CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 19: September 13, 2017

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19.1 Recap

In the last lecture we had moved from Social welfare functions to Social choice functions and tried to motivate the social choice setting with some real examples like voting. As the setting has changed, we need to redefine the axioms because we observe that IIA does not make any sense as the alternatives are deterministic and there is no social ordering anymore. So in this lecture we define some new axioms and we notice some similarity between these axioms and the axioms defined in the Social welfare setting. Before going into the lecture we recall the definition of Social Choice function as follows

Definition 19.1 *A Social Choice function "f" is a mapping from the strict Preference profiles ($\mathcal{P}^n$) to the set alternative "$\mathcal{A}$"and is given as*

$$f : \mathcal{P}^n \rightarrow \mathcal{A} \tag{19.1}$$

The output of the Social choice function is a collective alternative instead of a collective ordering.

19.2 Axioms

We define the following axioms for the Social choice setting,

19.2.1 Pareto Domination

An alternative b pareto dominates a if for every agent $i \in \mathcal{N}$, b is better than a. It is formally defined as follows

Definition 19.2 *$\forall a, b \in \mathcal{A}$, a is pareto dominated by b if bP_ia, $\forall i \in \mathcal{N}$.*

We see that pareto domination is defined for a particular preference profile.

19.2.2 Pareto Efficient/Optimal(PE)

It is a property of Social choice function which states that if an alternative a is dominated, then the social outcome should not be equal to a. It can be written mathematically as

Definition 19.3 *An SCF f is pareto efficient if $\forall P, a \in \mathcal{A}$, if a is pareto dominated then $f(P) \neq a$*

19.2.3 Unanimity(Un)

An SCF f is unanimous if, the first alternative in the preference profile of all agents is same say a, then the outcome of the SCF should be equal to that alternative a. Mathematically,

Definition 19.4 *An SCF f is unanimous if $\forall P$ such that $P_1(1) = P_2(1) = \cdots = P_n(1) = a$, then $f(P) = a$.*

In the above definition the subscript of the prefence profile corresponds to the agent and the number in the braces corresponds to the ranking position in the preference profiles of each agent.

From the definitions of pareto efficiency and unanimity we can claim the following,

Claim 19.5 *Pareto efficiency $\implies$ Unanimity. In other words by deviating from the original notations, if **PE** is the collection of all SCF's that are Pareto efficient and **Un** is the collection of all SCF's that are Unanimous, then **PE** $\subset$ **Un**.*

Proof:
Suppose consider two alternatives $a,\ b \in \mathcal{A}$. By the if-condition of Unanimity, we say a is unanimous alternative if

$$a\ P_i\ b\ , \quad \forall i \in \mathcal{N} \ \text{ and } \ b \in \mathcal{A} \setminus \{a\}. \tag{19.2}$$

By taking all possible combinations of a, b and invoking the definition of Pareto Efficiency from (19.2) we can say that

$$a\ P_i\ b \overset{PE}{\implies} f(P) \neq b\ , \quad \forall i \in \mathcal{N} \ \text{ and } \ b \in \mathcal{A} \setminus \{a\} \tag{19.3}$$

This proves the claim that $PE \implies Un$. ■

Strict Example: From the definition of unanimity, when the top alternative in all the preference profiles is not same, then Unanimity is silent. In such cases choosing the social outcome that is strictly dominated will make the SCF not PE.

So by picking a dominated alternative when top alternatives are not same we obtain a strict example.

19.2.4 Onto ness (Onto)

It is the classic mathematical definition of onto function which states that for every alternative there exists a preference profile whose social outcome is that alternative.

Definition 19.6 *An SCF f is onto if $\forall a \in \mathcal{A}$, $\exists P \in \mathcal{P}^n$ such that $f(P) = a$.*

There exists a relation between unanimity and onto ness and it can be given as

Claim 19.7

$$\textbf{Un} \subset \textbf{Onto} \tag{19.4}$$

Proof: The proof is pretty simple as we can see that the set **Onto** is a collection of SCF's that are onto(i.e., for every alternative there exists a preference profile whose social outcome is equal to the chosen alternative) and the set **Un** is the set of SCF's that are unanimous (i.e., functions whose social outcome is the most preferred alternative by all the agents).
Clearly, we can see that the unanimous functions form a subset of SCF's that are onto. ■

Strict example: Consider a SCF whose outcome is as given below.
Whenever there is a consistency among the agents, the social outcome is the alternative which is obtained by shifting the consistent alternative by 1 i.e., if all agents choose a the outcome is b, if all agents pick b then the outcome is c. So that the given function will be an Onto function but it is not unanimous as it violates unanimity.

19.2.5 Truthfulness/Manipulability

As we are in the setting of social choice there is no notion of IIA, the additional property that we need is the property of truthfulness/manipulability. It is similar to the definition of manipulability in the social welfare setting where it is defined in terms of utility representation but here we talk it in terms of preference profiles. It is defined as

Definition 19.8 *An SCF is manipulable if* $\exists i,\ P_i,\ P_i'$ *and* P_{-i} *such that* $f(P_i', P_{-i}) P_i f(P_i, P_{-i})$.

It is explained as, Suppose there exists a preference profile $P = (P_i, P_{-i})$ and another profile where the preference of agent i is changing $P' = (P_i', P_{-i})$ such that by manipulating his profile he is getting better social choice outcome than by reporting his true preference profile.

The SCF f is non-manipulable or strategy proof (**SP**) if it is not manipulable by any agent at any profile.

19.2.6 Dominated Sets

There exists a structural connection between Strategy proof and notion similar to IIA that connects together different preference profiles. This is why we define the Dominated sets.

Definition 19.9 *Dominated set of a at P_i is defined as* $D(a, P_i) = \{b \in \mathcal{A} : aP_ib\}$

Verbally the dominated set is the collection of all such alternatives that are dominated by an alternative a under a given preference profile P_i.
Suppose consider the following preference profile P_i which is given as follows

$$P_i = \begin{matrix} a \\ b \\ c \\ d \end{matrix} \quad , \quad \text{then} \;\; D(b, P_i) = \{c, d\} \tag{19.5}$$

19.2.7 Monotonicity

Definition 19.10 *f is monotone if for profiles P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i')$ $\forall\, i \in \mathcal{N}$, then $f(P') = a$.*

Consider two preference profiles P and P' as shown in (19.6) and suppose we have $f(P) = a$,

$$\begin{array}{ccc} & P & \\ \hline a & & \\ . & & \\ . & a & a \\ . & . & . \\ . & . & . \end{array} \qquad \begin{array}{ccc} & P' & \\ \hline a & & \\ . & a & a \\ . & . & . \\ . & . & . \\ . & . & . \end{array} \tag{19.6}$$

We can see that the relative position of a is getting weakly better from P to P'. From this we can say that the elements a is dominating is increasing from P to P'. If this type of situation happens and if we have $f(P) = a$, then $f(P') = a$.

Theorem 19.11 *f is strategy proof if and only if f is monotone*

Note: The reason to prove this theorem is the technique which is used in the proof is used many times in the Social choice setting.

Proof: We first try to prove Strategy Proof $\implies$ Monotone. To show this given an SCF is strategy proof we need to show the if-part of the monotone part. To do this ,

Consider two profiles P and P' such that $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i') \;\; \forall \; i \in \mathcal{N}$. To show that f is monotone we break the transition from P to P' into multiple stages in each stage we will be changing the preference of one agent at a time and it is shown below

$$\begin{array}{ccccccccc} P_1, P_2, \cdots, P_n & \longrightarrow & P_1', P_2, \cdots, P_n & \longrightarrow & P_1', P_2', \cdots, P_n & \longrightarrow & P_1, P_2, \cdots, P_{k-1}', P_k', \cdots, P_n & \longrightarrow & P_1', P_2', \cdots, P_n' \\ P = P^{(0)} & & P^{(1)} & & P^{(2)} \quad \cdots & & P^{(k)} & \cdots & P^{(n)} \end{array} \tag{19.7}$$

Claim 19.12 *The claim we are going to make is for every transition the social outcome should remain same. It can be represented as, $f(P^{(k)}) = a \;\; \forall \, k = 1, 2, \cdots n$.*

Proof: We prove the claim by contradiction, suppose $\exists \, k$ such that

$$\begin{aligned} f\left(P^{(k-1)}\right) &= a, & P_1', P_2', \cdots, P_{k-1}', P_k, \cdots, P_n \to P^{(k-1)} \quad \text{and} \\ f\left(P^{(k)}\right) &= b \neq a, & P_1', P_2', \cdots, P_k', P_{k+1}, \cdots, P_n \to P^{(k)} \end{aligned} \tag{19.8}$$

But given the if part of monotone property i.e., by moving from P to P' the relative position of a is weakly increasing . Because of the assumptions we made, three different cases are possible

- Case-1: aP_kb and $aP_k'b$, then for player k if he report his profile as P_k' the social outcome is not in favour of him so that he would always report his preference as P_k to get better social outcome. $P_k' \to P_k$.
- Case-2: bP_ka and $bP_k'a$, then for player k reporting his preference as P_k will give better social outcome than reporting P_k'. $P_k \to P_k'$
- Case-3: bP_ka and $aP_k'b$, then the player k will manipulate in both directions that is if his true preference order was P_k, then he will manipulate it as P_k' and if his true preference order was P_k' then he would report P_k to get better social outcome to him.

- Case-4: It is not possible because of our assumption that by moving from P to P' the position of a is weakly increasing.

From all the above cases we can see that the SCF f is not Strategic Proof which is a contradiction $\rightarrow\leftarrow$ ■

Now we need to prove the reverse part that is Strategic Proof $\Leftarrow$ Monotonicity.

We prove this as !SP $\Rightarrow$!MONO.

We prove this part by contradiction. Suppose $\exists\ f$ such that it is not Strategic Proof but it is Monotone.

Not Strategic Proof $\implies$ Manipulable, that means $\exists\ i$, P_i, P'_i, P_{-i} such that $f(P'_i, P_{-i}) P_i f(P_i, P_{-i})$ let the social outcomes be $f(P'_i, P_{-i}) = b$ and $f(P_i, P_{-i}) = a$, then we construct another preference profile P'' such that $P'' = (P''_i, P_{-i})$ and also P'' has the alternatives a and b in the top two positions which is shown in (19.9)

$$\begin{array}{rcll} P'' & = & (P''_i & , P_{-i}) \\ & & b & . \\ & & a & . \quad P''_i(1) = b \text{ and } P''_i(2) = a. \\ & & . & . \\ & & . & . \end{array} \tag{19.9}$$

As we have defined two different preference profiles P' and P'' we look at each of the transitions separately and apply the definition of monotonicity.

- Now going from $P \rightarrow P''$ we note that in P, $bP_ia\ \ \forall i \in \mathcal{N}$ i.e., only the position of a is getting better. It can be written in terms of Dominated sets as

$$D(a, P_i) \subseteq D(a, P''_i) \quad \forall i \in \mathcal{N}. \tag{19.10}$$

  From the assumptions we have f is monotone. Therefore from the definition of monotonicity we have $f(P'') = a$.

- Now going from $P' \rightarrow P$ we have the social outcome at P' as $f(P'_i, P_{-i}) = b$ and as we know that the position of b is getting relatively better which implies that $D(b, P'_i) \subseteq D(b, P''_i)\ \ \forall i \in \mathcal{N}$. By the definition of monotonicity we get $f(P'') = b \neq a$ which is a contradiction $\rightarrow\leftarrow$

Hence we have shown that f is strategy proof if and only if f is monotone. All the axioms of Social choice setting can be shown in the figure 19.1. ■

Lemma 19.13 *If f is motone+onto then it is Pareto efficient.*

Proof: The proof is similar to that of Theorem-19.11.

Suppose we assume that the above statement does not hold i.e., f is monotone+onto but not Pareto efficient, then it can be mathematically written as

$$\exists\, a, b, P \quad \text{such that} \quad bP_i a,\ \forall i \in \mathcal{N} \quad \text{but as } f \text{ in not PE} \implies f(P) = a.$$

As f is onto, from the definition of onto ness

$$\exists\, P' \quad \text{such that} \quad f(P') = b.$$

Now, we construct P'' as shown

$$\begin{array}{c} P'' \\ \hline \\ b \\ a \end{array} \qquad P_i(1) = b \quad \text{and} \quad P_i(2) = a \quad \forall i \in \mathcal{N} \tag{19.11}$$

Now looking at two transitions and from the definition of monotonicity, we have

- $P \to P''$ we see that $f(P) = a$ and moving from P to P' the relative position of a is getting better i.e., $D(a, P_i) \subseteq D(a, P_i'')\ \ \forall i \in \mathcal{N}$, then by monotonicity $f(P'') = a$.
- $P' \to P''$ we see that $f(P') = b$ and moving from P' to P'' we can see that $D(b, P_i') \subseteq D(b, P_i'')\ \ \forall i \in \mathcal{N}$, by monotonicity $f(P'') = b$. Which is a contradiction $\rightarrow\leftarrow$.

From this we can see the big picture that the notion of all the three properties of Social choice functions is different as shown in Figure 19.1 but under the property of Strategy proof they all are falling into same set. It is shown in Figure 19.2.

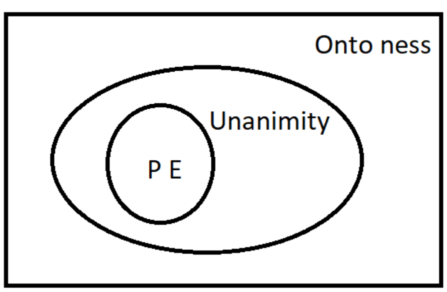

Figure 19.1: Representation of all the properties of Social choice function

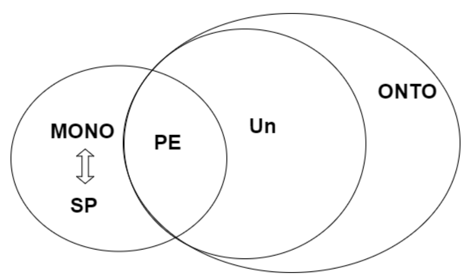

Figure 19.2: The big picture of all axioms of Social Choice setting

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