Financial Modeling with Matlab
Portfolio Optimization and Simulation

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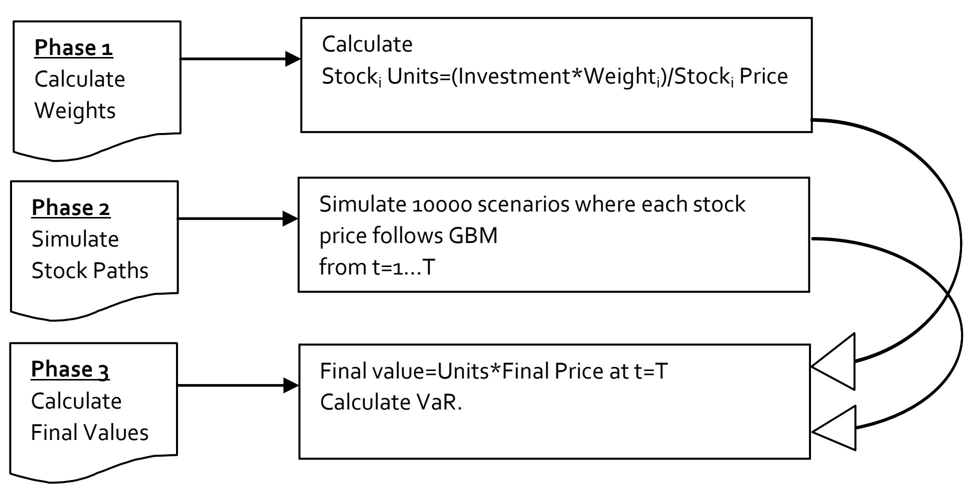
Briefs:

The purpose of the program is to demonstrate portfolio optimization and simulation. Using Harry Markowitz portfolio theory, the program computes the optimal weight for a given risk/return. Using Monte Carlo simulation with Cholesky decomposition, the program simulates portfolio value in the next ***T*** period. Finally, Value-at-Risk is computed.

Programming Steps:

There are 3 phases in this program. First, for comparison purpose, the optimum weight of investment for each stock is calculated for (1) equally weighted portfolio, (2) minimum variance portfolio, and (3) required rate of return portfolio. Given an initial sum of investment and the latest stock prices, the number of units for each stock can be calculated for each portfolio investment style. The second phase of the program is to simulate the price paths of each stock by using Monte Carlo simulation and Cholesky decomposition technique. Given that we have units of investment from phase 1 and each stock price path from phase 2, the final phase is to calculate portfolio value and its value-at-risk at the end of period.



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Problem Statement:

Suppose you have an initial investment of 1,000,000 baht and you would like to invest in 5 stocks. How much, in percentage of your initial investment, should you invest in each stock in order to achieve (1) the least risky return, or, (2) the required rate of return?

A. Portfolio Optimization

In order to answer the problem, we simply setup an objective function and find the optimum solution for such function. Before we unveil the objective function equation, we will discuss a few financial concepts first..

A.1 Portfolio Risk and Return Calculation

A simple calculation would help us understand the problem. Suppose you have these information tables.

Stock	Investment($)	Weight(Xi)	E(Ri)
CPN	6,000	0.60	2.91%
CPALL	4,000	0.40	4.09%
	10,000		

Stock	σ^2
CPN	2.42%
CPALL	1.02%

σ_{12}	ρ_{12}
0.0115	0.73243

Where σ^2 is variance of stock return, σ_{12} is covariance between 2 stock returns, and ρ_{12} is correlation between 2 stock returns. In order to calculate portfolio returns and risk, we apply the following calculations.

2-Asset Portfolio Return: $R_p = X_1R_1 + X_2R_2$

2-Asset Portfolio Risk:

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}$$

$$\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_1\rho_{12}\sigma_1\sigma_2$$

Note

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}$$

Hence, the portfolio return and risk from the above table are

R_p=(0.6)(2.91%)+(0.4)(4.09%)=3.38%

σ_p=sqrt[(0.6)^2(2.91%)^2+(0.4)^2 (4.09%)^2+2(0.6)(0.4)(0.0115)]=12.59%

=sqrt[(0.6)^2(2.91%)^2+(0.4)^2 (4.09%)^2+2(0.6)(0.4)(0.73243)(sqrt(2.42%))(sqrt(1.02%))]=12.59%

The generalized form would be...

$$R_p = \sum_{i=1}^{N} X_i R_i \qquad \sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + 2\sum_{i \neq j}^{N} X_i X_j \sigma_{ij} \qquad \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + 2\sum_{i \neq j}^{N} X_i X_j \sigma_{ij} \rho_{ij} \sigma_i \sigma_j$$

The covariance and correlation between 2 stocks are taken as given, which means we cannot do anything about it. However, if we assign a different set of weight, we would achieve different outputs of risk and return. We are also need to exhaust all our initial investment as well. Therefore, the objective function is **minimizing the portfolio variance(σ_p^2)** subject to summation of weight equal to 1($\sum$**x**$_i$=1).

$$\begin{aligned} &Min \quad \sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + 2\sum_{i \neq j}^{N} X_i X_j \sigma_{ij} \\ &s.t. \quad \sum_{i=1}^{N} X_i = 1 \end{aligned}$$

However, as more assets are added to the portfolios, solving polynomial algebra would be cumbersome. It is best to transform it to matrix system. A simple example would explain everything.

A.2 Portfolio Risk and Return Calculation: Matrix Form

The following explain how to transform linear algebra to matrix algebra

$$R_p = X_1 R_1 + X_2 R_2$$

$$\sigma_p^2 = X_1^2 \sigma_1^2 + X_2^2 \sigma_2^2 + 2 X_1 X_2 \sigma_{12}$$

Let **X** be a column vector of asset weight, **R** be a column vector of expected return and **Σ** be variance covariance matrix. The above polynomial system can be re-written as..

$$R_p = X'R$$

$$\sigma_p^2 = X' \sum X$$

Where

$$R_p = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \qquad \sigma_p = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now, consider 3-asset portfolio with minimizing portfolio variance problem.

$$\begin{aligned} \min_{x_A, x_B, x_C} \sigma_p &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC} \\ &st.\ x_A + x_B + x_C = 1 \end{aligned}$$

$$\begin{aligned} L(x_A, x_B, x_C, \lambda) = {} & x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC} \\ & + \lambda(x_A + x_B + x_C - 1) \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{x_A} &= 2x_A \sigma_A^2 + 2x_B \sigma_{AB} + 2x_C \sigma_{AC} + \lambda = 0 \\ \frac{\partial L}{x_B} &= 2x_A \sigma_{AB} + 2x_B \sigma_B^2 + 2x_C \sigma_{BC} + \lambda = 0 \\ \frac{\partial L}{x_C} &= 2x_A \sigma_{AC} + 2x_B \sigma_{BC} + 2x_C \sigma_C^2 + \lambda = 0 \\ \frac{\partial L}{\lambda} &= x_A + x_B + x_C - 1 = 0 \end{aligned}$$

With the same logic, we can transform the 3-asset portfolio variance minimization problem from linear algebra to matrix algebra as follow

$$\min_X \sigma_p^2 = X'\sum X \ st.\ X'1 = 1$$

$$\frac{\partial L}{X} = \begin{pmatrix} 2\sigma_A^2 & 2\sigma_{AB} & 2\sigma_{AC} & 1 \\ 2\sigma_{AB} & 2\sigma_B^2 & 2\sigma_{BC} & 1 \\ 2\sigma_{AC} & 2\sigma_{BC} & 2\sigma_C^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_A \\ x_B \\ x_C \\ \lambda \end{pmatrix} = \begin{pmatrix} 2\sigma_A^2 & 2\sigma_{AB} & 2\sigma_{AC} & 1 \\ 2\sigma_{AB} & 2\sigma_B^2 & 2\sigma_{BC} & 1 \\ 2\sigma_{AC} & 2\sigma_{BC} & 2\sigma_C^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence, to solve the system with 1 constraint, a generalized form would be

$$\begin{pmatrix} X \\ \lambda \end{pmatrix} = \begin{pmatrix} 2\Sigma & 1 \\ 1' & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is what we call "***Minimum Variance Portfolio***", the weight matrix X is the weight that would yield minimum variance.

Required Rate Return Portfolio

Suppose, instead of achieving minimum variance portfolio, we would like to add one more requirement, says we require a return of 5% from the portfolio. We can achieve this by simply adding a new constraint ($\sum \mathbf{x}_i \mathbf{R}_i = \mathbf{R}_q$). Hence, in a matrix form, it would be..

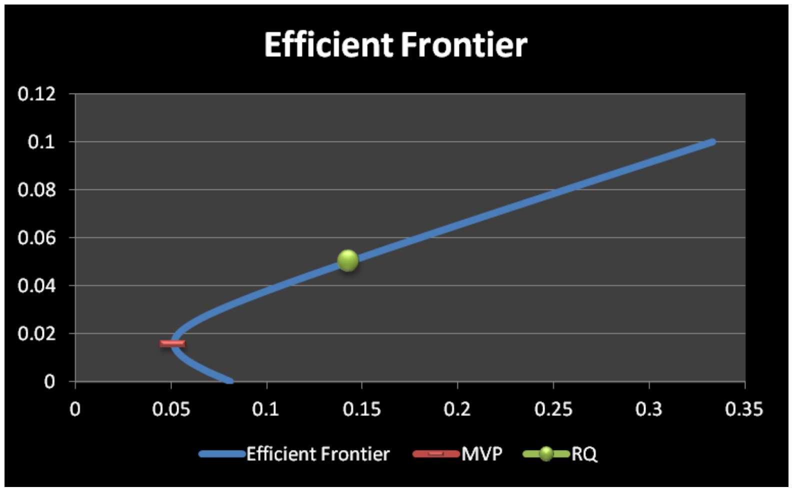
$$\min_X \sigma_p^2 = X'\sum X \ st.\ \ X'R = R_q \ and \ X'1 = 1$$

$$\frac{\partial L}{X} = \begin{pmatrix} 2\Sigma & R & 1 \\ R' & 0 & 0 \\ 1' & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ R_q \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} X \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2\Sigma & R & 1 \\ R' & 0 & 0 \\ 1' & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R_q \\ 1 \end{pmatrix}$$

The solution of the system will not only give us the weight that give us minimum variance but also the return of Rq%. This is what we call "***Frontier Portfolio***".

Efficient Frontier



Where x-Axis is risk(σ_p) and y-Axis is return. On the right hand side of the parabola curve is called "***Feasible Portfolios***", that is, all possible weights that would yield various risks and returns lie on the right hand side of this parabola. However, a **risk adverse** investor would choose the weight that gives him/her **highest return given the same risk**, or, **lowest risk given the same return**. Hence, only the weights along the curve line are efficient. All of them together are called **efficient frontier**.

B. Portfolio Simulation

In this section, we explain Monte Carlo Simulation with one single stock and show that the portfolio simulation inherits the same spirit but with a touch of Cholesky decomposition technique.

B.1 Monte Carlo Simulation: One Single Stock

Assume that stock prices follow Geometric Brownian Motion, GBM. In 1 short period (dt), we calculate stock price as follow.

$$S_{dt} = S_0 \times e^{[(\mu - \frac{\sigma^2}{2})dt + \sigma\sqrt{dt}\varepsilon]}, \varepsilon \sim N(0,1)$$

We apply the same calculation to the next period (dt+1) where $S_{dt+1} = S_{dt} \times e^{[(\mu - \frac{\sigma^2}{2})dt + \sigma\sqrt{dt}\varepsilon]}$ and so on for other ***T*** periods to follow. The same process of calculation is simulated for as many as ***S*** simulation scenarios. The final price of each scenario is then used for other calculation, for example multiple by unit of stocks to get the final valuation. The average of ***S*** simulation values is the solution to Monte Carlo Simulation of one single stock.

B.2 Monte Carlo Simulation: Portfolio

A portfolio contains multiple stocks. If stocks have zero correlation with one another, we can assume that $\varepsilon \sim N(0,1)$. However, it is usually not the case. We will use Cholesky decomposition technique to solve to problem.

Suppose we have 3-asset portfolio. The **1 period** stock prices for 3 stocks in this portfolio is calculated as follow

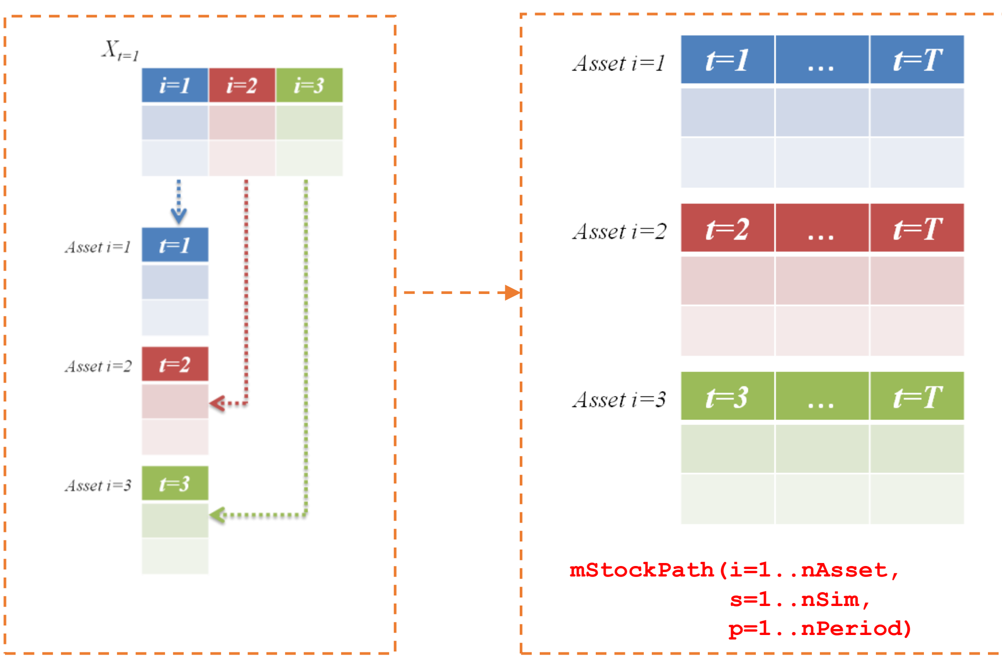
1. Calculate correlation matrix from stock return series.
2. Create an **upper Cholesky matrix** from correlation matrix.
3. Create a vector of **Z=[ε_1 ε_2 ε_3]** where $\varepsilon \sim N(0,1)$
4. Multiple the vector Z with the upper Cholesky matrix, yield **X=[x_1 x_2 x_3]**
5. For each stock price ***i=1..3***, calculate next period price with

$$S_{i,dt} = S_{i,0} \times e^{[(\mu_i - \frac{\sigma_i^2}{2})dt + \sigma_i \sqrt{dt} x_i]}$$

The above process is a calculation for 1 simulation, 1 period ahead. For **S** simulations, 1 period ahead, we can calculate as follow..

$$X_{[s=1..S,i=1..3} = Z_{[s=1..S,i=1..3]} * Cholesky_{[3,3]}$$

As a result, X is the matrix which contains multiplication factors for each stock ***i*** in each simulation ***s***. Hence for multiple ***T*** period, we apply the same calculation and prepare asset price path as follow..



C. Portfolio Value-at-Risk and Calculation Results

Once we obtain prices at the end of period, we calculate portfolio values in all ***S*** scenarios. From ***S*** portfolio values; we rank all data in ascending order, and then pick out the value at the 1st percentile. This value is called Value-at-Risk, ***VaR***.

Assumption and Setups

- 1,000,000
- Number of period ahead: 100
- Number of simulation: 10,000
- [4/30/2004-8/31/2014] Monthly Returns: **ADVANC, BBL, AOT, BIGC, PTT**
- Required Rate=5%
- Short sell is allowed, buy and sell fraction of stock is possible.
- Return correlation and covariance are constant.
- Assume investors are risk averse.

Below are the final results from calculations.

Portfolio Weights

	EQ	MVP	RQ=5%
ADVANC	20%	50%	13%
BBL	20%	15%	-229%
AOT	20%	-8%	82%
BIGC	20%	25%	260%
PTT	20%	17%	-27%

Portfolio Risk, Returns and VaR

	Return	Risk	VaR(1%)	Loss(1M-VaR)
EQ	1.09%	6.09%	872,021.24	127,978.76
MVP	1.01%	5.36%	890,060.79	109,939.21
RQ= 5%	5.00%	22.09%	573,661.71	426,338.29

D. Conclusion

The results show that **minimum variance portfolio** give us the smallest loss number. This means that, of 100 periods, there is 1% chance that the portfolio will lose its value as much as (109,939.21). Both minimum variance and equally weighted portfolio yield similar results in term of return, risk and VaR. Required Rate=5%, on the other hand yields 5% return as expected but generate as much as 22% risk. The result from simulation also show that, even thought the chance is just 1%, but if the loss is to occur, it would be as much as (426,338.29) which is approximately 42% of the portfolio value.