Defn: (Pareto Optimality)

A mechanism ~~is~~ $(f,(p_1,\dots,p_n))$ is Pareto Optimal if at every type profile $\theta \in \Theta$, $\nexists\, b \neq f(\theta)$ and payment $(\pi_1,\dots,\pi_n)$ ~~s.t.~~ having $\sum_{i\in N}\pi_i \geq \sum_{i\in N} p_i(\theta)$ s.t.

$$v_i(b,\theta_i) - \pi_i \geq v_i(f(\theta),\theta_i) - p_i(\theta) \quad \forall i \in N$$

with strict inequality for at least one $i \in N$.

- Pareto optimality is ~~max~~ meaningless if there is no restriction on the payment. A designer can always put excessive subsidy so that ~~the~~ every agent is better off. Hence we need the constraint on payment so that it has to ~~spend~~ ask/raise ~~the~~ at least the amount from the agent as the original mechanism.

Theorem: A mechanism $(f,(p_1,\dots,p_n))$ is Pareto optimal iff it is ~~ae~~ allocatively efficient.

Proof: $(\Rightarrow)$ We'll show $!AE \Rightarrow !PO$.

$!AE$: $\exists\, b \neq f(\theta)$ s.t. $\sum_{i\in N} v_i(b,\theta_i) > \sum_{i\in N} v_i(f(\theta),\theta_i)$

consider $\delta = \left[\sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i)\right]$

define $\pi_i = v_i(b,\theta_i) - v_i(f(\theta),\theta_i) + p_i(\theta) - \delta/n$

clearly $\left[v_i(b,\theta_i) - \pi_i\right] - \left[v_i(f(\theta),\theta_i) - p_i(\theta)\right] = \delta/n > 0 \quad \forall i \in N$

also $\sum_{i\in N}\pi_i = \sum_{i\in N} p_i(\theta)$. f is not PO.

$\Leftarrow$ again will show $\underset{\circ}{!}$PO $\Rightarrow$ $\underset{\circ}{!}$AE

$\underset{\circ}{!}$PO, $\exists\, b, \underline{\pi}$ s.t. $\sum_{i \in N} \pi_i \geq \sum p_i(\theta)$

and $v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - p_i(\theta)$

$\forall i \in N$ and strict for some $j \in N$

Sum them,

$$\sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} \pi_i > \sum_{i \in N} v_i(f(\theta), \theta_i) - \sum_{i \in N} p_i(\theta)$$

$$\Rightarrow \sum v_i(b, \theta_i) - \sum v_i(f(\theta), \theta_i) > \underbrace{\sum \pi_i - \sum p_i(\theta)}_{\geq 0}.$$

f is $\underset{\circ}{!}$AE. $\square$

Efficient rule is implementable

A fundamental result in mechanism design in quasi-linear domain is that efficient allocation rule is implementable.

- Implication: We can always pick allocations to maximize social welfare

e.g. in single object allocation, second price auction does this. But this idea can be generalized.

A class of mechanisms that implement the efficient allocation rule is Groves class of payments (Groves 1973). [Corresponding direct mechanism is called Groves mech.]

Efficient rule: $f^{Eff}(\theta) \in \underset{a \in A}{\arg\max} \sum_{i \in N} v_i(a, \theta_i)$

Groves payment:

$$p_i^G(\theta_i, \underline{\theta}_{-i}) = h_i(\underline{\theta}_{-i}) - \sum_{j \neq i} v_j(f^{Eff}(\theta_i, \underline{\theta}_{-i}), \theta_j)$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is any arbitrary function.

<u>Example</u>: Consider single indivisible object allocation. Value of agents 1, 2, 3, 4 are 10, 8, 6, 4 when they are allocated the object, zero otherwise. Clearly, the efficient allocation in this case is to allocate the item to player 1.

Let $h_i(\theta_{-i}) = 10 \ \forall \theta_{-i}$, then the payment is

~~to cha~~ $p_1 = 10, \quad p_2 = p_3 = p_4 = 0$

charge player 1, 10 and nothing to the others.

But there could be many others. $h_i(\theta_{-i}) = \sum_{j \neq i} \theta_j$ gives

agent 1 $= 9 - 0 = 9$

2 $= 10 - 10 = 0$

3 $= 11 - 10 = 1$

4 $= 12 - 10 = 2$

Looks very ~~wei~~ surprising payments, but it still ensures truthfulness.

<u>Theorem</u>: Groves mechanisms are DSIC.

<u>Proof</u>: Consider the agent i, true type θ_i, ~~his reporting~~ reported type $\hat{\theta}_i$, reported type of other agents θ_{-i}. The ~~outco~~ allocations

$$f^{Eff}(\theta_i, \theta_{-i}) = a, \quad f^{Eff}(\hat{\theta}_i, \theta_{-i}) = b \text{ (say)}$$

utility when agent i reports θ_i

$$v_i(f^{Eff}(\theta_i, \theta_{-i}), \theta_i) - p_i^G(\theta_i, \theta_{-i})$$

$$= v_i(f^{Eff}(\theta_i, \theta_{-i}), \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(f^{Eff}(\theta_i, \theta_{-i}), \theta_j)$$

$$= \sum_{j=1}^{n} v_j \left(f^{Eff}(\theta_i, \underline{\theta}_{-i}), \theta_j \right) - h_i(\underline{\theta}_{-i})$$

$$\geq \sum_{j=1}^{n} v_j \left(f^{Eff}(\hat{\theta}_i, \underline{\theta}_{-i}), \theta_j \right) - h_i(\underline{\theta}_{-i})$$

$$= v_i\left(f^{Eff}(\hat{\theta}_i, \underline{\theta}_{-i}), \theta_i\right) - \underbrace{\left(h_i(\underline{\theta}_{-i}) - \sum_{j \neq i} v_j \left(f^{Eff}(\hat{\theta}_i, \underline{\theta}_{-i}), \theta_j\right)\right)}_{p_i^G(\hat{\theta}_i, \underline{\theta}_{-i})}$$

= utility when agent i reports $\hat{\theta}_i$.

Are there other payment rules that implement efficient allocation? We will address this formally later. But a quick answer is that it depends on the restriction/ assumption on the space of valuations.

The Vickrey-Clarke-Groves Mechanism (VCG)

An interesting mechanism in the Groves class. Commonly known as pivotal mechanism. (V'61, C'71, G'73) Characterized by a unique $h_i(\cdot)$ function.

$$h_i(\underline{\theta}_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

This gives the payment:

$$p_i^{VCG}(\theta_i, \underline{\theta}_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j\left(f^{Eff}(\theta_i, \underline{\theta}_{-i}), \theta_j\right)$$

Note that, $p_i^{VCG}(\theta) \geq 0 \ \forall i \in N \ \forall \theta \in \Theta$.

No-subsidy and hence weak budget balanced.