Project: Assignment 2

Question 1(a)

Given:

- A **two agent** {1,2} model with **three alternatives** {a,b,c}
- An ***onto***, **strategyproof** SCF f with $f(P_1, P_2) = a$
- Domain of preferences is of **unrestricted strict preferences**

To show: $f(P_1^{'},P_2^{'}) = b$

P_1	P_2	$P_1^{'}$	$P_2^{'}$
a	c	b	a
b	b	a	b
c	a	c	c

Solution:

f is SP $\implies$ f **is MONO**.

Consider the preference profiles $P_1^{'}$ and P_2.
$\implies f(P_1^{'}, P_2) \in \{b, c\}$, since $f(P_1^{'}, P_2) \in \{P_1^{'}(1), P_2(1)\}$.
If $f(P_1^{'}, P_2) = c, \implies f(P_1, P_2) = c$ $(\because D(c, P_1^{'}) \subseteq D(c, P_1)$ and f is MONO)
$\Rightarrow\Leftarrow$

Therefore, $f(P_1^{'}, P_2) = b$

Construct a new preference profile $(P_1^{'}, P^{'})$ from $(P_1^{'}, P_2)$ as follows:

$P_1^{'}$	P_2	$P_1^{'}$	$P^{'}$
b	c	b	c
a	b	a	a
c	a	c	b

Clearly, $f(P_1^{'}, P^{'}) \in \{b, c\}$
$f(P_1^{'}, P^{'}) = c \implies f(P_1^{'}, P_2) = c$ $(\Rightarrow\Leftarrow)$
$\implies f(P_1^{'}, P^{'}) = b$

Note that $D(b, P^{'})(=\phi) \subseteq D(b, P_2^{'})(=\{c\})$
$\implies$ Due to monotonicity of f, $f(P_1^{'}, P_2^{'})$ must be b ($\because f(P_1^{'}, P^{'}) = b$)
Hence, $f(P_1^{'}, P_2^{'}) = b$

Question 1(b)

No, the earlier conclusion does not hold in this case. We have used a preference profile $c > a > b$, which is in violation of single peaked domain with respect to the given intrinsic ordering of the alternatives.

A mechanism that will guarantee $f(P_1^{'}, P_2^{'}) = a$, given the intrinsic ordering of alternatives is $f(P) = \min_{i \in N}\{P_i(1)\}$ ($\because a < b$ by the given ordering, and $P_{1'} = b$, $P_{2'} = a$)

Here, the minimum is taken with respect to the given intrinsic ordering $a < b < c$.

Question 2

Let the number of projects be p, i.e., $|X| = p$ (> 2)

Let the projects be labelled as $\{1, 2, ..., p\}$. Every agent $i \in N$ will have a linear ranking for the projects in X.
$\implies$ $|P| = p!$, where $P = \{P_1, P_2, ..., P_N\}$.

Let f be our SCF. The set of alternatives is $\mathcal{P}(X)$ (the power set of the set of projects.)

Ever agent i will have a **unique** preference order (not linear) for the elements in $\mathcal{P}(X)$, according to P_i (The preference order over $\mathcal{P}(X)$ will be unique since P_i is unique and fixed for every agent i).

$\implies$ Cardinality of domain for f = Number of linear orderings over the set $X = p!$

$\implies$ We have **restricted domain** for our SCF, as an unrestricted domain has all possible linear orderings available to be chosen by the agents.

Number of all possible linear orderings of $\mathcal{P}(X) = (2p - 1)! > p!$ $\forall p > 2$

GS theorem needs unrestricted preferences.

$\implies$ Gibbard-Satterthwaite result does not apply here.

Question 3

Yes, the **median voter SCF is group strategy-proof**.

Explanation:

Let f be a median voter SCF. Let K be a subset of agents (N in total) who misreport their preference. Let $f(P) = a$.
There can be four cases from here.

- **Case 1**: K contains the player whose peak preference is picked

  In this case, there is no reason for the player to manipulate.

- **Case 2**: $P_i(1) < a \ \forall i \in K$ (All the agents in K lie to the left of the median)

  If all the agents in K shift their preference to further left of a, the median won't change. If the agents in K shift their preference to the right of a, from (P_K, P_{-K}) to (P_K^i, P_{-K}), such that $a < a^{'}$ for some $a^{'} \in P_K^{'}$, it will not be a profitable manipulation ($\because$ for agent i whose peak preference is now $a^{'}$, it implies that $P_i(1) < a < P_i^{'}(1)$, and by the definition of single peaked preference, $a = f(P_i, P_{-i}^{'}) P_i f(P_i^{'}, P_{-i})$).

- **Case 3**: $a < P_i(1) \ \forall i \in K$ (All the agents in K lie to the right of the median)

  By an argument similar to the one in case 2, we can say that there will be no profitable manipulation for all the agents in K.

- **Case 4**: Agents in K lie to both left and right of the median

  In this case, some agents will satisfy case 2, and the others will satisfy case 3. Hence, moving the median to either left or right will not be profitable for the whole group.

Hence, f is group strategy-proof.