CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 5: Characterization Theorem for MSNE

Lecturer: Swaprava Nath *Scribe(s): Gundeep Arora*

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5.1 Recap

Continuing our discussion from the previous lecture, in which we had defined the notion of mixed strategy nash equilibrium for a multi-player one-shot non-cooperative game. Let us first revisit the definitions,

Definition 5.1 ***Mixed Strategy***

From [MAS], Given a player i with S_i as the set of pure strategies, a mixed strategy σ_i of player i is a probability distribution over S_i. That is, $\sigma_i : S_i \rightarrow [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
More formally, from [Narahari], let (N,A,u) be a normal-form game, and for any set X let $\Delta(X)$ be the set of all probability distributions over X. Then the set of mixed strategies for player i is $S_i = \Delta(A_i)$

The observation that pure strategies are special cases of mixed strategies, is trivial.

Definition 5.2 ***Mixed strategy profile***
Again from [MAS], a mixed strategy profile is simply the set of cartesian product of the individual mixed strategy sets, $S_1 \times S_2 \times \cdots S_n$

The utility of a mixed strategy profile is defined as,

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i, s_{-i} \in S} \sigma(s_i, s_{-i}) u_i(s_i, s_{-i})$$

Definition 5.3 ***Mixed Strategy Nash Equilibrium***
A strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) > u_i(\sigma_i^{'}, \sigma_{-i}^*) \quad \forall \sigma_i^* \in \Delta(S_i) \quad \forall i \in N$$

The above inequality states that under the assumption of rationality, in case of MSNE, there is no reason for any player i to deviate from σ_i^* in response to σ_{-i}^*, where $u_i(\cdot)$ is the expected utility of the strategy profile.

We also looked at a couple of examples where an MSNE existed while a pure strategy nash equilibrium (PSNE) did not. Towards the end, we looked at the characterization theorem of MSNE. In this lecture, we shall prove the theorem and revisit the examples of the previous lecture with the lens of this theorem.

5.2 Characterization Theorem of MSNE

For completeness, we shall restate the theorem from the last lecture. This theorem lists down the essential characteristics of a mixed strategy nash equilibrium.

Theorem 5.4 *A strategy profile* $(\sigma_i^*, \sigma_{-i}^*)$ *is a MSNE iff* $\quad \forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ *is same* $\forall s_i \in \delta(\sigma_i^*)$
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i^{'}, \sigma_{-i}^*) \quad \forall s_i \in \delta(\sigma_i^*), \quad s_i^{'} \notin \delta(\sigma_i^*)$

Before we prove this theorem, we shall state an observation,

Remark 5.5 *The maximum utility strategy will always be a part of the support of the MSNE strategy, as otherwise we could create another strategy with all the probability mass on the maximum one and this strategy would have a better utility.*

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

Proof: There willl be two parts to the proof, where we first assume that given a strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE, the two conditions hold true and vice-versa.

1. Given $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE,

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i^*, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{5.1}$$

Also, the definition of expected utility for the given strategy profile we have,

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{5.2}$$

Equating the 5.1 and 5.2, we see that the expectation and the maximum value of a set are equal. This can happen only when either the set is singleton or all the elements take the same value. This proves the first condition mentioned.
For the second condition, we proceed by the idea of contradiction. Suppose the condition does not hold, i.e.

$$\exists s_i \in \delta(\sigma_i^*), \quad s_i^{'} \notin \delta(\sigma_I^*) \quad s.t. \quad u_i(s_i, \sigma_{-1}^*) \leq u_i(s_i^{'}, \sigma_{-i}^*)$$

In such a case, we can always choose a strategy for player i, where

$$\begin{aligned} \sigma_i^{'}(s_i^{'}) &= 1 \\ \sigma_i^{'}(s_i) &= 0 \\ u_i(\sigma_i^*, \sigma_{-1}^*) &\leq u_i(\sigma_i^{'}, \sigma_{-i}^*) \end{aligned} \tag{5.3}$$

The above equation states that the player i can change his strategy profile from σ_i^* to $\sigma_i^{'}$ which is a contradiction to the given statement that $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE. This proves our second condition as well.

2. Now we assume that given the two conditions of the characterization theorem hold and try to prove that the strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE.

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_i^*) \\ &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-1}^*) \\ &\geq u_i(\sigma_i, \sigma_{-1}^*) \quad \forall \sigma_i \in \Delta(S_i) \end{aligned} \tag{5.4}$$

The final inequality states that the givens strategy profile that satisfies the two conditions is better than any other available strategy profile for player i. Hence the strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE.

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5.3 Some Examples for MSNE

With the proof of the characterization theorem done, we shall now look at some examples with the lens of that theorem and argue whether possible strategy profiles are MSNE or not based on their satisfiability of the two conditions of the theorem.

5.3.1 The Penalty Shoot-out Game

In the penalty shoot-out game, that we looked at last time, we had concluded that no pure strategy nash equilibrium existed for that game. The game poses the utility functions of either player hitting or missing the target.

1 \2	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

Let the probability of player 1 choosing H be p. Similarly, for player 2, let that probability be q. We shall now look at some of the strategy profiles and argue if it can be an MSNE or not.

- $k =$**({H},{H})** : In this case, $u_1(k) = 1$ and $u_2(k) = -1$. The for the given choice of strategy for player 1, the player 2, can choose {H} and have a utility $u_2\{\text{H}\}, \{\text{H}\}) = 1$, which is more than $u_2(k)$. We can establish the same result for each of the other pure strategy profiles will not be a MSNE. This is just what we had seen in the previous lecture, where we had concluded that no pure strategy nash equilibrium existed for this game. These pure strategy profiles also do not satisfy the second condition of the characterization theorem.

- $k =$**({H},{H,T})** : In this case, $u_1(k) = 1$ and

$$\begin{aligned} u_2(\{\text{H}\}, \{\text{H}\}) &= -1 = 0 \\ u_2(\{\text{H}\}, \{\text{T}\}) &= 1 \\ &= 2p - 1 \end{aligned} \tag{5.5}$$

For this to be a MSNE, by the first condition of the characterization equation, we have

$$\begin{aligned} u_2(\{\text{H}\},\{\text{H}\}) &= u_2(\{\text{H}\},\{\text{T}\}) \\ -1 &\neq 1 \end{aligned} \tag{5.6}$$

This hence cannot be a MSNE. By using the argument of symmetry for this game, we can argue that any strategy profile that has a pure strategy for any user will not be a nash equilibrium.

- $k =$**({H,T},{H,T})** : In this case,

$$\begin{aligned} u_2(\{\text{H,T}\},\{\text{H}\}) &= (-1)\cdot p + 1\cdot(1-p) \\ &= 1-2p \\ u_2(\{\text{H,T}\},\{\text{T}\}) &= 1\cdot p + (-1)\cdot(1-p) \\ &= 2p-1 \\ u_1(\{\text{H}\},\{\text{H,T}\}) &= 1\cdot q + (-1)\cdot(1-q) \\ &= 2q-1 \\ u_1(\{\text{T}\},\{\text{H,T}\}) &= (-1)\cdot q + 1\cdot(1-q) \\ &= 1-2q \end{aligned} \tag{5.7}$$

For this strategy for this to be a valid MSNE, from the characterization theorem that we just proved, we have the following,

$$\begin{aligned} u_2(\{\text{H,T}\},\{\text{H}\}) &= u_2(\{\text{H,T}\},\{\text{T}\}) \\ 1-2q &= 2q-1 \\ q &= \frac{1}{2} \\ u_1(\{\text{H}\},\{\text{H,T}\}) &= u_1(\{\text{T}\},\{\text{H,T}\}) \\ 2p-1 &= 1-2p \\ p &= \frac{1}{2} \end{aligned} \tag{5.8}$$

Since, the system of equations have a solution, the the strategy profile $(\{\text{H}:\frac{1}{2},\text{T}:\frac{1}{2}\},\{\text{H}:\frac{1}{2},\text{T}:\frac{1}{2}\})$ is a MSNE. The first condition is vacuously satisfied.

5.3.2 Game-Selection Problem

We also looked at the problem, where the two friends who had different preferences towards going for a game while still going together to have a good time. More formally, the game with the utility function is represented as :

1 \2	F	C
F	(2,1)	(0,0)
C	(0,0)	(1,2)

Like the previous example, let the probability of player 1 choosing H be p. Similarly, for player 2, let that probability be q. We shall now look at some of the strategy profiles and argue if it can be an MSNE or not.

- **Pure strategy** : Consider a pure strategy, $k =$({F},{F}). For this strategy profile, we have $u_1(k) = 2$ and $u_2(k) = 1$. This is a PSNE, hence definitely a MSNE. As we had seen in the previous lecture, the profile k =({C},{C}) is also a MSNE. Both these profiles trivially satisfy the two conditions of the characterization theorem.

- **({F},{F,C})** : For this strategy profile,

$$\begin{aligned} u_2(\{\text{F}\},\{\text{F}\}) &= 1 \\ u_2(\{\text{F}\},\{\text{C}\}) &= 0 \end{aligned} \tag{5.9}$$

Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({C},{F,C})** : For this strategy profile,

$$\begin{aligned} u_2(\{\text{C}\},\{\text{F}\}) &= 0 \\ u_2(\{\text{C}\},\{\text{C}\}) &= 2 \end{aligned} \tag{5.10}$$

Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({F,C},{F})** : For this strategy profile,

$$\begin{aligned} u_1(\{\text{F}\},\{\text{F}\}) &= 2 \\ u_1(\{\text{C}\},\{\text{F}\}) &= 0 \end{aligned} \tag{5.11}$$

Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({F,C},{C})** : For this strategy profile,

$$\begin{aligned} u_1(\{\text{C}\},\{\text{C}\}) &= 1 \\ u_1(\{\text{F}\},\{\text{C}\}) &= 0 \end{aligned} \tag{5.12}$$

Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({F,C},{F,C})** : In this case,

$$\begin{aligned} u_2(\{\text{F,C}\},\{\text{F}\}) &= 1 \cdot p + 0 \cdot (1-p) \\ &= p \\ u_2(\{\text{F,C}\},\{\text{C}\}) &= 0 \cdot p + 2 \cdot (1-p) \\ &= 2 - 2p \\ u_1(\{\text{F}\},\{\text{F,C}\}) &= 2 \cdot q + 0 \cdot (1-p) \\ &= 2q \\ u_1(\{\text{C}\},\{\text{F,C}\}) &= 0 \cdot q + 1 \cdot (1-q) \\ &= 1 - q \end{aligned} \tag{5.13}$$

For this strategy for this to be a valid MSNE, from the characterization theorem that we just proved, we have the following,

$$\begin{aligned} u_2(\{\text{F,C}\},\{\text{F}\}) &= u_2(\{\text{F,C}\},\{\text{C}\}) \\ p &= 2 - 2p \\ p &= \frac{1}{3} \\ u_1(\{\text{F}\},\{\text{F,C}\}) &= u_1(\{\text{C}\},\{\text{F,C}\}) \\ 2q &= 1 - q \\ p &= \frac{1}{3} \end{aligned} \tag{5.14}$$

Since, the system of equations have a solution, the the strategy profile $(\{\mathrm{F} : \frac{1}{3}, \mathrm{T} : \frac{2}{3}\}, \{\mathrm{F} : \frac{2}{3}, \mathrm{C} : \frac{1}{3}\})$ is a MSNE. The first condition is vacuously satisfied.

This problem can be extended to have the following form,

1 \2	F	C	D
F	(2,1)	(0,0)	(1,1)
C	(0,0)	(1,2)	(2,0)

For such a case, a mixed strategy for player two, will generally look like, $\{\mathrm{F} : \mathrm{q}_1; \mathrm{C} : \mathrm{q}_2; \mathrm{D} : 1 - \mathrm{q}_1 - \mathrm{q}_2\}$. Proceeding similar to the original example, we have,

$$\begin{aligned} u_1(\{\mathrm{F}\}, \{\mathrm{F,C,D}\}) &= u_1(\{\mathrm{C}\}, \{\mathrm{F,C,D}\}) \\ 2 \cdot q_1 + 0 \cdot q_2 + 1 \cdot (1 - q_1 - q_2) &= 0 \cdot q_1 + 1 \cdot q_2 + 2 \cdot (1 - q_1 - q_2) \\ 1 + q_1 - q_2 &= 2 - 2q_1 - q_2 \\ q_1 &= \frac{1}{3} \end{aligned} \tag{5.15}$$

However, we do not have any specific value for q_2, so any value of $\mathrm{q}_2 between [\frac{2}{3,1]}$. The solution for the player 1 remains the same.

5.4 General Principle for finding MSNE

As we saw in the two examples above, in order to evaluate the nash equilibrium. we enumerated all the possible supports of the cartesian product, $\mathrm{S}_1, S_2, \cdots, S_n$, where S_i is the set of all possible options for player i and then used the two conditions to check if it was a MSNE or not. This means that the number of supports, K, that need to be enumerated are,

$$\mathrm{K} = (2^{|\mathrm{S}_1|} - 1) \times (2^{|\mathrm{S}_2|} - 1) \times \cdots \times (2^{|\mathrm{S}_\mathrm{n}|} - 1)$$

More formally put, the problem that we are trying to solve here is, given a support $X_i \subseteq S_i$ and a support profile $X_1 \times X_2 \times \cdots X_n$, we have

$$w_i = \sum_{s_{-i} \in S_{-i}} \prod_{j \neq i} \sigma_j(s_j) u_i(s_i, s_{-i}) \quad \forall s_i \in X_i \quad \forall i \in N$$

From the first condition of the characterization theorem, $u_i(s_i, \sigma_{-i})$ should be same for all $s_i \in \delta(\sigma_i)$,

$$w_i \geq \sum_{s_{-i} \in S_{-i}} \prod_{j \neq i} \sigma_j(s_j) u_i(s_i, s_{-i}) \quad \forall s_i \in S_i \setminus X_i \quad \forall i \in N$$

So for every player $i \in N$, we will have a non-linear (if N¿2) equality and a non-linear inequality condition for a given support profile, making it very hard to solve, given that the number of support profiles grow exponentially with the size of the action set. Finding the Nash equilibrium is an active area of research. For two player zero-sum game, we can have a linear program, the equalities to be solved turn out to be simple and hence we have effiecient algorithms for it. There are algorithms like Lemke-Houson Algorithms that are proven to be optimal for two player non-zero sum game. We shall look at the existence of the Nash equilibrium in the next lecture. Here is a citation, just for fun [CW87].

References

[MAS] Yoav Shoham and Kevin Leyton-Brown, "Multi Agent Systems Algorithmic, Game-Theoretic, and Logical Foundations,"

[Narahari] Y. Narahari, "Mixed Strategies and Mixed Strategy Nash Equilibrium," *Game Theory, Lecture Notes*, 2012