1. Find the indicated derivative or value.

(a) (8 points) $y = \dfrac{\sec(5x)}{\tan x - 2}$ find $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{(\tan x - 2)\dfrac{d(\sec 5x)}{dx} - \sec 5x \cdot \dfrac{d}{dx}(\tan x - 2)}{(\tan x - 2)^2} \qquad \left[\because \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{(\tan x - 2)(5\sec 5x \tan 5x) - \sec 5x(\sec^2 x)}{(\tan x - 2)^2}$$

$$= \frac{5\tan x \sec 5x \tan 5x - 10 \sec 5x \tan 5x - \sec^2 x \sec 5x}{(\tan x - 2)^2}$$

(b) (8 points) $f(x) = 2\arctan(3x)$, find $f'(1)$

We know $\dfrac{d(\tan^{-1} x)}{dx} = \dfrac{1}{1+x^2}$

$$f'(x) = \frac{2}{1+(3x)^2} \cdot \frac{d(3x)}{dx}$$

$$= \frac{2}{1+9x^2}(3) = \frac{6}{1+9x^2}$$

$$f'(1) = \frac{6}{1+9(1)^2} = \frac{6}{10} = \frac{3}{5}$$

-0

2. Find the indicated derivative or value.

(a) (8 points) $f(x)=\ln\left[\frac{x^3-2}{\sqrt{x+7}}\right]$, $f'(x)$

$f(x) = \ln(x^3-2) - \ln(x+7)^{1/2}$

$= \ln(x^3-2) - \frac{1}{2}\ln(x+7)$

differentiating with respect to x,

$f'(x) = \frac{1}{x^3-2}\cdot\frac{d}{dx}(x^3-2) - \frac{1}{2(x+7)}\frac{d}{dx}(x+7)$

$= \frac{3x^2}{x^3-2} - \frac{1}{2x+14} = \frac{(3x^2)(2x+14) - x^3+2}{(x^3-2)(2x+14)}$

(b) (8 points) $y=e^x\cos(2x)$, find $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = \cos 2x\cdot\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\cos 2x)$

$= e^x\cos 2x + e^x(-2\sin 2x)$

$\frac{dy}{dx} = e^x(\cos 2x - 2\sin 2x)$

$\frac{d^2y}{dx^2} = (\cos 2x - 2\sin 2x)\frac{d}{dx}(e^x) + e^x\cdot\frac{d}{dx}(\cos 2x - 2\sin 2x)$

$= e^x(\cos 2x - 2\sin 2x) + e^x(-2\sin 2x - 4\cos 2x)$

$= e^x(\cos 2x - 2\sin 2x - 2\sin 2x - 4\cos 2x)$

$= e^x(-3\cos 2x - 4\sin 2x)$

$= -e^x(3\cos 2x + 4\sin 2x)$

-0

3. (8 points) Use implicit differentiation to find $\frac{dy}{dx}$.

$x^2y - xy^3 = 2$

differentiating with respect to x,

$$y \cdot \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(y) - \left(y^3 \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(y^3) \right) = \frac{d}{dx}(2)$$

$$2xy + x^2 \frac{dy}{dx} - y^3 - 3xy^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 - 3xy^2) = y^3 - 2xy$$

$$\frac{dy}{dx} = \frac{y^3 - 2xy}{x^2 - 3xy^2}$$

4. (8 points) Use logarithmic differentiation to find $\frac{dy}{dx}$.

$y = (3x+4)^x$

Applying "ln" on both sides,

$$\ln y = \ln (3x+4)^x$$

$$\ln y = x \ln (3x+4)$$

differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(3x+4) \cdot \frac{d(x)}{dx} + x \cdot \frac{d}{dx}(\ln(3x+4))$$

$$= \ln(3x+4) + \frac{x}{3x+4} \cdot (3)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(3x+4) + \frac{3x}{3x+4}$$

$$\frac{dy}{dx} = y \left[\ln(3x+4) + \frac{3x}{3x+4} \right]$$

$$\frac{dy}{dx} = (3x+4)^x \left(\ln(3x+4) + \frac{3x}{3x+4} \right)$$

-0

5. (8 pts) Suppose $y = 2x^3 + 4x$ where x and y are functions of t.

If $\frac{dx}{dt} = 0.1$, find $\frac{dy}{dt}$ when $x = 2$

differentiative "y" with respect to "t",

$$\frac{dy}{dt} = 6x^2 \frac{dx}{dt} + 4\frac{dx}{dt}$$

plugging in values for x,

$$\frac{dy}{dt} = 6(4)(0\cdot1) + 4(0\cdot1)$$

$$= 2\cdot4 + 0\cdot4 = 2\cdot8$$

$$\therefore \frac{dy}{dt} = 2\cdot8$$

6. (8 pts) Find the linearization, $L(x)$, of the function $f(x) = \arcsin\left(\frac{x}{2}\right)$ at the point $\left(1, \frac{\pi}{6}\right)$

$$f(x) = \sin^{-1}\left(\frac{x}{2}\right) \quad \text{at } (1, \pi/6)$$

$$L(x) = f(a) + f'(a)(x - a)$$

$$(a, f(a)) = \left(1, \frac{\pi}{6}\right)$$

$$f(1) = \sin^{-1}\left(\frac{1}{2}\right) = \pi/6$$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2\sqrt{1-\frac{x^2}{4}}} = \frac{2}{2\sqrt{4-x^2}} = \frac{1}{\sqrt{4-x^2}}$$

$$f'(1) = \frac{1}{\sqrt{4-1^2}} = \frac{1}{\sqrt{3}}$$

$$L(x) = \pi/6 + \frac{1}{\sqrt{3}}(x-1)$$

$$L(x) = \frac{x}{\sqrt{3}} + \pi/6 - \frac{1}{\sqrt{3}}$$

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7. (a) (3 points) Fill in the blanks appropriately to complete the Mean Value Theorem.

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed in interval [a, b].

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) (7 pts) Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the mean value theorem.

$f(x) = x^3 - 9x$ on $[-3, 6]$

As $f(x)$ is a polynomial, it is continuous on $(-\infty, \infty)$

Hence it is continuous on the interval $[-3, 6]$

$f'(x) = 3x^2 - 9$

(exists everywhere)

As $f'(x)$ is differentiable on $(-\infty, \infty)$, $f(x)$ is differentiable on $(-3, 6)$.

There is a c in $(-3, 6)$ such that

$$f'(c) = \frac{f(6) - f(-3)}{6 - (-3)}$$

$f(-3) = -27 + 27 = 0$; $f(6) = 216 - 54 = 162$

$$3c^2 - 9 = \frac{162 - 0}{9} = 18$$

$3(c^2 - 3) = 18$

$c^2 - 3 = 6$

$c^2 = 9$

$c = \pm 3$

We omit $c = -3$ as it does not exist in the interval $(-3, 6)$

$\therefore c = 3$

0

8. (8 pts) Find the open intervals on which the graph is concave upward and the open intervals on which the graph is concave downward. Give the coordinates of any point(s) of inflection.

$$f(x)=\frac{x^5}{10}+x^4+3x^3+2x+7$$

To find concavity, we have to f''(x).

If f''(x) > 0, the function is concave upwards

If f''(x) < 0, the function is concave downwards

$$f'(x) = \frac{5x^4}{10} + 4x^3 + 9x^2 + 2$$

$$f''(x) = \frac{20x^3}{10} + 12x^2 + 18x = 2x^3+12x^2+18x$$

$$= 2x(x^2+6x+9)$$

We check where f''(x) = 0 or does not exist to find possible inflection points. Inflection points are points where concavity changes

$$f''(x) = (2x)(x+3)^2 = 0$$

$$x = 0 \ ; \quad x = -3$$

Interval	$2x$	$(x-3)^2$	$f''(x)$
$(-\infty,-3)$	$-$	$+$	$-$ (Concave down)
$(-3,0)$	$-$	$+$	$-$ (concave down)
$(0,\infty)$	$+$	$+$	$+$ (concave up)

Therefore, the function is concave down on the interval $(-\infty, 0)$

The function is concave upwards on the interval $(0,\infty)$

There is only one inflection point at $x=0$, as the concavity changes from concave down to concave up.

$(0, f(0)) = (0,7)$ is the inflection point.

$x=-3$ is not an inflection point as the function does not change concavity there.

9. (6 pts) Evaluate the limit.

$$\lim_{x\to 1}\frac{1-e^{x^2-1}}{x^4-1} = \frac{1-e^{1-1}}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

It is of the form $\frac{0}{0}$. Hence we can use L'hopital's rule to find the limit.

Applying l'hopital's rule to,

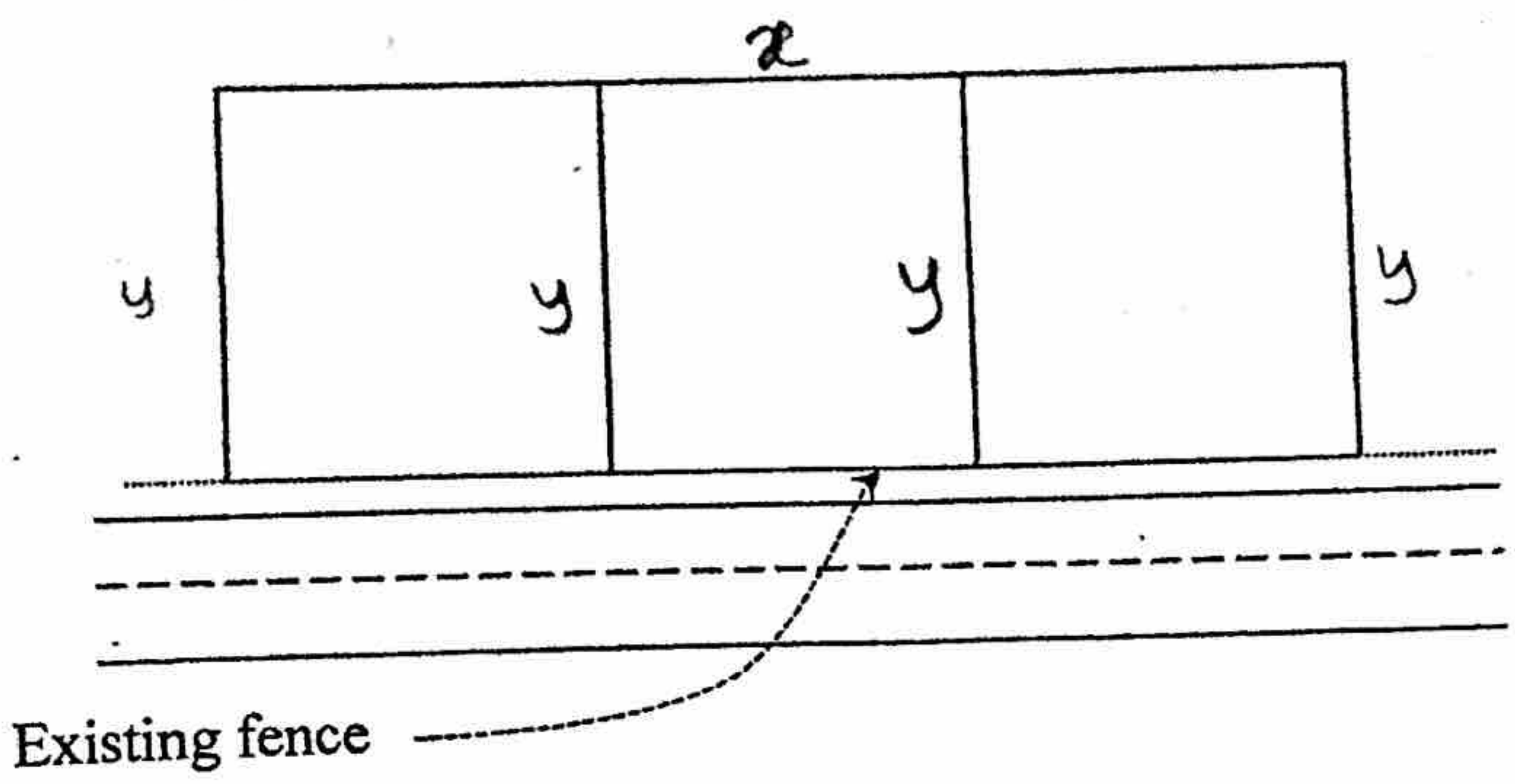
$$\lim_{x\to 1}\frac{1-e^{x^2-1}}{x^4-1} = \lim_{x\to 1}\frac{-e^{x^2-1}(2x)}{4x^3}$$

$$= \frac{-e^{1-1}(2)(1)}{4(1)^3} = \frac{-2}{4} = -\frac{1}{2}$$

Hence, using l'hôpital's rule,

$$\lim_{x\to 1}\frac{1-e^{x^2-1}}{x^4-1} = -\frac{1}{2}$$

10. (12 pts) A rancher has decided to fence 25,600 square meters of pasture in a rectangular plot along a straight highway. The plot will be divided into 3 subplots of equal size as shown. A fence exists along the highway thus only three sides of the perimeter and two partitions require fence. What are the dimensions of the rectangular plot that will require the least amount of fence?



From the question,

Perimeter $P = x + 4y$

Area of pasture $= xy = 25{,}600 \text{ m}^2$

$y = \dfrac{25600 \text{ m}^2}{x}$ or $x = \dfrac{25600 \text{ m}^2}{y}$

Now, $P(y) = \dfrac{25600}{y} + 4y$

Differentiating with respect to y,

$P'(y) = \dfrac{-25600}{y^2} + 4$

$\dfrac{-25600}{y^2} + 4 = 0$ (as we have to find minimum amount of fence)

$25600 = 4y^2 \Rightarrow y^2 = 6400$

$y = 80$ (as length cannot be negative)

$x = \dfrac{25600}{80} = 320$

[continued next page]

We have to check to make sure we have the least amount of fence

$$P(x) = x + \frac{4(25600)}{x}$$

$$= x + \frac{102400}{x}$$

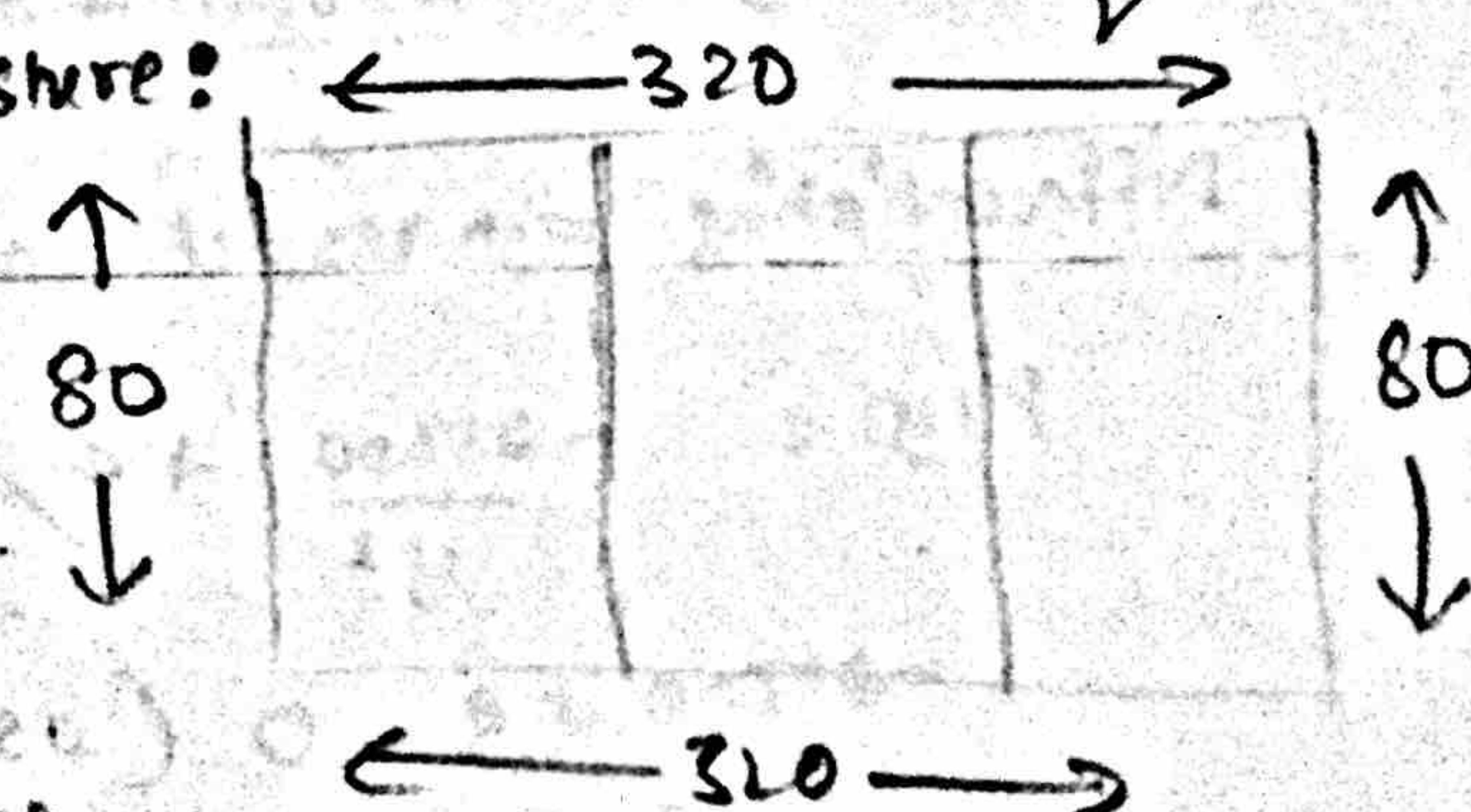
$$P'(x) = 1 - \frac{102,400}{x^2}$$

on the interval $(-\infty, 320)$, $f'(x) < 0$, so its decreasing on $(-\infty, 320)$

On the interval $(320, \infty)$, $f'(x) > 0$, so its increasing on $(320, \infty)$

So we have a local minimum at $x = 320$ and $y = 80$

So the dimensions of the pasture:



length = 80 m

width = 320 m

So we have the least amount of fence with the dimensions 320 m and 80 m.

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