CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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15.1 Recap of Mechanism Design basics

In the previous class, we discussed that in Mechanism Design we look at the game from the designer's perspective. Our motive is to design games such that they follow certain properties in their equilibria. Typical examples of where mechanisms are used include Auctions, Voting and Matchings. Notice that in each of these settings, the players have private information. Formally, we model the setting as follows:

15.1.1 Setup

- $N = \{1, 2, \cdots, n\}$ - A set of *players*
- X - A set of *outcomes*
- Θ_i - Set of *types* of player i. This is used to model the private information of player i.
- $u_i : X \times \Theta_i \to \mathbb{R}$ - The *utility* that player i gets. This depends on the outcome and also his true type. (Private Value Model).

15.2 Social Choice Function

One question that we might ask in the above setting is the following: How do we decide the outcome? This is essential since the utility of each player depends on the outcome. For any player (say player i), the only thing that is known is i's own type, say $\theta_i \in \Theta_i$. The central authority (mechanism designer or whoever conducts the game) gets these types and now must decide what outcome to choose. This leads to another question. What should the outcome depend on? It makes sense define the output based on the types of each player, since this is what the central authority knows. For this, we define a function that maps set of type profiles to set of outcomes and we call it the *Social Choice Function* $f : \Theta \to X$, where $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. That is, given a type profile $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$, the outcome is given by $f(\theta)$.

15.2.1 Examples of Social Choice Functions

The function f defined above is very generic. One possible *Social Choice Function (SCF)* simply ignores the types and outputs a specific action $a \in X$. On the other hand, we may have other Social Choice Functions that might be more meaningful. Some commonly used SCFs are described below.

1. **Utilitarian**: A SCF $f : \Theta \to X$ is *Utilitarian* if it *always* chooses the outcome that maximizes the sum of utilities of each player. Formally,
$$f(\theta_i, \theta_{-i}) = \arg\max_{a \in X} \sum_{i \in N} u_i(a, \theta_i) \qquad \forall \theta_i \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$$
In the above, $\sum_{i \in N} u_i(a, \theta_i)$ is also called the *Social Welfare* on the outcome a. Therefore, the utilitarian SCF chooses the outcome that maximizes the social welfare.

2. **Egalitarian**: A SCF $f : \Theta \to X$ is *Egalitarian* if it *always* chooses the outcome that maximizes the minimum utility that any player gets on that outcome. Formally,
$$f(\theta_i, \theta_{-i}) = \arg\max_{a \in X} \left[\min_{i \in N} \; u_i(a, \theta_i) \right] \qquad \forall \theta_i \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$$
Note that in an egalitarian SCF, we look at minimum utility that any player gets under an outcome and then we choose the outcome that maximizes this.

15.3 Mechanisms

Definition 15.1 *(**Mechanism**) A **Mechanism** $\mathcal{M}$ is a collection of* message spaces*, $(M_1, M_2, \cdots, M_n)$ and a* decision rule $g : \bigtimes_{i \in N} M_i \to X$*. It is typically denoted as $\mathcal{M} = (M_1, M_2, \cdots, M_n, g)$ or $\mathcal{M} = \langle M, g \rangle$, where $M = \bigtimes_{i \in N} M_i$.*

The idea of message spaces has been kept abstract on purpose to allow the model to be generic. One can think of each message space as a set of actions that a player can take. Players communicate with the central authority using messages and these messages are used by the decision function g to decide upon an outcome. There is a special case in which the central authority simply asks the players to report their types. Such a mechanism is called a *Direct Mechanism* and is defined as follows.

Definition 15.2 *(**Direct Mechanism**) A mechanism $\mathcal{M} = \langle M, g \rangle$ is a **Direct Mechanism** if $M_i = \Theta_i, \forall i \in N$ and g is the Social Choice Function f. Therefore, $\langle \Theta, f \rangle$ is a direct mechanism.*

Typically, the message that a player sends to the central authority depends on the player's type so we can also think of *message mapping* $m_i : \Theta_i \to M_i$ as a strategy. Therefore, a player's action depends on the type that player receives. Suppose player i has received a type $\theta_i \in \Theta_i$. We now give the following definition:

Definition 15.3 *(**Weakly Dominant Message**) Let $\mathcal{M} = \langle M, g \rangle$ be a mechanism. A message $m \in M_i$ is a **weakly dominant** <u>for player i at θ_i</u> if,*
$$u_i\big(g(m, m_{-i}),\, \theta_i\big) \geq u_i\big(g(m', m_{-i}),\, \theta_i\big) \qquad \forall m' \in M_i, \forall m_{-i} \in M_{-i}$$

Note that we define a message $m \in M_i$ as a weakly dominant *for a player i when his type is θ_i*. A weakly dominant message is optimal for player i (given θ_i) irrespective of what the other players do.

Definition 15.4 *(Implemented in Dominant Strategies) A SCF $f : \Theta \to X$ is **implemented in dominant strategies** by a mechanism $\mathcal{M} = \langle M, g \rangle$ if the following two conditions hold:*

1. *$\forall i \in N, \exists m_i : \Theta_i \to M_i$ such that $\forall \theta_i \in \Theta_i$, $m_i(\theta_i)$ is weakly dominant for player i at θ_i*
2. *$g(m_i(\theta_i), m_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}) \qquad \forall \theta \in \Theta$*

In the above case, we also say that f is *Dominant Strategy Implementable* (DSI) and $\mathcal{M} = \langle M, g\rangle$ implements f.

Definition 15.5 *(Strategy-proof or Dominant Strategy Incentive Compatible) A* direct mechanism $\langle \Theta, f\rangle$ *is* ***Strategy-proof*** *or* ***Dominant Strategy Incentive Compatible*** *(DSIC) if*

$$u_i\big(f(\theta_i, \hat{\theta}_{-i}),\ \theta_i\big) \geq u_i\big(f(\theta_i', \hat{\theta}_{-i}),\ \theta_i\big) \qquad \forall \theta_i, \theta_i' \in \Theta_i,\ \forall \hat{\theta}_{-i} \in \Theta_{-i},\ \forall i \in N$$

Strategy-proof says that no matter what the other players do (i.e. whether or not they reveal truthfully), revealing your type truthfully is at least as good as every other strategy (recall that a strategy is a message mapping). Formally, $m_i(\theta_i) = \theta_i$ is a weakly dominant message for player i, $\forall \theta_i \in \Theta_i,\ \forall i \in N$.

15.4 Revelation Principle for DSI SCFs

Theorem 15.6 *If there exists an indirect mechanism that implements a SCF f in dominant strategies, then there exists a direct mechanism to implement f (i.e., f is DSIC).*

Proof: Let f be implemented in dominant strategies by the indirect mechanism $\mathcal{M} = \langle M, g\rangle$.
$\implies \forall i \in N,\ \ \exists m_i : \Theta_i \to M_i$ such that $m_i(\theta_i)$ is weakly dominant for player i at θ_i, $(\forall \theta_i \in \Theta_i)$
$\implies \forall i \in N,\ \ u_i\big(g(m_i(\theta_i), m''_{-i}),\ \theta_i\big) \geq u_i\big(g(m', m''_{-i}),\ \theta_i\big) \qquad \forall m' \in M_i, \forall m''_{-i} \in M_{-i}, \forall \theta \in \Theta$

$$\text{and} \quad g\big(m_i(\theta_i), m_{-i}(\theta_{-i})\big) = f(\theta_i, \theta_{-i}) \quad \forall \theta \in \Theta$$

Since the above equations hold for all $\forall m' \in M_i, \forall m_{-i} \in M_{-i}, \forall \theta \in \Theta$, they must also hold when $m' = m_i(\theta_i')$ and $m''_{-i} = m_{-i}(\theta_{-i})$. Plugging this into the equation, we get

$$\forall i \in N,\ \ u_i\Bigg(g\big(m_i(\theta_i), m_{-i}(\theta_{-i})\big),\ \theta_i\Bigg) \geq u_i\Bigg(g\big(m_i(\theta_i'), m_{-i}(\theta_{-i})\big),\ \theta_i\Bigg) \qquad \forall \theta_i' \in \Theta_i, \forall \theta \in \Theta$$

Note that $g\big(m_i(\theta_i), m_{-i}(\theta_{-i})\big) = f(\theta_i, \theta_{-i})$ and $g\big(m_i(\theta_i'), m_{-i}(\theta_{-i})\big) = f(\theta_i', \theta_{-i})$ since f is DSI. This gives,

$$\forall i \in N,\ u_i\big(f(\theta_i, \theta_{-i})\big) \geq u_i\big(f(\theta_i', \theta_{-i})\big) \qquad \forall \theta_i' \in \Theta_i, \forall \theta \in \Theta.$$

This is precisely the definition of Strategy-proof Social Choice Functions. Therefore, f is DSIC. ∎

15.5 Summary

This lecture formalized the setting from a mechanism designer's perspective by defining a Social Choice Function. The SCF determines the outcome that is taken and is a function of the players' reported types. We then defined indirect mechanisms, where players communicate with the central authority via messages and the central authority determines the outcome using a decision rule (g). A special case of this is when the central authority asks players to report their types and the decision rule is simply the SCF (f). We then defined weakly dominant messages as those which maximize player i's utility when his type is θ_i, irrespective of how the other players play. We later defined what it means for a SCF to be implemented in Dominant Strategies and talked about Strategy-proof SCFs or Dominant Strategy Incentive Compatible (DSIC) SCFs. We concluded by stating and proving the Revelation Theorem which says that if f is implemented by an indirect mechanism in Dominant Strategies, then there is also a direct mechanism that implements f.