CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 33: November 1, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

33.1 Recap

We have studied some basic properties and results dealing with convex functions. We will use two lemmas from the previous lecture (reproduced here for completeness) to prove Myerson's characterization theorem.

Lemma 33.1 *Let $g : \mathbb{I} \to \mathbb{R}$ be a convex function. Let $\phi : \mathbb{I} \to \mathbb{R}$ such that $\phi(z) \in \partial g(z), \forall z \in \mathbb{I}$. Then $\forall x, y \in \mathbb{I}$ such that $x > y$, we have $\phi(x) \geqslant \phi(y)$.*

Lemma 33.2 *Let $g : \mathbb{I} \to \mathbb{R}$ be a convex function. Then for any $x, y \in \mathbb{I}$,*

$$g(x) = g(y) + \int_y^x \phi(z)dz$$

where $\phi : \mathbb{I} \to \mathbb{R}$ is such that $\phi(z) \in \partial g(z), \forall z \in \mathbb{I}$.

33.2 Monotonicity and Characterization of DSIC Mechanisms for Single Item Auctions

Definition 33.3 (Non-decreasingness) *An allocation rule is* non-decreasing *if for every agent $\forall\ i \in N$ and $\forall\ t_{-i} \in T_{-i}$ we have $f_i(t_i, t_{-i}) \geqslant f_i(s_i, t_{-i}), \forall\ t_i, s_i \in T_i$ with $t_i > s_i$.*

In words, if the types of other agents are held fixed, then the probability of allocation of the object to an agent weakly increases with the increase in her valuation. We now present the main characterization theorem of this section.

Theorem 33.4 (Myerson 1981) *Suppose $T_i = [0, b_i], \forall\ i \in N$ and the valuations are in product form. An allocation rule $f : T \mapsto \Delta A$ and a payment rule $(p_1, p_2, ..., p_n)$ is DSIC iff*

1. *The allocation f is non-decreasing, and,*
2. *Payment is given by*

$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i.$$

Proof: ($\Rightarrow$) Consider the utility of agent i when her types are t_i and s_i respectively and other agents' types are t_{-i}.

$$\begin{aligned} u_i(t_i, t_{-i}) &= t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \\ u_i(s_i, t_{-i}) &= s_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) \end{aligned}$$

Since (f, p) is DSIC, it must hold that $\forall s_i, t_i, t_{-i}$

$$\begin{aligned} u_i(t_i, t_{-i}) &= t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \\ &\geqslant t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) \\ &= s_i f_i(s_i, t_{-i}) + f_i(s_i, t_{-i})(t_i - s_i) - p_i(s_i, t_{-i}) \\ &= u_i(s_i, t_{-i}) + f_i(s_i, t_{-i})(t_i - s_i). \end{aligned}$$

Where the first inequality results from the fact that (f, p) is DSIC and the following equalities result from some algebraic manipulation. Define $g(t_i) := u_i(t_i, t_{-i})$, and $\phi(t_i) = f_i(t_i, t_{-i})$. Hence, the above inequality can be written as

$$g(t_i) \geqslant g(s_i) + \phi(s_i)(t_i - s_i).$$

Therefore, we conclude that $\phi(s_i)$ is a subgradient of g at s_i.

Convexity of g: Pick $x_i, z_i \in T_i$ and define $y_i = \lambda x_i + (1 - \lambda) z_i$ where $\lambda \in [0, 1]$. DSIC implies

$$\begin{aligned} g(x_i) &\geqslant g(y_i) + \phi(y_i)(x_i - y_i) \\ g(z_i) &\geqslant g(y_i) + \phi(y_i)(z_i - y_i) \end{aligned}$$

Multiplying the first equation with λ and the second with $(1 - \lambda)$ and adding we get

$$\begin{aligned} &\lambda g(x_i) + (1 - \lambda) g(z_i) \geqslant g(y_i) + \phi(y_i)(\lambda x_i + (1 - \lambda) z_i - y_i) \\ \Rightarrow\ &\lambda g(x_i) + (1 - \lambda) g(z_i) \geqslant g(y_i) \end{aligned}$$

Where the implication holds since $\lambda x_i + (1 - \lambda) z_i - y_i = 0$. This proves that g is convex.

Having proved that g is convex, we can apply Lemma 33.1. As $\phi \equiv f_i(\cdot, t_{-i})$ is a subgradient of $g \equiv u_i(\cdot, t_{-i})$, which is convex, we conclude that $f_i(\cdot, t_{-i})$ is non-decreasing. This proves part 1 of the implication of the theorem.

By Lemma 33.2, we have

$$g(t_i) = g(0) + \int_0^{t_i} \phi(x_i)\ dx_i$$

$$\Rightarrow \qquad u_i(t_i, t_{-i}) = u_i(0, t_{-i}) + \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

$$\Rightarrow \qquad t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = -p_i(0, t_{-i}) + \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

$$\Rightarrow \qquad p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i.$$

This proves part 2 of the implication.

($\Leftarrow$) For the other direction, it is given that allocation rule is non-decreasing and payment rule is given by

$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

We need to prove that (f, p) is DSIC.

Let t_i be the true type and s_i be the reported type of agent i, and the types of the other agents are t_{-i}. The utilities of the agent in the true and reported types are respectively

$$t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(0, t_{-i}) - t_i f_i(t_i, t_{-i}) + \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i$$

$$t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) = t_i f_i(s_i, t_{-i}) - p_i(0, t_{-i}) - s_i f_i(s_i, t_{-i}) + \int_0^{s_i} f_i(x_i, t_{-i})\ dx_i$$

The difference between the two utilities is given by

$$t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) - [t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i})] = (s_i - t_i) f_i(s_i - t_i) + \int_{s_i}^{t_i} f_i(x_i, t_{-i})\ dx_i \geqslant 0$$

The inequality results from the fact that f is non-decreasing. Since s_i, t_i, t_{-i} are arbitrary, we conclude that the mechanism is DSIC. Figure 33.1 illustrates this fact for a specific allocation rule and the corresponding payment.

When the reported type is greater than the true type, i.e., $s_i > t_i$, the RHS of the equality is the shaded region in the second subfigure. Similarly, the RHS of the equality is the shaded region in the third subfigure when $s_i < t_i$.

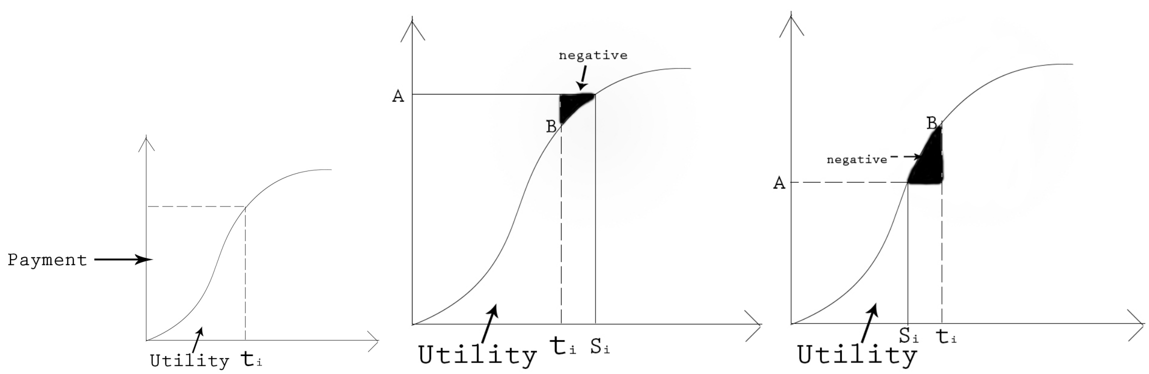

Figure 33.1: Illustration of the incentive compatibility of the payment rule

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