Properties of SCF (Recap)

Pareto Efficient (PE):

$\forall P \in \mathcal{P}$ and $\forall b \in A$ if $\exists a$ s.t. $a P_i b \ \forall i \in N$ then $f(P) \neq b$

Unanimous (UN):

$\forall P$ with $P_1(1) = P_2(1) = \cdots = P_n(1) = a$, $f(P) = a$.

Onto (ONTO):

$\forall a \in A$, $\exists P \in \mathcal{P}$ s.t. $f(P) = a$.

As before, $PE \subset UN \subset ONTO$

Claim: Let p_{min} and p_{max} are the leftmost and rightmost peaks of P according to $<$, then f is PE iff

$f(P) \in [p_{min}, p_{max}]$.

Proof: $\Rightarrow$ Suppose f is PE but $f(P) \notin [p_{min}, p_{max}]$. then $f(P)$ is either ~~small~~ at the left of p_{min} or at the right of p_{max}. Consider $f(P) < p_{min}$, but then every agent prefers p_{min} over $f(P)$, a contradiction to PE. Similar argument for $p_{max} < f(P)$.

$\Leftarrow$ if $f(P) \in [p_{min}, p_{max}]$ every other alternative $b \neq f(P)$ in $[p_{min}, p_{max}]$ will either be closer to p_{min} (and farther from p_{max}) or vice-versa. But then the "if" condition of the PE ~~def~~ definition is never triggered for those b's. Hence PE is vacuously satisfied.

(22-2)

Monotonicity (Recap)

If for two preference profiles P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i') \ \forall i \in N$, then $f(P') = a$.

- $D(a, P_i) = \{b \in A : a P_i b\}$. dominated set.

We will see results similar to the unrestricted preferences but the proofs will differ as we do not have the flexibility of arbitrary preference profile construction.

Thm: f is SP $\Rightarrow$ f is MONO

Exactly the same proof as before.
However the ~~other direction may not hold~~ - the construction of our previous result is not always feasible for the converse.
Home work: find a counterexample of the converse / OR prove the converse.

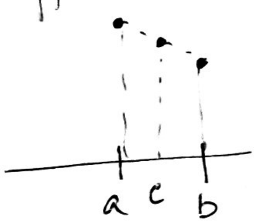
Theorem: ~~Let~~ Let $f : \mathcal{S}^n \to A$ is a SP SCF. Then

$$f \text{ is ONTO} \Leftrightarrow f \text{ is UN} \Leftrightarrow f \text{ is PE}$$

Proof: only need to show ONTO $\Rightarrow$ PE if f is SP.

Suppose not. $\exists\, a, b$ s.t. $a P_i b \ \forall i \in N$ but $f(P) = b$.

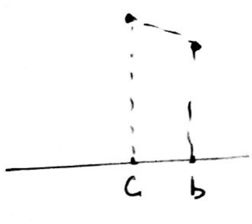
P



Since P_i's are single peaked $\exists$ another alternative $c \in A$ which is a neighbor of b, s.t. $c P_i b \ \forall i \in N$

c could be a itself.

P''



ONTO $\Rightarrow$ $\exists\, P'$ s.t. $f(P') = c$

construct P'' s.t. $\left.\begin{array}{l} P_i''(1) = c \\ P_i''(2) = b \end{array}\right\} \forall i \in N$

$P' \to P''$ MONO $\Rightarrow f(P'') = c$

$P \to P''$ " $\Rightarrow f(P'') = b$ Contradiction.

Since ~~the~~ for single-peaked preferences, there are non-dictatorial SCFs, we want to focus only on them.

~~Definition (Anonymity)~~

Anonymity

permutation of agents $\sigma : N \to N$.

We apply a permutation σ to a profile P to construct another profile as: the preference ordering of i goes to agent $\sigma(i)$ in the new profile. We denote this new profile as P^σ.

Example: $N = \{1, 2, 3\}$ $\sigma : \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$.

P_1	P_2	P_3	P_1^σ	P_2^σ	P_3^σ
a	b	b	b	a	b
b	a	c	c	b	a
c	c	a	a	c	c

Anonymity requires that the social outcome should not matter for agent renaming.

Defn: An SCF $f : \mathcal{S}^n \to A$ is anonymous (ANON) if for every profile P and for every permutation of the agents σ, $f(P^\sigma) = f(P)$.

Note: Dictatorship is not anonymous.

Theorem (Moulin 1980)

A SP SCF f is onto and anonymous iff it is a median voter SCF.

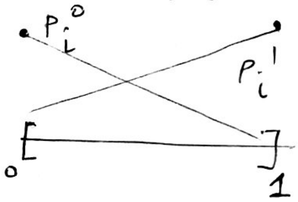
Proof: $\Leftarrow$ Median voter SCF is SP (see previous theorem),

- it is onto, put all voters peaks at the same alternative/location and phantoms are all at zero, then that location is the outcome
- it is anonymous; if we permute the agents with the peaks unchanged, the outcome does not change.

$\Rightarrow$ Given $f : \mathcal{S}^n \to A$ is SP, ONTO and ANON.

define P_i^0 : agent i's preference where the peak is at the leftmost point wrt $<$.

P_i^1 : same with rightmost point wrt $<$.



y_j's are phantom peaks, $j = 1, \dots, n-1$.

pick y_j's as follows

$$y_j = f(\underbrace{P_1^0, P_2^0, \dots, P_{n-j}^0}_{(n-j)\text{ peaks at leftmost}}, \underbrace{P_{n-j+1}^1, \dots, P_n^1}_{j \text{ peaks at rightmost}})$$

which agents have which peaks do not matter because of unanimity.

Claim: $y_j \leq y_{j+1}$ $\quad j = 1, \dots, n-2$ [peaks are non decreasing]

$$y_{j+1} = f(P_1^0, \dots, P_{n-j-1}^0, P_{n-j}^1, P_{n-j+1}^1, \dots, P_n^1)$$

SP $\Rightarrow$ $y_j \, P_{n-j}^0 \, y_{j+1}$ but P_{n-j}^0 is single peaked

$\Rightarrow$ $y_j \leq y_{j+1}$