Lec 30

<u>Theorem</u>: If an allocation solves ~~Rank-by-revenue solves~~ the winner determination problem then it must be a rank-by-revenue mechanism.

<u>Proof</u>: Assume for contradiction, allocation x is optimal, bids of 1 and 2 are such that

$$eCTR_1 \cdot b_1 > eCTR_2 \cdot b_2$$

But 2 is placed above 1, WLOG $x_2 = 1$, $x_1 = 2$

Consider a different allocation x' with all agents except 1 and 2 getting the same position but $x_1' = 1$, $x_2' = 2$

$$\sum_{i \in N} \hat{v}_i(x') = pos_1 (eCTR_1 \cdot b_1) + pos_2 (eCTR_2 \cdot b_2) + \sum_{i \neq 1,2} pos_{x_i} (eCTR_i \cdot b_i)$$

$$\sum_{i \in N} \hat{v}_i(x) = pos_1 (eCTR_2 \cdot b_2) + pos_2 (eCTR_1 \cdot b_1) + \sum_{i \neq 1,2} pos_{x_i} (eCTR_i \cdot b_i)$$

$$\sum \hat{v}_i(x') - \sum \hat{v}_i(x)$$
$$= \underbrace{(pos_1 - pos_2)}_{>0} \underbrace{(eCTR_1 b_1 - eCTR_2 b_2)}_{>0} > 0$$

x ~~is~~ does not solve the winner determination problem.

Advantage: The winner determination problem is poly-time.

Now we have an allocation mechanism, we need payments to implement it in DSIC.

Natural candidate: VCG payment (used in Twitter, Facebook ad)

VCG in position auction:

Given bids $(b_1, \dots, b_n)$, WLOG ordered such that

$$eCTR_1\, b_1 \geq eCTR_2\, b_2 \geq \cdots \geq eCTR_n\, b_n$$

- allocation x^* is s.t. $x_i^* = i$, $i = 1, \dots, n$
- payment is VCG, define

$$x_{-i}^* \in \operatorname*{argmax}_{x} \sum_{j \neq i} \hat{v}_j(x)$$

[Recall: $\hat{v}_j(x) = pos_{x_j}(eCTR_j \cdot b_j)$]

$$\begin{aligned}
p_i^{VCG}(b) &= \sum_{j \neq i} \hat{v}_j(x_{-i}^*) - \sum_{j \neq i} v_j(x^*) \\
&= \sum_{j=i}^{n-1} pos_j (eCTR_{j+1} \cdot b_{j+1}) - \sum_{j=i}^{n-1} pos_{j+1} (eCTR_{j+1} \cdot b_{j+1}) \\
&= \sum_{j=i}^{n-1} (pos_j - pos_{j+1})(eCTR_{j+1} \cdot b_{j+1}) \quad \forall i = 1, \dots, n-1 \\
&= 0, \text{ for } i = n.
\end{aligned}$$

Total expected payment, to convert to pay-per-click, need to normalize with $pos_i\, eCTR_i$

$$\text{pay-per-click} = \frac{1}{pos_i\, eCTR_i} \cdot p_i^{VCG}(b).$$

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Quite obviously, This ~~pay~~ allocation and payment is DSIC when eCTR's are accurate.

~~However,~~ However, GSP (generalized second price)
~~Example:~~ auction is used by Google, Bing etc.
which charges every agent the next expected bid.

This is simple, easy to explain payments, has similarities with second price auction, but has serious flaws/limitations.

Example:

Revenue before position effect	v_i	$eCTR_i$	pos_j
2	10	0.2	1
4	8	0.5	0.2
4.2	6	0.7	0.1

Both VCG and GSP allocates the slots to solve the winner determination problem, i.e., rank-by-revenue.

allocation: slot 1 → player 3
slot 2 → player 2
slot 3 → player 1.

VCG

payment of 1^{st} slot (pl 3)

$$\sum_{j=1}^{2} (p_j - p_{j+1})\, eCTR_{j+1}\, b_{j+1}$$

$= 0.8 \times 4 + 0.1 \times 2 = 3.4$

payment of 2^{nd} slot (pl. 2)

$= 0.1 \times 2 = 0.2$

3^{rd} slot (pl. 1) $= 0$.

utility of 1^{st} slot (pl 3)

$4.2 \times 1 - 3.4 = 0.8$

utility of 2^{nd} slot (pl 2)

$4 \times 0.2 - 0.2 = 0.6$

3^{rd} slot $= 2 \times 0.1 - 0 = 0.2$

GSP

payment of 1^{st} slot (pl 3)

$= 4 \times 0.2 = 0.8$

2nd slot (pl 2)

$= 2 \times 0.1 = 0.2$

utility

1^{st} slot (pl 3) $= 4.2 \times 1 - 0.8$

$= 3.4$

2nd slot (pl 2) $= 4 \times 0.2 - 0.2$

$= 0.6$

What if player 2 overbids

say bids 8.6 to change the allocation

Under VCG

payment will become $= 0.8 \times 4.2 + 0.1 \times 2 = 3.36 + 0.2$

$= 3.56$

payoff $= 4 \times 1 - 3.56 = 0.44$

under GSP

payment will become $= 4.2 \times 0.2 = 0.84$

payoff $= 4 \times 1 - 0.84 = 3.16$

GSP is not truthful.

Good points of VCG

1. DSIC - hence very low cognitive load on bidders
2. Never runs into deficit in some settings. - charges the marginal contribution to the other agents as payments and gets her own marginal contribution as payoff
3. Never charges a losing agent
4. Individually rational to participate - nobody loses money.