CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2018**

Project: Assignment 2

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1.1 Problem 1

We are given a two agent model with alternatives a,b,c. f is an onto SCF with f(P_1,P_2) = a.

P_1	P_2	$P_1^{'}$	$P_2^{'}$
a	c	b	a
b	b	a	b
c	a	c	c

1. Assuming the domain of preferences is of unrestricted strict preferences and f is strategy proof. Consider the following preference profiles, we first prove that $f(P_1, P_2) = a \implies f(P_1^{'}, \hat{P}_2) = b$.

P_1	P_2	$P_1^{'}$	$\hat{P}_2$	$P_1^{'}$	P_2
a	c	b	c	b	c
b	b	a	a	a	b
c	a	c	b	c	a

   We know that for any preference profile $(\bar{P}_1, \bar{P}_2), f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$. Therefore $f(P_1^{'}, \hat{P}_2) = b$ or c. To prove by contradiction, assume $f(P_1^{'}, \hat{P}_2) = c$.

   Consider transition from $(P_1^{'}, \hat{P}_2)$ to $(P_1^{'}, P_2)$:
   $D(c, P_1^{'}) \subseteq D(c, P_1^{'})$ and $D(c, \hat{P}_2) \subseteq D(c, P_2)$, $\therefore$ by monotonicity $f(P_1^{'}, \hat{P}_2) = c \implies f(P_1^{'}, P_2) = c$.
   In preference profile $(P_1^{'}, P_2)$ if agent 1 reports P_1 instead it gets a higher utility because it prefers a to c and $f(P_1, P_2) = a$. Hence manipulation is possible, which is a contradiction.
   Thus $\underline{f(P_1, P_2) = a \implies f(P_1^{'}, \hat{P}_2) = b}$.

   Now, to find $f(P_1^{'}, P_2^{'})$ we consider the following preference profiles:

$P_1^{'}$	$P_2^{'}$	$P_1^{'}$	$\hat{P}_2$
b	a	b	c
a	b	a	a
c	c	c	b

   $D(b, P_1^{'}) \subseteq D(b, P_1^{'})$ and $D(b, P_2^{'}) \subseteq D(b, \hat{P}_2)$,$\therefore$ by monotonicity, since $f(P_1^{'}, \hat{P}_2) = b$

   $$f(P_1^{'}, P_2^{'}) = b$$

2. In a single peaked preference domain with common order $a < b < c$, two preference profiles are invalid:

Invalid	
a	c
c	a
b	b

While proving the previous part we used $\hat{P}_2$ which is c,a,b. Thus, the previous proof cannot go through. Therefore the earlier conclusion can't hold.

A mechanism which would give $f(P_1^{'}, P_2^{'}) = a$ is :

$$f(P) = min_{i \in N}\{P_i(1)\}$$

1.2 Problem 2

X is the set of projects, each agent has a linear ordering P_i over the set X. A social choice function f takes preference profiles $P_i^{'}$ as input and gives a non empty subset of X as output. The preference profile $P_i^{'}$ is over all possible subsets of X.

Every agent will have a unique preference profile $P_1^{'}$ corresponding to its P_i which will be determined by the rule given. Let $\mid X \mid= k \geqslant 2$. The number of possible P_i are $k!$, Therefore the number of possible $P_i^{'}$ are also $k!$.

The total possible subsets of X i.e $\mid A \mid= 2^k - 1$. For the domain to be unrestricted $\mid P^{'} \mid= (2^k - 1)!$, which is not being fulfilled as $\mid P^{'} \mid= k! < 2^k - 1)!$ $\forall k \geqslant 2$. So, the domain is restricted. Since for the Gibbard-Satterwaite result to be applicable unrestricted preferences is a requirement, the result doesn't apply here.

1.3 Problem 3

Yes, the median voter Social Choice Function is group strategy-proof.

Proof:

Let f be a median voter SCF. Let K be a subset of agents (N in total) who misreport their preference. Let $f(P) = a$.
There can be four cases from here.

- **Case 1**: K contains the player whose peak preference is picked

  In this case, there is no reason for the player to manipulate.

- **Case 2**: $P_i(1) < a$ $\forall i \in K$ (All the agents in K lie to the left of the median)

  If all the agents in K shift their preference to further left of a, the median won't change. If the agents in K manipulate to report their peak to the further right of a, i.e $P_i^{'}(1) > a$ for some $i \in K$, it will not be a profitable deviation for agent i as $P_i(1) < a < P_i^{'}(1)$ and by the definition of single peaked preferences $a = f(P_i, P_{-i}) P_i f(P_i^{'}, P_{-i}^{'})$.

- **Case 3**: $a < P_i(1)$ $\forall i \in K$ (All the agents in K lie to the right of the median)

  Similar to Case 2, no agents in K can manipulate profitably.

- **Case 4**: Agents in K lie to both left and right of the median

  In this case, to change the median some agents will have to move their peak across it. Therefore there will be some player $i \in K$ such that $P_i(1) < a < P_i^{'}(1)$ or $P_i(1) > a > P_i^{'}(1)$. In either of these cases the deviation is not profitable as we saw in Case 2 and 3.

Hence, f is group strategy-proof.