Homework 3 due 10/20 Swastik Singh

Intro to Probability, First Edition by Anderson, Seppalainen, and Valko

Problems 2.5, 2.9, 2.15, 2.16, 2.35, 2.44, 2.50, 2.59

Problem 2.5: Have 2 urns. The first urn contains three balls labelled 1, 2, 3. Second urn has 4 balls labelled 2, 3, 4, 5. We choose an urn randomly so that picking Urn 1 is $\frac{1}{5}$ and the probability of choosing the Urn 2 is $\frac{4}{5}$. Then we sample a ball (uniformly @ Random) from the chosen urn. What is the probability we picked ball labelled 2?

$A = \{\text{Drawing a ball labelled } 2\}$

$\Omega = \{U1: \{1, 2, 3\}, U2: \{2, 3, 4, 5\}\}$

urn 1

return 1

$$P(A) = \begin{cases} \text{Urn } 1: \frac{1}{5}, & \text{Prob of } 2 = \frac{1}{3} \\ + \\ \text{Urn } 2: \frac{4}{5}, & \text{Prob of } 2 = \frac{1}{4} \end{cases}$$

$P(U1 \cap 2) = \frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$ $\qquad P(A) = \frac{1}{15} + \frac{3}{15} = \boxed{\frac{4}{15}}$

$P(U2 \cap 2) = \frac{4}{5} \cdot \frac{1}{4} = \frac{4}{20} = \frac{1}{5}$

$P(A) = P(A \mid U1)\, P(U1) + P(A \mid U2)\, P(U2)$

Problem 2.9: Same setup as 2.5. Suppose ball 3 was chosen, what is the probability it came from urn 2?

A = {Drawing 3}

$\Omega = \{U_1: \{1,2,3\}, U_2: \{2,3,4,5\}\}$

B = {Picked Urn 2}

Bayes Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

$$= \frac{\frac{1}{4}\cdot\frac{4}{5}}{\frac{1}{4}\cdot\frac{4}{5} + \frac{1}{3}\cdot\frac{1}{5}} = \frac{\frac{3}{15}}{\frac{4}{15}} = \boxed{\frac{3}{4}}$$

P(B)

Problem 2.15: Every Morning Ramona misses her bus w/ a probability $\frac{1}{10}$, Independently of other mornings. What's the probability that next week, she catches her bus on Mon, Tues, Thurs & misses it on Wed, & Fri?

Sampling w/ Replacement, w/ order

Mon · Tues · Wed · Thurs · Fri

$P(B^c) \cdot P(B^c) \cdot P(B) \cdot P(B^c) \cdot P(B)$

$(1-P(B)) \cdot (1-P(B)) \cdot \frac{1}{10} \cdot (1-P(B)) \cdot \frac{1}{10}$

$\left(\frac{9}{10}\right)^3 \cdot \left(\frac{1}{10}\right)^2 = \frac{729}{1000} \cdot \frac{1}{100} = \boxed{\frac{729}{100,000}}$

w/rep
w/ order

Problem 2.16: We flip a coin 3 times. for $i = 1, 2, 3$, let A_i be the event that among the first i coin flips we have an odd number of heads. Check whether A_1, A_2, A_3 are independent or not.

$0 =$ tails, $1 =$ heads

$\Omega = (\{s_1, s_2, s_3\} \in \{0, 1\})$

$\#\Omega = 8 = 2^3$

$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)$

Individual Prob.

$A_1 = \{$odd # of heads in first flip$\}$

$\#A_1 = 4 \quad P(A_1) = \frac{4}{8} = \frac{1}{2}$

$A_2 = \{$odd # of heads in 2nd flip$\}$

$\#A_2 = 4 \quad P(A_2) = \frac{4}{8} = \frac{1}{2}$

$A_3 = \{$odd # of heads in 3rd flip$\}$

$\#A_3 = 4 \quad P(A_3) = \frac{4}{8} = \frac{1}{2}$

$P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$

Pairwise

$P(A_1 \cap A_2) \stackrel{?}{=} P(A_1) \times P(A_2)$

$\frac{1}{4} \stackrel{?}{=} \frac{1}{2} \cdot \frac{1}{2} \rightarrow \frac{1}{4} \stackrel{\checkmark}{=} \frac{1}{4}$

$P(A_1 \cap A_3) \stackrel{?}{=} P(A_1) \times P(A_3)$ ✓

$\frac{1}{4} \stackrel{?}{=} \frac{1}{2} \times \frac{1}{2} \quad \frac{1}{4} = \frac{1}{4}$

Pairwise

$P(A_2 \cap A_3) \stackrel{?}{=} P(A_2) \times P(A_3)$ ✓

$\frac{1}{4} = 1/2 \times \frac{1}{2} \rightarrow \frac{1}{4} = \frac{1}{4}$

Mutual

$P(A_1 \cap A_2 \cap A_3) = \frac{1}{8}$

$P(A_1) \times P(A_2) \times P(A_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$ ✓

$A_1, A_2,$ and A_3 are Pairwise Independent & Mutually Independent

Order w/o replacement

Assumes fair deck

Not Independent

Problem 2.35: We shuffle a deck of cards & deal 2 cards. Find the probability that card 1 is a queen & card 2 is spade

Q = Queen is card 1

S = Spade is Car 2

$\#Q = 4$

$\#S = 13$

w/o replacement

Ω = 2 Cards from a card deck

$\#\Omega = 52 \cdot 51$

C: Queen of Spades = 1 — 12 other spades

D: Any other Queen = 3 — 13 other spades

$\#C = 12 \cdot 1$ — other spades, QoS

$\#D = 3 \cdot 13$ — Queens (w/o spades), # spades to choose from

$\to 39$

$\to 12$

$$P(Q \cap S) = \frac{\#C + \#D}{\#\Omega} = \frac{51}{52 \cdot 51} = \boxed{\frac{1}{52}}$$

Problem 2.44: We have 2 bins. Bin 1 has 6 blue marbles & 4 yellow. Bin 2 has 3 blue marbles & 4 yellows. Choose a bin @ random & then choose a marble from the bin.

a) If the marble is yellow, what is the probability the bin is bin 1?

A = Marble is yellow

B = Bin 1 is chosen

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)}$$

$$P(B|A) = \frac{\frac{4}{10} \cdot \frac{1}{2}}{\frac{4}{10} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2}} = \frac{4/10}{4/10 + 4/7} = \boxed{\frac{7}{17}}$$

b) Now suppose we put the marble from (a) back where it was drawn from & draw from same bin again. The 2nd draw is also yellow. What is the probability that we draw from bin 1?

Problem 2.49:
b) (Refer to previous page)
A = Ball 1 is yellow
B = Ball 2 is yellow
C = Bin 1 is selected

$$P(C \mid A, B) = \frac{P(A, B \mid C) \cdot P(C)}{P(A, B \mid C) \cdot P(C) + P(A, B \mid C') P(C')}$$

$$\frac{\frac{7}{10} \cdot \frac{4}{10} \cdot \frac{1}{2}}{\frac{7}{10} \cdot \frac{4}{10} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{1}{2}} = \boxed{\frac{49}{149}}$$

Problem 2.50: 3 prisoners A, B, C have been sentenced to die tomorrow. The king has chosen 1 of the 3 uniformly & @ Random to be pardoned tomorrow, while 2 unlucky ones head for the gallows. The guard knows who's been pardoned. Prisoner A begs the guard to name someone other than A himself who will be executed. he cajoles "Even if you tell me who of B & C is to die, I will not have gained any new knowlege because I know at least 1 of them is to die tomorrow." The guard is persuaded & reveals that B is to die tomorrow.

a) After recieving this info, does A still have a probability of $\frac{1}{3}$ of being pardoned?

A = A is pardoned
B = B is Pardoned
C = C is pardoned
$= \frac{1}{3} \to$ Initially

E = B is executed $= 1$

$$P(A \mid E) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{\frac{1}{6}}{3/6} = \boxed{\frac{1}{3}}$$

Yes, A still has a $\frac{1}{3}$ probability of being executed

If A, B & C are executed, Guard names B = $\frac{1}{2}$
If B, A & C are executed, Guard names B = 0
If C, A & B are executed, Guard names B = 1

Problem 2.50:

b) Prisoner A whispers this info to prisoner C. Prisoner C knows conditional Probability before turning to a life of crime & is now hopeful. What is his probability of being pardoned?

A = A is pardoned
B = B is Pardoned
C = C is pardoned

$\left.\begin{matrix} A = A \text{ is pardoned} \\ B = B \text{ is Pardoned} \\ C = C \text{ is pardoned} \end{matrix}\right\} \frac{1}{3}$ Before Info

E = B is named as Executed

~~D = B ?~~

$P(E|A) = \frac{1}{2}$

$P(E|C) = 1$

$$\frac{1 \cdot \frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}$$

Probability that C is pardoned is $\frac{2}{3}$

Problem 2.54: Let A & B be events w/ these properties. $0 < P(B) < 1$ & $P(A|B) = P(A|B^c) = \frac{1}{3}$

a) Is it possible to calculate P(A)? Declare it impossible or solve for P(A).

$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c) = \frac{1}{3}P(B) + \frac{1}{3} - \frac{1}{3}P(B) = \boxed{\frac{1}{3} = P(A)}$$

b) Are A & B independent, non-independent, or not possible to tell?

A & B are independent.

$\left.\begin{matrix} P(A) = P(A|B) \\ P(A) = P(A|B^c) \end{matrix}\right\}$ Defines independence

$P(A) = P(A|B) = P(A|B^c) = \frac{1}{3}$ } As shown in (a)