Bayesian Equilibria in Bayesian Games:

Ex. 1: Sealed bid auction

values of each bidder $\in [0,1]$, bids $\in [0,1]$

θ_i if the object is assigned to i. b_i – announcement

belief $f(\theta_2|\theta_1) = 1 \ \forall \theta_2 \in \Theta_2$, $f(\theta_1|\theta_2) = 1 \ \forall \theta_1 \in \Theta_1$

Common prior $f(\theta_1,\theta_2) = 1 \ \forall (\theta_1,\theta_2) \in [0,1]^2$

utilities are determined by the rule of auction

1.1 First price auction: ~~Players pay what they bid~~

If $b_1 \geq b_2$ player 1 wins and pays his bid, player 2 pays 0.

$b_1 < b_2$ player 2 ... – – –, player 1 ....

$$u_1(b_1,b_2,\theta_1,\theta_2) = (\theta_1 - b_1)\, I\{b_1 \geq b_2\}$$
$$u_2(b_1,b_2,\theta_1,\theta_2) = (\theta_2 - b_2)\, I\{b_1 < b_2\}$$

$b_1 = s_1(\theta_1)$, $b_2 = s_2(\theta_2)$, assume $s_i(\theta_i) = \alpha_i \theta_i$, $i = 1,2$, $\alpha_i > 0$

To find BE, we need to find the s_i^* (or α_i^*) that maximizes the ex-interim utility of player i

$$\underset{\sigma_i}{\text{maximize}} \ \mathbb{E}\left[u_i(\sigma_i, \sigma_{-i}^*, \theta_i, \theta_{-i}) \mid \theta_i\right]$$

For player 1, this problem reduces to

$$\int_0^1 f(\theta_2|\theta_1)(\theta_1 - b_1)\, I\{b_1 \geq \alpha_2\theta_2\}\, d\theta_2$$

$$= (\theta_1 - b_1)\frac{b_1}{\alpha_2} \qquad \text{maximize w.r.t. } b_1 \in [0, \alpha_2]$$

$$b_1 = \frac{\theta_1}{2} \text{ if } \alpha_2 > \frac{\theta_1}{2}$$

else α_2

$$s_1^*(\theta_1) = \min\left\{\frac{\theta_1}{2}, \alpha_2\right\}$$

Similarly for player 2

$$s_2^*(\theta_2) = \min\left\{\frac{\theta_2}{2}, \alpha_1\right\}$$

$\alpha_1 = \alpha_2 = 1/2$, then $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ is a BE with uniform prior.

(14-2)

What if highest bidder wins, but pays the losing bid.

1.2 Second price auction:

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2) I\{b_1 \geq b_2\}$$
$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1) I\{b_1 < b_2\}$$

Player 1's bidding problem is to maximize:

$$\int_0^1 f(\theta_2|\theta_1)(\theta_1 - s_2(\theta_2)) I\{b_1 \geq s_2(\theta_2)\}\, d\theta_2$$

$$= \int_0^1 1 \cdot (\theta_1 - \alpha_2\theta_2) I\{\theta_2 \leq \frac{b_1}{\alpha_2}\}\, d\theta_2$$

$$= \int_0^{b_1/\alpha_2} (\theta_1 - \alpha_2\theta_2)\, d\theta_2 = \theta_1 \cdot \frac{b_1}{\alpha_2} - \alpha_2 \cdot \frac{b_1^2}{2\alpha_2^2} = \frac{1}{\alpha_2}\left(b_1\theta_1 - \frac{b_1^2}{2}\right)$$

maximized when $b_1 = \theta_1$.

similarly for $b_2 = \theta_2$.

~~In particular~~ If the distributions of θ_1 and θ_2 are independent, but arbitrary

$$\int_0^{b_1/\alpha_2} f(\theta_2)(\theta_1 - \alpha_2\theta_2)\, d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \underbrace{\int_0^{b_1/\alpha_2} \theta_2 f(\theta_2)\, d\theta_2}$$

$$\int_0^{b_1/\alpha_2} \theta_2 f(\theta_2)\, d\theta_2 = \left[\theta_2 F(\theta_2)\right]_0^{b_1/\alpha_2} - \int_0^{b_1/\alpha_2} F(\theta_2)\, d\theta_2$$

$$= \frac{b_1}{\alpha_2} F\left(\frac{b_1}{\alpha_2}\right) - \int_0^{b_1/\alpha_2} F(\theta_2)\, d\theta_2$$

$$\theta_1 F\left(\frac{b_1}{\alpha_2}\right) - b_1 F\left(\frac{b_1}{\alpha_2}\right) + \alpha_2 \int_0 F(\theta_2)\, d\theta_2$$

differentiating

$$= \theta_1 f\left(\frac{b_1}{\alpha_2}\right)\frac{1}{\alpha_2} - \cancel{F\left(\frac{b_1}{\alpha_2}\right)} - b_1 f\left(\frac{b_1}{\alpha_2}\right)\frac{1}{\alpha_2} + \cancel{\alpha_2} \cancel{F\left(\frac{b_1}{\alpha_2}\right)} \frac{1}{\cancel{\alpha_2}}$$

$$= f\left(\frac{b_1}{\alpha_2}\right)\frac{1}{\alpha_2}(\theta_1 - b_1) = 0 \Rightarrow b_1 = \theta_1.$$

for any independent prior, ~~revealing~~ bidding true value is a BE.

Mechanism Design

Game Theory: Agent perspective, predictive approach

Mechanism Design: Designer's perspective, prescriptive approach.

Examples: (1) Matching students to universities

(2) Auction – goal is to give an object to a buyer who values ~~it~~ it the most

(3) Spectrum license.

(4) Voting

General setup

$N = \{1, 2, \ldots, n\}$ set of agents

X = set of outcomes

Θ_i = set of private information of ~~a~~ agent i

– types, $\theta_i \in \Theta_i$

$u_i : X \times \Theta_i \to \mathbb{R}$ → independent/private value model

$X \times \Theta \to \mathbb{R}$ → interdependent value model.

Examples: (1) Voting: X is the set of candidates

θ_i is the ranking over these candidates for player i.

u_i is any vNM utility function consistent with this ranking.

$X = \{a, b, c\}$, $\theta_i = \succcurlyeq_i$ s.t. $a \succcurlyeq_i b \succcurlyeq_i c$ then

$u_i(a) \geqslant u_i(b) \geqslant u_i(c)$

(2) Single object allocation:

an outcome $x \in X$ is a tuple $(\underline{a}, \underline{p})$ (allocation, payment).

$\underline{a} = (a_1, \ldots, a_n)$, $\underline{p} = (p_1, \ldots, p_n)$

$a_i \in \{0,1\}$, $\sum_{i=1}^{n} a_i \leq 1$. $\theta_i \in \mathbb{R}$, value for the object.

$u_i(x, \theta_i) = u_i((\underline{a}, \underline{p}), \theta_i) = a_i \theta_i - p_i$

③ Choosing a public project → building bridge, park, road etc.

$x \in X$ is a choice of project and the tax assigned to invidividuals. ~~xeo~~ $x = (a, \underline{p})$ $\quad a \in A = \{\text{bridge}, \text{park}, \dots\}$

a ↑ scalar

type of individuals $\theta_i : A \to \mathbb{R}$, $\theta_i \in \mathbb{R}^{|A|}$ if A is finite.

$$u_i(x, \theta_i) = u_i((a, \underline{p}), \theta_i) = \theta_i(a) - p_i$$

④ A set of public projects

$$\underline{a} \in \{0, 1\}^{|A|} \qquad a_i = 1 \Leftrightarrow \text{project } i \text{ is undertaken}$$

$$\theta_i \in \mathbb{R}^{|A|}$$

$$u_i((\underline{a}, \underline{p}), \theta_i) = \underline{a}^T \underline{\theta_i} - p_i$$

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types of ③ and ④ are multidimensional.

In mechanism design, there is a goal of the designers. This says given a specific type profile of the agents, we want a specific outcome.

The goal of the designer is captured in the Social Choice Function

$$f : \Theta \to X$$