CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 1

Problem 1:

Part (a):

P_1	P_2	P_1'	P_2'	P_1	P_2''	P_1'	P_2''
a	c	b	a	a	c	b	c
b	b	a	b	b	a	a	a
c	a	c	c	c	b	c	b

- Given f is strategy proof $\implies$ f is MONOTONE.
- We know that for any preference profile $(P_1, P_2), f(P_1, P_2) \in \{P_1(1), P_2(1)\}$.
- For the above preference profile $(P_1', P_2'), f(P_1', P_2') \in \{P_1'(1), P_2'(1)\}$ i.e. $\{b, a\}$.
- Lets suppose $f(P_1', P_2') = a$. We know that $f(P_1, P_2) = a$.
- Consider the transition $(P_1, P_2) \rightarrow (P_1, P_2'')$.
- Since f is MONOTONE and $D(a, P_1) \subseteq D(a, P_1)$ and $D(a, P_2) \subseteq D(a, P_2'')$, we have $f(P_1, P_2'') = a$.
- Now consider the transition $(P_1, P_2'') \rightarrow (P_1', P_2'')$.
- $f(P_1', P_2'') \in \{b, c\}$. Let $f(P_1', P_2'') = c$. Then Player 1 can manipulate his preference P_1' and report P_1 to obtain a more favourable outcome for him i.e. a (Since $f(P_1, P_2'') = a$ which is more preferable for him).
- But given that f is strategy proof. So $f(P_1', P_2'') = b$.
- Now consider the transition $(P_1', P_2'') \rightarrow (P_1', P_2')$.
- Since f is MONOTONE and $D(b, P_1') \subseteq D(b, P_1')$ and $D(b, P_2'') \subseteq D(b, P_2')$, we have $f(P_1', P_2') = b$.

But this is a contradiction .
Therefore $f(P_1', P_2') = b$.

Part (b):

- No.
- Given (P_1, P_2) and (P_1', P_2') are two profiles as defined above and $f(P_1, P_2) = a$.
- If f is strategy proof then $f(P_1', P_2') = b$ is not necessary.

- In the above proof we have considered an intermediate preference $P_2^{''} = \{c, a, b\}$. But this preference is not valid in case of Single-peaked preference domain. Therefore the proof that we used above to prove that $f(P_1^{'}, P_2^{'}) = b$ does not hold here.

One mechanism design that guarantees that both $f(P_1, P_2) = a$ and $f(P_1^{'}, P_2^{'}) = a$ is by defining f as follows Let $(\hat{P}_1, \hat{P}_2)$ be a preference profile.

$$f(\hat{P}_1, \hat{P}_2) = min\{\hat{P}_1(1), \hat{P}_2(1)\}$$

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- This scf f is clearly ONTO as we can obtain preference profiles $(\hat{P}_1, \hat{P}_2)$ such that $\hat{P}_1(1) = x, \hat{P}_2(1) = x$ where $x \in \{a, b, c\}$, Then the above defined scf on this preference profile outputs x.

This scf is strategy proof as well because:

- Let $f(\hat{P}_1, \hat{P}_2) = min\{\hat{P}_1(1), \hat{P}_2(1)\} = x, x \in \{a, b, c\}$. Then either of $\hat{P}_1(1)$, or $\hat{P}_2(1) = x$ say $\hat{P}_1(1)$, and the other $\hat{P}_2(1) \geq x$.
- If $\hat{P}_2(1) = x$, Then he is getting the preferred outcome and therefore no need for manipulation.
- Else if $\hat{P}_1(1) = y > x$ and if player 2 manipulates his preference by reporting a preference $\geq y$, then the outcome remains the same x and so no advantage to him.
- Else if he reports a preference $z < x$, then outcome becomes z. But his preference to this outcome is less than his preference for x (Since single peaked preference).

Therefore the defined scf f is both ONTO and strategy proof.

Problem 2:

Gibbard Satterthwaite Theorem Statement: Let the set of alternatives A be such that$|A| > 3$. If the social choice function $f : P^n \to A$ is onto and strategyproof then f is dictatorial.
We know that the GS theorem needs unrestricted preferences.
Since the scf f is defined over a non-empty subset of projects, the number of actions we have are $2^{|X|} - 1$.
Let P_i be the linear ordering over the set of projects X.
Then for any pair of subsets of projects $S, T \subseteq X, S$ is preferred to T if the highest ranked project in S (according to P_i) is better than the highest ranked project in T- if these two projects are the same, then S and T are indifferent.
Therefore all possible permutations of the actions are not possible for a player.

Let A, B be two subsets of X.
Let the highest preference in A, B be respectively A_1, B_1.
If $A_1 = B_1$ then both the subsets have equal preference or else they can be ordered.
Since $|X| \geq 2$, subsets can be found with equal preference.

All possible linear orderings are not available to be chosen by the agents in this case (restricted preferences).
Therefore Gibbard Satterthwaite Theorem may not be applicable.

Construction of a scf f such that it is not dictatorial

Let $P_i^{'}$ be the preference in the new domain.
Agent i has a linear ordering P_i over the set of projects X. The domain for the construction of scf takes all the possible subsets of X. In the new domain of alternatives, $P_i^{'}(1)$ consists of all the subsets of X which contain $P_i(1)$.

Lets define the scf as follows:
Define X_i as follows:
Let $P_i^{'}(1)$ be the top most preference of player i from the projects. If $P_i^{'}(1) = \{a_{i1}, ..a_{ik}\}$ with preferences in P_i as $a_{i1} > .. > a_{ik}$, then $X_i = a_{i1}$.

$$f(P_1^{'}(1), ..., P_n^{'}(1)) = \bigcup_{i=1}^{n} X_i$$

To show f is ONTO:

To obtain an outcome $a = \{a_1, .., a_k\}$, select a preference profile $P^{'}$ such that for every project $a_j, j \in \{1, ..k\}$, $P_i^{'}(1) = \{a_j\}$ for some $i \in \{1, ..n\}$ and the other preferences of player i can be anything.

$$f(P_1^{'}(1), ..., P_n^{'}(1)) = \bigcup_{i=1}^{k} a_i$$

Therefore f is onto.

To show f is strategyproof

f gives an outcome which is the union of the top preferences of all the players.
We can see that if a player reports his preference profile correctly then the outcome that is obtained is definitely favourable to him.
If a player i misreports his preference then if any other player has reported his preference as his top preference, then the outcome that is obtained is favourable to him, or else not. Therefore the defined scf f is strategyproof.

To show that f is anonymous:

The outcome does not depend on the players, but only on the preferences of the players reported. Therefore the defined scf f is anonymous, which clearly implies that f is not dictatorial.
Therefore we found a scf that is both ONTO and strategyproof but not dictatorial.

Problem 3:

Yes, the median voter SCF is group strategy-proof.
Proof: Sort the peaks from the preferences of all the players.
The player whose preference is same as the median does not wish to misreport his preference as he is already

getting the best outcome according to his preference.
Let $X = \{X_1, ..., X_{m1}\}$ be the set of players whose peaks are to the right of the median.
Any subset (group) of players in this set have 3 strategies:

- If all the players in this group misreport their peak preferences by increasing their peak preferences, then the median will not change. Therefore no advantage to any player in this group.
- If all the players in this group misreport their peak preferences to be less than the median, then the median will shift to the left which is way less preferred by these players than the original median. Therefore no advantage to any player in this group.
- If some of the players in this group misreport their preferences to be less than the median and some greater than the median then also the median will shift to the left which is also less preferred by these players than the median. Therefore no advantage to any player.

Similar is the case with the any subset of players $Y = \{Y_1, .., Y_{m2}\}$ whose peaks are less than the median.

What if some players from X and some from Y shift the median? In this case also the resultant outcome favours only one side of the players and the other side of the players are bestowed with an outcome that is more worse than the median which they do not like.

Finally a group of agents can shift a median if they can shift their peak to the other side of the median, and this will shift the outcome to the other side, which this agent will not like.

Therefore, the median voter SCF is group strategy-proof.