CS711: Introduction to Game Theory and Mechanism Design

Teacher: Swaprava Nath

Strategy, Rationality, Common Knowledge

Quick Recap

Game theory

- Analytical approach for predicting reasonable outcome

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- Fundamental building blocks: **players**, **strategies**, **utilities**

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- Fundamental building blocks: **players**, **strategies**, **utilities**
- Difference between **action** and **strategy**
- Key assumptions: **rationality** and **intelligence**

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  - Two player game: White and Black – 16 pieces each
  - Every piece has some legal moves – **actions**
  - The game progresses with each player taking turns and making legal moves – starts with White
  - Ends at
    - Win for White, if White captures the Black King
    - Win for Black, if Black captures the White King
    - Draw – if Black has no legal move but the King is not in check, both players agree to a draw, a board position where no player can win, ...

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- What is a **strategy**?

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Set of all game situations: H

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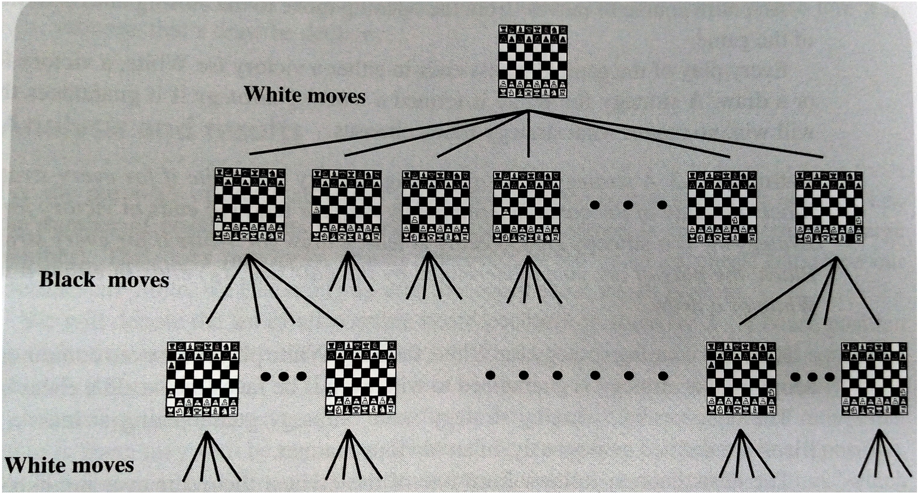
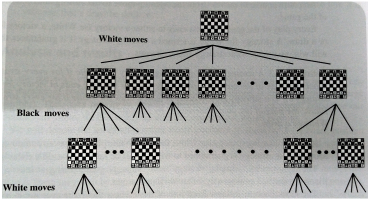


Image courtesy: Maschler et al., Game Theory.

Illustration



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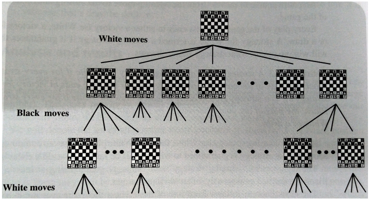

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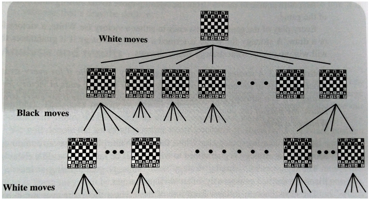

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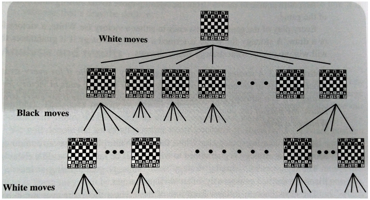

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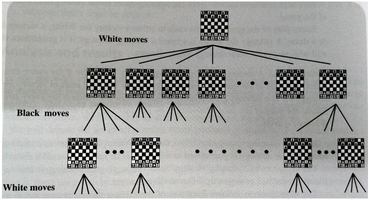

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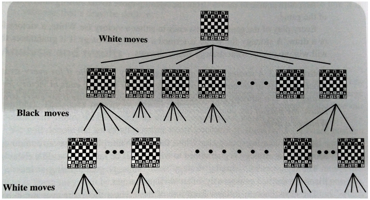

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- *plan of action* in a given *game situation*
- The complete plan is a strategy

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- this is a finite game – where does the game end? can the players guarantee a given end?

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- not obvious if such a strategy exists – this is a property of the mappings

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- chess will be a boring game if the answers were known

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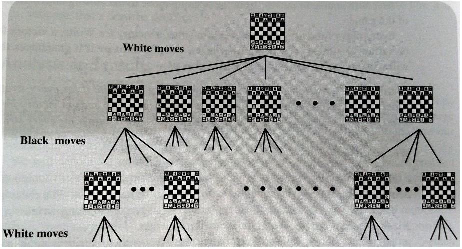

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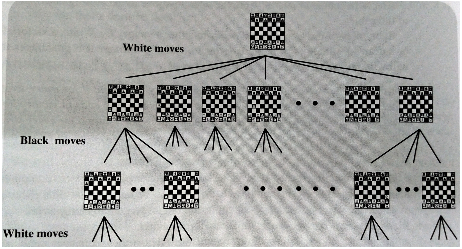

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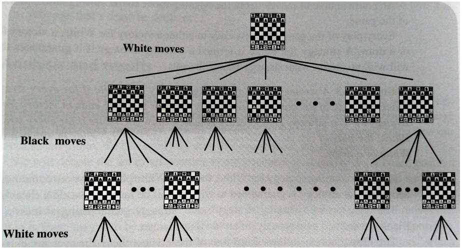

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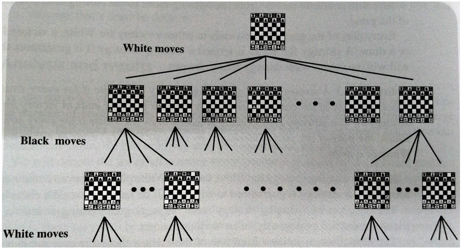

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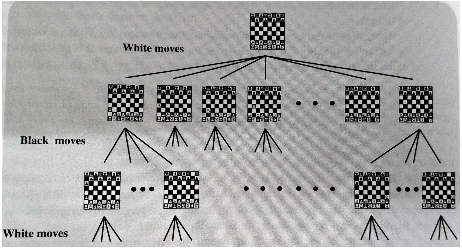

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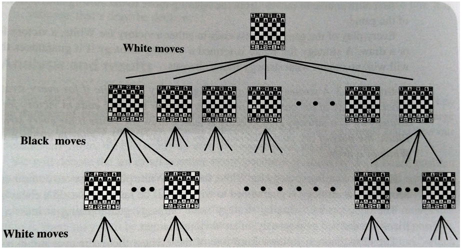

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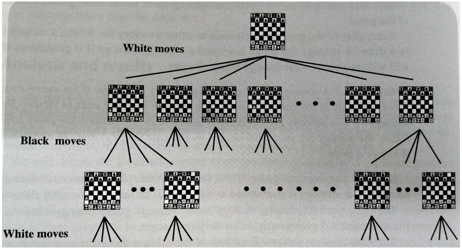

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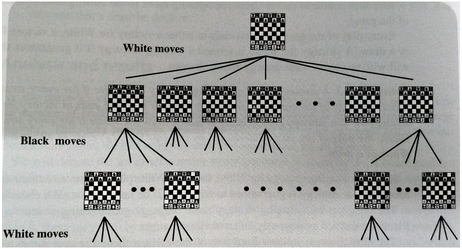

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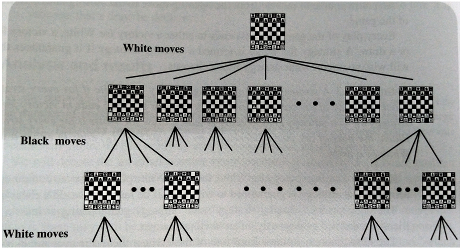

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$$\mathcal{F} = \{\Gamma(x) : x \in H\}$$

collection of all subgames defined by the subtrees of the game of chess

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Every game in $\mathcal{F}$ satisfies one and only one of the following statements:

1. *White has a winning strategy*
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both Kings are on board and game has ended implies a draw

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(i) if $\exists y_0 \in C(x)$ s.t. alternative (1) is true in $\Gamma(y_0)$, then (1) is true in $\Gamma(x)$ as well: W picks the action to reach y_0 augmented with the winning strategy at y_0

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B can watch W pick an action that reaches $y \in C(x)$ and then pick the strategy that ensures either win or at least a draw

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Exercise: prove this theorem when the length of the game is infinite (ex. 1.3, MSZ book)

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- Normal form / strategic form – appropriate for single shot games

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- if S_i is finite – the game is called a **finite game**

Example: Rock-Paper-Scissor

$\mathbf{1}\backslash\mathbf{2}$	Rock	Paper	Scissor
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
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- $N = \{1, 2\}$
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Players' Knowledge and Behavior

Definition (Rationality)

A player is *rational* if she picks actions to **maximize** her utility

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Definition (Common Knowledge)

A fact is a *common knowledge* if

1. All players know the fact, and
2. All players know that all other players know the fact, and
3. All players know that all other players know that all other players know the fact, and ... ad infinitum.

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- common knowledge percolates to the outcome in the following way

- If there were only one blue-eyed person, he would have seen that the other two had black eyes, realized that his eye color is blue (since sage is always correct), leaves at the end of day one– every other player understands this and stays back

Common Knowledge (Contd.)

- if there were two blue-eyed persons, then both of them will see one blue and one black eyed person, hope that he is not blue-eyed and wait till the second day if the other blue-eyed person leaves on day one

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Assumption

The fact that all players are rational and intelligent is a **common knowledge**

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$\mathbf{1 \backslash 2}$	L	M	R
U	1,0	1,3	3,2
D	-1,6	0,5	5,3

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Definition (Dominated Strategy)

A strategy $s'_i \in S_i$ of player i is **strictly dominated** if there exists another strategy s_i of i such that for every strategy profile $s_{-i} \in S_{-i}$ of the other players

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}).$$

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and there exists some $\tilde{s}_{-i} \in S_{-i}$ such that

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Definition (Dominant Strategy Equilibrium)

A strategy profile (s_i^*, s_{-i}^*) is a **strictly (weakly) dominant strategy equilibrium (SDSE (WDSE))** if s_i^* is a strictly (weakly) dominant strategy for every $i, i \in N$.

Domination (Contd.)

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	D	E
A	$5, 5$	$0, 5$
B	$5, 0$	$1, 1$
C	$4, 0$	$1, 1$