Lec17

~~Defn~~: Independence of Irrelevant Alternatives (IIA) (~~16-4~~) (17-1)

We ~~say~~ say $R_i, R_i' \in \mathcal{R}$ agree on $\{a,b\}$ if for agent i

$$a P_i b \Leftrightarrow a P_i' b$$

$$b P_i a \Leftrightarrow b P_i' a$$

$$a I_i b \Leftrightarrow a I_i' b$$

we denote this by $R_i|_{a,b} = R_i'|_{a,b}$

if this holds for every agent, $R|_{a,b} = R'|_{a,b}$.

Defn: An ASWF F satisfies IIA if $\forall a,b \in A$, ~~$\forall R, R' \in \mathcal{R}$~~

$$\left[R|_{a,b} = R'|_{a,b} \right] \Rightarrow \left[F(R)|_{a,b} = F(R')|_{a,b} \right]$$

if the relative positions of two alternatives are same in two different preference profiles — the outputs must have ~~also be~~ same ~~preference~~ relative positions.

Ex. Simple scoring rule $(s_1, s_2, \dots, s_m)$, $s_i > s_{i+1}$, $i=1,\dots,m-1$,

Plurality : $s_1 = 1$, $s_2 = \cdots = s_m = 0$

Does plurality satisfy IIA?

R				R'			
a	a	c	d	d	c	b	b
b	c	b	c	a	a	c	a
c	b	a	b	b	b	a	d
d	d	d	a	c	d	d	c

$a \, F^{PL}(R) \, b$

$b \, F^{Pl}(R') \, a$

$R|_{a,b} = R'|_{a,b}$

does dictatorship satisfy IIA?

Theorem (Arrow 1951)

Assume $|A| \geq 3$, If an ASWF F satisfies WP and IIA, it must be dictatorial.

Proof: Part 1 : Field Expansion Lemma

- if a group is decisive over over a pair of alternatives, it is decisive over all pairs of alternatives

Part 2 : Group Contraction Lemma

- if a group is decisive, there is a strict subset of that group is also decisive.

Defn: Let $F : \mathcal{R}^n \to \mathcal{R}$ be given, ~~satisfie~~

$G \subseteq N,\ G \neq \phi$

1. G is almost decisive over $\{a, b\}$ if

$$[a P_i b,\ \forall i \in G \text{ and } b P_j a\ \forall j \notin G]$$
$$\Rightarrow [a \hat{F}(R) b]$$

$\bar{D}_G(a,b)$: G is almost decisive over $\{a,b\}$ w.r.t. F.

2. G is decisive over $\{a,b\}$ if

$$[a P_i b\ \ \forall i \in G] \Rightarrow [a \hat{F}(R) b]$$

$D_G(a,b)$: G is decisive over $\{a,b\}$ w.r.t. F

Clearly, $D_G(a,b) \Rightarrow \bar{D}_G(a,b)$

Observe: ~~If a group G is decisive (or resp. almost) decisio over $\{a,b\}$ for some profile, it is decisive over $\{a,b\}$ for every profile. (IIA)~~

these notions are equivalent under WP and IIA F.

Field Expansion Lemma

Let F satisfy WP and IIA, then $\forall a, b, x, y$, $G \neq \phi$, $G \subseteq N$.
$a \neq b, x \neq y$

$$\bar{D}_G(a,b) \Rightarrow D_G(x,y) .$$

Proof: Cases to consider

① $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ i.e. $x = a$, $y \neq a, b$

② $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ i.e. $x \neq a,b$, $y = b$

③ $\bar{D}_G(a,b) \Rightarrow D_G(x,y)$ $x \neq a,b$, $y \neq a,b$

④ $\bar{D}_G(a,b) \Rightarrow D_G(x,a)$ $x \neq a,b$, $y = a$

⑤ $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$ $x = b$, $y \neq a,b$

⑥ $\bar{D}_G(a,b) \Rightarrow D_G(a,b)$

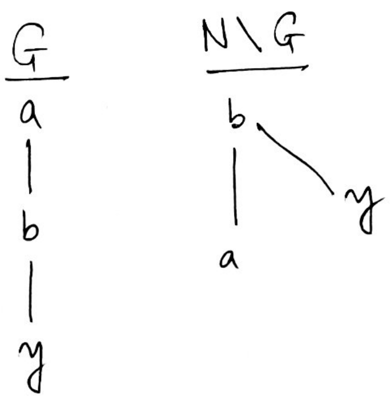
⑦ $\bar{D}_G(a,b) \Rightarrow D_G(b,a)$

Case① : Given $\bar{D}_G(a,b)$, need to show $D_G(a,y)$

i.e. pick arbitrary $R \in \mathcal{R}^n$ s.t. $a P_i y$ $\forall i \in G$

need to prove $a \hat{F}(R) y$.

Construct R'



ensure $R_i'|_{a,y} = R_i|_{a,y}$ $\forall i \in N$.

$\bar{D}_G(a,b) \Rightarrow a \hat{F}(R') b$

WP over b, y $\Rightarrow b \hat{F}(R') y$

$\xRightarrow{\text{transitivity}} a \hat{F}(R') y$

$\xRightarrow{\text{IIA}} a \hat{F}(R) y$

$\Rightarrow D_G(a,y)$.

(17-4)

Case ② : Given $\bar{D}_G(a,b)$, need to show $D_G(x,b)$

Pick R s.t. $x P_i b \ \forall i \in G$

need to show $x \hat{F}(R) b$

R'

G	$N \setminus G$
x	$x \quad b$
a	a
b	

$\cancel{R'_i|_{x,b,a} = R_i|_{x,a}}$

$$R_i'|_{x,b} = R_i|_{x,b}$$

$$\bar{D}_G(a,b) \Rightarrow a \hat{F}(R') b$$

$$\text{WP } x,a \Rightarrow x \hat{F}(R') a$$

$$\overset{\text{Transitivity}}{\Longrightarrow} x \hat{F}(R') b$$

$$\overset{\text{IIA}}{\Longrightarrow} x \hat{F}(R) b.$$

③ $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ case ① $y \neq a,b$

$\Rightarrow \bar{D}_G(a,y)$ definition

$\Rightarrow D_G(x,y)$ case ②

④ $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ case 2 $x \neq a,b$

$\Rightarrow \bar{D}_G(x,b)$

$\Rightarrow D_G(x,a)$ Case ①

⑤ $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ case ① $y \neq a,b$

$\Rightarrow \bar{D}_G(a,y)$

$\Rightarrow D_G(b,y)$ case ②

⑦ $\bar{D}_G(a,b) \overset{⑤}{\Rightarrow} D_G(b,y) \Rightarrow D_G(b,a)$
$y \neq a,b$

⑥ $\bar{D}_G(a,b) \overset{②}{\Rightarrow} D_G(x,b) \Rightarrow \bar{D}_G(x,b) \overset{②}{\Rightarrow} D_G(a,b)$
$x \neq a,b$