(Part 2) Given: $f(F, v)$ is a bargaining solution that satisfies all the five axioms

TST: $f(F, v) = f^N(F, v)$; $f^N(F,v) = \underset{\substack{(x_1,x_2)\in F \\ x_1 \geq v_1,\, x_2 \geq v_2}}{\operatorname{argmax}} (x_1 - v_1)(x_2 - v_2)$

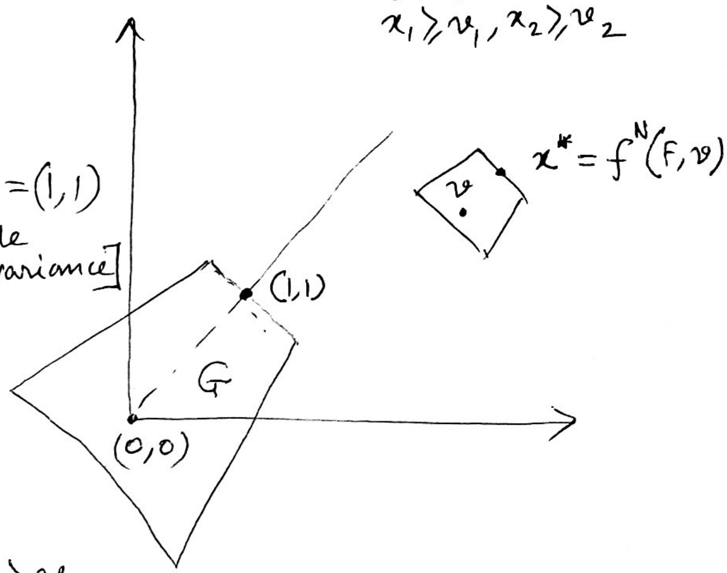
Plan: $f(F, v) = f^N(F, v)$

$\Leftrightarrow f(G, (0,0)) = f^N(G, (0,0)) = (1,1)$

[both satisfy scale covariance]

finally, need to show

$f(G, (0,0)) = (1,1)$



Since this is essential bargaining, i.e., $x_1^* > v_1$, $x_2^* > v_2$

$$L(x_1, x_2) = (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2)$$

$$\lambda_1 = \frac{1}{x_1^* - v_1}, \quad \lambda_2 = \frac{1}{x_2^* - v_2}, \quad \mu_1 = \frac{-v_1}{x_1^* - v_1}, \quad \mu_2 = \frac{-v_2}{x_2^* - v_2}$$

$$L(x_1, x_2) = \left(\frac{x_1 - v_1}{x_1^* - v_1}, \frac{x_2 - v_2}{x_2^* - v_2} \right)$$

$$L(v) = (0,0), \quad L(x^*) = (1,1)$$

$$G = \{ L(x) : x \in F \}$$

$$L(x^*) = f^N(G, (0,0)) = (1,1)$$

Claim: $y_1 + y_2 \leq 2$, $\forall (y_1, y_2) \in G$

Suppose not, then $\exists\ (y_1, y_2)$ s.t. $y_1 + y_2 > 2$

$f^N(G, (0,0))$ maximizer $y_1 y_2$

G is convex, as F was convex

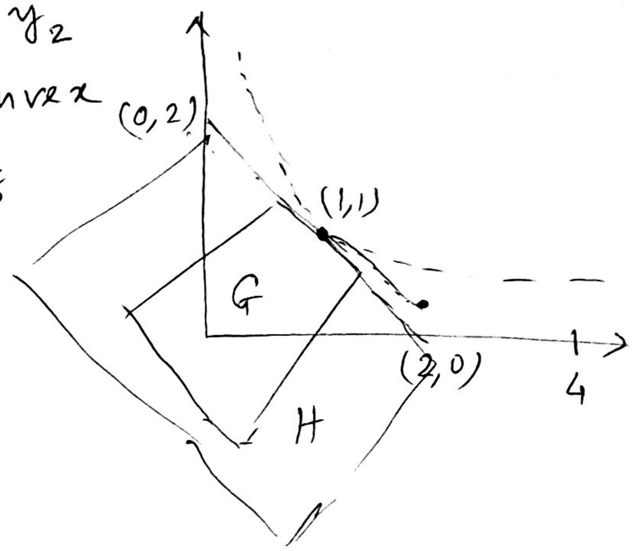
construct $\lambda y + (1-\lambda)(1,1) = z$

pick λ sufficiently small

s.t. The product $z_1 z_2 > 1$

which is a contradiction

to $f^N(G,(0,0))$ being 1.



Enclose G with H s.t. H is symmetric around $x_1 = x_2$ and $G \subseteq H$, with $(1,1)$ on the ~~Pareto~~ boundary of H.

- Strong Pareto Efficiency and symmetry $\Rightarrow$

$$f(H, (0,0)) = (1,1)$$

- IIA $\Rightarrow$ $\quad f(G,(0,0)) = (1,1)$ done!

Exercise: Extend the proof for inessential bargaining problem.

Exercise: Find at least one other solution for any combination of the three properties among SPE, Symmetry, Scale covariance, IIA.

Multi-person cooperative games

$(F, (v_1, \dots, v_n))$ defines the game in this setting

- Nash bargaining solution in this context
  does it reasonably capture the coalitional characteristics?

Examples:

(1) Divide the dollar — version 1:

$N = \{1, 2, 3\}$, want to divide a total wealth of 300.

Each player can propose a division so that the sum ≤ 300.

$F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 300\}$
feasible set

In this version of the game a division is implemented only if all agents agree to that division. Gets zero otherwise.

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{o/w} \end{cases}$$

Every player has equal power in this game.
The disagreement value is zero for every agent.
Hence the Nash bargaining solution is $(100, 100, 100)$
which looks perfectly reasonable — no agent or group can have a profitable deviation.

(2) Version 2:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ 0 & \end{cases}$$

disagreement point $v = (0, 0, 0)$ still.
Hence Nash bargaining solution remains $(100, 100, 100)$
But this does not look reasonable — group $\{1, 2\}$ can profitably deviate from this allocation.

Effective negotiation:

The members of a coalition of players can negotiate effectively (and form an effective coalition) if the players, on realizing that there is a feasible change in their strategies that can benefit all of them, actually makes such a change.

An n-person Nash bargaining solution would be relevant if the only coalition that can negotiate effectively is the grand coalition N.

(3) Version 3 (DTD):

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = x \\ & \text{or } s_1 = s_3 = x \\ 0 & \text{ow} \end{cases}$$

again the disagreement point is the same, however the Nash solution is far from reasonable, both $\{1,2\}$ and $\{1,3\}$ has a profitable deviation. Player 1 seems to have a lot more power in the decision making - therefore he can make very biased offers and other players have no choice but to accept it.

(4) Version 4 - majority voting game:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_j = s_k = (x_1, x_2, x_3) \text{ for some } j \neq k \\ 0 & \text{ow.} \end{cases}$$

Now again the negotiation may continue forever, since for every proposal among a pair of players, the third player has a better proposal for at least one of the players.

Need a better model for coalitional games with 3 or more players and better solution concepts.

Transferable Utility Games (TU Games)

Introduction of a fluid commodity that can transfer utility – this is called money. With this transfer being possible, we can define a cooperative game by a characteristic function

$$v : 2^{N} \to \mathbb{R} \quad , \quad N : \text{set of players}$$

$v(S)$: value of the coalition $S \subseteq N$

$v(\phi) = 0$.

Defn. A Transferable Utility (TU) game is given by the tuple (N, v) where N is the set of players and v is the characteristic function.

Examples: (1) DTD : Ver 1 :

$$v(\{1,2,3\}) = 300 \quad , \quad v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = v\{1,3\} = v(\{2,3\}) = 0$$

Ver 2 : $v(\{1,2\}) = v(\{1,2,3\}) = 300$, all others $= 0$

Ver 3 : $v(\{1,2\}) = v(\{1,3\}) = v(\{1\,2\,3\}) = 300$, others $= 0$

Ver 4 : $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 300$

(2) Minimum cost spaning tree game.

$v(1) = 10 - 5 = 5$, benefit – cost.

$v(2) = 10 - 1 = 9$

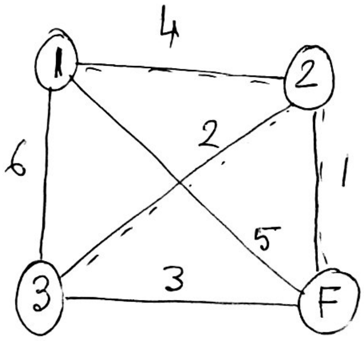
$v(3) = 10 - 3 = 7$

$v(1,2) = 20 - 5 = 15$

$v(2,3) = 20 - 3 = 17$

$v(1,3) = 20 - 8 = 12$

$v(1,2,3) = 30 - 7 = 23$



(3) Bankruptcy game (E,c) : $E \geq 0$ is the market value of an estate/company that went bankrupt. The vector c denotes the claim vector of different stakeholders of the estate, $c \in \mathbb{R}^n_{\geq 0}$

Value of a coalition $S \subseteq N$ is

$$v(S) = \Big[E - \sum_{i \in N \setminus S} c_i\Big]^+ \qquad x^+ := \max\{0, x\}$$

Say $N = \{1,2,3\}$, $c = (10, 50, 70)$, $E = 100$

$v(1) = 0$, $v(2) = 20$, $v(3) = 40$

$v(1,2) = 30$, $v(2,3) = 90$, $v(1,3) = 50$

$v(1,2,3) = 100$

Special classes of TU games

(1) Monotonic game : A TU game (N, v) is called monotonic if $v(C) \leq v(D) \quad \forall\, C \subseteq D \subseteq N$

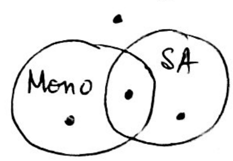
— almost all reasonable games

(2) Superadditive game : A TU game (N, v) is superadditive if

$$v(C \cup D) \geq v(C) + v(D) \quad , \forall\, C, D \subseteq N \text{ s.t. } C \cap D = \phi.$$

Monotonic and superadditive are independent features

Exercise: construct examples of all possibilities of monotonic and superadditive games.



(3) Convex games : A TU game is convex if

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D) \qquad \forall\, C, D \subseteq N$$

A convex game is always superadditive, but the converse is not true.

Proposition: (N, v) is convex iff $v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D) \quad \forall\, C \subseteq D \subseteq N \setminus \{i\}$ $\forall\, i \in N$.