Massachusetts Institute of Technology
6.046J/18.410J: Introduction to Algorithms
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Handout 9
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Master Theorem Worksheet

This is a worksheet to help you master solving recurrence relations using the Master Theorem. For each recurrence, either give the asympotic solution using the Master Theorem (state which case), or else state that the Master Theorem doesn't apply. You should be able to go through these **25** recurrences in **10** minutes.

Problem 1-1. $T(n) = 3T(n/2) + n^2$

$a \geq 1 \rightarrow 3 \geq 1$
$b > 1 \rightarrow 2 \geq 1$
$\log_2 3 \geq 1.58\ldots$

$c(n) = n^2 = \Omega(n^{\log_b(a)+\epsilon})$
$n^2 = \Omega(n^{\log_2(3)+\epsilon})$
$n^2 = n^{1.58+\epsilon}$

Case 3:
$T(n) = \Theta(c(n))$
$T(n) = \Theta(n^2)$

Problem 1-2. $T(n) = 7T(n/2) + n^2$

$a \geq 1 \rightarrow 7 \geq 1$
$b > 1 \rightarrow 2 > 1$
$\log_2 7 = 2.807$

$c(n) = n^2 = O(n^{2.807-\epsilon})$
$n^2 = n^{2.807-\epsilon}$

Case 1:
$T(n) = \Theta(n^{\log_b(a)})$
$T(n) = \Theta(n^{\log_2 7})$

Problem 1-3. $T(n) = 4T(n/2) + n^2$

$a \geq 1 \rightarrow 4 \geq 1$
$b > 1 \rightarrow 2 > 1$
$\log_2 4 = 2$

$c(n) = n^2 = \Theta(n^2)$
$n^2 = n^2$

Case 2:
$T(n) = \Theta(n^{\log_b(a)} \log(n))$
$T(n) = \Theta(n^2 \log(n))$

Problem 1-4. $T(n) = 3T(n/4) + n \lg n$

$a \geq 1 \rightarrow 3 \geq 1$
$b > 1 \rightarrow 4 > 1$
$\log_4 3 = 0.79$

$c(n) = n \log(n) = n^1 \log(n)$
$n^1 = n^{0.79+\epsilon}$

Case 3:
$T(n) = \Theta(c(n))$
$T(n) = \Theta(n \log n)$

Problem 1-5. $T(n) = 4T(n/2) + \lg n$

$a \geq 1 \rightarrow 4 \geq 1$
$b > 1 \rightarrow 2 > 1$
$\log_2 4$

$c(n) = \log n = n^0 \cdot \log n$
$n^0 = n^{2-\epsilon}$

Case 1
$T(n) = \Theta(n^{\log_b(a)})$
$T(n) = \Theta(n^2)$

Problem 1-6. $T(n) = T(n-1) + n$

$a \geq 1 \rightarrow 1 \geq 1$
$b > 1 \rightarrow 1 \not> 1$

Master Theorem does not apply as there is no recursion? or division idk right term

Problem 1-7. $T(n) = 4T(n/2) + n^2 \lg n$

$a \geq 1 \rightarrow 4 \geq 1$ $\quad c(n) = n^2 \log n$ $\quad$ Case 2:

$b > 1 \rightarrow 2 > 1$ $\quad n^2 = n^2$ $\quad T(n) = \Theta(n^{\log_b(a)} \log(n))$

$\log_2 4 = 2$ $\quad T(n) = \Theta(n^2 \log(n))$

Problem 1-8. $T(n) = 5T(n/2) + n^2 \lg n$

$a \geq 1 \rightarrow 5 \geq 1$ $\quad c(n) = n^2 \log n$ $\quad$ Case 1:

$b > 1 \rightarrow 2 > 1$ $\quad n^2 = n^{2.32 - \epsilon}$ $\quad T(n) = \Theta(n^{\log_b(a)})$

$\log_2 5 = 2.32$ $\quad T(n) = \Theta(n^{\log_2 5})$

Problem 1-9. $T(n) = 3T(n/3) + n/\lg n$

$a \geq 1 \rightarrow 3 \geq 1$ $\quad c(n) = n \cdot \frac{1}{\log n}$ $\quad$ Case 2:

$b > 1 \rightarrow 3 > 1$ $\quad T(n) = \Theta(n^{\log_b(a)} \log(n))$

$\log_3 3 = 1$ $\quad n^1 = n^1$ $\quad T(n) = \Theta(n \log(n))$

Problem 1-10. $T(n) = 2T(n/4) + c$

$a \geq 1 \rightarrow 2 \geq 1$ $\quad c(n) = c$ $\quad$ Case 1:

$b > 1 \rightarrow 4 > 1$ $\quad n^0 = n^{\frac{1}{2} - \epsilon}$ $\quad T(n) = \Theta(n^{\log_b(a)})$

$\log_4 2 = \frac{1}{2}$ $\quad T(n) = \Theta(n^{\frac{1}{2}})$

Problem 1-11. $T(n) = T(n/4) + \lg n$

$a \geq 1 \rightarrow 1 \geq 1$ $\quad c(n) = \log n \rightarrow n^0 \cdot \log n$ $\quad$ Case 2:

$b \geq 4 \rightarrow 4 > 1$ $\quad T(n) = \Theta(n^{\log_b(a)} \log(n))$

$\log_4 1 = 0$ $\quad n^0 = n^0$ $\quad T(n) = \Theta(\log(n))$

Problem 1-12. $T(n) = T(n/2) + T(n/4) + n^2$

$a \geq 1 \rightarrow$

$b > 1 \rightarrow$ Master Theorem does not apply.

Problem 1-13. $T(n) = 2T(n/4) + \lg n$

$a \geq 1 \rightarrow 2 \geq 1$ $\quad c(n) = \log n \cdot n^0$ $\quad$ Case 1:

$b \geq 1 \rightarrow 4 > 1$ $\quad n^0 = n^{\frac{1}{2} - \epsilon}$ $\quad T(n) = \Theta(n^{\log_b(a)})$

$\log_4 2 = \frac{1}{2}$ $\quad T(n) = \Theta(n^{\frac{1}{2}})$

Problem 1-14. $T(n) = 3T(n/3) + n \lg n$

$a \geq 1 \rightarrow 3 \geq 1$ $\quad c(n) = n \log n = n^1$ $\quad$ Case 2:

$b > 1 \rightarrow 3 > 1$ $\quad T(n) = \Theta(n^{\log_b(a)} \log(n))$

$n^1 = n^1$

$\log_3 3 = 1$ $\quad T(n) = \Theta(n \log(n))$

Problem 1-15. $T(n) = 8T((n - \sqrt{n})/4) + n^2$

$T\left(\frac{n}{4} - \frac{\sqrt{n}}{4}\right) + n^2$

Master Theorem doesn't apply?

Problem 1-16. $T(n) = 2T(n/4) + \sqrt{n}$

$a \geq 1 \rightarrow 2 \geq 1$
$b > 1 \rightarrow 4 > 1$

$\log_4 2 = \frac{1}{2}$

$n^{\frac{1}{2}} = n^{\frac{1}{2}}$

Case 2:
$T(n) = \Theta(n^{\log_b a} \log(n))$
$T(n) = \Theta(\sqrt{n} \log(n))$

Problem 1-17. $T(n) = 2T(n/4) + n^{0.51}$

$a \geq 1 \rightarrow 2 \geq 1$
$b > 1 \rightarrow 4 > 1$

$\log_4 2 = \frac{1}{2}$

$n^{0.51} = n^{\frac{1}{2}+\epsilon}$

Case 3:
$T(n) = \Theta(c(n))$
$T(n) = \Theta(n^{0.51})$

Problem 1-18. $T(n) = 16T(n/4) + n!$

$a \geq 1 \rightarrow 16 \geq 1$
$b > 1 \rightarrow 4 > 1$

$\log_4 16 = 2$

$n! \rightarrow n^1 (n-1) \cdot (n-2) \cdots$
$n^1 = n^{2-\epsilon}$

Case 1:
$T(n) = \Theta(n^{\log_b(a)})$
$T(n) = \Theta(n^2)$

Problem 1-19. $T(n) = 3T(n/2) + n$

$a \geq 1 \rightarrow 3 \geq 1$
$b > 1 \rightarrow 2 > 1$

$\log_2 3 = 1.58\ldots$

$n^1 = n^{1.58-\epsilon}$

Case 1:
$T(n) = \Theta(n^{\log_b a})$
$T(n) = \Theta(n^{1.58\ldots})$

Problem 1-20. $T(n) = 4T(n/2) + cn$

$a \geq 1 \rightarrow 4 \geq 1$
$b > 1 \rightarrow 2 > 1$

$\log_2 4 = 2$

$cn \Rightarrow n^1$
$n^1 = n^{2-\epsilon}$

Case 1:
$T(n) = \Theta(n^{\log_b a})$
$T(n) = \Theta(n^2)$

Problem 1-21. $T(n) = 3T(n/3) + n/2$

$a \geq 1 \rightarrow 3 \geq 1$
$b > 1 \rightarrow 3 > 1$

$\log_3 3 = 1$

$\frac{n}{2} = \frac{1}{2} n^1$
$n^1 = n^1$

Case 2:
$T(n) = \Theta(n^{\log_b(a)} \log(n))$
$T(n) = \Theta(n \log(n))$

Problem 1-22. $T(n) = 4T(n/2) + n/\lg n$

$a \geq 1 \Rightarrow 4 \geq 1$
$b > 1 \rightarrow 2 > 1$

$\log_2 4 = 2$

$n \cdot \frac{1}{\log n} = n^1$
$n^1 = n^{2-\epsilon}$

Case 1:
$T(n) = \Theta(n^{\log_b a})$
$T(n) = \Theta(n^2)$

Problem 1-23. $T(n) = 7T(n/3) + n^2$

$a \geq 1 \rightarrow 7 \geq 1$ $\quad n^2 = n^{1.77+\epsilon}$

$b > 1 \rightarrow 3 > 1$

$\log_3 7 = 1.77$

Case 3:

$T(n) = \Theta(c(n))$

$T(n) = \Theta(n^2)$

Problem 1-24. $T(n) = 8T(n/3) + 2^n$

$a \geq 1 \rightarrow 8 \geq 1$ $\quad n^0 \cdot 2^n$

$b > 1 \rightarrow 3 > 1$ $\quad n^0 = n^{1.89-\epsilon}$

$\log_3 8 = 1.89$

Case 1:

$T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{1.89-\epsilon})$

Problem 1-25. $T(n) = 16T(n/4) + n$

$a \geq 1 \rightarrow 16 \geq 1$ $\quad n^1 = n^{2-\epsilon}$

$b > 1 \rightarrow 4 > 1$

$\log_4 16 = 2$

Case 1:

$T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^2)$