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Homework 1, Due 10/6

Intro to Probability, First Edition
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Problems 1.5, 1.6, 1.7, 1.8, 1.20, 1.21, 1.27, 1.30
[Pages 30-34]

Problem 1.5: In one type of of State lottery 5 distinct numbers are picked from 1, 2, 3, ..., 40 uniformly at Random

a) Describe the sample space Ω and probability measure P to model this experiment

$$\boxed{\Omega = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in \{1, \dots, 40\}, x_i \neq x_j \text{ if } i \neq j\}}$$

Set of 5 numbers picked from 1-40 where each set of numbers is unique

Sampling w/o Replacement & Order doesn't Matter

$\# \Omega = \frac{n!}{k!(n-k)!} = \binom{40}{5}$

$\boxed{P(A) = \frac{\# A}{\# \Omega} = \frac{1}{\binom{40}{5}} = \frac{1}{658008}}$

b) What is the Probability that out of 5 picked numbers exactly 3 will be even? $\underbrace{\text{exactly 3 will be even}}_{A}$

Even = {all #s between 1-40 that are Even}
Odd = {All #s Between 1-40 that are Odd}

#Even = 20
#Odd = 20
3 even, means 2 odd

Sampling w/o Replacement & Order doesn't matter

$\left(\binom{20}{3} \cdot \binom{20}{2}\right) / \binom{40}{5} = \boxed{P(A) = \frac{475}{1443}}$

Problem 1.6: We have an Urn with 3 Green & 4 Yellow balls. We choose 2 Randomly & w/o Replacement. Let A be the event that there are 2 different colored balls.

a) Describe a possible Sample space with equally-likely outcomes & event A in the sample space.

Lets label the yellow balls from 1-4 & Green balls from 5-7. Now, we can pick the balls w/ order mattering.

$$\Omega = \{(x_i, x_j) \in \{1,2,3,4,5,6,7\}, i \neq j\}$$

$$A = \{x_i = \{1,2,3,4\} \overset{\text{and}}{\vee} x_j = \{5,6,7\} \text{ or } x_i = \{5,6,7\} \overset{\text{and}}{\vee} x_j = \{1,2,3,4\}\}$$

b) Compute P(A)

$$P(A) = \frac{\#A}{\#\Omega} \quad \frac{\binom{3}{1}\cdot\binom{4}{1}}{\binom{7}{2}} = \frac{3\cdot4}{\frac{7\cdot6}{2}} = \frac{2\cdot12}{42} = \frac{24}{42} = \boxed{\frac{4}{7} = P(A)}$$

Problem 1.7: We have an urn w/ 3 Green balls & 4 Yellow balls. We draw 3 w/o replacement

Order Matters

a) Find the probability that the colors we draw in order are Green, Yellow, Green

Label balls 1–7, Green balls: 1–3 & Yellow balls: 4–7

$A = \{G, Y, G\} \rightarrow \#A = 3 \cdot 4 \cdot 2 = 24$

$\Omega = \{(i, j, k) = \{1, \dots, 7\}, i \neq j, j \neq k, k \neq i\} \rightarrow \#\Omega = 7 \cdot 6 \cdot 5$

$\frac{7!}{4!}$

$P(A) = \frac{\#A}{\#\Omega} = \frac{24}{7 \cdot 6 \cdot 5} = \frac{4}{7 \cdot 5} = \boxed{\frac{4}{35} = P(A)}$

Problem 1.7 (cont) :

b) Find the probability that our sample of 3 balls contains 2 green balls & 1 yellow ball

$$P(A) = \frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = \boxed{\frac{12}{35} = P(A)}$$

Same sample space as a).

Problem 1.8: Suppose that a bag of scrabble tiles contains 5 Es, 4 As, 3 Ns, & 2 Bs. It is my turn & I draw 4 tiles from the bag w/o replacement. Assume the draw is Random. Let C be the event I got 2 Es, 1 A, & 1 N

Total: 14

a) Compute P(C) where tiles are drawn 1 by 1 & in an ordered sample

Es labelled: 1, 2, 3, 4, 5 ; As labeled: 6, 7, 8, 9 ; Ns labeled: 10, 11, 12 ; Bs labeled: 13, 14

$\Omega = \{(a_1, a_2, a_3, a_4) \; a_i \neq a_j, \; a_i \in \{1, \ldots, 14\}\}$,

$\# \Omega = 14 \cdot 13 \cdot 12 \cdot 11 = 24024$

Position for E $= \binom{4}{2} = 6$, $\underbrace{5}_{\text{Pos 1}} \cdot \overbrace{4}^{\text{Pos 2}}$ → Probability E

2 ways for A & Prob 4

1 way for N & Prob 3

$\left[\binom{4}{2} \cdot 5 \cdot 4\right] \cdot [4 \cdot 2] \cdot [1 \cdot 3]$

→ 2880

$P(A) = \frac{\# A}{\# \Omega} = \frac{2880}{24024} = \boxed{\frac{120}{1001} = P(C)}$

Problem 1.8 (Cont):

b) Compute P(C) by imagining the tiles are drawn all @ once as an unordered sample

$$P(C) = \frac{\binom{5}{2}\cdot\binom{4}{1}\cdot\binom{3}{1}}{\binom{14}{4}} = \boxed{\frac{120}{1001} = P(C)}$$

Same sample Space as a)

Problem 1.20: Roll a dice 4 times

a) Describe a sample space (Ω) & Probability measure (P) that model this experiment. To describe P, use $P(\omega)$ for each outcome $\omega \in \Omega$

Reasonable assumption: all equally likely

- we have a fair dice, labeled 1, 2, 3, 4, 5, 6.

$$\Omega = \{(x_1, x_2, x_3, x_4) : x_i \in \{1,2,3,4,5,6\}\}$$

$$\#\Omega = 6^4$$

$$P(\omega) = \frac{1}{6^4} = \frac{1}{1296}$$

b) Let A be the event that there are at least 2 5s among the 4 rolls. Let B be the event that there is at most 1 5 in the 4 rolls. Find P(A) & P(B) by finding total favorable : total as a ratio.

A = Only 2 → 5s + 3 → 5s + 4 5s

$$\binom{4}{2} 5 \cdot 5 + \binom{4}{3} \cdot 5 + 1$$

$$P(A) = \frac{\#A}{\#\Omega} = \frac{150+20+1}{1296} = \boxed{\frac{19}{144}}$$

Agrees w/ part b

B = 1 5s + 0 5s

$$\binom{4}{1} \cdot 5^3 + 5^4$$

$$P(B) = \frac{\#B}{\#\Omega} = \frac{500+625}{1296} = \boxed{\frac{125}{144}}$$

c) What is the set $A \cup B$? What Equality should P(A) and P(B) satisfy? Check

At least two 5s or at max one 5. → $\Omega = A \cup B$

Disjoint, so → $1 = P(\Omega) = P(A \cup B) = P(A) + P(B) = (125+19)/144 = 1$

Problem 1.21: Suppose an Urn has 3 black chips, 2 red, and 2 green chips. Draw 3 w/o replacement: A is the event that all chips are a different color

a) Compute P(A) by imagining it done as an ordered sample

Black: {1, 2, 3}, Red: {4, 5}, Green: {6, 7}

$$\#\Omega = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$$

$\Omega = \{x_1, x_2, x_3\} : x \in \{1, \ldots, 7\}$ for $x_i \neq x_j$ for $i \neq j\}$

3 black chip choices

$$\#A = \underbrace{3 \cdot 2 \cdot 2}_{\text{ways to choose}} \cdot \underbrace{3!}_{\text{ways to order}} = 12 \cdot 6 = 72$$

$$\boxed{P(A) = \frac{\#A}{\#\Omega} = \frac{12}{35}}$$

A = 1 black, 1 Red, 1 Green (any order)

b) Calculate P(A) as if this was an unordered sample.

$$\boxed{P(A) = \frac{\binom{3}{1} \cdot \binom{2}{1}\binom{2}{1}}{\binom{7}{3}} = \frac{12}{35}}$$

Sample space is the same as a)

stop

sample.

$$\boxed{P(A) = \frac{\binom{3}{1}\cdot\binom{2}{1}\binom{2}{1}}{\binom{7}{3}} = \frac{12}{35}}$$

Sample space is the same as a)

Total Possible Positions

Successful Positions

Problem 1.27: Suppose an Urn contains 7 chips labelled 1, . . ., 7. 3 chips are black, 2 are Red, 2 Green. Chips are drawn randomly, 1 at a time w/o replacement until the Urn is empty.

a) What is the Probability that the ith draw is labelled 5?

-Colors not Relevant.

$\Omega = \{(x_1, \dots x_7) : x\{1, \dots, 7\}\ x_i \neq x_j,\ i \neq j\}$ $\#\Omega = 7!$

$A =$ Probability of drawing 5 on ith draw $\#A = 6!$

$$\boxed{P(A) = \frac{\#A}{\#\Omega} = \frac{1}{7}}$$

Problem 1.27 (cont):

b) What is the probability that the ith draw is black?

$$P(B) = \frac{\# B}{\# \Omega} = \frac{3 \cdot 6!}{7!}$$

Same sample space as a)

Possible 3 Black chips @ position i

$P(\text{specific chip}) = \frac{6!}{7!} = \frac{1}{7}$

↳ @ position i

$$\boxed{P(\text{Black}) = 3 \cdot \frac{1}{7} = \frac{3}{7}}$$

Problem 1.30: 8 Rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture each other?

$\#\Omega = \binom{64}{8}$ → Total Sample space

$n = 64$ → Possible Placements

$k = 8$ → # of Rooks

$8 \times 8 = 8^2$

A = No rooks cross/capture each other

- 1 Row & Column Per Rook

$\#A = 8 \cdot 7 \cdot 6 \cdot \ldots \cdot 1 = (8!)$ ways that no squares overlap

$$\boxed{P(A) = \frac{8!}{\binom{64}{8}} = 9.1095 \times 10^{-6}}$$