CS711: Introduction to Game Theory and Mechanism Design
Indian Institute of Technology Kanpur
Homework Assignment Number 2

Student Name: Saim Wani
Roll Number: 160609
Date: November 3, 2018

QUESTION 1

a)

We know that $f(P_1', P_2') = b$ or $f(P_1', P_2') = a$ as $f(P_1', P_2') \in \{P_1'(1), P_2'(1)\}$. By considering a series of preference profiles, we will prove that $f(P_1', P_2') = a$ as follows:

1. Consider the profile $P = (P_1', P_2)$. Observe that $f(P_1', P_2)$ cannot be c as in transition from (P_1', P_2) to (P_1, P_2), the position of c weakly improves, so $f(P_1, P_2) = c$ by monotonicity. However, we know that $f(P_1, P_2) = a$, so this is a contradiction. Therefore, $f(P_1', P_2) = b$

2. Now consider a profile P_3 such that $P_3(1) = c; P_3(2) = a; P_3(3) = b$

$$P_3 = \begin{pmatrix} c \\ a \\ b \end{pmatrix}$$

   $f(P_1', P_3) = c$ or $f(P_1', P_3) = b$ as these are the top two alternatives. But position of c weakly improves from (P_1', P_3) to (P_1', P_2); therefore if $f(P_1', P_3) = c$, then $f(P_1', P_2) = c$ by monotonicity, but this is wrong as we already proved $f(P_1', P_2) = b$ above. Hence, (P_1', P_3) must be equal to b.

3. Now consider the transition from (P_1', P_3) to (P_1', P_2'). The position of b weakly improves and $f(P_1', P_3) = b$ as shown above. Therefore, $(P_1', P_2') = b$ by montonicity.

b)

If the preferences are generated from a single peaked preference domain, then the following preference profiles are not allowed:

$$P_3 = \begin{pmatrix} c \\ a \\ b \end{pmatrix}$$

$$P_4 = \begin{pmatrix} a \\ c \\ b \end{pmatrix}$$

In our proof for part a), we used the preference profile P_3 to come to our conclusion. However, as P_3 is not an allowed preference ordering anymore, that part of our proof above cannot go through.

Mechanism that has $f(P_1', P_2') = a$

The allowed preference orderings in single-peaked domain are:

$$P_w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$P_x = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

$$P_y = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

$$P_z = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

Now consider a median vector SCF with one phantom voter whose peak is at a. We can denote each preference by just its peak. By Moulin's Theorem, we know that this SCF is strategy-proof. We also need to show that it is *onto*.
$f(P_w, P_w) = \text{median}(a, a, a) = a$
$f(P_x, P_x) = \text{median}(c, c, a) = c$
$f(P_y, P_y) = \text{median}(b, b, a) = b$
Therefore, all outcomes are possible, so the mechanism is *onto*.

Now, $f(P_1, P_2)$ as described in question is $\text{median}(a, a, c) = a$. Similarly, $f(P_1', P_2') = \text{median}(a, b, a) = a$.
Therefore, the described median vector SCF is a mechanism that is *onto* and **strategy-proof**. Also, $f(P_1, P_2) = a$ and $f(P_1', P_2') = a$ as required by the problem.

CS711: Introduction to Game Theory and Mechanism Design
Indian Institute of Technology Kanpur
Homework Assignment Number 2

Student Name: Saim Wani
Roll Number: 160609
Date: November 3, 2018

QUESTION 2

The set of alternatives available is the set of all subsets of projects. For Gibbard-Satterthwaite result to be applicable, **all possible orders of alternatives for each agent should be possible**. As $|X| > 2$, we have two alternatives S and T such that $S \subset T$ and $S \neq T$. By definition of ordering described in the question, the preference ordering where S is ranked higher than T is not possible as the highest ranked project in S is also present in T. For example, consider a set T that contains the the most preferred project for player i, $P_i(1)$ and a set S that does not contain $P_i(1)$. Clearly, in no possible ordering can S be higher than T. This is a restriction on preferences and therefore Gibbard-Satterthwaite result will not apply.
Note: Their may be specific instances where Gibbard-Satterthwaite result may apply despite these restrictions, but it does not hold in general.

CS711: Introduction to Game Theory and Mechanism Design
Indian Institute of Technology Kanpur
Homework Assignment Number 2

Student Name: Saim Wani
Roll Number: 160609
Date: November 3, 2018

QUESTION 3

Yes, the median voter SCF is group strategy proof.
We need to consider only the peak preferences of all agents . So let us denote the preferences denoted only by their peaks, i.e., $P = (P_1(1), ..., P_i(1), ...P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and phantom peaks.

Case 1: For some agent $i \in K$, $P_i(1) = a$

Any change in preference profile such that $f(P'_K, P_{-K}) \neq f(P_K, P_{-K})$ cannot be beneficial for agent i. Therefore, $f(P'_K, P_{-K})P_i f(P_K, P_{-K})$ is not possible.

Case 2: All agents in K have their peak-preferences to the left of median a.

Now, to change the median, at least one agent has to shift her peak preference to the right of median a. If the peak preferences of all the agents stay on the same side of the median as they were before, the median cannot change.
The agents in K, by reporting their peak preference to the left of a cannot change the median, so they must report it to the right of a. Doing so will shift the median further right. Now consider an agent j whose peak preference lied to the left of the median and she moves it to the right of the median. That is, $a < P'_j(1)$. This implies that $P_j(1) < a < P'_j(1)$, and since P_j is a single peaked preference, $a = f(P_j, P_{-j})P_j f(P'_j, P_{-j})$. Thus $f(P_K, P_{-K})P_j f(P'_K, P_{-K})$ for agent j. Therefore, it is not profitable manipulation for agent j.

Case 3: All agents in K have their peak-preferences to the right of median a.

Exactly same as Case 2 above.

Case 4: Some peak preferences in K lie to the right of a and some to the left of K

Note that no agent in K has its peak-preferences equal to a as this was covered in Case 1 above.
Now, the agents in K, by mis-reporting their top preferences, can shift the median to the right, left or keep it fixed at a. If it remains at a, then $f(P'_K, P_{-K})P_i f(P_K, P_{-K})$ is not possible for any i.
Now consider the case when the median shifts to the right of a. Consider an agent j in K whose true peak preference lies to the left of a. This implies that $P_j(1) < a < P'_j(1)$, and since P_j is a single peaked preference, $f(P_K, P_{-K})P_j f(P'_K, P_{-K})$ for agent j. Therefore, it is not a profitable manipulation for agent j. Similar argument holds if the median shifts to the left of a.
Hence, median voter SCF is **group strategy-proof**.