Inverse differential operator and the particular Integral

The inverse differential operator $1/D$ is defined as:

$$\frac{1}{D}\phi(x) = \int \phi(x)\,dx$$

Eg: $$\frac{1}{D}x = \int x\,dx = \frac{x^2}{2}.$$

To obtain the P.I of a D.E $f(D)y = \phi(x)$

$$y_p = \frac{1}{f(D)}\phi(x)$$

Rules to obtain P.I of some standard functions

Consider a D.E $f(D)y = \phi(x)$.

The P.I y_p is given by $y_p = \frac{1}{f(D)}\phi(x)$.

Case I: when $\phi(x) = e^{ax}$.

Then:

(i) $y_p = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$; provided $f(a) \neq 0$.

(Replace D by a)

(ii) If $f(a) = 0$; $y_p = x.\frac{1}{f'(a)}e^{ax}$; $f'(a) \neq 0$

(iii) If $f(a) = f'(a) = 0$; $y_p = x^2\frac{1}{f''(a)}e^{ax}$; $f''(a) \neq 0$.

and so on.

Solve the following D.E

1. $y'' - 3y' + 2y = e^{3x}$.

Soln: Given D.E is $(D^2 - 3D + 2)y = e^{3x}$.

The soln is $y = y_c + y_p$

To find y_c :-

A.E :- $m^2 - 3m + 2 = 0$.

$m = 1, 2$.

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$.

To find y_p :-

$y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} e^{3x}$ (Replace D by a). $a = 3$.

Now; $f(D) = D^2 - 3D + 2$ $\therefore f(3) = 3^2 - 3\times 3 + 2 = 2 \neq 0$

$\therefore y_p = \frac{1}{2} e^{3x}$.

$\therefore$ The g.s is $y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}$.

2. $(D^2 + 3D - 4)y = 12 e^{2x}$.

Soln: Given D.E is $(D^2 + 3D - 4)y = 12e^{2x}$.

Soln is $y = y_c + y_p$.

To find y_c :- $y_c = c_1 e^{x} + c_2 e^{-4x}$.

To find y_p: $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} 12 e^{2x}$ (Replace D by $a=2$)

Now $f(D) = D^2 + 3D - 4 \Rightarrow f(2) = 4 + 6 - 4 = 6$.

$\therefore y_p = \frac{12 e^{2x}}{6} = 2e^{2x}$. $\therefore$ g.s is $y = c_1 e^{x} + c_2 e^{-4x} + 2e^{2x}$.

3. $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Solⁿ: Given D.E is $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Solⁿ is $y = y_c + y_p$.

To find y_c: AE : $m^3 - 6m^2 + 11m - 6 = 0$.

$m = 1, 2, 3$.

$\therefore \; y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

To find y_p: $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} (e^{-2x} + e^{-3x})$.

$$y_p = \frac{1}{f(D)} e^{-2x} + \frac{1}{f(D)} e^{-3x}.$$

Consider $\frac{1}{f(D)} e^{-2x}$ (Here $a = -2$)

$f(D) = D^3 - 6D^2 + 11D - 6 \Rightarrow f(-2) = -60$.

$\therefore \; \frac{1}{f(D)} e^{-2x} = \frac{-1}{60} e^{-2x}$.

& $\frac{1}{f(D)} e^{-3x}$ (Here $a = -3$).

$f(-3) = -120$. $\therefore \; \frac{1}{f(D)} e^{-3x} = \frac{-1}{120} e^{-3x}$.

$$\therefore \; y_p = -\frac{e^{-2x}}{60} - \frac{e^{-3x}}{120}.$$

$\therefore$ G.S is $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{60}\left(e^{-2x} + \frac{e^{-3x}}{2}\right)$.

(4)

4. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$.

Soluⁿ: Given D.E is $(D^2+4D+4)y = e^{-2x}$.

Soluⁿ is $y = y_c + y_p$

To find y_c: A.E is $m^2+4m+4=0$.

$m = -2, -2$

$\therefore y_c = \cancel{e^{-2x}} (c_1 + c_2 x)e^{-2x}$.

To find y_p: $y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)} e^{-2x}$ (Replace D by $a=-2$)

$f(D) = D^2+4D+4 \Rightarrow f(-2) = 0$.

$f'(D) = 2D+4 \Rightarrow f'(-2) = 0$

$\Rightarrow f''(2) = 2$

$f''(D) = 2$

$\therefore y_p = x^2 . \frac{e^{-2x}}{2} = \frac{x^2 e^{-2x}}{2}$.

$\therefore$ G.S is $y = \left(c_1 + c_2 x + \frac{x^2}{2}\right) e^{-2x}$.

5. $(D-3)^2 y = e^{3x}$.

$y = (c_1 + c_2 x)e^{3x} + \frac{x^2 e^{3x}}{2}$.

6. $(4D^2-1)y = e^{x/2} + 12e^{x} + 4$.

Solu: Given D.E is

$$(4D^2-1)y = e^{x/2} + 12e^{x} + 4.$$

solu is $y = y_c + y_p$.

To find y_c: AE: $4m^2 - 1 = 0$.

$$y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}.$$

To find y_p: $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} (e^{x/2} + 12e^{x} + 4)$

$$\therefore y_p = \frac{1}{f(D)} e^{x/2} + 12 \frac{1}{f(D)} e^{x} + 4 \cdot \frac{1}{f(D)} e^{0x}.$$

$\frac{1}{f(D)} e^{x/2}$: $(a=\frac{1}{2})$ $f(D) = 4D^2-1 \Rightarrow f(\frac{1}{2}) = 0$.

$f'(D) = 8D. \Rightarrow f'(\frac{1}{2}) = 8 \times \frac{1}{2} = 4$.

$$\therefore \frac{1}{f(D)} e^{x/2} = x \frac{e^{\frac{1}{2}x}}{4}.$$

$12 \frac{1}{f(D)} e^{x}$: $(a=1)$ $f(1) = 4-1 = 3$.

$$\therefore 12 \frac{1}{f(D)} e^{x} = 12 \times \frac{e^x}{3} = 4e^x.$$

$4 \cdot \frac{1}{f(D)} e^{0x}$: $(a=0)$ $f(0) = -1 \Rightarrow 4 \frac{1}{f(D)} e^{0x} = 4 \times \frac{e^{0x}}{-1}$

$$\therefore y_p = \frac{x e^{\frac{1}{2}x}}{4} + 4e^{x} - 4.$$

$$\therefore G.S = y = c_1 e^{x/2} + c_2 e^{-x/2} + \frac{x}{4} e^{\frac{1}{2}x} + 4e^{x} - 4$$

Case ii when $\phi(x) = \sin ax \mid \cos ax$.

Consider a D E $f(D)y = \sin ax$; $a = $ const

Then (i) $y_p = \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$; $f(-a^2) \neq 0$

(Replace D^2 by $-a^2$)

$D \sin ax = a \cos ax$
$D^2 \sin ax = -a^2 \sin ax$
$D^3 \sin ax = -a^3 \cos ax$
$D^4 \sin ax = a^4 \sin ax$

(ii) If $f(-a^2) = 0$ then

$y_p = x \cdot \frac{1}{f'(-a^2)} \sin ax$; $f'(-a^2) \neq 0$.

$\sin ax = \frac{-a^2}{D^2} \sin ax$.

$\frac{1}{D^2} \sin ax = \frac{1}{-a^2} \sin ax$

Solve the following D.E

1. $(D^2+16)y = 14 \cos 3x$.

Soln. Given $(D^2+16)y = 14 \cos 3x$.

Soln is $y = y_c + y_p$.

To find y_c: A.E: $m^2+16 = 0$. $\Rightarrow m = \pm 4i$ $(\alpha \pm i\beta)$

$\therefore y_c = e^{0x}\{c_1 \cos 4x + c_2 \sin 4x\}$.

To find y_p:

$y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} 14 \cos 3x$

(Replace D^2 by $-a^2 = -(3)^2 = -9$

$\therefore$ ~~$y_p = \frac{1}{}$~~ $f(D) = D^2+16 \Rightarrow f(-a^2) = -9+16$
$= 7$.

$\therefore \quad y_p = \dfrac{14\cos 3x}{7} = \underline{\underline{2\cos 3x}}$

Hence the G.S is $\quad y = c_1\cos 4x + c_2\sin 4x + \underline{\underline{2\cos 3x}}$

2. $(D^2+36)y = 4\cos 6x$.

Soln: Given D.E $\quad (D^2+36)y = 4\cos 6x$.

Soln is $\quad y = y_c + y_p$.

To find y_c: $\quad$ A.E: $m^2+36=0 \Rightarrow m = \pm 6i$

$\therefore \quad y_c = c_1\cos 6x + \underline{\underline{c_2\sin 6x}}$.

To find y_p: $\quad y_p = \dfrac{1}{f(D)}\phi(x) = \dfrac{1}{f(D)} 4\cos 6x$. $\quad a=6$

Replace D^2 by $-a^2 = -36$.

$f(D^2) = D^2+36 \Rightarrow f(-a^2) = -36+36 = 0$.

$f'(D^2) = 2D$

$\therefore \quad y_p = x \cdot \dfrac{1}{2D} 4\cos 6x = 2x \cdot \dfrac{1}{D}\cos 6x$.

$= 2x\int \cos 6x\, dx = 2x \cdot \dfrac{\sin 6x}{6} = \underline{\underline{\dfrac{x\sin 6x}{3}}}$

$\therefore$ G.S is $\quad y = c_1\cos 6x + \underline{\underline{c_2\sin 6x}} + \dfrac{x}{3}\sin 6x$

3. $(D^3+D^2+D+1)y = \cos 2x$.

Soln: Given D.E is $\quad (D^3+D^2+D+1)y = \cos 2x$.

Soln is $\quad y = y_c + y_p$.

8

A: E $\quad m^3 + m^2 + m + 1 = 0$

$m = -1, \pm i \quad (\alpha \pm i\beta, \alpha = 0 \quad \beta = 1)$

$y_c = C_1 e^{-x} + C_2 \cos x + C_3 \sin x.$

To find y_p:

$y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} \cos 2x. \qquad (a = 2)$

Replace D^2 by $-a^2 = -4.$

$f(D^2) = D^3 + D^2 + D + 1 = D^2 . D + D^2 + D + 1$

$\therefore f(-a^2) = -4.D + (-4) + D + 1$

$= -3(D+1).$

$\therefore y_p = \frac{1}{-3(D+1)} \cos 2x. \quad = -\frac{1}{3} \cdot \frac{1}{D+1} \cos 2x.$

$= -\frac{1}{3} \frac{1}{(D+1)(D-1)} \times (D-1) \cos 2x.$

Imp

$= -\frac{1}{3} \frac{(D-1)\cos 2x}{D^2 - 1}$

$= -\frac{1}{3} \left\{ \frac{-2 \sin 2x - \cos 2x}{-4 - 1} \right\}$

$\therefore y_p = -\frac{1}{15} (2 \sin 2x + \cos 2x)$

4. $y'' + 9y = \cos 2x \cos x$.

Sol: Given D.E is $(D^2+9)y = \cos 2x \cos x$.

$y_c = c_1 \cos 3x + c_2 \sin 3x$

To find y_p: $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} \cos 2x \cos x$.

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\Rightarrow \cos 2x \cos x = \frac{1}{2}[\cos 3x + \cos x]$

$\therefore \phi(x) = \frac{1}{2}(\cos 3x + \cos x)$

$\Rightarrow y_p = \frac{1}{2}\left\{ \frac{1}{f(D)} \cos 3x + \frac{1}{f(D)} \cos x \right\}$

Consider $\frac{1}{f(D)} \cos 3x$ (Replace D^2 by $-a^2$) $a = 3$.

$f(D^2) = D^2 + 9 \Rightarrow f(-a^2) = -9 + 9 = 0$.

$f'(D^2) = 2D$

$\therefore \frac{1}{f(D)} \cos 3x = x \cdot \frac{1}{2D} \cos 3x = \frac{x}{2} \int \cos 3x \, dx$

$= \frac{x}{2}\left\{ \frac{\sin 3x}{3} \right\} = \frac{x \sin 3x}{6}$.

Now consider $\frac{1}{f(D)} \cos x$ (Replace D^2 by $-a^2 = -1$)

$f(-1^2) = -1 + 9 = 8$.

$\therefore \frac{1}{f(D)} \cos x = \frac{1}{8} \cos x$.

$\therefore y_p = \frac{1}{2}\left\{ \frac{x \sin 3x}{6} + \frac{\cos x}{8} \right\}$.

Case iii: When $\phi(x) = x^m$; m is a +ve integer

P. I, $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} x^m$

$= (f(D))^{-1} x^m$.

Expand $(f(D))^{-1}$ is ascending powers of D as far as the term D^m and operate on x^m, term by term.

{ since (m+1)th & higher derivatives of x^m are zero's}

Note:

(i) $\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 + \cdots$

(ii) $\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots$

Solve the following D. E

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$.

Soln: Given D E is $(D^2 + D)y = x^2 + 2x + 4$.

Soln is $y = y_c + y_p$

To find y_c: A.E $m^2 + m = 0$. $m = 0, -1$.

$y_c = c_1 + c_2 e^{-x}$.

To find y_p: $y_p = \frac{1}{f(D)} (x^2 + 2x + 4)$

$$y_p = \frac{1}{D^2+D}(x^2+2x+4)$$

$$= \frac{1}{D(1+D)}(x^2+2x+4)$$

$$= \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2)(x^2+2x+4)$$

$$= \frac{1}{D}\left((x^2+2x+4)-(2x+2)+(2)\right)$$

$$= \frac{1}{D}(x^2+4) = \int (x^2+4)\,dx = \frac{x^3}{3}+4x.$$

$$\therefore\ y_p = \frac{x^3}{3}+4x.$$

$\therefore$ G. S is $\quad y = C_1 + C_2 e^{-x} + \frac{x^3}{3} + 4x$.

2. $\quad y'' + 3y' + 2y = 12x^2$

Soln: $\quad y_c = C_1 e^{-2x} + C_2 e^{-x}.$

To find y_p: $\quad y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)} 12x^2$

$$y_p = \frac{1}{D^2+3D+2}(12x^2)$$

$$= \frac{1}{2\left[1+\underbrace{\left(\frac{3}{2}D+\frac{D^2}{2}\right)}_{t}\right]}(12x^2)$$

$$= 6\left[1+\underbrace{\left(\frac{3}{2}D+\frac{D^2}{2}\right)}_{t}\right]^{-1} x^2$$

$$y_p = 6\left\{1 - \left(\tfrac{3}{2}D + \tfrac{D^2}{2}\right) + \left(\tfrac{3}{2}D + \tfrac{D^2}{2}\right)^2\right\} x^2$$

$$= 6\left\{1 - \tfrac{3}{2}D - \tfrac{D^2}{2} + \left(\tfrac{3}{2}D\right)^2 + 2\left(\tfrac{3}{2}D\right)\left(\tfrac{D^2}{2}\right) + \cancel{\left(\tfrac{D^2}{2}\right)^2}\right\} x^2$$

$$= 6\left\{1 - \tfrac{3}{2}D + \tfrac{7}{4}D^2\right\} x^2$$

$$-\tfrac{1}{2}D^2 + \tfrac{9}{4}D^2$$

$$\left(-\tfrac{1}{2} + \tfrac{9}{4}\right)D^2$$

$$\tfrac{7}{4}D^2$$

$$= 6\left\{x^2 - \tfrac{3}{2} \times 2x + \tfrac{7}{4} \times 2\right\}$$

$$y_p = 6x^2 - 18x + 21$$

$$\therefore GS = y = C_1 e^{-2x} + C_2 e^{-x} + 6x^2 - 18x + 21$$

3. $(D^2 + 2D + 2)y = 1 + 3x + x^2$

Soln: $y_c = e^{-x}\{c_1 \cos x + c_2 \sin x\}$

To find y_p: $y_p = \frac{1}{f(D)}(\phi(x)) = \frac{1}{f(D)}(1 + 3x + x^2)$

$$\therefore y_p = \frac{1}{D^2 + 2D + 2}(1 + 3x + x^2)$$

$$= \frac{1}{2\left[1 + \underbrace{\left(D + \tfrac{D^2}{2}\right)}_{t}\right]}(1 + 3x + x^2)$$

$$= \frac{1}{2}\left[1 - \left(D + \tfrac{D^2}{2}\right) + \left(D + \tfrac{D^2}{2}\right)^2\right](1 + 3x + x^2)$$

$$= \frac{1}{2}\left[1 - D - \tfrac{D^2}{2} + D^2\right](1 + 3x + x^2)$$

$$= \frac{1}{2}\left[1 - D + \tfrac{D^2}{2}\right](1 + 3x + x^2)$$

$$\therefore y_p = \frac{1}{2}\left\{(1 + 3x + x^2) - (3 + 2x) + \tfrac{1}{2}(2)\right\}$$

$$y_p = \frac{1}{2}\{x^2 + x - 1\}$$

$$G.S = e^{-x}\{C_1\cos x + C_2\sin x\} + \frac{1}{2}\{x^2 + x - 1\}$$

4. $$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3.$$

Sol^n: Given D.E is $(D^3 + 2D^2 + D)y = x^3$

$m = 0, -1, -1$

$$y_c = C_1 + (C_2 + C_3 x)e^{-x}$$

$$y_p = \frac{1}{f(D)}(x^3) = \frac{1}{D^3 + 2D^2 + D}x^3$$

$$= \frac{1}{D[1 + \underbrace{2D + D^2}_{t}]}x^3..$$

$$= \frac{1}{D}\{1 - (2D + D^2) + (2D + D^2)^2 - (2D + D^2)^3\}x^3$$

$$= \frac{1}{D}\{1 - 2D - D^2 + (2D)^2 + 2(2D)(D^2) + \cancel{(D^2)^2} \to 0 - ((2D)^3)\}x^3.$$

$$= \frac{1}{D}\{1 - 2D - D^2 + 4D^2 + 4D^3 - 8D^3\}x^3.$$

$$= \frac{1}{D}\{1 - 2D + 3D^2 - 4D^3\}x^3$$

$$= \left(\frac{1}{D} - 2 + 3D - 4D^2\right)x^3$$

$$= \int x^3 dx - 2x^3 + 3(3x^2) - 4(6x)$$

$$y_p = \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$$

Case iv :- When $\phi(x) = e^{ax} g(x)$ where $g(x)$ is a fun of x.

$$P.I = \frac{1}{f(D)} e^{ax} g(x) = e^{ax} \cdot \frac{1}{f(D+a)} g(x)$$

Solve the following D.E

1. $(D^2 - D - 2)y = 36x e^{2x}$.

Solu^n: Given D.E is $(D^2 - D - 2)y = 36x e^{2x}$.

Solu^n is $y = y_c + y_p$

To find y_c: A.E $= m^2 - m - 2 = 0$

$(m+1)(m-2) = 0$

$m = -1, 2$.

$$\therefore y_c = c_1 e^{-x} + c_2 e^{2x}$$

To find : $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} 36x e^{2x}$.

(This is in the form $e^{ax} g(x)$) (Here $a = 2$)

$$\therefore y_p = 36 e^{2x} \cdot \frac{1}{f(D+2)} x = \underbrace{36 e^{2x}}_{\text{outside the operator}} \cdot \frac{1}{f(D+2)} x$$

Consider $\frac{1}{f(D+2)} x$.

Now $f(D) = D^2 - D - 2 \Rightarrow f(D+2) = (D+2)^2 - (D+2) - 2$

$= D^2 + 4D + 4 - D - 2 - 2$

$= D^2 + 3D$.

$$\therefore \frac{1}{f(D+2)} x = \frac{1}{D^2+3D} x. \quad (\text{Expand } (f(D))^{-1})$$

$$= \frac{1}{3D\underbrace{(1 + D/3)}_{(1+t)}} x$$

$$\frac{1}{f(D+2)}x = \frac{1}{3D}(1+D/3)^{-1}x.$$

$$= \frac{1}{3D}\left[1 - D/3\right]x.$$

$$= \frac{1}{3}\left[\frac{1}{D} - \frac{1}{3}\right]x.$$

$$= \frac{1}{3}\left\{\int x\,dx - \frac{1}{3}x\right\}$$

$$\frac{1}{f(D+2)}x = \frac{1}{3}\left\{\frac{x^2}{2} - \frac{1}{3}x\right\} = \underline{\underline{\frac{x^2}{6} - \frac{x}{9}}}.$$

$$\therefore\ y_p = 36e^{2x}\left(\frac{x^2}{6} - \frac{x}{9}\right).$$

$$= \underline{\underline{(6x^2 - 4x)e^{2x}}}.$$

$$\therefore \text{G. S is } y = c_1e^{x} + c_2e^{2x} + \underline{\underline{(6x^2 - 4x)e^{2x}}}$$

2. $(D^2 - 3D + 2)y = xe^{3x}.$

Soln: $y_c = c_1e^{x} + c_2e^{2x}.$

To find y_p: $y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)}xe^{3x}$

(This is in the form $e^{ax}g(x)$) $(a=3)$

$$\therefore\ y_p = e^{3x}.\frac{1}{f(D+3)}x$$

Consider $\frac{1}{f(D+3)}x$:

Now $f(D) = D^2 - 3D + 2 \Rightarrow$ ~~$f(D+3) =$~~

$$f(D+3) = (D+3)^2 - 3(D+3) + 2$$

$$= D^2 + 6D + 9 - 3D - 9 + 2 = \underline{\underline{D^2 + 3D + 2}}$$

$f(D+3) = D^2 + 3D + 2$

$\therefore \frac{1}{f(D+3)} x = \frac{1}{D^2+3D+2} x.$

$= \frac{1}{2\left(1+\underbrace{\left(\tfrac{3}{2}D+\tfrac{D^2}{2}\right)}_{t}\right)} x.$

$= \frac{1}{2}\left\{1+\left(\tfrac{3}{2}D+\tfrac{D^2}{2}\right)\right\}^{-1} x.$

$= \frac{1}{2}\left\{1-\left(\tfrac{3}{2}D+\cancel{\tfrac{D^2}{2}}\right)\right\} x.$

$= \frac{1}{2}\left\{x - \frac{3}{2}.1\right\} = \frac{x}{2} - \frac{3}{4}.$

$\therefore \quad y_p = e^{3x}\left\{\frac{x}{2} - \frac{3}{4}\right\}.$

$\therefore$ G.S is $\quad y = c_1 e^{x} + c_2 e^{2x} + e^{3x}\left\{\frac{x}{2} - \frac{3}{4}\right\}.$

3. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x.$

Soln: $\quad y_c = c_1 e^{-2x} + c_2 e^{-3x}.$

To find y_p: $\quad y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)} e^{-2x}\sin 2x.$

This is in the form $e^{ax}g(x)$; Here $a = -2$

$\therefore \quad y_p = e^{ax}.\frac{1}{f(D+a)} g(x) = e^{-2x}.\frac{1}{f(D-2)}\sin 2x.$

Consider $\frac{1}{f(D-2)}\sin 2x.$

Here $f(D) = D^2 + 5D + 6$

$\Rightarrow f(D-2) = (D-2)^2 + 5(D-2) + 6 = D^2 + D.$

$$\frac{1}{f(D-2)} \sin 2x = \frac{1}{D^2+D} \sin 2x .$$

(Replace D^2 by $-a^2 = -4$).

$$\frac{1}{f(D-2)} \sin 2x = \frac{1}{-4+D} \sin 2x$$

$$= \frac{(D+4)}{(D-4)(D+4)} \sin 2x .$$

$$= \frac{2\cos 2x + 4\sin 2x}{D^2-4^2}.$$

$$\frac{1}{f(D-2)} \sin 2x = \frac{-1}{10}(\cos 2x + 2\sin 2x)$$

$$\therefore \quad y_p = -\frac{e^{-2x}}{10}(\cos 2x + 2\sin 2x).$$

$$\therefore \quad G.S = y = C_1 e^{-2x} + C_2 e^{-3x} - \frac{e^{-2x}}{10}(\cos 2x + 2\sin 2x)$$

4. $\dfrac{d^4y}{dx^4} - y = \cos x \cosh x .$

Sol^n: Given D.E is $(D^4-1)y = \cos x \cosh x .$

A.E $\quad m^4 - 1 = 0 \Rightarrow (m^2-1)(m^2+1) = 0 .$

$\therefore \quad m = \pm 1, \pm i \quad (\alpha = 0 \; \beta = 1)$

$$\therefore y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

To find y_p: $\quad y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)} \cos x \cosh x .$

$$\left\{ \cosh x = \frac{e^x + e^{-x}}{2} \; ; \; \sinh x = \frac{e^x - e^{-x}}{2} \right\}.$$

⑤

$\therefore \quad \phi(x) = \cos x \cosh x = \cos x\left(\frac{e^{x}+e^{-x}}{2}\right)$

$= \frac{1}{2}\left\{ e^{x}\cos x + e^{-x}\cos x \right\}.$

$\Rightarrow \quad y_p = \frac{1}{2}\left\{ \underbrace{\frac{1}{f(D)} e^{x}\cos x}_{\mathrm{I}} + \underbrace{\frac{1}{f(D)} e^{-x}\cos x}_{\mathrm{II}} \right\}.$

$\mathrm{I} \Rightarrow \frac{1}{f(D)} e^{x}\cos x = e^{x}.\frac{1}{f(D+1)}\cos x \qquad \{e^{ax}g(x)\}.$

$f(D) = D^4 - 1 \Rightarrow f(D+1) = (D+1)^4 - 1$

$= (D^4 + 4D^3 + 6D^2 + 4D + 1) - 1$

$= D^4 + 4D^3 + 6D^2 + 4D$

1

1 2 1

1 3 3 1

1 4 6 4 1

$\therefore \frac{1}{f(D+1)}\cos x = \frac{1}{D^4 + 4D^3 + 6D^2 + 4D}\cos x.$

(Replace D^2 by $-a^2 = -1^2 = -1$)

$\therefore \frac{1}{f(D+1)}\cos x = \frac{1}{(-1)^2 + 4D(-1) + 6(-1) + 4D.}\cos x$

$= \frac{-1}{5}\cos x.$

$\therefore \quad \mathrm{I} \Rightarrow \frac{1}{f(D)} e^{x}\cos x = \frac{-e^{x}}{5}\cos x$

$\mathrm{II} \Rightarrow -\frac{e^{-x}\cos x}{5}$ } HW.

$y_p = -\frac{1}{5}\cos x\cosh x.$

$\therefore$ G.S : $y =$

Solve the following D.E :

1. $\frac{d^2y}{dx^2} + 4y = 2e^x \sin^2 x$.

Soln: The Given D.E can be written as:

$(D^2+4)y = 2e^x \sin^2 x$.

Soln is $y = y_c + y_p$

To find y_c: A.E $m^2+4=0$

$m = \pm 2i$

$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x$.

To find y_p: $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} 2e^x \sin^2 x$.

$\phi(x) = 2e^x \sin^2 x = \cancel{2} e^x \left(\frac{1-\cos 2x}{\cancel{2}} \right) = e^x - e^x \cos 2x$

$\therefore y_p = \frac{1}{f(D)} (e^x - e^x \cos 2x) = \underbrace{\frac{1}{f(D)} e^x}_{I} + \underbrace{\frac{1}{f(D)} e^x \cos 2x}_{II}$

$I \Rightarrow \frac{1}{f(D)} e^x = \frac{1}{f(a)} e^x$ (Replace D by a) $a=1$

$f(D) = D^2+4 \Rightarrow f(a) = 1+4 = 5$

$\therefore I \Rightarrow \frac{e^x}{5}$.

$II \Rightarrow \frac{1}{f(D)} e^x \cos 2x = \frac{1}{f(D)} e^{ax} g(x) = e^{ax} \cdot \frac{1}{f(D+a)} g(x)$

Here $a = 1$

$f(D) = D^2+4 \Rightarrow f(D+1) = (D+1)^2+4$

D^2+2D+5

$$\mathrm{II} \Rightarrow e^{x} \cdot \frac{1}{D^2+2D+5} \cos 2x \qquad (\text{Replace } D^2 = -a^2 = -4)$$

$$= e^{x} \cdot \frac{1}{-4+2D+5} \cos 2x$$

$$= e^{x} \cdot \frac{1}{2D+1} \cos 2x = \frac{(2D-1)}{(2D)^2-1^2} \cdot \cos 2x.$$

$$= \frac{e^{x}\{2x - 2\sin 2x \cancel{+} \cos 2x\}}{4D^2-1}$$

$$\mathrm{II} = \frac{-e^{x}\{4\sin 2x + \cos 2x\}}{4\times -4-1} = -e^{x}\frac{(4\sin 2x + \cos 2x)}{17}.$$

$$\therefore y_p = \frac{e^{x}}{5} - \frac{e^{x}}{17}(4\sin 2x - \cos 2x)$$

$\therefore$ G.S is $y = C_1\cos 2x + C_2 \sin 2x + e^{x}\left\{\frac{1}{5} - \frac{4\sin 2x}{17} + \frac{\cos 2x}{17}\right\}$

2. $(D^2+3D+2)y = e^{-2x} + \cos 3x$.

soln: $m = -1, -2 \qquad y_c = C_1 e^{-x} + C_2 e^{-2x}$.

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)}(e^{-2x} + \cos 3x) = \underbrace{\frac{1}{f(D)} e^{-2x}}_{\mathrm{I}} + \underbrace{\frac{1}{f(D)} \cos 3x}_{\mathrm{II}}$$

$$\mathrm{I} \Rightarrow \frac{1}{f(D)} e^{-2x} = \frac{1}{f(a)} e^{-2x}.$$

$$f(D) = D^2 + 3D + 2 \Rightarrow f(-2) = 0.$$

$\therefore f'(D) = 2D+3 \Rightarrow f'(-2) = -4+3 = \underline{\underline{-1}}$

$\therefore \mathrm{I} \Rightarrow x\dfrac{e^{-2x}}{-1} = \underline{\underline{-xe^{-2x}}}$

$\mathrm{II} \Rightarrow \dfrac{1}{f(D)}\cos 3x$ (Replace D^2 by $-a^2 = -9$)

$= \dfrac{1}{-9+3D+2}\cos 3x = \dfrac{1}{3D-7}\cos 3x$

$= \dfrac{(3D+7)}{(3D)^2-7^2}\cos 3x = \dfrac{3\times -3\sin 3x + 7\cos 3x}{9\times(-9)-49}$

$= \dfrac{+1}{130}(9\sin 3x - 7\cos 3x)$

$\therefore y_p = -xe^{-2x} + \dfrac{1}{130}\underline{\underline{(9\sin 3x - 7\cos 3x)}}$

G. S $\Rightarrow y = c_1e^{-x} + (c_2 - x)e^{-2x} + \dfrac{1}{130}\underline{\underline{(9\sin 3x - 7\cos 3x)}}$

3. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$

Solⁿ: $y_c = c_1 + c_2e^x + c_3e^{-x}$.

$y_p = \underbrace{\dfrac{1}{f(D)}(2x+1)}_{\mathrm{I}} + \underbrace{\dfrac{1}{f(D)}4\cos x}_{\mathrm{II}} + \underbrace{\dfrac{1}{f(D)}(2e^x)}_{\mathrm{III}}$

$\mathrm{I} \Rightarrow \dfrac{1}{f(D)}(2x+1) = \dfrac{1}{D^3-D}(2x+1)$

$= \dfrac{1}{-D(1-D^2)}(2x+1) = \dfrac{-1}{D}[1-D^2]^{-1}(2x+1)$

$= \dfrac{-1}{D}\{1 + \cancel{D^2}^{0}\}(2x+1) = \dfrac{-1}{D}\{2x+1\}$

$I \Rightarrow -\int (2x+1)\,dx = -\left(x^2+x\right)$.

$II \Rightarrow \frac{1}{f(D)} 4\cos x$ (Replace D^2 by $-a^2 = -1$)

$f(D) = D^3 - D \Rightarrow f(-a^2) = D(-1) - D = -2D$.

$\therefore II \Rightarrow -\frac{1}{2D} 4\cos x = -2\int \cos x\,dx = -\cancel{2}\times \sin x$

$II \Rightarrow -2\sin x$.

$III \Rightarrow \frac{1}{f(D)} 2e^x$ (Replace D by $a = 1$)

$f(D) = D^3 - D \Rightarrow f(1) = 1 - 1 = 0$.

$f'(D) = 3D^2 - 1 \Rightarrow f'(1) = 3\times 1 - 1 = 2$.

$\therefore III \Rightarrow x \cdot \frac{2e^x}{2} \Rightarrow xe^x$.

$\therefore y_p = -(x^2+x) - 2\sin x + xe^x$

G.S: $y = c_1 + c_2 e^x + c_3 e^{-x} - (x^2+x) + xe^x - 2\sin x$.

4. $(D^2 - 6D + 25)y = e^{2x} + \sin x + x$.

5. $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$.

6. $y'' + 2y' + 5y = e^{-x}\sin 2x$.

7. $y'' - 2y' + y = xe^x + x$.