Assignment - 2 Solutions
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1 SOLUTION 1

1.1 Part 1.1

(1) From the question, we have $f(\{a, b, c\}, \{c, b, a\}) = a$. Since f is MONO, we get that $f(\{a, b, c\}, \{c, a, b\}) = a$, by the definition of MONO, as the first argument of f remains the same, and in the second, the dominating set of a strictly increases.

(2) Now, consider the case where $P_1^\star = \{b, a, c\}$ and $P_2^\star = \{c, a, b\}$. If $f(P_1^\star, P_2^\star) = c$, then the first player can change their preference to $\tilde{P_1}^\star = \{a, b, c\}$, and from the first paragraph $f(\{a, b, c\}, \{c, a, b\}) = a$, so we get $f(\tilde{P_1}^\star, P_2^\star) = a$, which contradicts the fact that f is SP. This implies that $f(P_1^\star, P_2^\star) = b$, as $f(P_1^\star, P_2^\star) \in \{b, c\}$ and $f(P_1^\star, P_2^\star) \neq c$ by the contradiction above.

(3) However, $f(P_1^\star, P_2^\star) = b \implies f({P_1}', {P_2}') = b$, by the monotonocity of f, since the dominated set of b in $P_2^\star$ is a strict subset of the dominated set of b in ${P_2}'$. (*Note that $P_1^\star = {P_1}'$. Here ${P_1}', {P_2}'$ are as in question.*)

1.2 Part 1.2

(1) No, the earlier conclusion doesn't hold here.

(2) In the earlier proof, we used the construction of $P_2^\star = \{c, a, b\}$, which is not possible here, as the preference is clearly not single peaked – if c is the most preferred alternative, then the second most preferred alternative has to be b – it cannot be a, since b is closer to c than a.

(3) A trivial mechanism that can have $f(P_1', P_2') = a$ is one that selects a if a is in the list of peaks, else it selects b if b is in the list of peaks, else it selects c; i.e., it selects the left-most peak from the list of peaks for the various preference profiles.

2 SOLUTION 2

The GS theorem doesn't apply here because the domain is not unrestricted – all preference profiles are not possible; also, indifference are allowed among various alternatives, which should also not be true, in general, for GS to hold. We will show both of these below: Let $X = \{p_1, p_2\}$. Then the possible subsets of X (excluding the empty set ϕ) are $X_1 = \{p_1\}$, $X_2 = \{p_2\}$ and $X_3 = \{p_1, p_2\}$.

For an agent with the linear ordering $p_1 > p_2$, X_1 and X_3 are indifferent. Further, for any agent with a linear ordering over p_1 and p_2, either the X_1, X_3 or X_2, X_3 are indifferent, and thus, we can never have either $X_1 > X_3$ or $X_2 > X_3$ for any preference profile for at least one agent, i.e. either an agent can't have $X_1 > X_3 > X_2$ as a preference profile, or it can't have $X_2 > X_3 > X_1$ as a preference profile.

Conclusion. Hence, the domain of preferences is not unrestricted, and indifference is allowed among the various alternatives. Because GS theorem requires, in general, that the indifference between various alternatives not be allowed, and that all preference profiles for each agent be possible, we can say that the GS theorem doesn't apply here.

3 SOLUTION 3

Let L represent the set of peaks to the left of (also called *less than*) the median peak M, not including the median peak, and R represent the set of peaks to the right of (also called *greater than*) the median peak M, not including the median peak. In the following lines, the *selected outcome* refers to the peak chosen by f.

LEMMA 1. *K consists of peaks either only to the left of the median peak or to the right of the median peak; i.e., either $K \subset R$ or $K \subset L$.*

PROOF. Consider the case where K contains elements from both L, R. In this case, changing the outcome selected by the function with necessarily lead to the worsening of the outcome for at least one of the members of K – if the selected outcome is greater than M, the peak from the left of M will have a worse outcome, and if the selected outcome is lesser than M, the peak from the right of M will have a worse outcome. Therefore, the peaks to be manipulated must lie either **all** in R or **all** in L. □

From Lemma 1, we get that either all peaks in K belong to the left of the median or to the right of the median. Suppose all members of the group K belong to R WLOG. Now, in order to change the selected outcome, at least one of the members in K must move to the left – since otherwise, the median remains unchanged by any internal movement of the peaks to its right. However, moving to the left can only lead to worsening the selected outcome for peaks belonging to R, as all of these peaks would prefer if the selected outcome would move to the left, and not the other way around. Thus, f is group-strategy proof.