CS711: Introduction to Game Theory and Mechanism Design	Jul-Nov 2018

Project: Assignment 2

1.1 Solution of Question 1

1.1.1 Part 1

Given : $f(P_1, P_2) = a$
Let us redefine strategy profiles P_1, P_2 by interchanging the priority of a and b for both agents 1 and 2 as follows.

P_1	P_2	$P_1^{"}$	$P_2^{"}$
a	c	b	c
b	b	a	a
c	a	c	b

Table 1.1: Preference profiles P_1, P_2 and redefined profiles $P_1^{"}, P_2^{"}$

Both $P_1^{"}, P_2^{"}$ are valid preferences since the domain of preferences is of unrestricted strict preferences.

Since $P_1^{"}, P_2^{"}$ are obtained by interchanging priority of a abd b for all agents priority we can get get the social choice value by interchanging a and b of $f(P_1, P_2)$.

$\rightarrow f(P_1^{"}, P_2^{"}) = b$

Following are the dominated sets defined for $P_1^{'}, P_2^{'}, P_1^{"}, P_2^{"}$
$D(b, P_1^{'}) = \{a, c\}$
$D(b, P_2^{'}) = \{c\}$
$D(b, P_1^{"}) = \{a, c\}$
$D(b, P_2^{"}) = \{\}$

From the above sets we say that $D(b, P_1^{"}) \subseteq D(b, P_1^{'})$ and $D(b, P_2^{"}) \subseteq D(b, P_2^{'})$

Using the monotonicity of social choice function f we get:
$f(P_1^{'}, P_2^{'}) = f(P_1^{"}, P_2^{"}) = b$

Hence prooved $f(P_1^{'}, P_2^{'}) = b$

1.1.2 Part 2

No, the earlier conclusion does not hold in this case.

Because for the proof in part 1 to be eligible in part 2 both $P_1^{"}, P_2^{"}$ should be valid. But since in part two, the preferences are generated from a single-peaked preference domain with the intrinsic ordering of the alternatives being $a < b < c$, $P_2^{"}$ is not a valid single-peaked preference hence the proof of part 1 can not be utilized here.

Let us define a direct mechanism that has $f(P_1^{'}, P_2^{'}) = a$.

For any preference profile $\bar{P}_1, \bar{P}_1$, SCF $f(\bar{P}_1, \bar{P}_1)$ is defined over S such that if $|s| = 1$ then that element of S is the SCF value for the given preferences.

$$f(\bar{P}_1, \bar{P}_1) = \begin{cases} a & \text{if } a \in \{\bar{P}_1(1), \bar{P}_2(1)\} \\ b & \text{if } a \notin \{\bar{P}_1(1), \bar{P}_2(1)\} \textbf{ and } b \in \{\bar{P}_1(1), \bar{P}_2(1)\} \\ c & \text{if } a, b \notin \{\bar{P}_1(1), \bar{P}_2(1)\} \textbf{ and } c \in \{\bar{P}_1(1), \bar{P}_2(1)\} \end{cases}$$

$f(P_1, P_2) = a$ ...(1)

The defined SCF function gives all three alternatives a, b and c as output for the following set of single-peaked preference frofiles hence SCF f is onto.

$P_1^{'}$	$P_2^{'}$	$P_1^{"}$	$P_2^{"}$	$P_1^{'''}$	$P_2^{'''}$
a	a	b	b	c	c
b	b	a	a	b	b
c	c	c	c	a	a

Table 1.2: The 3 preference profiles (from left to right) gives a,b and c SCF value respectively.

$f(\bar{P}_1, \bar{P}_1)$ is also strategyproof as shown below:

For any agent i

case 1: $\bar{P}_i(1) = a$

agent i gets his highest priority alternative as scf value so no need to collude.

case 2: $\bar{P}_i(1) = b$

if scf value is b no need to collude

else scf value is a, agent i can never change it to b whatever the agent i may report in the message

case 3: $\bar{P}_i(1) = c$

Similar to case 2

Considering the above cases we can say that f is strategyproof and we have already prooved it to be onto and together with equation (1) we can say the f satisfies all the required conditions.

1.2 Solution of Question 2

No, the Gibbard-Satterthwaite result will not apply here.

let us take a case where for all agents (2 agents in this example) and $|X| = 2$. Since $|X| = 2$ we can say that htere are only 3 distinct non-empty subsets let us denote them with S,T and U.
Where $X = \{x_1, x_2\}$, $S = \{x_1\}$, $T = \{x_2\}$, $U = \{x_1, x_2\}$.

For Gibbard-Satterthwaite result to apply here total number of alternatives are 3 which satisfies the condition for Gibbard-Satterthwaite. But there is also indifference for each agent i between two sets U and either T or S. Hence Gibbard-Satterthwaite result will not apply here.

We use **Veto** scoring method and the alternative with highest score is chosen as scf value. In case of tie, $S > T > U$ rule is followed. For both agents $P_i(1)$ and $P_i(2)$ are indifferent and one of these sets will be U. because U contains all the projects including highest priority project. and depending on which project has higher priority for agent i either S or T will be the other entry. Let us assume Score(a) to be the score of attribute a.

Let us define a SCF f as follows which is both onto and strategyproof.

$$f(P_1, P_1) = \begin{cases} U & \text{if } Score(U) > Score(S) \text{ and } Score(U) > Score(T) \\ T & \text{if } Score(T) > Score(S) \text{ and } Score(T) >= Score(U) \\ S & \text{if } Score(S) >= Score(T) \text{ and } Score(S) >= Score(U) \end{cases}$$

Considering the following 3 profile strategies we can say that f is onto.

P_1	P_2	P_1	P_2	P_1	P_2
S	U	U	T	U	U
U	S	T	U	S	T
T	T	S	S	T	S

Table 1.3: The 3 preference profiles (from left to right) gives S,T and U SCF value respectively.

For any agent i

case 1: $\bar{P}_i(1) = U$

if SCF value is U then no need to falsify the report. If SCF is S and $\bar{P}_i(2) = T$, This condition is not possible. similarly if SCF is T and $\bar{P}_i(2) = S$ This condition is not possible. And When SCF is equal to $\bar{P}_i(2)$ no need to modify the report.

case 2: $\bar{P}_i(1) = T$

if SCF value is U or T then no need to falsify the report. If SCF is S, This condition is not possible. So no need to modify the report.

case 3: $\bar{P}_i(1) = S$

If scf value is U or S no need to collude since. If SCF is T, This condition is not possible. So no need to modify the report.

Considering the above cases we can say that f is strategyproof and onto But it is not dictatorial as shown in Table 1.3.

1.3 Solution of Question 3

Yes, the median voter SCF is group strategy-proof. Assuming that no agent is willing to sacrifice itself for the benefit of the agents in K.

let us denote the preferences denoted only by their peaks, i.e., $P = (P_1(1), .., P_i(1), .., P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and the phantom peaks. Consider an agent i:

Case 1: If $P_k(1) = a, \forall k \in K$, then there is no reason for K to manipulate.

Case 2: If $P_k(1) < a, \forall k \in K$, then if all agents in K shifts their preference to further left of a (no agent manipulates their report to the right of a), the median will not change. If any one of K agents manipulates to report her peak to further right of a, i.e. $(P_k, P_k) \rightarrow (P_{k}, P_k)$ s.t. $a < P^{'}_k(1)$,this will imply that $P_k(1) < a < P^{'}_k(1)$, and since P_i is a single-peaked preference, we can say $a = f(P_k, P_k) P_k f(P^{'}_k, P_k)$. Thus, K has no profitable manipulation. similarly if more than one agent in K moves to the right of a then the new median will be further right of the median obtained when only single agent moved to the right of a.

Case 3: If $P_k(1) > a, \forall k \in K$, again by exactly symmetrical arguments, K has no profitable manipulation.

Case 4: When agents in K are on either side of median. Let $K1$ and $K2$ be two subsets of K such that $P_{k1}(1) < a, \forall k1 \in K1$ and $P_{k2}(1) > a, \forall k2 \in K2$. By manipulating the report new median either moves left to a or ritht to a. If new median moves right to a (can only be done by manipulating reports of K1), agents in K1 move further away from new median (which is loss-making manipulation for K1) hence K1 will not manipulate the report. similarly when new median moves to the left of a K2 does not get any profitable manipulation. hence there is no profitable manipulation.

hence the median vector SCF is group strategy-proof.