CS711: Introduction to Game Theory and Mechanism Design

Assignment 2

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Question 1

$P1$	$P2$	$P1'$	$P2'$
a	c	b	a
b	b	a	b
c	a	c	c

(a) **Given :**

- f is an **ONTO** and **strategyproof** SCF.
- $f(P_1, P_2) = a$
- Domain of preferences is unrestricted strict preferences.
- For any preference profile $(\bar{P}_1, \bar{P}_2), f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$
- Strategyproofness implies monotonicity.

To prove : $f(P_1', P_2') = b$

Proof :

$P1$	$P2$	$P1'$	$P2'$	$P1$	$P2''$	$P1''$	$P2''$
a	c	b	a	a	c	b	c
b	b	a	b	b	a	a	a
c	a	c	c	c	b	c	b

I construct two new preference profiles $(P1, P2'')$ and $(P1'', P2'')$ as shown above in table.

Consider transition from $(P1, P2)$ to $(P1, P2'')$. Preference for 'a' weakly improves for both agents and $f(P1, P2) =$ a. Hence by monotonicity $f(P1, P2'') =$ a.

Suppose $f(P1'', P2'') =$ c. However this violates the condition of strategyproofness since agent 1 can get 'a', by reporting a false preference $P1$, which it prefers more than 'c'.

Hence this a contradiction and using the given statement $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\} \forall (\bar{P}_1, \bar{P}_2)$ it can be said that $f(P1'', P2'') =$ b

Now consider transition from $(P1'', P2'')$ to $(P1', P2')$. Preference for 'b' weakly improves for both agents and $f(P1'', P2'') =$ b. Hence by monotonicity $f(P1', P2') =$ b.

(b) No, the earlier conclusion does not hold in this case. Earlier proof does not go through because under single-peaked preference domain restriction with the intrinsic ordering of the alternatives being $a < b < c$ we can not create preference $(P2'')$ as it violates the conditions of single peakedness.

Define an SCF $f : S^n \mapsto A$, where S is the set of single-peaked preferences w.r.t. the common order $a < b < c$, and A is the set of alternatives {a, b, c} as, for $P \in S^n$

$$f(P) = \min_{i \in \{1,2\}} \{P_i(1)\} \tag{1}$$

where minimum is taken w.r.t. the order relation $a < b < c$ and $P_i(1)$ is the most valued alternative for agent i. Hence the SCF picks the left-most peak among the peaks of the agents.

Above defined SCF provide a mechanism that has $f(P_1, P_2) = a$ and also $f(P_1', P_2') = a$.

Question 2

Let P_i' be the ordering over the set of subsets of projects of X for agent i obtained by extending P_i in the manner mentioned in question.

Consider two proper subsets $T, S \subset X$ defined as follows :

- $S = \{\text{a}\}$, where $a \in X$
- $T = X - S$

Here, $T \neq \phi$ as $|X| \geq 2$

Let us now consider two mutually exhaustive cases over set of all linear ordering P_i:

- $P_i(1) =$ 'a'. $\forall$ such P_i, S is strictly preferred over T and S is indifferent to X based on P_i'.
- $P_i(1) \neq$ 'a'. $\forall$ such P_i, T is strictly preferred over S and T is indifferent to X based on P_i'.

Since above two cases are exhaustive over the collection of all P_i and we do not get any P_i', where S is strictly preferred over X or T is strictly preferred over X we can conclude that P_i' come from restricted domain. Hence, the Gibbard-Satterthwaite Theorem is not applicable here.

Question 3

Yes, median voter SCF is group-strategy-proof.
For median voter SCF we need to consider only the peak preferences for all the agents. Let us denote the preferences denoted only by their peaks, i.e., $f(P) = (P_1(1), ..., P_i(1), ..., P_n(1))$ and let $f(P) = a \in A$, where A is the set of alternatives, is the median of those peaks and the phantom peaks.

Let a group of agents be $K \subseteq N$. To prove median voter SCF is group strategy-proof we need to consider following 4 cases :

- $P_i(1) = a \forall i \in K$. In such a case there is no reason for any agent 'i', belonging to group K, to manipulate as their most preferred alternative is the output of SCF. Hence, there does not exist a preference profile P'_K of agents in K such that $f(P'_K, P_{-K})P_i f(P_K, P_K)\forall i \in K$.
- $P_i(1) < a \forall i \in K$. In such a case if any agent 'i', belonging to group K, shifts her preference to further left of a, the median will not change and hence the output of SCF. If she manipulates to report her peak to further right of a, i.e. $(P_i, P_{-i}) \mapsto (P'_i, P_{-i})$ s.t. $a < P'_i(1)$, this will imply that $P_i(1) < a < P'_i(1)$, and since P_i is single peaked preference, by definition of single peakedness, $a = f(P_i, P_{-i})P_i f(P'_i, P_{-i})$. Thus, any agent 'i', belonging to group K, has no profitable manipulation. Hence, there does not exist a preference profile P'_K of agents in K such that $f(P'_K, P_{-K})P_i f(P_K, P_K)\forall i \in K$.
- $P_i(1) > a \forall i \in K$. In such a case if any agent 'i', belonging to group K, shifts her preference to further right of a, the median will not change and hence the output of SCF. If she manipulates to report her peak to further left of a, i.e. $(P_i, P_{-i}) \mapsto (P'_i, P_{-i}) s.t. a > P'_i(1)$, this will imply that $P'_i(1) < a < P_i(1)$, and since P_i is single peaked preference, by definition of single peakedness, $a = f(P_i, P_{-i})P_i f(P'_i, P_{-i})$. Thus, any agent 'i', belonging to group K, has no profitable manipulation. Hence, there does not exist a preference profile P'_K of agents in K such that $f(P'_K, P_{-K})P_i f(P_K, P_K)\forall i \in K$.
- $P_i(1) > a$ for some $i \in K$ and $P_i(1) < a$ for others. In such a case agents would not be able to cooperate among themselves. If a manipulation by any agent causes median to shifts to left of 'a' then by definition of single peakedness such an alternative would be less preferred than 'a' by those $i \in K$ whose $P_i(1) > a$. Similarly if a manipulation by any agent causes median to shifts to right of 'a' then by definition of single peakedness such an alternative would be less preferred than 'a' by those $i \in K$ whose $P_i(1) < a$. Hence, there does not exist a preference profile P'_K of agents in K such that $f(P'_K, P_{-K})P_i f(P_K, P_K)\forall i \in K$.

Thus, median voter SCF is group strategy-proof.