Core is a set-solution concept.

$v(\{i\}) = 0 \;\; \forall i = 1,2,3$, $v(1,2) = v(2,3) = 1$ $\quad v(1,3) = 2$

$v(1,2,3) = 3$.

$(2, 0.5, 0.5)$ is in the core.

If player 3 leave with the core share 0.5.
Does the other two players do a better division of the remaining money on table? E.g. (1, 2) may divide 2.5 as 1.25 each, which is not in core.

Davis - Maschler reduced game property

Defn: Let (N, v) be a TU game, let S be a non-empty coalition, let x be an efficient vector, i.e., $x(N) = v(N)$. The Davis - Maschler reduced game to S relative to x, denoted by (S, w_S^x) is the coalitional game with the set of players S and a coalition function.

$$w_S^x(R) = \begin{cases} \max\limits_{Q \subseteq N \setminus S} \{v(R \cup Q) - x(Q)\}, & \text{if } \phi \neq R \subset S \\ x(S) & , \text{if } R = S \\ 0 & \text{if } R = \phi \end{cases}$$

Idea: If players outside S accept the offer x then any $Q \subseteq N \setminus S$ will be happy with $x(Q)$.
The rest can be the worth of R, and R picks that Q which maximizes the leftover worth.
Of course, they can at most have $x(S)$ which they can divide in case $R = S$.

7-2

Consistency of the core

Defn: A set-solution concept ϕ satisfies the Davis-Maschler reduced game property if for every TU game (N,v), for every non-empty coalition $S \subseteq N$, and for every vector $x \in \phi(N,v)$, it holds that

$$(x_i)_{i\in S} \in \phi(S, w_S^x).$$

Remark: The reduced game property is a consistency property: if ~~these~~ the players believe in x, they will refrain from redistributing $x(S)$ since that is already a solution in the subgame reduced to S.

Theorem: The core satisfies the Davis-Maschler reduced game property.

Proof: Let x be a point in the core of (N,v) and let S be a non-empty coalition. We will show that $(x_i)_{i\in S}$ is in the core of (S, w_S^x).

Need to show: (1) $x(R) \geq w_S^x(R) \quad \forall R \subset S, R \neq \phi$

(2) $x(S) = w_S^x(S)$

(2) is satisfied by definition of w_S^x.

To show (1) consider $R \subset S$. By definition of w_S^x $\exists$ coalition $Q \subseteq N \setminus S$ s.t.

$$w_S^x(R) = v(R \cup Q) - x(Q)$$
$$= v(R\cup Q) - \left(x(R\cup Q) - x(R)\right)$$

since x is in core, $v(R\cup Q) \leq x(R \cup Q)$, hence

$$x(R) \geq w_S^x(R).$$

$\square$

Convex games revisited

Def: A coalitional game (N, v) is convex if for every pair of coalitions S and T,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \cdots \text{(1)}$$

Fact (Proof: exercise) If (N, v) is a convex game, then for every ~~ca~~ coalition $S \subseteq N$, the subgame (S, v) restricted to the players in S and v restricted to the power set of S, is also a convex game.

Convex games are characterized by the property that the marginal contribution of a player is larger in a larger coalition. Formally it is stated as:

Theorem: For any TU game (N, v) the following statements are equivalent

(1) (N, v) is a convex game

(2) For every $S \subseteq T \subseteq N$ and for every $R \subseteq N \setminus T$

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T).$$

(3) For every $S \subseteq T \subseteq N$ and $\forall\, i \in N \setminus T$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Remark: Any of these conditions can be used as definition.

Proof: We'll prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(7-2)

① ⇒ ② : Given (N, v) is a convex game

Suppose S, T be s.t. $S \subseteq T \subseteq N$ and $R \subseteq N \setminus T$

use convexity for $S \cup R$ and T

$$v(S \cup R) + v(T) \leq v(\underbrace{(S \cup R) \cup T}_{= S \cup T \cup R \,=\, T \cup R}) + v(\underbrace{(S \cup R) \cap T}_{= (S \cap T) \cup (R \cap T) \,=\, S \cup \phi \,=\, S})$$

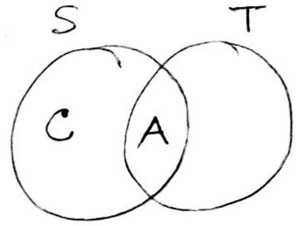
$$\Rightarrow v(S \cup R) - v(S) \leq v(T \cup R) - v(T).$$

② ⇒ ③ Obvious, put $R = \{i\}$.

③ ⇒ ① If $S \subseteq T$, then eqn. ① holds with equality.

Consider S and T are not contained in one another.

Define, $A := S \cap T$, $C = S \setminus T \neq \phi$



Let $C = \{i_1, i_2, \ldots, i_k\}$

Since $A \subseteq T$

$$A \cup \{i_1, \ldots, i_l\} \subseteq T \cup \{i_1, \ldots, i_l\} \quad \forall\, l = 0, 1, \ldots, k-1$$

Also $i_{l+1} \notin T \cup \{i_1, \ldots, i_l\}$

③ gives

$$v(A \cup \{i_1, \ldots, i_l, i_{l+1}\}) - v(A \cup \{i_1, \ldots, i_l\}) \leq v(T \cup \{i_1, \ldots, i_{l+1}\}) - v(T \cup \{i_1, \ldots, i_l\})$$

writing these inequalities for $l = 0, \ldots, k-1$ and summin

$$v(A \cup \{i_1\}) - v(A) \leq v(T \cup \{i_1\}) - v(T)$$

$$v(A \cup \{i_1, i_2\}) - v(A \cup \{i_1\}) \leq v(T \cup \{i_1, i_2\}) - v(T \cup \{i_1\})$$

$$\vdots$$

$$v(A \cup C) - v(A \cup \{i_1, \ldots, i_{k-1}\}) \leq v(T \cup C) - v(T \cup \{i_1, \ldots, i_{k-1}\})$$

$$\Rightarrow v(\underbrace{A \cup C}_{= S}) - v(\underbrace{A}_{= T \cap S}) \leq v(\underbrace{T \cup C}_{= T \cup S}) - v(T)$$

Convex games have non-empty core.

<u>Theorem</u>: Let (N,v) be a convex game, let x be the imputation

$$x_1 = v(1)$$
$$x_2 = v(1,2) - v(1)$$
$$\cdots$$
$$x_n = v(1,2,\ldots,n) - v(1,2,\ldots,n-1)$$

Then x is in the core of (N,v).

<u>Proof</u>: x is efficient

$$\sum_{i\in N} x_i = v(N).$$

Need to show $x(S) \geq v(S)$, $\forall S \subseteq N$.

Let $S = \{i_1, \ldots, i_k\}$ be an arbitrary coalition.

WLOG $i_1 < i_2 < \cdots < i_k$

$$\underbrace{\{i_1, i_2, \ldots, i_{j-1}\}}_{S} \subseteq \underbrace{\{1,2,\ldots,i_j - 1\}}_{T} \quad \forall j = 1,\ldots,k \qquad i_j \in N \setminus T$$

Implication ③ of previous theorem gives

$$v(1,2,\ldots,i_j) - v(1,2,\ldots,i_j - 1) \geq v(i_1,\ldots,i_j) - v(i_1,\ldots,i_{j-1})$$

Hence

$$x(S) = \sum_{j=1}^{k} x_{i_j}$$
$$= \sum_{j=1}^{k} \left[v(1,2,\ldots,i_j) - v(1,2,\ldots,i_j - 1)\right]$$
$$\geq \sum_{j=1}^{k} \left[v(i_1,\ldots,i_j) - v(i_1,\ldots,i_{j-1})\right]$$
$$= v(i_1,\ldots,i_k) = v(S). \qquad \square$$

The theorem shows if the agents are ordered lexicographically and correspondingly the x_i's are defined

$$x_i = v(1,2,\dots,i) - v(1,2,\dots,i-1)$$

Then x is in the core. But clearly the same construction holds for any permutation of the players, say $\pi = (i_1, i_2, \dots, i_n)$

$$w^{\pi} := (v(i_1), v(i_1,i_2) - v(i_1), \dots, v(N) - v(N \setminus \{i_n\}))$$

Description: Players enter a room in the order π and everyone is paid his/her marginal contribution. The imputation derived from this is w^{π}, and is in the core.

Other game classes having non-empty core.

- Spanning tree games
- Flow games.

Set-solution concept to point (single-value) solution concept.

① – Core has many solutions – what to expect

② – Core may not exist – some solution concept that is guaranteed to exist.

Shapley properties.