CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

1.1 Solution to Question 1

1.1.1 Part 1

It is given that the domain of preference is unrestricted. Hence we can construct any new preference profiles. Suppose we construct the following preference profiles:

P_1	P_2	P_1'	P_2'	P_1'	P_2''	P_1'	P_2
a	c	b	a	b	c	b	c
b	b	a	b	a	a	a	b
c	a	c	c	c	b	c	a

We are given that f is an onto strategyproof (implies monotonous) SCF with $f(P_1, P_2) = a$. Now $f(P_1', P_2') \in (P_1'(1), P_2'(1)) = \{b, a\}$. Let's assume that $f(P_1', P_2') = a$. We will later prove by contradiction that this cannot be the case. As $f(P_1', P_2') = a \implies f(P_1', P_2'') = c$. Because if this is not the case than player 2 can manipulate the outcome by chosing the preference profile P_2' instead of P_2''. By doing so, the new outcome would be a which is more preferred than b. Now $f(P_1', P_2) = c$ because the SCF is monotonous and the position of c has not changed in the transition from (P_1', P_2'') to (P_1', P_2). Finally consider the transition from (P_1', P_2) to (P_1, P_2). As $f(P_1', P_2) = c \implies f(P_1, P_2) = c$ because of monotonicity. But we are given that $f(P_1, P_2) = a$. This is a contradiction. Hence $f(P_1', P_2') = b$.

1.1.2 Part 2

1. No, the conclusion from Part 1 doesn't hold in the case that these preferences are generated from a single peaked preference domain. If we observe the proof of Part 1, we constructed a new preference ordering $P_2'' : c > a > b$ in order to prove the conclusion. But in the case of a single peaked preference domain with the intrinsic ordering of the alternatives being $a < b < c$, we can never have such a constructed preference ordering for any agent. Hence the conclusion of Part 1 fails here.

2. We need to design a mechanism that has $f(P_1', P_2') = a$. Consider an SCF that chooses as an outcome the top preference which is the leftmost given the intrinsic ordering of alternatives is $a < b < c$. According to this SCF $f(P_1', P_2') = a$. This SCF is clearly strategyproof too as the player whose top preference isn't the leftmost one, would have to move his peak preference to the left of the current peak. But this would be even less suitable for him as now the new outcome would be even further away from his peak preference than before. The player whose peak preference is the leftmost is already at the most optimum position and hence no reason whatsoever to manipulate his preference ordering.

1.2 Solution to Question 2

Notice that the agent i evaluates his preferences on non-empty subsets of X; i.e the alternatives available for him to rank in his preference ordering are all subsets of X except the null set. We are given that $|X| \geq 2$. Hence $\mathcal{P}(X)$ (the set of alternatives) contains atleast 3 elements ($2^{|X|} - 1$ elements in $\mathcal{P}(X)$). Thus number of available alternatives is greater than or equal to three. Hence the primary condition for applying the Gibbard-Satterthwaite result (i.e. $|A| \geq 3$) is valid. Now let us consider two sets: $S, T \in \mathcal{P}(X)$. Also consider that $S \subseteq T$. As S is a subset of T, all projects in S are also there in T. Now, there are only two possible cases; either both S and T have the same highest ranked project or the highest ranked project of T is better than S. Thus, by the definition of the preference ordering provided for any agent, i is either indifferent between S and T or he likes T to S. Thus there can no preference ordering where S would be ranked higher than T for any agent. **The Gibbard-Satterthwaite result doesn't apply when the domain is restricted; hence the result is not applicable here.**

1.3 Solution to Question 3

Let us assume for the time being that the median voter SCF is not group strategy-proof. Thus there exists a group of agents $K \subseteq N$ that can manipulate the decision in their favour **for all the agents in that group** by changing their reported preferences, i.e. for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P_K)P_i f(P_K, P_{-K})\ \forall i \in K$. Now consider the median voter SCF under single peaked domain model. If the entire group of K agents has the same peak preference which is also the median, then none of the group agents would want to manipulate the result, and hence there are no deviations in this case. Now consider this group in a general case under the median voter SCF. Some of the agents' peak preferences would lie to the left of the current median, whereas some would have their peak preferences to the right of the median. In order to manipulate the decision, this group needs to change the median peak. To do so, some of the agents of the group would have to shift their peak reported preferences from one side of the median to the other side of the median (For example if the group wants to manipulate the median to go the right, then some agents from the left side would have to cross the median and report their peak preferences to the right of the median. Also if they stay on the same side then the median is left unchanged). But this is even less preferable to this part of the group of agents as they have moved the new median (the updated decision) even further away from their true preferences. Thus they are worse off than before, and hence they don't have any incentive to manipulate the current decision. Note that the case where all the peaks of the group are to one side of the median is just a special case of the above generalized case itself. Hence, there is no such possible group that can manipulate the decision of the median SCF. **Hence it is group strategy proof. Hence, proved!**