Homework 4 Swastik Singh

Intro to Probability, FE

Problems: 3.2, 3.3, 3.11, 3.22, 3.30, 3.48, 2.55, 3.55

Problem 2.55: Peter & Mary take turns throwing 1 dart @ a dart board. Peter hits the Bullseye w/ prob p & mary hits w/ prob r. Whoever hits bullseye first wins. Peter throws first dart.

a) What is probability that Mary wins?

$M = \{\text{Mary wins}\} \to M_1 = \{\text{Mary hits bullseye}\}$

$M_2 = \{\text{Peter misses bullseye}\}$, $M_2^c = \{\text{Peter hits bullseye}\}$

$\underbrace{P(M_2)}_{\text{first throw}} = 1 - M_2^c = 1-p$ k = # throws it takes to win

$\underbrace{P(M_1)}_{\text{First throw}} = r$ Only throw, but what if it takes multiple throws?

$P(M_2) = (1-p)^{k-1} \cdot (1-p)$

$P(M_1) = (1-r)^{k-1} \cdot r$

$P(M) = \sum_{k=1}^{\infty} \left[\underbrace{(1-p)^{k-1} \cdot (1-r)^{k-1}}_{\text{Happens @ same time \& Independent}} \right] \cdot (1-p)r$

$$\boxed{P(M) = \frac{(1-p)r}{1-((1-p)(1-r))}}$$

b) X = number of times Mary throws a dart. Find possible values & PMF of X. Check the function & if it sums to 1. Is this a familiar named Distribution

$\boxed{P(X=0)=p}$ Peter wins w/o Mary throws

$$\boxed{\begin{aligned} P(X=0) &= p \\ P(X=k) &= 1-p \end{aligned}}$$

P = {Peter wins}

$P(X=k) = P(M) + P(P) = \sum_{k=1}^{\infty} [(1-p)(1-r)]^{k-1} \cdot (1-p)r + [(1-p)(1-r)]^k p$

If peter wins, he throws k+1 times.

$$\boxed{P(X=k) = \sum_{k=1}^{\infty} [(1-p)(1-r)]^{k-1} \cdot (1-p)(p+r-pr) = 1-p}$$

$1 \stackrel{?}{=} p + 1 - p$

$1 = 1$ ✓

c) Find conditional probability $P(X=k|M)$ for all values k of X.

$$P(X=k|M) = \frac{P(M \text{ wins on } k^{th})}{P(M)} = \frac{[(1-p)(1-r)]^{k-1}(1-p)r}{\frac{(1-p)r}{p+r-pr}}$$

$$\boxed{= [(1-p)(1-r)]^{k-1} \cdot (p+r-pr), \quad X \sim Geom(p+r-pr)}$$

Problem 3.2: Suppose the random variable X has probable values $\underbrace{\{1,2,3,4,5,6\}}_{k}$ & PMF: $p(k) = ck$

a) find c

$$1 = \sum_{k=1}^{k=6} ck = c\cdot 1 + c\cdot 2 + c\cdot 3 + c\cdot 4 + c\cdot 5 + c\cdot 6 = c\cdot 21$$

$$\boxed{c = \frac{1}{21}}$$

b) $P(X=\text{odd}) = p(1)+p(3)+p(5) = \frac{1}{21}\cdot 1 + \frac{1}{21}\cdot 3 + \frac{1}{21}\cdot 5 = \frac{9}{21}$

$= \boxed{\frac{3}{7}}$

Problem 3.3: Let X be a continuous random variable w/ density function:

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0, & \text{else} \end{cases}$$

$\lim_{x\to\infty}(-e^{-3x})$ $(-e^{0})$

a) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 3e^{-3x}\,dx = -e^{-3x}\Big|_0^{\infty} = 0 - (-1) = \boxed{1}$$

b) Calculate $P(-1 < X < 1)$

$$0\Big|_{-1}^{0} + \int_0^1 3e^{-3x}\,dx = 0 + e^{-3x}\Big|_0^1 = +(-e^{-3} - (-1)) = \boxed{1 - e^{-3}}$$

c) Calculate $P(X < 5)$

$$\int_{-\infty}^{5} f(x) = \int_0^5 3e^{-3x}\,dx = -e^{-3x}\Big|_0^5 = \boxed{1 - e^{-15}}$$

d) Calculate $P(2 < X < 4 \mid X < 5)$

$$\frac{\int_2^4 3e^{-3x}\,dx}{\int_0^5 3e^{-x}\,dx} = \frac{-e^{-3x}\Big|_2^4}{-e^{-3x}\Big|_0^5} = \boxed{\frac{-e^{-12} - e^{-6}}{1 - e^{-15}}}$$

Problem 3.11: Let Y be a random variable w/ Density f: $f(x) = \frac{2}{3}x$ for $x \in [1, 2]$ & $f(x) = 0$ otherwise. Compute $E((Y-1)^2)$

$(x-1)(x-1)$

$x^2 - x - x + 1$

$$E(X) = \int_{-\infty}^{\infty} f(x)\, x \, dx$$

$$E((Y-1)^2) = \int_1^2 (x-1)^2 \frac{2}{3}x \, dx$$

$$E((Y-1)^2) = \int_1^2 \frac{(x^2 - 2x + 1)2x}{3} dx = \frac{2}{3}\int^2 u^2(u+1)\,du$$

$$= \frac{2}{3} \cdot \int_2^{3} u^3 + u^2 \, du = \left[\frac{(x-1)^4}{4} + \frac{(x-1)^3}{3}\right]_1^2 \frac{2}{3}$$

$$= \left(\frac{1}{4} + \frac{1}{3}\right)\frac{2}{3} = \left(\frac{4}{12} + \frac{3}{12}\right)\frac{2}{3} = \boxed{\frac{7}{18}}$$

Problem 3.22: A fair coin Flipped 3 times.

X = # heads observed

a) Give possible values & Probability Mass Function

Values = $\{0, 1, 2, 3\}$ (# of heads) $p(H) = \frac{1}{2}$

$$\left.\begin{aligned} P(X=0) &= \frac{1}{2^3} \\ P(X=1) &= 3 \cdot \frac{1}{2^3} \\ P(X=2) &= 3 \cdot \frac{1}{2^3} \\ P(X=3) &= \frac{1}{2^3} \end{aligned}\right\} \text{Probability Mass Function}$$

b) $P(X \geq 1) = 1 - P(X=0) = 1 - \frac{1}{8} = \boxed{\frac{7}{8}}$

$P(X > 1) = 1 - P(X=0) - P(X=1) = 1 - \frac{1}{8} - \frac{3}{8} = \boxed{\frac{1}{2}}$

c) $E[X] = \sum_{x=0}^{3} np = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} =$

$\boxed{\frac{3}{2} = E[X],\ Var[X] = \frac{3}{4}}$

$Var[X] = E[X^2] - E[X]^2$

$3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

$E[X^2] = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3$

Problem 3.30: BBall player first success $= \frac{1}{2}$ if he misses, $s = \frac{1}{3}$, $t = \frac{1}{4}$, $f = \frac{1}{5}$, after 4 straight misses, he's taken out. Player shoots until success or his removal from game. X is # of misses until Removal or Successful shot.

a) Probability Mass Function

PMass F
$P(X=0) = \frac{1}{2} = \frac{1}{2}$
$P(X=1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} = 1/6$
$P(X=2) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{12} = 1/12$
$P(X=3) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} = \frac{1}{20} = 1/20$
$P(X=4) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5} = 1/5$

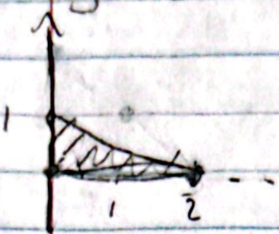
b) $E[X] = \sum_{n=0}^{4} n \cdot p$

$= \frac{1}{2} \cdot 0 + \frac{1}{6} + \frac{1}{12} \cdot 2 + \frac{1}{20} \cdot 3 + \frac{1}{5} \cdot 4$

$= 0 + \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{4}{5} = \frac{3+16}{20} + \frac{2}{6}$

$= \frac{19 \cdot 3}{60} + \frac{20}{60} = \boxed{\frac{77}{60}}$

Problem 3.48: Pick a random point inside a triangle w/ vertices (0,0), (2,0), (1,1). Compute the expectation of the distance of this point to the y-axis.



$P(E[\bar{x}], E[\bar{y}])$ = Center point of triangle

$\frac{\sum x}{\# x} = \bar{x}$ $\quad \frac{0+2+0}{3} = \bar{x}$ $\quad \boxed{\bar{x} = \frac{2}{3}}$ → Expected distance from y-axis

$\bar{y} = \frac{\sum y}{\# y} = \frac{0+0+1}{3} = \bar{y}$

Same Answer

Just to confirm:

Triangle → (x,y) → Random pt

distance to y-axis, R: $R = |x|$ $\quad 0 \le R \le 2$

$1 - \frac{x}{2}$ } Hypotenuse equation

cdf $= F_R(r) = P(R \le r)$

Goal: Calculate $E[R]$

For r within $0 < r < 2$

$F_R(r) = P(x \le r)$

→ Area denoted by $F_R(r)$
- Is a Trapezoid

Eqn for area of T $= \frac{1}{2}(b_1 + b_2)h$

$$\left.\begin{aligned} b_2 &= 1 \\ b_1 &= 1 - \frac{r}{2} \\ \text{height} &= r \end{aligned}\right\} \frac{(1 + 1 - \frac{r}{2})r}{2}$$

$$\frac{\frac{(1+1-\frac{r}{2})r}{2}}{1} = F_R(r) = r - \frac{r^2}{4}$$

→ Total Area $= \frac{bh}{2} = 1$

Derivative

PDF $= f_R(r) = F'_R(r) = 1 - \frac{r}{2}$ for $0 < r < 2$ & 0 otherwise

$$E[R] = \int_{-\infty}^{\infty} r f(r)\,dr = \int_0^2 r \cdot 1 - \frac{r}{2}\,dr = \int_0^2 r - \frac{r^2}{2}\,dr$$

$$= \left[\frac{r^2}{2} - \frac{r^3}{6}\right]_0^2 = 2 - \frac{8}{6} - 0 = \frac{4}{6} = \boxed{\frac{2}{3}}$$

Problem 3.55: Peter & Mary take turns throwing a dart 1 at a time. Peter hits bullseye w/ probability p, and Mary hits bullseye w/ probability r. Whomever hits bullseye first wins. Suppose Peter throws the first dart and let Y denote the number of darts thrown in a game. Find $E[Y]$

Peter Throws $= n+1$ } Peter Wins $= 2n+1$
Mary Throws $= n$ } Mary wins $= 2n$

$E[y]$ = Peter wins [First] + Mary Wins [First] + Both miss

$$E[Y] = p \cdot 1 + (1-p)2r + (1-p)(1-r)(2 + E[Y])$$

$$E[Y] = p + 2(1-p)r + (1-p)(1-r)2 + (1-p)(1-r)E[Y]$$

$$\underbrace{E[Y] - (1-p)(1-r)E[Y]}_{E[Y][1-(1-p)(1-r)]} = p + 2r(1-p) + 2(1-p)(1-r)$$

$1-(1-p)(1-r)$

$1-(1-r-p+pr)$

$r+p-pr$

$p + 2r - 2rp + 2 - 2r - 2p + 2rp$

$p - 2p + 2r - 2r - 2pr + 2rp + 2$

$-p + 0 + 0 + 2$

$2-p$

$$E[Y](r+p-pr) = p + 2r(1-p) + 2(1-r)(1-p)$$

$$\boxed{E[Y] = \frac{2-p}{r+p-pr}}$$