University of Chicago Professional Education

MSCA 37016

Advanced Linear Algebra for Machine Learning

Session 2

Shaddy Abado Ph.D.

Basic concepts Needed for this session

Linear combination

The sum of $av + bw$ *is a* ***linear combination***

Where

v and w are vectors

a and b are scalars

$$-0.5 * \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2 * \begin{bmatrix} 1 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

Linear Independency and Dependency

$$a\vec{v} + b\vec{u}$$

- Vectors v and u are **Independent** if no combination except $0\vec{v} + 0\vec{u}$ gives $\vec{0}$
- Vectors v and u are **Dependent** if there is a combination $a\vec{v} + b\vec{u}$ that gives $\vec{0}$

$$\begin{bmatrix}1\\0\end{bmatrix} \; and \begin{bmatrix}0\\1\end{bmatrix} \qquad \begin{bmatrix}1\\1\end{bmatrix} \; and \begin{bmatrix}2\\2\end{bmatrix}$$

Independent ***Dependent***

Upper and Lower Triangular Matrices

A **lower triangular** matrix is one in which all entries above the main diagonal are zero. Lower triangular matrices are often denoted by L.

$$L = \begin{bmatrix} \ell_{1,1} & & & & 0 \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

An **upper triangular** matrix is one in which all entries below the main diagonal are zero. Upper triangular matrices are often denoted by U.

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

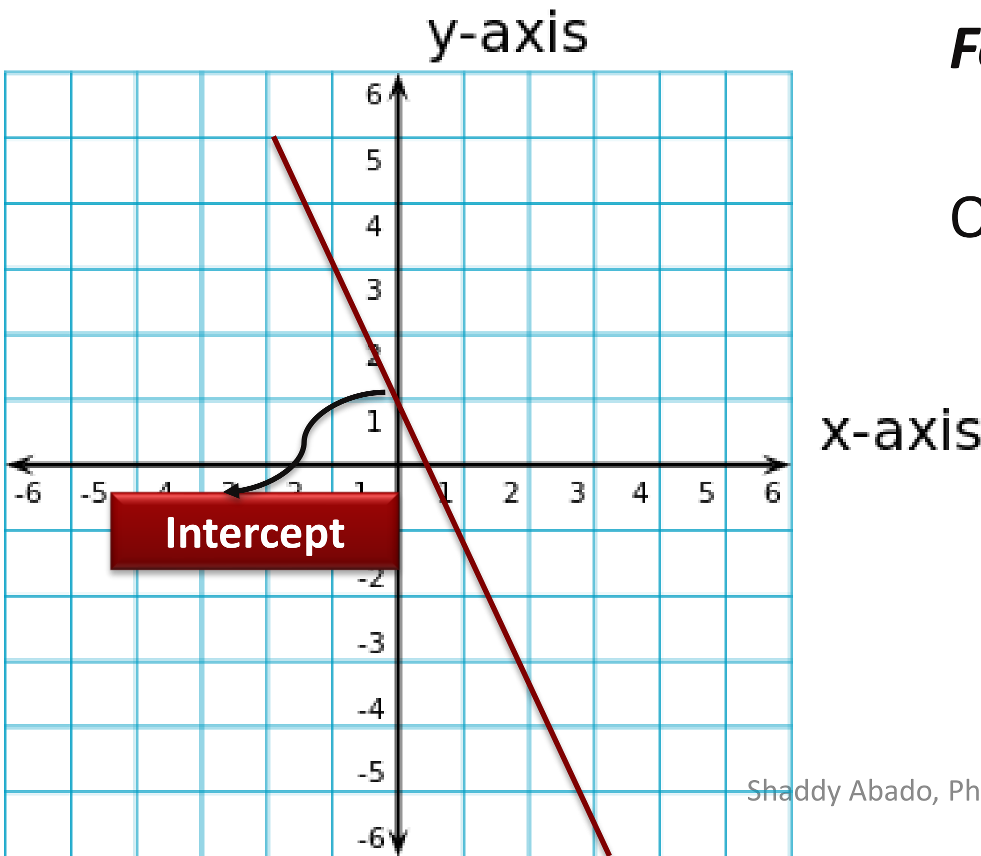
LINEAR EQUATIONS

THE UNIVERSITY OF CHICAGO

Linear Equation – Line (Two dimensional)

Each term is:

1) Constant or
2) the product of constant and a variable



For example:

$$2 * x + 1 * y = 1$$

Or

$$y = 1 - 2x$$

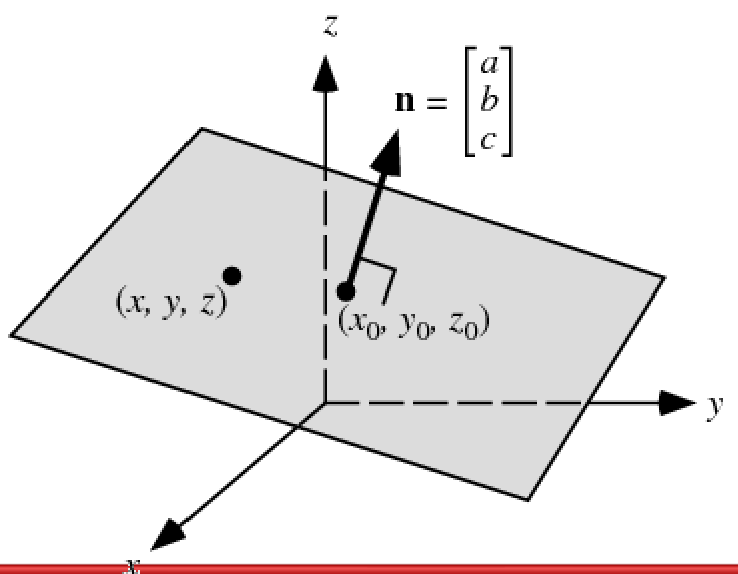
Intercept **slope**

Linear Equation – Plane (Three dimensional)

Each term is:

1) Constant or
2) the product of constant and a variable

$$ax + by + cz + d = 0$$



More than 3D Linear Equation → Hyperplane

System of Linear Equations

What if I have more than one equation?

System of Linear equations

Central problem of linear algebra is to solve a system of linear equations → Find the intersection

Example

$$\begin{cases} 3x + y = 0 \\ 2x + 7y + z = 2 \\ x + 2y - 2z = -4 \end{cases}$$

Three equations with three unknowns

$(x, y\ and\ z)$

System of Linear Equations

Is this a system of linear equations?

$$\begin{cases} 3x^2 + y^2 = 0 \\ 2x^2 + 7y^2 + z = 2 \\ x^2 + 2y^2 - 2z = -4 \end{cases}$$

$$Let$$
$$v = x^2$$
$$u = y^2$$

$$\begin{cases} 3v + u = 0 \\ 2v + 7u + z = 2 \\ v + 2u - 2z = -4 \end{cases}$$

Linear In weights

Motivation:

Existence and Uniqueness

$$\boldsymbol{A}_{mxn}x_{nx1} = b_{mx1}$$

Main question:

Existence and uniqueness

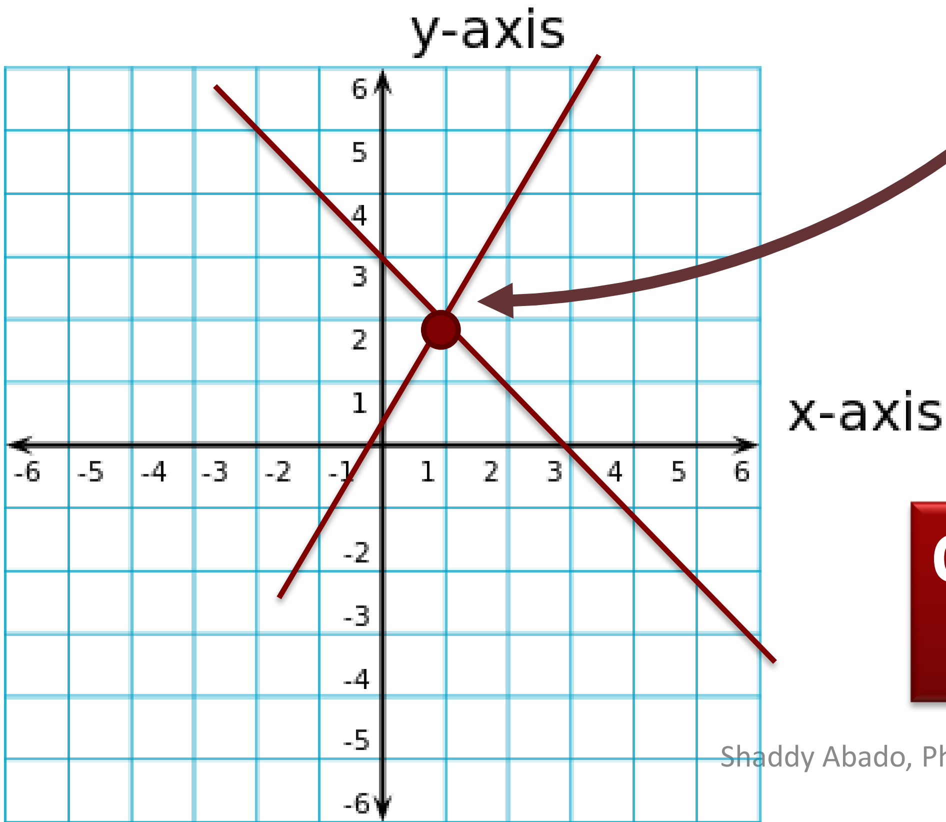
Is there a single solution, infinite solutions or no solution?

System of Linear Equations (2D) – Unique Solution

$$\begin{cases} x + y = 3 \\ 2x - y = 0 \end{cases}$$

Solution :

$$\boldsymbol{x = 1}$$
$$\boldsymbol{y = 2}$$



Intersecting point:
Solves both equations

One Intersecting point

System of Linear Equations (2D) – Infinite Number of Solutions

$$\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$$

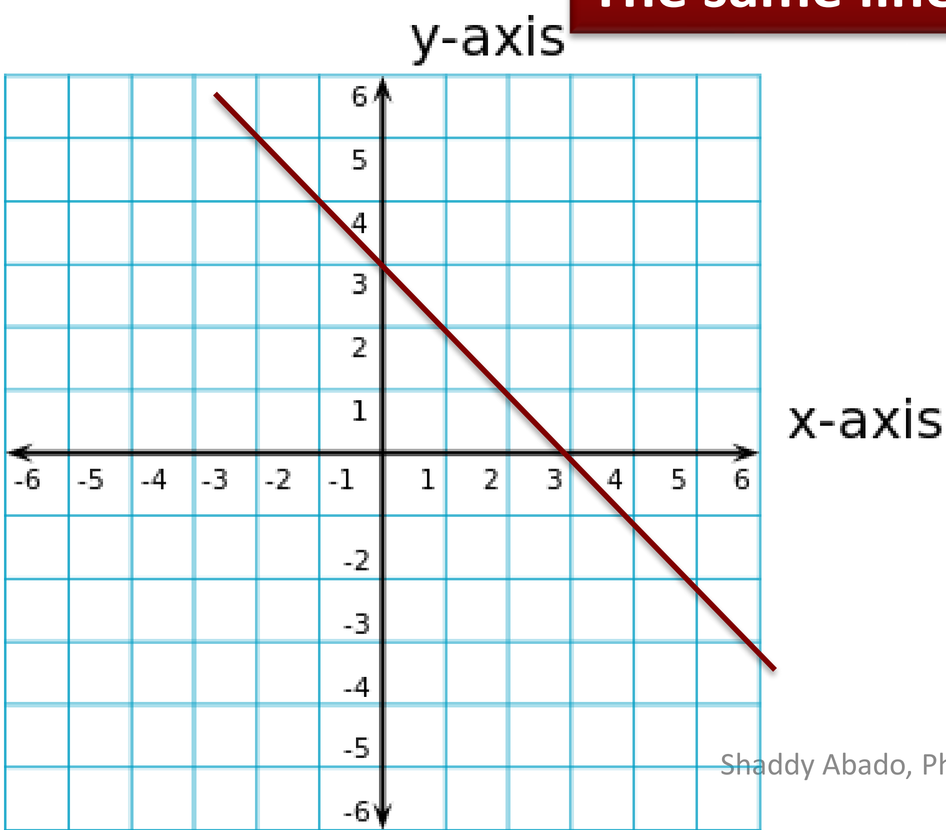
The same line

$\infty\ Solutions$

$Let\ x = t$

Then: $y = 3 - t$

$\boldsymbol{t\ - Free\ variable}$
$\boldsymbol{y - pivot\ variable}$



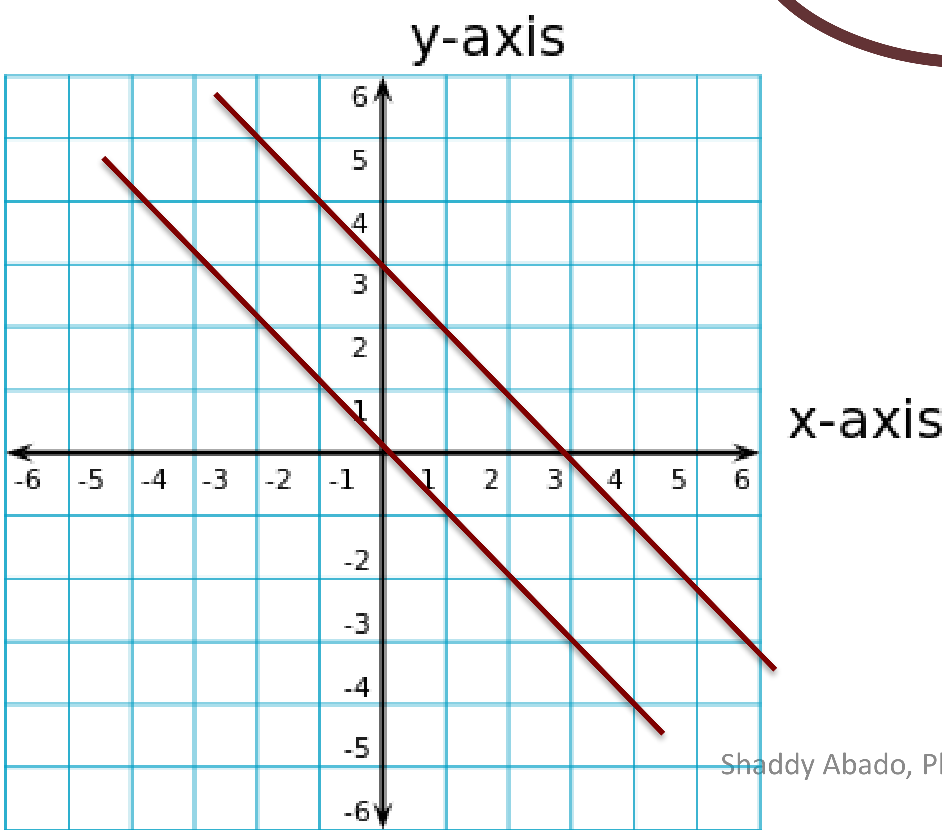
∞ Intersecting points

System of Linear Equations (2D) – No solution

$$\begin{cases} x + y = 3 \\ 2x + 2y = 0 \end{cases}$$

No solution

Parallel Lines



No Intersecting points

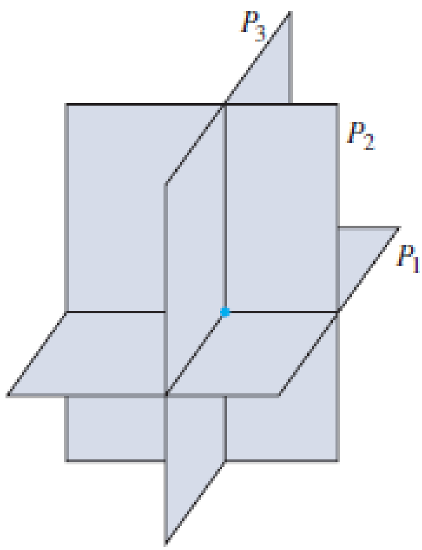
System of Linear Equations – 3D Planes

$$\begin{cases} 3x + y = 0 \\ 2x + 7y + z = 2 \\ x + 2y - 2z = -4 \end{cases}$$

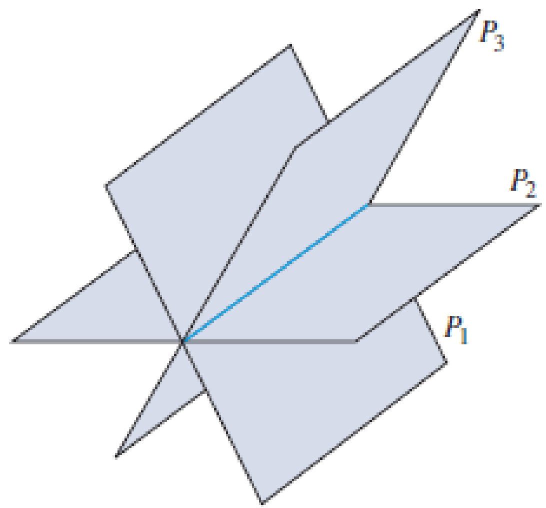
Intersection of three planes:

1. **Unique Solution(Point)**: The planes have a unique point of intersection.
2. **Line**: The planes intersect in a common line; any point on that line then gives a solution to the system of equations.
3. **Plane**: A plane of solutions, with two free parameters
4. **Infinite solutions:** All of R^3(Any point in R^3 is a solution) There are three free parameters.
5. **No solution**: Some of the equations are contradictory, so no solution exists.

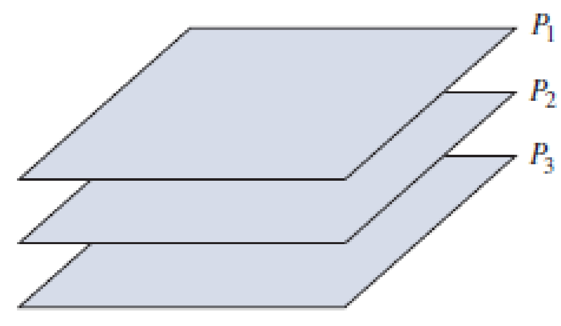
System of Linear Equations – 3D Planes



Unique Solution (Point)



Line



No solutions

System of Linear Equations - Example

Linear Equation $\boxed{Price = a * \#Doors + b * MPG + c * \#Seats}$

Car ID	# Doors	MPG	# Seats
A	4	30	5
B	4	35	2
C	2	20	2

Price ($K)
40
35
60

System of Linear Equations

$$4a + 30b + 5c = 40$$
$$4a + 35b + 2c = 35$$
$$2a + 20b + 2c = 60$$

Augmented Notation

$$\left(\begin{array}{ccc|c} 4 & 30 & 5 & 40 \\ 4 & 35 & 2 & 35 \\ 2 & 20 & 2 & 60 \end{array}\right)$$

$$a = -86$$
$$b = 9.8$$
$$c = 18$$

Linear Regression

Regression is a data mining function that predicts a numerical value

	Feature 1 (x)	Feature 2 (y)	Feature 3 (z)	....	Feature m
Sample 1					
Sample 2					
Sample 3					
. . .					
Sample n					

$$f(x, y, z, \dots) = w_0 + xw_1 + yw_1 + zw_3 + \cdots$$

$$Output = f(x, y, z, \dots) = Bias + x * w_1 + y * w_1 + z * w_3 + \cdots$$

Number of Equations → *Number of samples* → *"Constraints"*
Number of Unknowns → *Weights* → *"Degrees of Freedom"*

Overdetermined System of Equations

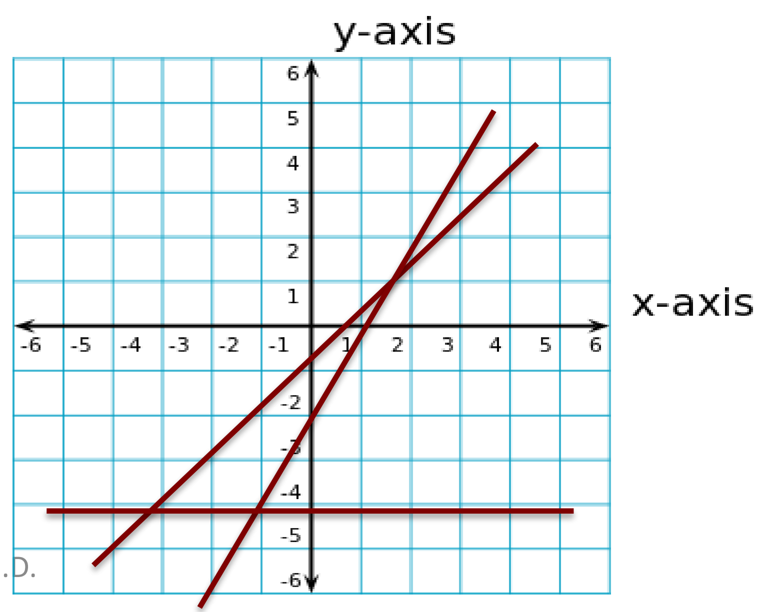
$$A_{mxn} x_{nx1} = b_{mx1}$$

$$m \ - Number\ of\ Equations$$
$$n \ - Number\ of\ Unknowns$$

Overdetermined System of Equations $m > n$
(# of Equations > # of Unknowns)

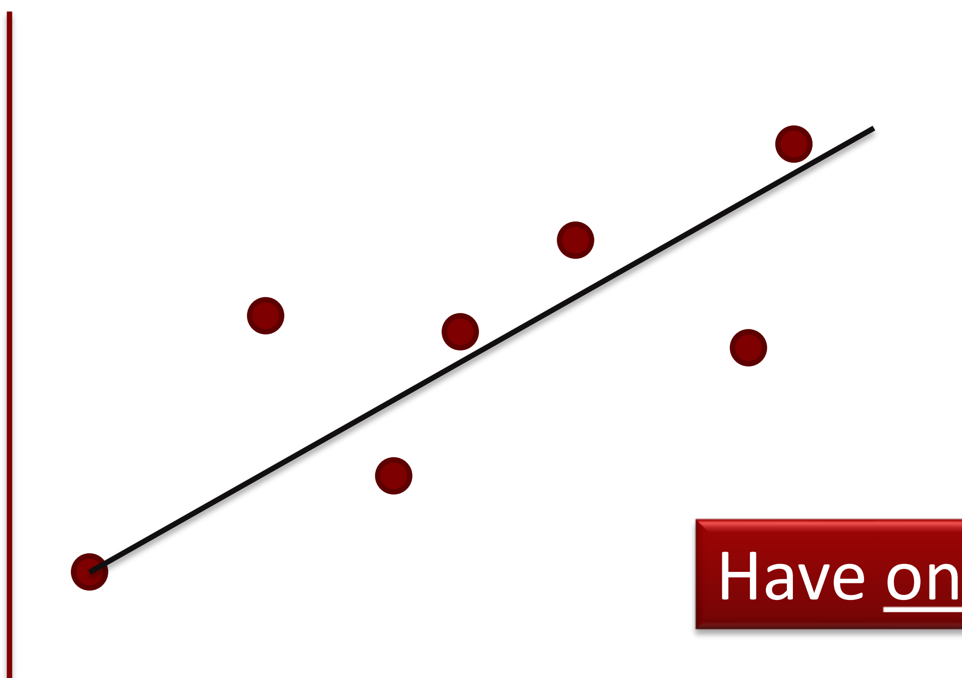
$$3x + 2y = -2$$
$$x - y = 0$$
$$y = -3$$

Have one or no solution



Overdetermined System of Equations

Overdetermined System of Equations $m > n$
(# of Equations > # of Unknowns)



$A\ x = b$

Narrow matrix

Have <u>one</u> or <u>no</u> solution

Underdetermined System of Equations

$$A_{mxn}x_{nx1} = b_{mx1}$$

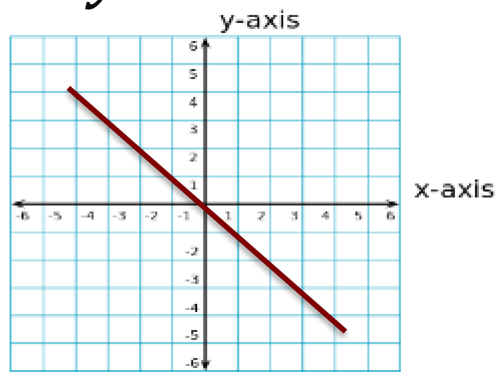
$$m \; - Number\ of\ Equations$$
$$n \; - Number\ of\ Unknowns$$

Underdetermined System of Equations $m < n$
(# of Equations < # of Unknowns)

$$3x + 2y + 2z = -2$$
$$x - y - z = 0$$

OR

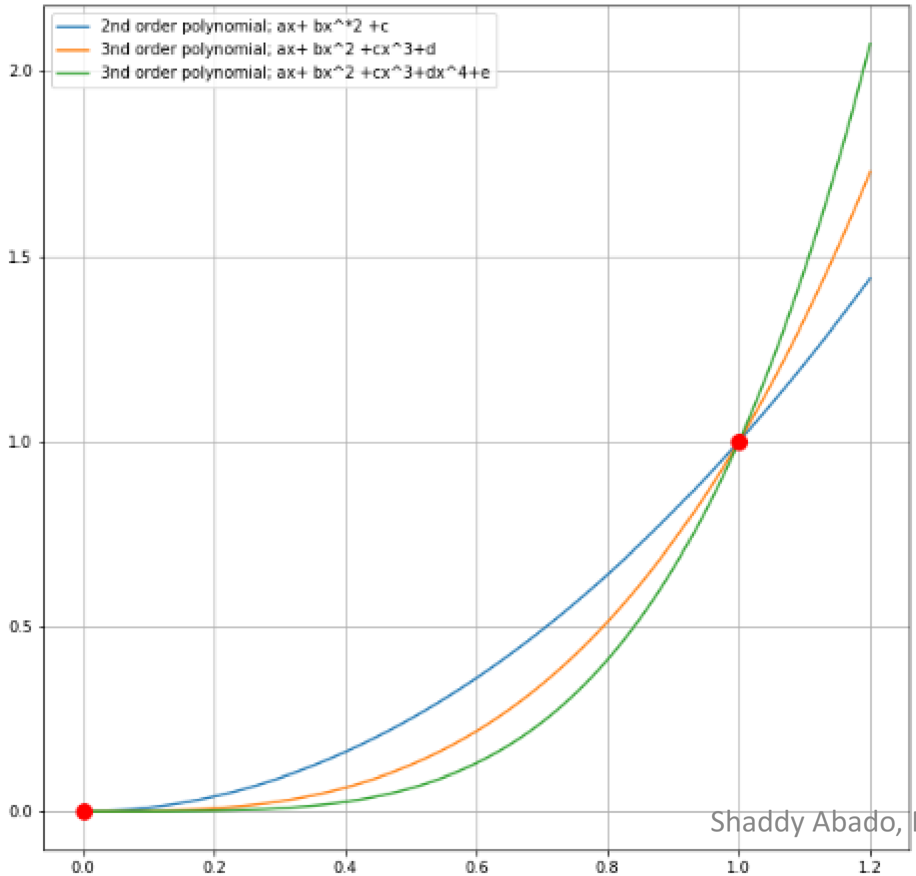
$$3x + 2y = 0$$



Infinite solutions

Underdetermined System of Equations

Underdetermined System of Equations $m < n$
(# of Equations < # of Unknowns)



A $x = b$

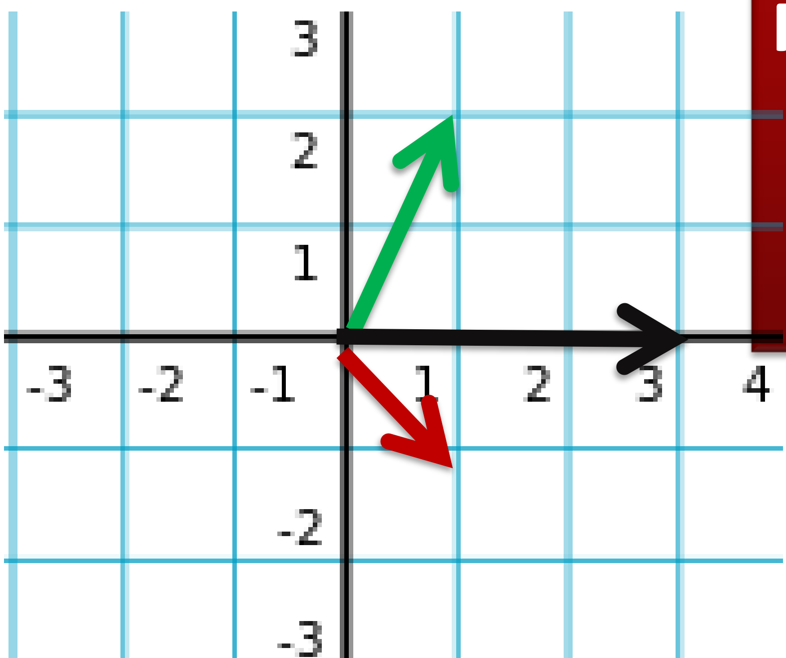
Wide matrix

Infinite solutions

System of Linear Equations – Vector Equation

Right side is linear combination of left side

$$\begin{aligned} x + y &= 3 \\ 2x - y &= 0 \end{aligned} \quad \Rightarrow \quad x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$



Problem:

Find the combination of those vectors that are equal to the vector on the right

Solution :

$$\boldsymbol{x = 1}$$

$$\boldsymbol{y = 2}$$

To solve this problem, we will use concepts from the previous session:
Scalar multiplication, vector addition, and linear combination

System of Linear Equations – Matrix Equation

$$x + y = 3$$
$$2x - y = 0$$

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$

Unknown

Known

Coefficient Matrix

$$\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$

Coefficient Matrix – Combination of column vectors

System of Linear Equations – $A\vec{x}=\vec{b}$

$$x + y = 3$$
$$2x - y = 0$$

Coefficient Matrix

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}$$

$$A\vec{x}=\vec{b}$$

In this session we will ask if there is a solution for $\boldsymbol{A}x = \boldsymbol{b}$ and how to find it.
(Which combination of $\boldsymbol{A}$ columns produces vector $\boldsymbol{b}$)

System of Linear Equations – Augmented Notation

$$\begin{cases} x + y = 3 \\ 2x - y = 0 \end{cases}$$

System of Linear Equations

Coefficient matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Matrix Notation

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \end{array}\right)$$

Augmented Notation

A $\vec{b}$

System of Linear Equations – $A\vec{x}=\vec{b}$

Unique solution

$$\begin{cases} x + y = 3 \\ 2x - y = 0 \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

***A* is invertible**

Columns of *A* are Independent

Inf. solutions

$$\begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

***A* is singular**

Columns of *A* are Dependent

No Solution

$$\begin{cases} x + 2y = 3 \\ 2x + 4y = 1 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

***A* is singular**

Columns of *A* are Dep and parallel

The coefficient matrix is singular if the columns are dependent, i.e., if there is not exactly one solution.

The idea of Elimination and Gaussian elimination

Elimination – Equation Form

$$\begin{cases} x + y = 3 \\ 2x - y = 0 \end{cases}$$

1. Multiply Eq.1 by 2 and Eq.2 by 1
2. Subtract to **eliminate x**

$$x + y = 3$$
$$3y = 6 \quad /3$$

$$x + y = 3$$
$$y = 2$$

Back substitute $y = 2$ back into Eq. 1

$$x + 2 = 3 \rightarrow x = 1$$

Solution **: $x = 1$ $y = 2$**

Elimination – Equation Form

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

Upper Triangular Matrix

$$\begin{aligned} x + y &= 3 \\ 2x - y &= 0 \end{aligned}$$

$$\begin{aligned} x + y &= 3 \\ 3y &= 6 \end{aligned}$$

$$\boldsymbol{A} \cdot \vec{x} = \vec{\boldsymbol{b}}$$

$$\boldsymbol{U} \cdot \vec{x} = \vec{\boldsymbol{c}}$$

Gaussian Elimination

Goal:

- Use **Gaussian elimination** to reduce the system of equations to **upper triangular form**
- Then solve the system from the bottom up using the **back-substitution** process

Apply Gaussian elimination to reduce the system of equations to upper triangular form

Elementary Row Operations (ERO)

There are three types of elementary row operations to transform a matrix into an upper triangular

1. Swapping two rows
2. Multiplying a row by a non-zero number
3. Adding a multiple of one row to another row

Recall:

$$Row \leftrightarrow Equation$$

Elementary Row Operations (ERO) –

Swapping two rows

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 4 & 0 & 1 & 1 \\ 2 & 3 & 1 & 3 \end{array}\right) \end{array} \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

$$L_1 \leftrightarrow L_2$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 4 & 0 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 3 & 1 & 3 \end{array}\right) \end{array} \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

Elementary Row Operations (ERO) –

Multiplying a row by a non-zero number

$$\begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 4 & 0 & 1 & 1 \\ 2 & 3 & 1 & 3 \end{array}\right) & \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array} \end{array}$$

$$2 * L_1 \rightarrow L_1$$

$$\begin{array}{cc} \left(\begin{array}{ccc|c} 2 & 6 & 6 & 4 \\ 4 & 0 & 1 & 1 \\ 2 & 3 & 1 & 3 \end{array}\right) & \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array} \end{array}$$

Elementary Row Operations (ERO) – Adding a multiple of one row to another row

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 4 & 0 & 1 & 1 \\ 2 & 3 & 1 & 3 \end{array}\right) \end{array} \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

$$-4 * L_1 + L_2 \rightarrow L_2$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & -12 & -11 & -7 \\ 2 & 3 & 1 & 3 \end{array}\right) \end{array} \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

Definitions:

Pivot & Multipliers

- **Pivot** – The 1st nonzero in the row that does the elimination
  - The left-most non-zero entry in each line.
  - The diagonal of the upper triangular matrix.
  - To solve n equations we want n pivots.

- **Multiplier** – Entry to eliminate divided by pivot

Example I: Gaussian Elimination

$$\begin{cases} 2x + 3y + 4z = 6 \\ 2x - 3y + 4z = 2 \\ 2x + 3y - 4z = 4 \end{cases}$$

Augmented Notation

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 2 & -3 & 4 & 2 \\ 2 & 3 & -4 & 4 \end{array}\right)$$

Example I: Gaussian Elimination

Pivot

Multiplier

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 4 & -6 & 8 & 4 \\ 1 & 3/2 & -2 & 2 \end{array}\right)$$

Recall

- Pivot – The 1st nonzero in the row that does the elimination
- Multiplier – Entry to eliminate divided by pivot

$$\frac{4}{2} * L_1 - L_2 \rightarrow L_2$$

$$\frac{1}{2} * L_1 - L_3 \rightarrow L_3$$

$$\boxed{2 * 4 - 8 = 0}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 0 & 12 & 0 & 8 \\ 0 & 0 & 4 & 1 \end{array}\right)$$

$$\boxed{2 * 2 - 4 = 0}$$

$$\boxed{1/2 * 6 - 2 = 1}$$

Example I: Gaussian Elimination

Pivot

Multiplier

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 0 & 12 & 0 & 8 \\ 0 & 0 & 4 & 1 \end{array}\right)$$

$L_2/12 \rightarrow L_2$

$L_3/4 \rightarrow L_3$

Row Echelon form

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 1/4 \end{array}\right)$$

Example I: Gaussian Elimination

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 4 & -6 & 8 & 2 \\ 1 & 3/2 & -2 & 2 \end{array}\right)$$

$$A \cdot \vec{x} = \vec{b}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 1/4 \end{array}\right)$$

$$U \cdot \vec{x} = \vec{c}$$

Example I: Gaussian Elimination

Back substitution

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 1/4 \end{array}\right) \end{array}$$

The matrix is now in echelon form (Also called triangular form)

We can stop here and solve using back substitution (i.e., Solve upper triangular matrix) by turning the augmented matrix back into equations.

y & z are known from previous steps

$$2x + 3y + 4z = 6 \Rightarrow x = 3/2$$

$$y = 2/3$$

$$z = 1/4$$

Recall
Multiplier – Entry to eliminate divided by pivot

Breakdown of Elimination

Nonsingular system of equations

The system of equations is nonsingular (i.e., invertible) if there is a full set of n pivots which are not equal to zero.

Problem:

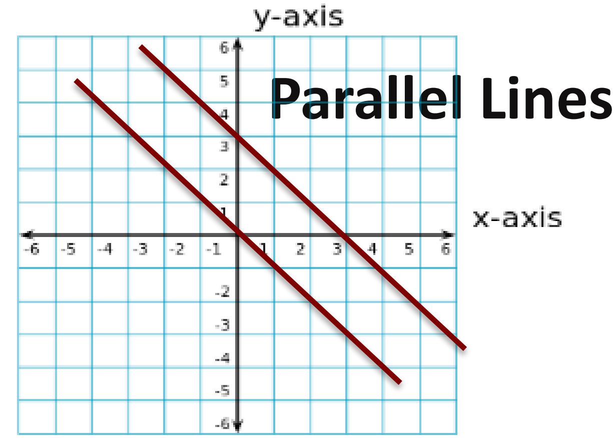
What if for n equations we don't get n pivots? Therefore, we can't find a full set of n pivots, and we need to divide by zero.

Three Possible Scenarios:

1. No solution $(0 \neq 0)$ ➔ i.e., $0 * y = 6$
2. Infinite solutions $(0 \ = \ 0)$ ➔ i.e., $0 * y = 0$
3. We may be able to resolve the problem by row exchange.

No Solution

$$\begin{cases} x + y = 3 \\ 2x + 2y = 1 \end{cases}$$



$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 1 \end{array}\right)$$

Recall
Multiplier – Entry to eliminate divided by pivot

$$\boxed{\frac{2}{1} * L_1 - L_2 \rightarrow L_2}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 5 \end{array}\right)$$

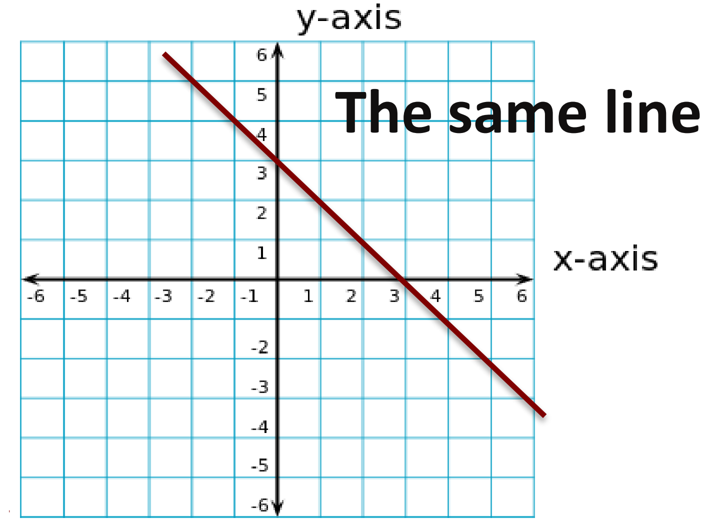
Zero is never allowed as a pivot

$$0 * y \ = 5$$

No Solution Exists: Singular Problem

Infinite Solutions

$$\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$$



$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right)$$

$$\frac{2}{1} * L_1 - L_2 \rightarrow L_2$$

Recall
Multiplier – Entry to eliminate divided by pivot

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\infty\ Solutions$$

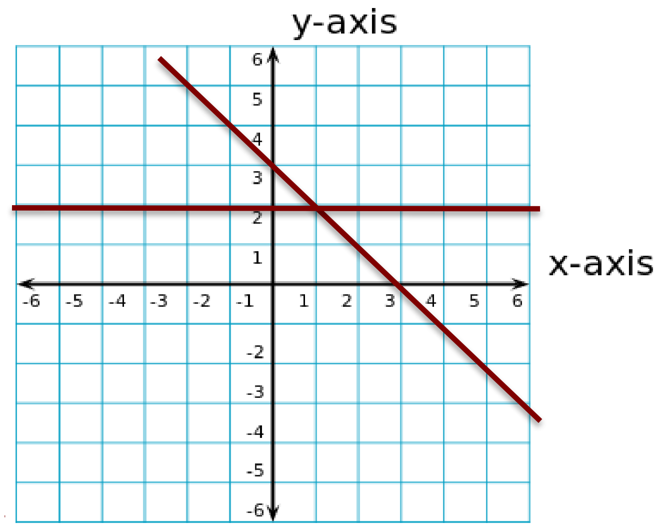
$$Let\ x = t$$

Then: $y = 3 - t$

$$0 * y = 0$$

Infinite solutions: Singular Problem

Unique Solution



$$y = 2$$
$$x + y = 3$$

$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 1 & 3 \end{array}\right)$$

$L_1 \leftrightarrow L_2$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

When possible, exchange rows if there is a nonzero below it.

$$x = 1;\ y = 2$$

Solve using back substitution

Unique solutions: Non Singular Problem

Example II: Gaussian Elimination

Pivot

Multiplier

$$\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right) \end{array}$$

Augmented Notation

Example II: Gaussian Elimination

Pivot

Multiplier

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right)$$

$\boxed{L_1 \leftrightarrow L_2}$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right) \end{array}$$

Example II: Gaussian Elimination

Pivot

Multiplier

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right)$$

$$\frac{2}{1} * L_1 - L_3 \rightarrow L_3$$

$$\frac{1}{1} * L_1 - L_4 \rightarrow L_4$$

Recall
Multiplier – Entry to eliminate divided by pivot

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & -3 & 3 & 6 \\ 0 & 6 & 6 & 1 & 21 \end{array}\right)$$

$$\boxed{2 * (1) - 2 = 0}$$

$$\boxed{1 * (2) - (-4) = 6}$$

Example II: Gaussian Elimination

Pivot

Multiplier

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & -3 & 3 & 6 \\ 0 & 6 & 6 & 1 & 21 \end{array}\right)$$

$$\boxed{\frac{6}{1} * L_2 - L_4 \rightarrow L_4}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & -3 & 3 & 6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right)$$

Recall
Multiplier – Entry to eliminate divided by pivot

$$-\frac{1}{3} * L_3 \rightarrow L_3$$
$$-\frac{1}{13} * L_4 \rightarrow L_4$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

Example II: Gaussian Elimination

Pivot

Multiplier

Row Echelon form

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Back Substitution

$$\begin{cases} x + 2y - z = 2 \Rightarrow x = -1 \\ y + z - 2w = -3 \Rightarrow y = 2 \\ z - w = -2 \Rightarrow z = 1 \\ w = 3 \end{cases}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

Check Answer

BREAK

DETERMINANTS

THE UNIVERSITY OF CHICAGO

Definition and Notation

- Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations.

(**Source:** mathworld.wolfram.com)

- Exist only for square matrices
- A single number
- Notation:

$$Det(A) \ \ or \ \ |A|$$

Note:
This isn't a matrix norm

Why do we care?

- It is useful to calculate the eigenvalues
- Determines when a solution exists to a system of linear equations (i.e., if matrix is invertible).

A system of linear equations has a unique *solution **if and only if*** the determinant of the system's matrix is nonzero (i.e., the matrix is nonsingular).

Otherwise, the matrix has no inverse (i.e., Singular)

How to Calculate the Determinant

1. **Pivot Formula** – Multiply the n pivots (times 1 or -1)
2. **"Big" Formula** - Add up $n!$ terms (times 1 or -1)
3. **Cofactor Formula** – Combine n smaller determinants (times 1 or -1)

Determinant of a $2x2$ Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Blue means positive $(+ad)$
Red means negative $(-bc)$

Source
http://www.mathsisfun.com/algebra/matrix-determinant.html
https://en.wikipedia.org/wiki/Determinant

Examples:

Determinant of a $2x2$ Matrix

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\det(A) = (1 * 6) - (4 * 5) = -14$$

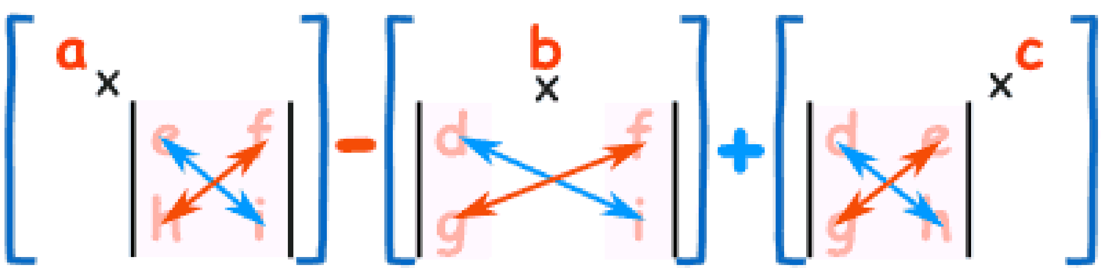
$$A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$$\det(A) = (2 * 8) - (4 * 4) = 0$$

Determinant of a $3x3$ Matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

1. Multiply *a* by the determinant of the 2 × 2 matrix that is not in a's row or column.
2. Likewise for *b*, and for *c*
3. Add them up, but remember that *b* has a negative sign!



$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Source
http://www.mathsisfun.com/algebra/matrix-determinant.html

Determinant of a $3x3$ Matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Minor Minor Minor

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Cofactor Cofactor Cofactor

Example:

Determinant of a $3x3$ Matrix

$$A = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{2} \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det(A) = \mathbf{3} * \begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} - \mathbf{1} * \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + \mathbf{2} * \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 3 * (-1 * 3 - 3 * 2) - 1 * (2 * 3 - 3 * 1) + 2 * (2 * 2 - (-1) * 1)$$

$$= 3 * (-9) - 1 * (3) + 2 * (5)$$

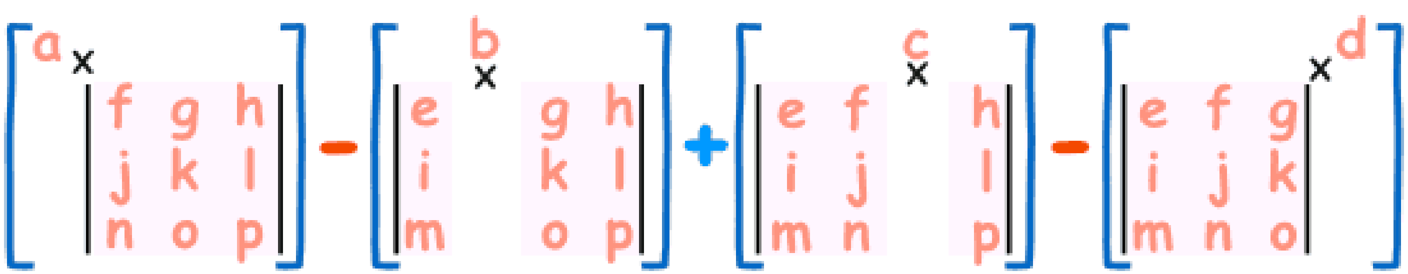
$$= -27 - 3 + 10$$

$$= -20$$

Determinant of a $4x4$ Matrix

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}.$$

+a times the determinant of the matrix that is **not** in **a**'s row or column,
-b times the determinant of the matrix that is **not** in **b**'s row or column,
+c times the determinant of the matrix that is **not** in **c**'s row or column,
-d times the determinant of the matrix that is **not** in **d**'s row or column,



Source
http://www.mathsisfun.com/algebra/matrix-determinant.html

Properties of Determinants

If A is singular (i.e., not invertible) then $\det(A) = 0$.
If A is invertible then $\det(A) \neq 0$

If two rows/columns of A are equal, then $\det(A) = 0$

A matrix with a row/column of zeros has $\det(A) = 0$

Properties of Determinants

The determinant of I_n is 1.

The determinant of a diagonal matrix is the product of its diagonal entries.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \qquad \det(A) = 1 * 2 = 2$$

If $\boldsymbol{A}$ is a triangular matrix, then its determinant equals the product of the diagonal entries.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}; \qquad \det(A) = 2 * 1 * 3 = 6$$

Determinants – Final Notes

- While the determinant is useful to some ends in linear algebra, most of the common problems are better solved without using the determinant at all.
- It is useful in solving linear systems of equations of small dimension but becomes much too cumbersome relative to other methods for commonly encountered large systems of linear algebraic equations.
- Further, while a zero value for the determinant almost always has significance, other values do not.

Reference: Lecture Notes on Mathematical Methods (By Mihir Sen and Joseph Power)

VECTOR SPACES
SUBSPACES, SPAN

THE UNIVERSITY OF CHICAGO

Homogeneous and non-Homogeneous System of Equations

$$\boldsymbol{A}x = \boldsymbol{b}$$

$b = 0$	$b \neq 0$
Homogeneous	**non-Homogeneous**
Special solutions	**Particular solution**
Null Space - $N(A)$	**Column Space - $C(A)$**
Uniqueness	**Existence**

Definition: Vector Space

In simple words:

A vector space V is a set of vectors on which the two-operations vector addition and scalar multiplication are defined.

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in the space and any two real numbers c and d, the vector $c\boldsymbol{v} + d\boldsymbol{w}$ is also in the vector space.

There are 8 conditions required of every vector space (See extra slides)

Example: $\mathcal{R}^n$space

Definition of $\mathcal{R}^n$space

The set of (column) vectors v with n real number components

For example

A vector space is a set V on which the two operations vector addition and scalar multiplication are defined.

$\mathcal{R}^1$ - $[x]$ **Line** (x or y axis)

$\mathcal{R}^2$ - The set of all vectors with exactly two real number components (i.e., $\begin{bmatrix} x \\ y \end{bmatrix}$)."$x - y$" ***Two–dimensional space***

$\mathcal{R}^3 - \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ***Three-dimensional space***

Additional Examples of Vector Spaces

- The vector space of all $3x3$ matrices (Similarly, any nxn matrices)
- The vector space of all real functions $f(x)$
- The vector space that consist only of a zero vector.
- Etc.

Why do we need to define vector spaces:

As we will see later on in this session, the solution of homogeneous linear equations ($Ax = 0$) forms a vector space → Nullspace

Examples of Non-Vector Spaces

The collection of vectors with exactly two *positive* real value components $\begin{bmatrix} a > 0 \\ b > 0 \end{bmatrix}$ is *not* a vector space. Why?

✓ The sum of any two vectors in that collection is again in the collection (*closed* under addition.)

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 > 0 \\ 6 > 0 \end{bmatrix}$$

X but multiplying any vector by, say, -4, gives a vector that's not in the collection (not closed under multiplication.)

$$-4 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 < 0 \\ -8 < 0 \end{bmatrix}$$

A vector space is a set V on which the two operations vector addition and scalar multiplication are defined.

Definition: Subspace

A vector space that is contained inside of another vector space is called a *subspace* of that space.

Let V be a vector space and let W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

- *W is nonempty*: The **zero vector** belongs to W.
- *Closure under addition and multiplication:* If u and v are any vectors in W, and c and d are real numbers, then the **liner combinations** $cv + dw$ is in the subspace W.

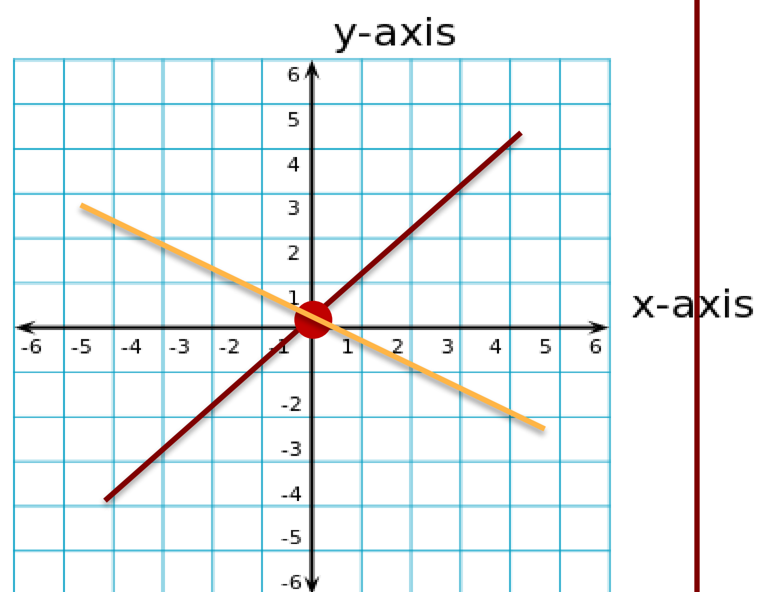
The main take away: Every subspace must contain the zero vector

Examples: Subspace

A vector space is a set V on which the two operations vector addition and scalar multiplication are defined.

The subspaces of $\mathcal{R}^2$ are:

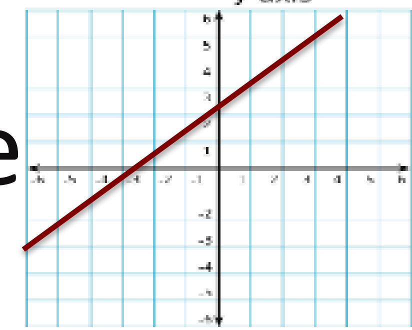
- all of $\mathcal{R}^2$,
- any line through the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and
- the *zero* vector alone ($\mathcal{Z}$).



The subspaces of $\mathcal{R}^3$ are:

- all of $\mathcal{R}^3$,
- any line through the origin
- any plane through the origin and
- the zero vector alone ($\mathcal{Z}$).

Examples of non-Subspace



Recall:

Every subspace must contain the zero vector

A line in $\mathcal{R}^2$ that does not pass through the origin is *not* a subspace of $\mathcal{R}^2$. Why?

Why?

Because multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line (i.e., subspace.)

Definition: Span of Vectors
A subset Spanned by a Set

If the vectors $v_1, \cdots, v_n$ are in a vector space V,
Then the set of all linear combinations of $v_1, \cdots, v_n$ is denoted by $span\{v_1, \cdots, v_n\}$

In simple words:
$span\{v_1, \cdots, v_n\}$ is the collection of all vectors that can be written in the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

Example

$$J = \begin{bmatrix} a - 2b \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{v_1}} + \mathrm{b} \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{v_2}} = a v_1 + b v_2$$

$$J = span\{v_1, v_2\}$$

COLUMN SPACE - $C(A)$

A_{mxn}

Definition: Column Space - $C(A)$

Given a matrix A with columns in $\mathcal{R}^m$, these columns and all their linear combinations form a subspace of $\mathcal{R}^m$.

$$C(A) = span\,\{v_1, \cdots, v_n\}$$

For example,

For $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 0 & 4 \end{bmatrix}$

The **column space** of $\boldsymbol{A}$ is the plane through the origin in $\mathcal{R}^3$ containing $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$\boldsymbol{A}_{mxn}x_{nx1} = b_{mx1}$$

Column Space - $C(\boldsymbol{A})$

Question:

Given a matrix $\boldsymbol{A}$, for what vectors $\boldsymbol{b}$ does $\boldsymbol{Ax} = \boldsymbol{b}$ have a solution $\boldsymbol{x}$?

Answer:

The system $\boldsymbol{A}x = \boldsymbol{b}$ is solvable <u>if and only if</u> $\boldsymbol{b}$ is in the column space of A (i.e., $C(\boldsymbol{A})$)

Core Idea:

The Column Space $C(\boldsymbol{A})$ describes all the attainable $\vec{b}$'s

$$A_{mxn} x_{nx1} = b_{mx1}$$

Column Space – Example I

Does $\boldsymbol{A}x = \boldsymbol{b}$ have a solution x for every $\boldsymbol{b}$?

$$\begin{cases} x = b_1 \\ 2x = b_2 \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solving $\boldsymbol{A}x = \boldsymbol{b}$ is equivalent to solving <u>two</u> linear equations with <u>one</u> unknown (i.e., over-determined.)

Recall: If there is a solution x to $\boldsymbol{A}x = \boldsymbol{b}$, then $\boldsymbol{b}$ must be a linear combination of the columns of $\boldsymbol{A}$.

<u>One</u> column <u>cannot</u> fill the entire <u>Two-Dimensional</u> vector space

Then $Ax = \boldsymbol{b}$ <u>does not</u> have a solution for every choice of $\boldsymbol{b}$ because some vectors $\boldsymbol{b}$ <u>cannot</u> be expressed as linear combinations of columns of A *(i.e., $C(\boldsymbol{A})$).*

$b = [2,4]$ is attainable (i.e., $x = 2$); $b = [2,5]$ is not attainable (i.e., no solution)

$$A_{mxn}x_{nx1} = b_{mx1}$$

Column Space – Additional Examples

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The column space $C(A)$ is a line (Col 2 =2*Col 1) ➔

$$C(A) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{➔}$$

$\boldsymbol{A}x = b$ is solvable when b is on the line

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The column space $C(\boldsymbol{A})$ is all $\mathbf{R}^2$ ➔

$$C(\boldsymbol{A}) = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \text{➔}$$

Every b is attainable

THE NULLSPACE OF A N(A) (SOLVING $Ax = 0$)

THE UNIVERSITY OF CHICAGO

$A_{mxn}x_{nx1} = 0_{mx1}$

Introduction:

Nullspace

This section is about finding the subspace containing all the solutions to $Ax = \vec{0}$

For invertible matrices

$x = \vec{0}$ **(Trivial solution)**

This is the only solution for invertible matrices (One solution)

For non-invertible matrices (i.e., Singular Matrices)

$x \neq \vec{0}$

Each solution belongs to the nullspace (More than one solution)

$A_{mxn} x_{nx1} = 0_{mx1}$

Definition:

Nullspace

The *nullspace* $N(\boldsymbol{A})$ of a matrix $\boldsymbol{A}$ is the collection of all solutions $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to the equation $\boldsymbol{A}x = \vec{0}$.

Definition - Special solutions:

The nullspace of $\boldsymbol{A}$ consists of all the combinations of the special solutions to $\boldsymbol{A}x = \vec{0}$

Again:

Only singular matrices have a nullspace that contains more than just the zero vector.

$A_{mxn}x_{nx1} = 0_{mx1}$

Example I:

Nullspace

What is the nullspace of A (i.e., $N(\boldsymbol{A})$)?

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Singular Matrix

$\det(\boldsymbol{A}) = \boldsymbol{0}$

Homogeneous Equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3 * L_1 - L_2 \rightarrow L_2$

$$\begin{array}{cc} \text{P} & \text{F} \end{array}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Under-determined

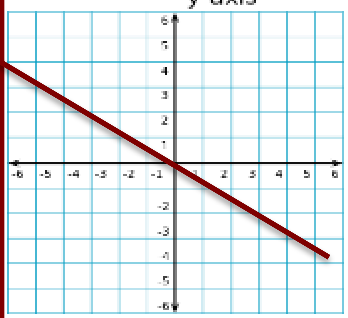
$$x + 2y = 0$$

$$0 = 0$$

Only one Equation w/ two unknowns

$$A_{mxn}x_{nx1} = 0_{mx1}$$

Example I: Nullspace



$$x + 2y = 0$$

Line Equation

- This <u>line</u> equation is the nullspace $N(A)$
- This is a <u>line</u> in $\mathcal{R}^2$

Special solution: $x = -2y$

Let $y = s$ ➔ (**Free variable**)

Then $x = -2s$ ➔ (**Pivot variable**)

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

For example

Let $s = 1$ ➔ $y = 1$ ➔ Then $x = -2$

Let $s = -9$ ➔ $y = -9$ ➔ Then $x = 18$

Etc.

Example I: Nullspace

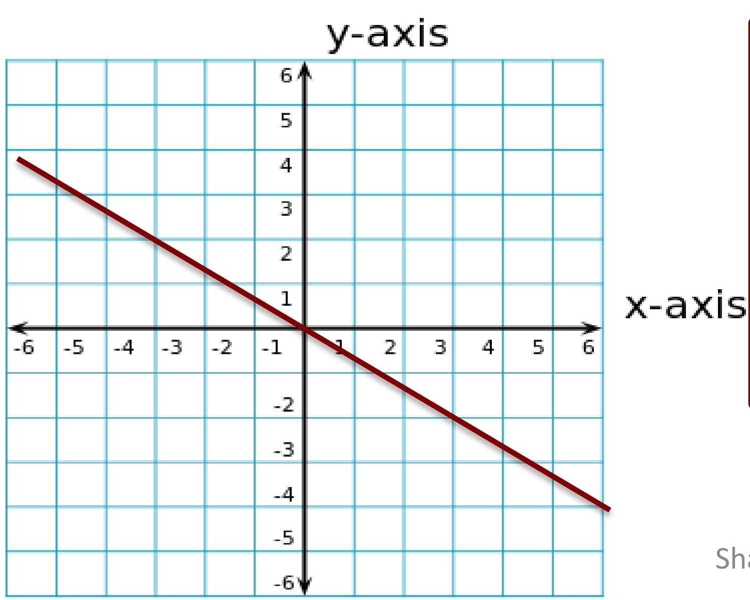
Special solution:

$$x = -2s$$
$$y = s$$

Special solution

The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of

$$s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



What does this mean?

Any point on the line (one <u>special solution</u>) is a solution to $Ax = 0$.

∞ Solutions

$$A_{mxn}x_{nx1} = 0_{mx1}$$

Example I: Nullspace

The nullspace of A contains all multiples of $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Does $\begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$ (i.e., $s = 1/2$) belong to the nullspace?

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1+1 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Yes

Does $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ belong to the nullspace?

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

No

Example II: Nullspace

Find the nullspace of matrix

Nonsingular Matrix
$\det(A) = 2$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \qquad -3L_1 + L_2 \to L_2$$

Step 1: Reduce A to its row echelon Form

Unique Solution

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{P} & \mathbf{P} \end{array} \\ \boldsymbol{U} = & \begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{2} \end{bmatrix} \end{array}$$

Both columns have pivots
(i.e., no free variable)

- $\boldsymbol{A}$ is <u>invertible</u> (Nonsingular)
- **Recall:** Only singular matrices have a nullspace that contains more than just the zero vector. Therefore, there is <u>no special solution</u>. ($\vec{0}$ is the only solution to $\boldsymbol{A}x = \vec{0}$ ➔ Trivial Solution)

THE RANK OF A MATRIX

THE UNIVERSITY OF CHICAGO

Definition- Matrix Rank

$$A_{mxn}x_{nx1} = b_{mx1}$$

Important

m and n **are the size of the matrix not the linear system**

The **rank** $\boldsymbol{r}$ of A equals the number of pivot columns

$(r \leq m \text{ and } r \leq n)$

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 0 & 0 \end{bmatrix} \quad Rank(A) = 1$$

Example II

$$\mathbf{1}x + 2y + 3z = 0 \quad Rank(A) = 1$$

Example III

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & 2 & \mathbf{2} \\ 0 & 0 & \mathbf{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad Rank(A) = 2$$

Example IV

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{2} \end{bmatrix} \quad Rank(A) = 2$$

Rank - Numpy

numpy.linalg.matrix_rank¶

numpy.linalg.matrix_rank(*M*, *tol=None*)

```python
import numpy as np
from numpy import linalg as LA


A1=np.array([[1,2],[3,6]])
A2=np.array([[1,2,3]])
A3=np.array([[1,2,2,2],[2,4,6,8],[3,6,8,10]])
A4=np.array([[1,2],[3,8]])

print('Rank of Matrix 1: ',LA.matrix_rank(A1),'\n')
print('Rank of Matrix 2: ',LA.matrix_rank(A2),'\n')
print('Rank of Matrix 3: ',LA.matrix_rank(A3),'\n')
print('Rank of Matrix 4: ',LA.matrix_rank(A4),'\n')
```

```
Rank of Matrix 1:  1

Rank of Matrix 2:  1

Rank of Matrix 3:  2

Rank of Matrix 4:  2
```

$A_{mxn}x_{nx1} = b_{mx1}$

Number of Free Variables

The number of free variables and number of special solutions is $(n - r)$

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 0 & 0 \end{bmatrix}$$

$2 - 1 = 1$

Example II

$$x + 2y + 3z = 0$$

$3 - 1 = 2$

Example III

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} & 2 & \mathbf{2} \\ 0 & 0 & \mathbf{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$4 - 2 = 2$

Example IV

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{2} \end{bmatrix}$$

$2 - 2 = 0$

Break

THE UNIVERSITY OF CHICAGO

LINEAR INDEPENDENT, BASES AND DIMENSION

THE UNIVERSITY OF CHICAGO

$$A_{mxn}x_{nx1} = b_{mx1}$$

Definition:

Linear Independence

A set of vectors $(v_1, v_2, \ldots, v_n)$ are linearly independent if the only set of coefficients $c_1, c_2, \ldots, c_n$ for which

$$v_1 * c_1 + \ldots + v_n c_n = 0$$

is $c_1 = c_2 = \ldots = c_n = 0$

The matrix $\boldsymbol{A}$ has linearly independent columns if and only if the $N(\boldsymbol{A})$ contains only the zero vector (i.e., $r = n$ and no free variable).

How to find out if columns are linearly independent:

Find the reduced row form of $\boldsymbol{A}$. If there are no free variables, the columns are linearly independent.

$A_{mxn}x_{nx1} = b_{mx1}$

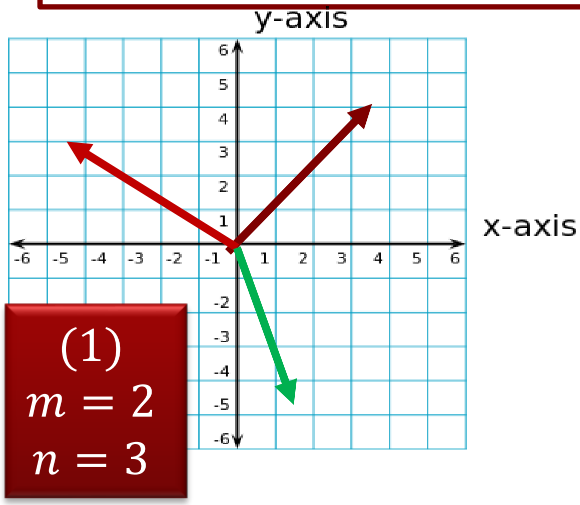
Linear Independence

Any set of n vectors in R^m must be linearly dependent if $n > m$ (i.e., Under-determined)

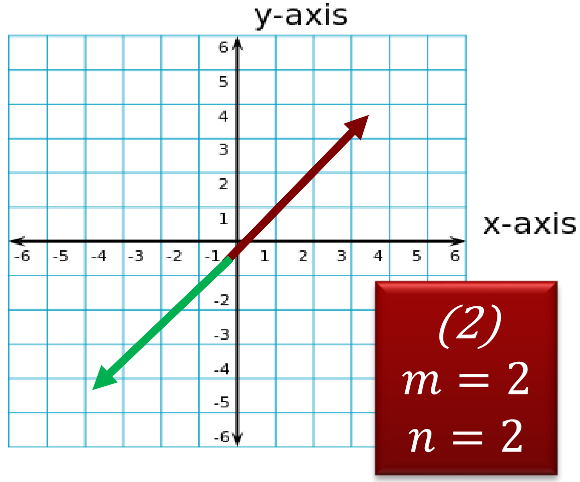
For example R^2 (i.e., $m = 2$):

(1) Three vectors ($n = 3$) are independent if they do <u>not</u> lie in the same <u>plane.</u>

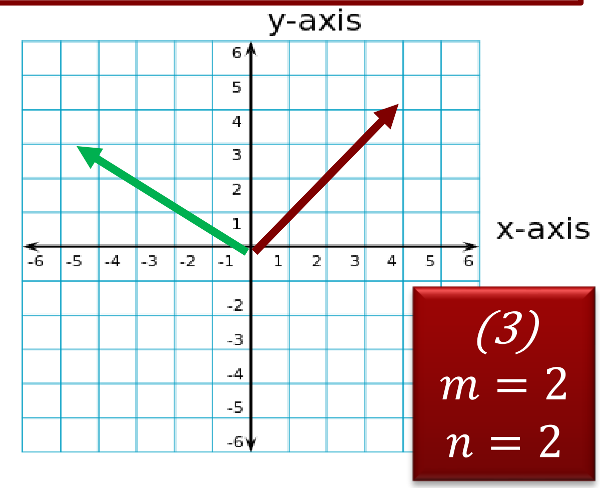
(2 & 3) Two vectors ($n = 2$) are independent if they do <u>not</u> lie on the same <u>line</u>.



Dependent



Dependent



Independent

Definition:

Spanning a space

Vectors $v_1, \cdots, v_n$ *span* a space when the space consists of all combinations of those vectors.

$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ Span the full $2D$ space R^2

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}3\\2\end{bmatrix}$ Span the full $2D$ space R^2

$\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-2\\-2\end{bmatrix}$ Only span a line in R^2

Motivation:
Vector Basis

Motivation:

Find <u>enough</u> Independent vectors to span the space (and not more).

Example for R^3 :

- <u>Two</u> vectors can't span all of R^3, even if they are independent.
- <u>Four</u> vectors can't be independent, even if they span R^3.

Definition:

Vector Basis

A *basis* for a vector space is a sequence of vectors that are

(1) Linearly independent and

(2) Span the space.

The main takeaway:

The basis of a space tells us everything we need to know about that space.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

The columns of the A and B form a basis for R^2

Notes: Basis

Note I:

The basis is <u>not unique</u>, but all bases have the same number of vectors.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

The columns of the A and B form a basis for R^2

Note II:

Every vector in the space is a unique combination of the basis vectors.

Basis - Why should we care?

- If we have a set of linearly dependent vectors, then we can keep a linearly independent subset and express the rest in terms of the linearly independent ones.
- Therefore, the number of linearly independent vectors (i.e., rank) is a measure of the information content in the set and compresses the set accordingly.

Basis - R^3 Examples

A *basis*

(1) Linearly independent and

(2) Span the space.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent but does not span R^3

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

A basis for span R^3

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

Spans R^3 but linearly dependent

Definition: Standard Basis

Linearly independent & orthogonal

$\boldsymbol{R^2}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [e_1 \quad e_2]$$

$\boldsymbol{R^3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [e_1 \quad e_2 \quad e_3]$$

$\boldsymbol{Etc}$.

$$A_{mxn}x_{nx1} = b_{mx1}$$

Definition:

Dimension of a Vector Space

Recall: The basis is not unique, but they all have the same number of vectors

The dimension of a vector space is the number of vectors in a basis.

So, there are exactly *n* vectors in every basis for $\mathbf{R}^n$

Rank is the dimension of the column space $C(\boldsymbol{A})$ of a matrix $\boldsymbol{A}$

$$\boldsymbol{Dim(C(A)) \;= r}$$

The dimension of the nullspace $N(A)$ is $n - r$

$$\boldsymbol{Dim(N(A)) \;= n - r}$$

(i.e., number of free variables)

LU Decomposition

What is Matrix Decomposition?

- A factorization/decomposition of a matrix into a product of canonical matrices
- Can be used to discover latent features underlying the interactions between different kinds of entities
- Many popular software programs have different routines to calculate these decompositions.

Prime factorization $30 = 2 * 3 * 5$

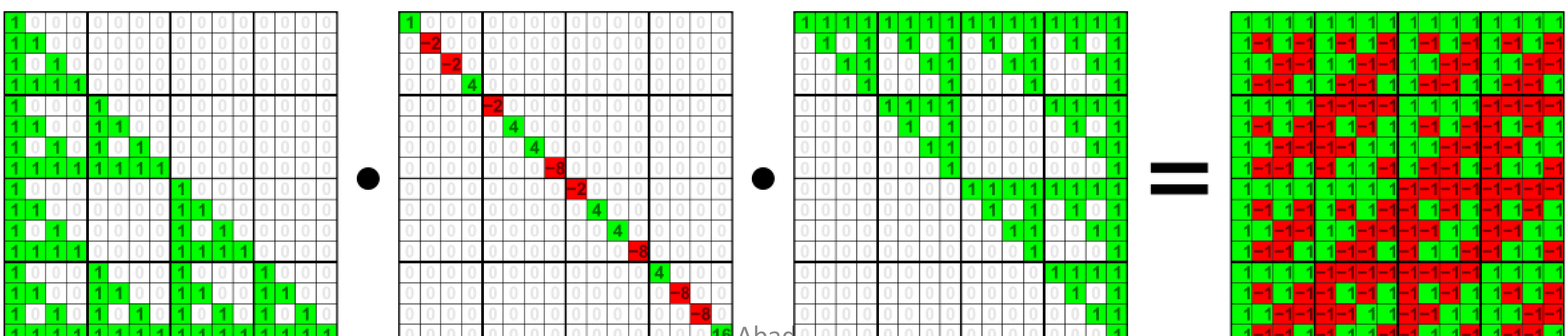


Image Source: https://en.wikipedia.org/wiki/LU_decomposition

Motivation: Matrix Decomposition

- Often performed for computational reasons: certain problems are easier to solve on a computer when the matrix is expressed in terms of its simpler constituents.
- Decompositions are widely used in computer algorithms for various computations, such as solving equations and finding eigenvalues.
- Makes the matrix easier to store and easier to access for computation.

Example: LU Decomposition

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right) \end{array}$$

LU Decomposition

Recall: Multiplier l – Entry to eliminate divided by pivot

$-\left(\frac{3}{2}\right) * L_1 - L_2 \to L_2$

$-(-1) * L_1 - L_3 \to L_3$

$(4) * L_2 - L_3 \to L_3$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

Multiplier

$-\frac{3}{2}$

-1

4

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

We can write A as the product of a lower triangular matrix (L) and an upper triangular matrix (U)

LU Decomposition

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

$\boldsymbol{A}$ $\boldsymbol{L}$ $\boldsymbol{U}$

Pivot

Multiplier

Produced by the elimination steps

$\boldsymbol{U}$ is an upper triangular matrix with the pivots on its diagonal.

LU Decomposition

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{A} \qquad\qquad \boldsymbol{L} \qquad\qquad \boldsymbol{U}$$

Pivot

Multiplier

$$-\frac{3}{2} * L_1 - L_2 \rightarrow L_2$$
$$-1 * L_1 - L_3 \rightarrow L_3$$
$$4 * L_2 - L_3 \rightarrow L_3$$

Produced by the elimination steps

Recall: Multiplier – Entry to eliminate divided by pivot

L is a lower triangular matrix with multipliers below its diagonal.

It contains the memory of Gaussian elimination (holds the numbers that multiplied the pivot before <u>subtracting</u> them from lower rows).

Motivation: LU Decomposition

Given a fixed left-hand side and more than one right-hand side (as is often the case), what is the most efficient way to solve the system of equations?

$$\boldsymbol{A}x = \boldsymbol{b}_i; \qquad i = 1 \ldots M$$

Car ID	# Doors	MPG	# Seats	Price ($K)
A	4	30	5	40
B	4	35	2	35
C	2	20	2	60

$$\left(\begin{array}{ccc|c} 4 & 30 & 5 & 40 \\ 4 & 35 & 2 & 35 \\ 2 & 20 & 2 & 60 \end{array}\right)$$

Car ID	# Doors	MPG	# Seats	Price ($K)
A	4	30	5	45
B	4	35	2	30
C	2	20	2	80

$$\left(\begin{array}{ccc|c} 4 & 30 & 5 & 45 \\ 4 & 35 & 2 & 30 \\ 2 & 20 & 2 & 80 \end{array}\right)$$

Motivation: LU Decomposition

Given a fixed left-hand side and more than one right-hand side (as is often the case), what is the most efficient way to solve the system of equations?

$$\boldsymbol{A}x = \boldsymbol{b}_i; \qquad i = 1 \ldots M$$

Idea:
Store the work required carrying out the elimination by storing the multipliers and pivots used to carry out the row operations.

The LU decomposition describe Gauss elimination in the most useful way.

Solution procedure given $\boldsymbol{A} = \boldsymbol{LU}$

Given

$$\boldsymbol{A}x = \boldsymbol{b_i}; \qquad i = 1 \dots M$$

Find

$$\boldsymbol{LU}x = \boldsymbol{b_i}; \qquad i = 1 \dots M$$

Then 1) solve for y **Then 2) solve for x**

$$\boldsymbol{L}y = \boldsymbol{b_i}; \quad where\ y = \boldsymbol{U}x \qquad i = 1 \dots M$$

Step 1 : Factor $\boldsymbol{A}$ into $\boldsymbol{LU}$ — **Row-reduction**

Step 2 : Solve $\boldsymbol{L}y = \boldsymbol{b_i}$ — **Forward substitution**

Step 3 : Solve $\boldsymbol{U}x = y$ — **Back substitution**

We still need to calculate x

We started with $\boldsymbol{A}x = \boldsymbol{b_i}$

Which we decomposed into $\boldsymbol{A} = \boldsymbol{LU}$

Therefore, $\boldsymbol{LU}x = \boldsymbol{b_i}$

Let $y = \boldsymbol{U}x$ ➔

Then $\boldsymbol{L}y = \boldsymbol{b_i}$ ➔ Find y

Forward substitution

$$L = \begin{bmatrix} \ell_{1,1} & & & & 0 \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

Forward subst.

We still need to calculate x

$$\boldsymbol{Ax = LUx = b_i}$$

Knowing y ➔ Use $y = \boldsymbol{U}x$ to find x

Back substitution

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

Back subst.

Back to our example: Forward substitution

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

Step 2 : Solve $\boldsymbol{L}y = \boldsymbol{b_i}$

Forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$\begin{cases} y_1 = 8 \\ y_2 = -11 + y_1 * \dfrac{3}{2} = 1 \\ y_3 = -3 + y_1 - 4y_2 = 1 \end{cases}$$

Forward subst.

Back to our example: Back substitution

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

Step 3 : Solve $\boldsymbol{U}x = \boldsymbol{y}$

Back substitution

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} 2x_1 + x_2 - x_3 = 8 \Rightarrow x_1 = 2 \\ \dfrac{x_2}{2} + \dfrac{x_3}{2} = 1 \Rightarrow x_2 = 3 \\ -x_3 = 1 \Rightarrow x_3 = -1 \end{cases}$$

Back subst.

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}$$

Different b

What if $\boldsymbol{b} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}$?

Step 2 : Solve $\boldsymbol{L}y = \boldsymbol{b_i}$

Forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}$$

$$y_1 = 4$$

$$y_2 = -7 + y_1 * \frac{3}{2} = -1$$

$$y_3 = -1 + y_1 - 4y_2 = 7$$

Forward subst.

Different b

$$
\begin{aligned}
2x + y - z &= 8 \\
-3x - y + 2z &= -11 \\
-2x + y + 2z &= {-3}
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix}
$$

Step 3 : Solve $\boldsymbol{U}x = y$

Back substitution

$$
\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}
$$

$$
2x_1 + x_2 - x_3 = 4 \Rightarrow x_1 = -4
$$

$$
\frac{x_2}{2} + \frac{x_3}{2} = -1 \Rightarrow x_2 = 5
$$

$$
-x_3 = 7 \Rightarrow x_3 = -7
$$

Back subst.

LU Decomposition - Python Example

LU Decompostion of a Matrix

```
import numpy as np
from scipy import linalg as la

# Define Coeff Matrix A
A=np.array([
            [2,1,-1]
            ,[-3,-1,2]
            ,[-2,1,2]
           ]) ## 3X3

# Define vectors b and c
b=np.array([8,-11,-3])
c=np.array([4,-7,-1])


### Step 1
lu, piv = la.lu_factor(A)


### Step 2
#?la.lu_solve

x_b = la.lu_solve((lu,piv), b) # Factorization of the coefficient matrix A, as given by lu_factor
x_c = la.lu_solve((lu,piv), c) # Factorization of the coefficient matrix A, as given by lu_factor

print("x for b ( Using lu_solve() ): \n {} \n".format(x_b))
print("x for c ( Using lu_solve() ): \n {} \n".format(x_c))
```

Step 1

Step 2

```
x for b ( Using lu_solve() ):
 [ 2.  3. -1.]

x for c ( Using lu_solve() ):
 [-4.  5. -7.]
```

EXTRA SLIDES

Carl Friedrich Gauss

Carl Friedrich Gauss

From Wikipedia, the free encyclopedia

"Gauss" redirects here. For things named after Carl Friedrich Gauss, see List of things named after Carl Friedrich Gauss. For other persons or things named Gauss, see Gauss (disambiguation).

Johann Carl Friedrich Gauss (/gaʊs/; German: *Gauß*, pronounced [gaʊs] (listen); Latin: *Carolus Fridericus Gauss*) (30 April 1777 Braunschweig – 23 February 1855 Göttingen) was a German mathematician who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics, astronomy, matrix theory, and optics.

Sometimes referred to as the *Princeps mathematicorum*[1] (Latin, "the foremost of mathematicians") and "greatest mathematician since antiquity", Gauss had an exceptional influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians.[2]

Johann Carl Friedrich Gauss

Contents

Source: https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

History - How ordinary elimination became Gaussian elimination

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Historia Mathematica 38 (2011) 163–218

HISTORIA MATHEMATICA

www.elsevier.com/locate/yhmat

How ordinary elimination became Gaussian elimination

Joseph F. Grcar

6059 Castlebrook Drive, Castro Valley, CA 94552-1645, USA

Available online 28 September 2010

Abstract

Newton, in notes that he would rather not have seen published, described a process for solving simultaneous equations that later authors applied specifically to linear equations. This method — which Euler did not recommend, which Legendre called "ordinary," and which Gauss called "common" — is now named after Gauss: "Gaussian" elimination. Gauss's name became associated with elimination through the adoption, by professional computers, of a specialized notation that Gauss devised for his own least-squares calculations. The notation allowed elimination to be viewed as a sequence of arithmetic operations that were repeatedly optimized for hand computing and eventually were described by matrices.

By Joseph F. Grcar; Historia Math., vol. 38, no. 2, pp. 163-218, 2011

Notation

Row Echelon Form

The leading coefficient (the first nonzero number from the left / pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Also called triangular form

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

Reduced Row Echelon Form (RREF)

All the leading coefficients are equal to 1 and are the only nonzero entry in its column.

Also called row canonical form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Naming Convention – Gaussian Elimination & Gauss-Jordan Elimination

Gaussian Elimination (Row Echelon Form)

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

Gauss-Jordan Elimination (Reduced Row Echelon Form)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Properties of Determinants

MOVE TO END

$$\det(cA_{NXN}) = c^N \det(A_{NXN})$$

$$\det(AB) = \det(A) * \det(B)$$

$$\det(A^{-1}) = {}^{1}\!/_{\det(A)}$$

$$\det(A^T) = \det(A)$$

*

Rules of Vector space - Axiom

Axiom	Meaning
Associativity of addition	$u + (v + w) = (u + v) + w$
Commutativity of addition	$u + v = v + u$
Identity element of addition	There exists an element $0 \in V$, called the zero vector, such that v + 0 = v for all $v \in V$.
Inverse elements of addition	For every $v \in V$, there exists an element $-v \in V$, called the additive inverse of v, such that $v + (-v) = 0$.
Compatibility of scalar multiplication with field multiplication	$a(bv) = (ab)v$
Identity element of scalar multiplication	$1v = v$
Distributivity of scalar multiplication with respect to vector addition	$a(u + v) = au + av$
Distributivity of scalar multiplication with respect to field addition	$(a + b)v = av + bv$

These 8 conditions are required of every vector space