CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Assignment: 2

1.1

1.1.1

P_1	P_2	P'_1	$\hat{P}_2$	P'_1	P'_2
a	c	b	c	b	a
b	b	a	a	a	b
c	a	c	b	c	c

The social Choice function f is Strategy-Proof (SP) and onto on the set of outcomes $\{a, b, c\}$ and $f(P) \in \{P_1(1), P_2(1)\}$ such that $f(P_1, P_2) = a$. We need to prove that $f(P'_1, P'_2) = b$. We prove this by contradiction. Assume that, $f(P'_1, P'_2) = a$. Consider the preference profile $(P'_1, \hat{P}_2)$. If $f(P'_1, \hat{P}_2) = b$, then Player2 can manipulate from $\hat{P}_2$ to P'_2 so that the outcome changes to a, because $a\hat{P}_2 b$. Thus for the SCF to remain Strategy-Proof, $f(P'_1, \hat{P}_2) = c$.

Consider the transition from $(P'_1, \hat{P}_2) \rightarrow (P_1, P_2)$. $D(c, P_1) = D(c, P'_1) = \phi$ and $D(c, \hat{P}_2) = D(c, P_2) = \{a, b\}$. Since $f(P'_1, \hat{P}_2) = c$ and every Strategy Proof SCF is also monotonic, by monotonicity of f, $f(P_1, P_2) = f(P'_1, \hat{P}_2) = c$. But, this is a contradiction to the fact that $f(P_1, P_2) = a$. Therefore, our assumption must have been incorrect and $f(P'_1, P'_2) = b$.

1.1.2

If the preferences are generated from a single-peaked preference with the intrinsic ordering of alternatives being $a < b < c$, then the domain of the social choice function gets restricted. The profiles P_1, P_2, P'_1, P'_2 are still admissible under this restriction however the profile $\hat{P}_2$ is not allowed since if b is the lowest preference then the outcomes (a, c) on its 2 sides are both peaks and the profile is no longer single-peaked. Since we cannot use any profile with minimum at b, our proof breaks down.

To construct a strategy-proof and onto SCF on single-peaked preference profile such that $f(P_1, P_2) = f(P'_1, P'_2) = a$, we can use the Median Voter Social Choice function. Every median voter SCF is Strategy-Proof and Onto and so is f. Now, to ensure that the outcome is a for profiles (P_1, P_2) and (P'_1, P'_2),the median voter SCF selects the leftmost peak among the 2 players' preference profiles. Thus, whenever a is a peak, which is the case for P_1 and P'_2, the outcome is a.

1.2

Gibbard-Satterthwaite Theorem states that for a SCF $f : \mathcal{P}^n \rightarrow \mathcal{A}$, where $\mathcal{A}$ is the set of outcomes and $\mathcal{P}$ is the set of all possible weak preferences(either strict or indifferent) of each voter over the set of outcomes $\mathcal{A}$, then if f is Strategy-Proof and onto and $\mathcal{A} \geq 3$, then it must be a dictatorial SCF. Now, for our social choice function, we try to define the set of outcomes $\mathcal{A}$ and the set of all weak preferences $\mathcal{P}$ over the set $\mathcal{A}$.

Let $S(X)$ be the set of all non-empty subsets of the set X. Then, the range of the SCF f is $S(X)$, thus $\mathcal{A} = S(X)$. $\mathcal{A} = 2^X - 1 \geq 3$ for $X \geq 2$.

Now, to define the domain of the SCF f, we define R_i as the preference of player i over $S(X)$ induced by the preference of player i (P_i) on X as defined in the problem.Thus, the set of all possible weak preferences R_i is the required set $\mathcal{P}$.

We will now show that the admissible preference profiles for this SCF come from a restricted domain. The set $X \in S(X)$. Since the peak of each player's preference over X ($P_i(1)$) lies in X, therefore for any set $T \in S(X), T \neq X$, $XR_iT, \forall i \in N$. Thus, there can be no possible preference profile for any player where the outcome X is less preferred any other outcome. This is a domain restriction on the set of all possible weak preferences over $\mathcal{A}$ and thus the Gibbard-Satterthwaite Theorem cannot be applied to this SCF.

1.3

Let f be a median voter SCF which selects the k^{th} peak from left.(a_k) The player whose peak is k^{th} from the left will never misreport its preference as it already has its most favorable outcome. Thus, no group containing the this player will be group-manipulable.

For any group $X \subset N$, which doesn't contain the player whose peak is the current outcome, the player which will try to change the outcome is never strictly better off. Let R_X be the number of players to the left of a_k who will misreport their peak to the right of a_k and let L_X be the number of players to the right of a_k who will misreport their peak to the left of a_k. If any non-empty set of players would want to change the outcome then $R_X \neq L_X$. To prove this statement, we analyse its converse. Thus, if the number of players to the right of a_k misreporting their peaks to the left of a_k is equal to number of players to the left of a_k misreporting their peaks to the right of a_k, then the number of elements to the right of a_k and to the left of a_k doesn't change. This implies that a_k is still is the k^{th} peak from the left and so the outcome of the median voter SCF would still remain a_k.

If $R_X > L_X$, then atleast 1 player to the left of a_k has manipulated to the right of a_k. Now, the k^{th} peak from left will always be to the right of a_k. For the players who manipulated from left of a_k to right of a_k, the new outcome is less preferred than a_k because their true peaks lie to the left of a_k and are now farther from the outcome than a_k as the preference profiles are single-peaked.

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Thus, manipulation can never leave every member of the group strictly better off and thus the SCF f is not group manipulable.