Contents

1	Relations and Functions	01 - 05
2	Inverse Trigonometric Functions	06 -08
3	Matrices	11 - 14
4	Determinants	15 -21
5	Continuity and Differentiability	22 - 28
6	Application of Differentiation	29 - 30
7	Integration	31 - 36
8	Application of Integration	37 - 38
9	Differential Equations	39 -39
10	Vectors	40 -44
11	Three Dimensional Geometry	45 - 49
12	Probability	50 -52
13	Linear Programming	53 - 58
14	List of Formulae	59 - 68

RELATIONS AND FUNCTIONS

TWO MARKS QUESTION

1. If $f: R \to R$ and $g: R \to R$ are the functions defined by $f(x) = \cos x$ and $g(x) = 3x^2$ then show that $fog \neq gof$

Solutions: Now, $fog(x) = f[g(x)]$

$= f[3x^2] = \cos(3x^2) \dots\dots\dots (1)$

Now, $gof(x) = g(f(x))$

$= g[\cos x] = 3\cos^2 x \dots\dots (2)$

From (1) and (2)

$\cos 3x^2 \neq 3\cos^2 x \Rightarrow fog \neq gof$

2. If $f(x) = 8x^3$ and $g(x) = x^{1/3}$ then find $fog(x)$ and $gof(x)$

Solutions: Now, $fog(x) = f[g(x)]$

$= f[x^{1/3}] = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$

Now, $gof(x) = g(f(x))$

$= g[8x^3] = (8x^3)^{1/3} = ((2x)^3)^{1/3} = 2x$

3. If $f: R \to R$ defined by $f(x) = (3 - x^3)^{\frac{1}{3}}$ then prove that $fof(x) = x$.

Solutions: $LHS = fof(x) = f[f(x)]$

$= f\left[(3 - x^3)^{\frac{1}{3}}\right] = \left(3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right)^{\frac{1}{3}}$

$= (3 - (3 - x^3))^{\frac{1}{3}} = (3 - 3 + x^3)^{\frac{1}{3}}$

$= (x^3)^{1/3} = x = RHS$

4. If $f: R \to R$ and $g: R \to R$ are the functions defined by $f(x) = |x|$, $g(x) = [x]$ then find $fog\left(-\frac{1}{2}\right)$ & $gof\left(-\frac{1}{2}\right)$.

Solutions : Now, $fog\left(-\frac{1}{2}\right) = f\left[g\left(-\frac{1}{2}\right)\right]$

$= f\left(\left[-\frac{1}{2}\right]\right) = f([-0.5]) = f(-1)$

$= |-1| = 1$

Now, $gof\left(-\frac{1}{2}\right) = g\left(f\left(-\frac{1}{2}\right)\right)$

$= g\left[\left|-\frac{1}{2}\right|\right] = g\left(\frac{1}{2}\right) = g(0.5) = [0.5] = 0$

THREE MARKS QUESTION

1. Determine whether the relation R in the set $A = \{1, 2, 3.....14\}$ defined as $R = \{(x, y) | 3x - y = 0\}$ is reflexive symmetric and transitive.

Solutions: Given , $R = \{(x, y) | 3x - y = 0\}$

$\Rightarrow R = \{(x, y) | y = 3x\}$

Given: $y = 3x$

If $x = 1 \Rightarrow y = 3(1) = 3 \in A$

If $x = 2 \Rightarrow y = 3(2) = 6 \in A$

If $x = 3 \Rightarrow y = 3(3) = 9 \in A$

If $x = 4 \Rightarrow y = 3(4) = 12 \in A$

If $x = 5 \Rightarrow y = 3(5) = 15 \notin A \dots\dots$

Now , $R = \{(1,3), (2,6), (3,9), (4,12)\}$

Reflexive : Here $(1,1) \notin R$

$\Rightarrow$ R is not reflexive.

Symmetric : Here $(1,3) \in R$ but $(3,1) \notin R$

$\Rightarrow$ R is not symmetric.

Transitive : Here $(1,3), (3,9) \in R$ but $(1,9) \notin R \Rightarrow$ R is not transitive.

2. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) | b = a + 1\}$ is reflexive or symmetric or transitive.

Solutions: Let $A = \{1,2,3,4,5,6\}$

Given , $R = \{(a, b) | b = a + 1\}$

Given: $b = a + 1$

If $a = 1 \Rightarrow b = 1 + 1 = 2 \in A$

If $a = 2 \Rightarrow b = 2 + 1 = 3 \in A$

If $a = 3 \Rightarrow b = 3 + 1 = 4 \in A$

If $a = 4 \Rightarrow b = 4 + 1 = 5 \in A$

If $a = 5 \Rightarrow b = 5 + 1 = 6 \in A$

If $a = 6 \Rightarrow b = 6 + 1 = 7 \notin A$

Now , $R = \{(1,2), (2,3), (3,4), (4,5), (5,6), \}$

Reflexive : Here $(1,1) \notin R$

$\Rightarrow$ R is not reflexive.

Symmetric : Here $(1,2) \in R$ but $(2,1) \notin R$

$\Rightarrow$ R is not symmetric.

Transitive : Here $(1,2), (2,3) \in R$ but $(1,3) \notin R \Rightarrow$ R is not transitive.

3. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a,b) | \, |a-b| \text{ is even}\}$ is an equivalence relation.

Solutions: Now, $R = \{(a,b) | \, |a-b| \text{ is even}\}$

Reflexive: Let $a \in A$

We know that $|0|$ is even $\Rightarrow |a-a|$ is even

$\Rightarrow (a,a) \in R, \forall\, a \in A \Rightarrow$ R is reflexive

Symmetric: Let $a, b \in A$ such that

$(a,b) \in R \Rightarrow |a-b|$ is even

$\Rightarrow |-(a-b)|$ is even $\Rightarrow |b-a|$ is even

$\Rightarrow (b,a) \in R \Rightarrow$ R is symmetric

Transitive: Let $a, b, c \in A$

such that $(a,b), (b,c) \in R$

$\Rightarrow |a-b|$ and $|b-c|$ is even

$\Rightarrow a-b$ and $b-c$ is even

$\Rightarrow a-b+b-c$ is even

$\Rightarrow a-c$ is even $\Rightarrow |a-c|$ is even

$\Rightarrow (a,c) \in R \Rightarrow$ R is transitive

Since R is reflexive symmetric and transitive

$\Rightarrow$ R is an equivalence relation.

4. Prove that the relation R in the set of integers Z defined by $R = \{x,y) | \, x-y \text{ is an integer}\}$ is an equivalence relation.

Solutions:

Given : $R = \{(x,y) | x-y \text{ is an integer}\}$

Reflexive: Let $x \in Z$

We know that 0 is an integer

$\Rightarrow x-x$ is an integer

$\Rightarrow (x,x) \in R, \forall\, x \in Z \Rightarrow$ R is reflexive

Symmetric: Let $x, y \in Z$ such that $(x,y) \in R$

$\Rightarrow x-y$ is an integer

$\Rightarrow -(x-y)$ is an integer

$\Rightarrow y-x$ is an integer

$\Rightarrow (y,x) \in R \Rightarrow$ R is symmetric

Transitive: Let $x, y, z \in Z$

such that $(x,y), (y,z) \in R$

$\Rightarrow x-y$ and $y-z$ is an integer

$\Rightarrow x-y+y-z$ is an integer

$\Rightarrow x-z$ is an integer $\Rightarrow (x,z) \in R$

$\Rightarrow$ R is transitive

Since R is reflexive symmetric and transitive

$\Rightarrow$ R is an equivalence relation.

5. Show that the relation R in the set z of integer given by $R = \{(x,y) | \; 2 \text{ divides } x-y\}$ is an equivalence relation

Solutions: Now, $R = \{(x,y) | 2 \; divides \; x-y\}$

$\Rightarrow R = \{(x,y) : 2 \mid x-y\}$

Reflexive: Let $x \in Z$

We know that $2 \; divides \; (x-x)$

$\Rightarrow (x,x) \in R, \forall\, x \in Z \Rightarrow$ R is reflexive

Symmetric: Let $x, y \in Z$

such that $(x,y) \in R \Rightarrow 2 \; divides \; x-y$

$\Rightarrow 2 \; divides - (x-y) \Rightarrow 2 \; divides \; y-x$

$\Rightarrow (y,x) \in R \Rightarrow$ R is symmetric

Transitive: Let $x, y, z \in Z$

such that $(x,y), (y,z) \in R$

$\Rightarrow 2 \; divides \; x-y$ and $2 \; divides \; y-z$

$\Rightarrow 2 \; divides \; x-y+y-z$

$\Rightarrow 2 \; divides \; x-z \Rightarrow (x,z) \in R$

$\Rightarrow$ R is transitive

Since R is reflexive symmetric and transitive

$\Rightarrow$ R is an equivalence relation.

6. Show that the relation R in the set $A = \{x/x \in z, 0 \le x \le 12\}$ is given by $R = \{(a,b) | \, |a-b| \text{ is a multiple of } 4\}$ is an equivalence relation.

Solutions:

Given , $R = \{(a,b) | \, |a-b| \text{ is a multiple of } 4\}$

Reflexive: Let $a \in A$

We know that

$\Rightarrow |a-a| = 0$ is a multiple of 4

$\Rightarrow (a,a) \in R, \forall\, a \in A \Rightarrow$ R is reflexive

Symmetric: Let $a, b \in A$

such that $(a,b) \in R$

$\Rightarrow |a-b|$ is a multiple of 4

$\Rightarrow |-(a-b)|$ is a multiple of 4

$\Rightarrow |b-a|$ is a multiple of 4

$\Rightarrow (b,a) \in R \Rightarrow$ R is symmetric

Transitive: Let $a, b, c \in A$

such that $(a,b), (b,c) \in R$

$\Rightarrow |a-b|$ and $|b-c|$ is a multiple of 4

$\Rightarrow a-b$ and $b-c$ is a multiple of 4

$\Rightarrow a-b+b-c$ is a multiple of 4

$\Rightarrow a-c$ is a multiple of 4

$\Rightarrow |a-c|$ is a multiple of 4

$\Rightarrow (a,c) \in R \Rightarrow$ R is transitive

Since R is reflexive symmetric and transitive

$\Rightarrow$ R is an equivalence relation.

7. Show that the relation R in R is defined as $R = \{a, b)/ a \leq b\}$ is reflexive and transitive but not symmetric.

Solutions: Given , $R = \{(a, b)/ a \leq b\}$

Reflexive: Let $a \in R$

We know that $a = a$

$\Rightarrow (a, a) \in R, \forall a \in R \Rightarrow$ R is reflexive

Symmetric: Let $1, 2 \in R$

such that $(1,2) \in R$ *because* $1 < 2$

Now, $2 > 1 \Rightarrow 2 \nleq 1 \Rightarrow (2,1) \notin R$

$\Rightarrow$ R is not symmetric

Transitive: Let $a, b, c \in R$

such that $(a, b), (b, c) \in R$

$\Rightarrow a \leq b \;\&\; b \leq c \Rightarrow a \leq c$

$\Rightarrow (a, c) \in R \Rightarrow$ R is transitive

$\therefore$ R is reflexive and transitive but not symmetric.

8. Show that the relation R in the set of real number R defined as $R = \{(a, b) / a \leq b^2\}$ is neither reflexive nor symmetric nor transitive

Solutions: Given , $R = \{(a, b)/ a \leq b^2\}$

Reflexive: Now $\frac{1}{2} \in R$ and $\frac{1}{2} > \left(\frac{1}{2}\right)^2$

$\Rightarrow \frac{1}{2} \nleq \left(\frac{1}{2}\right)^2 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \notin R,$

$\Rightarrow$ R is not reflexive

Symmetric: Now, $1 < 2^2 \Rightarrow (1,2) \in R$

but $2 > 1^2 \Rightarrow 2 \nleq 1^2 \Rightarrow (2,1) \notin R$

$\Rightarrow$ R is not symmetric

Transitive : Now, $8 < 3^2 \Rightarrow (8,3) \in R$

and $3 < 2^2 \Rightarrow (3,2) \in R$

But $8 > 2^2 \Rightarrow 8 \nleq 2^2 \Rightarrow (8,2) \notin R$

$\Rightarrow$ R is not transitive

$\therefore$ R is neither reflexive nor symmetric nor transitive

9. Show that the relation R in the set of real number R defined as $R = \{(a, b) / a \leq b^3\}$ is neither reflexive nor symmetric nor transitive

Solutions: Given , $R = \{(a, b)/ a \leq b^3\}$

Reflexive: Now $\frac{1}{2} \in R$ and $\frac{1}{2} > \left(\frac{1}{2}\right)^3$

$\Rightarrow \frac{1}{2} \nleq \left(\frac{1}{2}\right)^3 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \notin R,$

$\Rightarrow$ R is not reflexive

Symmetric: Now, $1 < 2^3 \Rightarrow (1,2) \in R$

but $2 > 1^3 \Rightarrow 2 \nleq 1^3 \Rightarrow (2,1) \notin R$

$\Rightarrow$ R is not symmetric

Transitive : Now, $10 < 3^3 \Rightarrow (10,3) \in R$

and $3 < 2^3 \Rightarrow (3,2) \in R$

But $10 > 2^3 \Rightarrow 10 \nleq 2^3 \Rightarrow (10,2) \notin R$

$\Rightarrow$ R is not transitive

$\therefore$ R is neither reflexive nor symmetric nor transitive

FIVE MARKS MARKS QUESTION

1. Let $f: R \to R$ be given by $f(x) = 4x + 3$. Show that f is invertible and find the inverse of f.

Solution: Given $f: R \to R$ is defined as $f(x) = 4x + 3$

Let $f(x) = y \Rightarrow 4x + 3 = y$

$\Rightarrow 4x = y - 3 \Rightarrow x = \frac{y-3}{4}$

Let $g: R \to R$ is defined as $g(x) = \frac{x-3}{4}$

Now, $fog(x) = f(g(x)) = f\left(\frac{x-3}{4}\right)$

$\Rightarrow fog = 4\left(\frac{x-3}{4}\right) + 3 = x - 3 + 3 = x$

$\Rightarrow fog = I_R$

Now, $gof(x) = g(f(x)) = g(4x + 3)$

$\Rightarrow gof = \frac{4x+3-3}{4} = \frac{4x}{4} = x$

$\Rightarrow gof = I_R$

Now, $fog = I_R$ and $gof = I_R \Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \frac{x-3}{4}$

2. Prove that the function $f: N \to Y$ defined by $f(x) = x^2$ where $Y = \{y \mid y = x^2, x \in N\}$ is invertible. Also find the inverse of f.

Solution: Given $f: N \to Y$ is defined as $f(x) = x^2$

Let $f(x) = y \Rightarrow x^2 = y \Rightarrow x = \sqrt{y}$

Let $g: Y \to N$ is defined as $g(x) = \sqrt{x}$

Now, $fog(x) = f(g(x)) = f(\sqrt{x})$

$\Rightarrow fog = (\sqrt{x})^2 = x \Rightarrow fog = I_y$

Now, $gof(x) = g(f(x)) = g(x^2)$

$\Rightarrow gof = \sqrt{x^2} = x \Rightarrow gof = I_N$

Now, $fog = I_Y$ and $gof = I_N$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \sqrt{x}$

3. Consider $f: R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$ where R_+ is the set of all non-negative real numbers.

Solution: Given $f: R_+ \to [4, \infty)$ is defined as $f(x) = x^2 + 4$

Let $f(x) = y \Rightarrow x^2 + 4 = y$

$\Rightarrow x^2 = y - 4 \Rightarrow x = \sqrt{y-4}$

Let $g: [4, \infty) \to R_+$ is defined as

$g(x) = \sqrt{x-4}$

Now, $fog(x) = f(g(x)) = f(\sqrt{x-4})$

$\Rightarrow fog = (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x$

$\Rightarrow fog = I_{[4,\infty)}$

Now, $gof(x) = g(f(x)) = g(x^2+4)$

$\Rightarrow gof = \sqrt{x^2+4-4} = \sqrt{x^2} = x$

$\Rightarrow gof = I_{R_+}$

Now, $fog = I_{[4,\infty)}$ and $gof = I_{R_+}$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \sqrt{x-4}$

4. Show that $f: [-1, 1] \to R$, given by $f(x) = \frac{x}{x+2}$ is invertible and also find the inverse of the function $f: [-1, 1] \to$ range of f.

Solution: Given $f: [-1,1] \to R$ is defined as $f(x) = \frac{x}{x+2}$

Let $f(x) = y \Rightarrow \frac{x}{x+2} = y \Rightarrow x = y(x+2)$

$\Rightarrow x = xy + 2y \Rightarrow x - xy = 2y$

$\Rightarrow x(1-y) = 2y \Rightarrow x = \frac{2y}{1-y}$

Let $g: R \to [-1,1]$ is defined as $g(x) = \frac{2x}{1-x}$

Now, $fog(x) = f(g(x)) = f\left(\frac{2x}{1-x}\right)$

$\Rightarrow fog = \frac{\frac{2x}{1-x}}{\frac{2x}{1-x}+2} = \frac{\frac{2x}{1-x}}{\frac{2x+2(1-x)}{1-x}} = \frac{2x}{2x+2-2x} = \frac{2x}{2} = x$

$\Rightarrow fog = I_R$

Now, $gof(x) = g(f(x)) = g\left(\frac{x}{x+2}\right)$

$\Rightarrow gof = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{\frac{2x}{x+2}}{\frac{x+2-x}{x+2}} = \frac{2x}{2} = x$

$\Rightarrow gof = I_{[-1,1]}$

Now, $fog = I_R$ and $gof = I_{[-1,1]}$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \frac{2x}{1-x}$

5. Let $f: R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ be a function defined as $f(x) = \frac{4x}{3x+4}$ then prove that f is invertible and also find f^{-1}.

Solution: Given $f: R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ is defined as $f(x) = \frac{4x}{3x+4}$

Let $f(x) = y \Rightarrow \frac{4x}{3x+4} = y$

$\Rightarrow 4x = y(3x+4) \Rightarrow 4x = 3xy + 4y$

$\Rightarrow 4x - 3xy = 4y \Rightarrow x(4-3y) = 4y$

$\Rightarrow x = \frac{4y}{4-3y}$

Let $g: R - \left\{\frac{4}{3}\right\} \to R - \left\{-\frac{4}{3}\right\}$ is defined as $g(x) = \frac{4x}{4-3x}$

Now, $fog(x) = f(g(x))$

$\Rightarrow fog = f\left(\frac{4x}{4-3x}\right) = \frac{4\left(\frac{4x}{4-3x}\right)}{3\left(\frac{4x}{4-3x}\right)+4} = \frac{\frac{16x}{4-3x}}{\frac{12x+4(4-3x)}{4-3x}}$

$\Rightarrow fog = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$

$\Rightarrow fog = I_{R-\left\{\frac{4}{3}\right\}}$

Now, $gof(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$

$\Rightarrow gof = \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{\frac{16x}{3x+4}}{\frac{4(3x+4)-12x}{3x+4}} = \frac{16x}{12x+16-12x}$

$\Rightarrow gof = \frac{16x}{16} = x \Rightarrow gof = I_{R-\left\{-\frac{4}{3}\right\}}$

Now, $fog = I_{R-\left\{\frac{4}{3}\right\}}$ and $gof = I_{R-\left\{-\frac{4}{3}\right\}}$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \frac{4x}{4-3x}$

6. If $f: R - \left\{-\frac{5}{2}\right\} \to R - \left\{\frac{3}{2}\right\}$ be the function defined by $f(x) = \frac{3x-5}{2x+5}$ then show that f is invertible & also find f^{-1}.

Solution: Given $f: R - \left\{-\frac{5}{2}\right\} \to R - \left\{\frac{3}{2}\right\}$ is defined as $f(x) = \frac{3x-5}{2x+5}$

Let $f(x) = y \Rightarrow \frac{3x-5}{2x+5} = y$

$\Rightarrow 3x - 5 = y(2x+5)$

$\Rightarrow 3x - 5 = 2xy + 5y$

$\Rightarrow 3x - 2xy = 5y + 5$

$\Rightarrow x(3-2y) = 4y \Rightarrow x = \frac{5y+5}{3-2y}$

Let $g: R - \left\{\frac{3}{2}\right\} \to R - \left\{-\frac{5}{2}\right\}$ is defined as $g(x) = \frac{5x+5}{3-2x}$

Now, $fog(x) = f(g(x)) = f\left(\frac{5x+5}{3-2x}\right)$

$\Rightarrow fog = \frac{3\left(\frac{5x+5}{3-2x}\right)-5}{2\left(\frac{5x+5}{3-2x}\right)+5} = \frac{\frac{3(5x+5)-5(3-2x)}{3-2x}}{\frac{2(5x+5)+5(3-2x)}{3-2x}}$

$\Rightarrow fog = \frac{15x+15-15+10x}{10x+10+15-10x} = \frac{25x}{25} = x$

$\Rightarrow fog = I_{R-\left\{\frac{3}{2}\right\}}$

Now, $gof(x) = g(f(x)) = g\left(\frac{3x-5}{2x+5}\right)$

$\Rightarrow gof = \frac{5\left(\frac{3x-5}{2x+5}\right)+5}{3-2\left(\frac{3x-5}{2x+5}\right)} = \frac{\frac{5(3x-5)+5(2x+5)}{2x+5}}{\frac{3(2x+5)-2(3x-5)}{2x+5}}$

$\Rightarrow gof = \frac{15x-25+10x+25}{6x+15-6x+10} = \frac{25x}{25} = x$

$\Rightarrow gof = I_{R-\left\{-\frac{5}{2}\right\}}$

Now, $fog = I_{R-\left\{\frac{3}{2}\right\}}$ and $gof = I_{R-\left\{-\frac{5}{2}\right\}}$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \frac{5x+5}{3-2x}$

7. Let $f: N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$ for some x in N, show that $f: N \to S$, where S is the range of f, is invertible. Find the inverse of f.

Solution: Given $f: N \to S$ is defined as $f(x) = 4x^2 + 12x + 15$

$\Rightarrow f(x) = (2x)^2 + 2(2x)(3) + 3^2 - 3^2 + 15$

$\Rightarrow f(x) = (2x+3)^2 - 9 + 15$

$\Rightarrow f(x) = (2x+3)^2 + 6$

Let $f(x) = y \Rightarrow (2x+3)^2 + 6 = y$

$\Rightarrow (2x+3)^2 = y - 6$

$\Rightarrow 2x + 3 = \sqrt{y-6} \Rightarrow 2x = \sqrt{y-6} - 3$

$\Rightarrow x = \frac{\sqrt{y-6}-3}{2}$

Let $g: S \to N$ is defined as $g(x) = \frac{\sqrt{x-6}-3}{2}$

Now, $fog(x) = f(g(x)) = f\left(\frac{\sqrt{x-6}-3}{2}\right)$

$\Rightarrow fog = \left(2\left(\frac{\sqrt{x-6}-3}{2}\right)+3\right)^2 + 6$

$\Rightarrow fog = \left(\sqrt{x-6} - 3 + 3\right)^2 + 6$

$\Rightarrow fog = \left(\sqrt{x-6}\right)^2 + 6 = x - 6 + 6 = x$

$\Rightarrow fog = I_S$

Now, $gof(x) = g(f(x))$

$\Rightarrow gof = g((2x+3)^2 + 6) = \frac{\sqrt{(2x+3)^2+6-6}-3}{2}$

$\Rightarrow gof = \frac{\sqrt{(2x+3)^2}-3}{2} = \frac{2x+3-3}{2} = \frac{2x}{2} = x$

$\Rightarrow gof = I_N$

Now, $fog = I_S$ and $gof = I_N$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \frac{\sqrt{x-6}-3}{2}$

8. Consider $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$.

Solution: Given $f: R_+ \to [-5, \infty)$ is defined as $f(x) = 9x^2 + 6x - 5$

$\Rightarrow f(x) = (3x)^2 + 2(3x)(1) + 1^2 - 1^2 - 5$

$\Rightarrow f(x) = (3x+1)^2 - 1 - 5$

$\Rightarrow f(x) = (3x+1)^2 - 6$

Let $f(x) = y \Rightarrow (3x+1)^2 - 6 = y$

$\Rightarrow (3x+1)^2 = y + 6$

$\Rightarrow 3x + 1 = \sqrt{y+6}$

$\Rightarrow 3x = \sqrt{y+6} - 1 \Rightarrow x = \frac{\sqrt{y+6}-1}{3}$

Let $g: [-5, \infty) \to R_+$ is defined as $g(x) = \frac{\sqrt{x+6}-1}{3}$

Now, $fog(x) = f(g(x)) = f\left(\frac{\sqrt{x+6}-1}{3}\right)$

$\Rightarrow fog(x) = \left(3\left(\frac{\sqrt{x+6}-1}{3}\right)+1\right)^2 - 6$

$\Rightarrow fog(x) = \left(\sqrt{x+6} - 1 + 1\right)^2 - 6$

$\Rightarrow fog(x) = \left(\sqrt{x+6}\right)^2 - 6 = x + 6 - 6 = x$

$\Rightarrow fog = I_{[-5,\infty)}$

Now, $gof(x) = g(f(x))$

$\Rightarrow gof(x) = g((3x+1)^2 - 6)$

$\Rightarrow gof(x) = \frac{\sqrt{(3x+1)^2-6+6}-1}{3}$

$\Rightarrow gof(x) = \frac{\sqrt{(3x+1)^2}-1}{3}$

$\Rightarrow gof(x) = \frac{3x+1-1}{3} = \frac{3x}{3} = x$

$\Rightarrow gof = I_{R_+}$

Now, $fog = I_{[-5,\infty)}$ and $gof = I_{R_+}$

$\Rightarrow f(x)$ is invertible and $f^{-1}(x)$ exist

$\Rightarrow f^{-1}(x) = g(x) = \frac{\sqrt{x+6}-1}{3}$

$\Rightarrow f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$

INVERSE TRIGONOMETRIC FUNCTIONS

TWO MARKS QUESTION

1. **Prove that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$**

   Solution: Let $y=\sin^{-1}x \ldots\ldots\ldots (1)$

   $\Rightarrow \sin y = x \Rightarrow \cos\left(\frac{\pi}{2}-y\right)=x$

   $\Rightarrow \frac{\pi}{2}-y=\cos^{-1}x \Rightarrow \frac{\pi}{2}=y+\cos^{-1}x$

   $\Rightarrow \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ from (1)

2. **Prove that $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$**

   Solution: Let $y=\tan^{-1}x \ldots\ldots\ldots (1)$

   $\Rightarrow \tan y = x \Rightarrow \cot\left(\frac{\pi}{2}-y\right)=x$

   $\Rightarrow \frac{\pi}{2}-y=\cot^{-1}x \Rightarrow \frac{\pi}{2}=y+\cot^{-1}x$

   $\Rightarrow \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ from (1)

3. **Prove that $\sec^{-1}x+\mathrm{cosec}^{-1}x=\frac{\pi}{2}$**

   Solution: Let $y=\sec^{-1}x \ldots\ldots\ldots (1)$

   $\Rightarrow \sec y = x \Rightarrow \mathrm{cosec}\left(\frac{\pi}{2}-y\right)=x$

   $\Rightarrow \frac{\pi}{2}-y=\mathrm{cosec}^{-1}x$

   $\Rightarrow \frac{\pi}{2}=y+\mathrm{cosec}^{-1}x$

   $\Rightarrow \sec^{-1}x+\mathrm{cosec}^{-1}x=\frac{\pi}{2}$ from (1)

4. **Prove that $\sin^{-1}(x)=cosec^{-1}\left(\frac{1}{x}\right)$**

   Solution: Let $y=\sin^{-1}x \ldots\ldots (1)$

   $\Rightarrow siny=x \Rightarrow \frac{1}{siny}=\frac{1}{x}$

   $\Rightarrow cosec\, y=\frac{1}{x} \Rightarrow y=cosec^{-1}\left(\frac{1}{x}\right)$

   $\Rightarrow \sin^{-1}x=cosec^{-1}\left(\frac{1}{x}\right)$ from (1)

5. **Prove that $\cos^{-1}(x)=sec^{-1}\left(\frac{1}{x}\right)$**

   Solution: Let $y=\cos^{-1}x \ldots\ldots (1)$

   $\Rightarrow \cos y=x \Rightarrow \frac{1}{\cos y}=\frac{1}{x}$

   $\Rightarrow sec\, y=\frac{1}{x} \Rightarrow y=sec^{-1}\left(\frac{1}{x}\right)$

   $\Rightarrow \cos^{-1}x=sec^{-1}\left(\frac{1}{x}\right)$ from (1)

6. **Prove that $\tan^{-1}(x)=cot^{-1}\left(\frac{1}{x}\right)$**

   Solution: Let $y=\tan^{-1}x \ldots\ldots (1)$

   $\Rightarrow \tan y=x \Rightarrow \frac{1}{\tan y}=\frac{1}{x}$

   $\Rightarrow cot\, y=\frac{1}{x} \Rightarrow y=\cot^{-1}\left(\frac{1}{x}\right)$

   $\Rightarrow \tan^{-1}x=\cot^{-1}\left(\frac{1}{x}\right)$ from (1)

7. **Prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$, $-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$**

   Solution: $Let\ x=sin\theta \Rightarrow \theta=\sin^{-1}x$

   $\text{LHS}=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

   $=\sin^{-1}\left(2sin\theta\sqrt{1-\sin^2\theta}\right)$

   $=\sin^{-1}\left(2sin\theta\sqrt{\cos^2\theta}\right)$

   $=\sin^{-1}(2sin\theta.cos\theta)=\sin^{-1}(sin2\theta)$

   $=2\theta=2\sin^{-1}x=RHS$

8. **Prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\cos^{-1}x$, $\frac{1}{\sqrt{2}}\le x\le 1$**

   Solution: $Let\ x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

   $\text{LHS}=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

   $=\sin^{-1}\left(2cos\theta\sqrt{1-\cos^2\theta}\right)$

   $=\sin^{-1}\left(2cos\theta\sqrt{\sin^2\theta}\right)$

   $=\sin^{-1}(2cos\theta.sin\theta)=\sin^{-1}(sin2\theta)$

   $=2\theta=2\cos^{-1}x=RHS$

9. **Prove that $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$, $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$**

   Solution: $Let\ x=sin\theta \Rightarrow \theta=\sin^{-1}x$

   $RHS=\sin^{-1}(3x-4x^3)$

   $=\sin^{-1}(3sin\theta-4\sin^3\theta)$

   $=\sin^{-1}(sin3\theta)=3\theta=3\sin^{-1}x=LHS$

10. **Prove that $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$, $x\in\left[\frac{1}{2},1\right]$**

    Solution: $Let\ x=cos\theta \Rightarrow \theta=\cos^{-1}x$

    $RHS=\cos^{-1}(4x^3-3x)$

    $=\cos^{-1}(4\cos^3\theta-3cos\theta)$

    $=\cos^{-1}(cos3\theta)=3\theta=3\cos^{-1}x=LHS$

11. **Write $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x\ne 0$ in the simplest form.**

    Solution: Let x=$tan\theta \Rightarrow \theta=\tan^{-1}x$

    Now, $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{x}\right)$

    $=\tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{tan\theta}\right)$

$= \tan^{-1}\left(\frac{sec\theta-1}{tan\theta}\right) = \tan^{-1}\left(\frac{\frac{1}{cos\theta}-1}{sin\theta/cos\theta}\right)$

$= \tan^{-1}\left(\frac{\frac{1-cos\theta}{cos\theta}}{\frac{sin\theta}{cos\theta}}\right) = \tan^{-1}\left(\frac{1-cos\theta}{sin\theta}\right)$

$= \tan^{-1}\left(\frac{sin\theta/2}{cos\theta/2}\right) = \tan^{-1}(tan\theta/2)$

$= \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$

12. Write $\tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right), 0 < x < \pi$ in the simplest form.

Solution:

$\tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}\right) = \tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right)$

$= \tan^{-1}(\tan x/2) = \frac{x}{2}$

13. Write $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right), 0 < x < \pi$ in the simplest form.

Solution:

$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right) = \tan^{-1}\left(\frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}\right)$

$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+1(\tan x)}\right)$

$= tan^{-1}(1) - tan^{-1}(tan\, x) = \frac{\pi}{4} - x$

14. Write $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right], \frac{a}{b}\tan x > -1$ in the simplest form.

Solution: $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right) =$

$\tan^{-1}\left(\frac{\frac{a\cos x}{b\cos x}-\frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x}+\frac{a\sin x}{b\cos x}}\right) = \tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)$

$= \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x)$

$= \tan^{-1}\left(\frac{a}{b}\right) - x$

15. Write $\tan^{-1}\left[\frac{3\cos x-4\sin x}{4\cos x+3\sin x}\right], \frac{3}{4}\tan x > -1$ in the simplest form.

Solution: $\tan^{-1}\left(\frac{3\cos x-4\sin x}{4\cos x+3\sin x}\right)$

$= \tan^{-1}\left(\frac{\frac{3\cos x}{4\cos x}-\frac{4\sin x}{4\cos x}}{\frac{4\cos x}{4\cos x}+\frac{3\sin x}{4\cos x}}\right) = \tan^{-1}\left(\frac{\frac{3}{4}-\tan x}{1+\frac{3}{4}\tan x}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}(\tan x)$

$= \tan^{-1}\left(\frac{3}{4}\right) - x$

16. Write $\cot^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$ in the simplest form.

Solution: Let $x = sec\theta \Rightarrow \theta = \sec^{-1}x$

Now, $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right)$

$= \cot^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right) = \cot^{-1}\left(\frac{1}{tan\theta}\right)$

$= \cot^{-1}(cot\theta) = \theta = \sec^{-1}x$

THREE MARKS QUESTION

17. Prove that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $x \geq 0, y \geq 0, xy < 1$

Solution: Let $tan^{-1}x = A, x \geq 0$

$\Rightarrow x = \tan A, 0 \leq A < \frac{\pi}{2}$

Let $tan^{-1}y = B, y \geq 0$

$\Rightarrow y = \tan B, 0 \leq B < \frac{\pi}{2}$

Now, $\tan(A+B) = \frac{\tan A + \tan B}{1-(\tan A)(\tan B)}$

$\Rightarrow \tan(A+B) = \frac{x+y}{1-xy}, 0 \leq (A+B) < \frac{\pi}{2}$

$\Rightarrow (A+B) = \tan^{-1}\frac{x+y}{1-xy}$

$\Rightarrow tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$

18. Prove that

$\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Solution: LHS = $\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \tan^{-1}\left(\frac{\frac{x(1-x^2)+2x}{1-x^2}}{\frac{1-x^2-2x^2}{1-x^2}}\right) = \tan^{-1}\left(\frac{x-x^3+2x}{1-3x^2}\right)$

$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = RHS$

19. Prove that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution: $LHS = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\frac{1}{3}}\right) = \tan^{-1}\left(\frac{\frac{3+2}{6}}{\frac{6-1}{6}}\right)$

$= \tan^{-1}\left(\frac{5}{5}\right) = \tan^{-1}(1) = \frac{\pi}{4} = RHS$

20. Prove that $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{2}{11}\right)+\tan^{-1}\left(\frac{4}{3}\right)=\frac{\pi}{2}$

Solution:

LHS = $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{2}{11}\right)+\tan^{-1}\left(\frac{4}{3}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\frac{2}{11}}\right)+\tan^{-1}\left(\frac{4}{3}\right)$

$=\tan^{-1}\left(\frac{\frac{11+4}{22}}{\frac{22-2}{22}}\right)+\tan^{-1}\left(\frac{4}{3}\right)$

$=\tan^{-1}\left(\frac{15}{20}\right)+\tan^{-1}\left(\frac{4}{3}\right)$

$=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{4}{3}\right)$

$w.k.t\,,\tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1}x$

$=\tan^{-1}\left(\frac{3}{4}\right)+\cot^{-1}\left(\frac{3}{4}\right)=\frac{\pi}{2}=RHS$

21. If $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ then find x

Solution: $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-(2x)(3x)}\right)=\frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2}=\tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \frac{5x}{1-6x^2}=1 \Rightarrow 5x=1-6x^2$

$\Rightarrow 6x^2+5x-1=0$

$\Rightarrow 6x^2+6x-x-1=0$

$\Rightarrow 6x(x+1)-1(x+1)=0$

$\Rightarrow (x+1)(6x-1)=0$

$\Rightarrow x+1=0 \text{ or } 6x-1=0$

$\Rightarrow x=-1 \text{ or } 6x=1 \Rightarrow x=\frac{1}{6}$

Since $x=-1$ is not a solution of the given equation,

$\Rightarrow x=\frac{1}{6}$ is the only solution.

22. If $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$ then find x.

Solution: $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\frac{x+}{x+2}}\right)=\frac{\pi}{4}$

$\Rightarrow \frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}=\tan\frac{\pi}{4}$

$\Rightarrow \frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-2^2-(x^2-1^2)}=1$

$\Rightarrow \frac{2x^2-4}{x^2-4-x^2+1}=1 \Rightarrow \frac{2x^2-4}{-3}=1$

$\Rightarrow 2x^2-4=-3 \Rightarrow 2x^2=-3+4$

$\Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2}\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

23. If $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ then find x

Solution: $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$

$\Rightarrow 1-x=\sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)$

$\Rightarrow 1-x=\cos(2\sin^{-1}x)$

$\Rightarrow 1-x=1-2\sin^2(\sin^{-1}x)$

$\Rightarrow -x=-2[sin(\sin^{-1}x)]^2$

$\Rightarrow x=2x^2 \Rightarrow x-2x^2=0$

$\Rightarrow x(1-2x)=0$

$\Rightarrow x=0 \text{ or } 1-2x=0$

$\Rightarrow x=0 \text{ or } 2x=1 \Rightarrow x=0 \text{ or } x=\frac{1}{2}$

Since $x=\frac{1}{2}$ is not a solution of the given equation, $\Rightarrow x=0$ is the only solution.

MATRICES

THREE MARKS QUESTION

1. **Find the inverse of $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ using elementary operations.**

Solution: Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

W.K.T , $A = IA$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$,

Now $R_2 \longrightarrow R_2 - 2R_1$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2-2 & -1-4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0-2 & 1-0 \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A$,

Now $R_2 \longrightarrow \frac{R_2}{-5}$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ \frac{0}{-5} & -\frac{5}{-5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{-5} & \frac{1}{-5} \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} A$,

Now $R_1 \longrightarrow R_1 - 2R_2$

$\Rightarrow \begin{pmatrix} 1-0 & 2-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{4}{5} & 0+\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} A$

$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}$

2. **Find the inverse of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ using elementary operations.**

Solution: Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

W.K.T , $A = IA$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$,

Now $R_2 \longrightarrow R_2 - 2R_1$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 2-2 & 3+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0-2 & 1-0 \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A$,

Now $R_2 \longrightarrow \frac{R_2}{5}$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ \frac{0}{5} & \frac{5}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$,

Now $R_1 \longrightarrow R_1 + R_2$

$\Rightarrow \begin{pmatrix} 1+0 & -1+1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{2}{5} & 0+\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$

$\Rightarrow A^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$

3. **Find the inverse of $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ using elementary operations.**

Solution: Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

W.K.T , $A = IA$

$\Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$,

Now $R_2 \longrightarrow R_2 - 2R_1$

$\Rightarrow \begin{pmatrix} 1 & 3 \\ 2-2 & 7-6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0-2 & 1-0 \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A$,

Now $R_1 \longrightarrow R_1 - 3R_2$

$\Rightarrow \begin{pmatrix} 1-0 & 3-3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+6 & 0-3 \\ -2 & 1 \end{pmatrix} A$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} A$

$\Rightarrow A^{-1} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$

4. **Find the inverse of $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ using elementary operations.**

Solution: Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

W.K.T , $A = IA$

$\Rightarrow \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$, Now $R_2 \longrightarrow R_2 - 2R_1$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 2-2 & 1+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0-2 & 1-0 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A,$$

Now $R_2 \longrightarrow \frac{R_2}{5}$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ \frac{0}{5} & \frac{5}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A,$$

Now $R_1 \longrightarrow R_1 + 2R_2$

$$\Rightarrow \begin{pmatrix} 1+0 & -2+2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{4}{5} & 0+\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} A$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

5. Find the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ using elementary operations.

Solution: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

W.K.T, $A = IA$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A,$$

Now $R_2 \longrightarrow R_2 - 3R_1$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3-3 & 4-6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0-3 & 1-0 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} A, \text{ Now } R_2 \longrightarrow \frac{R_2}{-2}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ \frac{0}{-2} & -\frac{2}{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{-2} & \frac{1}{-2} \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} A,$$

Now $R_1 \longrightarrow R_1 - 2R_2$

$$\Rightarrow \begin{pmatrix} 1-0 & 2-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-3 & 0+1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} A$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

6. Find the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ using elementary operations.

Solution: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

W.K.T, $A = IA \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A,$

Now $R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$$

Now $R_2 \longrightarrow R_2 - 2R_1$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2-2 & 1-2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-0 & 0-2 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} A, \text{ Now } R_2 \longrightarrow \frac{R_2}{-1}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ \frac{0}{-1} & \frac{-1}{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{-1} & \frac{-2}{-1} \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} A,$$

Now $R_1 \longrightarrow R_1 - R_2$

$$\Rightarrow \begin{pmatrix} 1-0 & 1-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0+1 & 1-2 \\ -1 & 2 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} A$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

7. Express the matrix $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$ as sum of symmetric and skew symmetric matrix.

Solution: Let $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

$$\Rightarrow A' = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$$

Now, $\begin{matrix}\text{symmetric part} \\ \text{of matrix A}\end{matrix} = \frac{1}{2}[A + A']$

$$= \frac{1}{2}\left[\begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 6 & 6 \\ 6 & -2 \end{pmatrix}$$

$$\Rightarrow \begin{matrix}\text{symmetric part} \\ \text{of matrix A}\end{matrix} = \begin{pmatrix} 3 & 3 \\ 3 & -1 \end{pmatrix}$$

Now, $\begin{matrix}\text{Skew symmetric part} \\ \text{of matrix A}\end{matrix} = \frac{1}{2}[A - A']$

$$= \frac{1}{2}\left[\begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix}\text{Skew symmetric part} \\ \text{of matrix A}\end{matrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

W.K.T $A = \frac{1}{2}[A + A'] + \frac{1}{2}[A - A']$

$$\Rightarrow A = \begin{pmatrix} 3 & 3 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

8. Express the matrix $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ as sum of symmetric and skew symmetric matrix.

Solution: Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \Rightarrow A'=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$

Now, $\begin{matrix}\text{symmetric part}\\ \text{of matrix A}\end{matrix}=\frac{1}{2}[A+A']$

$=\frac{1}{2}\left[\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}2 & 5\\ 5 & 8\end{pmatrix}$

$\Rightarrow \begin{matrix}\text{symmetric part}\\ \text{of matrix A}\end{matrix}=\begin{pmatrix}1 & \frac{5}{2}\\ \frac{5}{2} & 4\end{pmatrix}$

Now, $\begin{matrix}\text{Skew symmetric part}\\ \text{of matrix A}\end{matrix}=\frac{1}{2}[A-A']$

$=\frac{1}{2}\left[\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}-\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$\Rightarrow \begin{matrix}\text{Skew symmetric part}\\ \text{of matrix A}\end{matrix}=\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$

W.K.T, $A=\frac{1}{2}[A+A']+\frac{1}{2}[A-A']$

$\Rightarrow A=\begin{pmatrix}1 & \frac{5}{2}\\ \frac{5}{2} & 4\end{pmatrix}+\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$

9. Prove that inverse of a matrix if exist is unique.

Solution: Let A be a square matrix.

If possible, let B and C are the inverse of A

To prove inverse of a matrix is unique we shall prove that $B=A$

If B is the inverse of A

$\Rightarrow AB=BA=I$ ……… (1)

If C is the inverse of A

$\Rightarrow AC=CA=I$ ……… (2)

We know that $B=BI=B(AC)$ (From 2)

$\Rightarrow B=BA(C)$ (Associative law)

$\Rightarrow B=IC=C$ (From 1)

$\therefore$ *Inverse of a matrix is unique*

10. If A and B are invertible matrix of same order then prove that $(AB)^{-1}=B^{-1}A^{-1}$

Solution: Let A and B are invertible matrices. $\Rightarrow A^{-1}$ and B^{-1} exist

$\Rightarrow AA^{-1}=A^{-1}A=I$ and

$BB^{-1}=B^{-1}B=I$

Now by definition $(AB)(AB)^{-1}=I$

(Pre multiplying both sides by A^{-1})

$\Rightarrow A^{-1}(AB)(AB)^{-1}=A^{-1}I$

$\Rightarrow (A^{-1}A)B(AB)^{-1}=A^{-1}I$

(Associative law)

$\Rightarrow IB(AB)^{-1}=A^{-1}$

$\Rightarrow B(AB)^{-1}=A^{-1}$

(Pre multiplying both sides by B^{-1})

$\Rightarrow B^{-1}B(AB)^{-1}=B^{-1}A^{-1}$

$\Rightarrow I(AB)^{-1}=B^{-1}A^{-1}$

$\Rightarrow (AB)^{-1}=B^{-1}A^{-1}$

FIVE MARKS QUESTION

1. If $A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$, then show that $A^3-23A-40I=0$

Solution: Now,

$A^2=A.A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$

$=\begin{bmatrix}1+6+12 & 2-4+6 & 3+2+3\\ 3-6+4 & 6+4+2 & 9-2+1\\ 4+6+4 & 8-4+2 & 12+2+1\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}19 & 4 & 8\\ 1 & 12 & 8\\ 14 & 6 & 15\end{bmatrix}$

Now, $A^3=A^2A$

$=\begin{bmatrix}19 & 4 & 8\\ 1 & 12 & 8\\ 14 & 6 & 15\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$

$=\begin{bmatrix}19+12+32 & 38-8+16 & 57+4+8\\ 1+36+32 & 2-24+16 & 3+12+8\\ 14+18+60 & 28-12+30 & 42+6+15\end{bmatrix}$

$\Rightarrow A^3=\begin{bmatrix}63 & 46 & 69\\ 69 & -6 & 23\\ 92 & 46 & 63\end{bmatrix}$

LHS $=A^3-23A-40I$

$=\begin{bmatrix}63 & 46 & 69\\ 69 & -6 & 23\\ 92 & 46 & 63\end{bmatrix}-23\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}--40\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$=\begin{bmatrix}63 & 46 & 69\\ 69 & -6 & 23\\ 92 & 46 & 63\end{bmatrix}-\begin{bmatrix}23 & 46 & 69\\ 69 & -46 & 23\\ 92 & 46 & 23\end{bmatrix}-\begin{bmatrix}40 & 0 & 0\\ 0 & 40 & 0\\ 0 & 0 & 40\end{bmatrix}$

$$= \begin{bmatrix} 63-23-40 & 46-46-0 & 69-69-0 \\ 69-69-0 & -6+46-40 & 23-23-0 \\ 92-92-0 & 46-46-0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{RHS}$$

2. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ then prove that $A^3 - 6A^2 + 7A + 2I = 0$.

Solution: Now,

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$\text{Now } A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$$\text{LHS} = A^3 - 6A^2 + 7A + 2I$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{RHS}$$

3. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{pmatrix}$ then prove that $(AB)C = A(BC)$.

Solution: Now,

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$$

$$\text{Now, } (AB)C = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$\text{LHS} = (AB)C$$

$$= \begin{bmatrix} 4 & 4 & 7 & \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix} --(1)$$

$$\text{Now, } BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$$

$$\Rightarrow BC = \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

$$\text{RHS} = A(BC)$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-6 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$$\Rightarrow A(BC) = \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix} --(2)$$

From 1 and 2: $(AB)C = A(BC)$

4. If $A=\begin{pmatrix}0 & 6\\ -6 & 0\end{pmatrix}$, $B=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $C=\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$ then prove that $(A+B)C = AC + BC$.

Solution: Now $A+B$

$$=\begin{pmatrix}0 & 6\\ -6 & 0\end{pmatrix}+\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}2 & 7\\ -5 & 2\end{pmatrix}$$

Now, $AC=\begin{pmatrix}0 & 6\\ -6 & 0\end{pmatrix}\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$

$$=\begin{pmatrix}0+12 & 0+18\\ -18+0 & -24+0\end{pmatrix}=\begin{pmatrix}12 & 18\\ -18 & -24\end{pmatrix}$$

Now, $BC=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$

$$=\begin{pmatrix}6+2 & 8+3\\ 3+4 & 4+6\end{pmatrix}=\begin{pmatrix}8 & 11\\ 7 & 10\end{pmatrix}$$

Now, LHS $=(A+B)C$

$$=\begin{pmatrix}2 & 7\\ -5 & 2\end{pmatrix}\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$$

$$=\begin{pmatrix}6+14 & 8+21\\ -15+4 & -20+6\end{pmatrix}$$

$$=\begin{pmatrix}20 & 29\\ -11 & -14\end{pmatrix}\dots\dots\dots(1)$$

Now, RHS $= AC + BC$

$$=\begin{pmatrix}12 & 18\\ -18 & -24\end{pmatrix}+\begin{pmatrix}8 & 11\\ 7 & 10\end{pmatrix}$$

$$=\begin{pmatrix}20 & 29\\ -11 & -14\end{pmatrix}\dots\dots\dots(2)$$

From 1 and 2: $(A+B)C = AC + BC$

5. If $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$ and $C=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$. Calculate AB, AC and $A(B+C)$ and verify that $AB + AC = A(B+C)$.

Solution: Now, $AB=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$

$$=\begin{bmatrix}2+2 & 0+6\\ 4+1 & 0+3\end{bmatrix}=\begin{bmatrix}4 & 6\\ 5 & 3\end{bmatrix}$$

Now, $AC=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$

$$=\begin{bmatrix}1+4 & 1+6\\ 2+2 & 2+3\end{bmatrix}=\begin{bmatrix}5 & 7\\ 4 & 5\end{bmatrix}$$

Now, $B+C=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$

$$=\begin{bmatrix}2+1 & 0+1\\ 1+2 & 3+3\end{bmatrix}=\begin{bmatrix}3 & 1\\ 3 & 6\end{bmatrix}$$

Now, $LHS = AB + AC = \begin{bmatrix}4 & 6\\ 5 & 3\end{bmatrix}+\begin{bmatrix}5 & 7\\ 4 & 5\end{bmatrix}$

$$=\begin{bmatrix}4+5 & 6+7\\ 5+4 & 3+5\end{bmatrix}=\begin{bmatrix}9 & 13\\ 9 & 8\end{bmatrix}\dots\dots(1)$$

Now, $RHS = A(B+C) = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 3 & 6\end{bmatrix}$

$$=\begin{bmatrix}3+6 & 1+12\\ 6+3 & 2+6\end{bmatrix}=\begin{bmatrix}9 & 13\\ 9 & 8\end{bmatrix}\dots\dots(2)$$

From 1 and 2 : $AB + AC = A(B+C)$

6. If $A=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}$, and $C=\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$. Calculate AC, BC and $(A+B)C$. Also, verify that $(A+B)C = AC + BC$.

Solution: Now, $AC=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$

$$=\begin{bmatrix}0-12+21\\ -12+0+24\\ 14+16+0\end{bmatrix}=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}$$

Now, $BC=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$

$$=\begin{bmatrix}0-2+3\\ 2+0+6\\ 2-4+0\end{bmatrix}=\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}$$

Now, $A+B$

$$=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}+\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}$$

Now, $LHS = (A+B)C$

$$=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$$

$$=\begin{bmatrix}0-14+24\\ -10+0+30\\ 16+12+0\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix}\dots\dots\dots(1)$$

Now, $RHS = AC + BC$

$$=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}+\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix}\dots\dots\dots(2)$$

From 1 and 2 : $(A+B)C = AC + BC$

7. If $A=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}, B=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}4 & 1 & 2\\0 & 3 & 2\\1 & -2 & 3\end{bmatrix}$ then compute $(A+B)$ and $(B-C)$ also verify that $A+(B-C)=(A+B)-C$

Solution: Now, $A+B$

$$=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}+\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4 & 1 & -1\\9 & 2 & 7\\3 & -1 & 4\end{bmatrix}$$

Now, $B-C$

$$=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}-\begin{bmatrix}4 & 1 & 2\\0 & 3 & 2\\1 & -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -2 & 0\\4 & -1 & 3\\1 & 2 & 0\end{bmatrix}$$

Now, $LHS = A+(B-C)$

$$=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}+\begin{bmatrix}-1 & -2 & 0\\4 & -1 & 3\\1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & -3\\9 & -1 & 5\\2 & 1 & 1\end{bmatrix} \ldots\ldots\ldots\ldots (1)$$

Now, $RHS = (A+B)-C$

$$=\begin{bmatrix}4 & 1 & -1\\9 & 2 & 7\\3 & -1 & 4\end{bmatrix}-\begin{bmatrix}4 & 1 & 2\\0 & 3 & 2\\1 & 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & -3\\9 & -1 & 5\\2 & 1 & 1\end{bmatrix} \ldots\ldots\ldots\ldots (2)$$

From 1 and 2 : $A+(B-C)=(A+B)-C$

8. If $A=\begin{bmatrix}-2\\4\\5\end{bmatrix}, B=\begin{bmatrix}1 & 3 & -6\end{bmatrix}$ Verify that $(AB)' = B'A'$.

Solution: Now,

$$A=\begin{bmatrix}-2\\4\\5\end{bmatrix} \Rightarrow A' = \begin{bmatrix}-2 & 4 & 5\end{bmatrix} \text{ and}$$

$$B=\begin{bmatrix}1 & 3 & -6\end{bmatrix} \Rightarrow B' = \begin{bmatrix}1\\3\\-6\end{bmatrix}$$

$$\text{Now, } AB = \begin{bmatrix}-2\\4\\5\end{bmatrix}\begin{bmatrix}1 & 3 & -6\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -6 & 12\\4 & 12 & 24\\5 & 15 & -30\end{bmatrix}$$

Now, LHS = $(AB)'$

$$=\begin{bmatrix}-2 & 4 & 5\\-6 & 12 & 15\\12 & 24 & -30\end{bmatrix} \ldots\ldots\ldots (1)$$

$$\text{Now, RHS} = B'A' = \begin{bmatrix}1\\3\\-6\end{bmatrix}\begin{bmatrix}-2 & 4 & 5\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 4 & 5\\-6 & 12 & 15\\12 & 24 & -30\end{bmatrix} \ldots\ldots\ldots (2)$$

From 1 and 2 : $(AB)' = B'A'$

DETERMINANTS

TWO MARKS QUESTION

1. Find the area of the triangle whose vertices are $(3,8),(-4,2)$ *and* $(5,1)$ by using determinant method.

Solution: Let $A=(3,8)=(x_1,y_1)$,

$B=(-4,2)=(x_2,y_2)$ and

$C=(5,1)=(x_3,y_3)$

Now, Area of triangle ABC is given by

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}3 & 8 & 1\\ -4 & 2 & 1\\ 5 & 1 & 1\end{vmatrix}$$

$=\frac{1}{2}[3(2-1)-8(-4-5)+1(-4-10)]$

$=\frac{1}{2}[3(1)-8(-9)+1(-14)]$

$=\frac{1}{2}[3+72-14]=\frac{1}{2}(61)$

$\Rightarrow$ Area of triangle ABC $=\frac{61}{2}\ Sq.units$

2. Using determinant , find the area of the triangle whose vertices are $(1,0),(6,0)$ and $(4,3)$

Solution: Let $A=(1,0)=(x_1,y_1)$,

$B=(6,0)=(x_2,y_2)$ and

$C=(4,3)=(x_3,y_3)$

Now, Area of triangle ABC is given by

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}$$

$=\frac{1}{2}[1(0-3)-0(6-4)+1(18-0)]$

$=\frac{1}{2}[1(-3)-0(2)+1(18)]$

$=\frac{1}{2}[-3-0+18]=\frac{1}{2}(15)=\frac{15}{2}\ Sq.$units

$\Rightarrow$ Area of triangle ABC $=\frac{15}{2}\ Sq.units$

3. Find the area of the triangle whose vertices are$(-2,-3),(3,2)$ and $(-1,-8)$ by using determinants.

Solution: Let $A=(-2,-3)=(x_1,y_1)$,

$B=(3,2)=(x_2,y_2)$ and

$C=(-1,-8)=(x_3,y_3)$

Now, Area of triangle ABC is given by

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}-2 & -3 & 1\\ 3 & 2 & 1\\ -1 & -8 & 1\end{vmatrix}$$

$=\frac{1}{2}[-2(2+8)—3)(3+1)+1(-24+2)$

$=\frac{1}{2}[-2(10)+3(4)+1(-22)]$

$=\frac{1}{2}[-20+12-22]=\frac{1}{2}(-30)=-15$

Since area of triangle is positive

$\Rightarrow$ Area of triangle ABC $=15$ Sq.units.

4. Find the area of the triangle whose vertices are$(2,0),(-1,0)$ and $(0,3)$ by using determinants.

Solution: Let $A=(2,0)=(x_1,y_1)$,

$B=(-1,0)=(x_2,y_2)$ and

$C=(0,3)=(x_3,y_3)$

Now, Area of triangle ABC is given by

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}2 & 0 & 1\\ -1 & 0 & 1\\ 0 & 3 & 1\end{vmatrix}$$

$=\frac{1}{2}[2(0-3)-0(-1-0)+1(-3-0)]$

$=\frac{1}{2}[2(-3)-0(-1)+1(-3)]$

$=\frac{1}{2}(-9)=\frac{-9}{2}=\left|\frac{-9}{2}\right|=\frac{9}{2}$

$\Rightarrow$ Area of triangle ABC $==\frac{9}{2}$ Sq.units.

5. Find the area of the triangle whose vertices are$(2,7),(1,1)$ and $(10,8)$ by using determinants.

Solution: Let $A=(2,7)=(x_1,y_1)$,

$B=(1,1)=(x_2,y_2)$ and

$C=(10,8)=(x_3,y_3)$

Now, Area of triangle ABC is given by

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}2 & 7 & 1\\ 1 & 1 & 1\\ 10 & 8 & 1\end{vmatrix}$$

$=\frac{1}{2}[2(1-8)-7(1-10)+1(8-10)]$

$=\frac{1}{2}[2(-7)-7(-9)+1(-2)]$

$=\frac{1}{2}[-14+63-2]=\frac{1}{2}(47)=\frac{47}{2}$

$\Rightarrow$ Area of triangle ABC $=\frac{47}{2}$ Sq.units.

6. If the area of the triangle with vertices $(-2,0),(0,4)$ *and* $(0,k)$ is 4 square units, find the values of k using determinants.

Solution: Let $A=(-2,0)=(x_1,y_1)$,

$B=(0,4)=(x_2,y_2)$ and

$C=(0,k)=(x_3,y_3)$

Given: Area of $\Delta ABC = 4\ sq\ units$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = |4|$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & K & 1 \end{vmatrix} = \pm 4$$

$\Rightarrow -2(4-K)-0(0-0)+1(0-0)=\pm(2\times 4)$

$\Rightarrow -8+2k-0+0=\pm 8$

$\Rightarrow 2k-8=\pm 8 \Rightarrow 2k=8\pm 8$

$\Rightarrow 2k=8+8\ or\ 8-8$

$\Rightarrow 2k=16\ or\ 0 \Rightarrow k=8\ or\ 0$

7. Find values of k, if area of triangle is 4 sq. units and vertices are $(k,0),(4,0),(0,2)$ using determinants.

Solution: Let $A=(k,0)=(x_1,y_1)$,

$B=(4,0)=(x_2,y_2)$ and

$C=(0,2)=(x_3,y_3)$

Given: Area of $\Delta ABC = 4\ sq\ units$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = |4|$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4$$

$\Rightarrow k(0-2)-0(4-0)+1(8-0)=\pm(2\times 4)$

$\Rightarrow -2k-0+8=\pm 8 \Rightarrow 2k=8\mp 8$

$\Rightarrow 2k=8+8\ or\ 8-8 \Rightarrow 2k=16\ or\ 0$

$\Rightarrow k=8\ or\ 0$

8. Find values of k, if area of triangle is 35 sq. units and vertices are $(2,-6),(5,4),(k,4)$ using determinants.

Solution: Let $A=(2,-6)=(x_1,y_1)$,

$B=(5,4)=(x_2,y_2)$ and

$C=(k,4)=(x_3,y_3)$

Given: Area of $\Delta ABC = 4\ sq\ units$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = |35|$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35$$

$\Rightarrow 2(4-4)+6(5-k)+1(20-4k)=\pm(2\times 35)$

$\Rightarrow 2(0)+30-6k+20-4k=\pm 70$

$\Rightarrow 50-10k=\pm 70 \Rightarrow 50\mp 70=10k$

$\Rightarrow 10k=50+70\ or\ 50-70$

$\Rightarrow 10k=120\ or-20 \Rightarrow k=12\ or-2$

9. Find the equation of the line joining (1,2) and (3,6) using determinants.

Solution: Let $A=(1,2)=(x_1,y_1)$,

$B=(3,6)=(x_2,y_2)$

Now, Equation of line is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix}=0$$

$\Rightarrow x(2-6)-y(1-3)+1(6-6)=0$

$\Rightarrow x(-4)-y(-2)+1(0)=0$

$\Rightarrow -4x+2y=0 \Rightarrow 4x=2y$

$\Rightarrow 2x=y$

10. Find the equation of a line passing through $(1,3)$ *and* $(0,0)$ using determinants.

Solution: Let $A=(1,3)=(x_1,y_1)$,

$B=(0,0)=(x_2,y_2)$

Now, Equation of line is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}=0$$

$\Rightarrow x(3-0)-y(1-0)+1(0-0)=0$

$\Rightarrow x(3)-y(1)+1(0)=0$

$\Rightarrow 3x-y=0$

$\Rightarrow 3x=y$

11. Find the equation of a line passing through $(3,1)(9,3)$ using determinants.

Solution: Let $A=(3,1)=(x_1,y_1)$,

$B=(9,3)=(x_2,y_2)$

Now, Equation of line is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow x(1-3) - y(3-9) + 1(9-9) = 0$

$\Rightarrow x(-2) - y(-6) + 1(0) = 0$

$\Rightarrow -2x + 6y = 0 \Rightarrow 2x = 6y$

$\Rightarrow x = 3y$

12. Using determinant show that points $A(a, b+c), B(b, c+a)$ and $C(c, a+b)$ are collinear.

Solution: Let $A = (a, b+c) = (x_1, y_1)$,

$B = (b, c+a) = (x_2, y_2)$ and

$C = (c, a+b) = (x_3, y_3)$

Now, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$Now, C_1 \to C_1 + C_2$

$= \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix}$

(Here columns C_1 and C_3 are identical)

$= (a+b+c)(0) = 0$

$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ given three points are collinear

FIVE MARKS QUESTION

1. Solve the system of linear equations using matrix method $3x - 2y + 3z = 8$, $2x + y - z = 1$ and $4x - 3y + 2z = 4$.

Solution: Let $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

Now, $|A| = 3(2-3) + 2(4+4) + 3(-6-4)$

$= 3(-1) + 2(8) + 3(-10) = -3 + 16 - 30$

$\Rightarrow |A| = -17 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} \\ -\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 3 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \\ +\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(2-3) & -(4+4) & +(-6-4) \\ -(-4+9) & +(6-12) & -(-9+8) \\ +(2-3) & -(-3-6) & +(3+4) \end{bmatrix}$$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$

Now, $adj\ (A) =$ (cofactor matrix)'

$\Rightarrow adj\ (A) = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\ adjA = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

W.K.T ,

$X = A^{-1}B \Rightarrow X = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\Rightarrow x = 1, y = 2, z = 3.$

2. Solve the system of linear equations using matrix method $x - y + z = 4$, $2x + y - 3z = 0$ and $x + y + z = 2$

Solution: Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Now, $|A| = 1(1+3) - 1(2+3) + 1(2-1)$

$= 1(4) + 1(5) + 1(1) = 4 + 5 + 1 = 10$

$\Rightarrow |A| = 10 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix} +\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(1+3) & -(2+3) & +(2-1) \\ -(-1-1) & +(1-1) & -(1+1) \\ +(3-1) & -(-3-2) & +(1+2) \end{bmatrix}$$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$\Rightarrow adj\,(A) = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\, adjA = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

W.K.T ,

$X = A^{-1}B \Rightarrow X = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20-0+10 \\ 4-0+6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20/10 \\ -10/10 \\ 10/10 \end{bmatrix}$

$\Rightarrow x = 2, y = -1, z = 1.$

3. Solve the system of linear equations using matrix method $\mathbf{2x + 3y + 3z = 5}$, $\mathbf{x - 2y + z = -4}$ and $\mathbf{3x - y - 2z = 3}$

Solution: Let $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

Now, $|A| = 2(4+1) - 3(-2-3) + 3(-1+6)$

$= 2(5) - 3(-5) + 3(5) = 10 + 15 + 15$

$\Rightarrow |A| = 40 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix} +\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(4+1) & -(-2-3) & +(-1+6) \\ -(-6+3) & +(-4-9) & -(-2-9) \\ +(3+6) & -(2-3) & +(-4-3) \end{bmatrix}$$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$\Rightarrow adj\,(A) = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\, adjA = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

W.K.T ,

$X = A^{-1}B \Rightarrow X = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\Rightarrow x = 1, y = 2, z = -1.$

4. Solve the system of linear equations using matrix method $\mathbf{x - y + 2z = 7}$, $\mathbf{3x + 4y - 5z = {-}{-}5}$ & $\mathbf{2x - y + 3z = 12}$

Solution: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

Now, $|A| = 1(12-5) + 1(9+10) + 2(-3-8)$

$= 1(7) + 1(19) + 2(-11) = 7 + 19 - 22$

$\Rightarrow |A| = 4 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix} +\begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(12-5) & -(9+10) & +(-3-8) \\ -(-3+2) & +(3-4) & -(-1+2) \\ +(5-8) & -(-5-6) & +(4+3) \end{bmatrix}$$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$\Rightarrow adj\,(A) = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\, adjA = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$

W.K.T , $X = A^{-1} B$

$$\Rightarrow X = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 2 , y = 1 , z = 3 .$

5. Solve the system of linear equations using matrix method $2x + y + z = 1$, $x - 2y - z = \frac{3}{2}$ and $3y - 5z = 9$

Solution: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$

Now, $|A| = 2(10+3) - 1(-5-0) + 1(3-0)$

$= 2(13) - 1(-5) + 1(3) = 26 + 5 + 3$

$\Rightarrow |A| = 34 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix} +\begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(10+3) & -(-5-0) & +(3-0) \\ -(-5-3) & +(-10-0) & -(6-0) \\ +(-1+2) & -(-2-1) & +(-4-1) \end{bmatrix}$$

$$\Rightarrow \text{cofactor matrix of A} = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}$$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$$\Rightarrow adj\,(A) = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}\, adjA = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$

W.K.T ,

$$X = A^{-1} B \Rightarrow X = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

$\Rightarrow x = 1 , y = \frac{1}{2} , z = -\frac{3}{2}$.

6. Solve the system of linear equations using matrix method $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$, $\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$ and $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

Solution: Let $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$,

$X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Now,

$|A| = 2(120-45) - 3(-80-30) + 10(36+36)$

$= 2(75) - 3(-110) + 10(72)$

$\Rightarrow |A| = 150 + 330 + 720 = 1200 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix} +\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} & -\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} & +\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} \\ -\begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} & +\begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} \\ +\begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(120-45) & -(-80-30) & +(36+36) \\ -(-60-90) & +(-40-60) & -(18-18) \\ +(15+60) & -(10-40) & +(-12-12) \end{bmatrix}$$

$$= \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$$\Rightarrow adj\,(A) = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Now , $A^{-1} = \frac{1}{|A|}\, adjA = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

W.K.T ,

$X = A^{-1}B \Rightarrow X =$

$\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\1\\2\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1/x\\1/y\\1/z\end{bmatrix} = \frac{1}{1200}\begin{bmatrix}300+150+150\\440-100+60\\288+0-48\end{bmatrix}$

$\Rightarrow \begin{bmatrix}\frac{1}{x}\\\frac{1}{y}\\\frac{1}{z}\end{bmatrix} = \frac{1}{1200}\begin{bmatrix}600\\400\\240\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{1}{x}\\\frac{1}{y}\\\frac{1}{z}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{3}\\\frac{1}{5}\end{bmatrix}$

$\Rightarrow \frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5}$

$\Rightarrow x = 2, y = 3 \,\&\, z = 5.$

7. The Sum of three numbers is 6. If we multiply third number by 3 and added second number to it, we get 11. By adding first and third numbers, we get double of second number. Represent it algebraically and find the numbers using matrix method.

Solution: Let x, y, z are the numbers

$\Rightarrow x + y + z = 6, 3z + y = 11$ and $x + z = 2y$

$\Rightarrow x + y + z = 6,\ 0x + y + 3z = 11,$ and $x - 2y + z = 0$

Let $A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\1 & -2 & 1\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}, B = \begin{bmatrix}6\\11\\0\end{bmatrix}$

Now, $|A| = 1(1+6) - 1(0-3) + 1(0-1)$

$= 1(7) - 1(-3) + 1(-1) = 7 + 3 - 1 = 9$

$\Rightarrow |A| = 9 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix}+\begin{vmatrix}1 & 3\\-2 & 1\end{vmatrix} & -\begin{vmatrix}0 & 3\\1 & 1\end{vmatrix} & +\begin{vmatrix}0 & 1\\1 & -2\end{vmatrix}\\-\begin{vmatrix}1 & 1\\-2 & 1\end{vmatrix} & +\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} & -\begin{vmatrix}1 & 1\\1 & -2\end{vmatrix}\\+\begin{vmatrix}1 & 1\\1 & 3\end{vmatrix} & -\begin{vmatrix}1 & 1\\0 & 3\end{vmatrix} & +\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}+(1+6) & -(0-3) & +(0-1)\\-(1+2) & +(1-1) & -(-2-1)\\+(3-1) & -(3-0) & +(1-0)\end{bmatrix}$$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix}7 & 3 & -1\\-3 & 0 & 3\\2 & -3 & 1\end{bmatrix}$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})' = \begin{bmatrix}7 & -3 & 2\\3 & 0 & -3\\-1 & 3 & 1\end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\,adjA = \frac{1}{9}\begin{bmatrix}7 & -3 & 2\\3 & 0 & -3\\-1 & 3 & 1\end{bmatrix}$

W.K.T ,

$X = A^{-1}B \Rightarrow X = \frac{1}{9}\begin{bmatrix}7 & -3 & 2\\3 & 0 & -3\\-1 & 3 & 1\end{bmatrix}\begin{bmatrix}6\\11\\0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{9}\begin{bmatrix}42-33+0\\18+0+0\\-6+33+0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{9}\begin{bmatrix}9\\18\\27\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$\Rightarrow x = 1, y = 2, z = 3.$

$\Rightarrow$ The numbers are 1,2,3

8. The cost of 4kg onion, 3kg wheat and 2kg rice is Rs60.The cost of 2kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Solution: Let x be the amount of onion per kg , y be the amount of wheat per kg and z be the amount of rice per kg.

$\Rightarrow 4x + 3y + 2z = 60, 2x + 4y + 6z = 90$ & $6x + 2y + 3z = 70$

Let $A = \begin{bmatrix}4 & 3 & 2\\2 & 4 & 6\\6 & 2 & 3\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}, B = \begin{bmatrix}60\\90\\70\end{bmatrix}$

Now, $|A| = 4(12-12) - 3(6-36) + 2(4-24)$

$= 4(0) - 3(-30) + 2(-20) = 0 + 90 - 40$

$\Rightarrow |A| = 50 \neq 0$

Now, Cofactor matrix of $A =$

$$\begin{bmatrix}+\begin{vmatrix}4 & 6\\2 & 3\end{vmatrix} & -\begin{vmatrix}2 & 6\\6 & 3\end{vmatrix} & +\begin{vmatrix}2 & 4\\6 & 2\end{vmatrix}\\-\begin{vmatrix}3 & 2\\2 & 3\end{vmatrix} & +\begin{vmatrix}4 & 2\\6 & 3\end{vmatrix} & -\begin{vmatrix}4 & 3\\6 & 2\end{vmatrix}\\+\begin{vmatrix}3 & 2\\4 & 6\end{vmatrix} & -\begin{vmatrix}4 & 2\\2 & 6\end{vmatrix} & +\begin{vmatrix}4 & 3\\2 & 4\end{vmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}+(12-12) & -(6-36) & +(4-24)\\-(9-4) & +(12-12) & -(8-18)\\+(18-8) & -(24-4) & +(16-6)\end{bmatrix}$$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix}0 & 30 & -20\\-5 & 0 & 10\\10 & -20 & 10\end{bmatrix}$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$\Rightarrow adj\,(A) = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\,adjA = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$

W.K.T , $X = A^{-1}B$

$\Rightarrow X = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800-0-1400 \\ -1200+900+700 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$

$\Rightarrow x = 5\,, y = 8\,, z = 8\,.$

9. Use Product $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$ **to solve** $x - y + 2z = 1\,, 2y - 3z = 1$ **and** $3x - 2y + 4z = 2$

Solution: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Now, $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

$\Rightarrow A\begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix} = I$

$\Rightarrow A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$

W.K.T , $X = A^{-1}B$

$\Rightarrow X = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$

$\Rightarrow x = 0\,, y = 5\,, z = 3\,.$

10. If A $= \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ **, find** A^{-1}**. Using** A^{-1} **solve the system of equations** $2x - 3y + 5z = 11\,, 3x + 2y - 4z = -5$ **and** $x + y - 2z = -3$

Solution: Let $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Now, $|A| = 2(-4+4) - 3(-6+4) + 5(3-2)$

$= 2(0) + 3(-2) + 5(1) = 0 - 6 + 5$

$\Rightarrow |A| = -1 \neq 0$

Now, Cofactor matrix of A =

$\begin{bmatrix} +\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} +(-4+4) & -(-6+4) & +(3-2) \\ -(6-5) & +(-4-5) & -(2+3) \\ +(12-10) & -(-8-15) & +(4+9) \end{bmatrix}$

$\Rightarrow$ cofactor matrix of A $= \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$

$\Rightarrow adj\,(A) = (\text{cofactor matrix})'$

$\Rightarrow adj\,(A) = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|}\,adjA = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

W.K.T , $X = A^{-1}B$

$\Rightarrow X = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} 0+5-6 \\ 22+45-69 \\ 11+25-39 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\Rightarrow x = 1\,, y = 2\,, z = 3$

CONTINUITY AND DIFFERENTIABILITY

TWO MARKS QUESTION

1. **If $y + \sin y = \cos x$ $find \frac{dy}{dx}$.**

Solution: Given , $y + \sin y = \cos x$,

$Diff\ w\ r\ t\ x$

$\Rightarrow \frac{dy}{dx} + \cos y \frac{dy}{dx} = -\sin x$

$\Rightarrow \frac{dy}{dx}(1 + \cos y) = \frac{dy}{dx} = -\sin x$

$\Rightarrow \frac{dy}{dx}(1 + \cos y) = -\sin x$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{1+\cos y}$

2. **If $\sqrt{x} + \sqrt{y} = \sqrt{10}$ then show that $\frac{dy}{dx} + \sqrt{\frac{y}{x}} = 0$.**

Solution: Given , $\sqrt{x} + \sqrt{y} = 10$,

$Diff\ w\ r\ t\ x$

$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

$\Rightarrow \frac{dy}{dx} + \sqrt{\frac{y}{x}} = 0$

3. **Find $\frac{dy}{dx}$, $if\ x^2 + xy + y^2 = 100$**

Solution: Given , $x^2 + xy + y^2 = 100$,

$Diff\ w\ r\ t\ x$

$\Rightarrow 2x + x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$

$\Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y$

$\Rightarrow \frac{dy}{dx}(x + 2y) = -(2x - y)$

$\Rightarrow \frac{dy}{dx} = \frac{-(2x-y)}{x+2y}$

4. **Find $\frac{dy}{dx}$, if $ax + by^2 = \cos y$.**

Solution: $ax + by^2 = \cos y$, $Diff\ w\ r\ t\ x$

$\Rightarrow a(1) + b2y\frac{dy}{dx} = -siny\frac{dy}{dx}$

$\Rightarrow 2by\frac{dy}{dx} + \sin y\frac{dy}{dx} = -a$

$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a$

$\Rightarrow \frac{dy}{dx} = \frac{-a}{2by+\sin y}$

5. **Find $\frac{dy}{dx}$ if $2x + 3y = \sin y$**

Solution: $2x + 3y = \sin y$, $Diff\ w\ r\ t\ x$

$\Rightarrow 2(1) + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$

$\Rightarrow 3\frac{dy}{dx} - \cos y\frac{dy}{dx} = -2$

$\Rightarrow \frac{dy}{dx}(3 - \cos y) = -2 \Rightarrow \frac{dy}{dx} = \frac{-2}{3-\cos y}$

6. **If $y = x^x$ $then\ find \frac{dy}{dx}$**

Solution: $y = x^x$, Take log on both sides

$\Rightarrow \log y = logx^x \Rightarrow \log y = xlog\ x$,

$Diff\ w\ r\ t\ x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log(x)(1)$

$\Rightarrow \frac{dy}{dx} = y[1 + \log x]$

$\Rightarrow \frac{dy}{dx} = x^x(1 + \log x)$

7. **Find $\frac{dy}{dx}$. $if\ y = (log\ x)^{\cos x}$.**

Solution: $y = (\log x)^{\cos x}$,

Take log on both sides

$\Rightarrow \log y = \log[\log x]^{\cos x}$

$\Rightarrow \log y = \cos x . \log(\log x)$ $Diff\ w\ r\ t\ x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}[\log(\log x)] +$

$\log(\log x)\frac{d}{dx}(\cos x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x . \frac{1}{\log x}\frac{1}{x} + \log(\log x)(-\sin x)$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{xlog\ x} - \sin x \log(\log x)\right]$

$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{xlog\ x} - \sin x . \log(\log x)\right]$

8. **Find $\frac{dy}{dx}$, if $y = (\sin x)^x$, $x > 0$**

Solution: $y = (\sin x)^x$,Take log on both sides

$\Rightarrow \log y = \log(\sin x)^x$

$\Rightarrow \log y = x.\log(\sin x)$, $Diff\ w\ r\ t\ 'x'$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log(sinx)) + \log(\sin x)\frac{d}{dx}(x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{\sin x}(\cos x) + \log(\sin x)(1)$

$\Rightarrow \frac{dy}{dx} = y[x\cot x + \log(\sin x)]$

$\Rightarrow \frac{dy}{dx} = (\sin x)^x[xcot\ x + \log(\sin x)]$

9. Differentiate $\left(x+\frac{1}{x}\right)^x$ with respect to x

Solution: $y=\left(x+\frac{1}{x}\right)^x$,

Take log on both sides

$\Rightarrow \log y=\log\left(x+\frac{1}{x}\right)^x$

$\Rightarrow \log y = x\log\left(x+\frac{1}{x}\right)$, Diff $w\,r\,t\,x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]+\log\left(x+\frac{1}{x}\right)\frac{d}{dx}(x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=x\frac{1}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)(1)$

$\Rightarrow \frac{dy}{dx}=y\left[\frac{x}{\frac{x^2+1}{x}}\left(\frac{x^2-1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\right]$

$\Rightarrow \frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]$

10. If $x^y=a^x$ then prove $\frac{dy}{dx}=\frac{x\log a-y}{x\log x}$

Solution: $x^y=a^x$, Take log on both sides

$\Rightarrow \log x^y=\log a^x \Rightarrow y\log x = x\log a$,

Diff $w\,r\,t\,x$

$\Rightarrow y\frac{d}{dx}[\log x]+\log x\frac{d}{dx}(y)=\log a\frac{d}{dx}(x)$

$\Rightarrow y\frac{1}{x}+\log x\frac{dy}{dx}=\log a\ (1)$

$\Rightarrow \log x\frac{dy}{dx}=\log a-\frac{y}{x}$

$\Rightarrow \log x\frac{dy}{dx}=\frac{x\log a-y}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{x\log a-y}{x\log x}$

11. If $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$, $0<x<\frac{1}{\sqrt{2}}$ then find $\frac{dy}{dx}$

Solution: $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$,

Let $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

$\Rightarrow y=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)$

$\Rightarrow y=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$\Rightarrow y=\sec^{-1}(\sec 2\theta)$

$\Rightarrow y=2\theta \Rightarrow y=2\cos^{-1}x$, Diff $w\,r\,t\,x$

$\Rightarrow \frac{dy}{dx}=2\frac{-1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$.

12. If $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0<x<1$, then find $\frac{dy}{dx}$

Solution: $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$,

Let $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$\Rightarrow y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow y=\cos^{-1}(\cos 2\theta)$

$\Rightarrow y=2\theta \Rightarrow y=2\tan^{-1}x$,

Diff $w\,r\,t$ 'x'

$\Rightarrow \frac{dy}{dx}=2\frac{1}{1+x^2}=\frac{2}{1+x^2}$

13. If $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ then prove $\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$

Solution: $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$,

Let $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$

$\Rightarrow y=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow y=\sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$\Rightarrow y=\sin^{-1}(2\sin\theta.\cos\theta)$

$\Rightarrow y=\sin^{-1}(\sin 2\theta)$

$\Rightarrow y=2\theta \Rightarrow y=2\sin^{-1}x$,

Diff $w\,r\,t$ 'x'

$\Rightarrow \frac{dy}{dx}=2\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$

14. If $y=\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ then prove that $\frac{dy}{dx}=\frac{1}{2}$

Solution: $y=\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$,

$w\,k\,t\ \sin x=2\sin\frac{x}{2}.\cos\frac{x}{2}$

and $\cos x=2\cos^2\frac{x}{2}-1$

$\Rightarrow y=\tan^{-1}\left(\frac{2\sin\frac{x}{2}.\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$

$\Rightarrow y=\tan^{-1}(\tan x/2) \Rightarrow y=\frac{x}{2}$,

Diff $w\,r\,t$ 'x' $\Rightarrow \frac{dy}{dx}=\frac{1}{2}$.

THREE MARKS QUESTION

1. If $x=\sin t$, $y=\cos 2t$ then prove that $\frac{dy}{dx}=-4\sin t$

Solution: Now, $x=\sin t$, Diff $w\,r$ to t

$\Rightarrow \frac{dx}{dt}=\cos t$

Now, $y=\cos 2t$, Diff $w\,r$ to t

$\Rightarrow \frac{dy}{dt}=-\sin 2t(2)$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2sin2t}{cost} = \frac{-2(2sint\ cost)}{cost}$

$\Rightarrow \frac{dy}{dx} = -4sint$

2. If $x = a(\theta - \sin\theta)$ & $y = a(1 + \cos\theta)$. Prove that $\frac{dy}{dx} = -cot\frac{\theta}{2}$.

Solution: Now, $x = a(\theta - sin\theta)$,

$Diff\ w\ r\ to\ \theta \Rightarrow \frac{dx}{d\theta} = a(1 - cos\theta)$

Now, $y = a(1 + cos\,\theta)$, $Diff\ w\ r\ to\ \theta$

$\Rightarrow \frac{dy}{d\theta} = a(0 - sin\theta) = -asin\theta$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-asin\theta}{a(1-cos\theta)}$

$= \frac{-2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{1-\left(1-2\sin^2\frac{\theta}{2}\right)} = \frac{-2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -cot\frac{\theta}{2}$

3. If $x = a(\theta + \sin\theta)$ & $y = a(1 - \cos\theta)$. Prove that $\frac{dy}{dx} = tan\frac{\theta}{2}$.

Solution: Now, $x = a(\theta + sin\theta)$,

$Diff\ w\ r\ to\ \theta \Rightarrow \frac{dx}{d\theta} = a(1 + cos\theta)$

Now, $y = a(1 - cos\,\theta)$, $Diff\ w\ r\ to\ \theta$

$\frac{dy}{d\theta} = a\big(0 - (-sin\theta)\big) = asin\theta$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{asin\theta}{a(1+cos\theta)}$

$= \frac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)} = \frac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = tan\frac{\theta}{2}$

4. Find $\frac{dy}{dx}$ if $x = a(cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$.

Solution: Now, $x = a(cos\theta + \theta sin\theta)$,

$Diff\ w\ r\ to\ \theta$

$\Rightarrow \frac{dx}{d\theta} = a[-sin\theta + (\theta\cos\theta + sin\theta(1))]$

$\Rightarrow \frac{dx}{d\theta} = a(-sin\theta + \theta cos\theta + sin\theta)$

$\Rightarrow \frac{dx}{d\theta} = a\theta cos\theta$

Now,

$y = a(sin\theta - \theta cos\theta)$, $Diff\ w\ r\ to\ \theta$

$\Rightarrow \frac{dy}{d\theta} = a[cos\theta - (\theta(-sin\theta) + cos\theta(1))]$

$\Rightarrow \frac{dy}{d\theta} = a[cos\theta + \theta sin\theta - cos\theta]$

$\Rightarrow \frac{dy}{d\theta} = a\theta sin\theta$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta sin\theta}{a\theta cos\theta} = tan\theta$

5. If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$ then prove that $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

Now, $x = \sqrt{a^{\sin^{-1}t}}$ & $y = \sqrt{a^{\cos^{-1}t}}$

$\Rightarrow xy = \sqrt{a^{\sin^{-1}t}}\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow xy = \sqrt{a^{\sin^{-1}t}\ a^{\cos^{-1}t}}$

$\Rightarrow xy = \sqrt{a^{\sin^{-1}t+\cos^{-1}t}}$

$\Rightarrow xy = \sqrt{a^{\frac{\pi}{2}}}$, $Diff\ w\ r\ to\ x$

$\Rightarrow x\frac{dy}{dx} + y\frac{d(x)}{dx} = \frac{d}{dx}\sqrt{a^{\frac{\pi}{2}}}$

$\Rightarrow x\frac{dy}{dx} + y(1) = 0 \Rightarrow x\frac{dy}{dx} = -y$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

6. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, prove that $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$

Solution: Now, $x = a\cos^3\theta$, $Diff\ w\ r\ to\ \theta$

$\Rightarrow \frac{dx}{d\theta} = a3\cos^2\theta\,(-sin\theta)$

$\Rightarrow \frac{dx}{d\theta} = -3\,a\cos^2\theta\,\sin\theta$

Now, $y = a\sin^3\theta$, $Diff\ w\ r\ to\ \theta$

$\Rightarrow \frac{dy}{d\theta} = a3\sin^2\theta.cos\theta = 3\,a\sin^2\theta\,\cos\theta$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\,a\sin^2\theta\,\cos\theta}{-3\,a\cos^2\theta\,\sin\theta}$

$\Rightarrow \frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta}$ ……………(1)

Now, $x = a\cos^3\theta \Rightarrow \cos^3\theta = \frac{x}{a}$

$\Rightarrow \cos\theta = \sqrt[3]{\frac{x}{a}} \Rightarrow \cos\theta = \frac{\sqrt[3]{x}}{\sqrt[3]{a}}$ ……..(2)

Now, $y = a\sin^3\theta \Rightarrow \sin^3\theta = \frac{y}{a}$

$\Rightarrow \sin\theta = \sqrt[3]{\frac{y}{a}} \Rightarrow \sin\theta = \frac{\sqrt[3]{y}}{\sqrt[3]{a}}$ ……..(3)

From, (1), (2), (3)

$\Rightarrow \frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} = -\frac{\frac{\sqrt[3]{y}}{\sqrt[3]{a}}}{\frac{\sqrt[3]{x}}{\sqrt[3]{a}}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\sqrt[3]{\frac{y}{x}}$

7. Find $\frac{dy}{dx}$, if $= a\left(cos\,t + log\,tan\left(\frac{t}{2}\right)\right)$, $y = asin\,t$.

Solution: Now, $x = a(cost + \log\tan t/2)$,

$Diff\ w\ r\ to\ t$

$\Rightarrow \frac{dx}{dt} = a\left(-sint + \frac{1}{\tan\left(\frac{t}{2}\right)}(\sec^2 t/2)\frac{1}{2}\right)$

$\Rightarrow \frac{dx}{dt} = a\left(-sint + \frac{1}{\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)}}\frac{1}{\cos^2\left(\frac{t}{2}\right)}\frac{1}{2}\right)$

$\Rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$

$\Rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right)$

$\Rightarrow \frac{dx}{dt} = a\left(\frac{-\sin^2 t + 1}{\sin t}\right) \Rightarrow \frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right)$

Now, $y = asint$, $Diff\ w\ r\ to\ t$

$\Rightarrow \frac{dy}{dt} = a\cos t$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)}$

$\Rightarrow \frac{dy}{dx} = \cos t\frac{\sin t}{\cos^2 t} = \frac{\sin t}{\cos t} = \tan t$

8. Verify Rolle's theorem for the function $y = x^2 + 2, x \in [-2, 2]$.

Solution: Given $f(x) = x^2 + 2$

$\Rightarrow f(x)$ is a polynomial function,

$\Rightarrow f(x)$ is continuous in $[-2,2]$ and differentiable in $(-2,2)$

Now, Let $a = -2$ and $b = 2$

Now, $f(a) = f(-2) = (-2)^2 + 2$
$= 4 + 2 = 6$

Now, $f(b) = f(2) = 2^2 + 2 = 4 + 2 = 6$

Here $f(a) = f(b)$

Now, $f(x) = x^2 + 2$, $Diff\ w\ r\ to\ x$

$\Rightarrow f'(x) = 2x$

Let $f'(c) = 0 \Rightarrow 2c = 0 \Rightarrow c = 0$

Now, $c = 0 \in (-2,2)$ such that $f'(c) = 0$

$\therefore$ Rolle's theorem is verified.

9. Verify Rolles theorem for the function $f(x) = x^2 + 2x - 8,\ x \in [-4, 2]$.

Solution: Given $f(x) = x^2 + 2x - 8$,

$\Rightarrow f(x)$ is a polynomial function,

$\Rightarrow f(x)$ is continuous in $[-4,2]$ and differentiable in $(-4,2)$

Now, Let $a = -4$ and $b = 2$

Now, $f(a) = f(-4) = (-4)^2 + 2(-4) - 8$
$= 16 - 8 - 8 = 0$

Now, $f(b) = f(2) = 2^2 + 2(2) - 8$
$= 4 + 4 - 8 = 0$

Here $f(a) = f(b)$

Now, $f(x) = x^2 + 2x - 8$, $Diff\ w\ r\ to\ x$

$\Rightarrow f'(x) = 2x + 2$

Let $f'(c) = 0 \Rightarrow 2c + 2 = 0$

$\Rightarrow 2c = -2 \Rightarrow c = -1$

Now, $c = -1 \in (-4,2)$ such that $f'(c) = 0$

$\therefore$ Rolle's theorem is verified.

10. Verify mean value theorem if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$.

Solution: Given $f(x) = x^2 - 4x - 3$,

$\Rightarrow f(x)$ is a polynomial function,

$\Rightarrow f(x)$ is continuous in $[1,4]$ and differentiable in $(1,4)$

Now, given $a = 1$ and $b = 4$

Now, $f(a) = f(1) = (1)^2 - 4(1) - 3$
$= 1 - 4 - 3 = -6$ and

Now, $f(b) = f(4) = 4^2 - 4(4) - 3$
$= 16 - 16 - 3 = -3$

Now, $f(x) = x^2 - 4x - 3$, $Diff\ w\ r\ to\ x$

$\Rightarrow f'(x) = 2x - 4$

Let $f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow 2c - 4 = \frac{-3 - (-6)}{4 - 1}$

$\Rightarrow 2c - 4 = \frac{-3 + 6}{3} \Rightarrow c = \frac{3}{3} = 1$

Now, $c = 1 \in (1,4)$ such that

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\therefore$ Mean value theorem is verified.

11. **Verify Mean Value Theorem if $f(x)=x^3-5x^2-3x$ in the interval $[1,3]$.**

Solution: Given $f(x) = x^3-5x^2-3x$,

$\Rightarrow f(x)$ is a polynomial function,

$\Rightarrow f(x)$ is continuous in $[1,3]$ and differentiable in $(1,3)$

Now, Let $a=1$ and $b=3$

Now, $f(a)=f(1)=(1)^3-5(1)^2-3(1)$

$=1-5-3=-7$ and

Now, $f(b)=f(3)=3^3-5(3)^2-3(3)$

$=27-5(9)-9=27-45-9$

$=-27$

Now, $f(x)=x^3-5x^2-3x$,

$Diff\ w\ r\ to\ x \Rightarrow f'(x)=3x^2-10x-3$

Let $f'(c)=\frac{f(b)-f(a)}{b-a}$

$\Rightarrow 3c^2-10c-3=\frac{-27-(-7)}{3-1}$

$\Rightarrow 3c^2-10c-3=\frac{-27+7}{2}$

$\Rightarrow 3c^2-10c-3=\frac{-20}{2}=-10$

$\Rightarrow 3c^2-10c-3+10=0$

$\Rightarrow 3c^2-10c+7=0$

$\Rightarrow 3c^2-3c-7c+7=0$

$\Rightarrow 3c(c-1)-7(c-1)=0$

$\Rightarrow (c-1)(3c-7)=0$

$\Rightarrow c-1=0\ or\ 3c-7=0$

$\Rightarrow c=1\ or\ c=\frac{7}{3}$

Now, $c=\frac{7}{3}\in(1,3)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

$\therefore$ Mean value theorem is verified.

12. Verify mean value theorem for the function $f(x)=x^2$ in the interval $[2,4]$.

Solution: Given $f(x)=x^2$, $\Rightarrow f(x)$ is a polynomial function,

$\Rightarrow f(x)$ is continuous in $[2,4]$ and differentiable in $(2,4)$

Now, let $a=2$ and $b=4$

Now, $f(a)=f(2)=(2)^2=4$ and

$f(b)=f(4)=4^2=16$

Now, $f(x)=x^2$, $Diff\ w\ r\ to\ x$

$\Rightarrow f'(x)=2x$

Let $f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow 2c=\frac{16-4}{4-2}$

$\Rightarrow 2c=\frac{12}{2}\Rightarrow 2c=6\Rightarrow c=3$

Now, $c=3\in(2,4)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

$\therefore$ Mean value theorem is verified.

FOUR MARKS QUESTION

1. **Find the value of k, if $f(x)=\begin{cases}\frac{k\cos x}{\pi-2x} & if\ x\neq\frac{\pi}{2}\\ 3 & if\ x=\frac{\pi}{2}\end{cases}$ is continuous at $x=\frac{\pi}{2}$**

Solution: Given $f(x)$ is continuous at $x=\frac{\pi}{2}$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right) \Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi-2x}=3$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{k\sin\left(\frac{\pi}{2}-x\right)}{2\left(\frac{\pi}{2}-x\right)}=3 \Rightarrow \frac{k}{2}(1)=3$

$\Rightarrow k=6$

2. **Find the value of K so that the function $f(x)=\begin{cases}kx+1, & if\ x\le 5\\ 3x-5, & if\ x>5\end{cases}$ at $x=5$ is a continuous function.**

Solution: Given $f(x)$ is continuous at $x=5$

$\Rightarrow LHL=RHL \Rightarrow \lim_{x\to5^-} f(x)=\lim_{x\to5^+} f(x)$

$\Rightarrow \lim_{x\to5^-}(kx+1)=\lim_{x\to5^+} 3x-5$

$\Rightarrow 5k+1=3(5)-5$

$\Rightarrow 5k+1=15-5 \Rightarrow 5k+1=10$

$\Rightarrow 5k=10-1 \Rightarrow k=\frac{9}{5}$

3. **Find the values of a and b such that $f(x)=\begin{cases}5 & if\ x\le 2\\ ax+b & if\ 2<x<10\\ 21 & if\ x\ge 10\end{cases}$ is a continuous function.**

Solution: Given $f(x)$ is continuous at $x=2$

$\Rightarrow LHL=RHL \Rightarrow \lim_{x\to2^-} f(x)=\lim_{x\to2^+} f(x)$

$\Rightarrow \lim_{x\to2^-} 5=\lim_{x\to2^+}(ax+b)$

$\Rightarrow 5=2a+b \Rightarrow 2a+b=5 \ldots\ldots(1)$

Given $f(x)$ is continuous at $x=10$

$\Rightarrow LHL=RHL \Rightarrow \lim_{x\to10^-} f(x)=\lim_{x\to10^+} f(x)$

$\Rightarrow \lim_{x\to10^-}(ax+b)=\lim_{x\to10^+} 21$

$\Rightarrow 10a+b=21 \ldots\ldots\ldots\ldots(2)$

$eq\,(1) - eq\,(2)$

$\Rightarrow 2a + b - (10a + b) = 5 - 21$

$\Rightarrow 2a + b - 10a - b = -16$

$\Rightarrow -8a = -16 \Rightarrow a = \frac{16}{8} \Rightarrow \boxed{a = 2}$

$From\ eq(1) \Rightarrow 2a + b = 5$

$\Rightarrow 2(2) + b = 5 \Rightarrow b = 5 - 4$

$\Rightarrow \boxed{b = 1}$

4. Find the value of K, if $f(x) = \begin{cases} kx^2 & if\ x \le 2 \\ 3 & if\ x > 2 \end{cases}$ is continuous at $x = 2$.

Solution: Given $f(x)$ is continuous at $x = 2$

$\Rightarrow LHL = RHL \Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\Rightarrow \lim_{x\to 2^-} kx^2 = \lim_{x\to 2^+} 3$

$\Rightarrow k(2)^2 = 3 \Rightarrow 4k = 3$

$\Rightarrow k = \frac{3}{4}$

5. For what value of λ is the function $f(x) = \begin{cases} \lambda(x^2 - 2x) & if\ x \le 0 \\ 4x + 1 & if\ x > 0 \end{cases}$ continuous at $x = 0$?

Solution: Given $f(x)$ is continuous at $x = 0$

$\Rightarrow LHL = RHL \Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0^-} \lambda(x^2 - 2x) = \lim_{x\to 0^+} (4x + 1)$

$\Rightarrow \lambda(0^2 - 2(0)) = 4(0) + 1$

$\Rightarrow \lambda(0) = 0 + 1 \Rightarrow 0 = 1,$

It is not possible

$\therefore$ λ does not exist.

6. Find the relationship between 'a' and 'b' so that the function 'f' defined by $f(x) = \begin{cases} ax + 1 & if\ x \le 3 \\ bx + 3 & if\ x > 3 \end{cases}$ is continuous at $x = 3$.

Solution: Given $f(x)$ is continuous at $x = 3$

$\Rightarrow LHL = RHL \Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$\Rightarrow \lim_{x\to 3^-} ax + 1 = \lim_{x\to 3^+} bx + 3$

$\Rightarrow 3a + 1 = 3b + 3 \Rightarrow 3a = 3b + 3 - 1$

$\Rightarrow 3a = 3b + 2$

FIVE MARKS QUESTION

1. **If $y = (tan^{-1} x)^2$ then prove that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$**

Solution: Now, $y = (\tan^{-1} x)^2$,

$Diff\ w\,r\,t\ \ x$

$\Rightarrow y_1 = 2\tan^{-1} x \frac{1}{1+x^2}$ $(\times^{ly}\ by\ 1 + x^2)$

$\Rightarrow (1 + x^2)\, y_1 = 2\tan^{-1} x$, $Diff\ w\,r\,t\ 'x'$

$\Rightarrow (1 + x^2)y_2 + y_1(0 + 2x) = 2\frac{1}{1+x^2}$

$(\times^{ly}\ by\ 1 + x^2)$

$\Rightarrow (1 + x^2)[(1 + x^2)y_2 + 2xy_1] = 2$

$\Rightarrow (1 + x^2)^2 y_2 + 2x(1 + x^2)y_1 = 2$

$\Rightarrow (x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$

2. **If $y = sin^{-1} x$ then prove that $(1 - x^2)y_2 - xy_1 = 0$**

Solution: Now, $= \sin^{-1} x$, $Diff\ w\,r\,t\ \ x$

$\Rightarrow y_1 = \frac{1}{\sqrt{1-x^2}}$ $(\times^{ly}\ by \sqrt{1 - x^2})$

$\Rightarrow \sqrt{1 - x^2}\, y_1 = 1$ $Diff\ w\,r\,t\ \ x$

$\Rightarrow \sqrt{1 - x^2}\, y_2 + y_1\left(\frac{1}{2\sqrt{1-x^2}}(-2x)\right) = 0$

$\Rightarrow \sqrt{1 - x^2}\, y_2 + y_1\left(\frac{-x}{\sqrt{1-x^2}}\right) = 0$

$(\times^{ly}\ by \sqrt{1 - x^2})$

$\Rightarrow \left(\sqrt{1 - x^2}\right)^2 y_2 + y_1(-x) = 0$

$\Rightarrow (1 - x^2)y_2 - xy_1 = 0$

3. **If $y = 3\,cos(log\,x) + 4\,sin(log\,x)$ then prove that $x^2y_2 + xy_1 + y = 0$**

Solution:

Now, $y = 3\cos(logx) + 4\sin(logx)$,

$Diff\ w\,r\,t\ \ x$

$\Rightarrow y_1 = 3(-\sin(logx))\frac{1}{x} + 4\cos(logx)\frac{1}{x}$

$(\times^{ly}\ by\ x)$

$\Rightarrow xy_1 = -3\sin(logx) + 4\cos(logx)$,

$Diff\ w\,r\,t\ \ x$

$\Rightarrow xy_2 + y_1(1) = -3\cos(logx)\frac{1}{x} + 4\left[-\sin(logx)\frac{1}{x}\right]$

$(\times^{ly}\ by\ x)$

$\Rightarrow x^2y_2 + xy_1 = -3\cos(logx) - 4\sin(logx)$

$\Rightarrow x^2y_2 + xy_1 = -(3\,cos(log\,x) + 4\,sin(log\,x))$

$\Rightarrow x^2y_2 + xy_1 = -y \Rightarrow x^2y_2 + xy_1 + y = 0$

4. **If $y = 500\,e^{7x} + 600\,e^{-7x}$ then prove that $y_2 = 49y$**

Solution: Now, $y = 500\,e^{7x} + 600\,e^{-7x}$,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow y_1 = 500\,e^{7x}(7) + 600e^{-7x}(-7)$

$\Rightarrow y_1 = 7(500\,e^{7x}) - 7(600\,e^{-7x})$,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow y_2 = 7(500\,e^{7x}\,(7)) - 7(600\,e^{-7x}(-7))$

$\Rightarrow y_2 = 49(500\,e^{7x}) + 49(600\,e^{-7x})$

$\Rightarrow y_2 = 49(500\,e^{7x} + 600\,e^{-7x})$

$\Rightarrow y_2 = 49y$

5. **If $y = Ae^{mx} + Be^{nx}$ then prove that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$**

Solution: Now, $y = Ae^{mx} + Be^{nx}$,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow \frac{dy}{dx} = Ae^{mx}(\mathrm{m}) + Be^{nx}(n)$

$\Rightarrow \frac{dy}{dx} = Ame^{mx} + Bne^{nx}$ ……..(1)

$Diff\ w\ r\ t\ \ x$

$\Rightarrow \frac{d^2y}{dx^2} = Ame^{mx}(\mathrm{m}) + Bne^{nx}(n)$

$\Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$ ………..(2)

Now, $LHS = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= (Am^2e^{mx} + Bn^2e^{nx})$
$\quad -(m+n)(Ame^{mx} + Bne^{nx})$
$\quad +mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx}$
$\quad -Bmne^{nx} - Amne^{mx} - Bn^2e^{nx}$
$\quad +Amne^{mx} + Bmne^{nx}$

$= 0 = \mathrm{RHS}$

6. **If $y = 3e^{2x} + 2e^{3x}$ then prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$**

Solution: Now, $y = 3e^{2x} + 2e^{3x}$,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow \frac{dy}{dx} = 3e^{2x}(2) + 2e^{3x}(3)$

$\Rightarrow \frac{dy}{dx} = 6e^{2x} + 6e^{3x}$ ……..(1)

$Diff\ w\ r\ t\ \ x$

$\Rightarrow \frac{d^2y}{dx^2} = 6e^{2x}(2) + 6e^{3x}(3)$

$\Rightarrow \frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x}$ ………..(2)

Now, $LHS = \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y$

$= (12e^{2x} + 18e^{3x}) - 5(6e^{2x} + 6e^{3x})$
$\quad +6(3e^{2x} + 2e^{3x})$

$= 12e^{2x} + 18e^{3x} - 30e^{2x} - 30e^{3x}$
$\quad + 18e^{2x} + 12e^{3x}$

$= e^{2x}(12 - 30 + 18) + e^{3x}(18 - 30 + 12)$

$= e^{2x}(0) + e^{3x}(0) = 0 = RHS$

7. **If $e^y\,(x+1) = 1$, prove that $\frac{dy}{dx} = -e^y$ and hence prove that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$**

Solution: Now, $e^y\,(x+1) = 1$ …..(1) ,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow e^y\,(1+0) + (x+1)e^y\frac{dy}{dx} = 0$

$\Rightarrow e^y + (x+1)e^y\frac{dy}{dx} = 0$

$\Rightarrow (x+1)e^y\frac{dy}{dx} = -e^y$

$\xrightarrow{from\ (1)} (1)\frac{dy}{dx} = -e^y$

$\Rightarrow \frac{dy}{dx} = -e^y$ ………….(2) $Diff\ w\ r\ t\ \ x$

$\Rightarrow \frac{d^2y}{dx^2} = -e^y\frac{dy}{dx}$

$\xrightarrow{from\ (2)} \frac{d^2y}{dx^2} = \frac{dy}{dx}\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

8. If $y = e^{a\cos^{-1}x}$ then prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Solution: Now, $y = e^{a\cos^{-1}x}$ ……..(1) ,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}x}a\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$\xrightarrow{from\ (1)} \frac{dy}{dx} = ya\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow \frac{dy}{dx} = -\frac{ay}{\sqrt{1-x^2}}$ $\quad (\times^{ly}\ by\sqrt{1-x^2})$

$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = -ay$ ……..(2) ,

$Diff\ w\ r\ t\ \ x$

$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{2\sqrt{1-x^2}}(-2x)\right) = -a\frac{dy}{dx}$

$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{-x}{\sqrt{1-x^2}}\right) = -a\frac{dy}{dx}$

$(\times^{ly}\ by\sqrt{1-x^2})$

$\Rightarrow \left(\sqrt{1-x^2}\right)^2\frac{d^2y}{dx^2} + \frac{dy}{dx}(-x) =$
$-a\sqrt{1-x^2}\,\frac{dy}{dx}$

$\Rightarrow (1-x^2)\,\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\,(-ay) = a^2y$

$\Rightarrow (1-x^2)\,\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

APPLICATION OF DIFFERENTIATION

FIVE MARKS QUESTION

1. **The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of an edge is 10 centimeters?**

Solution: Let x be the edge , S be the surface area and V be the volume of the cube.

Given: $\frac{dV}{dt} = 9\ cm^3/s$, $\frac{dS}{dt} =?$, when $x = 10cm$

W.K.T , $V = x^3$, $Diff\ w.r.t\ t$

$\Rightarrow \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow 9 = 3(10)^2 \frac{dx}{dt}$

$\Rightarrow 9 = 3(100)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{3(100)}$

$\Rightarrow \frac{dx}{dt} = \frac{3}{100} cm/s$

W.K.T , $S = 6x^2$, $Diff\ w.r.t\ t$

$\Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12(10)\frac{3}{100}$

$\Rightarrow \frac{dS}{dt} = \frac{12(3)}{10} = \frac{18}{5}\ cm^2/s$

2. **The volume of a cube is increasing at a rate of 8 cubic centimeters per second. How fast is the surface area increasing when the length of an edge is 12 centimeters?**

Solution: Let x be the edge , S be the surface area and V be the volume of the cube.

Given: $\frac{dV}{dt} = 8cm^3/s$, $\frac{dS}{dt} =?$, when $x = 12cm$

W.K.T , $V = x^3$, $Diff\ w.r.t\ t$

$\Rightarrow \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow 8 = 3(12)^2 \frac{dx}{dt}$

$\Rightarrow 8 = 3(144)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3(144)} = \frac{1}{3(18)}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{54} cm/s$

W.K.T , $S = 6x^2$, $Diff\ w.r.t\ t$

$\Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12(12)\frac{1}{54}$

$\Rightarrow \frac{dS}{dt} = \frac{12(2)}{9} = \frac{8}{3}\ cm^2/s$

3. **The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8cm$ and $y = 6cm$, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.**

Solution: Given: x is the length and y is the width of the rectangle.

Let A be the area and P be the Perimeter of the rectangle.

Given: $\frac{dx}{dt} = -5\ cm/min$, $\frac{dy}{dt} = 4\ cm/min$, $\frac{dA}{dt} =?$, $\frac{dP}{dt} =?$ $when\ x = 8cm , y = 6cm$

(a) W.K.T $P = 2(x+y)$, $Diff\ w\,r\,t\ 't'$

$\Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$

$\Rightarrow \frac{dP}{dt} = 2(-5+4) \Rightarrow \frac{dP}{dt} = 2(-1)$

$\Rightarrow \frac{dP}{dt} = -2cm/min$

∴ rate of perimeter decreasing $2cm/min$

(b) $W.K.T\ \ A = xy$, Diff $w\,r\,t\ 't'$

$\Rightarrow \frac{dA}{dt} = \ x\frac{dx}{dt} + y\frac{dx}{dt}$

$\Rightarrow \frac{dA}{dt} = 8\ (4) + 6(-5) = 32 - 30 = 2$

$\Rightarrow \frac{dA}{dt} = 2\ cm^2/min.$

∴ rate of area increasing $2cm^2\ /min.$

4. **The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When $x = 10cm$ and $y = 6cm$, find the rates of change of (a) the perimeter and (b) the area of the rectangle.**

Solution: Given: x is the length and y is the width of the rectangle.

Let A be the area and P be the Perimeter of the rectangle.

Given: $\frac{dx}{dt} = -3\ cm/min$, $\frac{dy}{dt} = 2\ cm/min$, $\frac{dA}{dt} =?$, $\frac{dP}{dt} =?$ $when\ x = 10\ cm , y = 6\ cm$

(a) W.K.T $P = 2(x+y)$, $Diff\ w\,r\,t\ 't'$

$\Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$

$\Rightarrow \frac{dP}{dt} = 2(-3+2) = 2(-1)$

$\Rightarrow \frac{dP}{dt} = -2cm/min$

∴ rate of perimeter decreasing $2cm/min$

(b) $W.K.T\ \ A = xy$, Diff $w\,r\,t\ 't'$

$\Rightarrow \frac{dA}{dt} = \ x\frac{dx}{dt} + y\frac{dx}{dt}$

$\Rightarrow \frac{dA}{dt} = 10\ (2) + 6(-3) = 20 - 18 = 2$

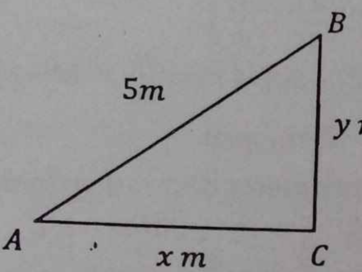
$\Rightarrow \frac{dA}{dt} = 2\ cm^2/min.$

∴ rate of area increasing $2cm^2\ /min.$

5. **A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?**

Solution: Let $AB = 5m$ be the length of ladder , Let $AC = x\ m$ and $CB = y\ m$

Given: $\frac{dx}{dt} = 2cm/sec,\ \frac{dy}{dt} =?$, when $x = 4\ m$



From figure, $AC^2 + CB^2 = AB^2$

$\Rightarrow x^2 + y^2 = 5^2$

$\Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 25 - 16$

$\Rightarrow y^2 = 9 \Rightarrow y = 3m$

Also we have $x^2 + y^2 = 5^2$, Diff $w\ r\ t\ 't'$

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt} = -\frac{2x}{2y}\frac{dx}{dt} = -\frac{x}{y}\frac{dx}{dt}$

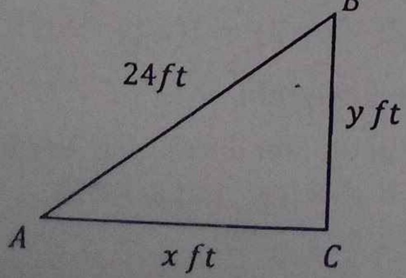
$\Rightarrow \frac{dy}{dt} == -\frac{4\ m}{3\ m}\ 2cm/sec$

$\Rightarrow \frac{dy}{dt} = -\frac{8}{3}\ cm/sec$

6. **A ladder 24 ft long leans against a vertical wall. The lower end is moving away at the rate of 3ft/sec. Find the rate at which the top of the ladder is moving down wars. If its foot is $8ft$ from the wall.**

Solution: Let $AB = 24ft$ be the length of ladder , Let $AC = x\ ft$ and $CB = y\ ft$

Given: $\frac{dx}{dt} = 3ft/sec,\ \frac{dy}{dt} =?$, $when\ x = 8ft$



From figure, $AC^2 + CB^2 = AB^2$

$\Rightarrow x^2 + y^2 = 24^2$

$\Rightarrow 8^2 + y^2 = 576 \Rightarrow y^2 = 576 - 64$

$\Rightarrow y^2 = 512 \Rightarrow y = \sqrt{512}\ ft = 16\sqrt{2}\ ft$

Also we have $x^2 + y^2 = 24^2$, Diff $w\ r\ t\ 't'$

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt} = -\frac{2x}{2y}\frac{dx}{dt} = -\frac{x}{y}\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt} = -\frac{8\ ft}{16\sqrt{2}\ ft}\ 3\ ft/sec$

$\Rightarrow \frac{dy}{dt} = -\frac{3}{2\sqrt{2}}\ ft/sec$

7. **Sand is pouring from a pipe at the rate of $12cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?**

Solution: Let h be the height, r be the base radius and V be the volume of the sand cone.

Given : $\frac{dV}{dt} = 12\ cm^3/sec,\ \frac{dh}{dt} =?$,

$when\ h = 4\ cm$

Given : $h = \frac{1}{6}r \Rightarrow r = 6h$

$W.K.T\ \ V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi(6h)^2 h$

$\Rightarrow V = \frac{1}{3}\pi 36h^3$

$\Rightarrow V = 12\pi h^3$, $Diff\ w\ r\ t\ 't'$

$\Rightarrow \frac{dv}{dt} = 12\pi 3h^2\frac{dh}{dt} \Rightarrow 12 = 12\pi 3\,(4)^2\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{12}{12\pi 48} \Rightarrow \frac{dh}{dt} = \frac{1}{48\pi}$

8. **A particle moves along the curve $6y = x^3 + 2$ then find the points on the curve at which the $y-$ co-ordinate is changing 8 times as fast as the x- co-ordinate.**

Solution: Let (x, y) be any point on the curve

Given : $\frac{dy}{dt} = 8\frac{dx}{dt}$

Now, $6y = x^3 + 2$, $Diff\ w\ r\ t\ 't'$

$\Rightarrow 6\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 6 \times 8\frac{dx}{dt} = 3x^2\frac{dx}{dt}$

$\Rightarrow 48 = 3x^2$,

$\Rightarrow x^2 = \frac{48}{3} \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Now, $6y = x^3 + 2 \Rightarrow y = \frac{x^3+2}{6}$

$If\ x = 4 \Rightarrow y = \frac{4^3+2}{6} = \frac{64+2}{6} = \frac{66}{6} = 11$

$If\ x = -4 \Rightarrow y = \frac{(-4)^3+2}{6} = \frac{-64+2}{6} = \frac{-62}{6} = -\frac{31}{3}$

$\therefore$ $(4,11)\ and\ \left(-4, -\frac{31}{3}\right)$ are the points on the curve.

INTEGRATION

THREE MARKS QUESTION

1. **Evaluate $\int \frac{x}{(x+1)(x+2)}$ dx.**

   Solution: Let , $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

   Where,

   $A = \left[\frac{x}{(x+2)}\right]_{x=-1} = \frac{-1}{(-1+2)} = -\frac{1}{1} = -1$

   $B = \left[\frac{x}{(x+1)}\right]_{x=-2} = \frac{-2}{(-2+1)} = \frac{-2}{-1} = 2$

   $\Rightarrow \frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

   $\Rightarrow \int \frac{x}{(x+1)(x+2)}\, dx = \int \frac{-1}{x+1}\, dx + \int \frac{2}{x+2} dx$

   $\Rightarrow \int \frac{x}{(x+1)(x+2)}\, dx = -\int \frac{1}{x+1}\, dx + 2\int \frac{1}{x+2} dx$

   $= -\log|x+1| + 2\log|x+2| + c$

2. **Find $\int \frac{x}{(x-1)(x-2)} dx$**

   Solution: Let , $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

   Where, $A = \left[\frac{x}{(x-2)}\right]_{x=1} = \frac{1}{(1-2)} = \frac{1}{-1} = -1$

   $B = \left[\frac{x}{(x-1)}\right]_{x=2} = \frac{2}{(2-1)} = \frac{2}{1} = 2$

   $\Rightarrow \frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$

   $\Rightarrow \int \frac{x}{(x-1)(x-2)}\, dx = \int \frac{-1}{x+1}\, dx + \int \frac{2}{x+2} dx$

   $\Rightarrow \int \frac{x}{(x-1)(x-2)}\, dx = -1\int \frac{1}{x-1}\, dx + 2\int \frac{1}{x-2} dx$

   $= -\log|x-1| + 2\log|x-2| + c$

3. **Evaluate: $\int \frac{2x}{x^2+3x+2} dx.$**

   Solution: $x^2 + 3x + 2 = x^2 + 2x + x + 2$

   $\Rightarrow x^2 + 3x + 2 = x(x+2) + 1(x+2)$

   $\Rightarrow x^2 + 3x + 2 = (x+1)(x+2)$

   Let , $\frac{2x}{x^2+3x+2} = \frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

   Where,

   $A = \left[\frac{2x}{(x+2)}\right]_{x=-1} = \frac{-2}{(-1+2)} = -\frac{2}{1} = -2$

   $B = \left[\frac{2x}{(x+1)}\right]_{x=-2} = \frac{-4}{(-2+1)} = \frac{-4}{-1} = 4$

   $\Rightarrow \frac{x}{(x+1)(x+2)} = \frac{-2}{x+1} + \frac{4}{x+2}$

   $\Rightarrow \int \frac{x}{(x+1)(x+2)}\, dx = \int \frac{-2}{x+1}\, dx + \int \frac{4}{x+2} dx$

   $\Rightarrow \int \frac{x}{(x+1)(x+2)}\, dx = -2\int \frac{1}{x+1}\, dx + 4\int \frac{1}{x+2} dx$

   $= -2\log|x+1| + 4\log|x+2| + c$

4. **Evaluate: $\int \frac{dx}{x(x^2+1)}$**

   Solution: Let , $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

   $\Rightarrow 1 = A(x^2+1) + (Bx+C)x$

   Where, $A = \left[\frac{1}{(x^2+1)}\right]_{x=0} = \frac{1}{(0+1)} = \frac{1}{1} = 1$

   Now, equating the coefficient of x^2

   $\Rightarrow 0 = A + B \Rightarrow B = -A \Rightarrow B = -1$

   Now, equating the coefficient of x

   $\Rightarrow 0 = 0 + C \Rightarrow C = 0$

   Now, $\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-1x+0}{x^2+1}$

   $\Rightarrow \frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$

   $\Rightarrow \int \frac{dx}{x(x^2+1)} = \int \frac{1}{x}\, dx - \int \frac{x}{x^2+1} dx$

   $= \int \frac{1}{x}\, dx - \frac{1}{2}\int \frac{2x}{x^2+1} dx$

   $\left(\because \int \frac{f'(x)}{f(x)} dx = \log f(x) + c\right)$

   $\Rightarrow \int \frac{dx}{x(x^2+1)} = \log x - \frac{1}{2}\log(x^2+1) + c$

5. **Evaluate $\int \tan^{-1} x\, dx$**

   Solution:

   Let , $\int \tan^{-1} x\, dx = \int \tan^{-1} x\ (1)\, dx$

   $= \int \underset{1st}{\tan^{-1} x}\ \underset{2nd}{(1)}\, dx$

   $= \tan^{-1} x \int 1\, dx - \int x \left(\frac{1}{1+x^2}\right) dx$

   $= \tan^{-1} x\ (x) - \int \frac{1}{2}\left(\frac{2x}{1+x^2}\right) dx$

   $= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$

   $= x\tan^{-1} x - \frac{1}{2}\log(1+x^2)$

6. **Integrate $x^2 e^x$ with respect to x**

   Solution: Let , $\int x^2 e^x dx = \int \underset{1st}{x^2}\underset{2nd}{e^x}\, dx$

   $= x^2 \int e^x\, dx - \int e^x(2x) dx$

   $= x^2 e^x - 2\int e^x x dx$

   $= x^2 e^x - 2\int \underset{1st}{x}\underset{2nd}{e^x}\, dx$

   $= x^2 e^x - 2(x\int e^x\, dx - \int e^x(1) dx)$

   $= x^2 e^x - 2(xe^x - e^x) + c$

   $= x^2 e^x - 2xe^x + 2e^x + c$

7. **Evaluate $\int x \tan^{-1} x \, dx$**

Solution:

Let , $\int x\tan^{-1}x\,dx = \int \tan^{-1}x\ (x)\ dx$

$= \int \underset{1st}{\tan^{-1}x}\ \underset{2nd}{(x)}\ dx$

$= \tan^{-1}x \int x\,dx - \int \frac{x^2}{2}\left(\frac{1}{1+x^2}\right)dx$

$= \tan^{-1}x\left(\frac{x^2}{2}\right) - \frac{1}{2}\int\left(\frac{x^2}{1+x^2}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(\frac{x^2+1-1}{1+x^2}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + c$

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c$

8. **Find $\int x^2 \log x\, dx$**

Solution:

Let , $\int x^2\log x\,dx = \int \underset{1st}{\log x}\ \underset{2nd}{(x^2)}\ dx$

$= \log x \int x^2\,dx - \int \frac{x^3}{3}\left(\frac{1}{x}\right)dx$

$= \log x\frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$

$= \left(\frac{x^3}{3}\right)\log x - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$

$= \left(\frac{x^3}{3}\right)\log x - \frac{x^3}{9} + c$

9. **Evaluate $\int x\log x\,dx$**

Solution:

Let , $\int x\log x\,dx = \int \underset{1st}{\log x}\ \underset{2nd}{(x)}\ dx$

$= \log x \int x\,dx - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx$

$= \log x\frac{x^2}{2} - \frac{1}{2}\int x\,dx$

$= \left(\frac{x^2}{2}\right)\log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$

$= \left(\frac{x^2}{2}\right)\log x - \frac{x^2}{4} + c$

10. **Integrate $x\,sec^2x$ with respect to x**

Solution:

Let , $\int x sec^2x\,dx = \int \underset{1st}{x}\ \underset{2nd}{(\sec^2 x)}\ dx$

$= x\int \sec^2 x\,dx - \int \tan x\ (1)dx$

$= x(\tan x) - \int \tan x\,dx$

$= x\tan x - (-\log|\cos x|) + c$

$= x\tan x + \log|\cos x| + c$

no number of A is related to any element of A

11. **Evaluate $\int e^x \sin x\ dx$**

Solution: Let , $I = \int e^x \sin x\ dx \ldots\ldots(1)$

$\Rightarrow I = \int \underset{1st}{\sin x}\ \underset{2nd}{(e^x)}\ dx$

$= \sin x \int e^x\,dx - \int e^x\ (\cos x)dx$

$\Rightarrow I = \sin x\ (e^x) - \int \underset{1st}{\cos x}\ \underset{2nd}{(e^x)}\ dx$

$= e^x\sin x - [\cos x \int e^x\,dx - \int e^x\ (-\sin x)dx]$

$\Rightarrow I = e^x\sin x - [\cos x\ e^x + \int e^x\ (\sin x)dx]$

$= e^x\sin x - [\cos x\ e^x + I]\quad (From\ (1))$

$\Rightarrow I = e^x\sin x - e^x\cos x - I$

$\Rightarrow I + I = e^x\sin x - e^x\cos x$

$\Rightarrow 2I = e^x(\sin x - \cos x)$

$\Rightarrow I = \frac{e^x}{2}(\sin x - \cos x) + c$

12. **Integrate $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$ with respect to x**

Solution: Let , $\cos^{-1}x = t \Rightarrow x = \cos t$,

$differentiate\ \cos^{-1}x = t$

$\Rightarrow -\frac{1}{\sqrt{1-x^2}}dx = dt \Rightarrow \frac{dx}{\sqrt{1-x^2}} = -dt$

Now, $\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}\ dx = \int \cos t\ (t)(-dt)$

$= -\int \underset{1st}{t}\ \underset{2nd}{(\cos t)}\ dt = -(t\int\cos t\,dt - \int(\sin t)dt$

$= -\big(t(\sin t) - (-\cos t)\big) + c$

$= -t(\sin t) - \cos t + c$

$= -(\cos^{-1}x)\sin(\cos^{-1}x) - x + c$

SIX MARKS QUESTION

1. **Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$ and hence integrate the following.**

   1) $\int_{-5}^{5}|x+2|\ dx$ 2) $\int_2^8|x-5|\ dx$

   3) $\int_0^4|x-1|\ dx$

Solution: Proof: Let $\int f(x)dx = F(x) + c$

LHS $= \int_a^b f(x)dx = [F(x)]_a^b$

$= F(b) - F(a)\ \ldots\ldots\ldots\ldots(1)$

RHS $= \int_a^c f(x)dx + \int_c^b f(x)dx$

$= [F(x)]_a^c + [F(x)]_c^b$

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)\ \ldots\ldots\ldots\ldots(2)$

From (1) and (2) ,

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

1) $\int_{-5}^{5}|x+2|\ dx$

$=\int_{-5}^{-2}|x+2|\ dx+\int_{-2}^{5}|x+2|\ dx$

$=\int_{-5}^{-2}-(x+2)\ dx+\int_{-2}^{5}(x+2)\ dx$

$=-\int_{-5}^{-2}(x+2)\ dx+\int_{-2}^{5}(x+2)\ dx$

$=-\left[\frac{(x+2)^2}{2}\right]_{-5}^{-2}+\left[\frac{(x+2)^2}{2}\right]_{-2}^{5}$

$=-\frac{1}{2}[(-2+2)^2-(-5+2)^2]$

$+\frac{1}{2}[(5+2)^2-(-2+2)^2]$

$=-\frac{1}{2}[(0)^2-(-3)^2]+\frac{1}{2}[(7)^2-(0)^2]$

$=-\frac{1}{2}(0-9)+\frac{1}{2}(49-0)$

$=\frac{9}{2}+\frac{49}{2}=\frac{9+49}{2}=\frac{58}{2}=29$

2) $\int_{2}^{8}|x-5|\ dx$

$=\int_{2}^{5}|x-5|\ dx+\int_{5}^{8}|x-5|\ dx$

$=\int_{2}^{5}-(x-5)\ dx+\int_{5}^{8}(x-5)\ dx$

$=-\int_{2}^{5}(x-5)\ dx+\int_{5}^{8}(x-5)\ dx$

$=-\left[\frac{(x-5)^2}{2}\right]_{2}^{5}+\left[\frac{(x-5)^2}{2}\right]_{5}^{8}$

$=-\frac{1}{2}[(5-5)^2-(2-5)^2]$

$+\frac{1}{2}[(8-5)^2-(5-5)^2]$

$=-\frac{1}{2}[(0)^2-(-3)^2]+\frac{1}{2}[(3)^2-(0)^2]$

$=-\frac{1}{2}(0-9)+\frac{1}{2}(9-0)$

$=\frac{9}{2}+\frac{9}{2}=\frac{9+9}{2}=\frac{18}{2}=9$

3) $\int_{0}^{4}|x-1|\ dx$

$=\int_{0}^{1}|x-1|\ dx+\int_{1}^{4}|x-1|\ dx$

$=\int_{0}^{1}-(x-1)\ dx+\int_{1}^{4}(x-1)\ dx$

$=-\int_{0}^{1}(x-1)\ dx+\int_{1}^{4}(x-1)\ dx$

$=-\left[\frac{(x-1)^2}{2}\right]_{0}^{1}+\left[\frac{(x-1)^2}{2}\right]_{1}^{4}$

$=-\frac{1}{2}[(1-1)^2-(0-1)^2]$

$+\frac{1}{2}[(4-1)^2-(1-1)^2]$

$=-\frac{1}{2}[(0)^2-(-1)^2]+\frac{1}{2}[(3)^2-(0)^2]$

$=-\frac{1}{2}(0-1)+\frac{1}{2}(9-0)$

$=\frac{1}{2}+\frac{9}{2}=\frac{1+9}{2}=\frac{10}{2}=5$

2. Prove that $\int_0^a f(x).dx=\int_0^a f(a-x).dx$ and hence evaluate

(1) $\int_0^{\frac{\pi}{4}}\log(1+\tan x)\ .dx.$

(2) $\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\ dx.$

(3) $\int_0^{\frac{\pi}{2}}2\log\sin x-\log\sin 2x)\ dx$

(4) $\int_0^{a}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

Solution: Proof: Let $a-x=t$, Differentiate

$\Rightarrow 0-dx=dt\Rightarrow dx=-dt$

If $x=0\Rightarrow t=a-0=a,$

If $x=a\Rightarrow t=a-a=0$

Now, $RHS=\int_0^a f(a-x)dx$

$=\int_a^0 f(t)(-dt)=-\int_a^0 f(t)dt$

$=\int_0^a f(t)dt\ =\ \int_0^a f(x)dx=LHS$

$\therefore\ \int_0^a f(x)dx=\int_0^a f(a-x)dx$

1) $\int_0^{\frac{\pi}{4}}\log(1+\tan x)\ dx.$

Let $I=\int_0^{\frac{\pi}{4}}\log(1+\tan x)\ dx$ …………..(1)

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]\ dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]\ dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left[1+\frac{1-\tan x}{1+(1)\tan x}\right]\ dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left[1+\frac{1-\tan x}{1+\tan x}\right]\ dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]\ dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left[\frac{2}{1+\tan x}\right]\ dx$ …………..(2)

Now, $eq(1)+eq(2)$

$\Rightarrow I+I=\int_0^{\frac{\pi}{4}}\log(1+\tan x)\ dx\ +\int_0^{\frac{\pi}{4}}\log\left[\frac{2}{1+\tan x}\right]\ dx$

$\Rightarrow 2I=\int_0^{\frac{\pi}{4}}\left[\log(1+\tan x)+\ \log\left[\frac{2}{1+\tan x}\right]\right]dx$

$\Rightarrow 2I=\int_0^{\frac{\pi}{4}}\log\left[(1+\tan x)\left(\frac{2}{1+\tan x}\right)\right]\ dx$

$\Rightarrow 2I=\int_0^{\frac{\pi}{4}}\log 2\ dx\Rightarrow 2I=\log 2\ \int_0^{\frac{\pi}{4}}1\,dx$

$\Rightarrow I=\frac{1}{2}\log 2\ [x]_0^{\frac{\pi}{4}}=\frac{1}{2}\log 2\left(\frac{\pi}{4}-0\right)$

$\Rightarrow I=\frac{1}{2}\log 2\left(\frac{\pi}{4}\right)=\frac{\pi}{8}\log 2$

(2) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx.$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \ldots\ldots\ldots\ldots (1)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \ldots\ldots\ldots\ldots (2)$

Now, $eq(1) + eq(2)$

$\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right] dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right] dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\, dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$

$\Rightarrow I = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$

(3) $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\, dx$

Let $I = \int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} [\log(\sin x)^2 - \log(2\sin x\cos x)]\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin^2 x}{2\sin x\cos x}\right) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin x}{2\cos x}\right) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\tan x\right) dx \ldots\ldots\ldots\ldots (1)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\tan\left(\frac{\pi}{2} - x\right)\right) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\cot x\right) dx \ldots\ldots\ldots (2)$

Now, $eq(1) + eq(2)$

$\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\tan x\right) dx + \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\cot x\right) dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left[\log\left(\frac{1}{2}\tan x\right) + \log\left(\frac{1}{2}\cot x\right)\right] dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left[\left(\frac{1}{2}\tan x\right)\left(\frac{1}{2}\cot x\right)\right] dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left[\frac{1}{4}\tan x\cot x\right] dx$

$\Rightarrow I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left[\frac{1}{4}\right] dx = \frac{1}{2}\log\left[\frac{1}{4}\right]\int_0^{\frac{\pi}{2}} 1\, dx$

$\Rightarrow I = \frac{1}{2}\log\left[\frac{1}{4}\right][x]_0^{\frac{\pi}{2}} = \frac{1}{2}\log\left[\frac{1}{4}\right]\left(\frac{\pi}{2} - 0\right)$

$\Rightarrow I = \frac{1}{2}\log\left[\frac{1}{4}\right]\frac{\pi}{2} = \frac{\pi}{4}\log\left(\frac{1}{4}\right)$

(4) $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$

Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx \ldots\ldots\ldots\ldots (1)$

$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}\, dx$

$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-a+x}}\, dx$

$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\, dx \ldots\ldots\ldots\ldots (2)$

Now, $eq(1) + eq(2)$

$\Rightarrow I + I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\, dx$

$\Rightarrow 2I = \int_0^a \left[\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\right] dx$

$\Rightarrow 2I = \int_0^a \left[\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\right] dx$

$\Rightarrow 2I = \int_0^a 1\, dx \Rightarrow 2I = [x]_0^a$

$\Rightarrow I = \frac{1}{2}(a - 0) = \frac{a}{2}$

3. Prove that $\int_a^b f(x).dx = \int_a^b f(a+b-x)\, dx$ and hence Evaluate

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx.$ (2) $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}\, dx$

Solution: Proof: Put $a + b - x = t$

$\Rightarrow 0 - dx = dt \Rightarrow dx = -dt$

If $x = a \Rightarrow t = a + b - a = b,$

If $x = b \Rightarrow t = a + b - b = a$

Now, $RHS = \int_a^b f(a+b-x)dx$

$= \int_b^a f(t)(-dt) = -\int_b^a f(t)dt$

$= \int_a^b f(t)dt = \int_a^b f(x)dx = LHS$

$\therefore \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx.$

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}\, dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}}}\, dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \ldots\ldots\ldots\ldots (1)$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\, dx$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \ldots\ldots\ldots (2)$$

Now, $eq(1) + eq(2)$

$$\Rightarrow I + I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$\Rightarrow 2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right] dx$$

$$\Rightarrow 2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right] dx$$

$$\Rightarrow 2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\, dx \Rightarrow 2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\Rightarrow I = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{1}{2}\left[\frac{2\pi - \pi}{6}\right] = \frac{1}{2}\left[\frac{\pi}{6}\right] = \frac{\pi}{12}$$

(2) $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}} dx$

Let $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}\, dx \ldots\ldots\ldots\ldots (1)$

$$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-(9-x)}+\sqrt{9-x}}\, dx$$

$$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-9+x}+\sqrt{9-x}}\, dx$$

$$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\, dx \ldots\ldots\ldots\ldots\ldots (2)$$

Now, $eq(1) + eq(2)$

$$\Rightarrow I + I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}\, dx + \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\, dx$$

$$\Rightarrow 2I = \int_3^6 \frac{\sqrt{x}+\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}\, dx$$

$$\Rightarrow 2I = \int_3^6 1\, dx$$

$$\Rightarrow 2I = [x]_3^6$$

$$\Rightarrow I = \frac{1}{2}(6-3) = \frac{3}{2}$$

4. Prove that

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f\ (x)\, dx\ , & \text{if } f(x) \text{ is an even} \\ 0 & , \text{ if } f(x) \text{ is an odd} \end{cases}$$

and hence

Evaluate (1) $\int_{-1}^{1} \sin^5 x \cos^4\ x\ dx$

(2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x) dx$

Solution: Proof:

$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \ldots\ldots (1)$

Let $x = -t$, in the first integral on the right hand side $\xrightarrow{Differentiate} dx = -dt$,

If $x = -a \Rightarrow t = -(-a) = a$ &

If $x = 0 \Rightarrow t = 0$

From (1)

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_a^0 f(-t)(-dt) + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = -\int_a^0 f(-t)dt + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_0^a f(-t)dt + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx \ldots (2)$$

Case 1: If $f(x)$ is even $\Rightarrow f(-x) = f(x)$

w.k.t $\int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

Case 2: If $f(x)$ is odd $\Rightarrow f(-x) = -f(x)$

w.k.t $\int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_0^a -f(x)dx + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = 0$$

From case 1 and case 2

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\quad f(x)\ is\ even \\ 0 & if\quad f(x)\ is\ odd \end{cases}$$

(1) $\int_{-1}^{1} \sin^5 x \cos^4\ x\ dx$

Let $I = \int_{-1}^{1} \sin^5 x \cos^4\ x\ dx$

Let $f(x) = \sin^5 x \cos^4\ x$

$$\Rightarrow f(-x) = \sin^5(-x) \cos^4\ (-x)$$

$$\Rightarrow f(-x) = (-\sin x)^5 \cos^4\ (x)$$

$$\Rightarrow f(-x) = -\sin^5 x \cos^4\ (x)\ = -f(x)$$

$\Rightarrow f(x)$ is odd function

$$\Rightarrow I = \int_{-1}^{1} \sin^5 x \cos^4\ x\ dx\ = 0$$

(2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x) dx$

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x) dx$

Let $f(x) = x^3 + x\cos x$

$$\Rightarrow f(-x) = (-x)^3 + (-x)\cos(-x)$$

$$\Rightarrow f(-x) = -x^3 - x\cos x$$

$$\Rightarrow f(-x) = -(x^3 + x\cos x) = -f(x)$$

$\Rightarrow f(x)$ is odd function

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x) dx = 0$$

5. **Prove that**

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ f(2a-x) = f(x) \\ 0 & if\ f(2a-x) = -f(x) \end{cases}$$

and hence evaluate

(1) $\int_0^{2\pi} \cos^5 x\, dx.$ **(2)** $\int_0^{\pi} |\cos x| dx$

Solution: Proof:

$\int_o^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_a^{2a} f(x)\,dx \ldots (1)$

Let $x = 2a - t$, in the second integral on the right hand side $\xRightarrow{Differentiate} dx = 0 - dt$

$\Rightarrow dx = -dt$

Now, $x = 2a - t \Rightarrow t = 2a - x$

If $x = a \Rightarrow t = 2a - a = a$ &

If $x = 2a \Rightarrow t = 2a - 2a = 0$

From eq (1)

$\int_o^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_a^{2a} f(x)\,dx$

$\Rightarrow \int_o^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_a^0 f(2a-t)\,(-dt)$

$\Rightarrow \int_o^{2a} f(x)\,dx = \int_0^a f(x)dx - \int_a^0 f(2a-t)\,(dt)$

$\Rightarrow \int_o^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_0^a f(2a-t)\,(dt)$

$\Rightarrow \int_o^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,(dx)$

Case 1 : $If \quad f(2a-x) = f(x)$

w.k.t $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$\Rightarrow \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$

$\Rightarrow \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$

Case 2: $If \quad f(2a-x) = -f(x)$

w.k.t $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$\Rightarrow \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a -f(x)dx$

$\Rightarrow \int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$

$\Rightarrow \int_0^{2a} f(x)dx = 0$

From case 1 and case 2

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ f(2a-x) = f(x) \\ 0 & if\ f(2a-x) = -f(x) \end{cases}$$

(1) $\int_0^{2\pi} \cos^5 x\, dx.$

Let $I = \int_0^{2\pi} \cos^5 x\, dx$

Let $f(x) = \cos^5 x$

$\Rightarrow f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x$

$\Rightarrow f(2\pi - x) = f(x)$

Now, $I = \int_0^{2\pi} \cos^5 x\, dx = 2\int_0^{\pi} \cos^5 x\, dx \ldots (1)$

Let $f(x) = \cos^5 x$

$\Rightarrow f(\pi - x) = \cos^5(\pi - x) = (-\cos x)^5$

$\Rightarrow f(\pi - x) = -\cos^5 x = -f(x)$

From (1), $I = 2\int_0^{\pi} \cos^5 x\, dx = 2(0) = 0$

(2) $\int_0^{\pi} |\cos x| dx$

Let $I = \int_0^{\pi} |\cos x| dx$

Let $f(x) = |\cos x|$

$\Rightarrow f(\pi - x) = |\cos(\pi - x)| = |-\cos x|$

$\Rightarrow f(\pi - x) = |\cos x| = f(x)$

Now, $I = \int_0^{\pi} |\cos x| dx = 2\int_0^{\frac{\pi}{2}} |\cos x| dx$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \cos x\, dx = 2[\sin x]_0^{\frac{\pi}{2}}$

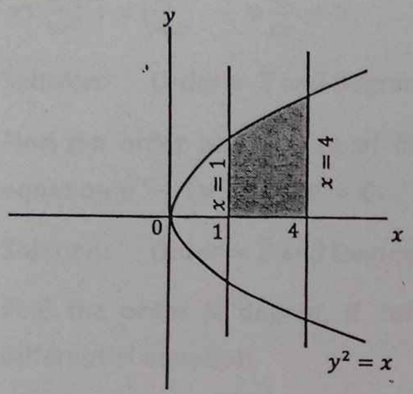
$\Rightarrow I = 2\left[\sin\frac{\pi}{2} - 0\right] = 2[1 - 0] = 2$

APPLICATION OF INTEGRATION (AREA)

THREE MARKS QUESTION

1. **Find the area of the region bounded by the curve $y^2 = x$ and the line $x = 1, \quad x = 4$ and the x-axis in the first quadrant.**

Solution:



Given: $y^2 = x \Rightarrow y = \sqrt{x}$

Now Area of the region $A = \int_a^b y\ dx$

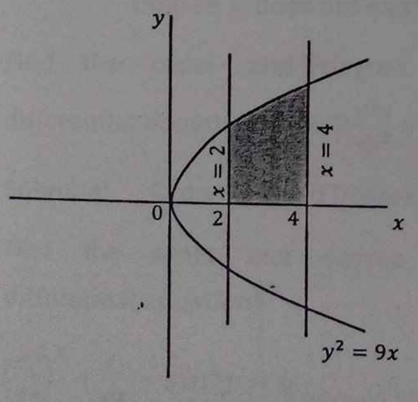
$\Rightarrow A = \int_1^4 \sqrt{x}\ dx = \frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_1^4$

$\Rightarrow A = \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{2}{3}\left[(2^2)^{\frac{3}{2}} - 1\right]$

$\Rightarrow A = \frac{2}{3}\left[8 - 1\right] = \frac{2}{3}[7] = \frac{14}{3}\ sq\ units$

2. **Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.**

Solution:



Given: $y^2 = 9x \Rightarrow y = \sqrt{9x} = 3\sqrt{x}$

Now Area of the region $A = \int_a^b y\ dx$

$\Rightarrow A = \int_2^4 3\sqrt{x}\ dx = 3\int_2^4 \sqrt{x}\ dx$

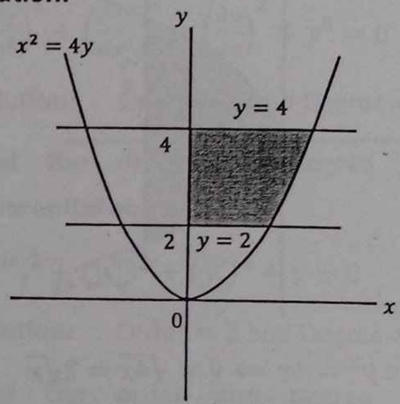
$\Rightarrow A = 3\frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_2^4$

$\Rightarrow A = 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right] = 2\left[(2^2)^{\frac{3}{2}} - (8)^{\frac{1}{2}}\right]$

$\Rightarrow A = 2\left[8 - \sqrt{8}\right] = 2\left[8 - 2\sqrt{2}\right]\ sq\ units$

3. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

Solution:



Given: $x^2 = 4y \Rightarrow x = \sqrt{4y} = 2\sqrt{y}$

Now Area of the region $A = \int_a^b x\ dy$

$\Rightarrow A = \int_2^4 2\sqrt{y}\ dy = 2\int_2^4 \sqrt{y}\ dy$

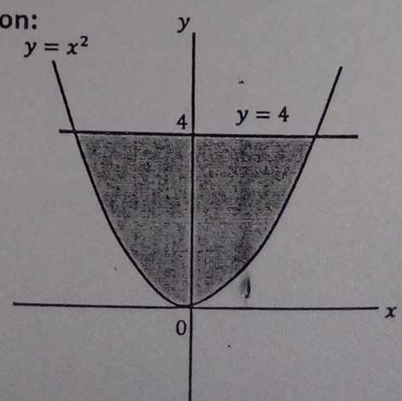
$\Rightarrow A = 2\frac{2}{3}\left[(y)^{\frac{3}{2}}\right]_2^4$

$\Rightarrow A = \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right] = \frac{4}{3}\left[(2^2)^{\frac{3}{2}} - (8)^{\frac{1}{2}}\right]$

$\Rightarrow A = \frac{4}{3}\left[8 - \sqrt{8}\right] = \frac{4}{3}\left[8 - 2\sqrt{2}\right]\ sq\ units$

4. **Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.**

Solution:



Given: $x^2 = y \Rightarrow x = \sqrt{y}$

Now Area of the region $A = 2\int_a^b x\, dy$

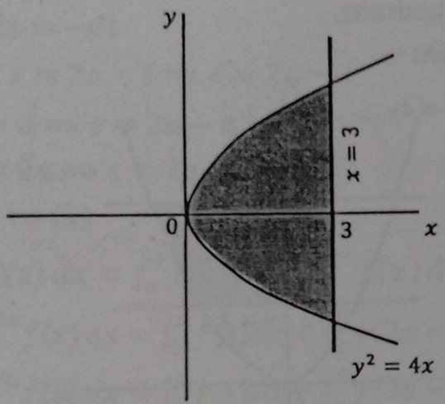
$\Rightarrow A = 2\int_0^4 \sqrt{y}\ dy = 2\frac{2}{3}\left[(y)^{\frac{3}{2}}\right]_0^4$

$\Rightarrow A = \frac{4}{3}\left[(4)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right] = \frac{4}{3}\left[(2^2)^{\frac{3}{2}} - 0\right]$

$\Rightarrow A = \frac{4}{3}[8 - 0] = \frac{4}{3}[8] = \frac{32}{3}\, sq\ units$

5. **Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.**

Solution:



Given: $y^2 = 4x \Rightarrow y = \sqrt{4x} = 2\sqrt{x}$

Now Area of the region $A = 2\int_a^b y\, dx$

$\Rightarrow A = 2\int_0^3 2\sqrt{x}\ dx = 4\int_0^3 \sqrt{x}\ dx$

$\Rightarrow A = 4\frac{2}{3}\left[(x)^{\frac{3}{2}}\right]_0^3$

$\Rightarrow A = \frac{8}{3}\left[(3)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right] = \frac{8}{3}\left[(27)^{\frac{1}{2}} - 0\right]$

$\Rightarrow A = \frac{8}{3}\left[\sqrt{27}\right] = \frac{8}{3}\left[3\sqrt{3}\right] = 8\sqrt{3}\, sq\ units$

DIFFERENTIAL EQUATION

TWO MARKS QUESTION

1. Find the order and degree of the differential equation

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0.$$

Solution: Order = 2 and Degree = 1

2. Find the order and degree of differential equation $y''+(y')^2+2y=0$.

Solution: Order = 2 and Degree = 1

3. Find the order & degree, if defined of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\frac{dy}{dx}+1=0.$$

Solution: Order = 2 and

Degree = does not exist

4. Find the order and degree (if defined) of the differential equation

$$\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$$

Solution: Order = 4 and

Degree = does not exist

5. Find the order and degree of the differential equation $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$

Solution: Order = 3 and Degree = 1

6. Find the order and degree of the differential equation,

$$\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}-\sin^2y=0$$

Solution: Order = 1 and Degree = 2

7. Find the order and degree, if defined, of the differential equation

$$\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$$

Solution: Order = 4 and

Degree = *does not exist*

8. Find the order and degree of the differential equation,

$$\left(\frac{d^3y}{dx^3}\right)^2+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^4+y^5=0$$

Solution: Order = 3 and Degree = 2

9. Find the order and degree of the differential equation,

$$(y''')^2+(y'')^3+(y')^4+y=0$$

Solution: Order = 3 and Degree = 2

10. Find the order and degree of the differential equation, $\left(\frac{ds}{dt}\right)^4+3s\left(\frac{d^2s}{dt^2}\right)=0$

Solution: Order = 2 and Degree = 1

VECTORS

TWO MARKS QUESTION

1. **Find the vector in the direction of vector $5\hat{\imath}-\hat{\jmath}+2\hat{k}$ which has magnitude 8 units.**

Solution: Let $\vec{a}=5\hat{\imath}-\hat{\jmath}+2\hat{k}$

$\Rightarrow |\vec{a}|=\sqrt{(5)^2+(-1)^2+(2)^2}$

$\Rightarrow |\vec{a}|=\sqrt{25+1+4}=\sqrt{30}$

Now unit vector along the direction of $\vec{a}$ is

$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{5\hat{\imath}-\hat{\jmath}+2\hat{k}}{\sqrt{30}}$

Now Vector of magnitude of 8 is

$8\hat{a}=8\left(\frac{5\hat{\imath}-\hat{\jmath}+2\hat{k}}{\sqrt{30}}\right)$

2. **Find the angle between the vectors $\vec{a}=\hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{b}=\hat{\imath}+\hat{\jmath}+\hat{k}$.**

Solution: Given $\vec{a}=\hat{\imath}+\hat{\jmath}-\hat{k}$

$\Rightarrow |\vec{a}|=\sqrt{(1)^2+(1)^2+(-1)^2}$

$\Rightarrow |\vec{a}|=\sqrt{1+1+1}=\sqrt{3}$

Given $\vec{b}=\hat{\imath}+\hat{\jmath}+\hat{k}$

$\Rightarrow |\vec{b}|=\sqrt{(1)^2+(1)^2+(1)^2}$

$\Rightarrow |\vec{b}|=\sqrt{1+1+1}=\sqrt{3}$

Now $\vec{a}.\vec{b}=(\hat{\imath}+\hat{\jmath}-\hat{k}).(\hat{\imath}-\hat{\jmath}+\hat{k})$

$\Rightarrow \vec{a}.\vec{b}=(1)(1)+(1)(-1)+(-1)(1)$

$\Rightarrow \vec{a}.\vec{b}=1-1-1=-1$

Let θ be the angle between $\vec{a}$ and $\vec{b}$ then

$cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{-1}{\sqrt{3}\sqrt{3}}=-\frac{1}{3}$

$\Rightarrow \theta=\cos^{-1}\left(\frac{-1}{3}\right)$

3. **Find the projection of the vector $\hat{\imath}+3\hat{\jmath}+7\hat{k}$ on the vector $7\hat{\imath}-\hat{\jmath}+8\hat{k}$.**

Solution: Let $\vec{a}=\hat{\imath}+3\hat{\jmath}+7\hat{k}$ and $\vec{b}=7\hat{\imath}-\hat{\jmath}+8\hat{k}$

W. K. T Projection of $\vec{a}$ on $\vec{b}$ is given by $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$=\frac{(\hat{\imath}+3\hat{\jmath}+7\hat{k}).(7\hat{\imath}-\hat{\jmath}+8\hat{k})}{|7\hat{\imath}-\hat{\jmath}+8\hat{k}|}$

$=\frac{(1)(7)+(3)(-1)+(7)(8)}{\sqrt{(7)^2+(-1)^2+(8)^2}}$

$=\frac{7-3+56}{\sqrt{49+9+64}}=\frac{60}{\sqrt{114}}$.

4. **Obtain the projection of the vector $\vec{a}=2\hat{\imath}+3\hat{\jmath}+2\hat{k}$ on the vector $\vec{b}=\hat{\imath}+2\hat{\jmath}+\hat{k}$.**

Solution: Let $\vec{a}=2\hat{\imath}+3\hat{\jmath}+2\hat{k}$ and $\vec{b}=\hat{\imath}+2\hat{\jmath}+\hat{k}$

W. K. T Projection of $\vec{a}$ on $\vec{b}$ is given by $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$=\frac{(2\hat{\imath}+3\hat{\jmath}+2\hat{k}).(\hat{\imath}+2\hat{\jmath}+\hat{k})}{|\hat{\imath}+2\hat{\jmath}+\hat{k}|}$

$=\frac{(2)(1)+(3)(2)+(2)(1)}{\sqrt{(1)^2+(2)^2+(1)^2}}$

$=\frac{2+6+2}{\sqrt{1+4+1}}=\frac{10}{\sqrt{6}}$

5. **If two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}|=2,\ |\vec{b}|=3$ & $\vec{a}.\vec{b}=4$ find $|\vec{a}-\vec{b}|$.**

Solution: Given: $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}.\vec{b}=4$

W.K.T $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}$

$\Rightarrow |\vec{a}-\vec{b}|^2=(2)^2+(3)^2-2(4)$

$\Rightarrow |\vec{a}-\vec{b}|^2=4+9-8$

$\Rightarrow |\vec{a}-\vec{b}|^2=5 \Rightarrow |\vec{a}-\vec{b}|=\sqrt{5}$.

6. **If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ then prove that $\vec{a}$ and $\vec{b}$ are perpendicular.**

Solution: Given: $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}$

$\Rightarrow 2\vec{a}.\vec{b}+2\vec{a}.\vec{b}=0$

$\Rightarrow 4\vec{a}.\vec{b}=0 \Rightarrow \vec{a}.\vec{b}=0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular.

7. **If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=8$ then find $|\vec{x}|$**

Solution:

Given: $\vec{a}$ is a unit vector $\Rightarrow |\vec{a}|=1$

Given: $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=8$

$\Rightarrow |\vec{x}|^2-|\vec{a}|^2=8$

$\Rightarrow |\vec{x}|^2-(1)^2=8$

$\Rightarrow |\vec{x}|^2=8+1=9$

$\Rightarrow |\vec{x}|^2=9 \Rightarrow |\vec{x}|=3$.

8. **Find $|\vec{a}|$ & $|\vec{b}|$, if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.**

Solution: Given $(\vec{a}+\vec{b})(\vec{a}-\vec{b}) = 8$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8 \Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8$

$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8 \Rightarrow 63|\vec{b}|^2 = 8$

$\Rightarrow |\vec{b}|^2 = \frac{8}{63} \Rightarrow |\vec{b}| = \frac{\sqrt{8}}{\sqrt{63}}$

Now, $|\vec{a}| = 8|\vec{b}| = 8\frac{\sqrt{8}}{\sqrt{63}}$

$\Rightarrow |\vec{a}| = \frac{8\sqrt{8}}{\sqrt{63}}$.

9. **Let $|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$ and $|\vec{a}\times\vec{b}| = 1$. Find the angle between $\vec{a}$ and $\vec{b}$.**

Solution: Given $|\vec{a}\times\vec{b}| = 1, |\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$ then

$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{1}{3\frac{\sqrt{2}}{3}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \Rightarrow \theta = \frac{\pi}{4}$

10. **Find the sine of the angle between the vectors $\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$.**

Solution: Let $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\Rightarrow |\vec{a}| = \sqrt{(1)^2 + (2)^2 + (2)^2}$

$\Rightarrow |\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$

Let $\vec{b} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

$\Rightarrow |\vec{b}| = \sqrt{(3)^2 + (2)^2 + (6)^2}$

$\Rightarrow |\vec{a}| = \sqrt{9+4+36} = \sqrt{49} = 7$

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 3 & 2 & 6 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i(12-4) - j(6-6) + k(2-6)$

$\Rightarrow \vec{a}\times\vec{b} = 8\hat{\imath} - 0\hat{\jmath} - 4\hat{k}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{(8)^2 + (0)^2 + (-4)^2}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{64+0+16} = \sqrt{80}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{16\times 5} = 4\sqrt{5}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$ then

$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{4\sqrt{5}}{3.7} = \frac{4\sqrt{5}}{21}$

11. **Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$, $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$.**

Solution: Given $\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$,

$\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$.

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i(1+4) - j(3-4) + \hat{k}(-3-1)$

$\Rightarrow \vec{a}\times\vec{b} = 5\hat{\imath} + \hat{\jmath} - 4\hat{k}$

Area of the parallelogram $= |\vec{a}\times\vec{b}|$

$= \sqrt{(5)^2 + (1)^2 + (-4)^2} = \sqrt{25+1+16}$

$= \sqrt{42}\ sq\ units$

12. **Find the area of parallelogram whose adjacent sides determine by the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$, and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$.**

Solution: Given $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$, $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$.

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i(1-1) - j(1+1) + k(-1-1)$

$\Rightarrow \vec{a}\times\vec{b} = 0\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$

Area of the parallelogram $= |\vec{a}\times\vec{b}|$

$= \sqrt{(0)^2 + (-2)^2 + (-2)^2} = \sqrt{0+4+4}$

$= \sqrt{8}\ sq\ units$

13. **Find the area of the parallelogram whose adjacent sides are given by vectors $\vec{a} = \hat{\imath} - \hat{\jmath} - 3\hat{k}$ and $\vec{b} = 2\hat{\imath} - 7\hat{\jmath} - \hat{k}$.**

Solution: Given $\vec{a} = \hat{\imath} - \hat{\jmath} - 3\hat{k}$

$\vec{b} = 2\hat{\imath} - 7\hat{\jmath} - \hat{k}$.

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & -3 \\ 2 & -7 & -1 \end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = i(1-21) - j(-1+6) + k(-7+2)$

$\Rightarrow \vec{a}\times\vec{b} = -20\hat{\imath} - 5\hat{\jmath} - 5\hat{k}$

Area of the parallelogram $= |\vec{a}\times\vec{b}|$

$= \sqrt{(-20)^2 + (-5)^2 + (-5)^2}$

$= \sqrt{400+25+25} = \sqrt{450}\ sq\ units$

14. **Find the area of the parallelogram whose adjacent sides are given by the vectors $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} - 7\hat{\jmath} + \hat{k}$.**

Solution: Given $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$

$\vec{b} = 2\hat{\imath} - 7\hat{\jmath} + \hat{k}$.

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$

$\Rightarrow \vec{a} \times \vec{b} = i(-1+21) - j(1-6) + k(-7+2)$

$\Rightarrow \vec{a} \times \vec{b} = 20\hat{\imath} + 5\hat{\jmath} - 5\hat{k}$

Area of the parallelogram $= |\vec{a} \times \vec{b}|$

$= \sqrt{(20)^2 + (5)^2 + (-5)^2}$

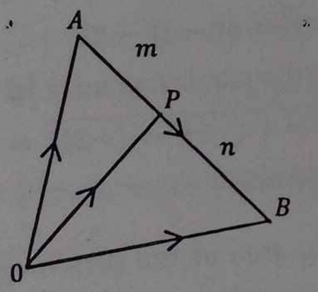
$= \sqrt{400 + 25 + 25}$

$= \sqrt{450}\ sq\ units$

THREE MARKS QUESTION

1. **Show that the position vector of the point P, which divides the line joining the points A and B having position vectors $\vec{a}$ and $\vec{b}$ internally in the ratio $m:n$ is $\frac{m\vec{b}+n\vec{a}}{m+n}$.**

Solution:



Let O be the origin. Let AB be the line segment and P divides AB internally in the ratio $m:n$.

Let P.V of $A = \overrightarrow{OA} = \vec{a}$

Let P.V of $B = \overrightarrow{OB} = \vec{b}$ and

Let P.V of $= \overrightarrow{OP} = \vec{r}$.

Now, $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{m}{n} \Rightarrow n\overrightarrow{AP} = m\overrightarrow{PB}$

$\Rightarrow n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$

$\Rightarrow n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$

$\Rightarrow n\vec{r} - n\vec{a} = m\vec{b} - m\vec{r}$

$\Rightarrow m\vec{r} + n\vec{r} = m\vec{b} + n\vec{a}$

$\Rightarrow \vec{r}(m+n) = m\vec{b} + n\vec{a} \Rightarrow \vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n}$.

2. **Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $-\hat{\imath} + \hat{\jmath} + \hat{k}$ respectively, in the ratio $2:1$**

(i) internally (ii) externally.

Solution:

Let P.V of the point $P = \vec{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and

Let P.V of the point $Q = \vec{b} = -\hat{\imath} + \hat{\jmath} + \hat{k}$

(i) Given R divides the line joining two points P and Q internally in the ratio 2:1

Now, P.V of $R = \frac{m\vec{b}+n\vec{a}}{m+n} = \frac{2\vec{b}+1\vec{a}}{2+1}$

$\Rightarrow$ P.V of $R = \frac{2(-\hat{\imath}+\hat{\jmath}+\hat{k})+(\hat{\imath}+2\hat{\jmath}-\hat{k})}{3}$

$\Rightarrow$ P.V of $R = \frac{-2\hat{\imath}+2\hat{\jmath}+2\hat{k}+\hat{\imath}+2\hat{\jmath}-\hat{k}}{3}$

$\Rightarrow$ P.V of $R = \frac{-\hat{\imath}+4\hat{\jmath}+\hat{k}}{3}$

(ii) Given R divides the line joining two points P and Q externally in the ratio 2:1

Now, P.V of $R = \frac{m\vec{b}-n\vec{a}}{m-n} = \frac{2\vec{b}-1\vec{a}}{2-1}$

$\Rightarrow$ P.V of $R = \frac{2(-\hat{\imath}+\hat{\jmath}+\hat{k})-(\hat{\imath}+2\hat{\jmath}-\hat{k})}{1}$

$\Rightarrow$ P.V of $R = -2\hat{\imath} + 2\hat{\jmath} + 2\hat{k} - \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$\Rightarrow$ P.V of $R = -3\hat{\imath} + 0\hat{\jmath} + 3\hat{k}$

3. **Find the unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$**

Solution: Given: $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ & $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Now, $\vec{a} + \vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k} + \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\Rightarrow \vec{a} + \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

Now, $\vec{a} - \vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k} - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$\Rightarrow \vec{a} - \vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k} - \hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

$\Rightarrow \vec{a} - \vec{b} = 0\hat{\imath} - \hat{\jmath} - 2\hat{k}$

Now, $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$

$= \hat{\imath}(-6+4) - \hat{\jmath}(-4-0) + \hat{k}(-2-0)$

$\Rightarrow (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

Now,

$|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})| = \sqrt{(-2)^2+(4)^2+(-2)^2}$

$= \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

Now, unit vector perpendicular to

$(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is $\hat{n} = \frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$

$\Rightarrow \hat{n} = \frac{-2\hat{\imath}+4\hat{\jmath}-2\hat{k}}{2\sqrt{6}}$

4. If $\vec{a} = 2\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{b} = -\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\vec{c} = 3\hat{\imath}+\hat{\jmath}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$. Then find the value of λ.

Solution: Now,

$\vec{a}+\lambda\vec{b} = 2\hat{\imath}+2\hat{\jmath}+3\hat{k}+\lambda(-\hat{\imath}+2\hat{\jmath}+\hat{k})$

$= 2\hat{\imath}+2\hat{\jmath}+3\hat{k}-\lambda\hat{\imath}+2\lambda\hat{\jmath}+\lambda\hat{k}$

$\vec{a}+\lambda\vec{b} = (2-\lambda)\hat{\imath}+(2+2\lambda)\hat{\jmath}+(3+\lambda)\hat{k}$

Given: $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$

$\Rightarrow (\vec{a}+\lambda\vec{b}).(\vec{c}) = 0$

$\Rightarrow [(2-\lambda)\hat{\imath}+(2+2\lambda)\hat{\jmath}+(3+\lambda)\hat{k}].(3\hat{\imath}+\hat{\jmath}+0\hat{k}) = 0$

$\Rightarrow (2-\lambda)3+(2+2\lambda)1+(3+\lambda)0 = 0$

$\Rightarrow 6-3\lambda+2+2\lambda = 0$

$\Rightarrow 8-\lambda = 0 \Rightarrow \lambda = 8$

5. If two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = 2$ $|\vec{b}| = 3$ and $\vec{a}.\vec{b} = 4$ find $|\vec{a}\times\vec{b}|$.

Solution: W.K.T

$|\vec{a}\times\vec{b}|^2 + (\vec{a}.\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

$\Rightarrow |\vec{a}\times\vec{b}|^2 + (4)^2 = (2)^2(3)^2$

$\Rightarrow |\vec{a}\times\vec{b}|^2 + 16 = 4\times 9$

$\Rightarrow |\vec{a}\times\vec{b}|^2 = 36-16 = 20$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{20}$

6. Three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c} = 0$. Evaluate the quantity $\mu = \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ if $|\vec{a}| = 1, |\vec{b}| = 4$ and $|\vec{c}| = 2$

Solution: Given $\vec{a}+\vec{b}+\vec{c} = 0$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = 0$, S.O.B.S

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow (1)^2+(4)^2+(2)^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow 1+16+4+2\mu = 0$

$\Rightarrow 21+2\mu = 0$

$\Rightarrow 2\mu = -21 \Rightarrow \mu = -\frac{21}{2}$.

7. Prove that $[\vec{a},\vec{b},\vec{c}+\vec{d}] = [\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{d}]$.

Solution: $LHS = [\vec{a}\ \ \vec{b}\ \ \vec{c}+\vec{d}]$

$= \vec{a}.[\vec{b}\times(\vec{c}+\vec{d})] = \vec{a}.[(\vec{b}\times\vec{c})+(\vec{b}\times\vec{d})]$

$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{d})$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{a}\ \ \vec{b}\ \ \vec{d}] = RHS$

8. Prove that

$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \ \vec{b}\ \ \vec{c}]$.

Solution: $LHS = [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b}).[(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{c})+(\vec{c}\times\vec{a})]$

$= (\vec{a}+\vec{b}).[(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{0})+(\vec{c}\times\vec{a})]$

$= (\vec{a}+\vec{b}).[(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})]$

$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})$
$\quad +\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$

$= \vec{a}.(\vec{b}\times\vec{c})+0+0+0+0+\vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{b}\ \ \vec{c}\ \ \vec{a}] = [\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{c}\ \ \vec{a}\ \ \vec{b}]$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 2[\vec{a}\ \ \vec{b}\ \ \vec{c}] = RHS$

9. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, prove that vectors $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ are coplanar.

Solution: $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$

$= (\vec{a}-\vec{b}).[(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})]$

$= (\vec{a}-\vec{b}).[(\vec{b}\times\vec{c})-(\vec{b}\times\vec{a})-(\vec{c}\times\vec{c})+(\vec{c}\times\vec{a})]$

$= (\vec{a}-\vec{b}).[(\vec{b}\times\vec{c})-(\vec{b}\times\vec{a})-(\vec{0})+(\vec{c}\times\vec{a})]$

$= (\vec{a}-\vec{b}).[(\vec{b}\times\vec{c})-(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})]$

$= \vec{a}.(\vec{b}\times\vec{c})-\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})$
$\quad -\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})-\vec{b}.(\vec{c}\times\vec{a})$

$= \vec{a}.(\vec{b}\times\vec{c})-0+0-0+0-\vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]-[\vec{b}\ \ \vec{c}\ \ \vec{a}] = [\vec{a}\ \ \vec{b}\ \ \vec{c}]-[\vec{c}\ \ \vec{a}\ \ \vec{b}]$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]-[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0$

$\Rightarrow [\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

$\Rightarrow \vec{a}-\vec{b}, \vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ are coplanar

10. Show that the four points with position vectors $4\hat{i}+8\hat{j}+12\hat{k}$, $2\hat{i}+4\hat{j}+6\hat{k}$, $3\hat{i}+5\hat{j}+4\hat{k}$ and $5\hat{i}+8\hat{j}+5\hat{k}$ are coplanar.

Solution: Let $\overrightarrow{OA}=4\hat{i}+8\hat{j}+12\hat{k}$,

$\overrightarrow{OB}=2\hat{i}+4\hat{j}+6\hat{k}$,

$\overrightarrow{OC}=3\hat{i}+5\hat{j}+4\hat{k}$,

$\overrightarrow{OD}=5\hat{i}+8\hat{j}+5\hat{k}$

Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB}=(2\hat{i}+4\hat{j}+6\hat{k})-(4\hat{i}+8\hat{j}+12\hat{k})$

$\Rightarrow \overrightarrow{AB}=2\hat{i}+4\hat{j}+6\hat{k}-4\hat{i}-8\hat{j}-12\hat{k}$

$\Rightarrow \overrightarrow{AB}=-2\hat{i}-4\hat{j}-6\hat{k}$

Now, $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$\Rightarrow \overrightarrow{AC}=(3\hat{i}+5\hat{j}+4\hat{k})-(4\hat{i}+8\hat{j}+12\hat{k})$

$\Rightarrow \overrightarrow{AC}=3\hat{i}+5\hat{j}+4\hat{k}-4\hat{i}-8\hat{j}-12\hat{k}$

$\Rightarrow \overrightarrow{AC}=-\hat{i}-3\hat{j}-8\hat{k}$

Now, $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$

$\Rightarrow \overrightarrow{AD}=(5\hat{i}+8\hat{j}+5\hat{k})-(4\hat{i}+8\hat{j}+12\hat{k})$

$\Rightarrow \overrightarrow{AD}=5\hat{i}+8\hat{j}+5\hat{k}-4\hat{i}-8\hat{j}-12\hat{k}$

$\Rightarrow \overrightarrow{AD}=\hat{i}+0\hat{j}-7\hat{k}$

Now, $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=\begin{vmatrix}-2 & -4 & -6\\ -1 & -3 & -8\\ 1 & 0 & -7\end{vmatrix}$

$=-2(21+0)-(-4)(7+8)-6(0+3)$

$=(-2)(21)+4(15)-6(3)$

$=-42+60-18=0$

$\Rightarrow \left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0$

$\therefore$ Given four points are coplanar.

11. Find λ if the vectors $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{i}+7\hat{j}+3\hat{k}$ are coplanar.

Solution: If $\vec{a}\ \ \vec{b}\ \ \vec{c}$ are coplanar then

$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

$\Rightarrow \begin{vmatrix}1 & 3 & 1\\ 2 & -1 & -1\\ \lambda & 7 & 3\end{vmatrix}=0$

$\Rightarrow 1(3+7)-3(6+\lambda)+1(14+\lambda)=0$

$\Rightarrow (1)(10)-3(6+\lambda)+1(14+\lambda)=0$

$\Rightarrow 10-18-3\lambda+14+\lambda=0$

$\Rightarrow 6-2\lambda=0 \Rightarrow 2\lambda=6 \Rightarrow \lambda=\frac{6}{2}=3$

12. Find x such that the four points $A(3,2,1)$, $B(4,x,5)$, $C(4,2,-2)$ and $D(6,5,-1)$ are co-planar.

Solution: Given: $A(3,2,1)$, $B(4,x,5)$, $C(4,2,-2)$ and $D(6,5,-1)$

Let $\overrightarrow{OA}=3\hat{i}+2\hat{j}+\hat{k}$,

$\overrightarrow{OB}=4\hat{i}+x\hat{j}+5\hat{k}$,

$\overrightarrow{OC}=4\hat{i}+2\hat{j}-2\hat{k}$,

$\overrightarrow{OD}=6\hat{i}+5\hat{j}-\hat{k}$

Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB}=(4\hat{i}+x\hat{j}+5\hat{k})-(3\hat{i}+2\hat{j}+\hat{k})$

$\Rightarrow \overrightarrow{AB}=4\hat{i}+x\hat{j}+5\hat{k}-3\hat{i}-2\hat{j}-\hat{k}$

$\Rightarrow \overrightarrow{AB}=\hat{i}+(x-2)\hat{j}+4\hat{k}$

Now, $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$\Rightarrow \overrightarrow{AC}=(4\hat{i}+2\hat{j}-2\hat{k})-(3\hat{i}+2\hat{j}+\hat{k})$

$\Rightarrow \overrightarrow{AC}=4\hat{i}+2\hat{j}-2\hat{k}-3\hat{i}-2\hat{j}-\hat{k}$

$\Rightarrow \overrightarrow{AC}=\hat{i}+0\hat{j}-3\hat{k}$

Now, $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$

$\Rightarrow \overrightarrow{AD}=(6\hat{i}+5\hat{j}-\hat{k})-(3\hat{i}+2\hat{j}+\hat{k})$

$\Rightarrow \overrightarrow{AD}=6\hat{i}+5\hat{j}-\hat{k}-3\hat{i}-2\hat{j}-\hat{k}$

$\Rightarrow \overrightarrow{AD}=3\hat{i}+3\hat{j}-2\hat{k}$

Given: four points are coplanar

$\Rightarrow \left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0$

$\Rightarrow \begin{vmatrix}1 & x-2 & 4\\ 1 & 0 & -3\\ 3 & 3 & -2\end{vmatrix}=0$

$\Rightarrow 1(0+9)-(x-2)(-2+9)+4(3-0)=0$

$\Rightarrow (1)(9)-(x-2)(7)+4(3)=0$

$\Rightarrow 9-7x+14+12=0$

$\Rightarrow -7x+35=0 \Rightarrow 7x=35 \Rightarrow x=\frac{35}{7}=5$

THREE DIMENSIONAL GEOMETRY

TWO MARKS QUESTION

1. **Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\vec{r} = \hat{\imath} + \hat{\jmath} + \lambda\,(2\hat{\imath} - \hat{\jmath} + \hat{k})$ and $\vec{r} = 2\hat{\imath} + \hat{\jmath} - \hat{k} + \mu\,(3\hat{\imath} - 5\hat{\jmath} + 2\hat{k})$**

Solution: Given: Equation of the line l_1 is

$\vec{r} = \hat{\imath} + \hat{\jmath} + \lambda\,(2\hat{\imath} - \hat{\jmath} + \hat{k})$

$\Rightarrow \vec{a_1} = \hat{\imath} + \hat{\jmath}$ and $\vec{b_1} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

Given: Equation of the line l_2 is

$\vec{r} = 2\hat{\imath} + \hat{\jmath} - \hat{k} + \mu\,(3\hat{\imath} - 5\hat{\jmath} + 2\hat{k})$

$\Rightarrow \vec{a_2} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b_2} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

W.K.T Shortest distance between two skew lines $d = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$ ………… (1)

Now, $\vec{a_2} - \vec{a_1} = (2\hat{\imath} + \hat{\jmath} - \hat{k}) - (\hat{\imath} + \hat{\jmath})$

$\Rightarrow \vec{a_2} - \vec{a_1} = 2\hat{\imath} + \hat{\jmath} - \hat{k} - \hat{\imath} - \hat{\jmath}$

$\Rightarrow \vec{a_2} - \vec{a_1} = \hat{\imath} + 0\hat{\jmath} - \hat{k}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$

$= \hat{\imath}(-2+5) - \hat{\jmath}(4-3) + \hat{k}(-10+3)$

$\Rightarrow \vec{b_1} \times \vec{b_2} = 3\hat{\imath} - \hat{\jmath} - 7\hat{k}$

Now, $|\vec{b_1} \times \vec{b_2}| = \sqrt{(3)^2 + (-1)^2 + (-7)^2}$

$\Rightarrow |\vec{b_1} \times \vec{b_2}| = \sqrt{9+1+49} = \sqrt{59}$

Now, $(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})$

$= (\hat{\imath} + 0\hat{\jmath} - \hat{k}).(3\hat{\imath} - \hat{\jmath} - 7\hat{k})$

$= 3 - 0 + 7 = 10$

Now, From (1)

$d = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\frac{10}{\sqrt{59}}\right| = \frac{10}{\sqrt{59}}$

2. **Find the shortest distance between the l_1 and l_2 given by $\vec{r} = \hat{\imath} + 2\hat{\jmath} + \hat{k} + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$ and $\vec{r} = 2\hat{\imath} - \hat{\jmath} - \hat{k} + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$**

Solution: Given: Equation of the line l_1 is

$\vec{r} = \hat{\imath} + 2\hat{\jmath} + \hat{k} + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$

$\Rightarrow \vec{a_1} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{b_1} = \hat{\imath} - \hat{\jmath} + \hat{k}$

Given: Equation of the line l_2 is

$\vec{r} = 2\hat{\imath} - \hat{\jmath} - \hat{k} + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$\Rightarrow \vec{a_2} = 2\hat{\imath} - \hat{\jmath} - \hat{k}$ and $\vec{b_2} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$

W.K.T Shortest distance between two skew lines $d = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$ ………… (1)

Now, $\vec{a_2} - \vec{a_1} = (2\hat{\imath} - \hat{\jmath} - \hat{k}) - (\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$\Rightarrow \vec{a_2} - \vec{a_1} = 2\hat{\imath} - \hat{\jmath} - \hat{k} - \hat{\imath} - 2\hat{\jmath} - \hat{k}$

$\Rightarrow \vec{a_2} - \vec{a_1} = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= \hat{\imath}(-2-1) - \hat{\jmath}(2-2) + \hat{k}(1+2)$

$\Rightarrow \vec{b_1} \times \vec{b_2} = -3\hat{\imath} - 0\hat{\jmath} + 3\hat{k}$

Now, $|\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + (0)^2 + (3)^2}$

$\Rightarrow |\vec{b_1} \times \vec{b_2}| = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}$

Now, $(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})$

$= (\hat{\imath} - 3\hat{\jmath} - 2\hat{k}).(-3\hat{\imath} - 0\hat{\jmath} + 3\hat{k})$

$= -3 + 0 - 6 = -9$

Now, From (1)

$d = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\frac{-9}{3\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$

3. **Find the distance between the lines $\mathbf{\vec{r}} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k} + \lambda\,(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$ and $\mathbf{\vec{r}} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k} + \mu\,(2\hat{\imath} + 3\hat{\jmath} + 6\,\hat{k})$.**

Solution: Given: Equation of the line l_1 is

$\vec{r} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k} + \lambda\,(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$

$\Rightarrow \vec{a_1} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$

Given: Equation of the line l_2 is

$\vec{r} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k} + \mu\,(2\hat{\imath} + 3\hat{\jmath} + 6\,\hat{k})$.

$\Rightarrow \vec{a_2} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$ & $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$

W.K.T Distance between two parallel lines

$d = \frac{|(\vec{a_2}-\vec{a_1})\times\vec{b}|}{|\vec{b}|}$ ………… (1)

Now, $|\vec{b}| = \sqrt{(2)^2 + (3)^2 + (6)^2}$

$\Rightarrow |\vec{b}| = \sqrt{4+9+36} = \sqrt{49} = 7$

Now,

$\vec{a_2} - \vec{a_1} = (3\hat{\imath} + 3\hat{\jmath} - 5\hat{k}) - (\hat{\imath} + 2\hat{\jmath} - 4\hat{k})$

$\Rightarrow \vec{a_2} - \vec{a_1} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k} - \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

$\Rightarrow \vec{a_2} - \vec{a_1} = 2\,\hat{\imath} + \hat{\jmath} - \hat{k}$

Now, $(\overrightarrow{a_2}-\overrightarrow{a_1})\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 1 & -1\\ 2 & 3 & 6\end{vmatrix}$

$=\hat{\imath}(6+3)-\hat{\jmath}(12+2)+\hat{k}(6-2)$

$\Rightarrow(\overrightarrow{a_2}-\overrightarrow{a_1})\times\vec{b}=9\hat{\imath}+14\hat{\jmath}+4\hat{k}$

Now,

$|(\overrightarrow{a_2}-\overrightarrow{a_1})\times\vec{b}|=\sqrt{(9)^2+(14)^2+(4)^2}$

$\Rightarrow|(\overrightarrow{a_2}-\overrightarrow{a_1})\times\vec{b}|=\sqrt{81+196+16}$

$\Rightarrow|(\overrightarrow{a_2}-\overrightarrow{a_1})\times\vec{b}|=\sqrt{293}$

Now, From (1)

$d=\frac{|(\overrightarrow{a_2}-\overrightarrow{a_1})\times\vec{b}|}{|\vec{b}|}=\frac{\sqrt{293}}{7}$

4. **Find the equation of the plane passing through the intersection of the planes $3x-y+2z-4=0$ & $x+y+z-2=0$ and the point $(2,2,1)$.**

Solution: Given equation of plane P_1 is $3x-y+2z-4=0$ and equation of plane P_2 is $x+y+z-2=0$

Now, Equation of plane passing through line of intersection is $P_1+\lambda P_2=0$

$\Rightarrow(3x-y+2z-4)+\lambda(x+y+z-2)=0$ ......(1)

Now, $(2,2,1)$ lie on equation (1)

$\Rightarrow 3(2)-2+2(1)-4+\lambda(2+2+1-2)=0$

$\Rightarrow 6-2+2-4+\lambda(3)=0$

$\Rightarrow 2+3\lambda=0 \Rightarrow 3\lambda=-2 \Rightarrow \lambda=\frac{-2}{3}$

From (1), Equation of plane is

$(3x-y+2z-4)+\lambda(x+y+z-2)=0$

$\Rightarrow(3x-y+2z-4)+\frac{-2}{3}(x+y+z-2)=0$

$(\times^{ly}\ by\ 3)$

$\Rightarrow 9x-3y+6z-12-2x-2y-2z+4=0$

$\Rightarrow 7x-5y+4z-8=0$

5. **Find the equation of the plane through the line of intersection of the planes $\vec{r}.(2\hat{\imath}+2\hat{\jmath}-3\hat{k})=7$ & $\vec{r}.(2\hat{\imath}+5\hat{\jmath}+3\hat{k})=9$ & passing through the point $(2,1,3)$**

Solution: Given equation of plane P_1 is $\vec{r}.(2\hat{\imath}+2\hat{\jmath}-3\hat{k})=7$

$\Rightarrow 2x+2y-3z-7=0$

and equation of plane P_2 is

$\vec{r}.(2\hat{\imath}+5\hat{\jmath}+3\hat{k})=9$

$\Rightarrow 2x+5y+3z-9=0$

Now, Equation of plane passing through line of intersection is $P_1+\lambda P_2=0$

$\Rightarrow 2x+2y-3z-7+\lambda(2x+5y+3z-9)=0$ ......(1)

Now, $(2,1,3)$ lie on equation (1)

$\Rightarrow 2(2)+2(1)-3(3)-7+\lambda[2(2)+5(1)+3(3)-9]=0$

$\Rightarrow 4+2-9-7+\lambda(4+5+9-9)=0$

$\Rightarrow -10+9\lambda=0 \Rightarrow 9\lambda=10 \Rightarrow \lambda=\frac{10}{9}$

From (1), Equation of plane is

$2x+2y-3z-7+\lambda(2x+5y+3z-9)=0$

$\Rightarrow 2x+2y-3z-7+\frac{10}{9}(2x+5y+3z-9)=0$

$(\times^{ly}\ by\ 9)$

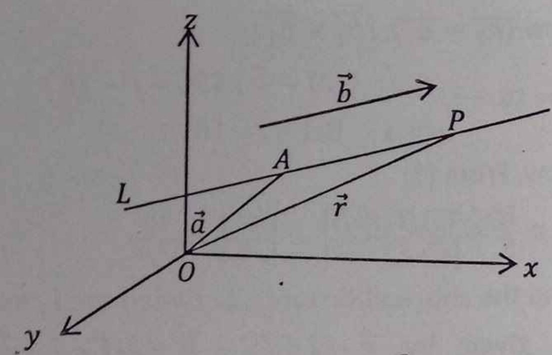
$\Rightarrow 18x+18y-27z-63+20x+50y+30z-90=0$

$\Rightarrow 38x+68y+3z-153=0$

FIVE MARKS QUESTION

1. **Derive the equation of the line in space, passing through a point and parallel to a vector both in vector and Cartesian form.**

Proof:



Vector form:

Let A be the given point and position vector of $A=\overrightarrow{OA}=\vec{a}$

Let $\vec{b}$ be the vector which is parallel to the line L

Let P be any point on the line and position vector of $P=\overrightarrow{OP}=\vec{r}$

Now, clearly $\overrightarrow{AP}$ is parallel to $\vec{b}$

$\Rightarrow \overrightarrow{AP} = \lambda\vec{b}$, where λ is a scalar

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda\vec{b}$

$\Rightarrow \vec{r} - \vec{a} = \lambda\vec{b} \Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$ is the equation of the line L in vector form.

Cartesian Form:

Let co-ordinates of the point A be (x_1, y_1, z_1)

$\Rightarrow$ PV of $A = \vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$

Let a, b, c are the direction ratios of $\vec{b}$

$\Rightarrow \vec{b} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Let co-ordinates of any point P be (x, y, z)

$\Rightarrow$ PV of $P = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now, Equation of the line L is $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda(a\hat{\imath} + b\hat{\jmath} + c\hat{k})$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda a\hat{\imath} + \lambda b\hat{\jmath} + \lambda c\hat{k}$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = (x_1 + \lambda a)\hat{\imath} + (y_1 + \lambda b)\hat{\jmath} + (z_1 + \lambda c)\hat{k}$

$\Rightarrow x = x_1 + \lambda a \quad , \quad y = y_1 + \lambda b \quad , \quad z = z_1 + \lambda c$

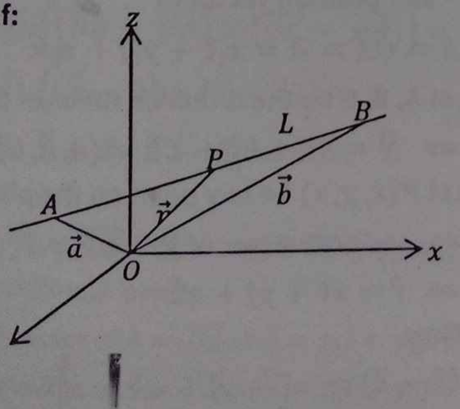
$\Rightarrow x - x_1 = \lambda a \quad , \quad y - y_1 = \lambda b \quad , \quad z - z_1 = \lambda c$

$\Rightarrow \frac{x-x_1}{a} = \lambda \quad , \quad \frac{y-y_1}{b} = \lambda \quad , \quad \frac{z-z_1}{c} = \lambda$

$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ is the equation of line L in Cartesian form

2. Derive the equation of the line in space, passing through two points both in vector and Cartesian forms.

Proof:



Vector Form:

Let A be the given point and position vector of $A = \overrightarrow{OA} = \vec{a}$

Let B be the given point and position vector of $B = \overrightarrow{OB} = \vec{b}$

Let P be any point on the line and position vector of $P = \overrightarrow{OP} = \vec{r}$

Now, clearly A, P & B are collinear

$\Rightarrow \overrightarrow{AP} = \lambda\overrightarrow{AB}$, where λ is a scalar

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ is the equation of the line L in vector form.

Cartesian Form:

Let co-ordinates of the point A be (x_1, y_1, z_1)

$\Rightarrow$ PV of $A = \vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$

Let co-ordinates of the point B be (x_2, y_2, z_2)

$\Rightarrow$ PV of $B = \vec{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

Let co-ordinates of any point P be (x, y, z)

$\Rightarrow$ PV of $P = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now, Equation of the line L is

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda\left(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})\right)$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} - x_1\hat{\imath} - y_1\hat{\jmath} - z_1\hat{k})$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda\left((x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}\right)$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda(x_2 - x_1)\hat{\imath} + \lambda(y_2 - y_1)\hat{\jmath} + \lambda(z_2 - z_1)\hat{k}$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = [x_1 + \lambda(x_2 - x_1)]\hat{\imath} + [y_1 + \lambda(y_2 - y_1)]\hat{\jmath} + [z_1 + \lambda(z_2 - z_1)]\hat{k}$

$\Rightarrow x = x_1 + \lambda(x_2 - x_1)$, $y = y_1 + \lambda(y_2 - y_1)$, $z = z_1 + \lambda(z_2 - z_1)$

$\Rightarrow x - x_1 = \lambda(x_2 - x_1)$, $y - y_1 = \lambda(y_2 - y_1)$, $z - z_1 = \lambda(z_2 - z_1)$

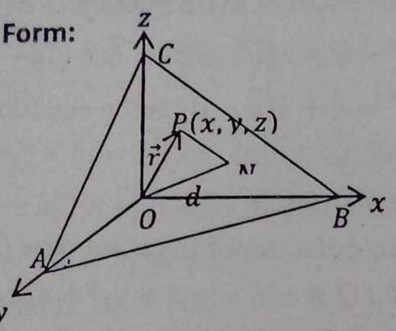
$\Rightarrow \frac{x-x_1}{x_2-x_1} = \lambda \quad , \quad \frac{y-y_1}{y_2-y_1} = \lambda \quad , \quad \frac{z-z_1}{z_2-z_1} = \lambda$

$\Rightarrow \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ is the equation of line L in Cartesian form.

3. Derive the equation of a plane in Normal form in both vector and Cartesian form.

Proof:

Vector Form:



Let ABC be the plane in space. Draw ON perpendicular to the plane and $ON = d$ be the normal to the plane.

Let $\hat{n}$ be the unit vector along the direction of $\overrightarrow{ON}$

$\Rightarrow \hat{n} = \frac{\overrightarrow{ON}}{|\overrightarrow{ON}|} = \frac{\overrightarrow{ON}}{d} \Rightarrow \overrightarrow{ON} = d\,\hat{n} \ldots\ldots (1)$

Let P be any point on the plane and position vector of $P = \overrightarrow{OP} = \vec{r}$

Now, NP is perpendicular to ON

$\Rightarrow \overrightarrow{NP} . \overrightarrow{ON} = 0 \Rightarrow (\overrightarrow{OP} - \overrightarrow{ON}) . \overrightarrow{ON} = 0$

$\Rightarrow \overrightarrow{OP} . \overrightarrow{ON} - \overrightarrow{ON} . \overrightarrow{ON} = 0$

$\Rightarrow \overrightarrow{OP} . \overrightarrow{ON} - |\overrightarrow{ON}|^2 = 0$

$\Rightarrow \overrightarrow{OP} . \overrightarrow{ON} = |\overrightarrow{ON}|^2$

$\Rightarrow \vec{r} . d\,\hat{n} = d^2 \Rightarrow \vec{r}.\hat{n} = d$

$\therefore$ Vector form of equation of plane is

$\vec{r}.\hat{n} = d$

Cartesian Form:

$P(x, y, z)$ be any point on the plane a

$\Rightarrow$ Position vector of $P = \overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Let $\hat{n}$ be the unit vector along the direction of the normal and l, m, n be the direction cosines of $\hat{n} \Rightarrow \hat{n} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$

Now,

Equation of the plane in normal form is $\vec{r}.\hat{n} = d$

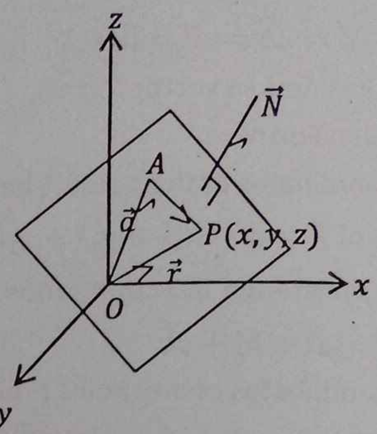
$\Rightarrow (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(l\hat{\imath} + m\hat{\jmath} + n\hat{k}) = d$

$\Rightarrow lx + my + nz = d$

$\therefore$ Cartesian form of equation of plane is $lx + my + nz = d$

4. Derive the equation of a plane passing through a point and perpendicular to a vector in both vector and Cartesian form.

Proof:



Vector Form:

Let plane passes through a point A with position vector $\overrightarrow{OA} = \vec{a}$ and perpendicular to $\vec{N}$

Let P be any point on the plane and position vector of $P = \overrightarrow{OP} = \vec{r}$

Now $\overrightarrow{AP}$ is perpendicular to $\vec{N}$

$\Rightarrow \overrightarrow{AP} . \vec{N} = 0 \Rightarrow (\overrightarrow{OP} - \overrightarrow{OA}).\vec{N} = 0$

$\Rightarrow (\vec{r} - \vec{a}).\vec{N} = 0$

Cartesian Form:

Let plane passes through a point $A(x_1, y_1, z_1)$ and position vector of

$A = \overrightarrow{OA} = \vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$

Let A, B, C be the direction ratios of the vector $\vec{N}$

$\Rightarrow \vec{N} = A\hat{\imath} + B\hat{\jmath} + C\hat{k} = (A, B, C)$

Let $P(x, y, z)$ be any point on the plane

$\Rightarrow$ position vector of $P = \overrightarrow{OP} = \vec{r}$

$\Rightarrow \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now,

$(\vec{r} - \vec{a}) = x\hat{\imath} + y\hat{\jmath} + z\hat{k} - x_1\hat{\imath} - y_1\hat{\jmath} - z_1\hat{k}$

$= (x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}$

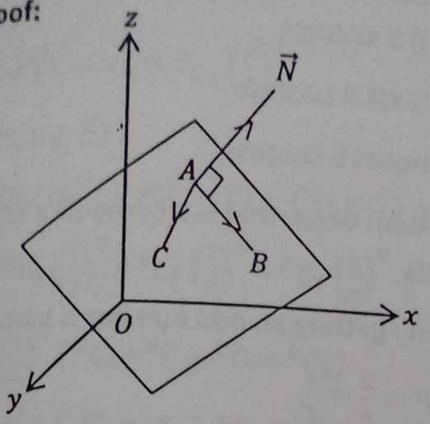
W.K.T Equation of the plane is $(\vec{r} - \vec{a}).\vec{N} = 0$

$\Rightarrow (x - x_1, y - y_1, z - z_1).(A, B, C) = 0$

$\Rightarrow A(x - x_1) + B(y - y_1) + C(Z - Z_1) = 0$

5. Derive the equation of a plane passing through three non collinear points in both vector and Cartesian form.

Proof:



Let A, B, C be any three non collinear points

Let Position vector of $A = \overrightarrow{OA} = \vec{a}$

Position vector of $B = \overrightarrow{OB} = \vec{b}$ &

Position vector of $C = \overrightarrow{OC} = \vec{c}$

Let $\vec{N}$ be the vector which is perpendicular to the plane

$\Rightarrow$ direction of the vector $\vec{N}$ is along the direction of $\overrightarrow{AB} \, X \, \overrightarrow{AC}$.

Now Equation of the plane passing through A and perpendicular to $\overrightarrow{AB} \, X \, \overrightarrow{AC}$ is

$(\vec{r} - \vec{a}).(\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

$\Rightarrow (\vec{r} - \vec{a}).\left((\overrightarrow{OB} - \overrightarrow{OA}) \times (\overrightarrow{OC} - \overrightarrow{OA})\right) = 0$

$\Rightarrow (\vec{r} - \vec{a}).\left((\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right) = 0$

$\Rightarrow [\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c} - \vec{a}] = 0$

Cartesian Form:

Let $A\,(x_1, y_1. z_1), B\,(x_2, y_2, z_2) \& C(x_3. y_3, z_3)$ be three non collinear points.

$\Rightarrow$ Position vector of $A = \overrightarrow{OA} = \vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$

Position vector of $B = \overrightarrow{OB} = \vec{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

Position vector of $C = \overrightarrow{OC} = \vec{c} = x_3\hat{\imath} + y_3\hat{\jmath} + z_3\hat{k}$

Let $P(x, y, z)$ be any point on the plane

Let Position vector of $P = \overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now,

$\vec{r} - \vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} - x_1\hat{\imath} - y_1\hat{\jmath} - z_1\hat{k}$

$\Rightarrow \vec{r} - \vec{a} = (x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}$

Now,

$\vec{b} - \vec{a} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} - x_1\hat{\imath} - y_1\hat{\jmath} - z_1\hat{k}$

$\Rightarrow \vec{b} - \vec{a} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

Now,

$\vec{c} - \vec{a} = x_3\hat{\imath} + y_3\hat{\jmath} + z_3\hat{k} - x_1\hat{\imath} - y_1\hat{\jmath} - z_1\hat{k}$

$\Rightarrow \vec{c} - \vec{a} = (x_3 - x_1)\hat{\imath} + (y_3 - y_1)\hat{\jmath} + (z_3 - z_1)\hat{k}$

Now, equation of the passing through A, B, & C is $[\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c} - \vec{a}] = 0$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

PROBABILITY

FIVE MARKS QUESTION

1. **If a fair coin is tossed 10 times, find the probability of**

   i) exactly six heads

   ii) atleast six heads

   iii) atmost six heads

   Solution: Given trial is a Bernoulli's trial with $n = 10$

   Given : getting head is success

   $\Rightarrow P = \frac{1}{2}$ and $q = 1 - P = 1 - \frac{1}{2} = \frac{1}{2}$

   W.K.T , $P(x = x) = n_{c_x}\, q^{n-x} p^x$

   $\Rightarrow P(x = x) = 10_{c_x}\left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x$

   $\Rightarrow P(x = x) = 10_{c_x}\left(\frac{1}{2}\right)^{10-x+x}$

   $\Rightarrow P(x = x)P(x = x) = 10_{c_x}\left(\frac{1}{2}\right)^{10}$

   i) $P(x = 6) = 10_{c_6}\left(\frac{1}{2}\right)^{10}$

   $= \frac{10!}{(10-6)!6!}\left(\frac{1}{2}\right)^{10} = \frac{10!}{4!6!}\left(\frac{1}{2}\right)^{10} = \frac{105}{512}$

   ii) $P(x \geq 6) = P(x = 6) + P(x = 7)$

   $+ P(x = 8) + P(x = 9) + P(x = 10)$

   $= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10}$

   $+ {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$

   $= \left(\frac{1}{2}\right)^{10}\left({}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right)$

   iii) $P(x \leq 6) = P(x = 0) + P(x = 1) + P(x = 2)$

   $+P(x = 3) + P(x = 4) + P(x = 5) + P(x = 6)$

   $= {}^{10}C_0\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10}$

   $+ {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10}$

   $= \left(\frac{1}{2}\right)^{10}\left({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5\right)$

2. **A die is thrown 6 times. If getting an odd number is a success. What is the probability of i) 5 success**

   ii) atleast 5 success

   iii) atmost 5 success

   Solution: Given trial is a Bernoulli's trial with $n = 6$

   Given : getting an odd number is success

   $\Rightarrow P = \frac{3}{6} = \frac{1}{2},$

   $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

   W.K.T , $P(x = x) = n_{c_x}\, q^{n-x} p^x$

   $\Rightarrow P(x = x) = 6_{c_x}\left(\frac{1}{2}\right)^{6-x}\left(\frac{1}{2}\right)^x$

   $\Rightarrow P(x = x) = 6_{c_x}\left(\frac{1}{2}\right)^6$

   i) $P(x = 5) = 6_{c_5}\left(\frac{1}{2}\right)^6 = 6\left(\frac{1}{64}\right) = \frac{3}{32}$

   ii) $P(x \geq 5) = P(x = 5) + P(x = 6)$

   $= 6_{c_5}\left(\frac{1}{2}\right)^6 + 6_{c_6}\left(\frac{1}{2}\right)^6$

   $= \left(\frac{1}{2}\right)^6\left(6_{c_5} + 6_{c_6}\right) = \frac{1}{64}(6 + 1) = \frac{7}{64}$

   iii) $P(x \leq 5) = 1 - P(x > 5) = 1 - P(x = 6)$

   $= 1 - 6_{c_6}\left(\frac{1}{2}\right)^6 = 1 - 1\left(\frac{1}{64}\right) = \frac{63}{64}$

3. **If a fair win is tossed 8 times find the probability of**

   i) atleast five heads

   ii) atmost five heads.

   Solution: Given trial is a Bernoulli's trial with $n = 8$

   Given : getting head is success

   $\Rightarrow P = \frac{1}{2}$ and $q = 1 - P = 1 - \frac{1}{2} = \frac{1}{2}$

   W.K.T , $P(x = x) = n_{c_x}\, q^{n-x}\, p^x$

$\Rightarrow P(x=x) = 8_{c_x}\left(\frac{1}{2}\right)^{8-x}\left(\frac{1}{2}\right)^{x}$

$\Rightarrow P(x=x) = 8_{c_x}\left(\frac{1}{2}\right)^{8-x+x}$

$\Rightarrow P(x=x) = 8_{c_x}\left(\frac{1}{2}\right)^{x}$

i) $P(x \geq 5)$

$= P(x=5)+P(x=6)+P(x=7)+P(x=8)$

$= {}^8C_5\left(\frac{1}{2}\right)^8 + {}^8C_6\left(\frac{1}{2}\right)^8 + {}^8C_7\left(\frac{1}{2}\right)^8 + {}^8C_8\left(\frac{1}{2}\right)^8$

$= \left(\frac{1}{2}\right)^8 ({}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8)$

ii) $P(x \leq 5) = 1 - P(x > 5)$

$= 1 - (P(x=6)+P(x=7)+P(x=8))$

$= 1 - \left({}^8C_6\left(\frac{1}{2}\right)^8 + {}^8C_7\left(\frac{1}{2}\right)^8 + {}^8C_8\left(\frac{1}{2}\right)^8\right)$

$= 1 - \left(\frac{1}{2}\right)^8 ({}^8C_6 + {}^8C_7 + {}^8C_8)$

4. **Find the probability of throwing atmost 2 sixes in 6 throws of a single die**

Solution: Given trial is a Bernoulli's trial with $n = 6$

Given : getting six is success

$\Rightarrow P = \frac{1}{6}$

$\Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

W.K.T , $P(x=x) = n_{c_x} q^{n-x} p^x$

$\Rightarrow P(x=x) = {}^6C_x\left(\frac{5}{6}\right)^{6-x}\left(\frac{1}{6}\right)^{x}$

Now, $P(x \leq 2)$

$= P(x=0)+P(x=1)+P(x=2)$

$= {}^6C_0\left(\frac{5}{6}\right)^{6-0}\left(\frac{1}{6}\right)^0 + {}^6C_1\left(\frac{5}{6}\right)^{6-1}\left(\frac{1}{6}\right)^1 + {}^6C_2\left(\frac{5}{6}\right)^{6-2}\left(\frac{1}{6}\right)^2$

$= 1\left(\frac{5}{6}\right)^6(1) + 6\left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)^1 + \frac{6\times5}{2}\left(\frac{5}{6}\right)^{6-2}\left(\frac{1}{6}\right)^2$

$= \left(\frac{5}{6}\right)^6 + 6\left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)^1 + 15\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^2$

$= \left(\frac{5}{6}\right)^4\left(\frac{25}{36} + 6\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + 15\left(\frac{1}{36}\right)\right)$

$= \left(\frac{5}{6}\right)^4\left(\frac{25+30+15}{36}\right) = \left(\frac{5}{6}\right)^4\left(\frac{70}{36}\right)$

5. **A pair of die is thrown 4 times. If getting a doublet is considering a success find the probability of 2 success**

Solution: Given trial is a Bernoulli's trial with $n = 4$

Given : getting doublet is success

$Doublet = \{(1,1),(2,2),(3,3) \dots (6,6)\}$

$Now, n(s) = 36, n(Doublet) = 4]$

$\Rightarrow P = \frac{6}{36} = \frac{1}{6},$

$\Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

W.K.T , $P(x=x) = n_{c_x} q^{n-x} p^x$

$\Rightarrow P(x=x) = {}^4C_x\left(\frac{5}{6}\right)^{4-x}\left(\frac{1}{6}\right)^{x}$

Now,

$P(x=2) = {}^4C_2\left(\frac{5}{6}\right)^{4-2}\left(\frac{1}{6}\right)^{2}$

$= \frac{4\times3}{2}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = 6\frac{25}{36}\times\frac{1}{36}$

$\Rightarrow P(x=2) = \frac{25}{216}$

6. **There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?**

Solution: Given trial is a Bernoulli's trial with $n = 10$

Given : getting defective item is success

$\Rightarrow P = 5\% = \frac{5}{100} = \frac{1}{20}$

$\Rightarrow q = 1 - \frac{1}{20} = \frac{19}{20}$

W.K.T , $P(x = x) = n_{c_x} q^{n-x} p^x$

$\Rightarrow P(x = x) = 10_{c_x} \left(\frac{19}{20}\right)^{10-x} \left(\frac{1}{20}\right)^x$

$\Rightarrow P(x \leq 1) = P(x = 0) + P(x = 1)$

$= {}^{10}C_0 \left(\frac{19}{20}\right)^{10-0} \left(\frac{1}{20}\right)^0 + {}^{10}C_1 \left(\frac{19}{20}\right)^{10-1} \left(\frac{1}{20}\right)^1$

$= 1\left(\frac{19}{20}\right)^{10} (1) + 10\left(\frac{19}{20}\right)^9 \left(\frac{1}{20}\right)$

$= \left(\frac{19}{20}\right)^9 \left(\frac{19}{20} + \frac{10}{20}\right) = \left(\frac{19}{20}\right)^9 \left(\frac{29}{20}\right)$

7. A person buys a lottery ticket in 50 lotteries in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that we will win a prize.

i) atleast once

ii) exactly once

iii) atleast twice?

Solution: Given trial is a Bernoulli's trial with $n = 50$

Given : Winning Prize is success

$\Rightarrow P = \frac{1}{100}$,

$\Rightarrow q = 1 - p = 1 - \frac{1}{100} = \frac{99}{100}$

W.K.T , $P(x = x) = n_{c_x} q^{n-x} p^x$

$\Rightarrow P(x = x) = 50_{c_x} \left(\frac{99}{100}\right)^{50-x} \left(\frac{1}{100}\right)^x$

i) $P(x \geq 1) = 1 - P(x = 0)$

$= 1 - {}^{50}C_0 \left(\frac{99}{100}\right)^{50-0} \left(\frac{1}{100}\right)^0$

$\Rightarrow P(x \geq 1) = 1 - \left(\frac{99}{100}\right)^{50}$

ii) $P(x = 1) = {}^{50}C_1 \left(\frac{99}{100}\right)^{50-1} \left(\frac{1}{100}\right)$

$\Rightarrow P(x \geq 1) = 50\left(\frac{99}{100}\right)^{49} \left(\frac{1}{100}\right)$

$\Rightarrow P(x \geq 1) = \frac{1}{2}\left(\frac{99}{100}\right)^{49}$

iii) $P(x \geq 2) = 1 - [P(x = 0) + P(x = 1)]$

$= 1 - \left[{}^{50}C_0 \left(\frac{99}{100}\right)^{50-0} \left(\frac{1}{100}\right)^0 + {}^{50}C_1 \left(\frac{99}{100}\right)^{50-1} \left(\frac{1}{100}\right)^1\right]$

$= 1 - \left[1\left(\frac{99}{100}\right)^{50} (1) + 50\left(\frac{99}{100}\right)^{49} \left(\frac{1}{100}\right)\right]$

$= 1 - \left[\left(\frac{99}{100}\right)^{50} + \frac{1}{2}\left(\frac{99}{100}\right)^{49}\right]$

LINEAR PROGRAMMING

SIX MARKS QUESTION

1. Minimize and Maximize $Z = -3x + 4y$ subject to the constraints $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$ and $y \geq 0$

Solution: To draw a line $x + 2y = 8$

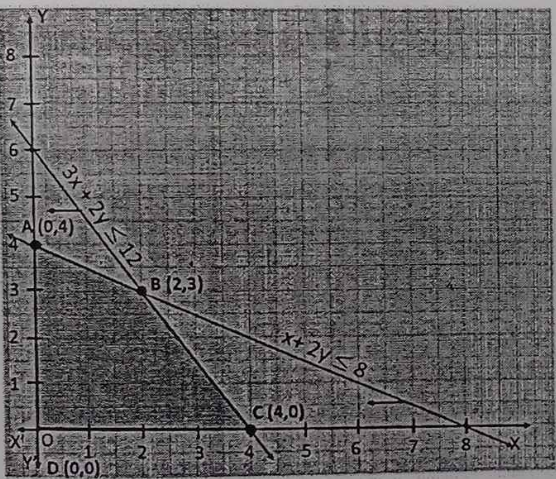
x	0	8
y	4	0

$\Rightarrow (0,4)$ *and* $(8,0)$ lie on the line $x + 2y = 8$

To draw a line $3x + 2y = 12$

x	0	4
y	6	0

$\Rightarrow (0,6)$ *and* $(4,0)$ lie on the line $3x + 2y = 12$



Here feasible region is bounded and corner points are $(0,0)$, $(4,0)$, $(2,3)$ and $(0,4)$

Now,

Corner points	$Z = -3x + 4y$
$(0,0)$	$Z = 0 + 0 = 0$
$(4,0)$	$Z = -12 + 0 = -12$
$(2,3)$	$Z = -6 + 12 = 6$
$(0,4)$	$Z = 0 + 16 = 16$

Now, $Z_{max} = 16$ and $Z_{min} = -12$

2. Minimize and Maximize $Z = 600x + 400y$ subject to the constraints $x + 2y \leq 12$, $2x + y \leq 12$, $4x + 5y \leq 20$, $x \geq 0$ and $y \geq 0$ graphical method.

Solution: To draw a line $x + 2y = 12$

x	0	12
y	6	0

$\Rightarrow (0,6)$ *and* $(12,0)$ lie on the line $x + 2y = 12$

To draw a line $2x + y = 12$

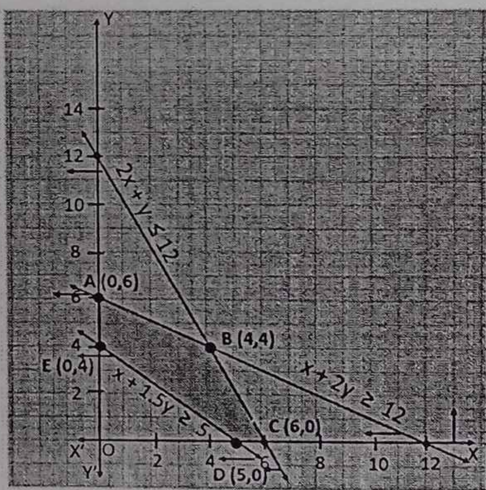
x	0	6
y	12	0

$\Rightarrow (0,12)$ *and* $(6,0)$ lie on the line $2x + y = 12$

To draw a line $4x + 5y = 20$

x	0	5
y	4	0

$\Rightarrow (0,4)$ *and* $(5,0)$ lie on the line $4x + 5y = 20$



Here feasible region is bounded and corner points are $(5,0)$, $(6,0)$, $(4,4)$ $(0,6)$ and $(0,4)$

Now,

Corner points	$Z = 600x + 400y$
$(5,0)$	$Z = 3000 + 0 = 3000$
$(6,0)$	$Z = 3600 + 0 = 3600$
$(4,4)$	$Z = 2400 + 1600 = 4000$
$(0,6)$	$Z = 0 + 2400 = 2400$
$(0,4)$	$Z = 0 + 1600 = 1600$

Now, $Z_{max} = 4000$ and $Z_{min} = 1600$

3. Minimize and Maximize $Z = 3x + 9y$ subject to the constraints $x + 3y \leq 60$, $x + y \geq 10$, $x \leq y$, $x \geq 0$ and $y \geq 0$

Solution: To draw a line $x + 3y = 60$

x	0	60
y	20	0

$\Rightarrow$ $(0,20)$ *and* $(60,0)$ lie on the line $x + 3y = 60$

To draw a line $x + y = 10$

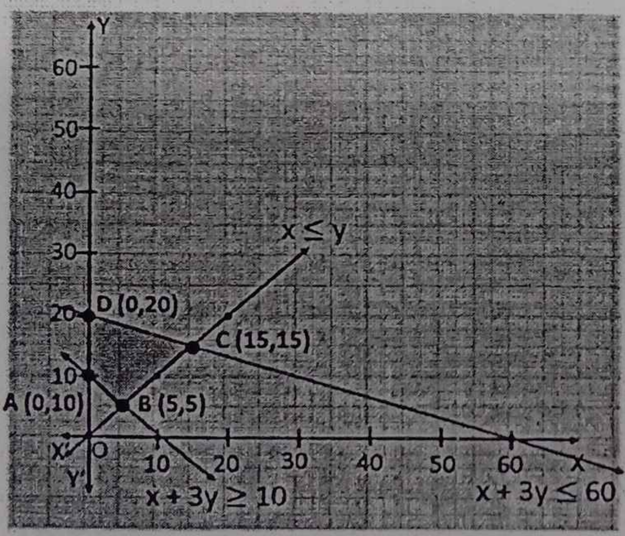
x	0	10
y	10	0

$\Rightarrow$ $(0,10)$ *and* $(10,0)$ lie on the line $x + y = 10$

To draw a line $x = y$

x	0	15
y	0	15

$\Rightarrow$ $(0,0)$ *and* $(15,15)$ lie on the line $x = y$



Here feasible region is bounded and corner points are $(0,10)$, $(0,20)$, $(15,15)$ and $(5,5)$

Now,

Corner points	$Z = 3x + 9y$
$(0,10)$	$Z = 0 + 90 = 90$
$(5,5)$	$Z = 15 + 45 = 60$
$(15,15)$	$Z = 45 + 135 = 180$
$(0,20)$	$Z = 0 + 180 = 180$

Now, $Z_{max} = 180$ and $Z_{min} = 60$

4. Minimize and maximize $z = 5x + 10y$, subject to the constraints $x + 2y \leq 120$, $x + y \geq 60, x - 2y \geq 0$ and $x \geq 0, y \geq 0$ by graphical method.

Solution: To draw a line $x + 2y = 120$

x	0	120
y	60	0

$\Rightarrow$ $(0,60)$ & $(120,0)$ lie on the line $x + 2y = 120$

To draw a line $x + y = 60$

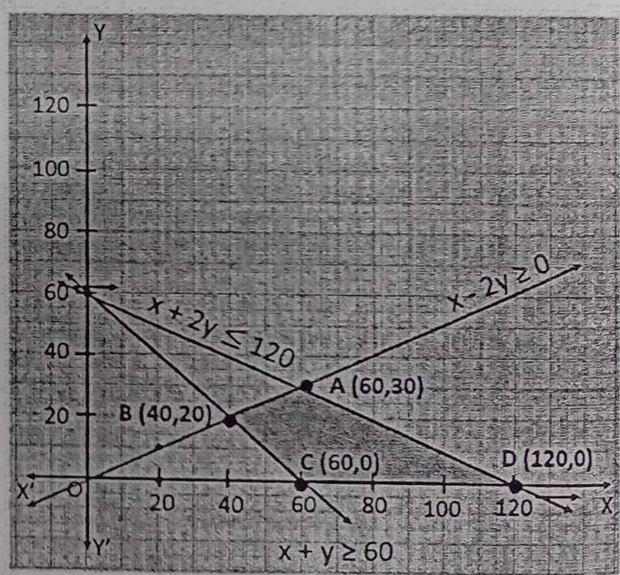
x	0	60
y	60	0

$\Rightarrow$ $(0,60)$ & $(60,0)$ lie on the line $x + y = 60$

To draw a line $x - 2y = 0$

x	0	60
y	0	30

$\Rightarrow$ $(0,0)$ & $(60,30)$ lie on the line $x - 2y = 0$



Here feasible region is bounded and corner points are $(60,0)$, $(120,0)$, $(60,30)$ and $(40,20)$

Now,

Corner points	$Z = 5x + 10y$
$(60,0)$	$Z = 300 + 0 = 300$
$(120,0)$	$Z = 600 + 0 = 600$
$(60,30)$	$Z = 300 + 300 = 600$
$(40,20)$	$Z = 200 + 200 = 400$

Now, $Z_{max} = 600$ and $Z_{min} = 300$

5. **Minimize and maximize $z = x + 2y$. subjected to constraints. $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x \geq 0, y \geq 0$**

Solution: To draw a line $x + 2y = 100$

x	0	100
y	50	0

$\Rightarrow$ (0,50) & (100,0) lie on the line $x + 2y = 100$

To draw a line $2x - y = 0$

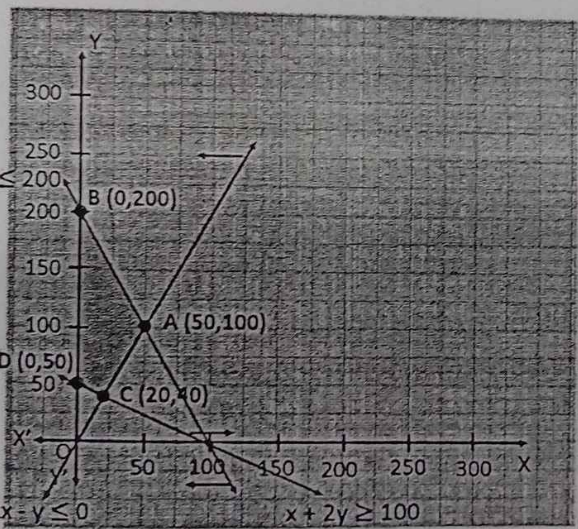
x	0	50
y	0	100

$\Rightarrow$ (0,0) & (60,0) lie on the line $2x - y = 0$

To draw a line $2x + y = 200$

x	0	100
y	200	0

$\Rightarrow$ (0,200) & (100,0) lie on the line $2x + y = 200$



Here feasible region is bounded and corner points are (0,50), (20,40), (50,100) & (0,200)

Now,

Corner points	$z = x + 2y$
(0,50)	$Z = 0 + 100 = 100$
(20,40)	$Z = 20 + 80 = 100$
(50,100)	$Z = 50 + 200 = 250$
(0,200)	$Z = 0 + 400 = 400$

Now , $Z_{max} = 400$ and $Z_{min} = 100$

6. **One kind of cake requires 200g of flour and 25g, of fat and another kind of cake requires 100 g flour and 50g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that three is no shortage of the other ingredients used in making the cakes.**

Solution:

	Cakes		
	I Kind	II Kind	
Flour	200	100	$\leq 5 \times 1000$
Fat	25	50	$\leq 1 \times 1000$

Let x be the number of first kind of cake and y be the number of second kind of cake

Objective function: $z = x + y$

Constraints:

$200x + 100y \leq 5000 \Rightarrow 2x + y \leq 50$

$25x + 50y \leq 1000 \Rightarrow x + 2y \leq 40$

$x \geq 0 , y \geq 0$

Now,

To draw a line $2x + y = 50$

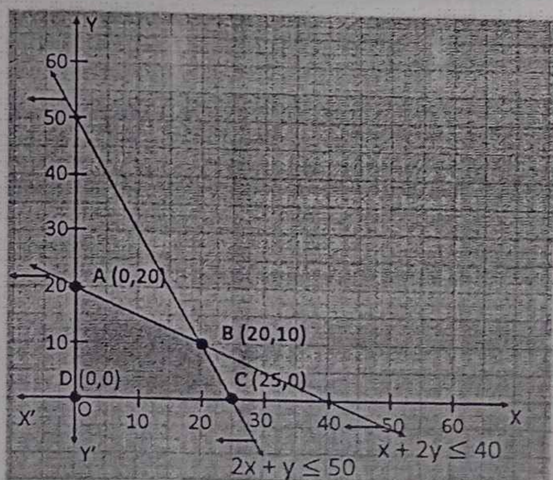
x	0	25
y	50	0

$\Rightarrow$ (0,50) & (25,0) lie on the line $2x + y = 50$

To draw a line $x + 2y = 40$

x	0	40
y	20	0

$\Rightarrow$ (0,20) & (40,0) lie on the line $x + 2y = 40$



Here feasible region is bounded and corner points are (0,0), (25,0), (20,10) & (0,20)

Now,

Corner points	$z = x + y$
(0,0)	$Z = 0 + 0 = 0$
(25,0)	$Z = 25 + 0 = 0$
(20,10)	$Z = 20 + 10 = 30$
(0,20)	$Z = 0 + 20 = 20$

Now , $Z_{max} = 30$

$\Rightarrow$ The maximum number of cakes which can be made is 30

7. A manufacturing company makes two models A and B of a product. Each piece of model A requires 9 labours hours for fabricating and 1 labour hour for finishing. Each piece of models B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing , the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs. 8,000 on each piece of model A and Rs. 12,000 on each piece of model B. How many pieces of model A and model B should be manufactured per week to realize a maximum profit? What is the maximum profit per week?

Solution:

	Models		
	A	B	
Fabricating	9	12	≤ 180
Finishing	1	3	≤ 30
Profit	8000	12000	

Let x be the number of pieces of model A and y be the number of pieces of model B

Objective function: $z = 8000x + 12000y$

Constraints:

$9x + 12y \leq 180 \Rightarrow 3x + 4y \leq 60$

$x + 3y \leq 30$ and $x \geq 0 , y \geq 0$

Now,

To draw a line $3x + 4y = 60$

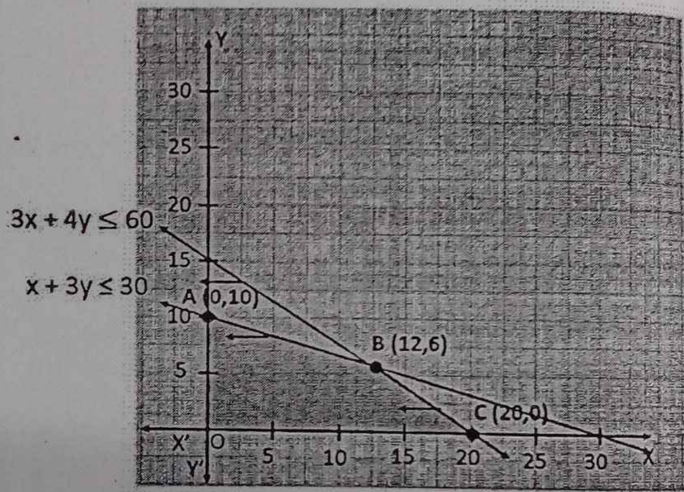
x	0	20
y	15	0

$\Rightarrow$ (0,15) & (20,0) lie on the line $3x + 4y = 60$

To draw a line $x + 3y = 30$

x	0	30
y	10	0

$\Rightarrow$ (0,10) & (30,0) lie on the line $x + 3y = 30$



Here feasible region is bounded and corner points are (0,0), (20,0), (12,6) & (0,10)

Now,

Corner points	$z = 8000x + 12000y$
(0,0)	$Z = 0 + 0 = 0$
(20,0)	$Z = 160000 + 0 = 160000$
(12,6)	$Z = 96000 + 72000 = 168000$
(0,10)	$Z = 0 + 120000 = 120000$

Now , $Z_{max} = Rs\ 168000$

$\Rightarrow$ The maximum profit of Rs 168000 is obtained when 12 pieces of model A and 6 pieces of model B are produces

8. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximize his profit, if he operates his machines for at the most 12 hours a day?

Solution:

	Nuts	Bolts	
Machine A	1	3	≤ 12
Machine B	3	1	≤ 12
Profit	17.50	7	

Let x be the number of packages of nuts and y be the number of packages of bolts.

Objective function: $z = 17.50x + 7y$

Constraints:

$x + 3y \leq 12$, $3x + y \leq 12$ & $x \geq 0$, $y \geq 0$

Now,

To draw a line $x + 3y = 12$

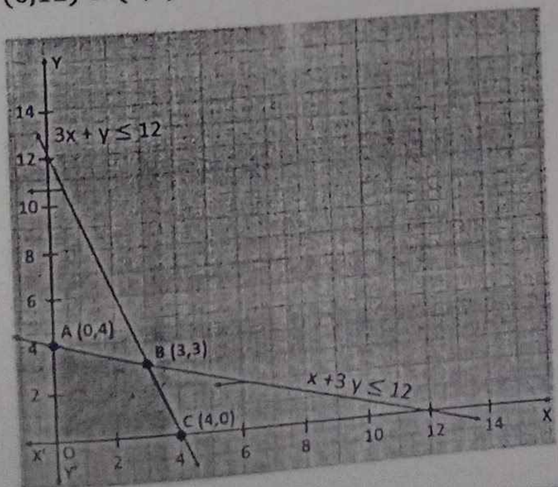
x	0	12
y	4	0

$\Rightarrow$ (0,4) & (12,0) lie on the line $x + 3y = 12$

To draw a line $3x + y = 12$

x	0	4
y	12	0

$\Rightarrow$ (0,12) & (4,0) lie on the line $3x + y = 12$



Here feasible region is bounded and corner points are $(0,0)$, $(4,0)$, $(3,3)$ & $(0,4)$

Now,

Corner points	$z = 17.50x + 7y$
(0,0)	$Z = 0 + 0 = 0$
(4,0)	$Z = 70 + 0 = 70$
(3,3)	$Z = 52.5 + 21 = 73.5$
(0,4)	$Z = 0 + 28 = 28$

Now , $Z_{max} = Rs\ 73.5$

$\Rightarrow$ The maximum profit of Rs 73.5 is obtained when 3 packages of nuts and 3 packages of bolts are produces

9. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit.

Solution:

	Screws		
Machines	A	B	
Automatic	4	6	$\leq 4 \times 60$
Hand Operated	6	3	$\leq 4 \times 60$
Profit	7	10	

Let x be the number of packages of screw A and y be the number of packages of screw B.

Objective function: $Z = 7x + 10y$

Constraints:

$4x + 6y \leq 240 \Rightarrow 2x + 3y \leq 120$

$6x + 3y \leq 240 \Rightarrow 2x + y \leq 80$

$x \geq 0, y \geq 0$

Now,

To draw a line $2x + 3y = 120$

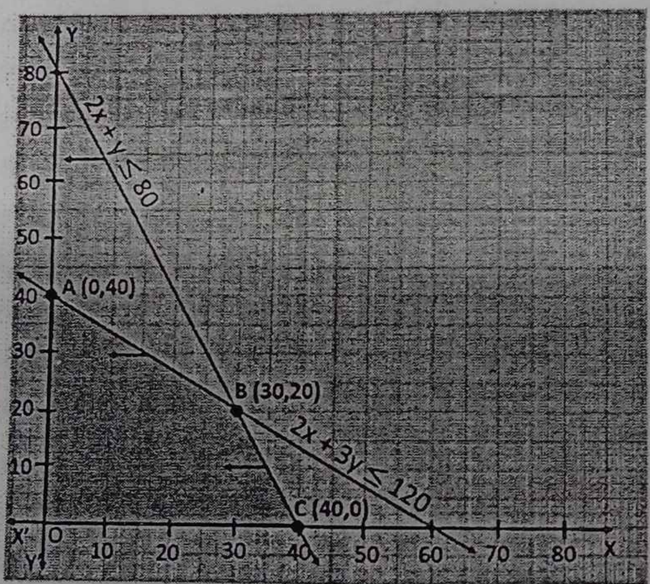
x	0	60
y	40	0

$\Rightarrow$ (0,40) & (60,0) lie on the line $2x + 3y = 120$

To draw a line $2x + y = 80$

x	0	40
y	80	0

$\Rightarrow$ (0,80) & (40,0) lie on the line $2x + y = 80$



Here feasible region is bounded and corner points are (0,0), (40,0), (30,20) & (0,40)

Now,

Corner points	$z = 7x + 10y$
(0,0)	$Z = 0 + 0 = 0$
(40,0)	$Z = 280 + 0 = 280$
(30,20)	$Z = 210 + 200 = 410$
(0,40)	$Z = 0 + 400 = 400$

Now, $Z_{max} = Rs\ 410$

$\Rightarrow$ The maximum profit of Rs 410 is obtained when 30 packages of screw A and 20 packages of screw B are produces

LIST OF FORMULAE

RELATIONS AND FUNCTIONS

i) Reflexive relation:- A relation R on set A is said to be reflexive if $(a,a) \in R$ for all $a \in A$

Example:- 1) Let $A = \{1,2,3,\}$ then $R = \{(1,1),(2,2),(3,3),(2,3)\}$ is reflexive

2) Let $A = \{1,2,3,\}$ then $R = \{(1,1),(2,2),\}$ is not reflexive relation

ii) Symmetric relation:- A relation R on set A is said to be symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$

Example:- 1) Let $A = \{1,2,3,\}$ then $R = \{(1,2),(2,1),(3,2),(2,3)\}$ is symmetric

2) Let $A = \{1,2,3,\}$ then $R = \{(1,2),(2,1),(3,2)\}$ is not symmetric

3) Let $A = \{1,2,3,\}$ then $R = \{(1,1)\}$ is symmetric relation

iii)Transitive relation:- A relation R on set A is said to be transitive if $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$

Example:- 1) Let $A = \{1,2,3,\}$ then $R = \{(2,3),(3,1),(2,1)\}$ is transitive relation

2) Let $A = \{1,2,3,\}$ then $R = \{(2,3)\}$ is transitive relation

3) Let $A = \{1,2,3,\}$ then $R = \{(1,2),(2,3),(1,3),(2,1)\}$ is not transitive

iv) Equivalence relation:- A relation R on set A is said to be equivalence if R is reflexive, symmetric and transitive.

Example:- 1) If $A = \{1,2,3,\}$ then $R = \{(1,1),(2,2),(3,3)\}$ is an equivalence relation

2) If $A = \{1,2,3\}$ then $R = \{((1,1),(2,2),(3,3),(1,2),(2,1))\}$ is an equivalence relation

INVERSE TRIGONOMETRIC FUNCTION

1. Domain and range of the inverse trigonometric function as follows

Functions	Domain	Range
$y = sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = tan^{-1}x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = sec^{-1}x$	$x \le -1, x \ge 1$	$0 \le y \le \pi \left(y \ne \frac{\pi}{2}\right)$
$y = cosec^{-1}x$	$x \le -1, x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2} (y \ne 0)$
$y = cot^{-1}x$	$-\infty < x < \infty$	$0 < y < \pi$

2. Properties Of Inverse Trigonometric Function :-

1. $sin\, sin^{-1} x = x, -1 \le x \le 1$ and $sin^{-1} sinx = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
2. $\cos cos^{-1} x = x, -1 \le x \le 1$ and $cos^{-1} cosx = x, 0 \le x \le 2\pi$
3. $\tan tan^{-1} x = x, -\infty < x < \infty$ and $\tan^{-1}\tan x = x, -\frac{\pi}{2} < x, < \frac{\pi}{2}$
4. $sin^{-1}(-x) = -sin^{-1}x, -1 \le x \le 1$
5. $cos^{-1}(-x) = \pi - cos^{-1}x\,, -1 \le x \le 1$
6. $tan^{-1}(-x) = -tan^{-1}x\,, -\infty < x < \infty$

7. $sec^{-1}(-x) = \pi - sec^{-1}x,\ x \leq -1, x \geq 1$
8. $cosec^{-1}(-x) = -cosec^{-1}x,\ x \leq -1, x \geq 1$
9. $cot^{-1}(-x) = \pi - cot^{-1}x,\ -\infty < x < \infty$
10. $\sin^{-1} x = cosec^{-1}\frac{1}{x}. -1 \leq x \leq 1 \ \&\ x \neq 0$
11. $\cos^{-1} x = sec^{-1}\frac{1}{x}. -1 \leq x \leq 1 \ \&\ x \neq 0$
12. $\sec^{-1} x = cos^{-1}\frac{1}{x},\ x \leq -1, x \geq 1$
13. $\text{cosec}^{-1} x = sin^{-1}\frac{1}{x},\ x \leq -1, x \geq 1$
14. $\tan^{-1}x = cot^{-1}\frac{1}{x},\ if\ x > 0$
15. $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}, -1 \leq x \leq 1$
16. $sec^{-1}x + cosec^{-1}x = \frac{\pi}{2}, x \leq -1, x \geq 1$
17. $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}, -\infty < x < \infty$

3. Standard Formulae :-

1. $tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $x \geq 0, y \geq 0\ and\ xy < 1.$
2. $tan^{-1}x + tan^{-1}y = \pi + tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $x \geq 0, y \geq 0\ and\ xy > 1.$
3. $2tan^{-1}x = tan^{-1}\left(\frac{2x}{1-x^2}\right)$ if $0 \leq x \leq 1$.
4. $tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$ if $x \geq 0, y \geq 0.$

MATRICES

i) **Diagonal matrix :-** A matrix in which all the elements except the principle diagonal elements are zero

ii) **Scalar matrix :-** A diagonal matrix in which all the principle diagonal elements are equal

iii) **Unit matrix or Identity matrix :-** A diagonal matrix in which each principle diagonal entries is one

DETERMINANTS

1. Area of the triangle formed by the vertices $(x_1, y_1), (x_2, y_2)$ & (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2. Equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

3. **Matrix method :-**

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(adjA)$ and solution is $X = A^{-1}B$