Lec 13

Two sided matching

- More widely used because of its application domain.

Examples:

- Marriage and ~~d~~ dating markets
- Medical residencies
- University - Student matching (IITJEE seat allocation)
- Job market : Employers and Candidates

We will refer to this setting as marriage problem

M : set of men

W : set of Women

For simplicity, we assume $|M| = |W|$, but this is not necessary. The results extend to more general settings too.

Every $m \in M$ has a strict preference P_m over W.

Similarly $w \in M$ has a strict preference P_w over M.

$$x \, P_m \, y \iff m \text{ strictly prefers } x \text{ over } y, \; x, y \in W.$$

A matching is a bijective mapping $\mu : M \to W$.

$\mu(m)$: woman matched to m

$\mu^{-1}(w)$: man matched to w.

Stable marriage problem

Stability in this context is slightly different from the one-sided matching model.

Example: $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$

Table 1:

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

A candidate matching:

$$\mu : \mu(m_1) = w_1 \,,\ \mu(m_2) = w_2 \,,\ \mu(m_3) = w_3$$

but, $w_2 \, P_{m_3} \, \mu(m_3)$ and $m_3 \, P_{w_2} \, \mu^{-1}(w_2)$

(m_3, w_2) can move out and block this matching.

Defn (Stability): A matching μ is pairwise unstable at a preference profile P if $\exists$ m, m' such that (a) $\mu(m') \, P_m \, \mu(m)$ and (b) $m \, P_{\mu(m')} \, m'$.

- The pair $(m, \mu(m'))$ is called a blocking pair of μ at P.
- If a matching μ has no blocking pairs at ~~any~~ a preference profile P, then it is called a pairwise stable matching at P.

Stable
$m_1 - w_1$
$m_2 - w_3$
$m_3 - w_2$

Questions:

1. Existence of pairwise stable matching?
2. Why pairwise? Contrast with the one-sided matching — a group of people can redistribute their initial endowments and be strictly better off. Group blocking.

Group blocking:

A coalition $S \subseteq (M \cup W)$ blocks a matching μ at a profile P if $\exists$ another matching μ' s.t.

(i) for all $m \in M \cap S$, $\mu'(m) \in W \cap S$, and for all $w \in W \cap S$, $\mu'^{-1}(w) \in M \cap S$, and

(ii) for all $m \in M \cap S$, $\mu'(m) \; P_m \; \mu(m)$ and for all $w \in W \cap S$, $\mu'^{-1}(w) \; P_w \; \mu^{-1}(w)$.

A matching μ is in the core of the induced coalitional game at a profile P if no coalition can block μ at P.

The following result shows that this condition is equivalent to pairwise blocking.

Theorem: A matching is pairwise stable at a profile iff it belongs to the core at that profile.

Pf: ($\Leftarrow$) direction is trivial. If no coalition of arbitrary size can block the matching, clearly a coalition of size 2 cannot block it — hence pairwise stable.

($\Rightarrow$)

(13-4)

($\Rightarrow$) Let μ be pairwise stable at P. For contradiction, assume μ is not in the core at P.

Then $\exists$ $S \subseteq (M \cup W)$ and a matching $\hat{\mu}$ such that for all $m \in M \cap S$ and $w \in W \cap S$ with $\hat{\mu}(m), \hat{\mu}^{-1}(w) \in S$ we have $\hat{\mu}(m)\ P_m\ \mu(m)$ and $\hat{\mu}^{-1}(w)\ P_w\ \mu^{-1}(w)$.

This means that $\exists$ $m \in M \cap S$ s.t. $\hat{\mu}(m) \in W \cap S$

Call $\hat{\mu}(m) = w$, hence

$$w\ P_m\ \mu(m) \quad \text{and} \quad m\ P_w\ \mu^{-1}(w)$$

hence (m, w) is a blocking pair of μ at P. This is a contradiction to μ being pairwise stable at P. □

Stable from now on will refer to pairwise stability.

Answering the other question of existence.

<u>Deferred Acceptance Algorithm</u> (Gale - Shapley)

A stable matching always exist in a marriage market. This is proved via exhibiting an algorithm to find such a matching.

2 versions : men-proposing and women-proposing

One-side of the market proposes the other side, and the proposed agent may accept or reject the offer.

Men-proposing Deferred Acceptance Algorithm

Step 1 : Every man proposes their top-ranked woman

Step 2: Every woman who got at least one proposal *tentatively* keeps the top man among the received proposals and rejects the rest.

Step 3: Every man who was rejected in the last round, proposed to the top woman who has not rejected him in earlier rounds.

Step 4: Every woman who gets at least one proposal, including the tentative accepted proposal tentatively keeps the top man and rejects the rest.

The process is repeated from step ~~5~~ 3 till each woman gets at least one proposal, at this point the tentative accepts become final accepts.

Example: Construct the men and Women proposing versions in table1

Illustration: see the app www.facebook.com/adstudmatch/

Remarks:

- Since each woman is allowed to keep only one proposal, no woman gets more than one man
- Similarly, if a man's proposal is tentatively accepted he is not allowed to propose more, that ensures one woman is assigned to one man.
- The algorithm terminates in finite steps since the set of woman a man proposes does not increase and strictly decreases for at least one man.
- This also shows that the algorithm terminates in a matching.

Stability and Optimality of the DA algorithm

Thm: At every preference profile, the DA algorithm terminates at a stable matching for that profile.

Proof: Consider men-proposing DA algorithm (similar proof for women-proposing) for a preference profile P.
Let μ be the matching of the DA algorithm.
Assume for contradiction, μ is not stable.
Hence $\exists\ m \in M$ and $w \in W$ s.t. (m, w) is a blocking pair. By assumption, ~~me~~ $w \neq \mu(m)$ ~~hence~~ and $w\ P_m\ \mu(m)$. Then in this algorithm m must have proposed w at some round and ~~was~~ have been rejected before being matched to $\mu(m)$. But w rejected m since she got a better proposal. Therefore $\mu^{-1}(w)\ P_w\ m$. This contradicts the fact that (m, w) is a blocking pair. $\square$

Questions:

1. Men-proposing and Women-proposing versions of DA ~~le~~ may lead to different stable matches. Is there a reason to prefer one?
2. How should we define a desirable criterion for selecting one stable matching?