Theorem (Nash 1951)

Every finite game has a (mixed) Nash equilibrium.

\# of players and strategies are finite.

Use a result from real analysis:

- Background:

A set $S \subseteq \mathbb{R}^n$ is convex if $\forall\, x, y \in S$, and $\forall\, \lambda \in [0,1]$
$$\lambda x + (1-\lambda) y \in S.$$

A set $S \subseteq \mathbb{R}^n$ is closed if it contains all its limit points. → points whose each neighborhood contains a point in S.

is bounded if $\exists\, x_0 \in \mathbb{R}^n$ and $R \in (0,\infty)$ s.t. $\forall\, x \in S$ $\|x - x_0\|_2 < R$.

A set $S \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

Brouwer's Fixed Point Theorem

If $S \subseteq \mathbb{R}^n$ is a convex and compact set and $T: S \to S$ is continuous, then T has a fixed point, i.e. $\exists$ a point x^* s.t. $T(x^*) = x^*$.

Proof of Nash theorem

define simplex $\Delta_k = \{ x \in \mathbb{R}^{k+1}_{\geq 0} : \sum_{i=1}^{k+1} x_i = 1 \}$

Consider two players (n-players case is an extension of the idea)

Player 1 : m strategies $1 \to m$

Player 2 : n strategies $1 \to n$

Player 1's (mixed) strategy is a point in Δ_{m-1}

Player 2's - - - - - - - Δ_{n-1}

Set of mixed strategy profiles $= \Delta_{m-1} \times \Delta_{n-1}$

For two players, the utilities can be expressed in terms of two matrices A and B, if $p \in \Delta_{m-1}$ and $q \in \Delta_{n-1}$ are the mixed strategies, then

$$u_1(p,q) = p^T A q, \quad u_2(p,q) = p^T B q$$

(6-2)

$$c_i(p,q) = \max\{A_i q - p^T A q, 0\} \geq 0 \qquad A_i : i^{th} \text{ row of } A$$

$$d_j(p,q) = \max\{p^T B_j - p^T B q, 0\} \geq 0 \qquad B_j : j^{th} \text{ col of } B$$

$$P_i(p,q) = \frac{p_i + c_i(p,q)}{1 + \sum_{k=1}^{m} c_k(p,q)} \; ; \; Q_j(p,q) = \frac{q_j + d_j(p,q)}{1 + \sum_{k=1}^{n} d_k(p,q)}$$

clearly, $P(p,q) \in \Delta_{m-1}$ and $Q(p,q) \in \Delta_{n-1}$

$$T(p,q) = (P(p,q), Q(p,q))$$

$T: \Delta_{m-1} \times \Delta_{n-1} \to \Delta_{m-1} \times \Delta_{n-1}$ ← convex and compact.

c_i, d_j's are continuous $\Rightarrow$ P_i's Q_j's cont. $\Rightarrow$ T is cont.

$\exists$ (p^*, q^*) s.t. $T(p^*, q^*) = (p^*, q^*)$ [Brouwer's]

<u>Claim</u>: $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$, $\sum_{k=1}^{n} d_k(p^*, q^*) = 0$

pf: suppose not, $\sum_{k=1}^{m} c_k(p^*, q^*) > 0$

since (p^*, q^*) is a fixed point of T

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \sum_{k=1}^{m} c_k(p^*, q^*)} \Rightarrow p_i\left(\sum_{k=1}^{m} c_k(p^*, q^*)\right) = c_i(p^*, q^*)$$

$$I = \{i : p_i^* > 0\} = \{i : c_i(p^*, q^*) > 0\} = \{i : A_i q^* > \underbrace{p^{*T} A q^*}_{=: u_1^*}\}$$

$$u_1^* = \sum_{i=1}^{m} p_i^* A_i q^* = \sum_{i \in I} p_i^* A_i q^* > \left(\sum_{i \in I} p_i^*\right) u_1^* = u_1^* \rightarrow\leftarrow$$

hence $\sum_{k=1}^{m} c_k(p^*, q^*) = 0 \Rightarrow c_k(p^*, q^*) = 0 \;\; \forall k = 1, \ldots, m$

$$\Rightarrow \quad A_i q^* \leq p^{*T} A q^* \Rightarrow \sum_{i=1}^{m} p_i' A_i q^* \leq p^{*T} A q^*$$

similarly for d_j and q^* $\Rightarrow (p^*, q^*)$ is a MSNE ■