CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 37: November 10, 2017

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37.1 Single object allocation with multiple bidders

Continuing our discussion on revenue optimization in single object allocation from the previous lecture, in which we had devised an optimization problem and characterized that in order to ensure the mechanism to be BIC, f had to be non-decreasing in expectation (NDE). The problem was formulated as:

$$\text{OPT1}: \quad \begin{aligned} &\max_f \int_T \left(\sum_{i \in N} w_i(t_i) \cdot f_i(t) \right) g(t) dt \\ &\text{s.t. } f \text{ is NDE} \end{aligned} \tag{37.1}$$

where $w_i(t)$ is the virtual valuation function, $f_i(t)$ is the allocation function for the type profile t and $g(t)$ is the prior on the type profile of agents. We also proved that if we impose the "regularity" condition on the virtual valuation function, the unrestricted version of the optimization problem above is maximized by an allocation function f^* that is NDE. The virtual valuation w_i was defined as regular if it was an increasing function. We shall now prove the lemma that states the above argument formally.

Lemma 37.1 *Suppose every agent's virtual valuations are regular. The solution of the constrained problem is same as the unconstrained problem.*

Proof: To obtain a solution for the unconstrained optimization problem, we observe that,

$$\sum_{i \in N} w_i(t_i) \cdot f_i(t) \tag{37.2}$$

is a convex combination of the virtual valuations weighted by the allocation function. This implies that the optimal solution would place all the weight on the $\{i \in N : w_i(t_i) \geqslant w_j(t_j) \forall j \in N\}$, with ties broken arbitrarily. We have to consider the fact that the virtual valuations can be negative and hence if

$$\begin{aligned} w_i(t) < 0 \quad \forall i \in N \implies f_i(t) = 0 \quad \forall i \in N \\ \text{else} \quad f_i(t) = \begin{cases} 1 & w_i(t_i) \geqslant w_j(t_j) \quad \forall j \in N \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{37.3}$$

We claim that f is non-decreasing. To see this, fix t_{-i}. If $t_i > s_i$, by regularity $w_i(t_i) > w_i(s_i)$. Since the allocation probability can only increase when the virtual valuation increases in the above allocation rule, we conclude that

$$f_i(t_i, t_{-i}) \geqslant f_i(s_i, t_{-i}) \quad t_i > s_i. \tag{37.4}$$

This proves that the given allocation function is non-decreasing. We had set out to prove that f was NDE, which was a weaker requirement. This concludes the proof. ■

Remark 37.2 *f is DSIC since it is non-decreasing.*

37.1.1 Payment Rule

We now look at the payment rule in this allocation. By Myerson's characterization and the fact that this is an optimal mechanism, i.e., IR and revenue maximizing, the allocation completely determines the payment. Define

$$\kappa_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\} \tag{37.5}$$

The above function lists down the minimum value type/bid of agent i, that ensures allocation to her. It is important to note that it includes the condition that the virtual valuation function of the agent be positive, for the allocation to take place. More formally $\kappa_i^*(t_{-i}) \geqslant w_i^{-1}(0)$. The payment is given by

$$\begin{aligned} p_i(t) &= t_i f_i(t) - \int_0^{t_i} f_i(x_i, t_{-i}) dx_i \\ &= \kappa_i^*(t_{-i}) f_i(t) \end{aligned} \tag{37.6}$$

Theorem 37.3 *Let virtual valuations are regular. For every type t*

$$\begin{aligned} &\textit{if} \quad w_i(t) < 0, \forall i \in N \implies f_i(t) = 0, \forall i \in N \\ &\textit{else} \quad f_i(t) = \begin{cases} 1 & w_i(t_i) \geqslant w_j(t_j) \quad \forall j \in N \\ 0 & \textit{otherwise} \end{cases} \end{aligned} \tag{37.7}$$

The payment is given by, $p_i(t) = \kappa_i^*(t_{-i}) f_i(t)$. Then (f, p) is a revenue optimal mechanism.

While we wanted to obtain an optimal mechanism that is BIC, IIR and randomized, thereby expanding the space of mechanisms, the one we just proved to be optimal is DSIC, IR and deterministic. However, the parameters of these DSIC, deterministic mechanisms need the knowledge of priors (the computation of w_i's need the priors). This is different from the Vickrey auction which never uses the prior. Such mechanisms are called prior-free mechanisms in literature.

37.2 Some Examples

Let us now look at some examples to illustrate the allocation and payment rules.

1. Consider an auction with two agents ($\mathcal{A}_1$ and $\mathcal{A}_2$), with uniform priors over type spaces $\mathcal{T}_1 = [0, 12]$, $\mathcal{T}_1 = [0, 18]$. The virtual valuations are given by,

$$\begin{aligned} w_1(t_1) &= t_1 - \frac{1 - G_1(t_1)}{g(t_1)} \\ &= t_1 - \frac{1 - \frac{t_1}{12}}{\frac{1}{12}} \\ &= 2t_1 - 12 \\ \text{similarly, } w_2(t_2) &= 2t_2 - 18 \end{aligned} \tag{37.8}$$

   where $G_1(t_1)$ is the cumulative distribution function of the uniform distribution prior of $\mathcal{A}_1$. Now, we shall take some values of the bids t_1, t_2 and look at the allocation function and the payment made by the agent in each case.

t_1	t_2	Allocated to	p_1	p_2	$w_1(t_1)$	$w_2(t_2)$
4	8	unsold	0	0	-4	-2
2	12	$\mathcal{A}_2$	0	9	-8	6
6	6	$\mathcal{A}_1$	6	0	0	-6
9	9	$\mathcal{A}_1$	6	0	6	0
8	15	$\mathcal{A}_2$	0	11	4	12

2. **Symmetric bidders**: As visible from the previous example, if the bidders were symmetric, i.e., they had the same prior distributions and hence the same CDF, their virtual valuation functions would have been the same.

$$\begin{aligned} T_i &= T \quad \forall i \in N \\ G_i &= G \quad \forall i \in N \\ \Rightarrow w_i &= w \end{aligned} \tag{37.9}$$

Now since, $w(t_i) > w(t_j) \iff t_i > t_j$, the object is allocated to the agent with highest valuation (and unsold if everyone has negative value for the object). The payment is given by

$$p_i(t) = \max\{w^{-1}(0), \max_{j \neq i} t_j\} \tag{37.10}$$

This is exactly the second price auction with a reserve price.

3. **Efficiency and optimality**: Efficiency requires to maximize the social welfare, while optimal mechanism requires to maximize the revenue. We now consider the impact revenue optimization has on the efficiency of the allocation function. Consider an auction with two agents, $\mathcal{A}_1$, $\mathcal{A}_2$ with uniform priors given as, $T_1 = [0, 10]$ and $T_2 = [0, 6]$. The virtual valuation function evaluates to,

$$\begin{aligned} w_1(t_1) &= 2t_1 - 10 \\ w_2(t_2) &= 2t_2 - 6 \\ 2t_1 - 10 &= 2t_2 - 6 \qquad \text{Decision boundary for the winning agent} \\ t_2 &= t_1 - 2 \end{aligned} \tag{37.11}$$

Now, in the figure presented, there is a region ($t_1 < 5, t_2 < 3$) where the product is not allocated to anyone, even though the types/bids of the agents are positive. This is not efficient.

Efficiency would require the allocation decision boundary to be the line $t_1 = t_2$ (allocate to $\mathcal{A}_1$ if $t_1 > t_2$), however, for the allocation rule in the optimal mechanism the decision boundary is the line $t_2 = t_1 - 2$, making the allocation in favor of $\mathcal{A}_2$ even though $t_1 > t_2$ (the shaded area in the figure), which is inefficient.

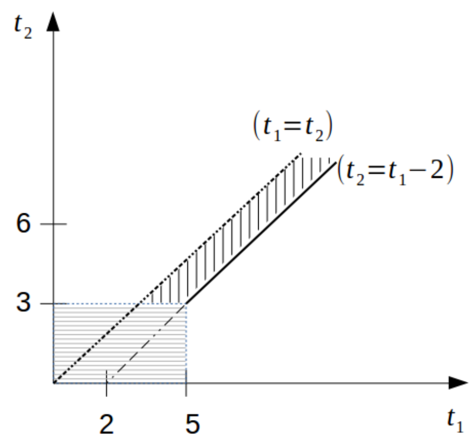

Figure 37.1: A plot of the bid and the allocation decision boundary of the optimal mechanism.