Algorithm: Top-trading cycle

Step 1: Set $M' = M$, $N' = N$ construct a directed graph G' with nodes N'.

- There is a directed edge from $i \in N'$ to $j \in N'$ iffs $P_i(1, M') = a^*(j)$.
- Allocate houses along every cycle of graph G'. i.e. if $(i^1, i^2, \ldots, i^p, i^1)$ is a directed cycle in G', set $a(i^1) = a^*(i^2), \ldots, a(i^p) = a^*(i^1)$.

let $\hat{N}'$ be the set of agents allocated houses

$\hat{M}'$

$N^2 = N' \setminus \hat{N}'$, $M^2 = M' \setminus \hat{M}'$

Step k: Continue to get G^k with nodes N^k

edge ~~between~~ from $i \in N^k$ ~~and~~ to $j \in N^k$ if

$P_i(1, M^k) = a^*(j)$

....

$N^{k+1} = N^k \setminus \hat{N}^k$, $M^{k+1} = M^k \setminus \hat{M}^k$.

Stop: if $N^{k+1} = \phi$. ELSE: REPEAT.

Theorem: TTC with fixed endowment mechanism is strategyproof and efficient.

Proof: Agent i is truthful, then gets room in round k (say)

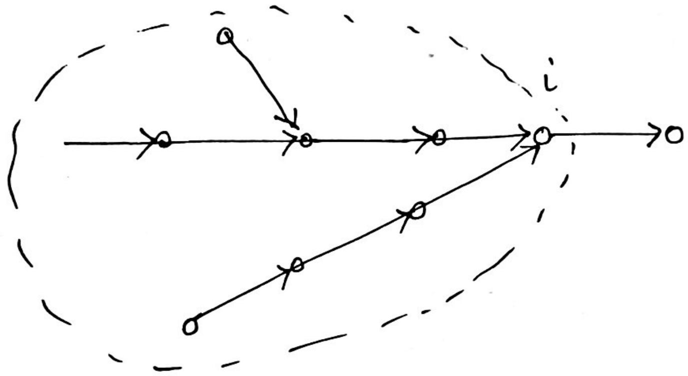
$H^k \rightarrow$ rooms allocated till round k (including that round)

i gets his best choice from $M \setminus H^{k-1}$.

How can agent i deviate:

1. i's deviation gives her a room on or after round k. Gets from $M \setminus H^{k-1}$ rooms — since the rounds till H^{k-1} are unaffected by the misreport — but truthful gives the best in this case, no reason to misreport.

2. i's deviation gives her a room in round $r < k$.



Π_i = {nodes that lead a path to i}

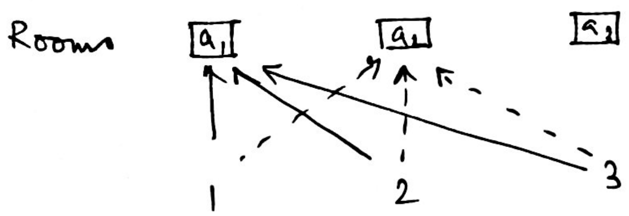
The only way i can change the allocation is by pointing to some node in Π_i

- note: i can't change the cycles by pointing to $M \setminus \Pi_i$
- why is this important? Since the room i is currently pointing to may not be available till round k, and we do not know what is her next best choice.

Claim: i's room is available till k, then all rooms of agents in Π_i are also available till k.

Hence the choice that i has at any round $r < k$ stays until making the choice in round k, which she'll do in truthful manner

TTC is not serial dictatorship



case 1 : $1 \to a_1$

case 2 : $2 \to a_2$ but not 1.

Efficiency : If this is not efficient, then there must exist some a' which is strictly better for some agent and gives the same house for others. Consider the first stage where in the TTC algorithm ~~gives a~~ differs from a', and $a'(i)\ P_i\ a(i)$ [by assumption]. Hence, $a(i)$ cannot be given to i – contradiction.

Stable house allocation with initial endowments

Stability ensures that when the agents have their initial endowments, and an allocation is suggested, there is ~~not~~ no better group deviation.

Previous example :

initial endowments : $a^*(1) = a_1$, $a^*(2) = a_3$, $a^*(3) = a_2$, $a^*(4) = a_4$, $a^*(5) = a_5$, $a^*(6) = a_6$.

P_3	P_4
a_1	a_2
a_4	
a_3	

If an allocation is proposed $a(i) = i$.

$a(3) = a_3$ and $a(4) = a_4$

$\{3,4\}$ can deviate and allocate a better choice.

3 gets a_4 and 4 gets a_2. ~~and~~ $a_4\ P_3\ a_3$, $a_2\ P_4\ a_4$

Such an allocation is not stable since the group $\{3,4\}$ ~~block~~ block such an allocation.

(12-4) Let a^* denote an initial endowment of agents.

a^S denote the allocation of $S \subseteq N$. — denotes the matchings of players in S with the ~~~~ houses available to players in S.

- A coalition $S \subseteq N$ can <u>block</u> a matching a at a preference profile P if $\exists$ a matching a^S s.t.

$$a^S(i) \; P_i \; a(i) \quad \text{or} \quad a^S(i) = a(i) \quad \text{for all } i \in N$$

and the strict preference occurs for at least one $j \in N$.

- A matching is in the <u>core</u> at a profile P if no coalition can block a at P.

- An SCF f is <u>stable</u> if $\forall P$, $f(P)$ is in the core at P.

<u>Note:</u> Stability implies efficiency.

Efficiency only requires that the grand coalition cannot block an allocation

<u>Ex:</u> Efficient but not stable.

$$a^*(1) = a_1 \quad , \quad a^*(2) = a_2, \quad a^*(3) = a_3$$

Agent 1, 2 : $a_1 \succ a_2 \succ a_3$

Agent 3 : $a_2 \succ a_1 \succ a_3$

$$a(1) = a_3 \quad , \quad a(2) = a_1 \quad , \quad a(3) = a_2$$

efficient as 2 and 3 gets their top choices.

not stable, since 1 can deviate and retain his house.

<u>Theorem</u>: The TTC mechanism is stable. Moreover, there is a unique core matching for every preference profile.

<u>Proof</u>: Suppose TTC is not stable. $\exists P$ s.t. matching produced by TTC is not in the core. Let coalition S blocks it. $\exists a^S$ s.t. $a^S(i) \, P_i \, a(i)$ or $a^S(i) = a(i)$ $\forall i \in S$ with at least one strict ~~ass~~ preference.

Let $T = \{ i \in S : a^S(i) \, P_i \, a(i) \}$ The set of all strict improvement individuals.

By assumption $T \neq \phi$.

~~Remind~~ Remember $\hat{N}^k$: people assigned houses in round k

$\hat{M}^k$: houses allocated in round k.

We will look at ~~the~~ how these people in S appear in these sets, i.e., the people $S \cap \hat{N}^k$.

Clearly, $S \cap \hat{N}^1$ are getting their top ranked houses, so they must not be in T, i.e., $S \cap \hat{N}^1 \subseteq S \setminus T$

We will use induction. Let $S \cap \hat{N}^k =: S^k$

~~Let $S^k \subseteq$~~

<u>Claim</u>: If $(S^1 \cup S^2 \cdots \cup S^{k-1}) \subseteq S \setminus T$, we show $S^k \subseteq S \setminus T$

all S^1 to S^{k-1}'s are subsets of $\hat{N}^1 \ldots \hat{N}^{k-1}$ respectively they got houses from $\hat{M}^1, \ldots, \hat{M}^{k-1}$. Hence $S \cap \hat{N}^k$ gets houses from $M \setminus (\hat{M}^1 \cup \hat{M}^2 \cdots \cup \hat{M}^{k-1})$

a gives the best available houses from $M \setminus (\hat{M}^1 \cup \hat{M}^2 \cdots \cup \hat{M}^{k-1})$ hence a^S cannot give them any better houses. Hence $S \cap \hat{N}^k \subseteq S \setminus T$. Hence $S = \bigcup_{k=1}^{K} S^k \subseteq S \setminus T \Rightarrow T = \phi$ $\square$

Uniqueness: Suppose TTC returns a and $\exists a' \neq a$ which is also in core.

Note: In $\hat{N}^1$ every agent gets their top choice.

Hence $a(i) = a'(i) \; \forall i \in \hat{N}^1$, because if not, the agents in $\hat{N}^1$ will block a'.

Now, induction is used.

Suppose $a(i) = a'(i) \; \forall \; i \in \hat{N}^1 \cup \hat{N}^2 \cup \dots \cup \hat{N}^{k-1}$

if $a(i) \neq a'(i)$ in $\hat{N}^k$, we see that every agent $i \in \hat{N}^k$ gets their top remaining houses. $M \setminus (\hat{M}^1 \cup \dots \cup \hat{M}^{k-1})$

~~Then $\hat{N}^k$ will block~~

Since all agents in $\hat{N}^1 \cup \hat{N}^2 \dots \cup \hat{N}^{k-1}$ get same houses in a and a', if there is any difference in a and a' in $\hat{N}^k$, then $\hat{N}^k$ blocks a'.

it must be $a(i) \; P_i \; a'(i)$

This contradicts that a' is a core matching $\square$

We can weaken the notion of stability to an individual level.

Defn: f is individually rational if at every profile P, the matching $f(P) \equiv a$ satisfies $a(i) \; P_i \; a^*(i)$ or $a(i) = a^*(i)$.

Clearly, stability implies individual rationality, since we want the single agent coalitions to be non-blocking.

Hence TTC satisfies individual rationality too. However, ~~the~~ this weaker condition along with the other two properties characterize TTC.

<u>Theorem</u>: A one sided matching mechanism is strategyproof, efficient, and individually rational iff it is a TTC mechanism.

<u>Generalized TTC mechanism</u>

It mixes the fixed priority and TTC in a convenient way.

It defines a priority order for every house – how the initial endowment will be transferred.

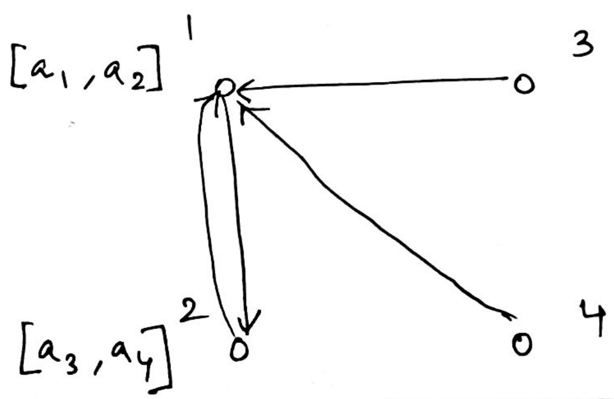
~~Every~~ $\sigma_j : N \to N$ for every $j \in M$

– one agent may endow more than one house

P_1	P_2	P_3	P_4
a_3	a_2	a_2	a_1
a_2	a_3	a_4	a_4
a_1	a_4	a_3	a_3
a_4	a_1	a_1	a_2

$\sigma_1 = \sigma_2 = (1, 2, 3, 4)$ for rooms a_1 and a_2

$\sigma_3 = \sigma_4 = (2, 1, 4, 3)$ for rooms $a_3 a_4$.



$4 \to a_4$

$3 \to a_1$

$1 \to a_3$, $2 \to a_2$

Theorem: GTTC is strategyproof and efficient.