in the group. If it increases, it certainly does not benifit agent i, as now $P_i(1) < a < f(P^{'}_K, P_{-K})$. Similarly, it does not benifit agent j if the median decreases. Thus manipulation in this case also does not benifit the entire group.

- Suppose $P_i^{'}(1) < P_i(1)$ and $P_j^{'}(1) > P_j(1)$, then the argument is symmetric to above.

Hence, we see that manipulation of preferences by any group never benifits the entire group, and the median voter SCF is group strategy-proof.