Topics in Game Theory and Collective Choice

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1 Some Background Results

- A set $S \subseteq R^n$ is convex if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$ $\lambda x + (1-\lambda)y \in S$
- A set $S \subseteq R^n$ is closed if it contains all its limit points (points whose each neighbour cointains a point in S)
- A set is bounded if $\exists x_0 \in R^n$ and $R \in (0, \infty)$ such that $\forall x \in \| x - x_0 \|_2 < R$
- A set is compact if it is closed and bounded

2 Brouwer's Fixed Point Theorem

If $S \in R^n$ is convex and compact set and T:$S \rightarrow S$ is continuous, then T has a fixed point, i.e. a point x^* s.t. $T(x^*) = x^*$

3 Nash Equilibrium

Finite Game: A game in which the number of players and the strategies are finite.

3.1 Nash Theorem (1951)

Every finite game has a (mixed) Nash Equilibrium.

3.2 Proof of Nash theorem

Define Simplex

$$\Delta_k = \{x \in R_{\geq 0}^{k+1} : \Sigma_{i=1}^{k+1} x_i = 1\}$$

Consider two players (n players case is an extension of this idea). Player 1 has m strategies labelled from 1 to m and player 2 has n strategies labelled from 1 to n. So player 1's mixed strategy is a point in Δ_{m-1} and Player 2's strategy is a point in Δ_{n-1}. Hence the set of mixed strategy profiles is a point in $\Delta_{m-1} \times \Delta_{n-1}$. For the players, the utilities can be expressed in terms of two

matrices A and B. If $p \in m\Delta_{m-1}$ and $q \in \Delta n - 1$ are mixed strategies then
$u_1(p,q) = p^T Aq, u_2(p,q) = p^T Bq$
$c_i(p,q) = max(\{A_i q - p^T Aq, 0\} \geq 0$ where A_i is i^{th} row of A
$d_j(p,q) = max(\{p^T B_j - p^T Bq, 0\} \geq 0$ where B_j is j^{th} col of B

$$P_i(p,q) = \frac{p_i + c_i(p,q)}{1 + \Sigma_{k=1}^m c_k(p,q)}; Q_j(p,q) = \frac{q_j + d_j(p,q)}{1 + \Sigma_{k=1}^n d_k(p,q)}$$

Clearly, $P(p,q) \in \Delta_{m-1}$ and $Q(p,q) \in \Delta_{n-1}$
$T(p,q) = (P(p,q), Q(p,q))$
$T : \Delta_{m-1} \times \Delta_{m-1} \to \Delta_{m-1} \times \Delta_{n-1}$ is convex and compact
If $c_i, d_j' s$ are continuous then $P_i' s$ and $Q_j' s$ are also continuous which implies that T is continuous.
By Brouwer's theorem,
$\exists (p^*, q^*)$ s.t. $T(p^*, q^*) = (p^*, q^*)$

Claim,

$$\Sigma_{k=1}^m c_k(p,q) = 0, \Sigma_{k=1}^n d_k(p,q) = 0$$

Proof of Claim Suppose the following does not hold i.e.$\Sigma_{k=1}^m c_k(p,q) > 0$,
Since (p^*, q^*) is a fixed point of T

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \Sigma_{k=1}^m c_k(p^*, q^*)} \Rightarrow p_i^*(\Sigma_{k=1}^m c_k(p^*, q^*)) = c_i(p^*, q^*)$$

$$I = \{i : p_i^* > 0\} = i : c_i(p^*, q^*) > 0 = \{i : A_i q^* > {p^*}^T A q^*\}$$

$$u_1^* = \Sigma_{i=1}^m p_i^* A_i q^* = \Sigma_{i \in I} p_i^* A_i q* > (\Sigma_{i \in I} p_i^*) u_1^* = u_1^*$$

$$\rightarrow\leftarrow$$

hence

$$\Sigma_{k=1}^m c_k(p^*, q^*) = 0$$

$$\Rightarrow c_k(p^*, q^*) = 0 \forall k = 1, 2, ..., m$$

$$A_i q* \leq {p^*}^T A q^*$$

$$\Rightarrow \Sigma_{i=1}^m p_i' A_i q^* \leq {p^*}^T A q^*$$

Similarly for d_j and q^*

$$(p^*, q^*) \text{ is a MSNE.}$$