So far we have discussed games with complete information.
The game was completely known – a common knowledge.
In some cases the actions of the other players – uncertain, but there was no uncertainty about which game was being played.

Incomplete Information Games

– When a player does not deterministically know which game is being played.
– Players receive private signals / types
– We'll discuss a special ~~sd~~ subclass called Bayesian games.

Example 1: Soccer game, two competing clubs.
Each can choose either to play to WIN or settle for a DRAW → but these are private signals to them, often caused by external factors, e.g., player injury, weather condition, ground condition etc.
– We call these condition/state as the type of the player.
– There a 4 possible types profiles: WW, WD, DW, DD. The payoff matrices are different for each type profile

WW

	A	D
A	1,1	2,0
D	0,2	0,0

WD

	A	D
A	2,1	3,0
D	1,1	1,0

DD

	A	D
A	0,0	1,0
D	0,1	-1,-1

Assumptions:
1. Number of players and actions remain same in all games – only utility changes
2. The probability of choosing different games is drawn from a common prior distribution.

(12-2)

Defn: Bayesian game is represented by

$$\left\langle N, (A_i)_{i\in N}, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta \in \times_{i\in N} \Theta_i =: \Theta} \right\rangle$$

this is a simplification, more generally, the action set may depend on types $A_i(\theta_i)$.

N: set of players

A_i: set of actions of player i

Θ_i: set of types of player i — play to WIN/DRAW e.g.

P: Common prior distribution over $\Theta = \times_{i\in N} \Theta_i$

with the restriction that $\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_i, \theta_{-i}) > 0 \;\; \forall \theta_i \in \Theta_i \;\; \forall i \in N$

marginals for every type is positive. Otherwise we prune the type set.

Γ_θ: NFG for the type profile $\theta \in \Theta$

$$\left\langle N, (A_i)_{i\in N}, (u_i(\theta))_{i\in N} \right\rangle$$

utility therefore is a function

$u_i : A \times \Theta \to \mathbb{R}$ $\qquad A = \times_{i\in N} A_i$

Game stages:

1. $\theta = (\theta_i, \theta_{-i})$ is chosen randomly according to P
2. Each player observes his/her own type θ_i.
3. They pick action $a_i \in A_i$
4. Player i's payoff $u_i((a_i, a_{-i}); \theta_i, \theta_{-i})$

Strategy : a plan to map state/type to action

Pure : $s_i : \Theta_i \to A_i$

Mixed : $\sigma_i : \Theta_i \to \Delta A_i$

Ex-ante utility : Expected utility before observing own type.

$$\mathcal{U}_i(\sigma) = \sum_{\theta \in \Theta} P(\theta)\, \mathcal{U}_i(\sigma \mid \theta)$$

$$= \sum_{\theta \in \Theta} P(\theta) \sum_{\substack{(a_1,\dots,a_n) \\ \in A}} \left(\prod_{j \in N} \sigma_j(\theta_j, a_j) \right) u_i(a_1,\dots,a_n; \theta_1,\dots,\theta_n) \quad \text{------} \; (1)$$

But the belief of player i after observing his own type θ_i changes according to Bayes rule on P

$$P(\theta_{-i} \mid \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})}$$

This is where positive marginal is crucial

Ex-interim utility : Expected utility after observing one's own type.

$$\mathcal{U}_i(\sigma \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i} \mid \theta_i)\, \mathcal{U}_i(\sigma \mid \theta) \quad \text{---} \; (2)$$

special case : independent types, observing θ_i doesn't give any information on θ_{-i}.

How are the two utilities related?

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i)\, \mathcal{U}_i(\sigma \mid \theta_i) \quad \text{----} \; (3)$$

Example 2 :

Two Player Bargaining Game.

Player 1 : seller, type : price at which he is willing to sell.
Player 2 : buyer, type : price at which he is willing to buy

types for both players : integers $\{1, \dots, 100\}$

bid any number between $\{1, \dots, 100\}$

If the bid of the seller smaller or equal to bid of buyer, trade happens at a price that is mean of the two bids. Else no trade.

$$N = \{1, 2\}, \quad \Theta_1 = \Theta_2 = \{1, 2, \dots, 100\}$$

$$A_1 = A_2 = \{1, 2, \dots, 100\}$$

$$P(\theta_2 | \theta_1) = \frac{1}{100} \quad \forall \theta_2 \in \Theta_2, \forall \theta_1 \in \Theta_1$$

$$P(\theta_1 | \theta_2) = \frac{1}{100} \quad \forall \theta_1 \in \Theta_1, \forall \theta_2 \in \Theta_2$$

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1 + a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{ow} \end{cases}$$

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1 + a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{ow} \end{cases}$$

The beliefs $P(\theta_2 | \theta_1)$ and $P(\theta_1 | \theta_2)$ are consistent with the prior $P(\theta_1, \theta_2) = \frac{1}{10000}$, $\forall \theta_1 \in \Theta_1, \theta_2 \in \Theta_2$

$$\Theta = \Theta_1 \times \Theta_2.$$

Sealed Bid Auction : values between $[0,1]$, bids also in $[0,1]$

2 bidders for one item.

allocation fn. $\quad o_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{ow} \end{cases}$

$$o_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 < b_2 \\ 0 & \text{ow} \end{cases}$$

belief is a distribution

$f_1(\theta_2 | \theta_1) = 1 \quad \theta_2 \in [0,1]$

$f_2(\theta_1 | \theta_2) = 1 \quad \theta_1 \in [0,1]$

$f(\theta_1, \theta_2) = 1 \quad (\theta_1, \theta_2) \in [0,1]^2$

$u_i(b_1, b_2; \theta_1, \theta_2) = o_i(b_1, b_2)(\theta_i - b_i)$