CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

7.1 Correlation Equilibrium

7.1.1 Motivation

In the previous lectures, we have been dealing with several types of Strategy Equilibriums and we finally arrived at the Mixed Strategy Nash Equilibrium which was the weakest and most general amongst all. The most special property of MSNE is that it always exists for any finite game and can be found by solving a finite number of equations. However, calculating the MSNE is computationally difficult and our search for another equilibrium leads to the **Correlation Equilibrium**.

Under Nash Equilibrium, each player chooses his strategy independent of the other player, which may not always lead to the best outcome. However if the players trust a third-party agent, who randomizes certain choices and suggests strategies to players accordingly, the outcomes can be significantly better. Such a strategy is called Correlation Strategy.

7.1.2 Definition

The correlation strategy is a Correlation equilibrium if it becomes self-enforcing, i.e. any player does not gain any advantage in deviating from the suggested strategy. Therefore, following the suggested Correlation Equilibrium strategy is the best response for all the players.
Note: Here the randomization process is a Common Knowledge.

Definition 7.1 *A* Correlation Strategy *is a mapping* $\Pi : S \mapsto [0,1]$ *such that* $\sum_{s \in S} \Pi(s) = 1$ *where* $S = S_1 \times S_2 \times \ldots \times S_n$ *and* S_i *represents the Strategy Profile of player i.*
Here, Π *is the joint probability distribution over the Strategy Profiles.*

Definition 7.2 *A* Correlated Equilibrium *is a correlated strategy* Π *such that* $\forall s_i, s_i'$ *and* $\forall i \in N$,

$$\sum_{s_{-i} \in S_{-i}} \Pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \Pi(s_i, s_{-i}) u_i(s_i', s_{-i}) \qquad \forall s_i' \in S_i. \tag{7.1}$$

This means that the player i does not gain any advantage in the expected utility if he deviates from the suggested correlated strategy Π, assuming all other players follow the suggested strategy.
The following examples would help you understand the definitions better.

7.1.3 Examples

7.1.3.1 Game Selection Problem

In the problem, two friends want to go to watch a game together, however Player 1 like Cricket more and Player 2 likes Football. The Utility Function is represented in the form:

1 \2	C	F
C	(2,1)	(0,0)
F	(0,0)	(1,2)

In MSNE, we saw that the expected utility of each player was $\frac{2}{3}$.
However if the Correlated strategy is such that $\Pi(C, C) = \frac{1}{2} = \Pi(F, F)$. If we assume that Player 1 is suggested to choose F, then the expected utility from following the suggestion is given as

$$\sum_{s_{-1}\in S_{-1}} P(s_{-1}|F)u_1(F, s_{-1}) = \frac{1}{P(F)}[P(F, C)u_1(F, C) + P(F, F)u_1(F, F)] = \frac{1}{\frac{1}{2}}\left[0 + \frac{1}{2}1\right] = 1 \tag{7.2}$$

where P(F) is the probability that F is suggested to Player 1, $P(s_{-1}|F)$ is the probability that s_{-1} is strategy of other players when 1 is suggested F and P(F,F) is probability that (F,F) is the strategy.
If Player 1 deviates from the strategy, then his expected utility is

$$\sum_{s_{-1}\in S_{-1}} P(s_{-1}|F)u_1(C, s_{-1}) = \frac{1}{P(F)}\,[P(C, F)u_1(C, C) + P(F, F)u_1(C, F)] = \frac{1}{\frac{1}{2}}[0 + 0] = 0. \tag{7.3}$$

Similarly, if C is suggested to Player 1, his expected utility is 2 when he follows the suggestion and 0 when he does not follow. This proves that the Correlated strategy here is a correlated equilibrium.
It is notable here that utility at the expected equilibrium is $\frac{1}{2}(1 + 2) = \frac{3}{2}$ as compared to $\frac{2}{3}$ in MSNE.

7.1.3.2 Traffic Accident Problem

In the problem, two people wish to cross the road. Their utilities are positive if they cross the road, by they cross together, they will collide. The Utility Function is represented in the form:

1 \2	Stop	Go
Stop	(0,0)	(1,2)
Go	(2,1)	(-10,-10)

The Correlated strategy is such that $\Pi(S, T) = \Pi(S, S) = \Pi(T, S) = \frac{1}{3}$. If we assume that Player 1 is suggested to choose Stop, then the expected utility from following the suggestion is given as

$$\sum_{s_{-1}\in S_{-1}} P(s_{-1}|S)u_1(S, s_{-1}) = \frac{1}{P(S)}[P(S, S)u_1(S, S) + P(S, G)u_1(S, G)] = \frac{1}{\frac{2}{3}}\left[0 + \frac{1}{3}1\right] = \frac{1}{2}. \tag{7.4}$$

where P(S) is the probability that S is suggested to Player 1, $P(s_{-1}|S)$ is the probability that s_{-1} is strategy of other players when 1 is suggested S and P(S,S) is probability that (S,S) is the suggested strategy.
If Player 1 deviates from the strategy, then his expected utility is

$$\sum_{s_{-1}\in S_{-1}} P(s_{-1}|S)u_1(G, s_{-1}) = \frac{1}{P(S)}[P(S, S)u_1(G, S)+P(S, G)u_1(G, G)] = \frac{1}{\frac{2}{3}}\left[\frac{1}{3}2 + \frac{1}{3}(-10)\right] = -4. \tag{7.5}$$

If G is suggested to 1, the expected utility is 2 on following the suggestion and 0 on deviating from the suggestion.

The Chicken Game presents a similar example as well.
Note: The Correlation Equilibrium is generally not unique for any game and depends upon the randomization process. In our example itself, $\Pi(S,T) = \Pi(T,S) = \frac{1}{2}$ is also a Correlation Equilibrium with expected utility of $\frac{3}{2}$ for each player as compared to 1 in the given example and $\frac{2}{13}$ for MSNE.

7.1.4 Interpretation

For $\bar{s_i}$ be the strategy suggested to player i, then it is a Correlation Equilibrium if $\forall i \in N$:

$$\sum_{s_{-i} \in S_{-i}} P(s_{-i}|\bar{s_i}) u_i(\bar{s_i}, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} P(s_{-i}|\bar{s_i}) u_i(s'_i, s_{-i}) \qquad \forall s'_i \in S_i. \tag{7.6}$$

$$\Rightarrow \sum_{s_{-i} \in S_{-i}} P(\bar{s_i}, s_{-i}) u_i(\bar{s_i}, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} P(\bar{s_i}, s_{-i}) u_i(s'_i, s_{-i}). \tag{7.7}$$

which translates to:
For every player, the expected utility while following the suggestion is always at least as good as the expected utility after deviating from the suggestion, given that all other players follow the suggestion.

Computing the Correlated Equilibrium:
As by 7.1, we know $\forall s_i \in S_i$, and $\forall i \in N$

$$\sum_{s_{-i} \in S_{-i}} \Pi(s) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \Pi(s) u_i(s'_i, s_{-i}) \qquad \forall s'_i \in S_i. \tag{7.8}$$

The total number of inequalities are nm^2
Here, s is the suggested strategy and $\Pi(s)$ is the probability of strategy s after randomization by the agent. Therefore

$$\begin{aligned} \Pi(s) \geq 0 \qquad & \{m^n \text{ inequalities}\} \\ \sum_{s \in S} Pi(s) = 1 \qquad & \{\text{This is the Feasibility L.P.}\} \end{aligned}$$

For any MSNE,the number of Support profiles are 2^{mn}
Therefore, no. of equations in MSNE (2^{mn}) is exponentially larger than CE (m^n), so is the computational time. It can also be proven that every MSNE is a CE. The whole situation can be represented as:

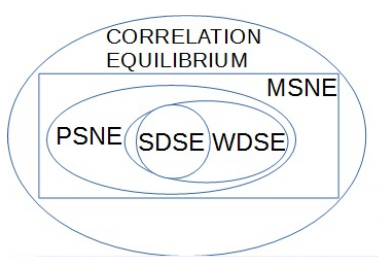

Figure 7.1: Representation of Equilibriums

7.2 Extensive Form Games

7.2.1 Perfect Information Extensive Form Games

Normal Form Games are when players take their actions simultaneously, the mutual actions then decide the outcome. On the other hand, in **Extensive form games**, players take actions depending upon the *history* of actions taken in the game, and the final action decides the outcome. While NFGs are easily represented in matrices, EFGs are best represented in a tree-like structure.
In a **Perfect Information EFG**, every player knows about the *history* of actions taken till that time. The game below is an example of PIEFG.

7.2.2 Example: Chocolate Division Game

The game is such that mother gives his son 2 chocolates (indivisible) to divide between him and his sister. After the brother divides chocolates, her sister may Accept the division or may Reject it- after which the mother would take away both the chocolates. This can be represented in the following form:

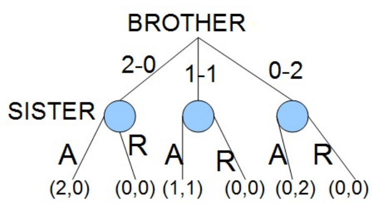

Figure 7.2: B-S Chocolate Game Representation

7.2.3 Notations

We now formally define the representation of a PIEFG as below:

$$< N, A, \chi, H, P, (u_i)_{i\in N} >, \qquad \text{where,}$$

N: Set of players A: Set of all possible Actions
H: Set of all *sequences of actions* (histories) satisfying:
1) empty sequence $\Phi \in H$.
2) If $h \in H$, any initial continuous sub-sequence of h, $h' \in$ H.
A history $h = (a^{(0)}, a^{(1)}, \ldots, a^{(T-1)})$ is *terminal* if $\exists\ a^{(T)} \in As.t.a^{(0)}, a^{(1)}, \ldots, a^{(T-1)}, a^{(T)}) \in H$.

Z : Set of all *terminal* histories. (Z$\in H$) χ : H\Z $\mapsto 2^A$ - Action Function
P : H\Z $\mapsto N$ Player Function $u_i : Z \mapsto \Re$ - utility of player i.

7.2.4 Representing the Chocolate Division Game

For the given game and equipped with our notations, we represent the game as

N = {B,S} $\quad$ A = $\{\ 2-0, 1-1, 0-2, A, R\}$
H = $\{\Phi, (2-0), (1-1), (0-2), (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$
Z = {(2-0,A), (2-0,R), (1-1,A), (1-1,R), (0-2,A), (0-2,R)}
$\chi(\Phi) = \{(2-0), (1-1), (0-2)\}$ $\quad$ $\chi(2-0) = \chi(1-1) = \chi(0-2) = \{A, R\}$
P(Φ) = B $\quad$ P(2-0) = P(1-1) = P(0-2) = S
$u_1(2-0, A) = 2 \quad u_1(1-1, A) = 1 \quad u_2(1-1, A) = 1 \quad u_2(0-2, A) = 2$
$u_1(0-2, A) = u_1(0-2, R) = u_1(1-1, R) = u_1(2-0, R) = 0$
$u_2(0-2, R) = u_2(1-1, R) = u_2(2-0, R) = u_2(2-0, A) = 0$

Here, the strategy of player i is given by the complete contingency plan.
$S_i = \chi \times (h) \quad \{h \in H : P(h) = i\}$
$S_1 = \{2-0, 1-1, 0-2\}$.
$S_2 = \{Y, N\} \times \{Y, N\} \times \{Y, N\} = \{YYY, YYN, YNY, YNN, NYY, NYN, NNY, NNN\}$

7.2.5 Representing PIEFG as NFG

In S_1 and S_2, it can be seen that S(Sister) has an altogether different strategy corresponding to each action of B. The situation with given S_1 and S_2 is same as that of a NFG, and hence can also be represented in the form of matrix. On observation, we see that this can be generalised for all PIEFG, i.e. each PIEFG can be represented as a NFG. For the given example, we can express the utility function as in the following table:

B \S	YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
2-0	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)
1-1	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)

In NFG, as the existence of Nash Equilibrium is guaranteed, we search for Pure strategies in the given game, which leads to some unintuitive solutions. The PSNEs are marked in **Bold**.

B \S	YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
2-0	**(2,0)**	**(2,0)**	**(2,0)**	**(2,0)**	(0,0)	(0,0)	**(0,0)**	**(0,0)**
1-1	(1,1)	(1,1)	(0,0)	(0,0)	**(1,1)**	**(1,1)**	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	**(0,2)**	(0,0)

Some of the results like {2-0,NNY}, {2-0,NNN} and {0-2} are not practically useful. As the representation is very clumsy and does not provide us with any advantage, the NFG representation is wasteful and the EFG representation is succinct for such cases. The example also forces us to look for some other equilibrium ideas as the Nash Equilibrium does not serve much purpose in this case.