Lecture 38: November 14, 2017

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38.1 Efficiency and Budget Balance

In this lecture, we will look at some selected results and their proof sketches that are related to some of the concepts we looked at during the course. The first thing we look at is the interplay of efficiency and budget balance in the quasi-linear setting.

The following result shows that if we want efficiency in 'rich enough' quasi-linear setting, the only class of payment rules that implements the efficient SCF is the Groves class of payments.

Recall that the Groves class of payment rules is given by

$$\begin{aligned} p_i(t) &= h_i(t_{-i}) - \sum_{j \neq i} t_j(f^*(t)) \\ \text{where} \quad f^*(t) &\in \arg\max_{a \in A} \sum_{i=1}^{n} t_i(a) \end{aligned} \tag{38.1}$$

Theorem 38.1 (Green and Laffont (1979), Holmström (1979)) *If the type space is* sufficiently rich*, every efficient and DSIC mechanism is a Groves mechanism.*

Proof sketch: [this exposition is due to Holmström (1979), see the paper for a complete treatment] Consider for simplicity that there are only two allocations $\mathcal{A} = \{a, b\}$. Let (f, p) be a DSIC and efficient mechanism. The valuation for agent i for the allocation a is given by $t_i(a)$. Hence the social welfare for these two allocations is $\sum_{i \in N} t_i(a)$ and $\sum_{i \in N} t_i(b)$. An efficient allocation rule maximizes the social welfare, hence if $\sum_{i \in N} t_i(a) > \sum_{i \in N} t_i(b)$, it will choose the allocation a. Let us now

- fix the valuations of all agents except i, i.e., t_{-i} is fixed, and
- fix the valuation of i at all the other allocations, i.e., b, $t_i(b)$.

We only change the valuation of agent i for allocation a keeping all the other terms of the social welfare fixed to observe what impact it has on the allocation rule. Clearly, $\exists$ a threshold $t_i^*(a)$, such that

$$\begin{aligned} &\forall t_i(a) \geqslant t_i^*(a) \quad \text{outcome is } a \\ &\forall t_i(a) < t_i^*(a) \quad \text{outcome is } b \end{aligned} \tag{38.2}$$

We know that if a mechanism is DSIC, and by changing the valuation of an agent if the allocation remains the same, the payment should not change for that agent. So, without loss of generality, payment of an agent can be denoted as a function of the allocation and the types of other agents. Since, here the types of all agents except i remains fixed, whenever the outcome is a, the payment can be denoted by p_{ia}. Consider a valuation $t_i^*(a) + \epsilon,\ \epsilon > 0$. By definition of DSIC

$$t_i^*(a) + \epsilon - p_{ia} \geqslant t_i(b) - p_{ib} \tag{38.3}$$

The RHS denotes the utility when agent i underreports his type below $t_i^*(a)$ and the allocation changes to b. Similarly, when the true type of i is $t_i^* a - \delta,\ \delta > 0$, DSIC implies

$$t_i^*(a) - \delta - p_{ia} \leqslant t_i(b) - p_{ib} \tag{38.4}$$

Here the LHS denotes the utility of agent i when he overreports his type and the allocation becomes a. Now since ϵ, δ are arbitrary, we take limits of them tending to zero, which gives

$$t_i^*(a) - p_{ia} = t_i(b) - p_{ib} \tag{38.5}$$

Now, by the definition of $t_i^*(a)$, it is the threshold where a starts becoming the efficient outcome. Hence the following balance equation holds.

$$\begin{aligned}
& & t_i^*(a) + \sum_{j \neq i} t_j(a) &= t_i(b) + \sum_{j \neq i} t_j(b) & \\
&\implies & p_{ia} - p_{ib} &= \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a) & \text{(substituting Eq. 38.5)} \\
&\implies & p_{ix} &= h_i(t_{-i}) - \sum_{j \neq i} t_j(x) &
\end{aligned}$$

The last implication holds as we have collected all the terms independent of t_i and defined that to be $h_i(t_{-i})$. Note that the difference in payment at two allocations hold for every agent and therefore the payment can only have an additive term $h_i(t_{-i})$ which is independent of t_i. This payment is same as the Groves payment rule. The proof can be extended to any finite number of allocations. ■

Theorem 38.2 (Green and Laffont (1979)) *No Groves mechanism is* budget balanced*, i.e.,* $\nexists p_i^G$ *such that* $\sum_{i \in N} p_i^G(t) = 0, \forall t \in T$.

Proof sketch:[see Green and Laffont (1979) for the complete proof] We outline the proof idea for two agents and two allocations $\{0, 1\}$ in a public project model, where 0 implies that the project is not undertaken and 1 implies that it is undertaken. For the case when allocation 0 is chosen, all agents have zero value.

For a contradiction, suppose $\exists h_i$ s.t. $\sum_{i \in N} p_i(t) = 0, \forall t \in T$. In the Groves class, the only flexibility we have is in the choice of h_i as the rest is fixed. Consider three numbers, w_1^+, w_1^-, w_2, where w_1^+, w_1^- are valuations of agent 1 and w_2 is the valuation of agent 2, such that

$$\begin{aligned}
w_1^- + w_2 &< 0 \quad \text{outcome is 0} \\
w_1^+ + w_2 &> 0 \quad \text{outcome is 1}
\end{aligned} \tag{38.6}$$

Now Groves payment at (w_1^+, w_2) satisfies

$$h_1(w_2) - w_2 + h_2(w_1^+) - w_1^+ = 0 \qquad \text{(project is undertaken and equality by BB assumption)}$$

Similarly

$$h_1(w_2) + h_2(w_1^-) = 0 \qquad \text{(project is not undertaken and equality by BB assumption)}$$

Subtracting one from the other, we get

$$w_2 = h_1(w_1^+) - h_2(w_1^-) - w_1^+.$$

Now the RHS of the above equation is completely independent of w_2. For any small change in w_2 such that the inequalities of Eq. 38.6 continues to hold, clearly the above equality cannot hold. This is a contradiction. ■ The two results can be summarized in the form of the following corollary.

Corollary 38.3 *If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and BB.*

38.2 Weakening DSIC for positive results

However, if we weaken the IC notion to Bayesian, we can have positive results. The following mechanism is due to d'Aspremont and Gerard-Varet (1979), Arrow (1979), and is called **dAGVA** mechanism. Under this mechanism, the allocation is still the efficient one. Payment is defined via priors. Define

$$\begin{aligned} \delta_i(t_i) &= \mathbb{E}_{t_{-i}|t_i} \sum_{j \neq i} t_j(a^*(t)) \\ a^*(t) &\in \arg\max_{a \in A} \sum_{i \in N} t_i(a) \end{aligned} \tag{38.7}$$

Payment of the mechanism is given by

$$p_i^{\text{dAGVA}}(t) = \frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i) \tag{38.8}$$

This payment implements the efficient allocation rule in the Bayesian Nash equilibrium.

$$\begin{aligned} &\mathbb{E}_{t_{-i}|t_i}[t_i(a^*(t) - p_i^{\text{dAGVA}}(t)] \\ &= \mathbb{E}_{t_{-i}|t_i} \sum_{j \in N} t_j(a^*(t)) - \mathbb{E}_{t_{-i}|t_i} \left[\frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) \right] \\ &\geqslant \mathbb{E}_{t_{-i}|t_i} \sum_{i \in N} t_j(a^*(t_i', t_{-i})) - \mathbb{E}_{t_{-i}|t_i} \left[\frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) \right] \\ &= \mathbb{E}_{t_{-i}|t_i}[t_i(a^*(t_i', t_{-i}) - p_i^{\text{dAGVA}}(t_i', t_{-i})] \end{aligned}$$

The first inequality holds by definition of a^*, and the rest of the equalities are obtained by reorganizing the payment expression. Also the sum of the payment of all the agents is given by

$$\begin{aligned} \sum_{i \in N} p_i^{d'AGVA}(t) &= \frac{1}{n-1} \sum_{i \in N} \sum_{j \neq i} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i) \\ &= \frac{n-1}{n-1} \sum_{j \in N} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i) = 0 \end{aligned}$$

Hence we have proved the following theorem.

Theorem 38.4 *The dAGVA mechanism is efficient, BIC and BB.*

But the dAGVA does not guarantee IIR. Turns out that the above properties along with IIR is impossible to satisfy even in bilateral trading problem, i.e., there is one buyer and one seller and one object, efficient trade happens when the sellers cost is below the valuation of the buyer.

Theorem 38.5 (Myerson and Satterthwaite (1983)) *In the bilateral trading problem, there is no mechanism that BIC, efficient, IIR and BB.*

38.3 Summary

We will now summarize the mechanism design space we discussed in this course via the following figures. The LHS denotes the preference domains and the RHS denotes the space of mechanisms – the arrows correspond to the necessity of a DSIC/BIC mechanisms to be in a certain class of mechanisms.

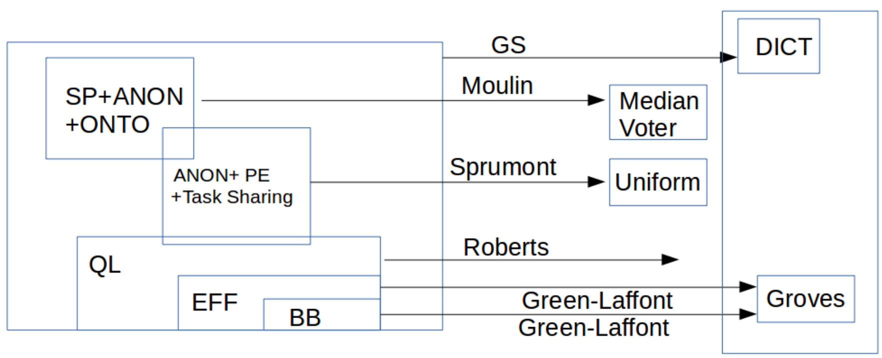

Figure 38.1: The map from valuation type to mechanism type for DSIC mechanisms only **(the last arrow goes to $\emptyset$)**

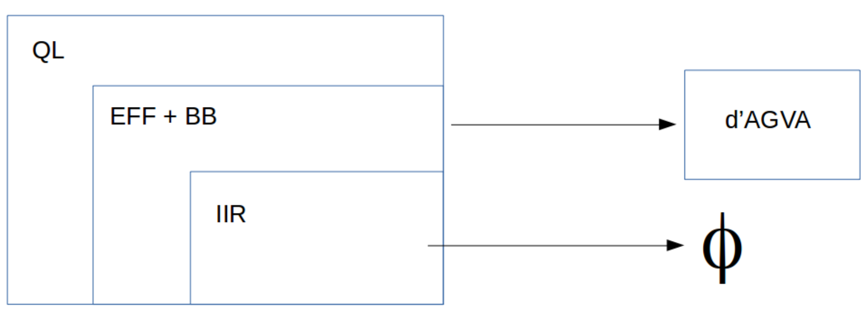

Figure 38.2: The map from valuation type to mechanism type for BIC mechanisms only

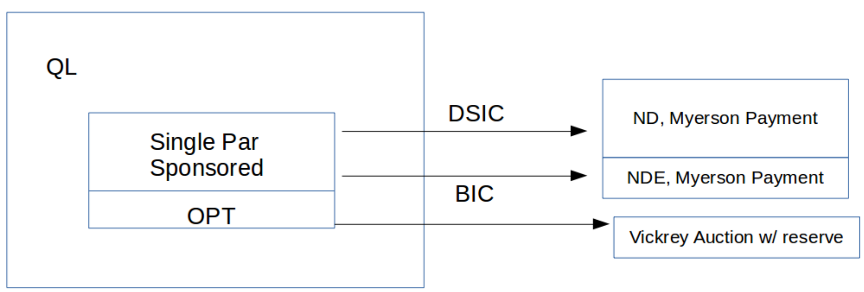

Figure 38.3: The map from valuation type to mechanism property