Monotonicity and Myerson's lemma.

Defn: An allocation rule is non-decreasing if for every agent $i \in N$ and $\forall \underline{t}_i \in \underline{T}_i$ we have

$$f_i(t_i, \underline{t}_i) \geq f_i(s_i, \underline{t}_i), \quad \forall s_i, t_i \in T_i \text{ with } t_i > s_i.$$

If the types of other agents are held fixed, the probability of allocation of the object does not decrease with increase in the value.

Theorem: (Myerson 1981)

Suppose $T_i = [0, b_i]$ $\forall i \in N$ and the valuations are in product form (~~single parameter and the allocation~~)

An allocation rule $f : T \to \Delta A$ and a payment rule $(p_1, \ldots, p_n)$ is DSIC iff

1. f is non-decreasing.
2. Payment is given by

$$p_i(t_i, \underline{t}_i) = p_i(0, \underline{t}_i) + t_i f_i(t_i, \underline{t}_i) - \int^{t_i} f_i(x, \underline{t}_i)\, dx$$

$\forall i \in N, \forall t_i \in T_i, \forall t_{-i} \in T_{-i}$.

Proof of ~~Mes~~ Myerson's theorem.

($\Rightarrow$) Consider the utility of agent i

$$u_i(t_i, \underline{t}_i) = t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)$$

and $$u_i(s_i, \underline{t}_i) = s_i f_i(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)$$

now, since $(f, \underline{p})$ is DSIC

$$u_i(t_i, \underline{t}_i) = t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)$$
$$\geq t_i f_i(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)$$
$$= s_i f_i(s_i, \underline{t}_i) + f_i(s_i, \underline{t}_i)(t_i - s_i) - p_i(s_i, \underline{t}_i)$$
$$= u_i(s_i, \underline{t}_i) + f_i(s_i, \underline{t}_i)(t_i - s_i)$$

define $g(t_i) = u_i(t_i, \underline{t}_i)$, $\phi(t_i) = f_i(t_i, \underline{t}_i)$

$$g(t_i) \geq g(s_i) + \phi(s_i)(t_i - s_i)$$

$\phi(s_i)$ is a subgradient of g at s_i --- (1)

Convexity of $g \equiv u_i(\cdot, \underline{t}_i)$

pick $x_i, z_i \in T_i$ define $y_i = \lambda x_i + (1-\lambda) z_i$, $\lambda \in (0,1)$.

~~DSIC implies~~

$$\cancel{g(x_i) \geq g(z_i) + \phi(z_i)(x_i - z_i)}$$
$$\cancel{g(z_i) \geq g(x_i) + \phi(x_i)(z_i - x_i)}$$

DSIC implies

$$g(x_i) \geq g(y_i) + \phi(y_i)(x_i - y_i) \qquad \times \lambda$$

$$g(z_i) \geq g(y_i) + \phi(y_i)(z_i - y_i) \qquad \times (1-\lambda)$$

$$\Rightarrow \lambda g(x_i) + (1-\lambda) g(z_i) \geq g(y_i) + \phi(y_i)\underbrace{[\lambda x_i + (1-\lambda) z_i - y_i]}_{=0}$$

$$= g(y_i)$$

$\Rightarrow g$ is convex.

Apply lemmas 3 and 4

lemma 3: ϕ is non-decreasing. proves part 1 of the claim $\phi \equiv f_i(\cdot, \underline{t}_i)$

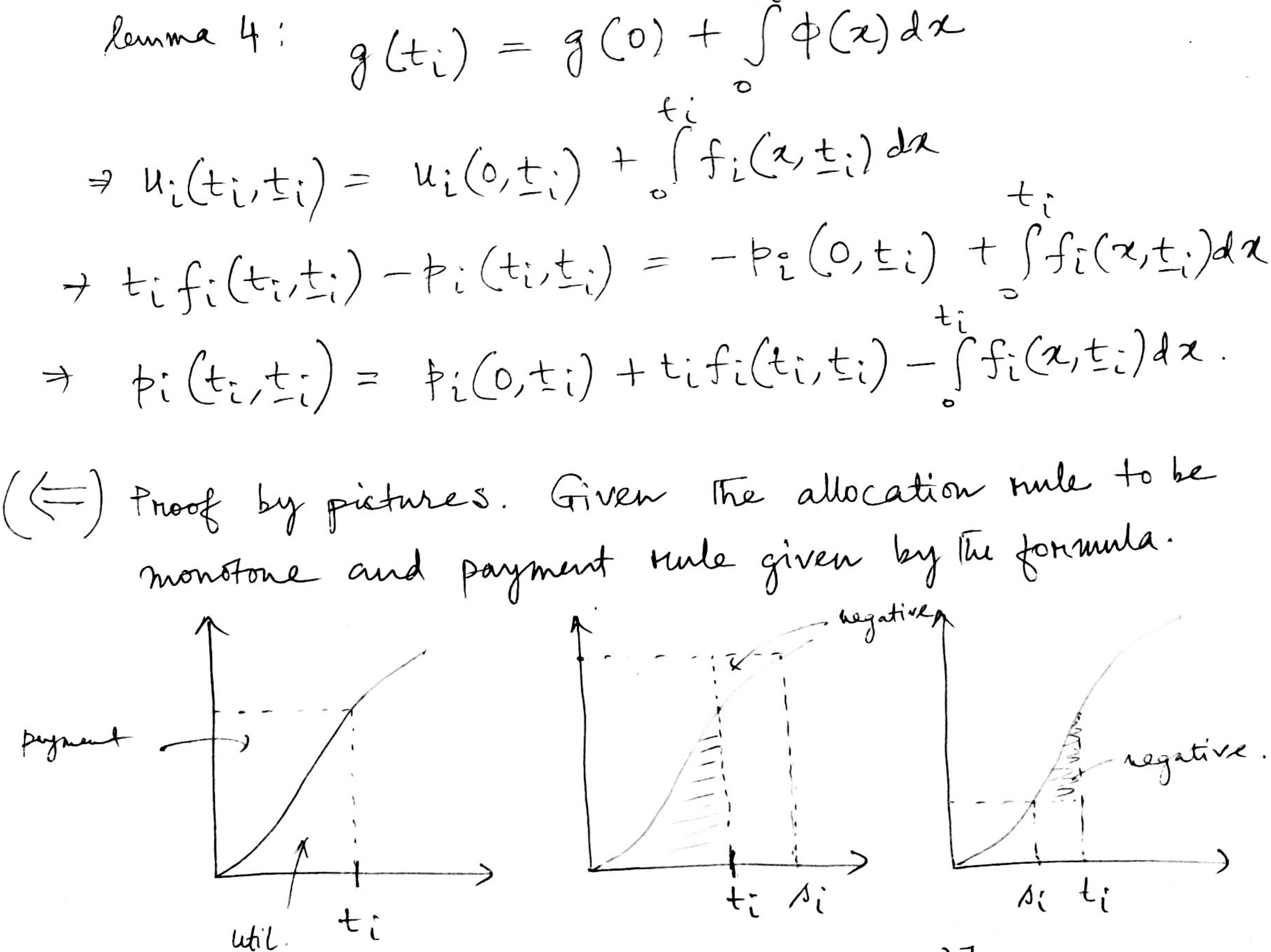
lemma 4: $$g(t_i) = g(0) + \int_0^{t_i} \phi(x)\,dx$$

$$\Rightarrow u_i(t_i, \underline{t}_i) = u_i(0, \underline{t}_i) + \int_0^{t_i} f_i(x, \underline{t}_i)\,dx$$

$$\Rightarrow t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i) = -p_i(0, \underline{t}_i) + \int_0^{t_i} f_i(x, \underline{t}_i)\,dx$$

$$\Rightarrow p_i(t_i, \underline{t}_i) = p_i(0, \underline{t}_i) + t_i f_i(t_i, \underline{t}_i) - \int_0^{t_i} f_i(x, \underline{t}_i)\,dx.$$

($\Leftarrow$) Proof by pictures. Given the allocation rule to be monotone and payment rule given by the formula.



$$[t_i f_i(t_i, \underline{t}_i) - p_i(t_i, \underline{t}_i)] - [t_i f_i(s_i, \underline{t}_i) - p_i(s_i, \underline{t}_i)]$$
$$= (s_i - t_i) f_i(s_i, \underline{t}_i) + \int_{s_i}^{t_i} f_i(x_i, \underline{t}_i)\,dx_i \geq 0 .$$