CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Assignment 2

Problem 1

(a)

1. Consider the following profiles:
   Suppose for contradiction $f(P^{''}) = c$ (it can either be 'a' or 'c' as f is onto and SP so outcome is first

   Table 1.1:

P_1	P_2	$P_1^{''}$	$P_2^{''}$	$P_1^{'''}$	$P_2^{'''}$
a	c	a	c	a	c
-	-	-	-	c	a
-	-	-	-	-	-

   preference of either agent)
   Now consider transition from (P_1, P_2) to $(P_1^{'''}, P_2^{'''})$. Preference for 'a' improves weakly, so $f(P^{'''}) = a$ since $f(P) = a$ and f is monotone (SP $\implies$ monotone). Next, consider transition from $(P_1^{''}, P_2^{''})$ to $(P_1^{'''}, P_2^{'''})$. Preference for 'c' improves weakly, so since $f(P^{''} = c$ thus $f(P^{'''}) = c$, but this is contradiction. Thus, $f(P^{''}) = a$ for any profile $P^{''}$ with $P_1^{''}(1) = a$ and $P_2^{''}(1) = c$.

2. Now consider the following profiles:
   Assume for contradiction that $f(P^{''}) = c$ (it can either be 'b' or 'c' as f is onto and SP so outcome is

   Table 1.2:

P_1	P_2	$P_1^{''}$	$P_2^{''}$	$P_1^{'''}$	P_2
a	c	b	c	b	c
-	-	-	-	a	-
-	-	-	-	-	-

   first preference of either agent). consider transition from $(P_1^{''}, P_2^{''})$ to $(P_1^{'''}, P_2)$, this case is similar to (P_1, P_2) and $(P_1^{''}, P_2^{''})$, so $f(P_1^{'''}, P_2) = c$ by our assumption. Now consider $P_1^{'''}, P_2$, if agent 1 reports P_1 instead of $P_1^{'''}$ then outcome is 'a' which she prefers more than 'c', as $f(P) = a$ and $f(P_1^{'''}, P_2) = c$. This contradicts that f is strategyproof. So, $f(P^{''}) = b$ for any $P^{''}$ with top preference for agent 1 and 2 being 'b' and 'c' respectively.

3. Finally consider profiles in Table 1.3:
   Consider transition from $(P_1^{''}, P_2^{''})$ to $(P_1^{'}, P_2^{'})$, preference for b improves for both the agents, so as f is montone, thus $f(P^{'}) = f(P^{''}) = b$.

Table 1.3:

$P_1^{'}$	$P_2^{'}$	$P_1^{''}$	$P_2^{''}$
b	a	b	c
a	b	-	-
c	c	-	-

(b)

Earlier conclusion won't hold in this case. As the preference profile $(P_1^{'''}, P_2^{'''})$ in table 1.1 is not allowed in single-peaked preference domain as it have 2 peaks at 'a' and 'c'.
Consider the following SCF f:
$f(P) = min_{i\in n}\{P_i(1)\}$
i.e. f picks the left-most peak among the peaks of agents. Clearly $f(P_1, P_2) = a$ and $f(P_1^{'}, P_2^{'}) = a$.

Problem 2

No, Gibbard-Satterthwaite result does not apply here.
Consider a set a projects $|X| = n$, so that SCF chooses from $|A| = 2^n - 1$ possible subsets. Let P_i be the preference profile over X for agent 'i', which is extended to $P_i^{'}$ over 'A'. Now, consider two sets: $A_1, A_2 \in A$ such that $A_1 \subset A_2$. Since any high ranked project in A_1 would be in A_2 too, so, any rational agent would never pick a profile where A_1 is more preferred than A_2. Thus, preferences are not unrestricted since all possible linear orderings in 'A' are not possible but GS theorem needs unrestricted preferences.

Problem 3

Yes, the median voter SCF is group strategy-proof.
Proof: Only the peak preferences of the agents need to be considered. Let $f(P) = a \in A$ be the medians of the peaks and phantom peaks. Clearly, for agent i with $P_i(1) = a$ need not manipulate.
Let $K \subseteq N$ agents decide to manipulate the outcome by shifting their medians where:

1. K_1 agents have their peaks on left of 'a'
2. K_2 $(= K - K_1)$ have their peaks at right of 'a'.

For K_1 agents on the left of 'a', the median does not changes if they shift their own peak to their left, so they will have to move their peak to right. Of these agents, let $K_1^{'} \subseteq K_1$ agents move their peak to the right of 'a'. Similarly, let $K_2^{'} \subseteq K_2$ agents move their peak to the left of 'a'.

1. Case 1: $K_1^{'} > K_2^{'}$. Clearly, new median is some $b \in A$ to the right of 'a'. But, for all the K_1 agents 'b' is less preferred than 'a' i.e. $aP_jb\forall j \in K_1$ as the preferences are over single peaked domain. So, this manipulation is not profitable for them.

2. Case 2: $K_1^{'} < K_2^{'}$. Similar to the previous case, the new outcome will be on left of 'a' and won't be profitable for K_2 agents on the right.

So, median voter SCF is group strategy-proof.