18.10 Exercises

18.7 Decompose the following game $(N; v)$ as a linear combination of carrier games. The set of players is $N = \{1, 2, 3, 4\}$, and the coalitional function is

$$\begin{gathered} v(1) = 6, \quad v(2) = 12, \quad v(3) = 0, \quad v(4) = 18, \\ v(1,2) = 24, \quad v(1,3) = 48, \quad v(1,4) = 60, \quad v(2,3) = 12, \\ v(2,4) = 32, \quad v(3,4) = 38 \\ v(1,2,3) = 120, \quad v(1,2,4) = 89, \quad v(1,3,4) = 150, \\ v(2,3,4) = 179, \quad v(1,2,3,4) = 240. \end{gathered}$$

18.8 Prove that the Shapley value is a linear solution concept: for every list of K games $((N; v_k))_{k=1}^K$, and every list of K real numbers $(\alpha_k)_{k=1}^K$:

$$\mathrm{Sh}\left(N; \sum_{k=1}^{K} \alpha_k v_k\right) = \sum_{k=1}^{K} \alpha_k \mathrm{Sh}(N; v_k). \tag{18.131}$$

18.9 In Theorem 18.15 (page 754), can any one of the properties (efficiency, symmetry, null player, or additivity) be replaced by covariance under strategic equivalence, while maintaining the conclusion of the theorem? Prove your answer.

18.10 A single-valued solution concept is a function associating an imputation with every coalitional game. One may also define solution concepts that are only defined for a subset of the class of coalitional games. Let $\mathcal{F}$ be a subset of the class of coalitional games. A *single-valued solution concept for* $\mathcal{F}$ is a function associating an imputation with each coalitional game in $\mathcal{F}$. The Shapley value is the only solution concept satisfying the four properties proposed by Shapley for the class of all coalitional games. In this exercise, we show that there exist families of coalitional games over which solution concepts different from the Shapley value that nevertheless satisfy Shapley's four properties can be defined.

A family $\mathcal{F}$ of coalitional games is called *additively closed* if for every pair of coalitional games $(N; v)$ and $(N; u)$ in $\mathcal{F}$, the game $(N; v + u)$ is also in $\mathcal{F}$.

Find a family of coalitional games that is additively closed and a single-valued solution concept defined over that family that satisfies the four Shapley properties but is not the Shapley value.

Explain why this exercise does not contradict Theorem 18.15 on page 754.

18.11 A coalitional game $(N; v)$ is called *additive* if every coalition S satisfies $v(S) = \sum_{i \in S} v(i)$. What is the Shapley value of each player i in an additive game?

18.12 Let $a \in \mathbb{R}^N$ be a vector. Compute the Shapley value of the coalitional game $(N; v)$ defined as follows:

$$v(S) := \left(\sum_{i \in S} a_i\right)^2, \quad \emptyset \neq S \subseteq N. \tag{18.132}$$

18.13 In this exercise, we present an algorithm for computing a solution concept. Given a coalitional game $(N; v)$:

(a) Choose a coalition whose worth is not 0, and divide this worth equally among the members of the coalition (this is called the *dividend* given to the members of the coalition).
(b) Subtract the worth of this coalition from the worth of every coalition containing it, or equal to it. This defines a new coalitional function (where subtracting a negative number is understood to be equivalent to adding the absolute value of that number).
(c) Repeat this process until there are no more coalitions whose worth is not 0.

For example, consider the game $(N; v)$ defined by the set of players $N = \{1, 2, 3\}$, and the coalitional function

$$v(1) = 6, \quad v(2) = 12, \quad v(3) = 18, \quad v(1,2) = 30, \quad v(1,3) = 60,$$
$$v(2,3) = 90, \quad v(1,2,3) = 120.$$

The following table summarizes a stage of the algorithm in each row, and includes the coalitional function at the beginning of that stage, the chosen coalition (whose worth is not 0), and the payoff given to each player at that stage. The last line presents the sum total of all payoffs received by each player.

Stage	1	2	3	1, 2	1, 3	2, 3	1, 2, 3	Coalition	1	2	3
1	6	12	18	30	60	90	120	1, 2	15	15	0
2	6	12	18	0	60	90	90	2	0	12	0
3	6	0	18	−12	60	78	78	1, 3	30	0	30
4	6	0	18	−12	0	78	18	1	6	0	0
5	0	0	18	−18	−6	78	12	3	0	0	18
6	0	0	0	−18	−24	60	−6	1, 2, 3	−2	−2	−2
7	0	0	0	−18	−24	60	0	1, 2	−9	−9	0
8	0	0	0	0	−24	60	18	1, 3	−12	0	−12
9	0	0	0	0	0	60	42	2, 3	0	30	30
10	0	0	0	0	0	0	−18	1, 2, 3	−6	−6	−6
11	0	0	0	0	0	0	0				
									22	40	58

Prove the following claims:

(a) This process always terminates.
(b) The total payoffs received by the players are the Shapley value of the game (and are therefore independent of the order in which the coalitions are chosen).

Remark 18.40 *The algorithm terminates in the least number of steps if we first choose the coalitions containing only one player, then the coalitions containing two players each, and so on. This process was first presented by John Harsanyi.* ♦

18.14 Compute the Shapley value of the game in Exercise 18.7, using the algorithm described in Exercise 18.13.

18.15 For every game $(N; v)$, define the *dual game* $(N; v^*)$ as follows:

$$v^*(S) = v(N) - v(N \setminus S). \tag{18.133}$$

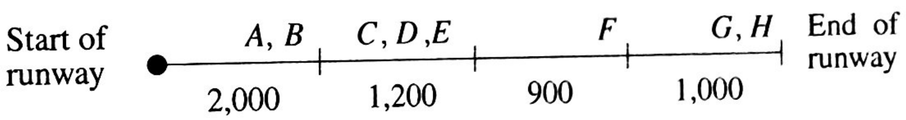
Prove the following claims:

(a) If $(N; v^*)$ is the dual game to $(N; v)$, then $(N; v)$ is the dual game to $(N; v^*)$.
(b) $\mathrm{Sh}(N; v) = \mathrm{Sh}(N; v^*)$.

18.16 The maintenance costs of airport runways are usually charged to the airlines landing planes at that airport. But light planes require shorter runways than heavy planes, and this raises the question of how to determine a fair allocation of maintenance costs among airlines with different types of planes.

Define a cost game $(N; c)$, where N is the set of all planes landing at an airport, and $c(S)$, for each coalition S, is the maintenance cost of the shortest runway that can accommodate all the planes in the coalition.

The following figure depicts an example in which eight planes, labeled A, B, C, D, E, F, G, and H land at an airport on a daily basis. Each plane requires the entire length of the runway up to (and including) the interval on which it is located in the figure. For example, plane F needs the first three segments of the runway. The weekly maintenance costs of each runway segment appear at the bottom of the figure. For example, $c(A, D, E) = 3{,}200$, $c(A) = 2{,}000$, and $c(C, F, G) = 5{,}100$.



Prove that if the Shapley value of this game is used to determine the allocation of costs, then the maintenance cost of each runway segment is borne equally by the planes using that segment.

For example, in the above figure,

$$\mathrm{Sh}_A(N; c) = \frac{2{,}000}{8} = 250, \tag{18.134}$$

$$\mathrm{Sh}_F(N; c) = \frac{2{,}000}{8} + \frac{1{,}200}{6} + \frac{900}{3} = 750. \tag{18.135}$$

18.17 This exercise considers maintenance costs associated with a road network connecting villages to a central township. The network is depicted as a tree, with the central township at the root of the tree. Each village is associated with a node of the tree, and there are additional nodes of the tree that represent road intersections. The villages vary in their numbers of inhabitants. An example appears in the following figure, which depicts six villages and two intersections; the number of inhabitants in each village appears in the figure, near that village's name, and each segment of road connecting two intersections, or connecting the township to an intersection, is labeled with that segment's maintenance cost.

A cost game $(N; c)$ is derived from the network, where N is the set of residents in all the villages (in this example $|N| = 200$), and for each coalition $S \subseteq N$, $c(S)$

Definition 18.42 *A function $P : \Gamma_U^* \to \mathbb{R}$ is called a* potential function *over Γ_U^* if for every $(N; v) \in \Gamma_U^*$ the sum of the marginal contributions equals $v(N)$:*

$$\sum_{i \in N} D_i P(N; v) = v(N). \tag{18.142}$$

Prove the following claims:

(a) For every nonempty set of players U, there is a unique potential function $P : \Gamma_U^* \to \mathbb{R}$.

(b) If P is a potential function, then for every game $(N; v) \in \Gamma_U^*$, and every $i \in N$,

$$D_i P(N; v) = \mathrm{Sh}_i(N; v). \tag{18.143}$$

18.27 Let $(N; v)$ be a simple monotonic game satisfying $v(N) = 1$. For each player i, define $B_i(N; v)$ to be the number[11] of coalitions S satisfying $v(S) = 0$ and $v(S \cup \{i\}) = 1$. The *Banzhaf value* of player i is defined to be

$$\mathrm{BZ}_i(N; v) := \frac{B_i(N; v)}{\sum_{j \in N} B_j(N; v)}. \tag{18.144}$$

Similarly to the Shapley–Shubik power index, the Banzhaf value also constitutes a power index, measuring the relative power of each player.

(a) Which of the following properties are satisfied by the Banzhaf value: efficiency, the null player property, additivity, marginality, symmetry?
(b) Compute the Banzhaf value of the game in Exercise 18.22.
(c) Find a formula for the Banzhaf value of the games in Exercises 18.23–18.24.
(d) Compute the Banzhaf value of the members of the United Nations Security Council, both in its pre-1965 structure and in its post-1965 structure (see Section 18.6.1 on page 765).

18.28 A cost game $(N; c)$ is called *convex* if

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T), \quad \forall S, T \subseteq N. \tag{18.145}$$

Prove that this property is equivalent to the following property:

$$c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T), \quad \forall i \in N, \forall S \subseteq T \subseteq N \setminus \{i\}. \tag{18.146}$$

Prove that the airport game in Exercise 18.16 is a convex cost game.

18.29 Construct a road network game, as in Exercise 18.17, for which the corresponding cost game is not convex.
Hint: Construct a road network in which the villages and the central township are geographically situated on a circle.

18.30 Let i be a null player in a coalitional game $(N; v)$. Compute the Hart–Mas-Colell reduced game over $N \setminus \{i\}$ relative to the Shapley value Sh.

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11 Such a coalition S is called a *swing* for player i.

20.17 Prove that the nucleolus is covariant under strategic equivalence: for every coalitional game $(N; v)$, for every set $K \subseteq \mathbb{R}^N$, for every $a > 0$, and every set $b \in \mathbb{R}^N$,

$$\mathcal{N}(N; av + b; aK + b) = a\mathcal{N}(N; v; K) + b. \tag{20.212}$$

20.18 Find a two-player coalitional game $(N; v)$, and a V-shaped set K, i.e., a set that is the union of two line segments sharing an edge point, such that the nucleolus of $(N; v)$ relative to K is the two edge points of K.

20.19 Compute the nucleolus of the weighted majority game $[q; 2, 2, 3, 3]$ for every quota $q > 0$.

20.20 Compute the nucleolus of the coalitional game $(N; v)$ where $N = \{1, 2, 3, 4\}$ and the coalitional function v is given by

$$v(S) = \begin{cases} i & \text{if } S = \{i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{20.213}$$

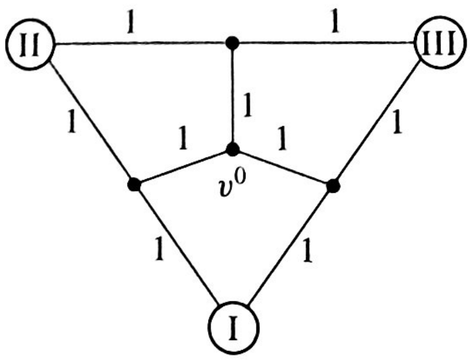
20.21 **My Aunt and I** Auntie Betty can complete a certain job together with me or with any of my three brothers, with the payment for the work being \$1,000, but she must choose one of the four of us. All four of us brothers together (without Auntie Betty) can also complete the same job.

(a) Describe this situation as a coalitional game.
(b) Compute the nucleolus of the game for the coalitional structure $\mathcal{B} = \{N\}$.
(c) Compute the nucleolus of the game for the coalitional structure

$$\mathcal{B} = \{\{\text{Auntie Betty, Me}\}, \{\text{Brother } A\}, \{\text{Brother } B\}, \{\text{Brother } C\}\}. \tag{20.214}$$

20.22 Define the nucleolus of a cost game $(N; c)$.

20.23 Compute the core and the nucleolus of the following spanning tree game (see Section 16.1.7, page 666). v^0 is the central point to which Players I, II, and III, who are physically located at the vertices of a triangle (as depicted in the next figure), wish to connect. The cost associated with every edge in the figure is one unit.



20.24 Let $(N; v)$ be a coalitional game, and let $\mathcal{B}$ be a coalitional structure. For every $\alpha > 0$, and every imputation $x \in \mathbb{R}^N$, define

$$f(x, \alpha) = \sum_{k=1}^{2^n} \alpha^k \theta_k(x). \tag{20.215}$$