Normal form game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$.

Pure strategy Nash Equilibrium: is a strategy profile (s_i^*, s_{-i}^*) s.t.

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i, \ \forall i \in N.$$

~~Observe~~ A Best response of agent i for the strategy profile s_{-i} of the other players is a strategy that gives ~~all~~ the maximum utility against the s_{-i} chosen by other players, i.e.

$$B_i(s_{-i}) = \{ s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \ \forall s_i' \in S_i \}$$

Pure strategy

Observe: Nash equilibrium (s_i^*, s_{-i}^*) is a strategy profile where $s_i^* \in B_i(s_{-i}^*) \ \forall i \in N$.

Example: The battle of sexes games — Two friends game

illustrate the best response part with this example.

1 \ 2	F	C
F	2, 1	0, 0
C	0, 0	1, 2

Q: Does PSNE always exist?

Matching coins game

1 \ 2	H	T
H	+1, −1	−1, +1
T	−1, +1	+1, −1

Weaker equilibrium concept:

Mixed strategy Nash equilibrium.

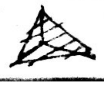
Mixed strategy: is a distribution σ_i over the strategies in S_i, i.e., $\sigma_i : S_i \to [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$ — illustrate with the example

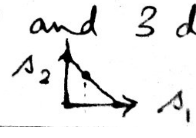
Notation: space of all mixed strategies

$$\Delta(S_i) = \{ p \in [0,1]^{|S_i|} : \sum_{s_i \in S_i} p(s_i) = 1 \}$$

$\sigma_i \in \Delta(S_i)$

meaning of a mixed strategy in 2 and 3 dimensions





Utility at a mixed strategy profile $(\sigma_i, \underline{\sigma}_{i})$

$$u_i(\sigma_i, \underline{\sigma}_i) = \sum_{s_n \in S_n} \sum_{s_1 \in S_1} \prod_{i \in N} \sigma_i(s_i)\, u_i(s_i, \underline{s}_i)$$

explain with the matching coins example.

Def^n: Mixed Strategy Nash Equilibrium is a mixed strategy profile $(\sigma_i^*, \underline{\sigma}_i^*)$ s.t. $u_i(\sigma_i^*, \underline{\sigma}_i^*) \geq u_i(\sigma_i', \underline{\sigma}_i^*)$ $\forall \sigma_i' \in \Delta(S_i)$ $\forall i \in N$.

Similarly can define a best response set and $\sigma_i^* \in B_i(\underline{\sigma}_i^*)$ $\forall i \in N$.

example $\left(\frac{1}{2}, \frac{1}{2}\right)\left(\frac{1}{2}, \frac{1}{2}\right)$ for the previous game.

Notation: $u_i(s_i, \underline{\sigma}_i) = \sum_{\underline{s}_i \in \underline{S}_i} \prod_{j \neq i} \sigma_j(s_j)\, u_i(s_i, \underline{s}_i)$

How did we come up with equilibrium. Is there a sysematic way to ~~do~~ find it?

<u>Def^n</u>: Support of a mixed strategy
Set of strategies / subset of the strategy space on which the mixed strategy σ_i has <u>positive</u> mass

~~Set~~ $\delta(\sigma_i) = \{ s_i \in S_i : \sigma_i(s_i) > 0 \}$

Theorem: (Characterization of a mixed strategy Nash eq.)
A mixed strategy profile $(\sigma_i^*, \underline{\sigma}_i^*)$ is a MSNE iff for all $i \in$

1. $u_i(s_i, \underline{\sigma}_i^*)$ is the same for all $s_i \in \delta(\sigma_i^*)$ and
2. $u_i(s_i, \underline{\sigma}_i^*) \geq u_i(s_i', \underline{\sigma}_i^*)$ $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$.

Fact: $\max\limits_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max\limits_{s_i \in S_i} u_i(s_i, \sigma_{-i})$ illustration.

$\Rightarrow$ Given $(\sigma_i^*, \sigma_{-i}^*)$ is a NE

Part ① (a) $u_i(\sigma_i^*, \sigma_{-i}^*) = \cancel{\max\limits_{\sigma_i \in \Delta(S_i)} u_i(} \sum\limits_{s_i \in S_i} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*)$

$$= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*) \quad \text{——— (a)}$$

(b) $u_i(\sigma_i^*, \sigma_{-i}^*) = \max\limits_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*)$

$$\overset{\text{Fact}}{=} \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

$$\overset{\text{claim}}{=} \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \quad \text{——— (b)}$$

Proof of claim, suppose not then $s_i^* \in S_i \setminus \delta(\sigma_i^*)$ is the point where the max happens.

~~post~~ and $u_i(s_i^*, \sigma_{-i}^*) > u_i(s_i, \sigma_{-i}^*) \quad \forall s_i \in \delta(\sigma_i^*)$

place all probability on $s_i^* \rightarrow \bar{\sigma}_i$ and $\bar{\sigma}_i, \sigma_{-i}^*$ breaks NE.

(a) = (b) Expectation = Max

happens only when every value in the support is equal.

Part ② suppose not, $\exists\, s_i' \notin \delta(\sigma_i^*)$ and $s_i \in \delta(\sigma_i^*)$

s.t. $u_i(s_i', \sigma_{-i}^*) > \underbrace{u_i(s_i, \sigma_{-i}^*)}_{\text{same for all}}$ $s_i \in \delta(\sigma_i^*)$

$\bar{\sigma}_i(s_i') = 1$

$\bar{\sigma}_i(s_i) = 0 \;\forall s_i \neq s_i'$

this ~~breaks~~ contradicts the MSNE of $(\sigma_i^*, \sigma_{-i}^*)$

4-4

$\Leftarrow$ Sufficiency.

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \underbrace{u_i(s_i, \sigma_{-i}^*)}_{= m_i(\sigma_{-i}^*)}$$

1

$$= m_i(\sigma_{-i}^*)$$

2

$$= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

Fact

$$= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*)$$

$$\geq u_i(\sigma_i, \sigma_{-i}^*) \quad \text{--- done!}$$

How is this result useful.

Compute NE of matching coins game

Battle of sexes game.

Extension

		q_1 F	q_2 C	$1-q_1-q_2$ D
p	F	2,1	0,0	1,1
$1-p$	C	0,0	1,2	2,0

$p = 2(1-p) \Rightarrow p$

$2q_1 + (1-q_1-q_2) = q_2 + 2(1-q_1-q_2)$

$\Rightarrow q_1 = 1/3$ anything for q_2 works

$q_2 + q_3 = 2/3$

in particular $q_2 = 0$

$q_3 = 2/3$

even though D is not in any PSNE.