CS711: Introduction to Game Theory and Mechanism Design	Jul-Nov 2018

Project: Assignment 2

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2.1 Problem 1

The given preference profiles are P and P' are:-

P1	P2
a	c
b	b
c	a

a

P1'	P2'
b	a
a	b
c	c

2.1.1 Part (a)

Let's assume that :-

$$f(P') = a$$

Now, consider the following preference profiles R, R', R".

R1	R2
b	c
a	b
c	a

R1'	R2'
b	c
a	a
c	b

R1"	R2"
b	c
c	b
a	a

In transition P' $\rightarrow$ R we see that the preference of 1 remains the same while 2 changes its preference. Also note that we have assumed $f(P') = a = P2'(1)$ (i.e; 2 is decisive in P').

In transition P $\rightarrow$ R we see that the preference of 2 remains the same while 1 changes its preference. Also note that $f(P) = a = P1(1)$ is given. (i.e; 1 is decisive in P).

Now suppose $f(R) = b = R1(1)$ despite $f(P') = a = P2'(1)$ as we had originally assumed. We need to show that this leads to a contradiction. We will do so by showing that the assumption $f(R) = b$ leads to contradictions in either of the cases where $f(R') = c$ and $f(R') = b$ (the only possible cases).

Firstly suppose $f(R') = c$. Consider the transition R' $\rightarrow$ R":-

$$f(R') = c$$

$$D(c, R'_{\mathrm{i}}) \subseteq D(c, R''_{\mathrm{i}}) \quad \forall \quad i$$

$$\text{So, } f(R'') = c \text{ (Since f is MONO)}$$

But now, consider the transition R $\rightarrow$ R":-

$$f(R) = b$$

$$D(b, R_{\text{i}}) \subseteq D(b, R''_{\text{i}}) \quad \forall \quad i$$

$$\text{So, } f(R'') = b \text{ (Since f is MONO)}$$

So we have contradiction. Hence $f(R') \neq c$.

And now suppose $f(R') = b$. But now

$$f(P') = a \text{ and } f(R') = b$$

$$\text{Also, } a\ R2'\ b \text{ (i.e; 2 prefers a to b in profile R')}$$

$$\text{Hence, } f(P1', P2')\ R2'\ f(P1', R2')$$

$$\text{Hence f is manipulable by 2.}$$

So we have a contradiction (since f is strategyproof and cannot be manipulated.) Hence $f(R') \neq b$.

So our assumption that $f(R) = b = R1(1)$ despite $f(P') = a = P2'(1)$ is wrong. Hence $f(P') = a = P2'(1)$ implies that $f(R) = c = R2(1)$ (i.e; 2 remains decisive in R).

Similarly we can show that $f(P) = a = P1(1)$ implies that $f(R) = b = R1(1)$. (i.e; 1 remains decisive in R)

Hence we have a contradiction. Thus our original assumption $f(P') = a$ is wrong.

Hence,

$$f(P') = b$$

2.1.2 Part (b)

The earlier conclusion DOES NOT hold in the case of single peaked preference domain.

The proof given in the Part (a) relies on the following result :-

If f(P) = P1(1) and P' is a preference ordering such that P1'(1) $\neq$ P1(1) or P2(1) and P2'(1) = P2(1) then f(P') = P1'(1).

This is the Case 2 of the proof of GS theorem.

Using this result we claimed that f(P') = a implies f(R) = c and f(P) = a implies f(R) = b to get the desired contradiction.

To prove this result we constructed a preference ordering R' and showed that when f(R) = b, both f(R') = c and f(R') = b leads to contradictions.

But if the preference ordering are generated from a single-peaked preference domain with the ordering $a < b < c$ then a preference ordering R' is NOT POSSIBLE since $a < b$ but a R2' b (violation of single peaked domain).

Hence the proof falls through.

Now consider the mechanism where SCF f is given as :-

$$f(P) = min_{i \in [1,2]} Pi(1)$$

The minimum taken here is wrt to the intrinsic ordering $a < b < c$.

Hence $f(P) = a < c$ and $f(P') = a < b$. Now we just need to show that such an f is strategyproof.

Since f is a Median Voter SCF (using appropriate set of phantom peaks), we conclude that f is strategy proof.

2.2 Problem 2

Let the projects in X be labelled 1,2,3 ... —X—.

Now, let the set of alternatives be :-

$$A = P(\{1, 2, 3....|X|\})$$

(P() denotes power-set)

Let $S, T \in A$ such that $S \subseteq T$. Clearly for an agent i with a set linear ordering P_i we have :-

Either TP_iS

Or agent i is indifferent between T and S

Thus we can never have a preference profile in which agent i ranks the subset S above T.

Thus the Domain is restricted.

GS theorem assumes that the domain of preferences is unrestricted. Hence GS theorem does not apply here.

2.3 Problem 3

Median Voter SCF is group strategy proof.

Let K be the group of people who might misreport their preferences from P_k to manipulate f. Following cases are the only ones possible.

CASE 1 :-

Suppose the SCF f chooses peak preference of an agent i in K. So no one in K has an incentive to misreport their preferences.

CASE 2 :-

Suppose that the peak preferences of all the agents in K lie to one side (say to the left WLOG) of the median outcome ; say x.

If the group K misreports their preferences and move away from x (i.e; report their peaks to further left) the median does not change and hence the group has no incentive to misreport preferences this way.

Now suppose the entire group K misreport their preferences and move their reported peaks to the right side of x. This changes the median to some new peak y to the right of x i.e; $x < y$ in the intrinsic ordering.

Now we can say that

$$f(P_k, P_{-k}) = x$$

$$f(P'_{\text{k}}, P_{\text{-k}}) = y$$

But in the original preference profile P, since the peaks of agents in group K are to the left of x, all these agents rank x above y i.e; xP_iy for all agents in K.

Thus the outcome by not misreporting their preferences :- $f(P_{\text{k}}, P_{\text{-k}}) = x$ is preferred by the group K in the original preference profile to the outcome by misreporting :- $f(P'_{\text{k}}, P_{\text{-k}}) = y$.

$$f(P_{\text{k}}, P_{\text{-k}})P_{\text{i}}f(P'_{\text{k}}, P_{\text{-k}})$$

Thus, the group has no incentive to misreport their preferences in this way.

CASE 3 :-

Suppose that the peak preferences of all the agents in K lie to either side of the median outcome ; say x and x is not the peak preference of any agent in the group K.

If the sub-group in K with peaks to the left of x misreport their peak further to the left and/or the sub-group in K with peaks to the right of x misreport their peak further to the right there is no change in the median. Hence the group K has no incentive to misreport their preferences this way.

Suppose an agent i in K misreports his peak from the left of x to the right (can be assumed WLOG) so that the new median y is the peak of some agent in K.We can see that $x < y$.

Since the peak of i in P lies to the left of x, i prefers x to y in the original(true) preference profile i.e; $xP_{\text{i}}y$. So :-

$$f(P_{\text{k}}, P_{\text{-k}})P_{\text{i}}f(P'_{\text{k}}, P_{\text{-k}})$$

Thus it is a non profitable deviation for agent i.

For a group to manipulate an SCF, misreporting peaks should be a profitable deviation for every agent in the group. Whereas in case 3 any misreport is guaranteed to be a non profitable deviation for the misreporting agent.

Hence a Median Voter SCF is group strategyproof.