CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

Question 1:

Part (a)

Lemma 1: Let $|A| \geq 3$, $N = \{1, 2\}$, and f is onto and strategyproof. Let $P : P_1(1) = a \neq b = P_2(1)$ and $P' : P_1'(1) = c \neq d = P_2'(1)$. Then

$$f(P) = a => f(P') = c$$

Also, given $c = a$ and $d = b$.
Proof: $c = a$ and $d = b$. Suppose for contradiction $f(P') = d = b$. We construct preference profiles as follows:

P_1	P_2	P_1'	P_2'	$\hat{P}1$	$\hat{P}_2$
a	b	a	b	a	b
$-$	$-$	$-$	$-$	b	a
$-$	$-$	$-$	$-$	$-$	$-$

Consider transition from (P_1, P_2) to $(\hat{P}_1, \hat{P}_2)$.Preference for a improves for both agents and $f(P_1, P_2) = a$. Thus by monotonicity, $f(\hat{P}_1, \hat{P}_2) = a$. Next consider transition from (P_1', P_2') to $(\hat{P}_1, \hat{P}_2)$. Preference for b improves for both agents and $f(P_1', P_2') = b$. Thus by monotonicity, $f(\hat{P}_1, \hat{P}_2) = b$. But, $a \neq b$. This gives us a contradiction. Thus, $f(P') = c = a$.

Lemma 2: Let $|A| \geq 3$, $N = \{1, 2\}$, and f is onto and strategyproof. Let $P : P_1(1) = a \neq b = P_2(1)$ and $P' : P_1'(1) = c \neq d = P_2'(1)$. Then

$$f(P) = a => f(P') = c$$

Also, given $c \neq a, b$ and $d = b$.
Proof: $c \neq a, b$ and $d = b$. Suppose for contradiction $f(P') = d = b$. We construct preference profiles as follows:

P_1	P_2	P_1'	P_2'	$\hat{P}1$	$\hat{P}_2$
a	b	c	b	c	b
$-$	$-$	$-$	$-$	a	$-$
$-$	$-$	$-$	$-$	$-$	$-$

First we consider transition from (P_1', P_2') to $(\hat{P}_1, P_2)$. Notice that this transition satisfies all constraints of Lemma 1. hence, $f(\hat{P}_1, P_2) = b$.
Consider preference profile $(\hat{P}_1, P_2)$. At this profile if agent 1 reports P_1 instead of $\hat{P}_1$, the outcome is a which she prefers more than the current outcome b, as $f(\hat{P}_1, P_2) = b$ and $f(P_1, P_2) = a$. This is a contradiction to f being strategyproof. Therefore, $f(P') = c$.

Now, to prove the required result.Consider the following profiles:

P_1	P_2	P'_1	P'_2	$\hat{P}1$	$\hat{P}_2$
a	c	b	a	b	c
b	b	a	b	a	$-$
c	a	c	c	c	$-$

We know that :$f(P'_1, P'_2) \epsilon \{b, a\}$.
Proof by contradiction: Let us assume $f(P'_1, P'_2) = a$. Consider transition from (P_1, P_2) to $(\hat{P}_1, \hat{P}_2)$, then from lemma 2 we have $f(\hat{P}_1, \hat{P}_2) = b$. Also on considering the transition from (P'_1, P'_2) to $(\hat{P}_1, \hat{P}_2)$, then from lemma 2 we have $f(\hat{P}_1, \hat{P}_2) = c$.But, $b \neq c$. Thus, we have a contradiction and our assumption was wrong. Thus, $f(P'_1, P'_2) = b$.

Part (b)

No, the result does not hold true because if we follow the same proof as done in case (a), then we will not get a single peak in lemma 1. Thus, the result does not hold true.
Mechanism that has $f(P'_1.P'_2) = a$:
SCF $f : S^3 \to A$, where S is the set of single peak single-peaked preferences w.r.t the common order $<$ as, foe $P \epsilon S^3$

$$f(P) = min_{t \epsilon N}\{P_t(1)\}$$

where minimum is taken w.r.t the order relation $<$. Hence, the SCF picks the left most peak among the peaks of the agents.

Question 2:

No, Gibbard-Satterthwaite result does not apply here because the result is not applicable on restricted domains. To prove this we find a preference which is not allowed.
Consider that we have just two alternatives in P_i which are a_1 and a_2. Now, since on extending the preference the alternatives are from the power set of P_i. Let us have the preference order to be:

$$\{a_1\} \succ \{a_2\} \succ \{a_1, a_2\}$$

Now, this is not possible if P_i has
(i) $a_1 \succ a_2$: Because in this case $\{a_1, a_2\} \succeq \{a_2\}$. Contradiction with above preference.
or
(ii) $a_2 \succ a_1$: Because in this case $\{a_2\} \succ \{a_1\}$. Contradiction with above preference.
So, the above preference is not allowed. Thus the domain is not restricted and the Gibbard-Satterthwaite result does not apply here.

Question 3:

Yes, the median voter SCF is a group strategy-proof.
Proof:
The function f is manipulable if all the agents of a group are becoming strictly better off by changing their profile to P'_K.
Let us us consider two exhaustive cases:

(Case I:) The preference P_K is that of one of the members of the group K. Now, since this agent will not become strictly better off on changing his preference, the SCF f is not manipulable in this case.

(Case II:) The preference P_K is not of anyone in the group K. Let $P_1, P_2, ..., P_j$ (let us call this group G_1) have their peaks to the left of P_k and let $P_{j+1}, P_{j+2}, ..., P_k$ (let us call this group G_2) have their peaks to the right of P_k. Now, as seen in the proof of Molin theorem, none of the members in group G_1 will benefit from moving their peaks to the right of P_K and also none of the members of G_2 will benefit from changing their preference peaks to the left of P_K. So, the group cannot change its preference to a P'_K s.t. all of the members of the group become better off. Thus, the SCF f is not manipulable in this case also.

Hence, proved.