CS-698w: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 27: Game Theory – Pareto Optimality

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

27.1 Pareto Optimality

A direct mechanism $(f, (p_1, p_2, \ldots, p_n))$ is pareto optimal if at every type profile $\theta \in \Theta$, $b \in A$ and payment $(\pi_1, \pi_2 \ldots, \pi_n)$ having $\sum_{i \in N} \pi_i \geq \sum_{i \in N} p_i$ such that,

$$v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - p_i(\theta) \quad \text{for all } i \in N$$

and strict for some $j \in N$

27.2 Allocatively Efficient

Recall that

$$f^{AE}(\theta) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$$

Theorem 27.1 *A mechanism (f,p) is pareto optimal iff it is Allocatively Efficient(AE).*

Proof: We first prove that if a mechanism is pareto optimal then it is AE. To do so we show that

$$!AE \quad \Rightarrow \quad ! \text{ Pareto Optimal}$$

Since f is not AE $\exists b \in A$ s.t

$$\sum_{i \in N} v_i(b, \theta_i) > \sum_{i \in N} v_i(f(\theta), \theta_i)$$

Let

$$\delta = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > 0$$

Define

$$\pi_i = v_i(b, \theta_i) - v_i(f(\theta), \theta_i) + p_i(\theta) - \delta/n$$

$$\Rightarrow \quad \delta/n = (v_i(b, \theta_i) - \pi_i) - (v_i(f(\theta), \theta_i) - p_i(\theta)) > 0$$

Also, $\sum_{i \in N} \pi_i > \sum_{i \in N} p_i$. Hence, (f, p) is not pareto optimal. Now we show that

$$!\text{Pareto Optimal} \quad \Rightarrow !\text{AE}$$

$\exists b, \pi$ such that,

$$\sum_{i \in N} \pi_i > \sum_{i \in N} p_i \quad \text{for all } \theta$$

and

$$v_i(b, \theta_i) - \pi_i \geq v_(f(\theta), \theta_i) - p_i(\theta) \text{ for all } i \in N \text{ and strict for some } j \in N$$

$$\begin{aligned}
& \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} \pi_i > \sum_{i \in N} v_i(f(\theta), \theta_i) - \sum_{i \in N} p_i \\
\Rightarrow \quad & \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > \sum_{i \in N} \pi_i - \sum_{i \in N} p_i \geq 0 \\
\Rightarrow \quad & f \text{ is not AE}
\end{aligned}$$

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27.3 Allocation

Efficient rule is implementable. $\exists p$ such that (f, p) is DSIC.

27.3.1 Groves class of payment

Groves payment

$$p_i{}^G(\theta) = h_{-i}(\theta_{-i}) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$$

where $h_i : \theta_i \mapsto R$ is arbitrary.

27.3.2 Example

There are 4 agents and one indivisible item. Value of the agents are 10, 8, 6 and 4 respectively.when they receive the item and zero otherwise.

$$h_{-i}(\theta_{-i}) = 10 \text{ for all } \theta_{-i} \text{ for all } i$$

Hence the payments are 10 , 0 , 0 and 0 respectively.

Theorem 27.2 *Groves mechanism is DSIC*

Proof: For Agent i, the type is θ_i and the reported type is $\hat{\theta}_i$. Also,

$$f^{AE}(\theta_i, \theta_{-i}) = a$$

and

$$f^{AE}(\hat{\theta}_i, \theta_{-i}) = a$$

Utility when agent i when he reports θ_i,

$$v_i(f^{AE}(\theta), \theta_i) - {p_i}^G(\theta) = v_i(f^{AE}(\theta), \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$$

$$\Rightarrow \qquad \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - h_i(\theta_{-i}) \geq \sum_{j \in N} v_j(b, \theta_j) - h_i(\theta_{-i})$$

27.4 The Vickrey Clarke Groves Mechanism

A mechanism in the Groves classs is also called pivotal mechanism. It is characterized by specific $h_i(\theta_{-i})$,

$$h_i(\theta_{-i}) = \arg\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

and

$$p_i(\theta) = \arg\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j) \geq 0$$

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