Axiomatic Bargaining

- Two agents
- negotiating on a mutually beneficial agreement which is self enforcing.
- Desirable properties : given by axioms

Axiom 1: Strong Efficiency

Problem setup : $\langle F, v \rangle$ - bargaining problem

F : feasible set , v : disagreement point.

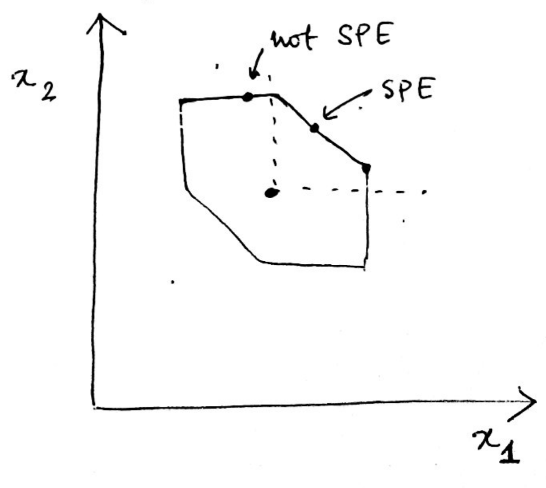
$f(F,v)$: bargaining solution

$f(F,v) = (f_1(F,v), f_2(F,v))$, $f_i(F,v) \in \mathbb{R}$, $i = 1,2$.

Given a feasible set F, we say an allocation $x = (x_1, x_2) \in F$ is strongly (Pareto) efficient if $\nexists$ another $y = (y_1, y_2) \in F$ s.t. $y_1 \geq x_1$ and $y_2 \geq x_2$ with the inequality being strict for at least one agent.

- an allocation $x = (x_1, x_2) \in F$ is weakly (Pareto) efficient if $\nexists$ another $y \in F$ s.t. $y_1 > x_1$ and $y_2 > x_2$.

We want the bargaining solution to be strongly efficient. Implies that there does not exist another allocation which will make both the players better off and at least one of them strictly.



Axiom 2 : Individual Rationality

$f(F,v) \geq v$

$\Rightarrow f_i(F,v) \geq v_i$, $\forall i = 1,2$.

(3-2)

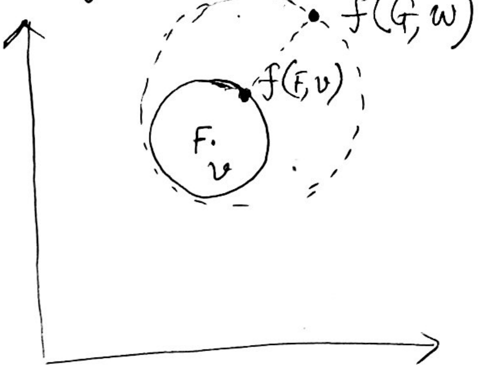
Axiom 3: Scale Covariance

Consider an affine transformation of the feasible space F, i.e., let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1, \lambda_2 > 0$ and $G := \{ (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F \}$

and $w := (\lambda_1 v_1 + \mu_1, \lambda_2 v_2 + \mu_2)$

[scaling and translating the feasible space and the disagreement point]

Then $(\lambda_1 f_1(F, v) + \mu_1, \lambda_2 f_2(F, v) + \mu_2)$ must be a solution of the scaled bargaining problem (G, w).

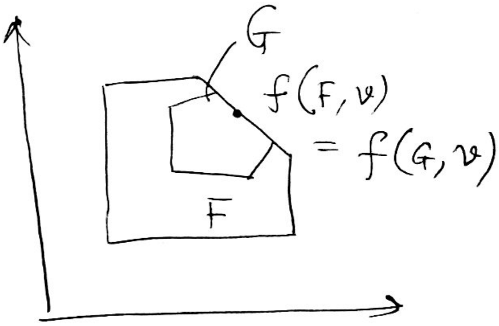


Axiom 4: Independence of Irrelevant Alternatives.

For any closed, convex set G

$G \subseteq F$ and $f(F, v) \in G$

$\Rightarrow f(G, v) = f(F, v)$.



Axiom 5: Symmetry

If positions of players 1 and 2 are symmetric, the solution should treat them symmetrically

$v_1 = v_2$ and $\{ (x_2, x_1) : (x_1, x_2) \in F \} \subseteq F \Rightarrow f_1(F, v) = f_2(F, v)$

The Nash Bargaining Solution

Thm: Given a two person bargaining problem (F, v), there exists a unique solution function f that satisfies axioms 1–5, and is given by

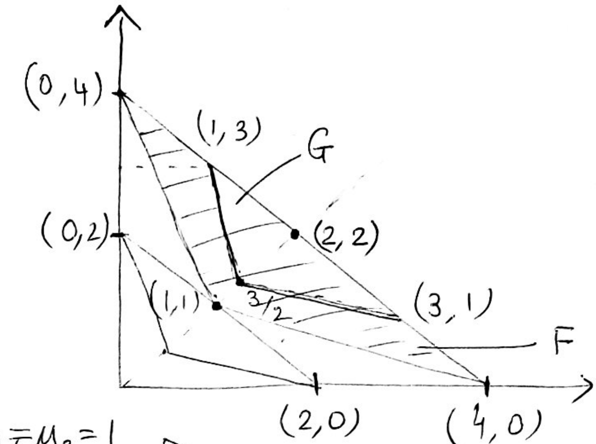
$$f(F,v) \in \underset{\substack{(x_1,x_2)\in F \\ x_1 \geq v_1,\ x_2 \geq v_2}}{\operatorname{argmax}} \left((x_1 - v_1)(x_2 - v_2)\right)$$

Illustrative example

F : convex hull of
$(4,0)$, $(1,1)$, $(0,4)$

$v = (1,1)$

$f(F,v) = (2,2)$ PE, IR, Sym.



Obtain G by $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\mu_1 = \mu_2 = 1$ → remains $(2,2)$ scale covariance

" H by $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\mu_1 = \mu_2 = 0$
→ $(1,1)$

Proof of Bargaining Theorem

We will consider a special (but almost general) subclass where $\exists$ at least one $y \in F$ s.t. $y_1 > v_1$ and $y_2 > v_2$. We call such bargaining problem as "essential" bargaining problem.

Defn: A function defined over a non-empty convex set $f : S \to \mathbb{R}$, S is convex and non-empty, is quasi-concave if $f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}$ $\forall x, y \in S, \forall \lambda \in [0,1]$

f is strictly convex if
$f(\lambda x + (1-\lambda) y) > \min\{f(x), f(y)\}$ $\forall x, y \in S$, $x \neq y$
$\forall \lambda \in (0,1)$

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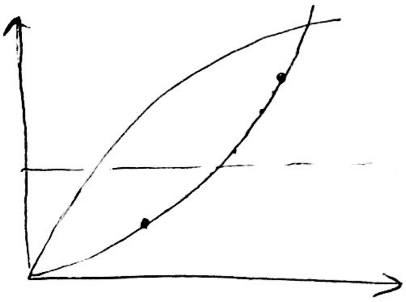
Alternative definition

upper contour set of f.

$U_a = \{x \in S : f(x) \geq a\}$

is (strictly) convex for every a, if f is (strictly) quasi-concave.

Examples of QC functions



Observation:

$N(x) = (x_1 - v_1)(x_2 - v_2)$ is strictly quasi-concave

for essential bargaining games in the region $x_1 \geq v_1, x_2 \geq v_2$.

- easier to see from the alternative definition.

The Nash Bargaining Theorem states that the five axioms are satisfied for the unique bargaining solution

$$f^N(F, v) = \arg\max_{(x_1,x_2) \in F \cap \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq v_1, x_2 \geq v_2\}} N(x) \quad \text{------ } (1)$$

$N(x)$ is called the Nash product.

Fact: A strict quasi-concave function has unique maxima.

Proof: (Part 1) Solution $x^* = (x_1^*, x_2^*)$ of (1) satisfies the 5 axioms

(1) Strong efficiency:

Given $(x_1^*, x_2^*) = \arg\max_{\substack{(x_1, x_2) \in F \\ x_1 \geq v_1, x_2 \geq v_2}} N(x)$

Suppose $\exists (\hat{x}_1, \hat{x}_2)$ s.t. $\hat{x}_1 \geq x_1^*$ and $\hat{x}_2 \geq x_2^*$ at least one of them is strict.

Since we consider essential bargaining problem

$N(x^*) > 0$, but by assumption

$N(\hat{x}_1, \hat{x}_2) > N(x^*) > 0$

which is a contradiction to the definition of x^*.

(2) Individual rationality is obvious from the definition of x^*.

(3) Scale covariance:

Consider $\lambda_1, \lambda_2 > 0$, μ_1, μ_2 and define

$$G = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) : (x_1, x_2) \in F\}$$

The Nash product problem in G

$$\max_{\substack{(y_1, y_2) \in G \\ y_1 \geq w_1, y_2 \geq w_2}} (y_1 - w_1)(y_2 - w_2) \quad \text{where } \begin{array}{l} w_1 = \lambda_1 v_1 + \mu_1 \\ w_2 = \lambda_2 v_2 + \mu_2 \end{array}$$

$$\Rightarrow \max_{\substack{(x_1, x_2) \in F \\ x_1 \geq v_1, x_2 \geq v_2}} \lambda_1 \lambda_2 (x_1 - v_1)(x_2 - v_2)$$

The maximum is attained at (x_1^*, x_2^*)

therefore the above problem attains maxima at $(\lambda_1 x_1^* + \mu_1, \lambda_2 x_2^* + \mu_2)$

unique as this is quasi-concave

(4) IIA: $G \subseteq F$ is convex and closed.

(x_1^*, x_2^*) is optimal to (F, v) and let (y_1^*, y_2^*) be optimal to (G, v), also $(x_1^*, x_2^*) \in G$.

Since $G \subseteq F$ $\quad N(x_1^*, x_2^*) \geq N(y_1^*, y_2^*)$

but y^* is optimal in G, $\Rightarrow$ $N(y_1^*, y_2^*) \geq N(x_1^*, x_2^*)$

$\Rightarrow N(x^*) = N(y^*)$

but the optima is unique $\Rightarrow x^* = y^*$.

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(5) Symmetry: Suppose F is symmetric, i.e.

$$\{(x_2, x_1) : (x_1, x_2) \in F\} = F \quad \text{and } v_1 = v_2 = v$$

by definition (x_1^*, x_2^*) maximizes $(x_1^* - v)(x_2^* - v) = N(x_1^*, x_2^*)$ which is same as $N(x_2^*, x_1^*)$. Since optima is unique $x_1^* = x_2^*$.

(Part 2) Given: $f(F, v)$ is a bargaining solution that satisfies all the five axioms

TST: $f(F, v) = f^N(F, v)$; $f^N(F, v) = \underset{\substack{(x_1, x_2) \in F \\ x_1 \geq v_1,\ x_2 \geq v_2}}{\operatorname{argmax}} (x_1 - v_1)(x_2 - v_2)$

Plan: $f(F, v) = f^N(F, v)$

$\Longleftrightarrow f(G, (0,0)) = f^N(G, (0,0)) = (1,1)$

[both satisfy scale covariance]

finally, need to show

$f(G, (0,0)) = (1,1)$