CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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34.1 Recap

Continuing our discussion from the previous lecture, in which we had defined the notion of monotonicity of an allocation rule and stated the Myerson's theorem.

Theorem 34.1 (Myerson 1981) *Suppose $T_i = [0, b_i], \forall\ i \in N$ and the valuations are in product form. An allocation rule $f : T \mapsto \Delta A$ and a payment rule $(p_1, p_2, ..., p_n)$ is DSIC iff*

1. *The allocation f is non-decreasing, and,*
2. *Payment is given by*

$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i.$$

In this lecture, we will look at the implications of the payment rule and a number of examples to illustrate the rule and compare this characterization result with that of Roberts' where only one specific class of allocation rules was implementable.

34.2 Discussions on the Payment Rule

Revenue Equivalence Consider the case where we have two different DSIC mechanisms given by (f, p) and (f, q) with the same allocation function f and different payment functions p and q. From the payment function characterization of Myerson's result, we see that the difference between the payments lies only in the constant terms, $p_i(0, t_{-i})$ and $q_i(0, t_{-i})$. Hence, we see that the following difference term is same for all the payments that implement f.

$$\begin{aligned} & p_i(t_i, t_{-i}) - p_i(s_i, t_{-i}) = q_i(t_i, t_{-i}) - q_i(s_i, t_{-i}) \quad \forall s_i \in T \\ \implies & \qquad p_i(t_i, t_{-i}) = q_i(t_i, t_{-i}) + h_i(t_{-i}) \end{aligned} \tag{34.1}$$

The term $h_i(t_{-i})$ is independent of i-th agents type t_i. So any two payments for an agent differs by a function that is independent of the type of that agent. Whenever any two payment rules that implement the same allocation rule that can be written in this form, we call the allocation rule to be **revenue equivalent**. The intuition is that for every payment rule the revenue generated is 'almost' equivalent.

Note that this is a much stronger statement than the fact that if any function $h_i(t_{-i})$ is added to the payment, that payment rule also implements the same allocation rule (we saw this before). This result says that for *every* payment rule that implements the same allocation rule must be related in this form.

Difference with Roberts' characterization theorem It is worth noting the difference of this result with the characterization provided by Roberts' theorem, that gives an explicit formula for the allocation rule. Roberts' result says that when the space of valuations is unrestricted, the allocation rule must be from the affine maximizer class. This is a precise functional form characterization unlike the 'implicit' characterization given by the Myerson's result, which says that the allocation should be non-decreasing, but does not give any structural guarantees.

Corollary 34.2 *An allocation rule is implementable iff it is non-decreasing.*

34.2.1 Some examples of non-decreasing allocation function

1. **Constant allocation:** This allocation is trivially non-decreasing. Payment given by the formula is constant, which can be zero. A special case of this allocation rule is **dictatorial allocation**, where the object is deterministically given to one agent, who is the *dictator*. The payment again is constant.

2. **Vickrey auction:** Here the allocation is efficient and the payment for each agent, fixing the other agents, is non-decreasing.
   allocation : $f_i(t) = \begin{cases} 1 & t_i > t_{-i}^{(2)} \\ 0 & t_i < t_{-i}^{(2)} \\ \alpha_i & t_i = t_{-i}^{(2)} \end{cases}$
   Where α_i is such that $\sum_i \alpha_i = 1$ and $\alpha_i \geqslant 0 \ \forall i \in N$, and $t_{-i}^{(2)} = \max_{j \neq i} t_j$. The characterization theorem says that a rule that breaks the tie in an arbitrary probabilistic way is DSIC. The allocation function can be seen as the subgradient of the utility function, given by:

   $$u(v_i) = \begin{cases} 0 & v_i \leqslant t_{-i}^{(2)} \\ v_i - p_i & v_i > t_{-i}^{(2)} \end{cases}$$

   Note that this function is not differentiable at $v_i = t_{-i}^{(2)}$ and therefore the subgradient at that point can be anything in $[0, 1]$. Therefore any $\alpha_i \in [0, 1]$ is a valid candidate for an implementable allocation.

   The payment rule (for the winning bidder, say player 1) can be written as:

   $$\begin{aligned} p(t_1, t_{-1}) &= 0 + t_1 \cdot f_1 - \int_{t_{-1}^{(2)}}^{t_1} f_1(x_1, t_{-1}) dx_1 \\ &= t_1 - (t_1 - t_{-1}^{(2)}) \\ &= t_{-1}^{(2)} \end{aligned} \tag{34.2}$$

3. **Efficient allocation with reserve**: The allocation here gives the bidder i the item if $t_i > \max(r, t_{-i}^{(2)})$ where r is the reserve price set by the auctioneer. The payment made by the winning bidder is $\max(r, t_{-i}^{(2)})$. Clearly, the allocation rule is non-decreasing. Here the item is not sold if no bid is higher than the reserve price.

4. **A not so common allocation rule**: Consider an allocation rule for two agents $N = \{1, 2\}$ and $A = \{a_0, a_1, a_2\}$ where the allocation a_0 refers to the item being unsold and a_i being the item alloted to player i. Give a type profile (t_1, t_2) the seller computes

   $$U(t) = \max\{2, t_1^2, t_2^3\} \tag{34.3}$$

The allocation proceeds as:

$$
\begin{aligned}
&a_0 \quad \text{if} \quad U(t) = 2 \quad \text{i.e.} \quad 2 > \max\{t_1^2, t_2^3\} \\
&a_1 \quad \text{if} \quad U(t) = t_1^2 \quad \text{i.e.} \quad t_1 > \sqrt{\max\{2, t_2^3\}} \\
&a_2 \quad \text{if} \quad U(t) = t_2^3 \quad \text{i.e.} \quad t_2 > \sqrt[3]{\max\{2, t_1^2\}}
\end{aligned} \tag{34.4}
$$

It is easy to see that the payments are zero, $\sqrt{\max\{2, t_2^3\}}$, and $\sqrt[3]{\max\{2, t_1^2\}}$ respectively for the three cases above.

34.3 Individual Rationality

Definition 34.3 (Individual Rationality) *A mechanism (f, p) is "ex-post"* individually rational *if*

$$t_i \cdot f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \geqslant 0 \quad \forall t_i \in T_i \quad \forall t_{-i} \in T_{-i} \quad \forall i \in N \tag{34.5}$$

Here "ex-post" refers to the idea that even after revealing everyone's type, participation is weakly preferable.

Lemma 34.4 *Suppose (f, p) is DSIC,*

1. *It is individually rational (IR) iff*

$$p_i(0, t_{-i}) \leqslant 0 \quad \forall i \in N \quad \forall t_{-i} \in T_{-i} \tag{34.6}$$

2. *It is IR and gives no subsidy, i.e. $p_i(t_i, t_{-i}) \geqslant 0\ \forall t_i \in T_i$ iff*

$$p_i(0, t_{-i}) = 0 \quad \forall i \in N \quad \forall t_{-i} \in T_{-i}. \tag{34.7}$$

Proof: We shall present the proof part by part, and both directions (if and only if) for each

1. **only if direction**: Assume

$$
\begin{aligned}
t_i \cdot f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) &\geqslant 0 \\
\text{for} \quad t_i = 0 \quad p_i(0, t_{-i}) &\leqslant 0
\end{aligned} \tag{34.8}
$$

if direction : Assume $p_i(0, t_{-i}) \leqslant 0$

$$t_i \cdot f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) - t_i \cdot f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i}) dx_i \geqslant 0 \tag{34.9}$$

Since in this case we assume $p_i(t_i, t_{-i}) \leqslant 0$, the above inequality holds.

2. **only if direction**: Assume the mechanism DSIC, IR and satisfies no subsidy hence,

$$p_i(0, t_{-i}) \geqslant 0 \tag{34.10}$$

but using the above proof, we have

$$
\begin{aligned}
&p_i(0, t_{-i}) \leqslant 0 \\
\Rightarrow\ &p_i(0, t_{-i}) = 0
\end{aligned} \tag{34.11}
$$

The other direction is obvious.

■

34.4 Some Non-Vickrey Auctions

1. **Redistribution**: Consider a case where the auction is Groves but not Vickrey auction. In this auction, the highest bidder wins and gets the object but the payment is such that everyone pays what they would have paid in a Vickrey auction, but is compensated some amount given by,

$$p_i(0, t_{-i}) = -\frac{1}{n} z_{-i}^{(2)} \tag{34.12}$$

where $z_{-i}^{(2)}$ = second highest among $\{t_j; j \neq i\}$. WLOG, assume that $t_1 > t_2 > \ldots > t_n$. So, we get

$$\begin{aligned}\text{payment of player 1} &= \frac{-1 \cdot t_3}{n} + t_2 \\ \text{payment of player 2} &= \frac{-1 \cdot t_3}{n} \\ \text{payment of every other agent} &= \frac{-1 \cdot t_2}{n} \\ \text{Hence, the sum of payment} &= \frac{2 \cdot (t_2 - t_3)}{n}\end{aligned}$$

The above expression tells us that such an auction is asymptotically budget balanced (as the surplus is redistributed as $n \to \infty$) while still being DSIC. The allocation however is still deterministic in nature and one can do something better by randomizing the allocation.

2. **Green-Laffont mechanism**: Give object to the highest bidder with probability $\left(1 - \frac{1}{n}\right)$, and to the second highest bidder with probability $\frac{1}{n}$. Say $t_1 > t_2 > \ldots > t_n$, and let

$$p_i(0, t_{-i}) = -\frac{1}{n} z_{-i}^{(2)} \tag{34.13}$$

where $z_{-i}^{(2)}$ = second highest among $\{t_j; j \neq i\}$

$$\text{payment of 1} = -\frac{1}{n} \cdot t_3 + \left(1 - \frac{1}{n}\right) \cdot t_1 - \frac{1}{n} \cdot (t_2 - t_3) - \left(1 - \frac{1}{n}\right) \cdot (t_1 - t_2) \tag{34.14}$$

$$\text{payment of 2} = \frac{-t_3}{n} + \frac{t_2}{n} - \frac{t_2 - t_3}{n} = 0 \tag{34.15}$$

$$\text{payment of every other agent} = -\frac{t_2}{n} \tag{34.16}$$

$$\text{Hence, the sum of payment} = \frac{n-2}{n} \cdot t_2 - (n-2) \cdot \frac{t_2}{n} = 0 \tag{34.17}$$

Hence such an allocation with the given payment mechanism is (exactly) budget balanced.