24-4 Lec 25

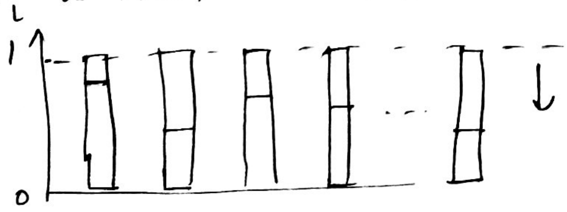
Uniform Rule SCF (Sprumont 1991)

- $f_i^u(P) = p_i$ if $\sum_{i \in N} p_i = 1$
- $f_i^u(P) = \max\{p_i, \mu(P)\}$ if $\sum_{i \in N} p_i < 1$.
- $f_i^u(P) = \min\{p_i, \lambda(P)\}$ if $\sum_{i \in N} p_i > 1$.

where, $\mu(P)$ solves $\sum_{i \in N} \max\{p_i, \mu(P)\} = 1$

and $\lambda(P)$ solves $\sum_{i \in N} \min\{p_i, \lambda(P)\} = 1$

Interpretation: of μ : take a thread that marks the level of task allocation for all agents. Start from top, i.e., every agent's allocation is 1. ~~Red~~ Keep moving the thread in uniform rate downwards until some agents p_i is hit. set the allocation for that agent to



$\sum_{i \in N} p_i < 1$

p_i and move the thread. Stop when the sum of all the allocations hit 1.

Do the reverse trick for λ.

Q: Is this rule PE, ANON, SP ?

Theorem (Sprumont 1991)

The uniform rule SCF is ANON, PE, and SP.

Proof: ANON is obvious – only the peaks matter not their owners, hence a permutation of the players with the same preference profile will return an allocation where the assignments/shares of task are correspondingly permuted.

PE is equivalent to checking the conditions

- $\sum_{i \in N} p_i < 1$, $f_i(P) \geq p_i$

  obvious since $f_i^u(P) = \max\{p_i, \mu(P)\} \geq p_i$

- $\sum_{i \in N} p_i > 1$, $f_i(P) \leq p_i$

  $f_i^u(P) = \min\{p_i, \lambda(P)\} \leq p_i$

- $\sum_{i \in N} p_i = 1$, $f_i^u(P) = p_i$.

Strategyproofness

- Case 1 : $\sum_{i \in N} p_i = 1$: every agent gets his peak, no reason to deviate

- Case 2 : $\sum_{i \in N} p_i < 1$: then $f_i^u(P) \geq p_i$

  only potential manipulable scenario

  $f_i^u(P) > p_i \Rightarrow \mu(P) > p_i$, i.e. the thread stopped before reaching p_i from top.

  ~~Agent~~ If agent i reports p_i' as his peak, it doesn't change his allocation until $p_i' > \mu(P)$, but that is ~~$p_i' > p_i$ which~~ worse than the share $\mu(P)$.

- Similar argument for $\sum_{i \in N} p_i > 1$ $\square$

However, the converse result is also true.

But the proof is skipped.

Thm: An SCF is SP, PE, and ANON iff it is the uniform rule.

Ref: Sprumont (1991): Division Problem with Single-peaked preferences.

Mechanism Design With Transfers

Quasi-linear Preferences

$$F : \Theta \longrightarrow X$$

Θ: space of type profiles

X: space of alternatives

Social Choice Function: using a different notation to avoid confusion.

Set of alternatives: $X = \{ x : x = (a, \pi) \}$

a : allocation $\in A$: set of allocations.

$\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n$: payments.

Examples of allocations

Ex 1: A public decision $a \in A = \{$Bridge, Park, Theater, ...$\}$

Ex 2: Private object allocation

Divisible good, say a cake ~~[0,1]~~ of unit length

$$a = (a_1, \ldots, a_n), \quad a_i \in [0,1], \quad \sum_{i \in N} a_i = 1$$

Ex 3: Single indivisible object allocation

$$a = (a_1, \ldots, a_n), \quad a_i \in \{0,1\} \quad \sum_{i \in N} a_i \leq 1$$

Ex 4: Partition of indivisible objects

S = set of objects

$$A = \{ (A_1, \ldots, A_n) : A_i \subseteq S \ \forall i \in N, \ A_i \cap A_j = \phi, \forall i \neq j \}$$

Type of agent i is $\theta_i \in \Theta_i$

this is private information of i.

Agents' takeaway from the allocation is defined via a function called valuation

Independent Private Values (IPV)

$$v_i : A \times \Theta_i \to \mathbb{R}$$

E.g. $v_i ((a_1, \ldots, a_i, \ldots, a_n), \theta_i)$

$$= \theta_i (a_i)$$

in ex. 3, this is agent i's value for the object if $a_i = 1$, zero otherwise.

for IPV, θ_i is a mapping $\theta_i : A \to \mathbb{R}$

sometimes the type is just denoted by v_i which is defined as this mapping WLOG in IPV context.

Payment function

$$p_i : \Theta \to \mathbb{R} \qquad \underline{p} = (p_1, \dots, p_n)$$

$$p_i(\theta_1, \dots, \theta_n) \in \mathbb{R}$$

Utility of an agent when the type profile is θ and the outcome is $(a, \underline{\pi})$ is given by

$$u_i((a, \underline{\pi}), \theta_i) = \underset{\substack{\uparrow \\ \text{possibly non linear} \\ \text{in allocation}}}{v_i(a, \theta_i)} - \underset{\substack{\uparrow \\ \text{linear in payment}}}{\pi_i}$$

[Quasi-linear utility]

Why is this a domain restriction?

The ~~out~~ alternative set

$$X = A \times \underbrace{\mathbb{R}}_{\text{payment}}$$

~~$P = \{ \underline{p} : p_i : \Theta \to \mathbb{R}, \forall i \in N \}$~~

Consider two alternatives

(a, π) and (a, π')

with $\pi_j = \pi_j' \ \forall j \neq i$, $\pi_i > \pi_i'$.

$$v_i(a) - \pi_i < v_i(a) - \pi_i'$$

for every quasi-linear preference of agent i

$$(a, \pi') \succ_i^{QL} (a, \pi)$$

This simple restriction opens up the opportunity of a lot of SCFs to ~~be~~ satisfy interesting properties.