Lec 23

23-1

Consider an arbitrary profile

$P = (P_1, P_2, \dots, P_n) \qquad P_i(1) = p_i$, peaks.

Goal: Show $f(P) = \text{med}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$

We can assume WLOG $p_1 \leq p_2 \leq \dots \leq p_n$ due to ANON.

say $a = \text{med}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$

Case 1: a is a phantom peak

say $a = y_j$, for some $j \in \{1, \dots, n-1\}$

is a median of $(2n-1)$ points, ~~hence~~ and $(j-1)$ phantom peaks are to the left of the median and $(n-1-j)$ are on the right (because of $y_j \leq y_{j+1}$). Hence

y_j

peaks $\begin{cases} (j-1) \text{ phantom} & (n-1-j) \text{ phantom} \\ (n-j) \text{ agent} & j \text{ agent} \end{cases}$

Hence

$$\boxed{p_1 \leq \dots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \dots \leq p_n} \quad (1)$$

Consider two profiles

$(P_1^0, P_2^0, \dots, P_{n-j}^0, P'_{n-j+1}, \dots, P'_n)$ $\qquad (P_1, P_2^0, \dots, P_n')$

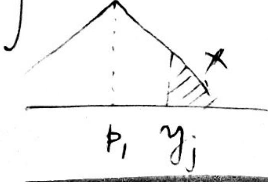
$\downarrow$ apply f $\qquad\qquad \downarrow f$

y_j (definition) $\qquad\qquad b$ (say)

f is SP $\Rightarrow y_j \, P_1^0 \, b \Rightarrow y_j \leq b$

" also $\Rightarrow b \, P_1 \, y_j$, but $p_1 \leq y_j \Rightarrow b \leq y_j$

$\Big\} \; b = y_j$



repeating the argument for the first $(n-j)$ agents, we get

$$f(P_1, \ldots, P_{n-j}, P'_{n-j+1}, \ldots, P'_n) = y_j$$

now consider $f(P_1, \ldots, P_{n-j}, P'_{n-j+1}, \ldots, P_n) = b$ (say)

$$\left.\begin{array}{l} y_j \, P'_n \, b \implies b \leq y_j \\ b \, P_n \, y_j \text{ and } y_j \leq P_n \implies y_j \leq b \end{array}\right\} b = y_j.$$

Case 2: a is an Agent peak

We prove this for 2 agents – the general case repeats this argument.

Claim: $N = \{1, 2\}$, let P and P' be s.t.

$$P_i(1) = P'_i(1) \ \forall i \in N \implies f(P) = f(P')$$

Proof: Let $a = P_1(1) = P'_1(1)$ and $b = P_2(1) = P'_2(1)$

$$f(P) = x \quad \text{and} \quad f(P'_1, P_2) = y$$

f is SP, hence $x \, P_1 \, y$ and $y \, P'_1 \, x$

since the peaks are same, if x, y fall on the same side of the peak, they must be same. The only other possibility is that x and y fall on different sides of the peak. We show that this is not possible.

WLOG $\quad x < a < y \quad$ and $\quad a < b$

since f is SP + ONTO $\iff$ f is SP + PE

and PE requires that $f(P) \in [a, b]$ but

~~x=~~ $f(P) = x < a$, which is a contradiction.

Repeat the argument for $(P'_1, P_2) \to (P'_1, P'_2)$ $\square$

Profile: $(P_1, P_2) = P$, $P_1(1) = a$, $P_2(1) = b$

y_1 is the phantom peak

by assumption $\cancel{\text{med}}(a, b, y_1)$ is an agent peak

WLOG let the median be a.

Assume for contradiction $f(P) = c \neq a$

By PE, c must be within a and b.

Two cases to consider: $b < a < y_1$ and $y_1 < a < b$

Case 1: $b < a < y_1$

By efficiency $c < a$

consider P_1' s.t. $P_1'(1) = a = P_1(1)$

and $y_1 \, P_1' \, c$ [possible since they are on other sides of the peak $P_1'(1)$]

Since $f(P) = c$

$\Rightarrow$ $f(P_1', P_2) = c$

by the claim

Now consider the profile $(P_1^{\shortmid}, P_2)$.

peak at the rightmost.

$P_2(1) = b < y_1 < P_1^{\shortmid}(1)$

The median of $(b, P_1^{\shortmid}(1), y_1)$ is y_1

and hence by the previous result,

$f(P_1^{\shortmid}, P_2) = y_1$

But since $y_1 \, P_1' \, c$ and $f(P_1', P_2) = c$

agent 1 will manipulate in (P_1', P_2) to $(P_1^{\shortmid}, P_2)$

which is a contradiction to f being SP.

Case 2: $y_1 < a < b$

PE $\Rightarrow a < c$

Construct P_1' s.t. $P_1'(1) = a = P_1(1)$

and $y_1 P_1' c$ [possible]

$f(P_1', P_2) = c$ [by claim]

Consider profile (P_1^0, P_2)

$P_1^0(1) < y_1 < b \Rightarrow f(P_1^0, P_2) = y_1$

but $y_1 P_1' c$, hence manipulable by agent 1. Contradiction □

The phantom voters/peaks are introduced so that the extreme preference conditions can be handled with a "fair" decision. E.g. if half the agents are at the extreme left and half are at the extreme right, ~~the~~ a fair distribution of phantom peaks may pick the median at somewhere in the center rather than at some extremal point.

Median voting rule is actually a class of voting rules.