An interpretation of the utility of the agent under VCG

$$u_i(\theta_i, \theta_{-i} | \theta_i) = v_i(f^{AE}(\theta_i, \theta_{-i}), \theta_i) - p_i^{VCG}(\theta_i, \theta_{-i})$$

$$= v_i(f^{AE}(\theta_i, \theta_{-i}), \theta_i) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

$$+ \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j)$$

$$= \sum_{j \in N} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

$$= \underbrace{\max_{a \in A} \sum_{j \in N} v_j(a, \theta_j)}_{\text{max SW in } i\text{'s presence}} - \underbrace{\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)}_{\text{max SW in } i\text{'s absence}}$$

= marginal contribution of i in the SW.

Illustration of VCG payment

(1) Single object allocation. Consider agent $i \in N$. Every agent has some value when the object is assigned to him — and zero otherwise. Efficiency requires that the object should go to the agent who values it the most. If i's value/bid is the maximum, he is assigned the object, but

$$\text{Payment} = \underbrace{\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)}_{\text{second highest bid}} - \underbrace{\sum_{j \neq i} v_j(f(\theta), \theta_j)}_{= 0 \text{ if } i\text{'s bid is highest}}$$

second price auction.

(2) What is Pivotal about the VCG payment? Consider a public project allocation.

First term : the social welfare of all the agents except agent i if i wasn't present

Second term : social welfare of the other agents in i's presence.

hence another interpretation of the payment : loss in social welfare of the other agents ~~of~~ ~~agent i~~ because of agent i's presence.

Example :

		Football	Library	Museum
~~Alice~~	A	0	70	50
~~Bob~~	B	~~100~~ 95	10	50
~~Carol~~	C	10	50	50

VCG allocation: efficient : M

$A = \{F, L, M\}$

A ~~B~~ pays $= 105 - 100 = 5$

B pays $= 120 - 100 = 20$

C pays $= 100 - 100 = 0$ ← non-pivotal agent

The agent whose presence changes the outcome are called pivotal. VCG charges only them.

③ Combinatorial allocation: Sale of multiple objects

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
v_1	0	8	6	12
v_2	0	9	4	14

type is the value itself

$$v_i(a, \theta_i) = \theta_i(a_i)$$

Efficient allocation $\{1\} \to$ Player 2, $\{2\} \to 1$

$$p_1^{VCG}(\theta_1, \theta_2) = \cancel{14 - 9 = 8}$$

$$\max_{a \in A} \sum_{j \neq 1} \theta_j(a) - \sum_{j \neq 1} \theta_j(f(v))$$

$$= 14 - 9 = 5 \quad , \text{payoff} = 6 - 5 = 1$$

$$p_2^{VCG}(\theta_1, \theta_2) = 12 - 6 = 6 \quad , \text{payoff} = 9 - 6 = 3$$

VCG mechanism in combinatorial auctions

VCG has several useful properties

① It is DSIC

② It is Efficient

③ Payments are non-negative (no-subsidy), hence WBB.

Combinatorial auctions:

$M = \{1, \ldots, m\}$ set of objects

Set of bundles $\Omega = \{S : S \subseteq M\}$

Type of agent i is $\theta_i : \Omega \to \mathbb{R}$

Hence $\theta_i(S)$, $S \in \Omega$, is the value of agent i for the bundle S. We assume $\theta_i(S) \geq 0 \ \forall S \in \Omega$

<all bundles are "goods">

An allocation in this case is a partition of the objects :

$$X = \{X_0, X_1, \ldots, X_n\}, \quad X_i \in \Omega$$

A is the collection of such X's. $X_i \cap X_j = \phi$ if $i \neq j$, $\bigcup_{i=0}^{n} X_i = M$.

X_0 is the set of unallocated objects and X_i is the bundle allocated to agent i. Assume $\theta_i(\phi) = 0$.

Note: the model is such that the valuations ~~are~~ have no externalities (selfish valuations)

$$v_i(X, \theta_i) = \theta_i(X_i) \quad \langle \text{does not depend on the allocation of the other agents} \rangle.$$

In the allocation of goods,

Claim 1: ~~Any agent~~ The VCG payment for an agent that gets no object in the efficient allocation is zero.

Fix θ, M.

Proof: Say agent i gets no object in the efficient allocation, i.e., $X \in \arg\max_{x \in A} \sum_{i \in N} v_i(x, \theta_i)$

$X_i = \phi$

VCG payment considers the allocation excluding agent i. Let that allocation is $Y \in \arg\max_{y \in A} \sum_{j \neq i} v_j(y, \theta_j)$

We have seen that VCG payment always yields

$p_i^{VCG}(\theta) \geq 0$ [no-subsidy]

But we can also write

$$p_i^{VCG}(\theta) = \sum_{j \neq i} v_j(Y, \theta_j) - \sum_{j \neq i} v_j(X, \theta_j)$$

adding $v_i(Y, \theta_i) \geq 0$

subtracting $v_i(X, \theta_i) = 0$

$$\leq \sum_{j \in N} v_j(Y, \theta_j) - \underbrace{\sum_{j \in N} v_j(X, \theta_j)}_{\text{max by definition}}$$

$$\leq 0$$

$$\boxed{p_i^{VCG}(\theta) = 0}$$

□