CS711: Introduction to Game Theory and Mechanism Design	**Jul-Nov 2018**

Project: Assignment 2

Question 1

We are given a two agent model with three alternatives {a, b, c}.

P_1	P_2	P_1'	P_2'
a	c	b	a
b	b	a	b
c	a	c	c

f is onto SCF with $f(P_1, P_2) = a$.

Part (a)

Claim 1 : In the below preference profile P'', $f(P_1'', P_2'') = a$.

P_1	P_2	P_1''	P_2''	$\hat{P}_1$	$\hat{P}_2$
a	c	a	c	a	c
b	b	-	-	c	a
c	a	-	-	b	b

Suppose for contradiction $f(P_1'', P_2'') = c$. Consider transition from (P_1, P_2) to $(\hat{P}_1, \hat{P}_2)$. Preference for a improves for both agents and $f(P_1, P_2) = a$. Thus by monotonicity, $f(\hat{P}_1, \hat{P}_2) = a$. Next consider transition from (P_1'', P_2'') to $(\hat{P}_1, \hat{P}_2)$. Preference for c improves for both agents and $f(P_1'', P_2'') = c$. Thus by monotonicity, $f(\hat{P}_1, \hat{P}_2) = c$. But $a \neq c$. This gives us a contradiction. Therefore, $f(P_1'', P_2'') = a$.

Claim 2 : In the below preference profile P'', $f(P_1'', P_2'') = b$.

P_1	P_2	P_1''	P_2''	$\hat{P}_1$	P_2
a	c	b	c	b	c
b	b	-	-	a	-
c	a	-	-	c	-

Suppose for contradiction $f(P_1'', P_2'') = c$. First we consider transition from (P_1'', P_2'') to $(\hat{P}_1, P_2)$. Notice that this transition satisfies all constraints of Claim 1. Hence, $f(\hat{P}_1, P_2) = c$.
Consider preference profile $(P_1'', P_2''))$. At this profile if agent 1 reports P_1 instead of $\hat{P}_1$, the outcome is a which she prefers more than the current outcome c, as $f(\hat{P}_1, P_2) = c$ and $f(P_1, P_2) = a$. This is a contradiction to f being strategyproof. Therefore, $f(P_1'', P_2'') = b$.

Now consider

Suppose for contradiction $f(P_1', P_2') = a$. We first consider transition from (P_1', P_2') to $(\hat{P}_1, \hat{P}_2)$. This transition follows the constraints of Claim 2. Hence, $f(\hat{P}_1, \hat{P}_2) = c$. Next, we consider transition from

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	c	b	a	b	c
b	b	a	b	a	-
c	a	c	c	c	-

(P_1, P_2) to $(\hat{P}_1, \hat{P}_2)$. This transition also follows the constraints of Claim 2. Hence $f(\hat{P}_1, \hat{P}_2) = b$. But $a \neq c$. We have a contradiction. $f(P_1^{'}, P_2^{'}) = b$.

Part (b)

Now the preferences are generated from a single-peaked preference domain with the common order < over the alternatives being a < b < c.

The earlier conclusion doesn't hold in this case.

The earlier proof can't go through because in proving earlier case we have constructed a profile $\hat{P}$ in claim 1 where agents preference violates the single-peaked preference domain.

Define a mechanism $< M, g >$ such that M = Θ and g = f and f : $S^3 \rightarrow$ {a,b,c} where S is the set of single-peaked preferences w.r.t. the common order < is defined as:

$$f(P) = \min_{i \in N}\{P_i(1)\}.$$

Where minimum is taken w.r.t. the order relation <. Hence the f picks the left-most peak among the peaks of the agents. We have proved in lecture notes that this f is SP.

The above f satisfies $f(P_1, P_2) = a$ and also gives $f(P_1^{'}, P_2^{'}) = a$

Question 2

In this setting, Gibbard-Satterthwaite result doesn't apply. Here I am going to show domain restriction in this setting.

Suppose a_1, a_2 ∈ X and in some profile P, agent i have preference order as $P_i(1) = \{a_1\}$, $P_i(2) = \{a_2\}$, $P_i(3) = \{a_1, a_2\}$. I am going to show that this preference order is not possible.

Consider following 2 cases:
Case 1 : $a_1 P_i a_2$ in linear ordering P_i of agent i. In this case clearly $P_i(3) P_i P_i(2)$. So, agent i can't have above defined preference order.

Case 2 : $a_2 P_i a_1$ in linear ordering P_i of agent i. In this case clearly $P_i(2) P_i P_i(1)$. So, agent i can't have above defined preference order.
So, agent i can't have above defined preference order which means this type of preference order doesn't exist in domain.

Question 3

Yes, the median voter SCF is group strategy-proof.

Proof : We need to consider only the peak preferences of all the agents. So let us denote the preferences denoted only by their peaks, i.e., P = $(P_1(1), .., P_i(1), .., P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and the phantom peaks. Consider group of agent $K \subseteq N$.

- Consider a case where $\exists i \in K$ such that $P_i(1) = a$. Then the group has no reason to manipulate it because if group tries to change this result such that $f(P_K^{'}, P_{-K}) = b \neq a$. Then $f(P_K, P_{-K}) P_i f(P_K^{'}, P_{-K})$ which violates the definition of group manipulable.
- Consider the case where $P_i(1) \neq a$, $\forall i \in K$. Suppose $K_1 = \{k : P_k(1) < a, k \in K\}$ i.e. set of agents in K whose peak is at the left of a. Similarly $K_2 = \{k : P_k(1) > a, k \in K\}$. Consider the following 2 sub cases:
  - Suppose $P_K{'}$ is such that $f(P_K^{'}, P_{-K}) = b < a$. Then $\exists k \in K_2$ who reports her peak to the further left of a which means $b < a < P_k(1)$. Since P_k is single peak preference which means $a = f(P_K, P_{-K}) P_k f(P_K^{'}, P_{-K}) = b$ thus violating the definition of group manipulable.
  - Suppose $P_K{'}$ is such that $f(P_K^{'}, P_{-K}) = b > a$. Similar proof as above.

Therefore, median voter SCF is not manipulable by the group of agents.