Lect 21

GS Theorem allows unrestricted preferences

$$f : \mathbb{P}^n \to A$$

One reason of such a restrictive result is that the domain of the SCF is large – a potential manipulator has many options to manipulate.

Look from the definition of strategyproofness.

$$f(P_i, P_{-i}) \; P_i \; f(P_i', P_{-i}) \quad \forall P_i, P_i' \in \mathbb{P}$$
$$\forall P_{-i} \in \mathbb{P}^{n-1}, \forall i \in N.$$

If we now reduce the set of preferences from $\mathbb{P}$ to some subset of $\mathbb{P}$, the SCFs that are truthful on $\mathbb{P}$ continues to be truthful, but the hope is that we may find more truthful SCFs.

Domain Restrictions

In this course, we'll see three domain restrictions

1. Single peaked preferences
2. Divisible good allocation
3. Quasi-linear preferences

Each of these domains have interesting non-dictatorial SCFs that are strategyproof.

Single-peaked preferences

One common order over the alternatives

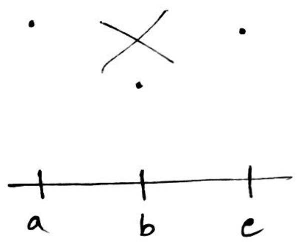
After the alternatives are ordered – agent preferences have a preferences that is single peaked wrt that order of the alternatives.

Motivating examples:

1. Facility location: School / Hospital / Post Office etc.
2. Political ideology: Left - Center - Right
3. Temperature sensing

Each of these examples have a natural ordering over the alternatives. - we denote this ordering with $<$ (as in real numbers) - this need not always be an ordering over the real numbers but can be any relation that is transitive and antisymmetric - for simplicity we'll discuss only alternatives on a real line.

<u>Example</u>: $\{a, b, c\}$ three facilities on a real line with $a < b < c$ (which means the locations of a, b, c)



preferences (valid)

a	b	b	c
b	a	c	b
c	c	a	a

preferences (invalid)

a	c
c	a
b	b

<u>Defn</u>: A preference ordering P_i (strict over A) of agent i is <u>single-peaked</u> w.r.t. $<$ if

- for all $b, c \in A$ with $b < c \leq P_i(1)$ we have

  $c \, P_i \, b$

- for all $b, c \in A$ with $P_i(1) \leq b < c$ we have

  $b \, P_i \, c$.

Let $\mathcal{S}$ be the set of single-peaked preferences

$\mathcal{S} \subset \mathcal{P}$ - we have seen this in the previous example

$f : \mathcal{S}^n \to A$ (social choice function)

An SCF f is *manipulable* if $\exists\, i, P_i, P_i'$ and P_{-i} s.t.

$$f(P_i', P_{-i}) \; P_i \; f(P_i, P_{-i})$$

f is strategyproof if it is not manipulable.

What does it imply?

$\forall\, P_i, P_i', P_{-i} \quad \forall i \in N$

either $\quad f(P_i, P_{-i}) = f(P_i', P_{-i})$

or $\quad f(P_i, P_{-i}) \; P_i \; f(P_i', P_{-i})$

but now P_i, P_{-i}'s belong to $\mathcal{S}$ and not $\mathcal{P}$

hence there are less number of conditions to be satisfied.

How does it circumvent GS theorem?

<u>Example</u>: pick the leftmost peak — mechanism 1

for the leftmost agent, no reason to misreport

for any other agent (on the right) the only way the outcome may change is to ~~push the~~ report a peak which is further left - but that is strictly worse for him.

Repeat this argument for any k-th peak from the left. Hence even a rightmost peak choosing SCF is strategyproof, so is the median $\left(k = \left\lfloor \frac{n}{2} \right\rfloor\right)$

Defn: (Median Voter SCF)

An SCF $f : \mathcal{S}^n \to A$ is a <u>median voter</u> SCF if there exists $B = (y_1, \ldots, y_{n-1})$ ~~agents~~ such that $f(P) = \text{median}(B, \text{peaks}(P))$ for all preference profiles $P \in \mathcal{S}$. The points in B are called the peaks of "phantom voters".

Note: B is fixed for f and does not change with P.

Why use phantom voters?

All the examples of leftmost, rightmost, median etc. can be taken into account in this single definition.

leftmost : all phantom peaks are on the leftmost point

rightmost : rightmost

Theorem: (Moulin 1980)

Every median voter SCF is strategyproof.

Proof: Only the peaks of the preferences matter.

If $P = (P_1, \ldots, P_n)$ and $f(P) = a$, agent i has no reason to manipulate if $P_i(1) = a$. If the peak $P_i(1)$ is to the left of a, the only way he can change the median is by reporting his peak to be right of a - which takes the median farther away from $P_i(1)$ - ~~hence~~ a similar argument holds when the peak is on the right of a. Hence strategyproof. $\square$

Note: Mean does not have this property.