Lec 20

f is SP + ONTO $\Leftrightarrow$ f is SP + UN $\Leftrightarrow$ f is SP + PE

Theorem (Gibbard '73, Satterthwaite '75)

Suppose $|A| \geq 3$. f is ONTO and SP $\Leftrightarrow$ f is dictatorial.

- $|A| = 2$, GS Theorem doesn't hold. Plurality with a fixed tie-breaking rule is SP, ONTO and non-dictatorial.
- $\mathcal{P}$: The preferences of every voter is unrestricted all possible permutations of the alternatives are feasible. This also implies that a manipulating agent has more options to misreport. If the domain had been restricted, then GS theorem may not hold.
- Indifference: in general, GS theorem may not hold. But in the proof, we will use some specific profile constructions, if they hold then GS characterization holds.
- Cardinalization: doesn't matter, as long as the ordinal order is maintained, GS Theorem will hold.

Proof: We'll follow a direct approach. First prove for $n=2$ and then use induction on the number of voters.

Ref: (Sen 2001)

Lemma: $|A| \geq 3$, $N = \{1, 2\}$, f is onto and SP, then for every preference profile P

$$f(P) \in \{P_1(1), P_2(1)\}$$

Proof: If $P_1(1) = P_2(1)$ then Unanimity implies $f(P) = P_1(1)$

Say $P_1(1) = a \neq b = P_2(1)$

for contradiction assume $f(P) = c \neq a, b$.

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
⋮	⋮	⋮	a	b	a	b	⋮
			⋮	⋮	⋮	⋮	
$\downarrow c$							

$f(P_1, P_2') \in \{a, b\}$ ∵ PE: every ~~other~~ alternative except b is Pareto dominated by a

if b, then 2 will manipulate from $P_2 \to P_2'$

hence $f(P_1, P_2') = a$

$f(P_1', P_2) = b$ (by similar arguments)

$P_1\, P_2' \longrightarrow P_1'\, P_2'$ should be a by MONO

$P_1'\, P_2 \longrightarrow P_1'\, P_2'$ - - - b by MONO. □

Lemma: $|A| \geq 3$, $N = \{1, 2\}$, f is ONTO and SP

Let P: $P_1(1) = a \neq b = P_2(1)$, P': $P_1'(1) = c$, $P_2'(1) = d$.

Then if $f(P) = a \Rightarrow f(P') = c$

if $f(P) = b \Rightarrow f(P') = d$

Proves dictatorship for two voters

Proof: If $c = d$, unanimity implies the lemma.

Why exhaustive		Cases	c	d
a	b	1	a	b
a	$\neq b, a$	2	$\neq a, b$	b
b	a	3	$\neq a, b$	$\neq b$
b	$\neq a, b$	4	a	$\neq a, b$
$\neq a, b$	b	5	b	$\neq a, b$
$\neq a, b$	$\neq b$	6	b	a

Enough to consider if $f(P) = a \Rightarrow f(P') = c$

The other case is symmetric.

Case 1: $c = a$, $d = b$

$P_1 P_2 \rightarrow \hat{P}_1 \hat{P}_2$

$\downarrow$ $\qquad \downarrow$

$a \qquad a$

say $f(P') = b$

$P_1' P_2' \rightarrow \hat{P}_1 \hat{P}_2$

$\downarrow$ $\qquad \downarrow$

$b \qquad b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	a	b	a	b
⋮	⋮	⋮	⋮	b	a
				⋮	⋮

Case 2: $c \neq a, b$ $\quad d = b$

say $f(P') = b$ for contradiction

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	$c \neq a, b$	b	c	b
⋮	⋮	⋮	⋮	a	⋮
				⋮	

Then $P_1' P_2' \rightarrow \hat{P}_1 P_2$ case 1

$\downarrow$ $\qquad \downarrow$

$b \qquad b$

agent 1 misreports from $\hat{P}_1 \rightarrow P_1$ since $f(P_1, P_2) \hat{P}_1 f(\hat{P}_1, P_2)$

Case 3: $c \neq a, b$, $d \neq b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	$c \neq a, b$	$d \neq b$	c	b

say $f(P') = d$

$P' \to \hat{P}$ $f(\hat{P}) = b$ (case 2)

$P \to \hat{P}$ $f(\hat{P}) = c$ (case 2)

Case 4: $c = a$, $d \neq b, a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	$c = a$	$d \neq a, b$	a	b

$f(P') = d$

$P' \to \hat{P}$ $f(\hat{P}) = b$ (case 2)

$P \to \hat{P}$ $f(\hat{P}) = a$ (case 1)

Case 5: $c = b$ $d \neq a, b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	$c = b$	$d \neq a, b$	a	d

$f(P') = d$

$P' \to \hat{P}$ $f(\hat{P}) = d$ (case 4)

$P \to \hat{P}$ $f(\hat{P}) = a$ (case 4)

Case 6: $c = b$ $d = a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	P_2'	$\tilde{P}_1$	P_2'
a	b	$c = b$	$d = a$	b	a	x	a
				x			

$f(P') = a$

$x \neq a, b$

$P' \to (\hat{P}_1\ P_2')$ $f(\hat{P}_1 P_1') = a$ (case 1)

$P \to (\tilde{P}_1\ P_2')$ $f(\tilde{P}_1 P_2') = x$ (case 3)

Player 1 manipulates

from $\hat{P}_1\ P_1' \longrightarrow \tilde{P}_1\ P_2'$

$x\ \hat{P}_1\ a$