Solution concepts - Questions for cooperative games

1. What coalitions will form? If a coalition S is formed, how does it divide the worth $v(S)$ among its members
2. What would a trusted mediator/arbitrator recommend to the players?

- Answer to 1 is hard. We will assume that the grand coalition forms (and find conditions when it is likely to form), and then ask how agents will divide the grand worth among themselves in a rational way.
- On the other hand, recommendation by an arbitrator is a point solution.

Imputation: (Share of the valuation among the players)

Defn: An <u>imputation</u> $x \in \mathbb{R}^n$ is a share of the players that satisfies

1. $x_i \geq v(\{i\}) \quad \forall i \in N$ ... individually rational
2. $\sum_{i \in N} x_i = v(N)$. — grand coalitionally rational.

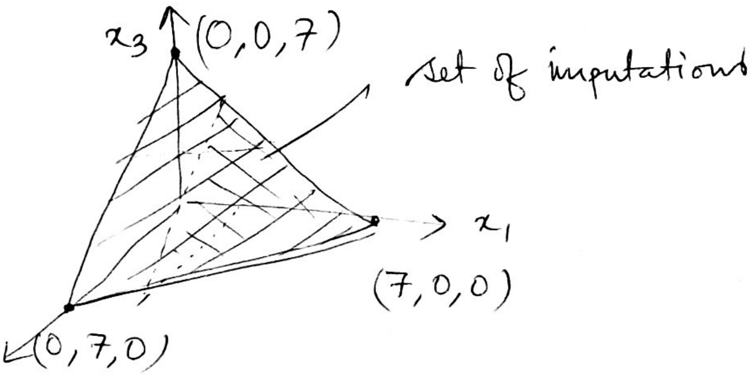
imputations are guaranteed?

<u>Example</u>: $N = \{1, 2, 3\}$

$v(1) = v(2) = v(3) = 0$

$v(1,2) = 2 \quad v(1,3) = 3, \; v(2,3) = 4, \; v(1,2,3) = 7$

if $v(1) = 2$



Defn:

An allocation $x \in \mathbb{R}^n$ is <u>coalitionally rational</u> if $\sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N$.

Note: this implies individual rationality.

Core:

Defn: An imputation is in the core if it is coalitionally rational, i.e.

① $\sum_{i \in S} x_i \geq v(S) \quad \forall S \subsetneq N$

② $\sum_{i \in N} x_i = v(N)$.

For the previous example

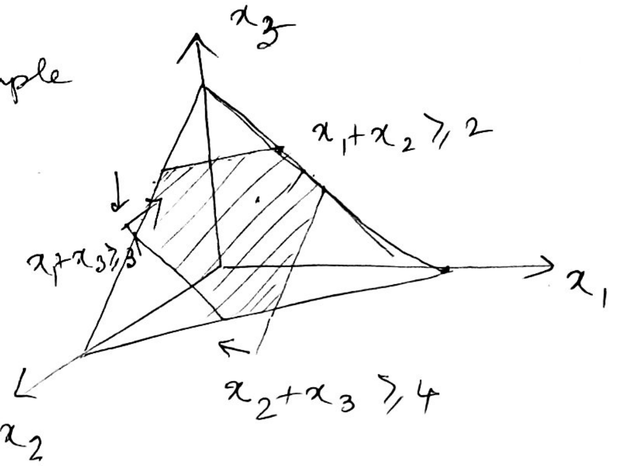
$x_1 + x_2 + x_3 = 7$

$x_1 + x_2 \geq 2$

$x_2 + x_3 \geq 4$

$x_1 + x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$



Core is a polytope.

Core for the previous examples

① DTD - ver 1:

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 300, x_1, x_2, x_3 \geq 0\}$$

ver 2: $C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 300, x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$.

ver 3:

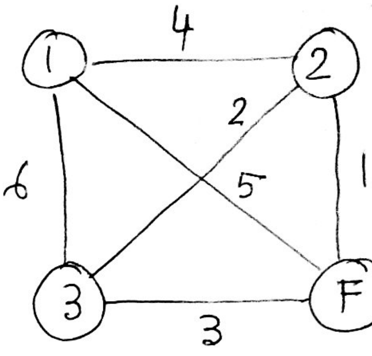
$$\left.\begin{array}{l} x_1 + x_2 + x_3 = 300 \\ x_1 + x_2 \geq 300 \\ x_1 + x_3 \geq 300 \\ x_1, x_2, x_3 \geq 0 \end{array}\right\} \Rightarrow x_2 = x_3 = 0$$

$(300, 0, 0)$ is the only point in the core.

ver 4:

$$\left.\begin{array}{ll} x_1 + x_2 + x_3 = 300 & \\ x_1 + x_2 \geq 300 & x_2 + x_3 \geq 300 \\ x_1 + x_3 \geq 300 & x_1, x_2, x_3 \geq 0 \end{array}\right\} \quad C(N, v) = \phi.$$

(2) Core for MST game



$v(1) = 5$ | $v(12) = 15$

$v(2) = 9$ | $v(13) = 12$

$v(3) = 7$ | $v(23) = 17$

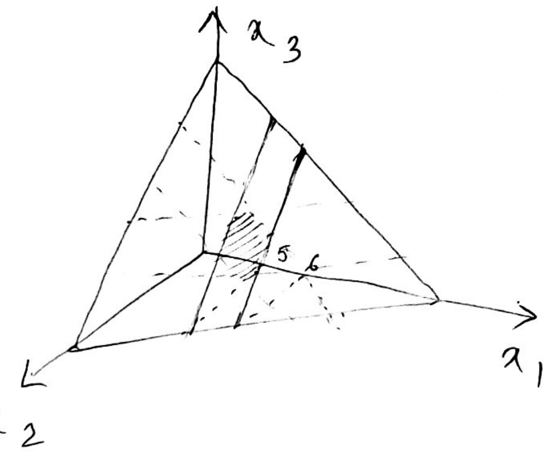
$v(123) = 23.$

$x_1 \geq 5$, $x_2 \geq 9$, $x_3 \geq 7$ $\qquad x_1 + x_2 + x_3 = 23$

$x_1 + x_2 \geq 15 \Rightarrow x_3 \leq 8$

$x_2 + x_3 \geq 17 \Rightarrow x_1 \leq 6$

$x_1 + x_3 \geq 12 \Rightarrow x_2 \leq 11$



uncountably many points in the core.

Exercise: core of the bankruptcy game.

For all such TU games, grand coalition is rational only if core is non-empty.

<u>Balanced collection of coalitions</u>

Consider $N = \{1, 2, 3\}$, an imputation x is in core iff the following holds:

$$\begin{aligned}
x_1 + x_2 + x_3 &= v(123) \\
x_1 + x_2 &\geq v(1,2) \\
x_1 + x_3 &\geq v(13) \\
x_2 + x_3 &\geq v(23) \\
x_1 &\geq v(1) \\
x_2 &\geq v(2) \\
x_3 &\geq v(3)
\end{aligned}$$

necessary condition means that we must have a solution for this system of inequalities, accordingly there will be restrictions on v.

Combining:

$$v(123) \geq v(1) + v(2) + v(3) \quad \text{--- } (1)$$

$$v(123) \geq v(12) + v(3) \quad \text{-- } (2)$$

$$v(123) \geq v(13) + v(2) \quad \text{-- } (3)$$

$$v(123) \geq v(23) + v(1) \quad \text{--- } (4)$$

$$v(123) \geq \tfrac{1}{2} v(12) + \tfrac{1}{2} v(13) + \tfrac{1}{2} v(23) \quad \text{-- } (5)$$

<u>necessary</u>

It can also be proved (needs work) that these are sufficient conditions for a 3-player game to have a non-empty core. (exercise).

Goal: generalize this for any number of players.

Collection of coalitions	coefficients on the RHS
$\{\{1\},\{2\},\{3\}\}$	1, 1, 1
$\{1,2\},\{3\}$	1, 1
$\{1,3\},\{2\}$	1, 1
$\{2,3\},\{1\}$	1, 1
$\{1,2\},\{2,3\},\{1,3\}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

consider 1–4, close to superadditivity but not quite, since this holds only for the grand coalition.

Incidence matrix $= I$ coefficients $= c$

$$\begin{array}{c} \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \\ 1 \end{array}$$

matrix rows are coalitions, columns are players

$$\begin{array}{c} \{1,2\} \\ \{3\} \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \end{array}$$

$$\begin{array}{c} \{1,3\} \\ \{2\} \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \end{array}$$

$$\begin{array}{c} \{2,3\} \\ \{1\} \end{array} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{array}{c} 1 \\ 1 \end{array}$$

$$\begin{array}{c c c}
\{1,2\} & \multirow{3}{*}{$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$} & 1/2 \\
\{1,3\} & & 1/2 \\
\{2,3\} & & 1/2
\end{array}$$

Observe that $C^T I = (1,1,1)$ in all cases.

A collection of coalitions that has a vector of positive coefficients satisfying this property is called a <u>balanced collection</u>. The coefficients are called balanced coefficients.

<u>Defn</u>: A collection of coalitions $\mathcal{D}$ is a balanced collection if $\exists$ a vector of positive numbers $(\delta_S)_{S \in \mathcal{D}}$ s.t.

$$\sum_{\{S \in \mathcal{D} : i \in S\}} \delta_S = 1, \quad \forall i \in \mathbb{N}.$$

$(\delta_S)_{S \in \mathcal{D}}$ is a vector of balancing weights for that collection of coalitions. If positive is replaced with nonnegative - weakly balanced collection/weights.

Balancing weights are like soft partition of every individual.

From the example, say a collection $\mathcal{D}_1 = \{\{1\},\{2\},\{3\}\}$ is balanced $\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = 1$

& $\mathcal{D}_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$ is balanced

$$\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = 1/2$$

Consider $\mathcal{D}_1 \cup \mathcal{D}_2 = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$

$$\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = \lambda \qquad \delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \tfrac{1}{2}(1-\lambda)$$

$\lambda \in [0,1]$ for balanced.

Claim: If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two balanced collections, then their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is also a balanced collection.

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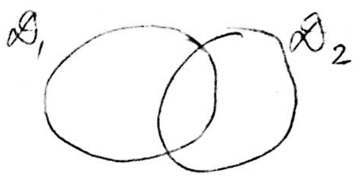
$$\sum_{S \in \mathcal{D}_1 \cup \mathcal{D}_2 : i \in S} \delta_S \overset{?}{=} 1 \quad , \text{find } \delta_S$$



Given $\exists\, \delta_S^{(1)} \ \forall S \in \mathcal{D}_1$ s.t. $\sum_{S \in \mathcal{D}_1 : i \in S} \delta_S^{(1)} = 1$

and $\exists\, \delta_S^{(2)} \ \forall S \in \mathcal{D}_2$ s.t. $\sum_{S \in \mathcal{D}_2 : i \in S} \delta_S^{(2)} = 1$

$$\sum_{S \in \mathcal{D}_1 \cup \mathcal{D}_2 : i \in S} \delta_S = \underbrace{\sum_{S \in \mathcal{D}_1 \setminus \mathcal{D}_2 : i \in S} \delta_S}_{\lambda \delta_S^{(1)}} + \underbrace{\sum_{S \in \mathcal{D}_2 \setminus \mathcal{D}_1 : i \in S} \delta_S}_{(1-\lambda)\delta_S^{(2)}} + \underbrace{\sum_{S \in \mathcal{D}_1 \cap \mathcal{D}_2 : i \in S} \delta_S}_{\lambda \delta_S^{(1)} + (1-\lambda)\delta_S^{(2)}}$$

$$= \lambda \sum_{S \in \mathcal{D}_1 : i \in S} \delta_S^{(1)} + (1-\lambda) \sum_{S \in \mathcal{D}_2 : i \in S} \delta_S^{(2)} = 1.$$

Consider the collection $\mathcal{D} = \{\{1,2\},\{1,3\}\}$

$\delta_{\{1,2\}}, \delta_{\{1,3\}}$ for 1: $\delta_{\{1,2\}} + \delta_{\{1,3\}} = 1$

for 2: $\delta_{\{1,2\}} = 1$, for 3: $\delta_{\{1,3\}} = 1$

infeasible, i.e., $\mathcal{D}$ is not a balanced collection.

$\mathcal{D} = \{\{1,3\},\{2,3\},\{1\}\}$ is weakly balanced but not ~~the~~ balanced collection.

$\delta_{\{1,3\}} + \delta_{\{1\}} = 1$

$\left.\begin{array}{l} \delta_{\{1,3\}} + \delta_{\{2,3\}} = 1 \\ \delta_{\{2,3\}} = 1 \end{array}\right\} \delta_{\{1,3\}} = 0, \ \delta_{\{1\}} = \delta_{\{2,3\}} = 1$

Relationship with the core

Theorem [Bondareva '63, Shapley '67] The necessary and sufficient condition for a coalitional game (N, v) to have a non-empty core is that for every balanced collection $\mathcal{D}$ of coalitions, and every vector of balancing weights $(\delta_S)_{S \in \mathcal{D}}$

$$v(N) \geq \sum_{S \in \mathcal{D}} \delta_S v(S)$$

The condition is also called balanced condition - and the game as balanced game.
An alternative statement says ~~the~~ that ~~game~~ a game has non-empty core iff the game is balanced.

Remark: ① The theorem holds even when balancedness condition is relaxed to weakly balanced, because the inequality holds for every balanced collection iff it holds for every weakly balanced collection.

② BS theorem is useful to find a counterexample of non-~~empty~~ empty core, also for guaranteeing non-empty core for a class of games.
- it is not very useful to find the core of a game.

To prove we will need <u>LP duality</u>.
A special set of balanced weights where

$\mathcal{D}^* = 2^N$ all subsets of N.

and call that set of balanced weights λ^*, balanced* weights, i.e.,
$$\sum_{\{S \subseteq N : i \in S\}} \lambda^*(S) = 1 \quad \forall i \in N.$$

EX: show that $\mathcal{D}^*$ is a balanced collection.

<u>Thm:</u> (BS, second formulation)
A ~~coalitional~~ TU game (N, v) has a non-empty core iff for all balanced* weights λ, we have
$$v(N) \geq \sum_{S \subseteq N} \lambda(S)\, v(S).$$

Check: DTD - ver 4 (majority)
$v(i) = 0 \ \forall i = 1,2,3$, $v(\{i,j\}) = 300$ $i \neq j$, $i,j \in N$, $v(123) = 300$
$\lambda(12) = \lambda(23) = \lambda(13) = 1/2$ → counterexample.