CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 22: October 4, 2017

Lecturer: Swaprava Nath *Scribe(s): Mayank Singour*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

22.1 Properties of SCF (Recap)

In previous lecture we discussed about the restricted domains (single peaked preferences in particular). Now, we will see some restricted domain specific properties, some of which are similar to the properties of unrestricted domain.
We also stated that each of the restricted subdomains has interesting non-dictatorial but strategyproof SCFs defined on it.

22.1.1 Pareto Efficiency (PE)

$\forall$ P $\in \wp$, $\forall$ b $\in$ A if $\exists$ a $\in$ A such that aP_ib $\forall$ i $\in$ N, then f(P) $\neq$ b

22.1.2 Unanimity (UN)

$\forall$ P with $P_1(1) = P_2(1) = \dots P_n(1) =$ a, then f(P)=a

22.1.3 Onto-ness (ONTO)

$\forall$ a $\in$ A $\exists$ P $\in \wp$ such that f(P) = a

$\longrightarrow$ As before, PE $\subset$ UN $\subset$ ONTO

Claim 22.1 *Let p_{min} and p_{max} are the leftmost and rightmost peaks of P according to $<$, then f is PE iff f(P) $\in$ [p_{min}, p_{max}]. Its proof follows from PE property.*

Proof: $\Rightarrow$ Suppose f is PE but f(P) $\notin$ [p_{min}, p_{max}], then f(P) is either at the left side of p_{min} or at the right side of p_{max}. Consider f(P)<p_{min}, but then every agent prefers p_{min} over f(P), a contradiction to PE. Similar argument can be given for p_{max}<f(P)

$\Leftarrow$ if f(P) $\in$ [p_{min}, p_{max}], every other alternative b $\neq$f(P) in [p_{min}, p_{max}] will either be close to p_{min}(and farther from p_{max}) or vice-versa. But then the "if" condition of PE definition is never triggered for those b's. Hence, PE is vacously satisfied. ∎

22.1.4 Monotonicity

If for two profiles P and P$'$ with f(P)= a and D(a,P_i)$\subseteq$ D(a,P_i') $\forall$ i $\in$ N, then f(P$'$)= a.

- where D(a,P_i)={b$\in$A : aP_ib} is dominated set.

22.2 Results for Restricted domain of Preferences

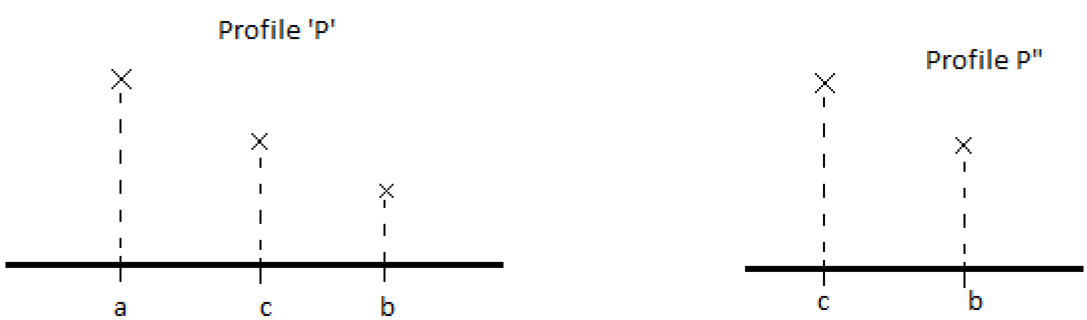
We will see some results similar to the unrestricted preferences but the proofs will differ as we do not have the flexibility of arbitrary preference profile construction. We are considering only single peaked preferences for now.

Theorem 22.2 *f is Strategy Proof (SP) $\Rightarrow$ f is Monotonous (MONO).*
This has exactly the same proof as we did for unrestricted preferences. However the construction of our previous result is not always feasible for the converse of this theorem.

Exercise: *Find a counterexample of the converse OR prove the converse.*

Theorem 22.3 *Let f: $S^n \mapsto A$ is SP Social Choice Function, then*
f is ONTO $\iff$ f is UN $\iff$ f is PE.

Proof: We only need to show f is ONTO $\Rightarrow$ f is PE, if f is SP.
Proving by Contradiction:
Suppose $\exists$ a,b such that aP_ib, $\forall$ i $\in$ N but f(P)=b.



Since P_i's (preferences) are single peaked, $\exists$ another alternative c $\in$ A such that cP_ib, $\forall$ i $\in$ N. Also, 'c' could be 'a' itself if there is no neighbour between 'a' and 'b'.

ONTO $\Rightarrow$ $\exists$ P$'$ such that f(P$'$)= c
Construct P'' such that P_i''(1)=c and P_i''(2)=b $\forall$ i $\in$ N.

Changing the Profile from P$'\longrightarrow P''$, by Monotonicity f(P'')= c.
Changing the Profile from P$\longrightarrow P''$, by Monotonicity f(P'')= b. Hence, Contradiction ■

22.3 Anonymity

Define permutation over the agents as σ : N $\mapsto$ N
We apply a permutation σ to a profile P to construct another profile as : the preference ordering of 'i' goes to another agent σ(i) in the new profile. We denote the new profile as P^{σ} .

Example: N={1,2,3} σ: σ(1)=2, σ(2)=3, σ(3)=1

P_1	P_2	P_3	P_1^{σ}	P_2^{σ}	P_3^{σ}
a	b	b	b	a	b
b	a	c	c	b	a
c	c	a	a	c	c

Anonymity requires that the social outcome should not matter for agent renaming.

Definition 22.4 *An SCF f: $S^n \mapsto A$ is anonymous(ANON) if for every profile 'P' and for every permutation of the agents σ, f(P^{σ})=f(P) .*
Note: *Dictatorship is not anonymous. Median Voter rule is anonymous.*

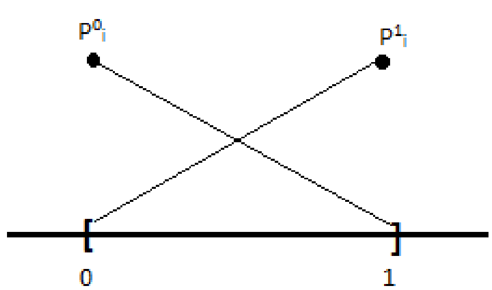
Theorem 22.5 (Moulin 1980) :*A Strategy Proof SCF 'f' is ONTO and Anonymous iff it is a median voter SCF.*

Proof: $\Leftarrow$ Median Voter SCF is SP (see theorem 21.9 of previous lecture), and

- - it is ONTO, since we can put all voters peaks at the same alternative/location and set all phantom peaks at zero, then that location is the outcome.
- - it is anonymous, since we can permute the agents with the peaks unchanged and the outcome will not change.

$\Rightarrow$ Given f: $S^n \mapsto$ A is SP, ONTO and ANON.
Define

- - P_i^0 : agent i's preference where the peak is at the leftmost point w.r.t. $<$.
- - P_i^1 : agent i's preference where the peak is at the rightmost point w.r.t. $<$.



Let y_j's are phantom peaks ; j= 1,2,..., n-1
Pick y_j's as follows :
y_j = f(P_1^0, P_2^0,..., P_{n-j}^0, P_{n-j+1}^1,..., P_n^1)
→ It does not matter which agents have which peaks because of unanimity.

Claim 22.6 $y_j \leq y_{j+1}$ *; j=1,2,...,n-2 [since peaks are non decreasing]*
Proving this Claim:
y_{j+1} *= f(*P_1^0, P_2^0,..., P_{n-j-1}^0, P_{n-j}^1, P_{n-j+1}^1,..., P_n^1*)*
SP $\Rightarrow y_j P_{n-j}^0 y_{j+1}$ *but* P_{n-j}^0 *is single peaked*
$\Rightarrow y_j \leq y_{j+1}$

We will continue to complete this proof of forward direction in the next lecture. ■

22.4 Summary

So, in this lecture we discussed some properties and results of single peaked preference restricted domain. The four properties namely PE, UN, ONTO and MONO are similar to what we saw in unrestricted domains. Then we discussed about Anonymity and Moulin theorem.