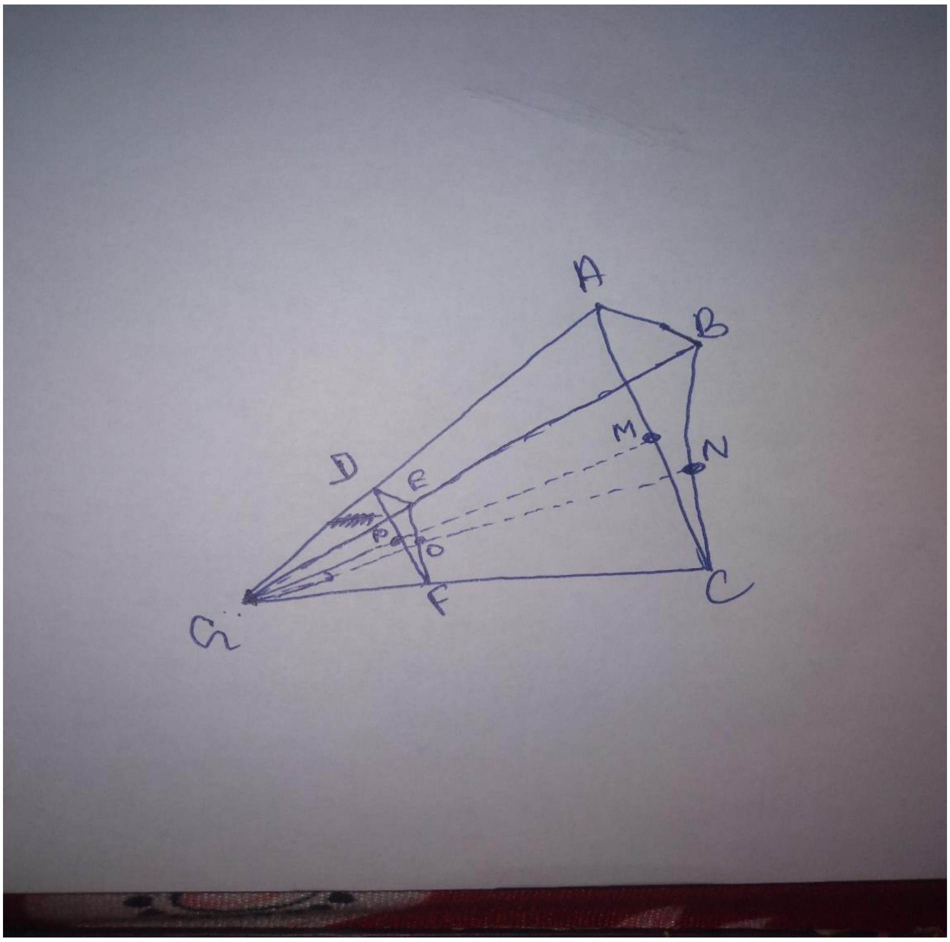

This is the geometric construction. Angle Y is between sides AC & BC. Angle x is between sides MG & NG. M is mid point of AC. N is mid point of BC. There is a similar smaller triangle created inside the construction but perfectly similar to Triangle ABC. This similar triangle is Triangle DEF. The lines MG passes through the mid point of line DF & NG line passes through mid of line EF.

Theorem:- "The angles x and y here are equal. No matter whatever degrees we take, they are always equal, in fact, they are congruent (almost)."

The crucial idea is that "sending $\triangle DEF$ up to $\triangle ABC$" is accomplished by a single dilation (homothety) centered at C. In particular:

1. Because $\triangle DEF$ is similar to $\triangle ABC$, there is some number $k > 1$ so that

$$\overline{CD} : \overline{CA} \ = \ \overline{CE} : \overline{CB} \ = \ \overline{CF} : 0 \quad (\text{in fact, } CF = 0 \text{ is the "trivial" statement } F \text{ lies on } BC),$$

but more conceptually, there is a single dilation (with center C) that carries
$D \mapsto A, \ E \mapsto B, \ F \mapsto C.$

In other words, if you choose the dilation (or "expansion") centered at C by factor

$$k \ = \ \frac{CA}{CD} \ = \ \frac{CB}{CE} \ = \ \frac{CC}{CF},$$

that same dilation sends

$$D \longmapsto A, \quad E \longmapsto B, \quad F \longmapsto C.$$

(Here we think of C "blowing up" so that the small triangle DEF becomes exactly the large triangle ABC.)

2. In particular, since M is the midpoint of DF, the dilation of ratio k (centered at C) must carry

$$M \ = \ \text{midpoint of } (D\,F) \quad \longmapsto \quad \text{midpoint of } (A\,C).$$

But the midpoint of AC is, by definition, $M' = \ \text{midpoint of } (AC)$. Thus

$$(\text{dilation at } C \text{ by factor } k): \quad M \ \longmapsto \ M',$$

where $M' =$ the midpoint of AC.

3. Likewise, since N is the midpoint of $E\,F$, the same dilation carries

$$N \ = \ \text{midpoint of } (E\,F) \quad \longmapsto \quad N' \ = \ \text{midpoint of } (B\,C).$$

(Because $E \mapsto B$ and $F \mapsto C$.)

4. Finally, that dilation also carries the point G (where the rays from C met DEF) up to the point C itself (in fact, C is a fixed point of the dilation). So under dilation:

$$C \ \mapsto \ C, \quad G \ \mapsto \ C, \quad M \ \mapsto \ M', \quad N \ \mapsto \ N'.$$

5. Because a dilation (homothety) preserves all angles through the center C, the angle

$$\angle MGN \quad \mapsto \quad \angle M'CN'.$$

But M' was chosen to be the midpoint of AC, and N' the midpoint of BC. Thus

$$\angle M'\,C\,N' \ = \ \angle(\text{line } CM')\,,\ (\text{line } CN') \ = \ \angle(\text{midpoint-to-}A\,\text{ray})\,,\ (\text{midpoint-to-}B\,\text{ray}).$$

Of course, the ray $C \to M'$ is just the same line-direction as $C \to A$ (because M' lies on AC), and the ray $C \to N'$ is the same direction as $C \to B$. Therefore

$$\angle M'CN' = \angle ACB = y.$$

6. Putting it all together:

$$\underbrace{\angle MGN}_{x} \xrightarrow[\text{dilation about } C]{\text{factor } k} \underbrace{\angle M'CN'}_{y} = \angle A\,C\,B.$$

Since dilation about C fixes C and sends G to C, it also carries the lines GM to CM' and GN to CN', so it carries $\angle MGN$ exactly onto $\angle M'CN'$. But $\angle M'CN' = \angle ACB$. Hence

$$x = \angle MGN = \angle M'CN' = \angle ACB = y.$$

> **Theorem (Midpoint-Homothety Angle Theorem).**
> If $\triangle DEF$ is similar to $\triangle ABC$ via a homothety centered at C, and M, N are midpoints of DF, EF respectively, then $\angle MGN = \angle ACB$, where G is the shared vertex on all three rays from C.