CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Assignment: 2

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Question 1

1(a)

We know that, $f(P) = a$, $f(P') \in \{b, c\}$ and f follows monotonicity and is strategyproof.
Let us have a profile P'' as shown below:

P''_1	P''_2
b	c
a	a
c	b

As, $f(P'') \in \{b, c\}$, we claim that the $f(P'') = b$. The reason is if $f(P'') \neq b$ but $f(P'') = c$, then by using the property of monotonicity of the SCF for the alternative c in P and P'', $f(P)$ must be c, which is a contradiction. Hence, $f(P'') = b$.
But, by using the property of monotonicity of the SCF for the alternative b, in P' and P'' it follows that, if $f(P'') = b$ then, $f(P') = f(P'') = b$. (proved)

1(b)

No, if these preferences are generated from a single-peaked preference domain with the intrinsic ordering of the alternatives being $a < b < c$. The earlier conclusion does not hold in this case.
The earlier proof uses the construction of P''. But, as $P2''$ is not single peaked, in this case such a P'' does not exist in the domain. Therefore, the earlier proof does not work here.
The median voter mechanism with a phantom peaks in a way that $f = min(P(1))$, in other words, if the SCF f picks the left-most peak as outcome, then $f(P) = f(P') = a$. And median voter SCF is strategyproof too.

Question 2

In this scenario, we cannot apply the Gibbard-Satterthwaite result.
According to the question, the set of alternatives, A' is the set of all non-empty subsets of projects, $= \{S : S \subseteq X\}$. If there are at least 2 projects then, the set of alternatives is at least 3. Now, consider two alternatives S and T such that $S \subseteq T$ and $S, T \in A'$. By definition of the preference ordering, any agent is either indifferent between S and T or prefers T to S. Hence, the preference ordering where S is ranked higher than T can never arise. This is a restriction of the domain and we cannot apply the Gibbard-Satterthwaite result here.

Question 3

Yes, the median voter SCF is group strategy-proof. The proof is similar to the proof that shows that the median voter SCF is SP (as done in class).
Let the preference profile is, P and the peaks of the agents are, $P_1(1), P_2(1), ..., P_n(1)$ and $f(P) = a \in A$ is the median of these peaks and the phantom peaks.
Consider a group of agents, $K = (i_1, i_2, ..., i_K)$, let P_K denotes the ordered set of peaks of the agents within group K. There can be 4 different cases for the locations of peaks of the agents in K with respect to the alternative a.
Case 1: If $P_K(1) = a$, that is, if all the agents in the group K have their peaks at a, Then there is no reason for the group K, to manipulate.

Case 2: If $P_K(1) < a$, that is, if every agent in K has her peak at the left of a on the axis representing the alternatives. Then, if the group K, shifts their preferences further left of a, the median with not change. The group can change the median only when atleast some of them reports their peaks to further right of a, but then the median will also shift to further right to a. In other words, say $P_K \to P'_K$, where P'_K is as $P_{K\backslash d}(1) < a < P_d(1)$ and $d \subseteq K$. Let the median of P'_K, P_{-K} be b, it must be the case that $a < b$. But, as the prefrences are single peaked, $\forall i \in K$,
$a = f(P_K, P_{-K}) P_i f(P'_K, P_{-K}) = b$. Thus, the group has no incentive to manipulate.

Case 3: If $P_K(1) > a$, the argument is similar to the case 2. The group has no incentive to manipulate.

Case 4: $P_{K\backslash d}(1) < a < P_d(1)$ and $d \subseteq K$. There may be two cases for $f(P'_K, P_{-K})$:

case 4a.) $f(P'_K, P_{-K}) < a$. then $\forall i \in K\backslash d$, $f(P'_K, P_{-K}) P_i f(P_K, P_{-K}) = a$ but, $\forall i \in d$, $a = f(P_K, P_{-K}) P_i f(P'_K, P_{-K})$.

case 4b.) $f(P'_K, P_{-K}) > a$. then $\forall i \in d$, $f(P'_K, P_{-K}) P_i f(P_K, P_{-K}) = a$ but, $\forall i \in K\backslash d$, $a = f(P_K, P_{-K}) P_i f(P'_K, P_{-K})$.
Therefore, the whole group has no incentive to manipulate.