Differential Calculus Review

02 December 2016

Limits and Continuity

1. Give the $\varepsilon-\delta$ definition for the limit statement; $\lim\limits_{x\to a} f(x)=L$.

2. Find a number δ such that if $|x-2|<\delta$, then $|4x-8|<\varepsilon$, where

   (a) $\varepsilon=0.1$ (b) $\varepsilon=0.005$

3. Find the limit, if it exists

   (a) $\lim\limits_{x\to 0}\dfrac{(2+x)^{-1}-2^{-1}}{x}$ (b) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2-9}}{2x-6}$

   (c) $\lim\limits_{x\to \pi/3^-}\left(\dfrac{1-\cos 3x}{\sin 3x}\right)$ (d) $\lim\limits_{x\to 0^+}\left(\arctan 1/x\right)$

   (e) $\lim\limits_{x\to 0^+}(e^{2x}+x)^{1/(3x)}$ (f) $\lim\limits_{x\to 0}\left(\dfrac{x-\tan 3x}{x+\tan 2x}\right)$

   slightly more challenging

   (g) $\lim\limits_{x\to\infty}\left[\sqrt{x^2+7x}-x\right]$ (h) $\lim\limits_{x\to\infty} x\ln\left(\dfrac{x^2}{x^2-1}\right)$

4. (a) For what value(s) of the constant *c* is the function *f* everywhere continuous?

   $$f(x)=\begin{cases} cx+7 & \text{if } x\le 2 \\ (cx)^2+1 & \text{if } x>2 \end{cases}$$

   (b) Determine where *f* is discontinuous and state which of the three conditions of continuity fail there.

   $$f(x)=\begin{cases} \sqrt{1-x} & \text{if } x\le 1 \\ 2+\ln e^{x-4} & \text{if } 1<x<5 \\ (3-x)^2 & \text{if } x>5 \end{cases}$$

Differentiation

1. State the definition of the derivative of a function at *a*, then use it to find the derivative of each function at the indicated point.

   (a) $f(x)=4x^2-x$ at $a=2$

   (b) $f(x)=\sqrt{2x+1}$ at $a=4$. Identify the domain of *f* and the domain of f'.

2. Given the curve $y=\sqrt{x}-\dfrac{1}{\sqrt{x}}$

   (a) Find an equation of the tangent line at the point with *x*-coordinate $x=4$.

   (b) (a) Find an equation of the normal line at the point with *x*-coordinate $x=4$.

3. An arrow is shot upward on mars from ground level with a velocity of 60 meters per second. Its height (in meters) after t seconds is given by $s(t) = 60t - 1.9t^2$. (a) what is the maximum height attained by the arrow? (b) With what velocity will the arrow hit mars?

4. Find the indicated derivative or value and simplify.

(a) $y = x^3(4 - 5x^2)$, $\dfrac{d^2y}{dx^2}$ (b) $y = \dfrac{(x^2-1)^{3/2}}{3x^3}$, $\dfrac{dy}{dx}$

(c) $f(x) = \sin^3\left(2x + \dfrac{\pi}{3}\right)$, $f'(\pi/2)$ (d) $y = \sqrt[3]{x + \sqrt[4]{x}}$, $\dfrac{dy}{dx}$

(e) $y = e^{2x}\cos 3x$, y'' (f) $y\ln x = x\ln y + 3$, $\dfrac{dy}{dx}$

(g) $y = x^{\sin x^2}$, $\dfrac{dy}{dx}$ (h) $g(x) = \displaystyle\int_1^{\sqrt{x}} \sqrt{t^3 - 1}\ dt$, $g'(x)$

(i) $f(x) = \cos^3\left(\dfrac{2}{x^2+3}\right)$, $f'(1)$ to 4 decimal places (j) $y = e^{2x}\sec 3x$, $\dfrac{d^2y}{dx^2}$

5. Related rates

(a) Assume *R* and *S* are differentiable functions of *t*. If *R* and *S* satisfy the equation; $3R^2 + 2S^3 = 14$, find $\dfrac{dR}{dt}$ at the instant when $R = 2$ and $\dfrac{dS}{dt} = 0.1$.

(b) A 5-meter ladder is leaning against the side of a house. The foot of the ladder is pulled away from the house at a rate of 0.4 m/sec. Determine how fast the top of the ladder is descending when the foot of the ladder is 3 meters from the house.

(c) The height of a cylinder is increasing at a rate of 2 centimeters per minute. Find the rate of change of the volume of the cylinder with respect to time when the height is 10 centimeters if the radius remains constant at 4 cm.

(d) A baseball diamond is a square with side 90 ft. A batter hits the ball and at the time he is half way to first base, his speed is 28 ft/sec. At what rate is his distance from second base decreasing when he is halfway to first base? Round the result to the nearest hundredth.

6 Determine the absolute extrema of the function on the indicated interval.

(a) $f(x) = x\sqrt{2x+6}$; $[-5/2, -1/2]$

(b) $f(x) = 5x^{2/3} + 2x^{5/3}$; $[-8, 8]$

7. State the Mean Value Theorem. Provide a sketch of a function that satisfies the hypotheses and indicate the location(s) on the curve that correspond to the conclusion of the theorem.

Determine whether the Mean Value Theorem applies to the function on the indicated interval and, if so, find all values '*c*' guaranteed by the theorem.

(a) $f(x) = \dfrac{7x-6}{x}$; $[1,6]$

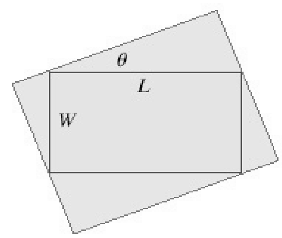
(b) $f(x) = \sin 2x - 2\sin x$; $[\pi, 2\pi]$

8. For the functions below,
   (a) determine the domain (unless otherwise specified)
   (b) find any intercepts and determine any asymptotes
   (c) compute $f'(x)$ and $f''(x)$
   (d) Determine the critical numbers, identify the open intervals on which *f* is increasing and those on which *f* is decreasing, find and classify any local extrema.
   (e) identify the open intervals on which *f* is concave up and those on which *f* is concave down, then determine the coordinates of any points of inflection.
   (f) sketch the graph of the function

   $f(x) = (x-2)^3(3x+14)$

   $f(x) = \dfrac{x^2 - 3x - 6}{x+2}$

9. (a) Find the point on the graph of $y = \sqrt{x+1}$ closest to the point (3, 0).
   (b) Find an equation of the line through the point (9,36) such that the area of the right triangle in the first quadrant determined by the line and the coordinate axes is a maximum.
   (c) Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L = 8 and width W = 3. Hint: write the area of the larger rectangle as a function of the angle theta.



   (d) A rectangular storage container with an open top is to have a volume of 288 m^3. The length of its base is twice the width. Find the dimensions of the container with minimal surface area.

10. (a) Find the differential of the function $y = x^2 \cos 2x$.

    (b) Find the linearization $L(x)$ of the function $f(x) = (x^2 - 1)^{2/3}$ at $a = 3$ and use it to approximate $f(2.5)$. i.e. compute $L(2.5)$

    (c) The measurement of the edge of a piece of square floor tile is found to be 12 inches with a possible error of 0.02 inches. Use differentials to approximate the maximum possible error in the calculated area and give the relative error of the calculation.

Integration

1. Estimate the area under the graph of $y = \sqrt{4 - x^3}$ over the interval [0,1.5] using three approximating rectangles and right endpoints.
2. Express the limit as a definite integral on the given interval.

$$\lim_{n\to\infty}\sum_{i=1}^{n} 3e^{r_i}\sin r_i \Delta r\ ;\ [-1,1]$$

3. Evaluate the integral by interpreting it in terms of area (note: you cannot find an antiderivative for the third term of the integrand by elementary methods)

$$\int_{-3}^{0}\left\lfloor x+3+\sqrt{9-x^2}\right\rfloor dx$$

4. Evaluate the integral

(a) $\displaystyle\int_{6}^{8}\frac{x}{\sqrt{100-x^2}}\,dx$

(b) $\displaystyle\int_{-5}^{2}\frac{x}{(x+6)^{2/3}}\,dx$

(c) $\displaystyle\int_{0}^{3}\frac{2x+3}{1+x^2}\,dx$

(d) $\displaystyle\int_{1}^{e^3}\frac{1-\ln x}{x}\,dx$

(e) $\displaystyle\int_{0}^{\pi/12}(\sin 3x+\cos 3x)^2\,dx$

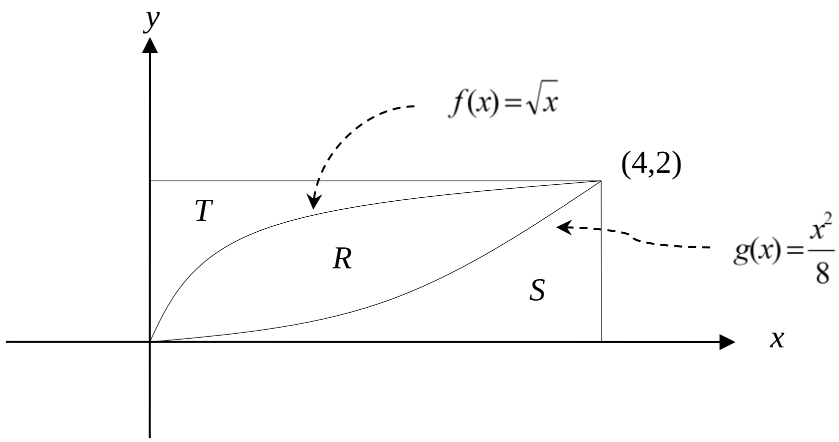
(f) $\displaystyle\int \tan^2 x\sec^2 x\,dx$

Applications of Integration

1. Find the area of the region bounded by the graphs of the given curves.
   (a) $y=x+2$ and $y=x^2$.
   (b) $y=\sin\pi x$ and $y=x^2-x$ on the interval $[0,1]$
   (c) $x=2y^2$ and $x=4+y^2$

2. Set up a definite integral(s) that represents the area of the triangle with vertices (0,0), (-2,3), and (2,1).

3. See the figure below.
   (a) Find the area of the region *R*.
   (b) Find the area of the region *T*.
   (c) Set up an integral that represents the volume of the solid created by rotating *R* about the x-axis.
   (d) Set up an integral that represents the volume of the solid created by rotating *R* about the *y*-axis.
   (e) Set up an integral that represents the volume of the solid created by rotating *S* about the *y*-axis.
   (f) Set up an integral that represents the volume of the solid created by rotating *T* about the line $x = -3$.



4. The cap of a sphere of radius 12 has been removed. Find the volume of the remainder of the sphere if the height *h* of the cap is 3 as shown.