CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2018**
Project: Assignment 2	

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Question 1

Part (a)

We firstly comment that since **f** is SP, it is MONO too according to the result in the lecture notes. We construct a new preference profile $P^{\hat{}}$ where the **-** means any legit alternative out of {a,b,c}.

P_1	P_2	$P_1^{'}$	$P_2^{'}$	$P_1^{\hat{}}$	$P_1^{\hat{}}$
a	c	b	a	b	c
b	b	a	b	a	-
c	a	c	c	-	-

Before we move on, we will prove 3 lemmas and then use them later.

Lemma 1.1 *Let P : $P_1(1) = a \neq c = P_2(1)$ and $P^{'}$: $P_1^{'}(1) = a \neq c = P_2^{'}(1)$. Then, if $f(P) = a$, then $f(P^{'}) = a$*

Proof: *We construct a new preference profile $P^{\hat{}}$ where the* **-** *means any legit alternative out of {a,b,c}.*

P_1	P_2	$P_1^{'}$	$P_2^{'}$	$P_1^{\hat{}}$	$P_2^{\hat{}}$
a	*c*	*a*	*c*	*a*	*c*
-	-	-	-	*c*	*a*
-	-	-	-	*b*	*b*

Please note that the third preference in $P^{\hat{}}$ HAS to be b as this is the only alternative left if we fix a and c in the top 2 positions which we want in our construction.

Let's assume $f(P^{'}) = c$.

A transition form $P^{'}$ to $P^{\hat{}}$ is clearly monotonic as the best position c can take for Player 1 is at the 2^{nd} spot which it is in $P_1^{\hat{}}$ and c is in the 1^{st} position in $P_2^{\hat{}}$ anyway. Thus since $f(P^{'}) = c$, $f(P^{\hat{}} = c)$.

A transition form P to $P^{\hat{}}$ is also monotonic as the best position c can take for Player 1 is at the 2^{nd} spot which it is in $P_1^{\hat{}}$ and c is in the 1^{st} position in $P_2^{\hat{}}$ anyway. Thus since $f(P) = a$, $f(P^{\hat{}}) = a)$

Clearly, we have a contradiction as $a \neq c$. Thus our assumption that $f(P^{'}) = c$ is incorrect. Thus $f(P^{'}) = a$.
■

Lemma 1.2 *Let P : $P_1(1) = a \neq c = P_2(1)$ and $P^{'}$: $P_1^{'}(1) = b \neq a \neq c = P_2^{'}(1)$. Then, if $f(P) = a$, then $f(P^{'}) = b$*

Proof: *We construct a new preference profile $P^{\hat{}}$ where $P_2^{\hat{}}$ is the same as P_2 and where the* **-** *means any legit alternative out of {a,b,c}.*

P_1	P_2	$P_1^{'}$	$P_2^{'}$	$P_1^{\hat{}}$	P_2
a	c	b	c	b	c
-	-	-	-	a	-
-	-	-	-	c	-

Please note that the third preference in $P_1^{\hat{}}$ HAS to be c as this is the only alternative left if we fix b and a in the top 2 positions which we want in our construction.

Let's assume $f(P^{'}) = c$.

Clearly, the transition from $P^{'}$ to $P^{\hat{}}$ follows Lemma 1.1. Thus, $f(P^{\hat{}}) = c$.

*Realise that $aP_1^{\hat{}}c$. However, we see $f(P^{\hat{}}) = c$ and a unilateral deviation of Player 1 to P_1 from $P_1^{\hat{}}$ changes the outcome to a. Thus Player 1 benefits from unilateral deviation which violates the fact that **f** is SP. Thus our assumption is wrong and $f(P^{'}) = b$* ■

Lemma 1.3 *Let $P : P_1(1) = b \neq a = P_2(1)$ and $P^{'} : P_1^{'}(1) = b \neq a \neq c = P_2^{'}(1)$. Then, if $f(P) = a$, then $f(P^{'}) = c$*

Proof: *We construct a new preference profile $P^{\hat{}}$ where $P_1^{\hat{}}$ is the same as P_1 and where the **-** means any legit alternative out of {a,b,c}.*

P_1	P_2	$P_1^{'}$	$P_2^{'}$	P_1	$P_2^{\hat{}}$
b	a	b	c	b	c
-	-	-	-	-	a
-	-	-	-	-	b

Please note that the third preference in $P_2^{\hat{}}$ HAS to be b as this is the only alternative left if we fix c and a in the top 2 positions which we want in our construction.

Let's assume $f(P^{'}) = b$.

Clearly, the transition from $P^{'}$ to $P^{\hat{}}$ follows Lemma 1.1. Thus, $f(P^{\hat{}}) = b$.

*Realise that $aP_2^{\hat{}}b$. However, we see $f(P) = a$ and a unilateral deviation of Player 2 to P_2 from $P_2^{\hat{}}$ changes the outcome from b to a. Thus Player 2 benefits from unilateral deviation which violates the fact that **f** is SP. Thus our assumption is wrong and $f(P^{'}) = c$* ■

Now, we get back to our original table and preferences. Here, we are given $f(P) = a$. Let $f(P^{'}) = a$. Now, the transition from $P^{'}$ to $P^{\hat{}}$ follows Lemma 1.3. Thus $f(P^{\hat{}}) = c$. But we also observe that the transition from P to $P^{\hat{}}$ follows Lemma 1.2. Thus $f(P^{\hat{}}) = b$. Since $b \neq c$, we reach a contradiction and thus our assumption that $f(P^{'}) = a$ is incorrect. Thus $f(P^{'}) = b$.

Part (b)

No, the earlier conclusion does not hold here.

To see exactly where the proof faulters, observe the proof of Lemma 1.3, which is used in part (a). The lemma requires the construction of the profile {c,a,b} which is clearly not single peaked.

A mechanism that can be proposed is the following :

$$f(P) = \min_{\forall i \in [2]} P_i(1)$$

i.e, picking the leftmost peak as the social outcome everytime. Clearly by this mechanism both $f(P) = a$ and $f(P^{'}) = a$ as $a < b$ in the intrinsic ordering.

Question 2

We define a few things :

- $\boldsymbol{A}$ *: The set of alternatives which is basically $2^{|\boldsymbol{X}|}$, the powerset of $\boldsymbol{X}$.*
- $R_i^{'}$ *: The extension of P_i defined over $\boldsymbol{A}$ as mentioned in the question*

Claim 1.4 *$R_i^{'}$ is restricted in its domain, i.e not all preference orderings over $\boldsymbol{A}$ are allowed.*
Proof: *Let $a_1, a_2 \in \boldsymbol{X}$. We then construct the following subsets of $\boldsymbol{X}$:*

- S_1 *: $\{a_1\}$*
- S_2 *: $\{a_2\}$*
- S_3 *: $\{a_1, a_2\}$*

Now we observe the ordering $S_1 \succ S_2 \succ S_3$ where $\succ$ denotes strict preference in accordance to $R_i^{'}$. We know that either $a_1 P_i a_2$(Case 1) or $a_2 P_i a_1$(Case 2).

- *Case 1: Then by the definition of $R_i^{'}$, since S_3 has a_1 as the best project and S_2 has a_2 as the best project, $S_2 \succ S_3$ is impossible.*
- *Case 2: Then by the definition of $R_i^{'}$, since S_1 has a_1 as the best project and S_2 has a_2 as the best project, $S_1 \succ S_2$ is impossible.*

Thus, for any arbitrary pair of projects a_1 and a_2, the ordering $S_1 \succ S_2 \succ S_3$ is not allowed in $R_i^{'}$. Thus $R_i^{'}$ is domain restricted. ∎

Since we know that the GS theorem needs the preferences to be unrestricted, it cannot be applied in this scenario.

Question 3

Yes, we claim that the median voter SCF is indeed group strategy proof.

Claim 1.5 *Median voter SCF is group strategy proof.*
Proof: *Let $f(P) = a =$ median of all agent peaks and phantom peaks.*

First, we consider all $K \subseteq N$ s.t $i \in K$ where $P_i(1) = a$.

*Clearly, agent **i** cannot do better than he is doing right now as he is achieving his peak preference. Thus, no matter how the agents deviate to other preferences, the clause $f(P_K^{'}, P_{-K})P_j f(P_K, P_{-K})$ will never be true for $j = i$. Thus, in this case, **f** is group SP.*

Now, we consider all $K \subseteq N$ s.t $\forall i \in K,\ P_i(1) \neq a$.

*Let $m_l + m_r =\mid K \mid$ s.t m_l denotes the number of agents in the group with their peaks to the left of **a** and m_r denotes the number of agents in the group with their peaks to the right of **a**.*

*Now, we can imagine that a new $P_K^{'}$ forms from P_K if atleast 1 agent deviates to a new preference. Now, within m_l, the only way agents can cause a change in outcome is if atleast 1 of them reports a peak to the right of **a**. Within m_r, the only way agents can cause a change in outcome is if atleast 1 of them reports a peak to the left of **a**.*

- *Case 1 : The number of deviating agents in $m_l >$ the number of deviating agents in m_r. In this case the median, i.e $f(P^{'}) = a^{'}$ lies to the right of **a**. Since preferences are single peaked, all agents in m_l,(whos peaks lie to the left of **a**) prefer **a** over $a^{'}$. Thus, the clause $f(P_K^{'}, P_{-K})P_j f(P_K, P_{-K})$ will not be true $\forall\ j \in\ m_l$.Thus, in this case, **f** is group SP.*
- *Case 2 : The number of deviating agents in $m_r >$ the number of deviating agents in m_l. In this case the median, i.e $f(P^{'}) = a^{'}$ lies to the left of **a**. Since preferences are single peaked, all agents in m_r,(whos peaks lie to the right of **a**) prefer **a** over $a^{'}$. Thus, the clause $f(P_K^{'}, P_{-K})P_j f(P_K, P_{-K})$ will not be true $\forall\ j \in\ m_r$.Thus, in this case, **f** is group SP.*
- *Case 3: The number of deviating agents in $m_l =$ the number of deviating agents in m_r. In this case the median doesnt change and $f(P^{'}) = a$. Thus, the clause $f(P_K^{'}, P_{-K})P_j f(P_K, P_{-K})$ will not be true $\forall\ j \in\ K$.Thus, in this case, **f** is group SP.*

*Thus, $\forall K \subseteq N$, **f** cannot be manipulated by K.*

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