17.11 Remarks

The concept of a balanced collection was introduced in Shapley [1967]. The proof appearing in Section 17.3.2 showing that the Bondareva–Shapley condition implies that the core is nonempty is due to Robert J. Aumann. Other proofs of this result appear in Bondareva [1963] and Shapley [1967]. The Weber set was introduced in Weber [1988].

Theorem 17.63 was first proved in Bird [1976]. The proof presented in this chapter is from Granot and Huberman [1981]. The results in Section 17.9 (page 724) and Exercise 17.65 are from Kalai and Zemel [1982b].

The definition of the reduced game to coalition S relative to preimputation x was introduced in Davis and Maschler [1965]. A different definition of the concept of a reduced game was introduced by Hart and Mas-Colell; we study the Hart–Mas-Colell reduced game in Chapter 18. The concept of a reasonable solution (Exercise 17.16) was introduced in Milnor [1952]. Exercise 17.17 is based on Huberman [1980]. The result in the exercise first appeared in Gillies [1953, 1959]. Exercise 17.18 is based on Schmeidler [1972]. The ε-core appearing in Exercise 17.33 was introduced in Shapley and Shubik [1966]. The intuitive meaning of this concept is that a deviation by members of a coalition S that leads to its formation requires information and imposes a cost, and the players will therefore not deviate to form a coalition S unless the profit from deviating is greater than this cost. The least core, and its geometric analysis, were introduced in Maschler, Peleg, and Shapley [1979]. Exercise 17.41 is from Kalai and Zemel [1982a]. Exercise 17.43 is from Aumann and Drèze [1975]. Exercise 17.58 is from Tamir [1991].

17.12 Exercises

17.1 Prove that the core is a convex set. That is, show that for any two imputations x, y in the core of a coalitional game $(N; v)$, and for all $\alpha \in [0, 1]$, the imputation $\alpha x + (1 - \alpha)y$ is also in the core of the game $(N; v)$.

17.2 (a) Give an example of a three-player coalitional game whose core is a triangle.
(b) Give an example of a three-player coalitional game whose core is a parallelogram.
(c) Give an example of a three-player coalitional game whose core is a pentagon.

17.3 Give an example of a monotonic game with an empty core.

17.4 Give an example of a superadditive game with an empty core.

17.5 Draw the cores of the following coalitional games. These games are 0-normalized, and in all of them $N = \{1, 2, 3\}$ and $v(N) = 90$.

(a) $v(1, 2) = 20, v(1, 3) = 30, v(2, 3) = 10$.
(b) $v(1, 2) = 30, v(1, 3) = 10, v(2, 3) = 80$.
(c) $v(1, 2) = 10, v(1, 3) = 20, v(2, 3) = 70$.
(d) $v(1, 2) = 50, v(1, 3) = 50, v(2, 3) = 50$.
(e) $v(1, 2) = 70, v(1, 3) = 80, v(2, 3) = 60$.

17.6 Draw the core of the following three-player coalitional game:

$$v(1) = 5, \quad v(2) = 10, \quad v(3) = 20, \quad v(1, 2) = 50, \quad v(1, 3) = 70,$$
$$v(2, 3) = 50, \quad v(1, 2, 3) = 90.$$

✓**17.7** Players i, j are *symmetric players* if for every coalition S that does not include any one of them,

$$v(S \cup \{i\}) = v(S \cup \{j\}). \tag{17.182}$$

(a) Prove that the symmetry relation between two players is transitive: if i and j are symmetric players, and j and k are symmetric players, then i and k are symmetric players.
(b) Show that if the core is nonempty, then there exists an imputation x in the core that grants every pair of symmetric players the same payoff, i.e., $x_i = x_j$ for every pair of symmetric players i, j.

✓**17.8** Let $(a_i)_{i \in N}$ be nonnegative real numbers. Let v be the coalitional function

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq k, \\ \sum_{i \in S} a_i & \text{if } |S| > k. \end{cases} \tag{17.183}$$

Compute the core of the game $(N; v)$ for every $k = 0, 1, \ldots, n$.

17.9 Prove that a three-player 0-normalized game whose core is nonempty, and satisfying $v(S) \geq 0$ for every coalition S, is monotonic. Is this true also for games with more than three players? Justify your answer. Does it hold true without the condition that $v(S) \geq 0$ for every coalition? Justify your answer.

✓**17.10** A player i in a coalitional game $(N; v)$ is a *null player* if for every coalition S,

$$v(S \cup \{i\}) = v(S). \tag{17.184}$$

In particular, by setting $S = \emptyset$, this implies that if player i is a null player then $v(i) = 0$. Show that if the core is nonempty, then $x_i = 0$ for every imputation x in the core, and every null player i.

✓**17.11** Let $(N; v)$ be a coalitional game satisfying the strong symmetry property: for every permutation π over the set of players, and every coalition $S \subseteq N$,

$$v(S) = v(\pi(S)), \tag{17.185}$$

where

$$\pi(S) = \{\pi(i) : i \in S\}. \tag{17.186}$$

Prove the following claims:

(a) The core of the game is nonempty if and only if for every coalition $S \subseteq N$,

$$v(S) \leq \frac{|S|}{n} v(N). \tag{17.187}$$

(b) If the core of the game is nonempty, and there exists a coalition $\emptyset \neq S \subset N$ satisfying $v(S) = \frac{|S|}{n} v(N)$, then the core contains only the imputation

$$\left(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}\right). \tag{17.188}$$

17.12 A player i in a simple game is a *veto player* if $v(S) = 0$ for every coalition S that does not contain i.

(a) Show that the core of a simple game satisfying $v(N) = 1$ contains every imputation x satisfying $x_i = 0$ for every player i who is not a veto player, and does not contain any other imputation. In other words, the only imputations in the core are those in which the set of veto players divide the worth of the grand coalition, $v(N)$, between them.

(b) Using part (a), find the core of the gloves game (Example 17.5 on page 690).

(c) Consider a simple majority game in which a coalition wins if and only if it has at least $\frac{n+1}{2}$ votes; that is, for every coalition $S \subseteq N$,

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq \frac{n+1}{2}, \\ 0 & \text{if } |S| < \frac{n+1}{2}. \end{cases} \tag{17.189}$$

What is the core of this game?

(d) What is the core of a simple coalitional game without veto players?

17.13 A *buyer–seller game* is a coalitional game in which the set of players N is the union of a set of buyers B and a set of sellers S (with these two sets disjoint from each other). The payoff function is defined by

$$v(T) := \min\{|T \cap B|, |T \cap S|\}, \quad \forall T \subseteq N. \tag{17.190}$$

Compute the core of this game. Check your answer against the gloves game (Example 17.5 on page 690).

17.14 Compute the core of the cost game $(N; c)$ in which $N = \{1, 2, 3, 4\}$ and the coalitional function c is

$$c(S) = \begin{cases} 0 & S = \emptyset, \\ 2 & \text{if } |S| = 2 \text{ or } |S| = 1, \\ 4 & \text{if } |S| = 3 \text{ or } |S| = 4. \end{cases} \tag{17.191}$$

17.15 Define the *dual game* of a coalitional game $(N; v)$ to be the coalitional game $(N; v^*)$ where

$$v^*(S) = v(N) - v(N \setminus S), \quad \forall S \subseteq N. \tag{17.192}$$

Is the core of a coalitional game $(N; v)$ nonempty if and only if the core of its dual $(N; v^*)$ is nonempty? Either prove this claim, or provide a counterexample.

17.16 Prove that every imputation x in the core of a coalitional game $(N; v)$ satisfies

$$x_i \leq \max_{S \subseteq N \setminus \{i\}} \{v(S \cup \{i\}) - v(S)\}, \quad \forall i \in N. \tag{17.193}$$

A solution satisfying this property is called a *reasonable solution*.

17.17 In this exercise, we will show that to compute the core of a coalitional game it suffices to know the worth of only some of the coalitions.

A coalition S is *inessential* in a coalitional game $(N; v)$ if there exists a partition $S_1, S_2, \ldots, S_r$ of S into nonempty coalitions such that $r \geq 2$ and $v(S) \leq \sum_{j=1}^{r} v(S_j)$. A coalition S that is not inessential is an *essential* coalition.

(a) Prove that if S is an inessential coalition, then there exists a partition $(S_j)_{j=1}^{r}$ of S into essential coalitions such that $v(S) \leq \sum_{j=1}^{r} v(S_j)$.

(b) Prove that an imputation x is in the core of the game $(N; v)$ if and only if (a) $x(N) = v(N)$, and (b) $x(S) \geq v(S)$ for every essential coalition S.

Let $(N; v)$ and $(N; u)$ be two coalitional games satisfying $v(S) = u(S)$ for every essential coalition S in $(N; v)$ or in $(N; u)$. Prove the following claims:

(c) A coalition S is essential in the game $(N; v)$ if and only if it is essential in the game $(N; u)$.

(d) Deduce that if $v(N) = u(N)$, then $\mathcal{C}(N; v) = \mathcal{C}(N; u)$.

(e) Prove that if the cores of the games $(N; v)$ and $(N; u)$ are nonempty, then $v(N) = u(N)$, and therefore by part (d), $\mathcal{C}(N; v) = \mathcal{C}(N; u)$.

(f) Show by example that it is possible for the core of the game $(N; v)$ to be nonempty while the core of the game $(N; u)$ is empty. In this case show, using part (d) above, that $v(N) \neq u(N)$.

17.18 A coalitional game $(N; v)$ with a nonempty core $\mathcal{C}(N; v)$ is an *exact game* if every coalition S satisfies $v(S) = \min_{x \in \mathcal{C}(N;v)} x(S)$. In other words, the worth of every coalition S equals the minimal total payoff, among the imputations in the core, that the members of S can get working together. In this exercise, we will show that for every game $(N; v)$ with a nonempty core there exists an exact game whose core equals the core of the original game $(N; v)$. In other words, the core of a coalitional game is also the core of an exact game. Moreover, we will show that every convex game is an exact game.

Let $(N; v)$ be a coalitional game with a nonempty core $\mathcal{C}(N; v)$. For every coalition $S \subseteq N$, define

$$v^E(S) := \min_{x \in \mathcal{C}(N;v)} x(S). \tag{17.194}$$

Answer the following questions:

(a) Prove that $v^E(S) \geq v(S)$ for every coalition $S \subseteq N$.

(b) Prove that $v^E(N) = v(N)$.

(c) Prove that $\mathcal{C}(N; v) = \mathcal{C}(N; v^E)$. Deduce that the coalitional game $(N; v^E)$ is exact.

17.19 Prove that if Equations (17.23)–(17.27) hold for a coalitional game $(N; v)$, where $N = \{1, 2, 3\}$, then the game has a nonempty core.
Guidance: Show that if $v(1, 2) + v(1, 3) \geq v(N) + v(1)$, then the imputation

$$(v(1, 2) + v(1, 3) - v(N), v(N) - v(1, 3), v(N) - v(1, 2)) \tag{17.195}$$

is in the core. If $v(1,2) + v(1,3) < v(N) + v(1)$ and $v(1,3) \geq v(1) + v(3)$, then the imputation

$$(v(1), v(N) - v(1,3), v(1,3) - v(1)) \tag{17.196}$$

is in the core. If $v(1,2) + v(1,3) < v(N) + v(1)$ and $v(1,3) < v(1) + v(3)$, then the imputation

$$(v(1), v(N) - v(1) - v(3), v(3)) \tag{17.197}$$

is in the core.

17.20 Prove that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two balanced collections, then their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is also a balanced collection.

17.21 Let $\mathcal{D}$ be a balanced collection of coalitions. Suppose that there is a player i contained in every coalition in $\mathcal{D}$. Prove that $\mathcal{D}$ contains a single coalition, $\mathcal{D} = \{N\}$.

17.22 Let $\mathcal{D}$ be a balanced collection of coalitions, and let $S \in \mathcal{D}$. Prove that there is a minimal balanced collection $\mathcal{T} \subseteq \mathcal{D}$ containing S. Deduce that $\mathcal{D}$ is the union of all the minimal balanced collections contained in $\mathcal{D}$.

17.23 Given a balanced collection $\mathcal{D}$ that is not minimal, and any coalition $S \in \mathcal{D}$, does there exist a minimal balanced collection $\mathcal{T} \subseteq \mathcal{D}$ that does not contain S? If so, prove it. If not, provide a counterexample.

17.24 Show that if $\mathcal{D}$ is a minimal balanced collection of coalitions, then the vectors $\{\chi^S, S \in \mathcal{D}\}$ (which are vectors in $\mathbb{R}^N$) are linearly independent.

17.25 Suppose that $|N| = 4$.

(a) Prove that $\{\{1\}, \{2\}, \{3\}, \{3,4\}, \{1,3,4\}\}$ is not a balanced collection.
(b) Prove that $\{\{1,2\}, \{1,3\}, \{1,4\}, \{3\}, \{4\}, \{2,3,4\}\}$ is a balanced collection, but is not a minimal balanced collection.

17.26 Prove or disprove the following:

(a) If $\mathcal{D}$ is a balanced collection of coalitions that is not minimal, then it has an infinite set of balancing weights.
(b) If $\mathcal{D}$ is a weakly balanced collection of coalitions that is not minimal, then it has an infinite set of balancing weights.

17.27 Show that if $N = \{1,2,3\}$, then the only minimal balanced collections of coalitions are: (a) $\{\{1,2,3\}\}$, (b) $\{\{1\},\{2\},\{3\}\}$, (c) $\{\{1,2\},\{3\}\}$, (d) $\{\{1,3\},\{2\}\}$, (e) $\{\{2,3\},\{1\}\}$, (f) $\{\{1,2\},\{1,3\},\{2,3\}\}$.

✓ **17.28** Prove that the two formulations of the Bondareva–Shapley Theorem, Theorem 17.14 (page 695) and Theorem 17.19 (page 701), are equivalent. To do so, show that the following two conditions are equivalent:

17.49 Let N be a set of players, let p_0 be a probability distribution over N, and let $\mathcal{B}$ be a partition of N into disjoint sets. Define a coalitional game $(N; v)$ by

$$v(S) := \sum_{\{B \in \mathcal{B}, B \subseteq S\}} \sum_{i \in B} p_0(i). \tag{17.203}$$

In words, $v(S)$ is the sum of the probabilities associated with the atoms of $\mathcal{B}$ that are contained in S. Let $C(p_0)$ be a set of probability distributions over N that are identical with p_0 over the elements of $\mathcal{B}$,

$$C(p_0) := \left\{ p \in \Delta(N) \colon \sum_{i \in B} p_i = \sum_{i \in B} p_0(i) \quad \forall B \in \mathcal{B} \right\}. \tag{17.204}$$

Prove the following claims:

(a) $v(N) = 1$ and $v(i) \geq 0$ for all $i \in N$. Deduce that the set of imputations $X(\mathcal{B}; v)$ is a subset of $\Delta(N)$. When does this inclusion hold as an equality?
(b) $(N; v)$ is a convex game.
(c) The core of $(N; v)$ equals $C(p_0)$.

17.50 Prove that if the coalitional game $(N; v)$ is strategically equivalent to the coalitional game $(N; w)$, and if $(N; v)$ is a convex game, then $(N; w)$ is also a convex game.

17.51 Let N be a set of players, and let $f : \mathbb{N} \to \mathbb{R}$ be a function. The function f is a *convex* function[9] if for every three natural numbers k, m, l satisfying $k \leq m \leq l$ and $k < l$,

$$f(m) \leq \frac{l-m}{l-k} f(k) + \frac{m-k}{l-k} f(l). \tag{17.205}$$

Let N be a set of players, and let $f : \mathbb{N} \to \mathbb{R}$ be a function. Define a coalitional game $(N; v)$ by

$$v(S) := f(|S|). \tag{17.206}$$

Prove that $(N; v)$ is a convex game if and only if f is a convex function.

✓**17.52** The *monotonic cover* of a coalitional game $(N; v)$ is the coalitional game $(N; \widetilde{v})$ defined by

$$\widetilde{v}(S) := \max_{R \subseteq S} v(R). \tag{17.207}$$

Prove that the monotonic cover of a convex game is a convex game.

17.53 Find a coalitional game that is not convex, and has a nonempty core that does not contain the Weber set.

17.54 Find a coalitional game in which all the vectors w^π defined in Equation (17.138) (page 720) are identical: $w^{\pi_1} = w^{\pi_2}$ for every pair of permuations π_1 and π_2 of the set of players N.

9 This is the discrete analogue to the definition of a convex function over $\mathbb{R}$, since $\frac{l-m}{l-k}k + \frac{m-k}{l-k}l = m$. Recall that a real-valued function g is convex if $g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$ for all x, y and for all $\alpha \in [0, 1]$.