CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

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2.1 Question 1

2.1.1 Part a

Given: f is ONTO and SP
To Show: $f(P_1', P_2') = b$
Proof: We prove this by contradiction.
Using the result proved in class, $f(P_1', P_2') \in \{a, b\}$.
Now assume,

$$f(P_1', P_2') = a$$

Now consider the following preference profiles:

P_1	P_2	P_1'	$\hat{P}_2$	P_1'	P_2'
a	c	b	c	b	a
b	b	a	a	a	b
c	a	c	b	c	c

Table 2.1: Preference Profiles

Now suppose $f(P_1', \hat{P}_2) = b$. If agent 2 falsely reports P_2' instead of $\hat{P}_2$, then the outcome changes to a, which is clearly more preferred over b by $\hat{P}_2$ ($a\hat{P}_2 b$). But f is strategy proof and hence non-manipulable. Thus our assumption was wrong and $f(P_1', \hat{P}_2) = c$. (Note: $f(P_1', \hat{P}_2) \in \{b, c\}$)
Consider the transition of profile from $(P_1', \hat{P}_2) \to (P_1, P_2)$. Clearly the dominated set of c remains same for both the agents and hence, by monotonicity (f is SP$\Rightarrow$MONO), $f(P_1, P_2) = c$. This contradicts the fact that $f(P_1, P_2) = a$.
Thus our assumption was wrong and $\underline{f(P_1', P_2') = b}$. ■

2.1.2 Part b

If the preferences are generated from a single-peaked preference domain with the intrinsic ordering of the alternatives being $a < b < c$, our conclusion from part a becomes invalid. This is because in our proof, we have used the preference profile $\hat{P}_2 = \left(\begin{smallmatrix} c \\ a \\ b \end{smallmatrix}\right)$, but the single-peaked preference domain invalidates the use of $\left(\begin{smallmatrix} c \\ a \\ b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a \\ c \\ b \end{smallmatrix}\right)$ as preferences.
We propose a mechanism where the SCF f chooses the leftmost peak among the peaks of agents, i.e.,

$$f(P_1, P_2) = \min\{P_1(1), P_2(1)\}$$

Following this mechanism, $f(P_1', P_2') = a$ ($\because a < b$). Now our provided mechanism will be valid if we prove that f is SP.
Proof: The agent with the left-most peak (WLOG assume agent 1 with $P_1(1)$ as its peak) will have no reason to manipulate the outcome. The other agent (agent 2) can affect the outcome only if it reports its new peak(say, $P_2'(1)$) to the left of $P_1(1)$. We thus have $P_2'(1) < P_1(1) < P_2(1)$, which implies $f(P_1, P_2) P_2 f(P_1, P_2')$. Thus, our proposed f is SP. ■

2.2 Question 2

We know that Gibbard-Satterthwaite setting demands that all possible ordering for each agent must be admissible in the domain. Thus in settings where preferences are restricted, Gibbard-Satterthwaite theorem does not hold.
Consider two distinct sets S and T such that $S \subset T$. Now, set T will contain all the projects of set S. So the highest ranked project of T will be atleast as good as the highest ranked project in S ($\because$ T already contains the highest ranked project of S). This means that any agent will always prefer T over S or will be indifferent to both. Never will the agent prefer S over T.
Hence we find that $\forall S, T \subseteq X, S \subset T$, no agent ever prefers S over T. Thus it is not possible to have all possible ordering over the domain for each agent and so Gibbard-Satterthwaite result will NOT apply here.

2.3 Question 3

Yes, the median vector SCF is group strategy proof.

Proof: We need to consider only the peak preferences of all the agents. So let us denote the preferences denoted only by their peaks, i.e., $P = (P_1(1), ..., P_i(1), ..., P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and the phantom peaks. Consider a group of agents $K = \{i_1, i_2, ..., i_k\}$.

- If all k agents have the same peak as the median, i.e., $P_{i_1} = P_{i_2} = ... = P_{i_k} = a$. Then there is no reason for any agent to manipulate.
- If all k agents lie to the left of the median, i.e., $P_{i_j} < a \ \forall j \in \{1, 2, ..., k\}$. Here the only method to affect the outcome is if some agents falsely report their peaks to be right of a. Let the new outcome be $b > a$. In this case, $P_{i_j} < a < b \ \forall j \in \{1, 2, ..., k\}$ which implies $f(P_K, P_{-K}) P_{i_j} f(P'_K, P_{-K}) \ \forall j \in \{1, 2, ..., k\}$.
- If all k agents lie to the right of the median, i.e., $P_{i_j} > a \ \forall j \in \{1, 2, ..., k\}$. Here the only method to affect the outcome is if some agents falsely report their peaks to be left of a. Let the new outcome be $b < a$. In this case, $P_{i_j} > a > b \ \forall j \in \{1, 2, ..., k\}$ which implies $f(P_K, P_{-K}) P_{i_j} f(P'_K, P_{-K}) \ \forall j \in \{1, 2, ..., k\}$.
- If some agents lie to the left of the median while some to the right. WLOG assume $P_{i_1} < P_{i_2} < ... < P_{i_l} < a < P_{i_{l+1}} < ... < P_{i_k}$ where $1 < l < k$. Suppose after some manipulation by the agent group, the new outcome becomes $b > a$. In this case $f(P_K, P_{-K}) P_{i_1} f(P'_K, P_{-K})$. Otherwise, if the new outcome was $b < a$, then $f(P_K, P_{-K}) P_{i_k} f(P'_K, P_{-K})$.

Clearly in none of the above cases, there was a valid profitable manipulation of f by the agent group. Thus f is group strategy proof. ■