CS-698W: Game Theory and Collective Choice	**Jul-Nov 2017**

Lecture 32: October 31, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

32.1 Single Object Auction Model

Let the type set of agent i be $T_i \subseteq \mathbb{R}$ and $t_i \in T_i$ be the value agent i gets if he wins the object.

An allocation a is a vector of length n, where a_i denotes the probability that i wins the object. The set of allocations is denoted by

$$\Delta A = \{a \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{n} a_i = 1\}$$

and the allocation rule is a function $f : T_1 \times T_2 \times \cdots \times T_n \longrightarrow \Delta A$ where $f_i(t_i, t_{-i})$ is the probability that i wins the object when type profile is (t_i, t_{-i}).

Given an allocation $a = (a_1, \ldots, a_n)$, the valuation of agent i is given by the product form $a_i.t_i$.

Note that the setting and results can be extended to setting where a_i is the amount allocated to agent i. For example, in sponsored search auctions, a_i is replaced by CTR_i.

32.1.1 Vickrey (Second Price) Auction

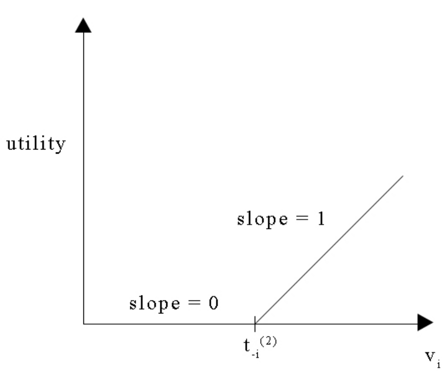

Figure 32.1: Utility of agent i

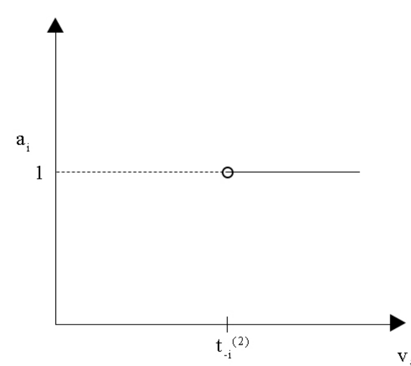

Figure 32.2: Allocation of agent i

Agent i wins if $v_i > t_{-i}^{(2)}$ and loses if $v_i < t_{-i}^{(2)}$ where v_i is the type of agent i, i.e., the value of the object if i wins. $t_{-i}^{(2)}$ is defined as $\max_{j \neq i} v_j$.

The utility u_i of agent i (Fig. 32.1) is defined as

$$u_i = \begin{cases} 0 & \text{if } v_i \leq t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & \text{if } v_i > t_{-i}^{(2)} \end{cases}$$

The utility is a convex function. The derivative of the utility function is 0 for $v_i < t_{-i}^{(2)}$ and 1 for $v_i > t_{-i}^{(2)}$, and is not differentiable at $v_i = t_{-i}^{(2)}$. Also, its derivative coincides with the probability of winning wherever it exists (Fig. 32.2).

32.2 Convex Analysis

We are interested in functions $g : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I} \subseteq \mathbb{R}$.

Definition 32.1 *: A function $g : \mathbb{I} \to \mathbb{R}$ is convex if for every $x, y \in \mathbb{I}$ and $\lambda \in [0, 1]$,*

$$\lambda g(x) + (1 - \lambda) g(y) \geq g\big(\lambda x + (1 - \lambda) y\big)$$

Facts:

1. Convex functions are continuous in the interior of its domain; jumps can only occur only at the boundaries.
2. Convex functions are differentiable "almost everywhere" in $\mathbb{I}$. Formally, there exists a $\mathbb{I}' \subseteq \mathbb{I}$ such that $\mathbb{I} \setminus \mathbb{I}'$ has countable points (has measure zero) and g is differentiable at every point of $\mathbb{I}'$.

32.2.1 Subgradient

Definition 32.2 *For any $x \in \mathbb{I}$, x^* is a subgradient of the function $g : \mathbb{I} \to \mathbb{R}$ at x if*

$$g(z) \geq g(x) + x^* g(z - x) \qquad \forall z \in \mathbb{I}$$

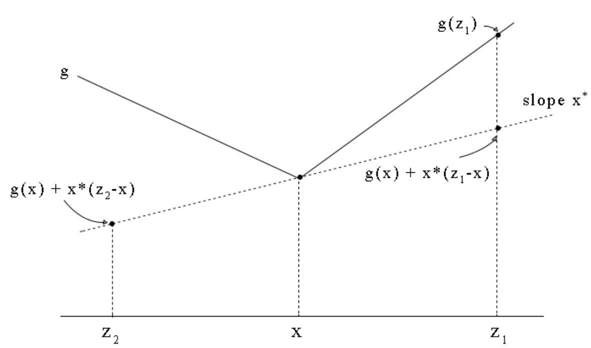

Figure 32.3: $g(x)$

Lemma 32.3 *Let $g : \mathbb{I} \to \mathbb{R}$ is a convex function. Suppose x is in the interior of $\mathbb{I}$ and g is differentiable at x. Then $g'(x)$ is the unique subgradient of g at x.*

Proof: Consider $z \in \mathbb{I}$ such that $z > x$ (a similar proof works if $z < x$). Consider h such that $(z-x) \geq h > 0$.

$$x + h = \frac{h}{z-x} z + \Big(1 - \frac{h}{z-x}\Big) x$$

Since g is convex,

$$\frac{h}{z-x}\ g(z) + \Big(1 - \frac{h}{z-x}\Big)\ g(x) \geq g(x+h)$$

$$\implies \frac{g(z) - g(x)}{z-x} \geq \frac{g(x+h) - g(x)}{h}$$

The above result holds for any $h > 0$. So when $h \to 0$,

$$g(z) - g(x) \geq g'(x)\ (z - x)$$

Hence $g'(x)$ is a subgradient of g at x.

Now, we need to show uniqueness. Say for contradiction, there exists another subgradient $x^* \neq g'(x)$ at x.

Case 1: $x^* > g'(x)$. By definition,

$$g(x+h) - g(x) \geq x^* h$$

$$\implies \frac{g(x+h) - g(x)}{h} \geq x^* > g'(x)$$

Taking limit as $h \to 0$

$$g'(x) \geq x^* > g'(x)$$

Contradiction.

Case 2: $x^* < g'(x)$. Can be proved similarly. ■

Lemma 32.4 *Let $g : \mathbb{I} \to \mathbb{R}$ be a convex function. Then for every $x \in \mathbb{I}$, the subgradient of g at x exists.*

Fact: For points in $\mathbb{I} \setminus \mathbb{I}'$, the set of subgradients at a point forms a convex set.

Define $g'_+(x) = \lim\limits_{\substack{z \to x \\ z \in \mathbb{I},\ z > x}} g'(x)$ and $g'_-(x) = \lim\limits_{\substack{z \to x \\ z \in \mathbb{I},\ z < x}} g'(x)$

The set of subgradients at $x \in \mathbb{I} \setminus \mathbb{I}'$ is $\big[g'_-(x), g'_+(x)\big]$

The set of subgradients of g at a point $x \in \mathbb{I}$ is denoted by $\partial g(x)$.

Lemma 32.3 says that $\partial g(x)$ is $\{g'(x)\}$ for $x \in \mathbb{I}'$ and Lemma 32.4 says that it's non-empty for all $x \in \mathbb{I}$.

Lemma 32.5 *Let $g : \mathbb{I} \to \mathbb{R}$ be a convex function. Let $\phi : \mathbb{I} \to \mathbb{R}$ such that $\phi(z) \in \partial g(z), \forall z \in \mathbb{I}$. Then $\forall x, y \in \mathbb{I}$ such that $x > y$, we have $\phi(x) \geq \phi(y)$.*

Proof: By definition,

$$g(x) \geq g(y) + \phi(y)\big(x - y\big)$$
$$g(y) \geq g(x) + \phi(x)\big(y - x\big)$$

Adding the above two equations, $\big(\phi(x) - \phi(y)\big)\big(x - y\big) \geq 0$

$$\implies \phi(x) \geq \phi(y) \qquad \text{(because } x > y\text{)}$$

■

If g was differentiable everywhere, then

$$g(x) = g(y) + \int_y^x g'(z)dz$$

An extension of this result holds for convex functions with subgradients.

Lemma 32.6 *Let $g : \mathbb{I} \to \mathbb{R}$ be a convex function. Then for any $x, y \in \mathbb{I}$,*

$$g(x) = g(y) + \int_y^x \phi(z)dz$$

where $\phi : \mathbb{I} \to \mathbb{R}$ is such that $\phi(z) \in \partial g(z), \forall z \in \mathbb{I}$