Introduction to Game Theory and Mechanism Design
Indian Institute of Technology Kanpur
Assignment Number 2

QUESTION 1

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Given,

$$\begin{aligned} \text{Number of agents,} \quad N &= \{1,2\} \\ \text{Set of alternatives,} \quad A &= \{a,b,c\} \\ \text{An SCF} \quad f &: ONTO \end{aligned}$$

P_1	P_2	P'_1	P'_2
a	c	b	a
b	b	a	b
c	a	c	c

$$f(P_1, P_2) = a$$

a. Unrestricted strict preference

Given f is Stategyproof.
For any preference profile $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$.

$$\therefore f(P'_1, P'_2) \in \{P'_1(1), P'_2(1) = \{a, b\}, \text{ as } P'_1(1) = b \text{ and } P'_2(1) = a$$

Lets assume a preference profile $(\hat{P}_1, \hat{P}_2)$, such that,

$\hat{P}_1$	$\hat{P}_2$
a	c
c	a
b	b

If we consider transition from $(P_1, P_2) \rightarrow (\hat{P}_1, \hat{P}_2)$, we can observe that $D(a, P_i) \subseteq D(a, \hat{P}_i), \forall i \in N$.

$$\therefore f(\hat{P}_1, \hat{P}_2) = a, \text{ From MONOTONICITY property} \qquad (1)$$

Next, we can have a preference profile $(P'_1, \hat{P}_2)$,

P'_1	$\hat{P}_2$
b	c
a	a
c	b

Let's assume, $f(P_1', \hat{P}_2) = c$, as $c \in \{P_1'(1), \hat{P}_2(1)\}$.
If we consider transformation from $(P_1', \hat{P}_2) \rightarrow (\hat{P}_1, \hat{P}_2)$, we can observe that $D(c, (P_1', \hat{P}_2)) \subseteq D(c, (\hat{P}_1, \hat{P}_2))$,

$$\implies f(\hat{P}_1, \hat{P}_2) = c, \text{ From MONOTONICITY property} \tag{2}$$

But, from equation 1 we know that $f(\hat{P}_1, \hat{P}_2) = a$. This contradicts our assumption that $f(P_1', \hat{P}_2) = c$.

$$\therefore f(P_1', \hat{P}_2) = b \tag{3}$$

Now, let's consider the transition from $(P_1', \hat{P}_2) \rightarrow (P_1', P_2')$, we can observe that $D(b, (P_1', \hat{P}_2)) \subseteq D(b, P_1', P_2')$.

$$\therefore f(P_1', P_2') = b, \text{ From MONOTONICITY property} \tag{4}$$

This concludes our proof that $f(P_1', P_2') = b$.

b. Single peaked prefrences

No, the conclusion of part a does not hold in this case,i.e.,

$$f(P_1', P_2') \neq b, \text{ instead } f(P_1', P_2') = a$$

This is because if we try to prove the same in this case we can not have a preference profiles with two peaks due to single peakedness restriction, eg. $\hat{P}_1 : aP_1cP_1b$, this has two peaks. By using remaining profiles, which follows single peakedness we can not prove that $f(P_1', P_2') = b$. The preference profiles that we used to show the solution are not feasible profiles in this case and by using the remaining profiles we cannot show the same as we did in unrestricted strict preference. Only remaining acceptable profile is $b < c < a$, on preference.

Let us take Median Voter SCF as mechanism in this case as Median Voter SCF is both Strategyproof and ONTO.
Given that intrinsic ordering of alternatives as: $a < b < c$.
We can choose B : set of peaks of phantom voters such that it satisfies,

$$f(P_1, P_2) = a$$

Let's say, $B = \{y\} : y \leq a$. The median of $\{y, a, b\}$ is a.

$$\therefore f(P_1, P_2) = a, \text{This is satisfied}$$

We can see that,

$$P_1'(1) = b \text{ and } P_2'(1) = a$$

For given $B = \{y\}$, the median of $\{y, a, b\}$ is a. So, due to Median Voter SCF rule,

$$f(P_1', P_2') = a$$

Hence, this mechanism shows that $f(P_1', P_2') = a$.

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QUESTION 2

Given,

$$X : \text{Set of projects, } |X| \geq 2$$

$$P_i : \text{Linear ordering over the set of projects } X \text{ for agent i}$$

$$A : \text{Set of alternatives.}$$

$$A = 2^{|X|}/\phi$$

$$f : \text{An SCF such that it chooses a non empty subset of projects}$$

Claim 1. Gibbard-Satterthwaite result does not apply to SCF f.

Proof : Gibbard-Satterthwaite is applied on settings which has unrestricted domain. If we prove that current setting is restricted domain, then Gibbard-Satterthwaite will not apply here.
Let's assume, $X = a, b$.

$$\therefore A = \{\{a\}, \{b\}, \{a, b\}\}$$

Here, $|A| \geq 3$, is satisfied as Gibbard-Satterthwaite theorem requires this condition.

The output set $\{a\}$ can never be preferred over $\{a, b\}$ as they are indifferent. Same for set $\{b\}$. This puts the restriction over the ordering of alternatives preferred over the set A, i.e., no agent i can have preference ordering over set A where $\{a\}$ is preferred over $\{a, b\}$. i.e., only possible orderings of alternatives are,

$$\begin{array}{ccccc} \{\{a,b\} & I_i & \{a\}\} & P_i & \{b\} \\ \{\{a,b\} & I_i & \{b\}\} & P_i & \{a\} \end{array}$$

For an unrestricted domain preferences, any ordering of alternatives should be possible.
$\therefore$ In this case it does not follow unrestricted preference domain.

This can be extended to any number of projects where, if

$$S \subset T, \text{ where } S, T \subseteq X, \forall |X| \geq 2$$

S can never be preferred over T. Which is nothing but domain restriction on ordering of alternatives.

Hence, Gibbard-Satterthwaite theorem does not apply here since f does not follow unrestricted preferences.

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QUESTION 3

Given that,

Single Peaked Domain model

N : Numbe of Agents

$\mathcal{S}$: Preference profiles following single peaked domain

P : Set of prefrence profiles of all agents, $P \in \mathcal{S}^n$

P_K : Preference profiles of group of agents, $K \subseteq N$

P_{-K} : Preference profiles of group of agents not in K

$P_i(1)$: Peak of agent i

Claim 2. Median Voter SCF f is group strategy-proof.

Note: This is not second claim for this question, instead it is first claim but in whole document it is second.
Proof : We know that Median Voter SCF is strategy-proof if only one agent deviates. We will need to proof in the following 3 cases that Median Voter SCF is group strategy-proof:

1. $f(P) = \min\{P_i(1)\}, \forall i \in N$
2. $f(P) = \max\{P_i(1)\}, \forall i \in N$
3. $f(P) = k^{th}$ peak in intrinsic ordering of the peaks $P_i(1) \forall i \in N$

Case 1 : $\boldsymbol{f(P) = \min\{P_i(1)\}, \forall i \in N}$

We have set of phantom peaks, $B = \{y_1, y_2, \ldots, y_{N-1}\}$ such that,

$$f(P) = a, \text{ where } a = \min P_i(1), \forall i \in N$$

$$y_i \leq a, \forall y_i \in B, \text{ as this will give } a \text{ as median}$$

Now, let's say we have a group $K \subseteq N$, such that $P_i(1) \neq a, \forall i \in K$ and from our assumption $a < P_i(1) \forall i \in K$ in intrinsic ordering.
Now the only way the median can be changed is when agents in K change there peaks to the left of a. Lets say the new set of preference profile is P' with changed peaks of group of agents in K.
Then, $f(P') = x, x < a$. The intrinsic ordering will follow,

$$f(P') < a < P_i(1), \forall i \in K$$

We can observe that the current output of Median Voter SCF is less preferred by the group of Agents $i \in K$, as the domain of preference is single peaked, due to which the alternative far from the peak will be less preferred.

Therefore, the group of agents will not change their peaks as after changing the output of median Voter SCF is less preferred than the current output of Median Voter SCF.

Case 2 : $\boldsymbol{f(P) = \max\{P_i(1)\}, \forall i \in N}$

We have set of phantom peaks, $B = \{y_1, y_2, \ldots, y_{N-1}\}$ such that,

$$f(P) = a, \text{ where } a = \max P_i(1), \forall i \in N$$

$$y_i \geq a, \forall y_i \in B, \text{ as this will give } a \text{ as median}$$

Now, let's say we have a group $K \subseteq N$, such that $P_i(1) \neq a, \forall i \in K$ and from our assumption $a > P_i(1) \forall i \in K$ in intrinsic ordering, i.e., it is the rightmost peak.
Now the only way the median can be changed is when agents in K change there peaks to the right of a. Lets say the new set of preference profile is P' with changed peaks of group of agents in K.
Then, $f(P') = x, x > a$. The intrinsic ordering will follow,

$$P_i(1) < a < f(P'), \forall i \in K$$

We can observe that the current output of Median Voter SCF is less preferred by the group of Agents $i \in K$, as the domain of preference is single peaked, due to which the alternative far from the peak will be less preferred.

Therefore, the group of agents will not change their peaks as after changing the output of median Voter SCF is less preferred than the current output of Median Voter SCF.

Case 3 : $\boldsymbol{f(P) = k^{th}}$ peak in intrinsic ordering of the peaks $\boldsymbol{P_i(1) \forall i \in N}$

Case 3.1 : When the group of agents having peaks on the right of $f(P)$, we can show by similar arguments as in case 1 that even after changing their peaks the median voter SCF output is less preferred than the original output, i.e., $f(P) P_i f(P')$.

Case 3.2 : When the group of agents having peaks on the left of $f(P)$, we can show by similar arguments as in case 2 that even after changing their peaks the median voter SCF output is less preferred than the original output, i.e., $f(P) P_i f(P')$.

Case 3.3 : When group of agents on both sides will try to change their peaks to change Median Voter SCF output.

If equal number of agents move on other side of $f(P)$ from their side, then the median will still be the same.

Let's assume an agent having peak on the left of $f(P)$ in intrinsic ordering change his

peak to close to $f(P)$ but still on same side,i.e., , $P'_l(1) = x, x < a$ and an agent having peak on the right side of $f(P)$ change his peak to the left of $f(P)$ and left to the peak of the agent who moved closer to $f(P)$ in intrinsic ordering,i.e., $P'_r(1) = z, z < x < a$. Lets consider new set of preference as P'. Here, $K = \{l, r\}$.
Now, in current scenario the new median will be x, therefore, $f(P') = x$. This is more preferred to agent l than $f(P) = a$ as it is closer to $P_l(1)$ which is due to single peaked property. But, the $f(P')$ is less preferred to agent r than $f(P) = a$ as it is farther to $P_r(1)$.

Therefore, agent r will not prefer changing his peak to $P_r(1) = z, z < x < a$.
This can be extended to multiple agents from either side trying to change the median, which will not give output preferred by all the agents in K than the original output.

Similar arguments can be used to show that it wont be preferred deviation if agents together try to shift median to the right of $f(P)$ by changing their peaks.

Hence, Median Voter SCF is group strategy proof.