CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

24.1 Recap

We discussed task allocation domain and the agent preferences were restricted to single-peaked over the share of the task. We wanted properties like Pareto efficiency, anonymity on top of strategyproofness in this domain. In this lecture, we will look at the uniform rule SCF introduced by Sprumont and study its properties.

24.2 Uniform rule SCF

Definition 24.1 (Uniform rule SCF) *In the task sharing domain, the* uniform rule SCF $f : \mathcal{S}^n \to A$ *is defined as*

- $f_i^u(P) = p_i$, *if* $\sum_{i \in N} p_i = 1$.
- $f_i^u(P) = \max\{p_i, \mu(P)\}$, *if* $\sum_{i \in N} p_i < 1$.
- $f_i^u(P) = \min\{p_i, \lambda(P)\}$, *if* $\sum_{i \in N} p_i > 1$.

Where $\mu(P)$ *solves*

$$\sum_{i \in N} \max\{p_i, \mu(P)\} = 1,$$

and $\lambda(P)$ *solves*

$$\sum_{i \in N} \min\{p_i, \lambda(P)\} = 1.$$

Interpretation of μ: Consider the image in Figure 24.1. The vertical bars denote the total share of the task for every agent and the solid lines within the bars denote the peak of the preferences of the agents. Consider a horizontal thread that denotes the share of tasks to every agent. Let is start from the top (shown as dotted line in the figure). Initially it assigns 1 unit of task to every agent – which is infeasible. But now imagine that the thread moves downward (as shown in the figure) and stops whenever some agent's peak is reached – that agent is assigned her peak share and others shares are according to the level of the thread. If the allocation is still infeasible, lower the thread further for all *other* agents. Continue doing this until a feasible allocation is reached. This is exactly the solution given by μ. For visualizing the solution of λ, a very similar exercise can be done with threads starting from bottom and moving upwards.

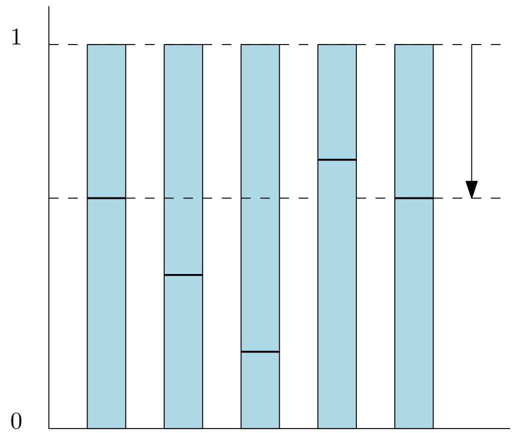

Figure 24.1: Illustration of μ

Theorem 24.2 (Sprumont (1991)) *An uniform rule SCF is* anonymous, Pareto efficient *and strategyproof.*

Proof: The anonymity of this rule is obvious, since the rule uses only the peaks of every agent and not the agent identities. Hence a permutation of the agents' preferences will return an allocation where the allocation of the tasks of the agents has been correspondingly permuted.

To show that this is Pareto efficient, we need to verify the conditions that characterize PE as shown in the previous section.We see that

- for $\sum_{i \in N} p_i < 1$, $f_i^u(P) = \max\{p_i, \mu(P)\} \geqslant p_i$,
- for $\sum_{i \in N} p_i > 1$, $f_i^u(P) = \min\{p_i, \lambda(P)\} \leqslant p_i$, and
- for $\sum_{i \in N} p_i = 1$, $f_i^u(P) = p_i$.

Therefore the uniform rule satisfies PE.

For strategyproofness, we consider the following cases.

Case 1, $\sum_{i \in N} p_i = 1$: In this case, the allocation rule gives every agent her peak, hence it is strategyproof.

Case 2, $\sum_{i \in N} p_i < 1$: In this case, $f_i^u(P) = \max\{p_i, \mu(P)\} \geqslant p_i$. The only potential manipulable scenario is when $f_i^u(P) > p_i$, which implies that $\mu(P) > p_i$, i.e., the thread stopped before reaching the peak p_i from above. The only way this allocation can be changed is by reporting a $p_i' > \mu(P)$. Clearly, that takes the allocation of agent i further away from p_i than $\mu(P)$. Since the preferences are single peaked for the agent i over the allocation of task, she strictly prefers the current allocation $\mu(P)$ than p_i'.

Case 3, $\sum_{i \in N} p_i > 1$: the proof here is similar to that of Case 2, with the argument reversed. Hence, we have proved the theorem. ■

The converse of the above theorem is also true, but we skip its proof. Interested reader can refer to the paper by Sprumont [S91].

Theorem 24.3 *An SCF is strategyproof, Pareto efficient, and anonymous if and only if it is an uniform rule.*

24.3 Mechanism Design with Transfers

We now consider a new restricted domain of preferences that allows transferring utility, which is widely used in many real-world domains. The transferred utility is called 'money', and for this reason this sub-domain is also called *mechanisms with money*. In particular, we will only consider the transfers in quasi-linear form, as defined in the following section.

24.3.1 Quasi-linear Utility Model

In this setting the social choice function will be denoted by

$$F : \Theta \mapsto X.$$

Where Θ denotes the set of type profiles and X denotes the set of outcomes. We deliberately use the uppercase F to denote the SCF to distinguish this from what we have discussed so far. The types of the agents are their private information. The set of outcomes in this setting is a collection of two objects: (1) the **allocation**, and (2) the **payment**. Formally,

$$\begin{aligned} &X = \{x : x = (a, \boldsymbol{\pi})\} \\ &a \in A : \text{ set of allocations, and} \\ &\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathbb{R}^n \text{ the payments.} \end{aligned}$$

Examples of Allocations: A few examples are as follows.

Example 1: A public decision, $a \in A = \{\text{Bridge, Park, Theater, Museum, ... }\}$

Example 2: Private object allocation, e.g., a homogeneous divisible object (cake) denoted by the real interval $[0, 1]$. An allocation of this object gives every agent some portion of it. Hence, $a = (a_1, \ldots, a_n)$, where $a_i \in [0, 1], \forall i \in N$ and $\sum_{i \in N} a_i \leqslant 1$.

Example 3: Single indivisible item allocation. Here the allocation vector is similar: $a = (a_1, \ldots, a_n)$, however, $a_i \in \{0, 1\}, \forall i \in N$ and $\sum_{i \in N} a_i \leqslant 1$. This implies that the item can either go entirely to exactly one agent or may go to none.

Example 4: Allocation of multiple indivisible objects. Let the set of objects be denoted by S. An allocation in this setting is a partition of these objects into $n+1$ groups (A_0 denoting unallocated objects), i.e.,

$$A = \{(A_0, A_1, \ldots, A_n) : A_i \subseteq S, \forall i \in N \cup \{0\}, \cup_{i \in N \cup \{0\}} A_i = S, A_i \cap A_j = \emptyset, \forall i \neq j\}.$$

The effect of the allocations are reflected in the valuation of every agent. Valuation v_i of agent i is a function of the allocation a and the *type* θ_i of every agent. This model of valuation is called **independent private values** (IPV).

$$v_i : A \times \Theta_i \mapsto \mathbb{R}.$$

It is called *private* since given the allocation, the valuation depends only on the private information of agent i. A more general model where it can depend on the entire vector of types given the allocation is called *interdependent* valuation.

In IPV setting, the notation of $v_i(a, \theta_i)$ is sometimes shortened to $\theta_i(a)$ or $v_i(a)$ to denote the same thing.

Payment function: This function is given by

$$p_i : \Theta \mapsto \mathbb{R}, \quad \boldsymbol{p} := (p_1, \ldots, p_n).$$

Utility: The utility of an agent in this model is given by the following expression when the type profile is θ and the outcome is $(a, \boldsymbol{\pi})$.

$$u_i((a, \boldsymbol{\pi}), \theta_i) := v_i(a, \theta_i) - \pi_i \qquad \textbf{(quasi-linear utilities)}$$

The utility is linear in payment but can potentially be non-linear in the allocation. Since allocation and payment completely determine the outcome in this setting, this utility model is called quasi-linear.

Why is this a domain restriction? Here the outcome set X is given by $A \times \mathbb{R}$. Consider two alternatives (a, π) and (a, π') with $\pi_j = \pi'_j, \forall j \neq i$ and $\pi_i > \pi'_i$. Due to the quasi-linear preferences, for every type θ_i of agent i, $\theta_i(a) - \pi_i < \theta_i(a) - \pi'_i$. Hence, for every quasi-linear preference $\succ_i^{\text{QL}}$ of i, $(a, \pi') \succ_i^{\text{QL}} (a, \pi)$. This implies that all possible orders over the alternatives is *not* admissible in this setting. Recall that the Gibbard-Satterthwaite setting demands that *all possible ordering over the outcomes much be admissible* in the domain. Only then the conclusion of dictatorship holds. Put in the current setting, it should mean that (a, π) *can be* more preferred than (a, π') by an agent. This is certainly not true in the quasi-linear setting – in particular, no agent can have a type where (a, π) is placed above (a, π'). This subtle domain restriction opens up the possibility of a lot of mechanisms to be strategyproof.

References

[S91] Y. Sprumont, "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule," *Econometrica*, Vol. 59, No. 2 (Mar., 1991), pp. 509-519.