Equilibrium Concepts

Ex-ante: before observing own type

Nash Equilibrium (σ^*, P)

$$\mathcal{U}_i(\sigma_i^*, \sigma_{-i}^*) \geq \mathcal{U}_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i', \forall i$$

[Expectation taken over θ]

Ex-~~post~~ interim: after observing own type

Bayesian Equilibrium (σ^*, P)

$$\mathcal{U}_i(\sigma_i^*, \sigma_{-i}^* | \theta_i) \geq \mathcal{U}_i(\sigma_i', \sigma_{-i}^* | \theta_i), \quad \forall \sigma_i' \ \forall \theta_i \in \Theta_i \ \forall i \in N.$$

this can be replaced by pure strategies a_i $\forall a_i \in A_i$

both definition are equivalent.

[Expectation taken over $P(\theta_{-i}|\theta_i)$]

Equivalence of the two equilibrium concepts

Theorem: In a finite Bayesian game, a strategy profile is a Bayesian equilibrium iff it is a Nash equilibrium.

$\Rightarrow$ Suppose (σ^*, P) is a Bayesian equilibrium

$$\mathcal{U}_i(\sigma_i^*, \sigma_{-i}^* | \theta_i) \geq \mathcal{U}_i(a_i, \sigma_{-i}^* | \theta_i), \quad \forall a_i \in A_i, \forall \theta_i \in \Theta_i \ \forall i \in N$$

$$\mathcal{U}_i(\sigma_i', \sigma_{-i}^*) \overset{\text{Eq. 3}}{=} \sum_{\theta_i \in \Theta_i} P(\theta_i)\, \mathcal{U}_i(\sigma_i'(\theta_i), \sigma_{-i}^* | \theta_i)$$

$$\leq \sum_{\theta_i \in \Theta_i} P(\theta_i)\, \mathcal{U}_i(\sigma_i^*, \sigma_{-i}^* | \theta_i)$$

$$= \mathcal{U}_i(\sigma_i^*, \sigma_{-i}^*)$$

Hence (σ^*, P) is a Nash equilibrium.

(13-2)

$\Leftarrow$ Suppose (σ^*, P) is a Nash equilibrium. Assume for contradiction that (σ^*, P) is not a Bayesian equilibrium.

$\exists\, a_i \in A_i$ and some $\theta_i \in \Theta_i$, some $i \in N$ s.t.

$$u_i(a_i, \sigma_{-i}^* \mid \theta_i) > u_i(\sigma_i^*, \sigma_{-i}^* \mid \theta_i) \quad \text{---} \; (1)$$

consider the strategy $\hat{\sigma}_i$ of i

$$\hat{\sigma}_i(\theta_i') = \sigma_i^*(\theta_i') \quad \forall\, \theta_i' \in \Theta_i \setminus \{\theta_i\}$$

$$\hat{\sigma}_i(\theta_i, a_i) = 1 \quad \text{and} \quad \hat{\sigma}_i(\theta_i, b_i) = 0 \;\; \forall\, b_i \in A_i \setminus \{a_i\}$$

$$u_i(\hat{\sigma}_i, \sigma_{-i}^*) = \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i, \sigma_{-i}^* \mid \tilde{\theta}_i)$$

$$= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i)\, u_i(\hat{\sigma}_i, \sigma_{-i}^* \mid \tilde{\theta}_i) + P(\theta_i)\, u_i(\hat{\sigma}_i, \sigma_{-i}^* \mid \theta_i)$$

$$> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i)\, u_i(\sigma_i^*, \sigma_{-i}^* \mid \tilde{\theta}_i) + P(\theta_i)\, u_i(\sigma_i^*, \sigma_{-i}^* \mid \theta_i)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*)$$

$\rightarrow\!\leftarrow$ Nash equilibrium of $(\sigma_i^*, \sigma_{-i}^*)$

$\square$

Existence of a Bayesian equilibrium.

Theorem: Every finite Bayesian game has a Bayesian equilibrium.

Proof: Idea: transform the Bayesian game into a complete information game treating each type a player.

$$N = \bigcup_{i \in N} \Theta_i = \{\theta_1^1, \theta_1^2, \ldots, \theta_1^{|\Theta_1|}, \theta_2^1, \ldots, \theta_2^{|\Theta_2|}, \ldots, \theta_n^1, \ldots, \theta_n^{|\Theta_n|}\}$$

This is finite by assumption.

$$A_{\theta_i} = A_i \quad \forall\, \theta_i \in \Theta_i \quad \forall i \in N.$$

$$u_{\theta_i}(a_{\theta_i}, a_{-\theta_i}) = \sum_{\underline{\theta}_{-i} \in \Theta_{-i}} P(\underline{\theta}_{-i} | \theta_i)\, u_i(a_i(\theta_i), \underline{a}_{-i}(\underline{\theta}_{-i}), \theta_i, \underline{\theta}_{-i})$$

Note: A mixed strategy of player θ_i, σ_{θ_i} is a probability distribution over ΔA_i, which is a mixed strategy of player i at type θ_i, $\sigma_i(\theta_i)$ in the original Bayesian game.

Hence a MSNE in the transformed game is a Bayesian equilibrium in the original game.

By Nash theorem MSNE exists in the transformed game $\Rightarrow$ Bayesian equilibrium exists in the original game. $\square$