CS698W: Game Theory and Collective Choice	**Jul-Nov 2017**

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8.1 Recap

Extensive form game is used to represent the finite sequential games. It can always be transformed into normal form game but doing so doesn't have the practical efficiency since the payoff are repeated in the induced normal form game and are proportional to the pure strategies available to reach to that particular payoff in the extensive form game. Also the normal form game can't be converted back to the perfect information extensive form game when they are played simultaneously. This shows that normal form game is a richer form of representation yet not Pareto superior.

Theorem 8.1 *Every finite perfect information extensive form game has a pure strategy Nash equilibrium.*

Intution: After every stage of the game, the player already has information about the action taken by the former player hence there is no logic in randomizing and mixing the strategies. Moreover the player also knows that the next player will be observing his action before playing hence the randomness doesn't lead to attain an equilibrium which is otherwise impossible. This will be obvious after the discussion of subgame perfect result and backward induction.

Examples of PIEFG: Chess, Tic-Tac-Toe,Bargaining.

Zermelo (1913) showed using an argument similar to EFG that if both the players in chess are infinitely rational and intelligent, chess must be a very boring game.

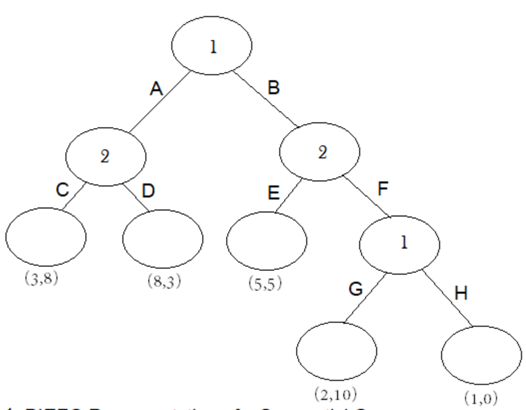


Fig:8.1 PIEFG Representation of a Sequential Game

In the given perfect information extensive normal game $\{(BH),(CE)\}$ is indeed a pure strategy Nash equilibrium but it is not so intitutive. Player 1 at his second choice node plays H instead of G whereas he will get more utility if he will play G. This suspicious behaviour of player 1 is termed as threat. Player 1 instead of optimising his own utility by choosing G gives more priority to pull down the utility of player 2 from 10 to 0. This activity of player 1 doesn't have any effect on equilibrium with the given scenario. But if player 2 doesn't consider the threat of player 1 credible and chooses F, will player 1 still choose H over G ?

Since the equilibrium $\{(BH), (CE)\}$ seems ambiguous. Therefore there is a need to refine the concept of equilibrium through defining the notion of a subgame to avoid such equilibrium inclusive of non-credible threat.

Definition 8.2 ***Subgame****: The extensive form game represented by the subtree at a node is called the subgame at that node.*

Definition 8.3 ***Subgame-Perfect Nash Equilibrium****: The subgame-perfect equilibria (SPE) of a game G are all strategy profiles s such that for any any subgame G' of G the restriction of s to G' is a Nash equilibrium of G'.*

Example: $\{(BH), (CE)\}$ is not a SPNE. Consider the subgame rooted at second decision node of player 1. The unique Nash equilibrium for the player 1 is to play G instead of H. Hence $\{(BH), (CE)\}$ is not a SPNE.

8.2 Computing SPNE: Backward Induction Algorithm

1. Start at the leaf having maximum depth.
2. For the player in the parent node find the action which maximises the utility for that player.
3. Retain that action and delete all the edges at that level, translate the utilities to the parent node.
4. Go up one level and repeat (1).
5. Stop if root is reached.

The algorithm is implemented as a single depth first traversal of the game tree. It identifies the strategy which has higher pay-off for the player in the bottom-most subgame tree and removes the other counter actions and updates the pay-off of the parent node and repeats the complete process again. The algorithm terminates when the action with higher pay-off among all the possible actions for the player at root is found out. In this manner the algorithm gives SPNE.

The algorithm is far much better than the present algorithms for finding out the Nash equilibria which requires exponential time in the size of normal form. Also the algorithm gives the SPNE for sure.

For complicated games like chess the backward induction algorithm is not efficient. Extensive form representation of chess has 10^{150} nodes and it is not feasible to apply backward induction brutally. Game softwares uses heuristic pruning for computer players and doesn't consider the parts of the game tree which can never be the candidates for equilibrium.

Exercise: Verify that $\{(AG), (CF)\}$ is the SPNE of the previous game.

Note: SPNE is always a PSNE but the reverse is not true.

8.3 Limitations of SPNE

Centipede Game: In this game two players makes alternate decisions, at each turn choosing between going down and ending the game or going across and continuing it except at the last node where going across also ends the game. The payoffs are constructed in such a way that the player achieves higher payoffs by choosing "down".Consider the last choice, at that point the best choice for the player is to go down. Going down is also the best choice for the other player in the previous choice point. By induction the same argument holds for all choice points.

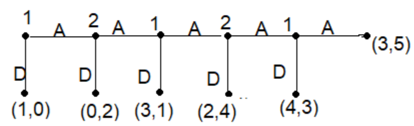


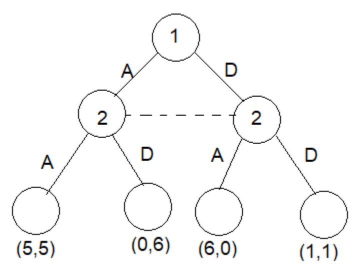
Fig:8.2 Centipede Game in PIEFG Representation

When this game was experimented on people it was found that people played "across" moves also until close to end of the game.

Suppose that you are the second player in the game, and in the first step of the game the first player plays across. The SPNE tells that you should go down, but the same analysis suggests that you would not have gotten to this choice point in the first place.

8.4 Imperfect-information extensive-form games

The PIEFG is not able to represent the simultaneous move games like neighboring kingdom's dilemma. Hence there is a need to move for a more general representation.



Imperfect-information extensive form game Representation of Neighboring Kingdom Dilemma

1 \ 2	A	D
A	5,5	0,6
D	6,0	1,1

Normal Form Representation

Definition 8.4 ***Imperfect-information extensive form game:*** *An imperfect-information extensive form game is a tuple* $(N, A, H, Z, \chi, P, (u_i)_{i \in N}, (I_i)_{i \in N})$*, where:* I_i *is a partition of* $\{h \in H \setminus Z\text{: } P(h)=i\}$ *with the property that if* $h, h' \in I_i$*, then* $\chi(h) = \chi(h')$ *or in words* ***Information Set*** *of a player is a set of player's decision nodes which are indistinguishable to him.*

In the above diagram $\mathrm{I}_1 = \{\phi\}$ $\mathrm{I}_2 = \{\{A\}, \{D\}\}$

Information set is always non-empty but it can be singleton as PIEFG is also a IIEFG with singleton information sets. IIEFG is a more richer representation and normal form game can be converted into IIEFG.