CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 28: October 18, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

28.1 Revisiting VCG

In VCG mechanism, the allocation and payment rules are given as follows.

$$f^{AE}(\theta) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$$
$$p_i^{VCG}(\theta) = \max_{b \in A} \sum_{j \neq i} v_j(b, \theta_j) \; - \; \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$$

Here f^{AE} is the allocation rule that maximizes the sum of the valuations (also called social welfare) of all the players. We call this rule *allocatively efficient* or just*efficient*. Consider the utility of player i

$$\begin{aligned}
& v_i(f^{AE}(\theta), \theta_i) - p_i^{VCG}(\theta) \\
&= v_i(f^{AE}(\theta), \theta_i) - \max_{b \in A} \sum_{j \neq i} v_j(b, \theta_j) \; + \; \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j) \\
&= \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - \max_{b \in A} \sum_{j \neq i} v_j(b, \theta_j)
\end{aligned}$$

The first term on the RHS of the last equality is the maximum social welfare. The second term is the maximum social welfare when i is absent. The difference gives us the marginal contribution of i in the social welfare. This is another way of interpreting the utility of an agent in the VCG mechanism.

28.2 Illustration of VCG payments

- **Single indivisible object allocation:** Consider an agent $i \in N$. Every agent has some value when the object is assigned to him, and zero otherwise. Efficiency requires the object must go to the agent who values it the most. So if i's bid is maximum, he is assigned the object. The payment is given by

  $$p_i^{VCG}(\theta) = \max_{a \in A} \sum_{j \neq i} v_j(b, \theta_j) \; - \; \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j).$$

  If agent i's bid is not the highest, i.e., she does not win the object, both the first and second term on the RHS gives the same number since that turns out to be the bid of the winning agent. Hence, agent i pays nothing.

  If agent i's bid is the highest, i.e., she wins the object, the first term on the RHS becomes the second highest bid, and the second term becomes zero since agents except i do not get the object in the efficient allocation. Thus the payment made by the highest bidder is equal to the second highest bid, and no other agent pays anything in this mechanism. This is precisely the second price auction that we have seen earlier. So, in the case of single object allocation, VCG mechanism is exactly the second price auction.

- **Public project allocation:** In this case, the first term of payment function is the social welfare of all agents except agent i if i were not present. The second term is the social welfare of other agents in i's presence. This gives another interpretation of the VCG payment: the loss in social welfare of other agents because of agent i's presence – and agent i is asked to compensate this loss. Consider the following example.

	Football	Library	Museum
A	0	70	50
B	95	10	50
C	10	50	50

The allocation set is $A = \{F, L, M\}$.
Efficient allocation: M.
A pays: 105 - 100 = 5.
Reason: without A's presence, best allocation would've been F, with total utility of other agents = 105. In A's presence, utility of other agents = 50+50 = 100. Difference = 5.
Similarly, we can find that
B pays: 120 - 100 = 20
C pays: 100 - 100 = 0

Observation: payment values of VCG are always positive (irrespective of utility function) – first term is always at least as large as second term.

We also notice that, in the above example, only those agents are charged a positive payment whose presence *changes the outcome*. These agents are called *pivotal* agents. A non-pivotal agent pays zero under VCG. This is why VCG is also called *pivotal* mechanism.

- **Combinatorial Allocation**: Sale of multiple objects. Consider two objects – 1 and 2. The valuations over all possible combinations of these objects are given as

	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
v_1	0	8	6	12
v_2	0	9	4	14

Efficient allocation - $\{1\}$ goes to player 2, and $\{2\}$ goes to player 1. Here, type is the value itself, so we can also write $v_i(a, \theta_i) = \theta_i(a)$.

$$p_1^{VCG}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq 1} \theta_j(a) \ - \ \sum_{j \neq 1} \theta_j(f(v)),$$

which gives $= 14 - 9 = 5$; Payoff $= 6 - 5 = 1$.
$p_2^{VCG}(\theta_1, \theta_2) = 12 - 6 = 6$; Payoff $= 9 - 6 = 3$.
Hence every agent gets a non-negative payoff – something we will discuss more later.

It is also instructive to verify if the agents tried to overbid their valuations to obtain both the objects, what payoff would they receive.

28.3 VCG mechanism in combinatorial auction

VCG mechanism has several useful properties:

1. It is DSIC.
2. It is efficient.
3. Payments are non-negative (hence no subsidy).

Combinatorial auction notation:

- $M = \{1, 2, \ldots, m\}$: set of objects.
- Set of bundles: $\Omega = \{S : S \subseteq M\}$.
- Type of agent i is $\theta_i : \Omega \mapsto \mathbb{R}$.

Hence $\theta_i(S)$, $S \in \Omega$ is the value of agent i for the bundle S. We assume $\theta_i(S) \geq 0 \ \forall S \in \Omega$, which means that every item is a 'good' (an item that gives negative valuation is called a 'bad').

An allocation of objects is given by $X = \{X_0, X_1, \ldots, X_n\},\ X_i \in \Omega,\ X_i \cap X_j = \emptyset$ if $i \neq j$, and $\cup_{i=0}^{N} X_i = M$. The set of allocations A is the collection of such Xs. X_0 is the set of uncallocated objects, X_i is the bundle allocated to i. Assume $\theta_i(\emptyset) = 0$.

We also assume that the valuations have no externalities (selfish valuations), ie $v_i(X, \theta_i) = \theta_i(X_i)$ (does not depend on the bundles of other agents).

Claim 28.1 *The payment for an agent who gets no object in the allocation is zero.*

Proof: Say agent i gets no object in the efficient allocation, ie $X \in \arg\max_{x \in A} \sum_{i \in N} v_i(x, \theta_i)$, and $X_i = \emptyset$.

VCG payment's first term considers allocation excluding agent i. Define $Y \in \arg\max_{y \in A} \sum_{j \neq i} v_j(y, \theta_j)$.

We have seen that if the allocation set remains unchanged, VCG payment is always non-negative , i.e., $p_i^{VCG} \geq 0$ (no-subsidy condition).

$$\begin{aligned} p_i^{VCG}(\theta) &= \sum_{j \neq i} v_j(Y, \theta_j) - \sum_{j \neq i} v_j(X, \theta_j) \\ &\leqslant \sum_{j \in N} v_j(Y, \theta_j) - \sum_{j \in N} v_j(X, \theta_j) \\ &\leqslant 0. \end{aligned}$$

The first inequality holds since we are adding $v_i(Y, \theta_i) \geqslant 0$ and subtracting $v_i(X, \theta_i) = 0$ – together a non-negative quantity. The second inequality holds since X maximizes the sum of the valuations by definition. Hence combining this with the no-subsidy condition, we conclude that $p_i^{VCG}(\theta) = 0$. ∎