CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 21: October 3, 2017

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21.1 Recap and need for domain restriction

In the previous lecture, we stated and proved the Gibberd-Satterthwaite theorem. Restating the theorem:

Theorem 21.1 (Gibbard(1973), Satterthwaite(1975)) *Suppose for the set of alternatives A, $|A| \geq 3$. If the social choice function $f : \mathcal{P}^n \to A$ is onto and strategyproof then f is dictatorial.*

Note that the setup for Gibbard-Satterthwaite theorem allows unrestricted preferences. One of the reasons for such a restrictive result is that the domain of the SCF is large, leaving a potential manipulation many options to possibly manipulate. Revisiting the major assumptions used by theorem 21.1 can help us better understand this:

- *(Unrestricted preferences)* For $|A| = m \geq 3$, $|\mathcal{P}| = m!$ *i.e.* all possible $m!$ linear orderings are available to be chosen by the agents.
- *(Strategyproofness)* We require the SCF to satisfy

$$\forall i \in N, \forall P_i^{'}, P_i \in \mathcal{P}, \forall P_{-i} \in \mathcal{P}^{n-1},$$
$$f(P_i, P_{-i}) P_i f(P_i^{'}, P_{-i}) \text{ or } f(P_i, P_{-i}) = f(P_i^{'}, P_{-i})$$

If we now reduce the set of linear ordering (over A) $\mathcal{P}$ to some subset of $\mathcal{P}$, the SCFs that are truthful on $\mathcal{P}$ continue to be truthful on the subset, but we hope to find more SCFs that are truthful and non-dictatorial on the new restricted domain. In this lecture, we will look at some of these restricted domains and in particular, single-peaked preferences.

21.2 Restricted domains

In this course, we will be studying the following three domain restrictions:

1. Single-peaked preferences
2. Divisible Object allocation
3. Quasilinear preferences (sometimes, also referred to as 'Mechanisms with money')

It is worth stating that each of these subdomains has interesting *non-dictatorial* but *strategyproof* SCFs defined on it. We will be first dealing with single-peaked preferences.

21.3 Single-peaked preferences

In this restricted domain, we set a single common order over the alternatives. Once the common order is chosen, we allow only those preferences from $\mathcal{P}$ which have a single peak with respect to that common order. We will soon be defining such preferences formally for a particular common order.
Note: The common order is a fixed order relation over the alternatives and is completely unrelated to individual preferences of the agents.

21.3.1 Motivating examples

1. Facility location: School, Hospital, Post-office etc.
2. Political ideology: Left, Center and Right
3. Temperature sensing: Highest preference to a single temperature and the preference decreases as one deviates from the peak on either sides of it.

Each of these examples have a natural ordering over the alternatives. We may place the alternatives on the real line $\mathbb{R}$ according to the common ordering $<$. The ordering need not always be an order relation on the real numbers, but can be any order relation that is *transitive* and *anti-symmetric*. For simplicity, we will consider only alternatives having an order on the real line.

21.3.1.1 A particular example

Let $A := \{a, b, c\}$ be the set of facilities ordered on a real line s.t. $a < b < c$ (referring to the locations of a, b, c). Note, here $|\mathcal{P}| = 6$ and the following table of valid and invalid preferences illustrates that for the set of single-peaked preferences $\mathcal{S}$, $|\mathcal{S}| = 4$. Also, clearly $\mathcal{S} \subset \mathcal{P}$

Table 21.1: Preferences

(a) Valid

a	b	b	c
b	a	c	b
c	c	a	a

(b) Invalid

a	c
c	a
b	b

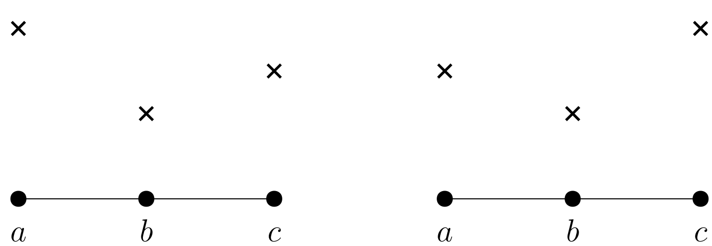

Figure 21.1: Invalid preference ordering

Definition 21.2 (Single-peaked preference order) *A preference ordering P_i (strict order over A) for agent i is single-peaked w.r.t. common order $<$ if*

- *$\forall b, c \in A$ with $b < c \leq P_i(1)$, we have cP_ib, where $P_i(1)$ is the peak preference for agent i*
- *$\forall b, c \in A$ with $P_i(1) \leq b < c$, we have bP_ic*

Let $\mathcal{S}$ be the set of single-peaked preferences. As illustrated by example 21.3.1.1, $\mathcal{S} \subset \mathcal{P}$. Let us define the SCF on the restricted domain, $f : \mathcal{S}^n \to A$

Definition 21.3 (Manipulability of an SCF) *An SCF f is said to be manipulable if $\exists i \in N, P_i^{'}, P_i \in \mathcal{S}, P_{-i} \in \mathcal{S}^{n-1}$ such that*

$$f(P_i^{'}, P_{-i})\ P_i\ f(P_i, P_{-i})$$

Definition 21.4 (Strategyproof) *An SCF f is said to be strategyproof if it is not manipulable. Thus, by definition, it implies*

$$\forall i \in N, \forall P_i^{'}, P_i \in \mathcal{S}, \forall P_{-i} \in \mathcal{S}^{n-1},$$
$$f(P_i, P_{-i}) P_i f(P_i^{'}, P_{-i}) \text{ or } f(P_i, P_{-i}) = f(P_i^{'}, P_{-i})$$

Note, now $P_i^{'}, P_i \in \mathcal{S}$, $P_{i-} \in \mathcal{S}^{n-1}$ and $\mathcal{S} \subset \mathcal{P}$ implying that we have lesser number of conditions to satisfy.

Now, the question is, how does an SCF defined on restricted domain of single-peaked preferences circumvent the Gibberd-Sattherthwaite theorem? We will illustrate this using an example social choice function.

21.3.2 Construction of SCF

Definition 21.5 *Define SCF $f : \mathcal{S}^n \to A$ ($\mathcal{S}$: set of single-peaked preferences) w.r.t. common order $<$ as : Pick the left-most peak among the peaks of the agents. Formally, for $P \in \mathcal{S}^n$*

$$f(P) = \min_{i \in N}\{P_i(1)\}$$

where minimum is taken w.r.t. the order relation $<$.

Claim 21.6 *f is strategyproof and non-dictatorial*
Proof: *For proving strategyproofness, let us first consider the agent having the peak preference as the left-most alternative. Clearly, she has no reason to deviate from her preference order.*
Now, let us take any other agent i having peak preference to the right of the left-most peak preference. In other words, take $i \in N$ s.t. $f(P) < P_i(1)$. The only possible manipulation she would want to cause is to change her peak preference to further left of $f(P)$ i.e. $(P_i, P_{-i}) \to (P_i^{'}, P_{-i})$ s.t.

$$P_i^{'}(1) \leq f(P) < P_i(1),$$
$$f(P_i^{'}, P_{-i}) = \min_{i \in N}\{P_1(1), P_2(1), .., P_i^{'}(1), , , , P_n(1)\} = P_i^{'}(1)$$

If $P_i^{'}(1) = f(P)$, clearly $f(P_i, P_{-i}) = f(P_i^{'}, P_{-i})$. Else, we will have

$$P_i^{'}(1) < f(P) < P_i(1)$$

And P_i is a single-peaked preference, thus, by definition 21.2, we get $f(P_i, P_{-i})\ P_i\ f(P_i^{'}, P_{-i})$. Hence, for any $\forall i \in N, \forall P_i^{'}, P_i \in \mathcal{S}, \forall P_{-i} \in \mathcal{S}^{n-1}$, we get either of the conclusions as stated in definition 21.3. Hence, f is strategyproof.
Note, since the agent choosing the left-most peak might vary among the agents, f is non-dictatorial. ■

Using similar arguments, we can prove that an SCF that picks the k^{th} alternatice from the left-most alternative will also be *strategyproof* and *non-dictatorial*, $\forall k \in \{1, 2, .., |A|\}$. In particular, it holds true for the right-most alternative and median ($k = \lfloor n/2 \rfloor$).

Definition 21.7 (Median Voter SCF) *An SCF $f : \mathcal{S}^n \to A$ is said to be a Median Voter SCF if $\exists B = (y_1, y_2, .., y_{n-1})$ s.t. $f(P) = median(B, peaks(P)), \forall P \in \mathcal{S}^n$. The points in B are called as peaks of "Phantom Voters" or "Phantom peaks".*

Note: B is fixed for a given f and does not change with $P \in \mathcal{S}^n$

21.3.2.1 Advantage of using Phantom Voters

All the examples stated above, i.e. left-most peak, right-most peak, median etc. can be characterized by a single definition of Median Voter SCF.

Claim 21.8 *The SCFs picking the left-most most peak, the right-most peak are median voter SCFs.*
Proof: *If $A = \{a_1, a_2, .., a_{|A|}\}$, let $a = \min_{w.r.t.<} A, b = \max_{w.r.t.<} A$*
Define $y_1, y_2, .., y_{n-1}, z_1, z_2, .., z_{n-1} \in \mathcal{S}^n$ s.t.

$$y_i(1) = a, \forall i \in N \setminus \{n\}$$
$$z_i(1) = b, \forall i \in N \setminus \{n\}$$

For the case of the left-most peak SCF, we can choose $B = (a, a, ..., a)$ which ensures that the median of points in B and peaks reported by the agents will always result in the minimum of the peaks w.r.t $<$ reported by the agents. For the case of right-most SCF, we can choose $B = (b, b, ..., b)$ and the proof follows as for the case of minimum. ■

Using similar arguments, we can prove that the other SCFs picking any k^{th} peak from left are also Median Vector SCFs.

Theorem 21.9 (Moulin(1980)) *Every median vector SCF is strategyproof.*

Proof: We need to consider only the peak preferences of all the agents. So let us assume $P = (P_1(1), .., P_i(1), .., P_n(1))$ to be the peaks and let $f(P) = a \in A$. Consider an agent i

- If $P_i(1) = a$, then there is no reason for i to manipulate.
- If $P_i(1) < a$, then if the agent shifts her preference to further left of a, the median will not change. If she manipulates to report her peak to further right of a, *i.e.* $(P_i, P_{-i}) \to (P_i^{'}, P_{-i})$ s.t. $a < P_i^{'}(1)$, this will imply that $P_i(1) < a < P_i^{'}(1)$, and since P_i is a single-peaked preference, by definition 21.2, $a = f(P_i, P_{-i})\ P_i\ f(P_i^{'}, P_{-i})$. Thus, f is not manipulable.
- If $a < P_i(1)$, again by exactly symmetrical arguments, f is not manipulable.

Hence, in each of the cases, f is not manipulable. Hence, f is strategyproof. ■

Note: Mean does not satisy the property used in the proof of theorem 21.8 *i.e.* changing the peak $P_i(1)$ of agent i on either sides of a will result in a change in the mean.

21.4 Summary

In this lecture, we revisited the Gibbard-Satterthwaite theorem and due to the restrictiveness of the result, we emphasized the need for domain restriction. Among the major domain restrictions, we looked at the single peaked preferences'. In particular, we looked at examples of SCFs that are *strategyproof* but *non-dictatorial* and defined the median voter SCF which characterizes a set of these examples.