1. (10 pts.) The graph of the function f is given below. Evaluate each limit if it is finite. Otherwise, write ∞, $-\infty$ or DNE.

(a) $\lim_{x\to -3^+} f(x)$

5

(b) $\lim_{x\to 0} f(x)$

DNE

(c) $f(-3)$

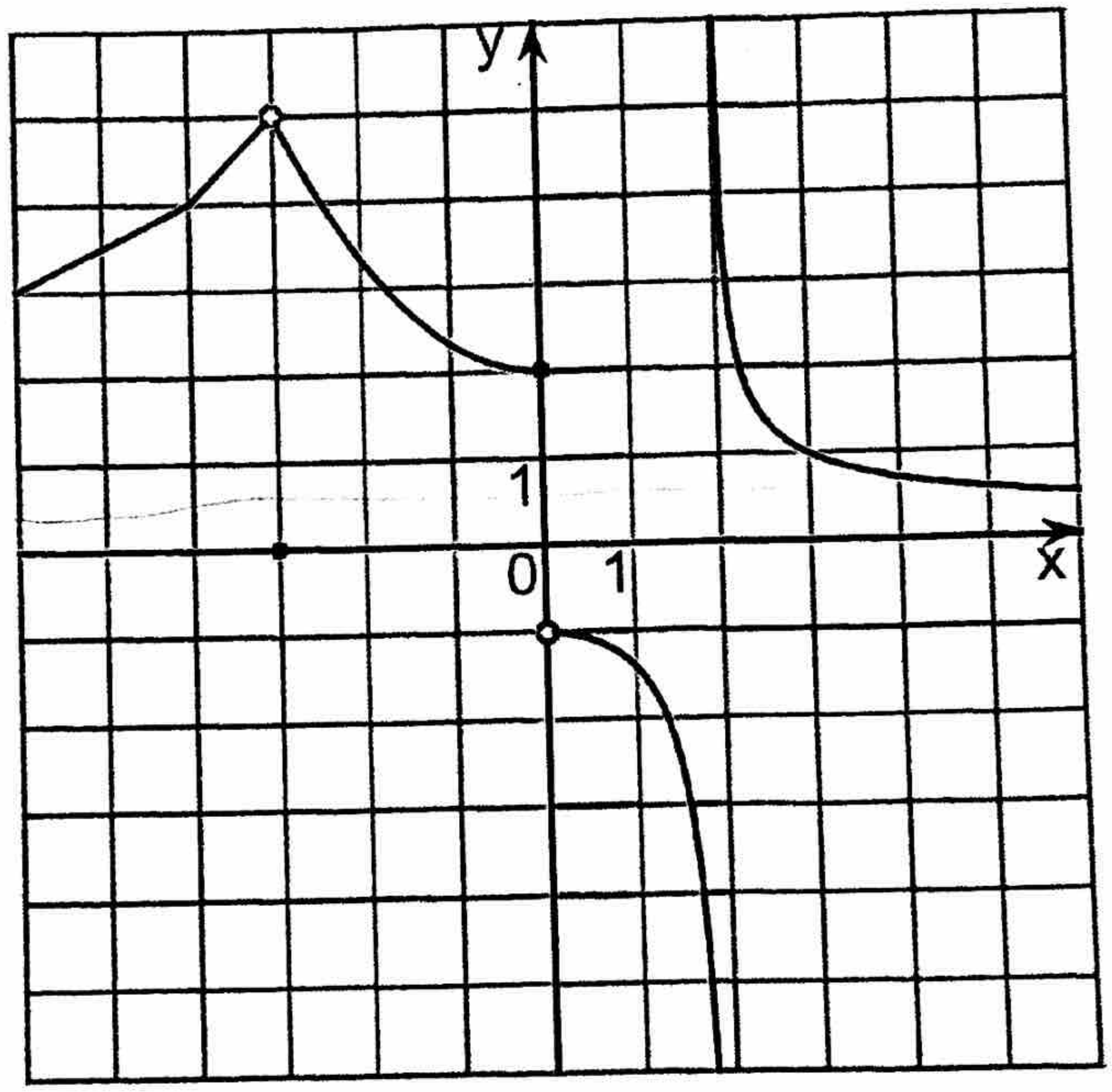
0

(d) $\lim_{x\to 2^-} f(x)$

$-\infty$

(e) $\lim_{x\to -4} f(x)$

4



2. Find the limit. Give the exact value in its simplest form. (L'Hospital's rule is not permitted.)

(a) (4 pts.) $\lim_{x\to 0} \frac{\ln(\tan x+1)}{x^3+8}$.

$$\lim_{x\to 0} \frac{\ln(\tan x+1)}{x^3+8} = \frac{\ln\left(\lim_{x\to 0} \tan x + 1\right)}{\lim_{x\to 0} x^3+8}$$

$$= \frac{\ln(0+1)}{0+8} = \frac{\ln 1}{8} = 0$$

$\to 0$

Find the limit. Give the exact value in its simplest form. (L'Hospital's rule is not permitted.)

(b) (6pts.) $\lim_{x\to-3} \frac{x^2+5x+6}{2x^2+5x-3}$.

$\Rightarrow \lim_{x\to-3} \frac{x^2+5x+6}{2x^2+5x-3} = \frac{0}{0}$, Hence we factorize it

$\lim_{x\to-3} \frac{x^2+3x+2x+6}{2x^2+6x-x-3} = \lim_{x\to-3} \frac{x(x+3)+2(x+3)}{2x(x+3)-1(x+3)} = \lim_{x\to-3} \frac{(x+2)\cancel{(x+3)}}{(2x-1)\cancel{(x+3)}}$

$= \lim_{x\to-3} \frac{x+2}{2x-1} = \frac{-3+2}{-6-1} = \frac{-1}{-7} = \frac{1}{7}$

(c) (4pts.) $\lim_{x\to\infty} \arctan(e^x)$.

as $\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} \arctan(e^x) = \cancel{\arctan(\infty)} = \pi/2$

(d) (6pts.) $\lim_{x\to0} \frac{\sin(7x)}{\sin(3x)}$.

$\lim_{x\to0} \frac{\sin 7x}{\sin 3x} = \lim_{x\to0} \frac{(7x)\sin 7x}{7x} \cdot \frac{3x}{\sin 3x} \cdot \frac{1}{3x}$

$= (7x) \lim_{x\to0} \frac{\sin 7x}{7x} \cdot \lim_{x\to0} \frac{3x}{\sin 3x} \cdot \frac{1}{3x}$

$= (7x) \cdot 1 \cdot 1 \cdot \frac{1}{(3x)} = \frac{7}{3}$ $\left[\text{AS } \lim_{x\to0} \frac{\sin x}{x} = \lim_{x\to0} \frac{x}{\sin x} = 1\right]$

Can't pull out X if X→0

−1

3. Given the limit statement $\lim_{x\to 3}(5x+2) = 17$.

(a) (3 pts.) Write the inequalities $|f(x) - L| < \epsilon$ and $0 < |x - a| < \delta$ as they pertain to this statement.

$0 < |x-3| < \delta$ and $|5x+2-17| < \varepsilon$

$\Rightarrow |5x-15| < \varepsilon$

(b) (4 pts) Illustrate the definition of the limit by finding a number δ that corresponds to $\epsilon = 0.1$.

$|x-3| < \delta$ $\quad |5(x-3)| < \varepsilon$

$5|x-3| < 5\delta$ $\quad 5|x-3| < \varepsilon$

By comparison, we choose $\varepsilon = 5\delta$

$\Rightarrow \delta = \frac{\varepsilon}{5}$

When $\varepsilon = 0.1$, $\delta = \frac{0.1}{5} = 0.02$

(c) (3 pts) Deduce a relationship between ϵ and δ that would allow you to compute δ for any ϵ.

We know $|x-3| < \delta$

$|5x-15| < \varepsilon$

$|5(x-3)| < \varepsilon$

$5|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{5}$; $|x-3| < \delta$

By comparison, we choose $\boxed{\delta = \frac{\varepsilon}{5}}$

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4. Let $f(x) = \begin{cases} \cos x & \text{if } x < 0; \\ ax + b & \text{if } 0 \le x < 3; \\ (x+1)^2 & \text{if } x \ge 3. \end{cases}$

i) (4 pts.) Evaluate the following limits:

(a) $\lim_{x \to 0^-} f(x)$.

$= \lim_{x \to 0^-} \cos x = 1$

(b) $\lim_{x \to 3^-} f(x)$.

$= \lim_{x \to 3^-} ax + b = 3a + b$

ii) (6 pts) Find the values of a and b that make f continuous everywhere.

For a function to be continuous,

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ & $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos x = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ax + b = b$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \implies b = 1$ — (1)

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax + b = 3a + b$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+1)^2 = 16$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) \implies 3a + b = 16$

From equation (1),

$3a + 1 = 16 \implies 3a = 15$

$\boxed{a = 5}$; $\boxed{b = 1}$

5. (14 pts.) Find all horizontal and vertical asymptotes of the function. Use appropriate limit statements to justify your conclusion and write the equation of any asymptote.

$$f(x) = \frac{\sqrt{3x^2+5}}{7-4x}.$$

$\frac{\sqrt{x^2(3+\frac{5}{x^2})}}{x(\frac{7}{x}-4)}$ $\quad -\frac{\sqrt{3}}{4}$

$|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$

Horizontal asymptotes

To find horizontal asymptotes, we check the values of $f(x)$ as $x \to \pm\infty$

when $x \to \infty$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{3x^2+5}}{7-4x}$$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{3+\frac{5}{x^2}}}{x(\frac{7}{x}-4)}$$

as $x \to +\infty$, $|x| = x$

$$\Rightarrow \lim_{x\to\infty} \frac{x\sqrt{3+\frac{5}{x^2}}}{x(7/x-4)}$$

$$= \frac{\sqrt{3}}{-4} = -\frac{\sqrt{3}}{4}$$

When $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{3x^2+5}}{7-4x}$$

$$= \lim_{x\to-\infty} \frac{|x|\sqrt{3+\frac{5}{x^2}}}{x(\frac{7}{x}-4)}$$

as $x \to -\infty$, $|x| = -x$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{3+\frac{5}{x^2}}}{x(\frac{7}{x}-4)}$$

$$= \frac{-\sqrt{3}}{-4} = \frac{\sqrt{3}}{4}$$

$\therefore$ $y = -\frac{\sqrt{3}}{4}$; $y = +\frac{\sqrt{3}}{4}$ are horizontal asymptotes of the given function

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6. (a) (3 pts.) Write the definition of the derivative of a function f.

We define the derivative of a function $f(x)$ as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) (8 pts.) Use the definition of the derivative to find the derivative of the function

$f(x) = \sqrt{2x-3}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)-3} - \sqrt{2x-3}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2x+2h-3} - \sqrt{2x-3}}{h} \times \frac{(\sqrt{2x+2h-3} + \sqrt{2x-3})}{(\sqrt{2x+2h-3} + \sqrt{2x-3})}$$

$$= \lim_{h \to 0} \frac{2x+2h-3-(2x-3)}{h(\sqrt{2x+2h-3} + \sqrt{2x-3})}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2x+2h-3} + \sqrt{2x-3}} = \frac{2}{2\sqrt{2x-3}}$$

$$\therefore \quad f'(x) = \frac{1}{\sqrt{2x-3}}$$

7. Find the derivatives of the following functions:

(a) (5 pts.) $f(x) = 6x^3 - \frac{5}{\sqrt{x}} + 3e^x + 1$.

$$f'(x) = 6\frac{d}{dx}(x^3) - 5\cdot\frac{d}{dx}(x^{-1/2}) + 3\frac{d}{dx}(e^x) + \frac{d}{dx}(1)$$

$$= 18x^2 - 5\left(-\frac{1}{2}\right)x^{-1/2-1} + 3e^x + 0$$

$$= 18x^2 + \frac{5}{2}x^{-3/2} + 3e^x$$

(b) (6 pts.) $f(x) = \frac{\tan x - 2}{\sec x}$.

$$f'(x) = \frac{\sec x \cdot \frac{d}{dx}(\tan x - 2) - (\tan x - 2)\frac{d}{dx}(\sec x)}{\sec^2 x}$$

$$= \frac{\sec x(\sec^2 x) - (\tan x - 2)(\sec x \tan x)}{\sec^2 x}$$

$$= \frac{\sec^2 x - (\tan x - 2)(\tan x)}{\sec x} = \frac{\sec^2 x - \tan^2 x + 2\tan x}{\sec x}$$

$$= \frac{1 + 2\tan x}{\sec x}$$

(c) (6 pts.) $f(x) = (x^2 + 2x + 1)(\cos x + \sin x)$.

$$f'(x) = (\cos x + \sin x)\cdot\frac{d}{dx}(x^2 + 2x + 1) + (x^2 + 2x + 1)\cdot\frac{d}{dx}(\cos x + \sin x)$$

$$= (\cos x + \sin x)[2x + 2] + (x^2 + 2x + 1)(\cos x - \sin x)$$

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-0

$$(\cos x + \sin x)(2x+2) + (x^2+2x+1)(\cos x - \sin x)$$

$$= 2x\cos x + 2\cos x + 2x\sin x + 2\sin x$$

$$+ x^2\cos x - x^2\sin x + 2x\cos x - 2x\sin x$$

$$+ \cos x - \sin x$$

$$= x^2\cos x - x^2\sin x + 4x\cos x + 3\cos x + \sin x$$

8. (8 pts.) Find an equation of the tangent line to the curve $y = \frac{x-5}{7-x}$ at the point $(6, 1)$.

To find the slope of tangent, we differentiate y with respect to x

$$\frac{dy}{dx} = \frac{(7-x)\cdot\frac{d}{dx}(x-5) - (x-5)\cdot\frac{d}{dx}(7-x)}{(7-x)^2}$$

$$= \frac{(7-x)(1) - (x-5)(-1)}{(7-x)^2} = \frac{7-x+x-5}{(7-x)^2} = \frac{2}{(7-x)^2}$$

at the point $(6,1)$,

$$\frac{dy}{dx} = \frac{2}{(7-6)^2} = 2 = m_t$$

Equation of tangent line: $y - y_1 = m_t(x - x_1)$

$$\Rightarrow y - 1 = 2(x-6)$$

$$y - 1 = 2x - 12$$

$y = 2x - 11$ is the equation of tangent line.