CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 12: Incomplete Information Games

Lecturer: Swaprava Nath *Scribe(s): Ankit Bhardwaj*

12.1 Introduction

In the complete information games that we have seen so far, the only uncertainties that could be introduced were the product of probabilistic choice of actions for different players. Even in that case, the pay-offs for different paths of a game were known to every player (was essentially common knowledge). In incomplete information games, as we shall see, that is not the case.

Definition 12.1 (Incomplete Information Games) *A game where players do not* ***deterministically*** *know which game they are playing. All players may have hidden information (types).*

We can understand the definition with the help of following example.

Example 12.2 ***A Soccer Game***

Let us say that there is a match scheduled between two competing clubs. The actions available to both the teams is the kind of game they decide to play. Let their choices be either to play attacking / aggressive or to play defensive. Let both these actions be denoted by A and D respectively.

Now, based on external random factors like weather at the day of game, unexpected player injuries and such; each team may have a hidden agenda. Say, when the teams are in a favourable position, they aim to **Win (W)** while in case of unfavourable position they aim to settle for a **Draw (D)**.

Note: The agendas of each team is unknown to the other team and is outside the control of either player (in this case, the teams). ***This private information dependent upon randomization is known as the type of each player***.

Consider the following utility distributions:

WW		
	A	D
A	1,1	2,0
D	0,2	0,0

WD		
	A	D
A	2,1	3,0
D	1,1	1,0

DW		
	A	D
A	1,2	1,1
D	0,3	0,1

DD		
	A	D
A	0,0	1,0
D	0,1	-1,-1

In this example, there are 4 possible profiles for the types of players: WW $\rightarrow$ type profile when both teams intend to win and so on. Thus, there are 4 different payoff matrices, one corresponding to each type profile.

Even if both players know all the 4 matrices, they don't know which matrix the game is going to be. Thus, they don't deterministically know which game they are playing.

Assumption 12.3 ***Uniformity in Action set and Player set***
The players and actions available to these players remain the same in different type profiles of the game - only the utility changes.

Assumption 12.4 ***Common Prior***
The type profiles are chosen from a common prior distribution **P**.

12.2 Bayesian Games

12.2.1 Formal Mathematical Definition

A Bayesian game is represented by the 5-tuple:

$$< N, (A_i)_{i \in N}, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in \times_{i \in N}(\Theta_i) = \Theta} >^1$$

where,
N $\rightarrow$ set of players or player set
A_i $\rightarrow$ action set of player i
Θ_i $\rightarrow$ set of types for player i –eg. Win / Draw
P $\rightarrow$ common prior distribution over $\Theta = \times_{i \in N}(\Theta_i)$ with the restriction that $P(\theta_i) > 0 \quad \forall \theta_i \in \Theta_i, \forall i \in N$
and
$\Gamma_\theta = < N, (A_i)_{i \in N}, (u_i(\theta)_{i \in N}) >$
where,
$u_i : AX\Theta \rightarrow \mathbb{R}$ $\quad A = \times_{i \in N} A_i$ (Normal Form Game utility function for each type profile).

12.2.2 Stages in Bayesian Games

- $\theta = (\theta_i, \theta_{-i})$ is chosen according to P
- Each player observes his / her own θ_i
- They pick action $a_i \in A_i$
- Player i's payoff is $u_i((a_i, a_{-i}); (\theta_i, \theta_{-i}))$

[1] This is a simplification. More generally, the Action Set may depend on types. $A_i(\theta_i)$

12.2.3 Strategies and Utilities

Strategy $\rightarrow$ A plan to map state/type to action
Pure Strategy $\rightarrow s_i : \Theta_i \rightarrow A_i$
Mixed Strategy $\rightarrow \sigma_i : \Theta_i \rightarrow \Delta(A_i)$

Ex-ante Utility

Definition 12.5 *Expected Utility before observing one's own type*

$$U_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) U_i(\sigma(\theta), \theta)$$

Thus,

$$U_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1,\ldots,a_n)\in A} (\Pi_{j\in N} \sigma_j(\theta_j, a_j)) u_i((a_i, a_{-i}); (\theta_i, \theta_{-i}))$$

We assume that once the player i observes his/her type, he/she will establish a belief according to Bayes rule on P as:

$$P(\theta_{-i}|\theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\widetilde{\theta_{-i}} \in \theta_{-i}} P(\theta_i, \widetilde{\theta_{-i}})}$$

Note: This is where the positive marginal is crucial.

Ex-interim Utility

Definition 12.6 *Expected utility after observing one's own type.*
It is to be noted that Ex-interim utility is most practical utility for analysis of incomplete information games.

$$U_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) U_i(\sigma(\theta), \theta)$$

Relation between the two

$$U_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma|\theta_i)$$

Example 12.7 ***Two player bargaining game***
Consider a game between two players where player 1 is the seller and player 2 is a buyer. Player 1 has a type variable which is equal to the minimum price at which he is willing to sell a commodity. For player 2, this hidden information is the maximum amount that the player is willing to pay.
For simplicity, let the types of both players be integers.
• $\Theta_1 = \Theta_2 = \{1,2, \ldots ,100\}$
• N = {1,2}
Assume that both players bid a number in {1,2,..100}. If the bid of the seller is less than or equal to the bid of the buyer, the sale happens. Else, there is no trade.
• $A_1 = A_2 = \{1,2,\ldots , 100\}$
Now,

- $P(\theta_2|\theta_1) = \frac{1}{100} \quad \forall\theta_2 \in \Theta_2, \forall\theta_1 \in \Theta_1$
- $P(\theta_1|\theta_2) = \frac{1}{100} \quad \forall\theta_1 \in \Theta_1, \forall\theta_2 \in \Theta_2$
- $u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1+a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$
- $u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1+a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$

The beliefs $P(\theta_2|\theta_1)$ and $P(\theta_1|\theta_2)$ are consistent with the prior $P(\theta) = \frac{1}{10000}, \forall\theta \in \Theta$, where $\Theta = \Theta_1 \times \Theta_2$.

Example 12.8 ***Sealed-Bid Auction***
In this game, we have a seller (who is not a player) willing to sell a commodity via an auction and two buyers (these are the competing players) who place sealed bids (secret to each other) on the commodity. The player who has bid more is given the commodity for the amount he/she has bid for.
values $\in [0,1] \implies$ type (θ_1, θ_2)
bids $\in [0,1] \implies$ actions (b_1, b_2)
allocation function:

$$O_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{otherwise} \end{cases}$$

$$O_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{otherwise} \end{cases}$$

The probability distribution over this continuous range is given by:

$$\left.\begin{array}{ll} \mathrm{f}_1(\theta_2|\theta_1) = 1 & \theta_2 \in [0,1] \\ \mathrm{f}_2(\theta_1|\theta_2) = 1 & \theta_1 \in [0,1] \end{array}\right\} f(\theta_1, \theta_2) = 1 \quad (\theta_1, \theta_2) \in [0,1]^2$$

The utility representation for the game can be:
$u_i(b_1, b_2; \theta_1, \theta_2) = O_i(b_1, b_2) * (\theta_i - b_i)$

Note : In this utility representation we assume that the valuation and bid currency are in the same metric (basically, we assume that the valuation/satisfaction of buyer for a commodity can be equated with money). Such utility representations are called as quasi-linear utility representation.