Game Theory and Mechanical Design Assignment 2

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Problem 1

Part(a)

Given the preference profiles-

P_1	P_2	P_1'	P_2'
a	c	b	a
b	b	a	b
c	a	c	c

We construct the preference profiles Q, Q' and Q'' such that preferences P_2 and Q_2, P_1' and Q_1' remain same.

Q_1	Q_2	Q_1'	Q_2'	Q_1''	Q_2''
b	c	b	c	b	c
a	b	a	a	c	b
c	a	c	b	a	a

Suppose $f(P_1', P_2') = a$.

Also suppose $f(Q_1', Q_2') = b$. Then-

$$\begin{gathered} f(P_1', P_2') = a \\ f(Q_1', Q_2') = b \\ \text{Also, } aQ_2'b \\ \implies f(P_1', P_2')Q_2'f(Q_1', Q_2') \\ \implies f \text{ is manipulable} \end{gathered}$$

This contradicts that f is strategy-proof. Hence $f(Q_1', Q_2') \neq b$.

Now suppose that $f(Q_1, Q_2) = b$. We will show that this assumption also leads to contradictions.

(i) Suppose $f(Q_1', Q_2') = c$. Then in $Q' \to Q''$, since $L_i(c, Q_i') \subseteq L_i(c, Q''_i)$ and f is MONO

$$\implies f(Q''_1, Q''_2) = c$$

But this is a contradiction. Hence $f(Q_1', Q_2') \neq c$.

(ii) When $Q \to Q''$, Since $f(Q_1, Q_2) = b$ and $L_i(b, Q_i) \subseteq L_i(b, Q''_i)$ and f is MONO

$$\implies f(Q''_1, Q''_2) = b$$

So the assumption $f(Q_1, Q_2) = b$ is wrong.
$\therefore f(P_1', P_2') = a \implies f(Q_1, Q_2) = c$.
By symmetrical arguments, we can show that $f(P_1, P_2) = a \implies f(Q_1, Q_2) = b$.
But this is a contradiction. Thus the assumption $f(P_1', P_2') = a$ is incorrect.

$\therefore \qquad f(P_1', P_2') = b$

Part(b)

No, the earlier conclusion does not hold in this case.
The proof does not go through because if the preferences are generated from a single peaked preference domain with the intrinsic ordering of the alternatives being $a < b < c$, it isn't possible to construct the preference profile (Q_1', Q_2') since $a \quad < \quad b$ whereas $aQ_2'b$.

A mechanism that can have $f(P_1', P_2') = a$ can be given as-

$$f(P) = min_{i \in N}[P_i(1)]$$

where the minimum is taken with respect to the order relation $a < b < c$. Therefore this SCF picks the left-most peak among the peaks of agents and hence-

$$f(P_1, P_2) = a$$
$$f(P_1', P_2') = a$$

Problem 2

Since the SCF chooses a non-empty subset of projects, therefore the preference domain is the power set of X.

Let $X = a_1, a_2, ..., a_n$ with the preference ordering P_i as $a_1 > a_2 > ... > a_n$. Consider two subsets $S, T \subseteq X$, which have a common element a_1. Then according to question, S and T are indifferent.

If indifferences are allowed among various alternative, then generally Gibbard-Satterthwaite theorem does not hold. Hence Gibbard-Satterthwaite result won't apply here.

Problem 3

Yes, the median voter SCF is group strategy-proof.

We need to consider only the peak preferences of all the agents. So let us denote the preferences denoted only by their peaks, i.e., $P = (P_1(1), .., P_i(1), .., P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and the phantom peaks. Consider a group of agents denoted by set G with cardinality $n(G)$. Four conditions arise-

1.) $P_i(1) = a \; \forall \; i \in G$ - There is no reason for any player in G to manipulate.

2.) $P_i(1) < a \; \forall \; i \in G$ - If some or all agents shift their peaks to the further left of a, the median will not change.
If some or all agents manipulate to report their peak to the further right of a, i.e.$(P_i, P_i) \rightarrow (P'_i, P_i)$ s.t. $a < P'_i(1)$then for any agent $i \in G$, , this will imply that the median gets shifted to the right of a, which the agent doesn't prefer to a.
Thus, the group G has no profitable manipulation.

3.) $P_i(1) > a \; \forall \; i \in G$ - Again by exactly symmetrical arguments, group G has no profitable manipulation.

4.) $P_i(1) < a$ for some $k(< n(G))$ players $\in G$ and $P_i(1) > a$ for $(n(G) - k)$ players $\in G$ - In this condition, by similar arguments as presented in 2^{nd} case, if agents of either side(say left) of a manipulate their peaks to move to the other side(say right) of a, then the agents on the other side(right) might benefit from the manipulation but the agents who manipulated (left) won't benefit from their own manipulation. If they instead manipulated to the same side(shifting their peaks further left to a), then median won't change and they again won't benefit.
Thus, the group of agents have no incentive to manipulate.
Hence, f is group strategy-proof.