Assignment 2

1.

(a)

P_1	P_2	$P_1^{'}$	$P_2^{'}$	$P_1^{''}$	$P_2^{''}$	$\hat{P}_1$	$\hat{P}_2$	$\tilde{P}_1$	$\tilde{P}_2$
a	c	b	a	b	c	b	c	b	c
b	b	a	b	a	-	-	-	c	b
c	a	c	c	c	-	-	-	a	a

f(P)=a (Given)

To Prove, f($P^{'}$)=b

We construct profiles as shown in table. Assume for contradiction that f($P^{'}$)=a.

First, consider the transition from P to $P^{''}$. Assume f($\hat{P}$)=c. Now assume that the transition from $\hat{P}$ to $P^{''}$. For f($\hat{P}$)=c, assume that f($P^{''}$)=b.

Now considering the transition from $P^{''}$ to $\tilde{P}$, preference for b improves for both the agents. Thus by monotonicity, f($\tilde{P}$)=b. Considering the transition from $\hat{P}$ to $\tilde{P}$, preference for c improves for both the agents. Thus by monotonicity, f($\tilde{P}$)=c. This is a contradiction. Hence, our assumption f($P^{''}$)=b was wrong. Hence, f($P^{''}$)=c.

For profile $P^{''}$, if agent 1 reports P_1 instead of $P_1^{''}$ then the outcome becomes a which he prefers more. This is a contradiction to f being strategyproof. Hence, our intial assumption of f($\hat{P}$)=c is wrong. Therefore, f($\hat{P}$)=b.

Now consider the transition from $P^{'}$ to $P^{''}$. It is same as the above mentioned case with agent 1 and 2 switched and b used in place of c. WLOG, we can say that f($\hat{P}$)=c. This gives a contradiction. Hence, our assumption of f($P^{'}$)=a is wrong. Since we know that for any preferemce profile (P_1, P_2), f(P_1, P_2) $\exists$ $\{P_1(1), P_2(1)\}$.

Hence, f($P^{'}$)=b

Thus, Proved.

(b)

This proof does not holds for this case. This is because when we are

considering transition from P' to P'' we obtain P'' for agent 2 as $P_2''(1)$=c, $P_2''(2)$=a and $P_2''(3)$=b. But as we know that $a < b < c$ this profile can't be achieved. Thus this proof does not hold.
If we consider an SCF f such that $f(P) = \min_{i \epsilon N}\{P_i(1)\}$.
In this case, f(P_1', P_2')=a.

2. Gibbard Satterthwaite Result won't apply here.

To prove this, let us assume, WLOG that for an agent i the linear ordering P_i over the set of projects X be such that $P_i(j) = x_j$ where $x_i \; \epsilon$ X. This linear ordering can be extended in such a way that for this agent P(1) contains all the subsets of x_1 in an indifferent manner. Similarily, P(2) will have all the subsets of x_2 apart from those present in P(1).

Thus, we can say that P(1) contains 2^{n-1} subsets as the alternatives or in general P(i) has 2^{n-i} such alternatives. It is quite obvious here that all the possible orderings of these alternatives/subsets is not possible (Most obvious example is the set itself which will be present in $P_i(1)$ no matter what linear ordering agent i has).

Therefore, the preferences present in this case are restricted preferences. GS theorem needs unrestricted preferences. Hence, in this case Gibbard Satterthwaite Theorem will not be valid.

3. Median voter SCF is group strategy-proof.

Only the peak of the preferences matter in this case so we will consider them only. Let the original outcome be a. Now let l number of agents have their $P(1) < a$, m have $P(1) = a$ and n have $P(1) > a$. Consider that upon manipulation the outcome changes to x.

For non zero number of agents trying to manipulate the SCF,only following cases are possible:

Case 1: $l = m = 0$, $n \neq 0$
If $x < a$ then for all the n agents, the outcome is worse than the previous outcome a. Hence, not possible. $x > a$ is not possible because even if the agents shift themselves to the right, the median won't change.
Hence, no manipulation for this case.

Case 2: $l = n = 0$, $m \neq 0$
In this case, there is no need for the agents to manipulate.

Case 3: $m = n = 0$, $l \neq 0$
This case is exactly symmetrical to Case 1. Hence no manipulation possible.

Case 4: $l = 0$, $m \neq 0$, $n \neq 0$
If $x \neq a$ then for these m agents it is less preferable. Hence, this manipulation is unfavourable.

Case 5: $m = 0$, $l \neq 0$, $n \neq 0$
If $x < a$ then for the n agents it becomes less preferable. Similarily, if $x > a$ it becomes less preferable for the l agents to the left. Thus not manipulable in this case as well.

Case 6: $n = 0$, $l \neq 0$, $m \neq 0$
This case is exactly symmetrical to Case 4. Hence no manipulation possible.

Case 7: $l \neq 0$, $m \neq 0$, $n \neq 0$
In this case as well, if $x \neq a$ then the m agents for which $P(1) = a$ will get a less preferable outcome. Hence, this case is not manipulable as well.

Thus, in any of the seven cases mentioned above the SCF is not group manipulable. Hence, we can conclude that median voter SCF is group strategy-proof.