CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 23: October 6, 2017

Lecturer: Swaprava Nath *Scribe(s): Debojyoti Dey*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

23.1 Recap

In the previous lecture we showed some important properties of social choice function in the restricted domain of single-peaked preferences. The claims we proved are as follows.

Lemma 23.1 *Let p_{min} and p_{max} are the leftmost and rightmost peaks according to order relation $<$. Then SCF f is PE if and only if $f(P) \in [p_{min}, p_{max}]$.*

Lemma 23.2 *f is SP $\implies$ f is MONO.*

Lemma 23.3 *Let $f : \mathcal{S}^n \to A$ is SP. Then f is ONTO $\iff$ f is UN $\iff$ f is PE.*

We also defined anonymous(ANON) SCF f which is independent of permutation of the agents for every preference profile P, that is, $f(P) = f(P^\sigma)$ where P^σ represents σ-permuted prefernces of P. We observed that a dictatorial SCF cannot be ANON.
Then we came to the crux of lecture by stating that Median Voting rule SCF is not dictatorial. In today's lecture, we will prove that an Median Voting rule is ANON and thus not dictatorial. We actually prove a stronger statement in the next section.

23.2 Median Voting Rule is not Dictatorial

Theorem 23.4 ***(Moulin,1980)*** *A SP SCF f is ONTO and ANON if and only if it is a Median Voting rule.*

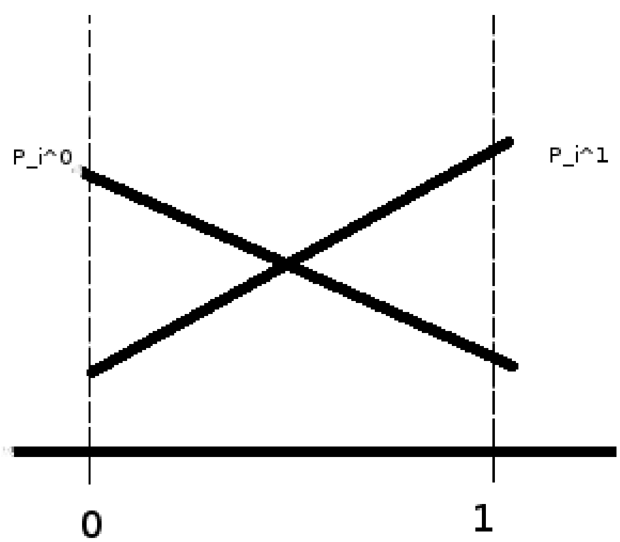

Figure 23.1: 0-1 Single peak preferences

Proof:
$\Leftarrow$ *Median voter SCF*

- *It is SP(see theorem 21.9 in previous lecture)*
- *it is ONTO. Put all voters' peak at same alternative/location and phantoms all at zero. Then that location is the outcome.*
- *it is anonymous, if we permute the agents with the peaks unchanged, the outcome does not change.*

$\Rightarrow$ *Given* $f : \mathcal{S}^n \to A$ *is SP,ONTO and ANON*

Define P_i^0 *as agents i's preference where the peak is at leftmost point wrt* $<$, *similarly* P_i^1 *where peak is at rightmost point.*
y_j*'s are phantom peaks for* $j = 1, 2, \ldots, n-1$ *such that,*

$$y_j = f(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1)$$

with $n-j$ *peaks at leftmost and* j *peaks at rightmost. Being ANON, it does not matter which agent has which peak.*

Claim 23.5 $y_j \leq y_{j+1}$ *for* $j = 1, 2, \ldots, n-2$, *i.e. phantom peaks are in non decreasing order wrt* $<$.

Proof: $y_{j+1} = f(P_1^0, P_2^0, \ldots, P_{n-j-1}^0, P_{n-j}^1, \ldots, P_n^1)$.

$$\begin{aligned} f \text{ is SP} &\implies y_j P_{n-j}^0 y_{j+1} \\ &\implies y_j \leq y_{j+1} \qquad \text{as } P_{n-j}^0 \text{ is peaked at } 0 \end{aligned}$$

■

Consider the arbitrary profile

$$P = (P_1, P_2, \ldots, P_n)$$
$$p_i := P_i(1) \text{ represents the peak of agent } i$$

Goal: Show $f(P) = med(p_1, p_2, \ldots, p_n, y_1, \ldots, y_{n-1})$.
We can assume wlog $p_1 \leq p_2 \leq \ldots \leq p_n$ *due to ANON. Say* $a = med(p_1, p_2, \ldots, p_n, y_1, \ldots, y_{n-1})$

Case 1: *a is a phantom peak*

Say $a = y_j$ *for some* $j \in 1, 2, \ldots, n-1$ *is a median of* $(2n-1)$ *points. There are* $(j-1)$ *phantom peaks to the left of the median and* $(n-1-j)$ *to the right, because of the fact* $y_j \leq y_{j+1}$ *we proved. So, there are* $(n-j)$ *agent peaks on the left. Hence the following holds,*

$$p_1 \leq \ldots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \ldots \leq p_n$$

Now consider two profiles, $(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1)$ *which gives* y_j *if* f *is applied on it(by definition) and* $(P_1, P_2^0, \ldots, P_n^1)$. *Assume that,*

$$f(P_1, P_2^0, \ldots, P_n^1) = b$$

We want to show that $b = y_j$.

$$f \text{ is SP} \implies y_j P_1^0 b \implies y_j \leq b$$

$$\begin{aligned} f \text{ is } SP &\implies bP_1y_j \text{ and it is known that } p_1 \le y_j \\ &\implies b \le y_j \end{aligned}$$

Combining the above two facts, $b = y_j$*. Repeating the argument for the first* $(n-j)$ *agents, we get*

$$f(P_1, P_2, \ldots, P_{n-j}, P^1_{n-j+1}, \ldots, P^1_n) = y_j$$

Now consider $f(P_1, \ldots, P_{n-j}, P^1_{n-j+1}, \ldots, P^1_{n-1}, P_n) = b$*(say). Using the SP property of* f*,*

$$\begin{aligned} &y_jP^1_nb \implies b \le y_j \\ &bP_ny_j \text{ and } y_j \le p_n \implies y_j \le b \end{aligned}$$

Combining the above two facts, $b = y_j$*. Repeating the arguments, we finally get,*

$$f(P_1, P_2, \ldots, P_{n-j}, P_{n-j+1}, \ldots, P_n) = y_j = a$$

which is the median.

Case 2: *a is agent peak*

We prove this for 2 agents. The general case repeats the argument.

Claim 23.6 *: N={1,2}, let* P *and* P' *be such that,* $P_i(1) = P'_i(1)\ \forall i \in N$*, then*

$$f(P) = f(P')$$

Proof: *Let* $a = P_1(1) = P'_1(1)$ *and* $b = P_2(1) = P'_2(1)$*. Also let* $f(P) = x$ *and* $f(P'_1, P_2) = y$*.*
f *is SP, hence* xP_1y *and* yP'_1x*. Since peaks are same, if* x *and* y *fall on the same side of the peak* $P_1(1)$*(equivalently* $P'_1(1)$*), they must be same. The only other possibility is that* x *and* y *fall on different sides of the peak. We show that this is not possible.*
wlog assume that, $x < a < y$ *and* $a < b$
Since f *is SP+ONTO* $\iff$ f *is SP+PE and PE requires that* $f(P) \in [a, b]$ *but* $f(P) = x < a$*, which is a contradiction.*
Repeat the argument for $(P'_1, P_2) \to (P'_1, P'_2)$ ■
Now consider the profile $P = (P_1, P_2)$ *such that* $P_1(1) = a$ *and* $P_2(1) = b$ *and* y_1 *be the phantom peak. By assumption,* $med(a, b, y_1)$ *is an agent peak. Wlog let the median be* a*. Assume for contradiction,* $f(P) = c \ne a$*.*

By PE, c *must lie within* a *and* b*. We consider the two cases,* $b < a < y_1$ *and* $y_1 < a < b$

Case A*:* $b < a < y_1$
By PE, $c < a$*. Construct* P'_1 *such that* $P'_1(1) = a = P_1(1)$ *and* $y_1P'_1c$ *[possible since* y_1 *and* c *are on different sides of the peak* $P'_1(1)$*]. Since* $f(P) = c$*,* $f(P'_1, P_2) = c$ *by the previous claim.*

Now consider the profile (P^1_1, P_2)*.*

$$P_2(1) = b < y_1 < P^1_1(1)$$

So the median of (b, P^1_1, y_1) *is* y_1*, which is a phantom, and hence by our result in Case 1,*

$$f(P^1_1, P_2) = y_1$$

By construction of P'_1*,*

$$\begin{aligned} y_1P'_1c &\implies f(P^1_1, P_2)\ P'_1\ f(P'_1, P_2) \\ &\implies f \text{ is not } SP \end{aligned}$$

a contradiction. Hence our assumption $f(P) \neq a$ is wrong.

***Case B**: $y_1 < a < b$*
By PE, $a < c$. Construct P'_1 such that $P'_1(1) = a = P_1(1)$ and $y_1 P'_1 c$ [possible since y_1 and c are on different sides of the peak $P'_1(1)$].

$$f(P) = c \implies f(P'_1, P_2) = c$$

Now consider the profile (P^0_1, P_2).

$$P^0_1(1) < y_1 < b = P_2(1) \implies f(P^0_1, P_2) = y_1$$

But by construction of P'_1,

$$\begin{aligned} y_1 P'_1 c &\implies f(P^0_1, P_2)\ P'_1\ f(P'_1, P_2) \\ &\implies f \text{ is not SP} \end{aligned}$$

a contradiction. Hence our assumption $f(P) \neq a$ is wrong. ■

23.3 Conclusion

In this lecture, we have proved the non dictatorial nature of Median Voter SCF by introducing phantom voters. The phantom voters/peaks are introduced so that the extreme preference conditions can be handled with a "fair" decision. For example, if half the agents are at the extreme left and other half is at the extreme right, a fair distribution of phantom peaks may lead to picking the median somewhere at the center rather than at some extreme point. Note that, Median voter SCF is actually a class of voting rules.