Lec 19

Pareto domination: An alternative a is Pareto dominated by b if $\forall i \in N$ $b P_i a$

Pareto Efficient: An SCF f is PE if for every preference profile P and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.

Unanimity: An SCF f is Un if for every preference profile P having $P_1(1) = P_2(1) = \cdots = P_n(1) = a$, $f(P) = a$.

Clearly, $PE \subset Un$

strictness example: for a profile that does not have the same top candidate - pick a dominated candidate.

Ontoness: f is onto if $\forall a \in A \; \exists P \in \mathcal{P}^n$ s.t. $f(P) = a$.

$Un \subset Onto$

Manipulability: An SCF f is manipulable if $\exists i$ and a profile P s.t.

$$f(P_i', P_{-i}) \; P_i \; f(P_i, P_{-i}) \text{ for some } P_i'.$$

f is strategyproof if is it not manipulable by any agent at any profile.

Implications of strategyproofness / non-manipulability

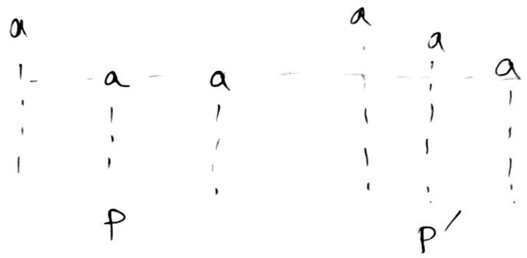
<u>Defn</u>: Dominated set of a at preference P_i

$$D(a, P_i) = \{b \in A : a P_i b\}$$

ex. $P_i = \begin{matrix} b \\ d \\ a \\ c \end{matrix}$ $\quad D(d, P_i) = \{a, c\}$

<u>Monotonicity</u>: f is <u>monotone</u> if for any two profiles P and P' with $f(P) = a$ and

$$D(a, P_i) \subseteq D(a, P_i') \ \forall i \in N \Rightarrow f(P') = a.$$



$f(P) = a \qquad\qquad f(P') = a$

<u>Theorem</u>: f is strategyproof $\Leftrightarrow$ f is monotone

Note the proof technique

<u>Proof</u>: SP $\Rightarrow$ MONO

Consider the 'if' condition of MONO

P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i')$ $\forall i \in N$.

Break the transition from P to P' into n stages.

$$(P_1 P_2 \cdots P_n) \quad (P_1' P_2 \cdots P_n) \quad (P_1' P_2' \cdots P_n) \cdots (P_1' \cdots P_k' P_{k+1} \cdots P_n) \quad (P_1' \cdots P_n')$$

$$P = P^{(0)} \longrightarrow P^{(1)} \longrightarrow P^{(2)} \longrightarrow P^{(k)} \longrightarrow P^{(n)} = P'$$

Claim: $f(P^{(k)}) = a \quad \forall k = 1, \ldots, n$

Suppose not $\quad \exists\, P^{(k-1)}, P^{(k)}$ s.t. $f(P^{(k-1)}) = a$, $f(P^{(k)}) = b \neq a$

$$P_1' \cdots P_{k-1}' \, P_k \cdots P_n \to a \qquad\qquad P_1' \cdots P_{k-1}' \, P_k' \, P_{k+1} \cdots P_n \to b$$

$$\begin{array}{c} a \\ \vdots \end{array}$$

$D(a, P_k) \subseteq D(a, P_k')$ i.e. a's position weakly increases

$a P_k b$ and $a P_k' b$ voter k misreports $P_k' \to P_k$

$b P_k a$ and $b P_k' a$ " " $P_k \to P_k'$

$b P_k a$ and $a P_k' b$ " " in both

contradiction to f being SP.

SP $\Leftarrow$ MONO

we'll show $\overset{\circ}{!}$SP $\Rightarrow \overset{\circ}{!}$MONO

suppose not, i.e. $\overset{\circ}{!}$SP and monotone.

$\overset{\circ}{!}$SP implies that $\exists\, i, P_i, P_i', P_{-i}$ s.t.

$$\underbrace{f(P_i', P_{-i})}_{=:\, b} \; P_i \; \underbrace{f(P_i, P_{-i})}_{=:\, a}$$

$b P_i a$

Construct P'' $\quad P_{-i}'' = P_{-i}$

$P_i''(1) = b$, $P_i''(2) = a$

$$\left(\begin{array}{cc} P_i'' & P_{-i} \\ b & \\ a & \\ \vdots & \end{array} \right)$$

$$\begin{array}{ccc} (P_i, P_{-i}) & \to & (P_i'', P_{-i}) \\ \downarrow & & \downarrow \\ a & & a \end{array}$$

$D(a, P_i) \subseteq D(a, P_i'')$

MONO $\Rightarrow f(P_i'', P_{-i}) = a$

$$\begin{array}{ccc} (P_i', P_{-i}) & \to & (P_i'', P_{-i}) \\ \downarrow & & \downarrow \\ b & & b \end{array}$$

$D(b, P_i') \subseteq D(b, P_i'')$

contradiction.

Lemma: f is MONO and ONTO $\Rightarrow$ f is PE

Suppose not, f is MONO + ONTO but not PE

$\exists\ a, b, P$ s.t. $b\, P_i\, a\ \forall i \in N$ but $f(P) = a$

ONTO : $\exists P'$ s.t. $f(P') = b$

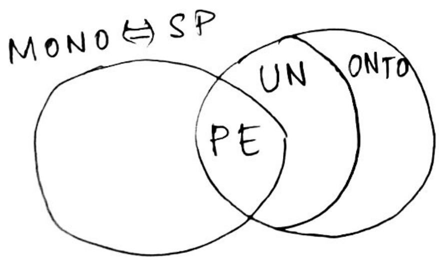
Construct P'' s.t. $\left.\begin{array}{l} P_i''(1) = b \\ P_i''(2) = a \end{array}\right\}\ \forall i \in N$

Clearly $D(b, P_i') \subseteq D(b, P_i'')\ \forall i \in N$

$\Rightarrow f(P'') = b$

But $D(a, P_i) \subseteq D(a, P_i'')\ \forall i \in N$

$\Rightarrow f(P'') = a$



Theorem: f is SP then f is PE $\Leftrightarrow$ f is UN $\Leftrightarrow$ f is ONTO

Gibbard - Satterthwaite ~~Result~~ Theorem (G'73, S'75)

Suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial.