Homework 5

Problem 320: Let $c>0$ & $X \sim \text{Unif}[0,c]$. Show $Y=c-X$ has the same cumulative distribution function as X & same density function.

$\frac{1}{b-a}$ $b=c$ $a=0$

$$PDF(x) = \begin{cases} \frac{1}{c} & \text{if } x \in [0,c] \\ 0 & \text{; otherwise} \end{cases}$$

$$CDF(x) = F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(x) \qquad \int_0^x \frac{1}{c}\,dx = \frac{x}{c} - 0$$

$$F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{x}{c} & \text{if } x \in [0,c] \\ 1 & \text{if } x>c \end{cases}$$

X can't be bigger than C

$$CDF(Y) = F_Y(y) = P(Y \le y) \qquad y = \text{Real } \#$$

$$= F_Y(y) = P(c-X \le y) = P(c-y \le X)$$

$$P(X \ge c-y) = 1 - P(X \le c-y) = 1 - F_X(c-y)$$

Same w/ diff variables

$y \in [0,c]$

$$F_Y(y) = 1 - F_X(c-y)$$

$$F_Y(y) = 1 - \frac{c-y}{c} = \frac{y}{c}$$

$$\boxed{F_Y = \frac{y}{c} = F_X(y)} \quad \in [0,c]$$

$$F_Y(y) = \begin{cases} 0, & y<0 \\ y/c & y \in [0,c] \\ 1 & y>c \end{cases}$$

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Problem 3.26: Suppose X is a discrete random variable w/ possible values $\{1, 2, 3, \ldots\}$ & the probability function $p_x[k] = \frac{c}{k \cdot (k+1)}$ w/ some constant $c > 0$

a) What is the value of c?

$$1 = PMF = \sum_{k=1}^{\infty} p_X(k) = c \cdot \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = c \cdot \lim_{z \to \infty} \sum_{k=1}^{z} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \cdots + \frac{1}{z} - \frac{1}{z+1} = 1 - \lim_{z \to \infty} \frac{1}{z+1} = 1 - c$$

$$PMF = 1 = c \cdot (1 - 0) \quad \boxed{c = 1}$$

b) Find $E[x]$

$$E[x] = \sum_{k=1}^{\infty} k \cdot p_x(k) = \sum_{k=1}^{\infty} \frac{1}{k+1} = \underbrace{\sum_{k=2}^{\infty} \frac{1}{k}}_{\text{Harmonic Sum}} = \boxed{\infty = E[x]}$$

$\frac{1}{k(k+1)}$

Problem 3.28: 5 closed boxes. 3 have nice prizes inside. The other 2 don't. You open boxes one at a time until you open one w/ a prize. Let X be # of boxes opened

a) Find PMF of X

$$PMF = \begin{cases} 0 \text{ if } x \leq 0 & \to 0 \text{ if } x \leq 0 \\ 3/5 \text{ if } x = 1 & \to 3/5 \text{ if } x = 1 \\ \frac{2}{5} \cdot \frac{3}{4} \text{ if } X = 2 & \to \frac{3}{10} \text{ if } x = 2 \\ \frac{1}{4} \cdot \frac{2}{3} \cdot 1 \text{ if } X = 3 & \to 1/10 \text{ if } x = 3 \end{cases}$$

b) Find $E[x] = 1 \cdot \frac{3}{5} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} = \frac{6}{10} + \frac{6}{10} + \frac{3}{10} = \boxed{\frac{3}{2} = E[x]}$

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Find Var[x]

Problem 3.28: Continued

$\boxed{\frac{9}{20} = Var[x]}$

c) $Var[X] = E[X^2] - E[X]^2$

$= E[X^2] - \frac{9}{4} = \frac{27}{10} - \frac{9}{4} = \frac{108}{40} - \frac{90}{40} = \frac{18}{40}$

$E[X^2] = 1 \cdot \frac{3}{5} + 4 \cdot \frac{3}{10} + 9 \cdot \frac{1}{10} = \frac{6}{10} + \frac{12}{10} + \frac{9}{10} = \frac{27}{10}$

d) Suppose inside each box w/ prize has 100$ & boxes w/ nothing costs 100$ what is gain/loss?

winnings

W = (Prize value) − (cost for Empty Boxes)

→ # boxes picked before winning

$100 - 100[X-1]$

$W = 200 - 100X$

$\boxed{\text{Expected wins} = 150 = 200 -}$

Problem 3.43: Choose a point uniformly @ random in a unit square [side length = 1]. Let X be the distance from point to nearest edge.

a) Find CDF $\quad 0 \le x \le \frac{1}{2}$ → Biggest short distance

1

Area of unit square

$1^2 - (1-2x)^2$

Area of smaller square

$\boxed{4x - 4x^2, \quad x \in [0, \tfrac{1}{2}]}$

$1 - (1 - 2x - 2x + 4x^2)$

b) Find PDF

$f_X(x) = F_X'(x) = 4x - 4x^2 = F_X(x)$

$\boxed{f(x) = 4 - 8x \text{ if } x[0, \tfrac{1}{2}], \text{ else } 0}$

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1

Problem 3.56: (Mean square of a Geometric Random Variable)
Let $X \sim Geom(p)$ & $q = 1-p$ Derive: $E(X^2) = \frac{1+q}{p^2}$

Identity: $E[X^2] \stackrel{?}{=} E[X] + E[X(X-1)]$

$P(X=k) = p \cdot q^{k-1}$ } Def of Geom(p)

$E[X] = \sum_{k=1}^{\infty} k \cdot P(X=k)$

$= p \sum_{k=1}^{\infty} k \cdot q^{k-1} = p \frac{1}{(1-q)^2} = p \cdot \frac{1}{(1-1+p)^2} = \frac{1}{p}$

$=1-0$

$E[X(X-1)] = \frac{2q}{p^2}$

PGF $= G(s) = E[s^X] = \sum_{k=0}^{\infty} s^k \cdot P(X=k)$

$G''(s) = E[X(X-1)\, s^{X-2}]$

$s=1$

$n=k-1$

$G(s) = \sum_{k=1}^{\infty} s^k \cdot p \cdot q^{k-1} = ps \sum_{k=1}^{\infty} (sq)^{k-1} = ps \sum_{n=0}^{\infty} (sq)^n$

$= \frac{ps}{1-sq} \quad |sq| < 1$

$G''(s) = \frac{2pq}{(1-sq)^3} = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$

$1+1-p = 1+q$

$E[X^2] = \frac{1}{p} + \frac{2q}{p^2} = \frac{p+2q}{p^2} = \frac{p+2(1-p)}{p^2} = \frac{2-p}{p^2} = \boxed{\frac{1+q}{p^2}}$

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Problem 3.57: Let $X \sim Geom(p)$ find expected value of $\frac{1}{X}$

$$E\left[\frac{1}{X}\right] = \sum_{k=1}^{\infty} \frac{1}{k} P(X=k) = \sum_{k=1}^{\infty} \frac{p(1-p)^{k-1}}{k} \qquad q = (1-p)$$

$$= p\sum_{k=1}^{\infty} \frac{q^{k-1}}{k} = p\sum_{n} \frac{q^n}{n+1} \qquad \frac{1}{n+1} = \int_0^1 x^n\,dx$$

$n = k-1$

$$= p\sum_{n=0}^{\infty} q^n \int_0^1 x^n = p\int_0^1 \underbrace{\left(\sum_{n=0}^{\infty} (qx)^n\right)}_{\text{Geom Series}} dx$$

$$= p\int_0^1 \frac{1}{1-qx} \quad \text{where } |qx| < 1.$$

$U = 1-qx$

$du = -q\,dx$

$$= p\int_1^p \frac{1}{U}\cdot\left(-\frac{1}{q}\right)du = -\frac{p}{q} \ln(U)\Big|_1^p = -\frac{p}{q}\ln p$$

$$\boxed{-\frac{p}{1-p}\ln(p) = E\left[\frac{1}{X}\right]}$$

Problem 3.58: Let $X \sim \text{Binom}(n, p)$, find the Expected value of $\frac{1}{1+X}$

$$E\left[\frac{1}{1+X}\right] = \sum_{k=0}^{n} \frac{1}{1+k} \cdot P(X=k)$$

$q = 1-p$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{1}{1+k} \cdot \binom{n}{k} p^k q^{n-k}$$

$\int_0^1 x^k \, dx$

$$= \sum_{k=0}^{n} \left(\int_0^1 x^k dx\right) \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} x^k\right) dx$$

$$= \int_0^1 \underbrace{\sum_{k=0}^{n} \binom{n}{k} (px)^k q^{n-k}}_{\text{Binomial Expansion}} \, dx$$

$$= \int_0^1 (q+px)^n \, dx \qquad u = q + px$$

$$= \int_{u=q}^{u=1} u^n \frac{du}{p} \qquad du = p\,dx$$

$$= \frac{1}{p}\left(\frac{u^{n+1}}{n+1}\right)\Bigg|_q^1 = \left(\frac{1}{n+1} - \frac{q^{n+1}}{n+1}\right)\frac{1}{p}$$

$$= \frac{1-q^{n+1}}{p(n+1)} = \boxed{\frac{1-(1-p)^{n+1}}{p(n+1)} = E\left[\frac{1}{1+X}\right]}$$

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