Homework 6

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Exercise 4.26: 100 randomly chosen individuals were intrviewed to estimate the unknown fraction $p \in (0,1)$ of the population that prefers whole milk over skim milk. The resulting estimate $\hat{p}$. With what confidence can we state the true p lies on the interval $\hat{p} \pm 0.1$?

$X \sim Bin(n, p)$

$n = 100$

$\sigma = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \frac{\sqrt{\hat{p}(1-\hat{p})}}{10}$

Confidence Interval $= \hat{p} \pm z \cdot \sigma$

$= \hat{p} \pm 0.1$

$z \cdot \sigma = 0.1$

$z \cdot \frac{\sqrt{\hat{p}(1-\hat{p})}}{10} = 0.1$

$z = \frac{1}{\sqrt{\hat{p}(1-\hat{p})}}$

Maximized when $\hat{p} = 0.5$

Conf Level $= 2\Phi(z) - 1$

$\frac{1}{\sqrt{(0.5)(0.5)}}$

$2\Phi(2) - 1 \approx \boxed{95.44\%}$

Exercise 4.33: In a call center, the number of recieved calls in a day can be modeled by a Poisson random variable. The average 0.5% the call center gets no calls. What is the average # of calls in a day?

Poisson $= P_X(k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}$

$\frac{0.5}{100} = 0.005$

$P_X(0) = e^{-\lambda} = 0.005$

$\lambda = -\ln 0.005$

$E[X] = \boxed{\lambda \approx 5.3}$

Exercise 4.36: How many Randomly chosen guests should I invite so that the probability of having a guest w/ the same birthday as mine is $2/3$

Assumed: Any day has equally likely birthday as others. Birthdays are independent, 365 days in a year

P(1^{st} person shares this birthday) $\geq 2/3$

$P(\text{no match}) = \frac{364}{365}$, $P(\text{No match w/ } n \text{ guests}) = \left(\frac{364}{365}\right)^n$

$P(1+ \text{ match}) = 1 - \left(\frac{364}{365}\right)^n$

$$1 - \left(\frac{364}{365}\right)^n \geq \frac{2}{3}$$

$$\frac{1}{3} \geq \left(\frac{364}{365}\right)^n$$

$$\ln\frac{1}{3} \geq n \ln\frac{364}{365}$$

$$\frac{\ln\frac{1}{3}}{\ln\frac{364}{365}} \leq n$$

$$\boxed{n \geq 401.}$$

Exercise 4.45: Estimate the probability that out of 10000 poker hands of 5 cards, we will see 0 four of a kind. Use Normal and Poisson (whichever is appropriate), justify the choice. $\boxed{P(k=0) = e^{-2.401} = 0.0907}$

Use Poisson → Large # of trials, w/ a low probability of success in each trial.

Total Possible hands: $\binom{52}{5} = 2598960$ $\qquad \frac{624}{2598960} = p$

4 of a kind: $13 \times 48 = 624$ $\qquad P(k=0) = \frac{e^{-\lambda} \cdot \lambda^k}{k!} = e^{-\lambda}$

$\lambda = n \cdot p = 10000 \cdot \frac{624}{2598960}$

store that sells a pawbroker

Exercise 9.49: Suppose you own a store for \$1000. You buy the stove for \$800. Life can be shown w/ exponential $1/10 = \lambda$, where units of time are years. If I went to offer a warranty that, if it breaks in r years. I will replace it (-\$800 for me). If the stove lasts longer than r years, then the warranty pays nothing. Let \$C be the extra charge for the warranty. Where is profit 0 for the warranty (given \$C, r). What is reasonable \$C & r. Why?

$E[X] = CP(X \geq C) + P(X \leq C-800)(C-800) = 0$

$X = C$ if $T > r$

$X = C-800$ if $T \leq r$

$P(T > r) = e^{-(1/10) \cdot r}$

$$\boxed{C \cdot e^{-r/10} + (C-800)(1-e^{-r/10}) = 0}$$

Pairs where C, r will satisfy this eqn

Reasonable \$C & r

My choice

r = 5 years → competitive edge over other stoves & helps user develop confidence in the product. Making this option attractive to buyers.

$C \geq \$315$

↳ to make \$0+ in profit

$$Ce^{-5/10} + (C-800)(1-e^{-5/10}) = 0$$

$$0 = 0.607 \cdot C + (C-800)(1-0.607)$$

$$315 = (0.607 + 1 - 0.607)C$$

$$\$315 = C$$

↳ no profit on extended warranty

Exercise 4.51: Customer Arrivals @ post office are a Poisson distribution. N w/ intensity $\lambda > 0$.
T_1 = first arrival, $t > 0$ that $N_t = 1$ what is T_1 w/ this info? Find: $P(T_1 \le s \mid N_t = 1)$ for all $s \ge 0$

$$P(T_1 \le s \mid N_t = 1) = \frac{P(T_1 \le s \cap N_t = 1)}{P(N_t = 1)}$$

$$P(N_s = k) = \frac{(\lambda s)^k e^{-\lambda s}}{k!}$$

Joint P
$$\left\{ \begin{array}{l} P(N_s = 1) = \lambda s\, e^{-\lambda s} \\ P(N_t - N_s = 0) = e^{-\lambda(t-s)} \end{array} \right\} \rightarrow \lambda s\, e^{-\lambda s} \times e^{-\lambda(t-s)} = \lambda s e^{-\lambda t}$$

$$P(N_t = 1) = \frac{(\lambda t)^1 e^{-\lambda t}}{1!} = \lambda t e^{-\lambda t}$$

$$\boxed{P(T_1 \le s \mid N_t = 1) = \begin{cases} 0, & s < 0 \\ \frac{s}{t}, & 0 \le s \le t \\ 1, & s \ge t \end{cases}}$$

Exercise 5.16: $X \sim \mathrm{Unif}\,[0,1]$. Find the nth moment of X.

a) Use $E[X^n] = \int_0^1 x^n \, dx$

$$\frac{x^{n+1}}{n+1}\Big|_0^1 = \boxed{\frac{1}{n+1} = E[X^n]}$$

b) Using result of (5.3)

$$M_X(t) = \begin{cases} 1, & t = 0 \\ \frac{e^t - 1}{t}, & t \ne 0 \end{cases}$$

to find nth derivative; coefficient of t^n

$$E[X^n] = M^{(n)}(0) = n! \cdot \frac{1}{(n+1)!} = \boxed{\frac{1}{n+1}}$$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}, \text{ thus } e^t - 1 = \sum_{k=1}^{\infty} \frac{t^k}{k!}$$

$$M_X(t) = \frac{1}{t} \cdot \sum_{k=1}^{\infty} \frac{t^k}{k!} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$$

Exercise 5.18 : Let $X \sim Geom(p)$

a) Compute moments generating function $M_X(t)$ of X

$$M_X(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} p(X=k)$$

$$\sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1} p = pe^t \sum_{k=1}^{\infty}[e^t(1-p)]^{k-1} = pe^t \sum_{n=0}^{\infty}(e^t(1-p))^n$$

$n = k-1$

$$M_X(t) = \frac{pe^t}{1-e^t(1-p)}$$

$$M_X(t) = \begin{cases} \frac{pe^t}{1-e^t(1-p)}, & t < \ln\left(\frac{1}{1-p}\right) \\ \infty, & \text{else} \end{cases}$$

$e^t < \frac{1}{1-p}$ $\quad t = \ln\left(\frac{1}{1-p}\right)$

b) $E[X] = M_X'(0)$

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t}$$ Quotient Rule

$$M_X'(t) = \frac{pe^t(1-(1-p)e^t) - pe^t(-(1-p)e^t)}{(1-(1-p)e^t)^2}$$

$$= \frac{pe^t}{(1-(1-p)e^t)^2}$$

$$M_X'(0) = \frac{p}{p^2} = \boxed{\frac{1}{p} = E[X]}$$

$E[X^2] = M_X''(0)$

$$\frac{pe^t}{(1-(1-p)e^t)^2} = M_X'(t)$$

$$M_X''(t) = \frac{pe^t[1-(1-p)e^t + 2e^t(1-p)]}{(1-(1-p)e^t)^3}$$

$$M_X''(0) = \frac{p \cdot (1+(1-p))}{p^3}$$

$$= \frac{2-p}{p^2}$$

$$Var[X] = E[X^2] - E[X]^2$$

$$Var[X] = \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$\boxed{Var[X] = \frac{1-p}{p^2}}$$