CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava.cse.iitk.ac.in.*

30.1 Recap

In the previous lecture, we proved that VCG is individually rational for allocation of goods. We also looked at internet advertising and position auctions to sell multiple ads on a webpage. The winner determination problem for position auction is restated for the purpose of this lecture.

30.2 Winner Determination Problem (continued)

Definition 30.1 (Winner Determination Problem (WDP)) *The optimal allocation x^* of the slots should be determined such that the sum of the reported valuations of the agents is maximized, i.e.,*

$$x^* = \arg\max_x \sum_{i \in N} \hat{v}_i(x). \tag{30.1}$$

We now prove a result on the winner determination problem.

Theorem 30.2 *If an allocation solves the winner determination problem, then it must be a rank-by-revenue mechanism.*

Recall that a rank-by-revenue assigns the slots according to the decreasing order of $eCTR_i \cdot b_i$, where $eCTR_i$ is the estimated quality of agent i's ad and b_i is his bid.

Proof: Assume for contradiction that allocation x is optimal, i.e., solves the WDP (Eq. 30.1) and bids of agents 1 and 2 are such that, $eCTR_1 \cdot b_1 > eCTR_2 \cdot b_2$, but, 2 is placed above 1. WLOG assume $x_2 = 1$ and $x_1 = 2$. Consider a different allocation x' with all the agents except 1 and 2 getting the same position, and $x'_1 = 1$ and $x'_2 = 2$. Therefore the social welfare is given by

$$\sum_{i \in N} \hat{v}_i(x') = pos_1(eCTR_1.b_1) + pos_2(eCTR_2.b_2) + \sum_{k \neq 1,2} pos_{x_k}(eCTR_k.b_k) \tag{30.2}$$

$$\sum_{i \in N} \hat{v}_i(x) = pos_2(eCTR_1.b_1) + pos_1(eCTR_2.b_2) + \sum_{k \neq 1,2} pos_{x_k}(eCTR_k.b_k) \tag{30.3}$$

Subtracting equation 30.3 from 30.3, we get

$$\sum_{i \in N} \hat{v}_i(x') - \sum_{i \in N} \hat{v}_i(x) = (pos_1 - pos_2)(eCTR_1 \cdot b_1 - eCTR_2 \cdot b_2).$$

As both factors in the RHS are positive, $\sum_{i\in N} \hat{v}_i(x') - \sum_{i\in N} \hat{v}_i(x) > 0$, which is a contradiction to x being optimal. This concludes the proof of the theorem. ■

Note: An advantage to be noted here is that the winner determination problem is poly-time.

30.3 VCG in Position Auction

After picking the efficient allocation (which is the solution of the WDP), we need payments to implement it in DSIC. The natural candidate for this is VCG payment. VCG payment is used by Twitter, Facebook for sponsored ads.

Given the bids, $(b_1, b_2, \ldots, b_n)$, ordered WLOG such that

$$eCTR_1 \cdot b_1 \geqslant eCTR_2 \cdot b_2 \geqslant \ldots \geqslant eCTR_n \cdot b_n.$$

Hence, the efficient allocation x^* is such that $x^*_i = i$ for $i = 1, 2, \ldots, n$. The payment used is VCG.

Define $x^*_{-i} \in \arg\max_x \sum_{j\neq i} \hat{v}_i(x)$. Now the payment of agent i according to VCG is given by

$$\begin{aligned} p_i^{VCG}(b) &= \sum_{j\neq i} \hat{v}_i(x^*_{-i}) - \sum_{j\neq i} \hat{v}_i(x^*) \\ &= \sum_{j=i}^{n-1} pos_j(eCTR_{j+1} \cdot b_{j+1}) - \sum_{j=i}^{n-1} pos_{j+1}(eCTR_{j+1} \cdot b_{j+1}) \end{aligned}$$

Therefore

$$p_i^{VCG}(b) = \begin{cases} \sum\limits_{j=i}^{n-1} (pos_j - pos_{j+1})(eCTR_{j+1}.b_{j+1}) & i = 1, 2, \ldots, n-1 \\ 0 & i = n \end{cases}$$

This gives us the total expected payment. To convert to payment per click, we need to normalize with $pos_i \cdot eCTR_i$. So payment per click $= \frac{1}{pos_i \cdot eCTR_i}\, p_i^{VCG}(b)$.

Observation: allocation w.r.t. rank-by-revenue and payment w.r.t. VCG is DSIC if the $eCTR$'s are accurate.

30.4 Generalized Second Price (GSP) in Position Auction

Generalized second price auction is a mechanism used by certain search engines e.g., Google, Bing etc. The allocation rule is same as rank-by-revenue, but the payment is such that every agent pays her next highest expected bid. This payment method is simple and easy to explain to advertisers, and has similarities with second price auction but also has serious limitations. Consider the following example

Expected revenue before position effect	v_i	$eCTR_i$	pos_j
2	10	0.2	1
4	8	0.5	0.2
4.2	6	0.7	0.1

Both VCG and GSP allocate the slots to solve the winner determination problem (i.e., rank-by-revenue allocation) with slot 1 going to player 3, slot 2 to player 2 and slot 3 to player 1.

For VCG, payment of slot 1 (for player 3) is $p_i^{VCG}(b) = \sum_{j=i}^{n-1}(p_j - p_{j+1})(eCTR_{j+1} \cdot b_{j+1}) = 0.8 \times 4 + 0.1 \times 2 = 3.4$.

Similarly payment for slot 2 (player 2) $= 0.1 \times 2 = 0.2$. Payment for slot 3 (player 1) is equal to 0.

Utility of slot 1 (player 3) $= 4.2 \times 1 - 3.4 = 0.8$.

Utility of slot 2 (player 2) $= 4 \times 0.2 - 0.2 = 0.6$.

Utility of slot 3 (player 1) $= 0.1 \times 2 + 0 = 0.2$.

For GSP:
Payment for slot 1 (player 3) $= 4 \times 0.2 = 0.8$. Payment for slot 2 (player 2) $= 2 \times 0.1 = 0.2$. Utility for Slot 1 $= 4.2 \times 1 - 0.8 = 3.4$. Utility for Slot 2 $= 4 \times 0.2 - 0.2 = 0.6$.

An interesting case to consider is if player 2 overbids. Say player 2 bids 8.6 to change the allocation. So $b_2 = 8.6$. In this case, the payment under VCG becomes $= 0.8 \times 4.2 + 0.1 \times 2 = 3.56$. The utility of player 2 (now assigned slot 1) $= 4 - 3.56 = 0.44$. We see that utility has decreased for player 2. This is expected because VCG is DSIC.

Under GSP, payment of player 2 $= 4.2 \times 0.2 = 0.84$. Utility of player 2 $= 4 \times 1 - 0.84$, which is more than that when she reports truthfully. This shows that GSP is em not DSIC.

30.5 Desirable properties of VCG

1. VCG is DSIC. Hence there is very low cognitive load on bidders.
2. VCG never runs into deficits (in some settings). It charges the marginal contribution to the other agents as payments and gets its own marginal contribution as payoff.
3. VCG never charges a losing agent.
4. It is individually rational for every agent to participate.