CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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24.1 Recap

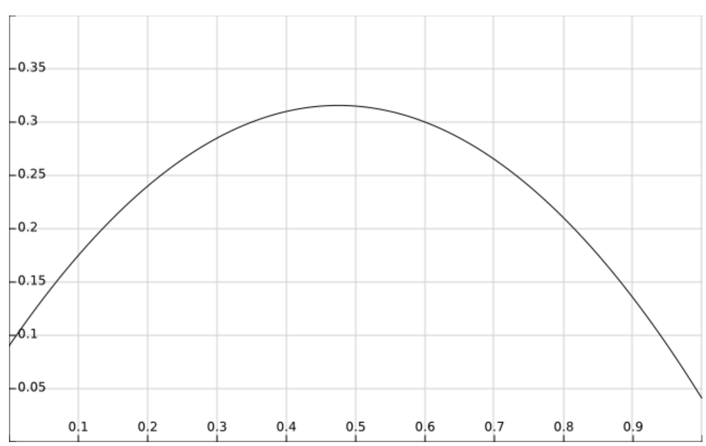
The previous classes focused on a domain restriction called *single-peaked preferences* where the agents had to submit a preference over their alternatives so that there was a single peak with reference to a pre-defined ordering on the real line. This can be extended to multiple dimensions by defining suitable orderings. The multi-dimensional case remains an open problem.

24.2 Task Allocation Problem

Consider a task that must be divided among a set of n agents ($N = \{1, 2, \ldots, n\}$). The task is such that the agent's payoff only depends on the amount or *share* of the task that the agent gets. In this case, we may represent the entire task as the unit interval $[0, 1]$ on the real line. We denote by s_i, the share of task that agent i gets. Note that the location of this share on the real line is irrelevant to us. Clearly, for all $i \in N$, we have $s_i \in [0, 1]$ and $\sum_{i \in N} s_i = 1$.

We give each agent a *reward* for the work that the agent does and this depends on the share of the task that the agent is assigned. Moreover, each agent has a particular *cost* that he/she incurs and this cost depends on the share too. For example, for agent i whose share of the task is s_i, the reward could be $w.s_i$ (where w is fixed) and the cost that agent i incurs could be $k_i s_i^2$. We can now define the *utility* that agent i gets when his share is s_i as, $u_i(s_i) = \textit{reward}(s_i) - \textit{cost}(s_i)$. In our example, this would be $u_i(s_i) = w.s_i - k_i s_i^2$. Note that in general, w could vary with i (but not s_i).

In the above example, the utility function would look something like this:



24.3 Single Peaked-ness

The above utility function can be used to define a preference over the agent's possible shares. Looking at the shape, one might easily start thinking of single-peaked preferences. But there is a catch. Firstly, an *alternative* is not the share of agent i in itself, but a tuple $(s_1, s_2, \ldots, s_n)$. So, the set of alternatives is $\mathcal{A} = \{(s_1, s_2, \ldots, s_n) : s_i \in [0,1] \textit{ and } \sum_{i \in N} s_i = 1\}$. Now, suppose that s_i^* is the share of agent i that optimizes his/her utility. Note that agent i is indifferent towards (s_i^*, s_{-i}) and (s_i^*, s'_{-i}) where $s_{-i} \neq s'_{-i}$. Recall that this indifference is not allowed in single-peaked preferences. The peak should be such that it is strictly prefered over all other alternatives.

However, there is still some structure to the preferences that we can look at. Observe that agent i has his/her own *peak* which represents the ideal share of the task that he/she would like. We would like to retain this structure. So we define $\mathcal{S}$ as the set of all such preferences such that the corresponding utility functions over $[0,1]$ are "single-peaked". Thus, we now define single peaked-ness over the share of an agent's tasks. We can now define a *social choice function* as $f : \mathcal{S}^n \to \mathcal{A}$.

Consider any $P \in \mathcal{S}^n$. We define $f(P) = (f_1(P), f_2(P), \ldots, f_n(P))$, where $f_i(P)$ is agent i's share of the task. Note that $f(P) \in \mathcal{A}$ so we have $\forall i, f_i(P) \in [0,1]$ and $\sum_{i \in N} f_i(P) = 1$. Also, we let p_i denote the peak of agent i, i.e., p_i is the (unique) peak of the preference P_i of agent i. We now look a familiar property of social choice functions.

24.4 Pareto Efficiency

Definition 24.1 *(Pareto Efficient SCF) A Social Choice Function f is Pareto Efficient (PE) if, $\forall P \in \mathcal{S}^n$, $\nexists a \in A$ such that, $aR_i f(P), \forall i \in N$ and $\exists j \in N, aP_j f(P)$.*

The above definition says that for any preference profile $P \in \mathcal{S}^n$ there should *not* exist any alternative a such that some agent j strictly prefers a to $f(P)$ and the other agents prefer a at least as much as $f(P)$. This very much resembles the if-condition of the *Strong Pareto* property of SCFs [See Lecture 16]. In informal terms, f is "optimal" in the sense that no other alternative is weakly preferred over $f(P), \forall P \in \mathcal{S}^n$. We can also say that no other alternative *pareto dominates* $f(P)$.

24.4.1 Characterizing Pareto Efficiency

Theorem 24.2 *(PE characterization) Let f be a social choice function in the above setting. Then, f is PE* iff *the following conditions hold for all $P \in \mathcal{S}^n$ (recall that p_i's denote the peaks for each agent):*

1. $\sum_{i \in N} p_i = 1 \implies f_i(P) = p_i, \ \forall i \in N.$

2. $\sum_{i \in N} p_i > 1 \implies f_i(P) \leq p_i, \ \forall i \in N.$

3. $\sum_{i \in N} p_i > 1 \implies f_i(P) \geq p_i, \ \forall i \in N.$

Proof: ($\implies$) Suppose f is PE. Let P be any preference profile in $\mathcal{S}^n$. Now we consider the three cases.

1. $\sum_{i \in N} p_i = 1$.
   Suppose $f_i(P) < p_i$ for some i (similar argument will hold for $>$). Then, there exists $j \in N \setminus \{i\}$ such that $f_j(P) > p_j$ (else we would have $\sum_{k \in N} f_k(P) < 1$). Note that if we increase the share of i by ϵ (> 0) and decrease the share of j by ϵ and keep the share of everyone else the same, we still get a valid alternative (for suitable values of ϵ). We define a new alternative in which we choose ϵ so that $f_i(P) < f_i(P) + \epsilon \leq p_i$ and $p_j \leq f_j(P) - \epsilon < f_j(P)$. Now, since nobody else's share has changed, they are indifferent to the new ordering. However, agents i and j are strictly better off since both of them move closer to their peaks. So, our new alternative *pareto dominates* $f(P)$. Therefore, f is not PE. ($\Rightarrow\Leftarrow$). Therefore, $f_i(P) = p_i, \ \forall i \in N$.

2. $\sum_{i \in N} p_i > 1$.
   Suppose $\exists i \in N$ such that $f_i(P) > p_i$. Then, there exists $j \in N \setminus \{i\}$ such that $f_j(P) < p_j$ (else we would have $\sum_{k \in N} f_k(P) > 1$). By the same ϵ-shift argument as last case, we can construct an alternative that pareto dominates $f(P)$. Therefore f is not PE. ($\Rightarrow\Leftarrow$)Therefore, $f_i(P) \leq p_i, \ \forall i \in N$.

3. $\sum_{i \in N} p_i < 1$.
   Suppose $\exists i \in N$ such that $f_i(P) < p_i$. Then, there exists $j \in N \setminus \{i\}$ such that $f_j(P) > p_j$ (else we would have $\sum_{k \in N} f_k(P) < 1$). Repeat the ϵ-shift argument in the two cases above to construct an alternative that pareto dominates $f(P)$. Therefore f is not PE. ($\Rightarrow\Leftarrow$)Therefore, $f_i(P) \geq p_i, \ \forall i \in N$.

Therefore, if f is PE then the three conditions must hold. This proves the if-part of the theorem.

($\Longleftarrow$) Now, suppose the three conditions hold. We take 3 cases as above.

1. $\sum_{i \in N} p_i = 1$ and $f_i(P) = p_i, \ \forall i \in N$.
   In this case, every agent receives gets peaks (by condition 1). Since peaks are most preferred for each agent individually, this outcome Pareto dominates all others. So in this case, f outputs an alternative that is not pareto dominated by any other alternative.

2. $\sum_{i \in N} p_i > 1$ and $f_i(P) \leq p_i, \ \forall i \in N$.
   To see that $f(P)$ is not pareto dominated, we only need to observe the following: since every agent is on the left side of their peaks ($f_i(P) \leq p_i$) and the sum of their shares sums to one, we cannot have an alternative where everyone moves "closer" to their peaks. If an agent moves closer to his/her peak, there must be another agent who moves away and therefore has strictly lower payoff. Therefore, there cannot exist an alternative that Pareto Dominates $f(P)$.

3. $\sum_{i \in N} p_i < 1$ and $f_i(P) \geq p_i, \ \forall i \in N$.
   This argument in this case is a repetition of the above argument.

Note that in all of the 3 cases, $\nexists a \in A$ such that a pareto dominates $f(P)$. Since the cases are exhaustive, we conclude that f is PE. ∎

24.5 Examples of Social Choice Functions

24.5.1 Sequential Dictator

In the Sequential Dictator (SD) social choice function, we fix a pre-determined order over the agents. Then we move through the ordering giving each agent his/her most preferred share (i.e., p_i). If we run out of shares of the task, then one agent gets below his preferred share and the agents in the ordering that follow, get 0. If we have reached the last agent in the ordering, we give that agent whatever share of the task is left.

Claim 24.3 *SD is Strategy-Proof.*

Proof: Let i be the first agent in the ordering who does not get her p_i, i.e. $s_i \neq p_i$. Consider 2 exhaustive cases:
Case 1 ($s_i < p_i$): Here, we must have that all agents after i in the ordering receive 0. Clearly, no one before i would want to change since they get their peaks. For i, increasing his p_i is no use since he cannot get above s_i. If he decreases p_i to below s_i then he gets something even less than s_i which is worse off as compared receiving s_i. For all agents after i, they cannot get anything other than 0 since the shares of task are already exhausted. So, truthful reporting weakly dominates every other strategy for all agents in this case.
Case 2 ($s_i < p_i$): The only possible case is that agent i is the last agent. Note that in this case, the other $n-1$ agents get their peaks so misreporting is strictly worse off for them. Agent i must accept whatever is left, irrespective of his preferences so misreporting won't give him any better payoff. Therefore, truthful reporting weakly dominates every other strategy in this case as well.
Therefore, SD is Strategy-Proof. ■

Claim 24.4 *SD is Pareto Efficient.*

Proof: Let i be the first agent in the ordering who does not get her p_i, i.e. $s_i \neq p_i$. The $i-1$ agents before i get their most preferred share and the ones after i get nothing (if i is not the last agent in the ordering). By the previous proof, we know that i gets more than his p_i iff i is the last agent in the ordering. In this case, decreasing i's share would mean increasing at least one of the $i-1$ agents' shares who currently have their peaks. All of those agents would be strictly worse off on this, so any such outcome cannot pareto dominate SD(P). Now, in case i gets less that p_i, giving more to i (or anyone after i in the ordering) would mean taking away from the agents who have their peaks. They would be strictly worse off and therefore such outcome cannot pareto dominate SD(P). Therefore, no outcome can Pareto Dominate SD(P) and SD is Pareto Efficient. ■

The SD social choice function is SP and PE but still seems "unfair" since it favours certain agents over others (the ones that come earlier in the ordering). As the name implies, the rule gives a preference over "dictators". We saw that in single-peaked preferences, a property called *anonimity* helped us increase the class of Strategy-proof functions beyond the Dictatorial bounds that we proved in the GS Theorem. Therefore, we define such a property here as well.

Definition 24.5 *(ANON) A Social Choice Function f is ANON if for all permutations $\sigma : N \to N$ and all preference profiles $P \in \mathcal{S}^n$, we have $f_i(P) = f_{\sigma(i)}(P^\sigma)$, where P^σ is the permutation of the P in accordance with σ [as in Lecture 22].*

Note that SD is SP and PE but not ANON.

24.5.2 Proportional

The Proportional SCF gives every agent a constant factor c of their peaks. So if agent i's peak is p_i, then agent i gets $c.p_i$. We must have, $\sum_{i \in N} c.p_i = 1 \implies c = \frac{1}{\sum_{i \in N} p_i}$.
Note that this SCF is clearly ANON since c is fixed once P is fixed and the agent's payoff depends purely on his peak, not his name or id. So if the preferences are permuted in the same way as the agents, the payoffs do now change.

Using the characterization theorem for PE (Theorem 24.2), we can show that this SCF is also PE. To see this consider 3 exhaustive cases. If $\sum_{i \in N} p_i = 1$ then $c = 1$ and everyone gets p_i. If $\sum_{i \in N} p_i < 1$, then $c > 1$ and everyone gets share $c.p_i > p_i$. Similarly, if $\sum_{i \in N} p_i > 1$ then $c < 1$ and everyone gets share $c.p_i < p_i$. Therefore, the three conditions of the theorem are satisfied and we conclude that the Proportional SCF is Pareto Efficient.

Proportional SCF is not Strategy-Proof in general. For instance consider the case when $\sum_{i \in N} p_i > 1$ so that $c < 1$. Everyone gets $c.p_i < p_i$. If agent j *increases* her peak slightly, then the sum increases and c decreases. Suppose s was the initial sum so that $c = 1/s$. On increasing p_j by ϵ, the sum becomes $s' = s + \epsilon$. The initial payoff of agent j is $u_j = c.p_j = p_j/s$ and her new payoff is $u'_j = c'.p'_j = (p_j + \epsilon)/(s + \epsilon)$. The ratio $\frac{u'_j}{u_j} = (\frac{p_j + \epsilon}{p_j})(\frac{s}{s + \epsilon}) = \frac{(1 + \frac{\epsilon}{p_j})}{(1 + \frac{\epsilon}{s})}$.

Since $p_j < s$, we must have $\frac{1}{p_j} > \frac{1}{s}$ and for $\epsilon > 0$, $\frac{\epsilon}{p_j} > \frac{\epsilon}{s}$. Plugging this into the equation above, we get that $u'_j > u_j$. Since she was initially to the left of the peak, we can choose ϵ so that $u'_j \leq p_j$ and so misreporting with an increased peak can be strictly better for agent j. Thus, the Proportional SCF is not Strategy-Proof.