CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2018**

Assignment 2

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Ans 1:

P_1	P_2	P'_1	P'_2	P'_1	P''_2	P'_1	P'''_2
a	c	b	a	b	c	b	c
b	b	a	b	a	a	a	b
c	a	c	c	c	b	c	a

Part(a): Let $f(P'_1, P'_2) = a$
Claim 1: $f(P'_1, P''_2) = c$**:**
Suppose this is not true, then $f(P'_1, P''_2) = b$ since $f(P'_1(1), P''_2(1)) \in \{P'_1, P''_2\}$. But then, $f(P'_1, P'_2) P''_2 f(P'_1, P''_2)$. This means that f in not strategy proof since player 2 is better off by choosing P'_2 in the preference profile (P'_1, P''_2). Hence, $f(P'_1, P''_2) = c$

Claim 2: $f(P'_1, P'''_2) = c$**:**
Suppose this is not true, then $f(P'_1, P'''_2) = b$ since $f(P'_1(1), P'''_2(1)) \in \{P'_1, P'''_2\}$. But then, $f(P'_1, P''_2) P'''_2 f(P'_1, P'''_2)$. This means that f in not strategy proof since player 2 is better off by choosing P''_2 in the preference profile (P'_1, P'''_2). Hence, $f(P''_1, P'''_2) = c$

But, because of claim 2, $f(P_1, P_2) P'_1 f(P'_1, P'''_2)$. This means the f is not strategy-proof because player 1 can manipulate by choosing P_1 in the preference profile (P'_1, P'''_2).. Therefore, our assumption $f(P'_1, P'_2 = a)$ is false. Hence, $f(P_1, P'_2) = b$ because $f(P_1, P'_2) \in \{P_1, P'_2\}$. Hence proved.

Part(b): The earlier proof will not work for this case, because the preference P''_2 can never exist for a player in single peaked preference where $a < b < c$. This is the reason, that is since the total number of preference profile are restrictive for the single peaked preference, that it doesn't hold GS result.

Yes, the social choice function which always gives the output the lowest peak among the players will give the output a. Since, this is also a median voter social choice function, hence f is strategy-proof and onto as well. Hence, this results holds for single-peaked preference.

Ans 2:

No, the Gibbard Satterthwaite theorem will not hold, since all the possible orderings over the alternative(subsets of X) are not possible. For eg: for any preference profile, the complete set X should always be at the top, as X will always contains the top preferred object in the linear ordering P_i over the set of projects X. Similarly, for any A $\supset$ B, A will always be weakly preferred over B. Hence, the GS theorem will not hold for this case.

Ans 3: Yes, the median voter SCF group is strategy-proof. Assume for contradiction that this is not strategy-proof. This implies that there exists the group of agents K such that
$f(P'_K, P_{-K})P_i f(P_K, P_{-K}) \forall i \in K$.

Case 1: All the agents in K has the peak preference less than $f(P_K, P_{-K})$:
Note that the agents in K can never decrease the value of $f(P_K, P_{-K})$ since all the agents has their peaks of lower value than that, and all the other players peak preference will not change. Hence $f(P'_K, P_{-K}) \geq f(P_K, P_{-K})$ or $f(P_K, P_{-K})P_i f(P'_K, P_{-K})$ for all $i \in K$. Hence case 1 is not possible.

Case 2: If all the agents in K has the peak preference greater than $f(P_K, P_{-K})$ then again by the similar arguments above $f(P'_K, P_{-K}) \leq f(P_K, P_{-K})$ or $f(P_K, P_{-K})P_i f(P'_K, P_{-K})$ for all $i \in K$. Hence case 2 is not possible.

Case 3: If even one agent(lets say i) in the K has the peak at $f(P_K, P_{-K})$ then $f(P_K, P_{-K})P_i f(P'_K, P_{-K})$ where $i \in K$. Hence case 3 is also not possible.

Case 4: If some agents in K has peak on left side of $f(P_K, P_{-K})$ **and some agents in K has peak on the right hand side.**
This means that $\exists i, j \in K$ such that $P_i(1) < f(P_K, P_{-K}) < P_j(1)$. So for any value which is not equal to $f(P_K, P_{-K})$ either i or j preference strictly decreases. Therefore, this case is also not possible.

Since case 1-4 are the exhaustive list of cases and neither of the case satisfies. Hence our assumption is wrong. This implies that median voter SCF is group strategy-proof.