CS-698W: Game Theory and Collective Choice	**Jul-Nov 2017**

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36.1 Recap

In the last lecture, we started discussing the optimal auction for a single object allocation. To build our intuition, we started with a single agent. That is, there is only one potential buyer for the object and the auctioneer wants to maximize the revenue earned (maximize the payment of that agent subject to some conditions). We ended our lecture with a result which gives a structure for the revenue function.
In this lecture, we will first prove the result. Then we will discuss the solution of *Optimal Revenue Problem* for the single-agent case. Finally, we will extend the result to the multi-agent scenario and try to obtain the solution of *Optimal Revenue Problem* in that setting as well.

36.2 Optimal Auction for a Single Agent

36.2.1 Struction of Revenue Function

We ended our last lecture with the following result:

Theorem 36.1 *For any implementable allocation rule f, the revenue earned is given by*

$$\Pi^f = \int_0^\beta w(t)f(t)g(t)dt$$

where $w(t) = \left(t - \frac{1-G(t)}{g(t)}\right)$, *which is known as the virtual valuation of an agent.*

Proof:

$$\begin{aligned}
\Pi^f &= \int_0^\beta p(t)g(t)dt && \text{(By Definition)}\\
&= \int_0^\beta [tf(t) - \int_0^t f(x)dx]g(t)dt && \text{(Mechanism is IC, IR and revenue is to be maximized)}\\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \left(\int_0^t f(x)dx\right) g(t)dt\\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \left(\int_x^\beta f(x)g(t)dt\right) dx && \text{(Changing order of integration of second term)}\\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \left(\int_x^\beta g(x)dx\right) f(t)dt && \text{(Exchanging the variables } t \text{ and } x\text{)}
\end{aligned}$$

$$= \int_0^\beta [tg(t) - \int_t^\beta g(x)dx]f(t)dt$$
$$= \int_0^\beta [tg(t) - (G(\beta) - G(t))]f(t)dt$$
$$= \int_0^\beta [tg(t) - (1 - G(t)]f(t)dt$$
$$= \int_0^\beta \left(t - \frac{1-G(t)}{g(t)}\right) g(t)f(t)dt$$
$$= \int_0^\beta w(t)g(t)f(t)dt \qquad \text{(where } w(t) = \left(t - \frac{1-G(t)}{g(t)}\right)\text{)}$$

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36.2.2 Optimal Revenue Problem

Now we want to find the allocation function which maximizes the revenue of the auctioneer. So we need to solve the following optimization problem :

$$OPT1 : \max_{f\,:\,f \text{ is non-decreasing}} \Pi^f$$

As we can see solving the above optimization problem is difficult. But solving the unconstrained version of the above problem is easier. The unconstrained optimization problem is:

$$OPT2 : \max_f \Pi^f$$

So is there any conditon under which the solution of $OPT1$ and $OPT2$ matches?
Turns out that it happens when we have the following assumption.

Assumption 36.2 *G satisfies the* Monotone Hazard Rate*(MHR) condition, i.e,* $\left(\frac{g(x)}{1-G(x)}\right)$ *is non-decreasing in* x*.*

Note 36.3 *Uniform Distribution, Exponential Distribution satisfies MHR condition.*

We state a fact concerning MHR condition without proof.

Fact 36.4 *If* G *satisfies MHR, then* $\exists$ *a unique* $x*$ *s.t.*

$$x* = \frac{1-G(x*)}{g(x*)} \implies w(x*) = 0$$

Observe that if G satisfies MHR condition, w is strictly increasing. Therefore for $x < x*$, w is negative while for $x > x*$, w is positive. This observation gives us an easy way to solve the unconstrained optimization problem $OPT2$.
For $t < x*$, since virtual valuation is 0, we want the allocation to be 0 as well intuitively. Also if we look at the expression of Π^f, we can see that for the value of integration to increase, we want the allocation to be 0

for $t < x*$, since $w(t) < 0$ in this domain. Conversely we want the allocation to be 1 for $t > x*$. Therefore the allocation function is given as:

$$f(t) = \begin{cases} 0 & \text{for } t < x* \\ \alpha \in (0,1) & \text{for } t = x* \\ 1 & \text{for } t > x* \end{cases} \tag{36.1}$$

Additionally we can see that the above allocation function is non-decreasing as well. Hence it is a valid solution to constrained optimization problem $OPT1$. Therefore, f defined as above is implementable in DSIC/BIC.

We now state a theorem to summarize the discussions we have had so far.

Theorem 36.5 *A mechanism (f, p) under MHR condition is optimal iff the following two conditions hold:*

1. *f is given by equation 36.1.*
2. *$\forall t \in T$, $p(t) = f(t)x*$*

36.3 Optimal Auction Mechanism for Multiple Agents

36.3.1 Struction of Revenue Function

We want the auction mechanism $M \equiv (f, p)$ to be BIC, IIR and $\Pi^M \geqslant \Pi^{M'}$ $\forall M'$.
BIC $\implies$ $f_i s$ is NDE (Non-decreasing in expectation) and Expected payment $\Pi_i(t_i)$ have a specefic formula.
IIR $\implies$ $\Pi_i(0) = 0$
Expected payment made by agent i is given by (assuming $T_i = [0, b_i]$):

$$\begin{aligned}
payment_i &= \int_0^{b_i} \Pi_i(t_i) g(t_i) dt_i \\
&= \int_0^{b_i} \left(t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x_i) dx_i \right) g(t_i) dt_i \\
&= \int_0^{b_i} \left(t_i - \frac{1 - G_i(t_i)}{g_i(t_i)} \right) g_i(t_i) \alpha_i(t_i) dt_i && \text{(Using proof given in previous section)} \\
&= \int_0^{b_i} {}_i(t_i) \alpha_i(t_i) g_i(t_i) dt_i && \text{where } \alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i}) dt_{-i} \\
&= \int_T w_i(t_i) f_i(t) g(t) dt && \text{where } T = \bigtimes_i^n T_i
\end{aligned}$$

Now, we have total revenue earned by the auctioneer as:

$$\begin{aligned}
\Pi^M &= \sum_{i \in N} payment_i \\
&= \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t) dt
\end{aligned}$$

36.3.2 Optimal Revenue Problem

Now we want to find the allocation function which maximizes the revenue of the auctioneer. So we need to solve the following constrained optimization problem :

$$OPT3 : \max_{f\,:\, f \text{ is NDE}} \Pi^M$$

Again we can see solving the above optimization problem is difficult. But solving the unconstrained version of the above problem is easier. The unconstrained optimization problem is:

$$OPT4 : \max_{f} \Pi^M$$

. We can observe that Π^M involves taking convex combination of virtual valuations of each agents. If we look at the expression of Π^M, we can easily conclude that to maximize the integration we need to effectively maximize the convex combination. We know that any convex combination $\sum_i \alpha_i x_i$ where $\sum_i \alpha_i = 1$ and $\alpha_i \in [0,1]$ is maximized (where α_is are variables) only when we set $\alpha_j = 1$ for $x_j : x_j \geqslant x_i$, $\forall i$ and set $\alpha_i = 0$ for $\forall i \neq j$. So following this scheme, we have unconstrained allocation of any agent i as:

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j) \forall j \\ 0 & \text{otherwise} \end{cases} \tag{36.2}$$

Of course, this is followed by arbitrary tie-breaking.
Clearly the allocation rule is **not** non-decreasing and potentially **not** non-decreasing even in expectation (because some of w_is can be decreasing w.r.t. t_is which might cause the corresponding agent to have its allocation reduced from 1 to 0 as result of having their type increased.).

Similar to single-agent case, we want the solution of $OPT3$ to be the same as $OPT4$. We can observe that the main reason why the two solutions don't match is because we don't have any guarantee for virtual valuation of agents (w_is).
Like in single-agent case, we make some assumptions regarding the virtual valuation of agents. Before stating the assumption we state a definition which will be used in the assumption.

Definition 36.6 ***(Regular Virtual Valuation)*** *A virtual valuation w_i is* ***regular*** *if $\forall s_i, t_i \in T_i$ with $s_i < t_i$, then $w_i(s_i) < w_i(t_i)$.*

This assumption regarding virtual valuations is weaker than the MHR condition. This can be stated in the form of the following lemma.

Lemma 36.7 *If the hazard rate $\frac{g_i(t_i)}{1-G_i(t_i)}$ is non-decreasing, then w_i is* ***regular****.*

Assumption 36.8 *The virtual valuations of all the agents are* ***regular****, i.e, w_i is regular $\forall i$.*

We can easily observe that the above mentioned assumption addresses the issue with virtual valuations mentioned earlier. The implication of this assumption is summarized in the following lemma 36.9.

Lemma 36.9 *If every agent's virtual valuation is* ***regular****, then the solution of the constrained optimization problem is the same as the unconstrained problem.*