CS711: Introduction to Game Theory and Mechanism Design **Jul-Nov 2018**

Project: Assignment 2

Question 1

We have the following information:

- f is *onto* and *strategyproof*. Also *strategyproofness* implies *monotonicity*.
- $f(P) = f(P_1, P_2) = a$.
- For any preference profile $\hat{P}$, $f(\hat{P}) \in \{\hat{P}_1(1), \hat{P}_2(1)\}$.

Question 1. (a)

P		P'		$\bar{P}$		$\hat{P}$	
P_1	P_2	P'_1	P'_2	$\bar{P}_1$	$\bar{P}_2$	$\hat{P}_1$	$\hat{P}_2$
a	c	b	a	a	c	b	c
b	b	a	b	b	a	a	a
c	a	c	c	c	b	c	b
a		b		a		b	

- Note that $\mathrm{D}(a, P) \subseteq \mathrm{D}(a, \bar{P})$. Thus, by monotonoicity, we get $f(\bar{P}) = f(P) = a$.
- We also know that $f(\hat{P}) \in \{b, c\}$. Assume $f(\hat{P}) = c$. Then Player 1 can report his preference profile as $\bar{P}_1$ and manipulate the outcome to be a which is strictly better than c in $\hat{P}_1$. Since f is *strategyproof*, we thus get, by contradiction, that $f(\hat{P}) = b$.
- Note that $\mathrm{D}(b, \hat{P}) \subseteq \mathrm{D}(b, P')$. Thus, by monotonicity, we get $f(P') = f(\hat{P}) = b$.

Thus, from the above case analyses and the conclusions drawn, we get the result that if $f(P) = a$, then $f(P') = b$.

Question 1. (b)

No, the earlier conclusion does not go through. It does not go through because to make the earlier conclusion, we relied on creating the profile preference $\hat{P}_2$. However, in case of an intrinsic ordering of $a < b < c$ in the alternatives, the profile preference $\hat{P}_2$ and $\bar{P}_2$ are not possible (not possible because of the intrinsic ordering as one cannot go from preference c to a without having b in between), which is why the proof does not go through and the conclusion reached does not hold.

To further show that the above conclusion does not hold, we provide an *onto* and *strategyproof* social choice function such that $f(P) = a$ and $f(P') = a$. We have already seen in class that the **Median Voter Social Choice Function** is both *strategyproof* and *onto*. Thus, we consider the mechanism with the following Social Choice Function:

$$f'(P_1, P_2) = \text{Median}(a, P_1(1), P_2(1))$$

The above is a Median Voter SCF with the phantom peak at a. Being a Median Voter SCF, we refer to the results proved in classroom that a Median Voter SCF in singly peaked preferences is both *onto* and *strategyproof*. Thus, we get:

$$\begin{aligned} f'(P) &= \text{Median}(a, a, c) \\ &= a \\ f'(P') &= \text{Median}(a, b, a) \\ &= a \end{aligned}$$

Thus, in this way we have constructed a Mechanism with $f' = \text{Median}(a, \text{peak}(P_1), \text{peak}(P_2))$ which is both *strategyproof* and *onto* and has the results $f'(P) = f'(P') = a$. Hence, the drawn conclusion of $f(P') = b$ does not hold in this case and both the reason for the proof not going through and a contradictory example showing $f'(P') = a$ are given above to show that.

Question 2

No, Gibbard-Satterthwaite result does not apply here. Note that there are $2^{|X|}$ alternatives. We call the original preference ordering of Agent i over X as P_i and the extended ordering over $\mathbb{P}(X)$ (Power Set of X) as P_i'. Some points that give reasonable hint at why the result might not hold are:

- **Indifference**: Consider $\Theta_x = \{S \mid S \subseteq X \ , \ xP_is, \ \forall s \in S\}$ for $x \in X$. Note that $|\Theta_x| \geq 1$ since $\{x\} \in \Theta_x$. Now, let $y \in X$ be the lowest alternative under the ordering P_i, i.e. $xP_iy \, \forall \, x \in X$ such that $x \neq y$. Then, we can see that $|\Theta_x| > 1$ for all $x \neq y$. Also note that if $S, T \in \Theta_x$ then S and T are indifferent under P_i'. This, combined with the fact that $\Theta_x > 1$ for all $x \neq y$ gives us that there are a number of alternatives from $\mathbb{P}(X)$ at each level except the lowest that are indifferent from each other under P_i'. Note that this requires that $|X| \geq 2$ (for existence of $x \neq y$) which is given in the question. We now refer to the classroom discussion of Gibbard-Satterthwaite result that it *generally* does not hold in the presence of Indifference.
- **Domain Restriction**: Note that $X \in \mathbb{P}(X)$. Also note that $X\hat{P}_i'S \quad \forall S \in \mathbb{P}(X)$ and $\forall \ \hat{P}_i$ with corresponding $\hat{P}_i'$ defined as the extension of $\hat{P}_i$ as per the question. In simpler terms, this means that one of the alternatives of $\mathbb{P}(X)$, which is X itself, will always be at the top of any preference ordering that is extended (given $|X| \geq 2$). This simply implies that the Domain of the extended preference profile is a restricted domain in which all permutations are not possible. We again refer to classroom discussion of Gibbard-Satterthwaite result that it does not hold in cases of Domain Restriction where all linear orderings are not possible, which is the case here.

The above two points give sufficient suspicion that Gibbard-Satterthwaite result might not hold in this case. We give a concrete example of a Social Choice Function which would show that Gibbard-Satterthwaite result does not hold. In particular, we give a Social Choice Function f that is *onto* and *strategyproof* and is also *anonymous*. Note that anonymity implies non-dictatorship, and hence if we can provide a SCF f which is *anonymous, onto and strategyproof*, it would mean that Gibbard-Satterthwaite result doesn't hold.

Counter Example to Gibbard-Satterthwaite result

Let number of agents be N, and suppose $N \geq |X|$.

We first give the first intuitive result that would help us come up with a counter example. Suppose the linear ordering for agent i is P_i. Now, P'_i is the preference order over $\mathbb{P}(X)$. Let K_i be the set of all subsets of X that are the top preferences in P'_i. Now, we give a result that follows from basic intuition:

$$\begin{aligned} S_i &= \{P_i(1)\} \\ &= \bigcap_{k \in K_i} k \end{aligned}$$

This holds because K_i is the set of all top preferences in P'_i. In particular, it would contain $\{P_i(1)\}$ and $P_i(1)$ would be contained in all members of K_i. This leads us to the above result. Thus, we can obtain the top element in the original preference ordering from the ordering over the subsets by above. We then define a SCF f as follows:

$$f(P') = \bigcup_{i=1}^{N} S_i$$

f is *onto*

Consider any outcome $Q \subseteq X$. Note that $|Q| \leq |X| \leq N$. Also note that given Q, we can write it as follows:

$$\begin{aligned} Q &= \bigcup_{i=1}^{N} Q_i \\ |Q_i| &= 1 \end{aligned}$$

Now we consider the following preference over alternatives:

$$P_i(1) = Q_i \qquad Q_i \text{ here means the only element inside } Q_i$$

Now consider the extension of P_i to be P'_i. It follows naively that $f(P') = Q$. Hence, f is *onto*.

f is *strategyproof*

Note that $P_i(1) \in f(P') \; \forall \; i \in N$. Hence, if a player (say player i) misreports his preference profile while reporting the top preference truthfully, he is neither better nor worse off than the current outcome. If he misreports his top preference, there are the following two cases:

- Some other player has the true top preference same as that of player i's true top preference. In this case, on misreporting his top preference, player i is neither better off nor worse off as the output of f would still contain the true top preference of i.
- If none of the other players have the true top preference same as that of player i's true top preference, then on misreporting his top preference, player i is worse off as the output of f will not contain the true top preference of i.

Thus, from above we see that if any player misreports his preference profile, he is never better off and in some case can actually be worse off. This proves that everyone would report their true preference profiles, that is f is *strategyproof*.

f is ***anonymous***

Note that the SCF $f(P')$ is obtained by doing a Union over the singleton preferences of every agent's top preferences. Since the output is obtained by a union, we can see that if the agents are permuted in any way, the Union of the top singleton preferences remains the same (since union is over all agents and unions are independent of permutations). Thus, we get that f is *anonymous*.

From these above proofs, we conclude that f is *onto*, *strategyproof* and *anonymous*. Since f is *anonymous*, it **cannot be dictatorial**. Thus, we can see that Gibbard-Satterthwaite result does not hold since $|X| \geq 2$, f is *onto* and *strategyproof* but *not dictatorial*.

Question 3

Consider an intrinsic ordering over the set of alternatives. Now consider a Median Voter Social Choice Function f. Let the number of agents be N. Let the true preference profile for all the players be P. Suppose that f is group manipulable and can be manipulated by a group of K individuals. This means that if these K individuals report P'_K instead of their true preference profiles P_K, then the outcome would be either strictly better off or not worse off for all the players in K, i.e. $f(P'_K, P_{-K})\ P_i\ f(P_K, P_{-K})\ \forall\ i \in K$ with at least one player better off and others not worse off. Let us consider the following three scenarios:

- **Case 1:** The Player whose peak is the outcome of f (say Player x) is contained in K. If K players change their preference profiles, there can be two scenarios:
  - The output of f does not change. This means that no player in K is better off or worse off.
  - The output of f changes. Then player x is worse off than before since his true peak was the original output and the output has changed so he has moved away from his peak.
- K contains players that have peaks only on one side of $f(P)$ (excluding the Median Peak $f(P)$). Then if these K players change their preferences, there are two cases:
  - The output of f does not change if the players in K keep their new peaks to their original side of $f(P)$. Thus, the output of f after misreporting stays $f(P)$. This means that no player in K is better off or worse off.
  - If some players in K cross over and misreport their top peaks to be on the other side of the Median output $f(P)$, then the new output of f, that is $f(P'_K, P_{-K})$, would move further away from the true peaks of these K players. This is because by misreporting their peaks to the other side, it is skewing the Median towards the misreported peaks which is away from the true peaks. Hence the players are worse off.
- K contains both types of players, that is players that have their peaks on one side of $f(P)$ and some players that have their peaks on the other side of $f(P)$. There are two cases here too:
  - If each player in K misreport their peaks on the same side as their original side, or cross over in pairs, the output of f does not change. Hence no player is better off or worse off.
  - If players in K misreport their peaks and change the outcome, then $f(P'_K, P_{-K})$ is on a different position than $f(P)$. Thus, the new outcome $f(P'_K, P_{-K})$ would be towards one side of the true outcome $f(P)$ and thus it would be better for some players that were originally on that side but worse off for the players on the other side. Thus, we see that when K players misreport their peaks, not all of them turn up to be better off and some of them are worse off.

Thus, by above exhaustive survey, we find that in all the cases, when any set of K players misreport their peaks, some players in K end up being worse off or no players end up being better off or worse off. Thus we get that no matter which K players choose to misreport their peaks in whichever way they want, it is not possible to improve the outcome for those K players. This shows that **Median Voter Social Choice Function f is group strategy-proof**.