Mechanism Design for selling a single object

Single object auction model:

Type set of agent i : $T_i \subseteq \mathbb{R}$

value t_i if he wins the object.

An allocation a is a vector of n length that represents the probability of winning the object by an agent. The set of allocations is denoted by $\Delta A = \{ a \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{n} a_i = 1 \}$

- the setting and results are extendable to setting where a_i is the amount of an object allocated to i – e.g. for sponsored search auctions a_i is replaced by the CTR_i

The allocation rule : $f : T_1 \times T_2 \times \cdots \times T_n \longrightarrow \Delta A$

Given an allocation $a \in \Delta A$, the valuation of agent i is given by $a_i \cdot t_i$ (t_i : type of i)
[valuation is in product form]

- $f_i(t_i, t_{-i})$ is the probability of winning the object for agent i when the type profile is (t_i, t_{-i})

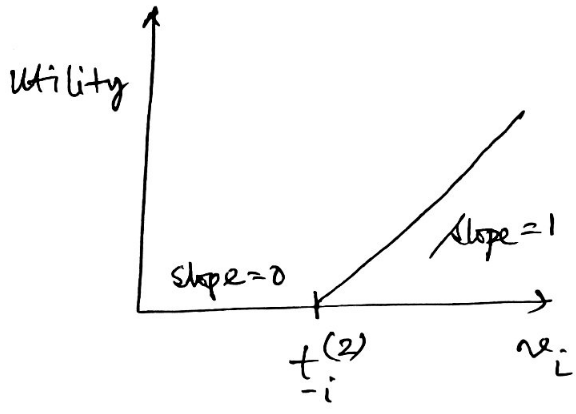
Recall: Vickrey auction / Second price auction:

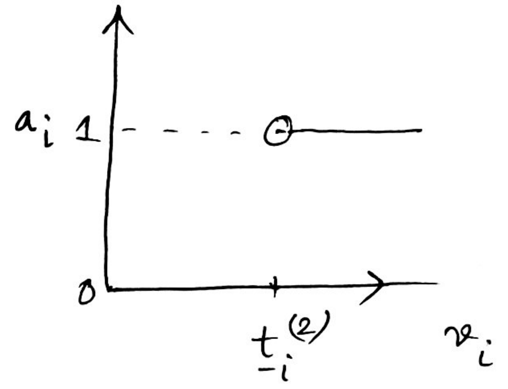
The type of agent i : v_i : value of the object if i wins.

define $t^{(2)}_{-i} = \max_{j \neq i} \{ v_j \}$.

Agent i wins if $v_i > t^{(2)}_{-i}$, loses if $v_i < t^{(2)}_{-i}$

$$\text{utility} = \begin{cases} 0 & v_i \leqslant t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & v_i > t_{-i}^{(2)} \end{cases}$$





utility is convex, derivative is zero $v_i < t_{-i}^{(2)}$ and 1 for $v_i > t_{-i}^{(2)}$, and is not differentiable at $v_i = t_{-i}^{(2)}$.

Whenever the derivative exists, it coincides with the probability of winning.

A convex function is differentiable "almost everywhere" — the fact the the derivative coincides with allocation probability holds almost everywhere.

Some facts from convex analysis:

We are interested in functions $g : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval.

Defn: A function $g : I \to \mathbb{R}$ is convex if for every $x, y \in I$ and $\lambda \in (0,1)$

$$\lambda g(x) + (1-\lambda) g(y) \geqslant g(\lambda x + (1-\lambda) y).$$

Facts

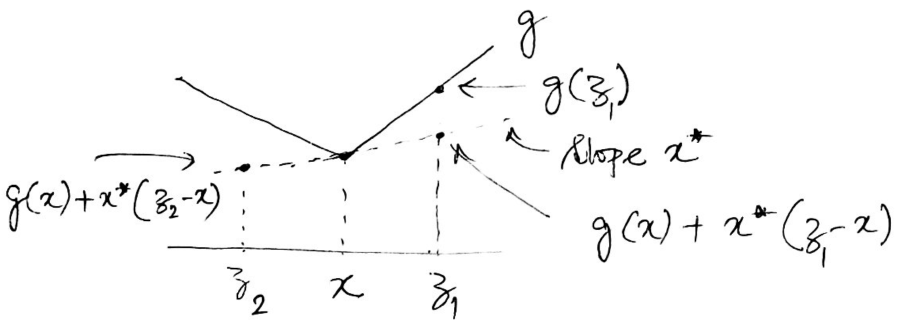
① — Convex functions are continuous in the interior of its domain, & jumps can occur only at the boundaries.

② — Convex functions are differentiable "almost everywhere" in I. formally, $\exists$ subset $I' \subseteq I$ s.t. $I \setminus I'$ has countable points (has measure zero) and g is differentiable at every point of I'.

If g is differentiable at some $x \in I$, we denote the derivative with $g'(x)$. The following definition extends the idea of gradient/derivative.

<u>Defn</u>: For any $x \in I$, x^* is a <u>subgradient</u> of the function $g: I \to \mathbb{R}$ at x if

$$g(z) \geq g(x) + x^*(z - x) \qquad \forall\, z \in I.$$

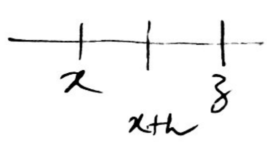


Lemma 1: Suppose $g: I \to \mathbb{R}$ is a convex function. Suppose x is in the interior of I and g is differentiable at x. Then $g'(x)$ is the unique subgradient of g at x.

(32-4)

Proof: Consider a $z \in I$ s.t. $z > x$ (A similar ~~Same~~ proof works if $z < x$). Consider h s.t. $(z-x) \geq h > 0$

$$x+h = \frac{h}{z-x} \cdot z + \left(1 - \frac{h}{z-x}\right) x$$



Since g is convex

$$\frac{h}{z-x} \cdot g(z) + \left(1 - \frac{h}{z-x}\right) g(x) \geq g(x+h)$$

$$\Rightarrow \quad \frac{g(z) - g(x)}{z-x} \geq \frac{g(x+h) - g(x)}{h} \qquad \text{true for any } h > 0$$

$$\xrightarrow{\lim_{h \to 0}} \quad g'(x)$$

$$g(z) - g(x) \geq g'(x)\,(z-x)$$

hence $g'(x)$ is a subgradient of g at x.

Now, we need to show uniqueness. Say for contradiction, there exists another subgradient $x^* \neq g'(x)$ at x.

Case 1: $x^* > g'(x)$. By definition

$$g(x+h) - g(x) \geq x^* h$$

$$\Rightarrow \quad \frac{g(x+h) - g(x)}{h} \geq x^* > g'(x) \qquad \text{holds for all } h > 0$$

taking lim as $h \to 0$

$$g'(x) \geq x^* > g'(x) \qquad \text{contradiction.}$$

Case 2: $x^* < g'(x)$: very similar exercise.

Lemma 2: Let $g : I \to \mathbb{R}$ be a convex function. Then for every $x \in I$ the subgradient of g at x exists.

Proof skipped.

— Facts: 1 For points in $I \setminus I'$ (g is not differentiable), this set is of measure zero, the set of subgradients at a point forms a convex set. Define $g'_+(x) = \lim_{\substack{z \to x \\ z \in I',\, z > x}} g'(z)$

and $g'_-(x) = \lim_{\substack{z \to x \\ z \in I',\, z < x}} g'(z)$.

The set of subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$

The set of subgradients of g at a point $x \in I$ is denoted by $\partial g(x)$.

Lemma 1 says that $\partial g(x)$ is $\{g'(x)\}$ for $x \in I'$, and Lemma 2 says that it is non-emtpy for all $x \in I$.

Lemma 3: Let $g : I \to \mathbb{R}$ be a convex function. Let $\phi : I \to \mathbb{R}$ such that $\phi(z) \in \partial g(z)$ $\forall z \in I$. Then for all $x, y \in I$ s.t. $x > y$, we have $\phi(x) \geq \phi(y)$.

Proof: By definition,

$$g(x) \geq g(y) + \phi(y)(x-y)$$

and

$$g(y) \geq g(x) + \phi(x)(y-x)$$

adding, $(\phi(x) - \phi(y))(x-y) \geq 0$

since $x > y \Rightarrow \phi(x) \geq \phi(y)$.

If g was differentiable everywhere, then

$$g(x) = g(y) + \int_y^x g'(z)\,dz$$

~~as~~ an extension of this result holds for convex functions with subgradients.

Lemma 4: Let $g : I \to \mathbb{R}$ be a convex function. Then for any $x, y \in I$

$$g(x) = g(y) + \int_y^x \phi(z)\,dz$$

where $\phi : I \to \mathbb{R}$ is s.t. $\phi(z) \in \delta g(z)$ $\forall z \in I$.