Single-peaked : on one dimension - one example of domain restriction.

Active research domains :

1. Multi-dimensional single-peaked - e.g. a cone on 2-D plane
2. More than one facility - valuations are non-additive.
   - Some results with randomized mechanisms achieving a fraction of social utility.

Private Good Allocation : Another restricted domain

Task sharing problem : Unit amount of task to be shared among n agents. Every agent's payoff depends only on the share he gets.

Say agent i gets $s_i \in [0,1]$ amount of the task and $\sum_{i \in N} s_i = 1$.

~~Supp~~ The allocation of the task has two effects

(a) Reward : if the wage per unit time is w then for t_i time of work agent i receives $w t_i$ reward

(b) cost : let the cost is quadratic : $k_i t_i^2$

hence the net payoff $= w t_i - k_i t_i^2$

maximized at $t_i^* = \dfrac{w}{2k_i}$

- Agents have one most preferred share of task but this is not single peaked according to the definition before - since this is not single-peaked over the alternatives.

(24-2)

Alternatives, $A = \{(s_1, \dots, s_n) : s_i \in [0,1], \sum_{i \in N} s_i = 1\}$

two alternatives having the same share for agent i will have same preference for i. Hence, this preference is not single-peaked.

- Call these preferences where every agent has a single ~~B~~ most preferred share of task as "single-peaked over the share of task" and denote with $\mathcal{S}$.

$$\text{SCF}: \quad f : \mathcal{S}^n \longrightarrow A.$$

$$P \in \mathcal{S}^n, \quad f(P) = (f_1(P), f_2(P), \dots, f_n(P))$$

$$f_i(P) \in [0,1] \ \forall i \in N, \qquad \sum_{i \in N} f_i(P) = 1.$$

- Let p_i be the most preferred share of player i – peak of P_i

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Pareto efficiency: There does not exist another share of tasks which is weakly preferred by every agent and strictly preferred by at least one.

$$\nexists a \in A \text{ s.t. } a \, R_i \, f(P) \ \forall i \in N \text{ and}$$

$$\exists j \text{ s.t. } a \, P_j \, f(P).$$

- If $\sum_{i \in N} p_i = 1$, then allocate the tasks according to the peaks of the agents. This is the unique PE allocation.

- If $\sum_{i \in N} p_i > 1$, $\exists$ at least one $k \in N$ s.t. $f_k(P) < p_k$

Q: Can there be an agent j s.t. $f_j(P) > p_j$?

- If so, then increasing k's share of task and reducing j's makes both players strictly better off.
  $\forall j \in N \quad f_j(P) \leq p_j$
- Similarly if $\sum_{i \in N} p_i < 1$, $\exists$ at least one $k \in N$ s.t.
  $$f_k(P) > p_k.$$
  $$\forall j \in N \quad f_j(P) \geq p_j.$$

Candidate SCFs

① Serial dictatorship: A predetermined sequence of agents is fixed. Each agent is given either ~~their~~ his peak share or given as much close to the peak as possible. The last agent in the sequence who gets positive share is given the leftover share.

[In case of $\sum_{i \in N} p_i < 1$, the last agent gets the whole leftover task share].

It is PE, SP, but very unfair to the later agents of the sequence – not ANON as well.

Defn. of anonymity is slightly different here: if the agent preferences are permutated, the shares will also get permutated accordingly.

② Proportional: Look at the peaks of every agent and give every agent a factor c of their peaks s.t. $c \sum_{i \in N} p_i = 1$. Every agent gets the same fraction/factor of the peaks.

Q: Is it ANON, PE, SP ?