One sided matching - object allocation mechanisms

Quick recap of social choice setting

- Agents have types $\theta_i \in \Theta_i$: private to agents
- Set of outcomes X
- Social choice function maps a type profile to an outcome

$$f : \Theta \to X$$

example: types are preferences over candidates (voting)

$$\theta_i = a \succ b \succ c \succ d .$$

The voting rule collects the preferences and selects a candidate. $X = A$: set of candidates.
In a voting setting, any preference order over the candidates is plausible, and therefore difficult to design truthful mechanisms.

Thm (Gibbard - Satterthwaite)
If $|A| \geq 3$ and ~~vote~~ preferences are unrestricted total orders, every onto and truthful social choice function must be dictatorial.

Restricted preferences and positive results

- Single-peaked preferences, mechanisms with transfers etc
- Another setting without money: object allocation mechanisms - one sided matching.

(11-2)

Setting of one sided matching

- $M = \{a_1, \dots, a_m\}$ finite set of objects
- $N = \{1, \dots, n\}$ set of agents, $m \geq n$.
- objects are houses, jobs, projects, positions,
- Each agent has a linear order over the objects
  - linear order: R_i of agent i

    complete : $\forall a, b \in M$ either $a R_i b$ or $b R_i a$

    transitive: if $a R_i b$ and $b R_i c \Rightarrow a R_i c$.

    anti-symmetric: if $a R_i b$ and $b R_i a \Rightarrow a = b$.

we will denote such linear orders with P_i

$(P_1, P_2, \dots, P_n)$ is a preference profile

- $\mathcal{M}$: set of all possible linear ~~ob~~ orders over M.
- $P_i(k, S)$ is the k-th top ~~alternati~~ object that belongs to $S \subseteq M$.

Departure from the classic G-S setting :

The preferences are over objects and not over alternatives.

Alternative in this setting is a matching / assignment of the objects to the agents.

A feasible matching is a mapping

$a : N \rightarrow M$ injective : distinct objects are allocated to distinct agents

set of alternatives A : collection of such mappings

$a(i) = j \in M$: object j assigned/matched to i.

Why is this a restricted domain?

- There cannot exist any preference profile where certain alternatives can have both permutations
  e.g. let a and b be two alternatives/matchings where player i gets the same object. He is indifferent between the alternatives, hence $\nexists P_i$ s.t. $a P_i b$ or $b P_i a$.
- Good news: G-S theorem does not hold anymore
- expect to have non-trivial truthful mechanisms.

Example: An SCF $f : \mathcal{M}^n \to A$. Define a fixed priority (serial dictatorship) mechanism.

A priority is a bijective mapping

$$\sigma : N \to N$$

The mechanism: every agent in the order σ picks her favorite object from the leftover list.

$$a(\sigma(i)) = P_{\sigma(i)}(1, N \setminus \{a(\sigma(1)), \ldots, a(\sigma(i-1))\})$$
$$i = 1, \ldots, n$$

$$a(\sigma(0)) = \phi.$$

$$f^{\sigma}(P) = a.$$

Remarks:

- A generalization of dictatorship
- Easy to see that this is strategyproof

<u>Defn</u>: An SCF $f : \mathcal{M}^n \to A$ is <u>strategyproof</u> (house alloc. model)

if $\quad f(P_i, \underline{P}_{-i})(i) \; P_i \; f(P_i', \underline{P}_{-i})(i)$

$$\forall\, P_i \in \mathcal{M} \;\; \forall\, \underline{P}_{-i} \in \mathcal{M}^{n-1} \;\; \forall\, i \in N.$$

– also, this is efficient in the following sense.

<u>Defn</u>: An SCF is <u>efficient</u> (house allocation model) if for all preference profiles P and all matchings a, if there exists another matching $a' \neq a$ s.t. either $\quad a'(i) \; P_i \; a(i)$ or $a'(i) = a(i) \;\; \forall i \in N$, then $\quad f(P) \neq a$.

Proposition: Every fixed priority SCF is strategyproof and efficient.

Pf: Let σ be any fixed priority. f^σ is the SCF

Fix ~~an~~ agent i, $\quad \sigma^+(i) = \{ j \in N : \sigma(j) < \sigma(i) \}$

set of ~~play~~ agents having higher priority than i.

Being truthful agent i gets $\quad P_i(1, M \setminus M^{\sigma^+(i)})$

$M^{\sigma^+(i)}$ – set of objects assigned to $\sigma^+(i)$.

By deviating, agent i cannot get any better since $M^{\sigma^+(i)}$ remains fixed.

<u>Efficiency</u>: Suppose ~~of~~ f^σ is not efficient.

$\exists\, P$ s.t. $f^\sigma(P) = a$. $\exists\, a' \neq a$ s.t.

$a'(i) \; P_i \; a(i)$ or $a'(i) = a(i) \;\; \forall i \in N$. Consider the first j s.t. $a'(j) \; P_j \; a(j)$. Since all before this were same $a'(j)$ was ~~avai~~ available to j – contradiction $\square$

However there are strategyproof SCFs that are not necessarily serial dictators.

Ex: $N = \{1, 2, 3\}$ $M = \{a_1, a_2, a_3\}$

The preference order changes depending on the top choice of a specific player.

$$\sigma = \begin{cases} (1, 2, 3) & \text{if } P_1(1) = a_1 \\ (2, 1, 3) & \text{if } P_1(1) \neq a_1 \end{cases}$$

Truthful: For a fixed priority, this is strategyproof, now 2 and 3 cannot change the priority, therefore it is strategyproof for them. Player 1 can,

if $P_1(1) = a_1$: she gets a_1

if $P_1(1) \neq a_1 \Rightarrow P_1(1) \in \{a_2, a_3\}$, the only way she can change the priority is by reporting a_1 as $P_1(1)$, but then she gets a_1 – which is at most her second choice. If she reported truthfully, she could get either top or second choice – hence this is strategyproof for agent 1 too.

Efficiency: similar argument as before.

For a given priority, the outcome is always efficient.

Top-trading cycle with fixed endowments

A different mechanism (actually a class of mechanisms) that is truthful and has other nice properties.

Assume $m = n$ for simplicity.

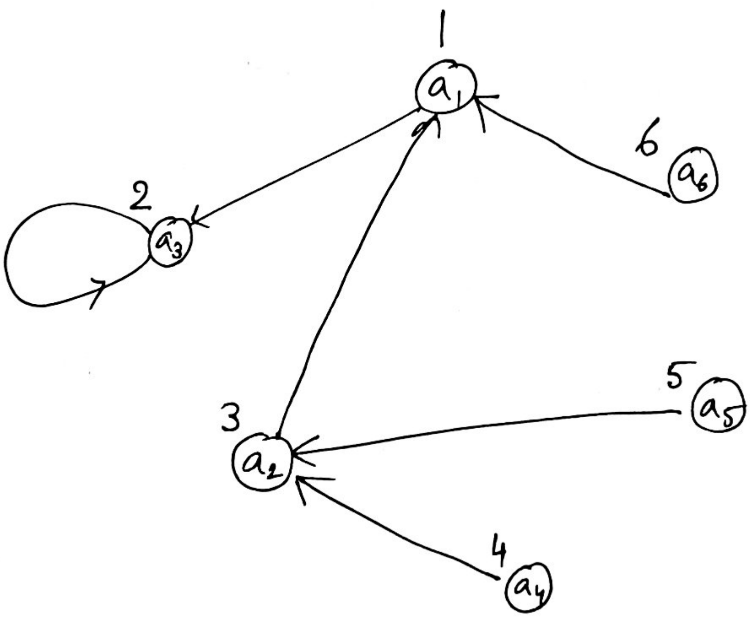
Initialization: Each agent is endowed with a house

Suppose there are 6 agents and are endowed with

a^* : $a^*(1) = a_1$, $a^*(2) = a_3$, $a^*(3) = a_2$, $a^*(4) = a_4$, $a^*(5) = 5$

$a^*(6) = a_6$

P_1	P_2	P_3	P_4	P_5	P_6
a_3	a_3	a_1	a_2	a_2	a_1
a_1	a_2	a_4	a_1	a_1	a_3
a_2	a_1	a_3	a_5	a_6	a_2
a_4	a_5	a_2	a_4	a_4	a_4
a_5	a_4	a_6	a_3	a_5	a_6
a_6	a_6	a_5	a_6	a_3	a_5



Initial graph of the TTC