CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 34: DSIC Mechanisms in Single Object Allocation

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

34.1 Recap

Continuing our discussion from the previous lecture, in which we had defined the notion of monotonicity of an allocation rule and stated the Myerson's Lemma:

Lemma 34.1 ***Myerson's Lemma***

Suppose $T_i = [0, b_i]$ $\forall i \in N$ and the valuations are in a product form . An allocation rule $f : T \to \Delta A$ and a payment rule is DSIC iff:

1. *f in non-decreasing*
2. *$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i \cdot f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i}) dx_i$ $\forall i \in N \quad \forall t_i \in T_i \quad \forall t_{-i} \in T_{-i}$*

So in this lecture we will look at the implications of the said payment rule and a number of examples around it and compare it with the Robert's Characterization where only one specific allocation function was true.

34.2 Payment Rule Discussions

Consider the case where we have two difference mechanisms given by (f, p) and (f, q) with the same allocation function f and different payment functions p and q. Looking at the payment function, the difference between the payments for these two mechanisms is the term $p_i(0, t_{-i})$. Since p and q implements f. So the entire flexibility of designing mechanisms is in this term and becomes visible by,

$$\begin{aligned} p_i(t_i, t_{-i}) - p_i(s_i, t_{-i}) &= q_i(t_i, t_{-i}) - q_i(s_i, t_{-i}) \quad \forall s_i \in T \\ p_i(t_i, t_{-i}) &= q_i(t_i, t_{-i}) + h_i(t_{-i}) \end{aligned} \tag{34.1}$$

The term $h_i(t_{-i})$ is independent of ith agents type t_i. So any two payment methods that implement f, by this characterization have to follow this rule. Hence we observe that,

1. Whenever any two payment rules that implement the same allocation rule that can be written in this form, we call the allocation rule to be **revenue equivalent**
2. The difference with Robert's Characterization Theorem, which gave us an explicit formula for f_i. It says that whenever the valuations are unrestricted, the allocation rule must be from the affine maximizer class. This functional form characterization by Robert's Theorem is different from the implicit property given by the Myerson's Lemma, where we have not given a form of the payment rule, but have mentioned a rule that it should follow.

Corollary 34.2 *An allocation rule is implementable iff it is non-decreasing.*

Some examples of the above allocation function are

1. **Constant allocation**: Payment is constant/zero , hence trivially non-decreasing
2. **Dictatorial Allocation**: The payment again here is constant, if we ensure monotonicity for the dictator's type
3. **Vickrey Auction**: Here the allocation is efficient and the payment for each agent, fixing the other agents, is non-decreasing.
   allocation : $f_i = \begin{cases} 1 & t_i \geq t_{-i}^{(2)} \\ 0 & t_i \leq t_{-i}^{(2)} \\ \alpha_i & t_i = t_{-i}^{(2)} \end{cases}$
   such that $\sum_i \alpha_i = 1$ and $\alpha_i \geq 0 \; \forall i \in N$. This tie-breaking is strategyproof and DSIC. payment $= t_{-i}^{(2)}$. The allocation function can be seen as the subgradient of the utility function here, which can be represented as:
   $u(v_i) = \begin{cases} 0 & v_i \leq t_{-i}^{(2)} \\ v_i & v_i > t_{-i}^{(2)} \end{cases}$ Now the payment rule (for the winning bidder, say player 1) can be written as :
   $$\begin{aligned} p(t_1, t_{-1}) &= 0 + t_1 \cdot f_1 - \int_{t_{-1}^{(2)}}^{t_1} f_1(x_1, t_{-1}) dx_1 \\ &= t_1 - (t_1 - t_{-1}^{(2)}) \\ &= t_{-1}^{(2)} \end{aligned} \tag{34.2}$$
4. **Efficient allocation with reserve**: The allocation here gives the bidder i the item if $t_i > max(r, t_{-i}^{(2)})$ where r is the reserve price set by the auctioneer. The payment made by the winning bidder is $max(r, t_{-i}^{(2)})$. This again is non-decreasing. Here the item is not sold if no bid is higher than the reserve price.
5. **A Not so common allocation rule**: Consider an allocation rule for two agents $N = \{1, 2\}$ and $A = \{a_0, a_1, a_2\}$ where the allocation a_0 refers to the item being unsold and a_i being the item alloted to player i. Give a type t_1,t_2 let the seller computes
   $$U(t) = max(2, t_1^2, t_2^3) \tag{34.3}$$
   The allocation proceeds as:
   $$\begin{aligned} &a_0 \quad \text{if} \quad U(t) = 2 \\ &a_1 \quad \text{if} \quad U(t) = t_1^2 \quad \text{i.e.} \quad t_1 > \sqrt[3]{max(2, t_2^3)} \\ &a_2 \quad \text{if} \quad U(t) = t_2^3 \quad \text{i.e.} \quad t_2 > \sqrt[2]{max(2, t_1^2)} \end{aligned} \tag{34.4}$$

34.3 Individual Rationality

Definition 34.3 ***Individual Rationality*** *: A mechanism (f, p) is "ex-post" individually rational if*

$$t_i \cdot f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \geq 0 \quad \forall t_i \in T_i \quad \forall t_{-i} \in T_{-i} \quad \forall i \in N \tag{34.5}$$

Here "ex-post" refers to the idea that even after revealing everyone's type, participation was weakly preferable.

Lemma 34.4 *Suppose (f, p) is DSIC,*

1. *It is Individually Rational (IR) iff*

$$p_i(0, t_{-i}) \leq 0 \quad \forall i \in N \quad \forall t_{-i} \in T_{-i} \tag{34.6}$$

2. *It is IR and gives no subsidy, i.e. $p_i(t_i, t_{-i}) \leq 0 \; \forall t_i \in T_i$ iff*

$$p_i(0, t_{-i}) = 0 \quad \forall i \in N \quad \forall t_{-i} \in T_{-i} \tag{34.7}$$

Proof: We shall present the proof part by part, and both directions (if and only if) for each

1. **if condition**: Assume

$$\begin{aligned} t_i \cdot f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) &\geq 0 \\ \text{for} \quad t_i = 0 \quad p_i(0, t_{-i}) &\leq 0 \end{aligned} \tag{34.8}$$

   only if condition : Assume $p_i(0, t_{-i}) \leq 0$

$$t_i \cdot f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) - t_i \cdot f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i}) dx_i \geq 0 \tag{34.9}$$

   Since in this case we assume $p_i(t_i, t_{-i}) \leq 0$, the above inequality holds.

2. **if condition**: Assume the mechanism DSIC, IR and satisfies no subsidy hence,

$$p_i(0, t_{-i}) \geq 0 \tag{34.10}$$

   but using the above proof, we have

$$\begin{aligned} p_i(0, t_{-i}) &\leq 0 \\ \Rightarrow p_i(0, t_{-i}) &= 0 \end{aligned} \tag{34.11}$$

   The opposite direction is straightforward.

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34.4 Some Non-Vickrey Acutions

1. **Redistribution**: Consider a case where the auction is Groves but not Vickrey Auction. In this auction, the highest bidder wins and gets the object but the payment is such that everyone is compensated some amount given by,

$$p_i(0, t_{-i}) = \frac{-1 \cdot t_{-i}^{(2)}}{n} \tag{34.12}$$

   where $t_{-i}^{(2)}$ = second highest among $\{t_j; j \neq i\}$
   WLOG, assume that $t_1 > t_2 > \cdots > t_n$

$$\text{So, payment of player } 1 = \frac{-1 * t_3}{n} + t_2 \tag{34.13}$$
$$\text{payment of player } 2 = \frac{-1 * t_3}{n} \tag{34.14}$$
$$\text{payment of other players} = \frac{-1 * t_2}{n} \tag{34.15}$$
$$\tag{34.16}$$
$$\text{Hence, the sum of payment} = \frac{2 \cdot (t_2 - t_3)}{n} \tag{34.17}$$
$$\tag{34.18}$$

The above expression tells us that such an auction is asymptotically budget balanced (as the surplus is redistributed as $n \to \infty$) while still being strategy-proof and DSIC. The allocation however is still deterministic in nature and let us see if we can do something better by randomizing the allocation.

2. **Green-Laffort mechanism**: Give object to the highest bidder with probability $1 - \frac{1}{n}$ and with the rest to the second highest bidder.
Say $t_1 > t_2 \cdots > t_n$, we have

$$p_i(0, t_{-i}) = \frac{-1 \cdot t_{-i}^{(2)}}{n} \tag{34.19}$$

where $t_{-i}^{(2)}$ = second highest among $\{t_j; j \neq i\}$

$$\text{payment of } 1 = \frac{1}{n} \cdot t_3 + (1 - \frac{1}{n}) \cdot t_1 - \frac{1}{n} \cdot (t_2 - t_3) - (1 - \frac{1}{n}) \cdot (t_1 - t_2) \tag{34.20}$$
$$\text{payment of } 2 = \frac{-t_3}{n} + \frac{t_2}{n} - \frac{t_2 - t_3}{n} = 0 \tag{34.21}$$
$$\text{payment of other} = -\frac{t_2}{n} \tag{34.22}$$
$$\text{Hence, the sum of payment} == \frac{n-2}{n} \cdot t_2 - (n-2) \cdot \frac{t_2}{n} \tag{34.23}$$

Hence such an allocation with the given payment mechanism is budget balanced.