In the context of QL preferences, SCF is split into two parts:

1. Allocation rule / decision rule

$$f : \Theta \to A$$ <yields the allocation given the type profile>

2. Payment rule:

$$p_i : \Theta \to \mathbb{R} \quad \forall i \in N$$

Examples of allocation rules

1. Constant rule, $f^c(\theta) = a \quad \forall \theta \in \Theta$.
2. Dictatorial rule, $f^D(\theta) \in \underset{a \in A}{\operatorname{argmax}}\, v_d(a, \theta_d)$, $\exists d$ $\forall \theta \in \Theta$.
3. Allocatively efficient rule / utilitarian rule:

$$f^{AE}(\theta) \in \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

Different from Pareto efficient (since the payment also needs to be accounted for). We will see the relationship between AE and PE later.

4. Weighted Efficient rule:

$$f^{WE}(\theta) \in \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} \lambda_i v_i(a, \theta_i), \quad \lambda_i \geq 0$$ not all zero.

5. Max-min / Egalitarian / Rawlsian rule

$$f^R(\theta) \in \underset{a \in A}{\operatorname{argmax}} \min_{i \in N} v_i(a, \theta_i)$$

Examples of payment rules

1. Feasible / Weak Budget Balanced / No-deficit:

$$\sum_{i \in N} p_i(\theta) \geq 0 \quad \forall \theta \in \Theta.$$

Doesnot need external supply of money to run the mechanism.

2. No-subsidy: if $p_i(\theta) \geq 0 \ \forall \theta \in \Theta, \forall i \in N$.
3. Budget Balanced: if $\sum_{i \in N} p_i(\theta) = 0$, for all $\theta \in \Theta$.

These properties on the payment rules are context dependent. Sometimes, when we are interested in only finding the truthful mechanisms, we do not impose such additional restrictions.

Incentive Compatibility:

SCF is decomposed into allocation rule and payment rule

$$F \equiv (f, (p_1, \cdots, p_n))$$

~~utility~~ The SCF takes as input only the reported types $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_n)$

Due to revelation principle, we are interested only in direct mechanisms. If $\hat{\theta}$ is reported ~~the~~ ~~as~~ and the true ~~valuation~~ type of i is θ_i then the payoff is

$$v_i(f(\hat{\theta}), \theta_i) - p_i(\hat{\theta}).$$

~~DSIC~~ -

<u>Defn</u>: A direct mechanism (f, p) is dominant strategy incentive compatible (DSIC) if $\forall i \in N$ $\forall \theta_{-i} \in \Theta_{-i}$, $\forall \theta_i, \theta_i' \in \Theta_i$

$$v_i(f(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i}) \geq$$
$$v_i(f(\theta_i', \theta_{-i}), \theta_i) - p_i(\theta_i', \theta_{-i}).$$

- DSIC – truthtelling is weakly DSE.

We say that p implements f is dominant strategies in this case. Or in general, f is implementable (by a payment rule).

In QL domain, we are more interested in the allocation rule ~~that~~ than the whole SCF (which includes the payment too), hence often there are less restriction put on the payment.

<u>Example</u>: $N = \{1, 2\}$, $\Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$

$f : \Theta_1 \times \Theta_2 \to A$ is the allocation rule.

To implement f in dominant strategies, we need ~~to~~ to satisfy the following conditions.

$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geq v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2)$$

$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geq v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2)$$

Similarly for player 2 $\qquad \forall \theta_2 \in \Theta_2$

$$v_2(f(\theta_1, \theta^H), \theta^H) - p_2(\theta_1, \theta^H) \geq v_2(f(\theta_1, \theta^L), \theta^H) - p_2(\theta_1, \theta^L)$$

$$v_2(f(\theta_1, \theta^L), \theta^L) - p_2(\theta_1, \theta^L) \geq v_2(f(\theta_1, \theta^H), \theta^L) - p_2(\theta_1, \theta^H)$$

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Properties of the payment:

① Say $(f, \underline{p})$ is incentive-compatible,

$$v_i(f(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i}) \geq$$
$$v_i(f(\theta_i', \theta_{-i}), \theta_i) - p_i(\theta_i', \theta_{-i}) \quad \forall \theta_{-i} \in \Theta_{-i},$$
$$\forall \theta_i, \theta_i' \in \Theta_i$$
$$\forall i \in N.$$

Consider another payment

$$q_i(\theta_i, \theta_{-i}) = p_i(\theta_i, \theta_{-i}) + h_i(\theta_{-i})$$

Q: Is $(f, \underline{q})$ incentive compatible?

Ans: Yes.

$$v_i(f(\theta_i, \theta_{-i}), \theta_i) - q_i(\theta_i, \theta_{-i})$$
$$= v_i(f(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i}) - h_i(\theta_{-i})$$
$$\geq v_i(f(\theta_i', \theta_{-i}), \theta_i) \underbrace{- p_i(\theta_i', \theta_{-i}) - h_i(\theta_{-i})}_{-q_i(\theta_i', \theta_{-i})}$$

If we can find one payment rule that implements f, there exists uncountably many payments that also implement f.

One can ask the converse question: when can the payments that implement an SCF differ by a factor?

② Implication of incentive compatibility on the payment

Suppose, the ~~outcome~~ allocation at (θ_i, θ_{-i}) and (θ_i', θ_{-i}) are same

$$f(\theta_i, \theta_{-i}) = f(\theta_i', \theta_{-i}) = a$$

$$\left.\begin{array}{l} v_i(a, \theta_i) - p_i(\theta_i, \theta_{-i}) \geq v_i(a, \theta_i) - p_i(\theta_i', \theta_{-i}) \\ v_i(a, \theta_i') - p_i(\theta_i', \theta_{-i}) \geq v_i(a, \theta_i') - p_i(\theta_i, \theta_{-i}) \end{array}\right\} \Rightarrow p_i(\theta_i, \theta_{-i}) = p_i(\theta_i', \theta_{-i})$$