 Proof of Bondareva - Shapley Theorem (second formulation)

Proof: Consider the following linear program to check the feasibility of the core

$$\text{minimize} \quad \sum_{i \in N} x_i \qquad \cdots (1)$$

$$\text{s.t.} \quad \sum_{i \in S} x_i \geq v(S) \qquad \forall S \subseteq N.$$

The value of this OPT problem is at least $v(N)$.

Claim: If there is a non-empty core then the $OPT = v(N)$.

if $OPT > v(N) \iff$ core empty

Consider the dual of (1)

$$\text{maximize} \sum_{S \subseteq N} \lambda(S) v(S)$$

$$\text{s.t.} \quad \sum_{S \subseteq N : i \in S} \lambda(S) = 1 \qquad \forall i \in N$$

$$\lambda(S) \geq 0 \quad \forall S \subseteq N.$$

The constraints are that of a balanced* weights.

Weak duality

$$\sum_{S \subseteq N} \lambda(S) v(S) \leq \underbrace{\sum_{i \in N} x_i^*}_{\text{Primal solution}} = \underbrace{v(N)}_{\text{core is non-empty}}$$

for all balanced* weights

□

A coalitional game satisfying B-S condition is called a balanced game, i.e., $\forall$ balanced weights λ

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S).$$

Market Games

Concentrate on coalitional games that arises naturally in practice — and apply the B–S theorem to prove non-empty cores.

- Producers $N = \{1, 2, \ldots, n\}$ trade L commodities
- Set of commodities, $\mathcal{C} = \{1, 2, \ldots, L\}$

  e.g., different kinds of raw materials

  wood, metal, human resources, expert consultation hours etc.

- A commodity vector is denotes by $x \in \mathbb{R}^L_{\geq 0}$

  x_j, $j = 1, \ldots, L$ denote the amount/quantity of commodity j. assuming these are fluid items.

  Refer this as a "bundle". Bundle of producer i is denoted by $x_i \in \mathbb{R}^L_{\geq 0}$, x_{ij} is the quantity of commodity j player i gets.

- Production/Utility function of producer i

  $$u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$$

  $u_i(x_i)$: The amount of money producer i can generate from the bundle x_i. [Ex. $p^T x_i$]

- Initial endowment of producer i is $a_i \in \mathbb{R}^L_{\geq 0}$

Coalitional strategy:

If a coalition S forms, the members trade commodities among themselves/pool them. The goal is to maximize the total money generated.

Total endowment of S, $\quad a(S) = \sum_{i \in S} a_i \in \mathbb{R}^L_{\geq 0}$

The coalition can only redistribute these items among its members, $x_i \in \mathbb{R}^L_{\geq 0}$ with

$$x(S) = \sum_{i \in S} x_i = a(S)$$

Hence, by redistributing the items They can generate a collective wealth of $\sum_{i \in S} u_i(x_i)$

Defn: A market is given by a vector $(N, C, (a_i, u_i)_{i\in N})$ where

- $N = \{1, 2, \ldots, n\}$ is the set of producers
- $C = \{1, 2, \ldots, L\}$ is the set of commodities
- $\forall i \in N$, $a_i \in \mathbb{R}^L_{\geq 0}$ is the initial endowment of producer i.
- $\forall i \in N$, $u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ is the production function of i.

Set of allocations for coalition S

$$X^S := \{(x_i)_{i\in S} : x_i \in \mathbb{R}^L_{\geq 0} \ \forall i \in S,\ x(S) = a(S)\}$$

Result: For every coalition S, X^S is compact [Closed and Bounded]

Assume all production functions are continuous.

Worth of ~~each~~ coalition S

$$v(S) = \max\left\{ \sum_{i\in S} u_i(x_i) : x = (x_i)_{i\in S} \in X^S \right\} \quad \text{-- (1)}$$

Since u_i's are continuous and X^S is compact the maxima is attained within X^S.

Example: $N = \{1, 2, 3\}$, $C = \{1, 2\}$

- $a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = (2, 2)$
- $u_1(x_1) = x_{11} + x_{12}$, $u_2(x_2) = x_{21} + 2x_{22}$

$u_3(x_3) = \sqrt{x_{31}} + \sqrt{x_{32}}$

$v(1) = 1$, $v(2) = 2$, $v(3) = 2\sqrt{2}$

Compute $v(123)$, leave $v(12), v(13), v(23)$ as exercise

Consider $\sum_{i=1}^{3} u_i(x_i)$ every unit of 1 contributes equally for producers 1 and 2, and for that of unit 2, producer 2's share contributes twice as that of 1. Hence in a maximum utility, $x_1 = (0,0)$ and the whole share of 1 can be transferred to 2.

$$v(123) = \max\left\{ x_{21} + 2x_{22} + \sqrt{3 - x_{21}} + \sqrt{3 - x_{22}} : 0 \leq x_{21} \leq 3,\ 0 \leq x_{22} \leq 3 \right\}$$

$$x_2 = \left(\frac{11}{4}, \frac{47}{16}\right), \quad x_3 = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Defn: A coalitional game (N, v) is a <u>market game</u> if $\exists\, L > 0$, $\forall i \in N$ $\exists\, a_i \in \mathbb{R}^L_{\geq 0}$, and $u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ continuous and concave $\forall i \in N$ s.t. eq. 1 is satisfied for all non-empty $S \subseteq N$.

Theorem (Shapley and Shubik (1969))
The core of a market game is non-empty.

Proof: We'll use B-S theorem to prove this result.
To prove: every market game is a balanced game.
Consider a market game $(N, \mathcal{C}, (a_i, u_i)_{i \in N})$.
Fix an arbitrary coalition S
let $x^S = (x_i^S)_{i \in S}$ be the allocation that maximizes $\sum_{i \in N} u_i(x_i^S)$ - by definition of u_i, $x^S \in X^S$.

We have

- $x_i^S \in \mathbb{R}^L_{\geq 0}$
- $x^S(S) = \sum_{i \in S} x_i^S = a(S)$
- $\sum_{i \in S} u_i(x_i^S) = v(S)$

(6-5)

let $\delta = (\delta_S)_{S \subseteq N}$ be a balanced weight vector (arbitrary)

$$v(N) \geq \sum_{S \subseteq N} \delta_S v(S) ,$$

Define, $$z_i := \sum_{\{S \subseteq N : i \in S\}} \delta_S x_i^S \in \mathbb{R}^L_{\geq 0}$$

Claim: z_i is a feasible bundle, i.e. $\sum_{i \in N} z_i = a(N)$

Pf: $$z(N) = \sum_{i \in N} z_i = \sum_{i \in N} \sum_{S \subseteq N : i \in S} \delta_S x_i^S .$$

$$= \cancel{\sum_{S \subseteq N : i \in S} \delta_S \sum_{i \in N} x_i^S} = \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \delta_S x_i^S$$

$$= \sum_{S \subseteq N} \sum_{i \in N} I\{i \in S\} \delta_S x_i^S$$

$$= \sum_{S \subseteq N} \sum_{i \in S} \delta_S x_i^S = \sum_{S \subseteq N} \delta_S \sum_{i \in S} x_i^S$$

$$= \sum_{S \subseteq N} \delta_S x^S(S) = \sum_{S \subseteq N} \delta_S a(S)$$

by definition of x^S

$$= \sum_{S \subseteq N} \delta_S \sum_{i \in S} a_i$$

$$= \sum_{i \in N} \sum_{S \subseteq N : i \in S} \delta_S a_i$$

$$= \sum_{i \in N} a_i \underbrace{\left(\sum_{S \subseteq N : i \in S} \delta_S \right)}_{=1} = a(N).$$

By definition of v

$$
\begin{aligned}
v(N) &\geq \sum_{i \in N} u_i(z_i) \\
&= \sum_{i \in N} u_i\Big(\sum_{S \subseteq N : i \in S} \delta_S x_i^S\Big) \\
&\geq \sum_{i \in N} \sum_{S \subseteq N : i \in S} \delta_S u_i(x_i^S) \\
&= \sum_{S \subseteq N} \sum_{i \in S} \delta_S u_i(x_i^S) \\
&= \sum_{S \subseteq N} \delta_S \sum_{i \in S} u_i(x_i^S) \\
&= \sum_{S \subseteq N} \delta_S v(S) \quad \text{[Balanced condition]}. \quad \square
\end{aligned}
$$

If the producers leave from market, leading to $(N, C, (a_i, u_i)_{i \in N})$ being reduced to $(S, C, (a_i, u_i)_{i \in S})$

- We can define a restriction of v in (N, v) to the v restricted to S which is same as $v(T)$ $\forall T \subseteq S$. Hence we can consider subgame (S, v) of the market game (N, v)

Let $(S, \tilde{v})$ be the reduced game, $\forall T \subseteq S$

$$
\begin{aligned}
\tilde{v}(T) &= \max\Big\{\sum_{i \in T} u_i(x_i) : x_i \in \mathbb{R}^L_{\geq 0}\ \forall i \in T, \sum_{i \in T} x_i = \sum_{i \in N} a_i\Big\} \\
&= v(T)
\end{aligned}
$$

Corollary [of Shapley-Shubik theorem]

If (N, v) is a market game, every subgame (S, v) of it is a market game, and in particular is balanced.

Such games are called totally balanced.

A coalitional game is totally balanced if every subgame of it has non-empty core.

Restatement of Shapley - Shubik:

Every market game is totally balanced.

The converse of this result is also true.

Theorem: Every totally balanced game is a market game.