

FUNDAMENTALS OF PHYSICS

Part One

Mechanics of Newtonian and Relativistic Motion



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Stefan Wurm

A·T·I·C·E

ATICE LLC, Albany NY

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Preface

Most colleges and universities around the world offer students a four-semester course in fundamental physics. These courses are usually obligatory for natural science or engineering students as well as for aspiring physicists and in many countries also for those studying to become high school physics teachers. For all those students, these two-year courses stretching over four semesters will form the basis of their physics education after which their path take different directions. The budding physicists among them will move on to take additional courses in various branches of theoretical physics or specific fields of physics, deepening their knowledge gained in the first two years in experimental and applied physics. For all others, such two year introductory courses in fundamental physics will remain a foundational element in their respective fields of specialization, be it in natural sciences other than physics, engineering or teaching. One can find such two-year college or university courses in fundamental physics labeled with various titles, but whatever their labels are, their objectives are usually quite similar. Physics is a fundamental science and as such it underlies many other sciences. Hence, it is essential for many students to master the foundations of physics over a broad range. Typically, the first of these four semester physics courses will introduce students to the fundamentals of motion physics, forces, and energy, essentially the breadth of Newtonian and relativistic mechanics, variously also including chapters on the physics of gases, fluids, vibrations and waves. The second semester course then often follows on with the physics of electrical phenomena, the fundamentals of electrostatic and electrodynamics. The third and fourth semester courses will usually focus either on the physics of heat, that is thermodynamics and aspects of statistical mechanics, and an introduction to atomic physics or on the physics of optics and optical phenomena. For most lecturers, optics is the natural third semester follow-up to the electrodynamics of the second semester, which then leaves thermodynamics, atomic and molecular physics for the fourth semester. Roughly, the present four-semester course of physics lectures, of which this volume comprises the first semester course, follows this scheme. There are of course many excellent physics textbooks out there, readily available to students. The present four semester physics lecture series does not seek to compete with any of those even though it inadvertently may do just that. What it however does seek is to provide an additional learning component which students can access easily in a hopeful highly readable and entertaining form.

Few students will enjoy slogging around heavy textbooks or working through electronic physics compendiums which essentially still only replicate the same textbook reading experience. However, an e-book physics lecture course such as the present one may be something students will be happy to take along wherever their activities may take them while they still have to prepare for their exams.

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1. The World of Natural Things

What is physics and what is physics about? The word physics, like quite a bit of the vocabulary of our modern sciences, originates with the ancient Greeks. It derives from the Ancient Greek word *physis* ($\varphi\omega\sigma\tau\sigma$) which has the meaning of nature with the Ancient Greek word *physike* ($\varphi\omega\sigma\kappa\tau\acute{e}$) denoting "natural things". Hence, physics is the science of natural things, or more precisely, the science of non-living natural things. Most definitions of physics emphasize three major aspects of this science:

- the understanding of the fundamental quantities which govern the behavior of matter such as force, momentum, or energy;
- the study of matter and how matter moves through space and time;
- and of course the understanding of the structure of matter itself.

To understand the world of natural things, physics requires a precise vocabulary to describe its experimental observations and formulate its theories regarding the laws of nature. This vocabulary derives of course from our everyday language but when physicists speak of things like the just mentioned ones such as mass, momentum, force, energy, space and time, they have a precise definition of the meaning of these words. In the remainder of this section we will look into some of these key concepts of physics and how they shape our knowledge of natural things, however, without going into too much detail. The objective is here to get a flavor of the bigger picture of physics and some of its fundamental concepts.

1.1 Forces and Energy

It may surprise many, but it is only a few hundred years ago that physicists began to discern more clearly what the nature of force actually is. From a practical aspect, there was never any doubt as to what constitutes a force and what its potential impact may be. For millenia, engineers had applied this knowledge to construct various kinds of

machines, mostly to lift weights but also to wage war. But it is only with Isaac Newton (1643 - 1727) that physicists gained a much clearer understanding of the concept of force and how forces actually act on matter. The concept of momentum did not exist in antiquity and how it relates to the concept of force is something physicists only began to fully appreciate with the discoveries of Newton. Likewise, a clearer understanding of the concept of energy and its role in physics is also of a more recent origin. What energy actually is, in what forms it can occur and how the energy of an object or the energy of a system of objects can change is central to our understanding of nature and to our ability to shape the world around us. From our everyday experience we all likely believe that we understand the mass property of a body but most of us are mistaken. Often what we interpret as the mass of a body is actually the weight of a body or better the weight force acting on the bodies mass. Physicists also learned that not everything that exists in nature has a mass and that with Albert Einstein's (1879 - 1955) famous $E = mc^2$ equation, where c is the velocity of light, an objects mass m is an expression of that objects energy content E .

Using the concepts of force, momentum, mass and energy, physicists can describe the movement of matter. Not that they could not do that by just observing how an object of matter moves through space without knowing anything about forces, momentum or energy by just tracing the position of an object as it traverses space. If they do the latter, physicists speak of kinematics, the study of object motion without having to know the forces acting on the object. When physicists speak of dynamics they mean the study of object motion in the case when one must know the acting forces in order to calculate how an object moves through space so one can predict its movements and what happens otherwise to an object which is subject to a given force. Today, physics knows four so-called fundamental forces and a number of other forces derived from them. These fundamental forces are

- the gravitational force
- the electromagnetic force
- the strong interaction or strong nuclear force
- the weak interaction or weak nuclear force

The gravitational force is the weakest of the four fundamental forces. However, when masses generating the gravitational force become very large, as is for example the case in the gravitational collapse of massive stars, it ultimately overcomes all other forces. The existence of neutron stars and black holes gives eloquent evidence of this.

Tab. 1.1: Classification of forces.

Force	Type	Attack-point	Source	Range
Gravitational force	imprinted force	heavy mass	heavy mass	$F \propto \frac{1}{r^2}$
Inertial force		inertial mass	accelerated movement relative to inertial system	∞
Elektromagnetic force		charges currents	charges currents	$F \propto \left(\frac{q}{r^2}, -\frac{q'}{r^2} \right)$
Elastic force, hydrostatic and hydrodynamic forces, plastic and viscous forces		matter, macroscopic bodies		$10^{-10} \text{ m} = 1 \text{ \AA}$
Sliding friction, static friction	reaction force			1 \AA
Forces between rigid bodies				1 \AA
Strong nuclear force	imprinted force	elementary particles	color charge	10^{-15} m
Weak nuclear force			weak charge	10^{-17} m

The range of the second strongest force, the electromagnetic interaction, scales just like that of the gravitational force with r^{-2} . However, it does not affect all masses but only those which carry an electrical charge. It is impossible to shield off gravitational forces; at best one can neutralize them in some cases by acceleration (such as in free fall). For electric charges and fields that is different as they can be shielded off. Just consider for example that there are certainly innumerable electrical charges on Earth and yet Earth as a whole is electrically neutral. On the cosmic length scale, therefore, only one single force eventually prevails and that is the gravitational force. But there are certainly electrical cosmic phenomena. A well-known example is the Earth's magnetic field, without which there would probably be no life on Earth. A much more powerful example are pulsars, highly magnetic rotating neutron stars which emit electromagnetic radiation of high energy and intensity along their polar axes.

We are all familiar with the effects of gravitational and electromagnetic forces on our planet as we experience both on a daily basis. This is not the case with the very short-

range nuclear forces, the strong and the weak interaction, because these only act on the subatomic length scale. The strong nuclear force it what binds protons and neutrons, the so-called nucleons, to form the nuclei of atoms. The strong nuclear force is strongly attractive at distances of about 0.5 times the diameter of the hydrogen nucleus but rapidly decreases and becomes negligible at distances reaching about 1.5 times the diameter of a hydrogen nucleus. At distances a little smaller than 0.5 times the diameter of the hydrogen nucleus the strong force becomes repulsive, thereby essentially defining the size of a nucleon, i.e., of a proton or a neutron. To wit, the diameter of a hydrogen nucleus, a single proton, is around 1.7 femtometer or $1.7 \cdot 10^{-15}$ meter. The weak nuclear force is important in nuclear reactions like those powering stars and it is the force behind radioactivity. The weak nuclear force is of an even shorter range than the strong nuclear force and acts on the so-called quarks, which are the particles making up protons and neutrons. The weak nuclear force is responsible for facilitating what is called the β^- -decay, the conversion of a neutron into a proton and the β^+ -decay, the conversion of a proton into a neutron.

The four fundamental forces are called imprinted forces because they act on a body from the outside, derived forces, on the other hand, are referred to as such because one finds on closer examination that these forces are macroscopic consequences of the microscopic effects of the fundamental forces. However, as the summary in tab. 1.1 shows, some derived forces are also counted among the imprinted forces. These include pseudo forces such as the force of inertia or elastic, hydrostatic, plastic, and viscous forces. The reason why these forces are counted among the imprinted forces even though they are derived forces is simply that they act on the body from the outside just like the gravitational force, the electromagnetic force, and the strong and weak nuclear force do. Sometimes one finds frictional forces listed among the imprinted forces as well. However, frictional forces are reaction forces, which means they are a reaction to the action that takes place when two bodies come into contact.

How are forces actually transmitted? For contact forces the answer is obvious as to how the respective forces are transmitted, i.e., through a physical contact. However, for forces acting at a distance, it is all but clear how one body can affect the motion of another body. According to Newton's law of gravity two massive bodies interact through a force proportional to each of the respective masses and inverse proportional to the distance between them, i.e.,

$$F \propto \frac{m \cdot M}{r^2}$$

Using this force law, physicists could provide explanations for phenomena that had been diligently observed and described, almost since the beginning of recorded history, but for which physics explanations were lacking. Such as why celestial bodies move through space as they do, or that gravity is the reason the Moon remains in its orbit around Earth, as it is in a continuous falling movement without ever falling onto Earth. However, right from the beginning it was clear to Newton and his contemporaries that despite all the practical success of this new force law there was a problem and that was action at a distance. How can for example the Sun affect the motion of a planet which orbits it at a many million kilometer distance? Because there seemingly was not medium transmitting the force of gravity between the Sun and a planet the interaction had to be instantaneous, regardless of how large the distance between the Sun and a planet actually was. The interaction between the Sun and Mercury, the planet closest to the Sun, had to be just as instantaneous as the interaction between the Sun and Saturn, the outermost planet known in Newtons time. Therefore, this interaction had to happen at infinite speed, or through some direct or indirect divine mediation as Newton presumably thought. Newton's law of gravity is not the only force law which has this problem. As tab. 1.1 shows, the electromagnetic force law has the same structure as Newtons law of gravity and therefore faces the same challenges with respect to action at a distance.

The solution to this problem were so-called field theories, i.e., force fields permeating space so that at each point in space a certain value could be assigned to a force. The first field theories evolved with electromagnetism as there physicists thought they had a medium, the so-called ether, which could provide the necessary substance they thought a force field required as they deemed it impossible for force fields to exist in empty space. However, once it became clear at the beginning of the twentieth century that electromagnetic waves could propagate in vacuum and that there was no need to postulate the existence of an ether, it also became clear that force fields could permeate a vacuum, i.e., exist in empty space. Consequently, gravitational fields could have just the same reality as electromagnetic fields and in 1917 Einstein published such a field theory of gravity, his General Theory.

Another issue that laws with the structure of Newtons law of gravity have is that if one considers point masses, which are essentially extension-less objects carrying a finite mass, then there lurks trouble once those point masses come very close together, i.e., once $r \rightarrow 0$. However, that is more of a theoretical problem as for bodies with extension, and that is what Newtonian mechanics usually deals with, that problem never surfaces. As we know today, Newtonian mechanics has its limitations and the physics of the very small is anyway governed by the laws of quantum mechanics.

In Ancient Greek energy (*ἐνέργεια*) denoted an activity, something being at work. Most modern definitions define the meaning of energy in physics as the capacity of doing work. Comparing what energy meant to the ancient Greeks as to what it means in modern physics, one could be forgiven for assuming that there is a connection but there really is not. The modern understanding of energy originates with the proposal by Gottfried Wilhelm Leibniz (1646 - 1716), Newton's contemporary and rival, of a living force (*vis viva* as he called it in Latin) which he argued was proportional to the product of the mass m of an object and its velocity v squared, i.e., $E \propto m \cdot v^2$. This form of energy differs from what we call kinetic energy today only by a factor of 2.

Energy can also exist in other forms such as thermal energy, electrical energy, chemical energy or nuclear energy just to name a few. As already Leibniz recognized, thermal energy, what we call heat, is associated with the motion of the respective constituent parts of matter, like the atoms or molecules in a gas. Thermal energy can thus be traced back to kinetic energy, a recognition which a generation after Leibniz would give rise to the development of kinetic gas theory. Other forms of energy are not associated with the motion of objects but with the potential for causing motion. Take for example chemical or nuclear energy, both of which are stores of energy in the form of chemical bonds and in the case of nuclear energy in the form of mass though Einsteins $E = mc^2$. When this potential energy is tapped into, like in a chemical or nuclear reaction, it becomes heat, i.e., thermal energy and with that kinetic energy. In turn, kinetic energy can become potential energy like when for example a massive object moves in a gravitational field or a charged object moves in an electromagnetic field.

To illustrate this energy transfer process, let us consider the curve which a ball describes when we throw it high in the air and as far away from us as we can. As we do that, we convert the potential energy we stored in our body through past food intake into muscle energy which in turn becomes the kinetic energy of the ball as it leaves our hand. As the ball rises, it slows down, i.e., it loses kinetic energy but at the same time, its potential energy in the gravity field of Earth increases as it rises. Once the vertical component of the kinetic energy has been spent as all of it has been converted into potential energy, the ball will stop rising and will begin to fall, thereby converting its potential energy back into kinetic energy. That process will stop once the ball hits the ground at which point its remaining kinetic energy is converted again into something else. This something else is of course the impression in the ground the ball will leave in the dirt. To create this impression requires work to be done as the Earth in that spot is densified, i.e., compacted vertically, and some material is pushed to the side laterally. The impression

the ball leaves in the dirt is hence a demonstration of the capacity of kinetic energy to do work, the work of moving soil.

Obviously, energy is something that has the latent capacity to do work and which can occur in two basic forms, kinetic and potential energy. While kinetic energy is always associated with the movement of a mass, i.e., there is just one manifestation of kinetic energy which is $E = m \cdot v^2 / 2$, there are many different forms of potential energy. Kinetic energy is in essence the additional energy an object possess because of its motion. It was Gaspard-Gustave de Coriolis (1892 - 1843) who in 1829 established the correct form of the equation describing the kinetic energy of an object, even though it was still referred to as *vis viva* at the time. The term kinetic energy did not originate until the middle of the nineteenth century when William Thomson (1824 - 1907), the later Lord Kelvin, began to use it. Shortly after that William John Macquorn Rankine (1820 - 1872), with Kelvin another pioneer of thermodynamics, introduced the term potential energy.

Already Leibniz noticed that in systems of several masses m_i with velocities v_i the sum of all the kinetic energies of the individual masses was a conserved quantity. About the same time Christiaan Huygens (1629 - 1695) reported a similar observation. Both of them observed, as modern physics puts it, that in a closed system where the respective masses interact through elastic collisions the total kinetic energy of the system is conserved. Energy conservation is fundamental to how natural things work but it was not until the mid-nineteenth century that the modern form of the energy conservation law became firmly established as one of the bedrocks of physics: Energy can be neither created nor destroyed but can only change from one form to another. The reason it took almost one hundred years from the observations of Leibniz and Huygens to the emergence of the modern law of conservation of energy is simply that heat and mechanical work were considered to be different things. Only once the equivalence of heat and mechanical work was demonstrated, such as in 1843 by James Prescott Joule (1818 - 1889) with his famous experiment, did it become clear that both were different forms of energy which could be converted into each other.

However one define a closed system, be it an experimental setup in a laboratory, be it our solar system or even the whole universe, the energy of a closed system never changes. All the energy a closed system will ever have is the energy with which it is endowed at its beginning. What happens in a closed system after that is only a matter of energy conversion and redistribution. We all know from experience that heat only flows from hot bodies to cold ones, never the other way. With the inevitable outcome that at some distant point in the future everything will have the same temperature.

1.2 The Concepts of a Space and Time

With some good justification one can point towards mathematics and physics as our two most important scientific endeavors. It has been progress in understanding how natural things behave, i.e., physics, articulated in the language of mathematics which has fundamentally reshaped our human world. When referring to “understanding how natural things work” one must keep a few things in mind to better appreciate the evolution of physics and why the architecture of the house of physics looks like what we have in front of us today and how modern universities introduce students entering the field of physics to this science. First and foremost, physics has been and continues to be an experimental science. It is how humans can experience natural things around them and observe other natural things farther away from them which for many centuries defined what was open to human questioning and exploration. We experience our world as a three-dimensional world of which we are a part for a certain time. This is just as true for modern humans today as it was true for our very distant ancestors when they began to seek a better understanding of the physical world around them. To describe how things move, how fast or how slow they move or what effect forces have on things requires a concept of space and time. Without being able to measure the dimension of length and the dimension of time it is impossible to tell where a body is in space or how fast it is moving through space. The measurement of length requires the existence of rulers and that of time the existence of clocks. Naturally, the first rulers our ancestors used derived from measures of the human body, i.e., they compared the size of objects to parts of the human body such as the human foot which eventually became the length of a standardized foot. To measure the passing of time our distant ancestors used natural clocks. The length of a day, even so it varies throughout the year, is one such clock, the time passing between the winter and the summer solstice just like any other seasonal clock of nature could also provide a measure of time. Then there is of course the human heartbeat which gives us a very personal perception of time. First simple clocks were sundials, sand clocks, water clocks or pendulums synchronized to the human pulse as Galileo Galilei (1564 - 1642) still used them in his experiments.

Today it seems obvious that the concepts of space and time we have are fundamental to our understanding of how objects move through space and time. However, for a long time that was not something which garnered a lot of interest from those trying to understand how matter moves through space and time. That only changed once what physicists could study went significantly beyond that which lay within the innate human space-time window of what we can observe unaided, i.e., with our natural senses only.

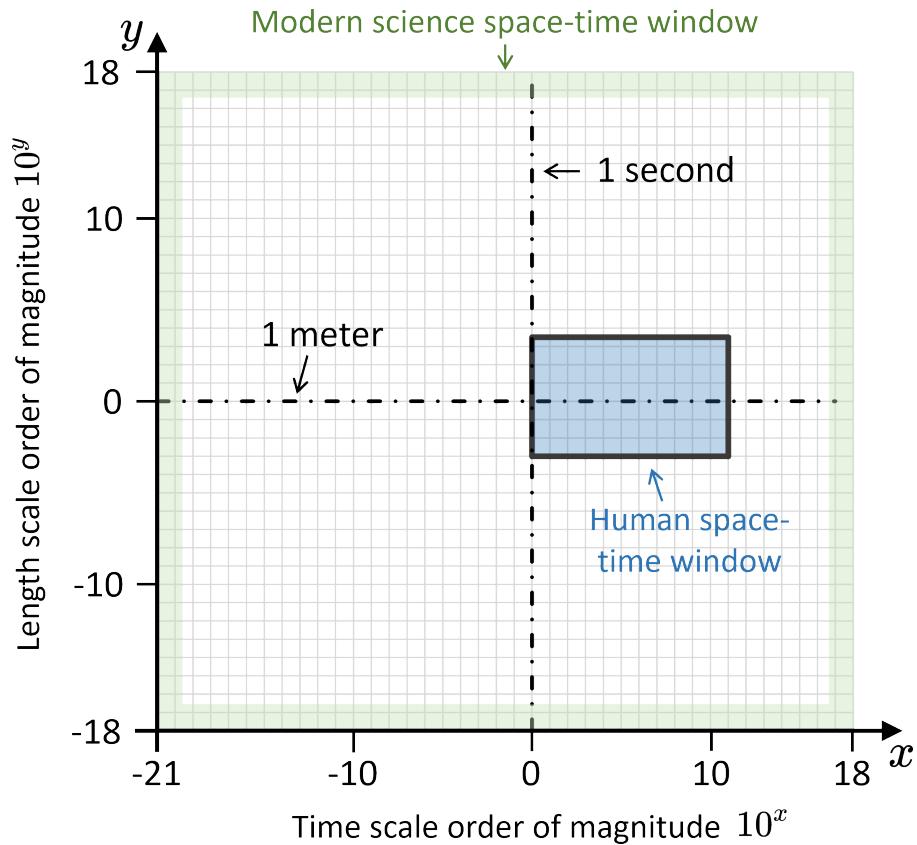


Fig. 1.1: The space-time horizon of modern science compared to the space-time horizon of human perception. The graph is a logarithmic representation of length and time scales, neighboring length and time scale units are separated by a factor of 10.

Without a doubt, modern humans live in a very different universe than their ancestors just a few generations ago and not because the universe has changed much since then. What changed is our perception of it. What we know today about the nature of space and time and the large scale structure of the universe we live in is totally different from what scientists at the end of the nineteenth century still considered as the true nature of the universe as well as that of space and time. Just to be clear, that only means that our scientific knowledge of the universe is much richer today than what scientists at the end of nineteenth century could ever imagine. But it does not by any means imply that we have a complete knowledge of the natural things in our universe. Far from it, for a large fraction of what seemingly makes up our universe we do not even know yet what it really is, we just have labels for it. Just like some of the Ancient Greeks were talking about atoms without knowing what they were, we are talking today about things like dark matter and dark energy without knowing what they really are. But just as in the case of atoms, this lack of understanding will not last forever.

With respect to the concepts of space and time it is important to understand how we can experience them. Physicist use the International System of Units (SI) (see [Appendix](#)), also known as the metric system, which measures length in meters [m] and time in seconds [s]. With our own senses, we probably can estimate distances from roughly one millimeter or one thousandth of a meter on the low side, to maybe several thousands of meters on the high side, spanning roughly six orders of magnitude. The human capability to sense time may stretch from a second to what we can call historical times, which we can generously put at a few thousand years, resulting in roughly eleven to twelve orders of magnitude in our human perception of time. This is essentially the space-time window with which the human exploration of the physical world began (see fig. [1.1](#)). However, right from the beginning, mathematics, the geometry of the Ancient Greeks, was put to good use to measure much larger distances, not directly but in an indirect way through mathematical deduction. The best example for that is likely how Eratosthenes of Cyrene (c. 276 – 195/194 BCE) for the first time determined the circumference of Earth. Eratosthenes realized that the curvature of the Earth results in different lengths of noon-shadows in Northern vs. Southern places which he measured at the Egyptian cities of Aswan and Alexandria. So, he used a length measure accessible to human perception, the length of a shadow at noon-time, to determine the circumference of Earth. This length scale, unless one views Earth form space, is otherwise not accessible to human perception. Using the tool of mathematics to infer from things known something about things unknown would continue to expand the space-time horizon of humans, but for many centuries only at a very slow pace and often it would not expand at all for a long time. All that began to change once human engineering capabilities began to provide science with ever better instruments to measure space and time.

It was instruments such as the first telescopes sufficiently powerful to observe the Moon's surface, or the first microscopes revealing the existence of bacteria, or increasingly improved chronometers for measuring time, thus allowing mariners eventually to measure longitude, which enabled the scientific revolution of the sixteenth and seventeenth centuries. That scientific revolution became the foundation for an industrial revolution which would provide ever better instruments and methods to measure time and space. This ratchet effect continues to work to this day as ever new scientific discoveries allow engineers to devise ever more accurate instruments and methods which in turn enable new scientific discoveries, and so forth. Over the last three centuries this has enormously increased the space-time horizon accessible to human exploration, on the very large scale as well as on the very small scale, from unimaginable short time intervals to equally unimaginable long stretches of time.

Today's scientists can measure the spatial dimensions of infinitesimally small things on the scale of elementary particles such as quarks as well as infinitely large things on the scale of the universe. This covers roughly the range from 10^{-18} meter to almost 100 billion light-years, or roughly 10^{18} meter, the size of the observable universe. On the time scale, humans have measured time intervals as short as how long it takes a photon to cross a hydrogen atom, on the order of 10^{-21} seconds, and on the long side, the age of our universe, some 13.8 billion years, a little less than 10^{18} seconds.

For most of human history, the nature of space and time was considered to be immutable. Once humanity managed to separate religious beliefs from scientific facts it became increasingly evident that the universe and with it space and time had existed for much longer than what religious scholars had told their flocks to believe. With science separating from religion, scientists and philosophers began to ask new questions. Do space and time exist separately from the objects that move through space and time? If there were no objects at all, would there still be a space? What would happen to time if nothing was there to mark its passage? In essence, such philosophical questions may not appear to be much different from the questions religious scholars had asked before. Like for example: What meaning has the existence of a divine being, be it the Christian God or some other deity, if there were no creation? However anyone may answer such questions, unlike in the past, today such questions have little to no bearing on the science of physics. This is quite unlike a few hundred years ago when theology could still dictate what natural sciences were allowed to explore and the Inquisition of the Catholic Church still censored, tortured and even killed scientists if they contradicted scripture.

Why do we perceive the world as we do? In that regard, it may not be just more entertaining but also more stimulating to consult "Flatland: A Romance of Many Dimensions", a satire published by the English author Edwin A. Abbott (1838 -1926) in 1884, than to search through usually much drier philosophical writings. As part of life on this planet the human species evolved in a three dimensional world. Therefore, one may wonder why we are even asking the question whether our human perceptions of space and time are innate or not. But not so fast. More recently humans have acquired a much more profound understanding of dimensional space and also of time. While the world of our everyday experience remains three-dimensional in space with a uniform passage of time, there are many aspects of physics where this is not the case any more. As humans expanded their understanding of the physical world beyond the window of the innate human space-time horizon sketched in fig. 1.1, it became clear that there were aspects of the physical world that require quite different concepts of space and time.

With good reason, many consider Issac Newton's contribution to our understanding of the physical world as the high point of the scientific revolution of the sixteenth- and seventeenth century which laid the foundation for the industrial revolution of the eighteenth century to follow. Newtonian physics provided not only explanations for many of the things humans could observe with their innate space-time perception of the physical world, it also could explain a number of things which humans began to explore outside of their innate space-time window. For almost three hundred years Newtonian physics reigned supreme and continues to this day to provide the tool to explain much of what happens in the physical world. However, from the outset it was clear that there were some challenges of which Newton was already well aware of.

Beginning with Newton, scientists and engineers applied Newtonian mechanics with much success to advance the science of physics itself as well as other sciences and importantly engineering. The result was that the scientific revolution of the sixteenth and seventeenth centuries paved the way for the eighteenth century industrial revolution. Until well into the nineteenth century, the science of physics had for all purposes been identical with understanding mechanics, be it the mechanics of how things worked on our planet, be it the celestial mechanics of planetary movements or be it the mechanics of engineering building the industrial world. However, that began to change in the late eighteenth and then certainly throughout the nineteenth century when the physics of charged particles as well as the physics of heat emerged as physics disciplines in their own rights. This culminated in the theory of electrodynamics pioneered by James Clark Maxwell (1831 - 1879) and the theory of statistical mechanics pioneered by scientists such as Ludwig Boltzmann (1844 - 1906), Josiah Willard Gibbs (1839 - 1903) as well as Maxwell.

Towards the end of the nineteenth century, Newtonian mechanics, electrodynamics and statistical mechanics increasingly looked like quite mature theories. Given this enormous progress, it seemed as if science should in principle be able to explain how pretty much everything in the world accessible to human exploration worked. That went so far that some notable physicists began to believe that everything which there was to discover in physics had been discovered. With no new theory being needed to explain things and the existing theories being considered pretty much complete, bright young minds therefore should rather study something else than physics. That was indeed the advice given to the young Max Planck (1858 - 1947), fortunately, he ignored it. That physics was far from being a finished building would become very clear soon enough as within the first quarter of the twentieth century physicists gained a complete new understanding of the nature of space, time and gravity as well as of the structure of matter.

Let's begin with space, time and gravity. A fundamental premise of electrodynamics is the constancy of the speed of light. On the other hand, there is another fundamental principle of physics, the so-called relativity principle. The relativity principle in its simple form states that the laws of nature in two reference frames that move relative to each other at uniform and constant speed have identical forms. Such reference frames that experience no acceleration, neither in linear or rotational form, are called inertial frames or inertial systems. According to the relativity principle, the laws of nature must produce the same results, i.e., equations, regardless of where in the universe they are studied as long as the respective measurements are done in inertial reference frames. Considering both principles equally as bed-rocks of physics, Albert Einstein came to the conclusion that in order for both to hold true, physicists must relinquish the concept of absolute space and time and accept the relativity of space and time.

Einstein's special theory of relativity published in 1905 showed us that time and space are both relative and that they are not independent, but make up the four coordinates of what came later to be called the space-time continuum. One of the theories prediction is time dilation, the fact that moving clocks slow down. Associated with time dilation is the so-called siblings-paradox. Surprisingly, instead of confusing people and being rejected at the time, this implication of Einstein's special theory of relativity was actually embraced by the popular culture back then. Here is how it goes. Of two identical twins one remains on Earth, and one leaves on a rocket ship, coasting for years in interstellar space at a uniform high speed, only to return to Earth to find his twin much more aged or even deceased if her or his trip took too long. The special theory of relativity provides the explanation for this seeming paradox. Einstein's later general theory of relativity helps the sibling on Earth to age a tiny bit slower than he would without gravity, but it cannot compete with the anti-aging effect his sibling traveling in space experiences. More than anything else, the siblings-paradox reflected a fascination of popular culture with what are seemingly science fiction aspects of science. But there is no science fiction here as many measurements have confirmed that moving clocks really do go slower.

Another prediction of the special theory of relativity was length contraction. This is actually something that Hendrik Antoon Lorentz (1853 - 1928) and George Francis FitzGerald (1851 - 1901) had been postulating earlier, however in a different context. In that context, Lorentz had also come up with what we refer to today as the so-called Lorentz transformations. Einstein derived these Lorentz transformations as well but in his case they were the logical consequences flowing from his approach that both, the principle of relativity and the constancy of the speed of light had to be preserved. The Lorentz

transformations tell us how at high speeds, the space and time coordinates of one inertial frame transform correctly into the space and time coordinates of another inertial frame moving with a constant speed relative to the first one. At speeds much slower than the speed of light, this Lorentz transformation reduces to the classical mechanics transformation between such inertial frames which is commonly referred to as the Galilean transformation. From a practical standpoint, at the time Einstein formulated his special theory of relativity, there was nothing were the Galilean transformation of classical mechanics would not have given the correct result. That has definitely changed as some of the important technology we all have come to rely on would not work without accounting for special relativity.

Time dilation and length contraction have measurable and real consequences, contingent on the inertial frame in which one considers a situation. A good and often used example is the muon, one of the two heavier siblings of the electron we know today. Muons are not stable and their half-life, the time after which half of the originally present muons in a sample decayed into other particles, is 2.2 microseconds or put differently 2.2 millionth of one second. Importantly, this half-life is as measured for the muon at rest or said differently, in the muon's own inertial frame. Muons are constantly generated in the upper atmosphere at altitudes of around fifteen kilometers. Muons have mass so they cannot move at the velocity of light but let us assume for the sake of argument that they had no mass and could travel at the speed of light. Even in that case, they would only be able to travel about 2.2 microseconds times 300 000 kilometers per second, which comes to 660 meters. So how can they reach Earth's surface where many more muons reach detectors that register their arrival than what could be expected from their short half-life? The answer is either time dilation or length contraction depending into which reference frame we put ourselves.

We will look at time dilation first. Muons are known to travel at about 99% of the speed of light. If one calculates how much slower at that speed time in the muon's rest frame passes, a quick estimate will give a factor of 50. That means the actual lifetime of muons as measured by an observer at rest on Earth is 50 times as long as the half-life of the muon in its rest frame. Therefore, muons can travel close to 50 times the 660-meter distance estimated above before half of them have decayed. Being able to travel about thirty-three kilometers from the perspective of an observer at rest on Earth, muons generated at about fifteen-kilometer altitude can easily reach Earth's surface.

Now we will look at length contraction, for which we are putting ourselves in the rest frame of the muon while it moves towards Earth at 99% of the velocity of light. For

someone moving in the muon's rest frame at 99% of the speed of light towards Earth, the distance between the muon generated at fifteen-kilometer altitude and Earth's surface is very much contracted. But by how much? Well, the fifteen-kilometer distance to Earth's surface from the muon's perspective has shrunk by the same factor 50 we calculated above. Fifteen kilometers divided by 50 equals 300 meters, which is a much shorter distance than the roughly 660 meters the muon can travel in its rest frame where its half-life is only 2.2 millionth of one second. Therefore, muons have no problem reaching Earth's surface as measured from their inertial frame either. The muon has become kind of a textbook example illustrating the practical consequences of the special theory of relativity. There are many more and often very practical examples that have demonstrated the reality of time dilation and length contraction since Einstein wrote his 1905 paper.

Another consequence of the special theory of relativity is somewhat harder to demonstrate but we have been at it for quite some time. Over the past century, we have built larger and larger particle accelerators to study the fundamental particles of nature and to explore physics under conditions, as they presumably existed very close to the beginnings of time when our universe was born. Einstein's special theory of relativity asserts that no massive particle can move as fast as the speed of light. The rest mass is the mass of an object like the muon in its own reference frame where it is at rest. As an object is accelerated to ever-higher speeds, another type of mass comes into play, called the dynamic mass. The latter is the mass of the moving object as observed from another inertial frame and it relates to the observed increase of its kinetic energy as the object is being accelerated. The higher the speed of the object, the less of the added increase in kinetic energy goes into increasing its speed and the more goes into adding to its dynamic mass. Eventually, all the added energy goes into increasing the dynamic mass of the object. The consequence is that no object can move faster than light and only massless objects such as photons, the quanta of light, can move at the speed of light. To accelerate massive objects with rest masses greater than zero to the speed of light would require infinite amounts of energy.

Inertial frames are an idealization. Practically everything in the universe is subject to forces resulting in acceleration or deceleration. Einstein knew he needed a more general theory allowing for transformations not just between inertial frames but between frames moving arbitrarily with respect to each other, including any type of acceleration. The result was Einstein's general theory of relativity, published 1917. Einstein was convinced that any theory allowing for transformation between mutually accelerated frames of reference, so-called non-inertial frames, would have to be a theory of gravity.

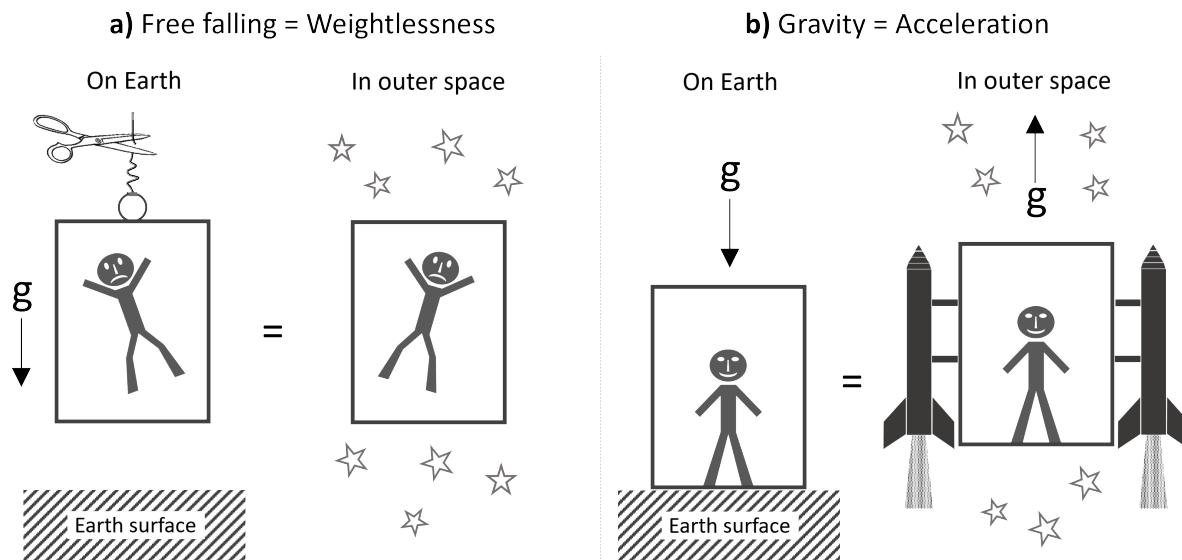


Fig. 1.2: (a) A person floating weightless in a closed elevator cannot know if she or he experiences weightlessness because of the elevator free falling in Earth's gravity field or because the elevator is in outer space with zero gravity. (b) A person pinned down to the floor by gravity in a closed elevator resting on Earth's surface cannot distinguish his or her situation from a person experiencing the same force in such an elevator accelerated at one g in outer space.

He came to this conclusion by a now famous Gedankenexperiment where he imagined that someone unlucky enough to find her- or himself in a free-falling elevator would experience a situation equivalent to being in a zero gravity environment (fig. 1.2a). If this person did not know that she or he was in a free-falling elevator there was no way this person could say if she or he experienced quasi zero gravity like in outer space or was unfortunately stuck in a free-falling elevator in Earth's gravity field. Next, Einstein considered the case when the elevator is not moving (fig. 1.2b). In this case, a person's feet are pinned down to the elevator floor by Earth's gravity. But then again, this situation also cannot be distinguished from an elevator in outer space with zero gravity but being accelerated in such a way and direction that the persons feet are similarly pinned down to the elevator floor. This convinced Einstein that gravity and acceleration are indistinguishable. This insight, that one cannot experimentally distinguish the effects of a gravitational field from those experienced in an accelerated frame of reference is Einstein's famous principle of equivalence.

In a twist to his Gedankenexperiment, Einstein thought about what would happen if one made a little hole in the side of the elevator to let a light beam pass through. If the elevator is greatly accelerated in the zero-gravity environment of outer space, let's call the direction upwards acknowledging there is no up or down in space, it will travel a

certain distance up before the light beam entering through the hole on one side can reach the elevator wall opposite to the hole. During the time it takes the light beam to travel this distance the elevator will have moved. Because of that, the light beam will hit the wall opposite the hole at a lower point than the hole drilled in the wall letting the light beam in. If one would now draw an imaginary line from the hole where the light beam entered, following its path to where it hit the opposite wall it would not be a straight line, but it would look bent, bent downward to stay in the picture. The result is that acceleration seems to bend light. And because the first part of the Gedankenexperiment showed that acceleration and gravity were indistinguishable it became clear to Einstein that gravity would have to bend light also.

But how can light be bent? As it turned out, the only way to achieve this is to bend space-time itself. Between any two points in space light will always travel along the shortest path, or put differently, in any geometry, light will always travel along what is the equivalent of a straight line in that geometry. Because of that, Einstein's general theory of relativity is a geometric description of how space-time curves in the presence of matter and how matter moves in a curved space-time. In Euclidean geometry, the shortest connection between two points on a planar surface is a line and the internal angles of a triangle on a planar surface add up to 180 degrees. In non-Euclidean geometry, as for example on the surface of spheres such as Earth, the shortest connection between two points is also a line but it is a curved one; and the angles of a triangle drawn on a sphere add up to more than 180 degrees. Light always travels a straight line in any geometry it just so happens that in the proximity of heavy masses, space-time geometry is not flat but curved.

It was only two years after Einstein made the prediction that large masses must bend light that a team led by the astronomer and physicist Arthur Eddington (1882 - 1944) was able to measure the bending of light. A number of other predictions Einstein's theory made have also been verified since then, most recently the existence of gravitational waves. Gravitational waves can be generated by extremely massive objects when they get too close to each other. If that happens, the resulting gravitational disturbance ripples through space-time as a gravitational wave, not unlike the circular waves rippling through the surface of an otherwise quiet pond after we throw a pebble into it. As gravitational waves propagate through space-time, they modulate it as they pass through. This is how we can detect them on Earth. Today we know that gravitational waves can be generated in the merger of two black holes, which was first observed in 2015, or when two neutron stars merge to form a black hole, which was discovered in 2017.

1.3 The Structure of Matter

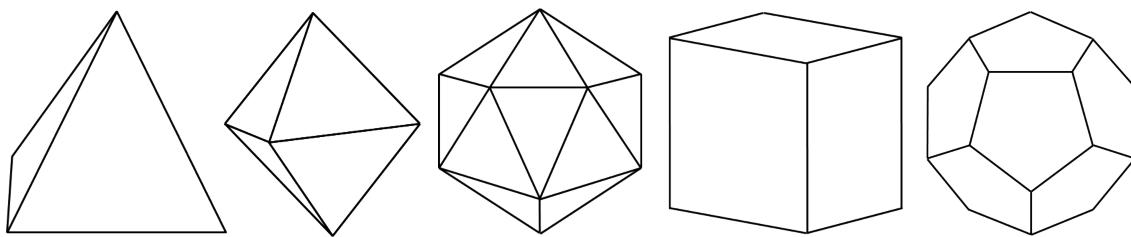


Fig. 1.3: The five regular polyhedra referred to as the five Platonic Solids. Plato believed the first four to be the fundamental building blocks of matter and the last one used by the gods to arrange the constellation of the heavens. Later, Aristotle assigned the fifth solid to the ether. From left to right: the tetrahedron (fire), the octahedron (air), the icosahedron (water), the cube (earth), and the dodecahedron (universe / ether).

Just as our understanding of natural things at the very large scale changed dramatically in the early twentieth century, so did our understanding of what constitutes matter, that is the structure of matter itself. Humans have of course always understood that among the natural things they dealt with existed certain differences. Metals are different from non-metals and among metals there are also differences. As humans experimented with metals to forge tools and weapons from them, they discovered bronze which gave rise to the Bronze Age around 3 000 BCE later to be followed by the Iron Age, the beginnings of which date in the Mediterranean to 1 200 BCE.

The concept of elements being the indivisible parts of what made up matter originates with the ancient Greeks who referred to them as elements or as atoms. Some elements, like the metals which could be found in their metallic state, e.g., such as silver, tin, gold, copper or lead, but also non-metals such as sulfur or carbon, have been known for a long time but were not recognized as elements. A little more than a generation before Aristotle (384 - 322 BCE), the philosopher Democritus (c. 460 – c. 370 BCE) had formulated his atomic theory of the universe. As maybe already the fifth century BCE philosopher Leucippus before him, Democritus held the view that everything is composed of atoms, which he thought to be physically indivisible, and the empty space between them through which atoms move. Those atoms he assumed to come in different shapes and sizes, thereby determining the various properties of the matter they constituted. We do not know exactly who first suggested what later became known as the Platonic solids (fig. 1.3) as the elemental parts of matter. Some assume it may have been Pythagoras (c. 570 - c. 495 BCE) others believe he only came up with three of the five solids and the other two were added in Plato's (c. 428 – 348 BCE) time.

The fascination the Platonic solids held for the ancient Greeks was no accident. One of the most famous theorems in all of mathematics that the ancient Greeks gave to us concerns regular polyhedra and it says that there are exactly five regular polyhedra, no more. These five regular polyhedra shown in fig. 1.3 are of course none other than the Platonic solids. Three of them are composed of equilateral triangular faces, the tetrahedron with four faces, the octahedron with eight faces, and the icosahedron with twenty faces. The dodecahedron has twelve sides made up of regular pentagons and the cube has of course six square faces. Plato thought that all matter was composed of them, and this is why we refer to them as the Platonic Solids. Four of the five solids he associated with what constituted the four elements of antiquity which were fire, air, water and earth. Plato speculated that the gods used the fifth solid for arranging the constellations of the heaven, meaning the universe itself. It was Aristotle who suggested to associate another element with the fifth solid, namely the ether, which he believed to permeate the universe. This ether eventually became the hypothetical medium in which physicists believed electromagnetic waves, i.e., light waves, propagated just as sound waves propagate for example in air. In 1887 the Michelson-Morley experiment demonstrated that there was just no evidence for Earth moving through something like an ether hypothetically permeating the universe. When not long after that Einstein's theory of special relativity showed that there was no need for an ether in the first place, physicists finally abandoned the ether hypothesis.

The quest to understand the structural composition of matter was for a long time limited or reduced to what was known as alchemy. The general objective of practitioners of alchemy was the purification of substances, specifically metals and its holy grail was the transmutation of substances, such as to convert base metals into gold. Among the practitioners of alchemy, which was often identified with magic, were notable scientists, among them for example Isaac Newton himself. Often, alchemy is ridiculed but at the time it was the state of the art to advance what was known about substances. It is not for nothing that chemistry derives its name from alchemy. With respect to the transmutation of base metals into gold, alchemists back then could not know that chemistry just does not have the tools to achieve that. However, their general contention with respect to transmutation was correct, as modern physics has demonstrated. It just requires a technology which alchemists back then could not have dreamed of and it is very expensive to make gold out of other metals.

The first chemical element was discovered in 1669 (phosphorus) and the list of 33 chemicals which Antoine-Laurent de Lavoisier (1743 - 1794), the discoverer of oxygen, published

in 1789 included already 23 elements in the modern sense. Today we know of 118 chemical elements, physicists usually refer to elements as atoms, and we also know that they are not elemental as they are made up of protons, neutrons and electrons. In turn, protons and neutrons are made up of even smaller particles. Physicists have become wardens of what is now a veritable elementary particle zoo and to describe the behavior of these particles they ended up developing a completely new branch of physics, quantum mechanics. Einstein's general theory describes physics on the very large scale, Newtonian mechanics describes physics on the scale of our human sense experiences and quantum mechanics describes the physics on the very small scale. Newtonian physics has in essence become a limiting case of Einstein's general theory, however, quantum mechanics is completely different from either.

When the theory of statistical mechanics was first conceived in the nineteenth century, the existence of atoms and molecules was still only an hypothesis. That began to change when in another 1905 publication Albert Einstein interpreted what was referred to as Brownian motion, named after its discoverer Robert Brown (1773 - 1858), as conclusive evidence for the existence of atoms. Only a few years after that the experimental evidence corroborating the existence of atoms became irrefutable. The chemical elements, i.e., the different kinds of atoms we know today number 118 which form what we usually get to learn as the periodic table of elements in high school. Tab. A.1 in the Appendix shows a version of this periodic table with its currently 118 elements, quite a few of which are not stable as they radioactively decay into other elements. The latter shows that transmutation, i.e., one chemical element changing into another, is not something outlandish but part of how natural things, i.e., physics, works. As modern physics has discovered a little more than one hundred years ago, those chemical elements are themselves made out of other more elementary buildings blocks, namely of positively charged protons, neutrons which carry no electric charge, and negatively charged electrons.

Protons and neutrons constitute the nucleus of a chemical element and determine its nuclear properties while the electrons bound to the nucleus determine its chemical properties. Protons and neutrons are made of other even smaller particles while the electron is for all we know an elementary particle. The branch of physics devoted to the understanding of the fundamental structure of matter is called particle physics. The study the fundamental structure of matter requires very large instruments, the largest scientific instruments, if that label is even still applicable, scientists have ever built. To study the building blocks of matter one must first create them and that is only possible by smashing subatomic particles at enormous energies into each other. Accelerating those

subatomic particles to such high energies requires huge machines, physicists call them particle accelerators. Increasingly, such machines can generate energy densities which are getting ever closer to the conditions which may have been present at the birth of our universe some 13.8 billion years ago. One could date the beginnings of particle physics with the discovery of the proton in 1917 by Ernest Rutherford (1871 - 1937) or with the discovery of the neutron by James Chadwick (1891 - 1974) in 1932. Certainly, by the 1930's particle physics was well on its way to become a physics discipline in its own right. When a theory predicts the existence of a certain elementary particle with a finite rest mass, such as a quark for example, Einstein's $E = mc^2$ tells physicists what the energy threshold is to create it. To confirm that such a theoretically predicted particle does exist, the energy density a particle accelerator can achieve in a particle collision must be so high that this threshold for particle creation can be overcome. If that is not the case one must build a larger accelerator which can achieve the required energy densities. In this way theory and experiment have predicted and confirmed the existence of a number of elementary particles which by the 1970's the community of particle physicists had integrated into what is known as the Standard Model of particle physics. With the Standard Model physicists have been able to explain three of the four fundamental forces: the electromagnetic force, the weak nuclear force and the strong nuclear force. Only the force of gravity cannot adequately be explained by the Standard Model. The standard model knows seventeen elementary particles (tab. 1.2) of which twelve are so-called matter particles (fermions) and of the remaining five (bosons) four are responsible for mediating the electromagnetic, weak and strong nuclear forces whereas the recently discovered fifth in that group, the so-called Higgs boson, gives mass to all other elementary particles.

Tab. 1.2: The 17 particles of the Standard Model

Fermions				Bosons	
Quarks	up	charm	top	gluon	Higgs-boson
	down	strange	bottom	photon	
Leptons	electron	muon	tau	Z-boson	
	electron neutrino	muon neutrino	tau neutrino	W-boson	

It is no accident that the rise of particle physics and quantum mechanics in the twentieth century parallel each other. Without the tools that quantum mechanics provides, the study of the very small, and that is what particle physics is, would be impossible. The laws of physics governing the very small are not just quite different from Newtonian

physics, there are things in the quantum world which do not exist in the macroscopic world. With good reason we can date the beginnings of quantum mechanics to the beginning of the last century when first in 1900 Max Planck introduced his radiation quanta to explain the measurement curve describing black body radiation and then Albert Einstein in 1905 provided an explanation of the photoelectric effect by introducing light quanta. That makes Einstein, who would come to consider quantum theory as a provisional answer but not as the final word with respect to the nature of things at the very small scale, one of the pioneers of early quantum mechanics. Despite Einstein's skepticism, quantum mechanics has become an immensely successful theory which has transformed our modern world just as Newtonian mechanics transformed the world of the eighteenth and nineteenth century. However, just as Newtons mechanics was not the final word in physics, maybe, despite all its success, this may also be the case with quantum mechanics, just as Einstein suspected.

Different from the theories of gravity, electromagnetism, the weak and the strong nuclear interaction, which are all theories to describe how a certain force works, quantum mechanics per se is not a theory about a single force. What quantum mechanics does is that it brings a new aspect to those other theories. The theories of weak and strong nuclear interaction are of course quantum theories as without quantum mechanics they could not have been devised. Bringing the quantum aspect to electrodynamics has given us quantum electrodynamics. Hence quantum mechanics has been reconciled or is inherent to three of the four fundamental forces; the odd one out is again the force of gravity. How the physics of the very large, i.e., gravity, and the physics of the very small connect is still an open question. There are some connections but they are few and so Einsteins general theory and quantum mechanics continue to stand apart. Until physicists understand how a theory of gravity and quantum mechanics can be brought together it is unlikely that the dream of a grand unified theory can come true which unlike the Standard Model then will also include gravity.

2. The Basics of Motion

To discuss even the most basic concepts of motion one has to first consider what it is of which one would like to describe its motion. In our everyday lives we perceive different kinds of motion all the time, that includes living things and non-living things. Even though physics concerns itself only with the non-living matter, that still leaves many different kinds of motion to consider and countless different objects that we can perceive as being in motion around us. That can be for example nearby objects such as the motion of a pebble, loosened by our step as we hike in the mountains, which we then see rolling downhill or the motion of the moon when we look into the night sky or even one of our solar system planets when we observe it with a sufficiently powerful telescope. In both cases, pebble or celestial object, this motion occurs due to the force of gravity. When we throw an object using our muscle force, we know intuitively that once we release the object, its motion will be governed by the energy we imparted to it, the air resistance it encounters and the force of gravity it is subject to. There are other forces we are familiar with which can cause objects to move as well, a simple example being magnets used to lift metallic objects off the ground. In the later case it is of course the electromagnetic force which is at work and which works against the force of gravity to lift an object. In all such cases where we observe the motion of objects we also intuitively know that the mass of an object is what determines to what extent a force can impact the motion of an object. Using just our muscle force, we likely will not be able to move a huge rock boulder at all, while we have no problem throwing a small pebble across a creek. Most would without a second thought assert that it is not the different size of those objects which makes it impossible for us to throw huge rock boulders like we can throw pebbles but that it is the much greater weight of rock boulders which prevents the limited force our muscles can exert from achieving that.

2.1 The Concept of a Point Mass

As we all know, objects can differ greatly in form, size or weight. The objective of physics is to give a description of motion that can then be applied in a general way to understand and describe the motion of all kinds of objects. To achieve that, the study of object motion begins with focusing on the commonalities one can abstract from observing the motion of bodies. Those commonalities are then the subject of further in depth studies of motion using an idealized object which physicists refer to as a point mass. The understanding of this idealized point mass motion physicists then apply to the study of motion of more complex objects which they consider to be made up of a greater number of point masses or to the study of many bodies which they consider to be systems of point masses. In this respect, physicists proceed just like mathematicians using the concept of the ideal space point as a fundamental object of geometry out of which they construct other mathematical objects such as circles or spheres or other more complex surfaces made up of such extension-less points. In essence, physicists use the same abstract concept of an extension-less point to which they however attach a mass to create what they call a point mass. For mathematicians, space points have to be extension-less in a much stricter sense than for physicists. For physicists to consider a point mass essentially extension-less means that the actual extension of the point mass is so small that it has no bearing on the physics governing point mass behavior as it moves freely or under the influence of various forces through space.

Under many circumstances elementary particles such as the electron are good examples for such point masses though there are conditions where this is not the case. As useful as point masses are, there are limitations to keep in mind. In that respect, just consider that a truly extension-less point mass would possess an infinite energy density. Within the framework of classical mechanics which deals with macroscopic bodies this fortunately does not matter. But it does matter in the quantum world where for example particles such as the electron must have a finite extension because they do have a finite mass. However, at this small scale we have already left the world of classical mechanics and to explore what the finite extension of an electron could possibly be, one has to enter the world of quantum mechanics.

The concept of a point mass is a useful idealization because the mechanics of point masses does not have to consider the extension of a physical object. This simplifies things quite a bit and it also provides the foundation for developing the mechanics of more complex bodies as those can be envisioned as being made up of countless point masses. In this

way, the mechanics of rigid bodies as well the mechanics of bodies which can change their shape and / or volume while being acted upon by an external force (so-called deformable bodies) follows naturally from the mechanics of point masses.

A point mass is characterized by its five properties: its three spatial coordinates, its time coordinate and its mass. The coordinate position of a point mass is measured relative to the coordinate origin of a frame of reference, like for example a Cartesian, polar or cylindrical coordinate system. Fig. 2.1 shows such a rigid coordinate system, a Cartesian coordinate system, in which the position of a point is being described by its position vector \mathbf{r} :

$$\mathbf{r} = \sum_{i=1}^3 \mathbf{e}_i x_i$$

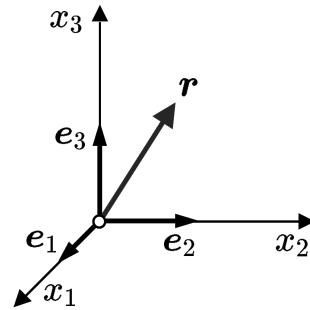


Fig. 2.1

The first basic quantity of mechanics is the spatial dimension of point mass motion traced in a rigid coordinate system by the point mass position vector \mathbf{r} . Evidently, that presupposes the existence of “length rulers”. The dimension of a coordinate x_i is its length, i.e., $x_i = [x_i]$, and this length is measured in units of meters (one meter = 1 m). In general, the coordinates of a point mass are functions of time. Hence

$$\mathbf{r} = \mathbf{r}(t) \quad \text{or respectivley} \quad x_i = x_i(t)$$

The second basic quantity of mechanics is thus the temporal dimension. Measuring time requires the existence of “clocks”. The dimension of time is its duration, i.e., $t = [t]$, and this duration is measured in units of seconds (one second = 1 s). The unspoken assumption with respect to the just introduced rulers to measure length and the clocks to measure the passing of time is the existence of continuous and linear metrics of length and of time. We know today, that the latter is not always a given when studying things outside classical mechanics. That is for example the case with black holes where the concept of time itself breaks down as one heads towards the singularity in space which the center of a black hole represents. However, the existence of continuous and linear metrics of length and of time is the underlying assumption of Newtonian mechanics. In classical mechanics, the position vector $\mathbf{r}(t)$ of a point mass describing its motion is a continuous function of time just like the trajectory which its motion describes in space is continuous, i.e., the point mass does not just vanish at one point and pop up at another point without traversing the stretch of space in between those two points in some way.

2.2 Kinematics

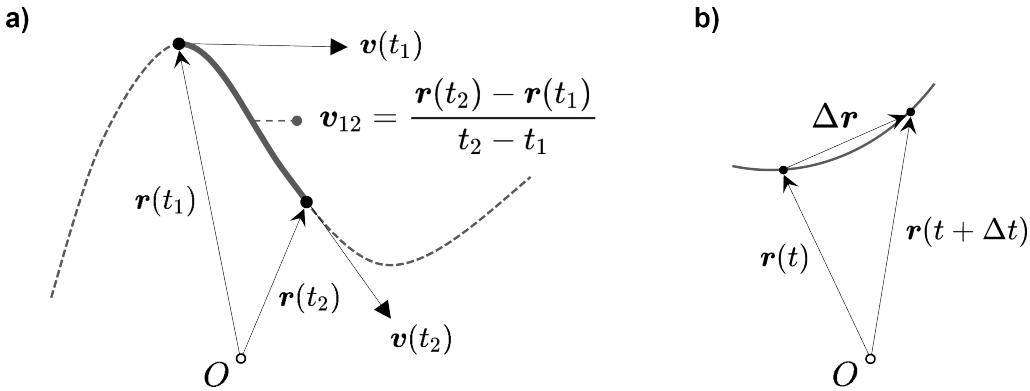


Fig. 2.2: a) Average velocity of a point mass between two positions along its trajectory. b) Transition from average velocity to instantaneous velocity as $\Delta t = t_2 - t_1$ approaches zero.

The objective of kinematics is to describe the movement of a point mass as it moves through space and time in simple geometric terms only. The velocity of a point mass can be deduced from knowing the position of the point mass as it moves through space at any given time. If we find the point mass at time t_1 at \mathbf{r}_1 and at time t_2 at \mathbf{r}_2 , this tells us that the point mass has moved from point \mathbf{r}_1 to point \mathbf{r}_2 with an average velocity

$$\mathbf{v}_{12} = \frac{\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)}{t_2 - t_1}$$

If the distance between the position vectors of the point mass at \mathbf{r}_1 and \mathbf{r}_2 is large, the point mass could have moved along the trajectory between \mathbf{r}_1 and \mathbf{r}_2 at speeds much greater and slower and still given us the average velocity \mathbf{v}_{12} . What we are really interested in is not the average velocity of the point mass between any two given position vectors describing the point mass motion through space. Rather, what we need to know to describe the movement of the point mass at any given point in time is its instantaneous velocity at any position vector $\mathbf{r}(t)$. We obtain the instantaneous velocity by shrinking the elapsed time $\Delta t = t_2 - t_1$ for measuring the average velocity \mathbf{v}_{12} to zero. The instantaneous velocity $\mathbf{v}(t)$ of a point mass is the limit of the average velocity \mathbf{v}_{12} as the elapsed time Δt approaches zero:

$$\mathbf{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} \quad (2.1)$$

Denoting the rate of change of a point mass position vector, i.e., the point mass instantaneous velocity $\mathbf{v}(t)$, with a dot over the position vector $\dot{\mathbf{r}}$, goes all the way back

to Isaac Newton; just like the more common notation for such differentials, i.e., $d\mathbf{r}/dt$, originated with Gottfried Wilhelm Leibniz. In physics, Newton's notation for the time derivative of a variable, has been widely adapted and in analogy to the first derivative, the second derivative with respect to time is indicated by two dots above the variable being differentiated.

The velocity of a point mass is a vector magnitude. The direction of the velocity vector is that of the tangent to the point mass trajectory at the point for which the velocity is calculated and the length of the velocity vector is given by its absolute value. The velocity vector can of course be decomposed to obtain for example the velocity components along any of the coordinate axes or it can be projected onto any arbitrarily oriented straight line to extract the velocity component along that line. Of course, velocities being vectors, vector rules apply to the addition of velocities.

In Cartesian coordinates, $\mathbf{r} = (x, y, z)$, the velocity vector has the three components

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

and its absolute value is given by

$$v = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} = \frac{ds}{dt}$$

where ds is the differential arc element of the trajectory. Instead of measuring velocity with respect to a specific coordinate system, $v(t) = ds/dt$ measures the magnitude of the velocity with respect to the trajectory itself, as the differential arc element ds which is run through in the time interval dt . Often physicist will also use the directional cosine representation of the velocity vector. With that the velocity vector can be written as

$$\mathbf{v} = v \cdot \begin{pmatrix} \cos \alpha_1 \\ \cos \alpha_2 \\ \cos \alpha_3 \end{pmatrix}$$

where

$$\cos \alpha_1 = \frac{v_x}{|\mathbf{v}|}, \quad \cos \alpha_2 = \frac{v_y}{|\mathbf{v}|}, \quad \cos \alpha_3 = \frac{v_z}{|\mathbf{v}|}$$

and

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 = 1$$

The most simple movement of a point mass is referred to as uniform linear motion. It describes the movement of a point mass covering equal distances in equal time intervals.

For a point mass moving in uniform linear motion the direction of the velocity vector and its absolute value are constant. Consider the case of uniform linear motion along one of the coordinate axes, for example along the x -axis. In this case the velocity vector has only one component, namely

$$v_x = \frac{dx}{dt} = \text{const} \quad \text{with} \quad v_y = v_z = 0$$

In that case we can easily trace the point mass position on the x -axis as a function of time by integrating

$$x(t) = \int_{x_0}^x dx = \int_{t_0}^t v_x dx = v_x \cdot t$$

where t_0 has been chosen such that $x(t_0 = 0) = x_0 = 0$ coincides with the coordinate system origin of our frame of reference S . This so-called laboratory system S is at rest with respect to both, the moving point mass and us as the observers of the point mass movement. Uniform linear motion of a point mass means that its position changes linearly with time along a straight line just as $x(t) = v_x \cdot t$ in the above example indicates.

An observer in a reference frame S' which moves with the velocity \mathbf{v}' relative to our reference frame S will of course measure a different velocity from what we will observe. Such reference frames moving relative to each other at constant velocities (that is without acceleration) are referred to as inertial frames of reference. The coordinate transfers between such inertial reference frames are known as Galilean transformations. Galileo Galilei was the first physicist to understand that a person in a uniformly moving frame, such as for example on a uniformly moving ship far out at sea, had no sense of such movement. Because of that, the observations of spatial distances, time differences, or angle measurements a person in such a uniformly moving frame makes must yield the same result as if those observations and measurements were made on land in a stationary laboratory. Fig. 2.3 illustrates how the position coordinates \mathbf{r} and \mathbf{r}' of a point P in the two inertial reference frames S and S' moving relative to each other at constant speed \mathbf{v}_0 are connected. For the coordinate and velocity transformations between the inertial reference frames S and S' we can read off the graph in fig. 2.3:

$$\left. \begin{array}{l} \mathbf{r} = \mathbf{v}_0 \cdot t + \mathbf{r}' \\ \mathbf{v} = \mathbf{v}_0 + \mathbf{v}' \end{array} \right\} \quad \text{Galilean transformation} \quad (2.2)$$

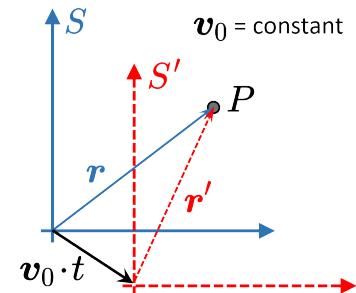


Fig. 2.3

The second equation for the velocity addition we obviously obtained by taking the time derivative of the first one. The two equations in eq. (2.2) are jointly referred to as the Galilean transformation. Galilei's fundamental insight that with respect to uniform linear motion the laws of mechanics are invariant is sometimes referred to as the Galilean relativity principle. Galilei's observation that if a ship moved with constant velocity on a perfectly calm ocean, its passengers, otherwise ignorant that they were aboard a ship, would have no way to tell if they were at rest on land or sailing the ocean has become a common experience for many. We take all this for granted when we for example travel on a plane moving at several hundred miles per hour. We walk through the aisle on a plane just like we do on Earth and when we drop something it falls to the ground in just the same way as anything does that we drop on Earth. An even more extreme example is Earth itself which orbits the Sun with a velocity of more than $100\,000 \text{ km h}^{-1}$ without impacting how for example an apple falls from a tree. What the Galilean relativity principle asserts is that there is no physical way to differentiate between a body moving in uniform linear motion and a body at rest. It certainly is possible to determine that two bodies are moving relative to each other. However, it is impossible to determine which one is moving and which one is at rest.

Importantly, Galilei also realized that uniform linear motion does not require a cause. A body that is moving in uniform linear motion will continue to move in uniform linear motion unless an external cause changes that. Such an external cause is of course connected to some kind of force. Hence, examples of uniform linear motion are rare as it can only occur far away from anything that might force an object to change its state of motion. Among the few examples for uniform linear motion one can find in nature are mass-less particles such as photons moving at light speed or nearly mass-less particles such as the neutrinos which move at speeds very close to the speed of light. Far away from stars and planetary bodies where the force of gravity exerted by such bodies on photons and neutrinos can be neglected they move indeed in uniform linear motion. However, it is just for such particles moving at very high speeds where the Galilean transformation with its simple addition of velocities breaks down. As we will discuss in the last chapter, this is because observations of spatial distances, time differences, or angle measurements actually do depend on the speed of the reference frame in which they are made. At velocities far below the speed of light these differences are so tiny that they just do not matter and hence, the Galilean transformation has served physicists very well for a long time. Only when they had to consider objects moving at much higher speeds did physicists discover that quite different rules apply for coordinate transformation and velocity addition (see section 12.3.1).

For objects such as a point mass moving in uniform linear motion, the direction of the velocity vector and its absolute value are both constant. Uniform linear motion is the most simple form of motion. Two other simple forms of movement arise in case that either the direction into which the velocity vector of a moving object points or the absolute value of its velocity changes, but not both at the same time. If the direction into which the velocity vector of a moving object points remains constant but its absolute value changes then the respective object moves on a straight line but in a non-uniform manner. It either decelerates or it accelerates along a straight line depending on whether the magnitude of the objects velocity decreases or increases. Conversely, if the absolute value of an objects velocity vector remains constant but its direction changes, then the objects is said to move uniformly, that is at constant speed, on an arbitrary curved trajectory whose shape is governed by the direction changes of the velocity vector. In either case, the point mass is subject to acceleration.

As illustrated in fig. 2.4, average acceleration and instantaneous acceleration are derived from changes of the velocity vector similar to how average velocity and instantaneous velocity were derived from changes of the position vector of a point mass over time. Just as the letter v stands for velocity, the letter a is usually the symbol used for acceleration. Accordingly, the average acceleration is given by

$$\mathbf{a}_{12} = \frac{\mathbf{v}_2(t_2) - \mathbf{v}_1(t_1)}{t_2 - t_1}$$

and the instantaneous acceleration is given by

$$\mathbf{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d^2 \mathbf{r}}{dt^2} = \ddot{\mathbf{r}} \quad (2.3)$$

In Cartesian coordinates, $\mathbf{r} = (x, y, z)$, the three components of the acceleration vector are

$$a_x = \frac{d^2 x}{dt^2}, \quad a_y = \frac{d^2 y}{dt^2}, \quad a_z = \frac{d^2 z}{dt^2}$$

In the simple case that the change in the velocity vector only impacts its absolute value but not its direction, the acceleration vector points in the same direction as the velocity vector. However, in general, the direction of the acceleration vector does not coincide with the tangential vector to the trajectory curve at the point mass position. Let $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ (see fig. 2.5) denote the unit vectors at the position of the point mass along the tangent of the trajectory curve and normal to it. Then we can write the velocity vector

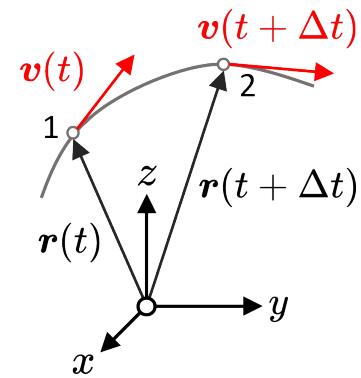


Fig. 2.4

as $v \cdot \hat{\mathbf{T}}$ where v is the absolute value of \mathbf{v} and split the acceleration vector in a tangential and a normal component with respect to the point mass trajectory:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d(v \cdot \hat{\mathbf{T}})}{dt} = \frac{dv}{dt} \cdot \hat{\mathbf{T}} + v \cdot \frac{d\hat{\mathbf{T}}}{dt}$$

Obviously, with $\hat{\mathbf{T}}$ being the tangential vector, the first component points in the direction of the tangential vector. As for the second component, we need to understand in which direction the time derivative of the tangential unit vector points. To determine that it helps to remember that $\hat{\mathbf{T}} \cdot \hat{\mathbf{T}} = 1$ is a constant and hence

$$\frac{d(\hat{\mathbf{T}} \cdot \hat{\mathbf{T}})}{dt} = 0 = \frac{d\hat{\mathbf{T}}}{dt} \cdot \hat{\mathbf{T}} + \hat{\mathbf{T}} \cdot \frac{d\hat{\mathbf{T}}}{dt} = 2 \cdot \hat{\mathbf{T}} \cdot \frac{d\hat{\mathbf{T}}}{dt}$$

The right hand side of this equation can only equal zero if the time derivative of the tangential vector is perpendicular to the tangential vector itself. Hence, the direction of the time derivative of the tangential vector points into the direction of the normal vector. With that, we can define the unit normal vector $\hat{\mathbf{N}}$ as the time derivative of the tangent vector divided by its absolute value:

$$\hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{dt} / \left\| \frac{d\hat{\mathbf{T}}}{dt} \right\| \quad \text{or} \quad \frac{d\hat{\mathbf{T}}}{dt} = \left\| \frac{d\hat{\mathbf{T}}}{dt} \right\| \cdot \hat{\mathbf{N}}$$

With that, we can now write the acceleration vector as the sum of its tangential and normal components

$$\mathbf{a}(t) = \mathbf{a}_{\hat{\mathbf{T}}} + \mathbf{a}_{\hat{\mathbf{N}}} = \frac{dv}{dt} \cdot \hat{\mathbf{T}} + v \cdot \left\| \frac{d\hat{\mathbf{T}}}{dt} \right\| \cdot \hat{\mathbf{N}}$$

As one learns in vector calculus, the arc length $s(\tau)$ of a curve defined between two points a and b (fig. 2.5), measuring the curve length between a and a point τ on the curve where $a \leq \tau \leq b$, is given by

$$s(\tau) = \int_a^\tau \|\dot{\mathbf{r}}(t)\| dt$$

With $\dot{\mathbf{r}}(t) = \mathbf{v}(t)$, the differential form of this equation then becomes

$$ds(t) = v(t) dt$$

Rewriting this equation one obtains

$$v(t) = \frac{ds(t)}{dt}$$

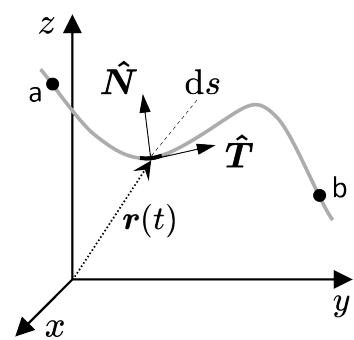


Fig. 2.5

Instead of measuring velocity with respect to a fixed coordinate frame, $v(t) = ds(t)/dt$ gives the absolute value of the point mass velocity with respect to its trajectory as it measures the infinitesimal arc segment ds traversed in the time interval dt . Multiplying this absolute velocity value with the tangent unit vector $\hat{\mathbf{T}}$ will then give the velocity vector \mathbf{v} . Now we rewrite the norm of the time derivative of the tangential unit vector to obtain

$$\left\| \frac{d\hat{\mathbf{T}}}{dt} \right\| = \left\| \frac{d}{ds} \cdot \frac{ds}{dt} \right\| = \left\| \frac{d\hat{\mathbf{T}}}{ds} \right\| \cdot v$$

Vector calculus tells us that the norm of the derivative of the unit tangent vector with respect to arc length is the absolute value of the curvature $\kappa(s)$ of the trajectory at the position of the point mass, the inverse of which is the radius of curvature $R(s)$:

$$|\kappa(s)| = \frac{1}{R(s)} = \left\| \frac{d\hat{\mathbf{T}}}{ds} \right\|$$

With that, the decomposition of the acceleration vector into a tangential and normal component becomes

$$\mathbf{a}(t) = \mathbf{a}_{\hat{\mathbf{T}}} + \mathbf{a}_{\hat{\mathbf{N}}} = \frac{dv}{dt} \cdot \hat{\mathbf{T}} + \frac{v^2}{R(s)} \cdot \hat{\mathbf{N}} \quad (2.4)$$

The second term, the normal component of acceleration the point mass is subject to, is at any position on the point mass trajectory always directed towards the center of curvature at that point. And its magnitude is for a given velocity inverse proportional to the radius of curvature at any given point of the trajectory.

2.2.1 The Kinematics of Planetary Motion

Kinematics provides the tools which helps us describe how a point mass moves on its trajectory curve by tracking and analyzing its position vector $\mathbf{r}(t)$, its velocity $\mathbf{v}(t)$ and its acceleration $\mathbf{a}(t)$, but it does not provide an explanation as to what makes a point mass move in the first place. Take for example the motion of the planets. As far as we can look back in recorded history, humans have sought to understand just how planets move and why they move the way they do.

While in Galilei's time it was still completely unknown what kind of force kept planets moving in their orbits, scientists had been gathering increasingly more accurate data of planet movements. This is what eventually enabled Johannes Kepler (1571 - 1630), with the help of Tycho Brahe's (1546 - 1601) data, to recognize that planets circle the Sun on elliptic orbits. To this day, high school physics students are introduced to Kepler's three laws but those laws tell us nothing about the force which causes planets to move in such

a way that they obey Kepler's laws. Kepler's three laws are likely the best example for what kinematics can achieve, nothing less than a correct understanding of the motion of planets. Kepler tried to understand what the force making the planets move on elliptical orbits around the Sun could look like. However, because he assumed its strength to be inverse proportional to the distance of a planet from the Sun, he did not succeed. It would only be a generation later when Isaac Newton's law of gravitation finally could explain what makes planets move according to Kepler's laws.

In the eyes of many of Galilei's and Kepler's contemporaries, understanding the motion of planets was still not so much a question as to what kept planets in their orbits but rather who moved those planets. That will sound ridiculous to (most) modern ears but this was only an admission that there was no physical explanation as to what made planets move the way they did. Angels supposedly moving the planets around, which may have been the favorite hypothesis in sixteenth century Western Europe, was possibly just as good an explanation to many as the prime mover which the Greek philosopher Aristotle had postulated some two thousand years earlier. Helped by the great Alexandrian mathematician and astronomer Claudius Ptolemaeus (c.100 - c.170), Aristotle's view would dominate the Western European conception of the universe once Ptolemy's work became available through translations. Aristotle's prime mover essentially morphed into a divine Christian force and angels pushing around planets were no less divine agents than the gods of antiquity doing the job before them. With that, divine forces such as angels continued to be viable explanations as to what kept the planets moving for many more centuries. Fig. 2.6 shows a simple sketch of the model of the universe which originated with Aristotle. Earth was viewed as sitting at the center of a number of so-called celestial spheres to which the respective planets were attached. In this universe the Moon was not an Earth satellite but a planet in its own right circling Earth on the innermost celestial sphere. Then followed the spheres of the known planets of which there were five in the sequence Mercury, Venus, Mars, Jupiter and Saturn sitting on celestial spheres two through seven. Then came the

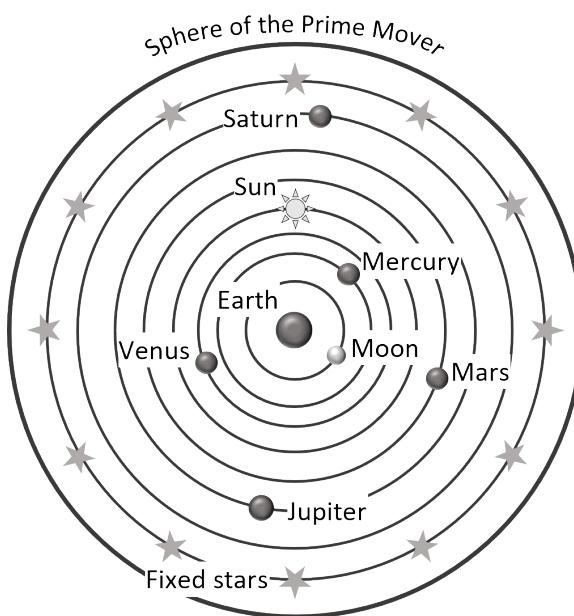


Fig. 2.6

fixed stars, thought to be attached to sphere number eight, followed by celestial sphere number nine, the sphere of Aristotle's prime mover. Ptolemy's astronomical world view, following Aristotle, kept Earth at the center of the universe and this worldview, reinforced by Christian dogma, held sway for some fifteen hundred years. Ptolemy, by the way, and others after him thought planets had a soul of their own which kept them in motion. The Aristotelian-Ptolemaic worldview only began to be seriously contested in the fifteenth century when in 1543 Nicolaus Copernicus (1473 - 1543), on his death bed, published his *De revolutionibus orbium coelestium* (*On the Revolutions of the Heavenly Spheres*).

In 1600 Kepler became Tycho Brahe's assistant in Prague where the latter held the position of imperial court astronomer of Emperor Rudolph II. It fell to Kepler, who became Tycho Brahe's successor in 1601, to complete the astronomical tables the emperor has commissioned from Tycho Brahe. It took him almost thirty years to finish this work which was eventually published in 1627 and is known today as the *Rudolphine Tables*, the star catalog containing the most accurate position measurements of stars and planets at the time. It was the work on these tables, built on Tycho Brahe's original compilations, which provided the basis for Kepler's seminal discoveries. After studying the planetary motion of Mars for a decade, he published *Astronomia Nova* (*New Astronomy*) in 1609. This work was nothing short of revolutionary as it did away with celestial spheres to which the movements of planets were thought to be tied until then and introduced the concept of planets moving freely through space. *Astronomia Nova* included the first two of Kepler's laws. After another decade of observations, Kepler published what became his third law in his 1619 treatise *Harmonices Mundi* (*Harmonies of the World*).

Galilei and Kepler differed in their approaches to astronomy as for the most Galilei was an observational while Kepler was much more of a mathematical astronomer. Galilei's arguments, derived from his observations, were more qualitative while Kepler's reasoning was more quantitative. South of the Alps, Galilei published his defense of the heliocentric worldview in his short treatise *Sidereus Nuncius* (*The Starry Messenger*) just one year after Kepler had completed his *Astronomia Nova*. The two also engaged in a limited correspondence sharing some of their views which at the time were still unacceptable to many of their academic colleagues. In 1610, Galilei completed the work on his famous *Dialogo*, which would finally bring down the inquisitions full force on him. The Catholic Church may have deemed Galilei's work as the greater threat to its geocentric worldview but Kepler's work was far more dangerous. Kepler's laws gave quantitative descriptions of planet movements, backed up by the most accurate astronomical data at the time. Here is what Kepler's three laws state:

Kepler's three laws

1. The planets move in elliptical orbits with the Sun at one focus.
2. The speed of a planet changes such that the time elapsed between two positions is always proportional to the area swept out in its orbital plane between those.
3. The ratio of the cube of the length of the semi-major axis of each planet's orbit to the square of time of its orbital period is the same for all planets.

Fig. 2.7 shows the elliptic orbit of a fictional planet with the Sun at one of the focal points. Most planetary orbits are much less elliptical than shown here. Earth's orbit around the Sun is much closer to a circle than to an ellipse. If drawn accurately, using the real relative length difference between the minor and major axis of Earth's elliptic orbit, the curve would very much look like a circle. In fact, on the largest scale that for practical matters is available in a book it would look indistinguishable

from a circle. Hence, almost all textbook drawings show Earth's orbit much more elliptical than it really is because otherwise it would convey the impression that Earth was orbiting the Sun in a circular instead of an elliptic orbit. A parameter called eccentricity measures how elliptic an ellipse really is. It is defined as the ratio between the length difference of major and minor axis and the length sum of those two axes. If they are of the same length, then the eccentricity is zero and we have a circle. The larger the length difference between those two axes, the closer the value of the eccentricity will be to one, but it will never reach it because a conical section with an eccentricity of one does not describe an ellipse anymore but a parabola. The eccentricity of the orbit of Earth is currently 0.017, so it is very close to a circle. The major axis of Earth's elliptic orbit around the Sun is less than 0.015% longer than its minor axis. Among the planets in our solar system, the orbit of Mercury, the planet closest to the Sun, has the largest eccentricity with a current value of 0.206 implying that its minor axis is about 2.19% shorter than its major axis. However, the first detailed observational data for Mercury became only available with the improved instrumentation of the eighteenth century. Among the planets known in Kepler's time, Mars has the second largest eccentricity with a current

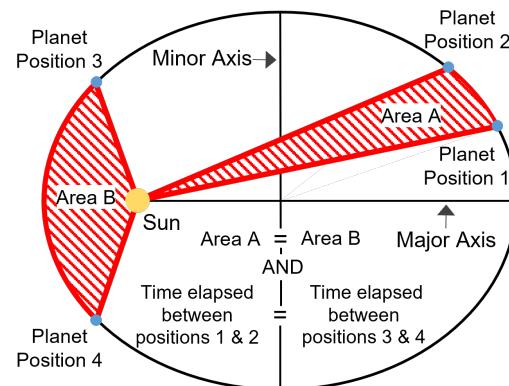


Fig. 2.7

value of 0.093, implying that the minor axis is about 0.44% shorter than the major axis of its elliptical orbit.¹ So Kepler picked well and his decade long gathering and evaluation of Mars orbital data was rewarded with insights that would change how humans perceive the cosmos.

2.3 Simple Motion

The magnitude of the acceleration's normal component in eq. (2.4) is tied to the shape of the trajectory on which the point mass traverses space. It is inverse proportional to the curvature $R(s)$ of the point mass trajectory. The greater the curvature $R(s)$ the smaller the acceleration's normal component will be in eq. (2.4) and vice versa. Under the umbrella of simple motion we will look at a few limit cases of eq. (2.4).

The limit cases of interest are when either the tangential or the normal acceleration component in eq. (2.4) vanishes. That is for example the case with $R(s) \rightarrow \infty$ because then the normal component of the acceleration vector vanishes and the shape of the curve becomes a straight line. If however the curvature $R(s)$ is a constant, that is every part of the point mass trajectory has the same curvature, then we have another simple limit case where the tangential component of acceleration vanishes and the point mass motion becomes circular.

2.3.1 Accelerated Linear Motion

Physicists refer to the case where $R(s) \rightarrow \infty$ and the point mass moves in a straight line as accelerated linear motion. In that case the velocity vector and the acceleration vector are either parallel or anti-parallel, i.e., they do not have to point in the same direction. If they point in the same direction, i.e., they are parallel, then we have accelerated linear motion; if they point in opposite directions, i.e., they are anti-parallel, then we have of course decelerated linear motion. Here we will consider the case where velocity and acceleration vector point in the same direction, that is in the direction the point mass is moving. To make things simple we will align one of the coordinate axes with the direction of motion. Consider for example such motion along the z -axes, that is $a_x = a_y = 0$ and $\hat{\mathbf{T}} = (0, 0, 1)$. The only remaining acceleration component is a_z with

$$a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

¹Of the planets discovered in later times, only Pluto has a greater eccentricity than Mars but because Pluto lost its planetary status in 2006, Mars again is planet number two in our solar system in terms of highest eccentricity.

In the case of

$$\frac{dv_z}{dt} = a_0 = \text{const} \quad \text{hence} \quad dv_z = a_0 dt$$

we are looking at a case of uniformly accelerated motion. For the velocity $v_z(t)$ we obtain by simple integration

$$v_z(t) = a_0 t + v_0 \quad \Rightarrow \quad dz = (a_0 t + v_0) dt$$

where v_0 is the velocity of the point mass at time $t = 0$. Integrating one more time we get for the point mass trajectory

$$z(t) = \frac{a_0 t^2}{2} + v_0 t + z_0 \quad (2.5)$$

where z_0 is the location of the point mass at $t = 0$. The first component in eq. (2.5) describes the accelerated motion of the point mass while the second component describes its uniform motion.

2.3.2 Inclined Plane and Free Fall Motion

Galilei was the first physicist to experimentally study motion in a systematic way. With a ruler for measuring the distance a ball had rolled down an inclined trough and using the beat of his pulse to mark equidistant time intervals, he discovered the basic relationship between distance D traveled and time t elapsed: $D \propto t^2$. That is exactly the relationship between distance traveled and time elapsed we just found for uniformly accelerated motion. Galilei's interest was to study the motion of free fall but in his time that was very difficult. There were just no instruments available to measure the rather short time it takes objects to reach ground, even when dropped from the highest towers that were accessible for such experiments back then. The inclined trough (fig. 2.8) allowed Galilei to study free fall in slow motion as he could adjust the incline such that it actually became possible to make sufficiently accurate time measurements. The force accelerating the ball in fig. 2.8

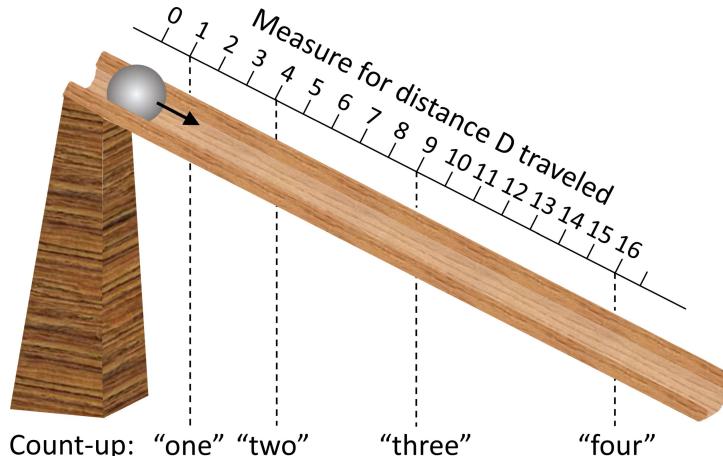


Fig. 2.8

is only the component of the weight force (gravity) parallel to the inclined trough, and that force is furthermore counteracted by friction; without friction the ball would not roll but slide. Reducing the incline angle α reduces the parallel component of the weight force and eventually the ball will stop moving. Because of the friction force, that however happens even before the incline angle α has reached zero. In his studies of acceleration Galilei found that regardless of the size of the ball (large vs small) or the ball material (heavy vs light) the acceleration due to gravity remained constant. He concluded that gravity accelerates all objects on Earth's surface at the same rate regardless of their size, shape, or mass.

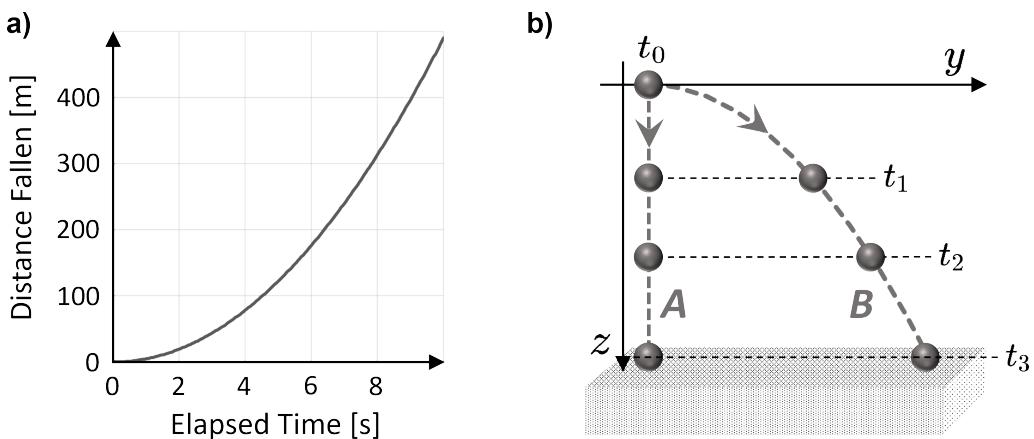


Fig. 2.9: (a) Distance fallen vs time elapsed for a free falling point mass. (b) Falling ball without (**A**) and with (**B**) initial horizontal velocity component.

Because of Earth's atmosphere this is not what we usually experience if we for example drop a feather and a marble from the same height, but it is exactly what happens when this is done in vacuum. For the acceleration due to gravity on Earth's surface Galilei's value of $g = 9.82 \text{ m s}^{-2}$ came very close to the modern-day value. The average value of g on Earth's surface $g = 9.81 \text{ m s}^{-2}$. Because Earth is not a perfect sphere, g is a little higher than that at the poles and a little lower at the equator. Locally, geology also leads to variations. Fig. 2.9a shows the graph of distance fallen vs time elapsed of a falling body for the average value of g .

As illustrated in eq. (2.5), accelerated motion and uniform motion are additive. A sufficiently heavy ball dropped vertically from a given height and a like ball launched simultaneously from the same height but with an additional horizontal velocity component will reach the ground at the same time. In a vacuum, this will always be the case regardless of the respective weight or size of the two objects. Both objects will always hit the ground at the same time, with the object that has the added horizontal motion component just landing farther away. For the trajectory of the object with the additional horizontal

motion component we look at the case shown in fig. 2.9b where the ball with trajectory B has a horizontal velocity component v_y moving it uniformly in positive y -axis direction with its y position given by:

$$y(t) = v_y t$$

The accelerated motion component of the ball on trajectory B is the same as the one for the ball on trajectory A . The tangent vector for this acceleration component in the negative z -axis direction is $\hat{\mathbf{T}} = (0, 0, -1)$, hence

$$z(t) = -\frac{g}{2}t^2$$

Using the above two equations to eliminate the time variable t we obtain the shape of the trajectory curve B

$$z(y) = -\frac{g}{2v_y^2}y^2$$

Hence, trajectory B of the ball with the added horizontal motion component is a parabola. As long as we ignore air resistance, the trajectory of any free falling body thrown in a homogeneous gravity field will always describe a parabola.

2.3.3 Ballistic Motion

If one adds to a free falling object not just a horizontal velocity as in the just discussed example but also a vertical velocity component, then one speaks of a ballistic motion in Earth's gravity field. Ballistic motion describes the throwing curve of an object, be it that of a football, a stone, or a projectile launched from a cannon. Here we will assume the object is thrown with an initial velocity \mathbf{v} in a constant gravity field and neglect things such as air resistance. To track point mass motion in two directions, x and z , it is convenient to use vector notation. Selecting the initial conditions such that the object is thrown at $t = 0$ from the height z_0 at an inclination α (see fig. 2.10) we have

$$\mathbf{r}_0 = \begin{pmatrix} 0 \\ z_0 \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} v_0 \cos \alpha \\ v_0 \sin \alpha \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

With \mathbf{a} being constant, integrating $d\mathbf{v} = \mathbf{a}dt$ and using the initial condition for $\mathbf{v}(t = 0) = \mathbf{v}_0$ gives

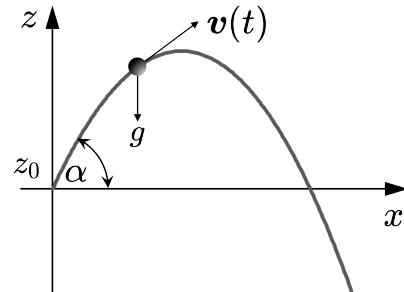


Fig. 2.10

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t = \begin{pmatrix} v_0 \cos \alpha \\ v_0 \sin \alpha - gt \end{pmatrix}$$

Integrating $d\mathbf{r} = \mathbf{v}dt$ and using the initial condition for $\mathbf{r}(t = 0)$, i.e., $x(0) = 0$ and $z(0) = z_0$, gives

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a}t^2 = \begin{pmatrix} v_0 t \cos \alpha \\ z_0 + v_0 t \sin \alpha - \frac{1}{2} g t^2 \end{pmatrix}$$

The time variable can be eliminated from $z(t)$ by using $x(t) = v_0 t \cos \alpha$ which gives the trajectory of the ballistic motion shown in fig. 2.10

$$z = z_0 + \tan \alpha \cdot x - \frac{g}{2v_0^2 \cos^2 \alpha} \cdot x^2 \quad (2.6)$$

The maximum height z_{max} the object will reach follows from solving $dz/dx = 0$ to determine $x_{z_{max}}$ and inserting it into eq. (2.6):

$$x_{z_{max}} = \frac{v_0^2}{2g} \sin 2\alpha \quad \Rightarrow \quad z_{max} = z_0 + \frac{v_0^2}{2g} \sin^2 \alpha$$

The object will hit the ground at $z(t_{tot}) = 0$ for a total flight time of

$$t_{tot} = \frac{1}{g} \left[v_0 \sin \alpha + \sqrt{v_0^2 \sin^2 \alpha + 2gz_0} \right]$$

To determine the distance the throw covers we insert t_{tot} into $x(t)$ and obtain

$$x(t_{tot}) = \frac{v_0}{g} \left[v_0 \sin \alpha + \sqrt{v_0^2 \sin^2 \alpha + 2gz_0} \right] \cos \alpha$$

Obviously, throwing the object at a higher initial speed v_0 will land it farther away, as will of course throwing it from a higher altitude. But at what angle do we need to throw the object to cover the greatest distance? To answer this question we need to find the extreme value of $x(t_{tot})$ as a function of α that means solving $\partial x(t_{tot})/\partial \alpha = 0$ for α . Leaving this calculation as an exercise we just look at the result which is that the throwing distance becomes maximal for

$$\sin \alpha_{x_{max}} = \frac{1}{\sqrt{2 + \frac{2gz_0}{v_0^2}}}$$

If we chose for simplicity $z_0 = 0$, that is the object is thrown on a level field where starting and landing positions are at the same altitude the angle where the throwing distance becomes maximal is 45° . Knowing any two of the three variables $x(t_{tot})$, v_0 , and α one can then determine the value of the third variable. For example, knowing α for a cannon and measuring how far the cannon ball traveled, the muzzle velocity can be calculated from the equation for $x(t_{tot})$. However, that only would apply if the cannon

were fired on a planet with no atmosphere. In the real world there are other factors such as air resistance to consider which will reduce $x(t_{tot})$ and change the trajectory shown in fig. 2.10.

2.3.4 Uniform Circular Motion

If not the normal component but the tangential component of the acceleration vector in eq. (2.4) vanishes we have another simple motion, the so-called uniform circular motion of a point mass. In this case the radius of curvature in the normal component becomes a constant, i.e., $R(s)$ equals the radius R of the circle the point mass is moving on. The absolute value of the point mass velocity is in this case constant (tangential acceleration is zero) but the direction of the velocity vector keeps changing.

Polar coordinates are of course the natural choice for describing circular motion:

$$x = |\mathbf{R}| \cos \varphi, \quad y = |\mathbf{R}| \sin \varphi$$

For uniform circular motion, the angular velocity, defined as the incremental increase of the angle φ with time, is constant:

$$\frac{d\varphi}{dt} = \omega = \text{const}$$

hence

$$\varphi(t) = \omega t$$

The angular velocity ω , having the dimension s^{-1} , is often referred to as angular frequency. As the point mass rotates counterclockwise, the direction of the radius vector $\mathbf{R}(t)$ changes while its absolute value R remains constant. The point mass coordinate positions and its corresponding coordinate vector as a function of time are given by

$$\begin{aligned} x(t) &= R \cos \omega t \\ y(t) &= R \sin \omega t \end{aligned} \quad \text{or} \quad \mathbf{R}(t) = R \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

The time it takes the radius vector $\mathbf{R}(t)$ to complete one full rotation until it points again into the same direction after sweeping the angle of 2π is

$$T = \frac{2\pi}{\omega}$$

T is called the period of uniform circular motion. The rotation frequency, i.e., the number of 2π -sweeps per second, is given by the inverse of the period T

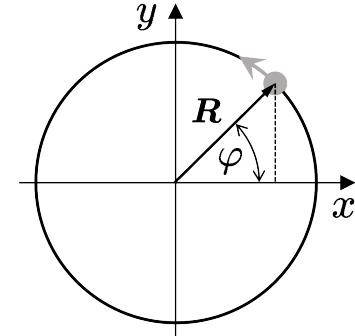


Fig. 2.11

$$f = \frac{1}{T} = \frac{2\pi}{\omega}$$

The direction and the absolute value of \mathbf{v} are given by

$$\mathbf{v} = \dot{\mathbf{R}} = R \begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix} \cdot \omega \quad \text{and} \quad v = \omega R$$

Inserting the absolute value $v = \omega R$ into eq. (2.4) the acceleration for uniform circular motion becomes ($dv/dt = 0$)

$$\mathbf{a}(t) = \frac{v^2}{R} \cdot \hat{\mathbf{N}} = \omega^2 R \cdot \hat{\mathbf{N}} = \omega^2 R \cdot \begin{pmatrix} -\cos \omega t \\ -\sin \omega t \end{pmatrix}$$

Here we have identified

$$\hat{\mathbf{N}} = \begin{pmatrix} -\cos \omega t \\ -\sin \omega t \end{pmatrix}$$

as the unit normal vector of acceleration. $\hat{\mathbf{N}}$ points in the opposite direction of \mathbf{R} , that is towards the center of the circle. We have not discussed forces yet but as we will see in the next chapter, the force acting on a point mass is given by the product of its mass m and its acceleration a . The force associated with uniform circular motion, the so-called centripetal force, points always towards the center of the circular motion and its magnitude is therefore given by

$$F = m \frac{v^2}{R} = m \omega^2 R$$

As shown in fig. 2.12, plotting the Cartesian coordinates $x(t)$ and $y(t)$ of a point mass in uniform circular motion against ωt , quickly reveals that uniform circular motion is a superposition of two harmonic oscillations. The velocity components of these two harmonic oscillation are

$$\dot{x} = -\omega R \sin \omega t \quad \text{and} \quad \dot{y} = \omega R \cos \omega t$$

with the corresponding acceleration components

$$\ddot{x} = -\omega^2 R \cos \omega t = -\omega^2 x \quad \text{and} \quad \ddot{y} = -\omega^2 R \sin \omega t = -\omega^2 y \quad (2.7)$$

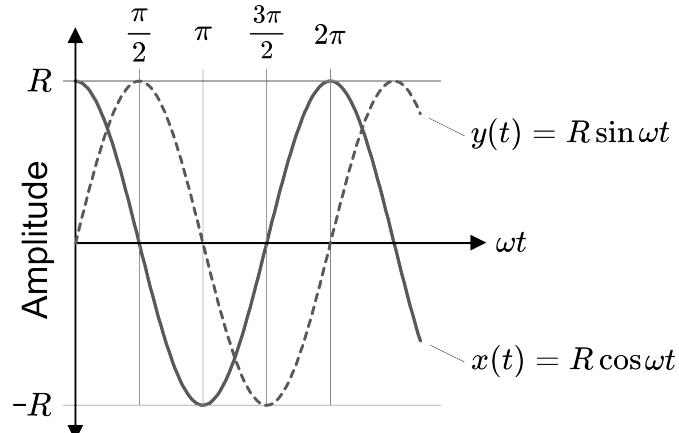


Fig. 2.12

proportional and opposed to the respective amplitude of oscillation. That uniform circular motion and harmonic motion are identical is easy to demonstrate experimentally. The pedaling of a cyclist as viewed from behind in the night will show the pedal reflectors moving up and down in an harmonic motion. By simple projection one can show that uniform circular motion is a harmonic motion. Many of us have seen this demonstrated in high school physics or science courses. But with our modern equipment we can do better than that. Instead of projecting the movement of an object in circular motion on a screen we can let this object project itself. As illustrated in fig. 2.13a, by simply attaching a small laser to the object in uniform circular motion we can have the object itself project its motion onto a high resolution pixelated light screen. As the laser dot moves vertically up and down on the pixelated detector we read out the respective pixels along that vertical at a frequency much greater than the angular velocity ω of the uniform circular motion. The resulting pixel values along this vertical we then display in real-time on a computer screen as a function of time as sketched in fig. 2.13b.

The equations in eq. (2.7) are equations of motion for harmonic oscillators. Harmonic oscillations are of great importance in many areas of physics and this is why we will take a closer look at them in section 10.1. But here we now have to move on from the kinematic study of motion to the dynamic study of motion.

From Kinematics to Dynamics

Kinematics gives a purely geometric description of point mass motion in terms of the point mass position vector, its velocity and its acceleration. There is no need in kinematics to know the forces which cause an object to move. Dynamics, a word which derives from the ancient Greek word for force, refers to the study of motion under the influence of forces. With kinematics we can learn how objects move, with dynamics we seek to understand why they move.

With regard to the latter it is not necessarily obvious which kinds of object motions actually do require a cause. This already became apparent to Galilei when he realized that uniform linear motion, that is motion for which both the velocity direction \mathbf{v} and

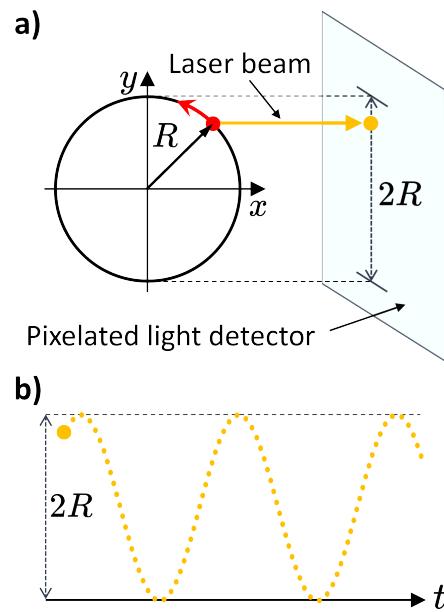


Fig. 2.13

the magnitude of velocity v do not change, does not require a cause. This discovery led him to formulate his law of inertia which states that a body left to itself will move uniformly in a linear direction. One should note that this also implies that a body at rest, i.e., $v = 0$, is just a special case of Galilei's law of inertia. Before Galilei, the general assumption had been that all horizontal motion required a direct cause. His experiments convinced Galilei that a body in motion would remain in motion unless a force (such as friction) caused it to come to rest.

Today, most will know Galilei's law of inertia as Newton's first law of motion but in this case, Galilei was first. Realizing that does not diminish in any way the contributions Newton made to our understanding of how objects move under the influence of forces to which we will turn next. Given Newton's many disputes with contemporaries regarding the primacy of his discoveries, like for example with Leibniz with respect to differential calculus, some may assume that Newton may have disputed Galilei's primacy regarding the law of inertia. Far from that, Newton gave Galilei full credit for his discoveries.

3. Newton's Laws of Motion

Newton was born in the year Galilei died. It was Galilei, who first recognized the significance of inertia and it was Newton studying Galilei's experimental work, who first realized that the concept of inertia was the fundamental principle on which to build an understanding of the dynamics of moving objects. The principle of inertia is the first of Newton's three laws of motion:

Newton's three laws of motion

1. ***Principle of inertia:*** "*Every body perseveres in its state of being at rest or of moving uniformly straight forward except insofar as it is compelled to change its state by forces impressed*". In modern form: Every body remains in its state of rest or in a state of linear motion at a constant speed unless acted upon by a force.
2. ***Principle of action:*** "*The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed*". In modern form: The change in the state of motion of any body is always proportional to the magnitude of the impacting force and it happens in the direction into which the force points.
3. ***Principle of reaction:*** "*To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts*". In modern form: Reaction always equals action. If the force \mathbf{F} acting on a first body has its origin in a second body then the force acting on the second body equals $-\mathbf{F}$.

Newton's second law, the principle of action, states that the force acting on a body is proportional to the acceleration the body experiences with the proportionality constant being the mass of the body:

$$\mathbf{F} = m_t \cdot \mathbf{a} \quad (3.1)$$

The mass of the object on which the force acts is labeled here with a subscript t , indicating that the mass which resists being accelerated or decelerated by the force is the so-called inertial mass.¹ Newton's third law states that for any two objects, for simplicity labeled objects 1 and object 2, the force \mathbf{F}_{12} which object 1 exerts on object 2 is equal and opposite to the force \mathbf{F}_{21} which object 2 exerts on object 1 (actio = reactio):

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \quad (3.2)$$

3.1 Force, Momentum and Mass

Before Newton, physics did not really have a solid concept of force, what physicists had were causes they identified with forces but not a unifying force concept. Similarly, with the concept of momentum. All that changed with Newton's three laws which gave physics the concepts of momentum and force which allowed physicists to correctly describe the movement of objects subject to known forces. More than that, they also gave physics its first law of conservation, the conservation of momentum. But let's take this one step at a time and look at how we practically understand forces, what do we mean when we speak of a force, what is its definition? We can for example pick up an object and get a sense of how heavy that object is. By doing this with several objects we can get an understanding of how the relative masses of objects compare. We can ask different people to throw the same object through the air then check how far the object will travel before hitting the ground again. In this way we get a sense of who is a stronger and who is a weaker thrower. You see, where this is going, we learn about something like weight and muscle force by comparison. Not all forces are of course alike.

We all experience various forces in our daily lives and those fall in two general categories, static forces and dynamic forces. Our distant ancestors were no less familiar with those two kinds of forces than we are today. They for example experienced the first kind when supporting the weight of something like a heavy stone and the second kind when they

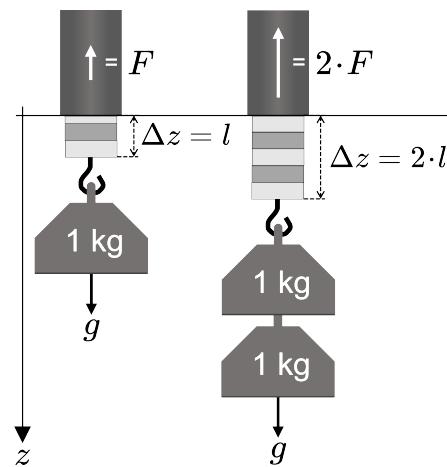


Fig. 3.1

¹The German word for inertia is “Trägheit” and because the index i is frequently used for numbering things it is better to use the t .

were throwing an object, like for example a spear. Without some understanding of static forces, humankind could never have achieved the engineering feats of antiquity or built the cathedrals of the middle ages. Static forces are weight forces. Their understanding allowed engineers for millennia to lift enormous weights through the use of levers and construct buildings which could carry their own enormous weight. A weight hanging from a spring balance (also called a Newton meter) is a simple example for a static force many will likely be familiar with.

Hanging a mass of 2 kg from such a spring balance (fig. 3.1) exerts twice the force of a 1 kg mass and will cause the spring to extend twice as much. This linear dependence between spring force and spring balance extension is known as Hooke's law, named after Robert Hooke (1635 - 1703), Newton's contemporary and sometime rival. The proportionality constant k depends on the spring material and because the spring force opposes the extension of the spring there is a minus sign in front of k in eq. (3.3). The extension of the spring balance is due to gravity's pull on the respective mass hanging from the spring balance, i.e., $F_{Gravity} = m_s \cdot g$ where m_s in the example of fig. 3.1 equals 1 kg and 2 kg, respectively. The mass of the object on which the force of gravity acts is labeled here with a subscript s , indicating that we are dealing here with the so-called heavy mass.² The force of gravity is the weight force which is what we measure when we weigh how heavy or how light an object is. The spring force is opposed to the force of gravity:

$$F_{Spring} = -k\Delta z = -F_{Gravity} = -m_s \cdot g \quad (3.3)$$

Instead of hanging weights from the spring balance one could of course also orient the spring balance horizontally and manually pull on it by force of hand to produce the same extensions as shown in the above example. In that case, it would be the muscle force of an arm balancing the spring force and not the force of gravity. Using their implicit understanding of static forces, i.e., forces acting on bodies at rest because of their masses, humans were able to build sophisticated levers and pulleys allowing them to lift and move weights much heavier than what human or animal muscle otherwise could achieve.

Understanding dynamic forces, i.e., forces associated with a body's motion, turned out to be much more challenging. Just like his views on astronomy, Aristotle's force concept influenced the thinking of natural scientists trying to understand the movement of bodies once his respective works had been rediscovered in twelfth-century Europe. According to Aristotle, the movement of a body required the constant application of a force. For example, if someone throws a ball, once it is in the air, Aristotle believed that some kind

²The German word for heavy is "schwer", hence the label s .

of force was required at every point of its curve to keep it moving. For Aristotle the explanation was that the object thrown continued to be moved by the surrounding air once it had left the hand of the person throwing it. This would be true for a body moving at constant speed without a force impacting its movement as well as for accelerated motion. The former is equivalent to a body at rest and hence only static forces apply while the latter is of course due to dynamic forces. From a practical standpoint, scholars of antiquity of course differentiated between what we would call today static and dynamic forces; they just did not realize that a body moving at constant speed without a force changing its direction was different from a body subjected to accelerated motion.

Scholars continued to struggle with Aristotle's force concept and it was only some eight hundred years later that they came up with the concept of a transmitted force. Among them were Aristotle scholars such as John Philoponus (490 - 570) or for example the very Simplicius (490 - 560) who became one of the three protagonists in Galilei's *Dialogo* in which he was the one arguing for the geocentric Aristotelian universe vs Galilei's heliocentric worldview. Men like Philoponus argued against Aristotle's position with respect to what kept a projectile moving through the air once it had left the hand of the thrower. They believed it impossible to impress a force on an object moving through the air and that the force keeping the projectile moving was transmitted onto the object in the moment it was thrown whence it became a property of the projectile. In the eleventh century, Ibn Sina (980 - 1037), in the West better known as Avicenna, also argued this view, believing that an object would keep moving until this impressed force, which he referred to not as a force but an inclination passed on to the projectile, was spent. But he also held the opinion that a projectile moving in vacuum would not stop moving unless it is acted upon, the very discovery Galileo would make some six hundred years later.

Towards the end of the scholastic period the fourteenth century philosopher Jean Buridan (c. 1301 - 1359/62) developed a theory of the motive force, i.e., the impressed force, which he called impetus and argued that $impetus = weight\ of\ projectile \times velocity$. To many, this definition of impetus will sound very similar to the modern concept of momentum and in some ways it is. In Buridan's view, an object in motion could only be stopped by the resistance of the air and the weight of the body which both oppose its impetus. How the latter could be the case with the impetus being proportional to the weight of the projectile is another story. According to scholastic scholars such as Buridan, the impetus was essentially transferred to the projectile by the so-called first mover, which is the person who launches the projectile. In addition to Buridan developing a mathematical theory of impetus, the fourteenth century also witnessed another step towards modernity: the

use of graphs to describe the laws of motion, a development pioneered by the philosopher Nicole Oresme (c. 1320/25 - 1382).

This was essentially the concept of force which Galilei inherited and which he began to change with his understanding of uniform linear motion, a motion which does not require a force at all. Galilei's discovery amounted not so much to a better understanding of the concept of force but of what inertia actually may be. The uniform linear motion, or inertial motion, of a body at constant speed v (including $v = 0$) will continue unless a force changes that. It is this insight on which Newton developed the modern understanding of the force concept.

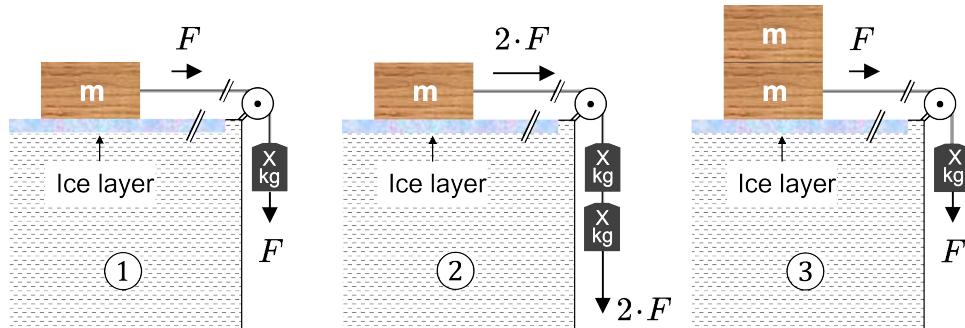


Fig. 3.2: A drum converts a vertical weight force into a horizontal force acting on blocks of wood sliding on a layer of ice without friction forces ($m \gg 2X \text{ kg}$).

Fig. 3.2 shows a simple way to study the relationship between force and acceleration using well known static weight forces to uniformly accelerate an object horizontally on a friction-less surface. The horizontal stretch must be of course much longer than shown here so one can accurately measure the velocity of the wood blocks as a function of time. In uniformly accelerated motion the velocity of an object increases by the same amount every time interval. Plotting velocity as a function of time produces a straight line, the slope of the line giving the value of the constant acceleration applied. For the three cases in fig. 3.2 the result of doing that is shown in fig. 3.3 where for convenience we have scaled things such that the slope in case ① of Fig. 3.3 equals 1. When we then look at the other two cases we will observe that the slope for case ② equals 2 while the slope for case ③ equals 0.5. Comparing the slopes of case ① and case ② in fig. 3.3 then tells us that applying twice the force

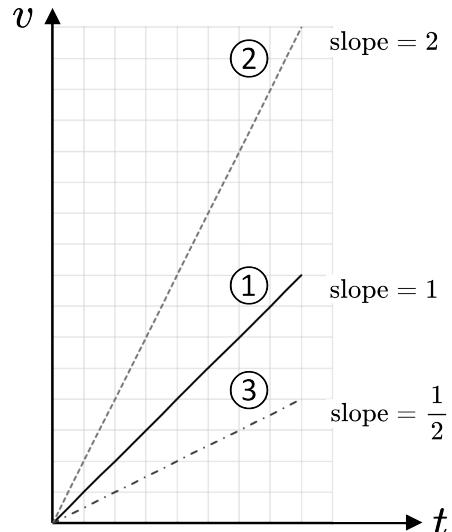


Fig. 3.3

will double the acceleration. From comparing cases ① and ③ it follows that applying the same force to pull the mass $2m$ instead of m halves the acceleration. The conclusion of the first observation tells us that the acceleration is proportional to the force applied while the second observation tells us that the acceleration is inverse proportional to the mass being accelerated:

$$\mathbf{a} \propto \mathbf{F} \quad \text{and} \quad \mathbf{a} \propto \frac{1}{m_t} \quad \Rightarrow \quad \mathbf{a} = \frac{\mathbf{F}}{m_t}$$

The mass is referred to as the inertial mass m_t because the greater the objects mass m_t , the greater the objects inertial resistance will be to acceleration because of $\mathbf{a} \propto m_t^{-1}$. The above equation is of course nothing else but Newton's famous second law of motion

$$\mathbf{F} = m_t \cdot \mathbf{a}$$

Nowhere did Newton write down exactly this equation as he formulated his second law somewhat different from how we do that today. But the meanings of Newton's and our modern version, which already was used by others in Newton's time, are the same. Isaac Newton's *Philosophiae Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy)* published in 1687, laid not only the foundation of classical mechanics but of modern science as we still very much define it today. Newton's force concept considered as a force every impact which led to a change in the state of motion of a body. As indicated above, this was very much the view that Galilei held. For Newton, force was not the property of a body like the impetus of scholastic scholars was but a measure of the strength and direction of interaction between two bodies. The way Newton put it, the change in the state of motion of a body of inertial mass m_t was characterized by the change of its momentum

$$\mathbf{p} = m_t \mathbf{v}$$

which was proportional to the impact of the force \mathbf{F} , or short $\Delta\mathbf{p} \propto \mathbf{F}\Delta t$ where Δt is the time interval for which the force was applied. In modern notation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m_t \mathbf{v})}{dt} = m_t \mathbf{a} \tag{3.4}$$

Here we have assumed that the mass of a body is a constant which in classical mechanics it is (but not in relativistic mechanics). The other new aspect Newton brought to the understanding of forces is that action always equals reaction. Hence, the force Earth's gravity exerts on the proverbial apple falling from a tree, which according to legend inspired Newton to think different about gravity, is equal in magnitude but opposed in direction to the force the apple in turn exerts on Earth.

For what we will discuss next we will drop the index for indicating the inertial mass. According to eq. (3.4) the change in the momentum of a point mass m subject to a force \mathbf{F} during the time interval dt and the respective change of its position during this time interval are given by

$$d\mathbf{p} = \mathbf{F} \cdot dt \quad \text{and} \quad d\mathbf{r} = \mathbf{v} \cdot dt$$

Eliminating dt from those two equations we get the equation

$$\frac{1}{m} \cdot \mathbf{p} \cdot d\mathbf{p} = \mathbf{F} \cdot d\mathbf{r}$$

Using the relationship

$$dp^2 = d(\mathbf{p} \cdot \mathbf{p}) = d\mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot d\mathbf{p} = 2\mathbf{p} \cdot d\mathbf{p}$$

we can rewrite this equation as

$$d\left(\frac{p^2}{2m}\right) = \mathbf{F} \cdot d\mathbf{r} \quad \text{or} \quad dT = \mathbf{F} \cdot d\mathbf{r} \quad (3.5)$$

On the right hand side of this equation we have the product of the constant force \mathbf{F} applied to the point mass to move it by the incremental distance $d\mathbf{r}$. In the next chapter we will see that this product of force times distance just corresponds to the incremental work dW required to move the mass this distance $d\mathbf{r}$. The work expended in this way to move the point mass has led to an incremental increase in the quantity on the left hand side which we will denote with the letter T . This quantity T which is associated with the velocity and the mass of the point mass is called the kinetic energy of the point mass:

$$T = \frac{p^2}{2m} = \frac{mv^2}{2} \quad (3.6)$$

According to eq. (3.6) any point mass m moving with a velocity v possess a kinetic energy T . If the point mass is subject to a force \mathbf{F} then this quantity T will according to eq. (3.5) incrementally change by the amount dT . If a point mass is not subject to a force then T remains constant. The latter is of course owed to Newton's first law which says that a body at rest or in uniform constant motion will stay at rest or in uniform constant motion unless it is subject to a force. Without a force acting on the body its velocity will stay constant and with that also its kinetic energy T will stay constant. What we just found applies also to systems of many point masses, like for example the atoms in an ideal gas where collisions between gas atoms are treated as elastic, which means that no kinetic energy is lost in atom-atom collisions. Hence for an ideal gas the total kinetic energy remains constant as long as no external forces are applied.

3.1.1 Superposition of Forces

Forces are vector quantities, something we all intuitively assume and understand as a force \mathbf{F} clearly is characterized by its strength, i.e., by $|\mathbf{F}|$ and by the direction of the force. This direction into which a force points is what is called its line of action sometimes also referred to as its line of application:

Line of action

The line of action of a force is the straight line that indicates the position of the force in space. It runs through the point at which the force is applied aligned with the direction into which the force vector points.

Two forces \mathbf{F} and \mathbf{F}' are said to be equivalent if and only if

$$|\mathbf{F}| = |\mathbf{F}'| \quad \text{and} \quad \mathbf{M} = \mathbf{M}'$$

where \mathbf{M} and \mathbf{M}' are the torques produced by \mathbf{F} and \mathbf{F}' with respect to an arbitrary reference point O . The torque with respect to a reference point O in space is the vector-or cross product of a force \mathbf{F} acting on a point P and the distance vector \mathbf{r}_{OP} , i.e.,

$$\mathbf{M} = \mathbf{r}_{OP} \times \mathbf{F} \quad \text{and} \quad \|\mathbf{M}\| = \|\mathbf{r}_{OP}\| \cdot \|\mathbf{F}\| \cdot \sin \theta$$

where $\sin \theta$ is the angle enclosed by \mathbf{r}_{OP} and \mathbf{F} . The direction of the torque vector \mathbf{M} is normal to the plane spanned by \mathbf{r}_{OP} and \mathbf{F} and it lies in a rotational axis running through the reference point O .

Torque associated with a line of action

To illustrate what it means that two forces are identical we consider fig. 3.4 which shows a force along a line of action at two different points labeled \mathbf{r}_1 and \mathbf{r}'_1 . The force vectors at these respective positions on the line of action are denoted by \mathbf{F}_1 and \mathbf{F}'_1 with $|\mathbf{F}_1| = |\mathbf{F}'_1|$. For the torque \mathbf{M}'_1 one then calculates:

$$\begin{aligned} \mathbf{M}'_1 &= \mathbf{r}'_1 \times \mathbf{F}'_1 = (\mathbf{r}_1 + \mathbf{r}'_1 - \mathbf{r}_1) \times \mathbf{F}_1 \\ &= \mathbf{r}_1 \times \mathbf{F}_1 + \underbrace{(\mathbf{r}'_1 - \mathbf{r}_1) \times \mathbf{F}_1}_{= 0} = \mathbf{M}_1 \end{aligned}$$

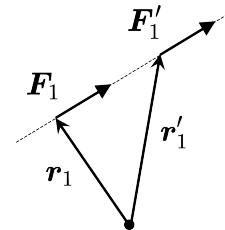


Fig. 3.4

Thus \mathbf{F}_1 and \mathbf{F}'_1 must lie on the same line of action and because of $|\mathbf{F}_1| = |\mathbf{F}'_1|$ it follows that $\mathbf{F}_1 = \mathbf{F}'_1$. For a body which is acted on by two forces pointing in the same direction but whose lines of action are not identical, meaning the forces are parallel, it follows with

$M_1 \neq M'_1$ that there will be a net torque acting on the body. Only force components that lie on the same line of action can be added or subtracted from each other, depending on whether they point in the same direction or not. However, one can move forces along their lines of action to a common point of intersection, i.e., where their lines of action cross. There one can then add the respective force components along a chosen coordinate axis which then becomes the resultant force in this coordinate direction. This process called force reduction is illustrated in fig. 3.5.

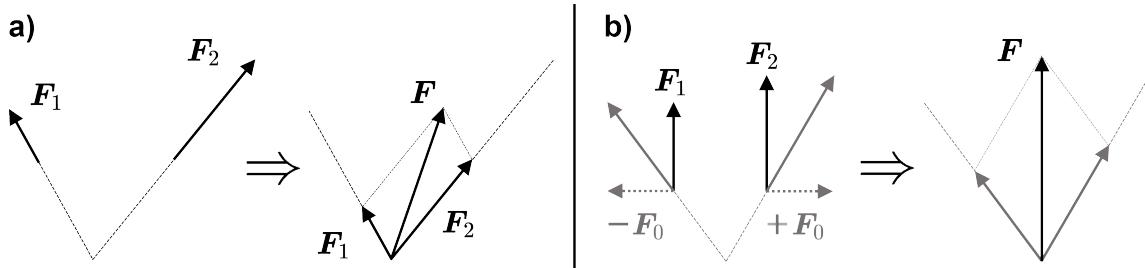


Fig. 3.5: Reduction of two coplanar force vectors: (a) the vectors are not parallel; (b) the vectors are parallel.

Fig. 3.5a shows the reduction of two coplanar, non-parallel force vectors \mathbf{F}_1 and \mathbf{F}_2 by tracing back the vectors to their point of intersection followed by subsequent addition of the two vectors to the sum vector \mathbf{F} . If \mathbf{F}_1 and \mathbf{F}_2 happen to be parallel vectors as shown in fig. 3.5b, one can generate a pair of non-parallel vectors from these parallel vectors by adding the zero vector with zero moment $-\mathbf{F}_0 + \mathbf{F}_0 = \mathbf{0}$, which then can be reduced to the sum vector \mathbf{F} in the same way as shown in fig. 3.5a. If \mathbf{F}_1 and \mathbf{F}_2 are however anti-parallel, i.e., they form a so-called force dipole, the trick with the zero-vector addition does not work and one needs to tackle the problem with more elaborate force reduction methods which are beyond the scope of interest here.³

The fact that one can move forces along their lines of action and add forces to resultant forces without changing the impact of those forces follows from a fundamental principle of physics, the so-called superposition principle of forces. As shown in fig. 3.6, using again the example of two forces, forces can be superimposed (added) to a resulting total force

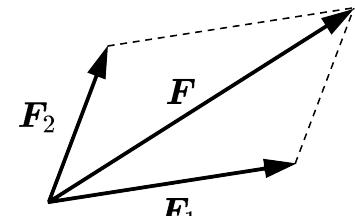


Fig. 3.6

$$\mathbf{F}(\mathbf{F}_1, \mathbf{F}_2) = \mathbf{F}_1 + \mathbf{F}_2 \quad (\text{linear})$$

³Those methods are part of engineering mechanics and can be found in standard textbooks dealing with the statics of rigid bodies

A force parallelogram for two forces as sketched in fig. 3.6 was already used by Newton. Importantly, as the starting point of the force vectors \mathbf{F}_1 and \mathbf{F}_2 could lie anywhere on their respective lines of action non-parallel vectors can always be moved to a shared point, i.e., where their lines of action cross each other, to create a force parallelogram such as fig. 3.6. The superposition principle for forces applies to linear force systems for which it allows the calculation of a resulting total force vector through the vector addition of all forces acting on a body. If there are N forces \mathbf{F}_i acting on a body, then their vector addition will result in a total force acting on the body of

$$\mathbf{F} = \sum_{i=1}^N \mathbf{F}_i \quad (3.7)$$

Eq. (3.7) is the general superposition principle for forces. Because of its great importance, this superposition principle is often referred to as Newton's fourth law.

3.1.2 Newton's Law of Gravity

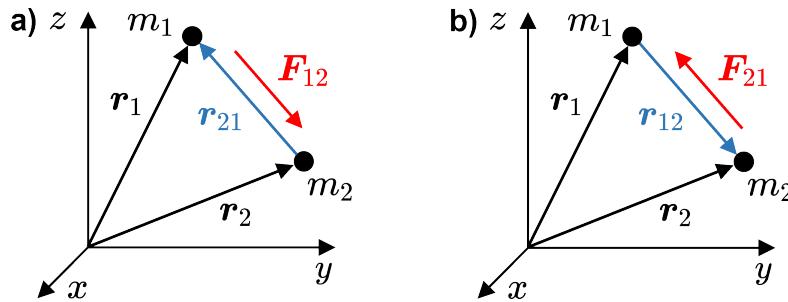


Fig. 3.7: Orientations of distance and force vectors in Newton's law of gravitation in vector form (see eq. (3.8) and eq. (3.9)): (a) for the force F_{12} exerted on m_1 by m_2 ; (b) for the force F_{21} exerted on m_2 by m_1 .

Isaac Newton's discovery of the attractive force which exists between any two massive objects in our universe, be it between the Moon and Earth or between an apple and Earth, ranks as one of the greatest science discoveries ever made. The gravitational force \mathbf{F}_{12} exerted on a point mass m_1 by a point mass m_2 (fig. 3.7a) which, as measured from the position of m_2 , is located a distance $\mathbf{r}_{21} = -\mathbf{r}_2 + \mathbf{r}_1$ away from m_1 , is given by

$$\mathbf{F}_{12} = -G \frac{m_1 m_2}{|\mathbf{r}_{21}|^3} \mathbf{r}_{21} = G \frac{m_1 m_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} \quad (3.8)$$

and the gravitational force \mathbf{F}_{21} exerted on point mass m_2 by point mass m_1 (fig. 3.7b) which, as measured from the position of m_1 , is located a distance $\mathbf{r}_{12} = -\mathbf{r}_1 + \mathbf{r}_2$ away from m_2 , is given by

$$\mathbf{F}_{21} = -G \frac{m_1 m_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} = G \frac{m_1 m_2}{|\mathbf{r}_{21}|^3} \mathbf{r}_{21} \quad (3.9)$$

In eq. (3.8) and eq. (3.9), G is a fundamental natural constant that determines the strength of gravity. With $\mathbf{r}_{12} = -\mathbf{r}_{21}$ it is obvious that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

applies. One can write the vector forms of Newton's gravity law in eq. (3.8) and eq. (3.9) without the minus sign in front by using \mathbf{r}_{12} instead of \mathbf{r}_{21} in eq. (3.8) and by using \mathbf{r}_{21} instead of \mathbf{r}_{12} in eq. (3.9). However, the convention is to write those equations with a minus sign in front to indicate that the respective force vector points into the opposite direction of the distance vector thereby clearly indicating that Newton's gravity force is attractive, i.e., it will result in the distance vector becoming shorter. Newton formulated his law of gravitation in 1687. He did not know what the value of the proportionality constant was, which physicists began to refer to as G in the late nineteenth century. Without knowing G or the masses of the planets or Earth, Newton could only work with ratios. More than 100 years after Newton's *Principia*, it was Henry Cavendish (1731 - 1810) who eventually succeeded in experimentally determining G in 1798.

According to [Johannes Kepler's third law](#), the ratio between the cube of the semi-major axis a of a planet's elliptical orbit and the square of its orbital period T is constant:

$$\frac{a^3}{T^2} = c$$

Simplifying assumption: Planetary orbits shall be circular with radius a (fig. 3.8). With that one obtains for the acceleration (due to the [centripetal force](#))

$$b = -\omega^2 a = -\frac{4\pi^2}{T^2} a = -4\pi^2 \frac{a}{T^2} = -\frac{c}{a^2} \quad (3.10)$$

The minus sign arises because the acceleration occurs in the direction opposite to where the radius vector points. If one now imagines the entire mass of the Earth being concentrated in its center, then the following applies to an object which orbits the center of the Earth at a distance equal to the Earth's radius R_E :

$$b_E = -\frac{c}{R_E^2} = -g$$

The value that this equation delivers for c can now be used in the corresponding equation for the Moon (radius R_M)

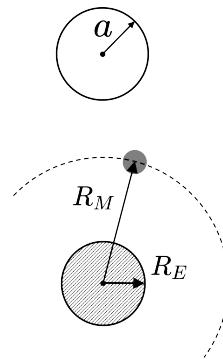


Fig. 3.8

$$b_M = -\frac{c}{R_M^2} = -g \left(\frac{R_E}{R_M} \right)^2$$

With eq. (3.10) and the revolution period of the Moon T_M also applies

$$b_M = -\frac{R_M}{T_M^2} 4\pi^2$$

With the last two equations, their left and hence their right sides being identical, it is possible to determine g from measurable quantities:

$$g = \frac{4\pi^2 R_M^3}{R_E^2 T_M^2} \quad (3.11)$$

If one inserts into this equation for R_M the mean Earth-Moon distance of 38 400 km, for the orbital period of the Moon T_M the duration between two perigee passages (27.56 days), and for R_E the mean Earth radius of 6 371 km, then one obtains for g the value 9.71 m s^{-2} . This result lies just a little more than 1% below the modern mean value of $g = 9.81 \text{ m s}^{-2}$. The above consideration also shows that the value of g depends on how far away one is from the center of the Earth because $g \propto R_E^{-2}$. The determination of G is a bit more difficult because one must know the mass of the Earth m_E . On the surface of the Earth applies

$$F = mg = G \frac{m_E m}{R_E^2}$$

The test mass m cancels out and the result for G is:

$$G = g \frac{R_E^2}{m_E} \quad (3.12)$$

If one uses $g = 9.81 \text{ m s}^{-2}$, $R_E = 6 371 \text{ km}$, and $m_E = 5.972 \cdot 10^{24} \text{ kg}$, then one obtains $G = 6.6675 \cdot 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$. However, that is just the reverse of what Henry Cavendish did. In his time, the radius of the Earth had been known for a long time, but not the mass of the Earth. With his torsion balance, Cavendish was able to determine the value of G for the first time in 1798 and that with great accuracy; his value for G differing only by 1.2% from the modern value of the gravitational constant $G = 6.6743 \cdot 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$. Knowing G and using eq. (3.12), Cavendish was then able to determine the mass of the Earth for the first time.

3.1.3 Inertial Mass and Heavy Mass

After space and time, the first and second basic quantities of mechanics, the inertial mass is the third basic quantity of mechanics. Physicists measure mass in units of kilograms

(one kilogram = 1 kg). The mass of a body is determined indirectly by measuring the impact of a force \mathbf{F} acting on the body. The magnitude of force itself has the dimension of mass \times length \times time $^{-2}$ and is measured in units of Newtons (N) where $1\text{ N} = 1\text{ kg}\cdot\text{m}\cdot\text{s}^{-2}$. Put into words: 1 N is the amount of force which will increase the acceleration of a mass of 1 kg by 1 m s^{-2} . A simple way to compare masses is to subject them to the application of the exact same force. In that case, according to eq. (3.4), one then can write:

$$\mathbf{F} = m_1 \mathbf{a}_1 = m_2 \mathbf{a}_2 = m_3 \mathbf{a}_3 = \dots$$

The relationships between the respective terms can be rewritten as

$$\frac{m_2}{m_1} = \frac{\mathbf{a}_1}{\mathbf{a}_2} \quad \text{and} \quad \frac{m_3}{m_1} = \frac{\mathbf{a}_1}{\mathbf{a}_3}$$

Hence, by measuring the acceleration which different masses experience under the same force one only has to select a specific mass m_1 as the standard to express m_2 and m_3 or any other mass in terms of m_1 . Of course, this mass comparison is done by measuring the weight force of the respective masses and humans have used standard weights for a long time. Since 1889 the modern weight standard was the weight of a kilogram mass of a cylinder made of a palladium iridium alloy kept at the International Bureau of Weights and Measures, in Sevres near Paris, France. To make this standard available widely many exact copies were distributed all over the globe so that a kilogram would have the same meaning, i.e., weight, in any part of the world. Only recently, in 2019, was the kilogram standard redefined in terms of a fundamental physical constant, the so-called Planck constant $h = 6.6262 \cdot 10^{-34} \text{ kg m}^2 \text{ s}^{-1}$.

In the above argument regarding the relationships between masses we were comparing inertial masses of objects. However, the kilogram standard physicists used for a long time compared those inertial masses to that of the heavy mass of the standard kilogram defined by its weight, i.e., by the weight force. When we determine the size of masses in everyday life, we actually do this in most cases by comparing the respective weight forces, such as with a beam balance. However, the weight force is not the mass of a body, as this is a property of matter itself, which is the inertial mass, but it is determined by the interaction between two bodies. With the help of a beam balance, e.g., one compares the interaction between a mass and the Earth's mass with the interaction of another mass with the Earth's mass. The weight force is location dependent:

$$\mathbf{F} = m_s \cdot g \cdot \hat{\mathbf{e}}$$

where g is the gravitational acceleration and the unit vector $\hat{\mathbf{e}}$ indicates the direction of the force. For a body in the gravitational field of Earth, the latter points in the

direction of the center of the Earth, i.e., the center of mass of the Earth. The weight force measures how “heavy” or “light” a body is, but it does not measure the mass of a body itself. Because of the Moon’s lower mass, gravity acceleration on its surface is about six times less than on Earth’s surface and therefore everything on the Moon is about six times “lighter” than on Earth. However, the mass of a body on the Moon is identical with the mass of the same body on Earth. Galilei’s discovery that all free falling objects in Earth’s gravity field experience the same acceleration $g = 9.81\text{m}\cdot\text{s}^{-2}$ defines the heavy mass m_s of objects on Earth:

$$m_s = \frac{F_W}{g}$$

where F_W is the weight force, i.e., the force of gravity. At the same time, according to Newton’s second law, for the acceleration a body experiences must hold

$$g = \frac{F_W}{m_t}$$

Combining those two equations shows that the following expression for g must hold:

$$g = \frac{F_W}{m_t} = \frac{g \cdot m_s}{m_t} \quad \Rightarrow \quad m_s = m_t$$

Importantly, $m_s = m_t$ must be true everywhere, regardless of what the local value of g is as one could have derived this equation just as well for objects falling in the Moon’s gravity field or any other gravity field for that matter. The requirement that m_s equals m_t is what physicists refer to as the equivalence principle. Newton was well aware of the importance of this principle and conducted his own measurements using a thread pendulum to test it. Much later, this very principle would convince Einstein that gravitation and acceleration are indistinguishable (see fig. 1.2) eventually leading him to formulate his general theory of relativity. Newton’s thread pendulum experiments enabled him to show with greater accuracy than what was possible with free fall experiments that the inertial and the heavy mass of an object were identical. Since then, many more experiments testing the equivalence principle with ever greater accuracy have been conducted and none so far has given any indication that inertial and heavy mass of an object are not identical. Among those experiments the igneous method used by the Hungarian physicist Baron Eötvös (1848 - 1919) stands out for its simplicity and high accuracy, showing that the ratio of m_s to m_t differed from 1 by less than 10^{-8} . Today, we can test the free fall of objects much more accurately than what was possible in Galilei’s time by observing free falling objects in satellites orbiting Earth. Such data tell us that the ratio of m_s to m_t differs from 1 by less than 10^{-15} . From here on we will not differentiate anymore between inertial mass and heavy mass but take their identity as a given.

3.2 Systems of Point Masses

So far we have only encountered situations where we had to deal with the motion of a single mass at a time, like with Newton's second law, or at most with the motion of two masses relative to each other under the influence of Newton's law of gravity. Looking around us we can certainly point to many examples of such situations of two-body interactions or we can abstract from what we find to something simpler which we then can treat as a two-body problem or even as a single-body problem. However, often that is not possible and we have to deal with many more masses than just two. Let us for example consider a system of N point masses m_i whose locations and velocities are given by their coordinate and velocity vectors \mathbf{r}_i and \mathbf{v}_i . The total mass M of such a system is then of course the sum of all its individual masses m_i , i.e.,

$$M = \sum_i^N m_i$$

The center of mass \mathbf{r}_{CM} of this system of point masses is then given by

$$\mathbf{r}_{CM} = \frac{1}{M} \sum_i^N m_i \mathbf{r}_i \quad (3.13)$$

and the center of mass velocity \mathbf{v}_{CM} is

$$\mathbf{v}_{CM} = \frac{d\mathbf{r}_{CM}}{dt} = \frac{1}{M} \sum_i^N m_i \mathbf{v}_i \quad (3.14)$$

For the total momentum \mathbf{p}_{CM} of the system of N point masses we can then write

$$\mathbf{p}_{CM} = M \mathbf{v}_{CM} = \sum_i^N p_i \quad (3.15)$$

Eq. (3.15) is known as the center of mass theorem which states that the total momentum of a system of point masses m_i is the same as that of a body with mass M moving with the velocity of the center of mass \mathbf{v}_{CM} . To illustrate the above equations lets look again at the two-body system of Earth and Moon. For this two body system, with some rearranging, eq. (3.13) reduces to (E = Earth, M = Moon)

$$m_E(\mathbf{r}_{CM} - \mathbf{r}_E) = m_M(\mathbf{r}_M - \mathbf{r}_{CM})$$

What this equation tells us is that the vectors $\mathbf{r}_{CM} - \mathbf{r}_E$ and $\mathbf{r}_M - \mathbf{r}_{CM}$ are parallel which means that the center of mass coordinate must lie on the straight line connecting Earth and Moon. If we rewrite this equation again into the form

$$\frac{\mathbf{r}_{CM} - \mathbf{r}_E}{\mathbf{r}_M - \mathbf{r}_{CM}} = \frac{m_M}{m_E}$$

we see that the center of mass coordinate divides this connecting line between Earth and Moon in the inverse ratio of their masses. If we insert the numbers, assuming that both Earth and Moon are point mass objects, we find that the center of mass position lies within Earth, about 73% along the way from Earth center to Earth's surface. That of course is owed to the fact that Earth's mass is so much larger than the mass of the Moon. Just as we do for an individual mass we can, according to the center of mass theorem of eq. (3.15), apply Newton's second law to the motion of the center of mass:

$$M \frac{d^2 \mathbf{r}_{CM}}{dt^2} = \frac{d\mathbf{p}_{CM}}{dt} = \mathbf{F}^{ext} \quad (3.16)$$

where the external force acting on the center of mass \mathbf{r}_{CM}

$$\mathbf{F}^{ext} = \sum_i^N \mathbf{F}_i^{ext} = \sum_i^N \frac{d\mathbf{p}_i}{dt}$$

is the sum over the forces \mathbf{F}_i^{ext} acting on the masses m_i . Without any external force acting on \mathbf{r}_{CM} , i.e., $\mathbf{F}^{ext} = 0$, the velocity of the center of mass remains constant just as Newton's first law requires.

The Center of Mass System

The center of mass system (CMS) is defined as the coordinate system whose origin resides in the center of mass of a system of point masses. Here we will denote the center of mass system with S' and coordinates and velocities or momenta in S' with \mathbf{r}' , \mathbf{v}' and \mathbf{p}' . Locating the coordinate origin in the center of mass means of course that in the center of mass system $\mathbf{r}_{CM} = 0$ in eq. (3.13), i.e.,

$$\sum_i^N m_i \mathbf{r}'_i = 0 \quad (3.17)$$

and with that

$$\sum_i^N m_i \mathbf{v}'_i = \sum_i^N \mathbf{p}'_i = 0 \quad (3.18)$$

When we make observations or measure something, we do that in the so-called laboratory system (LS) which is the coordinate system where we, as the observers, are at rest. We will denote the laboratory system as S and coordinates and velocities or momenta in S with \mathbf{r} , \mathbf{v} and \mathbf{p} . According to the Galilean relativity principle, as long as the coordinate systems S' and S move relative to each other in uniform linear motion, observers in S and S' can equally claim to be at rest or to be in motion, it does not make a difference. For that to be the case the external force \mathbf{F}^{ext} in eq. (3.16) must vanish because only then will

\mathbf{r}_{CM} be constant and the center of mass system will be in uniform linear motion relative to the laboratory system. Coordinates and velocities (and with that momenta) in the laboratory system S and in the center of mass system S' are then connected through the Galilean transformation of eq. (2.2) where the relative velocity of \mathbf{v}_0 in eq. (2.2) equals \mathbf{v}_{CM} :

$$\begin{aligned}\mathbf{r} &= \mathbf{v}_{CM} \cdot t + \mathbf{r}' \\ \mathbf{v} &= \mathbf{v}_{CM} + \mathbf{v}'\end{aligned}\tag{3.19}$$

3.2.1 The Conservation of Momentum

What we call today the momentum \mathbf{p} of an object, Newton defined as a quantity of motion (“quantitas motus” in his Principia) arising from the velocity and quantity of matter conjointly. One can look at the momentum, the product of mass and velocity

$$\mathbf{p} = m \cdot \mathbf{v}$$

as a measure of how difficult it is to stop a body. Newton used three different formulations for his third law. We already know the action-reaction principle. A second variant was what we now call the principle of conservation of linear momentum. Consider two interacting point masses, point mass 1 and point mass 2. The force \mathbf{F}_1 with which point mass 2 acts on point mass 1 is balanced by an equal but diametrically opposed force \mathbf{F}_2 with which point mass 1 acts upon point mass 2:

$$\mathbf{F}_1 = -\mathbf{F}_2 \quad \text{hence} \quad \dot{\mathbf{p}}_1 = -\dot{\mathbf{p}}_2$$

and therefore

$$\mathbf{p}_1 + \mathbf{p}_2 = \text{const}\tag{3.20}$$

The third formulation he used was the principle that the center of mass of a number of bodies does not change its state of motion because of any interaction between these bodies, the so-called internal forces of the system; it only changes its state of motion due to an external force being applied. In a way, this last formulation is the equivalent of the first law being applied to the center of mass of a closed system. Every point mass m_i in a system of point masses is subject to Newton's second law

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i$$

where \mathbf{p}_i is the momentum of the point mass m_i and \mathbf{F}_i is the force acting on it. In a closed system internal forces arise from the interaction between points masses which only include two-body interactions between any two given point masses:

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} \quad \text{where} \quad \mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

Different from that, external forces depend only on the coordinates of a point mass or on its velocity or are a function of time; but they do not depend on other point masses. Hence, for external forces F_i applies

$$\mathbf{F}_i \neq \sum \mathbf{F}_{ij}$$

Examples of external forces are the motion of a point mass in an electromagnetic field or in a gravitational field. Without the influence of external forces, the total momentum of a system of point masses is constant. Specifically, generalizing eq. (3.20) from two point masses to many point masses m_i it follows that

$$\sum_i \frac{d\mathbf{p}_i}{dt} = \underbrace{\frac{d}{dt} \sum_i \mathbf{p}_i}_{\mathbf{P}} = \frac{d\mathbf{P}}{dt} = \sum_{j \neq i} \mathbf{F}_{ij} = -\sum_{j \neq i} \mathbf{F}_{ji} \equiv 0 \quad (3.21)$$

From this equation it follows that the total momentum \mathbf{P} of a closed system of point masses cannot be changed by internal forces. This result is what physicists refer to as the momentum theorem.

Momentum theorem

The rate of change observed in the total momentum of a system of point masses is only caused by external forces. The total momentum of such a system cannot be changed by internal forces.

A comparison of eq. (3.21) with eq. (3.16) shows that in the case of $\mathbf{F}^{ext} = 0$ the equation of motion for the center of mass of a system of point masses becomes eq. (3.21). We derived eq. (3.21) in the laboratory system S . Accordingly, if the center of mass system S' is in a uniform linear motion with respect to S , then eq. (3.18) in S' is the equivalent of eq. (3.20) in S . There is one more observation we can make regarding eq. (3.21) or eq. (3.16) in the case where \mathbf{F}^{ext} vanishes. For $\mathbf{F}^{ext} = 0$ the total momentum is constant and with eq. (3.5) this also means that the total kinetic energy T or T_{CM} must stay constant, i.e.,

$$T = \frac{P^2}{2M} = const \quad \text{or respectively} \quad T = \frac{p_{CM}^2}{2M} = const$$

Because the total kinetic energy of a system of N point masses is the sum of the kinetic energies of all its point masses it follows that

$$T = \sum_i^N \frac{p_i^2}{2m_i} = \sum_i^N \frac{m_i v_i^2}{2} = \text{const} \quad (3.22)$$

Of course, in a system of many point masses there will also be collisions between point masses and those collisions can take different forms which is why kinetic energy is not always conserved in collision processes.

3.2.2 Collision Processes

Under collision processes we will understand here the collision between two collision partners, not more. For macroscopic collisions between objects such as billiard balls that may seem to be a good approach but how does that relate to what happens in systems of many point masses. As it turns out, to understand the physical properties of systems of many points masses one must first understand the physical properties of interactions between any two point masses. A prime example for this is the kinetic theory of gases which derives macroscopic gas properties such as gas pressure or temperature from the behavior of large ensembles of atoms or molecules on the microscopic level (see chapter 9). Among the basic assumptions it makes for the behavior of such gas particles is that their sizes are negligible, i.e., they are point masses, they are not subject to external forces and collisions are always collisions of two gas particles and those collisions are elastic.

In analyzing collision processes we will make use of the fact that in closed systems momentum and energy are conserved. As discussed in the preceding sections, in closed systems, that is systems which are not subject to external forces, the total momentum cannot change and the energy contained in the system cannot change either. To designate velocities and momenta of point masses after a collision we will put a tilde above the respective symbol. With \mathbf{v} and \mathbf{p} thus referring to before the collision and $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{p}}$ referring to after the collision we can write conservation of momentum and conservation of energy in the collision process of a point mass m_1 with a point mass m_2 as

$$\mathbf{p}_1 + \mathbf{p}_2 = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 \quad (3.23)$$

and

$$\frac{1}{2}m_1 \mathbf{v}_1^2 + \frac{1}{2}m_2 \mathbf{v}_2^2 = \frac{1}{2}m_1 \tilde{\mathbf{v}}_1^2 + \frac{1}{2}m_2 \tilde{\mathbf{v}}_2^2 + Q \quad (3.24)$$

The quantity Q in eq. (3.24) indicates that in the collision process some of the kinetic energy can be transformed into other kinds of energy. Depending on whether $Q = 0$, $Q > 0$ or $Q < 0$ we can differentiate between:

- Elastic collisions: The kinetic energy of the collision partners before and after the collision is the same, i.e., $Q = 0$.
- Inelastic collisions: The kinetic energy of the collision partners after the collision is lower than before the collision by an amount $Q > 0$ because some of the kinetic energy can be converted into internal energy. Examples are losses to electronic excitation energy in the case of colliding atoms or molecules. For the latter, losses to vibration or rotation energy are also possible. Kinetic energy losses in collisions of macroscopic objects can be due to such bodies being deformed in the collision with Q then being deformation energy.
- Superelastic collisions: The kinetic energy of the collision partners after the collision is higher than before the collision by an amount $Q < 0$. In that case internal energy of a collision partner is converted into kinetic energy. For molecules this can be electronic, vibration or rotation energy being converted into kinetic energy; in the case of macroscopic bodies it could be chemical energy stored in a collision partner which is released on impact.

In addition to categorizing collisions with respect to the value of Q , we differentiate them further as central impact collisions and oblique impact collisions. As the label insinuates, central impact collision are head-on collisions where two point masses happen to move before as well as after the collision along the very same line in space. This line in space is called the line of impact. In an oblique impact collision the two collision partners move at an angle with respect to the line of impact before the collision and after the collision. Fig. 3.9 sketches the situation for two masses on their path to a central impact collision and the same two masses on their way to a an oblique impact collision. In a central impact collisions we can treat the collision partners as dimensionless point masses. In oblique collisions this is not possible as such collisions cannot occur between extensionless point masses. Therefore, oblique collisions must be treated as collisions between extended objects.

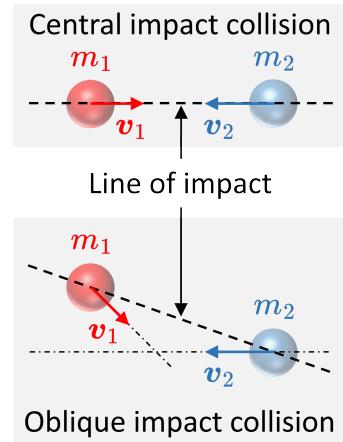


Fig. 3.9

In what follows next we will discuss a number of simple two body collision examples. All of them will be central impact collisions, inelastic ones and elastic ones. After that we will look into oblique collision examples.

Inelastic Collisions

We will begin by considering the special case of a completely inelastic central impact collision. In such a collision the impact “glues” the two masses together and from then on they move together as one combined mass. In the simple case shown in fig. 3.10 a mass m hits a mass m_0 at rest in a central impact collision. This being a one-dimensional problem, eq. (3.23), the equation for the conservation of total momentum, becomes

$$m \cdot v = (m + m_0) \tilde{v}_{inel}$$

hence

$$\tilde{v}_{inel} = \frac{m}{m + m_0} \cdot v \quad (*)$$

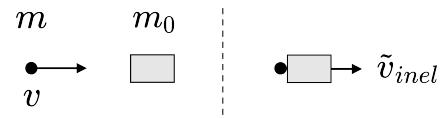


Fig. 3.10

Eq. (3.24), the equation for the conservation of energy takes the simple form

$$T = \frac{m \cdot v^2}{2} = \frac{(m + m_0) \cdot \tilde{v}_{inel}^2}{2} + Q$$

Inserting \tilde{v}_{inel} from (*) into this equation and solving for Q gives

$$Q = \frac{m_0}{m + m_0} \cdot \frac{m \cdot v^2}{2} = \frac{m_0}{m + m_0} \cdot T \quad (**)$$

If $m_0 = m$ then (*) tells us that $v_{inel} = v/2$. More generally, if $m_0 = n \cdot m$ then $v_{inel} = v/(1+n)$. From (**) we calculate the amount of kinetic energy converted into a different form of energy if $m_0 = m$ to be $Q = T/2$. For $m_0 = n \cdot m$ it is easy to see that the amount of kinetic energy being converted into a different form of energy will be $Q = (n/n+1) \cdot T$. Of course, if n becomes very large, i.e., $n \rightarrow \infty$, all of the kinetic energy will have been converted into different kinds of energy and v_{inel} will be equal to zero, like when for example one of those vehicles with human dummy passengers is purposefully run into a wall to test how well any car will protect human passengers in real accidents.

Next, we will look at what happens in a central impact inelastic collision if both collision partners are moving before the impact but on impact will still stick together and then move together as one combined mass. If the two collision partners of mass m_1 and m_2 have the velocities v_1 and v_2 the conservation of momentum in eq. (3.23) tells us that

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) \tilde{v}_{inel}$$

To calculate \tilde{v}_{inel} we make use of the [Galilean relativity principle](#) This principle tells us that if we move from observing the collision in a reference frame at rest to observing the collision in a reference frame moving at a constant speed then nothing changes, except

that we have to transform the coordinates when moving between reference frames. As long as the velocity of the moving reference frame is well below the speed of light c we can use the Galilean transformation of eq. (2.2) to add velocities. Denoting the coordinate systems at rest with S and the system moving with a velocity v_0 relative to S with S' , the respective velocities in the two coordinate systems are then connected by

$$v' = v - v_0$$

The trick we apply in the case when both collision partners are moving before the collision is simply to switch to a reference system where one of the collision partners is at rest before the collision. If we choose for example $v_0 = v_2$ then $v'_2 = 0$, i.e., mass m_2 will be at rest with respect to the moving system and mass m_1 will be moving with the velocity $v'_1 = v_1 - v_2$ with respect to the moving system. As a result, with respect to the moving reference system one can now apply the result from fig. 3.10 to obtain

$$\tilde{v}'_{inel} = \frac{m_1}{m_1 + m_2} \cdot (v_1 - v_2)$$

Now one only has to transform this result back via

$$\tilde{v}'_{inel} = \tilde{v}_{inel} - v_2$$

to obtain \tilde{v}_{inel} in the reference frame at rest:

$$\tilde{v}_{inel} = \frac{m_1}{m_1 + m_2} \cdot (v_1 - v_2) + v_2 = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

In the case of $m_1 = m_2$ one finds that $v_{inel} = (v_1 + v_2)/2$. More generally, if m_2 is an integer multiple of m_1 , i.e., $m_2 = n \cdot m_1$, then $v_{inel} = (v_1 + n \cdot v_2)/(n + 1)$. Calculating the equivalent of (**) in the case of a central impact inelastic collision of two moving objects one obtains for the amount of kinetic energy converted into other forms of energy

$$Q = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2$$

Evaluating this for the cases where $m_2 = n \cdot m_1$ one obtains $Q = m_1 (v_1 - v_2)^2 / 4$ for $n = 1$ and $Q = m_1 (v_1 - v_2)^2 \cdot n / 2(n + 1)$ for $n > 1$. For $v_2 = 0$ the results for v_{inel} and Q are of course identical with the above discussed case sketched in fig. 3.10. As one can read from the expression for Q , as long as $v_1 \neq v_2$ one is dealing with an inelastic collision. But what does it mean if $v_1 = v_2$, that is when Q becomes zero? To answer this question one can take either one of two perspectives: one looking at the situation before the collision happens and one looking at the situation after the collision happened. If one looks at it from the perspective after the collision the logic conclusion must be that the two masses must have stuck together all the time, i.e., a collision never occurred. Looking at it from the perspective before the collision when the two masses were separate, moving at

identical velocities, this means they will continue to stay separate, hence, a collision will never occur. Put differently, collisions where the collision partners are separate before the collision and then stick together after the collision can never be elastic collisions as per definition for such a collision $Q \neq 0$ must apply.

Of course, there are also inelastic collisions where on impact a fraction of the kinetic energy is converted into another form of energy but the colliding masses will not stick together. That could certainly be the case in collisions of atoms or molecules with kinetic energy being converted into internal electronic, vibration or rotation energy. But it could also be the case in collisions of macroscopic objects via a permanent deformation of one of the collision partners. There are many instances where understanding such collisions and the associated kinetic energy transfer into atomic or molecular excitation or the damage done to one or both of the collision partners is of great interest. The latter are for example specifically important in engineering mechanics but not so much in fundamental physics. For the former, inelastic collisions $Q \neq 0$ as well as elastic collision $Q = 0$ are of interest. For macroscopic objects, $Q = 0$ collisions do not imply that no deformation occurs, it only means that any such deformation is fully reversible.

Elastic Collisions

Collisions where Q is really zero are rare because some energy is almost always lost, like for example to heat. Nevertheless, we can imagine such a $Q = 0$ collision process and assume it exists. Just as physicists do in the kinetic gas theory where they treat atom collisions as elastic with atoms bouncing off each other (and off any walls) without kinetic energy being lost. Fig. 3.11 illustrates such a head-on elastic collision process where we have made the additional assumption that the two colliding balls are of equal mass, i.e., $m_1 = m_2 = m$. As shown in the top row of fig. 3.11, one of them, ball ①, comes from the left with velocity $v_1 = v$ and the other one, ball ②, comes from the right with velocity $v_2 = -v$. These two balls shall have the special elastic property that any deformation they experience is 100% reversible. The center of mass of this two ball system lies midway between the balls approaching each other and does not move. The center of mass velocity of this system is zero before the impact and will be zero after the impact. Hence, by default we are analyzing this collision situation in the center of mass system. On impact, the velocity of both balls is exactly zero as

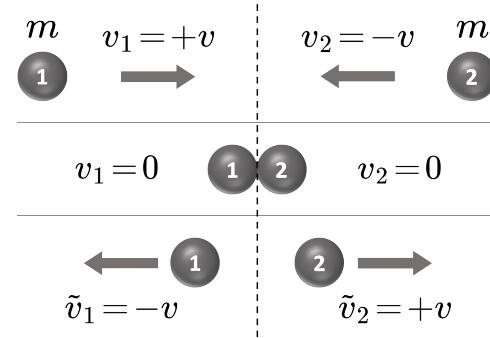


Fig. 3.11

shown in the middle row of fig. 3.11, i.e., $\mathbf{v}_1 = 0$ and $\mathbf{v}_2 = 0$. Therefore, at that moment, the kinetic energy of the system is exactly zero. But not for long because as shown in the bottom row of fig. 3.11, on impact the balls immediately rebound and head in the opposite direction from where they came from. Conservation of momentum and of energy (with $Q = 0$) require that the velocities after the impact are equal in magnitude to what they were before the impact but with their directions reversed. Therefore we have after the impact

$$\tilde{v}_1 = -v \quad \text{and} \quad \tilde{v}_2 = +v$$

How does that happen? Well, on impact the kinetic energy of the two balls is every so briefly converted into elastic deformation energy and immediately reconverted into kinetic energy as the deformation reverts. With a good slow motion camera one can observe this energy conversion process of kinetic energy becoming elastic energy and elastic energy being converted back into kinetic energy. This just happens too fast for us to observe in real time, without help from technology we just can infer what happened from the result we can observe.

Another simple inelastic collision case is the head-on collision of two balls of equal mass m where one was at rest before the collision. We will discuss this collision case with the help of fig. 3.12 which shows the collision process in the laboratory system fig. 3.12a and in the center of mass system fig. 3.12b. The two masses which we will label as ball ① and ball ② shall collide as sketched in fig. 3.12a where before the collision one of them is at rest while the other one moves towards it with the velocity v .

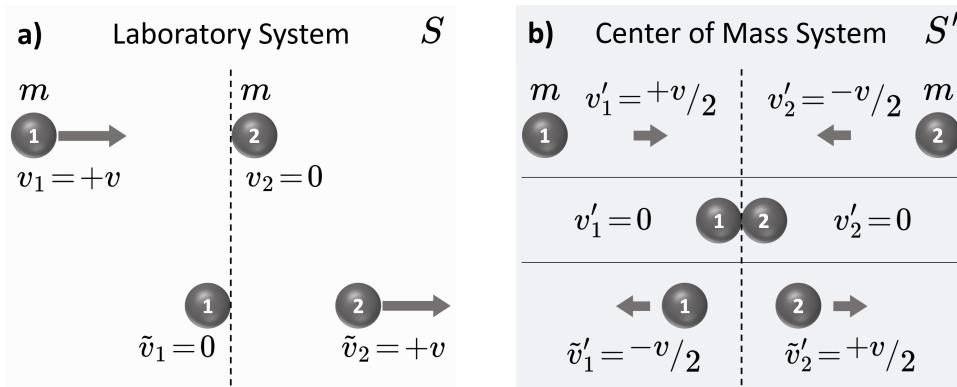


Fig. 3.12: Collision process in the laboratory system (a) and in the center of mass system (b).

Many will intuitively guess correctly or understand, billiard players certainly do, what happens when ball ① moving with a velocity v hits ball ② which is at rest. Billiard players know that after such a collision the velocity of ① we will be zero, i.e., $\tilde{v}_1 = 0$,

and the velocity of ball ② after the collision will be $\tilde{v}_2 = v$. So what happens in such a collision is that the two balls just exchange velocities. To understand why that is the case we need to look at how this collision happens for someone looking at it in the center of mass system. Because the balls are of equal mass we get for the center of mass velocity $v_{CM} = v/2$ and the velocities of the balls in the center of mass system before the collisions must be

$$v'_1 = \frac{v}{2} \quad \text{and} \quad v'_2 = -\frac{v}{2}$$

Hence, in the center of mass system S' this collision sketched in fig. 3.12b corresponds exactly to the collision illustrated in fig. 3.11 which we discussed in the laboratory system. The only difference being that instead of moving with velocities v and $-v$ the balls now move with velocities $v/2$ and $-v/2$. Hence, we know that in the center of mass system the two balls will have after the collision the velocities

$$\tilde{v}'_1 = -\frac{v}{2} \quad \text{and} \quad \tilde{v}'_2 = \frac{v}{2}$$

Transferring this result back into the laboratory system S we only have to add $v/2$ to these velocities to see that ball ① which was moving with the velocity v before the collision is now at rest, i.e., $\tilde{v}_1 = 0$ and ball ② which was at rest before the collision is now moving with the velocity $\tilde{v}_2 = v$. That is of course, as most will intuitively suspect, just what must happen in such a collision in the laboratory system S . What happens if m_2 is not at rest but also moves but with a velocity different from the velocity of m_1 ? Consider for example the situation where before the collision ball ① moves with a velocity $v_1 = 2v$ and ball ② moves with a velocity $v_2 = -v$. The center of mass velocity in this case is $v_{CM} = v/2$. With that the velocities of the two balls in the center of mass system before the collision are

$$v'_1 = 2v - \frac{1}{2}v = \frac{3}{2}v \quad \text{and} \quad v'_2 = -v - \frac{1}{2}v = -\frac{3}{2}v$$

Again we see that in the center of mass system the two masses have the same absolute velocity so we know that in the center of mass system the two masses will on impact exchange velocities just as in the case where one ball was at rest. It is easy to show that this kind of velocity exchange is generally what happens in elastic central impact collisions when the collision partners have the same mass. Just consider the motion of the center of mass of such a two body system. For simplicity we will assume that both masses move along the x -axis, with their respective coordinates being x_1 and x_2 . With that, the center of mass coordinate X is given by

$$X = \frac{mx_1 + mx_2}{m + m} = \frac{x_1 + x_2}{2}$$

and the center of mass velocity is

$$\dot{X} = \frac{\dot{x}_1 + \dot{x}_2}{2}$$

Center of mass conservation of momentum requires that the center of mass must continue to move with the same velocity and into the same direction before and after the collision. That however only works if the masses exchange velocities on impact. This holds true for any velocities the two masses may have had before the impact.

Non-Central Impact Collisions

In central impact collisions the direction of the relative motion of the collision partners coincides with the line of impact. Because of that we can treat such events as one-dimensional collision problems between dimensionless point masses. That changes for non-central impact collisions where the problem to solve becomes two-dimensional. We will begin by considering the case illustrated in fig. 3.13 which represents the most simple non-central impact collision scenario. The two collision partners, ball ① and ball ② in fig. 3.13, shall be of equal mass m and one of them, ball ②, shall be at rest before the collision. Ball ① moves parallel to the x -direction from the left towards ball ② with an impact parameter b . Furthermore, we will make two assumption in treating this collision scenario. First we assume that the two balls in fig. 3.13 are two perfectly smooth spheres. The second assumption we make is that on impact none of the total kinetic energy the two balls possessed before the collision is converted into rotation energy. The momentum of ball ① before the collision is $\mathbf{p}_1 = (p_{1x}, 0)$ and its momentum after the collision is $\tilde{\mathbf{p}}_1 = (\tilde{p}_{1x}, \tilde{p}_{1y})$. The momentum of ball ② before the collision is $\mathbf{p}_2 = (0, 0)$ and its momentum after the collision is $\tilde{\mathbf{p}}_2 = (\tilde{p}_{2x}, \tilde{p}_{2y})$. With eq. (3.23) we can write down the equation for the conservation of momentum as

$$\mathbf{p}_1 = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 \quad (3.25)$$

This equation tells us that the three momentum vectors \mathbf{p}_1 , $\tilde{\mathbf{p}}_1$ and $\tilde{\mathbf{p}}_2$ all lie in one plane. Because of that we can treat the collision sketched in fig. 3.13 as a two-dimensional problem. If ball ① were not at rest before the collision we could not do that. With eq. (3.23) the equation for the conservation of kinetic energy is

$$\frac{\mathbf{p}_1^2}{2m} = \frac{\tilde{\mathbf{p}}_1^2}{2m} + \frac{\tilde{\mathbf{p}}_2^2}{2m} \quad \text{or} \quad \mathbf{p}_1^2 = \tilde{\mathbf{p}}_1^2 + \tilde{\mathbf{p}}_2^2 \quad (3.26)$$

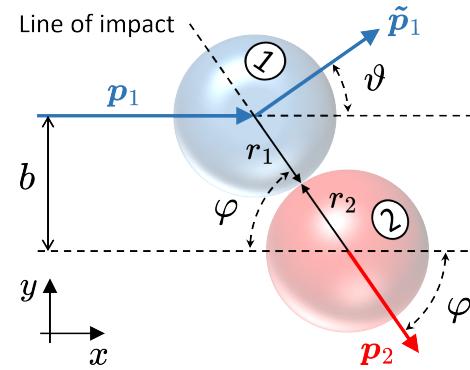


Fig. 3.13

Eq. (3.26) shows that in the specific case where the balls have the same mass, the equation for the conservation of kinetic energy turns into the Pythagorean theorem for the momentum vectors. With that, as illustrated in fig. 3.14, $\tilde{\mathbf{p}}_1$ and $\tilde{\mathbf{p}}_2$ are orthogonal. Every triangle inscribed into a circle in the way shown in fig. 3.14 for the situation sketched in fig. 3.13 will be a right angled triangle with $\vartheta + \varphi = 90^\circ$. The angle φ between the direction of the incoming ball ① and the line of impact is given by

$$\sin \varphi = \frac{b}{r_1 + r_2} \quad (3.27)$$

Hence, the two radii do not necessarily have to be equal. However, if $r_1 \neq r_2$ we must require that the line of impact lies within the collision plane, i.e., the plane defined by eq. (3.25). This ensures that the problem will remain a two-dimensional one and the above analysis is still applicable. Collisions where one of the collision partners is at rest before the collision are important in physics and that is why we will analyze this collision scenario now from the perspective of someone in the center of mass system. There, we will allow for the two collision partners to have different masses which means that the conservation of energy equation will not reduce to the Pythagorean theorem anymore as we saw in the case of equal masses in eq. (3.26). However, just as in the laboratory system this collision problem turned out to be a two-dimensional one it will also be a two-dimensional problem when we transfer to the center of mass system where the total momentum of the collision partners is zero.

Non-Central Impact Collision in the Center of Mass System

Before we can consider the specific collision scenario we are interested in, i.e., analyzing the collision of a ball moving with a velocity \mathbf{v} with a ball at rest in the center of mass system, we need to do some preparation work. In doing that we will stick with the notation introduced when we discussed the [center of mass system](#) earlier in this chapter. The reference frame S will again denote the laboratory system (LS) and the reference frame S' will again denote the center of mass system (CMS) which moves relative to LS with the constant velocity \mathbf{v}_{CM} . For the collision processes that interest us here we only need to consider two masses m_1 and m_2 . To differentiate between velocities and momenta before and after the collision we will stick with the notation we already introduced and label the respective quantities for the two masses m_1 and m_2 in the laboratory system S and the center of mass system S' as follows:

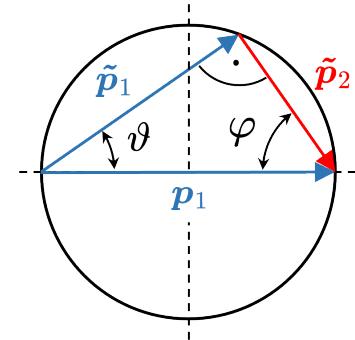


Fig. 3.14

$$\begin{aligned} \text{before: } & \left\{ \begin{array}{l} \text{in } S : \mathbf{v}_1, \mathbf{v}_2, \mathbf{p}_1, \mathbf{p}_2 \\ \text{in } S' : \mathbf{v}'_1, \mathbf{v}'_2, \mathbf{p}'_1, \mathbf{p}'_2 \end{array} \right. \\ \text{after: } & \left\{ \begin{array}{l} \text{in } S : \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \\ \text{in } S' : \tilde{\mathbf{v}}'_1, \tilde{\mathbf{v}}'_2, \tilde{\mathbf{p}}'_1, \tilde{\mathbf{p}}'_2 \end{array} \right. \end{aligned}$$

These “before” and “after” collision velocities and momenta in the laboratory system S and in the center of mass system S' are connected through the Galilean transformation of eq. (3.19). Using the above notation we can write in the laboratory system S for the conservation of momentum equation

$$\mathbf{p}_1 + \mathbf{p}_2 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = m_1 \tilde{\mathbf{v}}_1 + m_2 \tilde{\mathbf{v}}_2 = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 \quad (3.28)$$

and the conservation of kinetic energy equation in S becomes

$$\frac{m_1 \mathbf{v}_1^2}{2} + \frac{m_2 \mathbf{v}_2^2}{2} = \frac{m_1 \tilde{\mathbf{v}}_1^2}{2} + \frac{m_2 \tilde{\mathbf{v}}_2^2}{2} \quad (3.29)$$

In the center of mass system S' the total momentum is zero and therefore the equation for the conservation of momentum is

$$\mathbf{p}'_1 + \mathbf{p}'_2 = m_1 \mathbf{v}'_1 + m_2 \mathbf{v}'_2 = 0 = m_1 \tilde{\mathbf{v}}'_1 + m_2 \tilde{\mathbf{v}}'_2 = \tilde{\mathbf{p}}'_1 + \tilde{\mathbf{p}}'_2 \quad (3.30)$$

and for the conservation of kinetic energy in S' we have the equation

$$\frac{m_1 \mathbf{v}'_1^2}{2} + \frac{m_2 \mathbf{v}'_2^2}{2} = \frac{m_1 \tilde{\mathbf{v}}'_1^2}{2} + \frac{m_2 \tilde{\mathbf{v}}'_2^2}{2} \quad (3.31)$$

From the conservation of momentum eq. (3.30) follows that

$$\mathbf{v}'_1 = -\frac{m_2}{m_1} \mathbf{v}'_2 \quad \text{and} \quad \tilde{\mathbf{v}}'_1 = -\frac{m_2}{m_1} \tilde{\mathbf{v}}'_2 \quad (3.32)$$

Inserting this into eq. (3.31) to first replace \mathbf{v}'_2 with \mathbf{v}'_1 and $\tilde{\mathbf{v}}'_2$ with $\tilde{\mathbf{v}}'_1$ and then replace \mathbf{v}'_1 with \mathbf{v}'_2 and $\tilde{\mathbf{v}}'_1$ with $\tilde{\mathbf{v}}'_2$ we get the result that in the center of mass system

$$|\mathbf{v}'_1| = |\tilde{\mathbf{v}}'_1| \quad \text{and} \quad |\mathbf{v}'_2| = |\tilde{\mathbf{v}}'_2| \quad (3.33)$$

Evidently, for each collision partner the absolute value of its velocity after the collision is the same as before the collision. In the center of mass system the momentum vectors of the two masses before as well as after the collision must with eq. (3.30) point into exact opposite directions. Therefore we can write

$$\tilde{\mathbf{v}}'_1 = |\mathbf{v}'_1| \cdot \hat{\mathbf{u}} \quad \text{and} \quad \tilde{\mathbf{v}}'_2 = -|\mathbf{v}'_2| \cdot \hat{\mathbf{u}} \quad (3.34)$$

where $\hat{\mathbf{u}}$ is a unit vector pointing in one of the two opposing directions. Using eq. (3.14) for the definition of the center of mass velocity \mathbf{v}_{CM} and the Galilean transformation eq. (3.19) we can express the velocities of the two masses in the center of mass system before the collision as a function of the velocities before the collision in the laboratory system:

$$\mathbf{v}'_1 = \mathbf{v}_1 - \mathbf{v}_{CM} = \frac{m_2}{M} \cdot (\mathbf{v}_1 - \mathbf{v}_2)$$

$$\mathbf{v}'_2 = \mathbf{v}_2 - \mathbf{v}_{CM} = -\frac{m_1}{M} \cdot (\mathbf{v}_1 - \mathbf{v}_2)$$

With these two equations and using the relationships from eq. (3.34) we then obtain for the velocities after the collision in the laboratory system:

$$\tilde{\mathbf{v}}_1 = \tilde{\mathbf{v}}'_1 + \mathbf{v}_{CM} = |\mathbf{v}'_1| \cdot \hat{\mathbf{u}} + \mathbf{v}_{CM} = \frac{m_2}{M} |\mathbf{v}_1 - \mathbf{v}_2| \cdot \hat{\mathbf{u}} + \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{M} \quad (3.35a)$$

$$\tilde{\mathbf{v}}_2 = \tilde{\mathbf{v}}'_2 + \mathbf{v}_{CM} = -|\mathbf{v}'_2| \cdot \hat{\mathbf{u}} + \mathbf{v}_{CM} = -\frac{m_1}{M} |\mathbf{v}_1 - \mathbf{v}_2| \cdot \hat{\mathbf{u}} + \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{M} \quad (3.35b)$$

With that we are ready to revisit the collision scenario of fig. 3.13 we discussed before. However, this time, we will analyze the collision in the center of mass system. Regardless of the momentum of the incoming ball ① in the laboratory system with respect to ball ② at rest, in the center of mass system the incoming momenta of both balls will always be of equal magnitude and diametrically opposed as sketched in fig. 3.15. The assumption we made in the case of fig. 3.13 with respect to both balls shall apply again. They are perfectly smooth and $Q = 0$ in eq. (3.24), i.e., no kinetic energy is converted into rotation energy. What happens in the center of mass system on impact is that for each ball the component of the momentum which is perpendicular to the line of impact remains unchanged and the component which is parallel to the line of impact becomes inverted. With $\tilde{\mathbf{p}}'_1 \parallel = -\mathbf{p}'_1 \parallel$ and $\tilde{\mathbf{p}}'_2 \parallel = -\mathbf{p}'_2 \parallel$ we can read off from fig. 3.15 for ball ① and ball ② the following relationship between their respective momenta before and after the collision

$$\tilde{\mathbf{p}}'_1 = \mathbf{p}'_1 + 2 \cdot \tilde{\mathbf{p}}'_1 \parallel = \mathbf{p}'_1 - 2 \cdot \mathbf{p}'_1 \parallel \quad (3.36a)$$

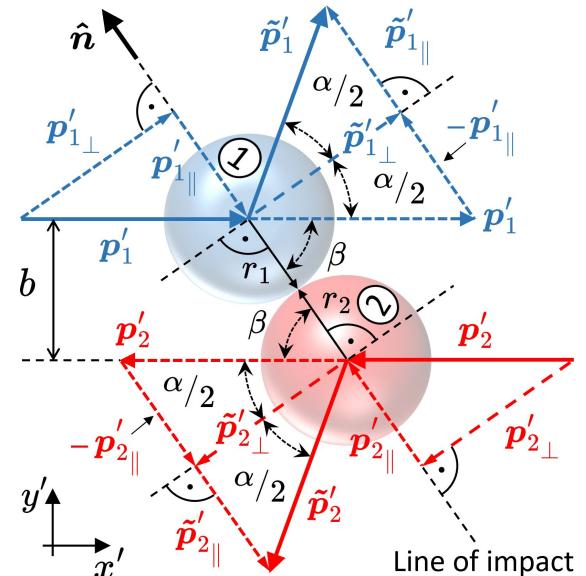


Fig. 3.15

and

$$\tilde{\mathbf{p}}'_2 = \mathbf{p}'_2 + 2 \cdot \tilde{\mathbf{p}}'_{2\parallel} = \mathbf{p}'_2 - 2 \cdot \mathbf{p}'_{2\parallel} \quad (3.36b)$$

If we denote with $\hat{\mathbf{n}}$ the unit vector in the direction of the line of impact, pointing from the center of ball ① to the center of ball ②, we can express the momentum components parallel to the line of impact before the collision in terms of the unit vector $\hat{\mathbf{n}}$. The magnitudes of the parallel momentum components of ball ① and ball ② in the center of mass system are given by

$$|\mathbf{p}'_{1\parallel}| = |\mathbf{p}'_1| \cos \beta \quad \text{and} \quad |\mathbf{p}'_{2\parallel}| = |\mathbf{p}'_2| \cos \beta$$

With $\mathbf{p}'_{1\parallel}$ being anti-parallel to $\hat{\mathbf{n}}$ and $\mathbf{p}'_{2\parallel}$ being parallel to $\hat{\mathbf{n}}$ we can rewrite eq. (3.36a) and eq. (3.36b) as

$$\tilde{\mathbf{p}}'_1 = \mathbf{p}'_1 + 2 \cdot \hat{\mathbf{n}} |\mathbf{p}'_1| \cos \beta \quad \text{and} \quad \tilde{\mathbf{p}}'_2 = \mathbf{p}'_2 - 2 \cdot \hat{\mathbf{n}} |\mathbf{p}'_2| \cos \beta$$

Now we rearrange the right sides of these two equations such that they begin to look like eq. (3.34). We do that by pulling out the respective mass times the absolute value of the respective velocity as a factor on the right sides and then divide both sides by the respective mass. The results we get are

$$\tilde{\mathbf{v}}'_1 = |\mathbf{v}'_1| \left[2\hat{\mathbf{n}} \cos \beta + \frac{\mathbf{v}'_1}{|\mathbf{v}'_1|} \right] \quad \text{and} \quad \tilde{\mathbf{v}}'_2 = -|\mathbf{v}'_2| \left[2\hat{\mathbf{n}} \cos \beta - \frac{\mathbf{v}'_2}{|\mathbf{v}'_2|} \right]$$

What we observe first is that in these two equations the expressions enclosed in square brackets are identical. The second terms inside the brackets are the unit vectors $\hat{\mathbf{v}}'_1$ and $\hat{\mathbf{v}}'_2$ in the directions of \mathbf{v}'_1 and \mathbf{v}'_2 . But we already know that $\hat{\mathbf{v}}'_1 = -\hat{\mathbf{v}}'_2$ as they must be diametrically opposed. Keeping that in mind, if we now compare these two equations with the equations in eq. (3.34) we find that the identical expression inside the square brackets is just the unit vector introduced in eq. (3.34):

$$\hat{\mathbf{u}} = 2\hat{\mathbf{n}} \cos \beta + \frac{\mathbf{v}'_1}{|\mathbf{v}'_1|} = 2\hat{\mathbf{n}} \cos \beta - \frac{\mathbf{v}'_2}{|\mathbf{v}'_2|}$$

To calculate $\hat{\mathbf{u}}$ we could choose to evaluate either of these two expressions, here we will go with the first one. For the unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}'_1$ we read from fig. 3.15

$$\hat{\mathbf{n}} = \begin{pmatrix} -\cos \beta \\ \sin \beta \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{v}}'_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

With that $\hat{\mathbf{u}}$ becomes

$$\hat{\mathbf{u}} = \begin{pmatrix} 1 - 2 \cos^2 \beta \\ 2 \sin \beta \cos \beta \end{pmatrix} = \begin{pmatrix} -\cos 2\beta \\ \sin 2\beta \end{pmatrix} \quad (3.37)$$

In the example fig. 3.13 we saw that the scattering angles ϑ and φ in the laboratory system were connected by $\vartheta + \varphi = 90^\circ$. For the scattering angles α and β in the center of mass system we can determine from fig. 3.15 that they are connected by the relationship $\alpha/2 + \beta = 90^\circ$. With $2\beta = \pi - \alpha$ the unit vector $\hat{\mathbf{u}}$ in fig. 3.15 as a function of the scattering angle α is given by:

$$\hat{\mathbf{u}} = \begin{pmatrix} -\cos 2\beta \\ \sin 2\beta \end{pmatrix} = \begin{pmatrix} -\cos \alpha \\ \sin \alpha \end{pmatrix}$$

The angles α and β are determined by the collision geometry in fig. 3.15, i.e., by the impact parameter b and the radii r_1 and r_2 of the balls via

$$\sin \beta = \sin \frac{\alpha}{2} = \frac{b}{r_1 + r_2} \quad (3.38)$$

This relationship in the center of mass system between scattering angle β and impact parameter b and ball radii is identical to eq. (3.27) which we derived in the laboratory system between the scattering angle φ and these parameters. The right hand sides of eq. (3.27) and eq. (3.38) express a geometry relationship which can only be positive definite and so eq. (3.27) and eq. (3.38) only tells us something about the absolute values $|\varphi|$ and $|\beta|$, namely that they must be identical. With eq. (3.38) we can rewrite the unit vector $\hat{\mathbf{n}}$ pointing in the direction of the line of impact as

$$\hat{\mathbf{n}} = \frac{1}{r_1 + r_2} \begin{pmatrix} -\sqrt{(r_1 + r_2)^2 + d^2} \\ d \end{pmatrix}$$

Now that we have determined all relevant quantities for the collision in the center of mass system we can transfer the results back to the laboratory system. In the laboratory system the velocities of the balls ① and ② are

$$\mathbf{v}_1 = \begin{pmatrix} v \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Inserting these velocities and the unit vector $\hat{\mathbf{u}}$ from eq. (3.37) in eq. (3.35a) and eq. (3.35b) we get for the velocities of the two balls in the laboratory system after the collision

$$\tilde{\mathbf{v}}_1 = \frac{v}{M} \begin{pmatrix} m_1 - m_2 \cos 2\beta \\ m_2 \sin 2\beta \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{v}}_2 = \frac{2m_1 v \cos \beta}{M} \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix}$$

As illustrated in the momentum diagrams in fig. 3.16, ϑ and φ designate the scattering angles in the laboratory system where ϑ denotes the deflection of ball ① with respect to its incoming direction and φ denotes the deflection of ball ②, also with respect to the incoming direction of ball ①. With the post collision velocities $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ determined we can calculate the angles ϑ and φ from

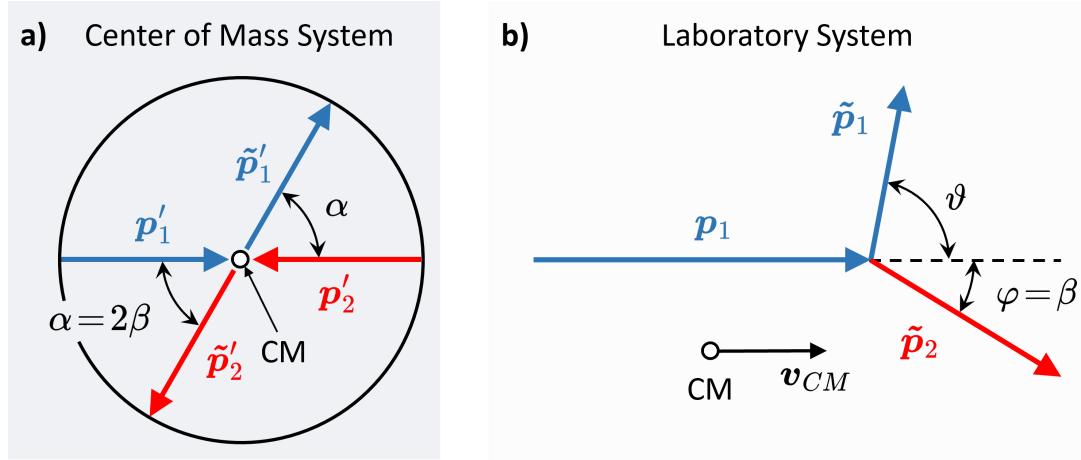


Fig. 3.16: Momentum diagrams of the non-central collision from fig. 3.15 in the center of mass system (a) and in the laboratory system (b). In this example $r_1 = r_2$, the collision parameter is set to $b = r_1 = r_2$ and $m_2 = 1.5 \cdot m_1$.

$$\tan \varphi = \frac{\tilde{v}_{2y}}{\tilde{v}_{2x}} \quad \text{and} \quad \tan \vartheta = \frac{\tilde{v}_{1y}}{\tilde{v}_{1x}}$$

Inserting the respective velocity components these equations become

$$\tan \varphi = -\frac{\sin \beta}{\cos \beta} = -\tan \beta \quad \text{and} \quad \tan \vartheta = \frac{m_2 \sin 2\beta}{m_1 - m_2 \cos 2\beta} \quad (3.39)$$

With the tangent being an odd function we find that $\varphi = -\beta$, meaning that the scattering direction of ball ② is to the lower right. Actually, ball ② will always be deflected to the lower right as φ is only a function of the collision parameters and not of the masses of the collision partners. Because collisions will only happen for impact parameters $b < r_1 + r_2$ the range of $\sin \beta$ is limited to $0 \leq \sin \beta \leq 1$ and with that β can only take the values $0 \leq \beta \leq \pi/2$. Hence, φ can only take values between $0 \leq \varphi \leq -\pi/2$.

Different from φ , ϑ does depend on the mass ratio of the collision partners. ϑ assumes its maximum and minimum values in the extreme cases $m_1 \gg m_2$ and $m_2 \gg m_1$

$$m_1 \gg m_2 \implies \tilde{v}_1 \rightarrow v \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \vartheta \rightarrow 0$$

$$m_2 \gg m_1 \implies \tilde{v}_1 \rightarrow v \cdot \begin{pmatrix} -\cos 2\beta \\ \sin 2\beta \end{pmatrix} \stackrel{\beta=0}{\implies} \vartheta \rightarrow \pi$$

The first case, $m_1 \gg m_2$, is that of forward scattering where the motion of a massive ball ① is only minimally impacted by a much lighter ball ②. In the second case, $m_2 \gg m_1$, ball ② is so much more massive than ball ① that the latter in the case of central impact gets back scattered in the direction from where it came from.

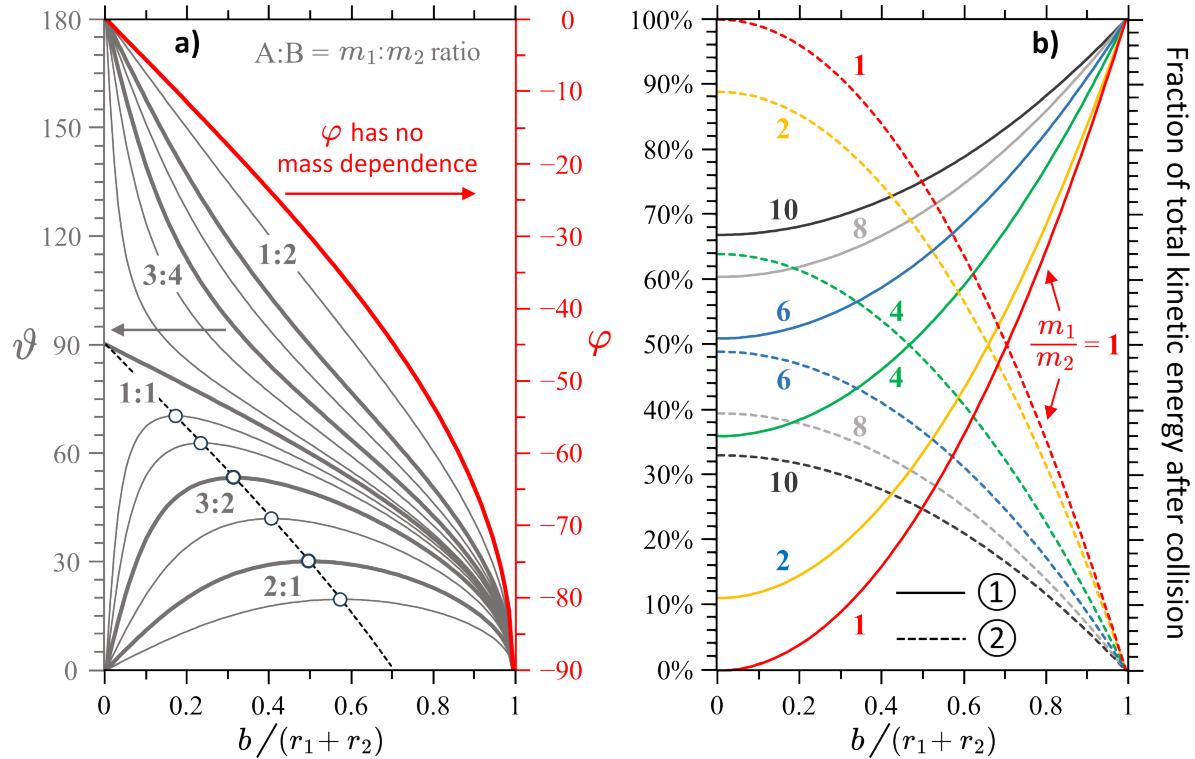


Fig. 3.17: (a) Scattering angles ϑ and φ as functions of the scattering geometry parameter $\sin \beta = b/(r_1 + r_2)$ for various mass ratios m_1/m_2 . (b) Partition of the total kinetic energy between balls ① (solid lines) and ② (dashed lines) after impact also as a function of the scattering geometry parameter for selected mass ratios m_1/m_2 .

In the case of $m_1 = m_2$ the equation for ϑ becomes

$$\tan \vartheta = \frac{\sin 2\beta}{1 - \cos 2\beta} = \frac{\cos \beta}{\sin \beta}$$

Therefore, $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$, the unit vectors in the directions of $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ in the case of $m_1 = m_2$ are

$$\tilde{\mathbf{v}}_1 = \begin{pmatrix} \sin \beta \\ \cos \beta \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{v}}_2 = \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix}$$

and with that

$$\tilde{\mathbf{v}}_1 \cdot \tilde{\mathbf{v}}_2 = 0 \quad \Rightarrow \quad \tilde{\mathbf{v}}_1 \perp \tilde{\mathbf{v}}_2$$

and therefore it follows for $m_1 = m_2$ that

$$\vartheta + \varphi = \frac{\pi}{2}$$

This is of course just as it should be as we found this result already earlier when discussing the non-central impact collision of equal masses (see fig. 3.14). Fig. 3.17a shows the

dependence of ϑ and φ on the scattering geometry parameter $\sin \beta$ from eq. (3.38) for various mass ratios $m_1:m_2$. As we already know from eq. (3.39), φ does not depend on the masses of the two collision partners and therefore there is only one curve in fig. 3.17a for φ as a function of $b/(r_1 + r_2)$. φ can range between $0 \leq \varphi \leq -90^\circ$. In the case of central impact, i.e., $\varphi = 0$ and $\vartheta = 90^\circ$, the velocity vectors of the two balls before and after the collisions are coincident with the line of impact and their relative directions and magnitudes depend on the mass ratio $m_1:m_2$. What a scattering angle of $\vartheta = 90^\circ$ means we can glean from fig. 3.17b which shows the partition of total kinetic energy between the two balls after the collision and how it depends on the mass ratio $m_1:m_2$. In the case of $m_1 = m_2$ we already know that for a central impact the balls exchange velocities, i.e., after the collision the kinetic energy of ball ① will be zero and the kinetic energy of ball ② will be equal to the kinetic energy ball ① had before the collision. This is just what fig. 3.17b shows in the case of $b/(r_1 + r_2) = 0$ and $m_1 = m_2$. For $m_1 > m_2$, ball ① will retain increasingly more of its kinetic energy and less will be transferred to ball ② as both balls now will move in the direction of where ball ① was headed. Of course the reverse happens in the case of $m_1 < m_2$. Fig. 3.17b does not show curves for which $m_1 < m_2$ but clearly, the energy partitioning is symmetrical with respect to switching ball ① and ball ②. However, fig. 3.17a tells us that for $m_1 < m_2$, ball ① will now move in the direction opposite to where it came from while ball ② will move in the direction into which ball ① was moving before the collision. As fig. 3.17a shows, for $m_1 > m_2$ we will observe forward scattering of m_1 ($0^\circ \leq \vartheta < 90^\circ$) while for $m_1 < m_2$ we will observe m_1 backward scattering from m_2 ($90^\circ < \vartheta \leq 180^\circ$).

For $m_1 > m_2$ the scattering angle curves for ϑ in fig. 3.17a go through a maximum which shifts towards lower values of $b/(r_1 + r_2)$ as the mass ratio $m_1:m_2$ approaches the value 1. To find those extreme values we take the derivative of $\arctan [\vartheta(\beta)]$ with respect to β in the interval $0^\circ < \vartheta \leq 90^\circ$ and determine its zeros which leads us to the equation

$$\frac{d}{d\beta} \arctan [\vartheta(\beta)] = \frac{1}{1 + \vartheta^2(\beta)} \cdot \frac{d}{d\beta} \vartheta^2(\beta) = 0$$

A quick calculation shows that the expression on the left will only become zero for $\cos 2\beta = m_2/m_1$. Hence, we find that in the range $0^\circ < \vartheta \leq 90^\circ$ the ϑ -curves in fig. 3.17a peak at

$$\sin \beta = \frac{b}{r_1 + r_2} = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_2}{m_1}}$$

The dashed curve in fig. 3.17a shows where the respective maximum values of the ϑ -curves fall. It intersects the ϑ -axis at $\vartheta = 90^\circ$ and the $\sin \beta$ -axis at $\sin \beta = 1/\sqrt{2}$.

Before we close this section and with that this chapter, we need to briefly come back to inelastic collisions. With the discussion above about elastic non-central impact collisions, which for the impact parameter $b = 0$ become central impact collisions, we saw that in the center of mass system the momentum vectors before as well as after the collision are diametrically opposed. That of course must also be the case in an inelastic collision as in the center of mass system the total momentum before as well as after the collision must be zero. Hence, while the scattering angle will depend on the specific scattering situation in any given inelastic collision scenario, the momentum vectors after an inelastic collision in the center of mass system will always be of equal magnitude and point in opposite direction. However, as sketched in fig. 3.18, the momentum vectors after the collision will be shorter than before the collision because of $Q > 0$ in eq. (3.24). With the total momentum in the center of mass system equal to zero eq. (3.32) holds for $Q \neq 0$ just as it does for $Q = 1$. What changes is eq. (3.32) which with $Q > 0$ becomes

$$\mathbf{v}'_1^2 = \tilde{\mathbf{v}}'_1^2 + \frac{m_2}{m_1 m_1 + m_2} \frac{2Q}{m_1 m_1 + m_2} \quad \text{and} \quad \mathbf{v}'_2^2 = \tilde{\mathbf{v}}'_2^2 + \frac{m_1}{m_2 m_1 + m_2} \frac{2Q}{m_2 m_1 + m_2}$$

Therefore eq. (3.34) will be replaced by

$$\tilde{\mathbf{v}}'_1 = \mathbf{u} \sqrt{\mathbf{v}'_1^2 - \frac{m_2}{m_1 m_1 + m_2} \frac{2Q}{m_1 m_1 + m_2}} \quad \text{and} \quad \tilde{\mathbf{v}}'_2 = -\mathbf{u} \sqrt{\mathbf{v}'_2^2 - \frac{m_1}{m_2 m_1 + m_2} \frac{2Q}{m_2 m_1 + m_2}}$$

If the energy loss occurring in the collision has no impact on the collision geometry or scattering behavior of m_1 and m_2 , the direction of the unit vector \mathbf{u} will be the same as if no energy loss happened at all and we can just transfer the results with respect to the scattering angles φ and ϑ from the case $Q = 0$ which we just discussed at length. Of course, the total kinetic energy which m_1 and m_2 possessed before the collision and the kinetic energies which m_1 and m_2 will have after the collision are going to be different from what fig. 3.17b shows for the case $Q = 0$ as we must account for the energy loss Q .

We have not discussed superelastic collisions here, that is collisions where $Q < 0$. Clearly, in such collisions the momentum vectors in the center of mass system would be longer after the collision than they were before the collision. However, in the center of mass system they would still be diametrically opposed before as well as after the collision just

Center of Mass System

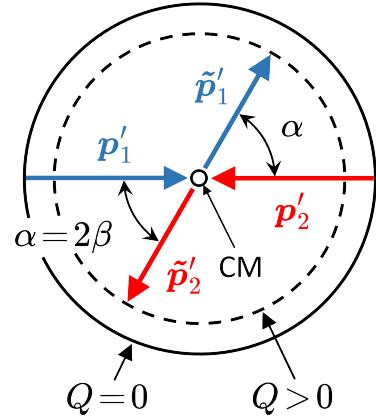


Fig. 3.18

as we discussed this above for inelastic collisions with $Q > 0$. Everything we found out with respect to $Q > 0$ collisions applies just as well to $Q < 0$ collisions, we only must replace Q with $-Q$.

4. Energy, Potentials and Fields

4.1 Energy, Work and Power

When we refer to velocity, acceleration, or mass we usually have no problem to explain what those things are. For energy this is different. We know different forms of energy but we have no explanation as to what energy really is. We can point to sources of energy and measure the quantity of energy but there is no specific property an object carries around which we can point to and say that this is energy. As it is, since Albert Einstein famous discovery more than a century ago, we know that mass multiplied by the speed of light squared happens to be a form of energy, the famous $E = mc^2$. Humans have learned to access this energy through nuclear fission, the process heat of which generates the electrical energy nuclear reactors provide. We have also used this energy source to create weapons far more destructive than what human had ever known before, nuclear bombs. So far we have not been able to use this relationship between energy and mass to produce energy through nuclear fusion, just like stars do that. However, we have learned to use nuclear fusion to build even more devastating weapons than atomic bombs, thermonuclear weapons. Nuclear energy is one of six forms of energy we know, the others being chemical, electrical, radiant, mechanical, and thermal energy.

Even though we do not know what energy is, we know that the quantity of energy in a closed system never changes. One form of energy can change into another form of energy but the amount of energy in a closed system remains constant. In many ways this is part of our daily experience, when we warm our food we convert electrical or chemical energy into thermal energy which cooks our food and of course warms the surrounding air volume. Our cars convert chemical or electrical energy into mechanical energy and when we hit the brakes, that mechanical energy is converted into heat energy or, as can be the case in new cars, it is even partially converted back into electrical energy. Our Sun is powered by nuclear energy, part of which is converted into radiant energy

which organisms such as plants convert into chemical energy and which we nowadays also convert into electrical energy. Because conservation of energy is so fundamental in physics, lets frame it here

The law of conservation of energy

In an isolated system the total energy remains constant. Energy can neither be created nor destroyed, it can only be converted from one form of energy to another.

As an immediate consequence of the law of conservation of energy it follows that it is impossible to construct so-called perpetum mobile machines, that is machines which can keep going indefinitely without any external source of energy. Similar to alchemists, practically the scientific ancestors of today's chemists, searching for the philosopher's stone which they believed would allow them to turn ordinary metals into gold, physicists and tinkerers kept devising countless machines which they hoped to proof out as perpetum mobiles. While modern physics has discovered that it is possible to convert one chemical element into another, it has also proven that a perpetuum mobile is a definite impossibility. Hence, patent offices will outright reject patent applications claiming some sort of perpetum mobile as evaluating such claims would just be a waste of time.

4.1.1 Potential Energy

When lifting an object, we use our muscle force to counteract the weight force $F = mg$. We all know by experience that it costs energy to lift a weight, and the heavier the weight, the more energy is required to lift it. Hence, the energy change of an object changing its position in a gravity field must be proportional to its mass. Further, we also know that if we lifted the same object not on Earth but on the Moon's surface that this would be easier because the Moon's gravity is lower. Therefore, the energy change of any object lifted in a gravity field must be proportional to the weight force acting on it. Putting both observations together, the energy change of an object lifted from the ground must be proportional to the weight force $F = mg$.

But what about the height to which we lift an object, how does it factor into an objects energy? If we for example store something heavy onto a shelf that has a lower board one meter above ground and a higher board at two meter above ground we intuitively understand that our lifting work will impart on two otherwise exactly identical objects different amounts of energy if we lift one of them to the lower and the other one to the higher board.

The kind of energy which depends on the position of an object in a force field relative to something else, here with respect to Earth's surface, is what physicists call potential energy which they usually denote with the letter U . To understand how this potential energy U of bodies changes when lifting them onto a shelf we will have the process filmed from the other side of the shelf so we are not in the way and then pick the four frames shown in fig. 4.1 for further examination.

In frame ① of the filmed process shown in fig. 4.1, two heavy balls **A** and **B** of equal weight sit on the ground floor next to each other and both levels of the shelf, one at the height of $h = 1\text{ m}$ and the other at the height of $h = 2\text{ m}$, are still empty. The situation for both balls in frame ① is identical, hence, their potential energy should also be identical. In frame ② we see that ball **A** has been lifted onto the first level of the shelf at $h = 1\text{ m}$. Both balls continue to have the same weight and gravity at one meter height we can consider to be identical with gravity on the ground. The only parameter in which their situations differ is that they sit now at different heights. Whatever potential energy we imparted on ball **A** by lifting it must be proportional to the height to which we have moved it. Frame ③ captures us halfway, i.e., at the first level, lifting ball **B** to the second level of the shelf. At this very instant, the situation for both balls is identical again and hence their respective potential energies should be identical. Finally, in frame ④ the lifting of ball **B** to the second level has been completed. Now, what can we say about the potential energies of the two balls shown in frame ④. Nothing changed for ball **A** since frame ② and his potential energy should still be proportional to $h = 1\text{ m}$ when compared to the situation in frame ①. As for ball **B**, in frame ④ we see the same step repeated as in frame ③, the only difference being that the very same lift for ball **B** started at $h = 0\text{ m}$ in frame ③ whereas in frame ④ it started at $h = 1\text{ m}$. Hence, in frame ④ ball **B** must have twice the potential energy of ball **B** in frame ③. Putting all of the above together, the potential energy U of an object we lift in Earth gravity field to the height h must then be given by

$$U = mgh$$

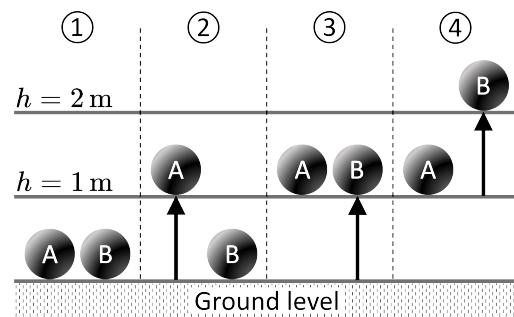


Fig. 4.1

Potential energy gets its name from the fact that it is essentially stored energy which can be released. When we lift an object we increase its potential energy and when we release this object this potential energy is converted into kinetic energy as the object falls to

the ground. The kinetic energy with which the object then hits the ground will again be converted into other forms of energy. If we drop an egg we know what will happen, the kinetic energy will be converted into an egg mess; if we drop a bouncy ball, the kinetic energy will be converted into elastic energy and then reconverted into kinetic energy which in turn becomes potential energy only to become kinetic energy again and so forth. Of course, there are also energy losses in the process and eventually the bouncy ball will stop to bounce. The potential energy which a massive object possesses because of its position in the gravitational field of Earth is just one type of potential energy. A similar case is the potential energy a charged object will possess as it finds itself at different positions in the electrical field of another charged object. Gravitational and electromagnetic fields are generated by massive or charged bodies so in these cases we are really talking about the potential energy being a function of the relative position of objects to each other. There are other forms of potential energy where this is different, such as with the mechanical potential energy of an extended or compressed spring or the potential energy stored in an object in the form of chemical energy.

4.1.2 Work and Power

Lifting weights such as the two balls in the above example is what we call work. In the case example above we converted some of our body's energy into the potential energy of two balls, thereby depleting the amount of energy stored in our body. To replenish our body's energy content we eat, this is how organisms such as us store energy. Doing work is the equivalent of transferring some of this energy to other objects. In general, the quantity of work we do is determined by the force we have to overcome, in the above example the weight force, and by the distance we are pushing against this force. Forces have of course a direction and if we move an object in a direction orthogonal to that force we are not doing any work. In the above example, if we move any of the balls laterally on the ground level or on the lower or upper board of the shelf, we are not doing any work as the lateral component of the weight force is zero. Of course, this requires us to assume that there is zero friction and we also have to neglect that to change the state of motion of one of the balls from being at

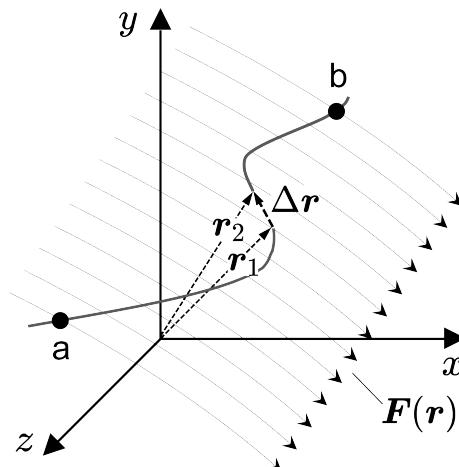


Fig. 4.2

rest to rolling requires energy. But the later is not potential energy but what physicists call kinetic energy which we impart on the ball by transferring a lateral movement of our hand into a lateral movement of the ball. Fig. 4.2 shows a point mass moving from point **a** to point **b** in a force field $\mathbf{F}(\mathbf{r})$. For the incremental work ΔW done by incrementally moving the point mass from point \mathbf{r}_1 to \mathbf{r}_2 and thereby covering the distance $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ in the force field $\mathbf{F}(\mathbf{r})$ we can write

$$\Delta W = \mathbf{F}(\mathbf{r})\Delta\mathbf{r}$$

where $\mathbf{F}(\mathbf{r})$ denotes the force vector at the position \mathbf{r} midway between \mathbf{r}_1 and \mathbf{r}_2 . Dividing the path between the points **a** and **b** in many small path increments and adding everything up we can write for the total work done to move the point mass from **a** to **b**

$$W = \sum_{i=1}^N \Delta W_i = \sum_{i=1}^N \mathbf{F}(\mathbf{r}_i)\Delta\mathbf{r}_i$$

where $\Delta\mathbf{r}_1$ is the first path increment after point **a** and $\Delta\mathbf{r}_N$ is the last path increment before point **b**. In the limit $N \rightarrow \infty$ the sum in the equation above becomes the so-called work integral

$$W = \int_a^b \mathbf{F}(\mathbf{r})d\mathbf{r} \quad (4.1)$$

Using this formula one can calculate the work being done on a mass moved in a gravity field like we just did in the example above. But it works just as well to calculate for example the work done on an electrically charged particle moving in an electromagnetic field. In the first case one refers to the respective changes in potential energy due to the work done as gravitational potential energy changes whereas in the case of moving a charge in an electrostatic force field of another charge, physicists speak of electrical potential energy. For an example illustrating mechanical work we only need to look at mechanical springs and their restoring forces which we already discussed earlier. With the spring force being $F = -kz$, the work done by the spring to balance the force weight of a mass extending the spring by z is

$$W = \int_0^z F dz' = -k \int_0^z z' dz' = -\frac{kz^2}{2}$$

The work done carries a minus sign because the work done does not result in the spring losing energy but gaining energy. This stored potential energy transferred to the spring is of course positive and therefore we have for the potential energy of the spring

$$U_{Spring} = -W = \frac{kz^2}{2} \quad (4.2)$$

The above works also in reverse, i.e., instead of extending the spring we can also compress it. In the case of compression, the extension will be negative, but since we will have to integrate now from $-z$ to 0 the resulting work is again negative as it should be, because regardless if we extend or compress a spring the energy it stores increases. The result is that the potential energy of a spring as a function of the displacement from its unloaded position is a parabola.

The integrand in eq. (4.1) is the vector product of the force \mathbf{F} acting on a point mass and the incremental distance $d\mathbf{r}$ the point mass is being moved in the respective force field. Evidently, only motion of the point mass in the direction of the force contributes to the work being done but not any point mass motion in a direction perpendicular to the acting force \mathbf{F} . Taking again the force of gravity which in our coordinate system of choice has the components $\mathbf{F} = (0, 0, -mg)$ only motion along the z axis will contribute to the work integral in eq. (4.1). Therefore, moving a point mass in Earth's gravity field from a height z_1 to a height z_2 above ground requires the work

$$W = \int_{z_1}^{z_2} \mathbf{F}(\mathbf{r}) d\mathbf{r} = \int_{z_1}^{z_2} \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

from which follows

$$W = -mg(z_2 - z_1) = -mgh$$

with $h = z_2 - z_1$ being the height difference to which the point mass has been lifted against the force of gravity. This energy expended to do the work of lifting the point mass has increased its potential energy by $U_{Gravity}$. Because the net energy of the closed system must stay the same, i.e., $W + \Delta U_{Gravity} = 0$ must hold, it follows that

$$\Delta U_{Gravity} = mgh = -W$$

To summarize: With respect to the potential energy of a mass being moved in a gravitational field, all that matters is the vertical distance we lift or lower an object in Earth's gravity field as only that will change the potential energy of the object and not any lateral movements. Trivially, if we first lift / lower an object and then lower / lift it back to its original position in the gravity field its potential energy must be identical to the potential energy the object had before we began to move it around. Forces which like the force of gravity give rise to this property physicists call conservative forces.

Conservative force

The work done by a conservative force in moving a mass between two points is independent of the path. For a closed path the work done is exactly zero, i.e., no energy is lost or in different words: on a closed path energy is conserved, hence the label conservative force.

With respect to a reference point \mathbf{r}_0 where the potential energy of a point mass shall be by definition zero the potential energy as a function of the point mass position \mathbf{r} is given by

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\xi) d\xi \quad (4.3)$$

The potential energy $U(\mathbf{r})$ gained in the gravitational field is just the negative of the work $W(\mathbf{r})$ which is a measure of the energy that had to be expended working against the force of gravity to move the point mass from \mathbf{r}_0 to \mathbf{r} . With

$$W(\mathbf{r}) + U(\mathbf{r}) = 0$$

the total energy does not change. $U(\mathbf{r}) = -W(\mathbf{r})$ is nothing but a reflection of the conservation of energy and holds for all conservative forces. Turned around, this obviously also makes the spring force discussed above a conservative force even though the mechanical potential energy of the spring is not related to any force field such as the gravitational or the electromagnetic forces are.

In eq. (4.3), $U(\mathbf{r})$ is a function of the upper integration limit \mathbf{r} of the integral. Because of that the respective force components $\mathbf{F} = (F_x, F_y, F_z)$ are given by

$$F_x = -\frac{\partial U(x, y, z)}{\partial x} \quad ; \quad F_y = -\frac{\partial U(x, y, z)}{\partial y} \quad ; \quad F_z = -\frac{\partial U(x, y, z)}{\partial z}$$

or respectively in vector form

$$\mathbf{F} = -\text{grad } U(\mathbf{r}) \quad (4.4)$$

The rate of change of the potential energy $U(r(t))$ is given by

$$\begin{aligned} \frac{dU(\mathbf{r}(t))}{dt} &= \frac{d}{dt} U(x(t), y(t), z(t)) \\ &= \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt} = \text{grad } U \cdot \frac{d\mathbf{r}}{dt} = \nabla U \cdot \mathbf{v} \end{aligned}$$

With $\mathbf{F} = -\text{grad } U(\mathbf{r})$ this becomes

$$\frac{dU}{dt} = \nabla U \cdot \mathbf{v} = -\mathbf{F} \cdot \mathbf{v} \quad (4.5)$$

The measure of how fast work is being done or how fast various forms of energy are being produced or consumed in a given amount of time is what physicists refer to as power, usually denoted with the letter P . Taking the example of work we find that (with \mathbf{F} independent of time) power equals force times velocity:

$$P = \frac{dW}{dt} = \frac{\mathbf{F} d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v} \quad (4.6)$$

Therefore

$$\frac{dU}{dt} = -P = -\frac{dW}{dt}$$

4.1.3 Kinetic Energy

Now, as sketched in fig. 4.3, we will consider a mass m that shall move with the constant speed v_0 and on which a constant acceleration a shall act beginning at the point in time $t = 0$:

$$\ddot{x} = a \quad \text{and} \quad \dot{x} = v(t) = a \cdot t + v_0$$

Integrating again over time gives the result we already know from eq. (2.5), which is that at time t the mass m will be at the location

$$x(t) = x_0 + v_0 t + \frac{1}{2} a \cdot t^2$$

Here x_0 is the location where the mass m was at the time $t = 0$. Without restriction, one can choose $x_0 = 0$ and $v_0 = 0$. In the time interval t , the mass m then has traveled the distance

$$x(t) = \frac{1}{2} a \cdot t^2$$

and moves with the velocity

$$v(t) = a \cdot t \quad \Rightarrow \quad t = \frac{v(t)}{a}$$

If one inserts this expression for t into the equation for the distance traveled, one obtains

$$x(t) = \frac{1}{2} \frac{v^2}{a}$$

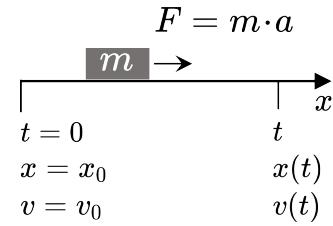


Fig. 4.3

The energy E_{kin} which the mass m possesses at time t is equivalent to the work required to move it from x_0 to $x(t)$ (no friction losses). This work is the product of the force that had to be expended times the length of the distance over which this was necessary, i.e.,

$$E_{kin} = F \cdot x(t) = ma \cdot \frac{1}{2} \frac{v^2}{a} = \frac{1}{2} mv^2 = T \quad (4.7)$$

The quantity E_{kin} , often also denoted by T , is the so-called kinetic energy that the mass m possesses at the time t which in this case is its translation energy. As we will see later, kinetic energy is also associated with the rotation of a body. The derivation given here for the kinetic energy associated with the translation of a point mass shows that the work done has been completely converted into kinetic energy of the point mass. This is a variant of the law of conservation of energy which states that the total energy in a closed system is constant. In a closed system, energy is neither lost nor can energy be gained, it can only be converted from one form of energy to another form of energy. With the assumption that the mass m of the point mass is a constant, the time derivative of the kinetic energy T becomes

$$\frac{dT}{dt} = \frac{d}{dt} \left(\frac{m}{2} \mathbf{v}^2 \right) = m \cdot \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = m \mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v} = P(t) \quad (4.8)$$

where $P(t)$ with eq. (4.6) denotes the power. Hence, eq. (4.8) tells us that the rate of change of the kinetic energy of a point mass equals the power expended by the force acting on the point mass. A comparison of eq. (4.8) and eq. (4.5) shows that

$$\frac{dT}{dt} = - \frac{dU}{dt} \quad \Rightarrow \quad \frac{d(T+U)}{dt} = 0$$

Evidently, the rate of change of the sum of kinetic and potential energy of a point mass is zero. Hence, the sum of kinetic and potential energy of a point mass is a constant which again is a specific statement of the law of conservation of energy. In other words: E_0 , the sum of the kinetic and potential energy of a point mass at the location \mathbf{r}_0 at the time t_0 is the same as E_1 , the total energy of the point mass at the location \mathbf{r}_1 at the time t_1

$$T + U = E \quad ; \quad E = E_0 = E_1 = const \quad (4.9)$$

Just as the potential energy of a point mass being moved in a gravitational field from \mathbf{r}_0 to \mathbf{r} is given by eq. (4.3), one obtains the change in kinetic energy associated with this movement through integration of eq. (4.8):

$$T(t_1) - T(t_0) = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \mathbf{v}(t) dt = \int_{t_0}^{t_1} P(t) dt \quad (4.10)$$

where t_0 is the time at which the point mass was located at \mathbf{r}_0 and t_1 is the time the point mass arrived at \mathbf{r} . How long it takes the point mass to get from \mathbf{r}_0 to \mathbf{r} is something we can calculate using eq. (4.9). For simplicity, let's consider a point mass moving parallel to the x -axis. Then it follows from eq. (4.9) that

$$\frac{m}{2} \dot{x}(t)^2 + U(x(t)) = E = \text{const}$$

Solving for the velocity $\dot{x}(t)$ and separation of the variables gives

$$\dot{x}(t) = \sqrt{\frac{2}{m}[E - U(x(t))]} \quad \text{and} \quad \frac{dx}{\sqrt{\frac{2}{m}[E - U(x)]}} = dt$$

Simple integration then yields

$$t - t_0 = \int_{x_0}^x \frac{d\xi}{\sqrt{\frac{2}{m}[E - U(\xi)]}} \quad \rightarrow \quad t = t(x) \Rightarrow x = x(t)$$

This shows how in principle one can obtain the trajectory curve of a point mass moving in a gravitational field from eq. (4.9). To be able to do that one must however know $U(\mathbf{r})$. Of course, this also applies to movements in other force fields generated by conservative forces such as for example for the movements of charged particles in an electromagnetic field.

4.2 Potentials and Fields

Newton's law of gravity (see section 3.1.2) has one peculiarity that already troubled Newton and his contemporaries and that is action at a distance. Because there is no medium transmitting the force of gravity, the mutual interaction of the two masses is instantaneous even if they are arbitrarily far away from each other; any slight change in the position of one planet would for example have to affect the other planet immediately in what would later be called action at a distance. In Newton's own words:¹

“...tis unconceivable, that inanimate brute matter should (without a divine impression) operate upon & affect other matter without mutual contact: as it must, if gravitation be essential and inherent in it.”

¹Quoted from a letter Newton wrote 1692/93 to the classical scholar Richard Bentley (1662-1742).

We know today that Newton and his contemporaries were right in harboring suspicions regarding action at a distance. However, Newton's equation worked exceedingly well and does so to this day throughout the galaxy where it is still a good enough approximation to solve most problems with it. Coulomb's law, discovered by Charles Augustine de Coulomb (1736 - 1806) in 1785 and named after him, describing the force acting between two electric point charges has a similar structure to Newton's law of gravity:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}|^3} \mathbf{r} = q_1 \mathbf{E}(\mathbf{r}) \quad (4.11)$$

Here, $\mathbf{E}(\mathbf{r})$ is the electric field generated by a stationary charge q_2 located at the coordinate origin which the charge q_1 experiences at the position \mathbf{r} . Strictly speaking, in electrostatics there are no moving charges and the way to understand Coulomb's law is that one of the charges, let's call it q , is just a tiny test charge which is being put in different locations in the electrostatic field produced by a stationary charge Q . In this way, one probes the strength of the electric field originating from the position of Q at those specific locations. If q is a tiny enough charge, so it does not alter the electric field produced by Q , then we can also consider q as a moving test charge in the electrostatic field produced by Q . While that strictly is not electrostatics any more, electrodynamic effects due to the tiny moving charge are considered negligible and hence solving the equations of motion assuming that Coulomb's law applies is a good approximation. With respect to Newton's law we do not have to consider any such constraints as the masses m_1 and m_2 in Newton's law (see eq. (3.7) and eq. (3.8)) can be arbitrarily large and can move with respect to each other without impacting the validity of Newton's law. Even though Coulomb's law and Newton's law are structurally similar they differ in this respect. Another difference is of course that in contrast to the gravitational force, which only knows one charge (namely the mass), the electromagnetic interaction knows positive and negative charges. The gravitational force always has an attractive effect, the Coulomb force can have both an attractive and a repulsive effect, and it has no effect at all on uncharged bodies.

To stress the similarities with Newton's law we can write the vector form of Coulomb's law in terms of the mutual interaction of both charges just as we did in section 3.1.2 for Newton's law. Keeping the same convention as used in section 3.1.2, \mathbf{F}_{12} shall be the force which a charge q_2 exerts on a charge q_1 which finds itself at the position \mathbf{r}_{21} as measured from the position of q_2 . Since the force is repulsive if both charges are positive or both charges are negative as shown in fig. 4.4a, we must not put a minus sign in front of the force equation, i.e.,

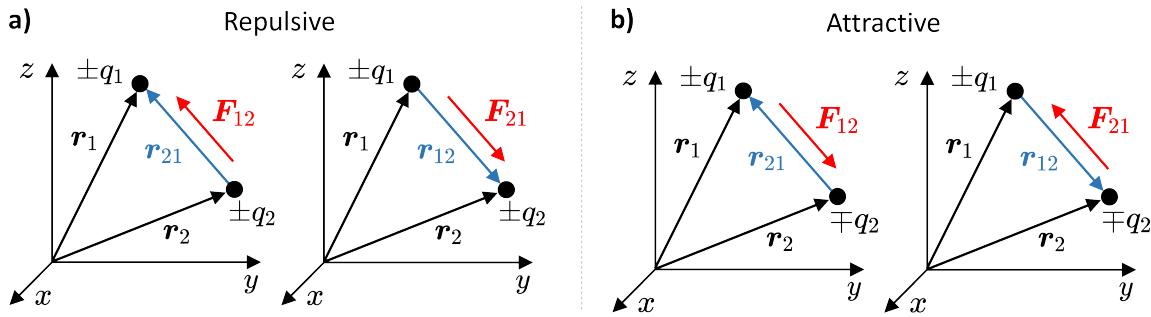


Fig. 4.4: Orientations of distance and force vectors in Coulomb's law in vector form: (a) for same polarity charges; (b) for opposite polarity charges. (Compare the respective diagrams for Newton's law in fig. 3.7.)

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{21}|^3} \mathbf{r}_{21} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} \quad (4.12)$$

$$\mathbf{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{21}|^3} \mathbf{r}_{21} \quad (4.13)$$

For unlike charges as shown in fig. 4.4a the force becomes attractive as the respective equation for \mathbf{F}_{12} and \mathbf{F}_{21} become

$$\mathbf{F}_{12} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{21}|^3} \mathbf{r}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} \quad (4.14)$$

$$\mathbf{F}_{21} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_{21}|^3} \mathbf{r}_{21} \quad (4.15)$$

Obviously, it follows in either case, repulsive or attractive interaction, that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

As long as we keep the limitations with respect to the size of the test charge in mind there is no difference between Newton's law and Coulomb's law except that Newton's force of gravity is always attractive while the Coulomb force can be attractive or repulsive. Another commonality is that just like Newton's law, Coulomb's law seemingly required action at a distance. However, with the development of the theory of electromagnetism it became increasingly clear that nothing could move faster than the speed of light. Since action at a distance with respect to Newton's and Coulomb's laws was instantaneous interaction between masses or charges however far separated the speed of interaction had to be infinite. The solution to this problem was to assume the existence of gravitational and electromagnetic fields permeating space which at every point would have a test mass or a test charge experience the strength of the gravitational potential or the electrical potential at the respective position of the test mass or test charge. In this way, instead

for the interaction having to travel at infinite speed as required by action at a distance, the interaction between two masses or charges happens through intermediating fields permeating space.

Such fields permeating space can be scalar fields, vector fields or tensor fields as the respective field at each point in space is either characterized by a scalar, a vector or a tensor. The gravity potential is an example for a scalar field as at every point in space its value is a scalar value. At Earth's surface, the gravity field generated by Earth's mass has the strength g ; as one moves away from Earth, the value of g decreases (see eq. (3.7)). Just like the gravitational potential energy is defined as the amount of work required to move a test mass in a gravity field from point A to point B (see eq. (4.1)), one can define an electrostatic potential energy as the amount of work required to move an electric test charge from point A to point B in an electrostatic field. Hence the electrostatic potential is associated with a scalar field just as the gravitational potential. Different from that, the electromagnetic field is an example of a vector field as at each point in space it is associated with two vector quantities, the electric field strength $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field strength $\mathbf{B}(\mathbf{r}, t)$, or more precise, with the magnitude $|\mathbf{E}|$ and the direction of \mathbf{E} and the magnitude $|\mathbf{B}|$ and the direction of \mathbf{B} . Examples for a tensor fields are rigid bodies under stress and / or strain which are quantified at each point in the body by a stress and / or strain tensor.

In this chapter we restrict our discussion to a basic understanding of the field concept in the context of the scalar fields associated with gravitational and the electrostatic interaction, i.e., the gravitational and the electrostatic potential fields. On the side, we will also learn a little about the respective vector fields associated with the gravity force and the electrostatic force.

We will begin with the gravity force. To keep things simple we will locate a large mass M in whose gravitational field a smaller mass m shall move at the coordinate origin. With eq. (4.3) and the force of gravity \mathbf{F}_G , the potential energy U_G of the mass m associated with the gravity field is then given by:

$$U_G(\mathbf{r}) = - \int_0^r \mathbf{F}_G(\boldsymbol{\xi}) d\boldsymbol{\xi} = m \int_0^r G \frac{M}{|\boldsymbol{\xi}|^3} \cdot \boldsymbol{\xi} d\boldsymbol{\xi} = m\Psi(\mathbf{r}) \quad (4.16)$$

Similarly, using eq. (4.11) for the Coulomb force \mathbf{F}_C we can write the potential energy U_E of a test charge associated with the electrostatic field generated by a charge Q located at the coordinate origin as

$$U_E(\mathbf{r}) = - \int_0^r \mathbf{F}_C(\boldsymbol{\xi}) d\boldsymbol{\xi} = -q \int_{r_0}^r \mathbf{E}(\boldsymbol{\xi}) d\boldsymbol{\xi} = q\Phi(\mathbf{r}) \quad (4.17)$$

where the electric field $\mathbf{E}(\mathbf{r})$ generated by a stationary point charge Q is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{|\mathbf{r}|^3} \cdot \mathbf{r} \quad (4.18)$$

$\mathbf{E}(\mathbf{r})$ is a vector field while $\Phi(\mathbf{r})$ in eq. (4.17) is a scalar field which reflects the electrostatic potential a charge q will experience at any given point of the electric field $\mathbf{E}(\mathbf{r})$ produced by the stationary point charge Q . Can we associate a likewise vector field with gravity? Yes, indeed we can. Just as we can write the force acting on an electric charge q in an electrostatic field as

$$\mathbf{F}_C(\mathbf{r}) = q\mathbf{E}(\mathbf{r})$$

we can write the gravitational force acting on a mass m as

$$\mathbf{F}_G(\mathbf{r}) = m\mathbf{g}(\mathbf{r})$$

Usually, we write the equation for the force of gravity on Earth surface as $\mathbf{F}_G = m \cdot g \cdot \hat{\mathbf{e}}_z$ where we indicate the direction of gravity with a unit vector pointing towards the center of Earth. However, we already found with eq. (3.11) that g is not a constant but scales with r^{-2} where r is the distance from Earth's surface. Hence, limiting our observations to a simple system like the mass of an apple m and the mass of Earth M we can write down a gravitational vector field for Earth in the form of

$$\mathbf{g}(\mathbf{r}) = -G \frac{M}{|\mathbf{r}|^3} \cdot \mathbf{r} \quad (4.19)$$

Assuming Earth's mass to be a point mass located at Earth's center and plugging in $M = M_E = 5.97 \cdot 10^{24} \text{ kg}$ for Earth's mass we get for $|\mathbf{g}(\mathbf{r})|$ at the position of Earth's surface, i.e., $r = R_E = 6371 \text{ km}$ the correct value $g = 9.81 \text{ m s}^{-2}$. As we saw earlier, eq. (4.3), of which eq. (4.16) is a reformulation, implied that $\mathbf{F}_G(\mathbf{r}) = -\nabla U_G(\mathbf{r})$. If we restrict ourselves to Earth and masses in Earth's gravity field we can write this with eq. (4.19) as

$$\mathbf{F}_G(\mathbf{r}) = m\mathbf{g}(\mathbf{r}) = -\nabla U_G(\mathbf{r}) = -m\nabla\Psi(\mathbf{r}) \quad (4.20)$$

Similarly, it follows from eq. (4.17) that

$$\mathbf{F}_C(\mathbf{r}) = q\mathbf{E}(\mathbf{r}) = -\nabla U_E(\mathbf{r}) = -q\nabla\Phi(\mathbf{r}) \quad (4.21)$$

Hence, the relationships between the respective scalar and vector fields are

$$\mathbf{g}(\mathbf{r}) = -\nabla\Psi(\mathbf{r}) \quad (4.22)$$

and

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \quad (4.23)$$

The gravity potential and the Coulomb potential are examples of scalar fields whereas the force of gravity and the Coulomb force are both examples of vector fields. Vector fields which can be written as the gradient of a scalar field, like $\mathbf{F}_G(\mathbf{r})$ and $\mathbf{F}_C(\mathbf{r})$ in eq. (4.20) and eq. (4.21), are also referred to as gradient fields. Saying that a force is the gradient of a scalar field is the same as saying a force is conservative because the line integral over a closed path in a gradient field always vanishes. The vector fields $\mathbf{F}_G(\mathbf{r})$ and $\mathbf{F}_C(\mathbf{r})$ are both of the form

$$\mathbf{F}(\mathbf{r}) = -\alpha \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} \quad (4.24)$$

where α is always positive for the force of gravity and can be positive or negative for the Coulomb force. At each point (x, y, z) in space a force vector $\mathbf{F}(x, y, z)$ is associated with the force field which has a direction and a magnitude

$$\mathbf{F}(x, y, z) = -\frac{\alpha}{(x^2 + y^2 + z^2)^{3/2}} \cdot (x, y, z)$$

and

$$|\mathbf{F}(x, y, z)| = \frac{|\alpha|}{x^2 + y^2 + z^2}$$

Fig. 4.5 illustrates the force field $\mathbf{F}(x, y, z)$ in the case of gravity or of an attractive Coulomb force. The arrows in fig. 4.5 mark the direction of the force at a given point (x, y, z) in space and their length and size indicates the magnitude of the force at the respective point in space. The object at the center of the force field sketched in fig. 4.5, i.e., the source of the force field, can be a mass m , a charge q or possibly both, i.e., a mass carrying a charge. Evidently, the potential $V(r)$ generating the force field in eq. (4.24) via $\mathbf{F}(\mathbf{r}) = -\nabla V(r)$ must possess a spherical symmetry and its value at any given position (x, y, z) does not depend on the direction of \mathbf{r} but only on its magnitude, i.e., $r = |\mathbf{r}|$:

$$V(r) = -\frac{\alpha}{r} \quad (4.25)$$

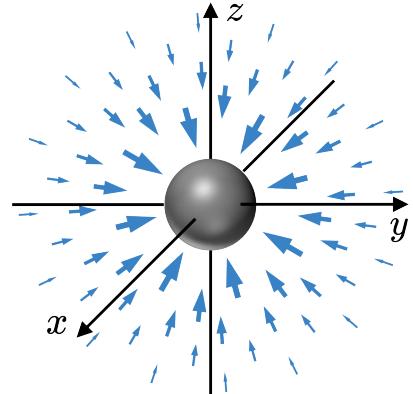


Fig. 4.5

Depending on whether $\alpha > 0$ or $\alpha < 0$ the potential $V(r)$ is either attractive or repulsive as sketched in fig. 4.6 which shows the potential energy $V(r)$ as a function of r . $V(r)$ is a so-called central potential as the potential energy of an object which is subject to the respective force field, as illustrated in fig. 4.6, only depends on the distance from the origin of the central potential. Correspondingly, force fields which are the gradients of such central potentials are so-called central forces as their lines of action are always directed towards a center of force as sketched in fig. 4.5.

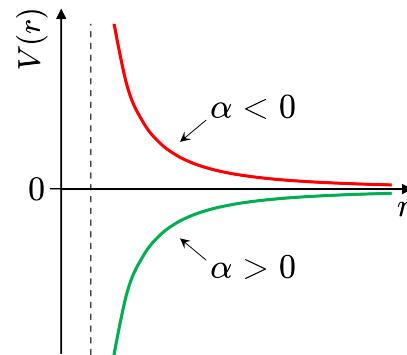


Fig. 4.6

Conservative central force

A central force is conservative if its strength only depends on the magnitude of the distance $|\mathbf{r}|$ from the center of the force.

Hence, for potentials giving rise to conservative central forces applies

$$U(\mathbf{r}) = V(|\mathbf{r}|) = V(r)$$

and to conservative forces themselves applies

$$\mathbf{F} = -\text{grad } V(r) = -\frac{\partial V}{\partial r} \cdot \text{grad } r$$

In Cartesian coordinates with

$$r = \sqrt{x^2 + y^2 + z^2}$$

one obtains e.g., for the derivative with respect to the x -coordinate

$$\frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

and analogously for the derivative with respect to y and z . With that follows

$$\frac{\partial V}{\partial r} \cdot \text{grad } r = \frac{\partial V}{\partial r} \cdot \frac{\mathbf{r}}{r}$$

which for a potential $V(r)$ in the form of eq. (4.25) gives rise to the force equation of eq. (4.24), i.e.,

$$\mathbf{F}(\mathbf{r}) = -\frac{\alpha}{r^2} \cdot \frac{\mathbf{r}}{r}$$

4.3 Examples of Fields

The implicit assumption in the above introduction into the concept of fields has been that the source generating gravitational or electrostatic fields are point masses or point charges. Also, we restricted the discussion to one such dimensionless object generating a field and to the attractive or repulsive force between two such point like objects. The elementary electrical charges such as electrons or protons are indeed very tiny objects and the assumption of point like charges is a good one in such cases. But we know from experience that larger objects can either be electrically neutral or carry a net negative or positive charge charges. In the latter two cases these net charges are the result of an object carrying either more of the many tiny negative charges than of the tiny positive charges or the other way around. In either case, the result of this imbalance is a macroscopic charge distribution associated with a much larger object than the point charge we discussed so far. As for masses, our daily experience is dominated by large masses and one large mass in particular which is of course Earth. But out there in the universe are much larger masses, beginning with the Sun which possess 333 060 times the mass of Earth. The Sun's density and its mass are quite different from those of Earth because Earth is a rocky planet and the Sun is still about 91% hydrogen, a little less than 8.9% helium with the rest consisting of various fusion products. There exist many more massive stars in the universe than the Sun whose mass makes up about 99.8% of the mass of our solar system. But stars are not the most massive objects in our universe, that title belongs to the so-called black holes which it what remains when very massive stars collapse under their own gravity. In a sense, this closes the circle as black holes are singularities in space, i.e., point like objects possessing an enormous mass which bend space around a black hole to the extent that not even light can escape anymore; that is why we call them black holes.

Fig. 4.5 shows a sketch of the spherically symmetric force field generated by a single point mass or a single negative point charge as the convention is that the force arrow points towards a negative and away from a positive charge. Knowing the shape of the field generated by a single point mass or charge, what does the force field generated by two point masses or charges look like? To understand that we consider the situation sketched in Fig. 4.7 with two masses M_1 and M_2 at the positions $z = d/2$ and $z = -d/2$ where we probe the strength of

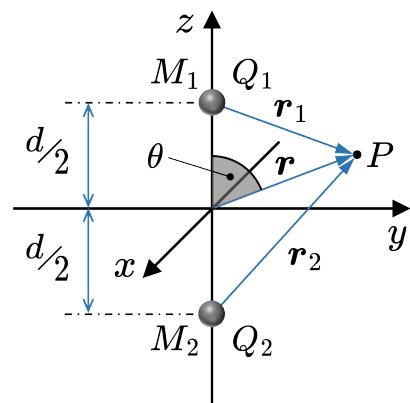


Fig. 4.7

the scalar field $\Psi(\mathbf{r})$ field at a position P . Because for conservative central forces the gravity potential $U_G(\mathbf{r})$ in eq. (4.20) is given by $V(r)$ from eq. (4.25), the scalar potential $\Psi = \Psi(r)$ in eq. (4.20) and eq. (4.22) must also only be a function of the distance r from the source generating the scalar field but not of the direction of \mathbf{r} . Hence, we can write the scalar field generated by the two masses M_1 and M_2 at the position of P as

$$\Psi_P(r) = \Psi_1(r) + \Psi_2(r) = -\frac{GM_1}{\sqrt{x^2 + y^2 + (z - d/2)^2}} - \frac{GM_2}{\sqrt{x^2 + y^2 + (z + d/2)^2}}$$

That we can write the resulting $\Psi_P(r)$ as the superposition of $\Psi_1(r)$ generated by M_1 and $\Psi_2(r)$ generated by M_2 is owed to the fact that the resultant force a test mass at P would experience is the superposition of the two forces exerted by each of the two masses on this test mass. Evidently, if we reduce the separation d of the two masses to zero we thereby combine both masses at the coordinate origin and the resulting scalar field will be just that of a single object with a combined mass $M_1 + M_2$. For distances $r \gg d$, we can expect the scalar field $\Psi_P(r)$ generated by the two masses M_1 and M_2 to increasingly look like the scalar field a single mass of the size $M_1 + M_2$ would generate. At distances $r \gg d$, the so-called far field, we can approximate the square roots

$$\sqrt{x^2 + y^2 + \left(z \mp \frac{d}{2}\right)^2} = \sqrt{x^2 + y^2 + z^2 \mp zd + \frac{d^2}{4}} = r \cdot \sqrt{1 \mp \frac{zd}{r^2} + \frac{d^2}{4r^2}}$$

by dropping the term $\frac{d^2}{4r^2}$ altogether and by using for $x \ll 1$ the approximation

$$\frac{1}{\sqrt{1 \mp x}} \approx 1 \pm \frac{x}{2}$$

we get the so-called far field approximation for the potentials $\Psi_{1/2}$

$$\Psi_{1/2} = -\frac{GM_{1/2}}{r \cdot \sqrt{1 \mp \frac{zd}{r^2} + \frac{d^2}{4r^2}}} \approx -\frac{GM_{1/2}}{r} \cdot \left(1 \pm \frac{zd}{2r^2}\right)$$

With this and $M_1 = M_2 = M$ $\Psi_P(r)$ becomes

$$\begin{aligned} \Psi_P(r) &= \Psi_1(r) + \Psi_2(r) \\ &= -\frac{GM}{r} \cdot \left(1 + \frac{zd}{2r^2}\right) - \frac{GM}{r} \cdot \left(1 - \frac{zd}{2r^2}\right) = -\frac{2GM}{r} \end{aligned}$$

As expected, far away from M_1 and M_2 , i.e., for $r \gg d$, the scalar field $\Psi_P(r)$ at P is that of the combined mass. For $M_1 = M_2 = M$, the center of this scalar field where the combined mass is “located” is the coordinate the origin, but obviously, for $M_1 \neq M_2$ it must be the center of mass of M_1 and M_2 .

With that, we have a basic understanding of what the gravity potential $\Psi_P(r)$ looks like far away from M_1 and M_2 , i.e., for distances $r \gg d$ in fig. 4.7. But what about the near field, i.e., for distances where $r \approx d$? We can assume that very close to either of those masses the spherically symmetric gravity field generated by the respective mass will dominate. Furthermore, we know that, given the symmetry of the problem, the shape of the resulting gravity field produced by the superposition of the gravity fields of M_1 and M_2 , will have to possess rotational symmetry with respect to each of the coordinate axes. To get a better picture of what this rotational symmetry looks like we can pick one of the coordinate axes, lets say the z -axis, and sketch what the two dimensional contour plot of the gravity field looks like for example in the yz -plane. The latter we do by plotting curves where $\Psi(x = 0, y, z) = const.$. As an example we will pick as the two masses M_1 and M_2 the masses of the Earth and the Moon and because the Earth is so much bigger than the Moon we will shift the coordinate origin into the center of Earth. With that the expression for $\Psi(x = 0, y, z)$ becomes

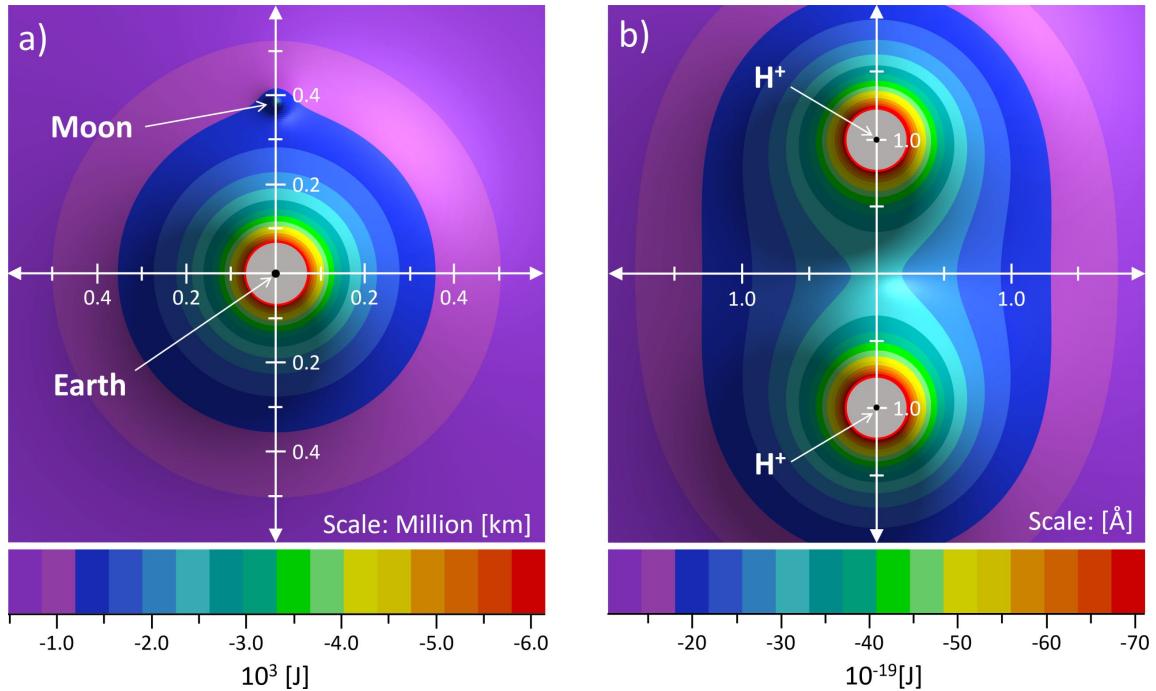


Fig. 4.8: Contour maps of the gravitational potential energy $U_G(r)$ of a 1 kg test mass moving in the Earth-Moon system (a) and the electrostatic potential energy $U_E(r)$ of the electron in the potential field generated by the two protons in a hydrogen atom (a). While the scale of the Earth dot is drawn to size the protons are orders of magnitudes smaller than the dots symbolizing their positions here.

$$\Psi(0, x, z) = -\frac{GM_E}{\sqrt{y^2 + z^2}} - \frac{GM_M}{\sqrt{y^2 + (z - r_{EM})^2}}$$

where M_E and M_M are the masses of Earth and Moon and r_{EM} is the average Earth-Moon distance of 384 400 km. Fig. 4.8a does not depict the contour map of the gravity potential but that of the gravitational potential energy $U_G(0, y, z)$ of a test mass of $m = 1$ kg moving in the gravity potential generated by the Earth-Moon system, i.e.,

$$U_G(0, y, z) = m \cdot \Psi_P(0, x, z) = \text{const}$$

Within the gray disk around the position of Earth the equipotential lines lie so close that they cannot be resolved on the scale of the plot shown here. That is of course a consequence of the gravity potential and hence the potential energy of the test mass scaling with r^{-1} . As fig. 4.7 shows, instead of applying the above analysis to the fields generated by two masses M_1 and M_2 we can just as well apply it to the fields generated by two charges Q_1 and Q_2 . For the respective electrical potential generated by two charges Q_1 and Q_2 in fig. 4.7 we can write

$$\Phi(0, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{Q_1}{\sqrt{y^2 + (z - d/2)^2}} + \frac{Q_2}{\sqrt{y^2 + (z + d/2)^2}} \right)$$

Here, because of the symmetry of the problem, we left the coordinate origin mid-way between the two charges. The hydrogen molecular ion H_2^+ is the simplest molecular ion consisting of two protons, each carrying a charge of $+e$ and one electron carrying a charge of $-e$. The equilibrium distance between the two protons (H^+) in this molecule's ground state is about 2 Å. Fig. 4.8b shows the contour plot of the electrical potential energy $U_E(0, y, z)$ of the electron moving in the potential field generated by the two protons which is given by

$$U_E(0, y, z) = -q \cdot \Phi_P(0, x, z) = \text{const}$$

As with the contour map in fig. 4.8a, the gray disks around the protons in fig. 4.8b indicate areas where the equipotential lines lie too close as to be resolved on the scale of the plot shown here. Given the symmetry of the hydrogen molecular ion, the electrical potential energy map outside the yz -plane is obtained by rotating the contour map shown in fig. 4.8b around the vertical symmetry axis. Similarly, one can generate a three-dimensional version of the gravitational potential energy contour map in fig. 4.8a by rotation of the map calculated for the yz -plane around the vertical symmetry axis.

In both examples of fig. 4.8 the potential is attractive. Hence the forces exerted on the 1 kg test mass in fig. 4.8a and the electron in fig. 4.8b point towards the masses and the charges generating the respective potentials. Of course, because the mass of Earth is much greater than the mass of the Moon, it is only in the vicinity of the Moon that

the force vectors will directly point towards it, hence, the force field in fig. 4.8a will be quite asymmetric with respect to the two masses generating the gravity potential. Closer to the Moon or behind the Moon the gravity of the Moon will dominate but the closer the test mass moves towards Earth the more the gravity potential will be dominated by the much larger mass of Earth. This is different for the electron moving in the electric potential shown in fig. 4.8b, where the fields generated by the two protons mirror each other. With eq. (4.3) we know that moving the test mass or the electron incrementally along a path $d\mathbf{s}$ is associated with an incremental change in potential energy of

$$dU = -\mathbf{F} \cdot d\mathbf{s}$$

If we choose the path $d\mathbf{s}$ to follow an equipotential line, dU must of course be zero as the very definition of the equipotential line is just that, $dU = 0$. That however means that if we move the test mass in fig. 4.8a or the electron in fig. 4.8b along an equipotential line, $\mathbf{F} \cdot d\mathbf{s}$ must be zero. A vector product is only zero if one of the components is zero or if two vectors are perpendicular to each other. Since neither \mathbf{F} nor $d\mathbf{s}$ is zero that means that the force component parallel to $d\mathbf{s}$, i.e., the force component tangential to the equipotential line, must vanish. Hence, if we move the test mass or the electron along an equipotential line, the acting force \mathbf{F} will at every point always be perpendicular to the tangent to the equipotential line. With that it is easy to picture how the force fields in fig. 4.8a and fig. 4.8b must look like as the lines of force pointing towards the two masses or the two protons must always be perpendicular to the respective equipotential lines.

What we just learned about the far field of the scalar field produced by two masses can easily be expanded to understand what scalar potentials of other similarly symmetric mass arrangements look like. Take for example the spherical mass shell of constant density. A cut through such a mass shell of mass M and volume V along the yz -plane is shown in fig. 4.9. As sketched, we imagine the shell to be divided into many small identical mass elements m , each of which faces a mass m^* on the opposite side of the shell. For each of these pairs m_i and m_i^* the same situation exists as fig. 4.7 shows for a single pair. Hence, the far field of the spherical mass shell must be equal to that of a single mass M located at the coordinate origin, the center of mass of the spherical mass shell. But what does the scalar field look like inside the mass shell?

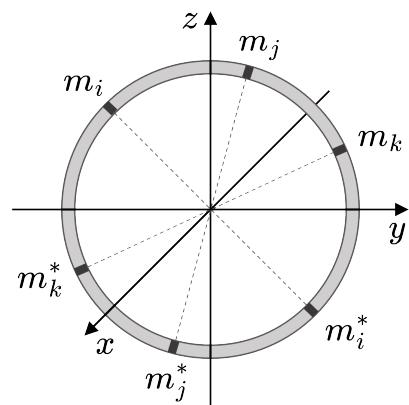


Fig. 4.9

To understand that we will consider an infinitesimal thin shell like the grey mass shell sketched in fig. 4.10. Of course, for this illustration we had to give the shell some thickness but for calculation purposes we consider it infinitesimal thin. The drawing makes use of the fact that this problem is highly symmetrical as it will look just the same whichever way we orient the cut as long as the cutting plane contains the center of the shell. Here we choose the cutting plane to be the yz -plane with the y -axis oriented such that it goes through the point P for which we want to calculate the scalar potential generated by the mass shell. For that we determine the contribution to the scalar potential of the masses at a set distance s from P . As one can see from fig. 4.10, the shell area dA containing those masses is that of a ring shell whose radius is $R \cdot \sin \theta$ and whose width is given by the arc element $R \cdot d\theta$. With the total mass of the shell being M and the surface area of the infinitesimal shell being $A = 4\pi R^2$ we can calculate the mass dm of this infinitesimal ring element to be

$$dm = \frac{dA}{A} M = \frac{2\pi \cdot R \cdot \sin \theta \cdot R \cdot d\theta}{4\pi R^2} M = \frac{\sin \theta \cdot d\theta}{2} M$$

The scalar potential generated by such a ring shell of mass dm at the position P is then

$$\Psi_P(r, s, \theta) = -GM \frac{\sin \theta \cdot d\theta}{2s}$$

To determine $\Psi_P(r)$ outside the mass shell we now only have to add up all ring shells with distances $r - R \leq s \leq r + R$ from P . To do that we will eliminate $\sin \theta \cdot d\theta$, which we can do by reading off fig. 4.10 the relationship

$$\begin{aligned} s^2 &= (r - R \cos \theta)^2 + R^2 \sin^2 \theta \\ &= r^2 + R^2 - 2Rr \cos \theta \end{aligned}$$

Differentiating this expression with respect to θ and rearranging shows that

$$\frac{\sin \theta d\theta}{s} = \frac{ds}{Rr}$$

With that we can now write the contribution to $\Psi_P(r, s)$ of a ring shell at the distance s from P as

$$\Psi_P(r, s) = -\frac{GM}{2Rr} ds$$

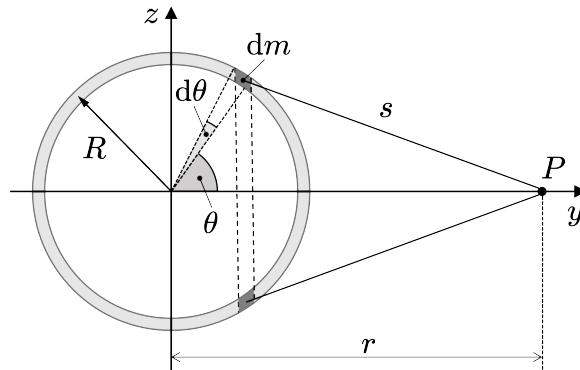


Fig. 4.10

First, we will calculate $\Psi_P(r)$ outside the mass shell, i.e., for $r > R$. To do that we sum up all contributions from $s = r - R$ to $s = r + R$

$$\Psi_P(r > R) = -\frac{GM}{2Rr} \int_{r-R}^{r+R} ds = -\frac{GM}{2Rr} [(r+R) - (r-R)] = -\frac{GM}{r}$$

This is of course just the result that we already derived using the simple argument sketched in fig. 4.9, only here we obtained the result by calculation. The advantage of doing this calculation is that we can now also determine $\Psi_P(r < R)$, i.e., the scalar field for a position P inside the mass shell, something we cannot really deduce from fig. 4.9 in a similar way as for $\Psi_P(r > R)$. For a P on the y axis inside the mass shell, s runs from $R - r$ to $R + r$ and with that

$$\Psi_P(r < R) = -\frac{GM}{2Rr} \int_{R-r}^{R+r} ds = -\frac{GM}{2Rr} [(R+r) - (R-r)] = -\frac{GM}{R}$$

Obviously, the gravity potential $\Psi_P(r < R)$ inside the mass shell is constant. Instead of considering the scalar and vector fields generated inside and outside of a mass shell of radius R we could just as well have looked at a charge carrying sphere and would have obtained an analogous result, i.e.,

$$\Phi_P(r > R) = \frac{Q}{4\pi\epsilon_0 r} \quad \text{and} \quad \Phi_P(r < R) = \frac{Q}{4\pi\epsilon_0 R}$$

The charge carrying sphere could either be a non-conductor carrying a uniform surface charge or it could be a conductor like a metallic sphere but in either case the sphere would be electrically isolated. From eq. (4.22) and eq. (4.23) it follows for the force field of a mass shell and the electric field of a charge shell that

$$\mathbf{g}(\mathbf{r} > \mathbf{R}) = -\frac{GM}{r^2} \cdot \hat{\mathbf{r}} \quad \text{and} \quad \mathbf{g}(\mathbf{r} < \mathbf{R}) = 0$$

and

$$\mathbf{E}(\mathbf{r} > \mathbf{R}) = \frac{Q}{4\pi\epsilon_0 r^2} \cdot \hat{\mathbf{r}} \quad \text{and} \quad \mathbf{E}(\mathbf{r} < \mathbf{R}) = 0$$

Fig. 4.11a shows the shape of the scalar field $\Psi(r)$ for locations outside the shell as well as inside the shell. $\Psi(r)$ describes the gravity potential as a function of its distance r from the center of a mass shell of radius R . A test mass m_0 at a location $P(r)$ will hence possess the gravitational potential energy of $U_G(r) = m_0\Psi(r)$ in the gravity potential generated by the mass shell. The gradient of the scalar field $\Psi(r)$, the gravity field $\mathbf{g}(\mathbf{r})$, is zero inside the shell and outside the shell this vector field is spherically symmetric with the force vectors associated with the gravity field generated by the mass shell all

pointing towards the center of the shell but terminating at its circumference, i.e., at $|r| = R$. Fig. 4.11b shows the corresponding situation for the scalar field $\Phi(r)$, the electrical potential generated by a spherical shell carrying a positive charge Q . A test charge q_0 at a location $P(r)$ will possess the electrical potential energy of $U_E(r) = q_0\Phi(r)$ in the electrical potential generated by a charged shell. The gradient of $\Phi(r)$, the electric field $\mathbf{E}(r)$, vanishes inside the shell as well and since the charge Q is positive the electrical field vectors point away from the charge carrying shell. If we had chosen Q to be negative, the electrical potential would carry a negative sign and hence its shape would correspond to the shape of the gravity potential shown in fig. 4.11a; in that case the electrical field vectors would also point towards the shell and terminate on its surface just as the force vectors of the gravity field do in fig. 4.11a.

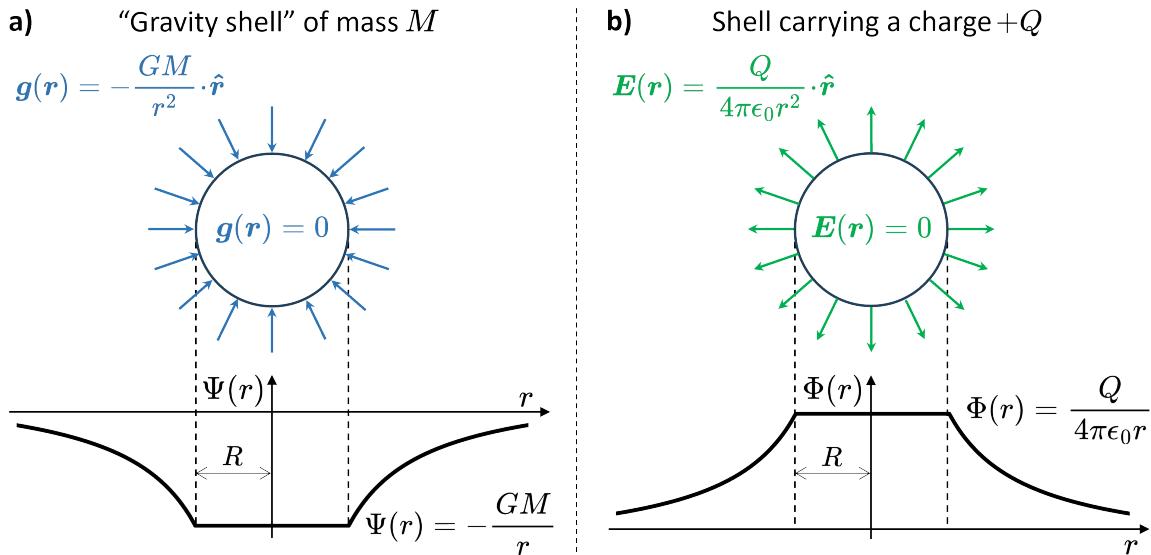


Fig. 4.11: Scalar field and force field inside and outside of a “gravity shell” (a) and scalar field and electrical field associated with a positively charged hollow sphere carrying a charge Q (b). For a negatively charged hollow sphere the electrical field inside would of course also vanish but the arrows of the electric field would point towards the hollow sphere as $\Phi(r) \rightarrow -\Phi(r)$.

The special situation of the "gravity shell" sketched in fig. 4.11a illustrates the so-called shell theorem which was already known to and proofed by Newton. Its two corollaries are

1. A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.
2. If the body is a hollow shell, no net gravitational force is exerted by the shell on any object inside, regardless of the object's location within the shell.

The above examples of a mass shell and a charge carrying shell highlight the analogies between the scalar gravity potential field and the gravity field on the one side and the scalar electrostatic potential and the electrostatic vector field on the other side. But there are of course fundamental differences, foremost of course that the force of gravity is always attractive whereas the electrostatic force can be attractive or repulsive. Also, it must be pointed out that while the gravity field inside a gravity shell vanishes just as the electric field inside a charged shell does, the situations are quite different. A conducting shell will, e.g., always shield the inside of the shell from electromagnetic fields outside the shell, something we are all familiar with when our cell phone reception disappears once we enter for example an elevator. The latter being a more or less perfect metal cage will just act like a metal shell and shield the inside from electromagnetic waves. This is different with gravity where the gravity potentials generated by various masses cannot be shielded against and always add up. In fig. 4.11a the field inside the mass shell vanishes because the respective forces acting on a point mass inside the shell cancel each other. As long as we view the mass shell in isolation the gravity field inside the shell will vanish. If we however locate another mass somewhere close to the shell than its gravity potential will add itself to the gravity potential generated by the mass shell. If the gradient of this added gravity potential is not also constant inside the mass shell a net gradient of the resultant gravity potential will exist and hence a test mass inside the mass shell will experience a net force.

Using the shell theorem we can now consider a situation like the one illustrated in fig. 4.12 where we drop a ball in a hole which we somehow have managed to drill through our planet. Such a feat would be of course extraordinarily difficult to achieve, as drilling a hole through the center of the Earth to the other side of our planet is a much more difficult task than what the planet drills in science fiction movies make us believe. In fact, it is likely impossible as the inside of Earth is not in a solid but in a molten state. However, lets just assume that we could do it and then lets step to the rim of the hole we just produced and drop a ball straight down towards the other side of Earth. What will happen as the ball falls towards the center of Earth. Importantly, we can apply the shell theorem because we drilled such a tiny hole whose diameter is so small that we can still consider the punctured shell with the radius of Earth still to

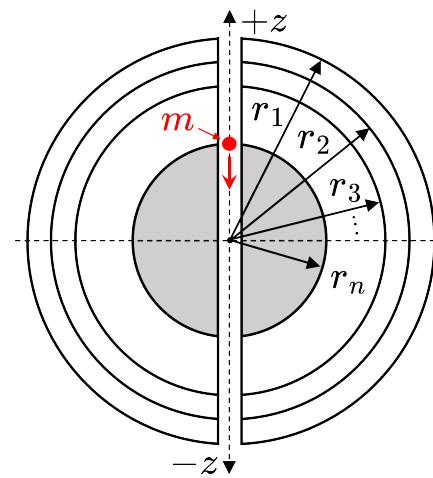


Fig. 4.12

resemble a complete mass shell. As we drop the ball and it falls, passing shell after shell, any shell it leaves behind, i.e., above it, has no more bearing on our falling ball as the gravity field it generates inside its shell is zero. And that is the case not just for the first shell, labeled r_1 in fig. 4.12 but all following shells, i.e., r_2 and r_3 and so on. For our ball only the gravity of the Earth volume within the radius r_n where it finds itself at any moment will be relevant. Once the ball arrives at the center of Earth it will of course continue on its path but now the gravity of the inner sphere beneath it will slow it down while the respective shells above it will not matter until the ball gets there. To make matters simple we will assume that Earth shall have a constant density of $\bar{\rho}_E = M_E/V_E = 5.51 \cdot 10^3 \text{ kg m}^{-3} = 5.51 \text{ g cm}^{-3}$. Orienting our coordinate system such that where we drop the ball is the Earth surface, i.e., $z = R_E$, and the center of Earth shall lie at $z = 0$ we can write for the gravity potential acting on the falling ball

$$\Psi(r) = \Psi(z) = -\frac{GM(z)}{z} = -\frac{G}{z} \cdot \bar{\rho}_E \cdot \frac{4\pi z^3}{3} = -\frac{4\pi G \bar{\rho}_E}{3} \cdot z^2$$

With that the gravity field becomes

$$\mathbf{g}(r) = \nabla \Psi(r) = \frac{\partial \Psi}{\partial z} \cdot \hat{z} = -\frac{8\pi G \bar{\rho}_E}{3} \cdot z$$

and hence the equation of motion for the motion of the ball in z -direction is

$$F = m\ddot{z} = -m \cdot \frac{8\pi G \bar{\rho}_E}{3} \cdot z = -k \cdot z$$

This however is nothing else but eq. (2.7), the equation of motion for an harmonic oscillator. Such equations of motion, which we will study in more detail in section 10.1, have the solution

$$z_0 = z_1 \cos \omega_0 t \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Here, obviously $z_1 = R_E$ and with that ω_0 and the associated period T of a complete oscillation are given by

$$\omega_0 = \sqrt{\frac{8\pi G \bar{\rho}_E}{3}} = 1.76 \cdot 10^{-3} \text{ s}^{-1}$$

and

$$T = \frac{2\pi}{\omega_0} = 3570 \text{ s} \approx 60 \text{ min}$$

It takes the ball about one hour to fall through Earth and arrive at the other side of the planet where it comes to a stop and falls back into Earth. As the expression for ω_0 shows, the frequency and with that the period T are independent of the mass of the falling ball. Essentially, this hypothetical example of a body falling through the center of the Earth

and on to the other side of the planet is an extreme version of Galileo's falling time experiments. These were intended to prove that, regardless of a bodies mass, all bodies fall at the same speed in Earth's gravity field; assuming of course that the differences in air resistance for the respective bodies are negligible.

In the above examples we considered the gravity fields of large extended objects such as Earth. We found for objects with spherical symmetry, and that applies to many bodies in the universe, that we can treat them with respect to their gravitational impact on other bodies as if all of their mass was concentrated at the center of mass. For a sphere of uniform density or spheres having a varying density with a shell-like structure such as Earth, this is of course the center of the sphere. Some may rightly point out that Earth is not exactly a sphere but a rotational ellipsoid. However, even that is not quite correct as its mass is not distributed evenly within the body of Earth which results in gravitational anomalies. But for what interests us here that does not really matter as from far away the approximation of a spherical Earth is still a very good one.

Similarly, if we just look at how very far away objects impact the movement of for example our solar system within our galaxy, the Milky Way Galaxy, we do not consider the impact of individual celestial bodies, i.e., unless they are extremely massive such a black holes. Instead, we seek to understand how our solar system moves with respect to the center of gravity of far away star systems whose internal relative distances are much smaller than the distance of their center of gravity to our solar system. Just as the gravity potential generated by two masses (compare fig. 4.12) at a point $P(r)$ is the sum of the gravity potentials generated by each of the two masses, the gravity potential generated at $P(r)$ by N masses is the sum over the respective gravity potentials generated by each of the N masses:

$$\Psi_P(r) = \sum_{j=1}^N \Psi_j(r) = -G \sum_{j=1}^N \frac{m_j}{|\mathbf{r} - \mathbf{r}_j|}$$

Now, if $r \gg r_j$, then one can approximate the gravity potential generated at $P(r)$ by

$$\Psi_P(r) \approx -\frac{G}{|\mathbf{r} - \mathbf{r}_{CM}|} \sum_{j=1}^N m_j \quad \text{where} \quad \mathbf{r}_{CM} = \sum_{j=1}^N m_j \mathbf{r}_j \Bigg/ \sum_{j=1}^N m_j$$

is the center of mass coordinate of the N masses. In this way one treats many far away masses as a single combined mass, located at the center of mass coordinate position \mathbf{r}_{CM} . This is not very different from how we approached the problems of highly symmetric mass distributions discussed in this chapter for homogeneous gravity fields where the center of mass and the center of gravity are one and the same. More challenging are

in-between situations where there is neither a highly symmetric mass distribution within a single object or where there are multiple separate masses which are just not far enough away from where one would like to evaluate the gravity potential they generate. As a consequence, one has to deal with gravity fields which are non-uniform across a mass distribution of interest and hence the center of mass and the center of gravity no longer coincide for such mass distributions. In these cases one either has to integrate over a non-constant density distributions within an object or one has to add individual discrete $\Psi_j(r)$ for each mass m_j to calculate $\Psi_P(r)$. If one has to deal with inhomogenous gravity fields, things get more complicated but this is anyway beyond the scope of what we can consider here.

5. Rotating Motion

The study of rotating motions has a history as long as the study of linear motion. Newton for example thought in his “Principia” no different about the preservation of rotating motion than he did about the preservation of linear motion, In his judgment, a spinning top

“...whose parts, by their cohesion, are perpetually drawn aside from rectilinear motions, does not cease its rotation otherwise than it is retarded by the air.”

A change of state in the linear motion of an object moving at constant speed requires a force and so does a change of state in the rotational motion of an object. When considering the rotational motion of objects, physicists differentiate between two types: the rotational motion of a point mass around a fixed point and the self-rotation of a solid body. With the emergence of quantum mechanics an additional rotational motion became important, the spin many elementary particles possess. While one could consider this spin also as a self-rotation, it is something entirely different. Here we will first focus on the rotational motion of point masses around a fixed point in space and then consider the self-rotation of extended objects; the spin of elementary particles is something we will discuss in detail in the fourth volume of this physics course.

Leaving the spin of elementary particles aside, rotational motion is always the rotational motion with respect to an axis. We describe, e.g., the rotational motion of a planet around a star as the motion of a point mass. That is, a point mass possessing the planets mass located at its center of mass, rotating around an axis which goes through the star’s center of mass and which is perpendicular to the orbital plane the planet sweeps around the star. Different from that is the rotation of a planet such as Earth around its own axis, clearly an example of self-rotation. But this self-rotation of Earth we can actually picture as simultaneous rotations of the many individual point masses which make up our planet. In either case, the fundamental concept in analyzing those rotational motions is that of the angular momentum and to it we will turn first.

5.1 Angular Momentum

Angular momentum gets its name of course from an analogy of rotational motion with linear motion. Linear momentum \mathbf{p} is the product of mass m and velocity \mathbf{v} whereas angular momentum \mathbf{L} can be associated with the angular velocity of a mass rotating around a fixed point outside itself or with a massive object rotating around an axis lying within the object, and frequently also with both. A rate of change in linear momentum $\dot{\mathbf{p}} \neq 0$ gives rise to a force \mathbf{F} (Newton's second law). Accordingly, a rate of change $\dot{\mathbf{L}} \neq 0$ gives rise to a moment of force, i.e., a torque \mathbf{M} :

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}}$$

and

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \dot{\mathbf{p}} = m \left[\underbrace{(\dot{\mathbf{r}} \times \dot{\mathbf{r}})}_{=0} + (\mathbf{r} \times \ddot{\mathbf{r}}) \right] = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} = \frac{d\mathbf{L}}{dt} \quad (5.1)$$

It is a commonly held misconception that the linear motion of a body is not associated with an angular momentum. To understand the angular momentum associated with a body in linear motion consider the sketch in fig. 5.1 which shows a mass m moving at constant speed \mathbf{v} parallel to the x -axis at a distance $y = h$. With m moving at constant speed we just have to pick three equal time intervals in which the mass m per definition will travel the very same distance, namely the stretches $a = b = c$ highlighted in fig. 5.1. The area of each triangle is given by its respective base times the height distance to the triangle's peak. Hence, with the bases of the triangles having the same length and their height being identical the triangle areas must be identical:

$$\frac{ah}{2} = \frac{bh}{2} = \frac{ch}{2} \quad \text{or} \quad A = B = C$$

Evidently, the vector from the coordinate origin to the position of the moving mass m sweeps identical areas in identical time intervals. However, the respective angles they sweep during those identical time intervals are different as $\varphi_a < \varphi_b$ and $\varphi_b > \varphi_c$ and hence the angular velocity $\dot{\varphi} \neq 0$. With the mass m approaching from $x \approx -\infty$, $\dot{\varphi}$ will at first be very small but continually increase as the mass m gets closer to the origin O .

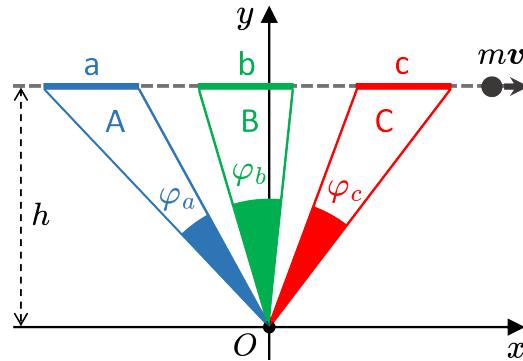


Fig. 5.1

However, once it crosses the y -axis at a distance h from the origin O , $\dot{\varphi}$ will continually decrease as the mass m moves away from the origin O towards $x \approx +\infty$. What is the angular momentum that corresponds to this change in angular velocity? First we look at eq. (5.1). With the velocity \mathbf{v} of the mass constant, obviously $\dot{\mathbf{p}} = 0$. Eq. (5.1) tells us in this case that $\dot{\mathbf{L}} = 0$ must be zero, i.e., $\mathbf{L} = \text{const}$. To determine this constant value of \mathbf{L} we calculate it in fig. 5.1 for $x = 0$, i.e., where $|\mathbf{r}| = h$. With $v = |\mathbf{v}|$ constant, the magnitude of the angular momentum is $|\mathbf{L}| = h \cdot m \cdot v$. But what about the direction of \mathbf{L} ? For that we have to write down the respective vectors to obtain:

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} \times \begin{pmatrix} v_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -mhv_x \end{pmatrix}$$

Hence the angular momentum vector points in the negative z -direction and has the length mhv . This simple example not only showed us that an angular momentum is associated with linear motion but that the angular momentum of a point mass in linear motion is a conserved quantity, i.e., it is a constant of motion. Of course, this angular momentum depends on the vantage point from which we observe the motion as the value of h depends on our choice of origin. After looking at the motion of a mass moving at constant speed without being subject to any force whatsoever we will next look at the motion of a point mass in a central force field.

Central force

A force whose line of action is always directed towards a center of force. |||

Hence, for central forces applies

$$\mathbf{F}(\mathbf{r}) = F(r) \cdot \frac{\mathbf{r}}{r}$$

From this follows with eq. (5.1)

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}(\mathbf{r}) = \frac{F(r)}{r} \underbrace{(\mathbf{r} \times \mathbf{r})}_{= 0} = 0$$

This means that for central forces \mathbf{L} is a conserved quantity, i.e., it is a constant of motion

$$\mathbf{L} = m\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \mathbf{L}_0 = \text{const} \quad (5.2)$$

The areal velocity, i.e., the area A which the position vector \mathbf{r} of a point mass sweeps per unit time, is given by

$$\frac{dA}{dt} = \frac{1}{2} \left| \mathbf{r} \times \frac{d\mathbf{r}}{dt} \right|$$

For central forces, the following therefore applies

$$\frac{dA}{dt} = \frac{|\mathbf{L}|}{2m} = \text{const} \quad (5.3)$$

At the beginning of this section, we considered the linear motion of a point mass not subject to any force which moves with a constant velocity \mathbf{v} parallel to one of the coordinate axes, which we chose to be the x -axis. There we found that the radius vector from the coordinate origin to the point mass position will sweep equal areas in equal time intervals. Here we find with eq. (5.3) that for a point mass moving in a central force field its radius vector from the force center to the point mass position will also sweep equal areas in equal time intervals. This is of course nothing else but Johannes Kepler's second law: The radius vector between the Sun and a planet sweeps in identical time intervals areas of the same size.

Next we will consider point mass motion not just in central force fields but in conservative central force fields. With what we have learned in the previous chapter at the end of section 4.2 we can write for the total energy of a point mass moving in a conservative central force field

$$E = T + V = \frac{m}{2} \dot{\mathbf{r}}^2 + V(r) \quad (5.4)$$

Rather than measuring the velocity of a point mass with respect to a fixed coordinate system, it is in many cases advantageous to measure it with respect to the trajectory itself. Thus, one has

$$E = \frac{m}{2} \dot{s}^2 + V(r)$$

where ds is the infinitesimal arc segment of the trajectory sketched in fig. 5.2 through which the point mass runs in the period dt . From fig. 5.2 one can see that

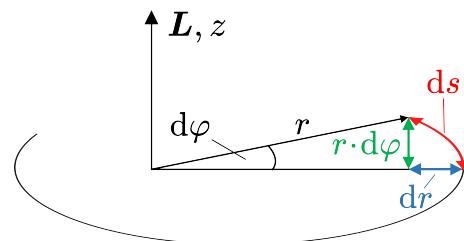


Fig. 5.2

$$ds = \sqrt{dr^2 + (r d\varphi)^2} \quad (5.5)$$

and therefore

$$E = \frac{m}{2} \left(\frac{\sqrt{dr^2 + (r d\varphi)^2}}{dt} \right)^2 + V = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) + V \quad (5.6)$$

It must hold that (conservation of angular momentum)

$$|\mathbf{L}| = l = mr(r\dot{\varphi}) = mr^2\dot{\varphi} = \text{const}$$

Hence, the angular velocity is

$$\dot{\varphi} = \frac{l}{mr^2} \quad (5.7)$$

With that, eq. (5.6) becomes

$$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = \frac{m}{2}\dot{r}^2 + V_{Cf}(r) + V(r) \quad (5.8)$$

where

$$V_{Cf} = \frac{l^2}{2mr^2} = \frac{m}{2}(\dot{\varphi} \times \mathbf{r})^2 \quad (5.9)$$

is the so-called centrifugal potential. The associated force in radial direction, the central force F_C , is given by

$$F_C = -\frac{\partial V_{Cf}}{\partial r} = \frac{l^2}{mr^3} = m\dot{\varphi}^2 r \quad (5.10)$$

Using $U(r) = V_{Cf}(r) + V(r)$ one can rewrite eq. (5.8) with the result

$$\frac{m\dot{r}^2}{2} = E - U(r)$$

and therefore

$$\dot{r} = \frac{dr}{dt} = \pm \sqrt{\frac{2}{m}[E - U(r)]} \quad (5.11)$$

With

$$\frac{d\varphi}{dt} = \frac{l}{mr^2}$$

and

$$r(t) = r(\varphi(t))$$

follows

$$\frac{dr}{dt} = \frac{dr}{d\varphi} \frac{d\varphi}{dt} = \frac{dr}{d\varphi} \frac{l}{mr^2} = r' \cdot \frac{l}{mr^2} \quad (5.12)$$

and therefore

$$\frac{dr}{d\varphi} = r' = \pm \frac{mr^2}{l} \sqrt{\frac{2}{m}[E - U(r)]} \quad (5.13)$$

5.2 Kepler Motion

The term Kepler motion refers to the motion of a point mass under the influence of the gravitational force. That means that for the force acting between a point mass m and a mass M , Newton's law of gravitation applies

$$\mathbf{F} = -G \frac{M \cdot m}{r^3} \mathbf{r}$$

With $\mathbf{F} = -\text{grad } V(r)$ the gravitational potential of Kepler motion is

$$V(r) = -G \frac{M \cdot m}{r}$$

Thus the potential $U(r) = V_{Cf}(r) + V(r)$ in eq. (5.8) becomes

$$U(r) = \frac{l^2}{2mr^2} - G \frac{M \cdot m}{r}$$

and, by using dr/dt from eq. (5.12), eq. (5.8) therefore becomes

$$E = \frac{m}{2} \left(\frac{l}{mr^2} r' \right)^2 + \frac{l^2}{2mr^2} - G \frac{M \cdot m}{r}$$

To solve this equation we use the substitution

$$\xi = \frac{1}{r} \quad ; \quad \xi' = \frac{d\xi}{d\varphi} = -\frac{1}{r^2} \frac{dr}{d\varphi} = -\frac{1}{r^2} \cdot r'$$

With this substitution the above equation becomes

$$E = \frac{l^2}{2m} \cdot \xi'^2 + \frac{l^2}{2m} \cdot \xi^2 - GM \cdot m \cdot \xi$$

By completing the square and rearranging the terms one obtains

$$\xi'^2 + \left(\xi - G \frac{M \cdot m^2}{l^2} \right)^2 = \frac{2mE}{l^2} + \left(G \frac{M \cdot m^2}{l^2} \right)^2 \quad (5.14)$$

With the abbreviations

$$\frac{1}{p} = G \frac{M \cdot m^2}{l^2} \quad \text{and} \quad \frac{1}{p} \cdot \epsilon = \frac{1}{p} \cdot \sqrt{\frac{2mE}{l^2} p^2 + 1} \quad (5.15)$$

the significance of which will become apparent in the following, eq. (5.14) reads

$$\xi'^2 + \underbrace{\left(\xi - \frac{1}{p} \right)^2}_{\eta} = \frac{\epsilon^2}{p^2}$$

This differential equation has the form

$$\eta'^2 + \eta^2 = \lambda^2 \quad \text{and the solution} \quad \eta = \eta_0 \cos(\varphi - \varphi_0)$$

where $\eta_0 = \pm(\epsilon/p)$ and with a suitable choice of the initial conditions $\varphi_0 = 0$. With that one obtains as the solution of eq. (5.14)

$$\xi = \frac{1}{r} = \frac{1}{p}(1 \pm \epsilon \cos \varphi)$$

or respectively the trajectory curve

$$r = \frac{p}{1 \pm \epsilon \cos \varphi} \quad (5.16)$$

This equation describes the trajectory curve of a conic section and especially for $\epsilon < 1$ it is the trajectory curve of an ellipse in polar coordinate representation. Hence, this equation states nothing different than Kepler's first law: The planets move in elliptical orbits with the Sun sitting in one focal point. The minus sign means that the coordinate origin lies in the left focal point (as shown in fig. 5.3b) and the plus sign that it lies in the right focal point.

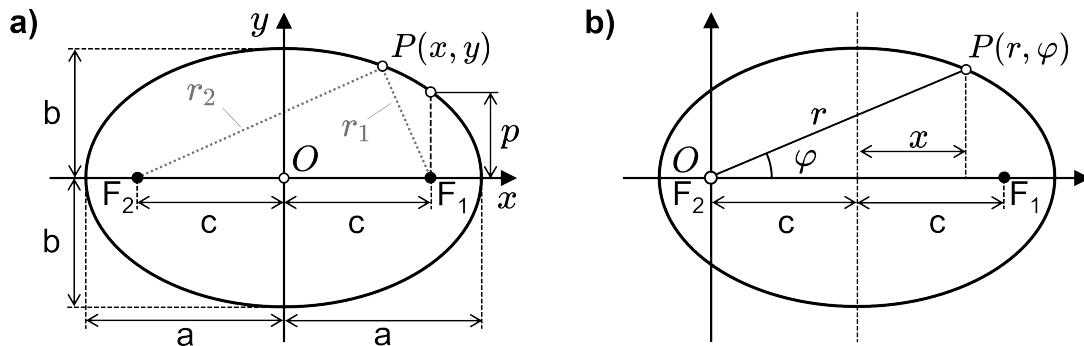


Fig. 5.3: Ellipse in Cartesian (a) and polar coordinates (b). In polar coordinates, one has the choice of placing the origin as shown here in the left focal point - minus sign in eq. (5.16) - or in the right focal point - plus sign in eq. (5.16).

Trajectory curve of the ellipse

An ellipse is defined as the set of all points for which the sum of the distances to two given points, the so-called focal points F_1 and F_2 separated by a distance $2c$ in fig. 5.3, is constant:

$$r_1 + r_2 = 2a \quad (5.17)$$

where a is the semi-major axis of the ellipse. For the relationship between the semi-major and the semi-minor axis, one can read from fig. 5.3 that

$$a^2 = c^2 + b^2$$

For r_1 and r_2 one can also read from fig. 5.3a in Cartesian coordinates

$$r_1^2 = (x - c)^2 + y^2 \quad \text{and} \quad r_2^2 = (x + c)^2 + y^2 \quad (5.18)$$

If one inserts this into eq. (5.17), one obtains after some calculation the equation describing an ellipse in Cartesian coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (5.19)$$

With eq. (5.17) also applies

$$r_2^2 - r_1^2 = (r_2 + r_1)(r_2 - r_1) = 2a(r_2 - r_1)$$

But it also follows from eq. (5.18) that

$$r_2^2 - r_1^2 = 4cx = 2a \frac{2cx}{a}$$

A comparison of the right-hand sides of the two equations yields

$$r_2 - r_1 = \frac{2cx}{a}$$

and with eq. (5.17) it therefore holds that

$$r_1 = a - \frac{cx}{a} \quad \text{and} \quad r_2 = a + \frac{cx}{a}$$

By setting $r = r_2$ for the polar coordinate representation, one transfers the coordinate origin into the left focal point of the ellipse (fig. 5.3b). With $x = r \cos \varphi - c$, the corresponding transfer of the x -coordinate of an ellipse point from fig. 5.3a into fig. 5.3b, one obtains with

$$r = a + \frac{cx}{a} = a + \frac{c}{a}(r \cos \varphi - c) = \frac{a^2 - c^2}{a} + \frac{c}{a}r \cos \varphi = \frac{b^2}{a} + \frac{c}{a}r \cos \varphi$$

the polar coordinate equation of the ellipse with the origin in the left focal point

$$r = \frac{p}{1 - \epsilon \cos \varphi} \quad (5.20)$$

with the ellipse parameters

$$p = \frac{b^2}{a} \quad \text{and} \quad \epsilon = \frac{c}{a} \quad (\text{eccentricity}) \quad (5.21)$$

The semi-major and semi-minor axes of the ellipse, a and b , as well as the distance c of the focal points from the center of the ellipse can be expressed by p and ϵ :

$$a = \frac{p}{1 - \epsilon^2} \quad ; \quad c = \frac{p\epsilon}{1 - \epsilon^2} \quad ; \quad b = \frac{p^2}{1 - \epsilon^2}$$

By inserting p and ϵ from eq. (5.15) it follows from these equations that

$$\frac{1}{a - c} = \frac{1 + \epsilon}{p} = G \frac{M \cdot m^2}{l^2} + \sqrt{\frac{2mE}{l^2} + \left(G \frac{M \cdot m^2}{l^2}\right)^2} \quad (5.22a)$$

and

$$\frac{1}{a+c} = \frac{1-\epsilon}{p} = G \frac{M \cdot m^2}{l^2} - \sqrt{\frac{2mE}{l^2} + \left(G \frac{M \cdot m^2}{l^2}\right)^2} \quad (5.22b)$$

Multiplying eq. (5.22a) by eq. (5.22b) yields

$$\frac{1}{a^2 - c^2} = \frac{1}{b^2} = \frac{1}{p^2} - \frac{\epsilon^2}{p^2} = \frac{2mE}{l^2} \quad (5.23)$$

and addition of eq. (5.22a) and eq. (5.22b) yields

$$\frac{2a}{a^2 - c^2} = \frac{2a}{b^2} = \frac{2}{p} = 2G \frac{M \cdot m^2}{l^2} \quad (5.24)$$

Division of eq. (5.23) by eq. (5.24) shows that

$$E = G \frac{M \cdot m}{2} \cdot \frac{1 - \epsilon^2}{p} = G \frac{M \cdot m}{2a} \quad (5.25)$$

With eq. (5.5) and the area of the ellipse, that is with

$$l = \frac{2mA}{\tau} \quad \text{and} \quad A = \pi \cdot a \cdot b$$

as well as with eq. (5.24), it follows that

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{G \cdot M} \quad (5.26)$$

Here, τ is the time required for a complete orbit of the point mass on the elliptical trajectory curve. Eq. (5.26) states that the ratio of the square of the orbital period to the third power of the semi-major axis depends only on the mass M and not on the mass m of the point mass in orbit. This ratio is therefore identical for all point masses m on any arbitrary elliptical trajectory curve with the mass M sitting in one of the focal points. Obviously, this is nothing but a restatement of Kepler's third law: The square of the orbital period divided by the third power of the semi-major axis is a constant for all planets orbiting the Sun. This relationship between the orbital period and orbital radius is not only valid in our solar system but it is valid for all planets that move on elliptical orbits around their star.

As eq. (5.25) shows, the trajectory curve of the ellipse has the property that the total energy can be expressed through the semi-major axis a of the ellipse. Eq. (5.25) can be formulated more generally as

$$E = -\frac{\alpha}{2a} \quad (5.27)$$

Eq. (5.27) states that for an elliptical trajectory curve the energy only depends on the semi-major axis. Each motion tracing an elliptical trajectory curve with the same a corresponds to the same total energy, independent of the eccentricity ϵ of the respective elliptical trajectory curve. Here specifically, the case of the gravitational potential was considered for which

$$V(r) = -\frac{\alpha}{r} = -G \frac{M \cdot m}{r}$$

applies with $\alpha = G \cdot M \cdot m$. But eq. (5.27) also applies, for example, to the Coulomb potential between two electric charges q_1 and q_2

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 \cdot q_2}{r} \quad \text{with} \quad \alpha = -\frac{1}{4\pi\epsilon_0} q_1 q_2$$

For the Coulomb potential α can be positive or negative. For example, if the charges q_1 and q_2 are both positive or both negative, then $\alpha < 0$. But if q_1 is positive and q_2 is negative or vice versa, then $\alpha > 0$.

In both cases, the gravitational potential and the Coulomb potential, the total energy can be determined using the simple formula in eq. (5.27). If $\alpha > 0$, then there is an attractive force and $E < 0$. With that $\epsilon < 1$ and the trajectory curve is an ellipse. The trajectory curve remains an ellipse as long as $0 < \epsilon < 1$ and at $\epsilon = 0$ the ellipse becomes a circle. If $\alpha = 0$ then $E = 0$ and thus $\epsilon = 1$ and the trajectory curve is thus a parabola. If, on the other hand, $\alpha < 0$, then one is dealing with a repulsive force and because of $E > 0$ it follows that $\epsilon > 1$ and the trajectory is therefore a hyperbola.

5.2.1 Bounded and Unbounded Motion

For bounded motion, the total energy of the relative motion is $E < 0$ and the trajectory curve is an ellipse or a circle. For unbounded motion, i.e., for $E \geq 0$, the trajectory curve is a parabola or a hyperbola. First, the bounded motion on an elliptical trajectory curve will be considered as sketched in fig. 5.3b. However, now Earth shall sit at the origin of the coordinate system, i.e., the left focal point of the ellipse, and a satellite shall orbit it on the respective trajectory curve. The trajectory curve is given by

$$r(\varphi) = \frac{p}{1 - \epsilon \cos \varphi}$$

and the total energy by

$$E = \frac{m_S}{2} \dot{r}^2 + U(r) = \frac{m_S}{2} \dot{r}^2 + V_{Cf}(r) + V(r)$$

with

$$V_{Cf}(r) = \frac{l^2}{2m_S r^2} \quad \text{and} \quad V(r) = -G \frac{M_E \cdot m_S}{r}$$

where M_E is the mass of the Earth and m_S is the mass of the satellite. Fig. 5.4 shows the shape of the potential curve $U(r)$, the sum of the attracting potential $V(r)$ and the repelling potential $V_{Cf}(r)$. If the angular momentum $l = |L|$ is too small, then the repelling force caused by it will be too small for a body to be able to move in an orbit around the Earth. The minimum distance r_{min} that a body has on an elliptical orbit around the Earth can be read from fig. 5.3b with eq. (5.15)

$$r_{min} = r(\pi) = \frac{p}{1 + \epsilon} = \frac{l^2}{GM_E m_S (1 + \epsilon)}$$

So that a body does not plunge back onto Earth $r_{min} > R_E$ must apply. Therefore, the required minimum angular momentum must be

$$l > \sqrt{GM_E m_S (1 + \epsilon) R_E} \quad (5.28)$$

The smallest possible value of l results for $\epsilon = 0$, i.e., for a circular orbit. For motion in a circular orbit with radius r , the following applies to the angular momentum $l = |L|$ and the tangential component of the velocity of the orbiting body v_φ

$$l = mr^2\dot{\varphi} \quad \text{and} \quad v_\varphi = r\dot{\varphi} \quad (5.29)$$

Inserting these relationships for the limiting case $r = R_E$ and $m = m_S$ in eq. (5.28) yields for v_φ

$$v_\varphi > \sqrt{\frac{GM_E}{R_E}} = 7.9 \text{ km}\cdot\text{s}^{-1} = v^{(1)} \quad (5.30)$$

$v_\varphi = 7.9 \text{ km}\cdot\text{s}^{-1}$ is the so-called first cosmic velocity $v^{(1)}$. If Earth had no atmosphere and its spherical surface were perfectly flat, one could throw a pebble just above Earth's surface in such a way as to impart it only with the tangential velocity component $v^{(1)}$ and thereby launch it into a stable orbit just above Earth's surface. However, because of the Earth's atmosphere, satellites must orbit our planet in a circular orbit at a much greater distance from the Earth's surface. The angular momentum for the respective desired circular orbit with radius R can be obtained analogously to eq. (5.28) and correspondingly

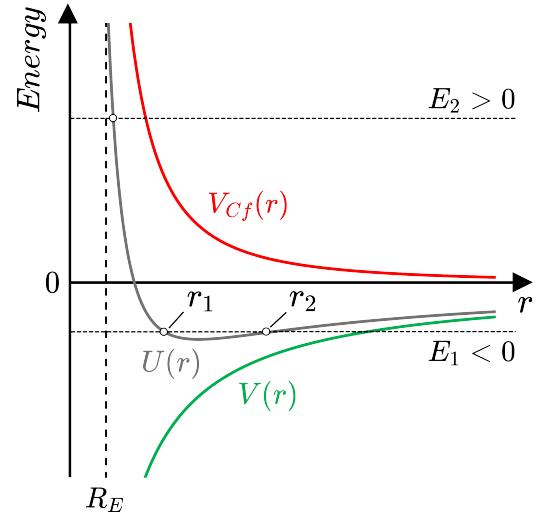


Fig. 5.4

one obtains the necessary tangential velocity. For a circular orbit, the distances r_1 and r_2 coincide and one is presented with the situation sketched in fig. 5.5.

Of particular interest is the so-called geostationary orbit, i.e., an orbit for which a satellite stays always above the same point on the Earth's surface. From eq. (5.28) with R_{geo} instead of R_E and eq. (5.30) with $\dot{\varphi} = 2\pi/1$ day one can calculate for the geostationary orbit

$$R_{geo} = \frac{G \cdot M_E \cdot 1 \text{ day}}{2\pi} = 42.2 \text{ km}$$

The geostationary orbit is therefore located at a height of approximately 35.9 km above the Earth's surface. The energy required to launch a satellite into such a geostationary bound orbit is considerable. This is of course due to the fact that with the satellite one must also accelerate the much heavier launch vehicle, most of its weight being the fuel it carries, to a correspondingly high speed. To understand what it takes to completely escape Earth's gravitational field, one must consider the energy required to bring a body from the Earth's surface into a specific orbit of radius R . The required energy is given by

$$\Delta E = G \cdot M_E \cdot m \cdot \left(\frac{1}{R_E} - \frac{1}{R} \right)$$

To completely escape Earth's gravitational field, i.e., $R \rightarrow \infty$, requires at least the kinetic energy

$$\Delta E = \frac{m}{2} \cdot v_{2E}^2 = G \cdot M_E \cdot m \cdot \frac{1}{R_E} \quad (5.31)$$

Hence, the escape velocity from the Earth's gravitational field, also known as the second cosmic velocity $v^{(2)}$, is given by

$$v_{2E} = \sqrt{\frac{2GM_E}{R_E}} = 11.2 \text{ km} \cdot \text{s}^{-1} = v^{(2)} \quad (5.32)$$

In relation to the elliptical trajectory curve in fig. 5.3b, $R \rightarrow \infty$ in eq. (5.31) means that the point on the trajectory curve with the greatest distance from the focal point moves further and further away and finally vanishes into infinity. For eq. (5.20) must therefore apply

$$r(0) = \frac{p}{1 - \epsilon} \rightarrow \infty \quad \Rightarrow \quad \epsilon \rightarrow 1$$

So, the elliptical orbit breaks up and becomes a parabola. Just as there is an escape velocity for escaping Earth's gravity, there is a corresponding escape velocity for leaving

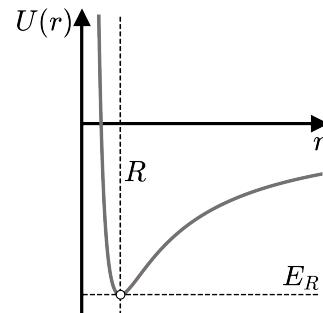


Fig. 5.5

our solar system. In order to calculate the escape velocity from the Sun at the location of the Earth, one replaces the mass of the Earth M_E and the radius of the Earth R_E in eq. (5.32) with the mass of the Sun M_S and the mean distance between the Sun and the Earth R_{SE} . With that one calculates the escape velocity from the Sun in Earth's orbital trajectory to be $v_{2S} = 42.1 \text{ km}\cdot\text{s}^{-1}$. The Earth orbits the Sun with a speed of $v_E \approx 29.8 \text{ km}\cdot\text{s}^{-1}$. This intrinsic speed of Earth reduces of course the escape velocity for leaving the solar system and therefore must be subtracted from v_{2S} . Finally, one also has to consider the effect of Earth's gravitation, that is v_{2E} . With all of this taken into account, one obtains for the third cosmic velocity $v^{(3)}$, the speed that one has to reach in order to leave the solar system from the position of Earth

$$v^{(3)} = \sqrt{(v_{2S} - v_E)^2 + v_{2E}^2} \approx 16.6 \text{ km}\cdot\text{s}^{-1} \quad (5.33)$$

Trajectory curve of the hyperbola

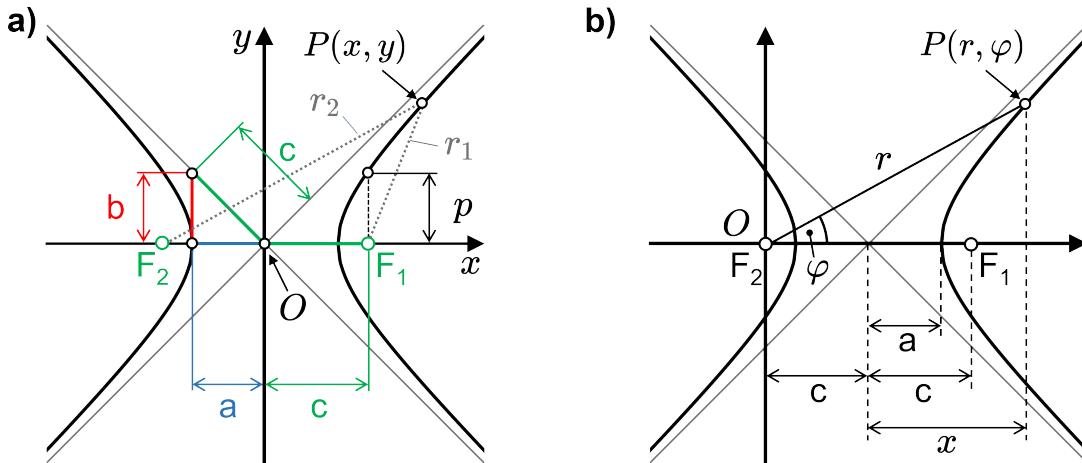


Fig. 5.6: Hyperbola in Cartesian (a) and polar coordinates (b). In polar coordinates, one has the choice of placing the origin as shown here in the left focal point - minus sign in eq. (5.16) - or in the right focal point - plus sign in eq. (5.16).

As with the ellipse, with the hyperbola one also has the choice of placing the origin of the coordinate system either in the left or the right focal point. Here, as sketched in fig. 5.6, the left focal point is chosen again to describe the right branch of the hyperbola. For every point on the right branch of the hyperbola applies $r_2 - r_1 = 2a$ and for the left branch of the hyperbola $r_2 - r_1 = -2a$ applies. For all points on the hyperbola applies in Cartesian coordinates (fig. 5.6a)

$$r_2 - r_1 = \sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a \quad (5.34)$$

With that and with $c^2 = a^2 + b^2$, the equation of the hyperbola can be derived in Cartesian coordinates

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (5.35)$$

Analogously to the procedure for the ellipse equation, by comparing the right-hand sides of

$$r_2^2 - r_1^2 = (r_2 - r_1)(r_2 + r_1) = \pm 2a(r_2 + r_1)$$

and

$$r_2^2 - r_1^2 = 4cx = 2a \frac{2cx}{a}$$

it follows that

$$r_2 + r_1 = \mp \frac{2cx}{a}$$

This gives with eq. (5.34) for the right branch of the hyperbola

$$r_1 = -a + \frac{cx}{a} \quad \text{and} \quad r_2 = a + \frac{cx}{a} \quad \text{with} \quad x \geq a$$

and for the left branch of the hyperbola

$$r_1 = a - \frac{cx}{a} \quad \text{and} \quad r_2 = -a - \frac{cx}{a} \quad \text{with} \quad x \leq -a$$

For the representation in polar coordinates with the origin in the left focal point as sketched in fig. 5.6b, $r = r_2$ is set again and one obtains with the transferred x -coordinate $x = r \cos \varphi - c$ of the right branch

$$r = a + \frac{cx}{a} = a + \frac{c}{a}(r \cos \varphi - c) = \frac{a^2 - c^2}{a} + \frac{c}{a}r \cos \varphi = -\frac{b^2}{a} + \frac{c}{a}r \cos \varphi$$

the equation for the right branch of the hyperbola in polar coordinates with the origin at the left focal point

$$r = \frac{p}{1 - \epsilon \cos \varphi} \quad (5.36)$$

with the hyperbola parameters

$$p = -\frac{b^2}{a} = -(\epsilon^2 - 1)a \quad \text{and} \quad \epsilon = \frac{c}{a} > 1 \quad (5.37)$$

To the left branch of the hyperbola eq. (5.36) also applies, but with the hyperbola parameters

$$p = \frac{b^2}{a} = (\epsilon^2 - 1)a \quad \text{and} \quad \epsilon = \frac{c}{a} > 1 \quad (5.38)$$

5.3 Scattering off a Central Potential

Scattering off a central potential is closely related to Kepler motion. In the simplest case, the scattering process describes the relative movement of two non-bound particles (i.e., $E > 0$) which approach each other coming from different directions, then at a small relative distance change their trajectories due to potential scattering, and after that move away from each other in different directions. Incidentally, the trajectory curve of this scattering process corresponds to a hyperbola branch of the Kepler motion.

In the following, a particle beam is considered in which all particles (point masses) have the same momentum \mathbf{p} . This particle beam shall be scattered at a central potential that has its center at the coordinate origin (fig. 5.7). A particle that is incident at a distance b from the z -axis with momentum \mathbf{p} is scattered by the angle θ and after scattering has the momentum \mathbf{p}' . The coordinate system in fig. 5.7 is rotated around the z -axis in such a way that the plane spanned by the two momentum vectors \mathbf{p} and \mathbf{p}' coincides with the yz -plane. The number of particles that fly through a cross-sectional area perpendicular to the xy -plane, i.e., the intensity I of the particle flux, shall be spatially constant. As can be seen from fig. 5.7, scattering off the central potential is rotationally symmetric. The number of particles dN entering the solid angle element $d\Omega = \sin \theta d\theta d\varphi$ (detector) around the scattering angle θ does not depend on φ ; scattering takes place respectively in planes for which $\varphi = \text{const}$. It is therefore sufficient to consider the problem in the scattering plane, in fig. 5.7 the yz -plane. The trajectory curve of a particle in this plane is only a function of the impact parameter b and the magnitude of the initial momentum $p = |\mathbf{p}|$, i.e., $\theta = \theta(b, p)$.

With considerations like those made in fig. 5.2 and the use of conservation of momentum and energy conservation, one can derive the equations for $\dot{\mathbf{r}}$ and $\dot{\vartheta}$; with the difference that the corresponding angle here is ϑ and not φ . Since the angle ϑ for the incident particle is larger than for the scattered particle one has in addition $\dot{\vartheta} < 0$ with

$$\dot{\vartheta} = -\frac{l}{mr^2} \quad (5.39)$$

and

$$\dot{r}^2 = \pm \frac{2}{m} \left(E - \frac{l^2}{2mr^2} - V(r) \right) \quad (5.40)$$

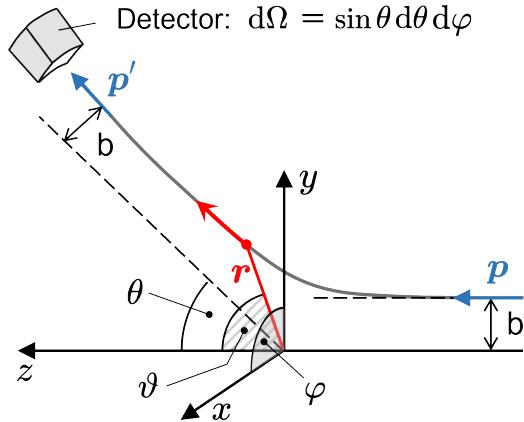


Fig. 5.7

Because of $\dot{\vartheta} < 0$, the sign in the equation equivalent to eq. (5.13) is reversed (i.e., from \pm to \mp)

$$\frac{dr}{d\vartheta} = \mp \frac{\sqrt{2m} \cdot r^2}{l} \left(E - \frac{l^2}{2mr^2} - V(r) \right)^{-1/2} \quad (5.41)$$

From this follows the equation for the scattering angle

$$d\vartheta = \mp \frac{l}{\sqrt{2m}} \left(E - \frac{l^2}{2mr^2} - V(r) \right)^{-1/2} \cdot r^{-2} \cdot dr \quad (5.42)$$

Integration from ϑ_0 to ϑ and r_0 to r yields

$$\vartheta - \vartheta_0 = \mp \frac{l}{\sqrt{2m}} \cdot \int_{r_0}^r \left(E - \frac{l^2}{2m\xi^2} - V(\xi) \right)^{-1/2} \cdot \xi^{-2} \cdot d\xi \quad (5.43)$$

When choosing ϑ_0 and r_0 , one uses the fact that the equation of motion is invariant with respect to time reversal $t \rightarrow -t$ and, theoretically, the scattering process can therefore run “backwards”, that is, the scattered particle becomes the incident particle and vice versa. This is reflected in the mirror symmetry of the trajectory curve (fig. 5.8) with respect to the straight line through the scattering center and the shortest distance r_0 of the scattered particle from the scattering center, the perihelion. Fig. 5.8 shows that $\alpha = \vartheta_0 - \theta$ and $\theta + 2\alpha = \pi$. Thus, for the angle ϑ_0 at which the particle is at the position of the perihelion the following applies:

$$\vartheta_0 = \frac{\theta}{2} + \frac{\pi}{2}$$

Now one considers the incident and the scattered branch separately. In eq. (5.40) \dot{r} is negative for the incident branch and positive for the scattered branch. The following applies to the values of ϑ and r :

$$\left. \begin{array}{l} r_0 \leq r \leq \infty \\ \vartheta_0 \leq \vartheta \leq \pi \end{array} \right\} \quad \begin{array}{c} \text{incident branch} \\ (\dot{r} < 0) \end{array} \quad \left. \begin{array}{l} r_0 \leq r \leq \infty \\ \theta \leq \vartheta \leq \vartheta_0 \end{array} \right\} \quad \begin{array}{c} \text{scattered branch} \\ (\dot{r} > 0) \end{array}$$

For the incident branch in eq. (5.43) applies

$$\pi - \left(\frac{\theta}{2} + \frac{\pi}{2} \right) = \frac{\pi}{2} - \frac{\theta}{2} = \frac{l}{\sqrt{2m}} \cdot \int_{r_0}^{\infty} \left(E - \frac{l^2}{2m\xi^2} - V(\xi) \right)^{-1/2} \cdot \xi^{-2} \cdot d\xi \quad (5.44)$$

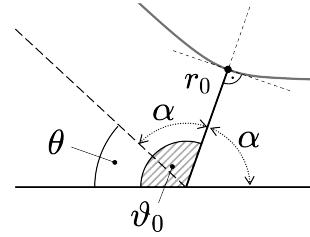


Fig. 5.8

$$\theta - \left(\frac{\theta}{2} + \frac{\pi}{2} \right) = \frac{\theta}{2} - \frac{\pi}{2} = -\frac{l}{\sqrt{2m}} \cdot \int_{r_0}^{\infty} \left(E - \frac{l^2}{2m\xi^2} - V(\xi) \right)^{-1/2} \cdot \xi^{-2} \cdot d\xi \quad (5.45)$$

With the magnitude of the angular momentum

$$l = |\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = b \cdot p = const$$

and the energy

$$E = \frac{p^2}{2m} = const$$

the equation for the scattered branch becomes

$$\frac{\theta}{2} - \frac{\pi}{2} = -b \cdot p \cdot \int_{r_0}^{\infty} \left(p^2 \left(1 - \frac{b^2}{\xi^2} \right) - 2mV(\xi) \right)^{-1/2} \cdot \xi^{-2} \cdot d\xi \quad (5.46)$$

With the substitution

$$u = \frac{b}{\xi} \quad , \quad du = -\frac{b}{\xi^2} d\xi$$

one obtains after transformation ($p^2 = 2mE$)

$$\theta = \pi - 2 \cdot \int_0^{b/r_0} \left(1 - u^2 - \frac{V(b/u)}{E} \right)^{-1/2} \cdot du \quad (5.47)$$

In order to be able to calculate the scattering angle from eq. (5.47) with a known impact parameter and known energy, r_0 must now be determined. At the perihelion $\dot{r} = 0$ must apply in eq. (5.40), i.e., (with $l^2 = b^2 p^2 = b^2 2mE$)

$$E - \frac{l^2}{2mr_0^2} - V(r_0) = E \left(1 - \frac{b^2}{r_0^2} \right) - V(r_0) = 0 \quad (5.48)$$

With the help of eq. (5.47) and eq. (5.48), one can for a given central potential $V(r)$ and energy E calculate the value of r_0 and with that the scattering angle θ as a function of the impact parameter and the energy E of the incident particle.

Rutherford scattering

Rutherford scattering, named after Ernest Rutherford, refers to the scattering of charged particles off a charged scattering center. The electrical potential (Coulomb potential) acting on a test charge q has the general form

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q \cdot q}{r} = -\frac{\alpha}{r} \quad (5.49)$$

where the constant prefactor is the Coulomb constant and Q is the electrical charge located at the scattering center. With Rutherford scattering, the electrical charges have

the same sign, i.e., $V(r)$ is positive and $\alpha < 0$ (repelling force). Inserting this into eq. (5.48) to determine r_0 we get

$$E\left(1 - \frac{b^2}{r_0^2}\right) + \frac{\alpha}{r_0} = 0$$

Multiplication with $\frac{r_0^2}{Eb^2}$ gives

$$\frac{r_0^2}{b^2} - 1 + \frac{r_0}{b} \frac{\alpha}{Eb} = 0$$

With the dimensionless quantities

$$\rho = \frac{r_0}{b} \quad \text{and} \quad \Lambda = \frac{\alpha}{2Eb}$$

one can, using completion of the square, rewrite this equation as

$$\rho^2 + 2\rho\Lambda + \Lambda^2 = 1 + \Lambda^2$$

and hence

$$\rho = -\Lambda \pm \sqrt{1 + \Lambda^2}$$

Since r_0 must be positive, the solution with the positive sign must be chosen, i.e.,

$$\rho = -\Lambda + \sqrt{1 + \Lambda^2} \tag{5.50}$$

With that, eq. (5.47) for the calculation of the scattering angle θ becomes

$$\theta = \pi - 2 \cdot \int_0^{1/\rho} (1 - u^2 + 2u\Lambda)^{-1/2} du \tag{5.51}$$

The expression in brackets under the integral can be transformed into

$$1 - u^2 - 2u\Lambda = (1 + \Lambda^2) \left[1 - \left(\frac{u - \Lambda}{\sqrt{1 + \Lambda^2}} \right)^2 \right]$$

With the substitution

$$x = \frac{u - \Lambda}{\sqrt{1 + \Lambda^2}} \quad , \quad dx = \frac{du}{\sqrt{1 + \Lambda^2}}$$

the integral in eq. (5.51) becomes

$$\int (1 - u^2 + 2u\Lambda)^{-1/2} du = \int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x = \arcsin \frac{u - \Lambda}{\sqrt{1 + \Lambda^2}}$$

One therefore obtains for the scattering angle θ in eq. (5.51)

$$\theta = \pi - 2 \arcsin \frac{u - \Lambda}{\sqrt{1 + \Lambda^2}} \Big|_0^{1/\rho} = \pi - 2 \left(\arcsin \frac{1/\rho - \Lambda}{\sqrt{1 + \Lambda^2}} - \arcsin \frac{-\Lambda}{\sqrt{1 + \Lambda^2}} \right)$$

Inserting ρ from eq. (5.50) into this expression gives

$$\begin{aligned}\theta &= \pi - 2 \left(\underbrace{\arcsin \left[\frac{1}{\sqrt{1+\Lambda^2}} \left(\frac{1+\Lambda^2 - \Lambda\sqrt{1+\Lambda^2}}{\sqrt{1+\Lambda^2} - \Lambda} \right) \right]}_{=1} - \arcsin \frac{-\Lambda}{\sqrt{1+\Lambda^2}} \right) \\ &= \pi - 2 \left(\underbrace{\arcsin 1}_{\pi/2} - \arcsin \frac{-\Lambda}{\sqrt{1+\Lambda^2}} \right) = 2 \arcsin \frac{-\Lambda}{\sqrt{1+\Lambda^2}}\end{aligned}$$

With that and using $\Lambda = \alpha/2Eb$ one obtains for the Rutherford scattering angle θ ($\alpha < 0$, $E > 0$ and thus $-\Lambda > 0$)

$$\theta = 2 \arcsin \frac{1}{\sqrt{1 + \frac{4E^2 b^2}{\alpha^2}}} \quad (5.52)$$

Fig. 5.9 shows the dependence of the scattering angle θ in eq. (5.52) on the impact parameter b in graphic form. In the case of a central impact ($b = 0$) one has direct backscattering ($\theta = \pi$) and the scattered branch is congruent with the incident branch. Eq. (5.52) shows that the argument of the arcsin function scales like b^{-1} . Because $\arcsin(x) \approx x$ applies for small x , $\theta(b)$

also scales for large impact parameters b like b^{-1} . The reason for this is the long range of potentials of the type $V(r) = \alpha/r$. Since the scattering angle θ in eq. (5.52) depends on the impact parameter b and the energy E in the same way, one obtains the same curve shape as for $\theta(b)$ in fig. 5.9 if one plots θ as a function of E .

In order to better illustrate the connection between Kepler motion and the trajectory curve of Rutherford scattering, the latter can be derived directly from Kepler motion. To do this one considers the plane trajectory curve of an incoming particle in polar coordinates. The particle shall come in from an “infinite” distance and shall be scattered at a central potential centered in the origin of the coordinate system. The radius vector from the origin of the central potential (origin of the coordinate system) to the incident particle shall be

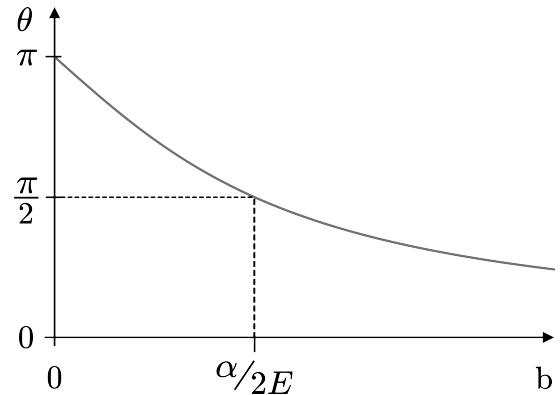


Fig. 5.9

$r = r \cdot \hat{u}$ (5.53)

where $\hat{\mathbf{u}}$ is the unit vector pointing in the direction of the particle. For $\hat{\mathbf{u}}$ holds

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1 \quad \text{and therefore} \quad \frac{\partial \hat{\mathbf{u}}^2}{\partial t} = 2\hat{\mathbf{u}} \cdot \dot{\hat{\mathbf{u}}} = 0 \quad (5.54)$$

For the velocity of the incoming particle, it follows from eq. (5.53)

$$\dot{\mathbf{r}} = \dot{r} \cdot \hat{\mathbf{u}} + r \cdot \dot{\hat{\mathbf{u}}}$$

and its angular momentum is thus

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \dot{\mathbf{r}}) = m(r\hat{\mathbf{u}} \times (\dot{r} \cdot \hat{\mathbf{u}} + r \cdot \dot{\hat{\mathbf{u}}})) = mr^2(\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}})$$

For the central potential, the force always acts along the radius vector, i.e.,

$$m\ddot{\mathbf{r}} = -\nabla V(r) = -\nabla\left(-\frac{\alpha}{r}\right) = -\frac{\alpha}{r^2}\hat{\mathbf{u}} \quad (\alpha < 0)$$

This directly implies angular momentum conservation in central potentials since

$$\frac{\partial \mathbf{L}}{\partial t} = m\frac{\partial}{\partial t}(\mathbf{r} \times \dot{\mathbf{r}}) = m\left(\underbrace{\dot{\mathbf{r}} \times \dot{\mathbf{r}}}_{=0} + \mathbf{r} \times \ddot{\mathbf{r}}\right) = -\frac{m\alpha}{r}\underbrace{(\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}})}_{=0} = 0$$

Now one uses the cross product of acceleration and angular momentum

$$\ddot{\mathbf{r}} \times \mathbf{L} = -\frac{\alpha}{mr^2}\hat{\mathbf{u}} \times mr^2(\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}) = -\alpha(\hat{\mathbf{u}} \times \hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}})$$

which can be rewritten using the Graßmann identity and eq. (5.54) to give

$$\ddot{\mathbf{r}} \times \mathbf{L} = -\alpha\left(\hat{\mathbf{u}}\underbrace{(\hat{\mathbf{u}} \cdot \dot{\hat{\mathbf{u}}})}_{=0} - \dot{\hat{\mathbf{u}}}\underbrace{(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})}_{=1}\right) = \alpha\dot{\hat{\mathbf{u}}} \quad (5.55)$$

Eq. (5.55) can be integrated directly, since $\mathbf{L} = \text{const}$ and one obtains

$$\dot{\mathbf{r}} \times \mathbf{L} = \alpha\hat{\mathbf{u}} + \mathbf{c} \quad (5.56)$$

where the vector \mathbf{c} is a constant of integration. The vector equation eq. (5.56) can be converted into a scalar equation by multiplying from the left with $\mathbf{r} = r\hat{\mathbf{u}}$. Because the scalar triple product is commutative, one obtains

$$\mathbf{r}(\dot{\mathbf{r}} \times \mathbf{L}) = (\mathbf{r} \times \dot{\mathbf{r}})\mathbf{L} = \frac{1}{m}\mathbf{L}\mathbf{L} = \alpha r\hat{\mathbf{u}}\hat{\mathbf{u}} + \mathbf{r}\mathbf{c}$$

If φ is the angle between the vectors \mathbf{r} and \mathbf{c} and $l = |\mathbf{L}|$, resolving this equation for r results in (with $\alpha < 0$ replaced by $-\alpha$)

$$r = \frac{p}{1 - \epsilon \cos \varphi} \quad (5.57)$$

with the parameters

$$p = -\frac{l^2}{m|\alpha|} \quad \text{and} \quad \epsilon = \frac{c}{|\alpha|} \quad (5.58)$$

Eq. (5.57) is the trajectory curve of a conic section. The fact that the conic section parameter p in eq. (5.58) is negative already suggests that one is dealing with a hyperbola. In order to verify that $\epsilon > 1$ holds, one has to determine c . One obtains the value of c by considering the initial conditions. If these are set as sketched in fig. 5.7, then the incoming particle comes from the $+z$ -direction with the velocity v_∞ on a straight line which runs parallel to the z -axis at a distance b ; the trajectory curve lies in the yz -plane. At very large distances, i.e., when the incident particle is still “infinitely” far away from the scattering center, the following applies to the total energy of the relative motion:

$$E = \frac{m}{2} \cdot v_\infty^2 \quad (5.59)$$

Without the influence of the scattering center, the incoming particle with momentum mv_∞ would intersect the y -axis on a straight-line path at b . With the unit vectors of the coordinate system $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ the particle has thus the angular momentum (see also the discussion about the angular momentum associated with linear motion at constant speed at the beginning of section 5.1)

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = b \cdot \mathbf{e}_y \times mv_\infty \cdot \mathbf{e}_z = mbv_\infty \cdot \mathbf{e}_x \quad (5.60)$$

The angular momentum is a conserved quantity. Therefore, eq. (5.60) also applies to the incoming particle still “infinitely” far away from the scattering center. For this case, one now evaluates eq. (5.56) with eq. (5.60) and thus obtains the initial condition

$$\dot{\mathbf{r}} \times \mathbf{L} = \dot{\mathbf{r}} \cdot \mathbf{e}_z \times mbv_\infty \cdot \mathbf{e}_x = mbv_\infty^2 \cdot \mathbf{e}_y = \alpha \cdot \mathbf{e}_z + \mathbf{c}$$

The unit vectors \mathbf{e}_y and \mathbf{e}_z are perpendicular to each other and therefore the absolute value of \mathbf{c} is

$$c = \sqrt{\alpha^2 + (mbv_\infty^2)^2}$$

For ϵ in eq. (5.58) this results in

$$\epsilon = \frac{c}{|\alpha|} = \sqrt{1 + \left(\frac{mbv_\infty^2}{\alpha}\right)^2} > 1$$

Hence, the trajectory curve of Rutherford scattering (eq. (5.57)) is indeed that of a hyperbola. With the incident particle at a large distance from the scattering center, the total energy is given by eq. (5.59) and thus ϵ becomes

$$\epsilon = \sqrt{1 + \frac{4E^2b^2}{\alpha^2}} \quad (5.61)$$

A comparison with eq. (5.52) shows that for the scattering angle θ holds

$$\theta = 2 \arcsin \frac{1}{\epsilon} \quad (5.62)$$

From eq. (5.61) one can now derive the relationship between the impact parameter b and the hyperbola parameters. Solving eq. (5.61) for b^2 gives

$$b^2 = \frac{\alpha^2}{4E^2}(\epsilon^2 - 1) \quad (5.63)$$

and from eq. (5.58) it follows with the help of eq. (5.59) and eq. (5.60) that

$$p = -\frac{l^2}{m|\alpha|} = -\frac{mv_\infty^2 b^2}{|\alpha|} = -\frac{2Eb^2}{|\alpha|} \quad \Rightarrow \quad \frac{\alpha^2}{4E^2} = \frac{b^4}{p^2}$$

The latter inserted in eq. (5.63) for b^2 gives for the connection between the impact parameter b and the hyperbolic parameters p and ϵ

$$b = \frac{p}{\sqrt{\epsilon^2 - 1}} \quad (5.64)$$

5.3.1 The Effective Cross Section

The effective cross section σ is a measure for the probability of an interaction between two particles. In the simplest case, σ is identical with the geometric cross section. A simple example of the latter is an umbrella. If one holds an umbrella above one's head when it rains, assuming the raindrops fall vertically, no raindrops will hit the ground in a small circle around oneself. The area of this circle around one's feet represents the geometric cross section of the umbrella. However, if the umbrella has holes, then it no longer interacts with all the raindrops aimed at it from above and some pass through it unhindered - the effective cross section of the umbrella has therefore become smaller.

In order to determine the cross-section for Rutherford scattering, one considers again a particle stream which runs parallel to the z -axis towards a scattering center (fig. 5.10). Through the interaction with the scattering center, particles whose impact parameter is between b and $b + db$ are being scattered into an angle element between θ and $\theta + d\theta$. Because the scattering potential (e.g., the Coulomb potential) does not depend on the angle ϕ , particles whose impact parameters fall within the annulus with radius b and thickness db will all be scattered into the solid angle element $d\Omega$ integrated over ϕ .

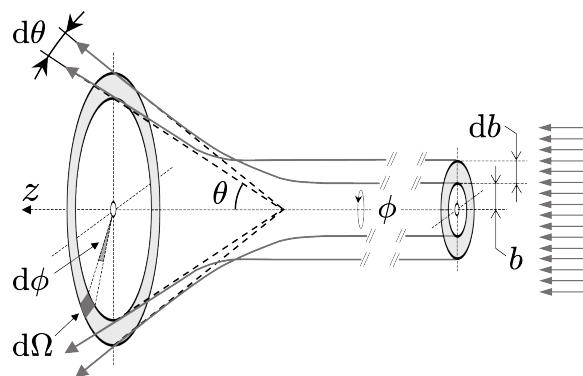


Fig. 5.10

The annulus with radius b and thickness db has the area

$$d\sigma = 2\pi b db$$

and the solid angle integrated over ϕ is

$$d\Omega = 2\pi \sin \theta d\theta$$

With that, the differential effective cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| \quad (5.65)$$

Since the cross section must always be positive (it has the measure of an area) and $db/d\theta$ can also assume negative values, eq. (5.65) uses its absolute magnitude (b is always positive and with $0 \leq \theta \leq \pi$ it always holds that $\sin \theta \geq 0$). From eq. (5.52) one obtains for b

$$b = \frac{\alpha}{2E} \cot \frac{\theta}{2} \quad (5.66)$$

and with that

$$\frac{db}{d\theta} = -\frac{\alpha}{4E} \frac{1}{\sin^2 \theta / 2} \quad (5.67)$$

Inserting eq. (5.66) and eq. (5.67) into eq. (5.65) yields the differential effective cross section as a function of the scattering angle

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E} \right)^2 \frac{1}{\sin^4 \theta / 2} \quad (5.68)$$

What is remarkable about eq. (5.68) is that the differential effective cross section depends on α^2 and not on α . Because of that, in the case of Coulomb scattering eq. (5.68) is independent of the sign of the electrical charges. The total effective cross section σ_{tot} is obtained by integration over the solid angle $d\Omega = 2\pi \sin \theta d\theta$, i.e.,

$$\sigma_{tot} = 2\pi \left(\frac{\alpha}{4E} \right)^2 \int_0^\pi \frac{\sin \theta}{\sin^4 \theta / 2} d\theta \quad (5.69)$$

The integrand in eq. (5.69) diverges for $\theta \rightarrow 0$. For small θ holds $\sin \theta \approx \theta$ and $\sin \theta / 2 \approx \theta / 2$, the result of which is that the integrand diverges for $\theta \rightarrow 0$ like θ^{-3} and that the respective primitive function $\sigma_{tot}(\theta)$ therefore diverges like θ^{-2} . This divergence at small scattering angles or large impact parameters is caused by the long range of central potentials of the type $V(r) = -\alpha/r$.

5.4 Angular Momentum of Many Point Masses

In this chapter we first considered the angular momentum of a single point mass in linear motion at constant speed. Then we saw the critical role angular momentum plays in understanding the motion of a point mass in the conservative central force field of another point mass, i.e., the movement of planets around the Sun, satellites around Earth, of rockets leaving Earth or the solar system. Finally, in the last section we saw the role angular momentum plays in the scattering process of one point mass off the repulsive central potential of another point mass. Such was the case when Rutherford and his team in 1911 bombarded a gold foil with He^{+2} atoms, so-called α -particles, thereby experimentally confirming the existence of atoms, in this case of gold atoms, through elastic scattering of α -particles off gold nuclei. In all three cases we looked into, we found that angular momentum is a conserved quantity, i.e., a constant of motion. So what about the angular momentum of systems of many point masses?

To address this question we will first return to the conservation of linear momentum and the momentum theorem introduced in section 3.2.1. There we found (eq. (3.21)) that without the influence of external forces, the total momentum of a system of point masses is constant. In what follows we will label external forces with a superscript “*ext*” and internal forces with a superscript “*int*”. The conservation of total momentum for a system of point masses on which no external forces act is then with eq. (3.21) given by

$$\sum_i \frac{d\mathbf{p}_i}{dt} = \frac{d}{dt} \underbrace{\sum_i \mathbf{p}_i}_{\mathbf{P}} = \frac{d\mathbf{P}}{dt} = \sum_{j \neq i} \mathbf{F}_{ij}^{int} \equiv 0 \quad (5.70)$$

where \mathbf{p}_i is the momentum of a point mass and \mathbf{F}_{ij}^{int} the two-body interaction between any two given point masses. With \mathbf{r}_{CM} denoting the center of mass coordinate of the system of point masses

$$\mathbf{r}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \quad \text{and} \quad \frac{d\mathbf{r}_{CM}}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i} = \frac{\mathbf{P}}{M}$$

the rate of change of the total linear momentum is given by $\dot{\mathbf{P}} = M \ddot{\mathbf{r}}_{CM}$ where M is the total mass of the system of point masses. With that we can rewrite eq. (5.70) as

$$\frac{d\mathbf{P}}{dt} = M \frac{d^2 \mathbf{r}_{CM}}{dt^2} = 0 \quad (5.71)$$

Eq. (5.71) is of course a restatement of Newton's first law, that a body will not change its state of rest or of uniform linear motion unless an external force acts on it. The difference is here that it applies to the center of mass of a system of point masses, each of which may move under internal forces in various ways with the center of mass not changing its state of motion. With the center of mass remaining unaffected by any rotational movement point masses in the system may make it is also clear that there can be only one rotational axis for a system of point masses on which no external forces act and that is an axis through the center of mass of the point mass system.

To change the state of motion of the center of mass of a system of point masses requires the action of an external force. With that the center of mass motion will be governed by Newton's second law, i.e.,

$$\frac{d\mathbf{P}}{dt} = M \frac{d^2 \mathbf{r}_{CM}}{dt^2} = \sum_i \mathbf{F}_i^{ext}$$

where \mathbf{F}_i^{ext} is the external force acting on point mass m_i . Next we will look at the force equation for individual point masses in a system of point masses which is subject to external and internal forces. The simple example of a system of three point masses m_1 , m_2 and m_3 in fig. 5.11 shows the respective internal and external forces acting on each of the point masses. The choice of coordinate origin in fig. 5.11 measuring the positions of the respective point masses is arbitrary; it could be the center of mass of the three point masses but it could also be any other reference point we might chose. Reading off the respective forces and distances from fig. 5.11 we get for the force momentum with respect to the coordinate origin, i.e., the torque exerted on each of the three point masses

$$\mathbf{M}_1 = \frac{d\mathbf{L}_1}{dt} = \mathbf{r}_1 \times \mathbf{F}_1 = \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{ext})$$

$$\mathbf{M}_2 = \frac{d\mathbf{L}_2}{dt} = \mathbf{r}_2 \times \mathbf{F}_2 = \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_2^{ext})$$

$$\mathbf{M}_3 = \frac{d\mathbf{L}_3}{dt} = \mathbf{r}_3 \times \mathbf{F}_3 = \mathbf{r}_3 \times (\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_3^{ext})$$

With some rearranging, using the fact that for the internal forces applies $\mathbf{F}_{12} = -\mathbf{F}_{21}$, $\mathbf{F}_{23} = -\mathbf{F}_{32}$ and $\mathbf{F}_{13} = -\mathbf{F}_{31}$, as well as using the vector identities (see insert in fig. 5.11) $\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_{32} = \mathbf{r}_2 - \mathbf{r}_3$ and $\mathbf{r}_{31} = \mathbf{r}_1 - \mathbf{r}_3$ we get for the total torque exerted on the

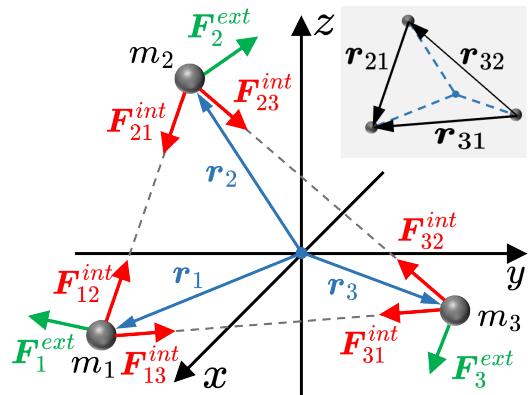


Fig. 5.11

system of three point masses

$$\begin{aligned}\mathbf{M} &= \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \frac{d}{dt}(\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3) \\ &= [\mathbf{r}_{21} \times \mathbf{F}_{12}] + [\mathbf{r}_{32} \times \mathbf{F}_{23}] + [\mathbf{r}_{31} \times \mathbf{F}_{13}] + \sum_{i=1}^3 [\mathbf{r}_i \times \mathbf{F}_i^{ext}]\end{aligned}$$

The vector \mathbf{r}_{21} is of course antiparallel to \mathbf{F}_{12} as is \mathbf{r}_{32} to \mathbf{F}_{23} and \mathbf{r}_{31} to \mathbf{F}_{13} . Hence, all vector products related to internal forces vanish and what remains is

$$\mathbf{M} = \frac{d\mathbf{L}}{dt} = \sum_{i=1}^3 [\mathbf{r}_i \times \mathbf{F}_i^{ext}]$$

where

$$\mathbf{L} = \sum_{i=1}^3 [\mathbf{r}_i \times \mathbf{p}_i]$$

Evidently, we could have included many more point masses in the system than the three in the example of fig. 5.11 and obtained for a system of n point masses the equations

$$\mathbf{M} = \frac{d\mathbf{L}}{dt} = \sum_{i=1}^n [\mathbf{r}_i \times \mathbf{F}_i^{ext}] \quad (5.72)$$

and

$$\mathbf{L} = \sum_i^n \mathbf{L}_i = \sum_{i=1}^n [\mathbf{r}_i \times \mathbf{p}_i] \quad (5.73)$$

From eq. (5.72) follows directly that without external forces acting on the system of point masses the angular momentum is a constant. This is exactly what we for example found when looking at the motion of planets around the Sun as the only forces at work in Kepler motion are the mutual gravitational attractions between the Sun and any given planet and between the planets themselves. The solar system itself is essentially a system of point masses where only the internal forces of mutual gravitational attraction are at work and hence the angular momentum of our solar system, i.e., its direction and its magnitude, is a conserved quantity. Even though more than 99% of our solar systems mass is contained in the mass of the Sun, it is the planets which carry toady about 98% of the solar systems angular momentum. That is of course simply due to the fact that angular momentum scales linearly with distance. Hence, even the mass of a small planet many million kilometers away from the Sun contributes much more to the solar system angular momentum than a similar mass at the periphery of the Sun, some 696 000 kilometers from the Sun's center.

The Earth-Moon system is another interesting example for the conservation of angular momentum. The Moon was created some 4 500 million years ago in a glancing collision between a planet about the size of Mars, scientists refer to it as Theia, and Earth, which at the time had accumulated about 90% of the mass it has today. We do not know how fast Earth rotated itself when it collided with Theia neither do we know how much this collision may have changed Earth's rotational speed but what we know is that as a consequence of the collision, the Moon formed some 24 000 km from Earth's center and today orbits Earth at an average distance of 384 400 km. We also know that due to the lack of external forces impacting the Earth-Moon system after this collision the angular momentum of the Earth-Moon system is practically still the same.¹ So far we have only considered the angular momentum associated with the rotation of a point mass with respect to a given reference point. But as indicated in fig. 5.12, for the Earth-Moon system we also have contributions to the total angular momentum from the self-rotation of Earth and of the Moon. Hence, the total angular momentum of the Earth-Moon system is the sum of the angular momentum due to the Moon orbiting Earth plus the angular momentum components associated with the self-rotations of the Earth and of the Moon, i.e.,

$$\mathbf{L}_{\text{Total}}^{\text{Earth-Moon}} = \mathbf{L}_{\text{Moon}}^{\text{Orbit}} + \mathbf{L}_{\text{Earth}}^{\text{Self}} + \mathbf{L}_{\text{Moon}}^{\text{Self}}$$

The Moon has moved away from Earth since it was formed and today is 16 times farther away from Earth's center than when it formed some 4 500 million years ago. $\mathbf{L}_{\text{Moon}}^{\text{Orbit}}$, the contribution to the total angular momentum of the Earth-Moon system by the orbiting Moon has very much increased, meaning that the contributions by the self-rotations of the Earth and the Moon must have decreased significantly. Because Earth is so much bigger than the Moon most of the decrease comes from a slower rotating Earth. Of course, there are still many uncertainties about the details of the Moon's formation so we do not know the exact decrease in $\mathbf{L}_{\text{Earth}}^{\text{Self}}$ but scientists can estimate ranges for it. Based on that and the fact that today Earth completes one rotation around its axis in 24 hours, it has been estimated that just after the Moon had formed, an Earth day may

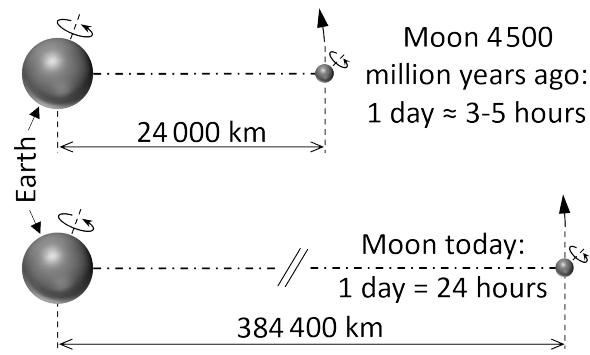


Fig. 5.12

¹Asteroid impacts are of course external forces but considering the size of Earth, their impact on the Earth-Moon system angular momentum is negligible.

not have had more than 3-5 hours. Fossil records give us additional confirmation that Earth's rotation is slowing down. Comparisons of growth lines of fossil corals that grew 350 million years ago with modern corals show that back then an Earth year had around 385 days. Hence, an Earth day then was about twenty-three hours long. And from tidal cycles recorded in sedimentary rocks in places such as southern Australia 620 million years ago and in Utah 900 million years ago, scientists calculated a day's length back then to be 21.9 and 18.9 hours, respectively. So, the trend is clear, the farther back we look in time the shorter Earth days were, the faster Earth rotated, and the closer the Moon was to Earth. Today we can actually measure how fast the Moon moves away from Earth. In the 1970's, Apollo astronauts set up mirrors on the Moons surface with the help of which we can bounce laser light from a source on Earth off the Moon surface. By measuring the time it takes the laser light to travel to the Moon and back to Earth we can precisely know the Moon's distance from Earth. That distance has increased by an average of 3.82 centimeters every year since the first measurements were taken in the early 1970's. Earth rotation around its axis an example, though not a perfect one, of rigid body rotation to which we will turn our attention next.

6. Rigid Body Rotation

6.1 The Tensor of Inertia

As sketched in fig. 6.1, let \mathbf{r}_P be the vector to the origin of the body-fixed reference system at P and \mathbf{r}_K be the vector of a point mass at position K within the rigid body with respect to the coordinate origin O . Then

$$\mathbf{r}_K = \mathbf{r}_P + \mathbf{r}_{PK} \quad (6.1a)$$

$$\mathbf{v}_K = \mathbf{v}_P + \mathbf{v}_{PK} \quad (6.1b)$$

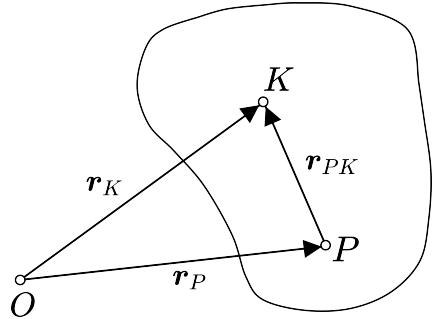


Fig. 6.1

The distances between any two point masses in the rigid body itself do not change (that is the definition of a rigid body) and thus

$$(\mathbf{r}_{PK})^2 = \mathbf{r}_{PK} \cdot \mathbf{r}_{PK} = const$$

With the time derivative of this expression it follows

$$0 = 2 \cdot \frac{d\mathbf{r}_{PK}}{dt} \cdot \mathbf{r}_{PK} \quad \Rightarrow \quad \mathbf{v}_{PK} \perp \mathbf{r}_{PK}$$

This allows eq. (6.1b) to be rewritten as

$$\mathbf{v}_K = \mathbf{v}_P + \underbrace{\omega \times \mathbf{r}_{PK}}_{\mathbf{v}_{PK}} \quad (6.2)$$

The total angular momentum of the rigid body is given by the sum of the angular momenta of all point masses of the body

$$\mathbf{L} = \sum_m m_K (\mathbf{r}_K \times \mathbf{v}_K) \rightarrow \int (\mathbf{r}_K \times \mathbf{v}_K) \underbrace{dm}_{\rho \cdot dV}$$

Specifically, if the origin of the body-fixed reference system is the vector to the center of mass, i.e.,

$$\mathbf{r}_P = \mathbf{r}_S \quad \text{and} \quad \mathbf{r}_{PK} = \mathbf{r}_{SK}$$

the total angular momentum of the rigid body becomes

$$\mathbf{L} = \int (\mathbf{r}_S + \mathbf{r}_{SK}) \times (\mathbf{v}_S + \boldsymbol{\omega} \times \mathbf{r}_{SK}) dm$$

where

$$dm = \rho \cdot dx_{SK} \cdot dy_{SK} \cdot dz_{SK}$$

With the mass $M = \int dm$ of the rigid body, this becomes

$$\mathbf{L} = (\mathbf{r}_S \times \mathbf{v}_S) \cdot M + \int (\mathbf{r}_S \times \boldsymbol{\omega} \times \mathbf{r}_{SK}) dm + \int \mathbf{r}_{SK} \times (\mathbf{v}_S + \boldsymbol{\omega} \times \mathbf{r}_{SK}) dm$$

As the center of mass coincides with the origin of the body-fixed reference system

$$\int \mathbf{r}_{SK} dm \equiv 0$$

and thus, integrals with integrands linear in \mathbf{r}_{SK} vanish and one obtains

$$\mathbf{L} = \underbrace{(\mathbf{r}_S \times \mathbf{v}_S) \cdot M}_{\mathbf{L}_S} + \underbrace{\int \mathbf{r}_{SK} \times (\boldsymbol{\omega} \times \mathbf{r}_{SK}) dm}_{\mathbf{L}_{rot}} \quad (6.3)$$

Here \mathbf{L}_S is the angular momentum due to the motion of the body's center of mass around the coordinate origin and \mathbf{L}_{rot} is the angular momentum due to the body's own rotation. This self-rotation of rigid bodies is also referred to as spin (not to be confused with the spin of elementary particles). With the help of the identity $a \times (b \times c) = (ac)b - (ab)c$, we can rewrite for \mathbf{L}_{rot} in eq. (6.3) the expression under the integral as

$$\mathbf{r}_{SK} \times (\boldsymbol{\omega} \times \mathbf{r}_{SK}) = \mathbf{r}_{SK}^2 \boldsymbol{\omega} - (\mathbf{r}_{SK} \cdot \boldsymbol{\omega}) \cdot \mathbf{r}_{SK}$$

and thereby obtain for the components of \mathbf{L}_{rot}

$$L_{j_{rot}} = \sum_{l=1}^3 \int \left(\sum_{i=1}^3 x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}} \right) dm \cdot \omega_l = \sum_{l=1}^3 J_{jl} \cdot \omega_l$$

where

$$J_{jl} = \int \left(\sum_{i=1}^3 x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}} \right) dm$$

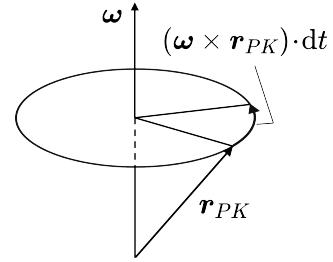


Fig. 6.2

depends only on the mass distribution of the body and the choice of reference system, i.e., the choice of \mathbf{r}_{SK} . The J_{jl} are the components of the symmetric tensor of inertia $\underline{\underline{\mathbf{J}}}$

$$\underline{\underline{\mathbf{J}}} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \quad (6.4)$$

and with it applies for \mathbf{L}_{rot}

$$\mathbf{L}_{rot} = \underline{\underline{\mathbf{J}}} \cdot \boldsymbol{\omega} \quad \text{or respectively} \quad \mathbf{L}_{rot} = \underline{\underline{\mathbf{J}}}^{(d)} \cdot \boldsymbol{\omega} \quad (6.5)$$

where the index d indicates that the notation $\underline{\underline{\mathbf{J}}}_d$ refers to the tensor of inertia of intrinsic rotation, i.e., to the body's spin ("Drall" in German). Hence eq. (6.3) becomes

$$\mathbf{L} = \mathbf{L}_S + \underline{\underline{\mathbf{J}}} \cdot \boldsymbol{\omega} = \mathbf{L}_S + \underline{\underline{\mathbf{J}}}^{(d)} \cdot \boldsymbol{\omega} \quad (6.6)$$

The diagonal components of $\underline{\underline{\mathbf{J}}}$, i.e., the J_{jj} , are called mass moments of inertia and the J_{jl} with $j \neq l$ are called products of inertia. If all J_{jl} with $j \neq l$ are equal to zero, then \mathbf{L}_{rot} points in the same direction as $\boldsymbol{\omega}$, otherwise, which is the more general case, \mathbf{L}_{rot} and $\boldsymbol{\omega}$ point in different directions.

On the calculation of the J_{jl} :

$$J_{11} = \int (\mathbf{r}_{SK}^2 - x_{1_{SK}}^2) dm = \int (y_{SK}^2 + z_{SK}^2) dm$$

J_{22} and J_{33} are obtained by cyclic permutation.

$$J_{12} = - \int x_{1_{SK}} x_{2_{SK}} dm = - \int x_{SK} y_{SK} dm = J_{21}$$

The remainder of the J_{jl} with $j \neq l$ is obtained again by cyclic permutation.

In summary, one obtains:

$$\begin{aligned} J_{11} &= \int (y_{SK}^2 + z_{SK}^2) dm & J_{12} = J_{21} &= - \int x_{SK} y_{SK} dm \\ J_{22} &= \int (x_{SK}^2 + z_{SK}^2) dm & J_{13} = J_{31} &= - \int x_{SK} z_{SK} dm \\ J_{33} &= \int (x_{SK}^2 + y_{SK}^2) dm & J_{23} = J_{32} &= - \int y_{SK} z_{SK} dm \end{aligned} \quad (6.7)$$

Adding the mass moments of inertia in eq. (6.7) results in

$$J_{11} + J_{22} + J_{33} = 2 \int (x_{SK}^2 + y_{SK}^2 + z_{SK}^2) dm = 2 \int \mathbf{r}_{SK}^2 dm \quad (6.8)$$

The integral depends only on the reference point and not on the choice of the reference system, that is, the coordinate origin. Therefore, the integral remains unchanged when the coordinate system is rotated.

Mass moment of inertia conservation

The sum of the three mass moments of inertia of a rigid body is a conserved quantity under rotation of the coordinate system.

The following triangle inequalities can also be read from eq. (6.7):

$$J_{11} + J_{22} \geq J_{33}, \quad J_{22} + J_{33} \geq J_{11}, \quad J_{33} + J_{11} \geq J_{22} \quad (6.9)$$

Mass moment of inertia triangle inequality

The sum of two mass moments of inertia of a rigid body is always greater than or at least equal to the third mass moment of inertia.

Huygens-Steiner theorem

The tensor of inertia $\underline{\underline{J}}$ of a body does depend on the choice of the reference point relative to the center of mass of the body. As sketched in fig. 6.3, the reference point P in the rigid body shall be shifted with respect to the body's center of mass S , but the reference system, i.e., the coordinate system, shall remain unchanged:

$$\mathbf{r}_{PK} = \mathbf{r}_{PS} + \mathbf{r}_{SK} \quad (*)$$

If an axis of rotation running through the center of mass is shifted parallel to run through a new reference point P , then the following applies to the components of $\underline{\underline{J}}$ with respect to this shifted axis of rotation:

$$J_{Pj} = \underbrace{\int \left(\sum_{i=1}^3 x_{iPK}^2 \delta_{jl} - x_{jPK} x_{lPK} \right) dm}_{Int} \quad (6.10)$$

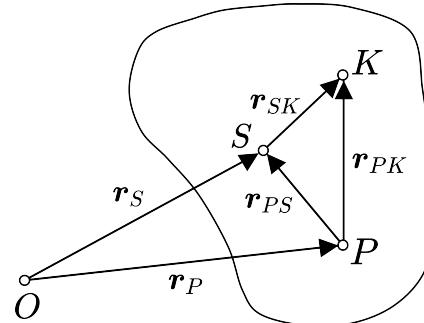


Fig. 6.3

In the integrand “ Int ” we replace \mathbf{r}_{PK} by the relation $(*)$ and get

$$Int = \sum_{i=1}^3 (x_{iPS} + x_{iSK})^2 \delta_{jl} - (x_{jPK} + x_{jSK})(x_{lPK} + x_{lSK})$$

After multiplying and rearranging this becomes

$$\begin{aligned} Int &= \sum_{i=1}^3 x_{iPS}^2 \delta_{jl} - x_{jPS} x_{lPS} + \sum_{i=1}^3 x_{iSK}^2 \delta_{jl} - x_{jSK} x_{lSK} \\ &\quad + 2 \sum_{i=1}^3 x_{iPS} x_{iSK} \delta_{jl} - x_{jSK} x_{lPS} - x_{jPS} x_{lSK} \end{aligned}$$

With this reshaped integrand eq. (6.10) becomes

$$\begin{aligned} J_{P_{jl}} &= \underbrace{\int \left(\sum_{i=1}^3 x_{iPS}^2 \delta_{jl} - x_{jPS} x_{lPS} \right) dm}_{J_{PS}} + \underbrace{\int \left(\sum_{i=1}^3 x_{iSK}^2 \delta_{jl} - x_{jSK} x_{lSK} \right) dm}_{J_{S_{jl}}} \\ &\quad + 2\delta_{jl} \sum_{i=1}^3 x_{iPS} \underbrace{\int x_{iSK} dm}_{=0} - x_{lPS} \underbrace{\int x_{jSK} dm}_{=0} - x_{jPS} \underbrace{\int x_{lSK} dm}_{=0} \end{aligned}$$

The last three integrals on the left side of this equation disappear because for integration in the center of mass system, i.e., the origin of the coordinate system lies in the body's center of mass, of course it applies again

$$\int \mathbf{r}_{SK} dm \equiv 0$$

Thus, the components of the tensor of inertia for an axis of rotation through a point P , which runs at a distance of \mathbf{r}_{PS} parallel to the axis of rotation through the center of mass S , are then given by

$$J_{P_{jl}} = J_{PS} + J_{S_{jl}} \quad (\text{Steiner theorem}) \quad (6.11)$$

In this equation the

$$J_{S_{jl}} = M \cdot \left(\sum_{i=1}^3 x_{iSK}^2 \delta_{jl} - x_{jSK} x_{lSK} \right)$$

are the components of the tensor of inertia for the parallel axis of rotation running through the center of mass and

$$J_{PS} = M \cdot \left(\sum_{i=1}^3 x_{iPS}^2 \delta_{jl} - x_{jPS} x_{lPS} \right)$$

tells us how much larger the components $J_{P_{jl}}$ of the tensor of inertia are for the rotation axis running through the point P as compared to the parallel axis of rotation running through the center of mass.

In summary, one obtains for the components of $\underline{\underline{J}}$ with respect to the new axis of rotation the relationships known under the label Huygens-Steiner theorem, named after Christiaan Huygens and Jakob Steiner (1796 - 1863):

$$\begin{aligned} J_{P_{11}} &= M(y_{PS}^2 + z_{PS}^2) + J_{11} \quad , \quad J_{P_{12}/P_{21}} = M \cdot x_{PS} y_{PS} + J_{12/21} \\ J_{P_{22}} &= M(x_{PS}^2 + z_{PS}^2) + J_{22} \quad , \quad J_{P_{13}/P_{31}} = M \cdot x_{PS} z_{PS} + J_{13/31} \\ J_{P_{33}} &= M(x_{PS}^2 + y_{PS}^2) + J_{22} \quad , \quad J_{P_{23}/P_{32}} = M \cdot y_{PS} z_{PS} + J_{23/32} \end{aligned} \quad (6.12)$$

In particular, the tensor of inertia $\underline{\underline{J}}$ for a rotation axis that does not run through the center of mass is always larger than for a rotation axis that runs parallel to it through the body's center of mass. This is a direct consequence of eq. (6.11) since $J_{S_{ij}}$ is always positive.

Huygens-Steiner theorem

The mass moments of inertia of a rigid body for rotational axes running through its center of mass are always smaller than those for parallel rotational axes that do not run through the center of mass.

The Huygens-Steiner theorem states that the tensor of inertia $\underline{\underline{J}}$ of a rigid body will change when the body's reference point is displaced from the body's center of mass. However, the angular velocity remains unchanged since it describes the rotation of the rigid body with respect to the coordinate system and the latter is independent of the choice of the body reference point.

6.2 Euler's Equations

With eq. (6.2) the kinetic energy of a rigid body is given by

$$T = \frac{1}{2} \int [\mathbf{v}_P + (\boldsymbol{\omega} \times \mathbf{r}_{PK})]^2 dm \quad \text{with} \quad dm = \rho \cdot dx_{SK} dy_{SK} dz_{SK}$$

If one calculates from the center of mass, that is $P = S$, then

$$\mathbf{r}_{PK} \rightarrow \mathbf{r}_{SK} \quad \text{and with} \quad \int \mathbf{r}_{SK} dm = 0$$

one obtains for the kinetic energy

$$T = \frac{1}{2} \left[M \cdot \mathbf{v}_S^2 + 2 \cdot \underbrace{\int \mathbf{v}_S \cdot (\boldsymbol{\omega} \times \mathbf{r}_{SK}) dm}_{= 0} + \int (\underbrace{\boldsymbol{\omega} \times \mathbf{r}_{SK}}_{\mathbf{a}}) (\boldsymbol{\omega} \times \mathbf{r}_{SK}) dm \right]$$

The scalar triple product $\mathbf{a} \cdot (\boldsymbol{\omega} \times \mathbf{r}_{SK})$ under the integral in the last term can be rewritten as

$$\mathbf{a} \cdot (\boldsymbol{\omega} \times \mathbf{r}_{SK}) = \mathbf{r}_{SK} \cdot (\mathbf{a} \times \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot (\mathbf{r}_{SK} \times \mathbf{a})$$

With this and with the help of eq. (6.3) and eq. (6.5) one gets for the kinetic energy of a rigid body:

$$T = \frac{1}{2}M\mathbf{v}_S^2 + \frac{1}{2}\boldsymbol{\omega} \mathbf{L}_{rot} = \frac{1}{2}M\mathbf{v}_S^2 + \frac{1}{2}\boldsymbol{\omega} \underline{\underline{\mathbf{J}}} \boldsymbol{\omega} \quad (6.13)$$

The total kinetic energy of a rigid body is the sum of two terms, a translation term T_T and a rotation term T_R . If a rigid body rotates around an axis D whose direction is given by the unit vector $\hat{\mathbf{n}}_D$, then $\boldsymbol{\omega} = \omega \hat{\mathbf{n}}_D$ and therefore one obtains for the rotation term T_R in eq. (6.13)

$$T_R = \frac{1}{2}\hat{\mathbf{n}}_D \underline{\underline{\mathbf{J}}} \hat{\mathbf{n}}_D \cdot \omega^2 = \frac{1}{2}J_D \cdot \omega^2$$

where J_D is the moment of inertia for rotation around the axis of rotation D. A comparison with T_T shows that the tensor of inertia plays the same role for T_R as the inertial mass M does for T_T .

A symmetric tensor ($A_{ij} = A_{ji}$) like the tensor of inertia $\underline{\underline{\mathbf{J}}}$ has three real eigenvalues, the so-called principal moments of inertia of a body, with mutually perpendicular eigenvectors, the so-called principal axes of inertia of the body.

Rotating cuboid

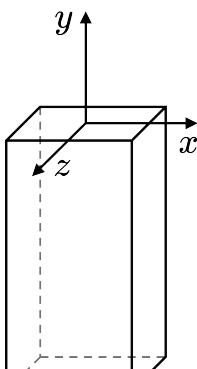


Fig. 6.4

In relation to a body-fixed principal axis system, such as for the cuboid shown in fig. 6.4, the off-diagonal elements of $\underline{\underline{\mathbf{J}}}$ vanish. The diagonal elements, i.e., the mass moments of inertia of the cuboid, then correspond to its principal moments of inertia. With

$$J_{xx} = A \quad , \quad J_{yy} = B \quad , \quad J_{zz} = C$$

$\boldsymbol{\omega} \underline{\underline{\mathbf{J}}} \boldsymbol{\omega}$ in eq. (6.13) becomes

$$\boldsymbol{\omega} \underline{\underline{\mathbf{J}}} \boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = A(\omega_x^2 + \omega_z^2) + B\omega_y^2$$

In the case $\mathbf{v}_S = 0$ the total kinetic energy of the cuboid is rotational energy, i.e.,

$$T = T_R = \frac{1}{2}A[(\omega_x^2 + \omega_z^2) + B\omega_y^2]$$

In the following, one considers a spatially fixed reference system x, y, z with origin O and a second reference system x', y', z' with origin O' moving relative to the first one. The position vectors of any point in the respective coordinate systems are linked by

$$\mathbf{r} = \mathbf{r}_{O'} + \mathbf{r}'$$

The motion of the O' -system relative to the O -system can be a translation or a rotation, or both together.

Now let \mathbf{A} be a vector quantity in the space-fixed coordinate system with origin O

$$\mathbf{A} = \sum_{i=1}^3 A_i \hat{\mathbf{e}}_i$$

The time derivative of \mathbf{A} is

$$\frac{d\mathbf{A}}{dt} = \sum_{i=1}^3 \frac{dA_i}{dt} \hat{\mathbf{e}}_i + \sum_{i=1}^3 A_i \frac{d\hat{\mathbf{e}}_i}{dt} \quad (6.14)$$

With $\frac{d\hat{\mathbf{e}}_i}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}_i$ this becomes

$$\frac{d\mathbf{A}}{dt} = \underbrace{\sum_{i=1}^3 \dot{A}_i \hat{\mathbf{e}}_i}_{\frac{d'\mathbf{A}}{dt}} + \boldsymbol{\omega} \times \mathbf{A} \quad (6.15)$$

Here $d'\mathbf{A}/dt$ is the rate of change of the vector \mathbf{A} as observed from the reference system in motion, i.e.,

$$\frac{d'\mathbf{A}}{dt} = \begin{pmatrix} \dot{A}_{x'} \\ \dot{A}_{y'} \\ \dot{A}_{z'} \end{pmatrix}$$

That this is actually the case becomes understandable when, as shown in fig. 6.5, the motion of the O' -system relative to the O -system is limited to a rotation and at the same time $O' = O$. If then e.g., \mathbf{A} is constant, i.e., $d'\mathbf{A}/dt = 0$, it follows that \mathbf{A} will move in the O system on a circle whose plane is perpendicular to $\boldsymbol{\omega}$, the vector of the rotational speed of the O' system. In that case \mathbf{A} moves in the O system with the velocity $\boldsymbol{\omega} \times \mathbf{A}$.

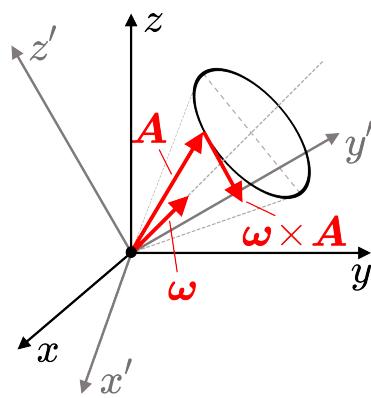


Fig. 6.5

If one now transfers this consideration to the angular momentum vector \mathbf{L} with center of mass velocity $\mathbf{v}_S = 0$, then the situation is the same as in fig. 6.5. That means

$$\frac{d\mathbf{L}}{dt} = \frac{d'\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} \quad (6.16)$$

With eq. (6.5) one thereby obtains the vector form of the so-called Euler equations, named after Leonhard Euler (1707 - 1783), sometimes also referred to as Euler's gyroscope equations, for the torque $\mathbf{M} = d\mathbf{L}/dt$:

$$\mathbf{M} = \underline{\underline{\mathbf{J}}} \cdot \frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\underline{\underline{\mathbf{J}}} \boldsymbol{\omega}) \quad (6.17)$$

For example, in coordinate form the Euler equations for the cuboid from fig. 6.4 are given by

$$\begin{aligned} M_x &= A\dot{\omega}_x - (B - A)\omega_y\omega_z \\ M_y &= B\dot{\omega}_y \\ M_z &= A\dot{\omega}_z - (A - B)\omega_x\omega_y \end{aligned} \quad (6.18)$$

Tab. 6.1: Important relationships in rigid body dynamics (S = center of mass).

Angular momentum:	$\mathbf{L} = \mathbf{L}_S + \mathbf{L}_{rot} = m \cdot \mathbf{r}_S \times \mathbf{v}_S + \underline{\underline{\mathbf{J}}} \cdot \boldsymbol{\omega}$
Tensor of inertia:	$J_{ij} = \int (\mathbf{r}^2 \delta_{ij} - x_i x_j) dm$
Huygens-Steiner theorem:	$J_{P_{ij}} = m(\mathbf{r}_{PS}^2 \delta_{ij} - x_{PSi} x_{PSj}) + J_{S_{ij}}$
Kinetic energy:	$T = \frac{1}{2} m \mathbf{v}_S^2 + \frac{1}{2} \boldsymbol{\omega} \underline{\underline{\mathbf{J}}} \boldsymbol{\omega}$
Rate of change of vector quantities:	$\frac{d\mathbf{A}}{dt} = \frac{d'A}{dt} + \boldsymbol{\omega} \times \mathbf{A}, \quad \frac{d\boldsymbol{\omega}}{dt} = \frac{d'\boldsymbol{\omega}}{dt}$
Rotating part of \mathbf{L} :	$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d'L}{dt} + \boldsymbol{\omega} \times \mathbf{L} \\ &= \underline{\underline{\mathbf{J}}} \frac{d'\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\underline{\underline{\mathbf{J}}} \cdot \boldsymbol{\omega}) \\ &= \underline{\underline{\mathbf{J}}} \frac{d\boldsymbol{\omega}}{dt} + \frac{d\underline{\underline{\mathbf{J}}}}{dt} \cdot \boldsymbol{\omega} = \mathbf{M} \end{aligned}$

6.2.1 Euler Angles

Leonhard Euler also introduced the three angles named after him, which are frequently used to describe the orientation of a rigid body with respect to a fixed coordinate system. Fig. 6.6 shows the rotation of a body-fixed coordinate system O' with the axes x'_1 , x'_2 and x'_3 with respect to a space-fixed coordinate system O with the axes x_1 , x_2 and x_3 . The line of intersection of the x_1x_2 -plane with the $x'_1x'_2$ -plane is the so-called line of nodes, in fig. 6.6 the straight line K . The Euler angles α , β and γ in fig. 6.6 are defined as follows:

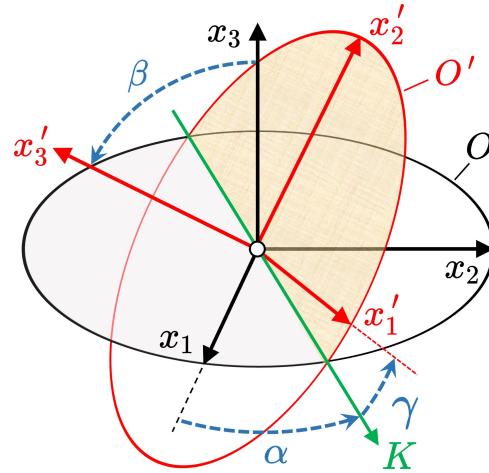


Fig. 6.6

α : The angle between the x_1 -axis and the line of nodes K .

β : The angle between the x_3 - and the x'_3 -axis.

γ : The angle between the line of nodes K and the x'_1 -axis.

The importance of the Euler angles derives from the fact that with their help a spatially fixed coordinate system O can be transformed into a body-fixed coordinate system O' by carrying out the three successive rotations \mathbf{R}^α , \mathbf{R}^β and \mathbf{R}^γ in fixed (none-interchangeable) order.

- In the first step, the system O is rotated by the angle α around the x_3 -axis thereby aligning the x_1 -axis to be coincident with the line of nodes K . The rotation matrix for this operation is

$$\mathbf{R}^\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- In the second step, this rotated system O is then rotated again by the angle β around the line of nodes K resulting in the x_3 -axis becoming coincident with the x'_3 -axis. The rotation matrix for this operation is

$$\mathbf{R}^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix}$$

- In the third step, this new O system is then rotated by the angle γ around the x'_3 -axis, thus making the x_1 -axis, which through step one is already coincident with the line of nodes, coincident with the x'_1 -axis. The rotation matrix for this operation is

$$\mathbf{R}^\gamma = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

After the execution of the third step, i.e., after the operation

$$\mathbf{R}^{\alpha\beta\gamma} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or respectively with the abbreviations $C_\alpha = \cos \alpha$, $S_\alpha = \sin \alpha$, $C_\beta = \cos \beta$, $S_\beta = \sin \beta$, $C_\gamma = \cos \gamma$ and $S_\gamma = \sin \gamma$

$$\mathbf{R}^{\alpha\beta\gamma} = \begin{pmatrix} C_\alpha C_\gamma - S_\alpha C_\beta S_\gamma & -C_\alpha S_\gamma - S_\alpha C_\beta C_\gamma & S_\alpha S_\beta \\ S_\alpha C_\gamma + C_\alpha C_\beta S_\gamma & -S_\alpha S_\gamma + C_\alpha C_\beta C_\gamma & -C_\alpha S_\beta \\ S_\beta S_\gamma & S_\beta C_\gamma & C_\beta \end{pmatrix} \quad (6.19)$$

O is now congruent with O' . The operation (notation from now on: $\mathbf{R} = \mathbf{R}^{\alpha\beta\gamma}$)

$$\hat{\mathbf{e}}'_i = \mathbf{R}\hat{\mathbf{e}}_i \quad i = 1, 2, 3 \quad (6.20)$$

converts the base vectors $\hat{\mathbf{e}}_i$ of the space-fixed coordinate system O into the base vectors $\hat{\mathbf{e}}'_i$ of the body-fixed coordinate system O' . For the components R_{ij} of the rotation matrix \mathbf{R} applies

$$R_{ij} = \hat{\mathbf{e}}_i^T \mathbf{R} \hat{\mathbf{e}}_j \quad (6.21)$$

which can be rewritten with eq. (6.20) as

$$R_{ij} = \hat{\mathbf{e}}_i^T \hat{\mathbf{e}}'_j \quad (6.22)$$

The components R_{ij} are each respectively the cosine of the angle between the x_i -axis in the spatially fixed coordinate system O and the x'_j -axis in the body-fixed coordinate system O' . For the component representation of a vector \mathbf{b} in the coordinate systems O and O' applies

$$\mathbf{b} = b_1 \hat{\mathbf{e}}_1 + b_2 \hat{\mathbf{e}}_2 + b_3 \hat{\mathbf{e}}_3 = b'_1 \hat{\mathbf{e}}'_1 + b'_2 \hat{\mathbf{e}}'_2 + b'_3 \hat{\mathbf{e}}'_3$$

For the components b'_j in the body-fixed coordinate system O' one has with eq. (6.22)

$$b'_j = \sum_{i=1}^3 b_i \hat{e}_i \hat{e}'_j = \sum_{i=1}^3 b_i R_{ij}$$

In matrix notation

$$\begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{that means} \quad \mathbf{b}' = \mathbf{R}^T \mathbf{b} \quad (6.23)$$

where \mathbf{R}^T is the transposed matrix of \mathbf{R} . Eq. (6.20) shows that \mathbf{R} is the rotation matrix of the unit vectors from O to O' and eq. (6.23) shows that \mathbf{R}^T is the rotation matrix of the vector components from O to O' .

$$\mathbf{R}^T = \begin{pmatrix} C_\alpha C_\gamma - S_\alpha C_\beta S_\gamma & S_\alpha C_\gamma + C_\alpha C_\beta S_\gamma & S_\beta S_\gamma \\ -C_\alpha S_\gamma - S_\alpha C_\beta C_\gamma & -S_\alpha S_\gamma + C_\alpha C_\beta C_\gamma & S_\beta C_\gamma \\ S_\alpha S_\beta & -C_\alpha S_\beta & C_\beta \end{pmatrix}$$

As can be easily checked, $\det \mathbf{R} = 0$. Therefore, an inverse matrix \mathbf{R}^{-1} exists. Since the column vectors of \mathbf{R} are pairwise orthogonal, $\mathbf{R}^{-1} \equiv \mathbf{R}^T$ holds true. This means that by multiplying eq. (6.23) with \mathbf{R} it follows that

$$\mathbf{R} \mathbf{b}' = \mathbf{R} \mathbf{R}^T \mathbf{b} = \mathbf{R} \mathbf{R}^{-1} \mathbf{b} \quad \text{and hence} \quad \mathbf{b} = \mathbf{R} \mathbf{b}'$$

In order to be able to calculate the position of a body from its rotational velocity, one must know the relationship between the rotational velocities ω' in the body-fixed coordinate system O' and the rates of change of the Euler angles $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$. In what follows, this relationship will be established by determining the respective axes of rotation of the three individual rotations in the coordinate system O' : around the x_3 -axis, the line of nodes K , and the x'_3 -axis. One then obtains the rotation vector components for ω' by multiplying the corresponding unit vectors \hat{e}_3 , \hat{e}_K and \hat{e}'_3 with the associated angular velocities $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$.

With the unit vectors \hat{e}'_1 , \hat{e}'_2 and \hat{e}'_3 in O' , \hat{e}_1 , \hat{e}_2 and \hat{e}_3 in O , as well as the unit vector \hat{e}_K in the direction of the line of nodes, the Euler angles are given by (directional cosines)

$$\cos \alpha = \hat{e}_1 \hat{e}_K \quad ; \quad \cos \beta = \hat{e}_3 \hat{e}'_3 \quad ; \quad \cos \gamma = \hat{e}'_1 \hat{e}_K \quad (6.24)$$

For rotations around the x_3 -axis (first step) in the coordinate system O' , one must express \hat{e}_3 as a function of \hat{e}'_1 , \hat{e}'_2 and \hat{e}'_3 . From fig. 6.6 it is immediately apparent that the

projection of $\hat{\mathbf{e}}_3$ onto the x'_3 -axis, i.e., the $\hat{\mathbf{e}}'_3$ component of $\hat{\mathbf{e}}_3$, is equal to $\cos \beta$. However, one can also get this result simply from eq. (6.24) by multiplying the expression for $\cos \beta$ with $\hat{\mathbf{e}}'_3$. The $\hat{\mathbf{e}}'_1$ and $\hat{\mathbf{e}}'_2$ components of $\hat{\mathbf{e}}_3$ follow from the projection of $\hat{\mathbf{e}}_3$ onto the $x'_1x'_2$ -plane. The absolute value of this projection is $\cos(90^\circ - \beta)$, i.e., $\sin \beta$. If one were to rotate the x_1 -axis around the x_3 -axis until the x_1 -axis becomes coincident with the line of nodes K , then one would find that the projection of the unit vector $\hat{\mathbf{e}}_3$ onto the $x'_1x'_2$ -plane lies in the x_2x_3 -plane, i.e., this projected vector and x'_2 enclose the angle γ . Thus, the components of $\hat{\mathbf{e}}_3$ in O' are

$$\hat{\mathbf{e}}_3 = \sin \beta \sin \gamma \cdot \hat{\mathbf{e}}'_1 + \sin \beta \cos \gamma \cdot \hat{\mathbf{e}}'_2 + \cos \beta \cdot \hat{\mathbf{e}}'_3 \quad (6.25)$$

For rotations around the line of nodes K (second step) in the coordinate system O' , $\hat{\mathbf{e}}_K$ must be expressed as a function of $\hat{\mathbf{e}}'_1$, $\hat{\mathbf{e}}'_2$ and $\hat{\mathbf{e}}'_3$. The unit vector $\hat{\mathbf{e}}_K$ in the direction of the line of nodes can be expressed in the coordinates of O as well as the coordinates of O' . As can easily be seen from fig. 6.6

$$\hat{\mathbf{e}}_K = \cos \alpha \cdot \hat{\mathbf{e}}_1 - \sin \alpha \cdot \hat{\mathbf{e}}_2 = \cos \gamma \cdot \hat{\mathbf{e}}'_1 - \sin \gamma \cdot \hat{\mathbf{e}}'_2 \quad (6.26)$$

The axis relevant for the rotation about $\hat{\mathbf{e}}'_3$ in O' (third step) is of course $\hat{\mathbf{e}}'_3$ itself. With that and with the axes of rotation $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_K$ as determined for O' through eq. (6.25) and eq. (6.26) one now obtains for the relationship between the rotational velocities ω' in the body-fixed coordinate system O' and the rate of change of the Euler angles $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$

$$\begin{aligned} \omega' &= \dot{\alpha} \cdot \hat{\mathbf{e}}_3 + \dot{\beta} \cdot \hat{\mathbf{e}}_K + \dot{\gamma} \cdot \hat{\mathbf{e}}'_3 \\ &= \dot{\alpha} [\sin \beta \sin \gamma \cdot \hat{\mathbf{e}}'_1 + \sin \beta \cos \gamma \cdot \hat{\mathbf{e}}'_2 + \cos \beta \cdot \hat{\mathbf{e}}'_3] + \dot{\beta} [\cos \gamma \cdot \hat{\mathbf{e}}'_1 - \sin \gamma \cdot \hat{\mathbf{e}}'_2] + \dot{\gamma} \cdot \hat{\mathbf{e}}'_3 \\ &= [\dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma] \cdot \hat{\mathbf{e}}'_1 + [\dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma] \cdot \hat{\mathbf{e}}'_2 + [\dot{\alpha} \cos \beta + \dot{\gamma}] \cdot \hat{\mathbf{e}}'_3 \end{aligned}$$

or respectively in matrix notation

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} = \begin{pmatrix} \sin \beta \sin \gamma & \cos \gamma & 0 \\ \sin \beta \cos \gamma & -\sin \gamma & 0 \\ \cos \beta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} \quad (6.27)$$

These equations are frequently referred to as kinematic Euler equations. By integrating eq. (6.27) one can in principle calculate the position of a body from its known angular velocities.

6.3 Gyroscope, Nutation, Precession

The unbalanced wheel

As sketched in fig. 6.7, due to an imbalance of the wheel, the wheel axis (axis 3) is no longer parallel to the axis of rotation ω but tilted with respect to it by an angle α . The axis of the running direction of the wheel (axis 2) no longer points straight ahead but sideways. The third coordinate axis (axis 1) points out of the image plane. The mass moments of inertia of the wheel in the directions of axis 1 and 2 are identical and denoted here by A ; the mass moment of inertia in the direction of the wheel axis, axis 3 shall be C . Thus, the tensor of inertia in the principal axis system is:

$$\underline{\underline{J}}^{(d)} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}$$

According to eq. (6.17), one thus obtains for the components of the torque

$$M_1 = A \frac{d\omega_1}{dt} + \omega_2 \omega_3 (C - A)$$

$$M_2 = A \frac{d\omega_2}{dt} + \omega_1 \omega_3 (A - C)$$

$$M_3 = A \frac{d\omega_3}{dt} + \omega_1 \omega_2 (A - A)$$

From fig. 6.7 one reads:

$$\omega_1 = 0 \quad ; \quad \omega_2 = \omega \sin \alpha \quad ; \quad \omega_3 = \omega \cos \alpha$$

That inserted into the components of the torque one gets:

$$M_1 = (C - A)\omega^2 \sin \alpha \cos \alpha = \frac{(C - A)}{2} \sin 2\alpha$$

$$M_2 = A\dot{\omega} \sin \alpha$$

$$M_3 = A\dot{\omega} \cos \alpha$$

For uniform rotation $\dot{\omega} = 0$, which means that only the M_1 component of the torque remains. The torque exerted by the rotating wheel via the wheel axle on the wheel bearing, the so-called gyroscopic moment \mathbf{M}^K , points in the opposite direction of the torque M_1 , hence

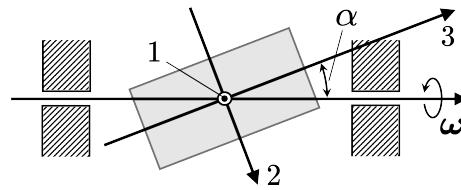


Fig. 6.7

$$M_1^K = -M_1 = \frac{(A - C)}{2} \sin 2\alpha$$

If $A > C$ then M_1^K increases the imbalance of the wheel, i.e., the angle α increases. Conversely, in the case $A < C$, as should be the case with tires, M_1^K counteracts the imbalance and works to reduce the angle α .

Torque-free motion

One speaks of a torque-free motion when the resulting total moment, i.e., the sum of all moments caused by all external forces acting on a rigid body, disappears. To examine torque-free motion one considers the ellipsoid of inertia (fig. 6.8) in the principal axis system. Of the kinetic energy of the rigid body (eq. (6.13)), only the kinetic energy due to its own rotation remains in the case of torque-free motion and without losses (e.g., due to friction), it remains constant. Hence:

$$T(\boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega}(\underline{\underline{J}} \cdot \boldsymbol{\omega}) = \frac{1}{2} \sum_{i,j}^3 \omega_i J_{ij} \omega_j = C \quad (6.28)$$

For the moments of inertia of the ellipsoid in fig. 6.8 applies

$$J_1 > J_2 > J_3$$

That inserted gives for the kinetic energy due to self-rotation

$$T(\boldsymbol{\omega}) = \frac{1}{2}(\omega_1^2 J_1 + \omega_2^2 J_2 + \omega_3^2 J_3) \quad (6.29)$$

With $L_i = J_i \omega_i$, $i = 1, 2, 3$ it follows

$$T(\boldsymbol{\omega}) = \frac{1}{2} \left(\frac{L_1^2}{J_1} + \frac{L_2^2}{J_2} + \frac{L_3^2}{J_3} \right) = \text{const} \quad (6.30)$$

that means for torque-free motion it must hold that

$$L^2 = L_1^2 + L_2^2 + L_3^2 = \text{const} \quad (6.31)$$

With $|\underline{\underline{L}}| = \text{const}$ it follows

$$\frac{dL}{dt} = 0 \quad (6.32)$$

The square of the angular momentum, i.e., its magnitude, is a conserved quantity in the body-fixed system.

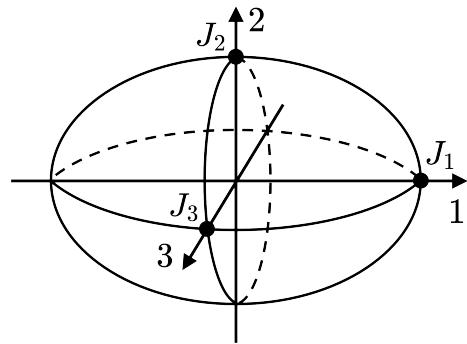


Fig. 6.8

In the following, the case of two-axis symmetry will be considered. In this specific case, it holds for the principal moments of inertia $A = J_{11}$ along the ξ_1 -axis, $B = J_{22}$ along the ξ_2 -axis and $C = J_{33}$ along the ξ_3 -axis that:

$$A = B \neq C$$

As sketched in fig. 6.9, the figure axis is identical with the coordinate axis ξ_3 . ω_F denotes the rotation around the figure axis and ω_N the rotation around the axis of nutation. Between the vectors of the angular velocities $\boldsymbol{\omega}$, ω_F and ω_N and the components of the angular velocity along the coordinate axes ξ_1 and ξ_3 the relationship

$$\boldsymbol{\omega} = \omega_N + \omega_F = \omega_1 \hat{\xi}_1 + \omega_3 \hat{\xi}_3$$

holds. As fig. 6.9 shows, for the angle Θ applies

$$\left. \begin{array}{l} \omega_1 = \omega \sin \Theta \\ \omega_3 = \omega \cos \Theta \end{array} \right\} \quad \tan \Theta = \frac{\omega_1}{\omega_3} \quad (6.33)$$

The following relationships for the angle Ψ can be read from fig. 6.9

$$\tan \Psi = \frac{L_1}{L_3} = \frac{A\omega_1}{C\omega_3} = \frac{\omega_1}{\omega_3 - \omega_F} \quad (6.34)$$

and therefore

$$\omega_F = \frac{A - C}{A} \cdot \omega_3 \quad (6.35)$$

For example, the following applies to the Earth: $\frac{A - C}{A} \approx \frac{1}{305}$

From the Euler equations eq. (6.18) for torque-free motion follows:

$$A\dot{\omega}_1 + \omega_2\omega_3(C - A) = M_1 = 0$$

$$A\dot{\omega}_2 + \omega_1\omega_3(A - C) = M_2 = 0$$

$$C\dot{\omega}_3 + \omega_1\omega_2(A - A) = M_3 = 0$$

From the third equation follows immediately

$$\omega_3 = \omega_3^0 = \text{const} \quad (6.36)$$

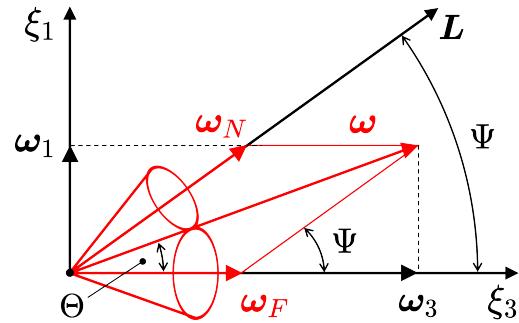


Fig. 6.9

This means that the projection of ω onto the figure axis is a time invariant. If one now makes use of the relationship from eq. (6.35) in the first two of Euler's equations, one obtains

$$\dot{\omega}_1 = \omega_2 \omega_F \quad \text{and} \quad \dot{\omega}_2 = -\omega_1 \omega_F \quad (6.37)$$

Taking the time derivatives in eq. (6.37) and inserting the respective original equations yields two differential equations, one for ω_1 and one for ω_2

$$\ddot{\omega}_1 + \omega_F^2 \omega_1 = 0 \quad \text{and} \quad \ddot{\omega}_2 + \omega_F^2 \omega_2 = 0 \quad (6.38)$$

These differential equations are equations of motion for harmonic oscillators (see section 10.1). The general solutions of eq. (6.37) are thus

$$\omega_1(t) = \alpha \sin(\omega_F t + \beta) \quad \text{and} \quad \omega_2(t) = \alpha \cos(\omega_F t + \beta) \quad (6.39)$$

It follows from eq. (6.36) that the absolute value of ω must be constant. That means it must hold that (see fig. 6.10)

$$|\omega| = \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\alpha^2 + \omega_3^0} = \text{const} \quad (6.40)$$

The projection of $\omega(t)$ onto the $\xi_1 \xi_2$ -plane, the plane perpendicular to the figure axis ξ_3 , therefore describes a circle in the $\xi_1 \xi_2$ -plane with radius α . The circular cone which ω describes is the so-called pole cone (fig. 6.10). The opening angle of this pole cone is given by eq. (6.33) and the angular velocity with which this rotation of ω occurs is ω_F from eq. (6.35) with $\omega_3 = \omega_3^0$. This circular movement of the axis of rotation of a rigid body around its figure axis is the so-called nutation motion. In the case of Earth, one obtains for the period of this nutation motion, referred to as Chandler's period after its discoverer, the value

$$T_F = \frac{2\pi}{\omega_F} = \frac{A}{A - C} \underbrace{\frac{2\pi}{\omega_3^0}}_{1 \text{ day}} \approx 305 \text{ days}$$

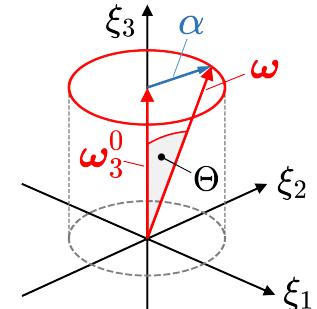


Fig. 6.10

However, in reality T_F is not ≈ 305 days long, but $T_F \approx 433$ days. The reason for this deviation lies in the earlier assumption that the Earth is a rigid body. But that is not the case, because the Earth is an elastic body and in parts it is liquid.

The coordinate system (ξ_1, ξ_2, ξ_3) in fig. 6.9 is the Earth body's principal axis system. With the help of eq. (6.27) one can determine the Euler angles, denoted here by ϕ , ϑ and ψ , with respect to the spatially fixed coordinate system (x_1, x_2, x_3) :

$$\boldsymbol{\omega} = \begin{pmatrix} \alpha \sin(\omega_F t + \beta) \\ \alpha \cos(\omega_F t + \beta) \\ \omega_3^0 \end{pmatrix} = \begin{pmatrix} \sin \vartheta \sin \psi & \cos \psi & 0 \\ \sin \vartheta \cos \psi & -\sin \psi & 0 \\ \cos \vartheta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\vartheta} \\ \dot{\psi} \end{pmatrix} \quad (6.41)$$

It is best to choose the space-fixed coordinate system (x_1, x_2, x_3) such that $\hat{\mathbf{x}}_3$ coincides with the direction of the angular momentum vector \mathbf{L} , i.e., $\hat{\mathbf{x}}_3 \parallel \mathbf{L}$. With eq. (6.25), the representation of \hat{x}_3 in the body's own coordinate system becomes

$$\hat{\mathbf{x}}_3 = \sin \vartheta \sin \psi \cdot \hat{\mathbf{\xi}}_1 + \sin \vartheta \cos \psi \cdot \hat{\mathbf{\xi}}_2 + \cos \vartheta \cdot \hat{\mathbf{\xi}}_3 \quad (6.42)$$

For torque-free motion, the magnitude of the angular momentum is a constant of motion (eq. (6.32)). For the components of \mathbf{L} in the principal axis system therefore applies:

$$\mathbf{L} = L \cdot (\sin \vartheta \sin \psi, \sin \vartheta \cos \psi, \cos \vartheta) \quad (6.43)$$

With eq. (6.41) and eq. (6.43) one obtains for $\mathbf{L} = \underline{\mathbf{J}} \cdot \boldsymbol{\omega}$ in the principal axis system the system of equations

$$\begin{aligned} L \sin \vartheta \sin \psi &= A \dot{\phi} \sin \vartheta \sin \psi + A \dot{\vartheta} \cos \psi \\ L \sin \vartheta \cos \psi &= A \dot{\phi} \sin \vartheta \cos \psi - A \dot{\vartheta} \sin \psi \\ L \cos \vartheta &= C \dot{\phi} \cos \vartheta + C \dot{\psi} \end{aligned} \quad (6.44)$$

By multiplying the first of these three equations by $\cos \psi$ and subtracting from it the second equation multiplied by $\sin \psi$, one can eliminate $\dot{\phi}$ and the angle ψ , thereby obtaining the relationship

$$0 = A \dot{\vartheta} (\cos^2 \psi + \sin^2 \psi) = A \dot{\vartheta}$$

From that it follows

$$\vartheta = \text{const} = \vartheta_0 \quad (6.45)$$

and for the components of $\boldsymbol{\omega}$ from eq. (6.41) one therefore gets

$$\begin{aligned} \omega_1 &= \alpha \sin(\omega_F t + \beta) = \dot{\phi} \sin \vartheta_0 \sin \psi \\ \omega_2 &= \alpha \cos(\omega_F t + \beta) = \dot{\phi} \sin \vartheta_0 \cos \psi \\ \omega_3 &= \omega_3^0 = \dot{\phi} \cos \vartheta_0 + \dot{\psi} \end{aligned} \quad (6.46)$$

A possible approach to determine $\phi(t)$ and $\psi(t)$ follows from the consideration of

$$\omega_1 \cos \psi - \omega_2 \sin \psi = 0$$

Thus, according to eq. (6.46), it must also apply

$$\alpha \sin(\omega_F t + \beta) \cos \psi - \alpha \cos(\omega_F t + \beta) \sin \psi = \alpha \sin(\omega_F t + \beta - \psi) = 0$$

This condition is fulfilled for two values of $\psi(t)$

$$\psi(t) = \begin{cases} \omega_F t + \beta \\ \omega_F t + \beta + \pi \end{cases} \quad (6.47)$$

Inserting eq. (6.47) into the third equation of eq. (6.46) yields

$$\omega_3^0 = \dot{\phi} \cos \vartheta_0 + \dot{\psi} = \dot{\phi} \cos \vartheta_0 + \omega_F$$

With eq. (6.35) it follows for $\dot{\phi}$

$$\dot{\phi} = \frac{\omega_3^0 - \omega_F}{\cos \vartheta_0} = \frac{C}{A \cos \vartheta_0} \omega_3^0 = \text{const}$$

and hence

$$\phi(t) = \frac{C}{A \cos \vartheta_0} \cdot t + \phi_0 \quad (6.48)$$

The value of α can now be determined by setting $\omega_F t + \beta = 0$ for ω_2 in eq. (6.46). Then, according to eq. (6.47) $\psi = 0$ or $\psi = \pi$, i.e., $\cos \psi = \pm 1$, and one obtains for α the two solutions

$$\alpha = \pm \dot{\phi} \sin \vartheta_0 = \pm \frac{C}{A} \omega_3^0 \tan \vartheta_0 \quad (6.49)$$

Of course, given the geometric interpretation of α (see eq. (6.40)) only the positive sign makes sense. There is an alternative way to determine $\phi(t)$ and $\psi(t)$ by multiplying the first equation in eq. (6.44) by $\sin \psi$ and then adding the second equation in eq. (6.44) multiplied by $\cos \psi$ to it. Therewith, one can eliminate $\dot{\vartheta}$ and the angles ϑ and ψ and thus obtains

$$A \dot{\phi} = L$$

From that it follows

$$\phi(t) = \frac{L}{A} \cdot t + \phi_0 \quad (6.50)$$

One can obtain $\dot{\psi}$, for example, with the help of eq. (6.35) and eq. (6.45) by inserting eq. (6.50) into the third equation of eq. (6.44):

$$\dot{\psi} = \left(\frac{L}{C} - \frac{L}{A} \right) \cos \vartheta_0 = \frac{L}{C} \frac{A - C}{A} \cos \vartheta_0 = \frac{L}{C} \frac{\omega_F}{\omega_3^0} \cos \vartheta_0 \quad (6.51)$$

However, $\dot{\psi}$ can also be obtained from the equation for ω_3 in eq. (6.46)

$$\dot{\psi} = \omega_3^0 - \dot{\phi} \cos \vartheta_0 = \omega_3^0 - \frac{L}{A} \cos \vartheta_0 \quad (6.52)$$

A comparison of eq. (6.52) with eq. (6.51) shows that it must be true that

$$\omega_3^0 = \frac{L}{C} \cos \vartheta_0 \quad (6.53)$$

Inserting this into eq. (6.51) yields $\dot{\psi} = \omega_F$ and therefore

$$\psi(t) = \omega_F t + \psi_0 \quad (6.54)$$

Now one can also determine the constants ϑ_0 and ψ_0 . Inserting eq. (6.50) into the first two equations of eq. (6.46) yields

$$\begin{aligned} \alpha \sin(\omega_F t + \beta) &= \frac{L}{A} \sin \vartheta_0 \sin \psi \\ \alpha \cos(\omega_F t + \beta) &= \frac{L}{A} \sin \vartheta_0 \cos \psi \end{aligned}$$

Dividing these two equations and a comparison with eq. (6.54) gives

$$\tan \psi = \tan(\omega_F t + \beta) \quad \Rightarrow \quad \psi_0 = \beta \quad (6.55)$$

As expected, $\psi_0 = \beta$ corresponds to the result of eq. (6.47). If one inserts this solution into the first equation of eq. (6.46) one finds that

$$\alpha \sin(\omega_F t + \beta) = \alpha \sin \psi = \frac{L}{A} \sin \vartheta_0 \sin \psi$$

With that applies

$$\sin \vartheta_0 = \frac{A}{L} \alpha$$

If one divides this equation by eq. (6.53), one eventually gets

$$\tan \vartheta_0 = \frac{A}{C} \frac{\alpha}{\omega_3^0} \quad (6.56)$$

This corresponds to the result of eq. (6.49) for α . In summary, one has for the Euler angles and the associated angular velocities

$$\begin{aligned}
 \phi(t) &= \frac{C}{A \cos \vartheta_0} \cdot t + \phi_0 & ; & \dot{\phi} = \frac{C}{A \cos \vartheta_0} \\
 \vartheta(t) &= \vartheta_0 = \arctan\left(\frac{A}{C} \frac{\alpha}{\omega_3^0}\right) & ; & \dot{\vartheta} = 0 \\
 \psi(t) &= \frac{A - C}{A} \cdot \omega_3^0 t + \beta & ; & \dot{\psi} = \frac{A - C}{A} \cdot \omega_3^0
 \end{aligned} \tag{6.57}$$

In these relationships one still finds five of the original six variables ϕ_0 , ϑ_0 , ψ_0 , α , β and ω_3^0 present, but these are no longer all independently freely selectable. Through eq. (6.49) or eq. (6.56) the parameters α , ϑ_0 and ω_3^0 are connected. Because of eq. (6.40), however, α and ω_3^0 cannot be chosen independently of each other. With the choice of $\hat{x}_3 \parallel \mathbf{L}$ made here, ϑ_0 became fixed and therefore α and ω_3^0 are no longer freely selectable. According to eq. (6.56) $\psi_0 = \beta$ applies and therefore only one of these two variables can be chosen freely; in eq. (6.57) this is β . From the original six variables ϕ_0 , ϑ_0 , ψ_0 , α , β and ω_3^0 only ϕ_0 and β remain therefore as free variables.

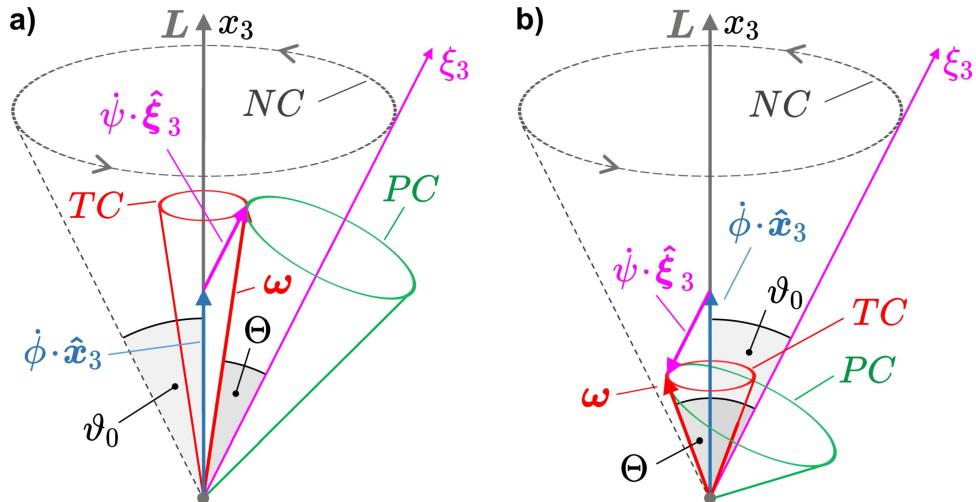


Fig. 6.11: Position of the angles, rotation vectors, nutation cone (NC), tracking cone (TK) and pole cone (PC) for the case $A > C$ (a) and the case $A < C$ (b).

In Fig. 6.11 the positions of the angles, the rotation vectors, the nutation cone, the tracking cone and the pole cone are sketched for two cases: In fig. 6.11a for $A > C$, such as is the case with the Earth; and in fig. 6.11b for the case $A < C$. With $\hat{x}_3 \parallel \mathbf{L}$, ϑ_0 is the angle between the figure axis ξ_3 and \mathbf{L} or respectively ξ_3 and x_3 (the angle ϑ_0 in fig. 6.11 is therefore identical to the angle Ψ in fig. 6.9). The figure axis rotates with the angular velocity $\dot{\phi}$ around the x_3 -axis and thereby traces out the nutation cone of the so-called free nutation. The associated rotation vector $\dot{\phi} \cdot \hat{x}_3$ is identical to ω_N in fig. 6.9. $\dot{\psi}$ is the angular velocity with which the rigid body under consideration (in fig. 6.11a the Earth)

rotates around the figure axis. The corresponding rotation vector $\dot{\psi} \cdot \hat{\xi}_3$ is either parallel or antiparallel to the ξ_3 -axis, depending on whether $A > C$ as in fig. 6.11a or $A < C$ as in fig. 6.11b. The rotation vector $\dot{\psi} \cdot \hat{\xi}_3$ is identical to ω_F in fig. 6.9. The total rotation vector ω in fig. 6.11 is thus

$$\omega = \dot{\phi} \cdot \hat{x}_3 + \dot{\psi} \cdot \hat{\xi}_3 = \omega_N + \omega_F$$

The rotation vector ω always lies in the $x_3\xi_3$ -plane and rotates with the figure axis around the direction of angular momentum \mathbf{L} or respectively around the x_3 -axis. In the process, ω traces out the so-called tracking cone around \mathbf{L} or respectively around the x_3 axis. The fixed angle between ω and the figure axis ξ_3 is the angle Θ from fig. 6.9. Θ is the opening angle of the so-called pole cone. While the figure axis performs its nutation motion with ω_N around \mathbf{L} or respectively the x_3 -axis, this pole cone rolls off on the tracking cone with ω_N . In the case $A > C$ (fig. 6.11a) the pole cone rolls off with its outer surface on the outer surface of the tracking cone. In the case $A < C$ (fig. 6.11b), however, the pole cone rolls off with its inner surface on the outer surface of the tracking cone.

Motion with non-vanishing torque

A simple example of the behavior of a rotating body when there is a non-vanishing torque is sketched in fig. 6.12. The figure shows a rotating wheel suspended on one side from a rope. The angular momentum vector \mathbf{L} of the rotating wheel lies in the wheel axis whose length is l_A . At one end, the wheel axle is held by the rope force, at the other end an equal but opposite weight force acts on the center of mass of the wheel.

The resulting torque \mathbf{M} is perpendicular to the plane spanned by the wheel axis and the weight force. If the wheel does not rotate, then it naturally will tilt down. However, if the wheel rotates, the torque \mathbf{M} perpendicular to \mathbf{L} causes a change in direction of the wheel axis, which is called precession. If the angular velocity ω with which the wheel rotates is large enough, then the wheel will rotate or respectively precess with the wheel axis in horizontal direction around the rope suspension. With the Euler equations in eq. (6.17) applies

$$\frac{d\mathbf{L}}{dt} = \underline{\underline{\mathbf{J}}} \frac{d\omega}{dt} + \omega \times (\underline{\underline{\mathbf{J}}} \cdot \omega) = \mathbf{M}$$

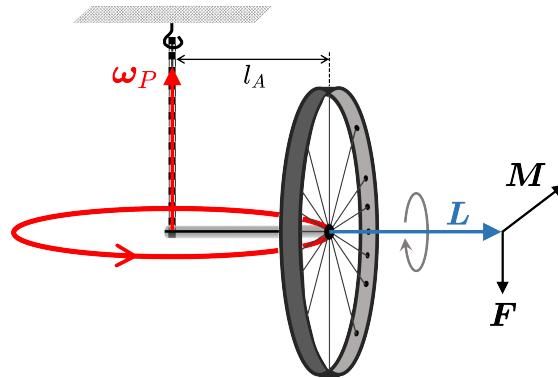


Fig. 6.12

As the direction of the rotation axis of the wheel changes, the rate of change of the angular momentum $d\mathbf{L}/dt$ remains perpendicular to the angular momentum vector \mathbf{L} and perpendicular to the angular momentum vector ω_P of precession. Therefore:

$$\frac{d\mathbf{L}}{dt} = \omega_P \times \mathbf{L} = \mathbf{M}$$

Without friction losses, the absolute value of \mathbf{L} will remain constant. With $|\mathbf{L}|$ and $|\mathbf{M}|$ each being constant, the magnitude of the angular velocity of the precession

$$|\omega_P| = \frac{|\mathbf{M}|}{|\mathbf{L}|}$$

is also constant. In this case one speaks of a so-called regular precession. As can easily be seen from fig. 6.13, the angular velocity of the precession will remain constant even when the wheel axis should move out of the horizontal position. For the torque applies

$$|\mathbf{M}| = l_A \cdot \cos \alpha \cdot |\mathbf{F}|$$

and for the rate of change of the angular momentum

$$\left| \frac{d\mathbf{L}}{dt} \right| = |\omega_P \times \mathbf{L}| = |\omega_P| |\mathbf{L}| \sin \beta = |\omega_P| |\mathbf{L}| \cos \alpha$$

The left-hand sides of the two equations are identical since $\mathbf{M} = d\mathbf{L}/dt$ and thus the corresponding right-hand sides of the two equations are also the same:

$$|\omega_P| |\mathbf{L}| \cos \alpha = l_A \cdot \cos \alpha \cdot |\mathbf{F}|$$

The $\cos \alpha$ term cancels out and with $\mathbf{L} = \underline{\mathbf{J}} \cdot \underline{\boldsymbol{\omega}}$ and $|\mathbf{F}| = m \cdot g$ one obtains

$$\omega_P = \frac{l_A \cdot m \cdot g}{I_A \cdot \omega}$$

where m is the mass of the wheel and I_A is the moment of inertia for rotation about the wheel axle. This expression is independent of α and given the length of the wheel axis (l_A), mass of the wheel (m) and mass distribution with respect to the wheel axis (I_A), the angular velocity of precession ω_P depends only on the rotational velocity of the wheel ω . However, the expression derived here for ω_P only applies if $\omega \gg \omega_P$.

In general, for a body which is subject to a non-zero moment of force (torque), the components of the torque in the body-fixed coordinate system will depend on the orientation of the body in the space-fixed coordinate system. This however means that one cannot any longer solve the Euler equations independently from the respective equations for the Euler angles ϕ , ϑ and ψ . To address this problem a different approach is required

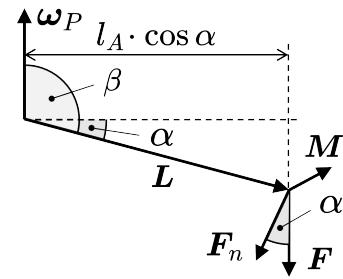


Fig. 6.13

such as the so-called Lagrange formalism which exceeds the scope to be discussed in an introductory course to mechanics such as this one.¹

Summarizing the above discussion, the gyroscopic motion is composed of the superposition of three different motions (compare fig. 6.11):

$\psi(t)$: Rotation of the gyroscope around the body-fixed figure axis with rotation vector $\dot{\psi}(t) \cdot \hat{\xi}_3$.

$\phi(t)$: Rotation of the figure axis around the space-fixed coordinate x_3 with rotation vector $\dot{\phi}(t) \cdot \hat{x}_3$. This motion is called precession.

$\vartheta(t)$: Periodic change of the angle $\vartheta(t)$ between the figure axis ξ_3 and the spatially fixed coordinate x_3 when there is a non-vanishing torque. This motion is then called nutation. In the case of torque-free motion, $\vartheta(t)$ does not vary but is constant, i.e., $\vartheta(t) = \vartheta_0$; in this case one speaks of free nutation.

Fig. 6.14 illustrates the precession and nutation motions using the Earth as an example. In spherical coordinates the direction of the figure axis in the spatially fixed coordinate system is

$$\hat{\xi}_3 = \sin \phi \sin \vartheta \hat{x}_1 - \cos \phi \sin \vartheta \hat{x}_2 + \cos \vartheta \hat{x}_3$$

If one now follows the motion of the figure axis on a spherical surface as a function of time, i.e., $\xi_3(t)$, then one can represent nutation and precession as depicted in fig. 6.14. In fig. 6.14, N_W is the range between which $\vartheta(t)$ oscillates. For Earth, it takes about 26 000 years, 25 772 to be exact, for one full cycle of its precession to complete. Largely, this precession of Earth's rotational axis is caused by the combined gravitational pull of Sun and Moon exerted on Earth's equatorial bulge; gravitational interactions with other planets matter as well but to a much lesser degree. As for Earth's nutation, it takes about 41 000 years for the tilt of our planet's axis to complete a cycle, i.e., for $\vartheta(t)$ to vary from its maximum value to its minimum value and back again to its maximum value. In the past, over such a 41 000-year cycle, $\vartheta(t)$ has varied between a minimum value of $\vartheta = 22.1^\circ$ and a maximum value $\vartheta = 24.5^\circ$.

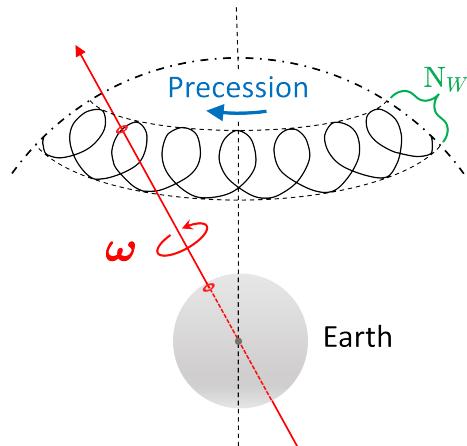


Fig. 6.14

¹There are many good textbooks discussing the Lagrange formalism in detail including "TECHNICAL MECHANICS - An Introduction", ISBN 978-1-951894-17-7, by the author of this book.

6.4 Moving Frames of Reference

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t)$$

$$\mathbf{r}'(t) = \sum_{i=1}^3 x'_i \hat{\mathbf{e}}'_i \quad \text{all time dependent}$$

$$\dot{\mathbf{r}}(t) = \dot{\mathbf{R}}(t) + \sum_{i=1}^3 \dot{x}'_i(t) \hat{\mathbf{e}}'_i(t) + (\boldsymbol{\omega} \times \mathbf{r}')$$

$$\ddot{\mathbf{r}}(t) = \ddot{\mathbf{R}}(t) + \sum_{i=1}^3 \ddot{x}'_i \hat{\mathbf{e}}'_i + 2 \sum_{i=1}^3 \dot{x}'_i \dot{\hat{\mathbf{e}}}'_i + (\dot{\boldsymbol{\omega}} \times \mathbf{r}') + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]$$

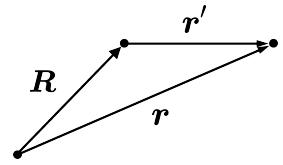


Fig. 6.15

If \mathbf{a} , \mathbf{A} , and \mathbf{a}' are the acceleration vectors in the corresponding coordinate directions, then

$$\ddot{\mathbf{r}}(t) = \mathbf{a} = \mathbf{A} + \underbrace{2(\boldsymbol{\omega} \times \mathbf{v}') + (\dot{\boldsymbol{\omega}} \times \mathbf{r}')}_{\substack{\text{Coriolis-} \\ \text{acceleration}}} + \underbrace{[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]}_{\substack{\text{Centrifugal} \\ \text{acceleration}}} \quad (6.58)$$

With $\mathbf{F} = m\mathbf{a}$ being the force acting on a mass m in a reference system at rest, the force $\mathbf{F}' = m\mathbf{a}'$ observed by someone in the moving system is given by

$$\mathbf{F}' = \mathbf{F} - \underbrace{m\{ \mathbf{A} + 2(\boldsymbol{\omega} \times \mathbf{v}') + (\dot{\boldsymbol{\omega}} \times \mathbf{r}') + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')] \}}_{\text{Inertial forces}} \quad (6.59)$$

The inertial forces in eq. (6.59) can only be made to disappear point by point because the gravitational force has a different \mathbf{r} -dependence. The Coriolis force and the centrifugal force are so-called apparent forces. The Coriolis force only acts on bodies moving in a rotating system, for a body at rest in a moving system, i.e., $\mathbf{v} = 0$, it does not exist. The centrifugal force exists of course for every body in a rotating system and disappears when the rotation stops, i.e., $\boldsymbol{\omega} = 0$. Hence, the Coriolis force and the centrifugal force are only experienced by bodies in a rotating system. In that respect they are similar to the inertial force which only bodies in an accelerated system experience. Apparent forces such as the Coriolis force, the centrifugal force or the inertial force depend on the reference frame. They do not exist for an observer outside the reference frame, that is for an observer who is not part of the respective rotating or linearly accelerated system being considered. Just consider a standing passenger in a bus moving at constant speed who is not holding on to a handrail. As long as there is no acceleration or deceleration

nothing undue will happen as the bus and the passenger both will continue to move with the same constant speed and hence no forces are at play. However, this changes when the bus suddenly accelerates or decelerates with the result that the passenger may either fly out through the back or the front window of the bus. When this happens, as far as the passenger is concerned, nothing has changed for she or he is still moving with the same constant velocity. Not so from the perspective of fellow travelers on the bus who see the passenger thrown out the back or the front window by an invisible force. Of course, something very similar will happen if the bus does not change speed at all but takes a sharp turn left or right; in that case the passenger will just fly out through a window on the left or the right side of the bus depending on the direction of the turn the bus takes. Here, an observer on the sidewalk will just see the passenger continuing in linear motion at constant speed as if no forces were acting on her or him.

Let's look at the Coriolis force and the centrifugal force associated with the rotation of a system O' with respect to O which is the system at rest. To do that we will pick a system that rotates at constant angular velocity ω , i.e., we are looking at a case of uniform circular motion of O' (see section 2.3.4). For a point mass at rest in the rotating system $\mathbf{v}' = 0$ and hence, the Coriolis force vanishes but the centrifugal force is still there. The total force \mathbf{F}' on a point mass that is at rest in the rotating system O' must be zero, hence for $\mathbf{v}' = 0$ eq. (6.59) becomes

$$0 = \mathbf{F}' = \mathbf{F} - m \cdot [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')] = \mathbf{F} - mr'_\perp \omega^2 \hat{\mathbf{r}}'_\perp$$

Here, we have used the fact that $\boldsymbol{\omega}$ points in the direction of the rotational axis running through O' and therefore only \mathbf{r}_\perp , the component of \mathbf{r} perpendicular to the axis of rotation contributes. Eq. (6.59) therefore reduces to

$$\mathbf{F} = mr'_\perp \omega^2 \hat{\mathbf{r}}'_\perp$$

This tells us that someone in O , the system at rest, sees a force of the magnitude $mr'_\perp \omega^2$ at work which is just the [centripetal force](#) we encountered earlier when we discussed uniform circular motion. From the perspective of an observer in O , it is this force which keeps the point mass in the rotating system O' at rest.

Considering again a system O' rotating at constant angular velocity $\boldsymbol{\omega}$ where a point mass now moves in a radial direction from O' outwards, i.e., its velocity \mathbf{v}' is parallel to its position vector \mathbf{r}' . The Coriolis force \mathbf{F}_C in \mathbf{F}' acting on this moving point mass, is perpendicular to $\boldsymbol{\omega}$ and \mathbf{v}' , i.e., it is perpendicular to \mathbf{r}' . For an outside observer this looks like the point mass is subject to a momentum

$$\begin{aligned}
 \mathbf{M} &= \mathbf{r}' \times \mathbf{F}_C = 2m\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{v}') \\
 &= 2m(\underbrace{\boldsymbol{\omega}(\mathbf{r}' \cdot \mathbf{v}')}_{\mathbf{r}' \cdot \mathbf{v}'} - \underbrace{\mathbf{v}'(\mathbf{r}' \cdot \boldsymbol{\omega})}_{0}) = 2mr'v'\boldsymbol{\omega}
 \end{aligned}$$

A moment of force, i.e., a torque is always the result of a change in angular momentum \mathbf{L} . The angular momentum associated with the rotation of the radially outward moving point mass as observed from O must therefore be given by

$$\mathbf{L} = mr'^2\boldsymbol{\omega}$$

so that its rate of change becomes

$$\frac{d\mathbf{L}}{dt} = 2mr'v'\boldsymbol{\omega} = \mathbf{M}$$

The weather map

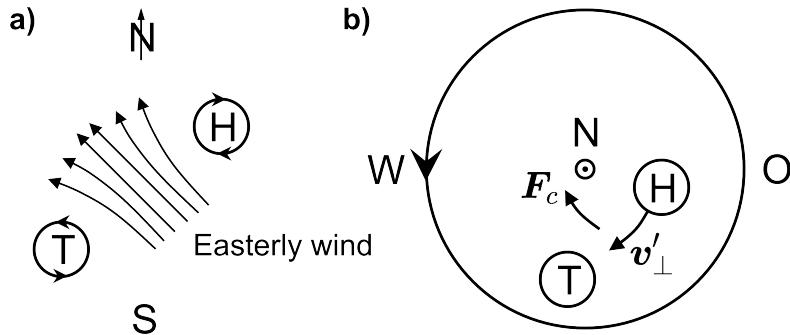


Fig. 6.16: (a) Easterly wind formation in the northern hemisphere. (b) Coriolis force \mathbf{F}_C in the northern hemisphere. The relative motion \mathbf{v}'_\perp occurs perpendicular to \mathbf{F}_C .

A high-pressure area in the atmosphere is characterized by the fact that air masses flow from higher air layers towards the Earth's surface and lead to an increase in pressure there. The air masses escaping from this high-pressure area above the Earth's surface are deflected by the Coriolis force to the right in the northern hemisphere and to the left in the southern hemisphere. Thus, high-pressure areas are surrounded by right-hand vortices in the northern hemisphere and left-hand vortices in the southern hemisphere. For low-pressure areas, exactly the opposite is the case. Low-pressure areas are characterized by the fact that near-surface air masses flow into an area with lower air pressure and rise there. So the velocity vector of these air masses points upwards while in high-pressure areas it points downwards. Therefore, low-pressure areas are surrounded by left-hand vortices in the northern hemisphere and by right-hand vortices in the southern hemisphere. Fig. 6.16a illustrates how an easterly wind can be created by adjacent high- and low-pressure areas in the northern hemisphere.

East deviation in free fall

At small angular velocities ω , terms proportional to ω^2 , i.e., the centrifugal force, can be neglected in eq. (6.59). Under this condition (in a first approximation), the equation of motion for free fall reads:

$$\ddot{\mathbf{r}}' = -g\hat{\mathbf{e}}_{z'} - 2(\boldsymbol{\omega} \times \mathbf{v}')$$

Fig. 6.17 shows the situation for free fall in the northern hemisphere at latitude φ . The y' -coordinate points out of the image plane in fig. 6.17.

For the rotation vector $\boldsymbol{\omega}$ one can see that:

$$\boldsymbol{\omega} = |\boldsymbol{\omega}| \cdot \begin{pmatrix} \cos \varphi \\ 0 \\ \sin \varphi \end{pmatrix}$$

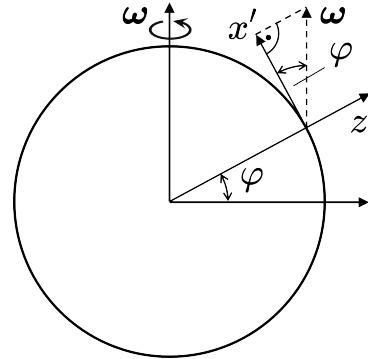


Fig. 6.17

The equations of motion are thus in a first approximation ($\boldsymbol{\omega}$ small):

$$\begin{pmatrix} \ddot{x}' \\ \ddot{y}' \\ \ddot{z}' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - 2 \cdot \begin{pmatrix} \omega \cos \varphi \\ 0 \\ \omega \sin \varphi \end{pmatrix} \times \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix}$$

or respectively

$$\left. \begin{aligned} \ddot{x}' &= 2\omega \sin \varphi \cdot \dot{y}' \\ \ddot{y}' &= -2\omega \sin \varphi \cdot \dot{x}' + 2\omega \cos \varphi \cdot \dot{z}' \\ \ddot{z}' &= -g + 2\omega \cos \varphi \cdot \dot{y}' \end{aligned} \right\} \quad (6.60)$$

In zeroth-order approximation $\omega \approx 0$, i.e., the Coriolis force falls away and thus

$$\dot{z}'_0 = -g \cdot t \quad ; \quad z'_0 = h - \frac{g}{2} \cdot t^2 \quad ; \quad \dot{x}'_0 = \dot{y}'_0 = 0$$

If one inserts the result from the zeroth-order approximation into eq. (6.60) - the first-order approximation - then only the equation for the y' -coordinate remains.

$$\ddot{y}' = -2\omega \cos \varphi \cdot g \cdot t$$

The y' -coordinate in fig. 6.17 runs from west to east along the constant latitude φ . A value for $y' \neq 0$ means that there will be a deviation along the direction of the latitude during free fall due to the Coriolis force. Integrating once yields

$$\dot{y}' = -\omega \cos \varphi \cdot g \cdot t^2$$

and integrating one more time gives for the y' -coordinate

$$y' = -\omega \cos \varphi \cdot g \cdot \frac{t^3}{3}$$

From the zeroth-order approximation one can determine the fall time from the initial height h to the Earth's surface $z'_0 = 0$. Inserting this gives for the deviation in y' -direction during free-fall in the northern hemisphere

$$y' = -\frac{2\sqrt{2}}{3} \cdot h \cdot \sqrt{\frac{h}{g}} \cdot \omega \cos \varphi$$

The minus sign means that the direction of the deviation points into the image plane of fig. 6.17, i.e., in the direction of the negative y' -axis. Hence there is an easterly deviation for free falling objects in the northern hemisphere. To illustrate: With a drop-height of 500 m at a latitude of $\varphi = 45^\circ$, this easterly deviation is ≈ -0.17 m.

7. Derived Forces

The chapter title refers to forces which we view as mechanical in nature because this is how we experience them. However, all of them are derived forces in the sense of how this term was defined in the introductory chapter. Most of them are counted among the imprinted forces listed in tab. 1.1 while others such as the friction force belong to the reaction forces. In what follows we will first discuss friction forces, then look into the role of forces in elastostatics, after which we will derive the Hooke's equations underlying Hooke's law before we finally turn our attention to forces in elastodynamics.

7.1 Frictional Forces

In our daily life we encounter frictional forces everywhere. Without them, many things in our world which we take for granted would not work. The best example is probably walking. We all know how difficult it can be to walk without or with little frictional force when we, e.g., step onto an icy surface. Without friction, we would also not be able to drive our cars around a curve either. It is the frictional force between the tire surfaces and the road surface that makes it possible for us to steer our cars around curves, frequently at high speed. As we know, even that has its limits. If a car enters a curve with too high a velocity and the centrifugal force becomes greater than the maximum frictional force keeping the car on the road, then the car will quickly leave the roadway completely. Frictional forces are contact forces, they arise where two bodies touch. When touching, bodies can be at rest, then one speaks of static friction, or they can move relative to one another, then one speaks of sliding friction. Static and sliding friction, also referred to as Coulomb's friction after Charles Augustin de Coulomb, are both examples of so-called external friction, i.e., the friction between rigid bodies. Different from that is the so-called internal friction which underpins a body's viscosity and a body's toughness. Here we will look first at examples of external friction and then turn to internal friction when we look at bodies in fluids.

7.1.1 Static and Gliding Friction

Static friction will be considered first. Fig. 7.1 shows a block of mass m on an inclined plane. The angle of inclination α of the plane shall be set in such a way that the block does not slide but remains stuck. The weight force \mathbf{F}_G acts in the center of mass S of the block and its component normal to the contact surface \mathbf{F}_G^n is compensated by an opposite force of equal magnitude \mathbf{F}_N , the force with which the inclined plane opposes the block. Because of the latter, \mathbf{F}_N is also called the constraint force of the inclined plane. The pressure exerted by this constraint force \mathbf{F}_N causes a frictional force \mathbf{F}_R in the contact plane which counteracts the tangential component of the weight force \mathbf{F}_G^t . As long as \mathbf{F}_G^t is smaller than the maximum value that the frictional force can assume for the system of block material and material of the inclined plane, \mathbf{F}_R will adjust itself just so that $\mathbf{F}_R + \mathbf{F}_G^t = 0$. However, if \mathbf{F}_G^t exceeds the maximum value that \mathbf{F}_R can assume, then the block begins to slide. The equilibrium condition for the maximum value of $F_R = |\mathbf{F}_R|$ is

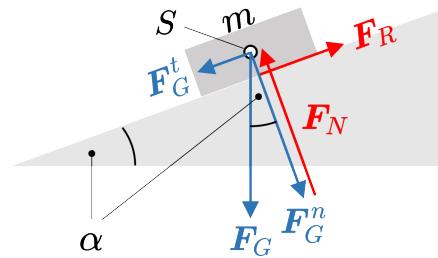


Fig. 7.1

$$F_R - m \cdot g \cdot \sin \alpha_{max} = 0 \quad (7.1)$$

where α_{max} is the angle at which the block just does not begin to slide. Static friction is a reaction force in response to the pressure exerted by F_N . The maximum value that F_R can assume will therefore depend on F_N :

$$F_R = \mu_0 \cdot F_N \quad (7.2)$$

where μ_0 is the so-called coefficient of static friction. The maximum value of the static friction on the inclined plane is therefore given by

$$F_R = \mu_0 \cdot m \cdot g \cdot \cos \alpha_{max} \quad (7.3)$$

From eq. (7.1) and eq. (7.3) it follows for μ_0

$$\mu_0 = \tan \alpha_{max} \quad (7.4)$$

Intriguingly, the inclined plane thus offers a very simple way to determine μ_0 by measuring α_{max} for any arbitrary combination of block material and material of the inclined plane.

The angle $\rho_0 = \alpha_{max}$ is the opening angle of the so-called friction cone (fig. 7.2). This opening angle is formed below the contact plane by the vectors of the normal force \mathbf{F}_N and the maximum frictional force \mathbf{F}_R^{max} which combine through vector addition to a virtual equivalent force lying in the cone shell. As long as the resultant of all external forces remains above the plane of contact and within this friction cone or in this cone shell, the body under consideration remains at rest. In the case of the inclined plane, \mathbf{F}_G^t and \mathbf{F}_G^n add up to the resulting external force, the weight force \mathbf{F}_G . The same argument also applies if, for example, $\alpha = 0$, with the difference being that in this case $\mathbf{F}_G^n = \mathbf{F}_G$ and \mathbf{F}_G^t , for example, is replaced by the displacement force with which one tries to move the body from its rest position. Just as the maximum static friction force is proportional to the normal force F_N , the sliding friction force is also proportional to F_N . The proportionality factor for sliding friction is the sliding friction coefficient μ

$$F_R = \mu \cdot F_N \quad (7.5)$$

In most cases, the coefficient of dynamic friction (sliding friction) μ is smaller than the coefficient of static friction μ_0 . That is, once the static friction between two bodies is overcome, less force is required to maintain the relative motion than was needed to start it. Tab. 7.1 lists μ_0 and μ values for some material compositions of practical importance. The values measured for such material pairings depend on the specific surface properties of the bodies in contact. Therefore, in the literature one often does not find single values for μ_0 and μ but value ranges.

Tab. 7.1: Coefficients of static and dynamic friction μ_0 and μ for some selected material pairings.

Material pair	μ_0 - Static friction		μ_0 - Gliding friction	
	dry	greased	dry	greased
Steel on steel	0.15 - 0.30	0.10 - 0.12	0.10 - 0.12	0.01 - 0.07
Steel on ice	0.03	—	0.01	—
Wood on wood	0.40 - 0.60	0.16	0.20 - 0.40	0.05 - 0.10
Leather on metal	0.30 - 0.50	0.16	0.30	0.15
Rubber on asphalt	0.70 - 0.90	0.10 ¹⁾	0.50 - 0.60	0.05 ¹⁾

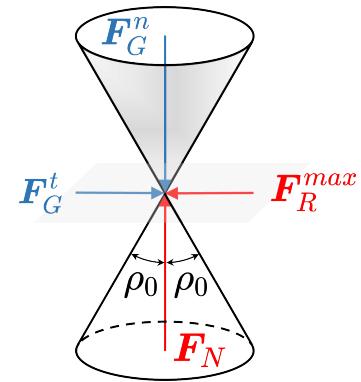


Fig. 7.2

¹⁾ Values on ice, not greased.

7.1.2 Rope Friction

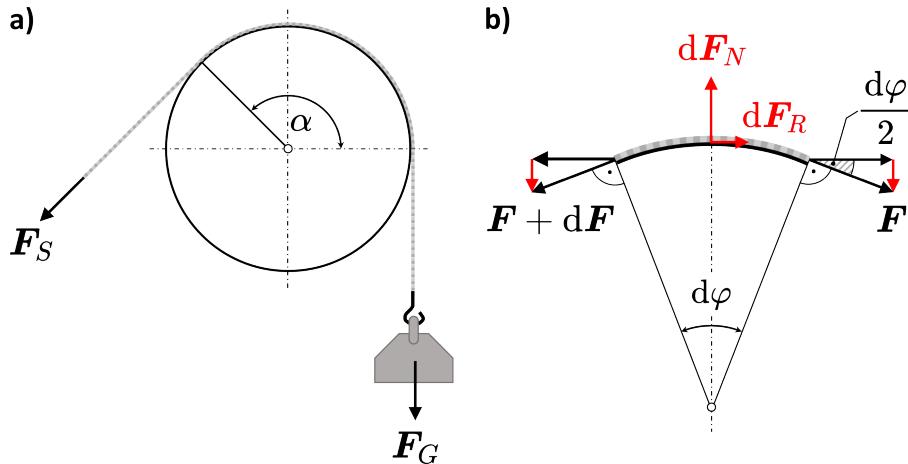


Fig. 7.3: Rope running over a cylinder surface: forces acting on the rope (a) and forces acting on the rope element with angle $d\varphi$ (b).

Consider a rope that runs over a fixed cylindrical disk that is anchored non-rotatably (fig. 7.3a). At one end of the rope hangs a weight with the weight force \mathbf{F}_G ; at the other end of the rope someone pulls with the rope force \mathbf{F}_S just enough to keep the weight in balance. The force \mathbf{F}_S required for this depends on the wrap angle α . The forces \mathbf{F}_G and $\mathbf{F}_S(\alpha)$ are tangential forces on the cylinder disk. To derive the equation which describes rope friction, one considers an infinitesimal angle element $d\varphi$ of the cylinder disk over which the rope runs (fig. 7.3b). As a result of the forces acting on the left and right of the respective piece of rope, it stretches over the cylinder disk and is pressed against its surface. Now one assumes that the resulting pressure distribution over the angular segment $d\varphi$ is uniform. In doing so one views this pressure distribution as caused by a single force $d\mathbf{F}_N$ applied to the mid-point of the contact area at $d\varphi/2$. As can be seen from fig. 7.3b, in a first approximation ($\sin \varphi \approx \varphi$) one can write

$$dF_N = 2 \cdot F \cdot \sin \frac{d\varphi}{2} = F \cdot d\varphi \quad (7.6)$$

If, as sketched in fig. 7.3b, the rope force at one end is slightly greater (by $d\mathbf{F}$) than at the other end, then the rope slides in the direction of the slightly greater force. Since the coefficient of static friction μ_0 for the material pair of rope and cylinder surface is not equal to zero, $d\mathbf{F}_N$ causes at $d\varphi/2$ a frictional force $d\mathbf{F}_R$ in the contact plane between the rope and the surface of the cylinder disk. Let the force difference $d\mathbf{F}$ between the two rope ends in fig. 7.3b now be just large enough that $dF = dF_R$. Then the equilibrium condition for maximum static friction is fulfilled and it holds that

$$dF = dF_R = \mu_0 \cdot dF_N = \mu_0 \cdot F \cdot d\varphi \quad (7.7)$$

With that one now has a differential equation for the tangential force F and the angle φ . This differential equation can be solved by separating the variables F and φ and subsequent integration. The angle φ runs from 0 to the respective wrap angle α and F from F_0 to F_α , so

$$\int_{F_0}^{F_\alpha} \frac{dF}{F} = \ln F_\alpha - \ln F_0 = \mu_0 \cdot \int_0^\alpha d\varphi = \mu_0 \cdot \alpha$$

From this follows the basic equation for rope friction

$$F_\alpha = F_0 \cdot e^{\mu_0 \alpha} \quad (\text{Euler-Eytelwein equation}) \quad (7.8)$$

This equation states that a smaller force F_0 (\mathbf{F}_G in fig. 7.3a) is sufficient to compensate a larger force F_α (\mathbf{F}_S in fig. 7.3a). Eq. (7.8) formulates an equilibrium condition: only when F_α reaches the value in eq. (7.8) can the weight force no longer compensate the force F_α and the rope will slide to the left, thereby lifting the weight. Eq. (7.8) was derived for the case that in the sketch of fig. 7.3 $\mathbf{F}_S(\alpha) > \mathbf{F}_G$ (by $d\mathbf{F}$). If, however, the reverse case occurs, that is $\mathbf{F}_S(\alpha) < \mathbf{F}_G$ (again by $d\mathbf{F}$), then the arrow of $d\mathbf{F}_R$ in fig. 7.3b points in the opposite direction and one then finds that the following holds:

$$F_\alpha = F_0 \cdot e^{-\mu_0 \alpha} \quad (7.9)$$

Hence, eq. (7.8) is the equation for lifting a load and eq. (7.9) for lowering a load. With the construction in fig. 7.3, the practical aspect is of course of interest, i.e., how much less force is required to lift a weight by means of a winch as in fig. 7.3a, compared to lifting it straight up. The maximum angle α in fig. 7.3a is π . With a static friction coefficient of $\mu_0 = 0.35$, e.g., this results in a force reduction by a factor of approx. 3.0. It is thus possible to compensate for a force that is 3.0 times as large. But much more is possible. If the rope is looped around the cylindrical disk several times, each loop increases the wrap angle α by a factor 2π and thus increases the force reduction by a further factor 9.0. With a rope that is wound around a bollard several times very large forces can be compensated.

With eq. (7.7) rope friction is a case of static friction just as the mass on an inclined plane is either a case of static or gliding friction, depending on its state of motion. In addition to static and sliding friction there is a third kind of mechanical friction which is rolling friction, the resistance a rolling wheel encounters where the friction force is given by $F_R = \mu_R F_N$ where μ_R is the rolling friction coefficient.

7.1.3 Fluid Friction

Fluid friction occurs when the layers of a fluid move relative to each other and is the result of molecular interactions between the molecules making up the fluid. A special case are the boundary layers of a fluid where the liquid interacts with a solid like for example the walls of a pipe. The interaction of such fluid boundary layers with the surface layers bounding the fluid are usually stronger than the interaction between neighboring fluid layers within the fluid body itself. The measure for the strength of interaction between fluid layers is a fluids viscosity (see also section 8.6).

The dynamic or shear viscosity η of a fluid provides a measure of the resistance to motion between two neighboring layers of a fluid. Another measure is the kinematic or volume viscosity λ of a fluid which is the ratio of η to the density ρ of a fluid. From a practical standpoint, dynamic viscosity is a measure of the resistance with which a fluid opposes to flow under the impact of an external force. Different from that, kinematic viscosity is a measure of a fluids resistance to flow if the only force acting on it is gravity. The solid surface containing a fluid's flow, like the inner surface of a pipe for example, provides a drag to the flow of that fluid. Similarly, a solid body dropped into a fluid will experience a drag between its outer surface layer and the fluid layer bounding its surface which will slow it down. This drag is the result of a frictional force which slows the sinking of the body dropped into the fluid. If the velocity v with which the body sinks is low, this frictional force acting on a sphere of radius r_0 is given by

$$F_R^\eta = 6\pi\eta \cdot r_0 \cdot v \quad (7.10)$$

Eq. (7.10) is known as Stokes law, named after George G. Stokes (1819 - 1903), and the regime where friction forces are proportional to an objects velocity is referred to as friction according to Stokes. The fluid friction force F_R^η is called the viscous resistance of the fluid, therefore the superscript η . At higher velocities forces proportional to v^2 begin to slow down objects moving in fluids. A good indication that something has changed and another kind of friction begins to dominate is usually the visible trail of vortices an object moving at high speed through a fluid leaves behind it. Creating these circulating vortices costs energy and that energy is being produced as the fluid in front of the object is pushed out of the way by the object moving through the fluid.

To understand how the v^2 dependence of this additional friction force comes about consider the change in momentum the liquid volume in front of the moving object is subject to. Let dm be the fluid mass in front of the moving object which has to be moved out of the way per unit time. Moving the mass in front of the object out of the way means to

bring it up to the same speed as the moving object. Hence we can write for the change in momentum which the volume being moved out of the way is subjected to

$$\frac{dp}{dt} \approx \frac{d(mv)}{dt} = \frac{dm}{dt} \cdot v \quad (v = const)$$

Assuming that the density of the fluid does not change the quantity dm/dt is given by $\rho \cdot dV/dt$ where the volume change per unit time is the product of the objects cross section A perpendicular to its moving direction, e.g., the z -axis, times the velocity $v = dz/dt$ with which the object is moving in that direction. Therefore, the force slowing down the object moving through the fluid must be

$$\frac{dp}{dt} \approx \rho \cdot \frac{dV}{dt} \cdot v = \rho \cdot A \cdot \frac{dz}{dt} \cdot v = \rho \cdot A \cdot v^2$$

The force slowing down the movement of an object being dragged through a fluid must be the fluid friction force F_R^{in} working in the v^2 -regime. On close inspection one finds that at velocities exceeding the Stokes friction regime

$$F_R^{in} = \frac{1}{2} c \cdot \rho \cdot v^2 \cdot A \quad (7.11)$$

where the dimensionless constant c depends on the shape of the object. Interestingly, this frictional force does not depend any more on the fluids viscosity. F_R^{in} is called the inertial resistance of the fluid as it is the inertia of the fluid volume in front of the object which slows down the movement of the object. So how do we know when F_R^{in} will begin to overtake F_R^η ? With $c = 0.5$ for a sphere shaped object which has the cross section $A = r_0^2 \pi$ this happens when

$$\frac{F_R^{in}}{F_R^\eta} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \rho \cdot v^2 \cdot r_0^2 \pi}{\frac{6\pi\eta \cdot r_0 \cdot v}{24}} = \frac{1}{24} \frac{r_0 \rho v}{\eta} = \frac{1}{24} \text{Re} > 1$$

The dimensionless number $\text{Re} = \rho_0 \rho v / \eta$, the so-called Reynolds number, is a measure of the relative strength of viscous vs inertial resistance with which a given fluid will slow down the motion of a body. If $\text{Re} < 1$ then viscous resistance will dominate the motion of a body in a fluid. When fluid friction is dominated by viscous resistance one speaks of laminar flow because in that regime one can model fluid flow as the flow of parallel fluid layers gliding by each other without being disrupted. If Re in the above equation becomes comparable to the factor in front of it then inertial resistance will begin to take over. When that happens depends for a given fluid on the objects geometry. For $\text{Re} \gg 1$ one speaks of turbulent flow where fluid layers are disrupted and vortices of varying sizes begin to appear and interact with each other.

7.2 Forces in Elastostatics

Forces can act in the interior of a continuum body, and they can act on its surfaces. Forces acting on a volume element in the interior of a continuum body are exerted by its respective neighboring volume elements. Each volume element dV (fig. 7.4) within a continuum body transmits forces associated with the corresponding force density this volume element possesses. In one were to cut such a volume element dV free from its neighboring volume elements, these forces must continue to act on the sectional areas of dV so that the state of stress of the cut-out volume element will remain identical to the state of stress the volume element was subject to before it was cut free (sectional cutting principle).

For a continuum body the following applies:

- Force density $\mathbf{f}(\mathbf{r})$: Associated with volume element dV .
- Stress: Associated with sectional cut areas dA .

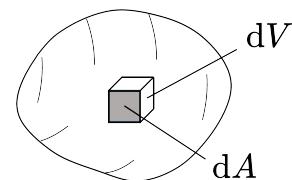


Fig. 7.4

From the force density $\mathbf{f}(\mathbf{r})$ follows the differential force $d\mathbf{F}$

$$d\mathbf{F} = \mathbf{f}(\mathbf{r})dV \quad (7.12)$$

In addition to internal forces, there are external forces acting on each volume element of the continuum body. An example of this is gravity:

$$\mathbf{f}_{gravity} = -\varrho \cdot g \cdot \mathbf{e}_z \quad (\varrho = \text{mass density})$$

For the stress of a section surface element dA of a cut volume element dV the following must apply:

$$\mathbf{P} dA = d\mathbf{F}$$

Here \mathbf{P} is the so-called stress vector with the dimension force / area. The total stress at a section surface element dA is broken down into a stress component perpendicular to this surface element, the so-called normal stress, and a stress component parallel to this surface element, the so-called shear stress.

The following applies to these stress components

- Shear stress: $|d\mathbf{F}_t| = \tau dA$
- Normal stress: $|d\mathbf{F}_n| = \sigma dA$

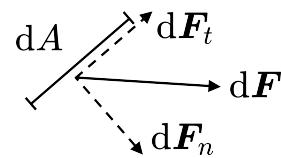


Fig. 7.5

The convention is that the Greek letter σ is used to designate normal stresses and the Greek letter τ is used for shear stresses. Positive normal stresses are referred to as tensile stresses and negative normal stresses are called compressive stresses. Tensile stresses always point in the direction of the surface normal vector, i.e., away from the section surface element, and vice versa for compressive stresses. Shear stresses can also be positive or negative. If a shearing force produces a positive moment with respect to the center of mass of the section surface element, i.e., a counter-clockwise rotation, then the shear stress is per definition positive; if it generates a negative moment, then the shear stress is negative.

Shear stresses are further broken down into their respective components along the coordinate axes lying in the section surface element. In this context the first index of τ stands for the orientation of the section surface element and the second index for the direction of the shear stress component. For example, τ_{xy} denotes a shear stress in a surface whose normal vector is parallel to the x -coordinate and which itself points in the direction of the y -coordinate in this surface.

7.2.1 The Stress Tensor

Let dA be the surface element with normal vector \mathbf{n} which separates the volume elements dV_1 and dV_2 in fig. 7.6 from each other. With \mathbf{P}_n as the associated stress vector, dV_1 then exerts a force on dV_2 proportional to dA :

$$d\mathbf{F}_{12} = \mathbf{P}_n dA$$

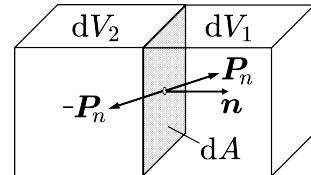


Fig. 7.6

And dV_2 exerts the same opposite force on dV_1 :

$$d\mathbf{F}_{21} = -\mathbf{P}_n dA$$

A cuboid volume element dV such as the one shown in fig. 7.7 has six neighboring volume elements. At each of the six boundary surfaces of dV there exists a corresponding equilibrium of forces with the respective neighboring volume element. To calculate the total force on such a volume element $dV = dx dy dz$, the forces transmitted at the respective boundary surfaces are added with the correct sign. For forces in the x -direction, these are the boundary surfaces

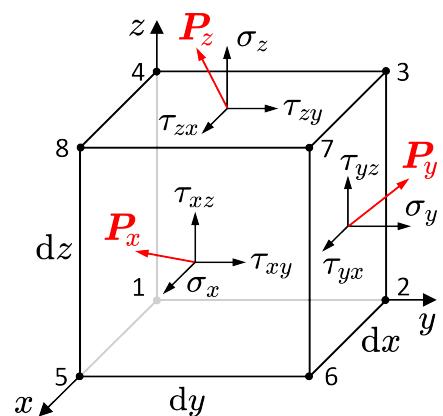


Fig. 7.7

normal to the x -axis in fig. 7.7 with the corner points 1-2-3-4 and 5-6-7-8. First one only considers the surface area 1-2-3-4 by itself. On both sides of this surface the same force acts, but it does so in opposite directions: namely $-\mathbf{P}_x dy dz$ on the outside of this surface (negative x -direction) and $\mathbf{P}_x dy dz$ on the inside of this surface (positive x -direction). If one now moves a copy of this surface along the x -axis in the direction of the surface 5-6-7-8, the acting force on both sides of the surface will change by an amount proportional to the displacement of this surface times the respective change of \mathbf{P}_x over the distance of that displacement. Virtually shifting this surface all the way to the position of the surface 5-6-7-8, the force transferred by dV in x -direction at this position becomes:

$$d\mathbf{F}_{5678} = \left(\mathbf{P}_x + \frac{\partial \mathbf{P}_x}{\partial x} dx \right) dy dz$$

The force transferred by dV at the position of the surface 1-2-3-4 still is:

$$d\mathbf{F}_{1234} = -\mathbf{P}_x dy dz$$

With that, the total force transmitted by dV in x -direction thus becomes:

$$d\mathbf{F}_{5678} + d\mathbf{F}_{1234} = \frac{\partial \mathbf{P}_x}{\partial x} dx dy dz$$

With similar considerations for the forces acting on the surfaces normal to the y - or z -axis, one obtains for the total force transmitted by the volume element dV in fig. 7.7

$$d\mathbf{F} = \left(\frac{\partial \mathbf{P}_x}{\partial x} + \frac{\partial \mathbf{P}_y}{\partial y} + \frac{\partial \mathbf{P}_z}{\partial z} \right) dx dy dz \quad (7.13)$$

With the axis unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ one can decompose the stress vectors \mathbf{P}_x , \mathbf{P}_y and \mathbf{P}_z into their directional components.

$$\begin{aligned} \mathbf{P}_x &= \hat{\mathbf{i}}\sigma_x + \hat{\mathbf{j}}\tau_{xy} + \hat{\mathbf{k}}\tau_{xz} \\ \mathbf{P}_y &= \hat{\mathbf{i}}\tau_{yx} + \hat{\mathbf{j}}\sigma_y + \hat{\mathbf{k}}\tau_{yz} \\ \mathbf{P}_z &= \hat{\mathbf{i}}\tau_{zx} + \hat{\mathbf{j}}\tau_{zy} + \hat{\mathbf{k}}\sigma_z \end{aligned} \quad (7.14)$$

Inserting this in eq. (7.13) yields for the force density $\mathbf{f}(\mathbf{r})$

$$\begin{aligned} \mathbf{f}(\mathbf{r}) = \frac{d\mathbf{F}}{dx dy dz} &= \hat{\mathbf{i}} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \\ &\quad \hat{\mathbf{j}} \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \\ &\quad \hat{\mathbf{k}} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) \end{aligned} \quad (7.15)$$

or respectively

$$\mathbf{f}(\mathbf{r}) = \nabla \underline{\underline{\sigma}} \quad \text{where} \quad \underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad (7.16)$$

is the stress tensor. The stress tensor $\underline{\underline{\sigma}}$ is a symmetric tensor of the second rank. For any arbitrary section surface element dA with normal vector \mathbf{n} one obtains the respective stress vector \mathbf{P}_n by multiplying $\underline{\underline{\sigma}}$ with \mathbf{n}

$$\mathbf{P}_n = \mathbf{n} \underline{\underline{\sigma}} \quad (7.17)$$

Because the stress tensor $\underline{\underline{\sigma}}$ is symmetric, it makes no difference for the determination of \mathbf{P}_n whether one multiplies the stress tensor with the normal vector \mathbf{n} from the left or from the right.

7.2.2 Strains

Strains are deformations of a continuum body due to changes in length and / or angle which cause the position vector $\mathbf{r} = (x, y, z)$ of a given point in the continuum body to be shifted by a displacement vector $\boldsymbol{\rho} = (\xi, \eta, \zeta)$. The displacement vector $\boldsymbol{\rho}$ of a deformation can indeed be constant, but more often the respective displacement coordinates ξ , η and ζ depend on the location, i.e., $\xi = \xi(x, y, z)$, $\eta = \eta(x, y, z)$ and $\zeta = \zeta(x, y, z)$. Naturally, strains are always the result of the effects of a force of some kind. Normal stresses cause stretching or compression of the continuum body, i.e., changes in length, while shear stresses cause so-called slips, i.e., changes in angle leading to distortions of a continuum body. An important part of the characterization of materials is the determination of strains as a function of applied stresses. The result of corresponding measurements are stress-strain diagrams which describe the strain behavior of a body under the applied stresses. A distinction is made between three different deformation domains:

- Linear elastic: The strain is proportional to the stress and it is reversible.
- Nonlinear elastic: The strain is no longer proportional to the stress but is still reversible.
- Plastic: The strain is no longer reversible, and the deformation will persist when the applied stress is removed.

In technical applications, the linear elastic domain is of great practical importance. In this domain, the respective changes in lengths and angles as compared to the initial lengths and initial angles are considered to be sufficiently small so that linear approximations are justified for calculating the resulting deformation. In the linear elastic domain, stresses (σ, τ) and strains, i.e., stretching or compression (ϵ) of a continuum body, as well as slips (γ), can be expressed through Hooke's laws:

$$\left. \begin{array}{l} \sigma = E \cdot \epsilon \quad \text{normal stresses} \\ \tau = G \cdot \gamma \quad \text{shear stresses} \end{array} \right\} \quad (7.18)$$

Here, the modulus of elasticity E , also referred to as Young's modulus, and the shear modulus G are material-specific parameters. The strain behavior of continuum bodies can be very complex, since different strains usually overlap. In general, however, it is often possible to decompose the complex state of strain of a body into a superposition of simpler states of strain. The latter include simple tension or compression situations, as well as bending and distortion.

Simple Tensile Strain

In the simple tensile test sketched in fig. 7.8, the weight attached to a hanging wire causes a uniaxial state of stress ($F = \sigma_z A$) in the wire which leads both to a change in the wire length l and to a change in the wire diameter d . The relative elongation of the wire ϵ_{zz} in the z -direction is given by

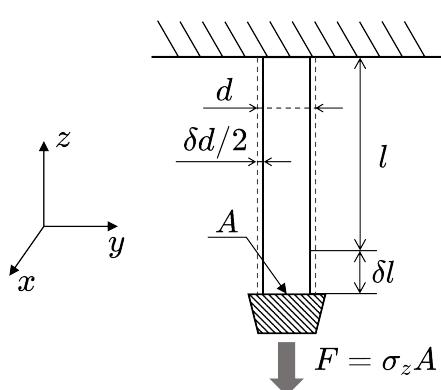


Fig. 7.8

$$\epsilon_{zz} = \frac{\delta l}{l} = \frac{\sigma_z}{E} \quad (7.19)$$

The decrease in wire diameter is described by the strain components ϵ_{xx} and ϵ_{yy} in the xy -plane.

$$\epsilon_{xx} = \epsilon_{yy} = \frac{\delta d}{d} = -\mu \epsilon_{zz} \quad (7.20)$$

where the so-called Poisson's ratio μ , also frequently referred to as Poisson's number, is a material-specific parameter. Just as with the components of the stress tensor, the indices of the components ϵ_{ik} of the strain tensor also describe the direction of the normal to the section plane (first index) and a directional coordinate in the section plane (second index).

Beam Bending

Let ρ_1 and ρ_2 be the radii of curvature that describe the expansion or compression of the infinitesimal volume element sketched in fig. 7.9 in dx - or dy -direction. The stretching and compression caused by the bending of the beam can be read from the sketch. ϵ_{xx} and ϵ_{yy} are the strain components in the x - and in the y -direction, respectively. The following applies to these:

$$\epsilon_{xx} = \frac{(\rho_1 + z)d\varphi - \rho_1 d\varphi}{dx} = z \frac{d\varphi}{dx} = \frac{z}{\rho_1}$$

$$\epsilon_{yy} = \frac{(\rho_2 - z)d\chi - \rho_2 d\chi}{dy} = -z \frac{d\chi}{dy} = \frac{z}{\rho_2}$$

$$\mu = \frac{\rho_1}{\rho_2}$$

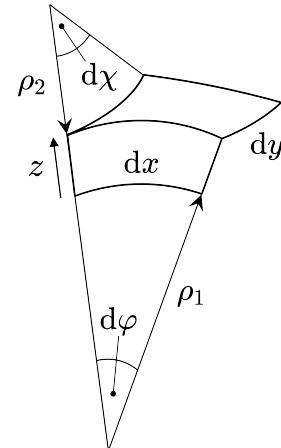


Fig. 7.9

This simple view of beam bending ignores the fact that there occurs stretching and compression in x - and y -direction and not stretching only in x -direction and compression only in y -direction. The line element dx which is closer to the angle $d\varphi$ experiences a compression as compared to the “unbent” state, while the dx line element lying further away from $d\varphi$ experiences a stretching as compared to the “unbent” state. Somewhere in between lies a line element which neither is bent nor stretched, the so called neutral fiber. The same applies to the line element dy in relation to the angle $d\chi$.

Distortions

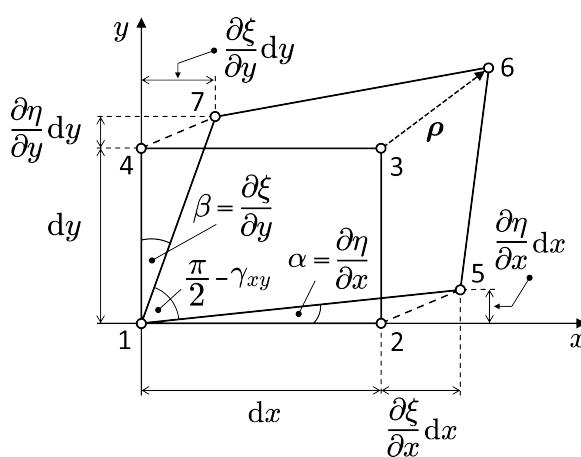


Fig. 7.10

First we consider a planar distortion such as in fig. 7.10. This distortion deforms the rectangle 1-2-3-4 into the polygon 1-5-6-7. The right angle between the segments $\overline{12}$ and $\overline{14}$ of the original rectangle becomes the angle

$$\angle_{412} \mapsto \frac{\pi}{2} - \gamma_{xy}$$

where $\gamma_{xy} = \alpha + \beta$ is the gliding angle. For the small deformations considered here, the angles α and β are very small and one can apply linear approximations

$$\alpha \approx \tan \alpha = \frac{\frac{\partial \eta}{\partial x} dx}{dx + \frac{\partial \xi}{\partial x} dx} = \frac{\frac{\partial \eta}{\partial x}}{1 + \frac{\partial \xi}{\partial x}} \left(1 + \frac{\partial \xi}{\partial x} + \dots\right) \approx \frac{\frac{\partial \eta}{\partial x}}{1}$$

and

$$\beta \approx \tan \beta = \frac{\frac{\partial \xi}{\partial y} dy}{dy + \frac{\partial \eta}{\partial y} dy} = \frac{\frac{\partial \xi}{\partial y}}{1 + \frac{\partial \eta}{\partial y}} \left(1 + \frac{\partial \eta}{\partial y} + \dots\right) \approx \frac{\frac{\partial \xi}{\partial y}}{1}$$

With that, one obtains for the gliding angle γ_{xy}

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

For the relative elongation, the ratio of the distorted to the original length, one can read from fig. 7.9 for x - and y -direction:

$$\epsilon_{xx} = \frac{\left(dx + \frac{\partial \xi}{\partial x} dx\right) - dx}{dx} = \frac{\partial \xi}{\partial x} \quad \text{and} \quad \epsilon_{yy} = \frac{\left(dy + \frac{\partial \eta}{\partial y} dy\right) - dy}{dy} = \frac{\partial \eta}{\partial y}$$

With $\gamma_{xy} = 2\epsilon_{xy}$ or $\gamma_{yx} = 2\epsilon_{yx}$, the distortion tensor in the plane becomes

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} \partial \xi / \partial x & \gamma_{xy} / 2 \\ \gamma_{yx} / 2 & \partial \eta / \partial y \end{bmatrix}$$

To describe the spatial distortion state, the result for the planar distortion can be easily extended and with the generalized coordinates

$$\boldsymbol{\rho}(\xi, \eta, \zeta) = \boldsymbol{\rho}(\xi_1, \xi_2, \xi_3)$$

one obtains for the spatial distortion tensor

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} ; \quad \epsilon_{ik} = \frac{1}{2} \left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i} \right) \quad i, k = 1, 2, 3 \quad (7.21)$$

The state of distortion which $\underline{\underline{\epsilon}}$ describes consists of length changes or elongations (stretching and compression) and shearing (angle changes). Elongations are described by the diagonal components of $\underline{\underline{\epsilon}}$ and shears are reflected by the mixed tensor components. Because of $\epsilon_{ik} = \epsilon_{ki}$, the distortion tensor is symmetric. The volume change of a continuum body associated with the respective elongations and shears can be easily determined in a linear approximation (i.e., for very small deformation):

$$\delta V = \prod_{i=1}^3 dx_i \left(1 + \frac{\partial \xi_i}{\partial x_i} \right) - dx_1 dx_2 dx_3$$

$$\delta V = dx_1 dx_2 dx_3 \sum_{i=1}^3 \frac{\partial \xi_i}{\partial x_i} + \text{nonlinear terms}$$

Thus, the relative change in volume is

$$\frac{\delta V}{V} = \sum_{i=1}^3 \frac{\partial \xi_i}{\partial x_i} = \sum_{i=1}^3 \epsilon_{ii} = \text{Sp}(\underline{\underline{\epsilon}}) \quad (7.22)$$

where $\text{Sp}(\underline{\underline{\epsilon}})$ is the trace of the distortion tensor.

7.3 Hooke's Law

With the basic understanding of stress and distortion that we now have, we can look into how the relationship between stress and distortion which we know as Hooke's law comes about. This law applies in the case of small distortions in isotropic and homogeneous elastic continuum bodies. Homogeneous means that the elastic properties of the considered body are the same in all its points, and it is called isotropic because the body's physical properties are identical in all directions. In such a continuum body, the stress at each of its body points is entirely determined by the respective distortion present, and the principal axes of stress and distortion coincide.

When an elastic body is being distorted, the work exerted in deforming it becomes stored as potential energy in its deformation. Accordingly, this process runs in reverse when the deformation heals as the stored potential energy is being used to do the work required to reverse the deformation. If U denotes the potential energy stored per volume unit dV , i.e., the potential energy density, then a volume element of the deformed body possesses the potential energy $U \cdot dV$. This potential energy density U of a distorted elastic body is always positive and equal to zero in the undistorted case. In the dV -neighborhood of each point of a distorted elastic body U is proportional to the respective local distortion and can be expressed by the components of the distortion tensor.

For the case of small strains, the restoring force is proportional to the stretching or compression components of the distortion tensor. On the other hand, the work done to distort the body and thus the potential energy density stored in a deformation, is approximately proportional to the acting force, again multiplied by the corresponding stretching or compression components. One boundary condition for the expansion of U in terms of the components of the distortion tensor $\underline{\underline{\epsilon}}$ is therefore that, in a first approximation, U must

be proportional to square products of stretching or compression components. Obviously, the best choice for this expansion is to use the deformation tensor in the principal axis system, i.e., $\underline{\underline{\epsilon}} \mapsto \underline{\underline{\epsilon}}^D$.

$$\underline{\underline{\epsilon}}^D = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \quad \text{with} \quad \text{Sp}(\underline{\underline{\epsilon}}^D) = \epsilon_1 + \epsilon_2 + \epsilon_3$$

In an isotropic body U must not change when the principal axes are swapped (e.g., ϵ_1 and ϵ_3). Hence, U must be a symmetric function of ϵ_1 , ϵ_2 , and ϵ_3 . In this regard, the sum of squares of the stretching and compression components is an obvious choice, because

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 \quad \text{is symmetric with respect to exchanging the } \epsilon_i$$

The square of the trace of $\underline{\underline{\epsilon}}^D$, itself an invariant tensor quantity, is also such a symmetric function of ϵ_i and in addition also contains mixed tensor elements.

$$[\text{Sp}(\underline{\underline{\epsilon}}^D)]^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + 2(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) \quad (7.23)$$

With $\epsilon_i\epsilon_k = \epsilon_k\epsilon_i$, eq. (7.23) contains all quadratic terms which one can possibly build from ϵ_1 , ϵ_2 , and ϵ_3 . Thus, all possible forms of U which satisfy the two conditions from above can be built from linear combinations of the square sum of the ϵ_i and the square of $\text{Sp}(\underline{\underline{\epsilon}}^D)$. Hence, it follows for U that:

$$U = G \left[\alpha(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) + \beta(\epsilon_1 + \epsilon_2 + \epsilon_3)^2 \right] \quad (7.24)$$

α and β are dimensionless constants and G is a dimensional material constant. So that $U \geq 0$ is always true, one must require $G \geq 0$ and α and β must be chosen such that

$$\frac{\alpha}{\beta} \geq -\frac{(\epsilon_1 + \epsilon_2 + \epsilon_3)^2}{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2}$$

For the following, one can without restriction set $\alpha = 1$. Now the stretching $d\epsilon_1$ of the cuboid shown in fig. 7.11 with the side lengths ds_1 , ds_2 and ds_3 shall be considered. The cuboid shown sits with its center at the coordinate origin and with its side faces perpendicular to the coordinate axes e_1 , e_2 , and e_3 which coincide with the principal axis directions of $\underline{\underline{\epsilon}}^D$. Apart from the stretching $d\epsilon_1$ in the direction of the e_1 axis, the cuboid experiences no further distortion. Due to stretching, the distance ds_1 is lengthened by the amount $ds_1 d\epsilon_1$ (remember: ϵ always represents a relative stretching

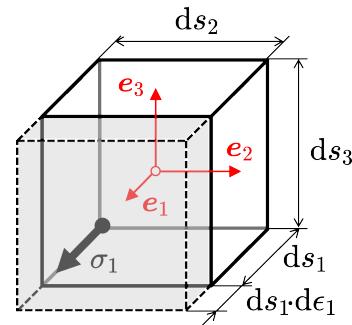


Fig. 7.11

$\Delta l/l$). Since ds_2 and ds_3 remain unchanged and the principal stress directions coincide with the principal directions of distortion, the work done to stretch the cuboid is

$$dA = \text{force} \times \text{distance} = \sigma_1 ds_2 ds_3 \times ds_1 d\epsilon_1$$

This work increases the potential energy $U \cdot dV = U \cdot ds_1 ds_2 ds_3$ of the cuboid by the equivalent amount $dU \cdot ds_1 ds_2 ds_3$. A simple comparison reveals that

$$dU = \sigma_1 \cdot d\epsilon_1 \equiv \frac{\partial U}{\partial \epsilon_1} \cdot d\epsilon_1$$

Using the parametrized form of the potential energy density of U from eq. (7.24) with $\alpha = 1$, one calculates for σ_1 :

$$\sigma_1 = \frac{\partial U}{\partial \epsilon_1} = 2G[\epsilon_1 + \beta(\epsilon_1 + \epsilon_2 + \epsilon_3)]$$

Since the trace of a tensor is an invariant quantity, i.e., $\text{Sp}(\underline{\underline{\epsilon}}) = \text{Sp}(\underline{\underline{\epsilon}}^D)$, this equation can also be written as

$$\sigma_1 = 2G[\epsilon_1 + \underbrace{2G\beta \cdot \text{Sp}(\underline{\underline{\epsilon}})}_{\text{constant}}] = 2G\epsilon_1 + \underbrace{2G\beta \cdot \text{Sp}(\underline{\underline{\epsilon}})}_{\text{constant}}$$

This means that no matter which coordinate transformation is carried out, the second term in this equation remains unchanged. Analogous to σ_1 , σ_2 and σ_3 can be expressed in a similar way as functions of the distortion components ϵ_1 , ϵ_2 , and ϵ_3 . Taken together, the relationships between the principal stresses σ_i and the principal distortions ϵ_i can be written in matrix notation as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = 2G \begin{bmatrix} 1 + \beta & \beta & \beta \\ \beta & 1 + \beta & \beta \\ \beta & \beta & 1 + \beta \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \quad (7.25)$$

or respectively $\sigma = 2G \cdot \underline{\underline{A}} \cdot \epsilon$. With the help of the inverse matrix

$$\underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}} \cdot \text{Adj}(\underline{\underline{A}}) = \frac{1}{1+3\beta} \begin{bmatrix} 1+2\beta & -\beta & -\beta \\ -\beta & 1+2\beta & -\beta \\ -\beta & -\beta & 1+2\beta \end{bmatrix}$$

one obtains the principal distortions ϵ_i as a function of the principal stresses σ_i from $\epsilon = (2G)^{-1} \cdot \underline{\underline{A}}^{-1} \cdot \sigma$. Carrying out this operation yields

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \frac{1}{2G(1+3\beta)} \begin{bmatrix} 1+2\beta & -\beta & -\beta \\ -\beta & 1+2\beta & -\beta \\ -\beta & -\beta & 1+2\beta \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \quad (7.26)$$

As can be seen from eq. (7.26), the stress σ_1 in the example of fig. 7.11 causes a stretching ϵ_1 of

$$\epsilon_1 = \frac{1 + 2\beta}{2G(1 + 3\beta)} \cdot \sigma_1 \quad (7.27)$$

At the same time, although σ_2 and σ_3 are zero, the stress σ_1 also causes a transverse contraction of the magnitude

$$\epsilon_2 = \epsilon_3 = -\frac{\beta}{2G(1 + 3\beta)} \cdot \sigma_1 \quad (7.28)$$

We have already encountered the constant G , the shear modulus, in eq. (7.18). This material-specific parameter, sometimes also referred to as slip modulus or torsional modulus, has the dimension of a pressure. A comparison of eq. (7.18) and eq. (7.27) shows that for the modulus of elasticity E , sometimes also referred to as Young's modulus, stress modulus, or tensile modulus, the following applies:

$$E = 2G \frac{1 + 3\beta}{1 + 2\beta} \quad (7.29)$$

From eq. (7.27) and eq. (7.28) one obtains for the Poisson ratio $\mu = \epsilon_2/\epsilon_1 = \epsilon_3/\epsilon_1$

$$\mu = \frac{\beta}{1 + 2\beta} \quad (7.30)$$

One therefore finds for the relationships between G , E and μ

$$E = 2G \cdot (\mu + 1) \quad , \quad G = \frac{1}{2} \frac{E}{\mu + 1} \quad , \quad \mu = \frac{E}{2G} - 1 \quad (7.31)$$

If one now replaces β in eq. (7.25) and eq. (7.26) by μ or respectively E , one obtains the well-known Hooke equations in the principal axes system, i.e., in a system in which the off-diagonal components of $\underline{\sigma}$ and $\underline{\epsilon}$ vanish:

$$\left. \begin{aligned} \sigma_1 &= \frac{E}{\mu + 1} \left[\epsilon_1 + \frac{\mu}{1 - 2\mu} \cdot (\epsilon_1 + \epsilon_2 + \epsilon_3) \right] = 2G \left[\epsilon_1 + \frac{\mu}{1 - 2\mu} \cdot \text{Sp}(\underline{\epsilon}) \right] \\ \sigma_2 &= \frac{E}{\mu + 1} \left[\epsilon_2 + \frac{\mu}{1 - 2\mu} \cdot (\epsilon_1 + \epsilon_2 + \epsilon_3) \right] = 2G \left[\epsilon_2 + \frac{\mu}{1 - 2\mu} \cdot \text{Sp}(\underline{\epsilon}) \right] \\ \sigma_3 &= \frac{E}{\mu + 1} \left[\epsilon_3 + \frac{\mu}{1 - 2\mu} \cdot (\epsilon_1 + \epsilon_2 + \epsilon_3) \right] = 2G \left[\epsilon_3 + \frac{\mu}{1 - 2\mu} \cdot \text{Sp}(\underline{\epsilon}) \right] \end{aligned} \right\} \quad (7.32)$$

or respectively

$$\left. \begin{aligned} \epsilon_1 &= \frac{\mu+1}{E} \left[\sigma_1 - \frac{\mu}{\mu+1} \cdot (\sigma_1 + \sigma_2 + \sigma_3) \right] = \frac{1}{2G} \left[\sigma_1 - \frac{\mu}{\mu+1} \cdot \text{Sp}(\underline{\underline{\sigma}}) \right] \\ \epsilon_2 &= \frac{\mu+1}{E} \left[\sigma_2 - \frac{\mu}{\mu+1} \cdot (\sigma_1 + \sigma_2 + \sigma_3) \right] = \frac{1}{2G} \left[\sigma_2 - \frac{\mu}{\mu+1} \cdot \text{Sp}(\underline{\underline{\sigma}}) \right] \\ \epsilon_3 &= \frac{\mu+1}{E} \left[\sigma_3 - \frac{\mu}{\mu+1} \cdot (\sigma_1 + \sigma_2 + \sigma_3) \right] = \frac{1}{2G} \left[\sigma_3 - \frac{\mu}{\mu+1} \cdot \text{Sp}(\underline{\underline{\sigma}}) \right] \end{aligned} \right\} \quad (7.33)$$

Leaving the principal axes system, the following applies to the components of the stress- and distortion tensor

$$\sigma_{ik} = 2G \left[\epsilon_{ik} + \frac{\mu}{1-2\mu} \left(\sum_{l=1}^3 \epsilon_{ll} \right) \delta_{ik} \right] \quad (7.34)$$

or respectively

$$\epsilon_{ik} = \frac{1}{2G} \left[\sigma_{ik} - \frac{\mu}{\mu+1} \left(\sum_{l=1}^3 \sigma_{ll} \right) \delta_{ik} \right] \quad (7.35)$$

For the off-diagonal elements ($i \neq k$) of the tensor components in eq. (7.34) and eq. (7.35) the following conventions apply:

$$\epsilon_{ik} = \frac{1}{2} \gamma_{ik} = \frac{\mu+1}{E} \sigma_{ik} = \frac{\mu+1}{E} \tau_{ik} ; \quad \gamma_{ik} = 2 \frac{\mu+1}{E} \tau_{ik} = \frac{1}{G} \tau_{ik}$$

Systems in which the off-diagonal components of $\underline{\underline{\sigma}}$ and $\underline{\underline{\epsilon}}$ vanish are those in which, for example, only compressive forces are present, such as is the case with the hydrostatic pressure P . There one has

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -P \quad \text{and} \quad \sigma_{ik} = 0 \quad \text{for} \quad i \neq k$$

The resulting volume change due to the hydrostatic pressure P is therefore given by

$$\frac{\delta V}{V} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = -3 \cdot \frac{1-2\mu}{E} P = -\kappa P = -\frac{P}{K}$$

where K is the compressibility modulus and κ is the compressibility. Eq. (7.34) connects the diagonal elements ϵ_{ii} of the distortion tensor (stretching and compression) and its off-diagonal elements i.e., the ϵ_{ik} with $i \neq k$ (slips). This relationship and the fact that the sum of stretching and compression ($\text{Sp}(\underline{\underline{\epsilon}})$) is proportional to the sum of the principal stresses ($\text{Sp}(\underline{\underline{\sigma}})$), i.e.,

$$\sum_K \epsilon_{KK} = \frac{\mu+1}{E} \sum_K \sigma_{KK} - \frac{3\mu}{E} \sum_K \sigma_{KK} = \frac{1-2\mu}{E} \sum_K \sigma_{KK} \quad (7.36)$$

are both very helpful when one needs to calculate stresses from strains.

7.4 Forces in Elastodynamics

Section 7.2 gave an introduction of how to describe forces acting in elastostatic situations which are by definition situations where elastic bodies are in a force equilibrium. In such equilibrium of force situations, the respective forces acting on an elastic body cancel each other out, the result of which is that the body does not move. Hence the term elastostatics. Different from that, elastodynamics considers situations in which this balance of forces is disturbed and the elastic body is therefore subject to acceleration by a resulting force. To understand how we can describe such situations we need to look at what happens to the elements of the stress tensor from eq. (7.16) if forces acting on an elastic body are not in balance. As we saw in section 7.2, the stress tensor has the components

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad (7.37)$$

Displacements or distortions in the direction of the spatially fixed coordinate axes (x, y, z) associated with the effect of an acting force are represented by the point mass coordinates (ξ_1, ξ_2, ξ_3) . At every point in the elastic body

$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}(\mathbf{x}_0, t)$$

and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ describes the local displacement caused around \mathbf{x}_0 where \mathbf{x}_0 refers to the respective coordinate in the elastic body if no forces were present. In order to determine the resultant force in a situation in which the forces acting on an elastic body are not in equilibrium, one considers what happens to the cuboid sketched in fig. 7.12 (also compare with fig. 7.7). Here, the forces acting on the cuboid shall only lead to a non-equilibrium of the stress and shear components in the x -direction, while the stress and shear components in the y - and z -directions shall be unaffected and shall continue to be in balance. For the resulting force F_x in the x -direction one reads from fig. 7.12

$$F_x = d\sigma_x dy dz + d\tau_{yx} dx dz + d\tau_{zx} dx dy + f_x dx dy dz \quad (7.38)$$

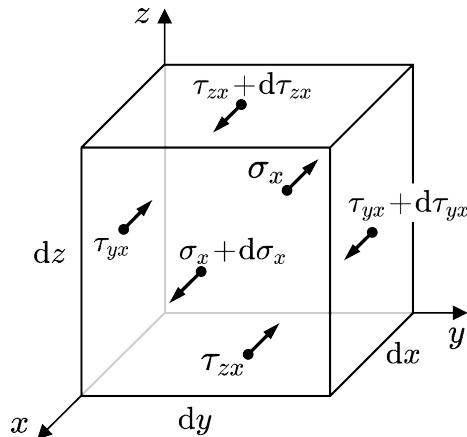


Fig. 7.12

In addition to the components of the stress tensor that act on the respective cuboid surfaces, the volume forces we already known from eq. (7.15) must also be taken into account. This happens in eq. (7.38) through the factor $f_x dx dy dz$, where f_x is the force density of the elastic body in x -direction. To begin with, the case of force equilibrium shall be considered once more.

The Equilibrium Case

In the case of equilibrium, the forces that are transferred according to eq. (7.38) from the volume element $dV = dx dy dz$ in x -direction disappear, i.e., $F_x = 0$. With $\tau_{xy} = \tau_{yx}$, it hence follows from eq. (7.38)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \quad (7.39)$$

With x, y, z fixed in space and the point mass coordinates (ξ_1, ξ_2, ξ_3) , this can be written in a generalized way as

$$\sum_k \frac{\partial \sigma_{ik}}{\partial \xi_k} + f_i = 0 \quad \text{with } i = 1, 2, 3 \quad (7.40)$$

For the coordinates $\xi_i = \xi_i(x, y, z, t)$ applies and for the volume forces $f_i = f_i(x, y, z, t)$. With the discussion of Hooke's law in section 7.3, the components of the stress tensor and of the strain tensor are with eq. (7.34) linked by the relationship

$$\sigma_{ik} = 2G \left[\epsilon_{ik} + \frac{\mu}{1-2\mu} \delta_{ik} \sum_{l=1}^3 \epsilon_{ll} \right] \quad (7.41)$$

where for the strain tensor with eq. (7.21) applies

$$\epsilon_{ik} = \frac{1}{2} \left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i} \right) \quad (7.42)$$

The derivatives of the components of the stress tensor in eq. (7.40) are determined using eq. (7.41) and eq. (7.42) as well as the following relationships

$$\begin{aligned} \sum_{l=1}^3 \epsilon_{ll} &= \sum_{l=1}^3 \frac{\partial \xi_l}{\partial x_l} = \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = \operatorname{div} \boldsymbol{\xi} \\ \sum_k \frac{\partial \epsilon_{ik}}{\partial x_k} &= \frac{1}{2} \left(\sum_k \frac{\partial^2 \xi_i}{\partial x_k^2} + \frac{\partial}{\partial x_i} \operatorname{div} \boldsymbol{\xi} \right) = \frac{1}{2} \left(\Delta \xi_i + \frac{\partial}{\partial x_i} \operatorname{div} \boldsymbol{\xi} \right) \\ \sum_{k=1}^3 \frac{\partial}{\partial x_k} \delta_{ik} (\operatorname{div} \boldsymbol{\xi}) &= \frac{\partial}{\partial x_i} (\operatorname{div} \boldsymbol{\xi}) \end{aligned}$$

Inserted into eq. (7.40) one obtains

$$G \left[\Delta \xi_i + \frac{1}{1-2\mu} \frac{\partial}{\partial x_i} (\operatorname{div} \boldsymbol{\xi}) \right] + f_i = 0 \quad \text{with } i = 1, 2, 3 \quad (7.43)$$

It is easy to see that

$$\operatorname{div} \boldsymbol{\xi} = \frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_3}{\partial x_3} = \epsilon_V$$

describes the (negative or positive) volume expansion of the elastic body. Insertion into eq. (7.43) gives the basic equations of the linear theory of elasticity

$$\Delta \xi_i + \frac{1}{1-2\mu} \frac{\partial \epsilon_V}{\partial x_i} + \frac{1}{G} f_i = 0 \quad \text{with } i = 1, 2, 3 \quad (7.44)$$

As the name already suggests, these three equations are linear in the displacement coordinates of the point masses, but they are also linear in the volume forces f_i . Consequently, the principle of superposition applies both to the displacements and to the volume forces. This is of course of great advantage for complicated problems if it is possible to represent them as linear superpositions of problems which are easier to solve. The next step is to consider the case in which there is no equilibrium of forces.

The Non-Equilibrium Case

In the above example (fig. 7.12) this means $F_x \neq 0$ in eq. (7.38). In general, there will not just be a single force in one coordinate direction but forces acting in different coordinate directions. If F_i with $i = 1, 2, 3$ denotes the forces in the three orthogonal coordinate directions then for each of those $F_i \neq 0$ Newton's law must apply. This means that the accelerated mass times the acceleration vector associated with a strain or a compression component for a given coordinate direction must equate the respective force in that coordinate direction. That means it must hold that

$$F_i = \rho dx dy dz \frac{\partial^2 \xi_i}{\partial t^2} \quad (7.45)$$

Analogous to the equilibrium case in eq. (7.40) with $F_i = 0$ ($i = 1, 2, 3$) this results in the equations:

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} = f_i + \sum_k \frac{\partial \sigma_{ik}}{\partial x_k} \quad \text{with } i = 1, 2, 3 \quad (7.46)$$

Hence, instead of the equation eq. (7.43) one obtains

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} = f_i + G \left[\Delta \xi_i + \frac{1}{1-2\mu} \frac{\partial}{\partial x_i} (\operatorname{div} \boldsymbol{\xi}) \right] \quad \text{with } i = 1, 2, 3 \quad (7.47)$$

Eq. (7.47) is the equation of motion for a differential volume element in an elastic body on which the volume forces f_i act. In the case of a fluid this equation becomes much simpler as the off-diagonal elements of $\underline{\underline{\sigma}}$ vanish.

8. The Motion of Fluids

Fluids can be incompressible liquids or compressible gases whose flow processes one can approach in two ways, by using the method of Lagrange or Euler's method. The Lagrange method, named after Joseph-Louis Lagrange (1736 - 1813), describes the trajectory curves of point masses by their position vectors, i.e., $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$ where \mathbf{r}_0 is the known position vector of a point mass at time t_0 . In the Lagrange description each point mass in the fluid carries its own properties and velocities and accelerations of point masses are obtained as usual by taking the time derivative once or twice, respectively. In the case of flow processes, however, it is often not so much of interest where a point mass is located, but what the velocity field looks like in which the point mass is moving. This is what the method of Euler provides with

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \quad ; \quad \rho = \rho(\mathbf{x}, t)$$

where \mathbf{v} is the velocity field, ρ is the local density and \mathbf{x} is a space-fixed coordinate. In the Euler description, physical quantities $a(\mathbf{x}, t)$, such as $\rho(\mathbf{x}, t)$, are each considered at a fixed point in space. The respective point in space is:

$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}(\mathbf{x}_0, t)$$

where the \mathbf{x}_0 are Lagrangian coordinates (as in the elastic equations). A point mass that moves along a trajectory curve \mathbf{x} is at every point in space both under the influence of the physical quantity $a(\mathbf{x}(\mathbf{x}_0, t), t)$ at the respective point in space, i.e., it registers the local strength or weakness of $a(\mathbf{x}(\mathbf{x}_0, t), t)$, but also experiences the rate of change of $a(\mathbf{x}(\mathbf{x}_0, t), t)$ at the respective location. The rate of change of a physical quantity $a(\mathbf{x}(\mathbf{x}_0, t), t)$ is therefore given by the total derivative with respect to time, often also called the substantial derivative. The term substantial derivative has a purely physical meaning. It expresses the fact that the substantial derivative describes the change in a physical quantity along the motion of the substance itself, i.e., the moving liquid or gas. The substantial derivative of a physical quantity $a(\mathbf{x}_0, t), t)$ is

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \underbrace{\frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial a}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial a}{\partial z} \cdot \frac{\partial z}{\partial t}}_{\substack{\text{local} \\ \text{part}}} + \underbrace{\frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial a}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial a}{\partial z} \cdot \frac{\partial z}{\partial t}}_{\substack{\text{convective part} \\ (\text{with fixed } \mathbf{x}_0)}}$$

This means that for every $a(\mathbf{x}, t)$ applies the relationship

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \mathbf{v} \cdot \nabla a = \frac{\partial a}{\partial t} + \mathbf{v} \cdot \text{grad } a \quad (8.1)$$

The substantial derivative of a physical quantity $a(\mathbf{x}_0, t)$ consists of two parts, the local change of this physical quantity at a given point in space and a so-called convective part of $a(\mathbf{x}_0, t)$. The convective part can be imagined as a location fixed set of markers distributed all over the fluid body, which at each point in space \mathbf{x}_0 indicate the magnitude of $a(\mathbf{x}_0, t)$. Sometimes a different notation of the substantive derivative is useful which makes use of the relationship

$$(\mathbf{a} \cdot \nabla) \mathbf{a} = \frac{1}{2} \nabla \mathbf{a}^2 - [\mathbf{a} \times (\nabla \times \mathbf{a})]$$

With that the substantial derivative becomes

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \mathbf{v} \cdot \nabla a = \frac{\partial a}{\partial t} + \frac{1}{2} \nabla \mathbf{a}^2 - [\mathbf{a} \times (\nabla \times \mathbf{a})] \quad (8.2)$$

8.1 The Hydrostatic Pressure Equation

Some problems in fluid motion can be solved within the confines of hydrostatics. Of particular interest there are the pressure forces exerted by liquids on vessel walls of various containers. To understand hydrostatic pressure, one considers a small deformable volume of a liquid body (fig. 8.1). In that case, one only must deal with:

- Normal stresses, no shear stresses; pressure varies only in the z -direction.
- The weight force.

For there to be a balance of forces, the pressure forces must vary with height. Mass and force are given by

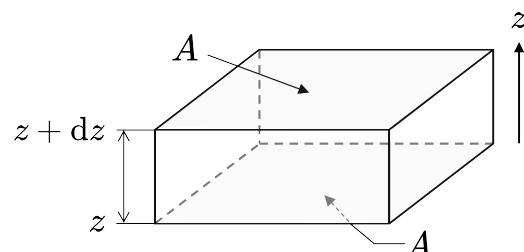


Fig. 8.1

$$dm = \rho \cdot A \cdot dz \quad (\text{mass}) \quad \text{and} \quad dF = dm \cdot g \quad (\text{force})$$

where ρ is the density of the liquid body. The force arises from the weight pressure and the equilibrium condition is

$$P(z + dz) \cdot A - P(z) \cdot A = -g \cdot \rho \cdot A \cdot dz$$

Since dz is very small, it is sufficient to consider only the first derivative in the Taylor expansion of $P(z + dz)$, i.e., $P(z + dz) \approx P(z) + P'(z) \cdot dz$. With that applies

$$\left(P(z) + \frac{\partial P}{\partial z} dz \right) \cdot A - P(z) \cdot A = -g \cdot \rho \cdot A \cdot dz$$

which means that

$$\frac{\partial P(z)}{\partial z} = -g \cdot \rho$$

Generally expressed with the force density $f(z)$ it follows that

$$\frac{\partial P(z)}{\partial z} = -g \cdot \rho = f(z) \quad (8.3)$$

or respectively in three dimensions

$$\nabla P(\mathbf{r}) = \mathbf{f}(\mathbf{r}) \quad (8.4)$$

For incompressible liquids $\rho = \rho_0$ is constant and therefore it holds:

$$P = P_0 - \rho \cdot g \cdot z \quad (8.5)$$

Eq. (8.4) is the basic equation of hydrostatics and eq. (8.5) is the so-called hydrostatic pressure equation. In addition to the graphic derivation just given, eq. (8.4) follows quite simply from eq. (7.40) because for a fluid body in equilibrium the shear stresses vanish, and only compressive stresses exist. Hence, for the stress tensor in eq. (7.40) it follows that $\sigma_{ij} = -P\delta_{ij}$ and thus eq. (7.40) becomes

$$\mathbf{f} = - \sum_k \frac{\partial \sigma_{ik}}{\partial x_k} = \sum_k \frac{\partial P}{\partial x_k} \delta_{ik} = \nabla P(\mathbf{r}) \quad (8.6)$$

For conservative forces like gravity, the force is the negative gradient of the potential. With the gravitational potential $U(z) = gz$ therefore applies (constant density ρ)

$$\mathbf{f} = -\rho \nabla U = -\rho \nabla(gz) \quad (8.7)$$

The hydrostatic pressure equation eq. (8.5) follows then directly from eq. (8.6) and eq. (8.7). In some cases, this approach can also be used to find solutions for non-static situations. One such case is for example that of a fluid in a centrifuge.

Fluids in a centrifuge

Consider the surface of an incompressible liquid in a centrifuge rotating with the angular velocity ω (fig. 8.2). The centrifugal force $r\omega^2$ acting on a volume element of the liquid at a distance r from the axis of rotation leads to the curved shape of the liquid surface in a rotating container many are familiar with. By choosing a coordinate system that rotates with the centrifuge, this problem can be treated like that of a static liquid on which a potential $U(r, z)$ acts:

$$U(r, z) = \rho g z - \frac{1}{2} \rho r^2 \omega^2 = \rho g \left(z - \frac{\rho r^2 \omega^2}{2g} \right)$$

With that the modified hydrostatic pressure equation becomes

$$P + \rho g \left(z - \frac{\rho r^2 \omega^2}{2g} \right) = \text{const}$$

The surface of the rotating liquid thus has the shape of a parabola

$$z = \frac{\rho r^2 \omega^2}{2g} + \text{const}$$

where the constant is determined by the ambient pressure $P = P_0$.

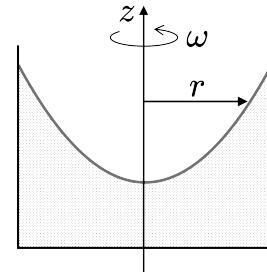


Fig. 8.2

8.2 The Continuity Equation

To derive the continuity equation of fluid motion we will use two different approaches. First we will use the Lagrange description. The conservation of mass tells us that

$$\frac{d\Delta m}{dt} = 0 = \frac{d\rho}{dt} \Delta V + \rho \frac{d\Delta V}{dt} \quad (8.8)$$

By replacing V in eq. (7.22) with the volume element ΔV and ξ_i with $d\xi_i$ one can express the differential of the volume element ΔV as a fraction of ΔV itself:

$$\frac{d\Delta V}{\Delta V} = \text{div}(d\xi)$$

Using this equation to replace $d\Delta V$ in eq. (8.8) then gives

$$\frac{d\rho}{dt} \Delta V + \rho \Delta V \text{div} \left(\frac{d\xi}{dt} \right) = 0$$

and with $d\xi/dt = \mathbf{v}$ follows

$$\frac{d\rho}{dt} + \rho \cdot \text{div } \mathbf{v} = 0$$

Using eq. (8.1) to replace the substantial derivative $d\rho/dt$, this equation becomes

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla \rho + \rho \cdot \nabla \mathbf{v}) = 0$$

giving us the continuity equation which links the rate at which the density ρ of a fluid volume element changes to the net fluid flow in / out of the volume element.

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0 \quad (8.9)$$

An alternative way to derive the continuity equation uses the theorem of Gauß to link the mass change in a volume element to the mass flow in / out of that volume element. As sketched in fig. 8.3, the fluid mass shall be given by

$$m = \int_V \rho dV$$

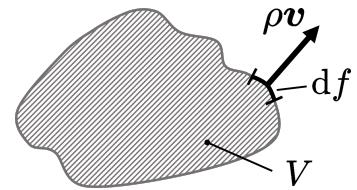


Fig. 8.3

For the mass loss or respectively mass gain of this volume V then must apply

$$\frac{\partial m}{\partial t} = \int_V \frac{\partial \rho}{\partial t} dV \quad (8.10)$$

To calculate the mass loss or respectively the mass gain per unit of time through the area A_V enclosing the volume V , one only requires the normal component of the flow $\rho \mathbf{v}$, i.e., $\mathbf{v} \cdot \hat{\mathbf{n}} = v_n$:

$$\frac{\partial m}{\partial t} - \iint_{A_V} \rho \cdot v_n \cdot df = 0 \quad (8.11)$$

According to the theorem of Gauß, for the surface integral in eq. (8.11) applies

$$\iint_{A_V} \rho \cdot v_n \cdot df = \int_V \operatorname{div}(\rho \mathbf{v}) dV$$

and with that also applies

$$\frac{\partial m}{\partial t} - \int_V \operatorname{div}(\rho \mathbf{v}) dV = 0 \quad (8.12)$$

A comparison of eq. (8.10) and eq. (8.12) shows that it must hold

$$\int_V \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right) dV = 0$$

This can only be true if the integrand vanishes, which means nothing other than that the continuity equation eq. (8.9)

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

must apply. In both derivations, when using the Lagrange approach or when using the theorem of Gauß, this continuity equation is a simple consequence of the conservation of mass. The mass which flows into a volume element in the fluid flow must leave this volume element or else the density of that volume element will change.

8.3 Streamlines and Stream Function

To describe flow processes, hydrodynamics distinguishes between streamlines, pathlines (i.e., trajectory lines or curves), streaklines and timelines.

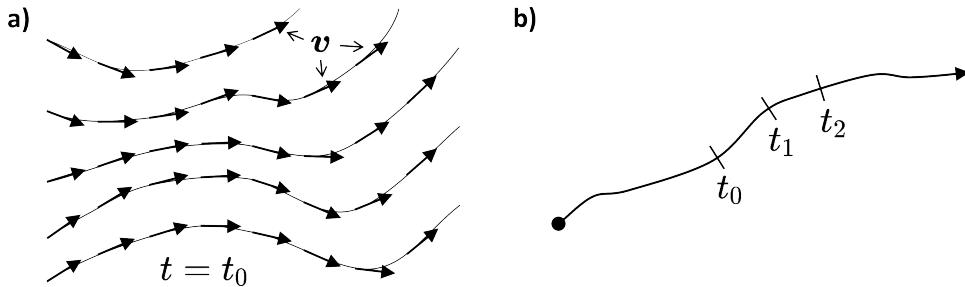


Fig. 8.4: (a) Streamlines provide a snapshot of the velocity field. The length of the \mathbf{v} -vectors at each tangent point (here drawn with the same length) depends on the local flow. (b) A pathline (trajectory) describes the path of a single particle as a function of time.

Pathlines (trajectories): These describe the path of a point mass or respectively of a liquid particle in a flow. Pathlines can easily be made visible by adding, for example, small cork particles to the flowing liquid.

Streamlines: Designate curves whose tangents at each curve point are identical to the respective velocity vectors \mathbf{v} of the flow. If one takes a snapshot of the flowing liquid at a point in time $t = t_0$ then the streamlines give an image of the velocity field of the flow.

Streaklines: Designate curves of point masses or respectively liquid particles that all run through a common point in space. A simple example for streaklines is the deployment of buoys at a fixed location in a liquid flow. If one connects the positions of the successively deployed buoys, one gets the streakline.

Timelines: Denote the connecting line between the current locations of particles whose locations were already known at an earlier point in time. To illustrate timelines, in the

simplest case, for example, light-weight sample particles are placed in the flow at different positions at the same time t_0 . At each later point in time t , the connecting line between the current positions of the sample particles then represents the timeline at the point in time t .

In the present context, the streamlines are of particular interest. Pathlines represent the Lagrange description of a flow (fig. 8.4b) while streamlines correspond to the Euler description of a flow (fig. 8.4a). In the case of a stationary flow, i.e., $\dot{\mathbf{v}} = 0$, the flow velocity is a time invariant (i.e., it is constant) at every location and thus streamline and pathline are identical.

In other words, a flow is always stationary if a sample particle, which is put into the liquid flow at different times but always at the same position of the flow, will move on the same pathline each time. If this is not the case, then one speaks of a so-called non-stationary flow. In the case of a non-stationary flow, it is no longer possible to determine from a given pathline the flow velocity present at a specific point in time. That, however, means nothing more than that for non-stationary flows, streamline and pathline are no longer identical.

In mathematical terms, the streamline definition given above states that the change in the position vector of the streamline curve at any point on the curve is always parallel to the velocity vector of the flow at that point. Hence, in every point of the streamline the vector product of the velocity field \mathbf{v} and the infinitesimal change $d\mathbf{r}$ of the position vector \mathbf{r} of the streamline curve must vanish:

$$\mathbf{v} \times d\mathbf{r} = 0 \quad (8.13)$$

To understand what, in the case of incompressible liquids and planar stationary flows, distinguishes two adjacent streamlines from one another, one considers the so-called stream function Ψ , also referred to as Stokes' stream function. If, for example, the coordinate system is chosen in such a way that the unit vector $\hat{\mathbf{e}}_z$ in z -direction is at the same time the normal vector of the plane through which the flow occurs (flow in the xy -plane), then Ψ is defined by the condition

$$\mathbf{v} = \nabla \times \Psi \hat{\mathbf{e}}_z = \nabla \Psi \times \hat{\mathbf{e}}_z \quad (8.14)$$

$$= \left(\frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial z} \right) \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}, 0 \right) = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix}$$

Hence, the velocity field \mathbf{v} can be calculated from the stream function Ψ . It immediately follows from the definition of Ψ that

$$\nabla\Psi \cdot \mathbf{v} = \nabla\Psi \cdot (\nabla\Psi \times \hat{\mathbf{e}}_z) = 0 \quad (8.15)$$

In other words: Along lines whose tangential vectors are parallel to the respective velocity vectors of the velocity field at every point, the stream function is always constant. This also follows directly by inserting eq. (8.14) into the definition of the streamline given in eq. (8.13). For a flow in the xy - plane with $d\mathbf{r} = (dx, dy, 0)$ this results in (Graßmann identity)

$$\mathbf{v} \times d\mathbf{r} = 0 = (\nabla\Psi \times \hat{\mathbf{e}}_z) \times d\mathbf{r} = (\nabla\Psi \cdot d\mathbf{r}) \cdot \hat{\mathbf{e}}_z - (\underbrace{\hat{\mathbf{e}}_z \cdot d\mathbf{r}}_{=0}) \cdot \nabla\Psi$$

Therefore, to streamlines must apply

$$\nabla\Psi \cdot d\mathbf{r} = \frac{\partial\Psi}{\partial x} dx + \frac{\partial\Psi}{\partial y} dy = d\Psi = 0$$

The stream function is constant along a streamline and streamlines are thus the contour lines (equipotential lines) of the stream function. That means of course that each streamline corresponds to a specific value of the stream function Ψ . For a better understanding of Ψ , the so-called specific discharge q shall be considered in the two-dimensional case (planar flow).

The specific discharge q is defined as the “liquid area” which per unit of time flows through the normal area in fig. 8.5 between the two streamlines $\Psi = \Psi_A$ and $\Psi = \Psi_B$. Hence

$$q = \int_A^B \mathbf{v} \cdot \hat{\mathbf{n}} ds$$

where ds is the infinitesimal line element along the connecting segment from A to B and $\hat{\mathbf{n}}$ is the normal

unit vector on this line element. In fig. 8.5, for the sake of simplicity, this connecting line is chosen as a straight line, but it can have any arbitrary shape. From fig. 8.5 one can read that

$$\hat{\mathbf{n}} ds = (dy, -dx)$$

Using eq. (8.14), this gives for q

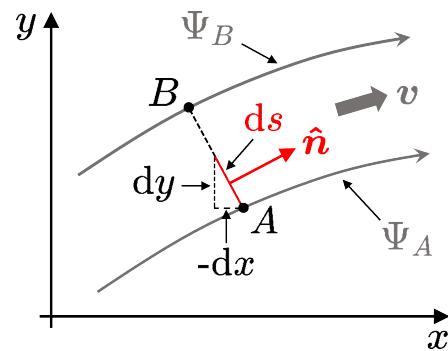


Fig. 8.5

$$q = \int_A^B (v_x dy - v_y dx) = \int_A^B d\Psi = \Psi_B - \Psi_A \quad (8.16)$$

With $\Psi_B = const$ and $\Psi_A = const$, the difference q is also constant. For incompressible liquids in a stationary flow the specific discharge is therefore constant. From that follows: If the distance between the streamlines $\Psi = \Psi_A$ and $\Psi = \Psi_B$ decreases / increases, then the flow velocity between the two streamlines increases / decreases. Another consequence of eq. (8.14) is

$$\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla \times \Psi \hat{\mathbf{e}}_z) = 0 \quad (8.17)$$

From this it follows that the liquid flow contains neither sources nor sinks. This means that streamlines can neither begin nor end inside the liquid under consideration but must be closed curves or run along the edge of the flow. Furthermore, it can be read from eq. (8.16) that the specific discharge vanishes through an area whose boundary consists of a streamline and a connecting line that begins and ends on the streamline ($\Psi_B = \Psi_A$). This applies in particular to a “connecting line” that runs along the streamline itself. That means, the specific discharge through a streamline is always zero. Streamlines act like impenetrable walls, which means that since there is no flow through a streamline itself, the flow rate between any two streamlines must vary with the distance between those streamlines.

8.4 Bernoulli's Equation

Lets consider the stationary flow of a fluid through a pipe as illustrated in fig. 8.6. The fluid mass flow at any position of this pipe is given by

$$\frac{dm}{dt} = \rho \cdot \frac{dV}{dt} = \rho \cdot A \cdot \frac{ds}{dt} = \rho \cdot A \cdot v$$

where ρ is the density, dV a volume element, A the tube cross section and ds the differential line element running though the center of each cross section along the x -axis. Therefore constant mass flow through the pipe in fig. 8.6 means that

$$\rho_0 v_0 A_0 = \rho_1 v_1 A_1 \quad (8.18)$$

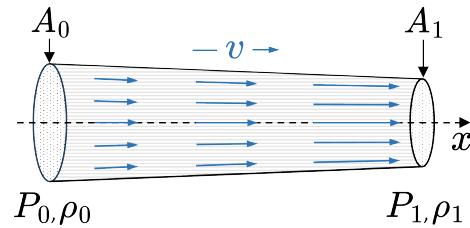


Fig. 8.6

Eq. (8.18) is of course nothing else but a simple restatement of the continuity equation discussed in section 8.2. For incompressible fluids with $\rho = \text{const}$, eq. (8.18) reduces to $v_0 A_0 = v_1 A_1$ and the streaming velocity of the fluids increases / decreases if the cross section of the tube gets narrower / wider. Eq. (8.18) also applies to gas flows but because gases are compressible, ρ is not constant and a change in pipe diameter does not only translate into a change in streaming velocity but can give rise to a density change. With the mass flow increasing as the pipe in fig. 8.6 narrows, pressure increases as well. That means the pressure on a fluid volume element in the pipe is higher on the side of the volume element which faces the narrower cross section of the pipe and lower on the side facing the wider cross section of the pipe. The force on each side of such a volume element is given by the respective pressure times the surface area, i.e., $F(x) = P(x) \cdot A$. With that the net force across a volume element becomes (compare eq. (7.13))

$$F_x = F(x + dx) - F(x) = -\frac{\partial P}{\partial x} \cdot dx \cdot A = -\frac{\partial P}{\partial x} \cdot dV \quad (8.19)$$

The minus sign arises because the force must be directed towards where the pressure decreases, i.e., where $\partial P/\partial x < 0$. Now that we know the force acting on a fluid volume element we can, with $dm = \rho dV$, write down its equation of motion as

$$\rho dV \frac{dv}{dx} = -\frac{\partial P}{\partial x} \cdot dV$$

which becomes

$$\rho \frac{dv}{dx} + \frac{\partial P}{\partial x} = 0 \quad (8.20)$$

What eq. (8.20) tells us is that any pressure change in a fluid is immediately transmitted everywhere through the fluid, a fact which is known as Pascals law, named after Blaise Pascal (1623 - 1662). The force in eq. (8.19) moves the fluid volume through the pipe and we know that this costs energy. With eq. (4.1) we know that the work expended in moving the fluid element from a point 1 to a point 2 along the positive x -axis through the pipe shown in fig. 8.6 must be

$$W = \int_1^2 F_x dx = -dV \int_1^2 \frac{\partial P}{\partial x} dx = (P_1 - P_2)dV$$

This work must equate the increase in kinetic energy of the fluid volume element which means that

$$\rho dV \left(\frac{v_2^2}{2} - \frac{v_1^2}{2} \right) = (P_1 - P_2)dV$$

from which follows

$$\frac{1}{2}\rho v_1^2 + P_1 = \frac{1}{2}\rho v_2^2 + P_2 \quad (8.21)$$

Given the situation depicted in fig. 8.6 where the pipe is orientated horizontally the potential energy of a fluid volume flowing from left to right does not change as its height above ground remains unchanged. However, often that is not the case as pipes also go vertically up and down. In that case we must include the respective potential energy change in eq. (8.21) which then gives us the common form of Bernoulli's equation:

$$\frac{1}{2}\rho v_1^2 + p_1 + \rho g z_1 = \frac{1}{2}\rho v_2^2 + p_2 + \rho g z_2 \quad (8.22)$$

or more generally

$$\frac{1}{2}\rho v^2 + p + \rho g z = \text{const} \quad (8.23)$$

Bernoulli's equation tells us that in a stationary fluid flow a change in pressure is always tied to a change in flow velocity and vice versa. A drop in pressure corresponds to an increase in flow velocity and an increase in pressure to a decrease in flow velocity.

The Bernoulli Paradox

A simple experimental demonstration of Bernoulli's equation is the so-called Bernoulli paradox sketched in fig. 8.7. At the end of a tube through which a gas or fluid stream is piped sits a disk from which a second disk is suspended such that it is free to move up or down in the z -direction. However, unlike the upper disk the lower disk has no middle outlet for the gas or fluid stream coming from the tube. Hence, the gas or fluid stream coming from the tube will be diverted sideways escaping through the slit opening between the disks. Now, most would likely venture that the result of this will be that the gas or fluid stream will push the lower disk away from the upper disk but just the opposite happens (and that is why this experiment is called the Bernoulli paradox). Of course, if a gas or fluid stream is piped in from the top of the tube, the air or fluid will escape between the two disks. However, with P_0 being the ambient pressure Bernoulli's equation tells us that the pressure between the two disks is given by

$$P_s = P_0 - \frac{\rho}{2}v^2$$

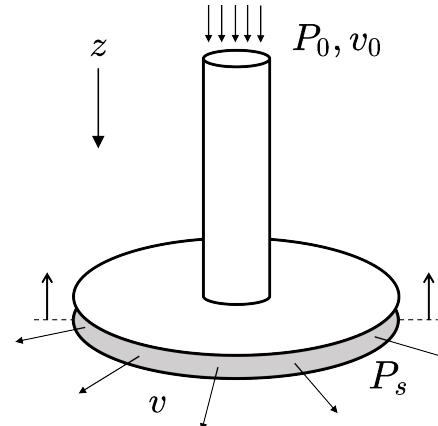


Fig. 8.7

where P_S is the pressure between the two disks and v is the velocity at which the air or fluid escapes from between the disks. Since P_0 is constant, this means that the pressure between the two disks decreases and the resulting negative pressure therefore pulls the lower disk upwards. If A is the area of the lower disk, then it levitates if

$$P_S \cdot A > m \cdot g$$

The Bernoulli paradox works with a gas stream in atmosphere and with a liquid stream where the apparatus is completely immersed in the liquid. Bernoulli's equation has many practical applications, one of them being in the field of aviation.

The Kutta-Joukowski Equation

Fig. 8.8 shows a simple sketch of the profile of an airplane wing and of the airflow around it. Far in front of the wing the air velocity (relative to the traveling plane) is uniform in x -direction and the same is the case for the air far behind the airplane wing. But the airflow around an airplane wing is quite different as air gets deflected by the wing in the positive and a little less in the negative z -direction. As a consequence, an air circulation forms around the wing profile which leaves the wing at the back of the wing profile. Usually this wing tip is shaped specifically to minimize vortices developing there when the air circulation reaches the end of the wing profile. The net result of all of this is that the air above the wing flows faster than below the wing. With the air density above and below the wing being the same this leads to a net lift force which is given by Bernoulli's equation which applies to the air flow above and below the wing to give a pressure differential:

$$P_2 - P_1 = \frac{1}{2} \rho v_1^2 - \frac{1}{2} \rho v_2^2$$

With the surface area under the wing profile being A the net lift force produced by this pressure difference is

$$F_z = A \cdot (P_2 - P_1) = A \cdot \rho \cdot u \cdot (v_1 - v_2) \quad \text{with} \quad u = \frac{1}{2}(v_1 + v_2)$$

where u is the velocity at which the aircraft travels. If the length of the wing is L and its profile width is B then $A = L \cdot B$ and the force lifting the airplane wing becomes

$$F_z = \rho \cdot u \cdot L \cdot B(v_1 - v_2) \approx \rho \cdot u \cdot L \cdot \Gamma \tag{8.24}$$

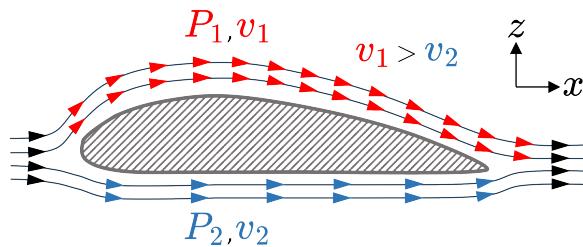


Fig. 8.8

where $\Gamma \approx B(v_1 - v_2)$ is the air circulation powering the lift of the wing. Eq. (8.24) is the so-called Kutta-Joukowski, the fundamental aerodynamic relationship describing the force enabling airplanes to fly. As a final example illustrating Bernoulli's equation we look at the so-called so-called Venturi tube.

The Venturi Tube

The Venturi tube, an example of which is sketched in fig. 8.9, is named after its discoverer Giovanni Battista Venturi (1746 - 1822). The dynamic pressure P_0 is at a maximum at the narrowest point of the Venturi tube and the hydrostatic pressure there is minimal. This leads to a negative pressure in the narrower tube where $P < P_0$ and $v > v_0$; there the air flows with higher velocity. The respective pressure difference is indicated by the different levels of liquid in the two arms of the U-tube. One can use this, for example, to measure the flow velocities of gases or liquids.

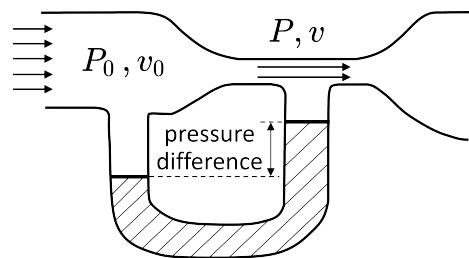


Fig. 8.9

8.5 Momentum Balance

An important concept of hydrodynamics is the flux tube, also known as streamline bundle (fig. 8.10). Here one first considers a so-called streamline surface, which is the surface that is spanned by the sum of all streamlines that go through a stationary line. If this stationary line happens to be a closed curve, then the respective streamline surface forms the mantle of a tube, i.e., of the flux tube. The surface normal vector of the entry or respectively exit area of the flux tube is

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{v}}{|\mathbf{v}|}$$

In a stationary flow, the position and shape of a flux tube does not change with time and therefore the continuity condition of eq. (8.18) applies:

$$\rho_0 \cdot v_0 \cdot A_0 = \rho \cdot v \cdot A$$

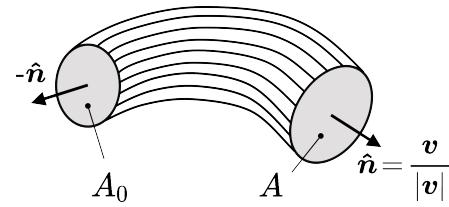


Fig. 8.10

Hence, the same mass must flow through each cross-sectional area of such a flux tube per unit of time. If the course of the flux tube is parameterized in such a way that the line

element ds runs through the centroid of each cross-sectional area of the flux tube, then the continuity condition becomes

$$\rho_0 \cdot v \cdot A = \rho \cdot A \cdot \frac{ds}{dt} = \rho \cdot \frac{dV}{dt} = \frac{dm}{dt} = \text{const}$$

To derive the continuity condition in the non-stationary flow case, one uses the conservation of mass and Newton's second law for incremental changes Δm and $\Delta \mathbf{p}$ of mass and momentum. It must apply:

$$\frac{d(\Delta m)}{dt} = 0 \quad (\text{mass})$$

$$\frac{d(\Delta \mathbf{p})}{dt} = \Delta \mathbf{F} \quad (\text{momentum})$$

Therefore, with eq. (7.45) and eq. (7.46), it applies component-wise

$$\begin{aligned} \frac{d(\Delta p_i)}{dt} &= \Delta F_i = f_i \Delta V + \left(\sum_{k=1}^3 \frac{\partial \sigma_{ik}}{\partial x_k} \right) \Delta V \\ &= \frac{d(\Delta m)}{dt} v_i + \Delta m \frac{dv_i}{dt} \end{aligned}$$

It follows with $d(\Delta m)/dt = 0$ and $\rho = \Delta m / \Delta V$

$$\rho \frac{dv_i}{dt} = f_i + \sum_{k=1}^3 \frac{\partial \sigma_{ik}}{\partial x_k} \quad (8.25)$$

But with eq. (8.1) it must also apply

$$\rho \frac{dv_i}{dt} = \rho \left(\frac{\partial v_i}{\partial t} + \mathbf{v} \cdot \nabla v_i \right) \quad (8.26)$$

Since the left sides of eq. (8.25) and eq. (8.26) are identical, the right sides must also be identical, i.e.,

$$f_i + \sum_{k=1}^3 \frac{\partial \sigma_{ik}}{\partial x_k} = \rho \frac{\partial v_i}{\partial t} + \rho \mathbf{v} \cdot \nabla v_i \quad (8.27)$$

Now one rewrites the continuity equation, i.e., eq. (8.9), to read

$$0 = \frac{\partial \rho}{\partial t} + \sum_{k=1}^3 \frac{\partial(\rho v_k)}{\partial x_k}$$

If one multiplies this equation by v_i and then adds the right side of it as a zero to the right side of eq. (8.27), then one obtains

$$f_i + \sum_{k=1}^3 \frac{\partial \sigma_{ik}}{\partial x_k} = \frac{\partial(\rho v_i)}{\partial t} + \sum_{k=1}^3 \frac{\partial(\rho v_i v_k)}{\partial x_k}$$

or respectively rewritten

$$f_i = \frac{\partial(\rho v_i)}{\partial t} + \sum_{k=1}^3 \frac{\partial}{\partial x_k} \underbrace{(\rho v_i v_k - \sigma_{ik})}_{\begin{array}{l} \text{momentum} \\ \text{density} \end{array}} \quad (8.28)$$

↓

momentum flux density = Π_{ik}

8.5.1 Microscopic Examination of Fluid Pressure

$$\sigma_{ik} = -P\delta_{ik} \quad ; \quad \sigma_{ik} = -\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

Assertion: For the momentum flux density applies

$$\Pi_{ik} = \rho \langle (v_i + u_i)(v_k + u_k) \rangle$$

Here the brackets around $\langle a \rangle$ denote taking the mean value of the physical quantity a averaged over the thermal motion of an ensemble (like the atoms or molecules of a gas) of which a is a physical measure. This thermal motion u_i is an additional thermal velocity for which applies $\langle u_i \rangle = 0$. Furthermore, if $i \neq k$, then

$$\langle u_i u_k \rangle = \langle u_i \rangle \langle u_k \rangle \quad \text{and therefore} \quad \langle u_i u_k \rangle = 0 \quad \text{for } i \neq k$$

In addition, the equipartition theorem applies (see section 9.1)

$$\frac{m}{2} \langle u_i^2 \rangle = \frac{k_B T}{2}$$

where k_B is the Boltzmann constant. This means that for Π_{ik}

$$\Pi_{ik} = \rho v_i v_k + \rho \langle u_i u_k \rangle = \rho v_i v_k + \rho \frac{k_B T}{m} \delta_{ik}$$

From the equation of state for the ideal gas (see eq. (9.1)) one obtains

$$P \cdot V = R \cdot T \quad ; \quad k_B = \frac{R}{N_A} \quad \rightarrow \quad P = \frac{N_A m}{V m} k_B T = \frac{\rho}{m} k_B T$$

and hence

$$\Pi_{ik} = \rho v_i v_k + P \delta_{ik} \quad (8.29)$$

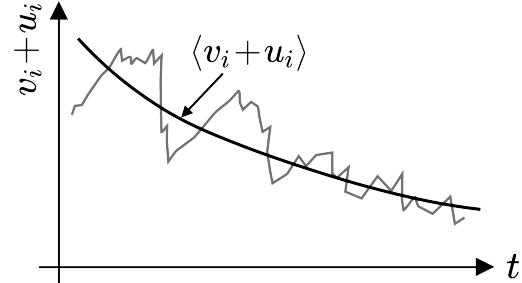


Fig. 8.11

8.5.2 Forces Acting on a Pipe

The stationary case is considered, i.e., in eq. (8.26)

$$\frac{\partial \rho v_i}{\partial t} = \frac{\partial \rho}{\partial t} = 0$$

and $f_i = 0$. With that must hold

$$\sum_{k=1}^3 \frac{\partial \Pi_{ik}}{\partial x_k} = 0 \quad (*)$$

The force on the cylindrical pipe wall is B_Z

$$\mathbf{F} = \iint_{B_Z} \underline{\underline{\sigma}} \hat{\mathbf{n}} dA$$

where $\hat{\mathbf{n}}$ is the normal unit vector of the pipe wall and dA is the corresponding differential surface element. For an indirect calculation of \mathbf{F} one considers the momentum flux density for the pipe volume. With the theorem of Gauß applies in general

$$\iiint \operatorname{div}_k \Pi_{i(k)} dV = \oint \sum_{k=1}^3 \Pi_{i(k)} n_k dA$$

However, according to the presumption in $(*)$

$$\iiint \sum_{k=1}^3 \frac{\partial \Pi_{ik}}{\partial x_k} dV = 0 = \iiint \operatorname{div}_k \Pi_{i(k)} dV$$

and with that applies

$$\oint \sum_{k=1}^3 \Pi_{i(k)} n_k dA = 0$$

For the total force on the walls of the considered pipe volume, i.e., including the forces on the two cross-sectional areas that limit the volume in the pipe at its two respective ends, one has

$$\iint_{B_Z} + \iint_{B_{A_0}} + \iint_{B_{A_1}} = \mathbf{F} + \iint_{B_{A_0}} + \iint_{B_{A_1}} = \oint \sum_{k=1}^3 \Pi_{i(k)} n_k dA = 0$$

From this it follows for the force impact on the pipe in fig. 8.12

$$\mathbf{F}_i = (\rho v_i v + P)_1 \cdot A_1 - (\rho v_i v + P)_0 \cdot A_0 \quad (8.30)$$

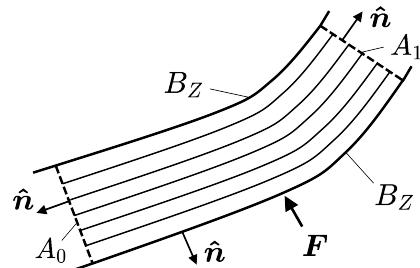


Fig. 8.12

8.5.3 The Euler Equation of Hydrodynamics

From the relationships

$$\sigma_{ik} = -P\delta_{ik} \quad \text{and} \quad \sum_k \frac{\partial \sigma_{ik}}{\partial x_k} = -\frac{\partial P}{\partial x_i} = -(\text{grad } P)_i$$

follows with eq. (8.26) the Euler equation of hydrodynamics in the form

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{f} - \text{grad } P = \mathbf{f} - \nabla P \quad (8.31)$$

or respectively rewritten also in the forms

$$\rho \frac{d\mathbf{v}}{dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \nabla P \quad (8.32)$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\rho}{2} \nabla \mathbf{v}^2 - \rho [\mathbf{v} \times (\nabla \times \mathbf{v})] = \mathbf{f} - \nabla P \quad (8.33)$$

As can be seen from the derivation of the Euler equation, frictional forces, which often play a significant role in practical cases, are not taken into account. The Euler equation only considers the flow of ideal incompressible liquids. However, without viscosity and without thermal conductivity, there are no energy losses in ideal liquids. The motion of an ideal liquid is therefore everywhere adiabatic and with that any change in state of such a liquid takes place without heat transfer.

8.6 Viscous Liquids

As discussed in section 7.1.3, a viscous liquid, unlike an ideal fluid, offers frictional resistance to any change in its shape. The viscosity of a fluid is a measure of how much it resists such change. With eq. (7.46) applies

$$\rho \frac{dv_i}{dt} = f_i + \sum_{k=1}^3 \frac{\partial \sigma_{ik}}{\partial x_k} ; \quad \sigma_{ik} = -P\delta_{ik}$$

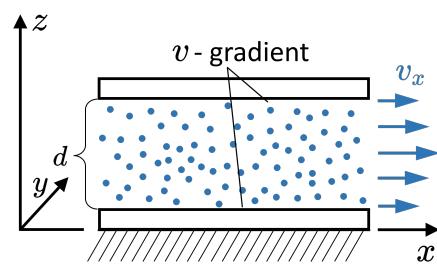


Fig. 8.13

Because of frictional resistance, a velocity gradient exists in a pipe (fig. 8.13) which is perpendicular to the flow direction. The flow velocity v_x is lower at the pipe wall than in the middle. For a given v_x , the velocity gradient is the larger, the smaller the diameter d of the tube.

$$\frac{\partial v_x}{\partial z} = \frac{v_x}{d}$$

With the dynamic viscosity η , also referred to as transverse viscosity (toughness), shear viscosity or first viscosity (see section 7.1.3), applies

$$\sigma_{xz} = \eta \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

The dynamic viscosity η of a substance is the ratio of shear stress σ_{xz} to the spatial velocity gradient $\nabla \mathbf{v}$. The reciprocal of it is referred to as the fluidity of a substance. The spatial velocity gradient describes the spatial rate of change of a fluid's velocity, and it is a second order tensor, also referred to as the velocity gradient tensor. In three dimensions it reads

$$\underline{\underline{\mathbf{l}}} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (8.34)$$

With its help, one can for example reformulate the expression for the substantial derivative of the velocity \mathbf{v} as

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \underline{\underline{\mathbf{l}}} \mathbf{v} \quad (8.35)$$

With eq. (8.2) it also applies to $\underline{\underline{\mathbf{l}}} \mathbf{v}$

$$\underline{\underline{\mathbf{l}}} \mathbf{v} = \frac{1}{2} \nabla \mathbf{v}^2 - [\mathbf{v} \times (\nabla \times \mathbf{v})] \quad (8.36)$$

Including the volume viscosity λ , also referred to as kinematic viscosity (see section 7.1.3), which takes into account the viscosity of fluids when their volume changes, the following applies in general to the stress tensor or respectively the distortion tensor

$$\sigma_{ik} = -P \delta_{ik} + \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + \lambda \delta_{ik} \nabla \cdot \mathbf{v} \quad (8.37)$$

and therefore

$$\rho \frac{dv_i}{dt} = f_i - \frac{\partial P}{\partial x_i} + \eta \Delta v_i + (\eta + \lambda) \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{v}) \quad (8.38)$$

or respectively in vector form

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{f} - \nabla P + \eta \Delta \mathbf{v} + (\eta + \lambda) \nabla (\nabla \cdot \mathbf{v}) \quad (8.39)$$

The transition to the Euler description for a physical vector quantity \mathbf{a} according to eq. (8.1) and insertion into eq. (8.39) finally yields the Navier-Stokes equation

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{f} - \nabla P + \eta \Delta \mathbf{v} + (\eta + \lambda) \nabla (\nabla \cdot \mathbf{v}) \quad (8.40)$$

To better understand what the velocity gradient in a pipe looks like, it is best to consider a specific configuration such as the stationary flow through a pipe as sketched in fig. 8.14. Stationary flow means

$$\frac{\partial \rho}{\partial t} = \frac{d\mathbf{v}}{dt} = 0$$

With the continuity equation (eq. (8.9)), this also means $\nabla \cdot \mathbf{v} = 0$ and thus the fluid under consideration is incompressible. It follows from eq. (8.39) with $\mathbf{f} = 0$ and $\lambda = 0$

$$0 = -\frac{1}{\rho} \nabla P + \frac{\eta}{\rho} \Delta \mathbf{v}$$

For the flow situation sketched in fig. 8.14 only the x -component of the equation is required:

$$\eta \Delta v_x(r) = \frac{\partial P}{\partial x} = \frac{P_l - P_0}{l} \quad (8.41)$$

The pressure gradient in the x -direction is given here by the pressure difference between the pressure $P_l = P(x = l)$ at the end of a piece of pipe of length l and the pressure $P_0 = P(x = 0)$ at the beginning of this piece of pipe divided by the length l of the piece of pipe. The problem of pipe flow is best treated in polar coordinates. Transforming Cartesian coordinates (x, y) into polar coordinates (r, φ) means for the Laplace operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \rightarrow \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \varphi} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi}$$

Eq. (8.41) only depends on r and thus eq. (8.41) becomes in polar coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_x(r)}{\partial r} \right) = \frac{1}{\eta} \frac{P_l - P_0}{l}$$

Integrating this equation once gives

$$r \frac{\partial v_x(r)}{\partial r} = \frac{r^2}{2\eta} \frac{P_l - P_0}{l} + C_1$$

For $r = 0$ the left side vanishes. So, the right-hand side must also disappear and with that follows $C_1 = 0$. Integrating one more time gives

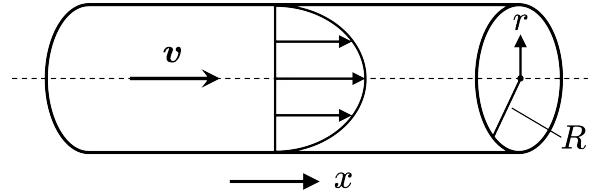


Fig. 8.14

$$v_x(r) = \frac{r^2}{4\eta} \frac{P_l - P_0}{l} + C_2$$

At the edge of the tube $r = R$ must apply, hence

$$v_x(R) = 0 = \frac{R^2}{4\eta} \frac{P_l - P_0}{l} + C_2$$

and therefore the value of C_2 becomes

$$C_2 = -\frac{R^2}{4\eta} \frac{P_l - P_0}{l}$$

With that, the result obtained for the radial dependence of the flow velocity in the pipe from fig. 8.14 is ($\Delta P = P_l - P_0$)

$$v_x(r) = \frac{r^2 - R^2}{4\eta} \cdot \frac{\Delta P}{l} \quad (8.42)$$

With the help of eq. (8.42) one can now calculate the flow rate $dV/dt = \dot{V}$, i.e., the volume of liquid flowing through the pipe with radius R and length L per unit of time, also referred to as the volume flow, by simple integration:

$$\begin{aligned} \dot{V} &= \int_0^R 2\pi r v_x(r) dr = \int_0^R 2\pi r \frac{r^2 - R^2}{4\eta} \cdot \frac{\Delta P}{l} dr \\ &= \frac{\pi}{2} \cdot \frac{\Delta P}{l} \left[\frac{r^2 R^2}{2} - \frac{r^4}{4} \right]_0^R = \frac{\pi}{8\eta} \cdot \frac{\Delta P}{l} \cdot R^4 \end{aligned}$$

The equation

$$\dot{V} = \frac{\pi}{8\eta} \cdot \frac{\Delta P}{l} \cdot R^4 \quad (\text{Hagen-Poiseuille law}) \quad (8.43)$$

named after Gotthilf Heinrich Ludwig Hagen (1797 - 1884) and Jean Léonard Marie Poiseuille (1797 - 1869) shows that the volume flow in a pipe with radius R varies linearly with the pressure difference over the pipe length and with the fourth power of the radius R . Prerequisite for the applicability of the Hagen-Poiseuille law is the existence of a stationary flow. Further constraints are that the boundary layers on the pipe wall can be neglected, i.e., the radius R must be large enough for this to apply, and the pipe must also be long enough for the pressure difference to be stable. The Hagen-Poiseuille law is not only useful for considering pipe flow in the case of liquids, but also for gas flow in pipes as long as the density of the gas is high enough. The latter is the case when the mean free path of the gas particles is much shorter than the radius of the pipe.

9. Thermal Motion

Just like liquids, gases are fluids but they are compressible fluids. In between liquids and gases there exist other states of matter which we describe as vapors or steams. Those are also gases but they are not what physicists call ideal gases. In vapors and steam the molecules and gases interact in other way than just by elastically bouncing of each other. In vapors and steams atoms and molecules condense or evaporate which means they are in the process of bonding with each other or breaking such bonds as they have formed. This is different for ideal gases in which such things do not happen because the gas temperature is so high that the thermal energy of the gas constituents is much higher than the respective bonding energy between two gas atoms or gas molecules would be; therefore, such bonds just cannot form. At a sufficiently high temperature and a sufficiently low density every gas behaves as an ideal gas. That certainly is true for those substances which we usually consider to be gaseous substances. Often it is also possible and useful to describe ensembles of other things as gases, like for example the electron gas but there other rules also kick in, i.e., the rules of quantum mechanics. Such quantum gases can behave quite different from normal gases even though the constituents of gases such as atoms and molecules are also subject to the rules of quantum mechanics. So why can we describe the behavior of gases from a classical perspective in the first place?

The correct answer is that the averaged properties of very large numbers of atoms and molecules which are all subject to quantum mechanics manifest themselves in the classical properties of a gas such as its temperature, pressure, or volume which we can all measure by classical means. Newtonian mechanics provided the understanding as to how forces must be applied so that certain things can happen and thereby it became the foundation of engineering mechanics. However, the machines engineers construct can usually only do the job they were built for if sufficient energy is at hand to drive their motions. Such energy is hard to generate by mechanical means. A simple steam engine, however, can generate plenty of it. Converting the thermal energy of a gas into the mechanical

energy of a moving piston was the practical engineering innovation which enabled the first industrial revolution. Steam is not an ideal gas but the discovery of the ideal gas law in 1834 marks the beginning of what would become kinetic gas theory, thermodynamics and eventually statistical mechanics whose rise mirrored and enabled the rise of the modern gas powered world of the nineteenth and early twenty-first century. The ideal gas law was not derived from theoretical considerations but was an empirically discovered relationships between gas pressure P , gas volume V and gas temperature T :

$$PV = nRT = nk_B N_A T = Nk_B T \quad (9.1)$$

The proportionality constant in the ideal gas law is either given by the product of the number of moles n of a substance and the gas constant R , or by the number of moles n times the Boltzmann constant k_B times the Avogadro number N_A , named after Amedeo Avogadro (1776 - 1856), or by the number of particles N times the Boltzmann constant k_B . This chapter is titled “Thermal Motion” and not “Kinetic Gas Theory”, “Thermodynamics”, or “Statistical Mechanics” as the purpose of this chapter is a more limited one, i.e., to provide a basic understanding as to how the macroscopic properties of a gas arise. That means to disregard, at least for the most, the specific atomic or molecular properties of the gas constituents and to focus on such aspects where Newtonian mechanics is sufficient to show how certain microscopic behaviors of atoms or molecules give rise to the macroscopic behavior of gases as we can observe them. The discussion of the statistical and quantum nature underlying the behavior of gases will have to wait until we have a better understanding of atomic and molecular physics which will be the subject of volume four of this physics course.

9.1 A Microscopic Examination of Gas Pressure

Section 8.5.1 gave a brief examination of the microscopic pressure of a fluid which made use of the equipartition theorem and the ideal gas law, both of which we introduced there without providing any further explanation. In this section we will do just that and to begin with lets look at the situation sketched in fig. 9.1 which shows a small gas volume which can be enlarged or shrunk by moving a piston. Lets assume that the piston can slide freely, i.e., without any friction. In that case, its position will adjust to an equilibrium position where the gas pressure on the piston from inside the gas volume v is equal to the gas pressure exerted on the piston from the gas

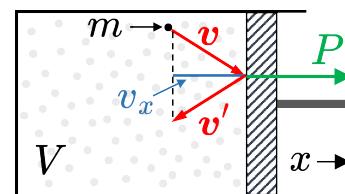


Fig. 9.1

outside the closed gas volume V . We can intuitively understand that this must be the case because on average the gas particles bouncing against the piston on the inside of the gas volume V impart the same momentum on the piston as do the gas particles bouncing off the outside surface of the piston. Now consider what would happen if on the outside of the enclosed gas volume there would be nothing, i.e., only empty space. In that case nothing would counter the momentum exerted from the gas particles inside the gas volume. The result of which would be that under the force exerted on the piston by the gas particles inside the volume V , the piston in fig. 9.1 would inexorably move to the right. An observer of this situation would accordingly ascribe this motion of the piston to the gas pressure given by

$$P = \frac{F}{A} \quad \text{or} \quad F = P \cdot A$$

where A is the cross section of the piston. If the gas pressure P results in the piston moving by a distance dx , the other obvious conclusion is that the gas particles do work to move the piston. With eq. (4.1) this work is given by

$$dW = -Fdx = -PdV \tag{9.2}$$

The minus sign arises here because if $dV < 0$, work is done to compress the gas, work that for example may result in the gas heating up, i.e., having more energy. On the other hand, if $dV > 0$, work is done by the gas which means energy is taken out of the gas. What more can we say about the force F ? If we consider one gas particle m_i bouncing off the piston wall inside the gas volume as shown in fig. 9.1 we see that in an elastic collision with the piston wall its momentum parallel to the piston surface will not change but its momentum normal to the piston surface will invert and the gas particle will experience a momentum change of $\Delta p_i = 2m_i v_x$. To quantify how many gas particles will bounce off the piston surface in such a manner we need to know how many gas particles in V are actually moving in the x -direction and how many of those are actually within striking distance so that they can hit the piston surface within a second. Let N be the total number of gas particles inside V and $dN(v_x)$ be the distribution of particles within N that have a velocity component in the x -direction. On average only half of $dN(v_x)$ will move in the positive x -direction while the other half will move in the negative x -direction. The number of such gas particles moving in the positive x -direction which are within striking distance of the piston surface, i.e., they are within the volume $A \cdot v_x \cdot \Delta t$ in front of the piston surface, as a fraction of all gas particles in V is then

$$\frac{1}{2} \frac{dN(v_x)}{V} A \cdot v_x \cdot \Delta t$$

Hence, assuming all gas particles have identical mass the total momentum transferred by gas particles with the velocity v_x is given by

$$\Delta p = \frac{1}{2} \frac{dN(v_x)}{V} A \cdot v_x \cdot 2mv_x \Delta t$$

and with that the pressure $dP(v_x)$ due to gas particles with the velocity v_x is given by

$$dP(v_x) = \frac{1}{A} \frac{\Delta p}{\Delta t} = mv_x^2 \frac{dN(v_x)}{V} \quad (9.3)$$

To get the total pressure we must sum this equation over all v_x -components, i.e., we must integrate the right-hand side over all v_x velocities. With $dN(v_x)$ being the number of particles with velocity components between v_x and $v_x + dv_x$, the mean velocity and the mean square velocity are given by

$$\langle v_x \rangle = \frac{1}{N} \int_{v_x=-\infty}^{v_x=+\infty} v_x dN(v_x) = 0 \quad \text{and} \quad \langle v_x^2 \rangle = \frac{1}{N} \int_{v_x=-\infty}^{v_x=+\infty} v_x^2 dN(v_x)$$

$\langle v_x \rangle$ vanishes because equal numbers of gas particles will be moving in positive and negative x -directions with the same absolute velocity v_x . Using the non-vanishing mean square velocity $\langle v_x^2 \rangle$, the integration of $dP(v_x)$ in eq. (9.3) over all velocities v_x yields

$$P = \int dP(v_x) = \frac{m}{V} \int_{v_x=-\infty}^{v_x=+\infty} v_x^2 dN(v_x) = \frac{m}{V} N \cdot \langle v_x^2 \rangle \quad (9.4)$$

Because there is no preferred direction for the gas medium enclosed in the volume V depicted in fig. 9.1, obviously $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$ and therefore with

$$v^2 = v_x^2 + v_y^2 + v_z^2 \quad \text{and} \quad \langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle$$

it follows that

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle$$

With that one can rewrite eq. (9.4) as

$$P = \frac{2}{3} \frac{N m \langle v^2 \rangle}{V} = \frac{2}{3} \frac{N}{V} \langle E_{kin} \rangle \quad (9.5)$$

where $\langle E_{kin} \rangle$ is the average kinetic energy of a gas particle. Before we move on from here it is important to keep in mind which restrictions we imposed in deriving this result. Most important is the restriction to only consider elastic collisions. All collisions between gas particles and the piston as well as all walls enclosing the volume V are presumed to be

fully elastic. However, that applies just as much to collisions between gas particles. This matters for molecular gases because unlike atoms, molecules posses additional internal degrees of freedom such as vibration and rotation. Eq. (9.5) is getting close to resemble the ideal gas law from eq. (9.1) but there remain two things to consider. First, in the above derivation we picked an arbitrary volume V so we somehow have to calibrate eq. (9.5). Second, the gas temperature does not show up in eq. (9.5) but there is instead a term relating to the average kinetic energy of a gas particle. The two, temperature and kinetic energy must of course be related but just how we will have to find out. Lets first calibrate the volume in eq. (9.5). At the beginning of the nineteenth century Avogadro discovered that equal volumes of ideal gases at the same temperature and pressure contain equal numbers of atoms or molecules. It took more than fifty years after Avogadro's discovery that the physicist Josef Loschmidt (1821 - 1895) was able to give an approximate estimate for that number (and that is why Avogadro's number N_A is sometimes called Loschmidt's number). One mole of a substance contains $N_A = 6.022 \cdot 10^{23}$ units of that substance. Hence, one mole of oxygen gas contains under standard conditions N_A oxygen atoms and so does one mole of argon gas or any other gas. With V_m identifying the volume of one mole of gas eq. (9.5) then becomes

$$PV_m = \frac{2}{3}N_A\langle E_{kin} \rangle \quad (9.6)$$

Comparing this equation with the empirical ideal gas law from eq. (9.1) (and using $V_m = V/n$ as well as $k_B = R/N_A$) one finds that

$$\frac{2}{3}N_A\langle E_{kin} \rangle = RT \quad \text{or} \quad \langle E_{kin} \rangle = \frac{3}{2}\frac{R}{N_A}T = \frac{3}{2}k_B T \quad (9.7)$$

With $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$ we also have $\langle E_x^{kin} \rangle = \langle E_y^{kin} \rangle = \langle E_z^{kin} \rangle$ which means that with each coordinate direction or degree of freedom a kinetic energy of $\langle E_{kin} \rangle / 3$ is associated. That however means that according to eq. (9.7) each degree of freedom is accorded an energy equivalent of $k_B T / 2$. This is of course nothing else but an example of the equipartition theorem which we used already in section 8.5.1. As eq. (9.1) and eq. (9.7) illustrate, Avogadro's number N_A , the gas constant R , and the Boltzmann constant k_B all express a specific way to connect the energy scale to the temperature scale. Historically, Avogadro's number N_A came first, then came the gas constant R and not long after that Boltzmann's constant k_B . While they are closely related, they serve somewhat different uses. Chemists are likely to use R with the dimension $\text{J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$ or N_A with the dimension mol^{-1} as they connect to molar properties whereas most physicists will likely use k_B with the dimension $\text{J} \cdot \text{K}^{-1}$.

9.2 Gas Temperature and Heat

Eq. (9.7) defines the macroscopic property of a gas which we call temperature by connecting it to the mean kinetic energy of a gas particle. Two gasses will have the same temperature if the mean kinetic energy of their respective gas particles is the same. This means that at a given temperature the gas particles in a gas which is made of heavier atoms or molecules will move slower. It also means that in a gas containing two different kinds of atoms m_1 and m_2 where for example $m_2 = 2m_1$ one essentially has two different ensembles of gas particles, one moving with an average square velocity $\langle v_1^2 \rangle$ and the other moving with an average square velocity $\langle v_2^2 \rangle = 0.5 \cdot \langle v_1^2 \rangle$. At first glance, it may seem surprising to find two such distinct ensembles of gas particles next to each other in the same gas volume. However, according to what we have learned about elastic collision processes in section 3.2.2, this is exactly the expected result.

According to eq. (9.7), temperature is a linear function of the average kinetic energy of a gas particle. At $T = 0$ the average kinetic energy of a gas particle is zero and vice versa. Because the average kinetic energy of a gas particle cannot be less than zero, $T = 0$ marks the low end of the temperature scale. However, this $T = 0$ has nothing to do with the $T = 0^\circ\text{C}$ or $T = 0^\circ\text{F}$. The Celsius scale is defined by the freezing point of water at $T = 0^\circ\text{C}$ and by the boiling point of water at $T = 100^\circ\text{C}$ at a pressure corresponding to the atmospheric pressure at sea level ($1\text{ atm} = 100\,325\text{ Pa}$). The historic definition of the Fahrenheit scale is a little murky but its modern definition is based on the $T = 180^\circ\text{F}$ degree separation of the freezing point of water at $T = 32^\circ\text{F}$ and the boiling point of water at $T = 212^\circ\text{F}$, both again measured at a pressure corresponding to the atmospheric pressure at sea level. In our everyday lives the Celsius and the Fahrenheit scale remain important but physics uses the absolute Kelvin scale where $T = 0$ in eq. (9.7) corresponds to $T = 0\text{ K} = -273.15^\circ\text{C} = -459.67^\circ\text{F}$. Eq. (9.7) also tells us that there exists no upper temperature limit because the average kinetic energy of a gas particle has no limit.

In our everyday language we use the words temperature and heat often interchangeably. However, in the vocabulary of physics temperature refers to the average kinetic energy of an atom or molecule in a given gas volume whereas heat refers to the total energy contained in this gas volume. If we have two volumes of gas which are characterized by the same temperature and pressure and volume, both will have the same measure of mean kinetic energy and both will contain equal amounts of heat. However, if we compare two volumes where both are at the same pressure and temperature but where one volume is twice the size of the other, the mean kinetic energy of molecules in these volumes will still

be the same but now one volume contains twice the amount of heat as the other. Heat always flows from bodies at high temperature to bodies at lower temperature. Processes where no heat is exchanged are called adiabatic processes whereas processes where the temperature does not change are so-called isothermal processes.

9.2.1 Atomic and Molecular Degrees of Freedom

For gases made of simple atoms the degrees of freedom are just the three translational degrees of freedom we discussed above. As fig. 9.2a illustrates, atoms can move along the three orthogonal coordinate directions but that is it. Unlike molecules, atoms have no other internal degrees of freedom in the direction of which they could move. Molecules however can rotate and they can vibrate. As fig. 9.2b shows for the CO-molecule, a diatomic molecule can rotate its molecular axis around any of the three orthogonal axes going through the center of mass of the molecule. However, only two of those rotational degrees of freedom are associated with an energy equivalent of $k_B T/2$ each. The rotation axis which lies in the molecular axis does not contribute. Because the mass of an atom is concentrated in its nucleus and atomic nuclei are very small the rotation around the axis parallel to the molecular axis of a diatomic molecule has a vanishing moment of inertia. As an analogy, consider the rotating cuboid we discussed in section 6.2 in the limit where the moment of inertia A vanishes because the cuboid is shrunk to an extremely thin rod running along the y -axis. In that case the associated rotational energy vanishes of course. Therefore, only rotations around the two rotational axes perpendicular to the molecular axis contribute. However, a diatomic molecule can also vibrate in the direction of its molecular axis as indicated in fig. 9.2b, which adds another degree of freedom and another factor of $k_B T/2$. In summary, a diatomic molecule possesses three translational, two rotational and one vibrational degree of freedom, i.e., in total six degrees of freedom and therefore a temperature increase in a diatomic molecular gas is channeled into six degrees of freedom.

If we consider a non-linear triatomic molecule, its three rotational degrees of freedom contribute a factor of $k_B T/2$ each and so do its three vibration modes. A molecule like water has therefore a total of nine degrees of freedom into or from which a temperature increase or decrease can channel or take energy. Molecules with more than three atoms

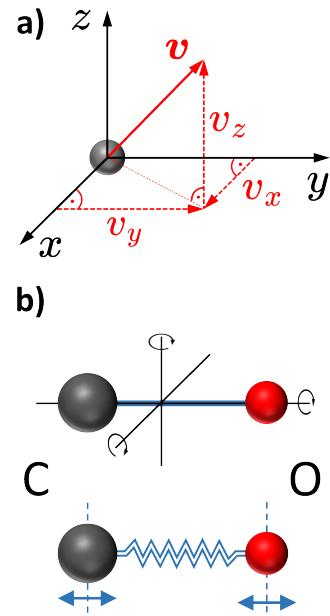


Fig. 9.2

all have three translational and three rotational degrees of freedom and an increasing number of vibrational degrees of freedom as the number of molecular bonds and the complexity of a molecule increase. In general, linear molecules with N atoms possess $3N - 5$ vibrational degrees of freedom while non-linear molecules with N atoms possess $3N - 6$ vibrational degrees of freedom. Hence, water with its three atoms has as a non-linear molecule three vibrational degrees of freedom while a linear tri-atomic molecule such as CO_2 has four vibrational degrees of freedom (two of which we will discuss in section 10.2.2). For all such molecules the equipartition theorem applies and if a molecule has f degrees of freedom a temperature increase channels energy into all those degrees of freedom just like a temperature decrease takes out energy from all of those degrees of freedom as each of those degrees of freedom is associated with the same internal energy equivalent $U = k_B T/2$. This internal energy comes in the form of average kinetic energy as discussed above for mono-atomic gases and also in the form of average rotational and average vibrational energies for molecular gases. Hence, a molecule with f degrees of freedom possess at equilibrium an average energy of

$$U = \frac{f}{2}k_B T \tag{9.8}$$

The term equilibrium here refers to the fact that if for example a gas is heated or cooled it takes some time until a body of gas has everywhere the same temperature. Just like gas atoms or gas molecules, atoms in a solid body also are subject to thermal motion, however, in a more restricted way. Atoms have no degrees of rotation in the gas phase and that does not change in a solid. Different from that, gas atoms which have also no vibrational degrees of freedom in the gas phase gain vibrational degrees of freedom in a solid as they have formed bonds with their neighbors. However, those vibrational degrees of freedom are usually referred to as degrees of freedom associate with changes in the kinetic and potential energy of an atom in a solid. One can picture atoms in solid as three dimensional oscillators which can oscillate in three orthogonal space dimensions. Just like spring-coupled linear oscillators (see section 10.2.2) have one kinetic energy degree of freedom and one potential energy degree of freedom associated with their linear motion so do atoms bound in solids have one degree of freedom associated with changes in their kinetic and potential energy with each coordinate direction. The former relates to how fast they oscillate and the second relates to how strongly they oscillate in any of the three orthogonal coordinate directions. Hence an atom in a solid has a total of six degrees of freedom vs three in the gas phase. Different from atoms, molecules bound in solids do not gain additional degrees freedom but usually loose all their rotational degrees of freedoms as their positions are now fixed in space. In a sense, atoms in a solid become part of a

very large molecule thereby gaining three degrees of freedom whereas molecules bound in solids become more like atoms bound in solids, loosing all their rotational degrees of freedom and only retaining their three degrees of freedom associated with kinetic energy changes in each of three orthogonal space directions and three vibrational degrees of freedom which are now associated with potential energy changes in each of the three orthogonal space directions. Energy partitioning applies to atoms and molecules in solids just as much as it does in the gas phase and therefore when heating or cooling a solid, energy is either channeled equally in or out of these six degrees of freedom an atom or molecule possesses as part of a solid body. If sufficient energy is channel into the thermal motion of atoms of molecules such that $k_B T/2$ for example exceeds the binding energy of an atom or molecule in a solid body then the solid body begins to melt.

So how about the degrees of freedom of atoms and molecules in the liquid phase? To understand that we only need to recall what those degrees of freedom actually stand for. Clearly, the gas model and the solid state model the above discussion about degrees of freedom invokes is a purely mechanistic one. The number of degrees of freedom of an atom or a molecule in the gas phase or in the solid phase equals the number of independent variables which are required to define the position of an atom or molecule with its constituent atoms in three dimensional space. This is no different for atoms or molecules in a liquid. Because bonds in liquids are usually too weak to restrict the motion of its constituent parts the number of degrees of freedom for a given kind of atom or molecule making up a liquid is usually the same as in its gas phase.

9.2.2 Heat Capacity of a Gas

Now that we have a better understanding of where the energy in a gas goes or from where it comes when we heat or cool a gas we can determine the amount of energy that is required to increase the temperature T of a gas from lets say T_1 to a temperature T_2 . To achieve that, the energy in each degree of freedom has to change from $k_B T_1/2$ to $k_B T_2/2$. Therefore, with $\Delta T = T_2 - T_1$ and eq. (9.8), the total energy required to raise the temperature of a gas by ΔT is given by

$$\Delta U = \frac{M}{m} \frac{f}{2} k_B \Delta T \quad (9.9)$$

Here M is the total mass of a body of gas and m is the mass of a gas atom or a gas molecule. Eq. (9.9) applies to gases, liquids and solid bodies alike. The quantity

$$C = \frac{\Delta U}{\Delta T} = \frac{M}{m} \frac{f}{2} k_B \quad (9.10)$$

is referred to as the heat capacity of a body. C is the energy required per Kelvin to raise the temperature of the gas. Often one will find $C = \Delta Q / \Delta T$ as the letter Q is frequently used to indicate that the source of energy to change the gas temperature is heat. Dividing the heat capacity labeled with a capital C by the total mass M gives the specific heat capacity labeled with a small c :

$$c = \frac{C}{M} = \frac{fk_B}{2m} \quad (9.11)$$

Heat capacity C and specific heat capacity c are material parameters. When measuring the heat capacity of a gas one must differentiate between the heat capacity measured at constant pressure, i.e., C_P or c_P and heat capacity measured at constant volume, i.e., C_V or c_V . Because liquids and solids expand or contract very little when heated or cooled this distinction only matter for gases. If one measures the heat capacity of a gas at constant pressure then this measurement includes the work done by the gas as it expands, therefore $C_P > C_V$ or respectively $c_P > c_V$. First, lets look at the most simple example, a mono-atomic gas. Each atom in the gas has three degrees of freedom and therefore a gas of N atoms at temperature T has the energy content

$$U = \frac{3}{2}Nk_B T \quad (9.12)$$

With eq. (9.10) the heat capacity of such a mono-atomic gas is given by

$$C_V = \left(\frac{dU}{dT} \right)_V = \frac{3}{2}Nk_B \quad (9.13)$$

or in differential form

$$dU = C_V \cdot dT \quad (9.14)$$

This equation assumes that the gas volume does not change and all the energy put into the mono-atomic gas goes towards increasing the energy contained in the three degrees of freedom of its atoms. However, as a gas volume such as in fig. 9.1 is being heated, the gas pushes against the piston. This means that not all the energy supplied goes into the internal degrees of freedom of the mono-atomic gas but some of it goes towards the work done by the gas to push the piston. If the gas pushes the piston which shall have a surface area A a distance dx , the work done by the gas equals

$$dW = Fdx = PAdx = PdV$$

This work being done by the gas must be included in the energy balance of eq. (9.13) which then becomes

$$dU = C_V dT + P dV \quad (9.15)$$

Using the ideal gas law as stated in eq. (9.1) at constant pressure P we can replace $P dV$ in eq. (9.15) with

$$P dV = N k_B dT$$

Using this expression we can rewrite eq. (9.15) as

$$dU = C_V dT + N k_B dT \quad (9.16)$$

from which follows that the heat capacity of a mono-atomic gas measured at constant pressure C_P is given by

$$C_P = \left(\frac{dU}{dT} \right)_P = C_V + N k_B \quad (9.17)$$

From eq. (9.13) and eq. (9.17) follows that for a mono-atomic gas

$$\gamma = \frac{C_P}{C_V} = \frac{c_P}{c_V} = \frac{5}{3}$$

This parameter γ is known as the heat capacity ratio, the ratio of specific heats or also as the adiabatic index. If one uses the general form of eq. (9.10) or eq. (9.11) then eq. (9.13) and eq. (9.17) become

$$C_V = \frac{f}{2} N k_B \quad \text{and} \quad C_P = C_V + N k_B$$

With that the heat capacity ratio γ becomes

$$\gamma = \frac{C_P}{C_V} = \frac{c_P}{c_V} = \frac{f+2}{f} \quad (9.18)$$

Heat capacities can of course be measured and thus it became quickly evident that predictions by the simple heat capacity theory derived above using the equipartition theorem and the ideal gas law differed materially from measurements. While for example calculated and experimentally derived values of γ did agree for some molecules, for others they did not. The temperature dependence of specific heat posed another serious challenge. Unless a phase change occurs in a substance, heat capacity is a smooth and continuous function which for low temperatures decreases rapidly. At a low enough temperatures every substance becomes a solid and the lighter the atomic or molecular constituents of that solid are the faster the heat capacity declines to become zero at $T = 0$ Kelvin. As it turns out, not all degrees of freedom are equal and rotational and vibrational degrees of freedom freeze out with dropping temperature, faster for lighter elements than for heavier ones.

9.3 Adiabatic and Isothermal Processes

Using the parameter γ , eq. (9.12) can be expressed in the more general form

$$U = \frac{1}{\gamma - 1} Nk_B T \quad (9.19)$$

which applies not just to mono-atomic gases but to molecular gases as well. For mono-atomic gases we just derived that $\gamma = 5/3$ and if we insert this we just get eq. (9.12). There are two particularly important ways to change the state of a gas, i.e., through an adiabatic or an isothermal process. In isothermal processes the temperature of the gas is kept constant whereas in adiabatic processes no heat transfer takes place. That means that in adiabatic processes all the energy supplied to the gas get channeled into its respective degrees of freedom. That mean all energy only goes towards increasing the internal energy of the gas and nowhere else. If a gas is compressed in an adiabatic way all of the energy goes towards increasing the internal energy of the gas and if a gas is expanded in an adiabatic way all of the energy to do that comes from the internal energy of the gas. Using the ideal gas equation eq. (9.1) to replace $Nk_B T$ with PV , eq. (9.19) becomes the equation for the energy change of a gas because of compression or expansion:

$$PV = (\gamma - 1)U \quad (9.20)$$

As just discussed above, γ is a function of temperature. Although we should always check in any given case if one can neglect the temperature dependence of γ , over the narrow temperature ranges usually of interest in adiabatic processes, this often is the case. Hence, assuming $\gamma = \text{const}$, a differential change dU in the internal energy of the gas is given by

$$dU = \frac{d(PV)}{\gamma - 1} = \frac{1}{\gamma - 1} (PdV + VdP) \quad (9.21)$$

From eq. (9.2) we know the work done in moving the piston in fig. 9.1, either by the gas when it expands or by the experimenter when compressing the gas. We also know that this work must go into or come out of the internal energy U of the gas. Hence

$$dU = dW = -PdV \quad (9.22)$$

From these two expressions for dU follows that

$$-PdV = \frac{1}{\gamma - 1} (PdV + VdP)$$

We can rewrite this equation into a more meaningful form giving us the differential equation

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0 \quad (9.23)$$

This differential equation has the simple solution

$$\ln P + \gamma \ln V = \ln (PV^\gamma) = \ln C_1$$

where we have given the integration constant the convenient form $\ln C_1$ so we can just take the exponential of both sides to yield

$$PV^\gamma = C_1 \quad (9.24)$$

Eq. (9.24) tells us that in an adiabatic compression or expansion where all the energy goes into internal energy of the gas, or comes out of the internal energy of the gas, the gas pressure P of an ideal gas scales with $V^{-\gamma}$. Instead of keeping T constant in an adiabatic process we can also choose to keep P or V constant. Just like for the pair P, V one can then derive analogous expressions for the pairs T, V or P, T to characterize an adiabatic expansion or compression. Taking the differential from of eq. (9.19) and eliminating the pressure P in eq. (9.22) by using the ideal gas law one obtains the expression

$$dU = \frac{1}{\gamma - 1} Nk_B dT = -P dV = \frac{Nk_B}{V} T dV$$

from which follows the differential equation

$$\frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0 \quad (9.25)$$

Simple integration gives us the solution

$$\ln T + (\gamma - 1) \ln V = \ln (TV^{\gamma-1}) = \ln C_2$$

and hence

$$TV^{\gamma-1} = C_2 \quad (9.26)$$

Finally, to describe an adiabatic process as a function of the variable pair P, T we use eq. (9.23) to replace dV/V in eq. (9.25) with dP/P which results in

$$\frac{dT}{T} - \frac{\gamma - 1}{\gamma} \frac{dP}{P} = 0 \quad (9.27)$$

Integrating this differential equation gives

$$\ln T - \frac{\gamma-1}{\gamma} \ln P = \ln \left(TP^{-\frac{\gamma-1}{\gamma}} \right) = \ln C_3$$

and with that follows

$$TP^{-\frac{\gamma-1}{\gamma}} = C_3 \quad (9.28)$$

To summarize, we found three state equations describing adiabatic processes where one of the variables P , V and T is kept constant:

$$PV^\gamma = C_1 \quad \text{with } T = \text{constant}$$

$$TV^{\gamma-1} = C_2 \quad \text{with } P = \text{constant}$$

$$TP^{-\frac{\gamma-1}{\gamma}} = C_3 \quad \text{with } V = \text{constant}$$

For an ideal mono-atomic gas with $\gamma = 5/3$ these equations become

$$PV^{\frac{5}{3}} = C_1 , \quad TV^{\frac{2}{3}} = C_2 , \quad TP^{-\frac{2}{5}} = C_3$$

Adiabatic processes are of course an idealization. There are few processes in nature where no heat is exchanged and energy is exclusively transferred into the internal energy of a gas by doing some work on it, like, e.g., compressing it, or where the internal energy of the gas is exclusively converted into work done by the gas as, e.g., when a gas expands. An example of an almost adiabatic process would be a compression of an ideal gas which happens so quickly that there is just no time for any appreciable heat transfer to happen on the time scale the compression takes place. The importance of adiabatic processes lies in the fact that they illustrate one limit case of the first law of thermodynamics which says that the total energy U of a system, like for example an ideal gas, is the sum of the heat energy Q put into the system and the work W done on the system. Here, in the context of the ideal gas we referenced the total energy of the ideal gas as its internal energy U for which must hold

$$U = Q + W$$

In an adiabatic process no heat is exchanged. This means that $\Delta Q = 0$ in an adiabatic process and hence $\Delta U = \Delta W$. That, however, means that the change in internal energy ΔU then only depends on the state of the system before the work is done and on the state of the system after the work is done but not on how the work is done, i.e., which path is taken to do the work. It is for that reason that adiabatic processes are important because they relate the value of the state variables P_0 , V_0 and T_0 before the process directly to their values P_1 , V_1 and T_1 after the process. If for example the temperature is kept constant, i.e., $T_1 = T_2$ in an adiabatic process, then we know that

$$P_0 V_0^\gamma = P_1 V_1^\gamma$$

Now lets move on from adiabatic process to consider isothermal processes. In isothermal processes heat exchange does take place, i.e., $\Delta Q \neq 0$, but it does take place in such a way that the temperature T of the system stays constant, i.e., $T = T_0$ at all times. Hence the ideal gas law becomes

$$PV = Nk_B T_0 = \text{const} \quad (9.29)$$

from which follows $\Delta(PV) = 0$. With eq. (9.12) we know that the energy content of an ideal gas is given by $U = fNk_B T$. Because the temperature is constant in isothermal processes the condition $\Delta(PV) = 0$ implies $\Delta U = 0$, i.e., as long as the number of degrees of freedom f does not change. So unless the ideal gas undergoes a phase change where f could change, an isothermal process is characterized by the fact that the total energy of the system does not change. This means that any work done by the system must decrease the heat energy of the system and any work W done on the system must increase it. Hence, for isothermal processes

$$\Delta U = 0 = \Delta Q + \Delta W \quad \Rightarrow \quad \Delta W = -\Delta Q$$

With eq. (9.2) the work done on the system or done by the system is given by $dW = -PdV$. Combining this with eq. (9.29) we then have for an isothermal process

$$dW = -\frac{Nk_B T_0}{V} dV$$

If, as shown in fig. 9.3, we denote the initial volume before the compression with V_i and the final volume after the completion of the isothermal compression with V_f then we get for the total work W_{if} done and the associated change in the systems heat Q_{if}

$$W_{if} = -Nk_B T_0 \int_{V_i}^{V_f} \frac{dV}{V} = -Nk_B T_0 \ln\left(\frac{V_f}{V_i}\right)$$

If the gas is compressed then $V_f/V_i < 1$ and $Q_{if} = -W_{if}$ will be positive meaning the work done to compress the gas increases the heat energy of the gas. The opposite is the case when the gas expands. In that case $V_f/V_i > 1$ and hence $Q_{if} = -W_{if}$ will be negative, meaning, the energy required to do the expansion work decreases the heat energy of the gas.

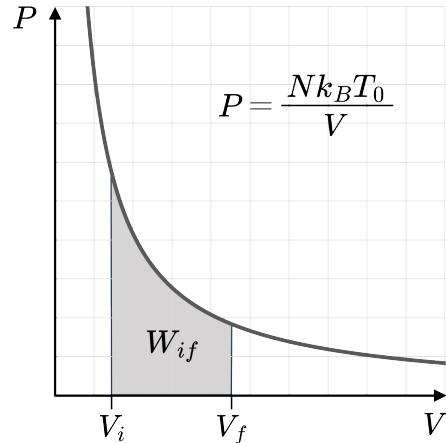


Fig. 9.3

9.3.1 Gas Compressibility

Generally, when the density ρ of a gas changes, gas pressure P and gas temperature T will also change. With the number of gas particles in the changing volume being constant, we can write the differential $dV(P, T)$ as

$$dV(P, T) = \left(\frac{\partial V}{\partial P}\right)_T dP + \left(\frac{\partial V}{\partial T}\right)_P dT \quad (9.30)$$

where the subscripts P and T indicate that those variables are held constant. The first term in eq. (9.30) is proportional to the quantity

$$\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_T \quad (9.31)$$

the so-called isothermal compressibility, which is a measure of how compressible a gas is at constant temperature. This isothermal compressibility is positive for a pressure increase which is usually associated with a shrinking volume, i.e., $\Delta V/V$ is negative. The second term in eq. (9.30) is proportional to the thermal expansion coefficient

$$\beta = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_P \quad (9.32)$$

which is positive as the gas volume expands with rising temperature and negative as it shrinks with decreasing temperature while the pressure is held constant. Keeping the number of gas particles N constant means keeping $\rho V = N$ constant and with that

$$d(\rho V) = \rho dV + V d\rho = 0 \quad \text{or} \quad \frac{d\rho}{\rho} = -\frac{dV}{V}$$

Using this relation as well as eq. (9.31) and eq. (9.32), eq. (9.30) can be rewritten such as to correlate the relative density change of a gas with its isothermal compressibility κ_T and its thermal expansion coefficient β :

$$\frac{d\rho}{\rho} = \kappa_T \cdot dP - \beta \cdot dT \quad (9.33)$$

With eq. (9.1), it follows that for an ideal gas

$$\kappa_T = \frac{1}{p} \quad \text{and} \quad \beta = \frac{1}{T} \quad (9.34)$$

and therefore the relative density change of an ideal gas is given by

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} \quad (9.35)$$

Now lets consider the isothermal and the adiabatic cases for a process that results in a density change of a gas. In the isothermal case, a density change happens so slowly that heat conduction between the gas volume and the outside effectively keeps the gas temperature T constant in which case eq. (9.33) becomes

$$\frac{d\rho}{\rho} = \kappa_T \cdot dP \quad (9.36)$$

In the adiabatic case the density change happens so rapidly that no heat flow can occur and the density change is given by

$$\frac{d\rho}{\rho} = \kappa_S \cdot dP \quad (9.37)$$

where

$$\kappa_S = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_S \quad (9.38)$$

is the so-called adiabatic compressibility. The subscript S refers to the entropy S of the system. We will discuss the concept of entropy in the fourth volume of this physics course in detail. Here it suffices to say that $S = \text{const}$ means that $\Delta Q = 0$ and the convention among physicists is to use the subscript S in this way to indicate that no heat is exchanged in a process. In an adiabatic process all energy goes towards increasing or decreasing the internal energy of the gas and nowhere else as expressed in eq. (9.14) and eq. (9.22). Combining these two equation we get

$$C_V dT = dU = -PdV \quad \Rightarrow \quad dT = -\frac{1}{C_V} PdV \quad (9.39)$$

From the ideal gas law eq. (9.1) follows

$$d(PV) = PdV + VdP = Nk_B dT$$

Using eq. (9.39) to replace dT on the right hand side of this equation one gets after a little rearrangement

$$\frac{dV}{V} = -\frac{C_V}{C_V + Nk_B} \frac{dP}{P}$$

Comparing this expression then with the definition of κ_S from eq. (9.38)

$$\frac{d\rho}{\rho} = -\frac{dV}{V} = \kappa_S \cdot dP$$

shows that

$$\kappa_S = \frac{1}{P} \frac{C_V}{C_V + Nk_B} \quad (9.40)$$

With eq. (9.17) and eq. (9.34) it then follows that

$$\frac{\kappa_S}{\kappa_T} = \frac{C_V}{C_P} = \frac{1}{\gamma} \quad (9.41)$$

9.3.2 The Barometric Height Formula

The barometric height formula describes the change in air pressure with the distance z from Earth's surface. With eq. (8.3) this change can be expressed as:

$$\frac{\partial P(z)}{\partial z} = -\rho \cdot g = -m \cdot n \cdot g$$

Here m stands for the particle mass and $n = N/V$ is the particle density with n being a function of z . Eq. (9.1), the general ideal gas equation provides the relationship between particle density n and pressure P

$$P \cdot V = N \cdot k_B \cdot T$$

and thus

$$P = n \cdot k_B \cdot T \quad \text{with} \quad n = \frac{N}{V}$$

Now one makes the assumption that $T = \text{constant}$ or respectively, that a temperature change of a few degrees should not matter much. With that follows

$$\frac{\partial P(z)}{\partial z} = -\frac{m \cdot g}{k_B T} \cdot P(z)$$

which after separation of the variables becomes

$$\frac{\partial P(z)}{P(z)} = -\frac{m \cdot g}{k_B T} \cdot \partial z$$

Simple integration of this equation from P_0 at $z = 0$ to $P(z)$ at the height z then yields the barometric height formula

$$P(z) = P_0 \cdot \exp\left(-\frac{m \cdot g \cdot z}{k_B T}\right) \tag{9.42}$$

The assumption that the temperature in the atmosphere is constant is of course only an approximation, limiting the validity of the barometric height formula just derived. For a broader approach, one uses a modified version of eq. (9.24), which replaces the volume V with $V = \rho \cdot N$ where N is kept constant as a given volume of air expands as it rises in the atmosphere, thereby decreasing the density ρ of the air in this rising volume. According to eq. (9.24)

$$P V^\gamma = P_0 V_0^\gamma \quad \Rightarrow \quad \frac{P}{P_0} = \left(\frac{V_0}{V}\right)^\gamma$$

from which follows with $V_0 = \rho_0 \cdot N$ and $V = \rho \cdot N$ the so-called adiabatic equation for the expansion of air in the atmosphere

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma \quad (9.43)$$

Now one uses a little trick by introducing the parameter integral

$$\mathcal{P}(P(\mathbf{r})) = \int_{P_0}^{P(\mathbf{r})} \frac{dP'}{\rho(P')} \quad (9.44)$$

of which one then calculates the spatial derivative with the help of

$$\frac{\partial \mathcal{P}(P(\mathbf{r}))}{\partial x_i} = \frac{\partial \mathcal{P}}{\partial P(\mathbf{r})} \frac{\partial P(\mathbf{r})}{\partial x_i} \quad (9.45)$$

and

$$\frac{\partial}{\partial P} \int_{P_0}^P \frac{dP'}{\rho(P')} = \frac{1}{\rho} \quad (9.46)$$

with the result that

$$\nabla \mathcal{P}(P(\mathbf{r})) = \frac{1}{\rho} \nabla P(\mathbf{r}) \quad (9.47)$$

The reason for introducing the parameter integral in eq. (9.44) is that it now follows from eq. (9.47) with the help of eq. (8.6) and eq. (8.7) that

$$\nabla \mathcal{P} = \frac{1}{\rho} \nabla P(\mathbf{r}) = \frac{1}{\rho} \mathbf{f} = -\nabla(U) \quad (9.48)$$

Hence, the result is that the sum of the parameter integral \mathcal{P} and the potential energy U is a constant

$$\mathcal{P} + U = \text{const} \quad (9.49)$$

The benefit of this relationship can be seen after inserting eq. (9.43) into eq. (9.44) and subsequent integration from which with $U = gz$ follows

$$\mathcal{P}(P(\mathbf{r})) = \frac{1}{\rho_0} \int_{P_0}^{P(\mathbf{r})} \left(\frac{P'}{P_0} \right)^{-\frac{1}{\gamma}} dP' = \frac{P_0}{\rho_0} \frac{\gamma}{\gamma-1} \left(\frac{P}{P_0} \right)^{\frac{\gamma-1}{\gamma}} = \text{const} - gz \quad (9.50)$$

With the help of the parameter integral \mathcal{P} one therefore obtains a modified version of the hydrostatic pressure equation eq. (8.5). To determine the constant, one uses the fact that $P = P_0$ must apply for the pressure on Earth's surface, i.e., for $z = 0$. Hence

$$\text{const} = \frac{P_0}{\rho_0} \frac{\gamma}{\gamma-1} = u$$

The dependence of the air pressure on height above Earth's surface is then given by

$$\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} = 1 - \frac{gz}{u} \quad (9.51)$$

The pressure in eq. (9.51) becomes zero at the critical height $z = u/g = h_c$. For air at room temperature $\gamma \approx 1.2$. With this value for γ , eq. (9.51) yields a critical height h_c of about 48 km. According to eq. (9.51) and eq. (9.43), the following applies to pressure and density as a function of height z above Earth's surface

$$\frac{P(z)}{P_0} = \left(1 - \frac{z}{h_c}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{and} \quad \frac{\rho(z)}{\rho_0} = \left(1 - \frac{z}{h_c}\right)^{\frac{1}{\gamma-1}} \quad (9.52)$$

With these two equations and the ideal gas equation one obtains for the dependence of the temperature on the height above the Earth's surface:

$$\frac{T(z)}{T_0} = 1 - \frac{z}{h_c} \quad \text{with} \quad T_0 = \frac{m \cdot P_0}{k_B \cdot \rho_0} \quad (9.53)$$

9.4 Thermal Distributions

The preceding discussion of the thermal motion of atoms or molecules may sound as if the atoms or molecules of a gas at a given temperature will all have the same kinetic energy, or for that matter, the same potential energy. However, that is not true. The microscopic discussion of gas pressure linked the temperature of a gas to the mean kinetic energy of its gas atoms or molecules. There will be gas atoms or molecules possessing higher or lower kinetic energies than the mean kinetic energy. The question is just how this distribution of gas particles as a function of kinetic energy looks like which then averages to the mean kinetic energy associated with the temperature of the respective gas. In the discussion of the barometric height formula we found that the temperature of the atmosphere varies with the height above Earth's surface. That is so because the density of the atmosphere decreases with height which in turn is a function of the potential energy of a gas atom or molecule varying with height.

9.4.1 The Boltzmann Distribution

Let's take another look at the barometric height formula in eq. (9.42). If we use the gas density $n = N/V$ instead of the gas pressure P as the variable as we keep the gas temperature T constant we can with $P(z) = n(z)k_B T$ and $P(z=0) = P_0 = n_0 k_B T$ rewrite eq. (9.42) as

$$n(z) = n_0 \cdot \exp\left(-\frac{m \cdot g \cdot z}{k_B T}\right) = n_0 \cdot \exp\left(-\frac{U(z)}{k_B T}\right) \quad (9.54)$$

The potential energy of a mass m at a height z above Earth's surface is given by $U(z) = mgz$. Hence, eq. (9.54) tells us how the density of the atmosphere varies for gas particles of mass m as a function of the local strength of the gravity field. The ratio of gas particles densities $n(z_1)$ and $n(z_2)$ at two different heights z_1 and z_2 is given by

$$\frac{n(z_1)}{n(z_2)} = \exp\left(-\frac{U(z_1) - U(z_2)}{k_B T}\right) \quad (9.55)$$

where $U(z_1)$ and $U(z_2)$ are the respective potential energies of gas particles with mass m at heights z_1 and z_2 . At constant temperature T , eq. (9.55) is only a function of the potential energy difference. Atoms and molecules higher up in the atmosphere have a higher potential energy than atoms and molecules at lower heights which means there will be fewer of them. While we used here the example of potential energy levels, eq. (9.55) also holds when considering other forms of energy. If we for example consider molecular rotation or molecular vibration, we will find less molecules in higher energy rotational or vibrational states than in lower energy vibrational or rotational energy states. For a gas in thermal equilibrium containing a total number of N molecules the number of such molecules in a given vibrational state N_{vib} with energy E_{vib} or a given rotational state N_{rot} with energy E_{rot} will be

$$N_{vib} = N \cdot \exp\left(-\frac{E_{vib}}{k_B T}\right) \quad \text{and} \quad N_{rot} = N \cdot \exp\left(-\frac{E_{rot}}{k_B T}\right) \quad (9.56)$$

For diatomic molecules such as O₂ or N₂ the first excited vibration levels lie some 200 to 300 meV above the vibrational ground state. At room temperature $k_B T \approx 26$ meV and therefore one will find the vast majority of molecules ($> 99.99\%$) in the vibrational ground state. Because the rotational energies of diatomic molecules are only a few meV, a much greater number of molecules will be in excited rotation levels. We could of course also have looked at the number of molecules in excited electronic states, or for that matter at atoms in excited electronic states. However, as atomic and molecular energy level separations are in the eV range, there will practically be no electronically excited molecules or atoms in a room temperature gas at thermal equilibrium.

What the examples in eq. (9.56) illustrate is that the factor $\exp(-E/k_B T)$ reflects a probability, i.e., the probability of finding at a given temperature a molecule at a certain vibrational or rotational energy level. The probability to find a molecule at an energy level E_1 vs finding it at an energy level E_2 only depends on the energy difference $E_1 - E_2$ of these levels. Similarly, we can read eq. (9.55) as the probability to

find a certain number of atoms or molecules at the height z_1 vs at the height z_2 which only depends on the difference in potential energy $U(z_1) - U(z_2)$. All of that is of course no coincidence. To demonstrate this lets stipulate that the probability $P(E_1)$ to find a particle at an energy level E_1 vs the probability $P(E_2)$ to find it at an energy level $E_2 > E_1$ shall only depend on the difference in energy between this states. Expressed in mathematical terms this means that

$$\frac{P(E_2)}{P(E_1)} = f(E_2 - E_1)$$

If we now insert an intermediate energy level E_i between the energy levels E_1 and E_2 we can write this equation as

$$\frac{P(E_2)}{P(E_1)} = \frac{P(E_2)}{P(E_i)} \cdot \frac{P(E_i)}{P(E_1)}$$

Therefore, the function $f(E_2 - E_1)$ must satisfy the following functional equation

$$f(E_2 - E_1) = f(E_2 - E_i)f(E_i - E_1)$$

which with $A = E_2 - E_i$ and $B = E_i - E_1$ we can rewrite as

$$f(A + B) = f(A)f(B)$$

However, this is just the functional equation which defines the exponential function. And with the exponential function being the only function satisfying this equation, $f = \exp$ must hold. Since the argument of the exponential function has to be dimensionless we must multiply the energy levels by some factor β where βE then becomes dimensionless. With that we have

$$\frac{P(E_2)}{P(E_1)} = f(\beta E_2 - \beta E_1) = \frac{e^{\beta E_2}}{e^{\beta E_1}} \quad (9.57)$$

To determine β one must only compare eq. (9.57) with eq. (9.54), eq. (9.55) or eq. (9.56) to find that $\beta = -1/k_B T$. If N denotes the total number of particles in a gas then the probability to find a fraction $N(E)/N$ of those particles with an energy E decreases exponentially, i.e.,

$$N(E) = N e^{-E/k_B T} \quad (9.58)$$

Distributions of this kind are called Boltzmann distributions. As just shown above, they arise from purely statistical probability considerations. Thermal motion and with that the physics of heat has a purely statistical nature and hence the physics of heat is statistical physics, a field of physics better known under the label of statistical thermodynamics or statistical mechanics. While at this point we cannot venture further into these areas of

theoretical physics, there are two more questions we must ask. First, there always must exist physics descriptions of such phenomena as follow from statistical considerations. So what does this physics description look like for the probability considerations we just used to derive the Boltzmann distribution? Second, what kind of velocity distribution follows from these statistical considerations?

To look at little more into the physics which give rise to the Boltzmann distribution lets consider again the barometric height formula. The density distribution as a function of height in eq. (9.54) at a given temperature T is the result of an equilibrium between two forces. On the one hand there is the force of gravity which results in gas particles falling towards the Earth's surface. On the other hand there is the imbalance in particle density which drives particle diffusion such as to correct this density imbalance. Lets consider two volumes separated by a membrane where on one side the gas density is higher than on the other side. We all know that once the membrane is removed and gas particles can freely move between the two volumes, the gas particle density will quickly become the same in both volumes. The same happens in a column of air with the difference being that the diffusion process working to equalize particle densities in the air column is opposed by the force of gravity which forces particles to sink towards Earth's surface. In section 7.1.3 we saw that for low sinking speeds of particles in a viscous fluid Stokes law, i.e., eq. (7.10), holds:

$$F_R^\eta = 6\pi\eta \cdot r_0 \cdot v$$

where F_R^η is the friction force, η is the viscosity of the fluid and v is the velocity with which a particle of radius r_0 moves in the viscous fluid. More generally, one can equate the force acting on a particle moving in a viscous fluid such as the atmosphere with the velocity at which such a particle is moving via

$$F_R^\eta = \frac{1}{\mu} \cdot v \quad \text{where} \quad \mu = \frac{1}{6\pi\eta \cdot r_0} \quad (9.59)$$

is the so-called mobility of a gas particle. If we associate v with the sinking velocity v_z of an air molecule in the atmosphere driven by the force of gravity F_G then this velocity will be given by $F_G + F_R^\eta = 0$, i.e.,

$$m \cdot g = -\mu^{-1} \cdot v_z \cdot \hat{\mathbf{z}} \quad \text{or} \quad v_z = -\mu \cdot m \cdot g \cdot \hat{\mathbf{z}}$$

With $n(z)$ being the particle density, gravity then gives rise to a stream of sinking gas particles which is given by

$$J_S = n(z) \cdot v_z = -n(z) \cdot \mu \cdot m \cdot g \cdot \hat{\mathbf{z}} \quad (9.60)$$

Opposing this sinking stream of particles is a diffusion stream of particles J_D driven by the gradient in particle density in z -direction

$$J_D = -D \nabla n(z) \cdot \hat{z} = -D \frac{dn(z)}{dz} \quad (9.61)$$

where D is the so-called diffusion coefficient. The minus sign in eq. (9.61) ensures that diffusion occurs in the direction of lower density, i.e., J_D is positive when $dn(z)/dz$ is negative. In thermal equilibrium the sinking stream of particles J_S and the diffusion stream of particles J_D must balance, i.e.,

$$J_S + J_D = 0$$

Inserting the expressions for J_S and J_D from eq. (9.60) and eq. (9.61) this requirement translates into

$$-n(z) \cdot \mu \cdot m \cdot g - D \frac{dn(z)}{dz} = 0$$

which after separation of the variables becomes

$$\frac{dn(z)}{n(z)} = -\frac{\mu \cdot m \cdot g}{D} dz$$

This differential equation has the solution

$$n(z) = n_0 \cdot \exp\left(-\frac{\mu \cdot m \cdot g \cdot z}{D}\right) \quad (9.62)$$

A comparison with eq. (9.54) shows that eq. (9.62) has the exact same form as the barometric height formula and provided that

$$D = \mu k_B T = \frac{k_B T}{6\pi\eta \cdot r_0} \quad (9.63)$$

it actually is just the barometric height formula. Eq. (9.63), the so-called Stokes-Einstein relation, shows how the diffusion coefficient of a particle of radius r_0 moving in a fluid with viscosity η depends on the temperature of the fluid.

In this section we explored the statistical nature underlying the Boltzmann distribution and we looked into how from a pure physics point of view the Boltzmann distribution arises in the case of the barometric height formula. In the course of the latter we introduced in eq. (9.59) the mobility μ of a particle, then the diffusion coefficient D as the proportionality factor in the diffusion equation eq. (9.61) and finally derived e.g. eq. (9.63), the Stokes-Einstein relation.

9.4.2 The Maxwell-Boltzmann Distribution

In our microscopic examination about the gas pressure of an ideal gas we had to determine the mean square velocity $\langle v^2 \rangle$. Knowing this quantity allowed us then with eq. (9.6) to derive a relationship between the pressure of an ideal gas and the mean kinetic energy of a gas particle in that gas. Now the question is: What is the distribution of gas particle velocities in such a gas that will result in any given ideal gas having such a mean square value of gas particle velocity. While on average gas particles will move with a velocity $\sqrt{\langle v^2 \rangle}$, many will move at higher velocities and many will move at lower velocities. What we also found is that $\langle v_i \rangle = 0$ where $i = 1, 2, 3$, which means that in velocity space the distribution of velocities is isotropic. Or as illustrated in fig. 9.4, if we pick out an absolute velocity value v from the velocity distribution, all velocity vectors \mathbf{v} with that magnitude form a sphere of radius $|v|$ with the density of velocity vectors penetrating that sphere being the same in all directions. Hence, the total number of gas particles with absolute velocity values between v and $v + dv$ must be proportional to the volume of the spherical shell between v and $v + dv$ which is given by $4\pi v^2 dv$. In an ideal gas, gas particle have only kinetic energy but no potential energy. Therefore the energy E in the Boltzmann distribution eq. (9.58) is only the kinetic energy of a gas particle. With that we can write for the number of particles $dN(v)$ we are likely to find with velocities between v and $v + dv$

$$dN(v) = N \cdot C \cdot 4\pi v^2 \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) dv = N \cdot f_{MB}(v) dv \quad (9.64)$$

where the proportionality constant C follows from the requirement that the sum over all velocities must equal the number of particles N in the gas, i.e.,

$$\int_{v=0}^{v=\infty} f_{MB}(v) dv = 4\pi C \int_0^\infty v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv = 1 \quad (9.65)$$

This definite integral is a standard integral which one can look up in integral tables where one finds that for $a > 0$ and $n > -1$

$$\int_0^\infty x^n e^{-ax^2} dx = \begin{cases} \frac{1 \cdot 3 \cdots (2k-1) \cdot \sqrt{\pi}}{2^{k+1} a^{k+0.5}} & \text{for } n = 2k \\ \frac{k!}{a^{k+1}} & \text{for } n = 2k+1 \end{cases} \quad (*)$$

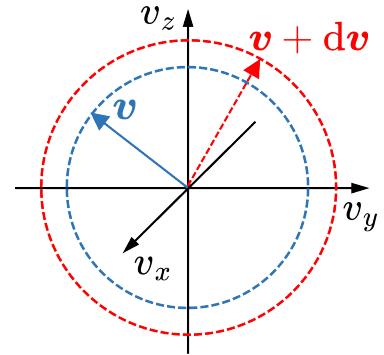


Fig. 9.4

Inserting for $n = 2$ and $a = m/2k_B T$ one then can determine the value of the constant C as

$$C = \left(\frac{m}{2\pi k_B T} \right)^{3/2}$$

Inserting this into eq. (9.64) one obtains the Maxwell-Boltzmann distribution function $f_{MB}(v)$ for the velocities of gas particles in an ideal gas:

$$f_{MB}(v) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T} \right)^{3/2} \cdot v^2 \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \quad (9.66)$$

Fig. 9.5 shows the Maxwell-Boltzmann distribution function $f_{MB}(v)$ for a gas of nitrogen molecules (the atmosphere is 70% nitrogen) at three different temperatures: at 100 K, at 300 K (room temperature) and at 1000 K. As one can easily verify by solving the optimization problem $df(v)/dv = 0$, these curves have their maximum values, i.e., the most probable velocity value at a given temperature, at

$$v_{max} = \sqrt{\frac{2k_B T}{m}}$$

At room temperature the most probable velocity for a nitrogen molecule is 422 m s^{-1} which corresponds to a kinetic energy of 26 meV. $f_{MB}(v)$ is not symmetric with respect to the most probable velocity as there are more molecules with higher and fewer molecules with lower velocities than v_{max} . The mean velocity $\langle v \rangle$ and the mean square velocity $\langle v^2 \rangle$ of the Maxwell-Boltzmann distribution follow from evaluating the integrals

$$\langle v \rangle = \int_0^\infty v f(v) dv \quad \text{and} \quad \langle v^2 \rangle = \int_0^\infty v^2 f(v) dv$$

Using (*) to evaluate these, the results are

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}} \quad \text{and} \quad \langle v^2 \rangle = \frac{3k_B T}{m}$$

The mean velocity $\langle v \rangle$ is always by a factor $2/\sqrt{\pi}$ greater than the most probable velocity v_{max} . Using the value of the mean square velocity $\langle v^2 \rangle$ obtained here we can calculate the mean kinetic energy of a gas particle as

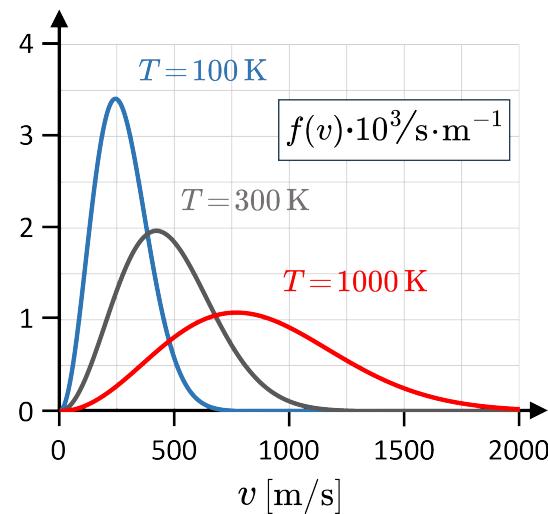


Fig. 9.5

$$\langle E_{kin} \rangle = \frac{m\langle v^2 \rangle}{2} = \frac{3}{2}k_B T$$

which is just the result we already derived in eq. (9.7). The asymmetry of $f_{MB}(v)$ gets more pronounced as the temperature increases, resulting in an ever longer tail of high energy molecules. The average velocity of a nitrogen molecule at room temperature is about 476 m s^{-1} or roughly 1714 km h^{-1} and quite a few nitrogen molecules zip around at speeds exceeding 1000 m s^{-1} or roughly 3600 km h^{-1} . However, even the energy of such fast moving gas particles is not sufficient to electronically excite other molecules or even ionize them on impact as the kinetic energy of a nitrogen molecule moving at 1000 m s^{-1} with 145 meV is barely sufficient to excite most diatomic molecules from their ground state vibration level to the first excited vibration level. Electronic excitation energies of molecules are in the several eV range which would require speeds of several thousand meter per second. How many nitrogen molecules have such velocities at room temperature? To answer this question we first rewrite $f_{MB}(v)dv$ with $dE = mvdv$ as a function of kinetic energy. From

$$f_{MB}(v)dv \rightarrow f_{MB}(E) \frac{dE}{\sqrt{2mE}} \quad \text{with} \quad dv = \frac{dE}{\sqrt{2mE}}$$

follows

$$f_{MB}(E) = \frac{2}{\sqrt{\pi}} \cdot (k_B T)^{-3/2} \cdot E^{1/2} \cdot e^{-E/k_B T}$$

Fig. 9.6 shows $f_{MB}(E)$ for the same three temperatures as fig. 9.5 does for $f_{MB}(v)$. Clearly, $f_{MB}(E)$ is a sharper distribution than $f_{MB}(v)$ and $f_{MB}(E)$ has the same asymmetry as $f_{MB}(v)$ which becomes more pronounced at higher temperatures. Because of

$$\int_0^\infty f_{MB}(v)dv = 1$$

follows

$$\int_0^\infty f_{MB}(E)dE = 1$$

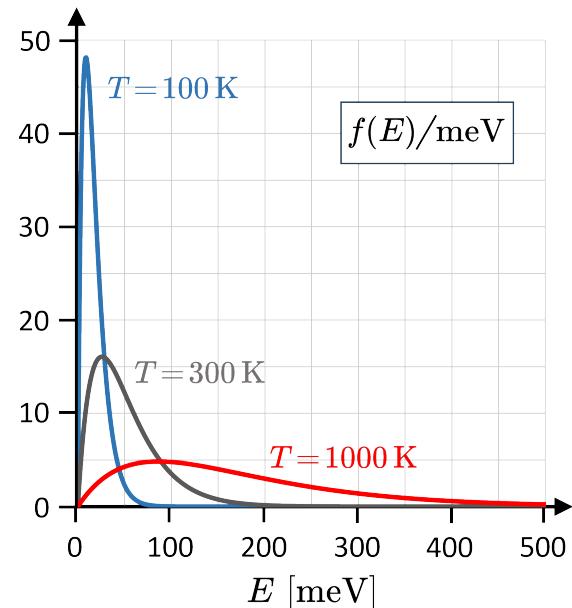


Fig. 9.6

and therefore we immediately know that

$$\int_0^{\infty} E^{1/2} e^{-E/k_B T} dE = \frac{\sqrt{\pi}}{2} (k_B T)^{3/2}$$

However, to get the number of all molecules with energies greater than a certain minimum energy E_0 we must evaluate the integral

$$\int_{E_0}^{\infty} f_{MB}(E) dE = \frac{2}{\sqrt{\pi}} \cdot (k_B T)^{-3/2} \int_{E_0}^{\infty} E^{1/2} \cdot e^{-E/k_B T} dE \quad (9.67)$$

Though this calculation requires a little more work one can show by repeated partial integration that

$$\int E^{1/2} \cdot e^{-E/k_B T} dE = - \left[\sqrt{E} + \frac{k_B T}{2\sqrt{E}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot n!} \left(\frac{k_B T}{E} \right)^n \right] k_B T e^{-E/k_B T}$$

Therefore the definite integral in eq. (9.67) becomes

$$\int_{E_0}^{\infty} f_{MB}(E) dE = \left[\frac{2}{\sqrt{\pi}} \sqrt{\frac{E_0}{k_B T}} + \frac{1}{\pi} \frac{k_B T}{E_0} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot n!} \left(\frac{k_B T}{E_0} \right)^n \right] e^{-E_0/k_B T}$$

We are only interested in the high energy tail of $f_{MB}(E)$ and for $E_0 \gg k_B T$ the second term in this expression quickly goes to zero. At room temperature $k_B T = 26 \text{ meV}$ which is $1 \text{ eV}/40$. Hence, as we are interested in $E_0 > 1 \text{ eV}$ the condition $E_0 \gg k_B T$ is certainly met and the area under the curve $f_{MB}(E)$ in the high energy tail is to a good approximation given by

$$\int_{E_0}^{\infty} f_{MB}(E) dE \approx \frac{2}{\sqrt{\pi}} \sqrt{\frac{E_0}{k_B T}} e^{-E_0/k_B T}$$

Inserting $E_0 = 1 \text{ eV}$ for $f_{MB}(E)$ at 300 K the result is that only a fraction of

$$\frac{2}{\sqrt{\pi}} \sqrt{40} e^{-40} = 3 \cdot 10^{-17}$$

gas particles possess an energy of $E_0 \geq 1 \text{ eV}$. If we take a cubic meter of gas particles at ambient pressure, i.e., $P = 1 \text{ atm}$, then this volume will contain some $2.7 \cdot 10^{25}$ molecules. Of those, only $9 \cdot 10^7$ will possess a kinetic energy $E_{kin} \geq 1 \text{ eV}$. Or put differently, on average each cubic millimeter in that volume contains just a little more than eleven such fast moving gas particles.

9.5 The Mean Free Path

Looking at the Maxwell-Boltzmann distribution we just discussed and how many gas particles move with obviously high velocities raises an obvious question: Why is it that it takes so long for the smell of for example good food to get from one room to another. If gas particles move so fast, should we not smell instantly anywhere in the home if something is burning on the stove? Yes, we should, but only if gas particles were able to travel long distances without colliding with other gas particles. As it is, gas particles collide frequently with other gas particles and instead of gas particles carrying such smells making it from the source to our noses in a bee-line they reach our olfactory organs much rather like drunken sailors on shore leave stumbling in random walk fashion. If gas particles would not collide so frequently, it would take a long time for gases to reach thermal equilibrium any time gas pressure or gas volume change or a gas is heated by an external source. If we increase the heat energy of a gas by compressing it, or decrease it by expanding it, the speed at which a new thermal equilibrium is reached within the gas volume depends on how frequently the atoms or molecules of a gas collide. One can characterize this frequency by either defining a mean time between atom-atom or molecule-molecule collisions or one can characterize this frequency by defining a mean free path for atoms and molecules between two subsequent collisions. This collision frequency depends of course on the density of the gas. However, before discussing the mean free path, it is good to first get a sense of the order of magnitude of the actual average distance \bar{d} between neighbor atoms and molecules. Of course, gas particles are always on the move but for the purpose of determining an average distance we can think of them as if they were frozen in space. Eq. (9.1) tells us that at a given temperature and pressure 1 mol of any given gas occupies the same volume, i.e., its molar volume V_m which contains exactly N_A atoms or molecules:

$$V_m = N_A \frac{k_B T}{P}$$

At the so-called standard conditions, i.e., at 0 °C and at sea level where the pressure is one atmosphere or 1 atm (in SI units: 273.15 K and $1.01325 \cdot 10^5$ Pa). At these conditions, the molar volume V_m of every gas is 22.4 liter. With that follows for the density n of the gas

$$n = N_A / 22.4 \cdot 10^{-3} \text{ m}^3 = 2.69 \cdot 10^{25} \text{ m}^{-3}$$

If we now consider a volume of 1 m³, this volume will contain $2.69 \cdot 10^{25}$ atoms or molecules. If we imagine that each of those atoms or molecules occupies its own tiny

space cube then the volume V_a of each of these cubes containing one atom or one molecule is just $1 \text{ m}^3 / 2.69 \cdot 10^{25}$ or $3.72 \cdot 10^4 \text{ \AA}^3$. Hence, the average distance \bar{d} between neighbor atoms or molecules in a gas under standard conditions is

$$\bar{d} = V_a^{1/3} = n^{-1/3} = 33 \text{ \AA} \quad (9.68)$$

For comparison, the bond length between two hydrogen atoms is 0.74 \AA and the bond length of an oxygen molecule is 1.21 \AA . The average distance between two atoms or molecules in a gas gives us a lower bound for the mean free path of an atom or molecule. Why? Well, if on average the closest neighbor atom or molecule is a distance \bar{d} away it is not possible for an atom or molecule on average to collide with another atom or molecule before having traveled the distance \bar{d} . However, that does not mean that an atom or molecule must collide with another atom or molecule after traveling the distance \bar{d} , meaning, the mean free path can be significantly longer than \bar{d} . How long the mean free path will be is a matter of probability. In section 5.3.1 we introduced the cross section σ as a measure of the probability that two particles will interact. In the simplest case this can be just the geometric cross section like the size of an atom but it usually is a little more complicated. That starts for example with the question of what the size of an atom or molecule really is.

While all such things can be calculated that usually requires quantum mechanical computations. Nevertheless, since as discussed in section 5.3.1 cross sections can be measured, we can just take such measured cross section values and use them to calculate within the confines of Newtonian mechanics. To do that we consider a stream $N(x)$ of gas particles which shall have only a velocity component in the x -direction and which, as sketched in fig. 9.7, we send through a volume containing like gas particles. After traversing a distance dx a certain number of the incoming gas particles will have undergone collisions with gas particles in the volume $dV = Adx$ where A is the cross section of the gas stream in the plane perpendicular to the x -direction. If we project all the gas particles in this volume onto the entrance surface of this volume then the sum of their respective geometric cross sections will cover a certain area of this surface A . If the scattering cross section for such collisions should be different from their geometric cross section then we will use that value which we will just call σ . If n is the density of gas particles in the volume dV then this volume will contain $n \cdot Adx$ gas particles, each of which has a cross section σ . The

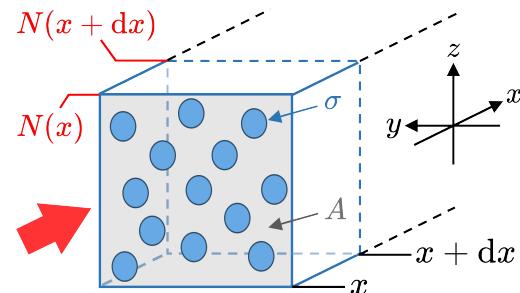


Fig. 9.7

sum of all those projected cross sections covers an area $n \cdot A \sigma dx$. The ratio of this area to the area A , i.e.,

$$\frac{n \cdot A \sigma dx}{A} = n \cdot \sigma dx$$

is then a measure for the fraction of gas particles in the stream $N(x)$ which will undergo a collision in the volume dV and which will be missing in the stream after traversing the distance dx . Therefore we can write

$$N(x + dx) - N(x) = dN = -N(x)n \cdot \sigma dx$$

which after separation of the variables becomes

$$\frac{dN}{N} = -n \cdot \sigma dx$$

This differential equation can be solved by simple integration with the result

$$N(x) = N(0)e^{-n\sigma x} \quad (9.69)$$

With eq. (9.69) one now defines as the mean free path length the distance after which the number of gas particles has declined by the factor $1/e$ meaning that the mean free path length l satisfies the equation $n \cdot \sigma \cdot l = 1$ and with that

$$l = \frac{1}{n \cdot \sigma} \quad (9.70)$$

With eq. (9.68) and eq. (9.70) we get for the ratio of the average distance \bar{d} between two gas particles to the mean free path l of a gas particle

$$\frac{\bar{d}}{l} = n^{-1/3} \cdot n \cdot \sigma = n^{2/3} \cdot \sigma$$

Because l must be greater or equal to \bar{d} , it follows for σ that

$$\sigma \leq n^{-2/3}$$

must hold. This means nothing else than that in our example from fig. 9.7 σ cannot be larger than the surface area $n^{-2/3}$. Going back to our example of a gas at standard conditions for which we calculated \bar{d} in eq. (9.68), we now can say that for such a gas

$$\sigma \leq 1.11 \cdot 10^{-17} \text{ m}^2 = 1.11 \cdot 10^3 \text{ Å}^2$$

must hold. This upper bound for σ is very much larger than any geometric cross section of atoms or molecules. If we use a hard sphere model then the geometric cross section of an atom or molecule will be $\pi \cdot d^2$ where d is the equivalent diameter of a hard sphere resulting in the respective measured scattering cross section. For air molecules like oxygen or nitrogen such hard sphere equivalent diameters are around 3 Å, leading to a cross

section estimate of $\sigma \approx 28 \text{ \AA}^2$. Using the ideal gas law eq. (9.1), we know that at room temperature and ambient pressure ($P = 1 \text{ atm}$) the density of air is $n = 2.69 \cdot 10^{25} \text{ m}^{-3}$. Therefore, at standard conditions the mean free path of an oxygen or nitrogen molecule in air will be about

$$l = \frac{1}{2.69 \cdot 10^{25} \cdot 28 \cdot 10^{-20}} \text{ m} = 1.32 \cdot 10^{-7} \text{ m} = 132 \text{ nm}$$

Instead of asking how far on average a gas particle travels before colliding with another gas particle, one can just as well ask how long on average it takes before a gas particle in a gas collides with another. Obviously, the result will look like

$$N(t) = N(0)e^{-t/\tau} \quad (9.71)$$

where $N(0)$ now is the number of gas particles in the stream at $t = 0$ traveling without a collision and τ is the time after which the number of such particles without collision will have decreased to $N(\tau) = N(0)/e$. The mean time between collisions τ and the mean free path l are linked by $l = v \cdot \tau$ where v is the velocity of a gas molecule. For the purpose of an estimate one can insert for v the mean velocity or the most probable velocity in the Maxwell-Boltzmann velocity distribution eq. (9.66). Here we use the mean velocity and with that the estimate for the mean time between collision for τ is

$$\tau \approx l \cdot \sqrt{\frac{\pi m}{8k_B T}}$$

For the example of air at standard conditions where we just calculated above $l = 132 \text{ nm}$ the estimate for τ becomes for nitrogen molecules with mass $m = 28 \text{ amu}$

$$\tau \approx 132 \cdot 10^{-9} \text{ m} \sqrt{\frac{\pi \cdot 28 \cdot 1.66 \cdot 10^{-27} \text{ kg}}{8 \cdot 1.38 \cdot 10^{-23} \text{ J} \cdot \text{K}^{-1} \cdot 300 \text{ K}}} = 2.8 \cdot 10^{-10} \text{ s} = 0.3 \text{ ns}$$

This estimate shows that the order of magnitude for the mean time between collisions τ is a fraction of a nanosecond. Hence, after only a fraction of a nanosecond the number of molecules not experiencing a collision has declined by a factor of $1/e$.

9.6 Brownian Motion

The atomic theory is an old one, going back to the Greek philosopher Democritus. The first evidence that atoms really existed, even though that was not recognized at the time, came from the botanist Robert Brown who observed in 1827 under the microscope how pollen suspended in water got pushed around by an invisible force in a seemingly random fashion. It was only with the development of the kinetic gas theory, the fundamentals of which this chapter introduced, that scientists began to attribute Brownian motion to

collisions with atoms and molecules surrounding such suspended microscopic particles. A first quantitative theory of Brownian motion was only proposed in 1905 by Einstein and then independently in 1906 by Marian Smoluchowski (1872 - 1917). The jiggling movement of a microscopic particle suspended in a gas or a liquid which one can observe is of course not due to any individual impact, rather, every single jiggle is the result of averaging over very many such collisions between the microscopic particle and fluid particles. These collisions are completely random and therefore the resulting motions of the microscopic particle in the three coordinate directions are uncorrelated and can be treated separately. To correlate the random motion of the suspended microscopic particle, lets say in the x -direction, with the temperature of the respective fluid, be it a liquid or gas it is suspended in, we will assume that the equipartition theorem applies to it in the same way as we discussed this for gas particles in section 9.1. This means that for the microscopic particle of mass m

$$\frac{1}{2}m\langle v_x^2 \rangle = \frac{1}{2}k_B T \quad (9.72a)$$

or respectively

$$m\bar{x^2} = k_B T \quad \text{or} \quad m\overline{\left(\frac{dx}{dt}\right)^2} = k_B T \quad (9.72b)$$

shall hold. In this equation the average in eq. (9.72b) is written in a different from than in eq. (9.72a). The reason for this is simply that the averages in eq. (9.72b) are indeed different from the average in eq. (9.72a) which we used before in section 9.1. In section 9.1 we used the average over many gas particles, the so-called ensemble average. The average in eq. (9.72b) is however a time average, which we indicate by a bar over the variable we are averaging. What we are claiming here is that if one took a snapshot of a gas and determined the average velocity by adding all the velocities the N gas particles have at this very moment and then divide this sum by N , the result is the same as if one just followed a single gas particle around and measured its velocity periodically over a very long time, added up all those values and then divided this sum by the number of measurements made to get the average velocity value. Put differently, if we were to observe a gas particle just long enough we would measure it having all the velocity values anyone gas particle has at a given moment in time in a gas containing very many gas particles. This equivalence of ensemble average and time average is a consequence of the so-called ergodic hypothesis, the discussion and proof of which is beyond the scope of this physics course. Here we just note, that it allows us to equate the time average of a velocity component of a single microscopic particle with a temperature. In addition to the assumption that Brownian motion is an ergodic process we will assume that the microscopic particle of mass m and

radius r_0 , in addition to being subject to the drag of the frictional force slowing it down according to Stokes law, shall also be subject to a completely random force F_{rdm} . This random force F_{rdm} shall be the net result of the random collisions with gas particles the microscopic particle is subjected to. With Newton we can then write down the equation of motion for the microscopic particle as

$$m \frac{d^2x}{dt^2} = -6\pi\eta \cdot r_0 \cdot \frac{dx}{dt} + F_{rdm} \quad (9.73)$$

where the first term on the right hand side comes from eq. (7.10), i.e., Stokes law, and the second term is the random force resulting from the collisions with gas particles. To solve eq. (9.73) we use the simple trick of multiplying it with x to give

$$mx \frac{d^2x}{dt^2} = -6\pi\eta \cdot r_0 \cdot x \frac{dx}{dt} + xF_{rdm}$$

which with

$$\frac{d}{dt} \left(x \frac{dx}{dt} \right) = \left(\frac{dx}{dt} \right)^2 + x \frac{d^2x}{dt^2} \quad \text{and} \quad x \frac{dx}{dt} = \frac{1}{2} \frac{d}{dt} x^2$$

then allows us to rewrite eq. (9.73) as

$$\frac{m}{2} \left(\frac{d^2}{dt^2} x^2 \right) - m \left(\frac{dx}{dt} \right)^2 = -3\pi\eta \cdot r_0 \cdot \left(\frac{d}{dt} x^2 \right) + xF_{rdm}$$

To make use of eq. (9.72b) we need the time averaged value of the velocity component in x -direction. Hence, we must consider the time averages for the above equation, i.e.,

$$\frac{m}{2} \cdot \overline{\left(\frac{d^2}{dt^2} x^2 \right)} - m \cdot \overline{\left(\frac{dx}{dt} \right)^2} = -3\pi\eta \cdot r_0 \cdot \overline{\left(\frac{d}{dt} x^2 \right)} + \overline{x F_{rdm}} \quad (9.74)$$

Because F_{rdm} is completely random, the time average of xF_{rdm} is zero and according to eq. (9.72b) the second component on the left side of eq. (9.74) is equal to $k_B T$. In many problems like this one, one can interchange differentiation and averaging, i.e.,

$$\overline{\left(\frac{d^2}{dt^2} x^2 \right)} = \frac{d^2}{dt^2} \overline{x^2} \quad \text{and} \quad \overline{\left(\frac{d}{dt} x^2 \right)} = \frac{d}{dt} \overline{x^2}$$

With that eq. (9.74) becomes

$$\frac{m}{2} \cdot \frac{d^2}{dt^2} \overline{x^2} + 3\pi\eta \cdot r_0 \cdot \frac{d}{dt} \overline{x^2} = k_B T \quad (9.75)$$

The standard way to solve this second order differential equation for the mean square displacement $\overline{x^2}$ is to substitute $\xi = (d/dt)\overline{x^2}$ resulting in the first order differential equation

$$\dot{\xi}(t) = -\frac{6\pi\eta \cdot r_0}{m} \xi(t) + \frac{2k_B T}{m} \quad (9.76)$$

The homogeneous version of this differential equation, i.e., without the second term on the right hand side, has the obvious solution

$$\xi_h(t) = C \exp\left(-\frac{6\pi\eta \cdot r_0}{m} \cdot t\right)$$

To find a special solution for the inhomogeneous differential equation one varies the constant, i.e., $C \rightarrow C(t)$, which inserted into eq. (9.76) results in the equation

$$\dot{C}(t) = \frac{2k_B T}{m} \exp\left(\frac{6\pi\eta \cdot r_0}{m} \cdot t\right)$$

from which follows

$$C(t) = \frac{2k_B T}{m} \int_0^t \exp\left(\frac{6\pi\eta \cdot r_0}{m} \cdot t'\right) dt' = \frac{k_B T}{3\pi\eta \cdot r_0} \exp\left(\frac{6\pi\eta \cdot r_0}{m} \cdot t\right)$$

Therefore, a special solution of the inhomogeneous differential equation eq. (9.76) is

$$\xi_s = C(t) \exp\left(-\frac{6\pi\eta \cdot r_0}{m} \cdot t\right) = \frac{k_B T}{3\pi\eta \cdot r_0}$$

With that follows the general solution of the inhomogenous differential equation eq. (9.76) as the sum of the special solution of the inhomogeneous differential equation and all possible solutions of the homogeneous differential equation, i.e., $\xi(t) = \xi_s + \xi_h$ and with that

$$\xi(t) = \frac{k_B T}{3\pi\eta \cdot r_0} + C \exp\left(-\frac{6\pi\eta \cdot r_0}{m} \cdot t\right) \quad (9.77)$$

Lets first consider the significance of the exponential term which decays with time. Assuming a typical microscopic particle with a one micrometer diameter, a density of 1 g cm^{-3} , in air or in water with the respective dynamic viscosity at room temperature of air and water being $1.83 \cdot 10^{-4} \text{ g cm}^{-1} \text{ s}^{-1}$ and $0.01 \text{ g cm}^{-1} \text{ s}^{-1}$ it is easy to see that the exponential term becomes very small very quickly, i.e., on the order of microseconds. Therefore, we can neglect the exponential term and by reinserting $\xi = (d/dt)\bar{x}^2$ eq. (9.77) then becomes

$$\frac{d}{dt}\bar{x}^2 = \frac{k_B T}{3\pi\eta \cdot r_0}$$

After a simple integration the final result for the average square displacement \bar{x}^2 of a microscopic particle suspended in a fluid, be it gaseous or liquid, after a time t has passed, is then given by

$$\bar{x}^2(t) = \frac{k_B T}{3\pi\eta \cdot r_0} \cdot t \quad (9.78)$$

Using the Stokes-Einstein relation from eq. (9.63) which connects the diffusion coefficient D with the mobility μ and the temperature T we can rewrite this equation as

$$\overline{x^2(t)} = 2D \cdot t = 2\mu k_B T \cdot t \quad (9.79)$$

Because the average square displacements in the three respective coordinate directions are completely independent from each other, we can move from the one dimensional case in eq. (9.78) to the two dimensional and three dimensional cases simply by noting that $\overline{x^2} = \overline{y^2} = \overline{z^2}$. Therefore to get the result for the two or three dimensional cases one just has to multiply eq. (9.77) with a factor 2 or 3 respectively. Eq. (9.78) and eq. (9.79) rewritten for motion in two dimensions therefore become

$$\overline{r^2(t)} = \frac{2k_B T}{3\pi\eta \cdot r_0} \cdot t = 4D \cdot t = 4\mu k_B T \cdot t \quad (\text{2-dimensional case})$$

and for motion in three dimensions one has the relations

$$\overline{r^2(t)} = \frac{k_B T}{\pi\eta \cdot r_0} \cdot t = 6D \cdot t = 6\mu k_B T \cdot t \quad (\text{3-dimensional case})$$

The average displacement of a microscopic particle over time is a measurable quantity. If one knows the mobility μ , i.e., the dynamic viscosity η of the fluid and the equivalent radius r_0 of the microparticle, then one can determine the value of the Boltzmann constant k_B and with that Avogadro's number N_A . Hence, the microscopic understanding of Brownian motion at the beginning of the nineteenth century enabled for the first time a direct experimental determination of k_B and N_A .

10. Oscillating Motion

At the end of section section 2.3.4 we discussed the basics of motion and found that uniform circular motion and harmonic motion are identical. Examples of harmonic motion, more frequently referred to as harmonic oscillations, are ubiquitous in nature. The reason for this is simple. If a force acts on a system in a stable equilibrium in such a way as to push it slightly out of that equilibrium, restoring forces intrinsic to the system will work to push the system back into equilibrium. As almost everything in nature is in a state of static or dynamic equilibrium, many systems respond to an external force causing a slight displacement from that equilibrium state with small harmonic oscillations around their equilibrium state.

10.1 Harmonic Oscillation

Eq. (3.1), the mathematical expression of Newton's second law, can be interpreted in three different ways. First, as a definition of a bodies mass m and how to determine it; second, as an expression of the force \mathbf{F} determining a bodies motion; and thirdly, as a means to determine the acceleration \mathbf{a} . The first interpretation is relevant as it provides the fundamental definition of inertial mass: The inertial mass is defined as the proportionality constant which links a specific force \mathbf{F} to a specific acceleration \mathbf{a} a body experiences. The second interpretation underlies kinematic analysis by asking the question: If a body of mass m changes its state of motion because of a force \mathbf{F} resulting in an acceleration \mathbf{a} , what does the bodies trajectory look like? The third interpretation leads to the integration of the equation of motion by asking the question: How can one understand why a body of mass m moves as it does when acted on by a force \mathbf{F} ? In the rest of this section we will look at this last question to get a better understanding of the motion of harmonic oscillation. The motion physicists refer to as harmonic oscillation is of fundamental importance in physics as it accurately models the behavior of many physical systems subject to a force which makes them oscillate

around a stable equilibrium. That includes for example the motion of pendula for small displacements, masses connected to springs and many other cases of mechanic vibrations. Harmonic oscillation however must not involve the physical movement of bodies as the electronic oscillating circuit exemplifies. Here we will look at the harmonic oscillation of masses suspended from springs for the case of simple harmonic oscillation, damped harmonic oscillation and forced harmonic oscillation.

10.1.1 Simple Harmonic Oscillation

Newton's second law tells us that the force driving the harmonic oscillation of a point mass m is proportional to the negative of the amplitude of the oscillation, the latter being the displacement of the point mass from its rest position. Choosing for example the z -direction as the coordinate of harmonic oscillation and a general proportionality constant k , this force is given by

$$F = m \frac{d^2z}{dt^2} = -kz \quad (10.1)$$

Any time we encounter such a force F acting on a point mass we now know that the respective point mass motion will always be that of an harmonic oscillator. One of the most simple examples for such a harmonic oscillator is that of a weight hanging from a spring. At rest, the spring stretches to the extent that it balances gravity as illustrated on the left side of fig. 10.1. Our choice of coordinates is such that the z -axis is parallel to the axis of the spring and in its rest position the point mass shall find itself at the position $z = 0$. Next, as shown on the right side of fig. 10.1, pulling the point mass vertically down, for example by hand, stretches the spring to the extent that the point mass position becomes $z = z_1$. Now, at the time $t_0 = 0$, we release the point mass at which moment the restoring force of the spring begins to drive the motion of the point mass. Provided that we have not extended the spring too much, Hooke's law applies and at t_0 the restoring force of the spring will be $F = -kz$ with $z(t_0) = z_1$. It was Robert Hooke who discovered that the force needed to extend or compress a spring will scale linearly with spring extension or compression as long as these remain small compared to the total possible compression or extension of the spring. In other words, as long as the spring deformation is elastic the spring will revert to its original state. Forces of the form $F = \pm k\Delta z$ are therefore

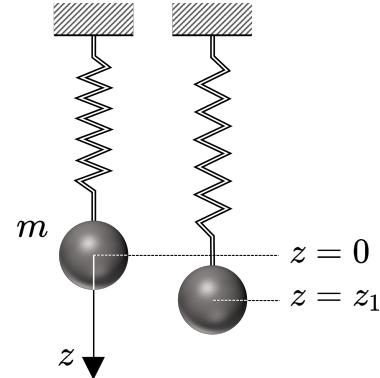


Fig. 10.1

referred to as elastic forces because once they are removed the temporary deformation, here of the spring, is self-reversing. The proportionality constant k is a material property and in the case of a spring it is called the spring constant. Hooke's law applies of course also to the point mass in its rest position in fig. 10.1. In that case the spring's restoring force balances gravity, the weight of the point mass, and hence:

$$mg - k\Delta z = 0 \quad \text{or} \quad \Delta z = \frac{mg}{k}$$

If the point mass is removed from the spring, the spring will contract by Δz . The two forces, the restoring force balancing gravity and the additional restoring force balancing the pulling force of our hand are additive. However, as only the pulling force of our hand is removed at t_0 it is only this force component which will drive the harmonic oscillation of the point mass. Extrapolating from the uniform harmonic motion discussed in section 2.3.4, the general solution for point mass motion driven by eq. (10.1) is

$$z = A \cos \omega_0 t + B \sin \omega_0 t \quad (10.2)$$

with the point mass velocity given by

$$\dot{z} = v_z = A\omega_0 \sin \omega_0 t - B\omega_0 \cos \omega_0 t$$

Here the initial conditions at $t_0 = 0$ are $z(t_0) = z_1$ for the point mass position and $v_z(t_0) = 0$ for its velocity. Therefore B must be 0 and A must equal z_1 . With that the equation describing the motion of the harmonic oscillator becomes

$$z(t) = z_1 \cos \omega_0 t$$

Inserting this into eq. (10.1) the frequency ω_0 of the harmonic oscillation for a system as shown in fig. 10.1 with spring constant k is given by

$$\omega_0 = \sqrt{\frac{k}{m}}$$

The frequency of the harmonic oscillation ω_0 only depends on the spring constant k and the oscillator mass m ; it does not depend on the amplitude of the oscillation. As illustrated in fig. 10.2, increasing the oscillator mass by a factor of 2 will reduce the oscillation frequency by a factor of $\sqrt{2}$. Doubling the spring constant using a spring with twice the spring constant or using two identical springs to support the point mass weight will result in multiplying the oscillation frequency by a factor of $\sqrt{2}$.

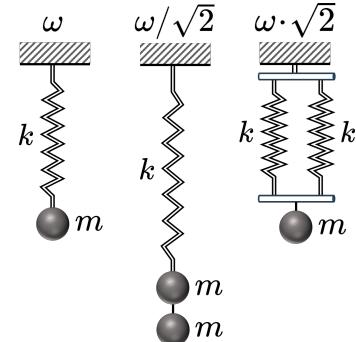


Fig. 10.2

Energy of Simple Harmonic Oscillation

In simple harmonic oscillation the kinetic energy is that of the point mass and the potential energy is that of the spring which we already know from eq. (4.2). Hence, the total energy $E = T + U$ of simple harmonic oscillation is

$$E = \frac{m\dot{z}^2}{2} + \frac{kz^2}{2} = \frac{kz_1^2}{2}$$

The total energy of simple harmonic oscillation is set at the time t_0 by the displacement $z_1 = z(t_0)$ from which the point mass is being released to freely oscillate around its rest position at $z_0 = 0$ (compare eq. (4.2)). As the point mass oscillates between $\pm z_1$ around its rest position at $z = 0$ the total energy does not change. At the turning points $z = \pm z_1$ the potential energy of the spring reaches its maximum values ($U_{max} = E$), while with $v(x = \pm z_1) = 0$ the kinetic energy becomes zero at these points ($T_{min} = 0$). When crossing the rest position at $z = z_0$ the potential energy of the spring becomes zero ($U_{min} = 0$), while the kinetic energy reaches its maximum ($T_{max} = E$). As shown in fig. 10.3, the shape of the potential energy curve of the spring is a parabola and with that, the shape of the kinetic energy curve of the spring must be an inverted parabola, ensuring that $E = T + U = const.$

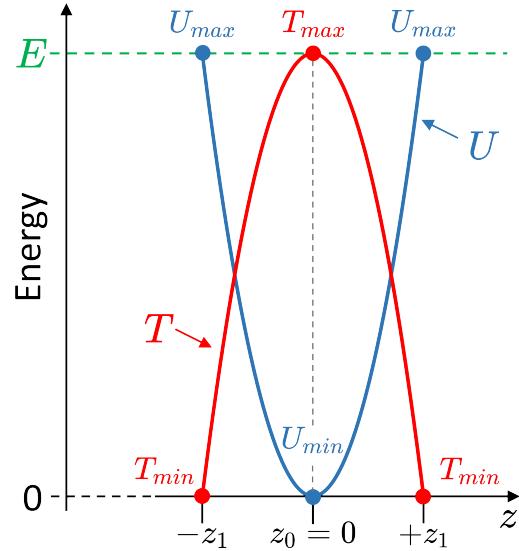


Fig. 10.3

10.1.2 Damped Harmonic Oscillation

The ideal harmonic oscillator just discussed will continue to oscillate for ever but that is of course not what usually happens, as our practical experience tells us. Over time, we will observe that the amplitude of the oscillation decreases and eventually, the oscillation will completely stop once the amplitude has decreased to zero. The motion of such a damped oscillation can often be described by introducing a simple exponential amplitude decay factor into the oscillator equation

$$z(t) = z_1 e^{-\beta t} \cos \omega t$$

The velocity of this damped oscillator motion is given by

$$\dot{z} = -z_1 e^{-\beta t} [\beta \cos \omega t + \omega \sin \omega t]$$

Differentiating one more time gives for the acceleration

$$\ddot{z} = z_1 e^{-\beta t} [\beta^2 \cos \omega t + 2\beta \omega \sin \omega t - \omega^2 \cos \omega t]$$

Adding inside the bracket on the left side a zero by inserting $0 = 2\beta^2 \cos \omega t - 2\beta^2 \cos \omega t$, the equation of motion can be rewritten as

$$\ddot{z} = -\omega^2 \left[1 + \frac{\beta^2}{\omega^2} \right] z - 2\beta \dot{z}$$

In the case of $\beta = 0$, this reduces of course to the equation for the undamped oscillation and ω will equal ω_0 of the undamped harmonic oscillator. If $\beta > 0$, then a dampening term proportional to the velocity of the oscillating point mass appears. Writing the force associated with damped oscillation as

$$m\ddot{z} = -kz + \text{damping term} = -m(\omega^2 + \beta^2)z - 2m\beta \dot{z} \quad (10.3)$$

shows that not just the amplitude of the damped oscillator decreases but that also its frequency ω is lower than that of the undamped oscillator ω_0 with

$$\omega = \sqrt{\frac{k}{m} - \beta^2} = \sqrt{\omega_0^2 - \beta^2}$$

If we assume that the decay constant β is much smaller than ω_0 , i.e., $\beta \ll \omega_0$, then

$$\omega = \omega_0 \sqrt{1 - \frac{\beta^2}{\omega_0^2}} \approx \omega_0 \quad (10.4)$$

and the equation of the damped oscillator motion translates into the force equation

$$F = -m\omega_0^2 z - 2m\beta v_z$$

The first term on the right side is the restoration force as we know it from the undamped harmonic oscillator with $k = m\omega_0^2$. The second term, the damping term, represents a friction force such as discussed in section 7.1. This friction force is proportional to the velocity of the moving point mass and it acts in the direction opposite to where the velocity vector points. The proportionality factor $2m\beta$ is called the friction coefficient γ and in general one finds such friction forces expressed as

$$\mathbf{F}_{\text{Friction}} = -\gamma \mathbf{v} \quad (10.5)$$

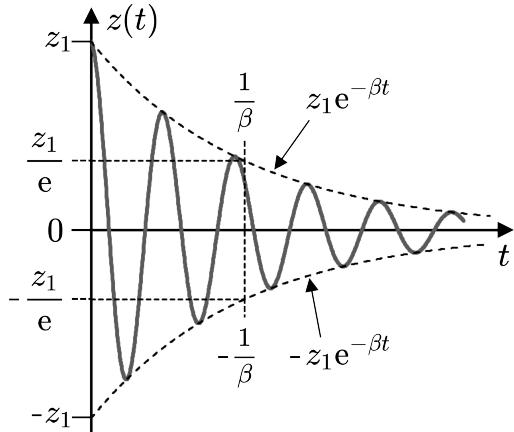


Fig. 10.4

where γ has to be determined experimentally. For the damped harmonic oscillation it can for example be determined from the observed decrease in amplitude (see fig. 10.4). A decrease in amplitude means of course that total energy is not conserved for the damped harmonic oscillator as a fraction of it is being dissipated through frictional forces.

Energy of Damped Harmonic Oscillation

Just as in the case of simple harmonic oscillation, the total energy of the damped harmonic oscillator at the release time of the point mass is determined by its initial amplitude, i.e., by its displacement z_1 from its rest position at $z = 0$. However, unlike for the simple harmonic oscillator, it does not stay this way for the damped harmonic oscillator because the amplitude of its turning points is time dependent and so is its total energy. With $z_1 \rightarrow z_1 e^{-\beta t}$ the total energy of the damped harmonic oscillator is not constant anymore but decreases exponentially.

$$E = \frac{1}{2} k z_1^2 e^{-2\beta t} = E_0 \cdot e^{-2\beta t}$$

Fig. 10.5 shows the exponential decay of the total energy E . At any point in time on this curve, energy is being converted from potential energy to kinetic energy and back, as long as $E(t) > 0$. A measure for the exponential energy decrease is the time τ it takes for E_0 to decrease by the factor e^{-1} . The shorter / longer τ , the faster / slower the damped harmonic oscillator loses its energy. Through β , this decay time τ is inverse proportional to the friction coefficient:

$$\tau = \frac{1}{2\beta} = \frac{1}{2} \frac{2m}{\gamma} = \frac{m}{\gamma} \quad (10.6)$$

Hence, by measuring τ and knowing the point mass m we can determine the friction coefficient γ . With γ und τ we now can rewrite eq. (10.3), the equation of motion of the damped oscillator, in its common forms as

$$m\ddot{z} + \gamma\dot{z} + kz = 0 \quad \text{or} \quad \ddot{z} + \frac{1}{\tau}\dot{z} + \omega_0^2 z = 0 \quad (10.7)$$

We will come back to this equation when we will discuss the damped forced oscillator. In most applications it is of great interest to know how fast an oscillator looses energy due to friction forces. This requires that we determine the rate at which the energy of an oscillator decreases as compared to the energy still stored in the oscillator. A quick calculation shows that this quantity is proportional to $-\tau^{-1}$, i.e.,

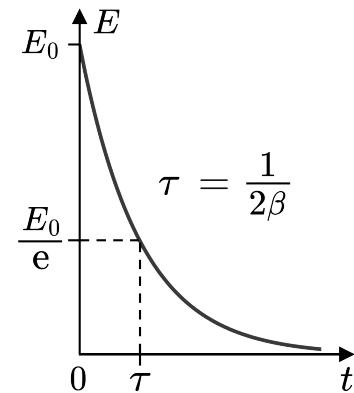


Fig. 10.5

$$\frac{1}{E} \frac{dE}{dt} = -2\beta = -\frac{1}{\tau}$$

The inverse absolute value of this quantity multiplied with the frequency, i.e., the amount of total energy an oscillator stores relative to the amount of energy it loses over the time ω^{-1} , is an important technical measure for the quality of an oscillator. This dimensionless parameter usually designated by the letter Q is given by

$$Q = \omega \cdot E \cdot \left| \frac{dE}{dt} \right|^{-1} = \omega \tau \quad (10.8)$$

Q is essentially a simple measure of oscillator damping. If Q is very high, the oscillator loses little energy because of dampening effects, if Q is low, it loses its energy quickly because of dampening. The frequency of damped harmonic oscillation we take from eq. (10.4) and with that Q becomes

$$Q = \omega_0 \tau \sqrt{1 - \frac{1}{(2\omega_0 \tau)^2}} \quad \Rightarrow \quad Q = \omega_0 \tau \quad (10.9)$$

10.1.3 Forced Harmonic Oscillation

Of course, it is possible to counteract the damping force by applying an external force and keep the harmonic oscillation going, despite the friction force dampening it. In this case one speaks of a forced harmonic oscillation. To keep things simple we will drop the dampening force here completely and only consider the impact of an additional external force being applied. With that the force equation to consider is

$$m\ddot{z} = -kz + F(t) = -m\omega_0^2 z + F(t) \quad (10.10)$$

where $F(t)$ is the external force which is periodically applied with a frequency ω

$$F(t) = F_0 \cos \omega t \quad \text{and} \quad \omega_0^2 = \frac{k}{m}$$

The frequency ω of the force can be different from ω_0 , the frequency of the ideal harmonic oscillator. Eq. (10.10) is an inhomogeneous differential equation. Its general solution is given by the solution of the homogeneous differential equation (that is the ideal harmonic oscillator) and a special solution of the inhomogeneous differential equation. The general solution for the homogeneous differential equation we already have with eq. (10.2). To find a special solution of the inhomogeneous differential equation we make use of the experimental observation that the oscillation of the displacement $z(t)$ will follow the oscillation of the applied force $F(t)$. However, such observations also show us that $z(t)$

does not follow $F(t)$ instantaneously but does so with a time lag. In the forced oscillator model we are considering here this is not the case because we did not include frictional forces. We will correct this later when we look into this phase shift between $z(t)$ and $F(t)$ caused by frictional forces in detail but for now we will ignore it. With that the general solution ansatz for $z(t)$ becomes

$$z(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

Inserting this approach into eq. (10.10) yields

$$-m\omega^2 z = -m\omega_0^2 z + F_0 \cos \omega t$$

Hence a special solution of eq. (10.10) is

$$z = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

and we get as the general solution for the motion of the forced oscillator

$$z = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

and for its velocity

$$\dot{z} = \omega A \sin \omega_0 t - \omega B \cos \omega_0 t + \frac{\omega F_0}{m(\omega_0^2 - \omega^2)} \sin \omega t$$

For simplicity, we will choose here for $t_0 = 0$ as initial condition that the point mass be in its rest position, i.e., $z(t_0) = 0$ and $\dot{z}(t_0) = 0$. Inserting this into the above equations for $z(t)$ and $\dot{z}(t)$ we get for the constants A and B

$$A = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0$$

With that the solution for the forced harmonic oscillator becomes

$$z = \frac{F_0}{m(\omega_0^2 - \omega^2)} [\cos \omega t - \cos \omega_0 t]$$

Using the trigonometric identity

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

and switching the sequence of the sine terms to rewrite the solution for the motion of the forced oscillator we get

$$z = \frac{-2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2} \quad (10.11)$$

This equation shows that the motion of forced harmonic oscillation has three components: a frequency dependent amplitude, a slower oscillating term, and a faster oscillating term.

To better understand this motion it helps to consider three cases: $\omega < \omega_0$, $\omega \approx \omega_0$ and $\omega > \omega_0$. For $\omega > \omega_0$ the amplitude is negative while for $\omega < \omega_0$ it is positive. This indicates that in the first case the force is directed opposite to the direction of the displacement of the point mass from its rest position while in the second case both point in the same direction.

$$\omega = 1.25 \cdot \omega_0$$



$$\omega = 1.15 \cdot \omega_0$$



$$\omega = 1.05 \cdot \omega_0$$

→ t

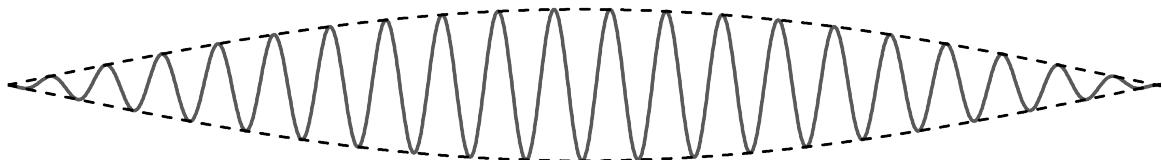


Fig. 10.6: Forced harmonic oscillation for three frequency ratios. Dashed lines outline the respective amplitude modulation.

Fig. 10.6 shows the amplitude behavior in forced harmonic oscillation for three examples where ω is 125% of ω_0 , 115% of ω_0 , and 105% of ω_0 . The dashed lines in fig. 10.6 outline the amplitude modulation by the first sine term in eq. (10.11), the smaller the difference between ω and ω_0 the slower the amplitude modulation. Because of the symmetry of the sine, this amplitude envelope would look the same if ω were 75% of ω_0 , 85% of ω_0 , and 95% of ω_0 instead of $\omega = 1.25\omega_0$, $\omega = 1.15\omega_0$ and $\omega = 1.05\omega_0$ as shown in fig. 10.6. However, the number of oscillations within the envelope would be somewhat smaller. As ω approaches ω_0 the amplitude in eq. (10.11) increases and it would go to infinity if $\omega = \omega_0$, which is not what happens with real world forced oscillators. The above solution eq. (10.11) does not apply for the resonance case of $\omega = \omega_0$ and we will soon see that in this case the amplitude rather increases linearly with t than becoming infinite. Of course, linear increases also have their limits in the real world and at some point, if the amplitude becomes too large, the spring will just brake. For $\omega \ll \omega_0$ the motion of the forced oscillator will ever closer resemble the motion of the ideal oscillator while for $\omega \gg \omega_0$ the amplitude will decrease ever more to the point where it will be difficult to perceive any movement at all.

10.1.4 Damped Forced Harmonic Oscillation

As we just saw, the solution we derived for the forced oscillator is not applicable in the case where $\omega = \omega_0$. This is because the equation of motion of the forced oscillator model which we used in eq. (10.10) did not include the friction term. We already know from eq. (10.7) what the equation for an oscillator subject to frictional forces looks like - it is the equation of motion of the damped oscillator. Now we will apply an external force to the damped oscillator model and explore what the solutions to the equation of motion for the damped forced oscillator look like. We begin with eq. (10.7) to which we add an external force in the same way we just did in the previous section for the simple harmonic oscillator (see eq. (10.10)):

$$\ddot{z} + \frac{1}{\tau} \dot{z} + \omega_0^2 z = \frac{F(t)}{m} \quad (10.12)$$

We will remind ourselves that in this equation

$$\omega_0^2 = \frac{k}{m} \quad \text{and} \quad \frac{1}{\tau} = \frac{\gamma}{m}$$

where γ is the friction coefficient we already know from eq. (10.6). To solve eq. (10.12) we will use a different method than we one we resorted to for solving eq. (10.10). We do that by switching to complex numbers, i.e., for the displacement $z(t)$ and the external force $F(t)$ we make the ansatz

$$F(t) = \hat{F} e^{i\omega t} \quad \text{and} \quad z = \hat{z} e^{i\omega t}$$

where \hat{F} and \hat{z} are complex amplitudes that do not depend on time. Inserting this in eq. (10.12) we get the complex equation

$$-\omega^2 \hat{z} e^{i\omega t} + i \frac{\omega}{\tau} \hat{z} e^{i\omega t} + \omega_0^2 \hat{z} e^{i\omega t} = \frac{\hat{F}}{m} e^{i\omega t} \quad (10.13)$$

Because all the exponentials cancel out, we can solve for \hat{z} using simple algebra to obtain

$$\hat{z} = \frac{\hat{F}}{m[(\omega_0^2 - \omega^2) + i(\omega/\tau)]} = \hat{F} \cdot R \quad (10.14)$$

where

$$R = \frac{1}{m[(\omega_0^2 - \omega^2) + i(\omega/\tau)]} = \rho e^{i\varphi} \quad (10.15)$$

is a complex number with amplitude ρ and phase φ . With the complex amplitude $\hat{F} = F_0 e^{i\theta}$ we can therefore rewrite eq. (10.14) as

$$\hat{z} = F_0 e^{i\theta} \rho e^{i\varphi} = \rho F_0 e^{i(\theta+\varphi)}$$

and thus obtain as the solution of the complex equation eq. (10.13):

$$\hat{z} e^{i\omega t} = \rho F_0 e^{i(\theta+\varphi)} e^{i\omega t} = \rho F_0 [\cos(\omega t + \theta + \varphi) + i \sin(\omega t + \theta + \varphi)]$$

What we are interested in is the real part of this expression which is the solution $z(t)$ of the forced oscillator equation eq. (10.12). Eq. (10.16) shows this real part, the displacement $z(t)$, along with the external force $F(t)$ driving this response.

$$\left. \begin{aligned} z(t) &= \rho F_0 \cos(\omega t + \theta + \varphi) \\ F(t) &= F_0 \cos(\omega t + \theta) \end{aligned} \right\} \quad (10.16)$$

Before we discuss this solution, consider for a moment how elegantly we obtained it through the use of the complex number approach. There was no need to solve tricky differential equations. All we required was some complex number algebra and trigonometry and after a few lines we had the solution in front of us. This just shows how powerful the complex number approach is to solve such problems. All we have to do at the end is to extract the physics solution which is the real part of the complex solution.

Eq. (10.16) shows that the time response of the displacement $z(t)$ is shifted in phase by φ with respect to the driving force $F(t)$. This is the time lag we referred to when we first discussed the forced oscillator in section 10.1.3 without considering the frictional force. To determine the value of φ we need to separate the real and imaginary parts of R in eq. (10.15). To accomplish this we first determine the value of ρ , which we do by calculating the square of the absolute value of R in eq. (10.15). With $|R|^2 = R \cdot \bar{R}$, where \bar{R} is the complex conjugate of R , we get from eq. (10.15) the two expressions

$$R \bar{R} = \frac{1}{m[(\omega_0^2 - \omega^2) + i(\omega/\tau)]} \frac{1}{m[(\omega_0^2 - \omega^2) - i(\omega/\tau)]}$$

and

$$R \bar{R} = \rho e^{i\varphi} \rho e^{-i\varphi} = \rho^2$$

from which follows that

$$\rho^2 = \frac{1}{m^2 [(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2]} \quad (10.17)$$

To get the value of φ we must compare the two expressions for R in eq. (10.15). First we can write R as

$$R = \frac{1}{m[(\omega_0^2 - \omega^2) + i(\omega/\tau)]} \cdot \frac{m[(\omega_0^2 - \omega^2) - i(\omega/\tau)]}{m[(\omega_0^2 - \omega^2) - i(\omega/\tau)]}$$

$$= \rho^2 m(\omega_0^2 - \omega^2) - i\rho^2 m(\omega/\tau)$$

but we can also write R as

$$R = \rho e^{i\varphi} = \rho \cos \varphi + i\rho \sin \varphi$$

By comparing the real and imaginary parts of these two expressions for R we see that

$$\cos \varphi = \rho m(\omega_0^2 - \omega^2) \quad \text{and} \quad \sin \varphi = -\rho m(\omega/\tau) \quad (10.18)$$

Hence, we get for $\tan \varphi$ the equation

$$\tan \varphi = \frac{-(\omega/\tau)}{\omega_0^2 - \omega^2} = \frac{-\gamma\omega}{m(\omega_0^2 - \omega^2)} \quad (10.19)$$

We can conclude immediately from eq. (10.19) that for $\gamma = 0$ we get $\varphi = 0$. Without frictional forces, the phase shift between $z(t)$ and $F(t)$ in eq. (10.16) vanishes and there is no time lag anymore between $z(t)$ and $F(t)$. Next, eq. (10.19) tells us that the phase shift φ will always be negative because $-\tan \varphi = \tan(-\varphi)$. Without restriction we can set the phase θ to zero which we have done in fig. 10.7a. With $\theta = 0$, $F(t)$ has its maximum values at $t = n\pi/\omega$, where n is an integer value. However, $z(t)$ reaches its maximum values at

$$t = \frac{1}{\omega}(n\pi + |\varphi|)$$

and therefore lags $F(t)$ by $\Delta t = |\varphi|/\omega$. Because of the frictional force the displacement $z(t)$ does not instantaneously respond to the external force $F(t)$ but does so only with a time lag of Δt . Fig. 10.7b shows $\varphi(\omega)$ for three different values of the dimensionless parameter $\omega_0\tau$. If we pull out this dimensionless parameter from eq. (10.19) and rewrite this equation to read

$$\tan \varphi = -\frac{1}{\omega_0\tau} \frac{\omega/\omega_0}{1 - (\omega/\omega_0)^2}$$

we can see that for $\omega_0\tau \rightarrow \infty$, $\tan \varphi \rightarrow 0$ while for $\omega_0\tau \rightarrow 0$, $\tan \varphi \rightarrow \pm\infty$ depending on whether $\omega < \omega_0$ or $\omega > \omega_0$. For $\omega_0\tau \rightarrow \infty$, $\varphi(\omega)$ becomes a step function with $\varphi = 0$ for $\omega < \omega_0$ and $\varphi = -\pi/2$ for $\omega > \omega_0$.

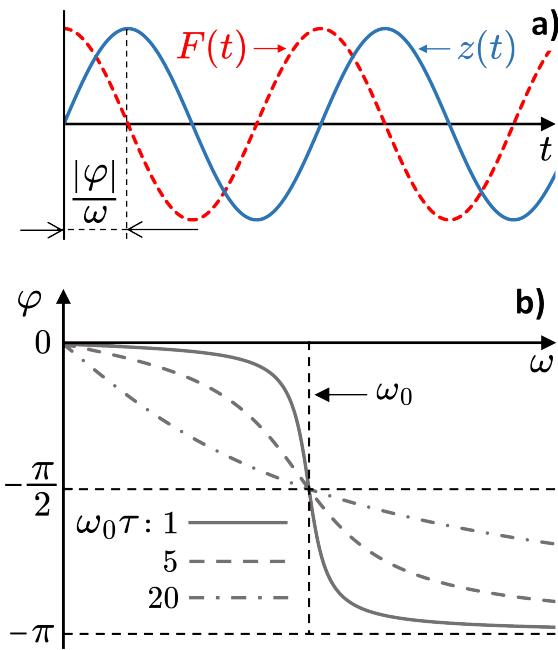


Fig. 10.7

Now let's look at the amplitude z_1 of $z(t)$ in eq. (10.16). With eq. (10.16) and eq. (10.17) we can write this amplitude as

$$z_1 = \frac{F_0}{m\sqrt{[(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2]}} \quad (10.20)$$

Fig. 10.8 shows z_1 as a function of ω for $\omega_0\tau = 20$, one of the cases for which fig. 10.7b shows the dependence of the phase shift φ on ω . For $\omega \ll \omega_0$, z_1 is practically independent of ω and for $\omega = 0$, z_1 reduces to

$$z_1(0) = \frac{F_0}{m\omega_0^2} \quad \text{or} \quad F_0 = k \cdot z_1$$

Except for the plus sign, this is Hooke's law. Solving the equation $dz_1/d\omega = 0$, a quick calculation shows that $z_1(\omega)$ has its maximum at

$$\omega_R = \omega_0 \sqrt{1 - \frac{1}{2\omega_0^2\tau^2}} \quad \xrightarrow{\omega_0\tau \gg 1} \quad \omega_R = \omega_0$$

The index R indicates, that ω_R is the so-called resonance frequency. Inserting the expression for ω_R into eq. (10.20) we obtain for the maximum amplitude value at $\omega = \omega_R$

$$z_1(\omega_R) = \frac{F_0\tau}{m\omega_0}$$

Comparing the amplitudes at resonance to the amplitude for $\omega = 0$ we see that

$$\begin{aligned} \frac{z_1(\omega_R)}{z_1(0)} &= \omega_0\tau \underset{\omega_0\tau \gg 1}{=} Q \\ &\quad \downarrow \end{aligned}$$

Hence, the Q -value of an oscillator as defined in eq. (10.8) and eq. (10.9) will tell us how much higher we can expect the amplitude of the oscillator at resonance to be as compared to its amplitude at $\omega = 0$. For $\omega > \omega_0$ the oscillator amplitude in fig. 10.8 quickly falls back to where it was before approaching ω_R from $\omega < \omega_R$ and then continues to slowly decrease to zero as $\omega \rightarrow \infty$. For the frequency width $\Delta\omega$ of the resonance at half-

point to its maximum value, the so-called full width at half maximum (FWHM) of the curve in fig. 10.8 we find

$$\omega^2 = \omega_0^2 \left(1 - \frac{1}{2\omega_0^2\tau^2}\right) \mp \frac{\omega_0}{\tau} \sqrt{3 + \frac{1}{4\omega_0^2\tau^2}}$$

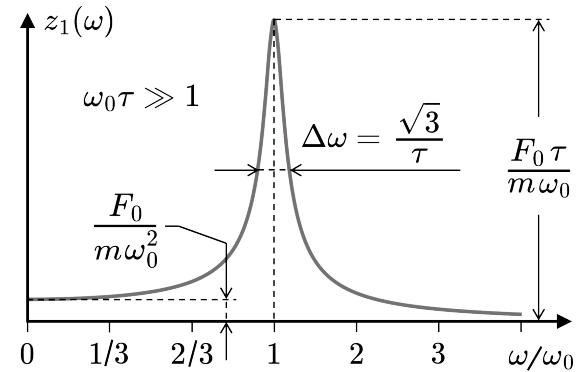


Fig. 10.8

For $\omega_0\tau \gg 1$ we can neglect all terms $(\omega_0\tau)^{-n}$ with $n > 1$ and thus get

$$\omega = \omega_0 \cdot \sqrt{1 \mp \frac{\sqrt{3}}{\omega_0\tau}} \stackrel{\omega_0\tau \gg 1}{\downarrow} \omega_0 \left(1 \mp \frac{\sqrt{3}}{2\omega_0\tau} \right) \Rightarrow \Delta\omega = \frac{\sqrt{3}}{\tau}$$

Energy of Forced Damped Harmonic Oscillation

By using an external force we can drive the harmonic oscillator to oscillate at frequencies ω quite different from its natural frequency ω_0 , its so-called eigenfrequency. Because of friction losses, the energy of the forced harmonic oscillator must be continuously replenished at whatever frequency we choose it to oscillate. Hence, when discussing the energy of this oscillator we really need to talk first about the power consumption required to keep it going. We know that the power we provide to the harmonic oscillator by applying an external force is with eq. (4.6) given by

$$P = \mathbf{F}\mathbf{v}$$

In the case of the forced oscillator the directions of the applied external force and that of the velocity with which the point mass moves fall on a straight line. With $F(t)$ from eq. (10.16) and \dot{z} calculated from $z(t)$ in eq. (10.16) and setting $\theta = 0$ we get for the power consumed by the oscillator

$$\begin{aligned} P(\omega, t) &= F(t) \cdot \dot{z} = -F_0 \cos(\omega t) \cdot \rho F_0 \omega \sin(\omega t + \varphi) \\ &= -\rho F_0^2 \omega \cos(\omega t) (\sin(\omega t) \cos \varphi + \cos(\omega t) \sin \varphi) \end{aligned}$$

With

$$\sin(2\omega t) = 2 \sin(\omega t) \cos(\omega t) \quad \text{and} \quad \cos^2(\omega t) = \frac{1}{2} + \frac{1}{2} \cos(2\omega t)$$

we can rewrite $P(\omega, t)$ as

$$P(\omega, t) = -\rho F_0^2 \omega \left(\frac{\cos \varphi}{2} \sin(2\omega t) + \frac{\sin \varphi}{2} \cos(2\omega t) + \frac{\sin \varphi}{2} \right) \quad (10.21)$$

What we are really interested is not the momentary value of $P(\omega, t)$ at any given time but its time averaged power value $\overline{P(\omega)}$ over many oscillation cycles. This we obtain by integration over a time period Δt and dividing the result by this time interval Δt and then taking the limit where $\Delta t \rightarrow \infty$, i.e.,

$$\overline{P(\omega)} = \lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t} \int_0^{\Delta t} P(\omega, t) dt \quad (10.22)$$

Inserting eq. (10.21) into eq. (10.22), obviously the time averages of the terms with $\sin(2\omega t)$ and $\cos(2\omega t)$ vanish and only the last term in eq. (10.21) contributes a non-zero value to $\overline{P(\omega)}$, hence

$$\overline{P(\omega)} = -\frac{1}{2}\rho F_0^2 \omega \sin \varphi = -\frac{1}{2}\rho F_0^2 \omega \cdot [-\rho m(\omega/\tau)] = \frac{m\rho^2 F_0^2 \omega^2}{2\tau} \quad (10.23)$$

where we used $\sin \varphi$ from eq. (10.18). If we now insert ρ^2 from eq. (10.17), $\overline{P(\omega)}$ becomes

$$\overline{P(\omega)} = \frac{F_0^2 \tau}{2m} \frac{\omega^2 / \tau^2}{(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2} \quad (10.24)$$

Fig. 10.9 shows $\overline{P(\omega)}$ for a narrow range around the maximum which lies at $\omega = \omega_0$, regardless of what the value of $\omega_0\tau$ is. This is different from what we found for the amplitude $z_1(\omega)$ where the maximum only lies at $\omega = \omega_0$ for $\omega_0\tau \gg 1$. The value of $\omega_0\tau$ chosen in fig. 10.9 is the same as that in fig. 10.8, i.e., $\omega_0\tau = 20$ but the frequency range displayed is much narrower (by a factor of 8).

What fig. 10.9 tells us is that we must tune the frequency of the external force to match the resonance frequency of the oscillator because the ability of the oscillator to absorb power is very much greater at resonance than for $\omega \neq \omega_0$. The average power consumption at the maximum of $\overline{P(\omega)}$ is

$$\overline{P(\omega)} = \frac{F_0^2 \tau}{2m}$$

When calculating the full width at half maximum (FWHM) $\Delta\omega$ of the $\overline{P(\omega)}$ curve we arrive at an equation for ω^2 similar to what we found for the amplitude curve, i.e.,

$$\omega^2 = \omega_0^2 \left(1 - \frac{1}{2\omega_0^2 \tau^2}\right) \mp \frac{\omega_0}{\tau} \sqrt{1 + \frac{1}{4\omega_0^2 \tau^2}}$$

Again we neglect for $\omega_0\tau \gg 1$ all terms $(\omega_0\tau)^{-n}$ with $n > 1$ and get in this case

$$\omega = \omega_0 \cdot \sqrt{1 \mp \frac{1}{\omega_0 \tau}} \underset{\omega_0 \tau \gg 1}{=} \omega_0 \left(1 \mp \frac{1}{2\omega_0 \tau}\right) \Rightarrow \Delta\omega = \frac{1}{\tau}$$

$\overline{P(\omega)}$ FWHM is therefore by a factor of $\sqrt{3}$ narrower than $z_1(\omega)$ FWHM. $\overline{P(\omega)}$ and $z_1(\omega)$ are both not symmetrical with respect to ω_0 but $\overline{P(\omega)}$ is more symmetrical than $z_1(\omega)$. With increasing $\omega_0\tau$ both curves become more and more symmetric.

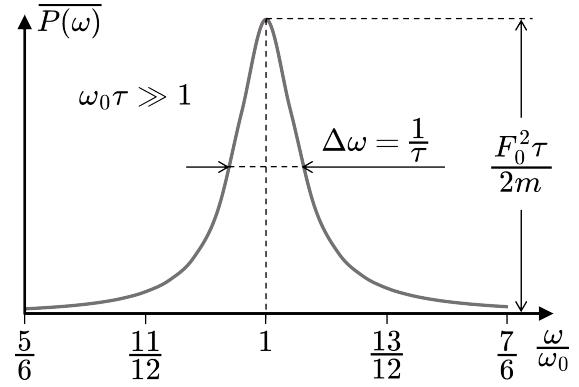


Fig. 10.9

Now we know the time average of the oscillator's power consumption but we do not know anything yet about the energy stored in the oscillator itself. Of course, our guess is that the energy stored in the oscillator will just look like what we found for the simple harmonic oscillator. To corroborate that we will determine $P(\omega, t)$ in a different way. Instead of calculating \dot{z} from eq. (10.16) and multiplying it with $F(t)$ from eq. (10.16) we can also calculate the power consumption of the oscillator by directly using $F(t)$ as expressed in eq. (10.12) and multiply it with \dot{z} . Going this route we get for $P(\omega, t)$ the equation

$$P(\omega, t) = F(t)\dot{z} = m\dot{z}\ddot{z} + m\omega_0^2 z\dot{z} + \frac{m}{\tau}\dot{z}\ddot{z} \quad (10.25)$$

where we have conveniently rearranged the terms on the right-hand side of the equation. Using the relationships

$$\dot{z}\ddot{z} = \frac{1}{2} \frac{d}{dt} \dot{z}^2 \quad \text{and} \quad z\dot{z} = \frac{1}{2} \frac{d}{dt} z^2$$

we can rewrite this equation to read

$$P(\omega, t) = \frac{d}{dt} \left(\frac{m\dot{z}^2}{2} + \frac{m\omega_0^2 z^2}{2} \right) + \frac{m}{\tau} \dot{z}^2 = \frac{d}{dt} (T + U) + \frac{m}{\tau} \dot{z}^2 \quad (10.26)$$

where T and U are the kinetic energy of the oscillating point mass and the potential energy of the spring, i.e., the total energy $E = T+U$ of the oscillator. To learn something about \dot{E} we will calculate its time averaged value. To do that we first rearrange this equation to read

$$\dot{E} = P(\omega, t) - \frac{m}{\tau} \dot{z}^2 \quad (10.27)$$

and then we calculate the time averaged value of the right-hand side of this equation. We already know from eq. (10.23) and eq. (10.27) the time averaged value of $P(\omega, t)$ so we only need to determine the time averaged value of the second term on the right hand side. Hence we need to calculate

$$\frac{m}{\tau} \lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t} \int_0^{\Delta t} \dot{z}^2 dt = \frac{m\rho^2 F_0^2 \omega^2}{\tau} \lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t} \int_0^{\Delta t} \sin^2(\omega t + \varphi) dt$$

By using the same trigonometric identities we used above to arrive at eq. (10.21), we can rewrite $\sin^2(\omega t + \varphi)$ as

$$\sin^2(\omega t + \varphi) = \frac{1}{2} - \frac{1}{4} \sin(2\omega t) \sin(2\varphi) - \frac{1}{2} \cos(2\omega t) \cos(2\varphi)$$

With the time averages of the $\sin(2\omega t)$ and $\cos(2\omega t)$ being zero, the time average of the second term on the right-hand side of eq. (10.27) becomes

$$\frac{m}{\tau} \lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t} \int_0^{\Delta t} \dot{z}^2 dt = \frac{m\rho^2 F_0^2 \omega^2}{2\tau} \quad (10.28)$$

That however is just the value of $\overline{P(\omega)}$ and therefore the time average of the right-hand side of eq. (10.27) is zero. With that we know that the time averaged value of \dot{E} must be zero which means that $E = \text{const.}$ Of course, the oscillator has to be switched on first, so there is a short ramp where E is not constant as the energy is being stored in the oscillator which it would have as a simple harmonic oscillator. After that the stored energy of the oscillator remains constant and all power consumption goes towards compensating friction losses, i.e., towards compensating the second term on the right-hand side of eq. (10.27). That of course makes sense as what happens is that any energy dissipated by the oscillator is replenished so that the energy $T+U$ stored in the oscillator does not change with time as it cycles between the kinetic energy of the moving point mass and the potential energy of the spring. While the time averaged value of \dot{E} is zero, the time averaged value of E is not. The energy of the oscillator is

$$E(\omega, t) = \frac{m\dot{z}^2}{2} + \frac{m\omega_0^2 z^2}{2}$$

From eq. (10.28) we already know the time averaged value of \dot{z}^2 , the first term on the right-hand side. In the same manner as for \dot{z}^2 , one can quickly determine that the time averaged value of z^2 is given by

$$\lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t} \int_0^{\Delta t} z^2 dt = \frac{m\rho^2 F_0^2 \omega_0^2}{4} \quad (10.29)$$

With that, the time averaged value of E becomes

$$\overline{E(\omega)} = \frac{m\rho^2 F_0^2}{4} (\omega^2 + \omega_0^2) \quad (10.30)$$

In eq. (10.8) we introduced a measure of oscillator efficiency called Q . There we said that Q is a measure of the energy stored in an oscillator relative to the amount of energy it loses over the time ω^{-1} . We now know what the time averaged value $\overline{E(\omega)}$ of the stored oscillator energy is and we found that the energy loss due to friction is just what gets replenished by $\overline{P(\omega)}$. Therefore, we can write Q from eq. (10.8) with eq. (10.23) and eq. (10.30) as

$$Q = \omega \frac{\overline{E(\omega)}}{\overline{P(\omega)}} = \omega \cdot \frac{\frac{m\rho^2 F_0^2}{4} (\omega^2 + \omega_0^2)}{\frac{m\rho^2 F_0^2 \omega^2}{2\tau}} = \frac{1}{2} \frac{\omega^2 + \omega_0^2}{\omega/\tau} \stackrel{\downarrow}{=} \omega_0 \tau \quad \omega \rightarrow \omega_0 \quad (10.31)$$

Sometimes Q is also defined as the inverse of the ratio between $\Delta\omega$, the FWHM of $\overline{P(\omega)}$, and the resonance frequency ω_0 which gives the same result, i.e.,

$$Q^{-1} = \frac{\Delta\omega}{\omega_0} = \frac{1}{\omega_0\tau}$$

While that definition allows a quick determination of Q by taking the measures of $\Delta\omega$ and ω_0 from a graph it does not tell us anything about the behavior of $Q(\omega)$ around the resonance.

10.2 Coupled Oscillations

The models of harmonic oscillation processes we just discussed help us explain many phenomena we encounter in nature. Resonance processes where energy is provided to a system at a frequency that matches the systems eigenfrequency are particularly important. If the frequencies match, energy transfer into the system becomes highly efficient. We find such resonant behavior at all scales of nature, ranging from quantum oscillators on the smallest scale to orbital resonances of planets at the very large scale. Often we can describe the physics of macroscopic systems by describing the interactions between many small objects such as atoms in solids by means of many small coupled oscillators. One simple macroscopic coupled oscillator system is that of the two spring-coupled pendula. But before we go there we need to take a look at the simple pendulum.

As illustrated in fig. 10.10, a simple pendulum consists of a mass m hung at the end of a string of length l from a fixed point P . For an incremental change in deflection angle $d\varphi$ the pendulum mass m moves the distance $l \cdot d\varphi$ driven by the restoring force component of the gravitational force mg perpendicular to l . With Newtons second law the equation of motion for the simple pendulum is therefore

$$ml \frac{d^2\varphi}{dt^2} = -mg \sin \varphi$$

If we consider only small deflections then we can replace $\sin \varphi$ with φ and the equation of motion for the simple pendulum becomes

$$\ddot{\varphi} = -\frac{g}{l}\varphi$$

A quick look at eq. (10.1) shows us that this is the equation of motion for a simple harmonic oscillator with a “spring constant” $k = mg/l$ and the solution is

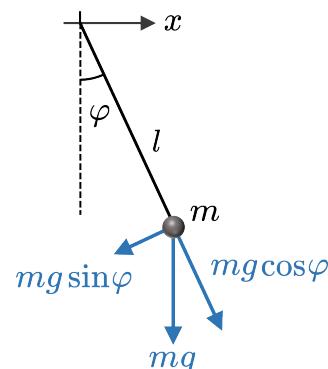


Fig. 10.10

$$\varphi = \varphi_0 \cos \omega_0 t \quad \text{where} \quad \omega_0 = \sqrt{\frac{g}{l}} \quad (10.32)$$

With that, the kinetic energy of this simple pendulum is given by

$$T = \frac{m}{2} \dot{x}^2 = \frac{m}{2} l^2 \dot{\varphi}^2 = \frac{m\omega_0^2}{2} l^2 \varphi_0^2 \sin^2(\omega_0 t)$$

where we have used the relationship $x = l\varphi$ which holds for small oscillations. For the potential energy U we read from fig. 10.10:

$$U = mgl(1 - \cos \varphi) = \frac{mgl}{2} \varphi^2 = \frac{m\omega_0^2}{2} l^2 \varphi_0^2 \cos^2(\omega_0 t)$$

where we have used $2 \cos^2 \varphi = 1 - \varphi^2$ which holds for small oscillations. For the total energy $E = T + U$ of the simple pendulum we therefore get

$$E = \frac{m\omega_0^2}{2} l^2 \varphi_0^2 \sin^2(\omega_0 t) + \frac{m\omega_0^2}{2} l^2 \varphi_0^2 \cos^2(\omega_0 t) = \frac{m\omega_0^2}{2} l^2 \varphi_0^2 \quad (10.33)$$

The total energy of the simple pendulum is constant and is set by the initial deflection φ_0 we give it, by the pendulum mass and by the length of the pendulum string. Of course, all of that can only be the case because we assumed that there are no friction losses.

10.2.1 Two Spring-Coupled Pendula

What the above quick look at the simple pendulum showed us is that for small deflections the motion of the simple pendulum is a harmonic one. Now we will take two such identical simple pendula and couple them with a spring as shown in fig. 10.11. As long as we keep the deflection for each pendulum small we have thus a model for two coupled harmonic oscillators. The gravity driven restoring force works for each of the pendula in fig. 10.11 in the same way as it does for the pendulum in fig. 10.10. However, the spring-coupling introduces an additional restoring force, the restoring force of the spring. Instead of the deflection angles φ_1 and φ_2 we will use the displacements x_1 and x_2 as the independent coordinates. To begin with, we can write down with Newton law the equations of motion for each of the uncoupled pendula just as we did for the simple pendulum above:

$$ml\ddot{\varphi}_{1,2} = -mg \sin \varphi_{1,2}$$

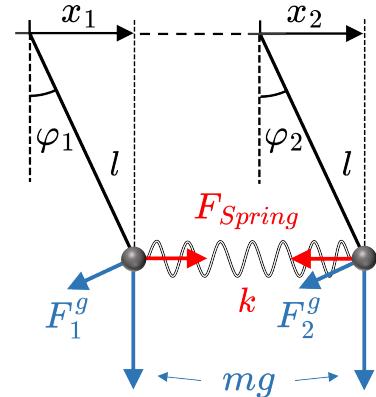


Fig. 10.11

With $x_{1,2} = l \sin \varphi_{1,2}$ and $\ddot{\varphi}_{1,2} = \ddot{x}_{1,2}/l$ this becomes with $\sin \varphi_{1,2} \approx \varphi_{1,2}$

$$m\ddot{x}_{1,2} = -\frac{mg}{l}x_{1,2}$$

To couple the two pendula, we have to add the restoring force of the spring which is

$$F_{Spring} = k\Delta x = k(x_2 - x_1)$$

As sketched in fig. 10.11, this force points in the direction of x_1 for the left pendulum and in the opposite direction of x_2 for the right pendulum. With that the equations of motion for the two pendula become

$$m\ddot{x}_1 = -\frac{mg}{l}x_1 + k(x_2 - x_1) \quad (10.34a)$$

$$m\ddot{x}_2 = -\frac{mg}{l}x_2 - k(x_2 - x_1) \quad (10.34b)$$

Rearranging eq. (10.34a) and eq. (10.34b) we get the two differential equations

$$\ddot{x}_1 + \left(\frac{g}{l} + \frac{k}{m} \right)x_1 - \frac{k}{m}x_2 = 0 \quad (10.35a)$$

$$\ddot{x}_2 + \left(\frac{g}{l} + \frac{k}{m} \right)x_2 - \frac{k}{m}x_1 = 0 \quad (10.35b)$$

These differential equations have the known solutions:

$$x_1 = A_1 e^{i\omega_1 t} \quad ; \quad x_2 = A_2 e^{i\omega_1 t} \quad \text{and} \quad x_1 = A_1 e^{i\omega_2 t} \quad ; \quad x_2 = A_2 e^{i\omega_2 t}$$

Inserting the solution approach $\omega = \omega_{1,2}$ into the equation of motion for x_1 (eq. (10.35a)) results in

$$-\omega^2 A_1 e^{i\omega t} + \left(\frac{k}{m} + \frac{g}{l} \right) A_1 e^{i\omega t} - \frac{k}{m} A_2 e^{i\omega t} = 0$$

from which follows

$$\omega^2 = \frac{k}{m} + \frac{g}{l} - \frac{k}{m} \frac{A_2}{A_1} \quad (10.36)$$

As can easily be checked, one gets the very same result if one alternatively inserts the above solution approach for x_1 and x_2 into the equation of motion for x_2 , i.e., eq. (10.35b). Eq. (10.36) will become useful if we can find another way to determine $\omega_{1,2}$ as then we can say something about the amplitude ratio $A_1 : A_2$. This we can achieve by moving from differential equations for x_1 and x_2 to differential equations for the so-called normal coordinates of the two pendulum system. To do that, we first subtract eq. (10.34b) from eq. (10.34a) and then add eq. (10.34a) and eq. (10.34b) to obtain two new equations:

$$m(\ddot{x}_1 - \ddot{x}_2) = -m\left(\frac{g}{l} + \frac{2k}{m}\right)(x_1 - x_2) \quad (10.37a)$$

$$m(\ddot{x}_1 + \ddot{x}_2) = -m\frac{g}{l}(x_1 + x_2) \quad (10.37b)$$

Now we introduce the normal coordinates λ_a and λ_b with

$$\lambda_a = x_1 - x_2 \quad \text{and} \quad \lambda_b = x_1 + x_2$$

The coordinates x_1 and x_2 expressed in normal coordinates are then

$$x_1 = \frac{\lambda_a + \lambda_b}{2} \quad \text{and} \quad x_2 = \frac{\lambda_b - \lambda_a}{2}$$

In normal coordinates eq. (10.37a) and eq. (10.37b) become

$$\ddot{\lambda}_a + \left(\frac{g}{l} + \frac{2k}{m}\right)\lambda_a = 0 \quad (10.38a)$$

$$\ddot{\lambda}_b + \frac{g}{l}\lambda_b = 0 \quad (10.38b)$$

Obviously, eq. (10.38a) and eq. (10.38b) describe harmonic oscillations in the normal coordinates λ_a and λ_b with oscillation frequencies

$$\omega_s = \sqrt{\frac{g}{l} + \frac{2k}{m}} \quad \text{and} \quad \omega_p = \sqrt{\frac{g}{l}}$$

The subscripts s and p for these frequencies indicate that, as sketched in fig. 10.12, ω_s refers to the symmetric or anti-phase oscillation mode with $A_1 = -A_2$ and ω_p refers to the parallel or in-phase oscillation mode with $A_1 = A_2$. We do know this now because ω_p and ω_s must be the solutions of eq. (10.36) and with that eq. (10.36) tells us that

$$\omega^2 = \begin{cases} \omega_1^2 = \omega_p^2 = \frac{g}{l} & \Rightarrow A_1 = A_2 \\ \omega_2^2 = \omega_s^2 = \frac{g}{l} + \frac{2k}{m} & \Rightarrow A_1 = -A_2 \end{cases}$$

To solve eq. (10.38a) and eq. (10.38b) we apply the solution approach we successfully used in the previous sections for such equations:

$$\lambda_a = A_a e^{i\omega_s t} \quad \text{and} \quad \lambda_b = A_b e^{i\omega_p t}$$

which gives us for x_1 and x_2

$$x_1 = \frac{1}{2}A_a e^{i\omega_s t} + \frac{1}{2}A_b e^{i\omega_p t} \quad (10.39a)$$

and

$$x_2 = \frac{1}{2}A_b e^{i\omega_p t} - \frac{1}{2}A_a e^{i\omega_s t} \quad (10.39b)$$

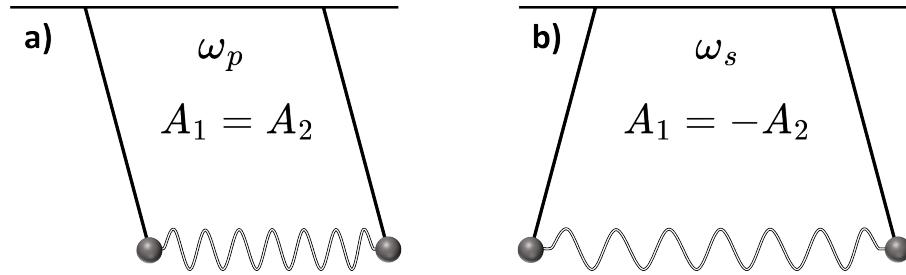


Fig. 10.12: Parallel or in-phase (a) and symmetric or anti-phase (b) solution of the equation of motion for two identical pendula coupled by a mechanical spring.

Comparing this solution with the initial solution of the differential equations eq. (10.35a) and eq. (10.35b) with $\omega_1 = \omega_p$ and $\omega_2 = \omega_s$, i.e.,

$$x_1 = A_1 e^{i\omega_p t} \quad ; \quad x_2 = A_2 e^{i\omega_p t} \quad \text{and} \quad x_1 = A_1 e^{i\omega_s t} \quad ; \quad x_2 = A_2 e^{i\omega_s t}$$

we see that at $t = 0$ must apply

$$A_a = A_1 - A_2 \quad \text{and} \quad A_b = A_1 + A_2$$

Inserting this into eq. (10.39a) and eq. (10.39b) yields

$$x_1 = \frac{1}{2} [A_1 e^{i\omega_s t} - A_2 e^{i\omega_s t} + A_1 e^{i\omega_p t} + A_2 e^{i\omega_p t}]$$

and

$$x_2 = \frac{1}{2} [A_1 e^{i\omega_p t} + A_2 e^{i\omega_p t} - A_1 e^{i\omega_s t} + A_2 e^{i\omega_s t}]$$

These equations can be rearranged into

$$x_1 = \frac{e^{i(\frac{\omega_p+\omega_s}{2})t}}{2} \left[A_1 \left(e^{i(\frac{\omega_s-\omega_p}{2})t} + e^{-i(\frac{\omega_s-\omega_p}{2})t} \right) - A_2 \left(e^{i(\frac{\omega_s-\omega_p}{2})t} - e^{-i(\frac{\omega_s-\omega_p}{2})t} \right) \right]$$

and

$$x_2 = \frac{e^{i(\frac{\omega_p+\omega_s}{2})t}}{2} \left[A_1 \left(e^{-i(\frac{\omega_s-\omega_p}{2})t} - e^{i(\frac{\omega_s-\omega_p}{2})t} \right) - A_2 \left(e^{i(\frac{\omega_s-\omega_p}{2})t} + e^{-i(\frac{\omega_s-\omega_p}{2})t} \right) \right]$$

If we now define

$$\frac{\omega_p + \omega_s}{2} = \bar{\omega} \quad \text{and} \quad \omega_s - \omega_p = \Delta\omega$$

then the equations for x_1 and x_2 become

$$x_1 = A_1 e^{i\bar{\omega}t} \cos\left(\frac{\Delta\omega}{2}t\right) - iA_2 e^{i\bar{\omega}t} \sin\left(\frac{\Delta\omega}{2}t\right)$$

and

$$x_2 = A_2 e^{i\bar{\omega}t} \cos\left(\frac{\Delta\omega}{2}t\right) - iA_1 e^{i\bar{\omega}t} \sin\left(\frac{\Delta\omega}{2}t\right)$$

The amplitudes A_1 and A_2 follow from the choice of initial conditions for $t = 0$:

$$x_1(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0 \quad \text{and} \quad x_2(0) = l\alpha$$

(remember: the angle α must be small). This means that pendulum number 1 is kept still, pendulum number 2 is moved to a deflection angle α and then both pendula are released at $t = 0$. From these initial conditions follows $A_1 = 0$ and $A_2 = l\alpha$. The equations for x_1 and x_2 are thus

$$x_1 = -il\alpha \sin\left(\frac{\Delta\omega}{2}t\right) [\cos(\bar{\omega}t) + i \sin(\bar{\omega}t)]$$

and

$$x_2 = l\alpha \cos\left(\frac{\Delta\omega}{2}t\right) [\cos(\bar{\omega}t) + i \sin(\bar{\omega}t)]$$

For the real part of the general solution, one thus obtains

$$x_1 = \left[l\alpha \sin\left(\frac{\Delta\omega}{2}t\right) \right] \sin(\bar{\omega}t) \tag{10.40a}$$

and

$$x_2 = \left[l\alpha \cos\left(\frac{\Delta\omega}{2}t\right) \right] \cos(\bar{\omega}t) \tag{10.40b}$$

with

$$\frac{\Delta\omega}{2} = \frac{\omega_2 - \omega_1}{2} = \frac{1}{2} \left(\sqrt{\frac{2k}{m} + \frac{g}{l}} - \sqrt{\frac{g}{l}} \right)$$

and

$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2} = \frac{1}{2} \left(\sqrt{\frac{2k}{m} + \frac{g}{l}} + \sqrt{\frac{g}{l}} \right)$$

The amplitude of the oscillations described by eq. (10.40a) and eq. (10.40b) is time-dependent with the frequency $\Delta\omega/2$ and both pendula oscillate with the frequency $\bar{\omega}$, however, out of phase by 90° . Because one amplitude oscillates with the sine (x_1) while the other oscillates with the cosine (x_2), one will be zero when the other one will reach its maximum value and vice versa. As a result of the coupling, the pendulum (x_2) deflected by α at the time $t = 0$ transfers its energy to the other pendulum (x_1) and comes to rest when the latter has reached its maximum amplitude; then the process is repeated in the reverse direction.

Energy of the Two Spring-Coupled Pendula Oscillation

The kinetic energy of the system is the sum of the kinetic energies of the two pendula

$$T = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2}$$

The potential energy of the system is the sum of the potential energies of the two pendulum masses and the potential energy of the spring. For the potential energies of the pendulum masses, one reads from fig. 10.11

$$U = mg(l - l \cos \varphi_1) + mg(l - l \cos \varphi_2) = 2mgl - mgl(\cos \varphi_1 + \cos \varphi_2)$$

The potential energy of the spring is

$$U_{Spring} = \frac{k}{2}l^2(\sin \varphi_1 - \sin \varphi_2)^2$$

Thus, the total energy of the two spring-coupled pendula system is

$$E = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{k}{2}l^2(\sin \varphi_1 + \sin \varphi_2)^2 + 2mgl - mgl(\cos \varphi_1 + \cos \varphi_2)$$

Using the approximation for small deflections (i.e., small oscillations)

$$\sin \varphi_{1,2} \approx \varphi_{1,2} \quad \text{and} \quad \cos \varphi_{1,2} \approx 1 - \frac{\varphi_{1,2}^2}{2}$$

With that the total energy E becomes

$$E = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{k}{2}l^2(\varphi_1 - \varphi_2)^2 + mgl\left(\frac{\varphi_1^2}{2} + \frac{\varphi_2^2}{2}\right)$$

Using $l\varphi_{1,2} = x_{1,2}$ to replace $\varphi_{1,2}$ with $x_{1,2}/l$, this equation becomes

$$E = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{k}{2}(x_1 - x_2)^2 + \frac{mg}{2l}(x_1^2 + x_2^2) \quad (10.41)$$

To evaluate this equation we need to insert the respective values for $x_{1,2}$ and $\dot{x}_{1,2}$. A quick calculation shows that:

$$\begin{aligned} \dot{x}_1^2 + \dot{x}_2^2 &= \frac{l^2\alpha^2}{2}(\omega_s^2 \sin^2(\omega_s t) + \omega_p^2 \sin^2(\omega_p t)) \\ (x_1 - x_2)^2 &= l^2\alpha^2 \cos^2(\omega_s t) \\ x_1^2 + x_2^2 &= \frac{l^2\alpha^2}{2}(1 + \cos^2(\omega_s t) \cos^2(\omega_p t) - \sin^2(\omega_s t) \sin^2(\omega_p t)) \end{aligned}$$

Inserting these expressions into eq. (10.41) then gives the equation

$$E = \frac{ml^2\alpha^2}{4} \cdot (\omega_s^2 + \omega_p^2) = \frac{ml^2\alpha^2}{2} \cdot \left(\frac{g}{l} + \frac{k}{m}\right) \quad (10.42)$$

Not surprisingly, because we did not consider friction losses, the energy of the two spring-coupled pendula system is constant. A comparison with the energy of the simple pendulum from eq. (10.33) shows that if the initial deflection is the same, the total energy of the simple pendulum and the two spring-coupled pendula would be identical if $\omega_s^2 + \omega_p^2$ were equal to $2\omega_0^2$. For the two spring-coupled pendula system the total energy is split between the two oscillation motions ω_s and ω_p .

10.2.2 Stretching Vibration of a Molecule

For the stretching vibration of a linear triatomic molecule, we consider the model of the CO₂ molecule in fig. 10.13. Even though the O-atom is about 30% heavier than the C-atom the radius of the C-atom is almost 40% larger than that of the O-atom because the oxygen nucleus with its eight protons pulls its electron in tighter

than the carbon nucleus with its six protons. In our CO₂ model the outer O-atoms of mass M are connected to the middle C-atom of mass m by two identical springs where each has the spring constant k . The equilibrium distance for the identical springs, i.e., the respective molecular distance when the springs are relaxed, shall be d . The natural choice of coordinates for this problem are the linear displacements (compression or extension of the springs) x_1 , x_2 and x_3 . The restoring forces of the two springs are proportional to $x_2 - x_1$ and $x_2 - x_3$. If $x_2 - x_1 > 0$, the left spring is stretched and it will pull the left O-atom to the right in the direction of x_1 and the middle C-atom will be pulled to the left in the negative x_2 direction. If $x_3 - x_2 > 0$, the right spring is stretched and it will pull the right O-atom in the negative x_3 direction and the middle C-atom in the positive x_2 direction. Restoring forces pointing in a positive coordinate direction are counted as positive, those pointing in a negative coordinate direction as negative. Thus the equations of motion for the three masses are given by

$$M\ddot{x}_1 = k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2)$$

$$M\ddot{x}_3 = -k(x_3 - x_2)$$

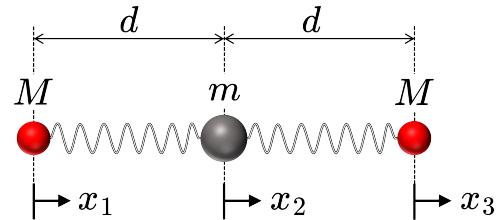


Fig. 10.13

As previously, we use the solution approach $x_i = A_i e^{i\omega t}$. Inserting this into the above equations gives us then a system of three linear equations in the coordinate displacements x_i which we write in matrix form as

$$\begin{bmatrix} k - M\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - M\omega^2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (10.43)$$

A matrix equation system of the form $\underline{\underline{A}}\underline{x} = \mathbf{0}$ has only a solution if the determinant of the matrix vanishes, i.e., $\det|\underline{\underline{A}}| = 0$. For eq. (10.43) to have a solution, this requirement translates into

$$\begin{aligned} 0 &= (k - M\omega^2)(2k - m\omega^2)(k - M\omega^2) - k^2(k - M\omega^2) - (k - M\omega^2)k^2 \\ &= (k - \omega^2 M)\omega^2 [\omega^2 m M - k(2M + m)] \end{aligned}$$

and one obtains the solutions (positive roots only)

$$\omega_1 = \sqrt{\frac{k}{M}} \quad ; \quad \omega_2 = 0 \quad ; \quad \omega_3 = \sqrt{k\left(\frac{2}{m} + \frac{1}{M}\right)} = \sqrt{\frac{k}{M}\left(1 + \frac{2}{\mu}\right)}$$

where $\mu = m/M$. To determine the eigenvectors and vibration modes of the molecule, one now inserts these eigenvalues into eq. (10.43) and applies the so-called Gauß-algorithm. With this algorithm one can transform the matrix in eq. (10.43) into an upper triangular matrix where in the last line the coefficients for all but the x_3 coordinate vanish which of course then provides us the solution for x_3 . In case all coefficients in the last line vanish we can select an arbitrary parameter as the solution for x_3 . Allowed matrix operations, i.e., such operations which will leave the solutions for x_1 , x_2 and x_3 unaffected, are:

1. Multiplication of a row by a scalar $\neq 0$.
2. Interchanging two rows.
3. Addition of a multiple of one row to another row

With that we can begin to solve the equation system eq. (10.43) for ω_1 :

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 0 & -k & 0 & 0 \\ -k & k(2 - \mu) & -k & 0 \\ 0 & -k & 0 & 0 \end{array} \xrightarrow{\quad} \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline k & k(\mu - 2) & k & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad ; \quad \begin{array}{l} -1 \times 2. \text{ row} \\ -1 \times 1. \text{ row} \\ 1. \text{ row} - 3. \text{ row} \end{array}$$

With backwards substitution one obtains for the eigenvalue

$$\omega_1 = \sqrt{\frac{k}{M}} \quad \text{the eigenvector} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A_1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

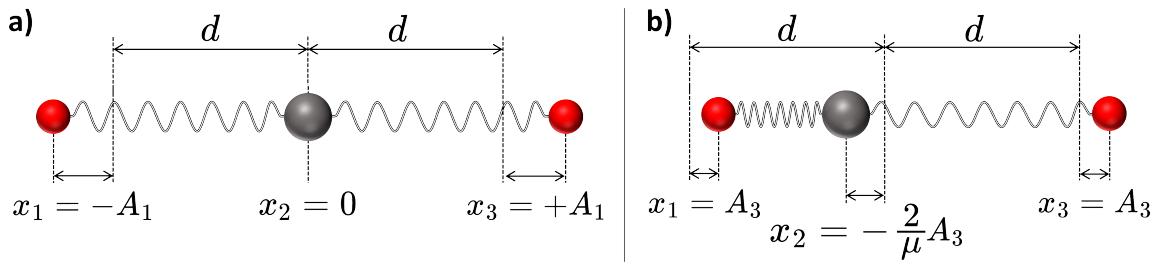


Fig. 10.14: Oscillation modes of a linear tri-atomic molecule: (a) outer atoms oscillate in anti-phase, while the central atom remains in its rest position; (b) outer atoms oscillate in phase with each other and in anti-phase to the middle atom.

As sketched in fig. 10.14a, the two outer atoms oscillate in anti-phase, while the central atom remains in its rest position. Since the mass of the central atom remains in its rest position, the motion of the outer atoms is independent of the mass ratio μ .

Now one inserts the eigenvalue ω_2 into eq. (10.43) and solves the equation system again by using the Gauß-algorithm:

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & k & 0 \end{array} \Rightarrow \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline k & -2k & k & 0 \\ 0 & k & -k & 0 \\ 0 & -k & k & 0 \end{array}; \begin{array}{l} -1 \times 2. \text{ row} \\ 2. \text{ row} + 1. \text{ row} \end{array}$$

$$\Rightarrow \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline k & -2k & k & 0 \\ 0 & k & -k & 0 \\ 0 & 0 & 0 & 0 \end{array}; \begin{array}{l} 2. \text{ row} + 3. \text{ row} \end{array}$$

With backwards substitution one obtains for the eigenvalue $\omega_2 = 0$ the eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The fact that all three masses experience the same deflection A_2 suggests that the molecule moves as a whole without any oscillation occurring, independently of the mass ratio μ . Of course, in this case, t can also simply be zero, that means nothing moves at all. Finally, one inserts the third eigenvalue ω_3 into eq. (10.43) and solves the equation system again by using the Gauß-algorithm:

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline -\frac{2k}{\mu} & -k & 0 & 0 \\ -k & -k\mu & -k & 0 \\ 0 & -k & -\frac{2k}{\mu} & 0 \end{array} \Rightarrow \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline k & k\mu & k & 0 \\ 0 & k & \frac{2k}{\mu} & 0 \\ 0 & -k & -\frac{2k}{\mu} & 0 \end{array}$$

; $-1 \times 2.$ row
 $1.$ row $- \frac{2}{\mu} \times 2.$ row

$$\Rightarrow \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline k & k\mu & k & 0 \\ 0 & k & \frac{2k}{\mu} & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

; $2.$ row $+ 3.$ row

With backwards substitution one obtains for the eigenvalue

$$\omega_3 = \sqrt{k \left(\frac{2}{m} + \frac{1}{M} \right)} \quad \text{the eigenvector} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A_3 \cdot \begin{pmatrix} 1 \\ -2/\mu \\ 1 \end{pmatrix}$$

In this case, as sketched in fig. 10.14b, the two outer atoms oscillate in phase with each other and in anti-phase (out of phase by 180°) to the middle atom. Here, the amplitude of the deflection of the central atom depends on the mass ratio μ . For example, if $\mu = 1$ then the middle atom oscillates with twice the amplitude of the outer atoms. If $\mu = 2$, i.e., the middle atom has twice the mass of an outer atom, all three atoms will oscillate with the same amplitude, however, the middle atom oscillates in anti-phase to the outer atoms.

Energy of the Stretching Vibration

The oscillation pictured in fig. 10.14a is called the symmetric stretching mode of the molecule and the oscillation shown in fig. 10.14b is referred to as its anti-symmetric stretching mode. To calculate the energies associated with those two stretching modes we first write down the solutions we obtained for $x_1(t)$, $x_2(t)$ and $x_3(t)$ which are the real parts of $x_i = A_i e^{i\omega t}$ for the respective frequencies. For the symmetric stretching mode in fig. 10.14a the solution is

$$x_1(t) = -A_1 \cos(\omega_1 t) \quad ; \quad x_2(t) = 0 \quad ; \quad x_3(t) = A_1 \cos(\omega_1 t)$$

and for the anti-symmetric stretching mode in fig. 10.14b the solution is

$$x_{1,2}(t) = A_3 \cos(\omega_3 t) \quad ; \quad x_2(t) = -\frac{2A_3}{\mu} \cos(\omega_3 t)$$

We will look first at the symmetric stretching mode. The kinetic energy in this case is given by

$$T_s = \frac{M}{2} \dot{x}_1^2 + \frac{M}{2} \dot{x}_3^2 = MA_1^2 \omega_1^2 \sin^2(\omega_1 t) = kA_1^2 \sin^2(\omega_1 t)$$

To get the potential energy we add the potential energies stored in the two springs

$$U_s = \frac{k(x_2 - x_1)^2}{2} + \frac{k(x_3 - x_2)^2}{2} = \frac{k}{2}(x_1^2 + x_3^2) = kA_1^2 \cos^2(\omega_1 t)$$

Therefore, the total energy E_s of the symmetric stretching mode is

$$E_s = T + U = kA_1^2 \sin^2(\omega_1 t) + kA_1^2 \cos^2(\omega_1 t) = kA_1^2$$

Now to the anti-symmetric stretching mode. In that case the kinetic energy is given by

$$\begin{aligned} T_a &= \frac{M}{2} \dot{x}_1^2 + \frac{m}{2} \dot{x}_2^2 + \frac{M}{2} \dot{x}_3^2 = \left(M + \frac{2m}{\mu^2} \right) A_3^2 \omega_3^2 \sin^2(\omega_3 t) \\ &= M \left(1 + \frac{2}{\mu} \right) A_3^2 \frac{k}{M} \left(1 + \frac{2}{\mu} \right) \sin^2(\omega_3 t) = kA_3^2 \left(1 + \frac{2}{\mu} \right)^2 \sin^2(\omega_3 t) \end{aligned}$$

To get the potential energy of the anti-symmetric stretching mode we again add the potential energies stored in the two springs, i.e.,

$$\begin{aligned} U_a &= \frac{k(x_2 - x_1)^2}{2} + \frac{k(x_3 - x_2)^2}{2} \\ &= \frac{k}{2} \left(-\frac{2}{\mu} - 1 \right)^2 A_3^2 \cos^2(\omega_3 t) + \frac{k}{2} \left(1 + \frac{2}{\mu} \right)^2 A^2 \cos^2(\omega_3 t) \\ &= kA_3^2 \left(1 + \frac{2}{\mu} \right)^2 \cos^2(\omega_3 t) \end{aligned}$$

With that, the total energy E_a of the anti-symmetric stretching mode is given by

$$E_a = T_a + U_a = kA_3^2 \left(1 + \frac{2}{\mu} \right)^2 (\sin^2(\omega_3 t) + \cos^2(\omega_3 t))$$

which simply becomes

$$E_a = kA_3^2 \left(1 + \frac{2}{\mu} \right)^2$$

Because our model does not include friction-type losses E_s and E_a are both constant. Of course, nothing oscillates for ever and our molecule model, even though it gives us a good basic understanding of molecular oscillations, can therefore not be the correct one. Hence, we cannot expect this model to give us the correct energy levels for the respective

vibration modes. This just is not possible in classical mechanics as the world of the very small, to which molecules certainly belong, requires the tools of quantum mechanics. But we can still compare our results to what measurements tell us to see how far off our classical approach is. The absorption frequencies of the vibration modes we just discussed lie in the infrared. However, only the anti-symmetric mode can be excited with infrared light. No change in dipole moment is associated with the symmetric mode and therefore light cannot couple with the molecule to excite this mode but it can be excited by other means such as the scattering of electrons or photons. For CO₂ the energy required to excite the symmetric mode is $E_s^m = 172 \text{ meV}$ and that for the anti-symmetric mode is $E_a^m = 291 \text{ meV}$. One meV stands for one millielectron volt, i.e., one thousandth of an electron volt ($1 \text{ eV} = 1.602 \cdot 10^{-19} \text{ J}$). Hence, these are really tiny energy amounts.

Quantum mechanics tells us that the energy absorbed in a molecular transition between two adjacent vibration levels of a molecule is given by $E_{vib} = \hbar\omega$ where \hbar is Planck's constant $h = 6.6262 \cdot 10^{-34} \text{ Js}$ divided by 2π . At the maximum displacement of the oxygen atoms in the symmetric vibration mode the kinetic energy is zero and the potential energy equals kA_1^2 . Hence, we can posit that

$$E_s^{vib} = \hbar\omega_1 = kA_1^2 = M\omega_1^2 A_1^2$$

from which follows that

$$A_1 = \sqrt{\frac{\hbar}{M\omega_1}}$$

The value of ω_1 we can calculate from the measured value E_s^m as

$$\omega_1 = \frac{E_s^m}{\hbar} = \frac{172 \cdot 10^{-3} \times 1.602 \cdot 10^{-19} \text{ J}}{1.055 \cdot 10^{-34} \text{ Js}} = 2.61 \cdot 10^{14} \text{ s}^{-1}$$

With that we get for the amplitude of the symmetric vibration

$$A_1 = \sqrt{\frac{1.055 \cdot 10^{-34} \text{ Js}}{16 \cdot 1.66 \cdot 10^{-27} \text{ kg} \times 2.61 \cdot 10^{14} \text{ s}^{-1}}} = 0.0039 \text{ nm}$$

With the length of the C-O bond being 0.113 nm this means that the amplitude of the symmetric vibration is about 3.5% of the C-O bond length. From $k = M\omega_1^2$ we can determine that the spring constant is

$$k = 16 \cdot 1.66 \cdot 10^{-27} \text{ kg} \times 6.81 \cdot 10^{28} \text{ s}^{-2} = 1.83 \cdot 10^3 \text{ Nm}^{-1}$$

Now let us go through the same exercise and see what results we get for the anti-symmetric vibration mode. Here we have

$$E_a^{vib} = \hbar\omega_3 = kA_3^2 \left(1 + \frac{2}{\mu}\right)^2 = M\omega_3^2 A_3^2 \left(1 + \frac{2}{\mu}\right)^1$$

from which follows

$$A_3 = \sqrt{\frac{\hbar}{M\omega_3}} \frac{\mu}{\mu + 2}$$

The value of ω_3 we can calculate from the measured value E_a^m as

$$\omega_3 = \frac{E_a^m}{\hbar} = \frac{291 \cdot 10^{-3} \times 1.602 \cdot 10^{-19} \text{ J}}{1.055 \cdot 10^{-34} \text{ Js}} = 4.42 \cdot 10^{14} \text{ s}^{-1}$$

With $\mu/(\mu + 2) = 3/11$ we then get for the amplitude of the anti-symmetric vibration

$$A_3 = \sqrt{\frac{1.055 \cdot 10^{-34} \text{ Js}}{16 \cdot 1.66 \cdot 10^{-27} \text{ kg} \times 4.42 \cdot 10^{14} \text{ s}^{-1}}} \frac{3}{11} = 0.0016 \text{ nm}$$

and for the spring constant k we get for the anti-symmetric vibration

$$k = \frac{M\mu}{\mu + 2} \omega_3^2 = \frac{3}{11} \cdot 16 \cdot 1.66 \cdot 10^{-27} \text{ kg} \times 1.96 \cdot 10^{29} \text{ s}^{-2} = 1.42 \cdot 10^3 \text{ Nm}^{-1}$$

The spring constant we calculate for the anti-symmetric vibration mode comes out some 22% lower than that for the symmetric vibration mode. Our model assumption was that the spring constants of all vibration modes are the same, but plucking experimental data in our model results shows us that this is not the case. Clearly, our classical harmonic oscillator model falls short in this respect. However, it still can give us a basic understanding of these two linear vibration modes of the CO₂ molecule. There is another simple oscillation mode of this tri-atomic molecule which is a bending oscillation. One can model this bending vibration in a similar way to what we just did for the linear vibration of the CO₂ molecule. The calculations are a little more cumbersome but a good exercise.

11. Wave Motion

Just as fundamental as the harmonic oscillator equations we discussed in the previous chapter and closely related to it are wave equations. Nature knows many different kinds of waves such as acoustic waves, fluid waves, plasma waves, light waves, gravitational waves and many more. Every kid is familiar with the circular water waves which a stone thrown in a tranquil pond creates, emanating from the point where the stone broke the waters surface. A different type of waves are the acoustic waves which we create when we speak. Those sound waves then propagate through the medium of air, are reflected or absorbed by different kinds of objects and eventually reach the ears of another person. There the eardrums converts acoustic waves into vibrations which they pass on to the bones of the inner ear where they are amplified before being passed on to the liquid inside the spiral cavities. The ripples thus produced in this liquid, i.e., waves, are then sensed by different kinds of hairs whose movement releases chemicals which leads to the generation of electrical signals. The latter are then transmitted to our brains which decodes them as the sounds we hear. Us being able to communicate involves therefore several kinds of waves: acoustic waves, liquid waves, mechanical waves (in the inner ear bones) and electrical waves (the electrical signals transmitted by nerve cells). In addition there are other kinds of waves such as electromagnetic waves which is how light propagates, waves in plasma - next to the gaseous, liquid, and solid states the fourth state of matter - or gravitational waves. Since the early twentieth century, with the discoveries of quantum mechanics, we know that every massive particle is also associated with a matter wave. The dual particle and wave character of objects is fundamental to our understanding of nature on the very small scale, i.e., the physics of molecules and atoms. However, this dual particle and wave character applies also to macroscopic objects but because the scale of the wavelength is so small compared to the size of object, the wave nature of large objects is completely negligible.

11.1 Simple Mechanical Waves

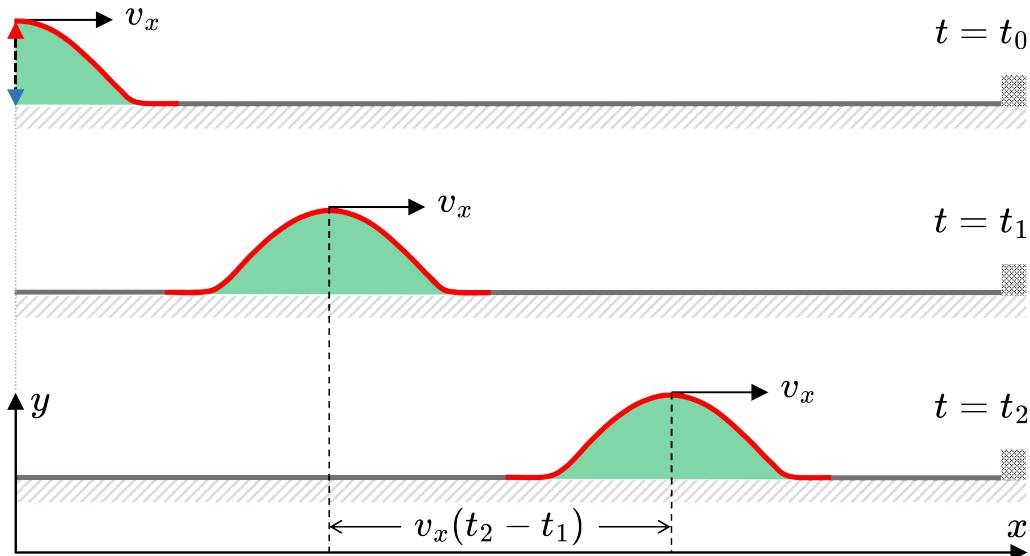


Fig. 11.1: Fixed to the floor on one end and stretched to tension, a rope quickly yanked up and down on its other end generates a pulse traveling in the rope.

Waves traveling along a rope under tension are likely a familiar experience and many will remember them from their childhood playground days when we all were likely playing around with ropes. If one ties down one end of a rope, pulls the rope straight so it is under slight tension, and then yanks the free end of the rope quickly up and down, a bulge-like distortion will travel the length of the rope towards the fixed end of the rope and then be reflected back from there. If one periodically yanks the rope up and down one can generate what is called a standing wave. Fig. 11.1 shows the even simpler situation where a rope is stretched on the floor with one end tied down and a pulse being generated at the other end which then travels the length of the rope. If we carefully observe from the sidelines what happens as a friend yanks the rope we will find the situation as sketched in fig. 11.1:

1. The velocity v_x with which the bulge travels the length of the rope is constant.
2. Along the length of the bulge each point in the rope oscillates vertically in y -direction, i.e., perpendicular to the x -direction in which the bulge travels with velocity v_x .
3. Only rope points along the length of the traveling bulge are vertically displaced.
4. Before the bulge reaches a position on the rope, that rope position is at rest and once it has passed, this rope position is at rest again.

Evidently, as the bulge generated in the rope shown in fig. 11.1 travels to the right, no part of the rope itself moves to the right. What travels is the vertical displacement generated at the left end of the rope, but once that vertical displacement has moved on to the right, the rope position the traveling bulge just passed is like it was before, i.e., at rest. What moves from left to right in fig. 11.1 is energy and not mass. What happens as the bulge travels the length of the rope is that in some way one piece of rope passes on the energy to the next piece of rope in front of it. This is not unlike what we encountered in section 10.2.1 when discussing two identical pendula coupled by a spring. What we found there is that in such a two pendula system energy is transferred from one pendulum to the other until one is in full swing while the other has stopped to oscillate at which point the process reverses. In a similar way we can imagine energy to be passed on between neighboring pieces of the rope with the difference being that the energy transfer happens only in one direction, i.e., in the travel direction of the bulge. When we say neighboring pieces of the rope we need to zoom into the rope matter to view things on the atomic level. On that scale we will indeed observe something that can be modeled as a system of coupled oscillators. Modeling interactions between atoms in solids in this way allows us to explain oscillations in solids, i.e., how for example vibration energy can move through a solid without transferring mass. If this energy transport involves a great number of atoms coupled in such a way as to move energy from one point to another without transferring mass one speaks of a wave.

The energy being transported in the example of fig. 11.1 is the distortion of the rope, i.e., the bulge which travels from left to right. This distortion does not change as the bulge travels the length of the rope. We can observe this traveling bulge from a reference frame S where we as well as our friend yanking the rope are at rest. But we can also take the position of an observer traveling with the bulge, i.e., chose a reference frame S' moving relative to S with the velocity v_x . Coordinates and velocities in S and S' are connected through a Galilean transformation

$$x' = x - v_x t \quad \text{and} \quad t' = t$$

For an observer in S the bulge created in fig. 11.1 at $t = t_0$ is clearly a rope distortion that travels to the right. An observer in S' , however, will describe the bulge as a stationary rope distortion $f(x')$. As fig. 11.2 illustrates, for an observer in S this stationary rope distortion then becomes the function

$$y(x, t) = f(x - v_x t)$$

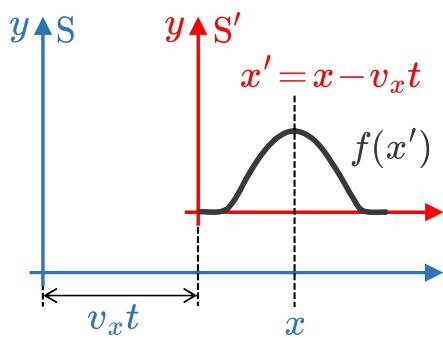


Fig. 11.2

which describes the stationary rope distortion as it moves to the right. To see that this really is the case consider the functions

$$y(x, t = 0) = f(x) \quad \text{and} \quad y(x, t_1) = f(x - v_x t_1)$$

At $t = 0$, the origins of S and S' shall coincide with each point on the distortion curve having the same coordinate position as measured in S and in S' , i.e., $y(x, t = 0) = f(x) = f(x') = y(x', t = 0)$. After a time $t = t_1$ the distortion in S' will still be $y(x', t_1) = f(x')$. However, for the observer in S this coordinate x' referencing the origin of S' has now become the coordinate $x = x' + v_x t_1$, which means the distortion has moved to the right without changing its form. Just as $y(x, t) = f(x - v_x t)$ describes such a distortion traveling to the right without changing its form, the function $y(x, t) = f(x + v_x t)$ describes the same stationary distortion traveling to the left.

Next, we will look into what governs the motion of $y(x, t)$. We already observed that as the distortion travels the length of the rope, none of the rope points move in the x -direction, they only oscillate in the y -direction. What are the forces which make them do that? To understand that we will assume that the rope shall have a constant diameter d and with that a constant cross section of $A = d^2\pi/4$ and then analyze the situation with the help of fig. 11.3 which

zooms in on a small piece of the rope of mass dm within the traveling bulge. We image this piece of rope to be cut free but retaining its place in the rope because we replace the connections to the rope parts left and right from dm with the forces required to keep dm in place as if the cut never happened. These forces are due to the axial tension of the rope and their magnitude is the same on the left and on the right side. The directions of the axial tension vectors σ are parallel to the respective surface normal vectors of the left and right ends of dm which point away from these surfaces. Hence these axial tension vectors point into opposite directions but because the end faces of dm are not exactly parallel the axial tension vectors cannot be anti-parallel. The magnitude of the force on the left and on the right side is the same, i.e., $F = \sigma \cdot A$. We are only interested in the net vertical force acting on the the rope piece dm . Hence, what we are looking for is

$$dF_y = F_y(x + dx) - F_y(x) = \frac{\partial F_y(x)}{\partial x} dx \quad (11.1)$$

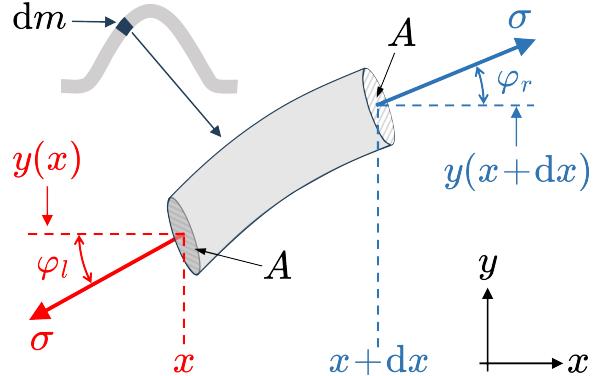


Fig. 11.3

For the force $F_y(x)$ we can read off from fig. 11.3

$$F_y(x) = \sigma A \sin \varphi_l$$

For small displacements, i.e., small angles, we can use the approximation

$$\sin \varphi_l \approx \tan \varphi_l = \frac{\partial y(x, t)}{\partial x}$$

With that we can express the net force dF_y in eq. (11.1) as

$$dF_y = \frac{\partial F_y(x)}{\partial x} dx = \frac{\partial}{\partial x} \left(\sigma A \frac{\partial y(x, t)}{\partial x} \right) dx = \sigma A \frac{\partial^2 y(x, t)}{\partial x^2} dx \quad (11.2)$$

With ρ being the density of the rope, we have $dm = \rho \cdot A \cdot dx$. Newton's second law tells us that the equation of motion in the y -direction for the rope piece dm is given by

$$dF_y = dm \frac{\partial^2 y(x, t)}{\partial t^2} = \rho A dx \frac{\partial^2 y(x, t)}{\partial t^2} \quad (11.3)$$

With the left sides of eq. (11.2) and eq. (11.3) being identical, their right sides must also be identical, hence

$$\frac{\partial^2 y(x, t)}{\partial t^2} = \frac{\sigma}{\rho} \frac{\partial^2 y(x, t)}{\partial x^2} \quad (11.4)$$

What this differential equation tells us is that the second time derivative of $y(x, t)$ is proportional to the second space derivative of $y(x, t)$ with respect to the x -coordinate. Differential equations of this kind are called wave equations because they describe the motion of a wave $y(x, t) = f(x \mp v_x t)$ which can travel in the positive x -direction (minus sign) or the negative x -direction (plus sign). With $y(x, t) = f(\xi)$ and $\xi = x \mp v_x t$ the derivatives in the differential equation become

$$\frac{\partial^2 y(x, t)}{\partial t^2} = \frac{\partial^2 f(\xi)}{\partial \xi^2} \left(\frac{\partial \xi}{\partial t} \right)^2 + \frac{\partial f(\xi)}{\partial \xi} \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 f(\xi)}{\partial \xi^2} v_x^2$$

and

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{\partial^2 f(\xi)}{\partial \xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{\partial f(\xi)}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 f(\xi)}{\partial \xi^2}$$

Inserting these derivatives into eq. (11.4) we obtain the relationship

$$v_x^2 = \frac{\sigma}{\rho} \quad (11.5)$$

which gives us the wave propagation velocity v_x as a function of the rope tension σ and the rope density ρ . We have not said anything yet about the specific waveforms, we just described the rope distortion as a bulge traveling the length of the rope. Now we will

consider a more specific situation where we not just yank the free end of the rope up and down a single time. To do that we will tie the left free end of the rope to a waveform generator which can modulate the amplitude of the yank we give the rope at that end to match any function or periodicity we may be interested in. As we do that we also lift the rope from the floor while keeping it at tension so it can freely oscillate without the constraint of the floor. So which yanking modulation should we chose? Well, just as the most oscillations one encounters in nature are of the harmonic form, most of the waveforms in nature are harmonic too. Hence, we will chose for the yanking amplitude a sine wave modulation. With that the waveform $f(x \pm v_x t)$ becomes

$$y(x, t) = f(x \pm v_x t) = y_0 \sin[k(x \pm v_x t)] \quad (11.6)$$

where y_0 is the wave amplitude, i.e., the yanking amplitude of our waveform generator, and k is the so-called wave number. The argument of the sine function must be dimensionless and therefore k must have the dimension of 1/length. The distance between two wave crests of the sine function is the wavelength λ . If we take a snapshot of the sine wave at $t = 0$ (fig. 11.4) we find that λ is determined by

$$k(x + \lambda) = kx + 2\pi \quad \Rightarrow \quad k = \frac{2\pi}{\lambda}$$

Hence, the wave number is a measure of how many crests fit into 2π . The shorter the wavelength, the more crests fit into 2π and vice versa. The time dependence of an harmonic oscillation is of course given by its angular frequency ω and with eq. (11.6) we therefore have the relationship $\omega t = kv_x t$ which means that the propagation speed of a wave is given by

$$v_x = \frac{\omega}{k} \quad \text{and with} \quad \omega = 2\pi f \quad \text{follows} \quad v_x = \lambda f$$

v_x is the so-called phase velocity of the wave. Each part of the wave moves with this velocity v_x , i.e., each part of the wave covers a distance of v_x meters in one second. ω is the angular frequency of the wave and f is the wave frequency. A stationary observer somewhere located on the x -axis will count $f = v_x/\lambda$ wave crests per second moving past her or his position. The phase change per second associated with those f wave crests passing by is $\omega = 2\pi f$ as 2π is the angular displacement between two wave crests. The lapsed time between two wave crests, i.e., one sine period, passing the observer is then $T = f^{-1} = 2\pi/\omega$.

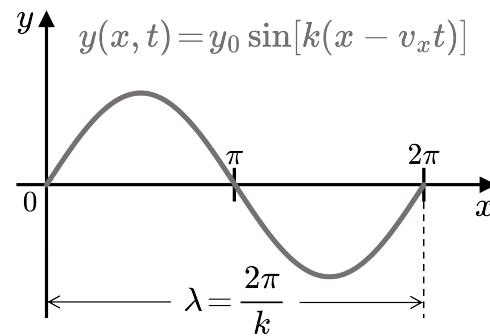


Fig. 11.4

Just as with harmonic oscillator motion, switching to complex numbers is also advantageous when one has to solve problems involving the propagation of waves. Hence, instead of using the sine or cosine functions one conveniently uses the complex exponential function to describe the motion of waves, i.e.,

$$y(x, t) = y_0 e^{i(kx \pm \omega t)} \quad (11.7)$$

and once the complex result has been found one identifies the real part of the complex result as the solution of the problem at hand. With eq. (11.4) we found the equation of motion for a wave form such as the one sketched in fig. 11.1 traveling along the length of the rope. But what happens when the wave form reaches the fixed end of the rope in fig. 11.1? For sure, the wave amplitude at the fixed end of the rope must vanish. Let's identify this fixed rope end with the position $x = 0$. Then at all times must hold

$$y(x = 0, t) = 0$$

One way to ensure that this condition always holds is to remove the fixed rope end point, virtually extend the rope, mirror the pulse $f(x - v_x t)$ in fig. 11.5 with respect to $x = 0$, invert it and then let it travel the other way as $g(x + v_x t)$. $f(x - v_x t)$ traveling the length of the rope to the right and its inverted mirror image $g(x + v_x t)$ traveling to the left will ensure that they cancel each other exactly at $x = 0$ as the function

$$y(x, t) = f(x - v_x t) + g(x + v_x t) \quad (11.8)$$

will always be zero at $x = 0$. At $x = 0$, $g(x + v_x t)$ turns into $f(x + v_x t)$, the wave form we see reflected at the fixed end of the rope and $f(x - v_x t)$ becomes its virtual mirrored and inverted $g(x - v_x t)$ which we of course cannot observe in reality. Hence, at the fixed end of the rope, i.e., at $x = 0$, the wave form is reflected such that the pulse becomes inverted and the wave travels back into the negative x -direction as $f(x + v_x t)$. The simple way in which we just added two waves in eq. (11.8) expresses an important principle of wave physics, the principle of superposition. According to this principle, when waves cross each other at a given point, the amplitude at that point is the sum of the amplitudes of the individual waves at that position. Because in fig. 11.5 the amplitude of the wave traveling to the left is at $x = 0$ just the negative of the amplitude of the wave traveling to

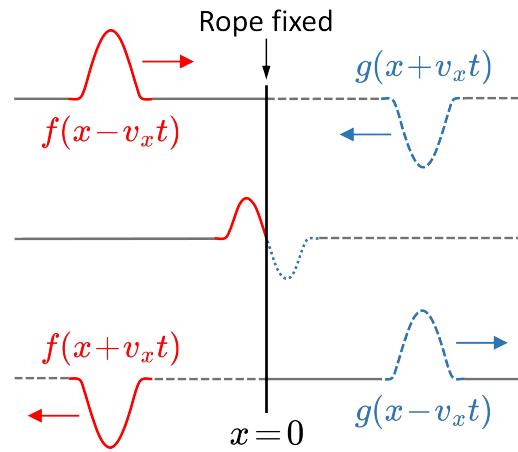


Fig. 11.5

the right, the two waves cancel each other at $x = 0$. Using the superposition principle we can simply add the sine wave produced by our continuously running waveform generator and the reflected wave it produces when being reflected at the fixed end of the rope. Because on reflection the wave is inverted we have to subtract the reflected wave from the incoming wave giving us the wave form

$$y(x, t) = y_0 \sin(kx - \omega t) - y_0 \sin(kx + \omega t) = -2y_0 \cos(\omega t) \sin(kx)$$

This is not the equation of a traveling wave anymore but, as illustrated in fig. 11.6, the waveform of a standing or stationary wave whose amplitude oscillates with $\cos(\omega t)$ between the stationary zeroes of this waveform. Fig. 11.6 shows the waveform $y_0 \sin(kx - \omega t)$ propagating to the right (solid red) and the reflected waveform $y_0 \sin(kx + \omega t)$ propagating to the left (dashed blue) at four different times. These zeroes of the resulting standing wave (solid black) are at those positions where $\sin(kx) = 0$, i.e., where $kx = n\pi$ with $n = 0, 1, 2, \dots$. The oscillating string of an instrument is likely the most familiar example of a standing wave. The strings of an instrument are tied down on both ends. Hence, only standing waves for which an integer multiple of $\lambda/2$ fits into the length L of a string are possible, i.e.,

$$L = n \cdot \frac{\lambda}{2} \quad \text{with} \quad n = 0, 1, 2, \dots$$

With $v_x = \lambda f$ and eq. (11.5) this tells us that a string of length L can only vibrate at frequencies

$$f = n \cdot \frac{v_x}{2L} = n \cdot \frac{\sqrt{\sigma/\rho}}{2L}$$

We all have likely watched someone tuning a string on an instrument or may have done that many times ourselves. When tuning an instrument string we change its tension and as this equation shows us, the frequency of the string we are tuning will change with the square root of the strings tension. By pressing down our fingers on different positions of a string instruments fingerboard, we are changing the frequency the string will vibrate at to a higher / lower frequency because we decrease / increase its effective length L .

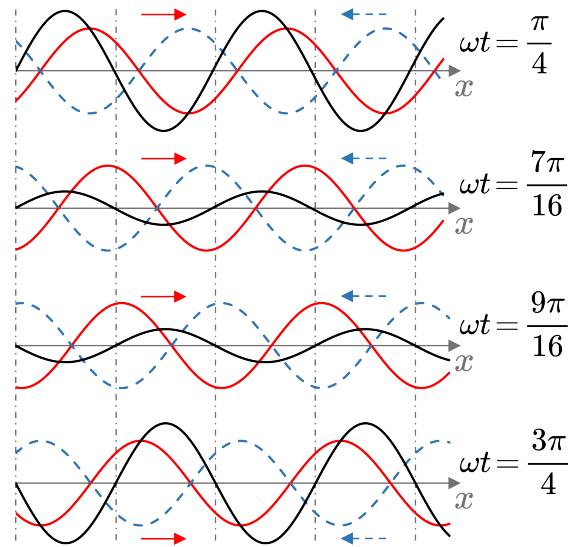


Fig. 11.6

11.2 Elastic Waves

In section 7.4 we discussed non-equilibrium force situations in elastic bodies and derived the equation of motion eq. (7.47) for a differential volume element in such an elastic body. To describe waves in elastic bodies, we use the complex exponential function and in particular consider elastic plane waves. With the amplitude $\xi^0 = (\xi_1^0, \xi_2^0, \xi_3^0)$ the general form of a plane wave is

$$\xi(x, y, z, t) = (\xi_1, \xi_2, \xi_3) = \text{Re} [\xi^0 \cdot e^{i(\mathbf{k}r - \omega t)}] \quad (11.9)$$

For a plane wave propagating in the x -direction as considered here, i.e., $\mathbf{k}r \rightarrow kx$, one determines the phase velocity at which this wave propagates by setting the expression in the exponential function constant. From

$$kx - \omega t = \varphi_0 = \text{const}$$

follows that

$$x = \frac{\varphi_0 + \omega t}{k} = \frac{\varphi_0}{k} + \frac{\omega t}{k} = \frac{\varphi_0}{k} + c_p t$$

where c_p is the phase velocity of the wave. The following relationships apply to c_p , wavelength λ and time period T of the plane wave:

$$\omega = c_p k \quad ; \quad \lambda = \frac{2\pi}{k} \quad ; \quad T = \frac{2\pi}{\omega} = \frac{1}{f}$$

If instead of the complex exponential wave function we just consider a simple sine wave

$$\sin(kx - \omega t) = \sin \varphi$$

with wavelength $\lambda = 2\pi/k$, we will accordingly find

$$x = \frac{\omega}{k} t \quad ; \quad \varphi = \text{const} \quad ; \quad \omega = c_p k$$

The previous section gave an introduction to simple mechanic waves using a rope wave as an example. A rope wave is a so-called transverse wave where wave oscillation occurs in a direction perpendicular to the propagation direction of the wave. There is however another type of wave, so-called longitudinal waves, where the oscillation occurs parallel to the propagation direction of the wave. Fig. 11.7 shows the basic characteristics

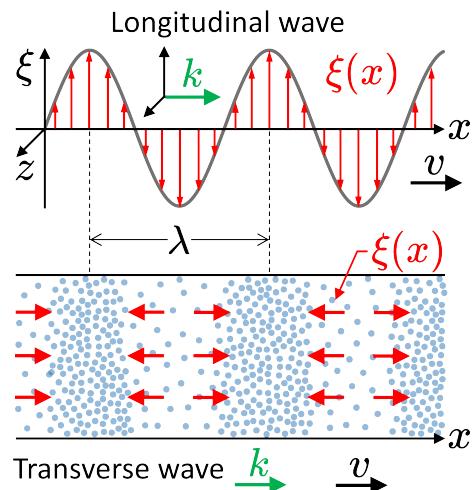


Fig. 11.7

for longitudinal and transverse waves. The best known example for longitudinal waves we are all familiar with are of course sound waves which, as sketched in fig. 11.7, are density waves.

First, we will consider transverse elastic waves. Transverse waves propagating in elastic media are also called shear waves because no volume change in the elastic body is associated with such a wave, just as is the case with a shearing motion. For such a wave, as shown in fig. 11.7, the direction of the wave oscillation $\boldsymbol{\xi}$ in eq. (11.9) must be orthogonal to the direction of propagation. In fig. 11.7 we have chosen a transverse wave that propagates in the positive x -direction and whose amplitude oscillates parallel to the y -direction, i.e., $\boldsymbol{\xi}^0 = (0, \xi_2^0, 0)$. For this transverse wave one can see immediately that

$$\operatorname{div} \boldsymbol{\xi} = \frac{\partial \xi_2}{\partial y} = \frac{\partial}{\partial y} (\xi_2^0 \cdot e^{i(kx - \omega t)}) = 0$$

As it is, $\operatorname{div} \boldsymbol{\xi}$ always vanishes for transverse waves as for them $\mathbf{k} \perp \boldsymbol{\xi}$ always holds. Hence, for a transverse wave propagating in x -direction eq. (7.47) becomes the wave equation

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} = f_i + G \Delta \xi_i$$

Inserting eq. (11.9) with $\boldsymbol{\xi}^0 = (0, \xi_2^0, 0)$ into this equation results in

$$-\rho \omega^2 \xi_2 = f_i - G k^2 \xi_2$$

Without external volume forces, i.e., in the case of $f_i = 0$, it follows

$$\omega^2 = \frac{G}{\rho} \cdot k^2 = c_t^2 \cdot k^2$$

where

$$c_t = \sqrt{c_{p,t}^2} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2\rho(\mu+1)}} \quad (11.10)$$

is the phase velocity of the transverse wave and the transformation in the last step made use of eq. (7.31).

Now lets consider longitudinal elastic waves. For such waves traveling in the x -direction obviously $\mathbf{k} \parallel \boldsymbol{\xi}$ and as sketched in fig. 11.7 the wave amplitude in eq. (11.9) must be $\boldsymbol{\xi}^0 = (\xi_1^0, 0, 0)$. With that follows

$$\operatorname{div} \boldsymbol{\xi} = \frac{\partial \xi_1}{\partial x} = -ik\xi_1 \quad \text{and} \quad \Delta \boldsymbol{\xi} = \frac{\partial^2 \xi_1}{\partial x^2} = -k^2 \xi_1$$

Therefore, for a longitudinal eq. (7.47) wave becomes

$$-\rho\omega^2\xi_1 = f_i + G \left[-k^2\xi_1 - \frac{ik}{1-2\mu} \frac{\partial}{\partial x} \xi_1 \right] = f_i - 2G \frac{1-\mu}{1-2\mu} k^2 \xi_1$$

Again considering the case without volume forces, i.e., $f_i = 0$, it follows

$$\omega^2 = \frac{2G}{\rho} \frac{1-\mu}{1-2\mu} k^2 = c_l^2 \cdot k^2$$

where

$$c_l = \sqrt{c_{p,l}^2} = \sqrt{\frac{2G}{\rho} \frac{1-\mu}{1-2\mu}} = \sqrt{\frac{E}{\rho} \frac{1-\mu}{(1+\mu)(1-2\mu)}} \quad (11.11)$$

is the phase velocity of the longitudinal wave. The last step in eq. (11.11) made again use of eq. (7.31). The considerations made here with respect to the propagation of transverse and longitudinal waves in elastic bodies assume that wave propagation takes place in an infinite medium. In practice, however, this is not the case. If the wavelength of a longitudinal wave is comparable to or greater than the transverse dimension of the medium, for example a thin rod, then something happens at the edge of the rod in which the longitudinal wave propagates. The lateral limitation of the rod causes a transverse contraction which is not prevented by surrounding matter, as is the case with an infinite medium. The density modulation caused by a longitudinal wave correlates with the stretching / compression in the x -direction:

$$\epsilon_x = \frac{1}{E} \sigma_x = \frac{\partial \xi_x}{\partial x}$$

From that, analogously to the previous treatment for an infinitely extended medium, the equation of motion follows directly from eq. (7.46) ($f_x = 0$)

$$\rho \frac{\partial^2 \xi_x}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} = E \frac{\partial^2 \xi_x}{\partial x^2}$$

Insertion of a longitudinal wave

$$\xi_x = \xi_0 e^{i(kx - \omega t)}$$

into this equation of motion gives for the phase velocity c_l^{rod} in the rod:

$$c_l^{rod} = \sqrt{\frac{E}{\rho}}$$

From the derivation of Poisson's number μ (see eq. (7.30)) one can see that μ can assume values between 0 and 0.5. This means that the phase velocity of the longitudinal wave in a thin rod is lower than the corresponding phase velocity in the infinite medium from eq. (11.11).

11.2.1 Sound Waves

Sound waves are density fluctuations propagating in an elastic body such as a gas, a liquid or a solid body. Therefore, one can use the equations governing the motion of fluids (see section 8.5.3) to derive the wave equation for sound waves. For the propagation of a pressure variation such as the one sketched in fig. 11.8 the Euler equation without internal forces, i.e., $\mathbf{f} = 0$ in eq. (8.31), and the continuity equation eq. (8.9) must apply. Hence:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

With eq. (8.1), the substantial derivative of the velocity for a sound wave propagating in an elastic medium in the x -direction (fig. 11.8) is given by

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \mathbf{v} \nabla v_x$$

The change in density caused by a sound wave is small compared to the density of the elastic body ($\delta\rho \ll \rho_0$). In addition, the rate of change and spatial changes in the velocity, $\partial\mathbf{v}/\partial t$ and $\nabla \cdot \mathbf{v}$ respectively, are small compared to \mathbf{v} . This enables the linearization of the Euler equation and the continuity equation in $\delta\rho$ and \mathbf{v} . It further applies:

$$P = P(\rho, T) = P(\rho, T(\rho)) \underset{\text{adiabatic}}{\downarrow} = P(\rho) \underset{\text{adiabatic}}{\downarrow}$$

With κ_S being the adiabatic compressibility from eq. (9.38), the gradient of P can therefore be rewritten as

$$\nabla P = \left(\frac{\partial P}{\partial \rho} \right)_{\text{ad. } \rho=\rho_0} \cdot \nabla(\delta\rho) = \frac{1}{\kappa_S \rho_0} \nabla(\delta\rho) \quad (11.12)$$

With that the continuity equation becomes

$$\frac{\partial(\delta\rho)}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0 \quad (11.13)$$

and the Euler equation becomes

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\kappa_S \rho_0} \nabla(\delta\rho) \quad (11.14)$$

From these two equations follows with $\nabla(\nabla(\delta\rho)) = \Delta(\delta\rho)$:

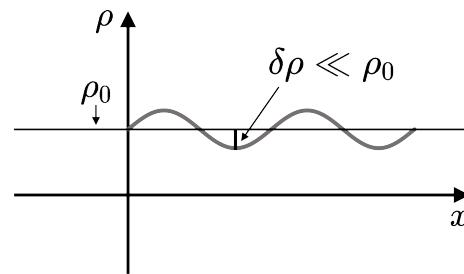


Fig. 11.8

$$\frac{\partial^2(\delta\rho)}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{v} = \frac{1}{\kappa_S \rho_0} \Delta(\delta\rho) = c_s^2 \Delta(\delta\rho) \quad (11.15)$$

where c_s^2 is the speed of sound. This equation is the wave equation in three dimensions which has the general form

$$\frac{\partial^2 \rho}{\partial t^2} = c_s^2 \Delta \rho \quad (11.16)$$

Eq. (11.16) is the wave equation for a density wave for which the appropriate solution approach is

$$\rho = \rho_0 + \delta\rho = \rho_0 + (\delta\rho)_0 e^{i(\mathbf{k}\mathbf{r}-\omega t)}$$

Calculating the respective derivatives

$$\begin{aligned} \frac{\partial(\delta\rho)}{\partial t} &= (\delta\rho)_0 (-i\omega) e^{i(\mathbf{k}\mathbf{r}-\omega t)} = -i\omega \delta\rho \quad ; \quad \frac{\partial(\delta\rho)}{\partial x} = i k_x \delta\rho \\ \frac{\partial^2(\delta\rho)}{\partial t^2} &= (i\omega)^2 \delta\rho \quad ; \quad \frac{\partial^2(\delta\rho)}{\partial x^2} = -k_x^2 \delta\rho \end{aligned}$$

and substituting into eq. (11.16) yields

$$\omega^2 = c_s^2 \mathbf{k}^2$$

As before, wavelength λ and the speed of sound c_s are linked via the relationship $\lambda f = c_s$. Hence, for the angular frequency ω and the wave vector \mathbf{k} applies

$$\omega = 2\pi f \quad \text{and} \quad |\mathbf{k}| = \frac{2\pi}{\lambda}$$

The wave vector is always perpendicular to the wavefront, i.e., to the line or surface of constant phase. In isotropic media, the direction of the wave vector \mathbf{k} is identical with the direction of wave propagation \mathbf{v} , i.e., $\mathbf{k} \parallel \mathbf{v}$. From the above derivation of the wave equation for a sound wave in an isotropic medium (eq. (11.13) and eq. (11.14) as well as the solution approach) we know that with

$$\mathbf{v} = \mathbf{v}_{0\perp} e^{i(\mathbf{k}\mathbf{r}-\omega t)} \quad \Rightarrow \quad \frac{\partial \mathbf{v}}{\partial t} = -i\omega \mathbf{v}$$

and therefore

$$\nabla(\delta\rho) = i\mathbf{k}\delta\rho = -\kappa_S \rho_0^2 \frac{\partial \mathbf{v}}{\partial t} = \kappa_S \rho_0^2 \cdot i\omega \mathbf{v}$$

which means that $\mathbf{v} \parallel \mathbf{k}$ must hold. Above, we found that for transverse waves, with $\boldsymbol{\xi}$ being the displacement measuring the wave amplitude, $\mathbf{k} \perp \boldsymbol{\xi}$ always holds. For sound waves, the displacement $\boldsymbol{\xi}$ points however in the direction of the density variation, i.e.,

the direction of sound propagation \mathbf{v} and with that for sound waves $\mathbf{k} \parallel \xi$ holds, as it must for longitudinal waves. Next we look at what happens if we make our observations from a “moving train”, i.e., we are moving relative to the source of the sound, or equally correct, the source of the sound is moving relative to us. In that case

$$\mathbf{r} \rightarrow \mathbf{r} - \mathbf{v}_0 t$$

and

$$e^{i(\mathbf{k}\mathbf{r}-\omega t)} \rightarrow e^{i(\mathbf{k}\mathbf{r}-\omega' t)}$$

with

$$\omega' = \omega + \mathbf{k}\mathbf{v}_0$$

This transformation describes a source of sound which moves with a relative velocity \mathbf{v}_0 with respect to the system of the observer which is at rest. Fig. 11.9 illustrates what happens in this situation. As the source of sound moves, the wavefronts of this sound will emanate from different points ever further removed from the origin on the straight line $x = v_0 t$. From fig. 11.9 one can read

$$v_0 = \frac{\Delta x}{\tau} \quad \text{and} \quad c_s = \frac{\Delta x}{\Delta \tau} \quad \Rightarrow \quad \frac{\Delta \tau}{\tau} = \frac{v_0}{c_s}$$

With

$$\tau' = \tau \pm \Delta \tau = \tau \left(1 \pm \frac{v_0}{c_s} \right)$$

follows accordingly

$$\omega' = \frac{\omega}{1 \pm v_0/c_s} \approx \omega \left(1 \mp \frac{v_0}{c_s} \right)$$

where the approximation in the last step applies for $v_0 \ll c_s$. As sketched in fig. 11.10, if $v_0 > c_s$, as is for example the case with airplanes that fly faster than the speed of sound, a so-called Mach cone, named after Ernst Mach (1838 - 1916), forms. The cone angle is

$$\sin \alpha = \frac{c_s}{v_0} = \frac{1}{M}$$

where M is the so-called Mach number. The faster an airplane flies the narrower this cone will become as the distance the airplane covers between any two sound waves being generated (their wavefronts indicated in fig. 11.10 by circles) will increase accordingly.

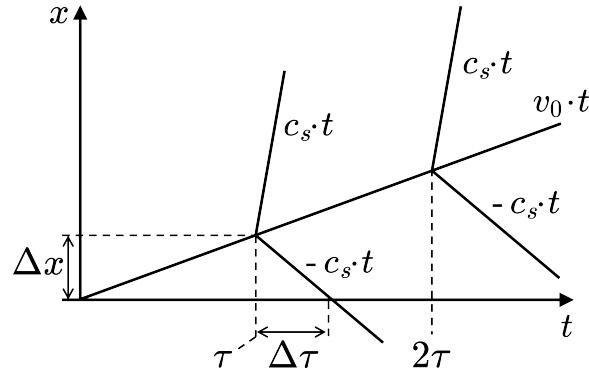


Fig. 11.9

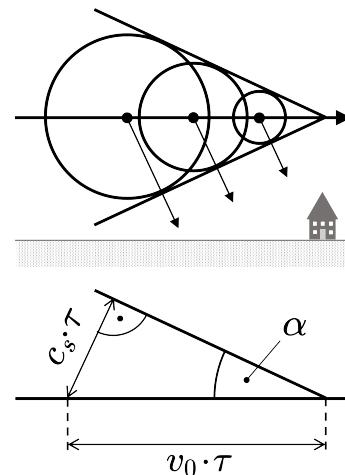


Fig. 11.10

11.2.2 The Doppler Effect

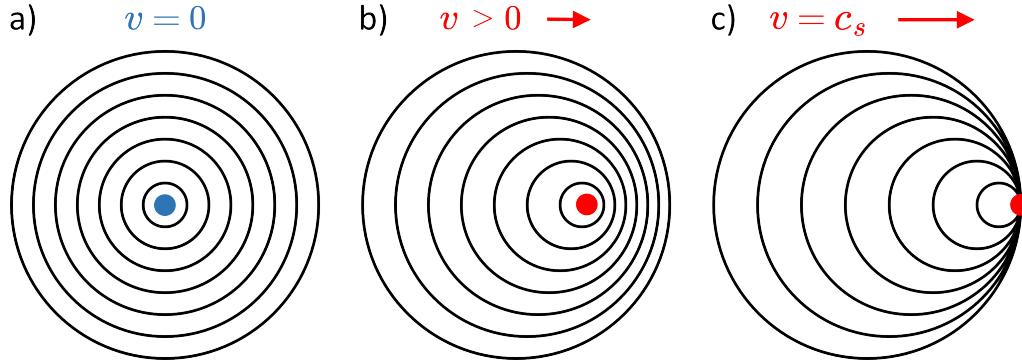


Fig. 11.11: Sound waves emanating from a sound source at rest (a), from a sound source moving at a velocity greater than zero but slower than the speed of sound (b) and from a sound source moving at the velocity of sound c_s (c).

The fact that an observer experiences a frequency shift if a sound source moves relative to her or to him is known as the Doppler effect, named after Christian Doppler (1803 - 1853), who first predicted it in the first half of the nineteenth century, albeit for light waves. We are all familiar with it when it comes to sound waves as we register the increasing pitch of a fire engine's siren when it approaches our position and the decreasing pitch as it moves away from us. The situation is illustrated in fig. 11.11 which depicts the waves fronts of spherical sound waves (sketched in two dimensions) emanating from a sound source at rest (fig. 11.11a), from a sound source moving at a velocity greater than zero but less than the speed of sound (fig. 11.11b) and a sound source moving at the speed of sound (fig. 11.11c). Clearly, in the first case it does not matter if an observer is located to the left or the right of the sound source as the distance between wavefronts that will reach the observer will be the same. That is different for the situation in fig. 11.11b where the distance between adjacent wavefronts will be less for an observer to the right of the source, i.e., in the direction the sound source is moving, than it will be for an observer located to the left to moving source. In fig. 11.11c this situation is even more pronounced and with the source moving at the speed of sound we have approached the situation we already discussed in fig. 11.10.

A sound wave will travel the distance between two wave crests in the time $T = \lambda_0/c_s$ where λ_0 shall be the wavelength in the case of fig. 11.11a where the sound source is at rest. Now, in the case of fig. 11.11b for an observer to the right of the sound source the wavelength separating two wave crests will appear shortened because when an observer registers one wave crest the next one will appear sooner than in fig. 11.11a because in the time T the sound source has moved towards the observer by the distance $v \cdot T = v/c_s$.

Accordingly, for an observer behind the moving source the distance between two wave crests has increased by that same amount. Hence, an observer in front or behind the moving source will register a wavelength of

$$\lambda = \lambda_0 - \frac{v}{f_0} \quad \text{or} \quad \lambda = \lambda_0 + \frac{v}{f_0}$$

respectively. The effective frequency registered by such observers is then given by

$$f = \frac{1}{1 \mp v/c_s} f_0 \tag{11.17}$$

where the minus sign stands for the source moving towards the observer and the plus sign for a source moving away from the observer. Eq. (11.17) is just the relationship we already encountered in the previous section, only that here it is expressed in terms of f instead of $\omega = 2\pi f$. This equation describes the frequency shift occurring because of the sound source itself moving and the observer being stationary. Interestingly, this frequency shift is different when the sound source is stationary and the observer moves. If an observer moves towards a stationary sound source she or he will register a higher pitch than what corresponds to the sound the source actually emanates because for an observer moving towards or moving away from a stationary sound source with a velocity v , the speed of sound will be higher or lower than c_s . Hence, the frequency a moving observer registers will be $f = (c_s \pm v)/\lambda_0$ where the plus sign is for an observer moving towards the source and the minus sign for an observer moving away from the source. Hence, expressed as a fraction of f_0 a moving observer will detect the frequency

$$f = \left(1 \pm \frac{v}{c_s}\right) f_0 \tag{11.18}$$

Eq. (11.17) describes the Doppler effect for a moving source and a stationary observer and eq. (11.18) the Doppler effect for a stationary sound source and a moving observer. Because for $v \ll c_s$

$$\frac{1}{1 \mp v/c_s} \approx 1 \pm \frac{v}{c_s}$$

the results of eq. (11.17) and eq. (11.18) will be identical for velocities $v \ll c_s$. However, for higher velocities the difference becomes quite noticeable as for example at $v = 0.5 c_s$, eq. (11.17) gives $f = 2 f_0$ and for eq. (11.18) the result is $f = 1.5 f_0$. Why is it not possible to switch reference systems between a sound source at rest and an observer at rest and come to the same result? Well, the medium of the sound in which the sound must propagate defines a preferred reference system and therefore a Galilean transformation does not work here as the two reference systems are not interchangeable.

11.2.3 Damped Sound Waves

If one neglects gravity, i.e., $\mathbf{f} = 0$, uses eq. (11.12) to replace ∇P with $\nabla \rho$, and keeps only terms linear in \mathbf{v} , then the Navier-Stokes equation, i.e., eq. (8.35), becomes

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\kappa_S \rho_0} \nabla \rho + \eta \Delta \mathbf{v} + (\eta + \lambda) \nabla (\nabla \cdot \mathbf{v}) \quad (11.19)$$

Taking the divergence of this equation by multiplying it from the left with ∇ gives

$$\frac{\partial}{\partial t} (\rho_0 \nabla \cdot \mathbf{v}) = -\frac{1}{\kappa_S \rho_0} \nabla \cdot (\nabla \rho) + \eta \cdot \underbrace{\nabla \cdot (\Delta \mathbf{v})}_{\parallel} + (\eta + \lambda) \nabla \cdot (\nabla (\nabla \cdot \mathbf{v})) \quad (11.20)$$

Now the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0$$

is used to replace the $\nabla \cdot \mathbf{v}$ terms in eq. (11.20) with the partial time derivative of ρ . One thus obtains the result

$$-\frac{\partial^2 \rho}{\partial t^2} = -\frac{1}{\kappa_S \rho_0} \Delta \rho - \frac{\eta}{\rho_0} \cdot \Delta \left(\frac{\partial \rho}{\partial t} \right) - \frac{\eta + \lambda}{\rho_0} \Delta \left(\frac{\partial \rho}{\partial t} \right) \quad (11.21)$$

By rearranging and inserting the speed of sound $c_s = 1/\sqrt{\kappa_S \rho_0}$, eq. (11.21) becomes the wave equation for damped sound waves

$$\frac{\partial^2 \rho}{\partial t^2} = c_s^2 \Delta \rho + \frac{2\eta + \lambda}{\rho_0} \Delta \left(\frac{\partial \rho}{\partial t} \right) \quad (11.22)$$

The standard solution approach for this type of equation is a plane wave:

$$\rho = \rho_0 + \delta \rho = \rho_0 + (\delta \rho)_0 \cdot \exp[i(kx - \omega t)]$$

By calculating the derivatives and inserting them into eq. (11.22) it follows that

$$\omega^2 = c_s^2 k^2 - i \frac{2\eta + \lambda}{\rho_0} k^2 \omega \quad (11.23)$$

One first solves eq. (11.23) for ω . By completing the square one gets (only the positive sign of the square root is of physical relevance)

$$\omega + i \frac{2\eta + \lambda}{2\rho_0} k^2 = c_s k \sqrt{1 - \left(\frac{2\eta + \lambda}{2\rho_0} \cdot \frac{k}{c_s} \right)^2} \quad (11.24)$$

For small x , one makes use of the approximations

$$(1 \pm x)^{1/2} \approx 1 \pm \frac{1}{2}x \quad \text{and} \quad (1 \pm x)^{-1/2} \approx 1 \mp \frac{1}{2}x$$

Using the first of these approximations for the case of a small k , eq. (11.24) becomes

$$\omega = c_s k - i \frac{2\eta + \lambda}{2\rho_0} k^2 \quad (11.25)$$

Inserted into eq. (11.23) this gives for the time dependence of the plane wave

$$\exp(-i\omega t) = \exp(-ic_s kt) \cdot \exp\left(-\frac{2\eta + \lambda}{2\rho_0} k^2 t\right) \quad (11.26)$$

Now one solves eq. (11.23) for k

$$k = \frac{\omega}{c_s} \left(1 - i \frac{2\eta + \lambda}{\rho_0 c_s} \cdot \frac{\omega}{c_s}\right)^{-1/2} \quad (11.27)$$

Using the second approximation from above for small ω , eq. (11.27) becomes

$$k = \frac{\omega}{c_s} + i \frac{2\eta + \lambda}{2\rho_0 c_s} \left(\frac{\omega}{c_s}\right)^2 \quad (11.28)$$

Inserted into eq. (11.23) this gives for the spatial dependence of the plane wave

$$\exp(-ikx) = \exp\left(-i\frac{\omega}{c_s} \cdot x\right) \cdot \exp\left[\underbrace{-\frac{2\eta + \lambda}{2\rho_0 c_s} \left(\frac{\omega}{c_s}\right)^2 \cdot x}_{= \alpha/2}\right] \quad (11.29)$$

Eq. (11.26) and eq. (11.29) show that sound waves propagating in a body with viscosity η and volume viscosity λ are subject to a temporal damping as well as a spatial damping (illustrated in fig. 11.12). The first viscosity or respectively dynamic viscosity η of a material is linked to its kinematic viscosity λ by the relationship $\eta = \lambda \cdot \rho$. Tab. 11.1 gives the corresponding values for three well-known substances.

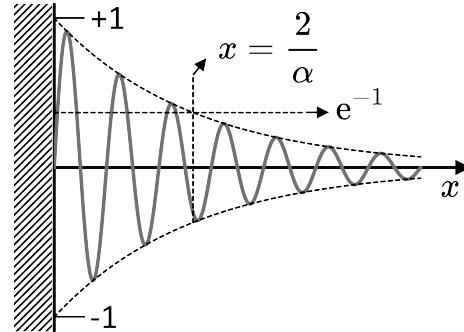


Fig. 11.12

Tab. 11.1: Density, dynamic and kinematic viscosity of air, water and glycerol at 20 °C.

Body	ρ [g·cm ⁻³]	η [g·cm ⁻¹ ·s ⁻¹]	λ [cm ² ·s ⁻¹]
Air	$1.20 \cdot 10^{-3}$	$1.83 \cdot 10^{-4}$	0.15
H ₂ O	1.00	0.01	0.01
Glycerol	1.26	14.12	11.2

11.3 Complex Waves and Wave Packets

11.3.1 Fourier Series

As we saw in section 11.1, a standing wave is the result of the addition of two periodic wave forms traveling in opposite direction where one is the inverted mirror image of the other. Superposition of waves is of course not limited to adding an incoming and a reflected wave. One can add any number of periodic wave forms differing from each other in frequencies and amplitudes. The superposition result will likely look more complicated than the standing wave we just discussed but nevertheless the result of this superposition will be another periodic wave form. This works also the other way around. Any periodic function, however complicated, can be described as a superposition of harmonic waves. This works for functions $F(x) = F(x + n \cdot \Delta x)$ with $n = 1, 2, 3, \dots$ which are periodic in space, functions $F(t) = F(t + n \cdot \Delta t)$ with $n = 1, 2, 3, \dots$ which are periodic in time or functions which are periodic in any other variable. All such functions can be represented as superpositions of harmonic waves which are called Fourier series, after Jean-Baptiste Joseph Fourier (1768 - 1830):

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nkx) + b_n \sin(nkx)] \quad (11.30a)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \quad (11.30b)$$

If we are for example tasked to represent an arbitrary periodic function $f(t)$ by its Fourier series, the challenge lies of course in determining the coefficient a_n and b_n . Only if we know all those coefficients, i.e., all correct weights for each sine or cosine in the Fourier series, will the superposition of all sines and cosines add up to the arbitrary function we seek to describe through a Fourier series. That task is actually easier than it may look at first glance. We just have to recognize that the average value of $f(t)$ over one period $T = 2\pi/\omega$ is just a_0 because over one period the average value for each sine or cosine in the series is exactly zero:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt \quad (11.31)$$

To determine the other coefficients we make use of the following definite integrals, the values of which one can determine by simple partial integration or by looking them up in integral tables (n and m integer)

$$\int_0^T \sin(n\omega t) \cos(m\omega t) dt = 0 \quad \text{for all } n, m \quad (11.32)$$

$$\left. \begin{array}{l} \int_0^T \cos(n\omega t) \cos(m\omega t) dt \\ \int_0^T \sin(n\omega t) \sin(m\omega t) dt \end{array} \right\} = \begin{cases} 0 & \text{if } n \neq m \\ \frac{T}{2} & \text{if } n = m \end{cases} \quad (11.33)$$

To determine all a_n and b_n with $n \geq 0$ for $f(t)$ in eq. (11.30b) we only have to calculate the value of the integrals of $f(t) \cos(n\omega t)$ and $f(t) \sin(n\omega t)$ over one period T to obtain

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t) dt \quad (11.34)$$

and

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t) dt \quad (11.35)$$

Now that we know how to determine the Fourier series coefficients of a periodic function $f(t)$ we will look at a simple example, the symmetrical square wave function. Fig. 11.13 shows such a square wave function plus three other curves which approximate the square wave function to different degrees. We will come to these other functions once we have determined the Fourier series of this square wave function. To calculate the respective Fourier series coefficients using eq. (11.31), eq. (11.34) and eq. (11.35) we split the integrals over the period T into two parts: A first part from $0 \leq t \leq T/2$ where $f(t) = h$ and a second part from $t/2 \leq t \leq T$ where $f(t) = -h$. Evaluating the respective integrals we then get for the coefficients

$$a_0 = \frac{1}{T} \int_0^{T/2} h dt - \frac{1}{T} \int_{T/2}^T h dt = \frac{h}{T} \left[\left(\frac{T}{2} - 0 \right) - \left(T - \frac{T}{2} \right) \right] = 0$$

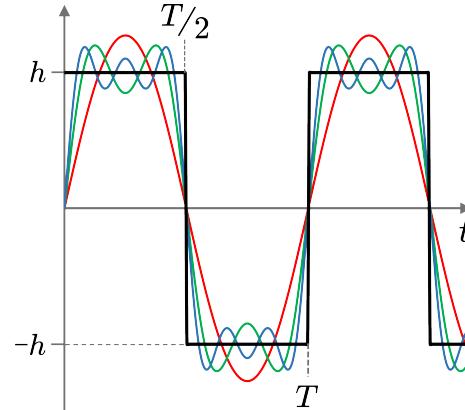


Fig. 11.13

$$\begin{aligned}
a_n &= \frac{2}{T} \int_0^{T/2} h \cos(n\omega t) dt - \frac{2}{T} \int_{T/2}^T h \cos(n\omega t) dt \\
&= \frac{2}{T} \frac{h}{n\omega} \sin(n\omega t) \Big|_0^{T/2} - \frac{2}{T} \frac{h}{n\omega} \sin(n\omega t) \Big|_{T/2}^T \quad \left/ T = \frac{2\pi}{\omega} \right. \\
&= \frac{h}{n\pi} [(\sin(n\pi) - \sin(0)) - (\sin(2n\pi) - \sin(n\pi))] = 0 \\
b_n &= \frac{2}{T} \int_0^{T/2} h \sin(n\omega t) dt - \frac{2}{T} \int_{T/2}^T h \sin(n\omega t) dt \\
&= -\frac{2}{T} \frac{h}{n\omega} \cos(n\omega t) \Big|_0^{T/2} + \frac{2}{T} \frac{h}{n\omega} \cos(n\omega t) \Big|_{T/2}^T \quad \left/ T = \frac{2\pi}{\omega} \right. \\
&= -\frac{h}{n\pi} [(\cos(n\pi) - \cos(0)) - (\cos(2n\pi) - \cos(n\pi))] \\
&= -\frac{h}{n\pi} [2\cos(n\pi) - 2] = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4h}{n\pi} & \text{for } n \text{ odd} \end{cases}
\end{aligned}$$

The Fourier series of the square function in fig. 11.13 is only composed of sines and at that, only of sines where the coefficient is odd, i.e., $n = 2k - 1$

$$f(t) = \frac{4h}{\pi} \left[\sin(\omega t) + \frac{1}{3} \sin(3\omega t) + \frac{1}{5} \sin(5\omega t) + \dots \right] = \frac{4h}{\pi} \sum_{k=1}^{\infty} \frac{\sin[(2k-1)\omega t]}{2k-1}$$

The three curves in fig. 11.13 showing various approximations represent the first terms of this Fourier series. The red curve is just the first series term, the green curve is the sum of the first and the second term and the blue curve is the sum of the first three terms in this Fourier series. Clearly, the more terms of the Fourier series we include, the better the resulting curve will reproduce the square wave form.

11.3.2 Wave Energy

Just like for an harmonic oscillator or any other moving mass, the energy of a wave is the sum of its kinetic energy and its potential energy. Take the example of the rope wave from section 11.1. Assuming we have a perfect rope we can define a linear mass density $\mu = \Delta m / \Delta x$ where Δm and Δx are the infinitesimal mass and length elements associated with an infinitesimal rope section. With the rope wave traveling in x -direction and the rope deflection occurring in y -direction the change in kinetic energy of such an infinitesimal rope element must be

$$\Delta T = \frac{1}{2} \Delta m \cdot v_y^2 = \frac{1}{2} \mu \Delta x \cdot v_y^2$$

which in the limits of $\Delta x \rightarrow 0$ becomes

$$dT = \lim_{\Delta x \rightarrow 0} \frac{1}{2} \mu \cdot v_y^2 \cdot \Delta x = \frac{1}{2} \mu \cdot v_y^2 \cdot dx$$

With the wave traveling in the positive x -direction given by $y(x, t) = y_0 \sin(kx - \omega t)$ the velocity v_y of any mass element in the rope oscillating in y -direction is given by

$$v_y = \frac{d}{dt} y(x, t) = -\omega y_0 \cos(kx - \omega t)$$

With that the kinetic energy dT becomes

$$dT = \frac{1}{2} \mu \cdot \omega^2 \cdot y_0^2 \cdot \cos^2(kx - \omega t) \cdot dx \quad (11.36)$$

To determine the corresponding potential energy of this oscillating rope element we look at it as a simple harmonic oscillator of mass Δm oscillating at an angular frequency ω . Associating with this oscillator the “spring constant” $k_{rope} = \Delta m \cdot \omega^2$ we get with eq. (4.2) for its potential energy

$$\Delta U = \frac{1}{2} \Delta m \cdot \omega^2 \cdot y^2 = \frac{1}{2} \mu \Delta x \cdot \omega^2 \cdot y^2$$

which in the limits of $\Delta x \rightarrow 0$ becomes

$$dU = \lim_{\Delta x \rightarrow 0} \frac{1}{2} \mu \cdot \omega^2 \cdot y^2 \cdot \Delta x = \frac{1}{2} \mu \cdot \omega^2 \cdot y^2 \cdot dx$$

Inserting for $y(x, t)$ the plane wave traveling in positive x -direction this becomes

$$dU = \frac{1}{2} \mu \cdot \omega^2 \cdot y_0^2 \cdot \sin^2(kx - \omega t) \cdot dx \quad (11.37)$$

With that the total change in energy dE associated with the motion of an infinitesimal mass element Δm of the rope as the wave propagates through the rope is given by

$$dE = dT + dU = \frac{1}{2} \mu \omega^2 y_0^2 dx \quad (11.38)$$

With dE/dx we have the energy of the wave per unit length. Because waves come at any number of length one standardizes the energy content of wave to the value contained in of one wavelength. Integrating eq. (11.38) from 0 to λ this standardized energy content of a wave is then given by

$$E_\lambda = \frac{1}{2} \mu \omega^2 y_0^2 \lambda \quad (11.39)$$

where the subscript λ tells us that we are looking at the energy contained within one wavelength of the wave. This wave energy is a function of a material parameter μ and

it is proportional to the square of the wave frequency ω and the square of the wave amplitude y_0 . With the oscillation period of the wave being T we know that the rate at which this wave energy travels is E_λ/T and therefore the average power of the wave at any point will be given by

$$\langle P \rangle = \frac{1}{2} \mu \omega^2 y_0^2 \frac{\lambda}{T} = \frac{1}{2} \mu \omega^2 y_0^2 v \quad (11.40)$$

where v is the phase velocity of the wave. With eq. (11.39) and eq. (11.40) we know the energy and average power associated with a mechanical wave such as a rope wave or a vibrating string, both of which are transverse waves but what about longitudinal waves such as sound waves. There, we need a different measure than the linear density μ we used to calculate the energy of a rope wave. Evidently, what we need to look at for a sound wave is the volume density measure, i.e., $\rho = \Delta m / \Delta V$. With a sound wave traveling in x direction given by $\xi(x, t) = \xi_0 \sin(kx - \omega t)$ the infinitesimal kinetic energy change associated with an infinitesimal volume change becomes

$$\Delta T = \frac{1}{2} \Delta m \cdot \left(\frac{d\xi}{dt} \right)^2 = \frac{1}{2} \rho \Delta V \cdot \left(\frac{d\xi}{dt} \right)^2$$

and using again eq. (4.2) we get for the respective infinitesimal potential energy change

$$\Delta U = \frac{1}{2} \Delta m \cdot \omega^2 \cdot \xi^2 = \frac{1}{2} \rho \Delta V \cdot \omega^2 \cdot \xi^2$$

In the limit $\Delta V \rightarrow 0$ these expressions for the kinetic and potential energy changes become

$$dT = \lim_{\Delta V \rightarrow 0} \frac{1}{2} \rho \cdot \left(\frac{d\xi}{dt} \right)^2 \Delta V = \frac{1}{2} \rho \cdot \left(\frac{d\xi}{dt} \right)^2 dV$$

and

$$dU = \lim_{\Delta V \rightarrow 0} \frac{1}{2} \rho \cdot \omega^2 \cdot \xi^2 \Delta V = \frac{1}{2} \rho \cdot \omega^2 \cdot \xi^2 dV$$

Inserting $\xi(x, t)$ and its time derivative the expressions for dT and dU become

$$dT = \frac{1}{2} \rho \omega^2 \xi_0^2 \cos^2(kx - \omega t) dV \quad (11.41)$$

and

$$dU = \frac{1}{2} \rho \omega^2 \xi_0^2 \sin^2(kx - \omega t) dV \quad (11.42)$$

and we get for the total energy change associated with the displacement of a volume element

$$dE = dT + dU = \frac{1}{2} \rho \omega^2 \xi_0^2 dV \quad (11.43)$$

If we consider the energy the wave carries through a cross section A perpendicular to the propagation direction of the wave then we can write with $dV = Adx$

$$dE = \frac{1}{2}\rho\omega^2\xi_0^2Adx$$

Looking again at the energy content within a single wavelength we get for the sound wave the result

$$E_\lambda = \frac{1}{2}\rho\omega^2\xi_0^2A\lambda \quad (11.44)$$

Just as for rope wave, the average power transferred by the sound wave is then given by E_λ divided by the period T of the wave, i.e.,

$$\langle P \rangle = \frac{1}{2}\rho\omega^2\xi_0^2\frac{\lambda}{T} = \frac{1}{2}\mu\omega^2\xi_0^2c_s \quad (11.45)$$

where c_s is the velocity of the sound wave. Now that we know how to calculate the energy contained in monochromatic longitudinal and transverse waves, the next question is how one can determine the energy of a wave containing more than one frequency, as for example the symmetrical square wave in fig. 11.13 does. The above shows that for longitudinal as well as transverse monochromatic waves, wave energy is proportional to the square of the wave amplitude. In section 11.3.1 we learned that any wave, however oddly it may be shaped and however many frequencies it may contain, can be described by a Fourier series. Monochromatic waves are just Fourier series where all coefficients but one are zero. Therefore, the energy of a wave must be proportional to the square of the Fourier series over one period. With eq. (11.31), eq. (11.34) and eq. (11.35) it is easy to see that the value of the square of the Fourier series in eq. (11.30b) over one period (remember, we measure wave energy over one wavelength) must be

$$\int_0^T [f(t)]^2 dt = Ta_0^2 + \frac{T}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad (11.46)$$

Eq. (11.46) has this simple form because with eq. (11.32) and eq. (11.33) the integrals over all the cross terms $f(t) \cdot f(t)$ vanish over one period and only the square terms remain. For example, in the case of the symmetric square wave sketched in fig. 11.13 for which $[f(t)]^2 = h^2$, eq. (11.46) becomes

$$h^2T = \frac{T}{2} \left(\frac{4h}{\pi} \right)^2 \sum_{k=1}^{\infty} \frac{1}{2k-1}$$

Incidentally, this result also tells us that the sum on the right side of this equation is just $\pi^2/8$.

11.3.3 Wave Packets

A Fourier series is composed of an infinite number of sine and cosine waves of discrete frequencies which are integer multiples of a basic frequency ω . It allows us to describe any periodic wave function. But what about non-periodic functions like a short pulse of wave forms, a wave packet? To understand how we can describe a wave packet we will first switch to the complex form of the Fourier series in eq. (11.30b) which is given by

$$f_T(t) = \sum_{n=-\infty}^{+\infty} f_n e^{in\omega t} = \sum_{n=-\infty}^{+\infty} f_n [\cos(n\omega t) + i \sin(n\omega t)] \quad (11.47)$$

where the subscript T indicates the period length of this periodic function. A simple comparison of eq. (11.30b) and eq. (11.47) shows that

$$f_0 = a_0 \quad ; \quad f_n = \frac{1}{2}(b_n - ia_n) \quad ; \quad f_{-n} = \frac{1}{2}(b_n + ia_n)$$

As can be easily checked, these complex Fourier coefficients are also given by

$$f_n = \frac{1}{T} \int_0^T f_T(\tau) e^{-in\omega\tau} d\tau = \frac{1}{T} \int_{-T/2}^{+T/2} f_T(\tau) e^{-in\omega\tau} d\tau$$

Using this expression for the Fourier coefficients and the equidistant frequency interval of the Fourier series

$$\Delta\omega = (n+1)\omega - n\omega = \omega = \frac{2\pi}{T}$$

we can rewrite eq. (11.47) as ($n\omega = \omega_n$)

$$f_T(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} \int_{-T/2}^{+T/2} f_T(\tau) e^{i\omega_n(t-\tau)} d\tau \cdot \Delta\omega \quad (11.48)$$

Up to here we are still looking at infinitely periodic functions. But our interest here is to understand what happens in the limit of $T \rightarrow \infty$, i.e., in the limit

$$f(t) = \lim_{T \rightarrow \infty} f_T(t) \quad (11.49)$$

where we transition from periodic to non-periodic functions. If we define

$$g_T(\omega) = \int_{-T/2}^{+T/2} f_T(\tau) e^{-i\omega\tau} d\tau \quad (11.50)$$

then we can write eq. (11.48) as ($\Delta\omega = \omega_{n+1} - \omega_n$)

$$f_T(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} g_T(\omega_n) e^{i\omega_n t} (\omega_{n+1} - \omega_n) \quad (11.51)$$

Eq. (11.51) is what mathematicians call a Riemann sum for the partition $\{\omega_n\}$. For very large periods, i.e., in the case of eq. (11.49), this sum becomes the integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\omega) e^{i\omega t} d\omega \quad (11.52)$$

where $g(\omega)$ is $g_T(\omega)$ from eq. (11.50) in the limit $T \rightarrow \infty$. If the respective integral for $T \rightarrow \infty$ exists for all real ω then

$$g(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \quad (11.53)$$

is the Fourier transform of $f(t)$ and with eq. (11.52) one also has the reverse Fourier transform. Similar to the transition from the Fourier series eq. (11.30b) to the Fourier transform of eq. (11.53) and reverse transform of eq. (11.52) for functions of the variables t and ω that we just went through, we can also transition Fourier series such as eq. (11.30a) which are functions of the variables x and k . We do that in the same way by moving from discrete values for k in eq. (11.30a) to a continuous k variable to obtain the respective Fourier transform

$$g(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \quad (11.54)$$

and the inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(k) e^{ikx} dk \quad (11.55)$$

Lets now look at examples, one each for the t - ω variable pair and the x - k variable pair. We will begin with the t - ω variable pair. Let $f(t)$, as sketched in fig. 11.14, be a single rectangular pulse with $f(t) = h$ for $-\tau \geq t \leq \tau$ and zero for all other times. The Fourier transform eq. (11.53) of this rectangular pulse reduces to

$$g(\omega) = h \cdot \int_{-\tau}^{+\tau} e^{-i\omega t} dt = -\frac{h}{i\omega} (e^{-i\omega\tau} - e^{i\omega\tau}) = \frac{2h}{\omega} \sin(\omega\tau) \quad (11.56)$$

In the case of $\omega = 0$ we need to calculate the limit value using L'Hôpital's rule

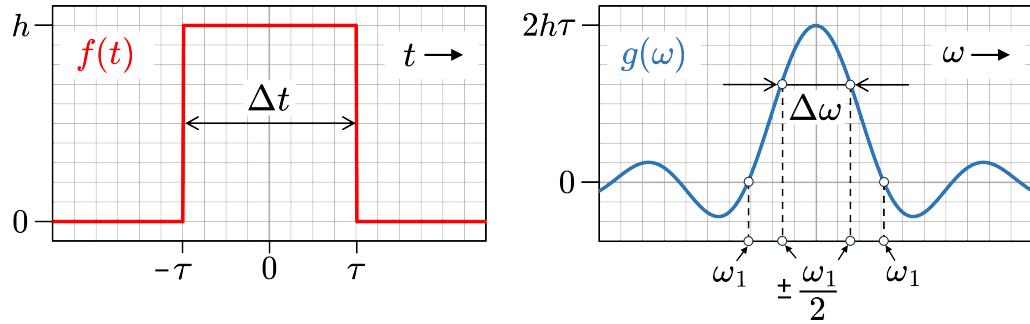


Fig. 11.14: Square wave pulse in the time domain (left) and the frequency domain (right).

$$\lim_{\omega \rightarrow 0} \frac{\sin(\omega\tau)}{\omega} \stackrel{\left(\frac{0}{0}\right)}{=} \lim_{\omega \rightarrow 0} \frac{\tau \cos(\omega\tau)}{1} = \tau$$

With the sinc function defined as

$$\text{sinc}(z) = \begin{cases} \frac{\sin(z)}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

we get for the Fourier transform of a square pulse of height \$h\$ and duration \$2\tau\$

$$g(\omega) = 2h\tau \text{sinc}(\omega\tau)$$

Determining the full width at half maximum of \$g(\omega)\$, i.e., \$\Delta\omega\$, is analytically not possible as \$\text{sinc } z = 1/2\$ has no analytic solution. However, as illustrated in fig. 11.14, we can use a different approximation as a measure for \$\Delta\omega\$. \$g(\omega)\$ has its first zeros at \$\omega_1 = \pm\pi/\tau\$. The value of \$g(\omega)\$ at \$\omega_1/2\$ is with

$$g\left(\pm\frac{\pi}{2\tau}\right) = \frac{4h\tau}{\pi} \approx 0.64 \cdot g(\omega = 0)$$

just a little higher than half the maximum \$g(\omega = 0) = 2h\tau\$. Hence, we can use \$\omega_1\$ as an approximation for \$\Delta\omega\$ and therefore use the condition for the first zeroes of \$g(\omega)\$ as a measure for \$\Delta\omega\$:

$$\Delta\omega \approx \omega_1 = \pm\pi/\tau$$

With \$\tau\$ being half the width of the rectangular pulse \$f(t)\$, i.e., \$\tau = \Delta t/2\$, this equation becomes

$$\Delta\omega\Delta t \approx 2\pi \tag{11.57}$$

Eq. (11.57) tells us that the shorter the duration \$\Delta t\$ of the rectangular pulse is, the broader will be the frequency spectrum \$\Delta\omega\$ of the oscillation the Fourier transform of

$f(t)$ gives us. Next we will look at a an example in the space / wave vector domain as described by eq. (11.54) and eq. (11.55). For that we pick the short cosine wave train shown in fig. 11.15 with wave vector k_0 where $f(x) = h \cdot \cos(k_0 x)$ for $-a \leq x \leq a$ and zero everywhere else. To calculate the Fourier transform of $f(x)$ it is best to use the complex waveform $\bar{f}(x) = \cos(k_0 x) + i \sin(k_0 x)$ instead of $f(x)$ and at the end just take the real part of the solution. Hence, we calculate

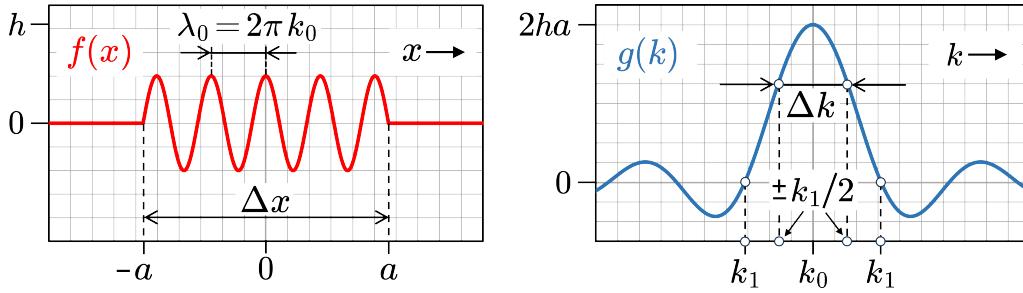


Fig. 11.15: Cosine wave packet in the space domain (left) and the wave vector domain (right).

$$g(k) = \operatorname{Re} \left\{ h \cdot \int_{-a}^{+a} e^{-i(k-k_0)x} dx \right\}$$

With eq. (11.56) we already know the solution for this integral, which is a real solution only, we just need to replace ω with $k - k_0$ and τ with a . Just as with the rectangular pulse, the Fourier transform of $f(x)$ in fig. 11.15 is also a sinc function:

$$g(k) = \begin{cases} 2h \frac{\sin[(k - k_0)a]}{k - k_0} & \text{if } k \neq k_0 \\ 2ha & \text{if } k = k_0 \end{cases}$$

Calculating the width of $g(k)$ in fig. 11.15 in the same way we calculated the width of $g(\omega)$ in fig. 11.14 we find the analogue of eq. (11.57) in the space / wave vector domain, i.e.,

$$\Delta x \Delta k \approx 2\pi \quad (11.58)$$

The shorter the length of the cosine pulse train is, the broader its wave vector spectrum becomes and vice versa. The relations in eq. (11.57) and eq. (11.58) express a fundamental property of the Fourier transformation of finite wave trains. Frequency ω and time t just as space x and wave vector k are so-called conjugate variables and eq. (11.57) and eq. (11.58) express a fundamental uncertainty relationship that exists for all such conjugate variable pairs. The more precise we know one of the conjugate variables, i.e.,

the smaller its Δ becomes, the less precise we will know the other conjugate variable as its Δ grows correspondingly larger.

11.3.4 Waves in Three Dimensions

Throughout this chapter, except for a brief section in the discussion of sound waves, we discussed wave phenomena for the most in terms of waves traveling in a given coordinate direction, be it simple sine waves such as in eq. (11.6)

$$y(x, t) = y_0 \sin[k(x \pm v_x t)]$$

or waves in complex notation such as in eq. (11.7)

$$y(x, t) = y_0 e^{i(kx \pm \omega t)}$$

For such waves, regardless if they are transverse or longitudinal waves, all of its points which possess the same phase or which have the same displacement lie on a plane perpendicular to the direction of propagation. Or phrased differently, in a plane perpendicular to the direction of wave propagation all points of such a wave have the same amplitude and the same phase. For that reason, such waves are called plane waves. To describe plane waves propagating in three dimensional space in any arbitrary direction one switches from the wave number k to the wave vector \mathbf{k} as we have done in this chapter on a number of occasions.

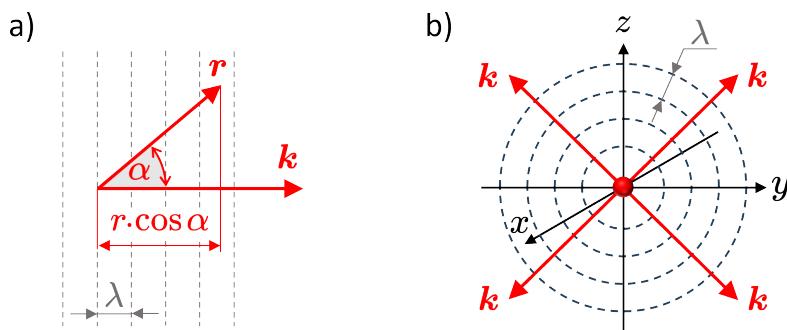


Fig. 11.16: Orientation of wave vectors in the case of a plane wave (a) and for circular or spherical waves (b).

As illustrated in fig. 11.16a, the wave vector of a plane wave points in the wave propagation direction. As sketched, to determine the phase of the wave at any arbitrary position \mathbf{r} one just needs to calculate the projection of \mathbf{r} onto the propagation direction of the wave, i.e., the direction of the wave vector. Hence, one must calculate

$$\mathbf{k} \cdot \mathbf{r} = k \cdot r \cos \alpha$$

All points for which $\mathbf{k} \cdot \mathbf{r} = \text{const}$ lie in one plane and therefore

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \sin(\mathbf{k}\mathbf{r} - \omega t)$$

and

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}$$

describe harmonic plane waves propagating in three dimensional space. In many cases, rather than a plane wave, we encounter what is called a spherical wave. If a wave is generated by a point source, like for example when we throw a pebble in a quiet pond or when an alarm clock wakes us in the morning, the wavefronts generated are not plane waves but circular or spherical waves as sketched in fig. 11.16b. If the medium in which such a wave propagates is isotropic the wave vector will be the same in all directions and just as for a plane wave it will always be orthogonal to the direction of propagation. As the wave expands, the energy in the wave is spread ever more thinly across the expanding wavefront, which means that the amplitude decreases with the distance from the wave source which we take to be the origin of the coordinate system. Hence, a simple harmonic spherical wave is described by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{A}_0}{r} e^{i(\mathbf{k}\mathbf{r} - \omega t)}$$

As we found in section 11.3.1, a wave's energy is proportional to the square of the wave amplitude. Hence, for a spherical wave the energy is proportional to r^{-2} , or put differently, it is inversely proportional to the expanding surface area of the wave's spherical wavefront. At great distances from the source of a spherical wave, its wavefront will appear to a local observer to become ever flatter and will ever more resemble a plane wave. Hence, in the limit of large distances from the source of a spherical wave we can locally treat spherical waves just like plane waves. Just like plane waves can travel back or forth, so can spherical waves. While spherical waves emanating from a wave source is what we commonly experience, the opposite is also possible. Instead of spherical waves traveling outwards from a common center they can also travel inwards towards a common center like as for example shock waves in an implosion do.

11.4 Huygens Principle

In the seventeenth century dispute regarding the nature of light, Newton's corpuscular theory of light did not prevail because, unlike the wave theory of light championed by his older rival and contemporary Huygens, it could not explain light diffraction. Light

diffraction is a pure wave phenomenon as it occurs when waves bend around an edge. Huygens wave theory could explain diffraction and Newton's corpuscular theory could not. It was only with the work of Max Planck and Albert Einstein, that the corpuscular aspect of light reasserted itself in physics and with the development of quantum mechanics in the early twentieth century it became clear that light possesses both, a corpuscular and a wave nature.

Huygens principle, in reference to the work of Augustin-Jean Fresnel (1788 - 1827) also referred to as the Huygens-Fresnel principle, simply states that each point on the wavefront of a propagating wave is the source of a so-called elemental wave. An elemental wave in two dimensions is a circular wave, like the wave that propagates through the surface of an otherwise quiet pond from the point where a pebble someone just threw into the pond broke the water surface. In three dimensions, elemental waves are of course spherical waves, originating from a single point in space with their wavefront propagating in all directions uniformly. Frequently, when using an elemental wave construction to understand the propagation of a wave, one uses only the half-sphere of the wave in the direction of where the wave propagates. Fig. 11.17a illustrates this for a plane wave propagating from left to right whereas fig. 11.17b shows for a circular wave full elemental waves emanating from all points of the current wavefront to then produce a new wavefront. The new wavefront generated by the elemental waves in the case of a plane wave is the tangent to all the elemental waves perpendicular to the direction of propagation; for a circular wave, the new wavefront it is the tangential curve enveloping all the elemental waves emanating from the outermost wavefront at any moment. Logically, in three dimensions the new wavefront is then the tangential surface to all the spherical elemental waves emanating from a given spherical wavefront at an earlier time, i.e., it is a spherical surface.

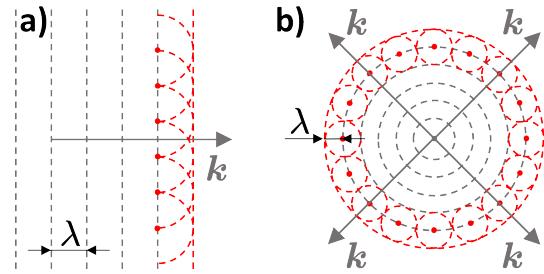


Fig. 11.17

First, we will consider wave reflection, like that of a light wave reflected from a perfect mirror as illustrated in fig. 11.18. A plane light wave of wavelength λ shall travel towards a mirror surface under an angle θ_1 with respect to the surface normal of the mirror.

11.4.1 Reflection and Refraction

First, we will consider wave reflection, like that of a light wave reflected from a perfect mirror as illustrated in fig. 11.18. A plane light wave of wavelength λ shall travel towards a mirror surface under an angle θ_1 with respect to the surface normal of the mirror.

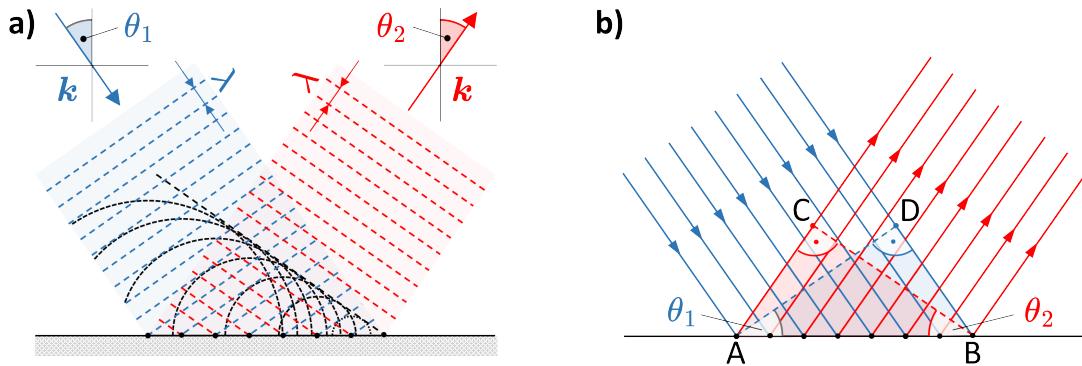


Fig. 11.18: Elemental wave construction after Huygens (a) and light ray sketch (b) for wave reflection.

As shown in the wavefront construction of fig. 11.18a, the point where a wavefront (dashed blue lines) hits the mirror surface first, is where the first elemental wave will start to propagate from the respective point on the mirror surface. The dots on the mirror surface indicate the points after which this wavefront has traveled another distance λ and where another elemental wave will have its origin. Of course, the points where an incoming wavefront successively hits the mirror surface are spaced infinitely dense on the mirror surface. Here we just picked a few selected elemental waves, i.e., those points on the mirror surface which are spaced such that between each of those points the wavefront will have traveled another distance λ . When the last such elemental wave just begins to spread, the elemental wave generated first by this incoming wavefront has already travel quite a distance. That distance is of course the same which a given waverfront travels between the time its first part hits the mirror and the time its last part hits the mirror. The tangential envelope of the elemental waves generated between the point where the wavefront hits the mirror first and where it hits it last then describes the reflected wavefront moving away from the mirror. For the incident angle of the wave θ_1 and the reflection angle θ_2 must hold $\theta_2 = \theta_1$.

To understand why that is so it is better to switch from the wavefront sketch in fig. 11.18a to the light ray sketch in fig. 11.18b. The light ray direction is of course orthogonal to the wavefront and fig. 11.18b shows the respective incoming and reflected light rays for the points we used as origins of elemental waves for the construction in fig. 11.18a. As is easy to check, the angles between mirror surface and the incoming and reflected wavefronts are the same as the angles θ_1 and θ_2 . The right angled triangle ABD in fig. 11.18b has as its hypotenuse the stretch of the mirror surface \overline{AB} which the light wave hits. Its sides are the distance \overline{AD} along the wavefront between the first light ray and the last light ray hitting the mirror surface and the distance \overline{DB} which the last ray has left to travel

towards the mirror after the first light ray hits the mirror. On the reflection side, the right angled triangle ABC has the same hypotenuse \overline{AB} . Its two sides are the distance \overline{AC} which the reflected light of the first light ray has traveled by the time the incoming last light ray hits the mirror surface and the distance \overline{BC} along the respective wavefront between this first reflected light ray to the point on the mirror where the last incoming light just hits the reflective surface. Because these two right angled triangles have the same hypotenuse and light traveling from D to B will cover the same distance as light traveling during the same time on the reflected stretch A to C obviously $\overline{AC} = \overline{DB}$. That however means that these two right angled triangles are congruent and with that must hold $\theta_2 = \theta_1$.

In the above derivation of the law of reflection, i.e., the angle of reflection is equal to the angle of incidence, we used a plane light wave being reflected off a flat mirror. However, the law of reflection holds for mechanical waves such as sound waves or water waves just as well. The difference is of course the very different nature of these different waves, light being an electromagnetic wave and sound and water waves being mechanical waves. The wavelength of light and the speed at which light propagates are orders of magnitude different from those of sound and water waves. Of course, light is just the name for a very narrow spectrum of electromagnetic waves which brackets the visible spectrum we are capable of detecting with our eyes. The spectrum of electromagnetic radiation spans many orders of magnitude in wavelength and for all of them the laws of reflection do apply in principle. However, in case the energy of the radiation becomes too high, light will increasingly be absorbed rather than reflected as photons then have enough energy to induce electronic transitions in the atoms making up a mirror. In addition to electromagnetic waves, sound waves or water waves there are many other kinds of waves. What for any of these can act as a mirror reflecting such waves differs widely as that depends on the nature of the waves one considers.

For wave refraction in the Huygens model we look at a plane light wave being refracted at the boundary of two media (fig. 11.19). The two media, for example air and glass, shall have refractive indexes n_1 and n_2 with $n_2 > n_1$. A higher refractive index means that light travels slower which in turn means that elemental waves generated at the boundary of the two media will propagate at a lower speed in the n_2 medium than they would in the n_1 medium. Because of that, the distance an elemental wave generated at the boundary of the media will expand in the n_2 medium is shorter than the distance λ which the incoming light wave travels in the n_1 medium between the creation of any two of the elemental waves sketched in fig. 11.19a.

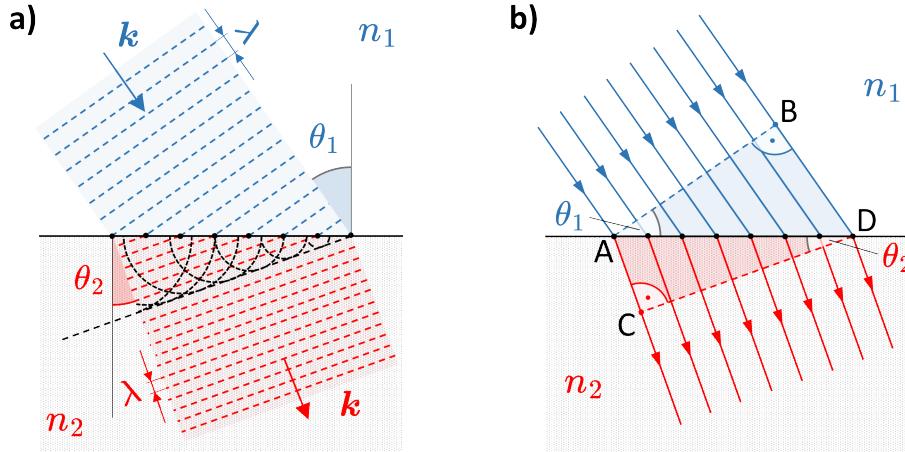


Fig. 11.19: Elemental wave construction after Huygens (a) and light ray sketch (b) for wave refraction.

In other words, the refractive angle θ_2 is smaller than the incident angle and the light wave is refracted towards the boundary surface normal. Had we chosen a situation where $n_1 > n_2$ then the situation would be reversed, the speed of light would be higher in the n_2 -medium meaning that $\theta_2 > \theta_1$ and the incoming wave would be refracted away from the surface normal. To derive the law of refraction, consider the light ray construction in fig. 11.19b with the two right angled triangles ADB and ADC which enclose the incident angle θ_1 in the n_1 medium and the refraction angle θ_2 in the n_2 medium. In the time it takes the last ray in n_1 to travel the distance \overline{BD} , the first light ray in n_2 travels the distance \overline{AC} . With the velocities of propagation in the n_1 - and n_2 medium being v_1 and v_2 (with $v_1 > v_2$), it then holds that $\overline{BD} = v_1 t$ and $\overline{AC} = v_2 t$. The triangles ADB and ADC have the same hypotenuse \overline{AD} and with that follows

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{\overline{BD}/\overline{AD}}{\overline{AC}/\overline{AD}} = \frac{\overline{BD}}{\overline{AC}} = \frac{v_1}{v_2} \quad (11.59)$$

The velocity of light in any given medium is given by $c_m = c_0/n_m$ where c_0 is the velocity of light in vacuum and n_m is the refractive index of the material in which light propagates. The refractive index of air under normal conditions (1 atm and 300 K) is with $n_{air} = 1.003$ practically the same as in vacuum. With that, $v_1 = c/n_1$ and $v_2 = c/n_2$ and therefore eq. (11.59) becomes Snell's law of refraction:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad (11.60)$$

Empirically, this law had been known for a long time but it was only in the seventeenth century that adequate proofs were given. One of them came through the application of Fermat's principle, after Pierre de Fermat (1607 - 1665), which states that light always

travels the path which takes the least time and the other one came by Huygens wave theory of light. From $n_m = c_0/c_m$ follows with $c = \lambda \cdot f$ also $n_m = \lambda_0/\lambda_m$ which means that the wavelength changes with the refractive index n_m and as can be seen in fig. 11.19a, is shorter in a medium with higher refractive index.

11.4.2 Diffraction and Interference

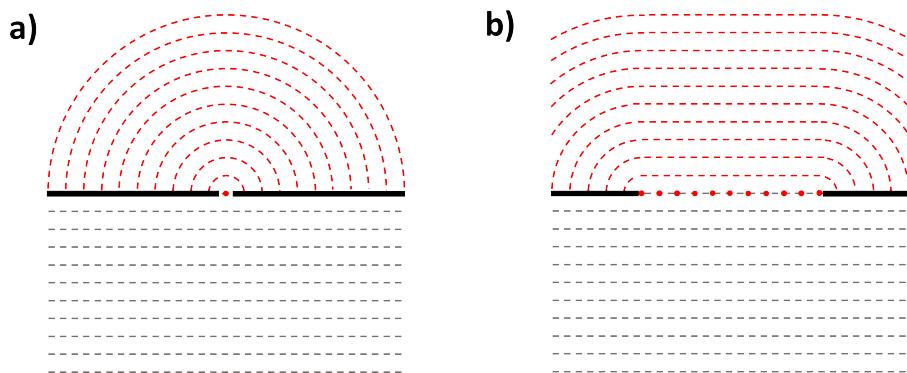


Fig. 11.20: (a) Diffraction of a planar wave at a slit so small that it “isolates” a single elemental wave. (b) At a broad slit most of the plane wave passes through and the wave only bends around the obstacle right at the edges of the opening.

One speaks of a wave being diffracted when it changes its direction on hitting an object by bending around the object. All of us have likely observed this with water waves when they continue to propagate behind an object after running into it. However, few will likely ask the question as to how it can be that a wave can travel into what we could call the shadow of the object. Instead of looking how a wave bends around the edge of an object it is easier, as shown in fig. 11.20a, to move the edges of two objects so close to each other that only a single elemental wave can pass through. Because of their long wavelength this is an experiment that is easy to set up with plane water waves propagating in a shallow tray against an obstacle with a small slit in the middle. The explanation for the circular wave pattern in which the water wave continues to propagate behind the slit is that this opening in the obstacle is so small that it isolates a single elemental wave which then continues to propagate as a circular wave behind the slit.

If one now opens up the slit, more and more elemental waves will be able to pass through and with the opening many times larger than the wavelength of the water wave one will find the plane wave passing through the opening undisturbed and only observe wave bending right at the edges of the opening. In the sketch in fig. 11.20b one can see that it essentially just one half of a single elemental wave which causes the wave to bend around the corner. As we will see, a fuller description of diffraction and interference in

the context of electrodynamics and optics in the second and third course volumes will show that it is even less than half of a single wave which bends around the corners. With regard to diffraction, Huygens principle leads to a correct description of the wavefront in the far field, i.e., far away from the object around which a wave bends, but it is incorrect in the near-field, i.e., at the corners. Nevertheless, this Huygens model of diffraction allows us to understand some simple everyday observations, like for example that we can hear around corners but that we cannot see around corners.

Imagine the opening in fig. 11.20b to be that of a door and the incoming plane wave being the sound wave generated by someone speaking in the direction of the door. Because of diffraction, someone standing behind the opening will be able to hear the speaker even when that person has no line of sight to the speaker. The reason this works with sound waves but not with light waves is that sound has a much greater wavelength than visible light. Humans with perfect hearing can detect sound waves roughly in the range between 20 Hz and $20 \cdot 10^3$ Hz, corresponding to a wavelength range of 1.7 cm to 17 m. An elemental wave with such a wavelength will have no problem bending around the corners of a door, filling the room with sound. Visible light covers roughly the wavelength range from 400 to 700 nm which is some 4 to 7 orders of magnitude shorter than the wavelength range sound covers. Therefore, light will only bend a few nanometers around a corner and not meters like sound waves do. A plane light wave will pass through a door practically undisturbed with very little bending, the result of which is that we only can see people standing in a direct line of sight behind a door opening but we cannot see people to the left and right of the opening where we have no direct line of sight.

Huygens wave propagation model is an amplitude model only. The dotted line of the wavefronts drawn up in the last several figures indicate the positions of the wave crests, i.e., the position of maximum amplitude. Huygens model says nothing about the phase of a wave but even without knowing the phases of two waves running into each other one can obtain a qualitative understanding of what happens in such a case and make some quantitative statements. For that one considers an ideal situation such as the one sketched in fig. 11.21 where a plan wave runs against an obstacle parallel to its wavefronts that has two small openings. These two openings, we will call them slits as they are so small that again only a single elemental wavefront can emanate from each slit, are separated by a distance d which is much larger than the slits. The two elemental waves in fig. 11.21 which generate the two circular waves behind the slits are created by the very same incoming plane wave. As furthermore the two slits are perfectly identical, they therefore generate two identical circular waves propagating behind the slits.

With that we can now simply add wave amplitudes to understand what happens. First, we note that at every intersection of a dashed red wavefront with a dashed blue wavefront the amplitudes of the respective wave crests add up to a bigger amplitude. Those positions are marked with grey doughnuts in fig. 11.21. Connecting successive doughnuts through a straight line running against a screen located behind and parallel to the obstacle at a distance L , every intersection of the screen with such a line marks a position on the screen of increased amplitude, or bright spots if we were to make this experiment with light. Next, we mark the points on the blue dashed wavefronts which fall precisely midway between two dashed red wavefronts. Why are we doing that? Well it is at such points where a wave crest of the dashed blue wave meets a wave trough of the dashed red wave because wave troughs are located exactly midway between wave crests. Hence, by marking such points in fig. 11.21 with grey dots we identify the positions where the resulting wave amplitude will be zero. By connecting successive grey dots and extending the connecting line to the screen we then can mark the positions on the screen where the two waves cancel each other; or mark the dark spots if we were to make this experiment with light. Instead of looking for points on dashed blue wavefronts which fall perfectly midway between two dashed red wavefronts, we could of course have just done the same exercise by looking for points on red dashed wavefronts which fall perfectly midway between two dashed blue wavefronts. However, the result would be the same and in order not to crowd the sketch in fig. 11.21 with too many marked spots we only used one such set to identify where the amplitude minima are located on the screen.

Above the screen in fig. 11.21 a series of numbers indicates the so-called interference order n , which is used to identify the respective positions of maximum and minimum amplitude / intensity on the screen. These numbers come in two series:

$$n = 0, \pm 1, \pm 2, \dots, \pm k \quad \text{and} \quad n = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots \pm \frac{2k+1}{2}$$

where k runs through all positive integer numbers. The value of n indicates the path length difference between a dashed blue wave and a dashed red wave at the respective

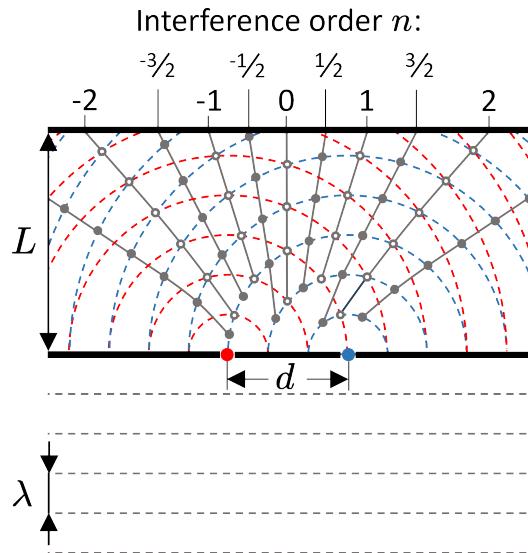


Fig. 11.21

position on the screen in fig. 11.21. The path length difference between the sources of the blue and red circular waves and the central bright spot with $n = 0$ is exactly zero. At the screen position labeled $n = 1$ the distance from the source of the dashed red wave minus the distance from the source of the dashed blue wave is exactly λ , for $n = 2$, it is 2λ ; and for $n = -1$ or $n = -2$ this distance is then exactly $-\lambda$ or -2λ . For the half integer values of n the difference between the distance the dashed red wave has to travel to get there and the distance the dashed blue wave has to travel to get there are then of course just half integer values of λ . If the travel distance is an integer multiple of λ then the crests and troughs of both waves are at the exact same position. If however that distance is a half-integer of λ then the crests of one wave align perfectly with the troughs of the other wave. In the first case, one speaks of constructive interference as an integer multiple of λ travel distance will produce a maximum amplitude or a bright spot on the screen. In the second case, one speaks of destructive interference because a half-integer multiple of λ in travel distance will produce a zero amplitude or a dark spot on the screen.

To determine where exactly on the screen the respective amplitude or intensity maxima and minima are located, one measures their distance from the position of $n = 0$ as a function of the angle θ as illustrated in fig. 11.22. This angle theta is defined by the orthogonal line connecting the midpoint between the two source points in the plane of the obstacle to the position of $n = 0$ on the screen and the line running from this midpoint to the respective position of interest on the screen.

The sketch in fig. 11.22 shows how to do this for the point $n = 1$ on the screen. First, one draws a line from the blue source point onto the line connecting the red source point and the $n = 1$ point such that the two lines intersect at a right angle. The stretch from the red source to that intersection point must just be λ as the path difference for $n = 1$ for red and blue wavefronts must equal λ . The opposite angle in this right angle triangle to the side with the length λ is θ . Instead of using $n = 1$ as the example in fig. 11.22 we could have chosen any other integer or half-integer value n of λ and the result would always be a similar construction. The difference being that we would then look at right angled triangles with path length differences of $n\lambda$ as the opposite sides to the angle θ and not λ as in fig. 11.22. With the right angle triangle connecting the path length difference to the angle θ we can now find the angle for any given interference order n using the simple formula

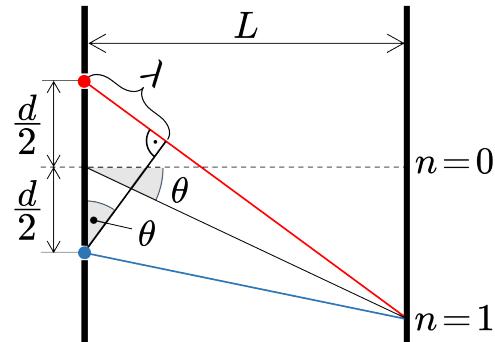


Fig. 11.22

$$d \cdot \sin \theta = n \cdot \lambda \quad (11.61)$$

We also know that the actual distance of any specific order on the screen from $n = 0$ must equal $z = L \tan \theta$. Knowing $\sin \theta$ we can then use this relationship to calculate z by using eq. (11.61) to rewrite $\tan \theta$ as

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{n\lambda}{\sqrt{d^2 - n^2\lambda^2}} = \frac{z}{L}$$

from which then follows

$$z = \frac{Ln\lambda}{\sqrt{d^2 - n^2\lambda^2}}$$

With this cursory discussion of what is commonly referred to as a double slit interference experiment we end our short excursion into reflection, refraction, diffraction and interference of waves. For a more detailed analysis of these phenomena we will need a better understanding of electrodynamics and optics which we do not have yet but which the second and third volumes of this course will focus on.

Before we end this chapter we need to say a little more about material (mechanical) and non-material (non-mechanical) transverse waves. This chapter referred to water waves as transverse waves but that only applies to water surface waves which, e.g., can exist as standing waves in the surface layer of an enclosed body of water. However, in general water surface waves are a mixture of longitudinal and transverse waves and in the depth of a water body only longitudinal waves can propagate. The reason for that is of course that the propagation of transverse waves requires the presence of shearing forces to drive motion perpendicular to the propagation of the wave and such forces do not exist in liquid or gaseous bodies. Hence, the propagation of transverse matter waves can only take place through solid bodies. However, different from that, non-material transverse waves such as electromagnetic radiation do not require the presence of a medium to propagate. That the latter is the case is something physicists only began to realize once they faced the challenge of having to reconcile the principle of relativity with the requirement of the velocity of light being constant regardless of whether one measured it in an inertial system at rest or an inertial system moving at a constant uniform velocity. It is to this challenge and how it was resolved to which we will turn next.

12. The Special Theory of Relativity

In the previous chapters we have been looking at moving frames of reference a number of times without saying much about this relative movement. Even though an obvious question is: How do we know that two observers in their respective reference frames moving with respect to each other, making observations of one and the same event, see the very same laws of nature at work, i.e., Newton's laws? What restrictions do Newton's laws impose on such reference frames?

12.1 The Relativity Principle

Newton's first law, the principle of inertia, states that a body at rest will stay at rest and a body in uniform straight motion will continue its uniform straight motion unless acted upon by an external force. Because of this principle, the moving reference frames from which two observers make their respective observations must move at constant relative speed with respect to each other. From the perspective of an observer in an accelerated reference frame, an object that is at rest or moves in uniform motion, as observed for example from a person in a reference frame that is not moving at all, will seemingly change its state of motion even though no outside force is acting on the object. Clearly a violation of the principle of inertia. Reference frames in which the principle of inertia holds true, physicists call them inertial reference frames or short inertial frames, therefore can only move with constant uniform speed relative to each other and they cannot be accelerated in anyway with respect to each other. That, of course, makes the rotating frames we discussed in the context of chapter 6 all non-inertial frames.

Any observer in a given inertial frame makes her or his observations of the laws of nature by measuring things like the position of a point mass, its velocity, or its acceleration. Those measurements allow the observer to describe the motion of the point mass and from that infer what the laws of nature must look like which make the point mass

move in such ways. Observers in two different inertial frames must therefore transform the coordinate positions and times which are the measurement basis for determining velocities and accelerations in either inertial frame such that they come to the same conclusions regarding the laws of nature at work.

The formulation of the relativity principle is a response to the challenge physicists faced with regard to the nature of space and time: if they were absolute or relative and if things like absolute motion existed or if there only was relative motion. The first to espouse the relativity principle was Galilei when he pronounced that aboard a ship traveling at uniform speed anyone inside a closed cabin would not be able to sense the motion of the ship. There would be no way for such a passenger to empirically determine whether she or he was on a ship sailing on a calm ocean at uniform speed or if she or he was on a ship moored in the harbor with the ship sitting still. Huygens, Newton's older contemporary, proposed a relativity principle which he believed did not necessitate the assumption that absolute space and absolute time did exist and he used his interpretation of the relativity principle to understand the conservation of momentum in collisions of hard bodies. Different from that, Newton thought of space and time as absolutes. To illustrate his relativity principle of mechanics Newton choose a somewhat different metaphor than Galilei but it expresses the same thought:

“The motions of bodies included in a given space are the same among themselves whether that space is at rest or moves uniformly in a straight line without circular motion.”

This quotation shows that Newton was thinking in terms of inertial frames even though he did not call his “given space” that way; he specifically excluded rotating spaces because rotating systems are associated with an acceleration and hence are non-inertial frames.

To understand how measurements of positions and time between inertial frames moving with respect to each other with a uniform velocity actually connect we consider the two inertial frames S and S' shown in fig. 12.1. The inertial frames S and S' move relative to each other with the velocity $\mathbf{v}_0 = \mathbf{r}_0/t$. With that, the positions \mathbf{r} and \mathbf{r}' of a point mass and the times t and t' where observers in the respective inertial frames S and S' will observe a point mass at a given position P are connected by

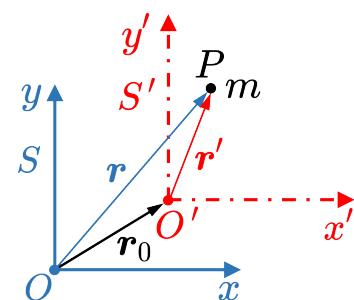


Fig. 12.1

$$\left. \begin{array}{l} \mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t \\ t' = t \end{array} \right\} \quad (12.1)$$

These equations, three for the respective spatial coordinate measurements and one for the respective time measurement in each inertial frame are the Galilean transformations we already encountered several times. The fact that in these transformations between two frames which move relative to each at constant uniform speed t' equals t , expresses the absoluteness of time in Newtonian mechanics; all clocks in all inertial frames measure the same absolute time t . When calculating velocities and accelerations of a moving point mass based on these equations for the respective systems, we make the assumption that both inertial frames possess the same metrics, meaning specifically that the spatial differentials are identical, i.e., $dx_i = dx'_i$ for $i = 1, 2, 3$; and of course, because time is absolute in Newtonian mechanics $dt = dt'$ must hold. With that follows from eq. (12.1) for the velocities

$$\frac{d\mathbf{r}'}{dt} = \frac{d\mathbf{r}}{dt} - \mathbf{v}_0 \quad \text{or} \quad \mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \quad (12.2)$$

Hence, for someone in the system S which is at rest, the uniform velocity \mathbf{v}_0 of the moving system S' just adds to the arbitrary velocity which the point mass may have. The reverse is the case for an observer in S' who will measure a point mass velocity reduced by the uniform velocity with which S' is moving itself. Of course, an observer in S' can take the position that her or his system is at rest and that the inertial frame S is moving with respect to her or his own inertial frame with a velocity $-\mathbf{v}_0 = -\mathbf{r}_0/t$. There is no preferred inertial frame, both are equal and both observers can claim that their system is at rest. This is all about relative motion and not about who moves and who sits still. For the accelerations observed in both systems follows from eq. (12.2)

$$\frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} \quad \text{or} \quad \mathbf{a}' = \mathbf{a} \quad (12.3)$$

That the acceleration of a point mass \mathbf{a}' observed by someone in the inertial frame S' equals the acceleration of that point mass \mathbf{a} observed by another person in S ensures that in both inertial frames Newtons law is the same. With

$$\mathbf{F}' = m\mathbf{a}' = m\mathbf{a} = \mathbf{F} \quad (12.4)$$

the force \mathbf{F}' causing the point mass to change its state of motion as observed by someone in S' is the same force as \mathbf{F} which for an observer in S causes the point mass to change its state of motion. This of course only works if the uniform velocity \mathbf{v}_0 with which S and

S' move relative to each other is constant, because only then follows eq. (12.3) and with that the dynamic equivalence of the inertial frames S and S' . Saying that inertial frames moving relative to each other at uniform speed are dynamically equivalent is another way of saying that Newton's laws and with that all of Newtonian mechanics looks the same in all inertial frames whose coordinates are connected through a Galilean transformation. Physicists in this case also speak of the Galilean covariance of Newtonian mechanics or Newtonian mechanics being covariant under Galilean transformations. One can also turn this definition around and define as Galilean frames such reference frames in which the laws of nature follow Newtonian mechanics. In that case, it turns out that Galilean frames are all those reference frames which move with constant uniform velocity with respect to a reference frame in which the laws of Newtonian mechanics hold true.

Expressed as Galilean transformation, the principal of relativity became a bedrock of classical mechanics. For quite some time after Newton, mechanics continued to be the only physics theory providing an understanding of the laws of nature. Physics was mechanics and with that the relativity principle of mechanics became the relativity principle of physics. Never did there arise any challenge to the relativity principle. However, that was to change and the validity of the relativity principle would be seriously questioned when the great nineteenth-century discoveries in the field of electricity and magnetism culminated eventually in Maxwell's theory of electromagnetism. Here was a complete and elegant new physics theory, no less formidable than classical mechanics, which came with its own laws, Maxwell's famous equations.

Given the fundamental nature of the principal of relativity, physicists expected Maxwell's equations just to be as covariant under Galilean transformations as the equations of classical mechanics were. However, they discovered that Maxwell's equations unlike Newton's equation in eq. (12.4) were not covariant under Galilean transformations as they changed their form under Galilean transformations. In different words, the electromagnetic theory was apparently add odds with the principle of relativity because - assuming the Galilean transformations were the correct way to convert coordinates from one inertial frame S to another inertial frame S' moving at constant uniform speed with respect to S - it turned out that their form depended on the inertial frame one chose. To many physicists this indicated that there must be something wrong with Maxwell's equations.

Maxwell's theory of electromagnetism clearly requires that electromagnetic waves, regardless of their wavelength and directions, always travel at the speed of light, i.e., whatever the value of speed of light in the respective medium is. In vacuum the speed of light equals $c = 2.9979 \cdot 10^8 \text{ m s}^{-1}$. To appreciate the problem physicists faced regarding the

new electromagnetic theory, given its inherent requirement of a constant speed of light and its apparent failure to comply with the relativity principle, consider the following situation. Two spaceships S and S' shall travel on parallel routes, both in what we will call the positive x -direction. Spaceship S' shall start its journey a little later than S . Unfortunately the captain of spaceship S finds out midway that he has an engine problem so he stops his spaceship to conduct repairs. When the captain on S sees the spaceship S' approaching he decides to send an emergency light signal out through his back window in the negative x -direction towards the spaceship S' which approaches his position with the uniform speed v_x . Clearly, from the perspective of the captain of spaceship S he sent a light signal with the speed c . However, for the captain on spaceship S' the speed of light is according to eq. (12.2) with c and v_x pointing in opposing directions:

$$c' = c - (-v_x) = c + v_x$$

If Galilean transformations are the correct way to transform time and spatial positions as observed in one inertial frame into the time and spatial positions in another inertial frame, then the apparent speed of light c' observed in the inertial frame S' must be higher than c . Another way to state this is that apparently Maxwell's equations in S' must be different from those in S . This is what the statement "Maxwell's equations are not covariant under Galilean transformations" means. There seemed to exist a preferred reference frame where Maxwell's equations took their most simple form and that had to be a system at rest. Physicists in the nineteenth century thought about the propagation of electromagnetic waves such as light in terms analogous to what they knew about the propagation of water waves or of acoustic waves which both require a medium for propagation. That medium for water waves is of course water and the acoustic waves we produce when we speak to one another we can only hear because they are carried by the medium of air around the speaker towards the direction in which she or he is speaking. Seemingly, waves requires a medium which provides the reference frame for their propagation and physicists thought the same was true for electromagnetic waves. The medium they had in mind was the so-called "light-bearing" ether, or "luminiferous" ether which filled the preferred inertial frame at rest with respect to all other inertial frames in which Maxwell's equations took their most simple form. Such a preferred immobile inertial frame carried another implication which is that there existed something like an absolute velocity measurement.

The concept of an ether goes all the way back to the Greek philosophers of antiquity and gained its prominence through Aristotle's endorsement. Back then the ether was the so-called fifth element, with water, fire, air and earth being the first four, and it was

thought to be the celestial substance out of which the heavens were made. This ether was in a sense the most perfect of the five elements and with the reception of Aristotle in Western Europe after the Dark Ages this fifth element became also referred to as the fifth essence, or in Latin “quinta essentia”, making it the quintessential element. The light-bearing ether in which physicists envisioned electromagnetic radiation to propagate had to have very peculiar properties. Electromagnetic waves are transverse waves, which means that their wave oscillations are perpendicular to the direction of wave propagation and not oscillations in the direction of wave propagation as is the case for longitudinal waves such as for example the density variations along the direction of which sound travels through the air. Nineteenth century physicists knew that transverse waves could not travel through liquid or gaseous media bodies, hence the only remaining option was that the ether had to be a solid. And it had to be a very peculiar solid as it seemingly did not interact with anything else. This vacuum-pervading ether was believed to fill all of interstellar space and seemingly Earth on its path around the Sun had to move through this ether. Many experiments were conceived to detect this motion of Earth through the ether but all failed to provide any evidence of its existence. The most convincing such failure to detect the ether was the seminal experiment of Albert A. Michelson (1852 - 1931) and Edward W. Morley (1838 - 1923) which they conducted in 1887.

The Michelson-Morley Experiment

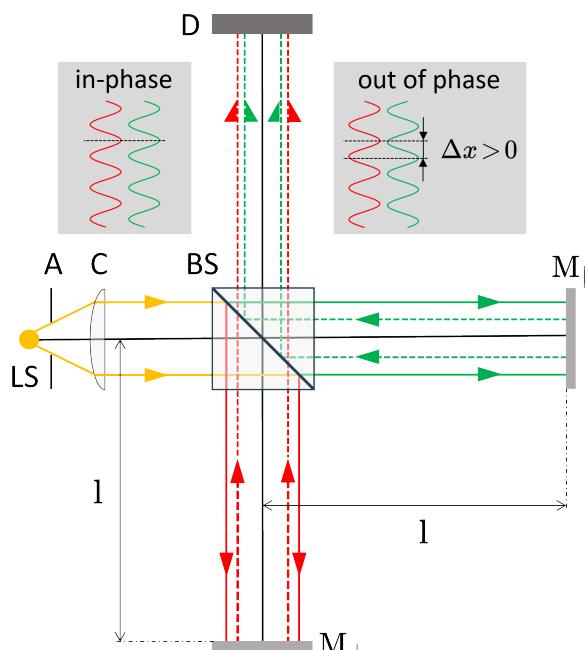


Fig. 12.2

In their experiment Michelson and Morley used an interferometer type as shown in fig. 12.2, which Michelson had pioneered. Such a Michelson interferometer, as it is called today, consists of a collimated light source (LS = light source; A = aperture; C = condenser) the light of which is sent via a beam splitter (BS) into two orthogonal interferometer arms of equal length l at the end of which sit the identical mirrors which we have labeled M_{\perp} and M_{\parallel} for reasons which will become clear below. The mirrors M_{\perp} and M_{\parallel} reflect the incoming light waves back to the beam splitter which now splits these light waves coming from the mirrors into a part that travels back to the source (not shown in fig. 12.2) and a

part which travels towards a detector D where the respective light waves coming from M_{\perp} and M_{\parallel} produce an interference pattern which depends on the phase difference which they accumulated along their separate path. To better visualize in fig. 12.2 the different legs the respective light waves travel in the interferometer, the light coming from the source and hitting the beam splitter is colored orange; the light traveling from the beam splitter towards M_{\perp} is indicated by solid red lines and the light traveling back from M_{\perp} via the beam splitter to the detector is indicated by a dashed red line; and the light traveling from the beam splitter to M_{\parallel} is indicated by a solid green line with the light reflected from M_{\parallel} via the beam splitter into the detector indicated by a dashed green line. The two inserts in fig. 12.2 illustrate the case where the light waves arriving from either arm of the interferometer are perfectly in-phase, i.e., wave crests and troughs are perfectly aligned and the case where a phase difference was accumulated and crests and troughs of the light wave coming from M_{\perp} are somewhat shifted with respect to the crests and troughs of the light wave coming from M_{\parallel} . Such a phase difference can only accumulate if it takes the light wave traveling the distance from the beam splitter to the mirror M_{\perp} and back longer or shorter than it takes the light wave traveling from the beam splitter to the mirror M_{\parallel} and back.

A different travel time along those perpendicular interferometer arms can have two reasons. The path length is somehow not exactly the same or there is a difference in how light travels along the orthogonal directions of the interferometer arms. Hence one must first make sure that the interferometer is perfectly aligned and that the path length of the interferometer arms are exactly the same which one can do by adjusting those distances just ever so minimally that the maximum light intensity is observed in the detector, which happens when the light waves coming from M_{\perp} and M_{\parallel} are in phase in the detector plane.

This is however easier said than done. Because of the low coherence of a white light source the fringe pattern is very sensitive to even the smallest changes in length. Just how sensitive such an instrument was to changes in the length of the interferometer arms, Michelson discovered with his first instrument which he used in 1881. Mechanical vibration were a big problem and so was keeping the instrument at a uniform temperature. The 1.2 m long interferometer arms of this instrument on the end of which sat the mirrors were made out of brass. Length changes of the interferometer arms associated with temperature differentials of only one-hundredth of a degree between the two brass arms were sufficient to destroy the fringe pattern in the detector. Learning from this experience, the instrument used in the Michelson-Morley experiment six years later in 1887 was mounted on a stone slab that floated in a trough filled with mercury.

Fig. 12.3 illustrates what we can expect to happen when the interferometer, together with all of Earth, moves with a velocity \mathbf{v} through the hypothetical ether, which itself is at rest. This figure also shows the reason why we labeled the mirrors as M_{\perp} and M_{\parallel} because the light bouncing from one of them, M_{\parallel} , moves parallel to the direction of \mathbf{v} while the light from the other mirror, M_{\perp} , travels at a normal direction with respect to \mathbf{v} . Fig. 12.3 shows two reference systems, S and S' . In S the ether is at rest and in S' , which moves with a relative speed \mathbf{v} with respect to S , the interferometer is at rest. Hence, if the velocity of light in S is c , then with eq. (12.1) the velocity of light c' in S' traveling the distance l in the direction of \mathbf{v} is given by

$$c' = c - v \quad \text{for the light wave from } P \rightarrow M_{\parallel}$$

and the velocity of light traveling the distance l in the opposite direction of \mathbf{v} is given by

$$c' = c + v \quad \text{for the light wave from } M_{\parallel} \rightarrow P'$$

Therefore, the time it takes the light wave to travel the distance $2l$ from the beam splitter at P to the mirror M_{\parallel} and back to P' is given by

$$t_{\parallel} = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2l}{c} \frac{1}{1 - v^2/c^2} \quad (12.5)$$

Evidently, in a system S' moving with a velocity of \mathbf{v} relative to the system S where the ether is at rest and the velocity of light is c , it takes the light longer to cover the distance $P-M_{\parallel}-P'$ than if the interferometer were at rest with respect to S , i.e., if $\mathbf{v} = 0$. The respective factor is

$$\left(\frac{t_{\parallel}}{(t_{\parallel})_{v=0}} \right)_{v \neq 0} = \frac{1}{1 - v^2/c^2}$$

Now what about light traveling along the other arm of the interferometer from P to M_{\perp} and back to P' ? To understand what happens there we look at the distance S' moves during the time t_{\perp} it takes for the light to travel the distance from P to M_{\perp} and back to

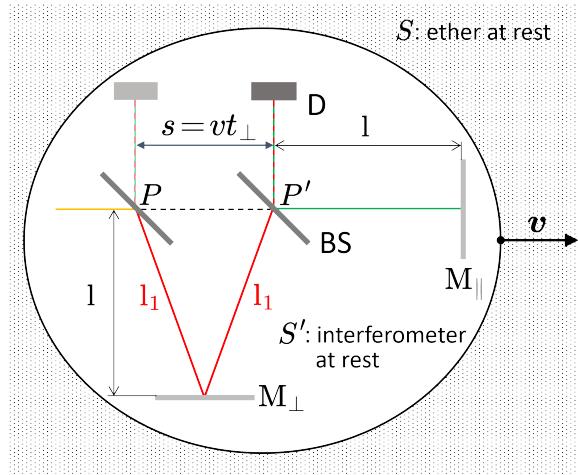


Fig. 12.3

P' . Obviously, at the time the light arrives back at the beam splitter, it has moved from P to P' covering the distance $s = v \cdot t_{\perp}$. Consequently, the light traveling from the beam splitter to M_{\perp} and back has not covered the distance $2l$ but $2l_1$. From fig. 12.3 we can read off the relationships

$$t_{\perp} = 2 \frac{l_1}{c} \quad \text{and} \quad l_1^2 = l^2 + \left(\frac{s}{2}\right)^2 = l^2 + \left(\frac{v \cdot t_{\perp}}{2}\right)^2$$

From these two equations we can eliminate l_1 to obtain for t_{\perp} the expression

$$t_{\perp} = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c} \frac{1}{\sqrt{1 - v^2/c^2}} \quad (12.6)$$

In the case of $\mathbf{v} \neq 0$ the travel time for light to cover the distance $P-M_{\perp}-P'$ increases therefore compared to $\mathbf{v} = 0$ by the fraction

$$\frac{(t_{\perp})_{v \neq 0}}{(t_{\perp})_{v=0}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

From eq. (12.5) and eq. (12.6) we see that for $\mathbf{v} \neq 0$ it takes light more time to cover the distance $P-M_{\parallel}-P'$ than to cover the distance $P-M_{\perp}-P'$. The respective factor is

$$\frac{t_{\parallel}}{t_{\perp}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

which for $\mathbf{v} \neq 0$ is greater than 1. With eq. (12.5) and eq. (12.6) we can now also calculate the difference in travel time Δt for light waves covering the distance $P-M_{\parallel}-P'$ vs covering the distance $P-M_{\perp}-P'$:

$$\Delta t = t_{\parallel} - t_{\perp} = \frac{2l}{c} \left(\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}} \right)$$

The relative velocity \mathbf{v} with which S' moves relative to S is the speed with which Earth supposedly ploughs through the ether as it orbits the Sun. With $v \approx 30 \text{ km s}^{-1}$, this speed is much less than the speed of light $c \approx 3 \cdot 10^5 \text{ km s}^{-1}$. The travel time difference Δt of light waves arriving at the detector plane from the two arms of the interferometer which results from the system S' moving relative to S is therefore very short. Comparable time differences could easily be the result of, e.g., the interferometer arms not being of the exact same length but differing in length by a few ten nanometers. A simple way to deal with this problem is to observe the relative shift in the interference pattern for two configurations of the interferometer with one just as shown in fig. 12.3 and a second configuration where the interferometer arms are rotated by 90° . In the rotated

interferometer the roles of the two interferometer arms are switched, the arm which was parallel to \mathbf{v} becomes the perpendicular arm and the arm which was perpendicular to \mathbf{v} becomes the parallel arm. With this trick small differences in length will not matter any more. In addition, the difference in travel time in the rotated configuration is just $-\Delta t$. With that the change in fringe pattern between those two configurations rotated by 90° will arise from a total travel time difference between the light waves in the interferometer arms of $2\Delta t$.

Calculating this total change in travel time of $2\Delta t$ between the two 90° -rotated configurations for the Michelson-Morley experiment which used an interferometer arm length of $l = 11 \text{ m}$ we get

$$2\Delta t \approx 7 \cdot 10^{-16} \text{ s}$$

For the actual measurements Michelson and Morley used a white light source but they used the yellow light of a sodium flame which emits light at wavelength of $\lambda = 589 \text{ nm}$ for aligning the interferometer. The period T of a single oscillation of light of this wavelength is

$$T = \frac{1}{f} = \frac{\lambda}{c} = 1.96 \cdot 10^{-15} \text{ s}$$

This means that the time difference Michelson and Morley sought to measure amounted to a travel distance difference of the two light waves corresponding to a ≈ 0.4 fraction of a full oscillation period. For the two light waves to move from in-phase to out of phase (see fig. 12.3) requires only a travel distance difference corresponding to half of a full oscillation period. Hence, the set-up of the Michelson-Morley experiment was more than sufficient to measure the expected changes in interference patterns when rotating the instrument between two positions at 90° -angles. The surprising result was however, that no time delay $2\Delta t$ between the light waves propagating in the two interferometer arms could be detected. The only conclusion physicists could draw from this was that either an ether did not exist or that something else compensated for the expected time delay between the two light waves traveling the arms of the interferometer in S' , moving with respect to the ether at rest in S . A proposal regarding the nature of such a compensation came from George Fitzgerald and Hendrik A. Lorentz.

Length Contraction and Time Dilation

Looking at equations eq. (12.5) and eq. (12.6) from which the time delay between the propagating light waves was calculated, there is only one other variable that could change and possibly compensate the expected time delay in some way, such that the time delay

vanishes. That variable is of course the length of the interferometer arms parallel and perpendicular to the direction of \mathbf{v} . With that in mind we rewrite equations eq. (12.5) and eq. (12.6) as

$$t_{\parallel} = \frac{2l_{\parallel}}{c} \frac{1}{1 - v^2/c^2} \quad \text{and} \quad t_{\perp} = \frac{2l_{\perp}}{c} \frac{1}{\sqrt{1 - v^2/c^2}}$$

Using these two equations we recalculate the expected time delay as

$$\Delta t = \frac{2}{c} \left(\frac{l_{\parallel}}{1 - v^2/c^2} - \frac{l_{\perp}}{\sqrt{1 - v^2/c^2}} \right)$$

Now, the time delay Δt obviously becomes zero if the expression between the brackets vanishes, i.e., if

$$l_{\parallel} = l_{\perp} \sqrt{1 - v^2/c^2}$$

To explain the negative outcome of the Michelson–Morley experiment and to rescue the stationary ether hypothesis, it was exactly this kind of length contraction which was separately proposed by FitzGerald in 1889 and by Lorentz, who developed the idea further, in 1892. According to it, the length of objects in motion would remain unaffected in the direction normal to the direction of motion but would shorten in the direction parallel to the direction of motion, i.e.,

$$l_{\perp} = l \quad \text{and} \quad l_{\parallel} = l \sqrt{1 - v^2/c^2} \quad (12.7)$$

If we now insert these values for l_{\parallel} and l_{\perp} into the modified version of eq. (12.5) and eq. (12.6) from above we get

$$t_{\parallel} = \frac{2l}{c} \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad t_{\perp} = \frac{2l}{c} \frac{1}{\sqrt{1 - v^2/c^2}}$$

With the introduction of the length contraction in the direction of motion, the ratios of t_{\parallel} and t_{\perp} at $v \neq 0$ to their respective values at $v = 0$ have become symmetrical:

$$\frac{(t_{\parallel})_{v \neq 0}}{(t_{\parallel})_{v=0}} = \frac{(t_{\perp})_{v \neq 0}}{(t_{\perp})_{v=0}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The travel time it takes a light beam to propagate parallel or normal to the direction of motion of S' increases by the same factor as compared to S' being at rest. This opens up the possibility of a very different interpretation of the Michelson–Morley experiment: Not only does the length of an object contract in the direction of motion of S' but somehow clocks in a moving S' also tick slower.

12.2 Relativistic Kinematics

The special theory of relativity was developed by Einstein in one of his four famous 1905 publications in which he introduced a new perspective on the relativity of space and time with regards to the electrodynamics of moving bodies. The challenge physics faced at the time in understanding the electrodynamics of moving bodies rooted in Maxwell's equations not being covariant under Galilean transformations but rather being covariant under Lorentz transformations. Einstein did not discover the Lorentz transformations and the effect of length contraction and time dilation had been introduced to deal with the successful “failure” of the Michelson-Morley experiment. However, it was Einstein who gave a comprehensive interpretation of these different aspects of the new electromagnetic theory in the context of a fundamentally different understanding of our concepts of time and space than what was the basis of Newtonian mechanics. By taking the relativity principle seriously and combining it with the constancy of the speed of light, Einstein gave physics a new understanding of the relativity principle, reasserting its validity as a fundamental concept of physics. It was not until 1917 when Einstein published his General Theory that his 1905 theory became referred to as the special theory of relativity, special in the sense that it assumes a flat space time, whereas his general theory is really a theory of gravity where masses bend space and curved space tells masses how to move. In this section on relativistic kinematics, the focus is on introducing and understanding its concept of relative space, time and simultaneity.

12.2.1 The Lorentz Transformation

Lorentz proposal of the length contraction in moving reference frames was a result of his effort to understand under which transformations Maxwell's equations remained covariant. With the new theory of electromagnetism being widely successful, not only in how it explained known phenomena but also in making predictions which could be tested by experiment, physicists believed it to be increasingly unlikely that Maxwell's equations could somehow be flawed because they were not covariant under Galilean transformations. Rather they began to consider that the applicability of Galilean transformations might have its limitations. Lorentz did indeed discover a new coordinate transformation between inertial frames under which Maxwell's equations remained covariant. For an inertial frame $S'(x', y', z', t')$ moving with the velocity v in x direction with respect to an inertial frame $S(x, y, z, t)$ this coordinate transformation, the Lorentz transformation for the three spatial coordinates and the time coordinate is given by:

$$\left. \begin{aligned} x' &= \frac{x - v \cdot t}{\sqrt{1 - v^2/c^2}} & , \quad y' = y & , \quad z' = z \\ t' &= \frac{t - v \cdot x/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \right\} \quad (12.8)$$

Obviously, in the limit case $v \ll c$, i.e., $v/c \rightarrow 0$, the Lorentz transformation becomes a Galilean transformation. Implicit in the Lorentz transformation are the concepts of length contraction in moving systems and that of local time, i.e., that clocks go slower in moving systems. Let's look at length contraction first and in doing so we will use the notation $\beta = v/c$ where opportune. How do we measure length? The answer is of course by using a ruler. We determine the coordinate position of an object by measuring its distance from the origin of the reference frame we are using. A person in S' measures the x' coordinate of an object located on the x' -axis in his reference frame by applying some kind of ruler and figuring out how many times this ruler fits between the origin of S' and the position of the object on the x' -axis and calls the result of his measurement the x' -coordinate of the object. However, from the perspective of someone in S the person in S' uses a ruler which is subject to the length contraction in eq. (12.7) and therefore does not measure the length between the origin and x' but measures x' shortened by the factor $x' \cdot \sqrt{1 - \beta^2}$. If the system S' moves with the velocity v parallel to the x -axis of S , the position of the coordinate origin of S' is $x = v \cdot t$ and the coordinate position x' a person measures in S' has in S the x coordinate

$$x = v \cdot t + x' \cdot \sqrt{1 - \beta^2} \quad \text{and therefore} \quad x' = \frac{x - v \cdot t}{\sqrt{1 - \beta^2}}$$

The above rationale for the length contraction follows directly from the Lorentz transformation if we consider a ruler in S' with the left and the right end of the ruler marked by x'_{left} and x'_{right} on the x' -axis, respectively. If we look at the ruler in S at the time $t = 0$ the respective positions of the rulers left and right ends will be the positions

$$x_{left} = x'_{left} \cdot \sqrt{1 - \beta^2} \quad \text{and} \quad x_{right} = x'_{right} \cdot \sqrt{1 - \beta^2}$$

With that we can calculate the length of the ruler in S' as observed from S :

$$\underbrace{(x_{right} - x_{left})}_{\text{ruler in } S} = \underbrace{(x'_{right} - x'_{left})}_{\text{ruler in } S'} \cdot \sqrt{1 - \beta^2}$$

This result shows again that from the perspective of an observer in S the person in S' uses a shortened ruler. Of course, from the perspective of an observer in S' who takes the position that his system is at rest the same applies to a ruler that a person in S uses. All

that matters is the absolute value of the relative velocity of S to S' , which is of course the same as for S' to S . Length contraction is built into the Lorentz transformation eq. (12.8), from whichever perspective we look at it. The same is true for time dilation. Consider a clock sitting at the origin of S' , i.e., at $x' = 0$. From eq. (12.8) follows that an observer in S will find this clock at the position $x = v \cdot t$ at the time

$$t = \frac{t'}{\sqrt{1 - \beta^2}} \quad (12.9)$$

Lets assume that a person in S' measures the time interval from $t' = 0$ to $t' = 1$ and calls it “1 second”. However, with eq. (12.9) an observer in S will come to the conclusion that this time interval which the person in S' calls “1 second” is actually of a somewhat longer duration, i.e., $1/\sqrt{1 - \beta^2}$ seconds. Inherent to the Lorentz transformation is of course that the velocity of light is constant in all reference frames moving relative to each other at a constant uniform speed. If in S light travels a distance $x = c \cdot t$, then for an observer in S' this light travels according to eq. (12.8) the distance

$$x' = \frac{(c - v)t}{\sqrt{1 - v^2/c^2}} \quad \text{in the time} \quad t' = \frac{(1 - v/c)t}{\sqrt{1 - v^2/c^2}}$$

Therefore, an observer in S' will find that $x' = c \cdot t'$ and the observers in S and S' both find that light propagates in their reference frames in exactly the same way.

12.2.2 Derivation of the Lorentz Transformation

We introduced the Lorentz transformation as the coordinate transformations under which Maxwell’s equations retain the same form in all coordinate system which move relative to each other at a constant uniform velocity. Lorentz discovered that Maxwell’s equations are covariant under the transformations named after him. However, these coordinate transformations do not derive from any property specific to the laws of electrodynamics but they arise from very general considerations about time and space. Maxwell’s equation just happen to be covariant under Lorentz transformation which is a good indication that they do describe the laws of nature correctly because in doing so they conform with the relativity of time and space, both of which as we know today are fundamental aspects of the universe we live in. This is different for Newton’s laws which are built on the concept of absolute space and time and because of that are only covariant under Galilean transformations but not under Lorentz transformations. As we know today, while being excellent approximations in most situations, Newtonian physics is not applicable at very high speeds and in situations of very high gravity.

While the coordinate transformations carry the name of Lorentz they are today much more associated with Einstein's special theory of relativity and not with Lorentz ether theory. This theory, to which also Henri Poincaré (1854 - 1912) contributed significantly, comes to the same conclusion with respect to the relativity of motion and the concept of local time vs. an absolute time. However, while dropping the concept of a material ether, which the Lorentz ether theory insisted could not be verified anyway, it kept the concept of a preferred rest frame which was akin to Newton's concept of space. Hence, the Lorentz ether theory could explain things like length contraction and time dilation while still adhering to the existence of a preferred rest frame filled with "Lorentz ether" of which it was not clear what it really was but that did not matter anymore.

It was clear to Lorentz, Poincaré and Einstein as well as a few others that the then existing concepts of space and time and our understanding of the meaning of things like simultaneity were poorly understood. This they sought to change and in doing so they chose a somewhat different path. Physicists like Lorentz and Poincaré approached the problem for the most from the perspective of addressing challenges arising in the theory of electrodynamics. Einstein, however, thought of space and time or simultaneity from a much more general perspective. He believed the principle of relativity and the constancy of the speed of light as being both equally fundamental. Lorentz' ether theory said the same about the latter but with its preferred reference frame at rest it had somewhat relinquished the principle of relativity.

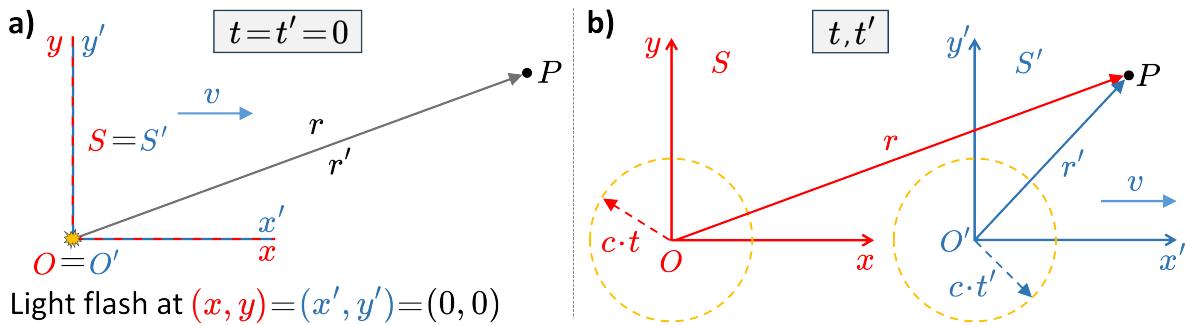


Fig. 12.4: (a) At $t = 0$ the coordinate origins of the inertial frames S and S' which moves along the x -direction from left to right with velocity v coincide and a light flash occurs at the origin. (b) Observers in each system see the light propagate in the same way as S' moves away from S . When does the light reach P in S and in S' ?

To derive the Lorentz transformation without any reference to the theory of electromagnetism we will consider the two situations sketched in fig. 12.4. Without any restriction, we can drop the third dimension and look at an inertial frame S' traveling with a constant uniform velocity v in the positive x -direction of an inertial frame S which is at

rest. Also, the x' -axis of S' shall intersect the y -axis of S at $y = 0$. Fig. 12.4a shows the moment $t = t' = 0$ where the origin of the inertial frame S' coming from the left coincides with the origin of the inertial frame S at rest. In just this moment when the two inertial frames are identical, i.e., $S(t) = S'(t')$ at $t = t' = 0$, a flash of light emerges from the shared origin of both frames. The spherical wavefront of the light wave coming from the point-like source propagates from the origin of both inertial frames in the same way with the radius of the wavefront in S equal to $c \cdot t$ and in S' equal to $c \cdot t'$ with c being the same constant velocity of light. As illustrated in fig. 12.4b, the question now is when the light in either system will reach the point P which at $t = t' = 0$ was located at the same position with respect to the origins of both inertial frames. However, as S' is moving towards P it seems that it will reach P in S' before the light in S can reach P . How can that be given that light travels at the speed of c in both frames? It looks like the only way in which light in both frames can reach P at the same moment in time is if in some manner the clocks in S' somehow tick slower than the clocks in S . Obviously, the distances $r = c \cdot t$ and $r' = c \cdot t'$ light travels from the origin of S and S' to P are different so t' must be different from t . The squares of the absolute value of those two distances are given by

$$x^2 + y^2 = c^2 t^2 \quad (12.10a)$$

and

$$x'^2 + y'^2 = c^2 t'^2 \quad (12.10b)$$

Using these two equations we can now find out how the coordinates between the two inertial frames S and S' transform. We do that by using the following parameterized approach:

$$x' = \gamma(x - vt) \quad ; \quad y' = y \quad ; \quad t' = a(t - bx) \quad (*)$$

where γ , a and b are now parameters we must determine. A quick comparison with the Galilean transformation in eq. (12.1) shows that the transformation approach from x to x' is just the respective Galilean transformation but for the scaling factor γ . Because S' moves with respect to S only in x -direction, obviously, y' must equal y , so no transformation needed there. While the transformation approach used for t to t' may look arbitrary it is just the most simple way to combine a linear approach that considers a change in t' due to the changing x coordinate of the origin of S' with a scaling factor. Inserting the parameterized approach into eq. (12.10b) we get the equation

$$\gamma^2(x - vt)^2 + y^2 = c^2 a^2(t - bx)^2$$

With a little algebra we can rewrite this equation as

$$x^2(\gamma^2 - c^2a^2b^2) - 2xt(\gamma^2v - bc^2a^2) + y^2 = c^2t^2\left(a^2 - \frac{\gamma^2v^2}{c^2}\right)$$

For this equation to become eq. (12.10a) it must apply for the coefficients that:

$$\gamma^2 - c^2a^2b^2 = 1 \quad \text{and} \quad \gamma^2v - bc^2a^2 = 0 \quad \text{and} \quad a^2 - \frac{\gamma^2v^2}{c^2} = 1$$

Solving this set of three equations for the three unknown parameters γ , a and b yields

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad ; \quad a = \gamma \quad ; \quad b = \frac{v}{c^2}$$

Inserting this into the parameterized approach (*) from above we obtain

$$x' = \frac{x - v \cdot t}{\sqrt{1 - v^2/c^2}} \quad ; \quad y' = y \quad ; \quad t' = \frac{t - v \cdot x/c^2}{\sqrt{1 - v^2/c^2}}$$

A quick comparison with eq. (12.8) shows that these are indeed the equations for the Lorentz transformation between two reference frames S and S' moving relative to each other with a constant uniform velocity in x -direction. The Lorentz transformation expresses a time dependent change of spatial coordinates as well as a space dependent change of the time coordinate. With the concept of absolute time gone and each inertial frame having its own local time the question of time synchronization arises. How can one synchronize clocks in different inertial frames so observers in those respective frames can agree when an event they all observed actually happened?

12.2.3 The Definition of Simultaneity

We all do have an intuitive understanding of what it means when we refer to two events as occurring simultaneously but this common sense understanding is too imprecise for physicists to work with. How can an observer in a given inertial frame judge if two events happening at the points A and B in space occurred at the same moment in time, i.e., if they happened simultaneously? The observer can for example construct two identical clocks and synchronize them perfectly and then place one of them at the position of A and put the other clock at the position of B . Smart as the observer is, she or he knows what kind of events will happen at A and B and hence she or he constructs the two clocks in such a way that they will stop at the exact moment the respective events happen. Let's think of the A -event and the B -event as for example being light flashes where the A -event will trigger a photocell in the clock at A and the B -event will trigger a photocell in the clock at B ; in either case, triggering the photocell shall stop the respective clock. In this

way the clock at A will record the time at which the A -event happened and the clock at B will record the time at which the B -event happened. After the events happened the observer can then go to position A and find when the clock at A stopped and then go to position B and find when the clock stopped there. By comparing those readings she or he can then judge if the clocks stopped at the same time. If they indeed stopped at the same time she or he can say that the A -event and the B -event happened simultaneously. While in principle this method works it is not very practical and hard to expand to more clocks in different positions.

What is required to test events for simultaneity is that we have the inertial frame S filled with stationary clocks which are all synchronized. Einstein himself suggested such a synchronization method which we will follow here. In the above described method we compared what Einstein called an A -position time with a B -position time but not a common time. To establish a common time we have to define what condition must be met such that the clocks in A and B are synchronized. For the latter

to be the case, following Einstein, we must require that the time it takes light to travel from the position of clock A to the position of clock B must equal the time it takes light to travel from the position of clock B to the position of clock A . The situation and the sequence of events are sketched in fig. 12.5. At the time t_A a light pulse is sent from the position of clock A to the position of clock B where it arrives at the time t_B . There, the light immediately bounces back from a mirror mounted at the position of B and travels back to the position of A where it arrives at the time t_A^* . With that, the definition for the synchronization of the two clocks at A and B translates into the requirement that

$$t_B - t_A = t_A^* - t_B \quad (12.11)$$

This definition of synchronicity is easily expandable to more clocks. Take for example another clock at position C . If clock C measures time synchronous to clock B it does so automatically also with respect to clock A . The same applies if clock C is synchronous to A because then it will also be automatically synchronous to B . This process can be repeated for as many clocks as are required with every newly added clock once being synchronous with one of the already existing synchronous clocks automatically being

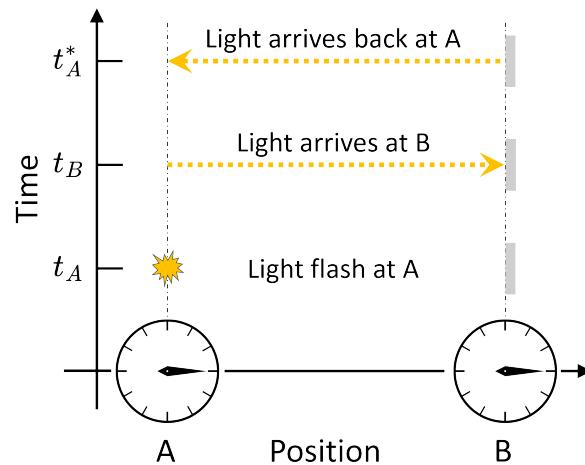


Fig. 12.5

synchronous with all other existing synchronous clocks. By setting up a reference clock at the origin of the inertial frame S we can synchronize all clocks in S with it. This of course only works because the velocity of light is constant in all directions.

Given the fact that each inertial frame has its own local time we can expect that what an observer in one inertial frame S defines as synchronous clocks in the just discussed manner may not be synchronous clocks at all from the perspective of an observer in an inertial frame S' moving relative to S with a uniform velocity v . Consider the situation sketched in fig. 12.6 where we have two synchronized clocks A and B in the inertial frame S which is at rest and two synchronized clocks C and D in the inertial frame S' which moves with a uniform velocity v relative to S . The synchronous clocks C and D shall be mounted at the ends of a measuring rod, i.e., a ruler, the length of which shall be identical to the separation of the synchronous clocks A and B in S . Further, we will require that at the moment the ends of the ruler in S' align with the distance markers of positions A and B in S , i.e., front and end of the ruler spatially coincide with the positions A and B in S , all four clocks shall show the same time. The question is: Can the clocks C and D in S' be synchronous in the same way as the clocks A and B in S are? Remember, the definition of synchronicity in the system at rest S is that it takes light the same time to travel from A towards B as it takes the light to travel back from B to A . Because S' in fig. 12.6 moves with the uniform velocity v to the right, it takes light longer to travel from C towards D than it takes the light to travel back from D to C . In the same way as we did in fig. 12.5 for the points A and B in S , we label in S' the point in time at which a light pulse begins to travel from C to D as t_C , the point in time the light pulse arrives at D and is reflected back to C as t_D and the point in time when it arrives back at C as t_C^* . Hence, the time intervals it takes the light pulse for the leg from C to D and then for the leg back to C are given by

$$t_D - t_C = \frac{r_{CD}}{c - v} \quad \text{and} \quad t_C^* - t_D = \frac{r_{CD}}{c + v}$$

where r_{CD} denotes the distance \overline{CD} . Evidently, unlike for the clocks at positions A and B in S , eq. (12.11) does not hold for the points C and D in S' because

$$\Delta t_{S'} = (t_D - t_C) - (t_C^* - t_D) = \frac{2vr_{CD}}{c^2 - v^2} > 0$$

We can look at this result from a different perspective if we remember that according to

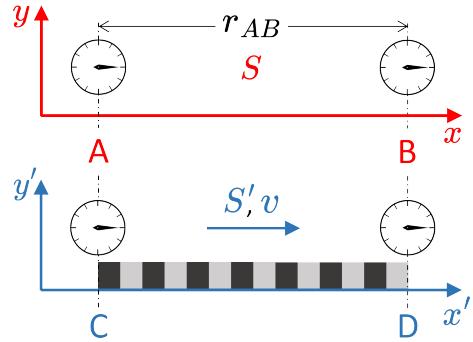


Fig. 12.6

eq. (12.7) the length of the ruler r_{CD} which the observer in S' measures is not r_{AB} but shortened by the factor $\sqrt{1 - \beta^2}$. Therefore, we can rewrite $\Delta t_{S'}$ as

$$\Delta t_{S'} = \frac{2v}{c^2 - v^2} \cdot r_{AB} \sqrt{1 - v^2/c^2} = 2 \cdot \frac{v \cdot r_{AB}/c^2}{\sqrt{1 - v^2/c^2}} \quad (12.12)$$

Eq. (12.12) tells us that clock C and clock D in S' are not synchronous according to the definition of eq. (12.11). As one can easily check, to make the two clocks synchronous the observer in S' can set clock C ahead by $\Delta t/2$, i.e., replace t_C and t_C^* with $t_C \rightarrow t_C + \Delta t/2$ and $t_C^* \rightarrow t_C^* + \Delta t/2$; or she or he can set clock D back by $\Delta t/2$, i.e., replace t_D with $t_D \rightarrow t_D - \Delta t/2$. How can we understand the result of eq. (12.12)?

When we examine the equation for the transformation of the time coordinate in eq. (12.8), we find that it is composed of two parts. The first part corresponds to the time dilation from eq. (12.8) and the second part very much looks like the right side of eq. (12.12) except for the factor 2. But that factor 2 simply arises because we had to consider the light in S' traveling the length of r_{AB} two times and instead of $2 \cdot r_{AB}$ we could have just written this length as r_{ABA} . Hence, the right side of eq. (12.12) is exactly the second term of the time coordinate transformation in the Lorentz transformation eq. (12.8). This is of course no coincidence as one can quickly check by considering a simultaneous event happening in S as viewed from the moving inertial frame S' . Lets say the simultaneous event happens in S at t_0 at the coordinates x_1 and x_2 . An observer in S' will say that those events were not simultaneous at all as one of them happened at time t'_1 and the other one at time t'_2 with

$$t'_1 = \frac{t_0 - v \cdot x_1/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad t'_2 = \frac{t_0 - v \cdot x_2/c^2}{\sqrt{1 - v^2/c^2}}$$

From the perspective of the observer in S' the two events are separated in time by

$$\Delta t_{S'} = t'_2 - t'_1 = \frac{v \cdot (x_1 - x_2)/c^2}{\sqrt{1 - v^2/c^2}}$$

With $x_1 - x_2 = 2 \cdot r_{AB}$ this becomes of course eq. (12.12). The second term of the Lorentz time coordinate transformation in eq. (12.8) ensures that simultaneous events in one inertial frame can only be simultaneous events in another frame if $x_1 - x_2 = 0$. This is why in the example illustrated in fig. 12.4a it was possible to synchronize the two coinciding inertial frames, or to synchronize the clocks A and C in fig. 12.6 at the moment when both inertial frames became spatially coincident. However, simultaneity at a distance ($x_1 \neq x_2$) observed in one inertial frame S will never be simultaneity at a distance in another inertial frame S' moving relative to S at constant uniform speed.

12.2.4 The Concept of Space-Time

The fact that the speed of light is the same in every inertial frame ensures that regardless of the relative speed at which any given inertial frame S' moves in constant uniform motion with respect to another inertial frame S , it always holds that

$$x^2 + y^2 + z^2 = c^2 t^2 \quad \text{and} \quad x'^2 + y'^2 + z'^2 = c^2 t'^2$$

A different way of saying this is to state that the conserved quantity

$$c^2 t^2 - x^2 - y^2 - z^2 = \text{const}$$

is the invariant metric of what the Mathematician Hermann Minkowski (1864 - 1909) introduced as the four-dimensional space-time referred to as Minkowski space. The coordinates of this 4-dimensional Euclidean space-time are a time coordinate multiplied with the velocity of light, so that all four vector components possess the same dimension, plus the three spatial coordinates of three-dimensional Euclidean space. The coordinates of such a four-dimensional vector in space-time, usually denoted by $\mathbf{X} = \mathbf{X}(x_0, x_1, x_2, x_3)$ and often also called just a four-vector, are given by

$$x_0 = ct \quad , \quad x_1 = x \quad , \quad x_2 = y \quad , \quad x_3 = z$$

The length of this four vector \mathbf{X} is a conserved quantity under Lorentz transformations LT, i.e., $\mathbf{X}' = \text{LT}\{\mathbf{X}\}$, or in components

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = x'_0^2 - x'_1^2 - x'_2^2 - x'_3^2 \quad (12.13)$$

As graphing in four dimensions is not possible, fig. 12.7 illustrates in a sketch what the Minkowski space looks like when we confine ourselves to motion in two spatial dimensions, as we pretty much did in most examples so far anyway. In "normal" space a point corresponds to a location but not to a specific time. That only changes when we associate for example a point mass being at a specific location P_1 with a specific time t_1 , thereby defining an event. As the point mass moves we can assign to another location P_2 on its path a different time t_2 , thereby defining another event: the point mass being at P_2 at time t_2 . Evidently, in between the events of the point mass being at P_1 at time t_1 and then being at P_2 at time t_2 something happened which we then can describe

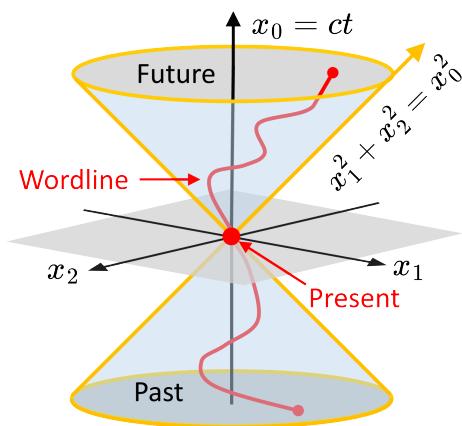


Fig. 12.7

by marking additional events associated with the point mass between t_1 and t_2 . This is very different in space-time. There we do not have to define an event as every single point is an event. When we trace a point mass in space time where we are the observer moving along with the point mass, each point on the trajectory represents an event taking place at a specific location at a specific time and the sequence of those is called a worldline, the worldline of the point mass and the observer moving along with it.

Because nothing can move faster than light, space-time with two space coordinates has a cone like structure. The tip of the future-cone begins at the origin of the coordinate system, at the event we call the "present" and it opens up towards the future. The event trajectories for all things traveling at speeds less than the speed of light must lie within the future-cone. Only things traveling at the speed of light c have a mantle-line trajectory, defined for the example in fig. 12.7 by $x_1^2 + x_2^2 = x_0^2$. Event trajectories within the cone mantle are referred to as time-like worldlines and event trajectories following the cone mantle line are called light-like curves because there at each point of space-time the velocity equals the velocity of light. Just like the future cones constrains the event space of future events which are potentially accessible from the present, the past-cone limits the event space of past events which can lead to the present. Worldlines can have only one direction and that direction points forward with respect to the time axis or put differently, two successive points (events) on the worldline must be separated by a $\Delta t > 0$.

The sketch in fig. 12.7 shows space time as perceived in a specific moment in time which we usually refer to as the present time. Now, as every event happening in space-time at a certain time possesses its own present time, a new future-cone and a new past cone connect to this present time. As sketched in fig. 12.8, space-time consists of infinitely many such light cone structures and at each new present time a cone constrains the event space of possible future events, of what can happen in the future, and it constrains the event space of possible past events that could have led to the present event. In this way, those infinitely many light cones define which events can either become causally connected in the future or have a possible causal connection in the past. This property of space time is referred to as the causal structure of space time: A present

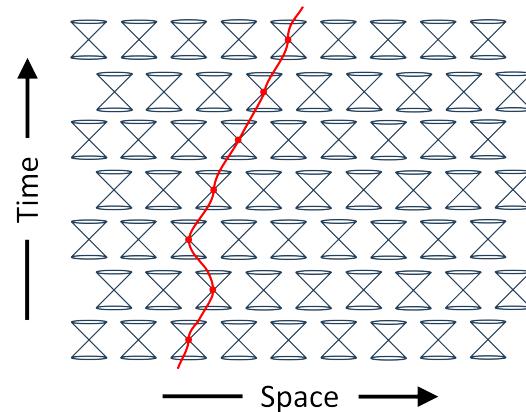


Fig. 12.8

event cannot be caused by an event that does not lie in its past-cone and a present event can not cause an event that does not lie in its future cone.

As for now we have only looked at space-time from the perspective of one inertial frame. What happens to space-time under a Lorentz transformation? To look into that we will simplify things even more by considering only movement in one space dimension. The coordinates in an inertial frame $S(x, ct)$ are connected to the coordinates in the inertial frame $S'(x', ct')$ which moves with respect to S with a constant uniform velocity v in x -direction by the Lorentz transformation eq. (12.8). However, here we will rewrite the respective two equations from eq. (12.8) using again the abbreviations $\beta = v/c$ and $\gamma^{-1} = \sqrt{1 - \beta^2}$ already introduced earlier. We also rearrange these two equations a little different to read

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma \cdot \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ ct \end{pmatrix} \quad (12.14)$$

with the inverse transformation being

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \gamma \cdot \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ ct' \end{pmatrix} \quad (12.15)$$

The way we rearranged this Lorentz transformation already suggests that we are looking at some kind of rotation of the light cone orientation in S' with respect to S . Let's consider again the moment at which S and S' are spatially coincident. Then we can determine the orientation of the x' -axis by determining the coordinates of two points on it with respect to S . Along the x' -axis we have of course $ct' = 0$ and for the two points on the x' -axis to determine its orientation we chose the obvious point $x = x' = 0$ and an arbitrary point like $x' = 1$. Inserting $(x' = 0, ct' = 0)$ into the inverse transformation eq. (12.15) trivially returns the coordinates $(x = 0, ct = 0)$ and inserting $(x' = 1, ct' = 0)$ gives us the coordinates $(x = \gamma, ct = \gamma\beta)$. Using the procedure for determining the orientation of the ct' -axis with the points $(x' = 0, ct' = 0)$ and $(x' = 0, ct' = 1)$ tells us that the ct' -axis runs through $(x = 0, ct = 0)$ and $(x = \gamma\beta, ct = \gamma)$. Hence, the x' -axis runs through the origin with a positive slope of β and the ct' -axis runs through the origin with a positive slope of β^{-1} . The respective angles

$$\varphi_1 = \arctan \beta = \arctan(v/c) \quad \text{and} \quad \varphi_2 = \arctan \beta^{-1} = \arctan(c/v)$$

are indicated in the example sketched in fig. 12.9. Unless the relative velocity v is a substantial fraction of the velocity of light c , these angles are very small. Therefore, the example were are using in fig. 12.9 is sketched for $v = 0.3 \cdot c$.

What one likely notices first in fig. 12.9 is that the x' -axis and the ct' -axis are no longer orthogonal. The angle between the x' -axis and the ct' -axis which for $v = 0$ is a right angle, closes like a pair of scissors as v increases. Eventually, with v approaching c the space axis and the time axis in S' will coincide as with $\beta \rightarrow 1$ obviously $\varphi_{1,2} \rightarrow \pi/4$. The line running at an angle of $\pi/4$ with respect to the x -axis through the origin of S is however just where light-like curves of the future-cone of S run. This makes good sense because with S' moving at the speed of light, its trajectory will have to fall within the mantle of the future-cone of S . But what

about the future-cone of S' itself? How does it transform? The short answer is the light-cone does not change at all, it looks the same from every present-point in space-time an observer happens to find her- or himself. Eq. (12.13) tells us that the metric $x_0^2 - x_1^2 - x_2^2 - x_3^2$ is an invariant. In 3-dimensional Euclidean space the invariant metric is spatial distance, meaning that $x^2 + y^2 + z^2 = R^2 = const$. That is of course the equation for a sphere of radius R with its center at the coordinate origin. In 4-dimensional space-time the equation $x_0^2 - x_1^2 - x_2^2 - x_3^2 = const$ defines a hyperbolic surface. The Lorentz invariance of $x_0^2 - x_1^2 - x_2^2 - x_3^2$ tells us that this surface is preserved under Lorentz transformations and therefore the light-cone does not change at all, it is the same in S and in S' . What changes is that time axis and space axis are rotated towards each other. The square of a space-time interval $(ct)^2 - x^2 - y^2 - z^2$ measures a distance. For $t = 0$ the square of this interval becomes negative and the corresponding length, the square root of a negative number, becomes imaginary. Events being separated by an imaginary interval are said to have a space-like interval between them. Conversely, if two events share the same spatial coordinates they are said to be separated by a time-like interval.

If we include more than the one space dimension we considered in the example of fig. 12.9 we would find that the 2-dimensional space surface (as illustrated in fig. 12.7) is tilted in an inertial frame S' which moves relative to an inertial frame S at rest (like the one depicted in fig. 12.7); the same would be the case if we included all three spatial dimensions. The 2-dimensional space surface in the x_1x_2 -plane in fig. 12.7 is also called the hypersurface of simultaneity. The tilt of this hypersurface of simultaneity allows us to quickly understand if two events which for an observer in one inertial frame happen

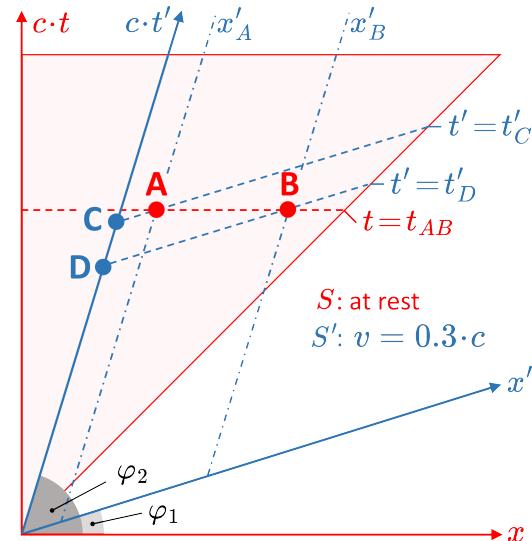


Fig. 12.9

simultaneously can be simultaneous events for an observer in another inertial frame. To illustrate this, we have added in the example of fig. 12.9 two events A and B happening in S . Each of those two events must of course lie within the future-cone of S . Trivially, two events are simultaneous events in 4-dimensional space-time if they have the same time coordinate, i.e., if they fall on a line running parallel to the space coordinate. In fig. 12.9 we picked for this time coordinate in S on which the events A and B lie the value $t = t_{AB}$. The respective space and time coordinates of the events A and B in S' we find by running lines parallel to the x' -axis and parallel to the ct' axis through the points A and B which gives us the space coordinates x'_A and x'_B as well as the time coordinates $t' = t'_C$ and $t' = t'_D$ for the events C and D in S' which correspond to the events A and B in S . Clearly, $t'_C \neq t'_D$ and the events C and D cannot be simultaneous in S' .

12.2.5 The Relativistic Doppler Effect

In section 11.2.2 we discussed the Doppler effect for acoustic waves. Now we will look at the relativistic Doppler effect which is irrelevant for acoustic waves as they travel so slowly but of course matters for light waves. As it is, to understand the relativistic Doppler effect we do not even have to know anything more about the particular nature of light waves, all we need to know is simply that they represent a kind of information transfer that happens at light speed. The situation illustrated in fig. 12.10 is similar to fig. 12.9 but here we are not looking for simultaneity of events but for the periodicity sequence of events, here light flashes, as they are registered by an observer at rest with the light source and an observer moving relative to the light source with a constant speed v on a worldline intersecting the x -axis at x_0 . The interval at which the light source flashes shall be τ . The light flash sent at $t = t_0$ travels on the world line $x = ct$ and intersects the worldline of the moving observer at (x_1, t_1) ; the second light flash sent at $t = \tau$ travels on the worldline $x = c(t - \tau)$ and intersects the worldline of the moving observer at (x_2, t_2) ; the third light flash sent at $t = 2\tau$ then travels on the worldline $x = c(t - 2\tau)$ and intersects the worldline of the moving observer at (x_3, t_3) and so forth.

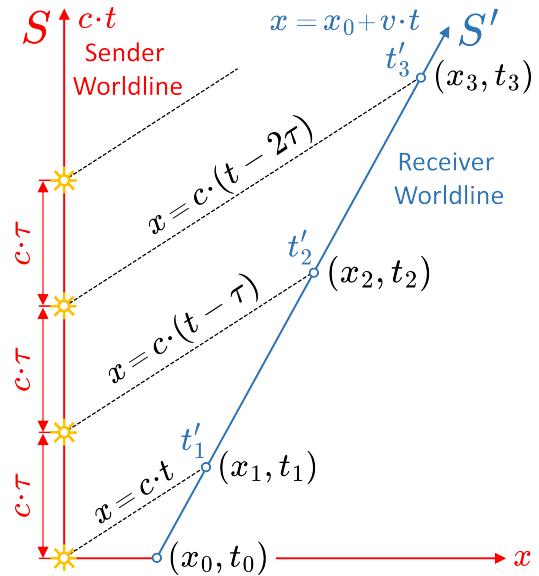


Fig. 12.10

The periodicity which the observer at rest with the light source registers is of course τ . The periodicity which the moving observer registers is given by the time differences $t_1 - t_0$, $t_2 - t_1$, $t_3 - t_2$, $t_n - t_{n-1}$ and so forth, which are defined by the intersection of the worldlines $x = c(t_n - (n-1)\tau)$ with the worldline of the moving observer. These intersections are given by

$$\begin{aligned} x_1 &= ct_1 = x_0 + vt_1 \quad \Rightarrow \quad t_1 = \frac{x_0}{c-v} \\ x_2 &= c(t_2 - \tau) = x_0 + vt_2 \quad \Rightarrow \quad t_2 = \frac{x_0 + c\tau}{c-v} \\ x_{n-1} &= c(t_{n-1} - (n-2)\tau) = x_0 + vt_{n-1} \quad \Rightarrow \quad t_{n-1} = \frac{x_0 + c(n-2)\tau}{c-v} \\ x_n &= c(t_n - (n-1)\tau) = x_0 + vt_n \quad \Rightarrow \quad t_n = \frac{x_0 + c(n-1)\tau}{c-v} \end{aligned}$$

From the last two equations follows for the time difference $t_n - t_{n-1}$ and the position difference $x_n - x_{n-1}$

$$t_n - t_{n-1} = \tau \frac{c}{c-v} \quad \text{and} \quad x_n - x_{n-1} = c\tau \frac{v}{c-v}$$

With that we have determined the time differences $t_n - t_{n-1}$ the observer at rest in S will measure which we must now convert to the time differences the moving observer in S' will measure which we get through the Lorentz transformation eq. (12.8):

$$\begin{aligned} t'_n - t'_{n-1} &= \frac{t_n - v \cdot x_n/c^2}{\sqrt{1 - v^2/c^2}} - \frac{t_{n-1} - v \cdot x_{n-1}/c^2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{\tau c}{c-v} \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \tau \cdot \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} = \tau \cdot \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} \end{aligned}$$

Therefore, while an observer in the reference frame S where the light source is at rest will register a light signal flashing with the frequency $f = 1/\tau$, the observer in the frame S' moving with a velocity v relative to S will register a signal with periodicity

$$f' = \frac{1}{t'_n - t'_{n-1}} = \frac{1}{\tau} \cdot \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}} = f \cdot \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}} \quad (12.16)$$

The ratio $f':f$ only depends on the relative velocity v of light source and observer. This is unlike what we found for the acoustic Doppler effect in section 11.2.2 where the result depends on whether the source is at rest and the observer moves or vice versa. The reason is simply that acoustic waves require a medium in which they can propagate and therefore there will always be a preferred reference system. Different from that, light

waves do not require a medium and therefore the Doppler effect for light waves only depends on the relative velocity of the respective reference frames. If we compare the relativistic result for light waves with the classical results we obtained for acoustic waves in the case of a source moving away from an observer at rest (the plus sign in eq. (11.17)) and in the case of an observer moving away from a light source at rest (the minus sign in eq. (11.18)), i.e.,

$$f_A = \frac{1}{1 + v/c} f \quad \text{and} \quad f_B = (1 - v/c) f$$

then we see that f' in eq. (12.16) is just the geometric mean of these classical results:

$$f' = \sqrt{f_A f_B} = f \cdot \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}$$

In the above derivation of the relativistic Doppler effect we have made no use of the fact that we are actually interested in how an observer in S and an observer in S' register the propagation of a traveling wave. We were just looking at how the period of a periodic event changes when moving from S to S' . Now however, we will look at a specific wave which, as illustrated in fig. 12.11, shall originate at the origin of the reference frame S' . As S' moves with a relative velocity v with respect to the stationary reference frame S , the light source at the origin of S' shall periodically emit its signal at an angle θ relative to the x' coordinate, i.e., the angle between the wave vector \mathbf{k}' and the x' -direction is θ . The period at which the light source flashes in S' is τ' but because of time dilation the observer in S will see this period stretched to

$$\tau = \frac{\tau'}{\sqrt{1 - v^2/c^2}}$$

In that time span the reference frame S' moves the distance $v\tau$ relative to S . To determine the wavelength λ one measures the distance between the emission of two light pulses in the propagation direction. As sketched in fig. 12.11, this distance is given by

$$\lambda = c\tau - v\tau \cos \theta = \frac{c\tau'}{\sqrt{1 - v^2/c^2}} \left(1 - \frac{v}{c} \cos \theta\right)$$

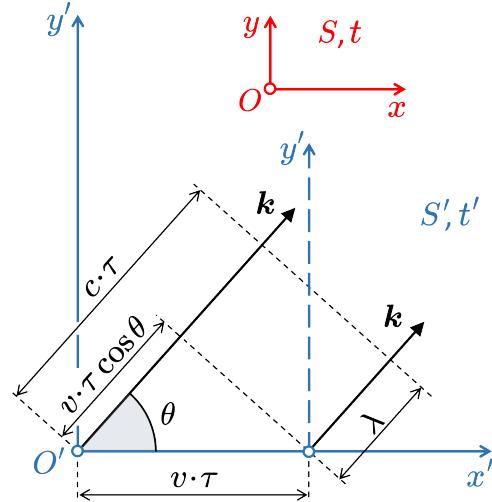


Fig. 12.11

With $\lambda' = c \cdot \tau'$ being the wavelength in S' we can write this as

$$\lambda = \frac{\lambda'}{\sqrt{1 - v^2/c^2}} \left(1 - \frac{v}{c} \cos \theta\right) \quad (12.17)$$

and with $c = \lambda f = \lambda' f'$ the angle dependent frequency shift is

$$f = f' \sqrt{1 - v^2/c^2} \cdot \left(1 - \frac{v}{c} \cos \theta\right)^{-1} \quad (12.18)$$

Evidently, for $\theta = 0$, i.e., when the direction of emission points forward, we just get the result of eq. (12.16). In that case, emission occurs parallel to the x' -axis (and with that also to the x -axis) as the source is moving towards the observer in S , who therefore will register a blue shift in wavelength:

$$\lambda = \lambda' \cdot \sqrt{\frac{1 - v/c}{1 + v/c}} \quad \text{and} \quad f = f' \cdot \sqrt{\frac{1 + v/c}{1 - v/c}}$$

Conversely, if $\theta = \pi$ we have backward emission, i.e., the direction of emission points in the opposite direction into which S' is moving and therefore an observer in S will register a red shift in wavelength

$$\lambda = \lambda' \cdot \sqrt{\frac{1 + v/c}{1 - v/c}} \quad \text{and} \quad f = f' \cdot \sqrt{\frac{1 - v/c}{1 + v/c}}$$

where the frequency shift correspond to the one of eq. (12.16) but with v replaced by $-v$. If $\theta = \pi/2$, i.e., the radiation is propagating perpendicular to the direction of the relative motion of S and S' , eq. (12.18) reduces to

$$f = f' \cdot \sqrt{1 - v^2/c^2} \quad (12.19)$$

and with

$$\lambda = \frac{\lambda'}{\sqrt{1 - v^2/c^2}} \quad (12.20)$$

an observer in S will detect a red shift in wavelength. The frequency shift in eq. (12.19) is known as the transverse Doppler effect. It is a purely relativistic effect and occurs simply because of time dilation which always exists when two reference frames move with respect to each other. Eq. (12.19) and eq. (12.20) presume that the observer at rest measures frequency / wavelength shifts at the moment when the light source emits at an angle of $\theta = \pi/2$ with respect to the x' -axis. However, one can also choose a different definition of the transverse Doppler effect, resulting in a blue and not a red shift.

To understand why, we need to consider the situation sketched in fig. 12.12. When the signal reaches the observer at the origin of S at the time τ , the signal source has moved the distance $v\tau$ on the x' -axis to a position where the line of sight from the source to the observer forms an angle θ_1 with the x' -axis. For the height h , the normal distance of the observer from the x' axis, one can read off the relationships

$$c\tau \sin \theta = h = d \sin \theta_1$$

and therefore the angle θ_1 is given by

$$\sin \theta_1 = \frac{c\tau}{d} \sin \theta$$

The distance d follows also from the law of cosines which tells us that

$$d^2 = (c\tau)^2 + (v\tau)^2 - 2vc\tau^2 \cos \theta$$

Inserting this the value of θ_1 is determined by the equation

$$\begin{aligned} \sin \theta_1 &= \frac{c\tau \sin \theta}{\sqrt{(c\tau)^2 + (v\tau)^2 - 2vc\tau^2 \cos \theta}} \\ &= \frac{\sin \theta}{\sqrt{1 + (v/c)^2 - 2(v/c) \cos \theta}} \end{aligned}$$

As a matter of definition, one now requires that at the moment the light reaches the observer at rest the angle θ_1 shall equal $\pi/2$. With this definition follows that

$$\sin^2 \theta = 1 + \frac{v^2}{c^2} - 2\frac{v}{c} \cos \theta$$

which leads to the equation

$$\cos^2 \theta + 2\frac{v}{c} \cos \theta + \frac{v^2}{c^2} = 0 \quad \Rightarrow \quad \cos \theta = \frac{v}{c}$$

Inserting $\cos \theta = v/c$ into eq. (12.17) we get for the wavelength shift due to the transverse Doppler effect

$$\lambda = \frac{\lambda'}{\sqrt{1 - v^2/c^2}} (1 - v^2/c^2) = \lambda' \sqrt{1 - v^2/c^2} \quad (12.21)$$

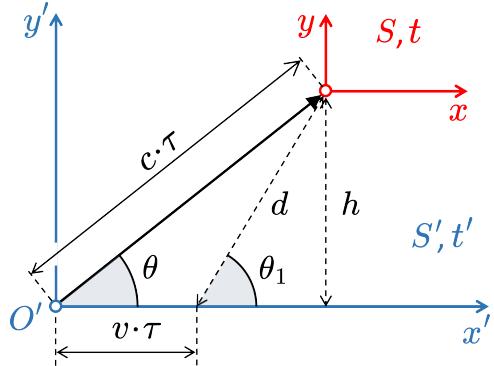


Fig. 12.12

Hence, with this definition of the transverse Doppler effect, the wavelength λ measured by the observer at rest is shorter than the wavelength λ' with which the source in S' emits the light. So how does one reconcile that in one definition of the transverse Doppler effect the observer at rest detects according to eq. (12.20) a red shift while with another definition the observer measures according to eq. (12.21) a blue shift? The solution is to use a symmetric definition of the transverse Doppler effect as sketched in fig. 12.13. If one requires that $\theta + \theta_1 = \pi$, i.e., the angle between the observer and the source at the time when the light reaches the observer shall be $\theta_1 = \pi - \theta$ then $\sin \theta_1 = \sin \theta$. In that case the above equation for $\sin \theta_1$ becomes

$$\sin \theta_1 = \sin \theta = \frac{\sin \theta}{\sqrt{1 + (v/c)^2 - 2(v/c) \cos \theta}}$$

This is only possible if

$$1 + \frac{v^2}{c^2} - 2\frac{v}{c} \cos \theta = 1 \quad \Rightarrow \quad \cos \theta = \frac{1}{2}\frac{v}{c}$$

Inserting $\cos \theta = v/2c$ into eq. (12.17) we get for the wavelength shift due to the transverse Doppler effect

$$\lambda = \frac{\lambda'}{\sqrt{1 - v^2/c^2}} \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

which one can rewrite as

$$\lambda = \frac{1}{2} \left(\frac{\lambda'}{\sqrt{1 - v^2/c^2}} + \lambda' \sqrt{1 - v^2/c^2} \right) \quad (12.22)$$

A comparison with eq. (12.20) and eq. (12.21) shows that this result is nothing else but the arithmetic mean of the red shift from eq. (12.20) and the blue shift from eq. (12.21). For low velocities, i.e., $v \ll c$, eq. (12.22) becomes $\lambda = \lambda'$ and the transverse Doppler effect vanishes in the classical limit, as it must. The existence of a purely relativistic Doppler effect due to time dilation, the transverse Doppler effect, was one of the unique predictions of Einstein special theory of relativity. When this effect was experimentally confirmed in 1938 that also was the first confirmation that time dilation is real.

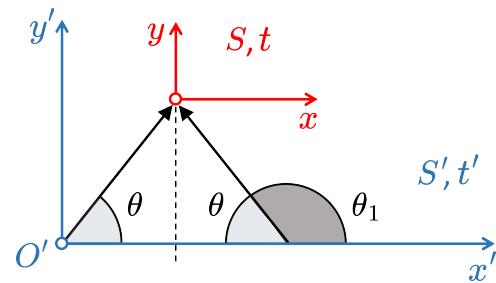


Fig. 12.13

12.3 Relativistic Dynamics

The first part of this chapter focused on the concepts of relative time and relative space which the special theory of relativity introduced to physics, replacing the absolute space and absolute time of Newtonian mechanics. This new understanding of space and time we applied to the description of object motion from the perspective of two observers in inertial frames moving with a constant uniform velocity with respect to each other. Now it is time to go beyond this kinematic description and consider the forces which make bodies move they way they do. How must Newton's laws change so they give a correct description of relativistic dynamics? This is what we will look into next. We will start with Newton's second law itself and introduce the required change in a formally simple way before we can justify it more fully once we understand how the addition of velocities and the conservation of momentum work in the relativistic case.

Using eq. (12.9) which relates the length of "one second" in an inertial frame S to the length of "one second" in an inertial frame S' which moves with the constant velocity v relative to S , we can express the time differential dt in S as a function of the time differential dt' in S' and vice versa:

$$\frac{dt}{dt'} = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (12.23)$$

Because relativistic velocity addition works different than in the classical case, the relativistic velocity \mathbf{v}_{rel} differs from the classical velocity \mathbf{v} . As we will find out later, the relationship between the relativistic velocity \mathbf{v}_{rel} and the classical velocity \mathbf{v} is

$$\mathbf{v}_{rel} = \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \frac{d\mathbf{r}}{dt} \frac{dt}{dt'} = \frac{d\mathbf{r}(t)}{dt'}$$

To get the relativistic velocity we need to take the time derivative with respect to the inertial frame where the moving point mass is at rest, i.e., eq. (12.23) comes into play. Obviously, for $v \ll c$ applies $dt' \rightarrow dt$ and $\mathbf{v}_{rel} \rightarrow \mathbf{v}$. With this relativistic velocity and m_0 the mass of point mass at rest, Newtons second law then becomes

$$\mathbf{F} = \frac{d}{dt} \left(\frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) = \frac{d\mathbf{p}}{dt} \quad (12.24)$$

where

$$\mathbf{p}_{rel} = m\mathbf{v} = \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad (12.25)$$

is the relativistic momentum. We can continue to use the classical velocity and the classical momentum in Newton's equation of motion by defining

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad (12.26)$$

as the dynamic mass of the moving point mass. The above equations are the correct relativistic results. The formally simple way to derive the relativistic equation of motion here did not give a justification as to why relativistic velocities have to be calculated differently. We will rectify this in section 12.3.3 once we have understood how velocity addition and momentum conservation works in the relativistic case. With the definition of the relativistic momentum in eq. (12.25) and the relativistic equation of motion eq. (12.24) we have now everything we need to begin our exploration of relativistic dynamics.

Fig. 12.14 shows the dependence of the dynamic mass to rest mass ratio $m:m_0$ as a function of v/c . Evidently, at $v = 0$ the dynamic mass m is the same as the rest mass m_0 . In this case, eq. (12.24) just reverts to $\mathbf{F} = m_0 \cdot \dot{\mathbf{v}}$. However, the greater v becomes the more the dynamic mass will increase and if v would equal c the dynamic mass would become infinite. That cannot happen of course. But how can that not happen? Let's take a look

back at section 4.1.3 where we found that a force which is continuously applied to a point mass will result in a continuously increasing kinetic energy of that point mass (see eq. (4.8)). In Newtonian mechanics a continuously increasing kinetic energy of a point mass translates into a continuously increasing velocity of that point mass. Different from relativistic mechanics, there is nothing built into Newtonian mechanics which would prevent a point mass from moving faster than the speed of light if a force were only applied for a sufficiently long time.

In Newtonian mechanics, the work done by a force on a point mass can only be channeled into an increase of its kinetic energy, i.e., its velocity. This is different for the relativistic version of Newton's law in eq. (12.24). There the work done by a force on a point mass cannot only translate into an increase of its velocity but also an increase of its mass. As fig. 12.14 illustrates, the speed of light seemingly represents an insurmountable barrier for a massive object. We already know that nothing can move faster than the velocity

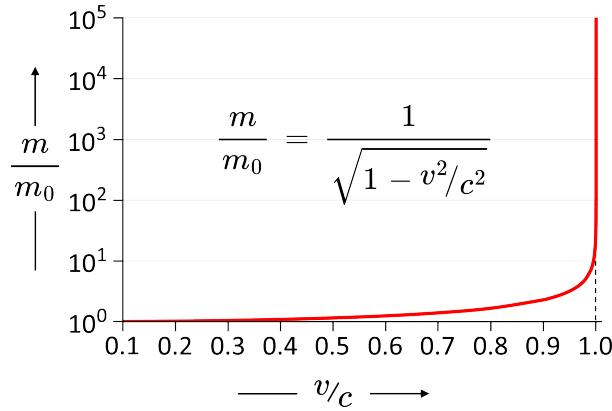


Fig. 12.14

of light c . That the expression for the dynamic mass in eq. (12.26) would go to infinity for $v = c$ tells us that we must refine this statement to read: No object possessing a mass can ever move at the speed of light, only massless objects, i.e., objects for which $m_0 = 0$, can move at the speed of light. Whenever we try to accelerate an object, a part of the energy we must expend to do that will not go into increasing the object's velocity but will increase its mass. For every incremental increase in the velocity of an object we must spend a bit more energy than for the previous incremental increase in velocity as with every such incremental increase the object will become more massive. As v gets closer to c the object becomes ever more massive until practically all of the energy we expend to increase the object's velocity is channeled into an increase of its dynamic mass. Since an object with mass can never reach the velocity of light its dynamic mass will increase ever more without ever becoming actually infinite while the velocity of the object will increase by ever smaller amounts without ever reaching the velocity of light. Among such objects are elementary particles such as electrons or protons which researchers accelerate to velocities v where $(c - v)/c$ is less than 10^{-8} . At this velocity the dynamic mass of the elementary particle is about a factor of 7500 higher than its rest mass. Accelerating elementary particles to such high speeds not only requires large facilities but copious amounts of energy.

How does the expression for the kinetic energy of an elementary particle moving at such high speeds differ from the kinetic energy of a non-relativistic particle. To understand that we use the fact that the kinetic energy the particle gains must equal the work expended to move it. With eq. (4.1) and eq. (12.24) the kinetic energy then becomes

$$T = \int \mathbf{F}(\mathbf{r}) d\mathbf{r} = \int \frac{d}{dt} \left(\frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) d\mathbf{r}$$

Using $d\mathbf{r} = \mathbf{v} dt$ this equation can be written as

$$T = \int d \left(\frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) \mathbf{v} = m_0 \int \mathbf{v} \cdot d \left(\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} \right)$$

Now we apply partial integration to obtain

$$T = m_0 \mathbf{v} \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} - m_0 \int \frac{\mathbf{v} d\mathbf{v}}{\sqrt{1 - v^2/c^2}}$$

To evaluate the second integral in this equation we make use of

$$dv^2 = d(\mathbf{v} \cdot \mathbf{v}) = d\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot d\mathbf{v} = 2 \mathbf{v} \cdot d\mathbf{v}$$

With that the equation for T becomes

$$\begin{aligned} T &= m_0 \frac{v^2}{\sqrt{1 - v^2/c^2}} - \frac{m_0 c^2}{2} \int \frac{d(v^2/c^2)}{\sqrt{1 - v^2/c^2}} \\ &= m_0 \frac{v^2}{\sqrt{1 - v^2/c^2}} + m_0 c^2 \sqrt{1 - v^2/c^2} + E_0 \\ &= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} + E_0 \end{aligned}$$

where we added E_0 as an integration constant because we are dealing with an indefinite integral. For $v = 0$ the kinetic energy T must also be zero from which follows that $E_0 = -m_0 c^2$. With that the expression for the relativistic kinetic energy becomes

$$T = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 \quad (12.27)$$

or using the expression for the dynamic mass m from eq. (12.26)

$$T = mc^2 - m_0 c^2 \quad (12.28)$$

To check how eq. (12.27) compares to the kinetic energy expression of Newtonian mechanics we use the first two terms of the Taylor expansion of the inverse square root in eq. (12.27) for $v/c \ll 1$ with the result

$$T_{v \ll c} = m_0 c^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} \right) - m_0 c^2 = \frac{1}{2} m_0 v^2$$

As expected, we get the result of Newtonian mechanics for the kinetic energy. What eq. (12.28) tells us is that mass is a form of energy, or as it is frequently put, it establishes the equivalence of energy and matter with the speed of light c being the proportionality constant. This is Einsteins famous $E = mc^2$. A change in mass equals a change in energy and a change in energy equals a change in mass. This equivalence of mass and energy also means that the kinetic energy as a function of v/c scales just like the dynamic mass. The scaling curve for the kinetic energy is identical to the scaling curve shown for the dynamic mass in fig. 12.14. It is this equivalence of mass and energy which keeps the nuclear furnaces of stars like our Sun burning by converting the mass loss occurring in nuclear reactions into energy. Sometimes not only a little mass is lost in a reaction but all of a particles mass gets converted into energy. So for example when a particle like the electron runs into its antiparticle, the positron, and all matter is converted into radiation energy in such an electron-positron annihilation.

We have not said much about the relativistic momentum for now and that is for a good reason. Before we can do that we need to get a better understanding as to how the addition of velocities works in the relativistic case. Once we have accomplished that, we will be in a position to consider how the conservation of momentum works in the relativistic case and that in turn will provide us a better explanation as to why the relativistic equation of motion, i.e., eq. (12.24), looks the way it does. Having said that, being able to derive eq. (12.24) in the qualitative way we just did and thereby getting a basic understanding of its implications does have its advantages.

12.3.1 Relativistic Addition of Velocities

The composition of velocities in Newtonian mechanics is simple. Let again S be the inertial frame at rest and S' be the inertial frame moving relative to S with a constant velocity of \mathbf{v}_0 . An object moving in S' with the velocity \mathbf{v}' will then from the perspective of S have the velocity

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_0 \quad (12.29)$$

As we already saw earlier with eq. (12.2), this result is a direct consequence of the Galilean transformation of coordinates. How does this simple addition of velocities change when we apply the Lorentz transformation to the respective space and time coordinates? To understand that we must know how the space differentials dx , dy , dz , and the time differential dt change as the velocity components of an object are defined as the respective space differential (dx , dy , or dz) divided by the time differential dt . To achieve that, we need the inverse Lorentz transformation, like we already used it in the discussion of the concept of space time (see eq. (12.15)), however, not for the coordinates themselves but for the coordinate differentials. Here, we will restrict ourselves again to consider motion in one direction only, i.e., along the x -axis or respectively x' -axis. This will make the calculations a lot simpler without loss of generality. We can copy directly from eq. (12.14) and eq. (12.15) to write down the Lorentz transformation as ($\beta = v_0/c$)

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \beta x/c) \end{aligned} \quad (12.30)$$

and the inverse Lorentz transformation as

$$\begin{aligned}
x &= \gamma(x' + \beta ct') \\
y &= y' \\
z &= z' \\
t &= \gamma(t' + \beta x'/c)
\end{aligned} \tag{12.31}$$

The inverse transformation is nothing else than the observer in S' insisting that she or he is at rest and S is moving relative to S' with a velocity of $-v_0$. The latter is of course the reason why the minus signs in the brackets of the first and last equations of eq. (12.30) have become plus signs. With the inverse transformation eq. (12.31) we can write the space and time differentials in S as

$$\begin{aligned}
dx &= \gamma \cdot dx' + \gamma \beta c \cdot dt' \\
dy &= dy' \\
dz &= dz' \\
dt &= \gamma \cdot dt' + (\gamma \beta / c) \cdot dx'
\end{aligned}$$

Using these differentials we can now calculate the velocity components of \mathbf{v} in S as functions of the velocity components of \mathbf{v}' in S'

$$\begin{aligned}
v_x &= \frac{dx}{dt} = \frac{\gamma \cdot dx' + \gamma \beta c \cdot dt'}{\gamma \cdot dt' + (\gamma \beta / c) \cdot dx'} = \frac{\frac{dx'}{dt'} + \beta c}{1 + \frac{\beta}{c} \frac{dx'}{dt'}} = \frac{v'_x + v_0}{1 + \frac{v'_x v_0}{c^2}} \\
v_y &= \frac{dy}{dt} = \frac{dy'}{\gamma \cdot dt' + (\gamma \beta / c) \cdot dx'} = \frac{\frac{dy'}{dt'}}{\gamma \left(1 + \frac{\beta}{c} \frac{dx'}{dt'}\right)} = \frac{v'_y \sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_x v_0}{c^2}} \\
v_z &= \frac{dz}{dt} = \frac{dz'}{\gamma \cdot dt' + (\gamma \beta / c) \cdot dx'} = \frac{\frac{dz'}{dt'}}{\gamma \left(1 + \frac{\beta}{c} \frac{dx'}{dt'}\right)} = \frac{v'_z \sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_x v_0}{c^2}}
\end{aligned}$$

To summarize: If an inertial frame S' moves with respect to another inertial frame S with a velocity $\mathbf{v}_0 = (v_0, 0, 0)$ and an object from the perspective of an observer in S' moves with the velocity $\mathbf{v}' = (v'_x, 0, 0)$, then an observer in S will determine that this object moves from her or his perspective with the velocity $\mathbf{v} = (v_x, v_y, v_z)$ where

$$v_x = \frac{v'_x + v_0}{1 + \frac{v'_x v_0}{c^2}}, \quad v_y = \frac{v'_y \sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_x v_0}{c^2}}, \quad v_z = \frac{v'_z \sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_x v_0}{c^2}} \tag{12.32}$$

Comparing this addition of velocities with eq. (12.29), the addition of velocities following from Galilean transformation, we first note that in the direction of \mathbf{v}_0 we get the result from the Galilean transformation divided by a relativistic correction factor. This correction factor depends on the velocity \mathbf{v}_0 and the velocity component of \mathbf{v}' parallel to \mathbf{v}_0 . For the velocity components of \mathbf{v}' orthogonal to \mathbf{v}_0 we have this very same correction factor plus an addition correction factor, the familiar $\sqrt{1 - \beta^2}$. Evidently, an observer in S will find that the velocity of an object that in S' only has a velocity component orthogonal to the velocity \mathbf{v}_0 will have velocity components orthogonal and parallel to \mathbf{v}_0 .

To better understand why this is the case we need to look into what happens if an object moves in S' with the velocity of light. Already Newton thought that light had a corpuscular nature but because of the successes of Huygens wave theory and later of Maxwell's electromagnetic wave theory, the dual nature of light only resurfaced again with the advent of quantum mechanics. Newton's light corpuscles became the massless fundamental particles of light we now call photons. Let us now assume that such a photon moves in S' in x' -direction, i.e., $\mathbf{v}' = (c, 0, 0)$. In this case eq. (12.32) tells us that $v_y = v_z = 0$ and an observer in S will hence see the photon move in the x -direction with the velocity

$$v_x = \frac{c + v_0}{1 + v_0/c} = c \cdot \frac{c + v}{c + v} = c$$

Evidently, regardless of what the velocity v_0 is, an observer in S will always see the photon move with the speed of light c , just like the observer in S' does. This should not be a surprise as the speed of light having the same constant value of c in all inertial frames was built into the Lorentz transformation. Now let us assume that a photon moves in S' orthogonal to \mathbf{v}_0 , e.g., in the y' -direction, i.e., $\mathbf{v}' = (0, c, 0)$. According to eq. (12.32), an observer in S will see the photon move with the velocity

$$\mathbf{v} = \begin{pmatrix} v_0 \\ c\sqrt{1 - \beta^2} \\ 0 \end{pmatrix} = \begin{pmatrix} v_0 \\ \sqrt{c^2 - v_0^2} \\ 0 \end{pmatrix}$$

As the sketch in fig. 12.15 illustrates, the photon in S is still moving with the speed of light c . However, it is not moving orthogonal to \mathbf{v}_0 anymore as it does in S' but now has a v_x -component in addition to the v_y -component. The direction of the photon

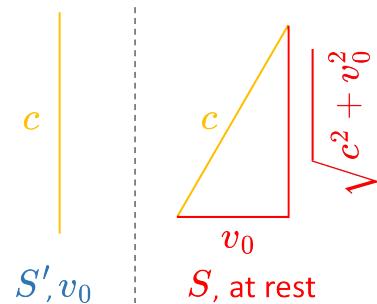


Fig. 12.15

velocity vector changes in S but not its magnitude as

$$|\mathbf{v}| = \sqrt{v_0^2 + (c^2 - v_0^2)} = c$$

regardless of the magnitude of the velocity v_0 . From the perspective of an observer in S only the direction of the photon changes but not its speed. In the extreme case of $v_0 = c$ the v_y -component in S will vanish completely and what was a photon moving orthogonal to \mathbf{v}_0 will be perceived by an observer in S as a photon moving parallel to \mathbf{v}_0 . What does that mean? Consider the observer in S' to be somewhere located on the positive x' -axis looking back towards the origin of S' where the photon which travels in the positive y' -direction shall have originated at the moment of spatial coincidence of S' and S . Obviously, this photon will never reach the eye of the observer in S' . But not so for an observer in S who similarly sits somewhere on his positive x -axis looking back to the origin of S . If $v_0 = c$ the photon will move along the x -axis and reach the observer. What this tells us is that due to the mixing of normal and parallel velocity components the observer in S will be able to see things which the observer in S' cannot see. Seemingly, with increasing velocity \mathbf{v}_0 the y' horizon increasingly bends towards the observer in S .

Another peculiarity of the relativistic velocity addition in eq. (12.32) is that unlike the Galilean velocity addition in eq. (12.29), it is not linear. Fig. 12.16 compares relativistic velocity addition with Galilean velocity addition for the particular case where $v'_x = v_0$. The horizontal axis in the graph of fig. 12.16 shows the velocities $v'_x = v_0$ as fractions of the velocity of light c and the vertical axis shows the resulting velocity v_x in the case we use Galilean velocity addition and in the case of relativistic velocity addition. As long as $v'_x = v_0 \ll c$ the results are practically identical. Up to velocities $v'_x = v_0$ of less than 10% of the velocity of light, i.e., $v'_x = v_0 \leq 0.1 \cdot c$, the Galilean value deviates by less than 1% from the relativistic result. At $v'_x = v_0 = 0.2 \cdot c$, this deviation has grown to 4%. At $v'_x = v_0 = 0.5 \cdot c$, this deviation has reached exactly 25% and from there on the Galilean velocity addition gives results exceeding the speed of light.

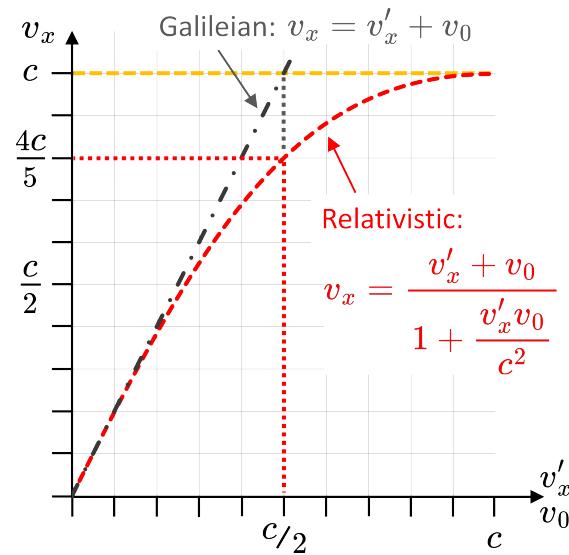


Fig. 12.16

The v_x -component in eq. (12.32) is symmetric with respect to v'_x and v_0 . That is to be expected as it should make no difference whether for example $v'_x = 0.2 \cdot c$ and $v_0 = 0.8 \cdot c$ or if $v'_x = 0.8 \cdot c$ and $v_0 = 0.2 \cdot c$. This symmetry is reflected in fig. 12.17 which shows a contour plot of v_x as a function of v'_x and v_0 , both ranging from 0 to c . Clearly, the contour map is symmetrical with respect to the diagonal where $v'_x = v_0$. Fig. 12.16 illustrates the non-linearity of relativistic velocity addition for the specific case where $v'_x = v_0$. The respective curve in fig. 12.16 corresponds to the diagonal line running from $v'_x = v_0 = 0$ to $v'_x = v_0 = c$ in fig. 12.17. The contour map also illustrates what this non-linearity of relativistic velocity addition looks like for velocities where $v'_x \neq v_0$. For $v'_x \ll c$ and $v_0 \ll c$ the contour lines run diagonally from upper left to lower right indicating that the relativistic velocity addition gives a result not much different from the Galilean result. With greater velocities the contour lines develop an increasing curvature which is indicative of the stronger deviation of the relativistic velocity addition from the Galilean result at higher velocities.

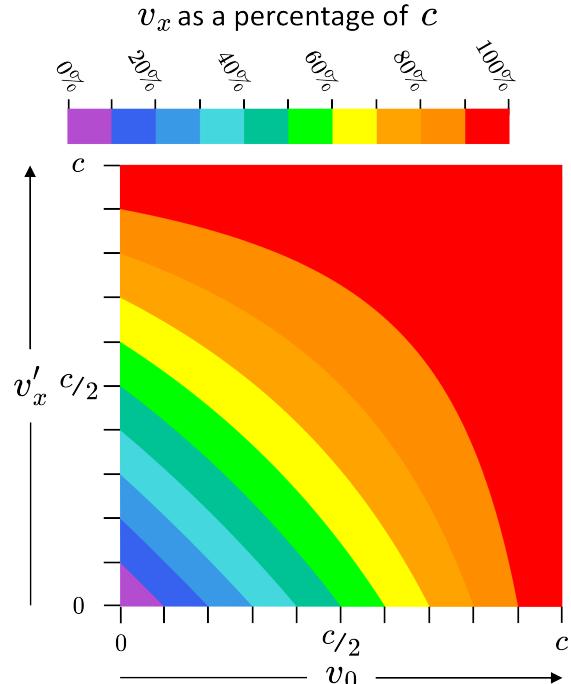


Fig. 12.17

12.3.2 Relativistic Momentum

Now that we have derived the equation governing the relativistic addition of velocities we can turn our attention to understanding how the relativistic momentum differs from the momentum of Newtonian mechanics. We will begin by revisiting the conservation of momentum discussion in section 3.2.1. There, we found that the total momentum \mathbf{P} of a closed system of point masses cannot be changed by internal forces, only external forces acting on such a closed system can change its total momentum. Without external forces the total momentum \mathbf{P} of such a closed system is therefore constant. Hence, without external forces, the rate for change of the total momentum \mathbf{P} of a closed system of point masses vanishes:

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt} \sum_i \mathbf{p}_i = \frac{d}{dt} \sum_i m_i \mathbf{v}_i \equiv 0 \quad (12.33)$$

If we move from the inertial frame at rest S to an inertial frame S' that moves relative to S with a velocity \mathbf{v}_0 then according to the Galilean velocity addition the total momentum in eq. (12.33) becomes

$$\mathbf{P}' = \sum_i m_i(\mathbf{v}_i + \mathbf{v}_0) = \mathbf{P} + v_0 \sum_i m_i$$

Because the second term with v_0 is a constant the total momentum in S' will be again a constant and the rate of change of total momentum is zero in S' just as it is in S , i.e., $\dot{\mathbf{P}}' = \dot{\mathbf{P}} = 0$. We already know that the Galilean velocity addition is not correct. Therefore, the conservation of momentum in the form discussed in section 3.2.1 cannot be correct either but can only be a good approximation for the correct relativistic form at speeds $v_0 \ll c$. But what happens to the conservation of momentum when we use instead the relativistic velocity addition from eq. (12.32) to calculate the total momentum in two inertial frames moving relative to each other with a velocity \mathbf{v}_0 ?

To understand that we will look at a collision process in the center of mass system S' and then transfer it back using relativistic velocity addition into a laboratory system of our choice. We discussed collision processes in section 3.2.2 and will consider here the special case where the two collision partners have equal mass, i.e., $m_1 = m_2 = m_0$, and both masses shall have velocities different from zero before the collision. Because of $m_1 = m_2$ we can replace the momentum vectors in fig. 3.16 with the velocity vectors as shown in fig. 12.18a for a scattering angle α . In the center of mass system S' an elastic collision process where $m_1 = m_2$ will always look as sketched in fig. 12.18a with $\mathbf{v}'_1 = -\mathbf{v}'_2$, $\tilde{\mathbf{v}}'_1 = -\tilde{\mathbf{v}}'_2$, $|\mathbf{v}'_1| = |\mathbf{v}'_2|$, $|\tilde{\mathbf{v}}'_1| = |\tilde{\mathbf{v}}'_2|$ and the scattering angle α determined by the specific scattering geometry. Fig. 12.18a is just a version of fig. 3.16a where m_1 equals m_2 and the same holds for fig. 12.18b with the difference that we have chosen there to rotate our perspective with respect to the S' coordinate axes of the center of mass system. We can always do that without changing the physics of what is happening. Here we have chosen to view the collision process from a perspective where in S' it becomes completely symmetrical with respect to the S' coordinate axes. From this perspective, sketched in fig. 12.18b, the velocity components of both masses in the x' -direction before and after the collision are the same and their

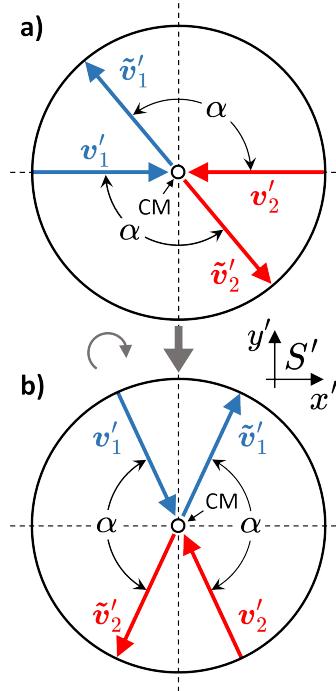


Fig. 12.18

respective y' -components are inverted by the collision. Hence, in fig. 12.18b for the velocity components in x' -direction holds

$$\tilde{v}'_{1x} = v'_{1x} \quad , \quad \tilde{v}'_{2x} = v'_{2x} \quad , \quad v'_{1x} = -v'_{2x} \quad , \quad \tilde{v}'_{1x} = -\tilde{v}'_{2x}$$

and for the velocity components in y' -direction holds

$$\tilde{v}'_{1y} = -v'_{1y} \quad , \quad \tilde{v}'_{2y} = -v'_{2y} \quad , \quad v'_{1y} = -v'_{2y} \quad , \quad \tilde{v}'_{1y} = -\tilde{v}'_{2y}$$

Using these relationships we can now calculate the total momentum in the laboratory system in x - and y -direction from the respective components in the center of mass system using the relativistic velocity addition from eq. (12.32). We will start with the x -direction where we get for P_x and \tilde{P}_x , the total momentum before and after the collision:

$$P_x = m_0(v_{1x} + v_{2x}) = m_0 \left[\frac{v'_{1x} + v_0}{1 + \frac{v'_{1x}v_0}{c^2}} + \frac{v'_{2x} + v_0}{1 + \frac{v'_{2x}v_0}{c^2}} \right]$$

Now we use $v'_{1x} = \tilde{v}'_{1x}$ and $v'_{2x} = \tilde{v}'_{2x}$ and with that P_x becomes

$$P_x = m_0 \left[\frac{\tilde{v}'_{1x} + v_0}{1 + \frac{\tilde{v}'_{1x}v_0}{c^2}} + \frac{\tilde{v}'_{2x} + v_0}{1 + \frac{\tilde{v}'_{2x}v_0}{c^2}} \right] = m_0(\tilde{v}_{1x} + \tilde{v}_{2x}) = \tilde{P}_x$$

With $\tilde{P}_x = P_x$, momentum conservation in the x -direction holds also in the laboratory system. With respect to momentum conservation in the y -direction we get with eq. (12.32) in the laboratory system for the total momentum before and after the collision P_y and \tilde{P}_y :

$$P_y = m_0(v_{1y} + v_{2y}) = m_0 \left[\frac{v'_{1y}\sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_{1y}v_0}{c^2}} + \frac{v'_{2y}\sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_{2y}v_0}{c^2}} \right]$$

Now we use $v'_{1y} = -\tilde{v}'_{1y}$, $v'_{1x} = \tilde{v}'_{1x}$, $v'_{2y} = -\tilde{v}'_{2y}$ and $v'_{2x} = \tilde{v}'_{2x}$. With that P_y becomes

$$P_y = m_0 \left[\frac{-\tilde{v}'_{1y}\sqrt{1 - v_0^2/c^2}}{1 + \frac{\tilde{v}'_{1y}v_0}{c^2}} - \frac{\tilde{v}'_{2y}\sqrt{1 - v_0^2/c^2}}{1 + \frac{\tilde{v}'_{2y}v_0}{c^2}} \right] = -m_0(\tilde{v}_{1y} + \tilde{v}_{2y}) = -\tilde{P}_y$$

Clearly, with $\tilde{P}_y = -P_y$, momentum conservation in the y -direction does not hold in the laboratory system. Evidently, when transferring from the center of mass system to a laboratory system moving with respect to the center of mass system with a velocity \mathbf{v}_0 the total momentum in the direction of \mathbf{v}_0 is an invariant under Lorentz transformation but not the total momentum component perpendicular to \mathbf{v}_0 . Consequently, the Newtonian

momentum $\mathbf{p} = m \cdot \mathbf{v}$ is not an invariant under Lorentz transformations, the definition of the relativistic momentum must be different from the Newtonian momentum so that the conservation of momentum will hold in any inertial frame and not just in one frame. With eq. (12.24) we already introduced this relativistic momentum albeit in a somewhat hand waving manner. Now we need to check if the expression of the relativistic momentum in eq. (12.24) is indeed invariant under Lorentz transformations. Using eq. (12.32) the relativistic momenta of the two masses become

$$\begin{aligned} p_{1y} &= \frac{m_0 v_{1y}}{\sqrt{1 - v_1^2/c^2}} = \frac{m_0 v_{1y}}{\sqrt{1 - (v_{1x}^2 + v_{1y}^2)/c^2}} \\ &= \frac{m_0 v'_{1y} \sqrt{1 - v_0^2/c^2}}{1 + \frac{v'_{1x} v_0}{c^2}} \left\{ 1 - \frac{1}{c^2} \left[\left(\frac{v'_{1x} + v_0}{1 + \frac{v'_{1x} v_0}{c^2}} \right)^2 + \frac{v'^2_{1y} (1 - v_0^2/c^2)}{\left(1 + \frac{v'_{1x} v_0}{c^2} \right)^2} \right] \right\}^{-1/2} \end{aligned}$$

The expression between the curly brackets we can simplify to

$$\begin{aligned} &\frac{1}{c^2 \left(1 + \frac{v'_{1x} v_0}{c^2} \right)^2} \left[\left(c + \frac{v'_{1x} v_0}{c} \right)^2 - (v'_{1x} + v_0)^2 - v'^2_{1y} (1 - v_0^2/c^2) \right] \\ &= \frac{(1 - v_0^2/c^2)}{c^2 \left(1 + \frac{v'_{1x} v_0}{c^2} \right)^2} (c^2 - v'^2_{1x} - v'^2_{1y}) = \frac{(1 - v_0^2/c^2)}{\left(1 + \frac{v'_{1x} v_0}{c^2} \right)^2} (1 - v'^2_{1y}/c^2) \end{aligned}$$

With that p_{1y} becomes

$$p_{1y} = \frac{m_0 v_{1y}}{\sqrt{1 - v_1^2/c^2}} = \frac{m_0 v'_{1y}}{\sqrt{1 - v'^2_{1y}/c^2}} = p'_{1y}$$

Evidently, the relativistic momentum is an invariant under Lorentz transformations. The relativistic momentum is the same in the laboratory system S as it is in the center of mass system S' moving relative to S with the velocity v_0 . Indeed, the relativistic momentum will be the same regardless in which inertial system we consider it and at which velocity this inertial system moves relative to other systems. Repeating the above calculation for p_{2y} we get

$$p_{2y} = \frac{m_0 v_{2y}}{\sqrt{1 - v_2^2/c^2}} = \frac{m_0 v'_{2y}}{\sqrt{1 - v'^2_{2y}/c^2}} = p'_{2y}$$

Because of $\tilde{v}'_{2y} = -\tilde{v}'_{1y}$ and $\tilde{v}'_{2x} = -\tilde{v}'_{1x}$ obviously v'^2_{1y} equals v'^2_{2y} and consequently we have $p'_{2y} = -p'_{1y}$ as well as $p_{2y} = -p_{1y}$, meaning that the total momentum before the collision in the center of mass system

$$P'_y = p'_{1y} + p'_{2y} = 0$$

and the total momentum before the collision in the laboratory system

$$P_y = p_{1y} + p_{2y} = 0$$

are both zero. Because the relativistic momentum is invariant under Lorentz transformations, the total momentum before the collision will be zero in any inertial frame. What about the momentum after the collision? Well, because all that changes in the collision sketched in fig. 12.18b is that the velocity components in the y -direction become inverted, total momentum will also be zero after the collision, in the center of mass system as well as in the laboratory system, i.e.,

$$\tilde{P}'_y = \tilde{p}'_{1y} + \tilde{p}'_{2y} = 0$$

and

$$\tilde{P}_y = \tilde{p}_{1y} + \tilde{p}_{2y} = 0$$

Again, because the relativistic momentum is invariant under Lorentz transformations, the total momentum after the collision will be zero in any inertial frame and therefore we will always have

$$P_y = \tilde{P}_y$$

in any frame and with that of course momentum conservation in the direction perpendicular to \mathbf{v}_0 . How does momentum conservation in the direction parallel to \mathbf{v}_0 change with the introduction of the relativistic momentum? Calculating the momenta p_{1x} and p_{2x} before the collision analogously to how we did that for p_{1y} and p_{2y} above, we find

$$\begin{aligned} p_{1x} &= \frac{m_0 v_{1x}}{\sqrt{1 - v_1^2/c^2}} = \frac{m_0 v_{1x}}{\sqrt{1 - (v_{1x}^2 + v_{1y}^2)/c^2}} = \frac{m_0 (v'_{1x} + v_0)}{\sqrt{1 - v_1'^2/c^2} \sqrt{1 - v_0^2/c^2}} \\ p_{2x} &= \frac{m_0 v_{2x}}{\sqrt{1 - v_2^2/c^2}} = \frac{m_0 v_{2x}}{\sqrt{1 - (v_{2x}^2 + v_{2y}^2)/c^2}} = \frac{m_0 (v'_{2x} + v_0)}{\sqrt{1 - v_2'^2/c^2} \sqrt{1 - v_0^2/c^2}} \end{aligned}$$

and therefore with $v'_{2x} = -v'_{1x}$ and $v_1'^2 = v_2'^2$ we have $p'_{2x} = -p'_{1x}$ as well as $p_{2x} = -p_{1x}$, meaning that the total momentum in x -direction before the collision in the center of mass system and the laboratory system is

$$P'_x = p'_{1x} + p'_{2x} = 0 \quad \text{and} \quad P_x = p_{1x} + p_{2x} = 0$$

In the same way we get for the total momentum after the collision in the center of mass system and the laboratory system

$$\tilde{P}'_x = \tilde{p}'_{1x} + \tilde{p}'_{2x} = 0 \quad \text{and} \quad \tilde{P}_x = \tilde{p}_{1x} + \tilde{p}_{2x} = 0$$

Like for the total momentum in the direction of the y -coordinate, we also find for the total momentum in x -direction that in any reference frame $P_x = \tilde{P}_x$. Taken together this shows with $\mathbf{P} = P_x \cdot \hat{\mathbf{x}} + P_y \cdot \hat{\mathbf{y}}$ that in all reference frames moving relative to each other at a constant speed \mathbf{v}_0 total momentum conservation, i.e., $\mathbf{P} = \tilde{\mathbf{P}}$ always holds.

The above discussion shows that the relativistic momentum ensures that conservation of momentum will hold in any inertial system and as one can quickly check, for speeds $v \ll c$ the relativistic momentum becomes the Newtonian momentum. But how does this specific equation for the relativistic momentum arise in the first place. To shed some light on that we will consider the situation sketched in fig. 12.19. An inertial frame S' shall move relative to our inertial system S which will be our reference frame. Think of a space ship passing by our position, a space station where we are waiting to get our own ship provisioned. Behind a window in the space ship S' a ball is moving straight down with a velocity $-v'_y$, hits a surface and then bounces straight up with an inverted velocity v'_y . How does this process look for us as we watch it from our platform as the space ship passes by us with a velocity u in the very moment when this happens inside it? For us, the ball is not moving straight down because in our reference frame it has a velocity component in the x -direction different from zero. What we see is the ball coming in towards the surface at an angle α and then bouncing back from the surface also at an angle α . We know that in S' the magnitude of the momentum change which the ball experiences in the process is given by

$$\Delta p'_y = 2 \cdot m_0 v'_y$$

We label the mass of the ball in S' with m_0 to indicate that it is at rest with respect to motion in the direction of u . Now what about the magnitude of the balls momentum change as judged from our position in S ? We know from eq. (12.32) that the velocity v_y in S is given by ($v_0 = u$)

$$v_y = \frac{v'_y \sqrt{1 - u^2/c^2}}{1 + \frac{v'_x u}{c^2}} \stackrel{v'_x = 0}{=} v'_y \sqrt{1 - u^2/c^2}$$

and therefore the magnitude of the balls momentum change measured by an observer like us in S must be

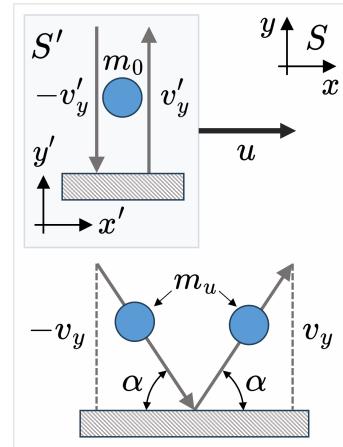


Fig. 12.19

$$\Delta p_y = 2 \cdot m_u v'_y \sqrt{1 - u^2/c^2}$$

Since in S the ball has a velocity component in the direction of the relative motion u we conveniently labeled the mass the observer in S sees moving with m_u to differentiate it from the mass m_0 in S' which only moves vertically. Now, because we require that an observer in S' and an observer in S must measure the same momentum change, i.e., Δp_y must equal $\Delta p'_y$, it follows that

$$m_u v'_y \sqrt{1 - u^2/c^2} = m_0 v'_y \quad \Rightarrow \quad m_u = \frac{m_0}{\sqrt{1 - u^2/c^2}}$$

That m_u differs from m_0 is a direct consequence of the relativistic addition of velocities. Not only the velocity component in the Newtonian momentum $\mathbf{p} = m\mathbf{v}$ equation is subject to the rules of relativistic vector addition but so is the mass. Using the dynamic mass m_u , sometimes also referred to as relativistic mass, we can formally write the relativistic momentum to look just like the Newtonian momentum as $\mathbf{p} = m_u \mathbf{v}$. With $u \rightarrow 0$ the relativistic momentum $\mathbf{p} = m_u \mathbf{v}$ becomes the Newtonian momentum $\mathbf{p} = m_0 \mathbf{v}$.

12.3.3 Relativistic Energy

The discussion of the concept of space-time in section 12.2.4 introduced the four-vector quantity

$$\mathbf{X}(x_0, x_1, x_2, x_3) = \mathbf{X}(ct, x, y, z) = \mathbf{X}(ct, \mathbf{r}) \tag{12.34}$$

whose length is conserved under Lorentz transformations. This means that for any two inertial frames S and S' we have

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

Using this four vector measuring the metric of space-time we can also define a velocity four-vector and a momentum four-vector. To obtain the velocity four vector we just have to differentiate the space-time four vector with respect to time. But with respect to what time? Every observer will measure time differently as she or he will determine the occurrence of events in their respective coordinate time. What we require, however, is a measure of the proper time that elapses between the occurrence of two events. This proper time we can measure with a clock that moves along the worldline connecting the two events. So lets do just that but in two dimensions as this is easier to illustrate.

First we will need a clock which in its rest frame will measure time such that the measurement will not depend on the motion of the reference frame itself. Fig. 12.20 shows such a clock at rest in the inertial frame S' . The clock consists of a base plate where an event E_A generates a light pulse in the horizontal base plate of the clock which then travels a distance d straight up until it hits a mirror attached to the top plate of the clock. From there the light pulse bounces straight back down traveling again the distance d until it hits the base plate where it triggers an event E_B . The time $\Delta t'$ elapsed between the two events E_A and E_B is the time it takes the light in S' to travel from the base plate, up to the mirror at the top of the clock and then back down to the base plate:

$$\Delta t' = \frac{2d}{c} \quad \text{or} \quad c\Delta t' = 2d = \Delta s \quad (*)$$

The inertial frame S' moves to the right with a uniform velocity u . From the perspective of an observer in S the light does not just travel straight up and then straight down but also travels in the direction of u . In case you wondered, yes, this situation is similar to what we discussed in fig. 12.19. However, here we are not dealing with a massive object like in fig. 12.19 but with light and for photons, the particles of light, $m_0 = 0$. For an observer in S the time elapsed between the events E_A and E_B will be Δt and with a sketch like in fig. 12.20 she or he will deduce using the Pythagorean theorem that

$$\left(\frac{u\Delta t}{2}\right)^2 + d^2 = \left(\frac{c\Delta t}{2}\right)^2 \quad \text{or} \quad (u\Delta t)^2 + (\Delta s)^2 = (c\Delta t)^2 \quad (**)$$

Fig. 12.21 shows the geometric interpretation of these equations in S and S' . With (*) and (**)

$$(c\Delta t')^2 = (\Delta s)^2 = (c\Delta t)^2 - (u\Delta t)^2$$

is the invariant space-time interval Δs and with $u\Delta t = \Delta x$ we can write the equation defining this invariant space-time interval as

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

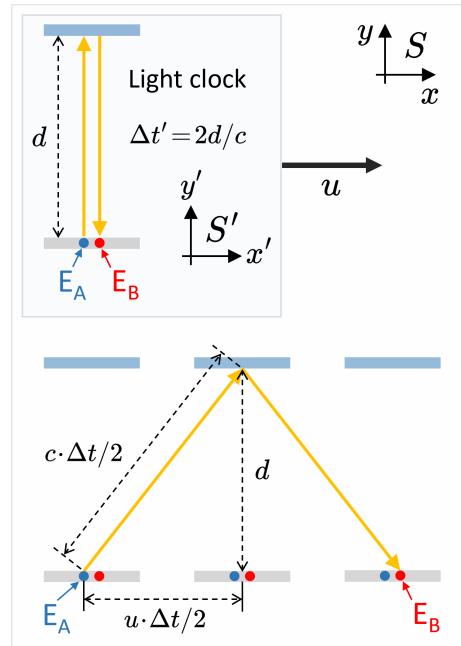


Fig. 12.20

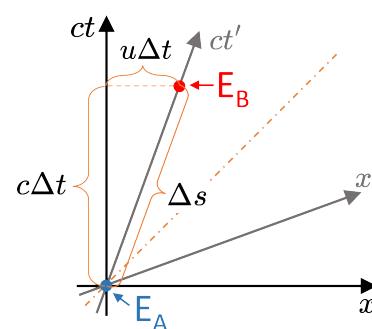


Fig. 12.21

In four-dimensional space-time this equation obviously becomes

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 + (\Delta z)^2$$

or in differential form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

The time measured in the inertial frame S' where the clock is at rest will always be the shortest time elapsed between two events. In any inertial system moving relative to S' , like in S , elapsed time will exceed what is measured in S' because from

$$(c\Delta t')^2 = (\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

follows

$$(c\Delta t)^2 = (c\Delta t')^2 + (\Delta x)^2$$

The time measured in the inertial frame in which the clock is at rest is called “proper time” and it is usually denoted by the letter τ to distinguish it from the symbols used to identify time coordinates in inertial systems. From eq. (12.9) we know that proper time and coordinate time are connected via

$$dt = \frac{d\tau}{\sqrt{1 - u^2/c^2}} = \gamma(u)d\tau \quad (12.35)$$

where u is the velocity with which the inertial frame with coordinate time t moves with respect to the frame where our light clock from fig. 12.20 is at rest. With that we can now calculate the four-velocity vector \mathbf{U} as the “proper time” derivative of the four-position vector \mathbf{X}

$$\mathbf{U} = \frac{d}{d\tau} \mathbf{X}(ct, \mathbf{r}) \quad \text{with} \quad \mathbf{r} = (x, y, z)$$

The time derivatives of the components of \mathbf{U} are

$$\frac{d(ct)}{d\tau} = c\gamma, \quad \frac{d\mathbf{r}}{d\tau} = \frac{dr}{dt} \frac{dt}{d\tau} = \mathbf{u}\gamma \quad \text{with} \quad \mathbf{u} = \frac{d\mathbf{r}}{dt}$$

and with that the four-velocity vector becomes

$$\mathbf{U} = \gamma(c, \mathbf{u}) = \frac{1}{\sqrt{1 - u^2/c^2}}(c, \mathbf{u}) \quad (12.36)$$

Now that we have this four-velocity vector, it is easy to determine the four-momentum vector. Momentum is always the product of velocity times mass. But which mass must we use to multiply the four-velocity vector with, the rest mass m_0 or the dynamic mass $m = \gamma(u)m_0$? Just as we determined the four-velocity using the “proper time” derivative

of the four-position vector we must calculate the four-momentum vector also in the rest frame of our light clock. That of course means we have to multiply \mathbf{U} with m_0 . Doing that we get the four-momentum vector

$$\mathbf{P} = m_0 \mathbf{U} = m_0 \gamma(c, \mathbf{u}) = \frac{m_0}{\sqrt{1 - u^2/c^2}}(c, \mathbf{u}) \quad (12.37)$$

The space component \mathbf{p} of this four-momentum vector is

$$\mathbf{p} = \frac{m_0 \mathbf{u}}{\sqrt{1 - u^2/c^2}} \quad (12.38)$$

Without giving a thought as to what the relativistic momentum \mathbf{p} must look like we get the correct expression. This happened simply by first taking the “proper time” derivative, i.e., the time derivative of the four-position vector \mathbf{X} in the reference frame where the light clock with which we measure time is at rest, thereby obtaining the four-velocity vector \mathbf{U} ; and second, by multiplying \mathbf{U} with m_0 , the mass in the reference frame where the light clock is at rest to obtain the four-momentum vector \mathbf{P} . Before discussing the time component of the four-momentum vector we will first calculate the four-force vector \mathbf{F} which we do by taking the “proper time” derivative of \mathbf{P}

$$\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = \frac{d\mathbf{P}}{dt} \frac{dt}{d\tau} = \gamma \frac{d}{dt}(m_0 \gamma c, m_0 \gamma \mathbf{u})$$

The space component of this four-force vector is $\gamma \cdot \mathbf{F}$ where the force \mathbf{F} is given by

$$\mathbf{F} = \frac{d}{dt}(m_0 \gamma \mathbf{u}) = \frac{d\mathbf{p}}{dt} \quad (12.39)$$

The time component of the four-force vector we will calculate explicitly so we can rewrite it in a way that gives us a better physics understanding of this component. To do that we first have to calculate the time derivative of γ :

$$\dot{\gamma} = \frac{d}{dt} \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{u/c^2}{\left(1 - u^2/c^2\right)^{3/2}} \dot{u} = \frac{u/c^2}{\left(1 - u^2/c^2\right)^{3/2}} \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{u} = \gamma^3 \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{c^2}$$

Next we consider the time derivative of the space component \mathbf{p} of the four-momentum vector

$$\frac{d\mathbf{p}}{dt} = m_0(\dot{\gamma}\mathbf{u} + \gamma\dot{\mathbf{u}}) = m_0\left(\gamma^3 \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{c^2} \mathbf{u} + \gamma\dot{\mathbf{u}}\right) = m_0\gamma\dot{\mathbf{u}}\left(\underbrace{\gamma^2 \frac{u^2}{c^2} + 1}_{\gamma^2}\right) = m_0\gamma^3\dot{\mathbf{u}}$$

Now we can express $\dot{\mathbf{u}}$ through the space component of the four-force vector $\gamma\mathbf{F}$ from eq. (12.39)

$$\dot{\mathbf{u}} = \frac{\dot{\mathbf{p}}}{m_0\gamma^3} = \frac{\mathbf{F}}{m_0\gamma^3}$$

With that we can now write the time component of the four-force vector as

$$m_0\dot{\gamma}c = m_0c\gamma^3 \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{c^2} = m_0\gamma^3 \frac{\mathbf{u}}{c} \frac{\mathbf{F}}{m_0\gamma^3} = \frac{\mathbf{F} \cdot \mathbf{u}}{c} \quad (12.40)$$

Inserting this into the four-force vector we finally get for \mathbf{F} :

$$\mathbf{F} = \gamma \left(\frac{\mathbf{F} \cdot \mathbf{u}}{c}, \mathbf{F} \right) = \frac{1}{\sqrt{1 - u^2/c^2}} \left(\frac{\mathbf{F} \cdot \mathbf{u}}{c}, \mathbf{F} \right) \quad (12.41)$$

The time component of the four-force vector looks familiar as we have seen the expression $\mathbf{F} \cdot \mathbf{u}$ before in section 4.1.2 where we discussed the concepts of work and power. There we saw that $\mathbf{F} \cdot \mathbf{u}$ equates the rate of change in the work required to move a massive object from one position to another. The work W expended to achieve that translates into kinetic and potential energy changes effected on the object:

$$\mathbf{F} \cdot \mathbf{u} = \frac{dW}{dt} = \frac{dE}{dt} = \frac{d(T + U)}{dt}$$

With that we can rewrite eq. (12.40) to express this time derivative of the four-momentum time component as

$$\frac{dm_0\gamma c}{dt} = \frac{1}{c} \frac{dE}{dt} \quad \Rightarrow \quad m_0\gamma c = \frac{E}{c}$$

This not only shows us that the time component of the four-momentum can be expressed as the total energy E divided by the speed of light c . It also shows us that the total energy of a point mass is given by

$$E = \frac{m_0c^2}{\sqrt{1 - u^2/c^2}} = mc^2 \quad (12.42)$$

Because the total energy E is the sum of kinetic energy T and potential energy U it follows from eq. (12.42) with eq. (12.28) that

$$E = mc^2 = T + U = mc^2 - m_0c^2 + U$$

and therefore the potential energy of a point mass is connected to its rest mass via

$$U = m_0c^2$$

This equation implies that a change in potential energy translates into a change of rest mass. Actually this is to be expected for a closed system. We know that the total energy of a closed system, that is a system not subject to external forces, is constant, i.e., $E = T + U = \text{const}$. That must also hold true in the relativistic case where it

translates into $E = mc^2 = \text{const}$, i.e., the total relativistic mass must remain constant. Because according to eq. (12.28) a change in kinetic energy equates to a change in mass it follows that the rest mass must change. Now that we know that the total relativistic energy is given by $E = mc^2$ we see that eq. (12.28) really reads $T = E - m_0c^2$ which means of course that the potential energy must be $U = m_0c^2$ and as the potential energy changes so must the rest mass m_0 .

Now that we know what the relativistic momentum and the relativistic total energy are, we can ask how they are connected. In Newtonian mechanics we know that for point mass m_0 moving with a velocity \mathbf{v} applies:

$$\mathbf{p} = m_0\mathbf{v} \quad \text{and} \quad T = \frac{m_0\mathbf{v}^2}{2} \quad \Rightarrow \quad T = \frac{\mathbf{p}^2}{2m_0}$$

To find the corresponding relationship between the relativistic momentum and the relativistic kinetic energy we first note that from eq. (12.42) follows $m = E/c^2$, hence we can simply write the relativistic momentum as

$$\mathbf{p} = m\mathbf{v} = \frac{E}{c^2}\mathbf{v}$$

We already found that the time component of the four-momentum vector can be expressed as $m_0\gamma c = E/c$. Therefore we can rewrite the four-momentum vector from eq. (12.37):

$$\mathbf{P} = m_0\gamma(c, \mathbf{v}) = \left(\frac{E}{c}, \mathbf{p}\right) = \left(\frac{E}{c}, \frac{E}{c^2}\mathbf{v}\right) \quad (12.43)$$

Four-vectors are invariant under Lorentz transformations which means that their length does not change when moving between inertial frames. Therefore, the quantity

$$\mathbf{P}^2 = \frac{E^2}{c^2} - \mathbf{p}^2$$

must be a constant. To determine this constant we just have to subtract $c^2\mathbf{p}^2$ from E^2 to obtain with eq. (12.38) and eq. (12.42)

$$E^2 - c^2\mathbf{p}^2 = \frac{m_0^2 c^4}{1 - v^2/c^2} - c^2 \frac{m_0^2 \mathbf{v}^2}{1 - v^2/c^2} = \frac{m_0^2 c^4}{1 - v^2/c^2} \left(1 - \frac{\mathbf{v}^2}{c^2}\right) = (m_0 c^2)^2$$

This result tells us that relativistic total energy and relativistic momentum are connected through the equation

$$E^2 = \mathbf{p}^2 c^2 + (m_0 c^2)^2 \quad (12.44)$$

Now we replace E with $E = T + U = T + m_0 c^2$ giving us

$$(T + m_0 c^2)^2 = \mathbf{p}^2 c^2 + (m_0 c^2)^2$$

from which follows the equation connecting relativistic kinetic energy and momentum (with always $T \geq 0$ we must take the positive root)

$$T = -m_0 c^2 + \sqrt{\mathbf{p}^2 c^2 + (m_0 c^2)^2} \quad (12.45)$$

So far we have restricted the discussion to point masses, i.e., objects which posses a rest mass m_0 . Clearly, equations eq. (12.38) for the relativistic momentum \mathbf{p} and eq. (12.42) for the total energy E cannot apply for massless particles such as photons for which not only the numerator is zero because of $m_0 = 0$ but also the denominator is zero as they move with the speed of light. However, what also applies for massless particles is the relationship between relativistic momentum and relativistic energy which we just discussed. For a particle with mass $m_0 = 0$, eq. (12.44) reduces to $E = p \cdot c$. Therefore we can ascribe to massless particles the momentum

$$p = \frac{E}{c}$$

Because of the relativistic energy mass relationship $E = mc^2$ in eq. (12.42) we can also formally ascribe a mass to a particle with $m_0 = 0$ with which we can rewrite its momentum as

$$p = mc$$

For the four-momentum vector of a massless particle we then can just use the expression derived in eq. (12.43) with an added constraint for the velocity \mathbf{v}

$$\mathbf{P} = \left(\frac{E}{c}, \frac{E}{c^2} \mathbf{v} \right) \quad \text{where} \quad v_x^2 + v_y^2 + v_z^2 = c^2$$

Appendix

International System of Units (SI)¹

Base Units

Base quantity	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	Ampere	A
Temperature (thermodynamic)	Kelvin	K
Substance amount	mole	mol
Luminous intensity	candela	cd

Derived Units (selection)

Derived quantity	Unit	Symbol	Dimension
Frequency	Hertz	Hz	s^{-1}
Force	Newton	N	$\text{kg}\cdot\text{m}\cdot\text{s}^{-2}$
Pressure	Pascal	Pa	$\text{N}\cdot\text{m}^{-2}$
Energy, work, heat	Joule	J	$\text{N}\cdot\text{m}$
Power	Watt	W	$\text{J}\cdot\text{s}^{-1}$
Electric charge	Coulomb	C	$\text{A}\cdot\text{s}$
Electric voltage	Volt	V	$\text{W}\cdot\text{A}^{-1}$
Electric resistance	Ohm	Ω	$\text{V}\cdot\text{A}^{-1}$
Electric capacitance	Farad	F	$\text{C}\cdot\text{V}^{-1}$
Electric conductance	Siemens	S	$\text{A}\cdot\text{V}^{-1}$
Magnetic flux	Weber	Wb	$\text{J}\cdot\text{A}^{-1}$
Magnetic flux density	Tesla	T	$\text{Wb}\cdot\text{m}^{-2}$
Inductance	Henry	H	$\text{V}\cdot\text{s}\cdot\text{A}^{-1}$
Angle	Radian	rad	—
Solid angle	Steradian	sr	—
Temperature relative to 273.15 K	Celsius	$^{\circ}\text{C}$	K
Luminous flux	Lumen	lm	$\text{cd}\cdot\text{sr}$
Illuminance	Lux	lx	$\text{lm}\cdot\text{m}^{-2}$
Radioactivity	Becquerel	Bq	s^{-1}

¹SI is the abbreviation for Système international d'unités

Derived quantity	Name	Symbol	Dimension
Absorbed dose	Gray	Gy	$\text{J}\cdot\text{kg}^{-1}$
Equivalent dose	Sievert	Sv	$\text{A}\cdot\text{kg}^{-1}$
Catalytic activity	Katal	kat	$\text{mol}\cdot\text{s}^{-1}$

SI Prefixes for Powers of Ten

Power	Prefix	Symbol	Power	Prefix	Symbol
10^{12}	tera	T	10^{-2}	centi	c
10^9	giga	G	10^{-3}	milli	m
10^6	mega	M	10^{-6}	micro	μ
10^3	kilo	k	10^{-9}	nano	n
10^2	hecto	h	10^{-12}	pico	p
10^1	deca	da	10^{-15}	femto	f
10^{-1}	deci	d	10^{-18}	atto	a

Unit Conversions

Length and area conversions:

Unit of measurement	Symbol	Conversion
Parallax second (Parsec)	pc	$1 \text{ pc} = 3.084 \cdot 10^{16} \text{ m}$
Light-year	ly	$1 \text{ ly} = 9.46 \cdot 10^{15} \text{ m}$
Astronomic unit	AU	$1 \text{ AE} = 1.496 \cdot 10^{11} \text{ m}$
English or US mile	mi	$1 \text{ mi} = 1.6093 \text{ km}$
Yard	yd	$1 \text{ yd} = 0.9144 \text{ m}$
Inch	in	$1 \text{ in} = 2.540 \text{ cm}$
Angstrom	\AA	$1 \text{ \AA} = 10^{-10} \text{ m}$
Fermi	F	$1 \text{ F} = 10^{-15} \text{ m}$
Barn	b	$1 \text{ b} = 10^{-28} \text{ m}^2$

Angle conversions:

Unit of measurement	Symbol	Conversion
Radian	rad	$1 \text{ rad} = 360^\circ / 2\pi = 57.2958^\circ$
Arc minute	$1'$	$1' = 1^\circ / 60 = 2.91 \cdot 10^{-4} \text{ rad}$
Arc second	$1''$	$1'' = 1^\circ / 3600 = 4.85 \cdot 10^{-6} \text{ rad}$

Time conversions:

Unit of measurement	Symbol	Conversion
Minute	min	1 min = 60 s
Hour	h	1 h = 3 600 s
Day	d	1 d = 86 400 s
Year	a	1 a = 365.26 d = $3.156 \cdot 10^7$ s

Mass conversions:

Unit of measurement	Symbol	Conversion
English or US pound	lb	1 lb = 0.4536 kg
Atomic mass unit	amu	1 amu = $1.6605 \cdot 10^{-27}$ kg

Force conversions:

Unit of measurement	Symbol	Conversion
Dyne	dyn	1 dyn = 10^{-5} N
Kilopond	kp	1 kp = $1 \text{ kg} \cdot g = 9.81$ N

Energy conversions:

Unit of measurement	Symbol	Conversion
Calorie	cal	1 cal = 4.187 J
Ergon	erg	1 erg = 10^{-7} J
Electronvolt	eV	1 eV = $1.602 \cdot 10^{-19}$ J

Mass-Energy equivalents:

Reference measure	Relation	Equivalence
Electron mass	$E = mc^2$	$m_e c^2 = 0.511 \text{ MeV}$
Proton mass	$E = mc^2$	$m_p c^2 = 1.6726 \text{ MeV}$
Neutron mass	$E = mc^2$	$m_n c^2 = 1.6749 \text{ MeV}$
Atomic mass	$E = mc^2$	$1 \text{ amu} \equiv 931.5 \text{ MeV}$
Temperature	$E = k_B T$	$1 \text{ K} \equiv 0.8618 \cdot 10^{-4} \text{ eV}$
Frequency	$E = hf$	$1 \text{ Hz} \equiv 4.136 \cdot 10^{-15} \text{ eV}$

Power conversions:

Unit of measurement	Symbol	Conversion
Horsepower	hp	1 hp = 735.5 W

Temperature conversions:

Unit of measurement	Symbol	Conversion
Celsius	$^{\circ}\text{C}$	$^{\circ}\text{C} = \text{K} - 273.15$
Fahrenheit	$^{\circ}\text{F}$	$^{\circ}\text{F} = 32 + (\text{K} - 273.15) \cdot 9/5$

Pressure conversions:

Unit of measurement	Symbol	Conversion
Bar	bar	$1 \text{ bar} = 10^5 \text{ Pa}$
Atmosphere (standard)	atm	$1 \text{ atm} = 1.013 \text{ bar} = 1.013 \cdot 10^5 \text{ Pa}$
Atmosphere (technical)	at	$1 \text{ at} = 0.981 \text{ bar} = 9.81 \cdot 10^4 \text{ Pa}$
Torr	Torr	$1 \text{ Torr} = 1.333 \text{ mbar} = 1.333 \text{ hPa}$
Millimeter of mercury	mm Hg	$1 \text{ mm Hg} \equiv 1 \text{ Torr}$

Fundamental Constants

Constant	Symbol	Value	Dimension
Elementary charge	e	$1.6022 \cdot 10^{-19}$	C
Vacuum speed of light	c	$2.9979 \cdot 10^8$	$\text{m} \cdot \text{s}^{-1}$
Planck constant	h	$6.6262 \cdot 10^{-34}$	$\text{J} \cdot \text{s}$
Reduced Planck constant	$\hbar = h/2\pi$	$1.0546 \cdot 10^{-34}$	$\text{J} \cdot \text{s}$
Vacuum magnetic permeability	μ_0	$1.2566 \cdot 10^{-6}$	$\text{N} \cdot \text{A}^{-2}$
Vacuum electric permittivity	$\epsilon_0 = 1/\mu_0 c^2$	$8.8542 \cdot 10^{-12}$	$\text{C}^2 \cdot \text{m}^{-2} \cdot \text{N}^{-1}$
Coulomb constant	$k_e = 1/4\pi\epsilon_0$	$8.9876 \cdot 10^9$	$\text{N} \cdot \text{m}^2 \cdot \text{C}^{-2}$
Electron rest mass	m_e	$9.1094 \cdot 10^{-31}$	kg
Proton rest mass	m_p	$1.6726 \cdot 10^{-31}$	kg
Neutron rest mass	m_n	$1.6749 \cdot 10^{-31}$	kg
Bohr radius	a_0	$5.2918 \cdot 10^{-11}$	m
Classical electron radius	r_e	$2.8179 \cdot 10^{-15}$	m
Avogadro number	N_A	$6.0221 \cdot 10^{23}$	mol^{-1}
Boltzmann constant	k_B	$1.3806 \cdot 10^{-23}$	$\text{J} \cdot \text{K}^{-1}$
Gas constant	$R = N_A k_B$	8.3145	$\text{J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Gravitational constant	G	$6.6743 \cdot 10^{-11}$	$\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Gravity acceleration (standard)	g_0	9.80665	$\text{m} \cdot \text{s}^{-2}$
Gravity acceleration (50°)	g	9.810	$\text{m} \cdot \text{s}^{-2}$
Gravity acceleration (poles)	g	9.832	$\text{m} \cdot \text{s}^{-2}$
Gravity acceleration (equator)	g	9.780	$\text{m} \cdot \text{s}^{-2}$

Tab. A.1: The periodic table of elements. The atomic weight is given in atomic mass units. If instead of the atomic weight a number in square brackets is given this number refers to the most stable isotope of an element.

		18																	
		2 Ne																	
		helium																	
		4.0026																	
		1s ²																	
1	H	hydrogen	1.0080	1s ¹															
2	Be	beryllium	9.0122	1s ² 2s ²															
3	Li	lithium	6.94	1s ² 2s ¹															
11	Na	sodium	22.990	(Ne) 3s ¹															
3	Mg	magnesium	24.305	(Ne) 3s ²															
19	K	potassium	39.098	(Ar) 4s ¹															
4	Ca	calcium	40.078	(Ar) 4s ²															
20	Sc	scandium	41.956	(Ar) 4s ²															
21	Ti	titanium	47.887	(Ar) 3d ¹ 4s ²															
22	V	vanadium	50.942	(Ar) 3d ² 4s ²															
23	Cr	chromium	51.956	(Ar) 3d ³ 4s ²															
24	Mn	manganese	54.938	(Ar) 3d ⁴ 4s ²															
25	Fe	iron	55.845	(Ar) 3d ⁵ 4s ²															
26	Co	cobalt	58.933	(Ar) 3d ⁶ 4s ²															
27	Ni	nickel	58.693	(Ar) 3d ⁷ 4s ²															
28	Cu	copper	63.546	(Ar) 3d ¹⁰ 4s ²															
29	Zn	zinc	65.38	3d ¹⁰ 4s ²															
30	Ga	gallium	69.723	(Ar) 3d ¹⁰ 4s ²															
31	Ge	germanium	72.630	(Ar) 3d ¹⁰ 4s ²															
32	As	arsenic	74.922	(Ar) 3d ¹⁰ 4s ²															
33	Sb	antimony	78.971	(Ar) 3d ¹⁰ 4s ²															
34	Se	selenium	78.971	(Ar) 3d ¹⁰ 4s ²															
35	Br	bromine	79.904	(Ar) 3d ¹⁰ 4s ²															
36	Kr	krypton	83.798	(Ar) 3d ¹⁰ 4s ²															
54	Xe	xenon	131.29	(Xe)															
55	Cs	cesium	132.91	(Xe) 6s ¹															
56	Ba	barium	137.33	(Xe) 6s ²															
57	Rb	rubidium	167.62	5s ²															
58	Sr	strontium	188.906	4d ¹ 5s ²															
59	Y	yttrium	183.84	4f ¹ 5d ¹ 6s ²															
60	Tc	technetium	191.924	4d ⁵ 5s ²															
61	Pt	platinum	196.07	5d ⁹ 6s ²															
62	Mo	molybdenum	197.95	4d ⁵ 5s ¹															
63	Nb	niobium	92.906	4d ⁵ 5s ¹															
64	Ta	tantalum	180.95	(Xe) 5d ³ 6s ²															
65	W	tungsten	183.84	4f ¹ 5d ⁴ 6s ²															
66	Re	rhodium	190.22	(Xe) 5d ⁶ 6s ²															
67	Os	osmium	196.07	(Xe) 5d ⁷ 6s ²															
68	Ir	iridium	195.97	(Xe) 5d ⁸ 6s ²															
69	Pt	platimum	196.97	(Xe) 5d ⁹ 6s ²															
70	Os	osmium	196.97	(Xe) 5d ¹⁰ 6s ²															
71	Ds	darmstadtium	(277)	5f ¹⁴ 6d ⁷ 7s ²															
72	Mt	meitnerium	(281)	5f ¹⁴ 6d ⁷ 7s ²															
73	Ta	taurium	(270)	(Rn) 5f ¹⁴ 6d ⁷ 7s ²															
74	W	bohrium	(268)	(Rn) 5f ¹⁴ 6d ⁷ 7s ²															
75	Sg	seaborgium	(270)	(Rn) 5f ¹⁴ 6d ⁷ 7s ²															
76	Bh	bogertium	(270)	(Rn) 5f ¹⁴ 6d ⁷ 7s ²															
77	Hs	hsamium	(269)	(Rn) 5f ¹⁴ 6d ⁷ 7s ²															
78	Ts	tsamium	(277)	(Rn) 5f ¹⁴ 6d ⁷ 7s ²															
79	La	lanthanum	138.91	(Xe) 5d ¹ 6s ²															
80	Ce	cerium	140.12	(Xe) 4f ¹ 6s ²															
81	Pr	praseodymium	140.91	(Xe) 4f ² 6s ²															
82	Nd	neodymium	144.24	(Xe) 4f ³ 6s ²															
83	Pm	promethium	145.0	(Xe) 4f ⁴ 6s ²															
84	Sm	samarium	150.36	(Xe) 4f ⁵ 6s ²															
85	Eu	europtium	151.36	(Xe) 4f ⁶ 6s ²															
86	Tb	terbium	158.93	(Xe) 4f ⁷ 6s ²															
87	Dy	diopsmium	162.50	(Xe) 4f ⁸ 6s ²															
88	Ho	holmium	164.93	(Xe) 4f ⁹ 6s ²															
89	Fm	ferdium	173.05	(Xe) 4f ¹⁰ 6s ²															
90	Th	thorium	232.04	(Rn) 5f ¹ 6d ¹ 7s ²															
91	Pa	protactinium	231.04	(Rn) 5f ⁶ 6d ¹ 7s ²															
92	U	neptunium	237.0	(Rn) 5f ⁶ 6d ¹ 7s ²															
93	Np	plutonium	244.0	(Rn) 5f ⁶ 6d ¹ 7s ²															
94	Am	americium	243.0	(Rn) 5f ⁷ 6s ²															
95	Pu	curium	247.0	(Rn) 5f ⁷ 6s ²															
96	Cm	curium	247.0	(Rn) 5f ⁷ 6s ²															
97	Bk	berkelium	247.0	(Rn) 5f ⁷ 6s ²															
98	Cf	einsteinium	251.0	(Rn) 5f ⁷ 6s ²															
99	Es	eserrium	252.0	(Rn) 5f ⁷ 6s ²															
100	Fm	fermium	257.0	(Rn) 5f ⁷ 6s ²															
101	Md	mekdeleium	258.0	(Rn) 5f ⁷ 6s ²															
102	No	nobelium	259.0	(Rn) 5f ⁷ 6s ²															
103	Lr	lawrencium	262.0	(Rn) 5f ⁷ 6s ²															

Period number
Group number

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