CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 18: 12^{th} September 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava.cse.iitk.ac.in .*

18.1 Recap

In the last class, we discussed about the Independence of Irrelevant Alternatives (IIA) and studied Scoring Rules. We then stated the Arrow's Impossibility Theorem (Arrow 1951).
We noted that to prove Arrow's Impossibility Theorem, we need to prove two lemmas.We proved the *Field Expansion Lemma* in the previous class. In this class we will prove the *Group Contraction Lemma.*

18.2 Continuing the proof of Arrow's impossibility result (Theorem 18.1)

Theorem 18.1 ***(Arrow 1951)*** *if $|A| \geq 3$, if an ASWF f satisfies Weak Pareto and IIA, then it must be dictatorial.*

Proof: Arrows Impossibility Theorem (contd.)
To complete the proof of Arrow's Impossibility Theorem, we state and prove the *Group Contraction Lemma*

Lemma 18.2 ***Group Contraction Lemma***
Let the group $G \subseteq N, G \neq \phi$ be decisive. Then $\exists G' \subseteq G$ which is also decisive.

Proof: Consider two cases on the size of G
If $|G| = 1$, the lemma trivially holds.
For $|G| \geq 2$,
Consider two subsets of G, namely $G_1 \subseteq G$ and $G_2 = G \setminus G_1$
Construct an ordering R like the example given

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

Observing all the agents in the set G, we see that a is strictly preferred over b. Thus, $aP_ib\forall i \in G$. and as G is known to be decisive,

$$a\hat{F}(R)b \tag{18.1}$$

Considering the social preferences of a and c, we have

Case 1: $a\hat{F}(R)c$
But if we consider G_1,

$$aP_ic\ \forall i \in G_1 \text{ and } cP_ja\ \forall j \notin G_1$$

$\forall R'$ where the above ordering holds,as F is IIA, $a\hat{F}(R)c$
Now by using Field Expansion Lemma,

$$\bar{D}_{G_1}(a,c) \Rightarrow G_1 \text{is decisive}$$

Case 2: $\neg\ a\hat{F}(R)c \Rightarrow c\hat{F}(R)a$
Also from (18.1) , $a\hat{F}(R)b \Rightarrow b\hat{F}(R)a$ (by using transitivity property)
But, if we consider G_2,

$$cP_ib\ \forall i \in G_2 \text{ and } bP_ic\ \forall j \notin G_2$$

Now again pick any arbitrary R',
$\forall R'$ where this is the case, using IIA again,$a\hat{F}(R)c$
So, G_2 is almost decisive over c and b ie, $\bar{D}_{G_b}(c,b)$
Again by Field Expansion Lemma we say that G_2 is decisive. ■

Now by using Weak Pareto,

N is decisive

Also by lemma 18.2 , $\exists i \in N$ such that $\{i\}$ is decisive.
As we have a singleton set which is decisive, we conclude that i is the dictator.
This proves Arrows Impossibility theorem. ■

Observation: The dictator (i) is unique.

18.3 Social Choice Setting

Arrow's Theorem tells us that we cannot hope to find a voting scheme that satisfies the notions of fairness we find desirable. However, the Arrowian welfare setup asks for a lot. Social ordering over all outcomes in achievable in a democratic way. One way to mitigate is to allow relaxing the ordering or looking at restricted preferences.
We will now consider the setting of "social choice" functions and identify only a single top-ranked outcome. Instead of a collective order, we aim for a collective alternative.

$$f: \mathscr{P}^n \to A$$

Where $\mathscr{P}^n$ is the set of linear order profiles (assuming strict preferences only)
A is a single alternative.
A representative case of this kind of social setting is voting.

18.3.1 Examples of Voting Protocols

This is a list of some voting protocols.

1. **Plurality**
   In plurality voting a single candidate is chosen for each agent and the candidate with highest number of votes is the winner. **Examples:** Voting system in India,USA,Britain etc.

2. **Plurality with runoff (two stages)**
   in this case the top 2 candidates from the first round of voting advance to the second round of voting.

In this round the highest voted candidate wins overall. **Examples:** French Presidential Election.

3. **Approval Voting**
Each voter casts a single vote for as many candidates as he wants. The candidate with the most votes is the winner. **Examples:** Approval Rating is used by the Mathematical Association Of America

4. **Scoring Rule**
Scores are assigned to candidates $(s_1, s_2, ..., s_m)$. The highest scoring candidate wins. Some scoring rules which we have seen in the previous lecture are.

   (a) Borda Count
   (b) Veto Rule
   (c) Dictatorial Rule

   We calculate scores as we calculated before but now we only consider the highest scorer as the winner.

5. **Maximin**
Candidates are visualized as points in a directed graph and are considered pairwise to assign points.Edges point from the winner to the loser in pairwise runoff. Edges are given weights according to the margin. The candidate with the largest margin of weights wins.

6. **Copeland**
Candidate with maximum pairwise wins is the winner.(similar to maximin)

None of the voting rules is better than the other. Thus, we have several voting rules to account for their limitations. Example : Plurality is limited due to loss of information about the complete ordering of the candidates. Plurality and scoring rules do not give a proportional representation of the whole population in the final outcome .

Pair-wise runoffs can lead to paradoxical situations.For example
Condorcet Paradox :
Suppose we have three candidates a,b,c and three voters whose preferences are as follows.

voter 1	voter 2	voter 3
a	c	b
b	a	c
c	b	a

In this case no candidate beats everyone else.

Definition 18.3 (Condorcet Consistency) *If there exists a candidate which is preferred over every candidate in pairwise runoffs, then he* should *be the winner.*

Note: Condorcet consistency does not hold for all voting schemes. Example : Scoring Rule

18.4 Summary

In this lecture we continued our proof to Arrow's Impossibility Theorem. We stated and proved the *Group Contraction Lemma* and then the theorem. We analyzed what this theorem means for Social Ordering setting and shifted to a more relaxed "Social Choice Setting". We then listed major voting rules in this setting. Finally we acknowledged the fact that no voting rule is better than the other by looking at Condorcet Paradox and Condorcet Consistency.

References

[1] Yoav Shoham and Kevin Leyton-Brown, "MULTIAGENT SYSTEMS: Algorithmic, Game-Theoretic,and Logical Foundations"