CS-XXX: Game Theory and Collective Choice **Jul-Nov 2017**

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13.1 Recap

In the previous lecture, for bayesian games, two different types of utilities were discussed: Ex-ante utility and Ex-interim utility. Ex-ante utility is the utility of any player before observing own type of profile and is expressed as:

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta\Theta} P(\theta) U_i(\sigma(\theta), \theta)$$

where, $\sigma(\theta) = (\sigma_1(\theta_1), \sigma_2(\theta_2), ..., \sigma_n(\theta_n))$

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta\in\Theta} P(\theta) \sum_{(a1,a2,...,a_n)\in A} (\prod_{j\in N} \sigma_j(\theta_j, a_j) u(a_1, ..., a_n, \theta_1, ..., \theta_n)) \tag{13.1}$$

And, while calculating Ex-intrim utility the player knows own type of profile and is expressed as:

$$U_i(\sigma|\theta_i) = \sum_{\theta_{-i}\in\Theta_i} P(\theta_{-i}|\theta_i) U_i(\sigma(\theta), \theta) \tag{13.2}$$

The relation between the two utilities is expressed as:

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta_i\in\Theta_i} P(\theta_i) U_i(\sigma|\theta_i). \tag{13.3}$$

13.2 Equilibrium Concepts

13.2.1 Nash Equilibrium

If the utility is *Ex-ante utility* (expected utility before observing own type) then (σ^*, P) is called a *nash equilibrium* if :

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i^{'}, \sigma_{-i}^*) \forall \sigma_i^{'}, \forall i \in N \tag{13.4}$$

This nash equilibrium is actually a bayesian nash equilibrium but, to make it different with the bayesian equilibrium let it say only nash equilibrium.

13.2.2 Bayesian Equilibrium

If the utility is *Ex-intrim utility* (expected utility after observing own type) then (σ^*, P) is called a *bayesian equilibrium* if :

$$U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \geq U_i(\sigma_i^{'}, \sigma_{-i}^*|\theta_i) \forall\theta_i \in \Theta_i, \forall\sigma_i^{'}, \forall i \in N \tag{13.5}$$

Therefore,

$$U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \geq U_i(a_i, \sigma_{-i}^*|\theta_i) \tag{13.6}$$

$\forall\theta_i \in \Theta_i, \forall a_i \in A_i, \forall i \in N$ and $a_i \in \delta(\sigma_i^{'})$

13.3 Equivalence of the two equilibrium concepts

Theorem 13.1 *In finite bayesian games (σ^*, P) is a bayesian equilibrium iff it is a Nash equilibrium.*

Proof: $/*$ 1) In finite bayesian games, (σ^*, P) is a bayesian equilibrium if it is a Nash equilibrium. $*/$
Suppose (σ^*, P) is BE then,

$$U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \geq U_i(\sigma_i^{'}, \sigma_{-i}^*|\theta_i) \forall\sigma_i, \forall i \in N, \forall\theta_i \in \Theta_i$$

and,

$$U_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \qquad \text{(from 13.1)}$$

but,

$$\sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \geq \sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma_i^{'}, \sigma_{-i}^*|\theta_i) \qquad \text{(using 13.2 and 13.5)}$$

$$\sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma_i^{'}, \sigma_{-i}^*|\theta_i) = U_i(\sigma_i^{'}, \sigma_{-i}^*)$$

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i^{'}, \sigma_{-i}^*)$$

Hence, if (σ^*, P) is bayesian equilibrium then its a nash equilibrium also. (1)

$/*$ 2) In finite bayesian games, (σ^*, P) is a Nash equilibrium if it is a bayesian equilibrium. $*/$

Suppose (σ^*, P) is a nash equilibrium. Assume for contradiction that (σ^*, P) is not a Bayesian equilibrium.
Then, $\exists a_i \in A_i$, some $\theta_i \in \Theta_i$, some $i \in N$ such that,

$$U_i(a_i, \sigma_{-i}^*|\theta_i) > U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \qquad \text{(negative of eq 13.6)}$$

consider the strategy $\sigma_i^{\wedge}$ of i,

$$\sigma_i^{\wedge}(\theta_i) = \sigma_i^*(\theta_i^{'}) \forall\theta_i^{'} \in \Theta_i \backslash \{\theta_i\}$$

$$\sigma_i^{\wedge}(\theta_i, a_i) = 1 \text{ and } \sigma_i^{\wedge}(\theta_i, b_i) = 0 \ \forall b_i \in A_i \backslash \{a_i\}$$

$$U_i(\sigma_i^{\wedge}, \sigma_{-i}^*) = \sum_{\theta_i^{\sim} \in \Theta_i} P(\theta_i^{\sim}) U_i(\sigma_i^{\wedge}, \sigma_{-i}^*|\theta_i^{\sim})$$

$$= \sum_{\theta_i^{\sim} \in \Theta_i \backslash \theta_i} P(\theta_i^{\sim}) U_i(\sigma_i^{\wedge}, \sigma_{-i}^* | \theta_i^{\sim}) + P(\theta_i) U_i(\sigma_i^{\wedge}, \sigma_{-i}^* | \theta_i)$$

but, as we have assumed that (σ_i^*, P) is not bayesian.
Therefore,

$$U_i(\sigma_i^{\wedge}, \sigma_{-i}^* | \theta_i) > U_i(\sigma_i^*, \sigma_{-i}^* | \theta_i)$$

and hence, $U_i(\sigma_i^{\wedge}, \sigma_{-i}^*) > U_i(\sigma_i^*, \sigma_{-i}^*)$, which can not be true as we know that $U_i(\sigma_i^*, \sigma_{-i}^*)$ is a nash equilibrium. Thus, whatever we have supposed was incorrect.
And, if (σ^*, P) is a nash equilibrium then it must be a bayesian equilibrium also. (2)
From (1) and (2), it is proved that,
"In finite bayesian games (σ^*, P) is a bayesian equilibrium iff it is a Nash equilibrium." ■

13.4 Existence of Bayesian Equilibrium

Theorem 13.2 *Every finite Bayesian game has a Bayesian equilibrium.*

Proof: *Idea:* Transform the bayesian game into a complete information game treating each type a player. The transformed game is as:

- $\bar{N} = \cup_{i \in N} \Theta_i = \{ \theta_1^1, \theta_1^2, \ldots, \theta_1^{|\Theta_1|}, \theta_2^1, \theta_2^2, \ldots, \theta_2^{|\Theta_2|}, \theta_n^1, \theta_n^2, \ldots, \theta_2^{|\Theta_n|} \}$ This is finite by assumption.
- $A_{\theta_i} = A_i \ \forall \theta_i \in \Theta_i$ and $\forall i \in N$
- $U_{\theta_i}(a_{\theta_i}, a_{\theta_i}) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i} | \theta_i) \ U_i(a_i(\theta_i), a_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$

Note: A mixed strategy of player θ_i, σ_{θ_i} is a probability distribution over ΔA_i, which is a mixed strategy of player i at type θ_i, $\sigma_i(\theta_i)$ in the original Bayesian game.
Hence, a MSNE in the transformed game is a Bayesian equilibrium in the original game.
Since, by Nash theorem MSNE exists in the transformed game, Bayesian equilibrium exists in the original game.

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