TECHNICAL MECHANICS

An Introduction

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Preface

Mechanics is the oldest part of the body of knowledge we call today physics and there are many good reasons to refer to the part of mechanics which is known as technical mechanics (in modern times also labeled as engineering mechanics) as the oldest part of mechanics. The problems that technical mechanics dealt with throughout its history were always of a practical nature, such as the stability of buildings, moving loads weighing tons or the use of hydro power. What has changed over time are the tools of the trade which are available to today's builders and engineers. On the one hand, there is the edifice of classical mechanics, which provides the theoretical foundation for the treatment of problems in technical mechanics. The origin of classical mechanics lies in the scientific revolution of the sixteenth and seventeenth centuries, and the building of this branch of theoretical physics seemed so complete towards the end of the nineteenth century that some physicists at the time thought there was nothing new to discover. This was not long before Albert Einstein published his thoughts on relativistic mechanics in 1905 and just a few more years before the development of quantum mechanics which began in earnest in the 1920's. Our modern world would not exist without the technology built on the insights of relativistic mechanics and quantum mechanics, but these areas of physics play no role in what concerns technical mechanics.

The field of technical mechanics can be subdivided in different ways. For example, by the type of objects being treated and their corresponding properties. Objects can exist in three physical states of matter: solid, liquid or gaseous. In technical mechanics, a distinction is made between the treatment of solid objects and fluida (the plural form of the Latin word for fluid), the latter being comprised of liquid and gaseous bodies. Solid objects can in turn be rigid, elastic or plastic and one then speaks of the statics of rigid bodies and elastostatics. The latter is also referred to as the science of material strength and usually also includes plastic deformations. In the case of fluida, liquids are incompressible and gaseous bodies are compressible. In addition, liquid bodies can also flow without friction or possess viscosity. As for gaseous bodies, they can for example behave as ideal or real gases. Liquid fluida are treated within the framework of hydromechanics in the sub-fields of hydrostatics and hydrodynamics. The same applies to gaseous fluida which are treated accordingly in aerostatics and aerodynamics within the framework of aeromechanics.

Another way of looking at technical mechanics is to differentiate it according to movement. In this respect, a fundamental distinction is made between kinematics, which describes the movement of a body without taking forces into account, and dynamics, which describes the movement of a body under the influence of forces. In kinematics one does not need to know the forces, it is completely sufficient to be able to correctly describe the trajectory of the movement of a body. In dynamics one must know the forces at work. If the respective forces are associated with the movement of an object, then one speaks of kinetics, if forces are described in a resting or uniformly moving system, then one speaks of statics.

In the literature on technical mechanics, one usually finds a mixture of these two classifications according to the state of aggregation and movement. This is no different here either. Because it is not possible to cover the entirety of technical mechanics in an introductory text as the one at hand which roughly contains the material for a one-semester lecture course, a selection has to be made. One has to decide what to include, what to leave out, and to which depth any chosen topic can be covered. Of course, this selection depends on the students for whom such an introductory course is intended. In the broadest sense, however, these students are prospective engineers in civil or mechanical engineering or physics engineers. The material in this volume is primarily aimed at the latter, but with slight cutbacks and additions it can also serve as a basis for students of civil engineering or mechanical engineering. The first part of the seven chapters of this volume deals with the forces acting on a point mass, the statics of rigid bodies and elastostatics within the framework of statics. In the second part, within the framework of dynamics, the kinetics of a point mass, the dynamics of rigid bodies, systems of point masses and continuum mechanics are treated. In the appendix, some basic mathematical aspects which have been used at various points in this volume are explained for quick reference.

Stefan Wurm

Albany, New York
September 2023

Part I

Statics

1. Forces Acting on a Point Mass

Technical mechanics uses two basic models to describe the physical properties of real bodies. One model is that of the point mass and the other one is that of the rigid body. In the model of the point mass, also sometimes referred to as mass point, one imagines the entire mass of a body combined in one point, its dimension being assumed to be so small that its location can be defined by the position of a point. In general, this point is the center of mass of the body.

If one looks for example at a car, then the volume or the dimensions of the car play no role in many physical questions, but its mass certainly does. The decisive variables when calculating the braking distance, i.e., the length of the distance after which a car comes to a standstill when braking hard, are the mass of the car and of course the speed with which it was moving when the braking process started; in this case, the dimensions of the car are irrelevant. However, if emergency braking is no longer able to stop a collision, then the dimensions of the automobile play a very important role in the event of an impact, especially for the safety of the passengers. Most of us are familiar with a less dramatic example where the dimensions of a vehicle's body are important - parking a car.

Definition

Point mass: For the description of the motion of a point mass, its extension can be completely neglected. The motion of a point mass is described solely by tracing its position as a function of time.

The model of the point mass is always advantageous when the dimensions of the body, i.e., its shape and volume, play no role in understanding the movement of a body. Examples of this are the trajectories of planets, i.e., their orbits. A more obvious example for many is the trajectory followed by a ball thrown in a game or sport. In these cases, the impact of a force on a body is described by the respective force impact on the point mass corresponding to it.

1.1 Kinematics

The sole objective of kinematics is to describe the movement of a point mass, nothing more. No inquiry whatsoever is made with respect to what causes a point mass to move. The coordinate system in which the movement of a point mass is being described is itself a rigid body.

Fig. 1.1 shows such a rigid coordinate system, a Cartesian coordinate system, in which the position of a point is being described by its position vector $\boldsymbol{r}$:

$$\boldsymbol{r} = \sum_{i=1}^{3} \boldsymbol{e}_i x_i$$

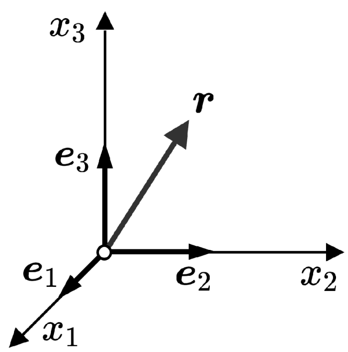

Fig. 1.1

The spatial dimension of which the rigid coordinate system takes measure and in which the trajectory of a point mass is being described by its position vector $\boldsymbol{r}$, is the first basic quantity of mechanics. Evidently, that presupposes the existence of "length rulers". The dimension of a coordinate x_i is its length, i.e., $x_i = [x_i]$, and this length is measured in units of meters (one meter $= 1\,\mathrm{m}$). In general, the coordinates of a point mass are a function of time. Hence

$$\boldsymbol{r} = \boldsymbol{r}(t) \qquad \text{or respectivley} \qquad x_i = x_i(t)$$

The second basic quantity of mechanics is thus the temporal dimension. Measuring time requires the existence of "clocks". The dimension of time is its duration, i.e., $t = [t]$, and this duration is measured in units of seconds (one second $= 1\,\mathrm{s}$).

With rulers for measuring the spatial dimension and clocks for measuring the temporal dimension, one can calculate the velocity $\boldsymbol{v}$ of a point mass relative to the reference system $\boldsymbol{e}_i$ from fig. 1.1. That of course presupposes that the position vector $\boldsymbol{r} = \boldsymbol{r}(t)$ in fig. 1.2 is a continuous function of time.

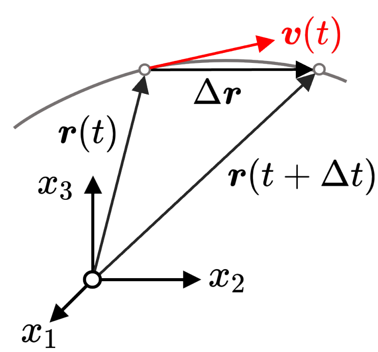

Fig. 1.2

$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} = \dot{\boldsymbol{r}}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

and

$$\dot{\boldsymbol{r}}(t) = \sum \boldsymbol{e}_i \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \sum \boldsymbol{e}_i \dot{x}_i$$

The magnitude of the velocity is the length of the vector $\boldsymbol{v}$, i.e.,

$$v = |\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} = \frac{\mathrm{d}s}{\mathrm{d}t}$$

where $\mathrm{d}s$ is the differential arc element of the trajectory. Instead of measuring velocity with respect to a specific coordinate system, $v(t) = \mathrm{d}s/\mathrm{d}t$ measures the magnitude of the velocity with respect to the trajectory itself, as the differential arc element $\mathrm{d}s$ which is run through in the time interval $\mathrm{d}t$.

Fig. 1.3 illustrates the case where the reference system O, in which the movement of a point mass is being considered, is itself moving with a constant speed relative to another reference system O'. If the origin O of the moving reference system, measured from the system at rest, has the position vector $\boldsymbol{r}_0$, then the position vector $\boldsymbol{r}_m$ of the point mass in the stationary reference system O' follows from the position vector $\boldsymbol{r}$ in the moving reference system via

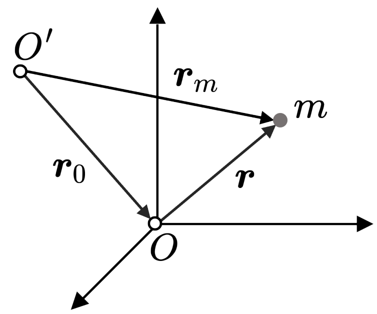

Fig. 1.3

$$\boldsymbol{r}_m - \boldsymbol{r}_0 = \boldsymbol{r}$$

$$\dot{\boldsymbol{r}}_m = \dot{\boldsymbol{r}}_0 + \sum_{i=1}^{3} \left(\dot{\boldsymbol{e}}_i x_i + \boldsymbol{e}_i \dot{x}_i\right)$$

For $\boldsymbol{r}_0$ the following applies

$$\boldsymbol{r}_0 = \boldsymbol{v}_0 \cdot t$$

where $\boldsymbol{v}_0$ is the constant speed with which the reference system O moves relative to the reference system O'.

The acceleration $\boldsymbol{a}(t)$ of a point mass (fig. 1.4) is given by the change in its velocity during a time interval Δt from a velocity $\boldsymbol{v}(t)$ to $\boldsymbol{v}(t+\Delta t)$:

$$\boldsymbol{a}(t) = \frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \dot{\boldsymbol{v}}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t+\Delta t) - \boldsymbol{v}(t)}{\Delta t}$$

and

$$\boldsymbol{a}(t) = \frac{\mathrm{d}^2\boldsymbol{r}(t)}{\mathrm{d}t^2} = \ddot{\boldsymbol{r}}(t)$$

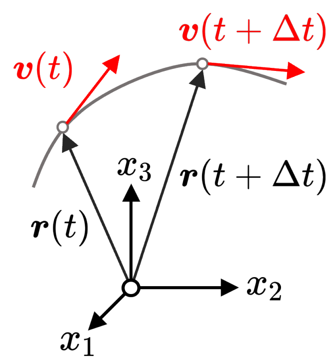

Fig. 1.4

Because of $\dot{\boldsymbol{v}}_0 = 0$, the acceleration is identical in reference systems which, as illustrated in fig. 1.3, move relative to each other with a constant speed $\boldsymbol{v}_0$.

1.2 Dynamics

The third basic quantity of technical mechanics is the (inertial) mass, which physicists measure in units of kilograms (one kilogram = 1 kg). In contrast to the dimensions of space and time, the determination of a quantity of mass does not presuppose something like the existence of a mass ruler or a mass clock. The mass of a body is determined indirectly by measuring the impact of a force $\boldsymbol{F}$ acting on the body. The magnitude of force itself has the dimension of mass × length × time^{-2} and it is measured in units of Newtons (N) where $1\,\mathrm{N} = 1\,\mathrm{kg{\cdot}m{\cdot}s^{-2}}$. Put into words: 1 N is the amount of force which will increase the acceleration of a mass of 1 kg by $1\,\mathrm{m{\cdot}s^{-2}}$.

When we determine the size of masses in everyday life, we do this in most cases by comparing the respective weight forces, such as with a beam balance. However, the weight force is not the mass of a body, as this is a property of matter itself, which is called inertial mass, but it is determined by the interaction between two bodies. With the help of a beam balance, e.g., one compares the interaction between a mass and the Earth's mass with the interaction of another mass with the Earth's mass. The weight force is location dependent:

$$\boldsymbol{F} = m_s \cdot g \cdot \hat{\boldsymbol{e}}$$

where g is the gravitational acceleration and the unit vector $\hat{\boldsymbol{e}}$ indicates the direction of the force. For a body in the gravitational field of Earth, the latter points in the direction of the center of the Earth, i.e., the center of mass of the Earth. The weight force measures how "heavy" or "light" a body is, but it does not measure the mass of a body itself. Because of the Moon's lower mass, gravity acceleration on its surface is about six times less than on Earth's surface and therefore everything on the Moon is about six times "lighter" than on Earth. However, the mass of a body on the Moon is identical with the mass of the same body on Earth.

The inertial mass is a property of matter itself and it is a measure of how much resistance a point mass will offer to any acceleration applied to it. To set a mass at rest in motion or to change the state of motion of a moving mass, requires the application of a force (against the inertia of the mass, hence the name). This is the statement of Newton's first law:

1 A body at rest remains at rest if no external forces act on it.

2 A body in uniform motion will continue to move at constant speed if no external forces are acting on it.

A force acting on a body will give it a certain acceleration depending on its inertial mass. If another body requires twice the force to achieve the same acceleration than it possesses twice the inertial mass. The inertial mass is the constant of proportionality in Newton's second law, i.e.,

$$\boldsymbol{F} = m_t \cdot \boldsymbol{a}$$

At first glance, one might suppose that what inertia (inertial mass) and weight force (heavy mass) measure reflect different physical properties of a body, but this is not true. In Einstein's general theory of relativity, the so-called equivalence principle, which refers to the indistinguishability of acceleration and gravitation, stipulates the equality of heavy and inertial mass and so far, not a single experiment has been able to show that this is not the case.

The collision of masses is a familiar experience to all of us as we encounter it in various forms in our daily life. Physicists categorize collisions between point masses in two basic ways, collisions which they call elastic collisions and collisions which they call inelastic collisions.

A completely inelastic central impact collision is illustrated in fig. 1.5. In this case, the impact has "glued" the two masses together in such a way that the two collision partners now move together as one mass. In the simplest case, as shown in fig. 1.5, a mass m hits a mass m_0 at rest in a central impact collision. For the total momentum before the collision and the total momentum after the collision applies

$$m \cdot v = (m + m_0) v_{inel}$$

hence

$$v_{inel} = \frac{m}{m + m_0} \cdot v \qquad \text{(scalar)}$$

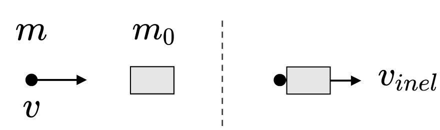

Fig. 1.5

The variable

$$\boldsymbol{p} = m\boldsymbol{v} = \boldsymbol{p}(t) \tag{1.1}$$

the product of mass and velocity, denotes the momentum of a point mass of mass m moving with the velocity $\boldsymbol{v}$. According to Newton's second law, the basic law of motion, it holds that:

$$\frac{\mathrm{d}\boldsymbol{p}(t)}{\mathrm{d}t} = \boldsymbol{F}\big(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), t\big) \tag{1.2}$$

The change in momentum of a point mass over time is equal to the force $\boldsymbol{F}$ acting on the point mass. The force itself can be a function of either of three variables, i.e., location, velocity, and time, or a combination of them.

The law of conservation of momentum applies to the total momentum within a so-called closed system. The conservation of momentum follows directly from Newton's second law, i.e., eq. (1.2), and Newton's third law. The latter states that in a system on which no external forces act, that is the definition of a closed system, there exists for every force $\boldsymbol{F}_i$ an equal but opposite force $-\boldsymbol{F}_i$. Thus, for a closed system with N mass points, the total force must be zero, i.e.,

$$\boldsymbol{F} = \sum_i^N \boldsymbol{F}_i = 0 \tag{1.3}$$

With eq. (1.2) this means that at the same time

$$\boldsymbol{p} = \sum_i^N \boldsymbol{p}_i = 0 \qquad \text{conservation of momentum} \tag{1.4}$$

In general, for two masses m_1 and m_2 that had the velocities v_1 and v_2 before a central impact inelastic collision, and move on together after the inelastic collision with the velocity v_{inel} the conservation of momentum applies in the form

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_{inel} \qquad \text{inelastic collision}$$

For the perfectly central impact elastic collision, the law of conservation of momentum reads

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \text{elastic collision}$$

where v_1 and v_2 as well as v_1' and v_2' are the respective velocities of the masses m_1 and m_2 before and after the elastic collision.

Now a mass m is being considered that moves with the speed v_0 and on which a constant acceleration a shall act beginning at time $t = 0$ (fig. 1.6). Hence it applies that

$$\ddot{x} = a \cdot t$$

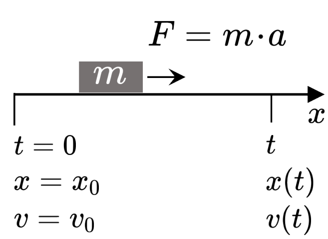

Fig. 1.6

Integrating twice over time gives the result that at time t the mass will be at the location

$$x(t) = x_0 + v_0 t + \frac{1}{2} a \cdot t^2$$

Here, x_0 is the location where the mass m was at the time $t = 0$. Without restriction, one can choose $x_0 = 0$ and $v_0 = 0$. In the time interval t, the mass m then has traveled the distance

$$x(t) = \frac{1}{2} a \cdot t^2$$

and moves with the velocity

$$v(t) = a \cdot t \qquad \text{that means it applies} \qquad t = \frac{v(t)}{a}$$

If one inserts this expression for t into the equation for the distance traveled, one obtains

$$x(t) = \frac{1}{2} \frac{v^2}{a}$$

The energy E_{kin} which the mass m possesses at time t is equivalent to the work required to move it from x_0 to $x(t)$ (no friction losses). This work is the product of the force that had to be expended times the length of the distance over which this was necessary, i.e.,

$$E_{kin} \;=\; F \cdot x(t) \;=\; ma \cdot \frac{1}{2} \frac{v^2}{a} \;=\; \frac{1}{2} m v^2$$

The quantity E_{kin}, often also denoted by T, is the kinetic energy that the mass m possesses at the time t. This derivation of the kinetic energy shows that the work done has been completely converted into kinetic energy of the point mass. This is a variant of the law of conservation of energy which states that the total energy in a closed system is constant. In a closed system, energy is neither lost nor can energy be gained, it can only be converted from one form of energy to another form of energy. In the case of a central impact inelastic collision, the following applies to the kinetic energy before and after the collision

$$\frac{1}{2} m_1 \cdot v_1^2 + \frac{1}{2} m_2 \cdot v_2^2 \;=\; \frac{1}{2} (m_1 + m_2) \cdot v_{inel}^2 + \Delta E \qquad \text{inelastic collision}$$

where $\Delta E > 0$ is the internal energy of the two collision partners. Therefore, in the case of an inelastic collision the kinetic energy is not conserved because a part of it is converted into internal energy (e.g., through elastic or plastic deformation of the collision partners). In contrast, the kinetic energy is conserved in a perfectly elastic collision

$$\frac{1}{2} m_1 \cdot v_1^2 + \frac{1}{2} m_2 \cdot v_2^2 \;=\; \frac{1}{2} m_1 \cdot v_1'^2 + \frac{1}{2} m_2 \cdot v_2'^2 \qquad \text{elastic collision}$$

1.2.1 Examples of Forces

Today, physics knows four fundamental forces and a number of forces derived from them. The four fundamental forces are:

- the gravitational force
- the electromagnetic force
- the strong interaction or strong nuclear force
- the weak interaction or weak nuclear force

Although the gravitational force is the weakest of the four fundamental forces, in the case of very large masses being involved, as is for example the case in the gravitational collapse of massive stars, it ultimately overcomes all other forces. The existence of neutron stars and black holes gives eloquent evidence of this.
The range of the second strongest force, the electromagnetic interaction, scales just like that of the gravitational force with r^{-2}. However, it does not affect all masses but only those which carry an electrical charge. It is impossible to shield off gravitational forces; at best one can neutralize them in some cases by appropriate acceleration (such as in free fall). For electric charges and fields that is different as they can be shielded off. Just consider for example that there are certainly innumerable electrical charges on Earth and yet Earth as a whole is electrically neutral. On the cosmic length scale, therefore, only one single force eventually prevails and that is the gravitational force. But there are certainly electrical cosmic phenomena. A well-known example is the Earth's magnetic field, without which there would probably be no life on Earth. A much more powerful example are pulsars, highly magnetic rotating neutron stars that emit electromagnetic radiation of high energy and intensity along their polar axes.
We are all familiar with the effects of gravitational and electromagnetic forces on our planet, we experience both on a daily basis. This is not the case with the very short-range nuclear forces, the strong and the weak interaction, because these only act on the subatomic length scale. The four fundamental forces are also called imprinted forces because they act on a body from the outside. As the summary in tab. 1.1 shows, a number of derived forces are also counted as imprinted forces in addition to these four fundamental imprinted forces.
While the four fundamental forces are called imprinted forces because they act on a body from the outside, derived forces, on the other hand, are referred to as such because one finds on closer examination that these forces are macroscopic consequences of the microscopic effects of the fundamental forces.

Tab. 1.1: Classification of forces.

<table>
<tr><th>Force</th><th>Type</th><th>Attack-point</th><th>Source</th><th>Range</th></tr>
<tr><td>Gravitational force</td><td rowspan="4">imprinted force</td><td>heavy mass</td><td>heavy mass</td><td>$F \propto \frac{1}{r^2}$</td></tr>
<tr><td>Inertial force</td><td>inertial mass</td><td>accelerated movement relative to inertial system</td><td>∞</td></tr>
<tr><td>Elektromagnetic force</td><td>charges currents</td><td>charges currents</td><td>$F \propto \left(\frac{q}{r^2}, -\frac{q'}{r^2}\right)$</td></tr>
<tr><td>Elastic force, hydrostatic and hydrodynamic forces, plastic and viscous forces</td><td colspan="2" rowspan="3">matter, macroscopic bodies</td><td>$10^{-10}\,\mathrm{m} = 1$ Å</td></tr>
<tr><td>Sliding friction, static friction</td><td rowspan="2">reaction force</td><td>1 Å</td></tr>
<tr><td>Forces between rigid bodies</td><td>1 Å</td></tr>
<tr><td>Strong nuclear force</td><td rowspan="2">imprinted force</td><td rowspan="2">elementary particles</td><td>color charge</td><td>$10^{-15}\,\mathrm{m}$</td></tr>
<tr><td>Weak nuclear force</td><td>weak charge</td><td>$10^{-17}\,\mathrm{m}$</td></tr>
</table>

The derived forces include, for example, pseudo forces such as the force of inertia or elastic, hydrostatic, plastic, and viscous forces. However, since these act on the body from the outside like the gravitational force, the electromagnetic force, and the strong and weak nuclear force, they are also counted among the so-called imprinted forces. Sometimes one finds frictional forces listed among the imprinted forces as well. However, frictional forces are reaction forces, which means they are a reaction to the action that takes place when two bodies come into contact. A body that is isolated, for example in interstellar space, does not experience any frictional forces, but if it falls to Earth as a meteorite, then this changes as soon as the body begins to immerse itself in the Earth's atmosphere. Falling meteorites, but also space capsules returning to Earth, hit the Earth's atmosphere at very high speed and without heat shields, space capsules would burn up just like shooting stars.

In technical mechanics, the gravitational force, the electromagnetic force and above all the macroscopic and microscopic forces derived from these two play a major role. Nuclear forces, however, do not matter in technical mechanics.

The gravitational force

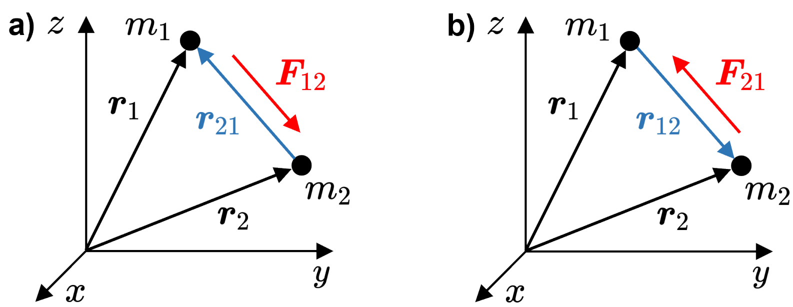

Fig. 1.7: Orientations of distance and force vectors in Newton's law of gravitation in vector form (see eq. (1.5a) and eq. (1.5b)): (a) for the force F_{12} exerted on m_1 by m_2; (b) for the force F_{21} exerted on m_2 by m_1.

For two masses m_1 and m_2 whose distance vector is $\boldsymbol{r}_{21} = -\boldsymbol{r}_2 + \boldsymbol{r}_1$ (fig. 1.7a), the gravitational force $\boldsymbol{F}_{12}$ which m_2 exerts on m_1 is given by

$$\boldsymbol{F}_{12} = -G\frac{m_1 m_2}{|\boldsymbol{r}_{21}|^3}\boldsymbol{r}_{21} = G\frac{m_1 m_2}{|\boldsymbol{r}_{12}|^3}\boldsymbol{r}_{12} \tag{1.5a}$$

and the gravitational force $\boldsymbol{F}_{21}$ exerted on m_2 by m_1 (fig. 1.7b) which, as measured from the position of m_1, is located a distance $\boldsymbol{r}_{12} = -\boldsymbol{r}_1 + \boldsymbol{r}_2$ away from m_2, is given by

$$\boldsymbol{F}_{21} = -G\frac{m_1 m_2}{|\boldsymbol{r}_{12}|^3}\boldsymbol{r}_{12} = G\frac{m_1 m_2}{|\boldsymbol{r}_{21}|^3}\boldsymbol{r}_{21} \tag{1.5b}$$

In eq. (1.5a) and eq. (1.5b), G is a fundamental natural constant that determines the strength of gravity. With $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$, obviously, $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ applies. One can write the vector forms of Newton's gravity law in eq. (1.5a) and eq. (1.5b) without the minus sign in front by using $\boldsymbol{r}_{12}$ instead of $\boldsymbol{r}_{21}$ in eq. (1.5a) and by using $\boldsymbol{r}_{21}$ instead of $\boldsymbol{r}_{12}$ in eq. (1.5b). However, the convention is to write those equations with a minus sign in front to indicate that the respective force vector points into the opposite direction of the distance vector thereby clearly indicating that Newtons gravity force is attractive, i.e., it will result in the distance vector becoming shorter.

Example 1.1 A historical perspective

After Newton formulated the law of gravitation in 1687 it then took more than 100 years before Henry Cavendish succeeded in experimentally determining G in 1798. According Johannes Kepler's third law, the ratio between the cube of the semi-major axis a of a planets elliptical orbit and the square of its orbital period T is constant:

$$\frac{a^3}{T^2} = c$$

Simplifying assumption: Planetary orbits shall be circular with radius a (fig. 1.8). With that one obtains for the acceleration (due to the centripetal force)

$$b = -\omega^2 a = -\frac{4\pi^2}{T^2} a = -4\pi^2 \frac{a}{T^2} = -\frac{c}{a^2} \tag{1.6}$$

The minus sign arises because the acceleration occurs in the direction opposite to where the radius vector points. If one now imagines the entire mass of the Earth being concentrated in its center, then the following applies to an object which orbits the center of the Earth at a distance equal to the Earth's radius R_E:

$$b_E = -\frac{c}{R_E^2} = -g$$

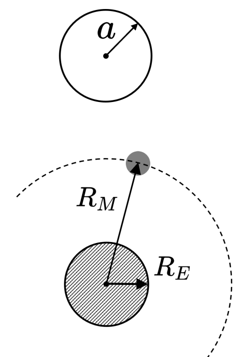

Fig. 1.8

The value that this equation delivers for c can now be used in the corresponding equation for the Moon (radius R_M)

$$b_M = -\frac{c}{R_M^2} = -g\left(\frac{R_E}{R_M}\right)^2$$

With eq. (1.6) and the revolution period of the Moon T_M it also applies

$$b_M = -\frac{R_M}{T_M^2} 4\pi^2$$

With the last two equations, their left and hence their right sides being identical, it is possible to determine g from measurable quantities:

$$g = \frac{4\pi^2 R_M^3}{R_E^2 T_M^2} \tag{1.7}$$

If one inserts into this equation for R_M the mean Earth-Moon distance of 38 400 km, for the orbital period of the Moon T_M the duration between two perigee passages (27.56 days), and for R_E the mean Earth radius of 6 371 km, then one obtains for g the value $9.71\,\mathrm{m\cdot s^{-2}}$. This is just a little more than 1% below the modern mean value of $g = 9.81\,\mathrm{m\cdot s^{-2}}$. With eq. (1.7) the value of g depends on how far away one is from the center of the Earth because $g \propto R_E^{-2}$. The determination of G is a bit more difficult because one must know the mass of the Earth m_E. On the surface of the Earth applies

$$F = mg = G\frac{m_E m}{R_E^2}$$

The test mass m cancels out and the result for G is:

$$G = g\frac{R_E^2}{m_E} \tag{1.8}$$

If one uses here $g = 9.81\,\mathrm{ms^{-2}}$, $R_E = 6\,371\,\mathrm{km}$, and $m_E = 5.972{\cdot}10^{24}\,\mathrm{kg}$, then one obtains $G = 6.6675{\cdot}10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$. However, that is just the reverse of what Henry Cavendish did. In his time, the radius of the Earth had been known for a long time, but not the mass of the Earth. With his torsion balance, Cavendish was able to determine the value of G for the first time in 1798 and that with great accuracy; his value for G differing only by 1.2% from the modern value of the gravitational constant $G = 6.6743 \cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$. Knowing G and using eq. (1.8), Cavendish was then able to determine the mass of the Earth for the first time.

The electromagnetic force

Analogous in its form to Newton's law of gravitation, Coulomb's law, discovered by Charles Augustine de Coulomb in 1785 and named after him, describes the force acting between two electric point charges.

$$\boldsymbol{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\boldsymbol{r}_{12}|^3}\boldsymbol{r}_{12} \tag{1.9}$$

In contrast to the gravitational force, which only knows one charge (namely the mass), the electromagnetic interaction knows positive and negative charges. The gravitational force always has an attractive effect, the Coulomb force can have both an attractive and a repulsive effect, and it has no effect at all on uncharged bodies.

Coulomb's law provides the foundation for electrostatics, i.e., for the physics of electric charges and charge distributions at rest. In electrodynamics, one considers the forces acting on a point charge that moves in an electromagnetic field. The force acting on such a point charge q moving with the velocity $\boldsymbol{v}$ in an electromagnetic field having the electric field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{B}$ is given by

$$\boldsymbol{F} = q\big(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\big) \tag{1.10}$$

This force equation for the so-called Lorentz force, named after Hendrik Antoon Lorentz, is often used to define the vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In reality, however, this is only possible in principle since each moving test charge q in eq. (1.10) generates its own $\boldsymbol{E}$ and $\boldsymbol{B}$ field. If in addition the test charge moves on a curved path, i.e., it experiences an acceleration, it will also lose energy by emitting radiation.

Frictional forces

The gravitational force and the electromagnetic force are both so-called field forces that act over long distances. The effect of their respective forces being mediated by the gravitational field or the electromagnetic field, with electromagnetic waves and gravitational waves both propagating at the speed of light. The situation is completely different for frictional forces because these are contact forces. Action at a distance as mediated by the respective fields of the gravitational force or the electromagnetic force does not exist for frictional forces. Frictional force is the term used to describe different forces by which two bodies in contact act as braking forces on each other. Frictional forces always work against the direction of movement or prevent the movement of bodies entirely. The latter is the case when the force acting on a body at rest is not sufficient to overcome the static friction keeping it in place. Friction is also the reason we can move by walking in the first place. One can easily determine how difficult it is to move without friction by stepping onto an icy surface.

We all know from our own experience that frictional forces increase with increasing speed when moving through air or water, for example. At low velocities, the frictional force is proportional to $\boldsymbol{v}$, one then speaks of friction according to Stokes, named after George Gabriel Stokes. At high speeds, the frictional force depends on $\boldsymbol{v}^2$, one then speaks of friction according to Newton. In the one-dimensional case, the velocity-dependent frictional force in the expansion according to Lord Rayleigh (born as John William Strutt) is given by

$$F = F(v) = \underbrace{\overbrace{F_0 \frac{v}{|v|}}^{\text{sliding friction}} \overbrace{- F_1 v - F_2 v|v|}^{\text{flow friction}}}_{\text{Rayleigh}} \overbrace{- \ldots}$$

Elastic forces

If not stretched or compressed too much, many bodies behave elastically. An example for this is the mechanical spring. Using an expansion with respect to the stretching or compression coordinate x of the spring, the acting force can be written as

$$F(x) = \overbrace{F_0 - F_1 x}^{\text{harmonic}} - \overbrace{\underset{\substack{\uparrow \\ \text{can have both signs}}}{F_2} x^2 - \ldots}^{\text{anharmonic}}$$

The linear term for which the force is proportional to the expansion or compression of the spring corresponds to the well-known law of Hooke, named after Robert Hooke. According to Hooke's law

$$F = m\ddot{x} = -k \cdot x \tag{1.11}$$

The acceleration of a body that is subject to a spring force is therefore directly proportional and opposite to the extension/compression of the spring from its rest position.

The kind of relationship expressed in eq. (1.11) is the basis of harmonic oscillations that play a significant role in physics and many practical applications.

Plastic forces

Deviations from Hooke's law do occur when the expansion or compression of a body exceeds its elasticity limits. In this case, the elastic expansion or compression turns into a plastic deformation.

Hydrostatic and hydrodynamic force

The term hydrostatic force refers to a force exerted by a liquid at rest. A well-known example is the force which the column of liquid above a submerged submarine exerts on its outer hull, i.e., on the pressure vessel of the submarine protecting the sailors on board. In contrast to hydrostatic forces, hydrodynamic forces are caused by moving liquids. The suction effect of water flowing down a drain is just as much an example for a hydrodynamic force as is the power of a river current.

1.2.2 Superposition of Forces

An important tool in physics is the so-called superposition principle. This applies to a number of physical quantities and forces are among them. As shown in fig. 1.9, using the example of two forces, forces can be superimposed (added) to a resulting total force

$$\boldsymbol{F}\big(\boldsymbol{F}_1, \boldsymbol{F}_2\big) \;=\; \boldsymbol{F}_1 + \boldsymbol{F}_2 \qquad \text{(linear)}$$

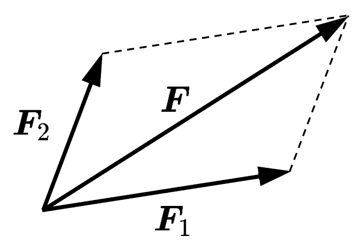

Fig. 1.9

A force parallelogram for two forces as sketched in fig. 1.9 was already used by Newton. It generally applies to linear systems that all forces acting on a body can be added vectorially to produce the resulting total force vector. If there are N forces $\boldsymbol{F}_i$ acting on a body, then their vector addition will yield the total force acting on the body $\boldsymbol{F}$ as

$$\boldsymbol{F} \;=\; \sum_{i=1}^{N} \boldsymbol{F}_i \tag{1.12}$$

Eq. (1.12) is the general superposition principle for forces. Because of its great importance, the superposition principle it is also often referred to as Newton's fourth law.

2. Statics of Rigid Bodies

Statics of rigid bodies, the theory of the balance of forces on rigid bodies, deals with non-deformable rigid bodies in force and torque (moment of force) equilibrium. This applies to bodies at rest, but also to bodies that are moving at constant speed and are viewed from a position in the respective moving reference system.

2.1 Kinematics

2.1.1 Geometry of Rigid Bodies

Naturally, in the geometry of rigid bodies, all distances and angles are fixed. The possible movements of a rigid body are determined by the so-called degrees of freedom available to it.

Definition

Degrees of freedom: The number of elements that must be specified to fix the position of a body in space.

Degrees of freedom in the two-dimensional case (plane):

For the rigid body (fig. 2.1) whose motion is restricted to movements in a plane, the position vector has two degrees of freedom with respect to translation, i.e., along its two orthogonal coordinate axes in $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ direction, and it can be rotated in the plane by an angle α.

Parameter	Degrees of freedom
$\boldsymbol{r}$	2
α	1
All parameters	3

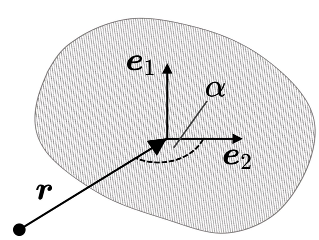

Fig. 2.1

Degrees of freedom in three-dimensional space:

For movements in three-dimensional space a rigid body has three orthogonal directions available for translation and it can be rotated around each of its three body axes (Euler angles α, β and γ).

Parameter	Degrees of freedom
$\boldsymbol{r}$	3
$\alpha\,,\ \beta\,,\ \gamma$	3
All parameters	6

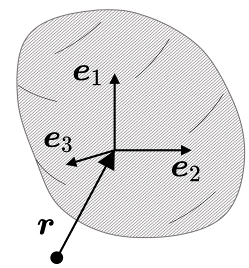

Fig. 2.2

From the above it follows that one needs three equations in the plane whereas in three-dimensional space six equations are required to fully determine the position of a rigid body. The state of motion of a rigid body is completely determined by its total momentum and its total angular momentum. Thus, the required equations must result from the consideration of the total momentum change and the total angular momentum change of the rigid body.

1) Total momentum change:

$$\sum_i \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \sum_i \boldsymbol{F}_i = \boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \tag{2.1}$$

From eq. (2.1) one obtains two equations in the plane and three in space.

2) Total angular momentum change: The total angular momentum is given by

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i \tag{2.2}$$

Therefore, the change in total angular momentum is

$$\sum_i \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum_i \underbrace{\dot{\boldsymbol{r}}_i \times m_i \dot{\boldsymbol{r}}_i}_{=\,0} + \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{M} \tag{2.3}$$

where $\boldsymbol{M}$ denotes the torque associated with the change in angular momentum. Eq. (2.3) provides one equation in the plane and three in three-dimensional space. In summary, the rates of change in total momentum and total angular momentum provide the required number of equations, three in the plane and six in three-dimensional space.

2.2 Forces and Torques

Definition

The line of action of a force is the straight line that indicates the position of the force in space. Two forces $\boldsymbol{F}$ and $\boldsymbol{F}'$ with associated torques $\boldsymbol{M}$ and $\boldsymbol{M}'$ are said to be equivalent if and only if

$$|\boldsymbol{F}| = |\boldsymbol{F}'| \qquad \text{and} \qquad \boldsymbol{M} = \boldsymbol{M}'$$

$\boldsymbol{F}$ and $\boldsymbol{F}'$ are then of the same absolute value and lie on the same line of action.

Example 2.1 **Torque associated with a line of action**

As sketched in fig. 2.3, one considers a force along a line of action at two different points labeled $\boldsymbol{r}_1$ and $\boldsymbol{r}_1'$. The force vectors at these respective positions on the line of action are denoted by $\boldsymbol{F}_1$ and $\boldsymbol{F}_1'$ with $|\boldsymbol{F}_1| = |\boldsymbol{F}_1'|$. For the torque $\boldsymbol{M}_1'$ one then calculates:

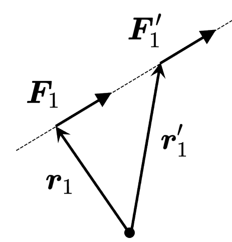

Fig. 2.3

$$\begin{aligned}\boldsymbol{M}_1' &= \boldsymbol{r}_1' \times \boldsymbol{F}_1' = (\boldsymbol{r}_1 + \boldsymbol{r}_1' - \boldsymbol{r}_1) \times \boldsymbol{F}_1 \\ &= \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \underbrace{(\boldsymbol{r}_1' - \boldsymbol{r}_1) \times \boldsymbol{F}_1}_{=0} = \boldsymbol{M}_1\end{aligned}$$

Thus $\boldsymbol{F}_1$ and $\boldsymbol{F}_1'$ must lie on the same line of action and because of $|\boldsymbol{F}_1| = |\boldsymbol{F}_1'|$ it follows that $\boldsymbol{F}_1 = \boldsymbol{F}_1'$. For a body which is acted on by two forces pointing in the same direction but whose lines of action are not identical, meaning the forces are parallel, it follows with $\boldsymbol{M}_1 \neq \boldsymbol{M}_1'$ that there will be a net torque acting on the body. Only force components that lie on the same line of action can be added or subtracted from each other, depending on whether they point in the same direction or not.

Example 2.2 **Reduction of coplanar vectors**

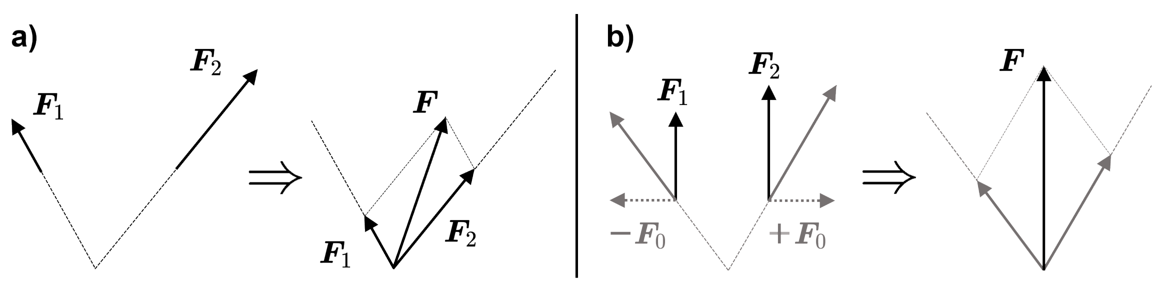

Fig. 2.4: Reduction of two coplanar vectors: (a) the vectors are not parallel; (b) the vectors are parallel.

The construction in fig. 2.4a shows the reduction of two coplanar, non-parallel vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ by tracing back the two vectors to their point of intersection followed by subsequent addition of the two vectors to the sum vector $\boldsymbol{F}$. If $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ happen to be parallel vectors as shown in fig. 2.4b, one can generate a pair of non-parallel vectors from these parallel vectors by adding the zero vector with zero moment $-\boldsymbol{F}_0 + \boldsymbol{F}_0 = \boldsymbol{0}$, which then can be reduced to the sum vector $\boldsymbol{F}$ in the same way as shown in fig. 2.4a. An exception to this is the so-called force pair, also known as a force dipole. If $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are anti-parallel, as in the case of the force dipole, the trick with the zero-vector addition does not work because this will then only generate a new anti-parallel vector pair.

To determine the "line of action" in the case of multiple forces acting on a body (fig. 2.5) one calculates:

$$\boldsymbol{F} = \sum_i \boldsymbol{F}_i$$

(S = center of mass of the rigid body)

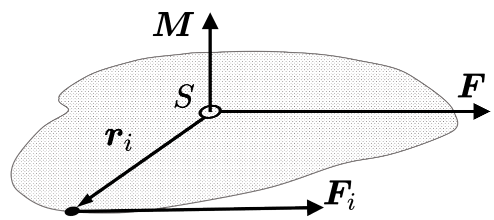

Fig. 2.5

Equations of motion for rigid bodies:

1) Total momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \boldsymbol{p}_i\right) = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \sum_i \boldsymbol{F}_i = \boldsymbol{F} \tag{2.4}$$

2) Total angular momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i\right) = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{M} \tag{2.5}$$

The concept of bound vectors is essential for calculating the statics of rigid bodies. Bound vectors are those vectors whose starting point is assigned to a specific point in space. Forces and moments of force (torques) belong to the physical quantities to which bound vectors are assigned. While bound vectors are assigned a starting point, these starting points can however be shifted in such a way that the new force and torque system created by the respective shifts will be equivalent to the original force and torque system. This is shown in ex. 2.1 for the displacement of a bound force vector along its line of action and in ex. 2.2 for the reduction of two coplanar vectors. The total torque does not change in these cases. As a matter of fact, force vectors are always bound in sterostatics. This is usually also the case for the vectors associated with torques, but not always.

Example 2.3 **Torque of the force dipole**

A force dipole consists of two opposing forces which have the same absolute value, and which lie on parallel lines of action. That means that the resulting force of a force dipole vanishes:

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 = 0$$

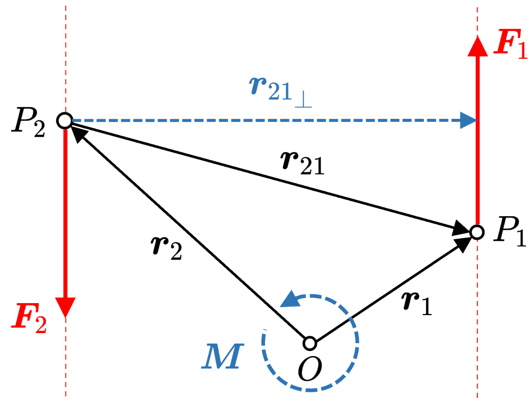

Fig. 2.6

With respect to the torque caused by $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in relation to a freely chosen point O, it follows from the force dipole in fig. 2.6 that

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 \\ &= (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{F}_1 = \boldsymbol{r}_{21} \times \boldsymbol{F}_1 \\ &= (-\boldsymbol{r}_1 + \boldsymbol{r}_2) \times \boldsymbol{F}_2 = \boldsymbol{r}_{12} \times \boldsymbol{F}_2\end{aligned}$$

and therefore

$$\boldsymbol{M} = \boldsymbol{r}_{21_\perp} \times \boldsymbol{F}_1 = \boldsymbol{r}_{12_\perp} \times \boldsymbol{F}_2$$

For the absolute value of the torque one then finds

$$|\boldsymbol{M}| = |\boldsymbol{r}_{21_\perp}| \cdot |\boldsymbol{F}_1| = |\boldsymbol{r}_{12_\perp}| \cdot |\boldsymbol{F}_2| \tag{2.6}$$

Hence, the torque of the force dipole depends only on the normal distance of the respective lines of action of the two vectors and not on the distance between their starting points. The torque $\boldsymbol{M}$ is therefore the same everywhere, independent of the location of the axis of rotation O. This also applies to axes of rotation that do not lie between the two lines of action of the forces but outside of this area. In other words, the starting point of the torque vector can be moved freely within the rigid body and therefore the torque of a force dipole is not a bound, but a free vector.

Because the torque of a force dipole is independent of the reference point, a force dipole can be shifted to any desired location in its plane of action without its effect, i.e., its torque, changing. A force dipole can always be replaced by its associated torque without changing its effect on a body. In contrast, a single force cannot be replaced by its torque, and it can only be shifted along its line of action without changing the torques associated with it.

2.2.1 Principle of Equivalence

The principle of equivalence states that two systems of bound force vectors are identical if the resulting total force $\boldsymbol{F}$ with respect to any arbitrary reference point P will produce the same resulting torque $\boldsymbol{M}$. According to the principle of equivalence, operations are therefore permitted that do not change the total torque of a system. These include:

- The displacement of the $\boldsymbol{F}_i$ along their respective lines of action (ex. 2.1).
- The addition of $\boldsymbol{F}_i$ with the same point of attack (ex. 2.2).
- The decomposition of $\boldsymbol{F}$ into components $\boldsymbol{F}_i$

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \dots$$

- The addition of zero vectors $\boldsymbol{F}_0$ and $\boldsymbol{M}_0$

$$\boldsymbol{F}_0 = \sum_i \boldsymbol{F}_i = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{M}_0 = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{0}$$

For the reduction of force systems, the displacement of forces along their lines of action, addition of forces or their decomposition, as well as the addition of zero-force vectors have already been discussed. Another important tool is the parallel displacement of forces.

Parallel displacement of a force in the plane

In order to satisfy the principle of equivalence, a torque must always be added when a force vector is displaced parallel to its line of action. That this is indeed the case, follows from considering the respective individual equivalence-preserving steps of the parallel shift in fig. 2.7.

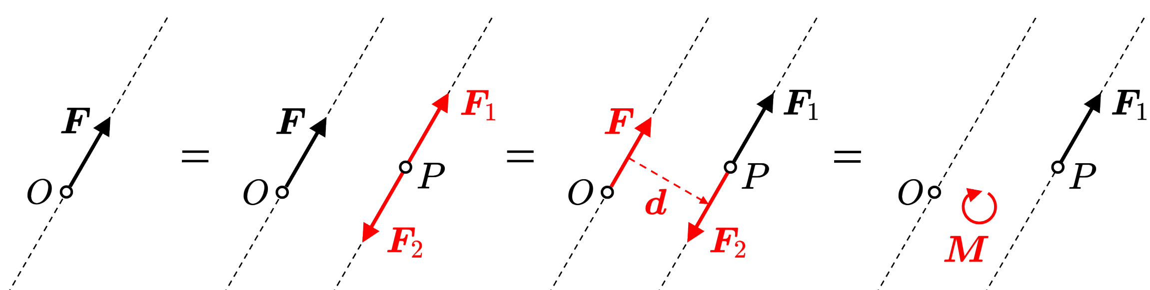

Fig. 2.7: Parallel force vector displacement according to the principle of equivalence.

First, on a line of action parallel to the line of action of the force vector $\boldsymbol{F}$ to be shifted, a zero-force vector is added at the desired distance d. The specific zero vector added consists of oppositely directed vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ with magnitudes chosen identical to that of $\boldsymbol{F}$. In the following step, $\boldsymbol{F}$ and the parallel offset and oppositely directed $\boldsymbol{F}_2$ are identified as the two force vectors of a force dipole. In the last step, the force dipole (see fig. 2.6) is replaced by its associated free torque $\boldsymbol{M}$ whose axis of rotation can be positioned arbitrarily. In summary, the individual force $\boldsymbol{F}$ is equivalent to the parallel displaced force $\boldsymbol{F}$ and a torque with magnitude $|\boldsymbol{M}| = d \cdot |\boldsymbol{F}|$.

Definition

For a force system $\big(\boldsymbol{F}_1, \boldsymbol{F}_2, \dots \boldsymbol{F}_n; \boldsymbol{r}_1, \boldsymbol{r}_2, \dots \boldsymbol{r}_n\big)$ the vectors

$$\boldsymbol{F}_R = \sum_{k=1}^{n} \boldsymbol{F}_k \qquad \text{und} \qquad \boldsymbol{M}_{R_P} = \sum_{k=1}^{n} (\boldsymbol{r}_k - \boldsymbol{r}_P) \times \boldsymbol{F}_k$$

are called resulting force and resulting torque with respect to the point P. The determination of $\boldsymbol{F}_R$ and $\boldsymbol{M}_{R_P}$ is called reduction.

Definition

Two systems of forces

$$\big(\boldsymbol{F}_1, \boldsymbol{F}_2, \dots \boldsymbol{F}_n; \boldsymbol{r}_1, \boldsymbol{r}_2, \dots \boldsymbol{r}_n\big) \quad \text{und} \quad \big(\boldsymbol{F}'_1, \boldsymbol{F}'_2, \dots \boldsymbol{F}'_n; \boldsymbol{r}'_1, \boldsymbol{r}'_2, \dots \boldsymbol{r}'_n\big)$$

are called equivalent if and only if it holds that

$$\boldsymbol{F}_R = \boldsymbol{F}'_R \qquad \text{and} \qquad \boldsymbol{M}_{R_P} = \boldsymbol{M}'_{R_P}$$

The reference point P for the resulting torque can be freely selected but must be identical for equivalent force systems. The resulting torque $\boldsymbol{M}_{R_P}$ depends on the selected reference point P. For example, if one considers two different reference points P_1 and P_2, then the resulting torques are

$$\boldsymbol{M}_{R_{P_1}} = \sum_{k=1}^{n} (\boldsymbol{r}_k - \boldsymbol{r}_{P_1}) \times \boldsymbol{F}_k \qquad \text{and} \qquad \boldsymbol{M}_{R_{P_2}} = \sum_{k=1}^{n} (\boldsymbol{r}_k - \boldsymbol{r}_{P_2}) \times \boldsymbol{F}_k$$

If one now adds the zero vector $\boldsymbol{r}_{P_1} - \boldsymbol{r}_{P_1} = \boldsymbol{0}$ to the expression for $\boldsymbol{M}_{R_{P_2}}$, then one gets

$$\begin{aligned}
\boldsymbol{M}_{R_{P_2}} &= \sum_{k=1}^{n} (\boldsymbol{r}_k - \boldsymbol{r}_{P_1} + \boldsymbol{r}_{P_1} - \boldsymbol{r}_{P_2}) \times \boldsymbol{F}_k \\
&= \sum_{k=1}^{n} (\boldsymbol{r}_k - \boldsymbol{r}_{P_1}) \times \boldsymbol{F}_k + \sum_{k=1}^{n} (\boldsymbol{r}_{P_1} - \boldsymbol{r}_{P_2}) \times \boldsymbol{F}_k \\
&= \boldsymbol{M}_{R_{P_1}} + (\boldsymbol{r}_{P_1} - \boldsymbol{r}_{P_2}) \times \sum_{k=1}^{n} \boldsymbol{F}_k
\end{aligned}$$

This means that the resulting torque $\boldsymbol{M}_{R_P}$ is only independent of the reference point P if $\sum \boldsymbol{F}_k = 0$ applies. This condition however implies that this sum of forces must represent a force dipole such as shown in ex. 2.3.

If for the resulting force and the resulting torque of a force system acting on a body it holds true that

$$\boldsymbol{F}_R = \sum_{k=1}^{n} \boldsymbol{F}_k = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{M}_{R_P} = \sum_{k=1}^{n} (\boldsymbol{r}_k - \boldsymbol{r}_P) \times \boldsymbol{F}_k = \boldsymbol{0}$$

then the system of forces and therefore the body are in a state of equilibrium. In which case the body then will either move at a constant speed or it will be at rest. This is of course nothing else but a restatement of Newton's first law (see section 1.2) in a different context.

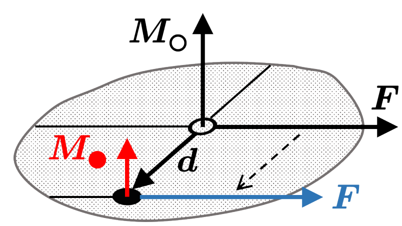

Fig. 2.8

An alternative approach to understanding the parallel displacement of a force in the plane is sketched in fig. 2.8. Here the force $\boldsymbol{F}$ acts on the center of mass of the depicted surface and $\boldsymbol{M}_\circ$ denotes the zero moment. In planar statics, all moments are perpendicular to the plane. The parallel displacement of the force $\boldsymbol{F}$ to a point P at a distance $\boldsymbol{d}$ from the center of mass does not generate a torque in P because this point lies in the line of action of the displaced force. Hence, for the respective torques it must hold that

$$\boldsymbol{M}_\bullet = \boldsymbol{M}_\circ + \boldsymbol{d} \times \boldsymbol{F} \overset{!}{=} 0$$

This equation has many solutions because of

$$\big(\boldsymbol{d} + \lambda\cdot\boldsymbol{F}\big) \times \boldsymbol{F} = \boldsymbol{d} \times \boldsymbol{F}$$

Hence, one requires that $\boldsymbol{d} \perp \boldsymbol{F}$ shall hold. With

$$\boldsymbol{F} \times \big(\boldsymbol{M}_\circ + \boldsymbol{d} \times \boldsymbol{F}\big) = 0$$

$$\boldsymbol{F} \times \big(\boldsymbol{d} \times \boldsymbol{F}\big) = -\boldsymbol{F} \times \boldsymbol{M}_\circ$$

$$(\boldsymbol{F}\cdot\boldsymbol{F})\cdot\boldsymbol{d} - \underbrace{(\boldsymbol{F}\cdot\boldsymbol{d})}_{=0}\cdot\boldsymbol{F} = -\boldsymbol{F} \times \boldsymbol{M}_\circ$$

it follows for $\boldsymbol{d}$ that

$$\boldsymbol{d} = -\frac{\boldsymbol{F} \times \boldsymbol{M}_\circ}{|\boldsymbol{F}|^2} \tag{2.7}$$

Since in planar statics $\boldsymbol{F}$ and $\boldsymbol{M}_{\bigcirc}$ are perpendicular to each other, this means nothing else but that the following must hold true:

$$|\boldsymbol{M}_{\bigcirc}| = |\boldsymbol{d}| \cdot |\boldsymbol{F}|$$

This expression is identical to eq. (2.6), the equation for the magnitude of the torque of a force dipole; but it was won here in a different way.

The above approach to determining the effect of a force shift parallel to a force's line of action can be easily transferred from the planar to the three-dimensional case. In three-dimensional space, as sketched in fig. 2.9, the zero torque $\boldsymbol{M}_{\bigcirc}$ is broken down into its respective components normal ($\boldsymbol{M}_{\perp}$) and parallel ($\boldsymbol{M}_{\parallel}$) to the plane in which $\boldsymbol{F}$ lies. In this way one obtains analogous to the derivation in the plane

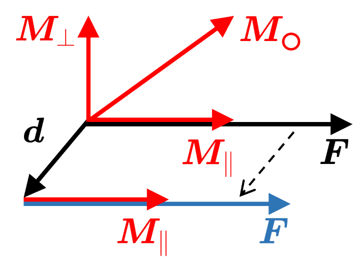

Fig. 2.9

$$\boldsymbol{d} = -\frac{\boldsymbol{F} \times \boldsymbol{M}_{\perp}}{|\boldsymbol{F}|^2} \tag{2.8}$$

and thus

$$|\boldsymbol{M}_{\perp}| = |\boldsymbol{d}| \cdot |\boldsymbol{F}|$$

In the case of a parallel displacement of a force vector in space, there only arises a torque $\boldsymbol{M}_{\perp}$ normal to the plane in which $\boldsymbol{F}$ lies. The force dipole belonging to $\boldsymbol{M}_{\perp}$ also lies in this plane. $\boldsymbol{M}_{\parallel}$, the component of the torque parallel to the plane in which the displaced force lies, remains unchanged by the parallel displacement of the force.

2.2.2 Reduction of Systems of Forces

The space- and vector diagram method

A more powerful tool for reducing forces than the simple methods shown in fig. 2.4 is the space- and vector diagram method ("Krafteck-Seileck" method in German; "Krafteck" = force corner, "Seileck" = rope corner). This method provides a simple way to graphically determine the resulting total force produced by a group of coplanar forces which do not act on a common point. Fig. 2.10 illustrates this procedure using the example of the reduction of three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ to the resulting total force $\boldsymbol{F}$. The starting point for the construction is the presence of the three force vectors $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$. In the first step of the procedure, the force vector diagram (the "Krafteck" or "force corner") on the right in fig. 2.10 is constructed from these.

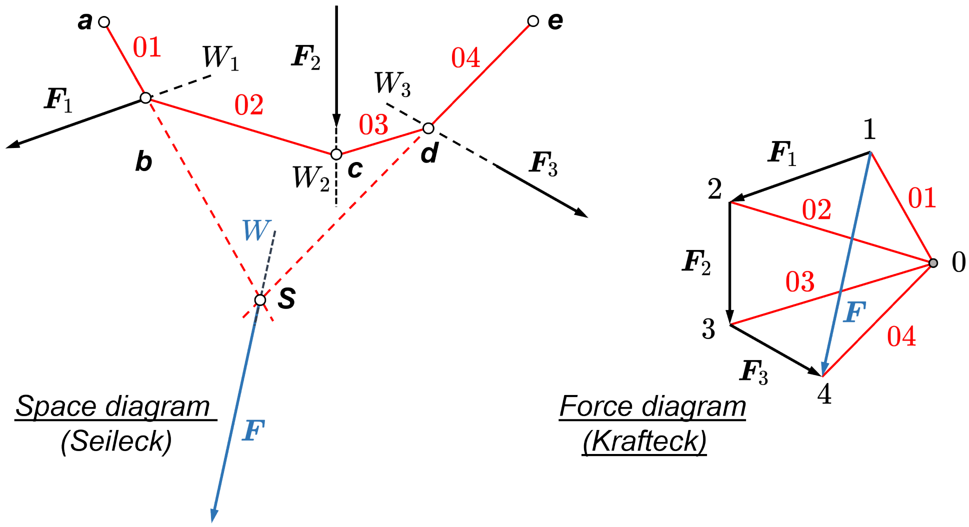

Fig. 2.10: The space- and vector diagram ("Krafteck-Seileck" in German) method for determining the resultant force of a group of co-planar forces with different attack points.

This is done by adding duplicates of the three force vectors from the space-diagram to form a force chain such that the vector between the beginning and end of this vector chain corresponds to the resulting total force. Then a point is chosen to the right of this force vector diagram, called its corner or pole denoted here by 0, from which lines are drawn to the respective start and end points of the three force vectors.

In the second step, the lines of action $\boldsymbol{W}_1$, $\boldsymbol{W}_2$, and $\boldsymbol{W}_3$ of the three forces are added to the space diagram. Which section of a line of action - lying before or after the respective force arrow - is required for the further construction of the space diagram depends on the relative position of the force vectors. In the third step, the connecting lines 01, 02, 03, and 04 are then transferred from the force diagram to the space diagram.

First, a line is drawn parallel to 01 from a point **a**, which is to the left of the line of action $\boldsymbol{W}_1$ of the force $\boldsymbol{F}_1$, to where it intersects $\boldsymbol{W}_1$. From this point of intersection **b** a line is now drawn parallel to 02 to where it intersects the line of action $\boldsymbol{W}_2$ of the force $\boldsymbol{F}_2$; this defines the point **c**. In the same way, a line is then drawn parallel to 03, starting at point **c**, until it intersects $\boldsymbol{W}_3$. Finally, from this point of intersection **d**, a parallel to 04 is drawn to the point **e**.

The resulting straight lines segments **a-b**, **b-c**, **c-d**, and **d-e** form a rope-like polygon, hence the German name "Seileck" (rope corner) for the space diagram. The intersection point **S** of the extensions of the two outer polygon segments **a-b** and **d-e** lies on the line of action of $\boldsymbol{F}$, the force resulting from $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$. The latter can now also be transferred from the force diagram to the space diagram as shown in fig. 2.10.

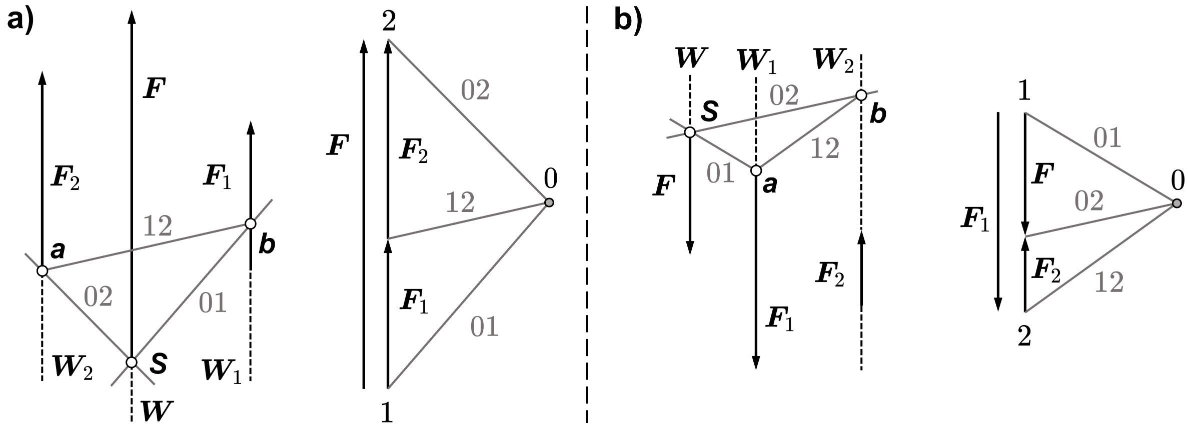

Fig. 2.11: Reduction of force couples using the space- and vector diagram method: (a) in the case of parallel and (b) in the case of anti-parallel force vectors.

With the help of the space- and vector diagram method anti-parallel force vectors can also be reduced, which is not possible with the zero-vector addition method already discussed. Fig. 2.11 shows the space- and vector diagram construction in the case of parallel (fig. 2.11a) and anti-parallel forces (fig. 2.11b).

Example 2.4 **Reduction of force systems in the gravitational field**

The total force $\boldsymbol{F}$ that acts on a rigid body consisting of individual masses m_i in the gravitational field is given by

$$\boldsymbol{F} = -\sum_i m_i g \boldsymbol{e}_z = -Mg\boldsymbol{e}_z$$

where

$$|\boldsymbol{e}_z| = 1 \quad \text{and} \quad \sum_i m_i = M$$

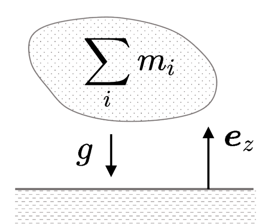

Fig. 2.12

With the position vectors $\boldsymbol{r}_i$ of the individual masses m_i, the torque $\boldsymbol{M}$ is

$$\boldsymbol{M} = -\sum_i \boldsymbol{r}_i \times \big(\underbrace{m_i g \boldsymbol{e}_z}_{\boldsymbol{F}_i}\big) = \boldsymbol{R} \times \big(\underbrace{Mg\boldsymbol{e}_z}_{\boldsymbol{F}}\big)$$

where

$$\boldsymbol{R} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i$$

is the position vector of the center of mass (CM) of the rigid body. CM is the point at which $\boldsymbol{F}$ attacks in the sense of an equivalent force system. Since the center of mass CM of a body is the point with respect to which the torque vanishes in the (constant) gravitational field this means that $\boldsymbol{M}$ and $\boldsymbol{F}$ have been replaced by a single force!

2.3 Bearings and Trusses

2.3.1 Bearings

Mechanical bearings enable or prevent movements in selected degrees of freedom between structural parts and transmit forces between them.

Definitions

Bearings: Connecting elements between different rigid construction parts which ensure the position and orientation of a construction.

Valence of a bearing: The valence of a bearing corresponds to the number of force- and torque components acting on it.

Kinematic determinacy: A mechanical bearing system is called kinematically determinate if it completely fixes the body's position (no wobbling). If the body can wobble, then the bearing is called kinematically indeterminate.

Static determinacy: A bearing system of a body is called statically determinate if the number of equilibrium conditions corresponds to the valence of its bearings. If the number of equilibrium conditions is less than the valence of the bearings, then the bearing system of the body is called statically indeterminate.

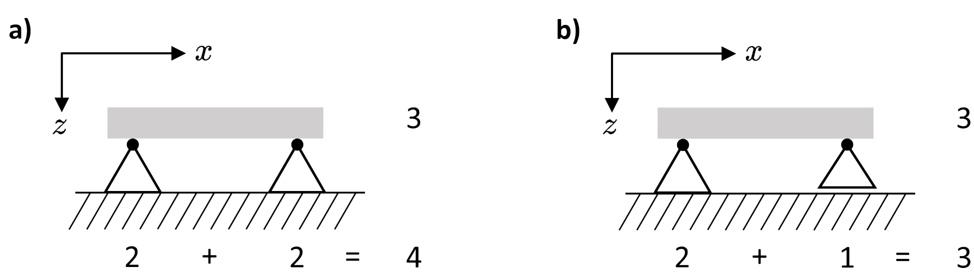

Fig. 2.13: A statically indeterminate (a) and a statically determinate (b) bearing system.

The number of equilibrium conditions of the body in fig. 2.13a is 3. Forces in the x and z direction can act on each of the two fixed pivot bearings. The valence of the bearing system is therefore 4 and thus greater than the number of equilibrium conditions. Consequently, the bearing system in Fig. 2.13a is statically indeterminate. In fig. 2.13b one of the fixed pivot bearings has been replaced by a movable pivot bearing on which forces can no longer act in the x direction. This reduces the valence of the bearing system to 3 which corresponds to the number of equilibrium conditions. The bearing system in fig. 2.13b is therefore statically determinate.

Example 2.5 **Bearing reaction forces**

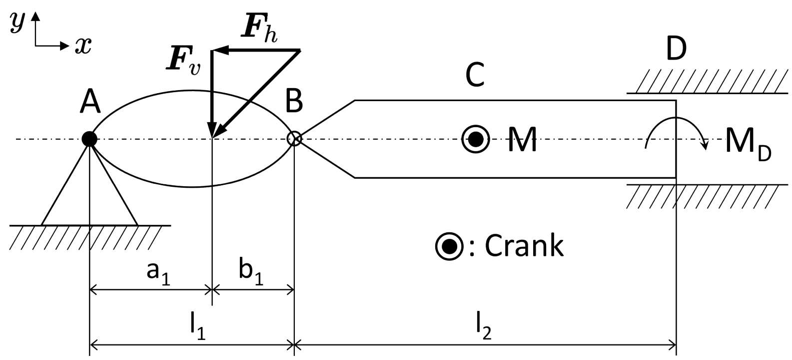

Fig. 2.14: Example for the calculation of bearing reaction forces.

The first calculation is carried out without the bearing B, i.e., B rigid as in fig. 2.15. The equations for forces and force moments can be read directly from fig. 2.15. For the forces one finds

$$F_x: \quad \mathrm{A}_H - F_h = 0$$

$$F_y: \quad \mathrm{A}_V - F_v - \mathrm{D}_V = 0$$

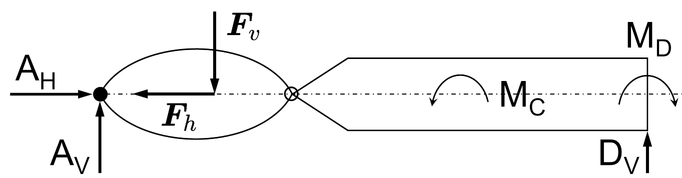

Fig. 2.15

and for the torques applies

$$M_{zA}: \quad a_1 F_v - M_c - (l_1 + l_2) D_V + M_D = 0$$

$$M_{zB}: \quad -M_c - l_2 D_V + M_D = 0 \qquad \text{(free movement around B)}$$

From the two equations for the force moments one obtains by subtraction:

$$a_1 F_v - l_1 D_V = 0$$

From the equations for the forces it follows:

$$\mathrm{A}_H = F_h \quad ; \quad \mathrm{A}_V = \frac{b_1}{l_1} F_v \quad ; \quad D_V = \frac{a_1}{l_1} F_v$$

Inserting the last equation into the condition for M_{zB} yields for the torque M_D

$$M_D = M_C + l_2 \frac{a_1}{l_1} F_v$$

Sectional cut in B:

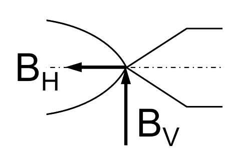

Fig. 2.16

$$\mathrm{A}_H - F_h + B_H = 0 \quad \Rightarrow \quad B_H = 0$$

$$\mathrm{A}_V - F_v + B_V = 0 \quad \Rightarrow \quad B_V = D_V$$

2.3.2 Trusses

In Central Europe, half-timbered construction goes back to the twelfth century, but it probably has a much longer history. A half-timbered construction, or more generally a truss construction, is generally understood to mean a wooden framework in which diagonally installed struts reduce the forces acting on the construction to forces directed downwards, i.e., normal forces. Trusses are not only important in timber construction, but also for steel and concrete constructions.

Definition
Trusses are constructions consisting of "bars", "nodes", and "bearings".

a) Bars are straight and rigid.

b) Bars are centrally connected in nodes through hinges.

c) Forces only attack in nodes.

Because of **b)** there are no torques in the nodes. An important assumption is that the bars themselves do not have a weight of their own; their weight is thought to be distributed over the nodes. This means that the bars of a truss do not experience any bending stress and forces always act only in the direction of the axis of a bar.

Definition

a) A truss is kinematically determinate if the nodes are fixed.

b) A truss is statically determinate if the equilibrium conditions are sufficient for bearing and "bar" forces.

If s denotes the number of bars and k the number of nodes of a truss and $\sum w_i$ the sum of the applied forces and force moments, then the following applies:

$$s + \sum w_i = 2k \qquad \text{in the plane}$$

and

$$s + \sum w_i = 3k \qquad \text{in three-dimensional space}$$

A truss can only be kinematically and statically determinate if the respective condition in the plane or in space is fulfilled. There are graphical and mathematical methods for calculating the bar forces, the so-called nodal method is one of them.

The nodal method

So that the truss sketched in fig. 2.17 is kinematically and statically determinate, the following must apply:

$$s + 3 \;=\; 2k$$

The result of counting bars and nodes is:

$$9 + 3 \;=\; 12$$

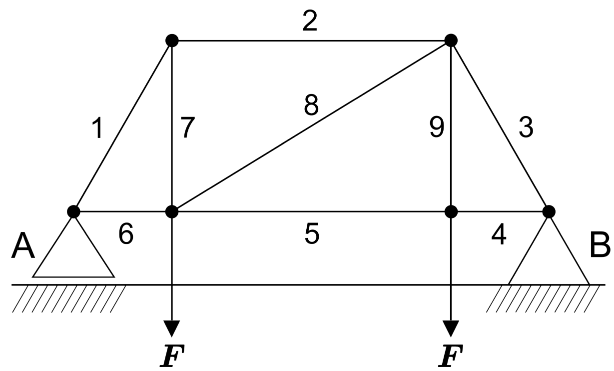

Fig. 2.17: Nodal method truss example.

The condition is therefore fulfilled. The nodal method considers individual nodes cut free from the truss. For each such node isolated from the truss, it must then hold that all forces acting on it cancel each other out, that is $\sum \boldsymbol{F} = 0$. Another important feature of the nodal method is that after the nodes have been cut free, the bar forces must point away from the nodes (if this were not the case, cutting the nodes free would cause the truss construction to collapse). For the planar truss under consideration here, the number of equilibrium conditions is $2k$ because for each node the forces acting in x- and in y-direction must be in equilibrium. The conditional equations for each of the six nodes can be read from fig. 2.18, which shows a sketch of the nodes I through VI cut out from the truss, along with their respective associated force arrows and angles.

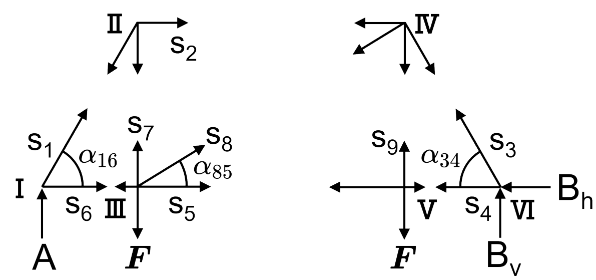

Fig. 2.18: Cut free nodes of the truss shown in fig. 2.17 (s_i = bar forces).

$$\textbf{I}\quad F_x:\quad s_1\cos\alpha_{16} + s_6 \;=\; 0 \qquad\qquad \textbf{II}\quad F_x:\quad -s_1\cos\alpha_{16} + s_2 \;=\; 0$$

$$F_y:\quad s_1\sin\alpha_{16} + A \;=\; 0 \qquad\qquad F_y:\quad -s_1\sin\alpha_{16} + s_7 \;=\; 0$$

$$\textbf{III}\quad F_x:\quad s_5 - s_6 + s_8\cos\alpha_{85} \;=\; 0 \qquad\qquad \textbf{IV}\quad F_x:\quad -s_2 - s_3\cos\alpha_{34} - s_8\cos\alpha_{85} \;=\; 0$$

$$F_y:\quad -s_7 - s_8\sin\alpha_{85} + F \;=\; 0 \qquad\qquad F_y:\quad s_9 + s_3\sin\alpha_{34} + s_8\sin\alpha_{85} \;=\; 0$$

$$\textbf{V}\quad F_x:\quad s_4 - s_5 \;=\; 0 \qquad\qquad \textbf{VI}\quad F_x:\quad -s_3\cos\alpha_{34} - s_4 - B_h \;=\; 0$$

$$F_y:\quad s_9 - F \;=\; 0 \qquad\qquad F_y:\quad -s_3\sin\alpha_{34} - B_v \;=\; 0$$

It follows immediately from these equations that

$$s_1 = -\frac{A}{\sin\alpha_{16}} \quad ; \quad s_6 = A\cot\alpha_{16} \quad ; \quad s_2 = -A\cot\alpha_{16} \quad ; \quad s_7 = A$$

$$s_8 = \frac{1}{\sin\alpha_{85}}(F - A) \quad ; \quad s_5 = A\cot\alpha_{16} - (F - A)\cot\alpha_{85} = s_4 \quad ; \quad s_9 = F$$

$$s_3 = \frac{1}{\sin\alpha_{34}}(A - 2F) \quad ; \quad B_v = 2F - A$$

If one now requires that no horizontal forces should act on the fixed pivot bearing, that is $B_h = 0$, then it follows

$$A(\cot\alpha_{16} + \cot\alpha_{34} + \cot\alpha_{85}) \;=\; F(2\cot\alpha_{34} + \cot\alpha_{85})$$

and with $\cot\alpha_{16} = \cot\alpha_{34}$ follows: $\quad A \;=\; F \;=\; B_v$

The sectional method

Another method for calculating the bar forces in a truss is the sectional method. With this method, the problem of calculating a bar force is partly reduced to solving a bearing problem. For this purpose, one cuts off part of the truss in a suitable node. Then, one considers the necessary equilibrium of moments which will ensure the partial truss remains stable after the cut. Fig. 2.19 illustrates this procedure for the truss to which the node procedure has been applied above.

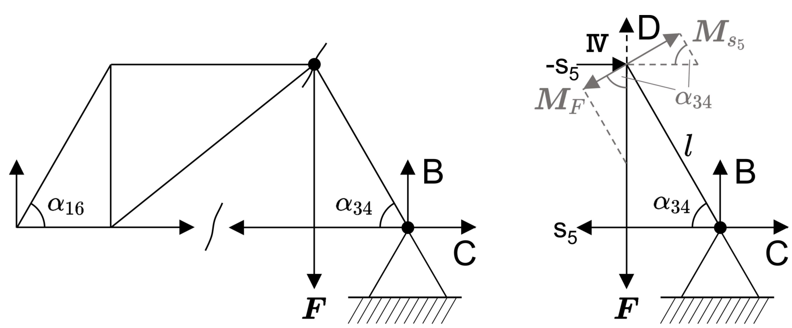

Fig. 2.19: Sectional method: (a) cuts to remove the partial truss from the truss shown in fig. 2.17; (b) free-standing partial framework in momentum equilibrium.

The bar forces of the two cut bars cancel each other out. Requiring that horizontal bearing forces must vanish means $C = 0$. With $B = F$ follows $D = 0$. For the partial truss in fig. 2.19b to remain free-standing after the cut, the torque exerted by the node IV on the pivot bearing via the lever l must disappear. This means it must apply:

$$M_{\mathrm{IV}} \;=\; l(F\cos\alpha_{34} - s_5\sin\alpha_{34}) \;=\; 0$$

and thus

$$s_5 = F\cot\alpha_{34}$$

Since $\alpha_{16} = \alpha_{34}$ this corresponds to the result from the nodal method.

Example 2.6 **Truss with load**

The truss under consideration in fig. 2.20 is made up of 6 equal-length massless bars and is firmly anchored at the points A and D. For this truss loaded in the node C with the force P, the tensile and compressive forces in the bars must be determined.

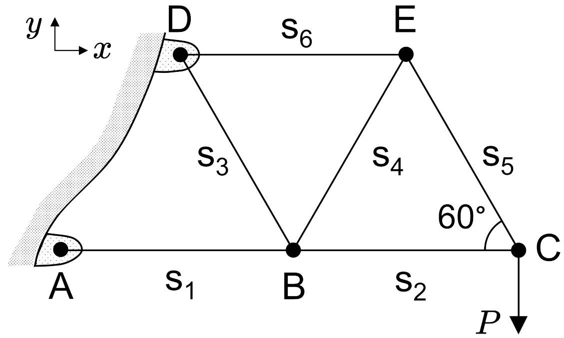

Fig. 2.20

In a first step, the truss is cut free at the points A and D and the fixed anchoring of these nodes is being replaced with horizontally and vertically acting forces F_{A_H} and F_{A_V} in A and F_{D_H} and F_{D_V} in D, respectively.

In a second step, the equations describing the balance of forces in the five nodes A, B, C, D and E are being formulated.

$$\mathrm{A}: \; F_x: \; s_1 + F_{A_H} = 0 \qquad F_y: \; F_{A_V} = 0$$

$$\mathrm{B}: \; F_x: \; s_2 - s_1 + s_4 \cos 60° - s_3 \cos 60° = 0 \qquad F_y: \; s_3 \sin 60° + s_4 \sin 60° = 0$$

$$\mathrm{C}: \; F_x: \; -s_2 + s_5 \cos 60° = 0 \qquad F_y: \; -P + s_5 \sin 60° = 0$$

$$\mathrm{E}: \; F_x: \; -s_6 - s_4 \cos 60° + s_5 \cos 60° = 0 \qquad F_y: \; -s_4 \sin 60° - s_5 \sin 60° = 0$$

$$\mathrm{D}: \; F_x: \; F_{D_H} + s_6 + s_3 \cos 60° = 0 \qquad F_y: \; F_{D_V} - s_3 \sin 60° = 0$$

By solving these equations, one can determine the tensile and compressive forces acting on the six bars ($\sin 60° = \sqrt{3}/2$ and $\cos 60° = 1/2$).

Compressive forces:

$$s_1 = -\sqrt{3}\cdot P \quad ; \quad s_2 = -\frac{1}{\sqrt{3}}\cdot P \quad ; \quad s_4 = -\frac{2}{\sqrt{3}}\cdot P$$

Tensile forces:

$$s_3 = \frac{2}{\sqrt{3}}\cdot P \quad ; \quad s_5 = \frac{2}{\sqrt{3}}\cdot P \quad ; \quad s_6 = \frac{2}{\sqrt{3}}\cdot P$$

In addition, one obtains for the forces in the nodes A and D

$$F_{A_H} = \sqrt{3}\cdot P \quad ; \quad F_{A_V} = 0 \quad ; \quad F_{D_H} = -\sqrt{3}\cdot P \quad ; \quad D_{A_V} = P$$

2.4 Frictional Forces

In our daily life we encounter frictional forces everywhere. Without them, many things in our world which we take for granted would not work. The best example is probably walking. We all know how difficult it can be to walk without or with little frictional force when we step onto an icy surface. Without friction, we would also not be able to drive our cars around a curve either. It is the frictional force between the tire surfaces and the road surface that makes it possible for us to steer our cars around curves, frequently at high speed. As we know, even that has its limits. If a car enters a curve with too high a velocity and the centrifugal force becomes greater than the maximum frictional force keeping the car on the road, then the car will quickly leave the roadway completely. Frictional forces are contact forces, they arise where two bodies touch. When touching, bodies can be at rest, then one speaks of static friction, or they can move relative to one another, then one speaks of sliding friction. Static and sliding friction as discussed here briefly, also referred to as Coulomb's friction after Charles Augustin de Coulomb, are both examples of so-called external friction, i.e., the friction between rigid bodies. Different from that is the so-called internal friction which underpins a body's viscosity and a body's toughness, and which will be discussed in a later section.

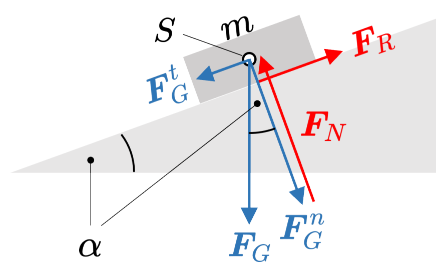

Fig. 2.21

Static friction will be considered first. Fig. 2.21 shows a block of mass m on an inclined plane. The angle of inclination α of the plane shall be set in such a way that the block does not slide but remains stuck. The weight force $\boldsymbol{F}_G$ acts in the center of mass S of the block and its component normal to the contact surface $\boldsymbol{F}_G^n$ is compensated by an opposite force of equal magnitude $\boldsymbol{F}_N$, the force with which the inclined plane opposes the block (in a later section we will encounter this forces again as the constraint force of the inclined plane). The pressure exerted by the normal force $\boldsymbol{F}_N$ causes a frictional force $\boldsymbol{F}_R$ in the contact plane which counteracts the tangential component of the weight force $\boldsymbol{F}_G^t$. As long as $\boldsymbol{F}_G^t$ is smaller than the maximum value that the frictional force can assume for the system of block material and material of the inclined plane, $\boldsymbol{F}_R$ will adjust itself just so that $\boldsymbol{F}_R + \boldsymbol{F}_G^t = 0$. However, if $\boldsymbol{F}_G^t$ exceeds the maximum value that $\boldsymbol{F}_R$ can assume, then the block begins to slide. The equilibrium condition for the maximum value of $F_R = |\boldsymbol{F}_R|$ is

$$F_R - m \cdot g \cdot \sin \alpha_{max} \;=\; 0 \tag{2.9}$$

where α_{max} is the angle at which the block just does not begin to slide. Static friction is a reaction force in response to the pressure exerted by F_N. The maximum value that F_R can assume will therefore depend on F_N:

$$F_R = \mu_0 \cdot F_N \tag{2.10}$$

where μ_0 is the so-called coefficient of static friction. The maximum value of the static friction on the inclined plane is therefore given by

$$F_R = \mu_0 \cdot m \cdot g \cdot \cos \alpha_{max} \tag{2.11}$$

From eq. (2.9) and eq. (2.11) it follows for μ_0

$$\mu_0 = \tan \alpha_{max} \tag{2.12}$$

Intriguingly, the inclined plane thus offers a very simple way to determine μ_0 by measuring α_{max} for any arbitrary combination of block material and material of the inclined plane.

The angle $\rho_0 = \alpha_{max}$ is the opening angle of the so-called friction cone (fig. 2.22). This opening angle is formed below the contact plane by the vectors of the normal force $\boldsymbol{F}_N$ and the maximum frictional force $\boldsymbol{F}_R^{max}$ which combine through vector addition to a virtual equivalent force lying in the cone shell. As long as the resultant of all external forces remains above the plane of contact and within this friction cone or in this cone shell, the body under consideration remains at rest. In the case of the inclined plane, $\boldsymbol{F}_G^t$ and $\boldsymbol{F}_G^n$ add up to the resulting external force, the weight force $\boldsymbol{F}_G$. The same argument also applies if, for example, $\alpha = 0$, with the difference being that in this case $\boldsymbol{F}_G^n = \boldsymbol{F}_G$ and $\boldsymbol{F}_G^t$, for example, is replaced by the displacement force with which one tries to move the body from its rest position.

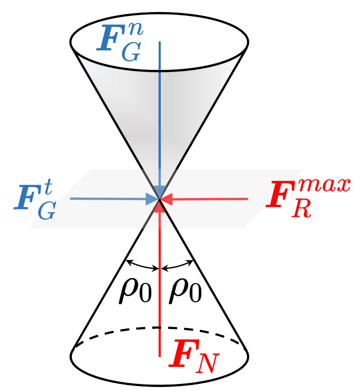

Fig. 2.22

Just as the maximum static friction force is proportional to the normal force F_N, the sliding friction force is also proportional to F_N. The proportionality factor for sliding friction is the sliding friction coefficient μ

$$F_R = \mu \cdot F_N \tag{2.13}$$

In most cases, the coefficient of dynamic friction (sliding friction) μ is smaller than the coefficient of static friction μ_0. That is, once the static friction between two bodies is overcome, less force is required to maintain the relative motion than was needed to start it. Tab. 7.1 lists μ_0 and μ values for some material compositions of practical importance. The values measured for such material pairings depend on the specific surface properties of the bodies in contact. Therefore, in the literature one often does not find single values for μ_0 and μ but value ranges.

Tab. 2.1: Coefficients of static and dynamic friction μ_0 and μ for some selected material pairings.

Material pair	**μ_0 - Static friction**		**μ_0 - Gliding friction**	
	dry	**greased**	**dry**	**greased**
Steel on steel	0.15 - 0.30	0.10 - 0.12	0.10 - 0.12	0.01 - 0.07
Steel on ice	0.03	–	0.01	–
Wood on wood	0.40 - 0.60	0.16	0.20 - 0.40	0.05 - 0.10
Leather on metal	0.30 - 0.50	0.16	0.30	0.15
Rubber on asphalt	0.70 - 0.90	0.10[1)]	0.50 - 0.60	0.05[1)]

[1)] Values on ice, not greased.

Example 2.7 Rope friction

Consider a rope that runs over a fixed cylindrical disk that is anchored non-rotatably (fig. 2.23a). At one end of the rope hangs a weight with the weight force $\boldsymbol{F}_G$; at the other end of the rope someone pulls with the rope force $\boldsymbol{F}_S$ just enough to keep the weight in balance. The force $\boldsymbol{F}_S$ required for this depends on the wrap angle α. The forces $\boldsymbol{F}_G$ and $\boldsymbol{F}_S(\alpha)$ are tangential forces on the cylinder disk.

To derive the equation which describes rope friction, one considers an infinitesimal angle element $\mathrm{d}\varphi$ of the cylinder disk over which the rope runs (fig. 2.23b). As a result of the forces acting on the left and right of the respective piece of rope, it stretches over the cylinder disk and is pressed against its surface. Now one assumes that the resulting pressure distribution over the angular segment $\mathrm{d}\varphi$ is uniform. In doing so one views this pressure distribution as caused by a single force $\mathrm{d}\boldsymbol{F}_N$ applied to the mid-point of the contact area (at $\mathrm{d}\varphi/2$). As can be seen from fig. 2.23b, in a first approximation ($\sin\varphi \approx \varphi$) one can write

$$\mathrm{d}F_N = 2\cdot F\cdot\sin\frac{\mathrm{d}\varphi}{2} = F\cdot\mathrm{d}\varphi \tag{2.14}$$

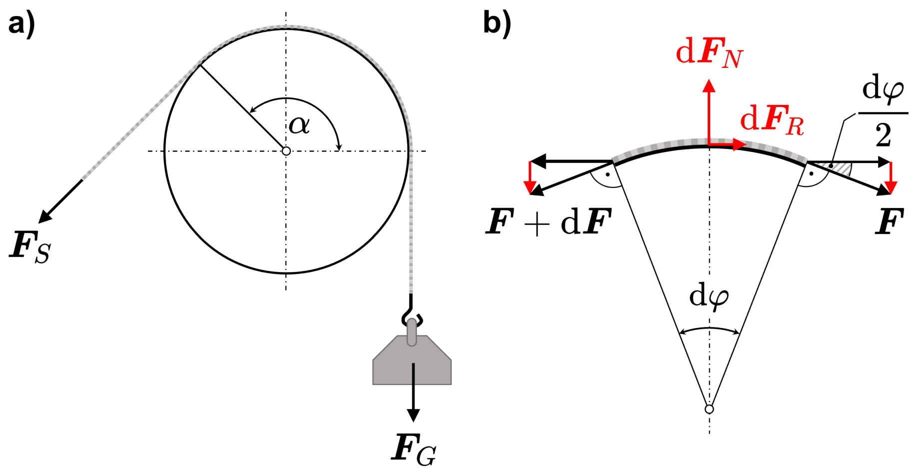

Fig. 2.23: Rope running over a cylinder surface: forces acting on the rope (a) and forces acting on the rope element with angle $\mathrm{d}\varphi$ (b).

If, as sketched in fig. 2.23b, the rope force at one end is slightly greater (by $\mathrm{d}\boldsymbol{F}$) than at the other end, then the rope slides in the direction of the slightly greater force. Since the coefficient of static friction μ_0 for the material pair of rope and cylinder surface is not equal to zero, $\mathrm{d}\boldsymbol{F}_N$ causes at $\mathrm{d}\varphi/2$ a frictional force $\mathrm{d}\boldsymbol{F}_R$ in the contact plane between the rope and the surface of the cylinder disk. Let the force difference $\mathrm{d}\boldsymbol{F}$ between the two rope ends in fig. 2.23b now be just large enough that $\mathrm{d}F = \mathrm{d}F_R$. Then the equilibrium condition for maximum static friction is fulfilled and it holds that

$$\mathrm{d}F \;=\; \mathrm{d}F_R \;=\; \mu_0 \cdot \mathrm{d}F_N \;=\; \mu_0 \cdot F \cdot \mathrm{d}\varphi \tag{2.15}$$

With that one now has a differential equation for the tangential force F and the angle φ. This differential equation can be solved by separating the variables F and φ and subsequent integration. The angle φ runs from 0 to the respective wrap angle α and F from F_0 to F_α, so

$$\int_{F_0}^{F_\alpha} \frac{\mathrm{d}F}{F} \;=\; \ln F_\alpha - \ln F_0 \;=\; \mu_0 \cdot \int_0^\alpha \mathrm{d}\varphi \;=\; \mu_0 \cdot \alpha$$

From this follows the basic equation for rope friction

$$F_\alpha \;=\; F_0 \cdot \mathrm{e}^{\mu_0 \alpha} \qquad \text{(Euler-Eytelwein equation)} \tag{2.16}$$

This equation states that a smaller force F_0 ($\boldsymbol{F}_G$ in fig. 2.23a) is sufficient to compensate a larger force F_α ($\boldsymbol{F}_S$ in fig. 2.23a). Eq. (2.16) formulates an equilibrium condition: only when F_α reaches the value in eq. (2.16) can the weight force no longer compensate the force F_α and the rope will slide to the left, thereby lifting the weight. Eq. (2.16) was derived for the case that in the sketch of fig. 2.23 $\boldsymbol{F}_S(\alpha) > \boldsymbol{F}_G$ (by d$\boldsymbol{F}$). If, however, the reverse case occurs, that is $\boldsymbol{F}_S(\alpha) < \boldsymbol{F}_G$ (again by d$\boldsymbol{F}$), then the arrow of d$\mathbf{F}_R$ in fig. 2.23b points in the opposite direction and one then finds that the following holds:

$$F_\alpha = F_0 \cdot \mathrm{e}^{-\mu_0 \alpha} \tag{2.17}$$

Hence, eq. (2.16) is the equation for lifting a load and eq. (2.17) for lowering a load. With the construction in fig. 2.23, the practical aspect is of course of interest, i.e., how much less force is required to lift a weight by means of a winch as in fig. 2.23a, compared to lifting it straight up. The maximum angle α in fig. 2.23a is π. With a static friction coefficient of $\mu_0 = 0.35$, for example, this results in a force reduction by a factor of approx. 3.0. It is thus possible to compensate for a force that is 3.0 times as large. But much more is possible. If the rope is looped around the cylindrical disk several times, each loop increases the wrap angle α by a factor 2π and thus increases the force reduction by a further factor 9.0. With a rope that is wound around a bollard several times, very large forces can be compensated, as is for example the case when mooring ships in a harbor.

3. Elastostatics

The field of elastostatics deals with the mechanics of reversibly deformable bodies under external loads. These loads can have various causes, but they are always small enough so that the deformation of a body will scale linearly with the load and will remain reversible. In the context of elastostatics, bodies are treated as continua, the respective discrete inner structure of which plays no role in the behavior of a body itself. Elastostatics, also known as strength of materials in German-speaking countries, is therefore a sub-field of continuum mechanics. Macroscopic parameters that describe the behavior of a body under load include for example stress, strain, torsion, the bending of a solid, or the equilibrium pressure in liquids.

3.1 Stresses

Forces can act in the interior of a continuum body, and they can act on its surfaces. Forces acting on a volume element in the interior of a continuum body are exerted by its respective neighboring volume elements. Each volume element $\mathrm{d}V$ (fig. 3.1) within a continuum body transmits forces associated with the corresponding force density this volume element possesses. In one were to cut such a volume element $\mathrm{d}V$ free from its neighboring volume elements, these forces must continue to act on the sectional areas of $\mathrm{d}V$ so that the state of stress of the cut-out volume element will remain identical to the state of stress the volume element was subject to before it was cut free (sectional cutting principle).

For a continuum body the following applies:

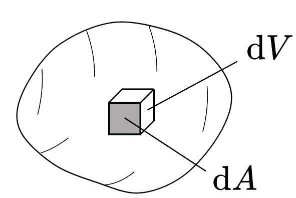

Fig. 3.1

— Force density $\boldsymbol{f}(\boldsymbol{r})$: Associated with volume element $\mathrm{d}V$.

— Stress: Associated with sectional cut areas $\mathrm{d}A$.

From the force density $\boldsymbol{f}(\boldsymbol{r})$ follows the differential force $\mathrm{d}\boldsymbol{F}$

$$\mathrm{d}\boldsymbol{F} \;=\; \boldsymbol{f}(\boldsymbol{r})\mathrm{d}V \tag{3.1}$$

In addition to internal forces, there are external forces acting on each volume element of the continuum body. An example of this is gravity:

$$\boldsymbol{f}_{gravity} \;=\; -\varrho\cdot g\cdot \boldsymbol{e}_z \qquad (\varrho = \text{ mass density})$$

For the stress of a section surface element $\mathrm{d}A$ of a cut volume element $\mathrm{d}V$ the following must apply:

$$\boldsymbol{P}\,\mathrm{d}A \;=\; \mathrm{d}\boldsymbol{F}$$

Here $\boldsymbol{P}$ is the so-called stress vector with the dimension force / area. The total stress at a section surface element $\mathrm{d}A$ is broken down into a stress component perpendicular to this surface element, the so-called normal stress, and a stress component parallel to this surface element, the so-called shear stress.

The following applies to these stress components

— Shear stress: $\quad |\mathrm{d}\boldsymbol{F}_t| \;=\; \tau\mathrm{d}A$

— Normal stress: $\quad |\mathrm{d}\boldsymbol{F}_n| \;=\; \sigma\mathrm{d}A$

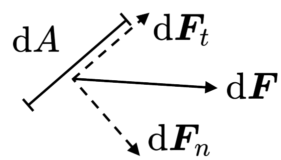

Fig. 3.2

The convention is that the Greek letter σ is used to designate normal stresses and the Greek letter τ is used for shear stresses. Positive normal stresses are referred to as tensile stresses and negative normal stresses are called compressive stresses. Tensile stresses always point in the direction of the surface normal vector, i.e., away from the section surface element, and vice versa for compressive stresses. Shear stresses can also be positive or negative. If a shearing force produces a positive moment with respect to the center of mass of the section surface element, i.e., a counter-clockwise rotation, then the shear stress is per definition positive; if it generates a negative moment, then the shear stress is negative.

Shear stresses are further broken down into their respective components along the coordinate axes lying in the section surface element. In this context the first index of τ stands for the orientation of the section surface element and the second index for the direction of the shear stress component. For example, τ_{xy} denotes a shear stress in a surface whose normal vector is parallel to the x-coordinate and which itself points in the direction of the y-coordinate in this surface.

3.1.1 On the Derivation of the Stress Tensor

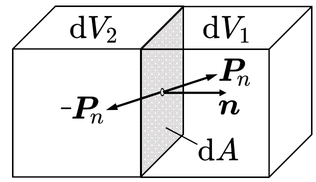

Fig. 3.3

Let $\mathrm{d}A$ be the surface element with normal vector $\boldsymbol{n}$ which separates the volume elements $\mathrm{d}V_1$ and $\mathrm{d}V_2$ in fig. 3.3 from each other. With $\boldsymbol{P}_n$ as the associated stress vector, $\mathrm{d}V_1$ then exerts a force on $\mathrm{d}V_2$ proportional to $\mathrm{d}A$:

$$\mathrm{d}\boldsymbol{F}_{12} = \boldsymbol{P}_n \mathrm{d}A$$

Conversely, $\mathrm{d}V_2$ exerts the same opposite force on $\mathrm{d}V_1$:

$$\mathrm{d}\boldsymbol{F}_{21} = -\boldsymbol{P}_n \mathrm{d}A$$

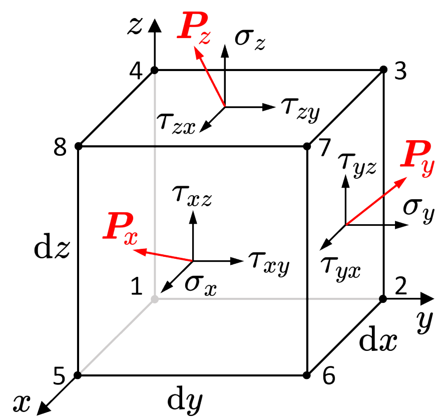

Fig. 3.4

A cuboid volume element $\mathrm{d}V$ such as the one shown in fig. 3.4 has six neighboring volume elements. At each of the six boundary surfaces of $\mathrm{d}V$ there exists a corresponding equilibrium of forces with the respective neighboring volume element. To calculate the total force on such a volume element $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ the forces transmitted at the respective boundary surfaces are added with the correct sign. For forces in the x-direction, these are the boundary surfaces normal to the x-axis in fig. 3.4 with the corner points 1-2-3-4 and 5-6-7-8. First one only considers the surface area 1-2-3-4 by itself. On both sides of this surface the same force acts, but it does so in opposite directions: namely $-\boldsymbol{P}_x\mathrm{d}y\mathrm{d}z$ on the outside of this surface (negative x-direction) and $\boldsymbol{P}_x\mathrm{d}y\mathrm{d}z$ on the inside of this surface (positive x-direction). If one now moves a copy of this surface along the x-axis in the direction of the surface 5-6-7-8, the acting force on both sides of the surface will changes by an amount proportional to the displacement of this surface times the respective change of $\boldsymbol{P}_x$ over the distance of that displacement. Virtually shifting this surface all the way to the position of the surface 5-6-7-8, the force transferred by $\mathrm{d}V$ in x-direction at this position becomes:

$$\mathrm{d}\boldsymbol{F}_{5678} = \left(\boldsymbol{P}_x + \frac{\partial \boldsymbol{P}_x}{\partial x}\mathrm{d}x\right)\mathrm{d}y\mathrm{d}z$$

The force transferred by $\mathrm{d}V$ at the position of the surface 1-2-3-4 still is:

$$\mathrm{d}\boldsymbol{F}_{1234} = -\boldsymbol{P}_x \mathrm{d}y\mathrm{d}z$$

With that, the total force transmitted by $\mathrm{d}V$ in x-direction thus becomes:

$$\mathrm{d}\boldsymbol{F}_{5678} + \mathrm{d}\boldsymbol{F}_{1234} = \frac{\partial \boldsymbol{P}_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

With similar considerations for the forces acting on the surfaces normal to the y- or z-axis, one obtains for the total force transmitted by the volume element $\mathrm{d}V$ in fig. 3.4

$$\mathrm{d}\boldsymbol{F} = \Big(\frac{\partial \boldsymbol{P}_x}{\partial x} + \frac{\partial \boldsymbol{P}_y}{\partial y} + \frac{\partial \boldsymbol{P}_z}{\partial z}\Big)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{3.2}$$

With the axis unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ one can decompose the stress vectors $\boldsymbol{P}_x$, $\boldsymbol{P}_y$ and $\boldsymbol{P}_z$ into their directional components.

$$\begin{aligned}\boldsymbol{P}_x &= \hat{\boldsymbol{i}}\sigma_x + \hat{\boldsymbol{j}}\tau_{xy} + \hat{\boldsymbol{k}}\tau_{xz} \\ \boldsymbol{P}_y &= \hat{\boldsymbol{i}}\tau_{yx} + \hat{\boldsymbol{j}}\sigma_y + \hat{\boldsymbol{k}}\tau_{yz} \\ \boldsymbol{P}_z &= \hat{\boldsymbol{i}}\tau_{zx} + \hat{\boldsymbol{j}}\tau_{zy} + \hat{\boldsymbol{k}}\sigma_z\end{aligned} \tag{3.3}$$

Inserting this in eq. (3.2) yields for the force density $\boldsymbol{f}(\boldsymbol{r})$

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{r}) = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} &= \hat{\boldsymbol{i}}\Big(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\Big) + \\ &\hat{\boldsymbol{j}}\Big(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\Big) + \\ &\hat{\boldsymbol{k}}\Big(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\Big)\end{aligned} \tag{3.4}$$

or respectively

$$\boldsymbol{f}(\boldsymbol{r}) = \nabla\underline{\underline{\boldsymbol{\sigma}}} \quad \text{where} \quad \underline{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix} \tag{3.5}$$

is the stress tensor. The stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$ is a symmetric tensor of the second rank. For any arbitrary section surface element $\mathrm{d}A$ with normal vector $\boldsymbol{n}$ one obtains the respective stress vector $\boldsymbol{P}_n$ by multiplying $\underline{\underline{\boldsymbol{\sigma}}}$ with $\boldsymbol{n}$

$$\boldsymbol{P}_n = \boldsymbol{n}\underline{\underline{\boldsymbol{\sigma}}} \tag{3.6}$$

Because the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$ is symmetric, it makes no difference for the determination of $\boldsymbol{P}_n$ whether one multiplies the stress tensor with the normal vector $\boldsymbol{n}$ from the left or from the right.

3.1.2 Planar Stress

To calculate the planar state of stress at a given point P of a continuum body, one uses the equilibrium of moments at this point P as well as the equilibrium of forces in the corresponding section surface element in which P lies. The situation is illustrated in fig. 3.5, which shows the section through a body with an arbitrary section angle φ. Now the state of stress must be determined at the point $P = \big(\mathrm{d}x/2, \mathrm{d}y/2\big)$ of this section surface element.

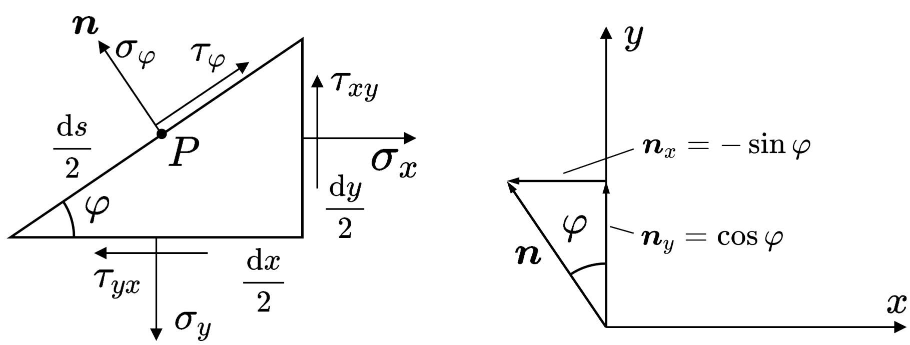

Fig. 3.5: Calculation of the stress at a point P in an arbitrary section surface.

First step: Determination of the equilibrium of moments in P.

$$\underset{\substack{\uparrow \\ \text{3-dim. case}}}{(\mathrm{d}z)}\left(\tau_{xy}\mathrm{d}y\frac{\mathrm{d}x}{2} - \tau_{yx}\mathrm{d}x\frac{\mathrm{d}y}{2}\right) = 0 \qquad \Rightarrow \qquad \tau_{xy} = \tau_{yx} \tag{3.7}$$

Generalized for the three-dimensional case:

$$\left.\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned}\right\} \tag{3.8}$$

Second step: The task is to establish the equations for the equilibrium of forces in the xy-plane. Equilibrium of forces in the xy-plane means that in fig. 3.5 the sum of all forces F_x in the x-direction must be zero and the same must apply to the sum of all forces F_y in y-direction. Hence

$$F_x: \quad \big(-\sigma_\varphi \sin\varphi + \tau_\varphi \cos\varphi\big)\mathrm{d}s\mathrm{d}z + \sigma_x \mathrm{d}y\mathrm{d}z - \tau_{yx}\mathrm{d}x\mathrm{d}z = 0$$

$$F_y: \quad \big(\sigma_\varphi \cos\varphi + \tau_\varphi \sin\varphi\big)\mathrm{d}s\mathrm{d}z + \tau_{xy}\mathrm{d}y\mathrm{d}z - \sigma_y \mathrm{d}x\mathrm{d}z = 0$$

Therefore, one has to solve the following system of equations

$$\left.\begin{aligned}\big(-\sigma_\varphi\sin\varphi+\tau_\varphi\cos\varphi\big)\mathrm{d}s\mathrm{d}z &= -\sigma_x\mathrm{d}y\mathrm{d}z+\tau_{yx}\mathrm{d}x\mathrm{d}z\\ \big(\sigma_\varphi\cos\varphi+\tau_\varphi\sin\varphi\big)\mathrm{d}s\mathrm{d}z &= -\tau_{xy}\mathrm{d}y\mathrm{d}z+\sigma_y\mathrm{d}x\mathrm{d}z\end{aligned}\right\} \tag{3.9}$$

From fig. 3.5 one can read that

$$\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \mathrm{d}s\cdot\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} = \mathrm{d}s\cdot\begin{pmatrix}n_y\\ -n_x\end{pmatrix} \tag{3.10}$$

With $\boldsymbol{F} = \boldsymbol{P}_n\mathrm{d}s\mathrm{d}z$ in P, $\tau_{yx} = \tau_{xy}$ from eq. (3.7) and with eq. (3.10) it follows from the right-hand sides of eq. (3.9) that

$$\boldsymbol{P}_n = \begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}\begin{pmatrix}n_x\\ n_y\end{pmatrix} = \boldsymbol{n}\underline{\underline{\boldsymbol{\sigma}}} \tag{3.11}$$

where $\underline{\underline{\boldsymbol{\sigma}}}$ is the symmetric stress tensor in the plane.

If one multiplies in eq. (3.9) F_x by $\sin\varphi$ and F_y by $\cos\varphi$ and then subtracts the results from each other, one obtains an equation for σ_φ. Conversely, if one multiplies F_x by $\cos\varphi$ and F_y by $\sin\varphi$ and adds the resulting equations, then one obtains an equation for τ_φ. If one now uses the identity $\tau_{xy} = \tau_{yx}$ obtained from the equilibrium of moments (eq. (3.7)) and eq. (3.10), then the equations for σ_φ and τ_φ turn into

$$\left.\begin{aligned}\sigma_\varphi &= \frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_y-\sigma_x}{2}\cos 2\varphi-\tau_{xy}\sin 2\varphi\\ \tau_\varphi &= \frac{\sigma_y-\sigma_x}{2}\sin 2\varphi+\tau_{xy}\cos 2\varphi\end{aligned}\right\} \tag{3.12}$$

The angle 2φ in eq. (3.12) can be eliminated by squaring the equations and adding the results. Thus one obtains

$$\left(\sigma_\varphi-\frac{\sigma_x+\sigma_y}{2}\right)^2+\tau_\varphi^2 = \underbrace{\left(\frac{\sigma_y-\sigma_x}{2}\right)^2+\tau_{xy}^2}_{r^2} \tag{3.13}$$

The right-hand side of this equation, a constant which is independent of φ, is denoted here as r^2 since eq. (3.13) is obviously the equation of a circle with the variables σ_φ and τ_φ. The x-coordinate of the circle center is shifted on the σ_φ-axis from the coordinate origin by $(\sigma_y+\sigma_x)/2$. Fig. 3.6 shows this so-called Mohr circle of stress. This circle describes the value pairs σ_φ and τ_φ which as a function of the section surface angle φ are being traversed in the $\sigma_\varphi\tau_\varphi$-plane. The values for σ_φ and τ_φ for $\varphi = 0°$ can be obtained from eq. (3.12). As can already be seen from fig. 3.5, $\varphi = 0°$ means a section perpendicular to the y-axis and consequently $\tau_{\varphi=0} = \tau_{xy}$ and $\sigma_{\varphi=0} = \sigma_y$. For positive or negative φ, the angle 2φ in eq. (3.12) rotates away from this radius vector counterclockwise or clockwise, respectively. At the points σ_1 and σ_2, i.e., at the angles $2\varphi_1$ and $2\varphi_2$, the shear stress τ_φ vanishes. σ_1 and σ_2 are the so-called principal stresses and φ_1 and φ_2 define the corresponding principal stress directions. For these angles one can read from fig. 3.6 (φ_1 is negative):

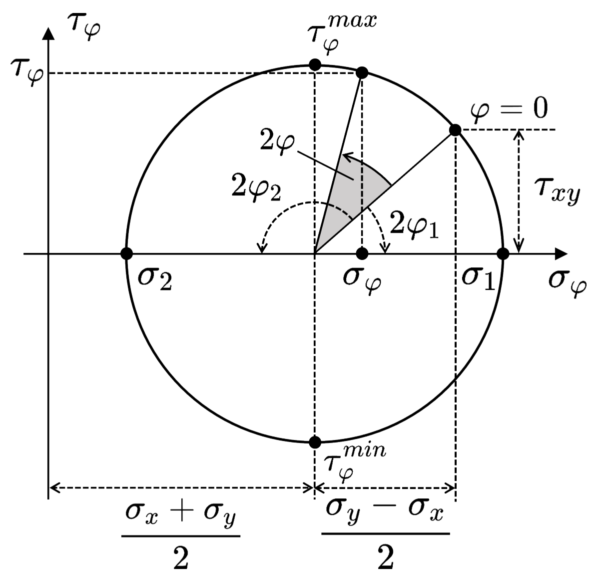

Fig. 3.6: Mohr's circle of stress.

$$2\varphi_2 - 2\varphi_1 = 180° = \pi \qquad \text{or respectively} \qquad \varphi_2 - \varphi_1 = \frac{\pi}{2}$$

The exact values of both angles can be obtained from eq. (3.12) by solving the optimization problem $\mathrm{d}\sigma_\varphi/\mathrm{d}\varphi = 0$. The maximum values of the shear stress are obtained analogously by solving $\mathrm{d}\tau_\varphi/\mathrm{d}\varphi = 0$. The 2φ angle between the respective extreme values of the normal stress, i.e., σ_1 and σ_2, and of the shear stress, i.e., τ_φ^{min} and τ_φ^{max}, is of course 90°. Therefore, the section surface angle of the maximum / minimum shear stress is rotated by 45° ($\pi/4$) compared to that of the maximum / minimum normal stress. In summary, for a plane stress state the following holds:

- There are always two mutually orthogonal principal stress directions. For these, the normal stresses assume extreme values and the shear stresses disappear.
- The two directions in which the shear stress has its extreme values are rotated by $\frac{\pi}{4}$ with respect to the respective principal stress directions.

3.1.3 The Stress Ellipsoid

An alternative method to Mohr's stress circle for the graphic representation of the state of stress in any given point of a continuum body is the so-called stress ellipsoid in the three-dimensional stress case or the stress ellipse in the planar stress case. The stress ellipse in fig. 3.7 is described by

$$\sum_{i,k} \sigma_{ik} x_i x_k \;=\; 1$$

In the principal axis representation:

$$\underbrace{\sigma_1}_{\frac{1}{a^2}} {x_1'}^2 + \underbrace{\sigma_2}_{\frac{1}{b^2}} {x_2'}^2 \;=\; 1$$

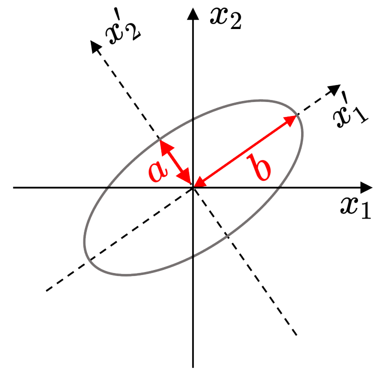

Fig. 3.7

The principal axes and principal stresses are provided by the eigenvalue equation

$$\underline{\underline{\boldsymbol{\sigma}}}\boldsymbol{x}^{(i)} \;=\; \sigma_i \boldsymbol{x}^{(i)} \quad \text{or written out} \quad \sum_i \sigma_{kl} x_l^{(i)} \;=\; \sigma_i x_k^{(i)} \tag{3.14}$$

where $\boldsymbol{x}^{(i)}$ are the eigenvectors and σ_i are the corresponding eigenvalues. The σ_{kl} are symmetric and the eigenvectors $\boldsymbol{x}^i$ are therefore all real and can be chosen to be pairwise orthogonal to each other.

$$\sum_k x_k^{(i)} x_k^{(j)} \;=\; \delta_{ij}$$

$$\underbrace{\begin{bmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \end{bmatrix}}_{\underline{\underline{\boldsymbol{x}}}' = \underline{\underline{\boldsymbol{x}}}^T} \times \underbrace{\begin{bmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{bmatrix}}_{\underline{\underline{\boldsymbol{x}}}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices $\underline{\underline{\boldsymbol{x}}}'$ and $\underline{\underline{\boldsymbol{x}}}^T$ are orthogonal to $\underline{\underline{\boldsymbol{x}}}$. Or in other words:

$$\underline{\underline{\boldsymbol{x}}}' = \underline{\underline{\boldsymbol{x}}}^T = \underline{\underline{\boldsymbol{x}}}^{-1}$$

The eigenvalue equations according to eq. (3.14) can be summarized as

$$\underline{\underline{\boldsymbol{\sigma}}}\,\underline{\underline{\boldsymbol{x}}} \;=\; \underline{\underline{\boldsymbol{\sigma}}}^D \underline{\underline{\boldsymbol{x}}} \;=\; \underline{\underline{\boldsymbol{x}}}\,\underline{\underline{\boldsymbol{\sigma}}}^D \tag{3.15}$$

where

$$\underline{\underline{\boldsymbol{\sigma}}}^D \;=\; \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

is the diagonalized stress tensor where all mixed tensor elements (the shear stresses) disappear, and the principal stresses are the eigenvalues in the tensor diagonal. By multiplying eq. (3.15) from the left with $\underline{\underline{\boldsymbol{x}}}'$ it follows directly that

$$\underline{\underline{\boldsymbol{\sigma}}}^D \;=\; \underline{\underline{\boldsymbol{x}}}'\underline{\underline{\boldsymbol{\sigma}}}\,\underline{\underline{\boldsymbol{x}}}$$

3.1.4 Forces and Moments in the 3-dimensional Case

The equations for the equilibrium of forces and moments of a volume element $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ are determined analogously to the 2-dimensional case (section 3.1.2). If f_x, f_y and f_z again denote volume forces in the corresponding axis directions, then for the cuboid in fig. 3.4 to be in a force equilibrium, the requirement is that the total force $\boldsymbol{F} = (F_x, F_y, F_z)$ must vanish in each coordinate direction. Hence

$$
\begin{aligned}
F_x &= 0 = \mathrm{d}\sigma_x \mathrm{d}y\mathrm{d}z + \mathrm{d}\tau_{yx}\mathrm{d}x\mathrm{d}z + \mathrm{d}\tau_{zx}\mathrm{d}x\mathrm{d}y + f_x\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
F_y &= 0 = \mathrm{d}\tau_{xy} \mathrm{d}y\mathrm{d}z + \mathrm{d}\sigma_{y}\mathrm{d}x\mathrm{d}z + \mathrm{d}\tau_{zy}\mathrm{d}x\mathrm{d}y + f_y\mathrm{d}x\mathrm{d}x\mathrm{d}y \\
F_z &= 0 = \mathrm{d}\tau_{xz} \mathrm{d}y\mathrm{d}z + \mathrm{d}\tau_{yx}\mathrm{d}x\mathrm{d}z + \mathrm{d}\sigma_{z}\mathrm{d}x\mathrm{d}y + f_z\mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}
$$

From that one obtains the equations

$$
\begin{aligned}
\frac{\mathrm{d}\sigma_x}{\mathrm{d}x} + \frac{\mathrm{d}\tau_{yx}}{\mathrm{d}y} + \frac{\mathrm{d}\tau_{zx}}{\mathrm{d}z} + f_x &= 0 \\
\frac{\mathrm{d}\tau_{xy}}{\mathrm{d}x} + \frac{\mathrm{d}\sigma_{y}}{\mathrm{d}y} + \frac{\mathrm{d}\tau_{zy}}{\mathrm{d}z} + f_y &= 0 \\
\frac{\mathrm{d}\tau_{xz}}{\mathrm{d}x} + \frac{\mathrm{d}\tau_{yx}}{\mathrm{d}y} + \frac{\mathrm{d}\sigma_{z}}{\mathrm{d}z} + f_z &= 0
\end{aligned}
$$

In summary and in general, the requirement for a balance of forces is

$$
\sum_k \frac{\partial \sigma_{ik}}{\partial x_k} + f_i = 0 \quad \text{with} \quad i = 1, 2, 3 \tag{3.16}
$$

For the equilibrium of moments with respect to the axes of rotation along σ_x, σ_y and σ_z through the center of the respective side faces of the cuboid in fig. 3.4 one finds for $\boldsymbol{M} = (M_x, M_y, M_z)$:

$$
\begin{aligned}
M_x &= \frac{1}{2}\big[\big(\hat{\boldsymbol{e}}_y \mathrm{d}y \times \hat{\boldsymbol{e}}_z \tau_{yz}\mathrm{d}x\mathrm{d}z\big) - \big(\hat{\boldsymbol{e}}_z \mathrm{d}z \times \hat{\boldsymbol{e}}_y \tau_{zy}\mathrm{d}x\mathrm{d}y\big)\big] = \frac{1}{2}(\tau_{yz} - \tau_{zy})\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \\
M_y &= \frac{1}{2}\big[\big(\hat{\boldsymbol{e}}_z \mathrm{d}z \times \hat{\boldsymbol{e}}_x \tau_{zx}\mathrm{d}x\mathrm{d}y\big) - \big(\hat{\boldsymbol{e}}_x \mathrm{d}x \times \hat{\boldsymbol{e}}_z \tau_{xz}\mathrm{d}y\mathrm{d}z\big)\big] = \frac{1}{2}(\tau_{zx} - \tau_{xz})\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \\
M_z &= \frac{1}{2}\big[\big(\hat{\boldsymbol{e}}_x \mathrm{d}x \times \hat{\boldsymbol{e}}_y \tau_{xy}\mathrm{d}y\mathrm{d}z\big) - \big(\hat{\boldsymbol{e}}_y \mathrm{d}y \times \hat{\boldsymbol{e}}_x \tau_{yx}\mathrm{d}x\mathrm{d}z\big)\big] = \frac{1}{2}(\tau_{xy} - \tau_{yx})\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0
\end{aligned}
$$

Not unexpectedly one gets the same result as in eq. (3.8), namely that $\tau_{yz} = \tau_{zy}$, $\tau_{zx} = \tau_{xz}$ and $\tau_{xy} = \tau_{yx}$ must apply so that all three components of $\boldsymbol{M}$ vanish. In other words, the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$ must be symmetric, i.e., $\sigma_{ik} = \sigma_{ki}$.

3.2 Strains

Strains are deformations of a continuum body due to changes in length and / or angle which cause the position vector $\boldsymbol{r} = (x, y, x)$ of a given point in the continuum body to be shifted by a displacement vector $\boldsymbol{\rho} = (\xi, \eta, \zeta)$. The displacement vector $\boldsymbol{\rho}$ of a deformation can indeed be constant, but more often the respective displacement coordinates ξ, η and ζ depend on the location, i.e., $\xi = \xi(x, y, z)$, $\eta = \eta(x, y, z)$ and $\zeta = \zeta(x, y, x)$. Naturally, strains are always the result of the effects of a force of some kind. Normal stresses cause stretching or compression of the continuum body, i.e., changes in length, while shear stresses cause so-called slips, i.e., changes in angle leading to distortions of a continuum body. An important part of the characterization of materials is the determination of strains as a function of applied stresses. The result of corresponding measurements are stress-strain diagrams which describe the strain behavior of a body under the applied stresses. A distinction is made between three different deformation domains:

- <u>Linear elastic:</u> The strain is proportional to the stress and it is reversible.
- <u>Nonlinear elastic:</u> The strain is no longer proportional to the stress but is still reversible.
- <u>Plastic:</u> The strain is no longer reversible, and the deformation will persist when the applied stress is removed.

In technical applications, the linear elastic domain is of great practical importance. In this domain, the respective changes in lengths and angles as compared to the initial lengths and initial angles are considered to be sufficiently small so that linear approximations are justified for calculating the resulting deformation. In the linear elastic domain, stresses (σ, τ) and strains, i.e., stretching or compression (ϵ) of a continuum body, as well as slips (γ), can be expressed through Hooke's laws:

$$\left.\begin{aligned} \sigma &= \mathrm{E} \cdot \epsilon \qquad \text{normal stresses} \\ \tau &= \mathrm{G} \cdot \gamma \qquad \text{shear stresses} \end{aligned}\right\} \tag{3.17}$$

Here, the modulus of elasticity E, also referred to as Young's modulus, and the shear modulus G are material-specific parameters. The strain behavior of continuum bodies can be very complex, since different strains usually overlap. In general, however, it is often possible to decompose the complex state of strain of a body into a superposition of simpler states of strain. The latter include simple tension or compression situations, as well as bending and distortion.

A) A simple tensile test

In the simple tensile test sketched in fig. 3.8, the weight attached to a hanging wire causes a uniaxial state of stress ($F = \sigma_z A$) in the wire which leads both to a change in the wire length l as well as to a change in the wire diameter d. The relative elongation of the wire ϵ_{zz} in the z-direction is given by

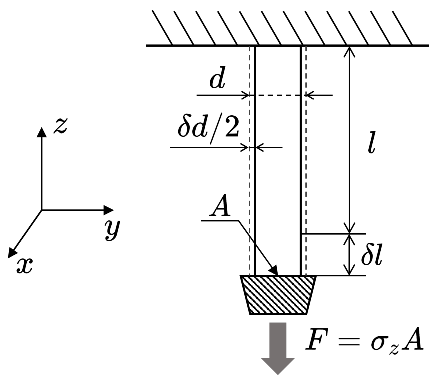

Fig. 3.8: Simple tensile test.

$$\epsilon_{zz} = \frac{\delta l}{l} = \frac{\sigma_z}{\mathrm{E}} \tag{3.18}$$

The decrease in wire diameter is described by the strain components ϵ_{xx} and ϵ_{yy} in the xy-plane.

$$\epsilon_{xx} = \epsilon_{yy} = \frac{\delta d}{d} = -\mu\epsilon_{zz} \tag{3.19}$$

where the so-called Poisson's ratio μ, also frequently referred to as Poisson's number, is a material-specific parameter. Just as with the components of the stress tensor, the indices of the components ϵ_{ik} of the strain tensor also describe the direction of the normal to the section plane (first index) and a directional coordinate in the section plane (second index).

B) Beam bending

Let ρ_1 and ρ_2 be the radii of curvature that describe the expansion or compression of the infinitesimal volume element sketched in fig. 3.9 in dx- or dy-direction. The stretching and compression caused by the bending of the beam can be read from the sketch. ϵ_{xx} and ϵ_{yy} are the strain components in the x- and in the y-direction, respectively. The following applies to these:

$$\epsilon_{xx} = \frac{(\rho_1 + z)\mathrm{d}\varphi - \rho_1\mathrm{d}\varphi}{\mathrm{d}x} = z\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{z}{\rho_1}$$

$$\epsilon_{yy} = \frac{(\rho_2 - z)\mathrm{d}\chi - \rho_2\mathrm{d}\chi}{\mathrm{d}y} = -z\frac{\mathrm{d}\chi}{\mathrm{d}y} = \frac{z}{\rho_2}$$

$$\mu = \frac{\rho_1}{\rho_2}$$

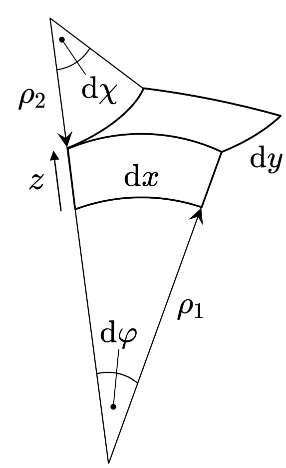

Fig. 3.9: Beam bending.

This simple view of beam bending ignores the fact that there occurs stretching and compression in x- and y-direction and not stretching only in x-direction and compression only in y-direction. The line element $\mathrm{d}x$ which is closer to the angle $\mathrm{d}\varphi$ experiences a compression as compared to the "unbent" state, while the $\mathrm{d}x$ line element lying further away from $\mathrm{d}\varphi$ experiences a stretching as compared to the "unbent" state. The same applies to the line element $\mathrm{d}y$ in relation to the angle $\mathrm{d}\chi$. In general, stretching and compression in bending refer to the so-called neutral fiber, i.e., to the line element in the loaded continuum body which corresponds to the line element in the unloaded state, i.e., in the "unbent" state. If this neutral fiber would lie in the middle of the x- and y-direction respectively, i.e., at $z/2$, then for both directions the result would be half the expansion and compression of the corresponding line elements as compared to the above results. However, the expansion differences between stretched and compressed line elements in the x- and y-direction remain the same.

C) Distortions

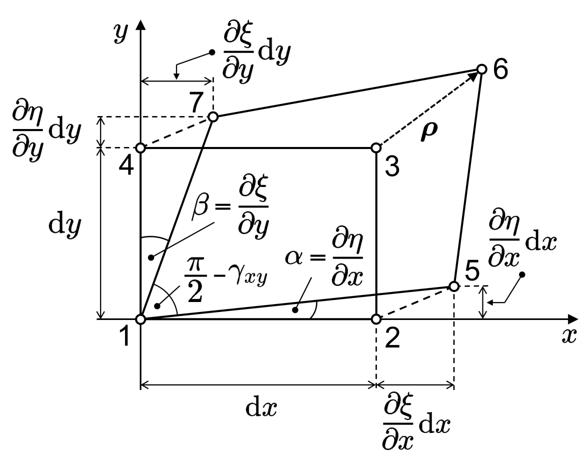

Fig. 3.10: Planar distortion.

First we consider a planar distortion such as in fig. 3.10. This distortion deforms the rectangle 1-2-3-4 into the square 1-5-6-7. The right angle between the segments $\overline{12}$ and $\overline{14}$ of the original rectangle becomes the angle

$$\measuredangle_{412} \longmapsto \frac{\pi}{2} - \gamma_{xy}$$

where $\gamma_{xy} = \alpha + \beta$ is the gliding angle. For the small deformations considered here, the angles α and β are very small and one can apply linear approximations

$$\alpha \approx \tan\alpha = \frac{\dfrac{\partial\eta}{\partial x}\mathrm{d}x}{\mathrm{d}x + \dfrac{\partial\xi}{\partial x}\mathrm{d}x} = \frac{\partial\eta}{\partial x}\left(1 + \frac{\partial\xi}{\partial x} + \dots\right) \approx \frac{\partial\eta}{\partial x}$$

and

$$\beta \approx \tan\beta = \frac{\dfrac{\partial\xi}{\partial y}\mathrm{d}x}{\mathrm{d}y + \dfrac{\partial\eta}{\partial y}\mathrm{d}y} = \frac{\partial\xi}{\partial y}\left(1 + \frac{\partial\eta}{\partial y} + \dots\right) \approx \frac{\partial\xi}{\partial y}$$

With that, one obtains for the gliding angle γ_{xy}

$$\gamma_{xy} = \frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}$$

For the relative elongation, the ratio of the distorted to the original length, one can read from fig. 3.9 for x- and y-direction:

$$\epsilon_{xx} = \frac{\left(\mathrm{d}x + \frac{\partial\xi}{\partial x}\mathrm{d}x\right) - \mathrm{d}x}{\mathrm{d}x} = \frac{\partial\xi}{\partial x} \quad \text{and} \quad \epsilon_{yy} = \frac{\left(\mathrm{d}y + \frac{\partial\eta}{\partial y}\mathrm{d}y\right) - \mathrm{d}y}{\mathrm{d}y} = \frac{\partial\eta}{\partial y}$$

With $\gamma_{xy} = 2\epsilon_{xy}$ or $\gamma_{yx} = 2\epsilon_{yx}$, the distortion tensor in the plane becomes

$$\underline{\underline{\boldsymbol{\epsilon}}} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} {}^{\partial\xi}/_{\partial x} & {}^{\gamma_{xy}}/_{2} \\ {}^{\gamma_{yx}}/_{2} & {}^{\partial\eta}/_{\partial y} \end{bmatrix}$$

To describe the spatial distortion state, the result for the planar distortion can be easily expanded and with the generalized coordinates

$$\boldsymbol{\rho}(\xi, \eta, \zeta) = \boldsymbol{\rho}(\xi_1, \xi_2, \xi_3)$$

one obtains for the spatial distortion tensor

$$\underline{\underline{\boldsymbol{\epsilon}}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \quad ; \quad \epsilon_{ik} = \frac{1}{2}\left(\frac{\partial\xi_i}{\partial x_k} + \frac{\partial\xi_k}{\partial x_i}\right) \quad i, k = 1, 2, 3 \tag{3.20}$$

The state of distortion which $\underline{\underline{\boldsymbol{\epsilon}}}$ describes consists of length changes or elongations (stretching and compression) and shearing (angle changes). Elongations are described by the diagonal components of $\underline{\underline{\boldsymbol{\epsilon}}}$ and shears are reflected by the mixed tensor components. Because of $\epsilon_{ik} = \epsilon_{ki}$, the distortion tensor is symmetric. The volume change of a continuum body associated with the respective elongations and shears can be easily determined in a linear approximation (i.e., for very small deformation):

$$\delta V = \prod_{i=1}^{3} \mathrm{d}x_i\left(1 + \frac{\partial\xi_i}{\partial x_i}\right) - \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$\delta V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \sum_{i=1}^{3} \frac{\partial\xi_i}{\partial x_i} + \text{nonlinear terms}$$

Thus, the relative change in volume is

$$\frac{\delta V}{V} = \sum_{i=1}^{3} \frac{\partial\xi_i}{\partial x_i} = \sum_{i=1}^{3} \epsilon_{ii} = \mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big) \tag{3.21}$$

where $\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big)$ is the trace of the distortion tensor.

3.3 On Hooke's Law

Incidentally, the previous section has already made use of Hooke's law in several instances. Hooke's law describes the relationship between stress and distortion in an isotropic and homogeneous elastic continuum body for the case of small distortions. In this context, homogeneous means that the elastic properties of the considered body are the same in all its points, and it is called isotropic because the body's physical properties are identical in all directions. In such a continuum body, the stress at each of its body points is entirely determined by the respective distortion present, and the principal axes of stress and distortion coincide.

When an elastic body is being distorted, the work exerted in deforming it becomes stored as potential energy in the deformation of the elastic body. Accordingly, this process runs in reverse when the deformation heals as the stored potential energy is being used to do the work required to reverse the deformation. If U denotes the potential energy stored per volume unit $\mathrm{d}V$, i.e., the potential energy density, then a volume element of the deformed body possesses the potential energy $U \cdot \mathrm{d}V$. This potential energy density U of a distorted elastic body is always positive and equal to zero in the undistorted case. In the $\mathrm{d}V$-neighborhood of each point of a distorted elastic body U is proportional to the respective local distortion and can be expressed by the components of the distortion tensor.

For the case of small strains, in a first approximation, the restoring force is proportional to the stretching or compression components of the distortion tensor. On the other hand, the work done to distort the body and thus the potential energy density stored in a deformation, is in the lowest approximation proportional to the acting force, again multiplied by the corresponding stretching or compression components. One boundary condition for the expansion of U in terms of the components of the distortion tensor $\underline{\underline{\boldsymbol{\epsilon}}}$ is therefore that, in a first approximation, U must be proportional to square products of stretching or compression components. Obviously, the best choice for this expansion is to use the deformation tensor in the principal axis system, i.e., $\underline{\underline{\boldsymbol{\epsilon}}} \mapsto \underline{\underline{\boldsymbol{\epsilon}}}^D$.

$$\underline{\underline{\boldsymbol{\epsilon}}}^D = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \quad \text{with} \quad \mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}^D\big) = \epsilon_1 + \epsilon_2 + \epsilon_3$$

In an isotropic body, the energy density must not change. A second condition for U is therefore that U must not change when the principal axes are swapped (e.g., ϵ_1 and ϵ_3).

This of course implies that U must be a symmetric function of ϵ_1, ϵ_2, and ϵ_3. In this regard, the sum of squares of the stretching and compression components is an obvious choice, because

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 \qquad \text{is symmetric with respect to exchanging the } \epsilon_i$$

The square of the trace of $\underline{\underline{\boldsymbol{\epsilon}}}^D$, itself an invariant tensor quantity, is also such a symmetric function of ϵ_i and in addition also contains mixed tensor elements.

$$\left[\mathrm{Sp}\left(\underline{\underline{\boldsymbol{\epsilon}}}^D\right)\right]^2 \;=\; \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + 2\left(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3\right) \tag{3.22}$$

With $\epsilon_i\epsilon_k \;=\; \epsilon_k\epsilon_i$, eq. (3.22) contains all quadratic terms which one can possibly build from ϵ_1, ϵ_2, and ϵ_3. Thus, all possible forms of U which satisfy the two conditions from above can be built from linear combinations of the square sum of the ϵ_i and the square of $\mathrm{Sp}\left(\underline{\underline{\boldsymbol{\epsilon}}}^D\right)$. Hence, it follows for U that:

$$U \;=\; \mathrm{G}\left[\alpha\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2\right) + \beta\left(\epsilon_1 + \epsilon_2 + \epsilon_3\right)^2\right] \tag{3.23}$$

where α and β are both dimensionless constants and G is a dimensional material constant. So that $U \geq 0$ is always true, one must require $\mathrm{G} \geq 0$ and α and β must be chosen such that

$$\frac{\alpha}{\beta} \geq -\frac{\left(\epsilon_1 + \epsilon_2 + \epsilon_3\right)^2}{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2}$$

For the following, one can without restriction set $\alpha \;=\; 1$. Now the stretching $\mathrm{d}\epsilon_1$ of the cuboid shown in fig. 3.11 with the side lengths $\mathrm{d}s_1$, $\mathrm{d}s_2$ and $\mathrm{d}s_3$ shall be considered. The cuboid shown sits with its center at the coordinate origin and with its side faces perpendicular to the coordinate axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ which coincide with the principal axis directions of $\underline{\underline{\boldsymbol{\epsilon}}}^D$. Apart from the stretching $\mathrm{d}\epsilon_1$ in the direction of the $\boldsymbol{e}_1$ axis, the cuboid experiences no further distortion. Due to stretching, the distance $\mathrm{d}s_1$ is lengthened by the amount $\mathrm{d}s_1\mathrm{d}\epsilon_1$ (remember: ϵ always represents a relative stretching $\Delta l/l$). Since $\mathrm{d}s_2$ and $\mathrm{d}s_3$ remain unchanged and the principal stress directions coincide with the principal directions of distortion, the work done to stretch the cuboid is

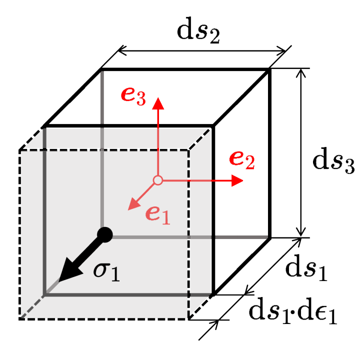

Fig. 3.11

$$\mathrm{d}A \;=\; \text{force} \times \text{distance} \;=\; \sigma_1 \mathrm{d}s_2 \mathrm{d}s_3 \times \mathrm{d}s_1 \mathrm{d}\epsilon_1$$

This work increases the potential energy $U \cdot \mathrm{d}V \;=\; U \cdot \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3$ of the cuboid by the equivalent amount $\mathrm{d}U \cdot \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3$. A simple comparison reveals that

$$\mathrm{d}U \;=\; \sigma_1 \cdot \mathrm{d}\epsilon_1 \;\equiv\; \frac{\partial U}{\partial \epsilon_1} \cdot \mathrm{d}\epsilon_1$$

Using the parametrized form of the potential energy density of U from eq. (3.23) with $\alpha = 1$, one calculates for σ_1:

$$\sigma_1 \;=\; \frac{\partial U}{\partial \epsilon_1} \;=\; 2\mathrm{G}\big[\epsilon_1 + \beta\big(\epsilon_1 + \epsilon_2 + \epsilon_3\big)\big]$$

Since the trace of a tensor is an invariant quantity, i.e., $\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big) = \mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}^D\big)$, this equation can also be written as

$$\sigma_1 \;=\; 2\mathrm{G}\big[\epsilon_1 + \beta \cdot \mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big)\big] \;=\; 2\mathrm{G}\epsilon_1 + \underbrace{2\mathrm{G}\beta \cdot \mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big)}_{\text{constant}}$$

This means that no matter which coordinate transformation is carried out, the second term in this equation remains unchanged. Analogous to σ_1, σ_2 and σ_3 can be expressed in a similar way as functions of the distortion components ϵ_1, ϵ_2, and ϵ_3. Taken together, the relationships between the principal stresses σ_i and the principal distortions ϵ_i can be written in matrix notation as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \;=\; 2\mathrm{G} \begin{bmatrix} 1+\beta & \beta & \beta \\ \beta & 1+\beta & \beta \\ \beta & \beta & 1+\beta \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \tag{3.24}$$

or respectively $\boldsymbol{\sigma} \;=\; 2\mathrm{G} \cdot \underline{\underline{\boldsymbol{A}}} \cdot \boldsymbol{\epsilon}$. With the help of the inverse matrix

$$\underline{\underline{\boldsymbol{A}}}^{-1} \;=\; \frac{1}{\det \underline{\underline{\boldsymbol{A}}}} \cdot \mathrm{Adj}\big(\underline{\underline{\boldsymbol{A}}}\big) \;=\; \frac{1}{1+3\beta} \begin{bmatrix} 1+2\beta & -\beta & -\beta \\ -\beta & 1+2\beta & -\beta \\ -\beta & -\beta & 1+2\beta \end{bmatrix}$$

one obtains the principal distortions ϵ_i as a function of the principal stresses σ_i from $\boldsymbol{\epsilon} \;=\; (2\mathrm{G})^{-1} \cdot \underline{\underline{\boldsymbol{A}}}^{-1} \cdot \boldsymbol{\sigma}$. Carrying out this operation yields

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \;=\; \frac{1}{2\mathrm{G}(1+3\beta)} \begin{bmatrix} 1+2\beta & -\beta & -\beta \\ -\beta & 1+2\beta & -\beta \\ -\beta & -\beta & 1+2\beta \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \tag{3.25}$$

As can be seen from eq. (3.25), the stress σ_1 in the example of fig. 3.11 causes a stretching ϵ_1 of

$$\epsilon_1 \;=\; \frac{1+2\beta}{2\mathrm{G}(1+3\beta)} \cdot \sigma_1 \tag{3.26}$$

At the same time, although σ_2 and σ_3 are zero, the stress σ_1 also causes a transverse contraction of the magnitude

$$\epsilon_2 = \epsilon_3 \;=\; -\frac{\beta}{2\mathrm{G}(1+3\beta)}\cdot\sigma_1 \tag{3.27}$$

We have already encountered the constant G, the shear modulus, in eq. (3.17). This material-specific parameter, sometimes also referred to as slip modulus or torsional modulus, has the dimension of a pressure. A comparison of eq. (3.17) and eq. (3.26) shows that for the modulus of elasticity E, sometimes also referred to as Young's modulus, stress modulus, or tensile modulus, the following applies:

$$\mathrm{E} \;=\; 2\mathrm{G}\frac{1+3\beta}{1+2\beta} \tag{3.28}$$

From eq. (3.26) and eq. (3.27) one obtains for the Poisson ratio $\mu = \epsilon_2/\epsilon_1 = \epsilon_3/\epsilon_1$

$$\mu \;=\; \frac{\beta}{1+2\beta} \tag{3.29}$$

One therefore finds for the relationships between G, E and μ

$$\mathrm{E} \;=\; 2\mathrm{G}\cdot(\mu+1) \quad , \quad \mathrm{G} \;=\; \frac{1}{2}\frac{\mathrm{E}}{\mu+1} \quad , \quad \mu \;=\; \frac{\mathrm{E}}{2\mathrm{G}} - 1 \tag{3.30}$$

If one now replaces β in eq. (3.24) and eq. (3.25) by μ or respectively E, one obtains the well-known Hooke equations in the principal axes system, i.e., in a system in which the off-diagonal components of $\underline{\underline{\boldsymbol{\sigma}}}$ and $\underline{\underline{\boldsymbol{\epsilon}}}$ vanish:

$$\left.\begin{aligned}
\sigma_1 &= \frac{\mathrm{E}}{\mu+1}\Big[\epsilon_1 + \frac{\mu}{1-2\mu}\cdot\big(\epsilon_1+\epsilon_2+\epsilon_3\big)\Big] = 2G\Big[\epsilon_1 + \frac{\mu}{1-2\mu}\cdot\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big)\Big]\\
\sigma_2 &= \frac{\mathrm{E}}{\mu+1}\Big[\epsilon_2 + \frac{\mu}{1-2\mu}\cdot\big(\epsilon_1+\epsilon_2+\epsilon_3\big)\Big] = 2G\Big[\epsilon_2 + \frac{\mu}{1-2\mu}\cdot\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big)\Big]\\
\sigma_3 &= \frac{\mathrm{E}}{\mu+1}\Big[\epsilon_3 + \frac{\mu}{1-2\mu}\cdot\big(\epsilon_1+\epsilon_2+\epsilon_3\big)\Big] = 2G\Big[\epsilon_3 + \frac{\mu}{1-2\mu}\cdot\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\epsilon}}}\big)\Big]
\end{aligned}\right\} \tag{3.31}$$

or respectively

$$\left.\begin{aligned}
\epsilon_1 &= \frac{\mu+1}{\mathrm{E}}\Big[\sigma_1 - \frac{\mu}{\mu+1}\cdot\big(\sigma_1+\sigma_2+\sigma_3\big)\Big] = \frac{1}{2G}\Big[\sigma_1 - \frac{\mu}{\mu+1}\cdot\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\sigma}}}\big)\Big]\\
\epsilon_2 &= \frac{\mu+1}{\mathrm{E}}\Big[\sigma_2 - \frac{\mu}{\mu+1}\cdot\big(\sigma_1+\sigma_2+\sigma_3\big)\Big] = \frac{1}{2G}\Big[\sigma_2 - \frac{\mu}{\mu+1}\cdot\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\sigma}}}\big)\Big]\\
\epsilon_3 &= \frac{\mu+1}{\mathrm{E}}\Big[\sigma_3 - \frac{\mu}{\mu+1}\cdot\big(\sigma_1+\sigma_2+\sigma_3\big)\Big] = \frac{1}{2G}\Big[\sigma_3 - \frac{\mu}{\mu+1}\cdot\mathrm{Sp}\big(\underline{\underline{\boldsymbol{\sigma}}}\big)\Big]
\end{aligned}\right\} \tag{3.32}$$

Leaving the principal axes system, the following applies to the components of the stress- and distortion tensor

$$\sigma_{ik} \;=\; 2\mathrm{G}\Big[\epsilon_{ik} + \frac{\mu}{1-2\mu}\Big(\sum_{l=1}^{3}\epsilon_{ll}\Big)\delta_{ik}\Big] \tag{3.33}$$

or respectively

$$\epsilon_{ik} \;=\; \frac{1}{2\mathrm{G}}\Big[\sigma_{ik} - \frac{\mu}{\mu+1}\Big(\sum_{l=1}^{3}\sigma_{ll}\Big)\delta_{ik}\Big] \tag{3.34}$$

For the off-diagonal elements ($i \neq k$) of the tensor components in eq. (3.33) and eq. (3.34) the following conventions apply

$$\epsilon_{ik} \;=\; \frac{1}{2}\gamma_{ik} \;=\; \frac{\mu+1}{\mathrm{E}}\sigma_{ik} \;=\; \frac{\mu+1}{\mathrm{E}}\tau_{ik} \quad ; \quad \gamma_{ik} \;=\; 2\frac{\mu+1}{\mathrm{E}}\tau_{ik} \;=\; \frac{1}{\mathrm{G}}\tau_{ik}$$

Systems in which the off-diagonal components of $\underline{\underline{\boldsymbol{\sigma}}}$ and $\underline{\underline{\boldsymbol{\epsilon}}}$ vanish are those in which, for example, only compressive forces are present, such as is the case with the hydrostatic pressure P. There one has

$$\sigma_{11} \;=\; \sigma_{22} \;=\; \sigma_{33} \;=\; -P \qquad \text{and} \qquad \sigma_{ik} = 0 \quad \text{for} \quad i \neq k$$

The resulting volume change due to the hydrostatic pressure P is therefore given by

$$\frac{\delta V}{V} \;=\; \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \;=\; -3 \cdot \frac{1-2\mu}{\mathrm{E}}P \;=\; -\kappa P \;=\; -\frac{P}{\mathrm{K}}$$

where K is the compressibility modulus and κ is the compressibility.

On the general case of calculating stresses from strains.

The stretching and compression components in the distortion tensor are the diagonal elements ϵ_{ii} and the tensor components above and below the diagonal, i.e., the ϵ_{ik} with $i \neq k$, correspond to slips. Two relationships are useful for calculating stresses from strains.

First, the sum of stretching and compression $(\mathrm{Sp}(\underline{\underline{\boldsymbol{\epsilon}}}))$ is proportional to the sum of the principal stresses $(\mathrm{Sp}(\underline{\underline{\boldsymbol{\sigma}}}))$, i.e.,

$$\sum_{K}\epsilon_{KK} \;=\; \frac{\mu+1}{\mathrm{E}}\sum_{K}\sigma_{KK} - \frac{3\mu}{\mathrm{E}}\sum_{K}\sigma_{KK} \;=\; \frac{1-2\mu}{\mathrm{E}}\sum_{K}\sigma_{KK} \tag{3.35}$$

Secondly, eq. (3.33) naturally still applies to the dependence of the stress components on stretching and compression.

3.3.1 Torsion of Cylindrical Rods

The following assumptions are being made for treating the torsion of cylindrical rods such as the one sketched in fig. 3.12 within the confines of Hooke's law:

- Rods are straight and have circular cross-sections which remain unchanged by torsion, i.e., the respective radius R is constant over the length L of a cylindrical rod.
- Torsion is caused solely by a torque whose vector lies in the axis of the rod and leads to a slip (shear stress) $\tau = \mathrm{G}\gamma$ at every point of the cross-sectional area of a rod.

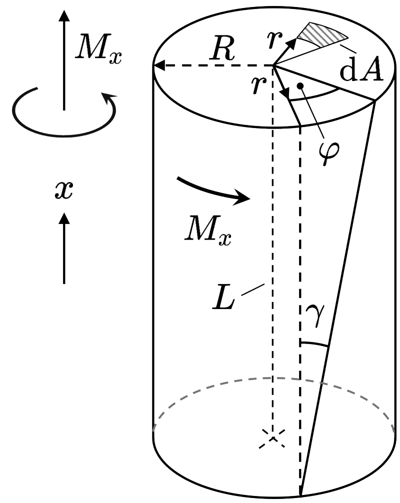

Fig. 3.12: Simple torsion.

These boundary conditions describe a simple torsion, the implication of which is that the cylinder cross-sections in fig. 3.12 are only rotated in the yz-plane during torsion, but not warped in the x-direction. In the torsion of the cylindrical rod shown in fig. 3.12 with the cylinder axis parallel to the x-axis, the torque vector M_x lies in the rod axis and points in the x-direction. This torque M_x causes a torsion of the cylindrical rod by the angle φ which is proportional to the product of the length of the cylindrical rod L and the torque M_x:

$$\varphi = c \cdot L \cdot M_x$$

For small slip angles, i.e., $\gamma \ll 1$, the relation $L\gamma = r\varphi$ applies at every point $0 \leq r \leq R$ along the cylinder radius R. Hence, with $\tau = \mathrm{G}\gamma$ it follows that:

$$\tau(r) = \frac{\varphi \cdot r}{L} \cdot \mathrm{G} \tag{3.36}$$

Each surface element $\mathrm{d}A$ of a cross-sectional area at a distance r from the rod axis is therefore subject to the shear stress $\tau(r)$. The resulting force, the shearing force $\mathrm{d}F = \tau(r)\mathrm{d}A$ attacking at a distance r, acts parallel to the tangent of the circular ring with radius r. The result for the torque M_x is

$$M_x = \int r\tau(r)\mathrm{d}A = \frac{\varphi \cdot \mathrm{G}}{L} \int r^2 \mathrm{d}A = \frac{\varphi \cdot \mathrm{G}}{L} I_p \tag{3.37}$$

where I_p is the polar area moment of inertia. The product of the shear modulus G and I_p is the so-called torsional stiffness. The larger $\mathrm{G}I_p$, the smaller the resulting torsion angle

φ at a given torque for a cylindrical rod of length L. With φ as the deflection, GI_p/L is the spring constant in Hooke's law. For a cylindrical rod, using the parameterization $\mathrm{d}A = 2\pi r \mathrm{d}r$ one finds for I_p

$$I_p = 2\pi \int_0^R r^3 \mathrm{d}r = \frac{\pi}{2} R^4$$

As can be seen, doubling the radius of a cylindrical rod increases the torsional stiffness by a factor of 16. The above consideration applies not only to cylindrical rods but also to cylindrical tubes with an inner radius R_i and an outer radius R_a. Only the value of I_p changes to

$$I_p = 2\pi \int_{R_i}^{R_a} r^3 \mathrm{d}r = \frac{\pi}{2}\left(R_a^4 - R_i^4\right)$$

As eq. (3.36) indicates, the shear stress increases linearly with the distance from the cylinder axis. A comparison of eq. (3.36) and eq. (3.37) shows that for the dependence of the shear stress τ on the torque M_x the following applies:

$$\tau = \frac{r \cdot M_x}{I_p} \qquad \text{with} \qquad \tau_{max} = \frac{R \cdot M_x}{I_p} = \frac{M_(x)}{W_p}$$

where W_p is the so-called polar resistance moment of the cylindrical rod. For a given torque, W_p provides a measure of how much resistance a cylindrical rod will offer to the development of internal stresses. The larger W_p becomes, the lower the maximum shear stress that can occur. For the example of the solid cylinder one has $W_p = \pi R^3/2$, i.e., if the cylinder radius doubles, the polar moment of resistance increases by a factor of factor 8. W_p and I_p are purely geometric quantities, which means they do not depend on the material properties of the cylinder.

3.3.2 Bending of Beams

A simple question: What is the magnitude of the stress in a horizontally clamped beam? Throughout the history of mechanics there have been various answers to this question. As far as we know, Galileo Galilei was one of the first scientists to systematically study the bending of beams. In 1638 he calculated the stress for the horizontally clamped beam in fig. 3.13 to be

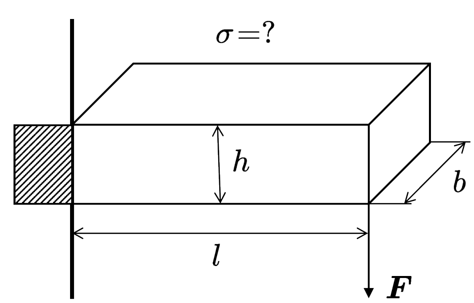

Fig. 3.13

$$\sigma = \frac{2\boldsymbol{F}l}{h^2 b} \qquad \text{(Galilei 1638)}$$

Then in 1684 Jakob Bernoulli calculated the stress to be

$$\sigma = \frac{3\boldsymbol{F}l}{h^2 b} \qquad \text{(Bernoulli 1694)}$$

And in 1744, Leonard Euler came to the conclusion that

$$\sigma = \frac{6\boldsymbol{F}l}{h^2 b} \qquad \text{(Euler 1744)}$$

For the analysis of beam bending to follow, the first assumption is that the beam under consideration is straight and slender, i.e., its cross-sectional dimensions are small compared to its length l. Second, the deformations caused by the beam bending shall everywhere be sufficiently small so that Hooke's law can be applied throughout the beam. A horizontally clamped beam (fig. 3.14a) will bend under the load of its own weight force, i.e., its own weight. As the beam bends under its own weight, the part of the beam above the neutral axis is stretched while the part below the neutral axis is compressed. The neutral fiber itself does not experience any stretching or compression, hence its name; its length is the same as it would be in the unloaded beam on which no forces are acting. The question is: where within the beam does this neutral fiber run?

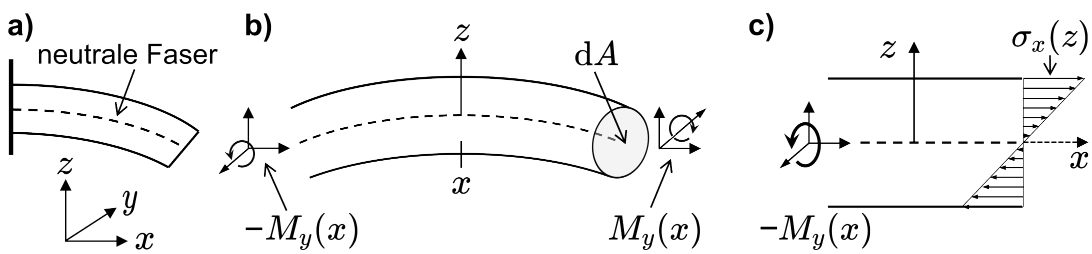

Fig. 3.14: (a) neutral fiber. (b) uniaxial bending, dA = normal surface. (c) equilibrium of forces and moments.

The situation shown in fig. 3.14a is equivalent to the straight or uniaxial bending sketched in fig. 3.14b. For the latter, the beam was mirrored with respect to the wall, thus extended into the negative x-direction; then the wall was removed. In fig. 3.14 the cross-sectional profile of the beam lies in the yz-plane and it is doubly symmetrical with respect to the y- and z-axis. This means that these axes are also the principal axes. The force of gravity acting in the z-direction thus causes a moment about the y-axis. In fig. 3.14b the bending moment M_y points in the direction of the $+y$-axis or respectively in the direction of the $-y$-axis for the mirrored part of the beam. Because the bending moment M_y must be transferred across the beam between neighboring profile cross-sections (section principle), the deformation caused by the moment $M_y(x)$ is the same at every point x. However, this is only possible if the axis of the bent beam assumes the shape of a circular arc. This means that fibers lying above the beam axis are stretched ($\epsilon > 0$, larger arc radius) and fibers lying below are compressed ($\epsilon < 0$, smaller arc radius). For the beam axis itself $\epsilon = 0$ applies. But that is exactly the definition of a neutral fiber, which means nothing other than that the beam axis is the neutral fiber. It follows that

$$\epsilon_x(z) \;=\; c^{(1)}\!\cdot z \quad \text{and} \quad \sigma_x(z) \;=\; c^{(2)}\!\cdot z \qquad \text{(law of straight lines)}$$

where $c^{(1)}$ and $c^{(2)} = \mathrm{E}c^{(1)}$ are constants of proportionality and $c^{(1)}$ is set such that $\epsilon_x(z)$ and $\sigma_x(z)$ is normalized with respect to the bottom or top fiber of the beam, i.e., $c^{(1)} = \epsilon_u/z_u$ or $c^{(1)} = \epsilon_o/z_o$. To find out where the neutral fiber runs, one considers the balance of forces for a cut-free piece of the beam, in fig. 3.14c at the right end of the beam. There, in the case of pure bending, the normal force must vanish. This means the following must hold:

$$\boldsymbol{F}_x \;=\; \int_A \sigma_x(z)\mathrm{d}A \;=\; c^{(2)}\!\cdot\!\int_A z\mathrm{d}A \;\equiv\; 0$$

With the definition of the centroid of the area A

$$z_S \;=\; \frac{1}{A}\int_A z\mathrm{d}A$$

this condition can be expressed as

$$\boldsymbol{F}_x \;=\; c^{(2)}\!\cdot\!\int_A z\mathrm{d}A \;=\; c^{(2)}\!\cdot A\cdot z_S \;\equiv\; 0$$

This condition is only met if $z_S = 0$. This means that in the case of pure uniaxial bending, the position of the neutral fiber runs through the center of the area of the doubly symmetrical (e.g. circular or rectangular) beam cross-section.

In order for the cut-free piece of beam in fig. 3.14c to be in equilibrium, not only must the normal force be zero, but the sum of the moments must also be zero. Hence, the following must apply:

$$\boldsymbol{M}_y(x) + \int_A z\sigma_x(z)\mathrm{d}A \;=\; c^{(2)}\!\cdot\!\int_A z^2\mathrm{d}A \;=\; c^{(2)}\!\cdot I_y$$

I_y, a so-called axial moment of inertia, is a purely geometric quantity that describes the dependence on the cross-section when bending beams:

$$\frac{\boldsymbol{M}_y(x)}{I_y}\cdot z \;=\; \sigma_x(z) \;=\; c^{(2)}\!\cdot z \tag{3.38}$$

With respect to the question asked at the beginning of this section regarding the magnitude of stress in a clamped rectangular beam of height h and width b (see fig. 3.13), one obtains for I_y ($\mathrm{d}A = \mathrm{d}y\mathrm{d}z$)

$$I_y \;=\; b\cdot\int\limits_{-h/2}^{+h/2} z^2\mathrm{d}z \;=\; \frac{b\cdot h^3}{12}$$

The maximum bending stress occurs at the outermost fibers of the beam, since the distance to the neutral fiber $z = \pm h/2$ is greatest for these fibers. With the applied force $\boldsymbol{F}_z$ and $\boldsymbol{M}_y(l) = \boldsymbol{F}_z\cdot l$, the maximum tensile or compressive stress for the clamped beam in fig. 3.13 is

$$\sigma_{max} \;=\; \pm\frac{\boldsymbol{F}_z\cdot l}{b\cdot h^3}\cdot 12\cdot\frac{h}{2} \;=\; \frac{6\boldsymbol{F}_z\cdot l}{h^2 b} \tag{3.39}$$

This corresponds to Euler's result of 1744 and thus the initial question about the stress of the clamped beam has been answered. Next comes the question regarding the bending line: What exactly does the curve describing the deformation of a beam under load look like?

The bending line

The bending line describes the displacement of the neutral fiber under load. To determine the shape of this curve, one considers the relative change in length in x-direction. From the sketch in fig. 3.15 one can read that the following applies to the uppermost fiber of the beam (stretching in the x-direction):

$$\epsilon_x(z) \;=\; \frac{(R+z)\mathrm{d}\varphi - R\mathrm{d}\varphi}{R\mathrm{d}\varphi} \;=\; \frac{z}{R} \;=\; \frac{\mathrm{d}\varphi}{\mathrm{d}x}\cdot z$$

If $w(x)$ is the graph of the bending line, then the following general explicit representation applies to the curvature κ of $w(x)$ at the position x:

$$\kappa \;=\; \frac{w''(x)}{\left(1+w'^2\right)^{3/2}}$$

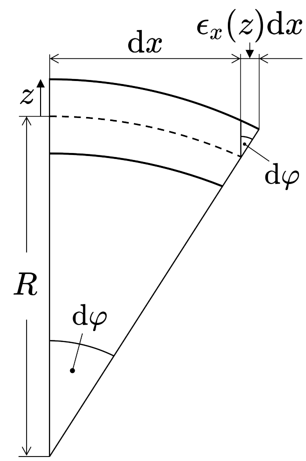

Fig. 3.15

For the negatively curved bending line considered here, one can read from fig. 3.15

$$\kappa \;=\; -\frac{1}{R} \;=\; \frac{w''(x)}{\left(1+w'^2\right)^{3/2}}$$

Hence, for small deformations ($w'(x) \ll 1$) and with eq. (3.38) $\epsilon_x(z)$ becomes

$$\epsilon_x(z) \;=\; -w''(x)\cdot z \;=\; \frac{\sigma_x(z)}{\mathrm{E}} \;=\; \frac{\boldsymbol{M}_y(x)}{\mathrm{E}\cdot I_y}\cdot z$$

The equation for the bending line is thus (Euler 1744)

$$w''(x) = -\frac{\boldsymbol{M}_y(x)}{\mathrm{E}\cdot I_y} \tag{3.40}$$

Example 3.1 **Bending line of the beam in fig. 3.13**

$$w''(x) = -\frac{\boldsymbol{F}_x(l-x)}{\mathrm{E}\cdot I_y}$$

This once integrated gives

$$w'(x) = -\frac{\boldsymbol{F}_x}{\mathrm{E}\cdot I_y}\left(lx - \frac{x^2}{2}\right) + c_1$$

The beam is firmly clamped horizontally on the left. The boundary conditions are

$$w(0) = 0 \quad , \quad w'(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

A further integration gives

$$w(x) = -\frac{\boldsymbol{F}_x}{\mathrm{E}\cdot I_y}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + c_2$$

From $w(0) = 0$ follows $c_2 = 0$ and one gets the equation for the bending line:

$$w(x) = -\frac{\boldsymbol{F}_x\cdot l}{\mathrm{E}\cdot I_y}\cdot\frac{x^2}{2}\left(1 - \frac{x}{3l}\right) \tag{3.41}$$

If instead of a constant force one must deal with an arbitrary force distribution with force density $f(x)$, one uses in eq. (3.40) for $\boldsymbol{M}_y(x)$

$$\boldsymbol{M}_y(x) = \int_x^l f(\xi)(\xi - x)\mathrm{d}\xi$$

Example 3.2 **Clamped beam with load**

Here one considers a clamped beam (fig. 3.16) of length l with a constant beam weight q per unit length at the end of which sits a mass m. First, one determines the total bending moment of the beam. With the bending moment M_m due to the mass m, the bending moment $M(x)$ at the position x of the bar is given by

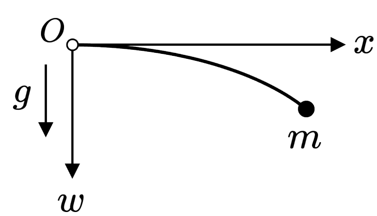

Fig. 3.16

$$M(x) = \int_x^l q(\xi - x)\mathrm{d}\xi + M_m$$

With $M_m(x) = m{\cdot}g{\cdot}(l - x)$ and integration one gets

$$M(x) \;=\; \frac{q}{2}{\cdot}x^2 - (q{\cdot}l + m{\cdot}g){\cdot}x + l\left(\frac{q{\cdot}l}{2} + m{\cdot}g\right)$$

The equation of the bending line (positive curvature) is thus

$$w''(x) \;=\; \frac{1}{\mathrm{E}{\cdot}I}\left[\frac{q}{2}{\cdot}x^2 - (q{\cdot}l + m{\cdot}g){\cdot}x + l\left(\frac{q{\cdot}l}{2} + m{\cdot}g\right)\right]$$

With the boundary conditions $w'(0) = 0$ and $w(0) = 0$ and integrating twice one gets

$$w(x) \;=\; \frac{1}{\mathrm{E}{\cdot}I}\left[\frac{q}{24}{\cdot}x^4 - (q{\cdot}l + m{\cdot}g){\cdot}\frac{x^3}{6} + l\left(\frac{q{\cdot}l}{2} + m{\cdot}g\right){\cdot}\frac{x^2}{2}\right]$$

If one neglects the mass of the clamped beam, i.e., $q = 0$, then the deflection $w(l)$ at the free end of the beam becomes

$$w(l) \;=\; \frac{m{\cdot}g{\cdot}l^3}{3{\cdot}\mathrm{E}{\cdot}I}$$

The oscillation frequency of the clamped beam can be found by using Hooke's law and the definition of the force constant $k = m\omega^2$

$$F \;=\; m{\cdot}g \;=\; k{\cdot}w(l) \qquad \text{and} \qquad m\omega^2 = \frac{m{\cdot}g}{w(l)}$$

For the oscillation frequency $f = \omega/2\pi$ and oscillation period $T = 1/f$ of the clamped beam it follows that

$$f \;=\; \frac{1}{2\pi}\sqrt{\frac{3{\cdot}\mathrm{E}{\cdot}I}{m{\cdot}l^3}} \qquad ; \qquad T \;=\; 2\pi\sqrt{\frac{m{\cdot}l^3}{3{\cdot}\mathrm{E}{\cdot}I}}$$

The buckling formula (Euler)

For the bending sketched in fig. 3.17, the force has the lever arm $w(x)$ around the y-axis

$$\boldsymbol{M} \;=\; \boldsymbol{F}w(x)$$

Instead of a torque with lever arm x, a torque with lever arm w now acts in eq. (3.40):

$$w''(x) \;=\; -\frac{F}{\mathrm{E}{\cdot}I}{\cdot}w(x) \;=\; -q^2w(x)$$

Approach: General solution of differential equations of this type by linear combination of $\sin qx$ and $\cos qx$, meaning

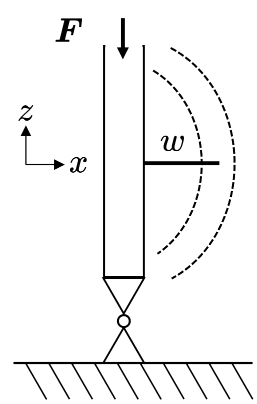

Fig. 3.17

$$w(x) \;=\; \alpha\sin qx + \beta\cos qx$$

Calculating the respective derivatives:

$$w'(x) = \alpha q \cos qx - \beta q \sin qx$$

$$w''(x) = -\alpha q^2 \sin qx - \beta q^2 \cos qx$$

Now determine α and β using the boundary conditions:

$$\text{I:} \quad w(0) = 0 \quad \Rightarrow \quad \beta = 0$$

$$\text{II:} \quad w(l) = 0 \quad \Rightarrow \quad \alpha \sin ql = 0 \quad \Rightarrow \quad \begin{cases} \alpha = 0 \quad \text{(no buckling)} \;\; w(x) = 0 \\ \qquad \text{or} \\ q \cdot l = n \cdot \pi \quad n \;\; \text{even} \end{cases}$$

This means that for buckling applies

$$q^2 l^2 = \frac{\boldsymbol{F}_{\text{buckling}} \cdot l^2}{\mathrm{E} \cdot I} = \pi^2 \qquad \text{hence} \qquad \boldsymbol{F}_{\text{buckling}} = \frac{\mathrm{E} \cdot I}{l^2} \pi^2$$

$\boldsymbol{F}_{\text{buckling}}$ is the so-called buckling load; starting from this force, a solution of the differential equation is possible that results in a bending.

If one now considers a non-centrally attacking force as sketched in fig. 3.18, then to the torque bending the beam the following applies:

$$\boldsymbol{M} = \boldsymbol{F}\big(w(x) + a\big)$$

and thus the differential equation to be solved becomes

$$w''(x) = -\frac{F}{\mathrm{E} \cdot I}\big(w(x) + a\big)$$

This equation can be rewritten to give

$$\big(w(x) + a\big)'' = -\frac{F}{\mathrm{E} \cdot I}\big(w(x) + a\big)$$

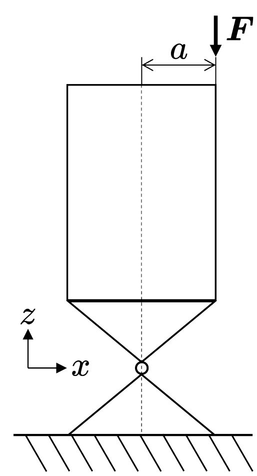

Fig. 3.18

Same solution approach as for the buckling just discussed:

$$w(x) + a = \alpha \sin qx + \beta \cos qx$$

The boundary conditions are now

$$\text{I:} \quad w(0) + a = 0 \quad \Rightarrow \quad \beta = a$$

$$\text{II:} \quad w(l) + a = 0 \quad \Rightarrow \quad \alpha \sin ql + \beta \cos ql = a \quad \Rightarrow \quad \alpha = a \cdot \frac{1 - \beta \cos q}{\sin ql}$$

With that one obtains for the deflection in the case of a not centrally applied force

$$w(x) = a\Big(-1 + \frac{1 - \beta \cos q}{\sin ql} \sin qx + \cos ql \Big)$$

3.4 Hydrostatics

The field of hydrostatics deals with liquids at rest. Of particular interest are the pressure forces exerted by liquids on the vessel walls of various containers. For the understanding of hydrostatic pressure, one considers a small deformable volume of a liquid body (fig. 3.19). In that case, one only must deal with:

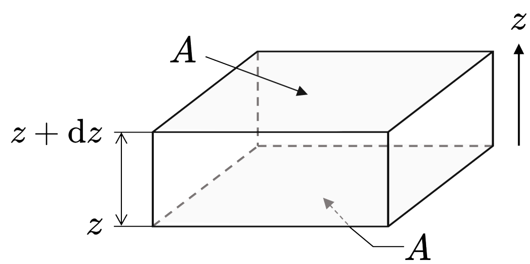

Fig. 3.19

- Normal stresses, no shear stresses; pressure varies only in the z-direction.
- The weight force.

In order for there to be a balance of forces, the pressure forces must vary with height. Mass and force are given by

$$\mathrm{d}m \;=\; \rho{\cdot}A{\cdot}\mathrm{d}z \quad \text{(mass)} \qquad \text{and} \qquad \mathrm{d}F \;=\; \mathrm{d}m{\cdot}g \quad \text{(force)}$$

where ρ is the density of the liquid body. The force arises from the weight pressure and the equilibrium condition is

$$P(z+\mathrm{d}z){\cdot}A - P(z){\cdot}A \;=\; -g{\cdot}\rho{\cdot}A{\cdot}\mathrm{d}z$$

Since $\mathrm{d}z$ is very small, it is sufficient to consider only the first derivative in the Taylor expansion of $P(z+\mathrm{d}z)$, i.e., $P(z+\mathrm{d}z) \approx P(z) + P'(z){\cdot}\mathrm{d}z$. With that applies

$$\left(P(z) + \frac{\partial P}{\partial z}\mathrm{d}z\right){\cdot}A - P(z){\cdot}A \;=\; -g{\cdot}\rho{\cdot}A{\cdot}\mathrm{d}z$$

which means that

$$\frac{\partial P(z)}{\partial z} \;=\; -g{\cdot}\rho$$

Generally expressed with the force density $f(z)$ it follows that

$$\frac{\partial P(z)}{\partial z} \;=\; -g{\cdot}\rho \;=\; f(z) \tag{3.42}$$

or respectively in three dimensions

$$\boldsymbol{\nabla} P(\boldsymbol{r}) \;=\; \boldsymbol{f}(\boldsymbol{r}) \tag{3.43}$$

For incompressible liquids $\rho = \rho_0$ is constant and therefore it holds:

$$P = P_0 - \rho \cdot g \cdot z \tag{3.44}$$

Eq. (3.43) is the basic equation of hydrostatics and eq. (3.44) is the so-called hydrostatic pressure equation. In addition to the graphic derivation just given, eq. (3.43) follows quite simply from eq. (3.16) because for a fluid body in equilibrium the shear stresses vanish, and only compressive stresses exist. Hence, for the stress tensor in eq. (3.16) it follows that $\sigma_{ij} = -P\delta_{ij}$ and thus eq. (3.16) becomes

$$\boldsymbol{f} = -\sum_k \frac{\partial \sigma_{ik}}{\partial x_k} = \sum_k \frac{\partial P}{\partial x_k}\delta_{ik} = \boldsymbol{\nabla} P(\boldsymbol{r}) \tag{3.45}$$

For conservative forces like gravity, the force is the negative gradient of the potential. With the gravitational potential $U(z) = gz$ therefore applies (constant density ρ)

$$\boldsymbol{f} = -\rho \boldsymbol{\nabla} U = -\rho \boldsymbol{\nabla}(gz) \tag{3.46}$$

The hydrostatic pressure equation eq. (3.44) follows then directly from eq. (3.45) and eq. (3.46). In some cases, this approach can also be used to find solutions for non-static situations. One such case is, for example, that of a fluid in a centrifuge.

Example 3.3 Fluids in a centrifuge

In this example, the object of interest is the surface of an incompressible liquid in a centrifuge which rotates with the angular velocity ω (fig. 3.20). The centrifugal force $r\omega^2$ acts on a volume element of the liquid at a distance r from the axis of rotation, which leads to the curved shape of the liquid surface in a rotating container many are familiar with. If one chooses the coordinate system in such a way that it rotates with the centrifuge, the problem can be treated like that of a static liquid on which the potential $U(r, z)$ acts:

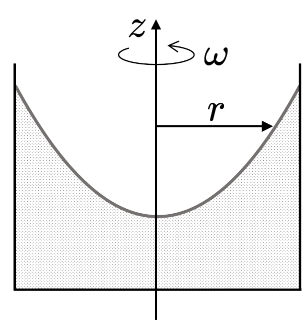

Fig. 3.20

$$U(r, z) = \rho g z - \frac{1}{2}\rho r^2 \omega^2 = \rho g \left(z - \frac{\rho r^2 \omega^2}{2g} \right)$$

With that the modified hydrostatic pressure equation becomes

$$P + \rho g \left(z - \frac{\rho r^2 \omega^2}{2g} \right) = const$$

The surface of the rotating liquid thus has the shape of a parabola

$$z = \frac{\rho r^2 \omega^2}{2g} + const$$

where the constant is determined by the ambient pressure $P = P_0$.

Buoyancy according to Archimedes

For a body in a liquid the buoyant force F_A results from the summation of the pressure forces over its surface. The buoyant force is just as large as the weight force F_G of the displaced liquid. The following applies to the points of attack of the two forces

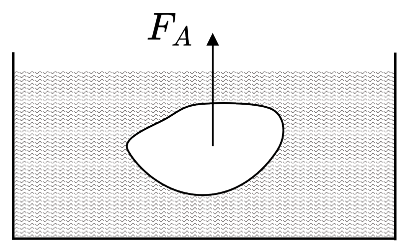

Fig. 3.21

- The buoyant force F_A acts on the center of mass of the displaced liquid body (designated SpA).
- The weight force F_G acts on the center of mass of the body (denoted SpG).

For a shipping vessel to float, it must be built in such a way that $F_G < F_A$ applies. A floating body like that of a shipping vessel sinks into the water just far enough for the buoyancy of the displaced liquid to compensate for the weight of the ship's body. So that no torque occurs in the "resting position" of the shipping vessel, i.e., when the sea is calm and there are no waves, SpA and SpG must lie on the vertical center axis of the ship. In addition, for a stable equilibrium in this rest position, SpG must lie lower than SpA. However, that alone is not sufficient for a shipping vessel to float stably. A ship must be constructed in such a way that when the ship is inclined in the water, which is the normal situation, the moment caused by the relative horizontal displacement of SpA and SpG will right the ship again.

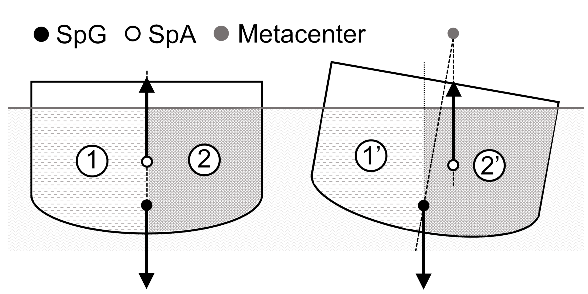

Fig. 3.22

Fig. 3.22 illustrates the respective situation. In the resting position, SpA and SpG both lie on the vertical central axis. To the left and to the right of the line perpendicular to the waterline on which the center of mass lies, the amount of water displaced is identical, the areas 1 and 2 are the same size. In an inclined position of the shipping vessel, this changes, and area 1' is smaller than area 2'. That, however, means nothing other than that SpA is shifted to the right. The resulting torque counteracts the inclined position of the shipping vessel. The point of intersection between the extension of the line of force through SpA and the center of mass axis of the ship is the so-called metacenter. If the metacenter comes to lie above SpG, as in fig. 3.22, then the torque generated when the ship is inclined will stabilize the ship. However, if the ship is constructed in such a way that with the same inclination as in fig. 3.22 SpA is not shifted to the right but to the left relative to SpG and the metacenter therefore comes to lie below SpG, then a torque arises that does not counteract the shipping vessels inclination but reinforces it. The floating position of a ship is hence always stable when the metacenter comes to rest above SpG.

3.5 Aerostatics

Gaseous and liquid bodies are collectively referred to as fluid bodies, or fluida (which is the plural form of the Latin word for fluid). Just as hydrostatics deals with incompressible fluida, aerostatics deals with gaseous bodies, which are compressible fluida. Much of hydrostatics can be transferred to aerostatics. Since for compressible fluida the density is not constant, an additional equation of state, for example the ideal gas equation, is required to treat such bodies.

Barometric height formula

The barometric height formula describes the change in air pressure with the distance z from the Earth's surface. With eq. (3.42) this change can be expressed as:

$$\frac{\partial P(z)}{\partial z} = -\underbrace{m \cdot n}_{\rho} \cdot g$$

Here m stands for the particle mass and n is the particle density with n being a function of z. The general gas equation provides the relationship between particle density n and pressure P

$$P \cdot V = N \cdot k_B \cdot T \tag{3.47}$$

and thus

$$P = n \cdot k_B \cdot T \qquad \text{with} \qquad n = \frac{N}{V} \tag{3.48}$$

Now one makes the assumption that $T =$ constant or respectively, that a temperature change of a few degrees should not matter much. With that follows

$$\frac{\partial P(z)}{\partial z} = -\frac{m \cdot g}{k_B T} \cdot P$$

From this follows for the barometric height formula

$$P = P_0 \cdot \exp\left(-\frac{m \cdot g \cdot z}{k_B T}\right) \tag{3.49}$$

The assumption that the temperature in the atmosphere is constant is of course only an approximation, limiting the validity of the barometric height formula just derived. For a broader approach, one uses the so-called adiabatic equation:

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \qquad \text{where} \qquad \gamma = \frac{c_P}{c_V} \tag{3.50}$$

is the adiabatic exponent, and c_P and c_V are the molar heat capacities at constant pressure P and constant volume V. If one defines now the parameter integral

$$\mathscr{P}(P(\boldsymbol{r})) = \int\limits_{P_0}^{P(\boldsymbol{r})} \frac{\mathrm{d}P'}{\rho(P')} \tag{3.51}$$

and calculates its spatial derivative with the help of

$$\frac{\partial \mathscr{P}(P(\boldsymbol{r}))}{\partial x_i} = \frac{\partial \mathscr{P}}{\partial P(\boldsymbol{r})} \frac{\partial P(\boldsymbol{r})}{\partial x_i} \tag{3.52}$$

and

$$\frac{\partial}{\partial P} \int\limits_{P_0}^{P} \frac{\mathrm{d}P'}{\rho(P')} = \frac{1}{\rho} \tag{3.53}$$

then one gets the result

$$\boldsymbol{\nabla} \mathscr{P}(P(\boldsymbol{r})) = \frac{1}{\rho} \boldsymbol{\nabla} P(\boldsymbol{r}) \tag{3.54}$$

The reason for introducing the parameter integral in eq. (3.51) is that it now follows from eq. (3.54) with the help of eq. (3.45) and eq. (3.46) that

$$\boldsymbol{\nabla} \mathscr{P} = \frac{1}{\rho} \boldsymbol{\nabla} P(\boldsymbol{r}) = \frac{1}{\rho} \boldsymbol{f} = -\boldsymbol{\nabla}(U) \tag{3.55}$$

Hence, the result is that the sum of the parameter integral $\mathscr{P}$ and the potential energy U is a constant

$$\mathscr{P} + U = const \tag{3.56}$$

The benefit of this relationship can be seen after inserting eq. (3.50) into eq. (3.51) and subsequent integration ($U = gz$):

$$\mathscr{P}(P(\boldsymbol{r})) = \frac{1}{\rho_0} \int\limits_{P_0}^{P(\boldsymbol{r})} \left(\frac{P'}{P_0}\right)^{-\frac{1}{\gamma}} \mathrm{d}P' = \frac{p_0}{\rho_0}\frac{\gamma}{\gamma-1}\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} = const - gz \tag{3.57}$$

With the help of the parameter integral $\mathscr{P}$ one therefore obtains a modified version of the hydrostatic pressure equation. To determine the constant, one uses the fact that $P = P_0$ applies for the pressure on Earth's surface ($z = 0$), hence

$$const = \frac{P_0}{\rho_0}\frac{\gamma}{\gamma-1} = u$$

The dependence of the air pressure on height above Earth's surface is then given by

$$\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} = 1 - \frac{gz}{u} \tag{3.58}$$

The pressure in eq. (3.58) becomes zero at the critical height $z = u/g = h_c$. For air at room temperature $\gamma = 1.2$. With this value for γ, eq. (3.58) yields a critical height h_c of about 48 km. According to eq. (3.58) and eq. (3.50), the following applies to pressure and density as a function of height above the Earth's surface

$$\frac{P(z)}{P_0} = \left(1 - \frac{z}{h_c}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{and} \qquad \frac{\rho(z)}{\rho_0} = \left(1 - \frac{z}{h_c}\right)^{\frac{1}{\gamma-1}} \tag{3.59}$$

With these two equations and the ideal gas equation one obtains for the dependence of the temperature on the height above the Earth's surface:

$$\frac{T(z)}{T_0} = 1 - \frac{z}{h_c} \qquad \text{with} \qquad T_0 = \frac{m\cdot P_0}{k_B\cdot\rho_0} \tag{3.60}$$

Part II

Dynamics

4. Point Mass Kinetics

Newton's second law, the momentum theorem of technical mechanics, states:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{r},\boldsymbol{v},t) = \frac{\mathrm{d}m\boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;; \quad m \text{ is mostly constant}$$

For now: No dependency of $\boldsymbol{F}$ on $\boldsymbol{v}$ and t. With that

$$\boldsymbol{F}(\boldsymbol{r}) = m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$$

This second-order differential equation can only be solved for conservative forces, that is, only in cases where the force is the gradient of a potential.

4.1 Energy Theorem of Point Mechanics

Let T denote the kinetic energy of a point mass:

$$T = \frac{m\cdot\boldsymbol{v}^2}{2}$$

With a constant mass m, the time derivative of T is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\boldsymbol{v}^2\right) = m\cdot\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a}\cdot\boldsymbol{v} = \boldsymbol{F}\cdot\boldsymbol{v} = L \tag{4.1}$$

By integration one obtains

$$T(t_1) - T(t_0) = \int_{t_0}^{t_1} \underbrace{\boldsymbol{F}(\boldsymbol{r},\ldots)\cdot\boldsymbol{v}(t)}_{L(t)}\,\mathrm{d}t$$

Definition

<u>Conservative Force:</u> The work done by a conservative force in moving a mass between two points is independent of the path. For a closed path the work done is exactly zero, i.e., no energy is lost or in different words: on a closed path energy is conserved, hence the label conservative force.

If $\boldsymbol{F}$ is a conservative force, then the value of the integral

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

commonly referred to as the work integral, is independent of the path. It follows, that the potential $U(\boldsymbol{r})$

$$U(\boldsymbol{r}) \;=\; -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{\xi}) \cdot \mathrm{d}\boldsymbol{\xi}$$

is a function of the upper integral limit! Thus the following applies to each coordinate or respectively the coordinate vector

$$F_x \;=\; -\frac{\partial U(x,y,x)}{\partial x} \qquad \text{or respectively} \qquad \boldsymbol{F} \;=\; -\mathrm{grad}\, U(\boldsymbol{r})$$

For the potential U as a function of time t and its change as a function of t one obtains

$$\begin{aligned}
\frac{\mathrm{d}U(\boldsymbol{r}(t))}{\mathrm{d}t} \;&=\; \frac{\mathrm{d}}{\mathrm{d}t} U\big(x(t), y(t), z(t)\big) \\
&=\; \frac{\partial U}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial U}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial U}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \;=\; \mathrm{grad}\, U \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \;=\; \boldsymbol{\nabla}\, U \cdot \boldsymbol{v}
\end{aligned}$$

With that holds

$$\frac{\mathrm{d}U}{\mathrm{d}t} \;=\; \nabla U \cdot \boldsymbol{v} \;=\; -\boldsymbol{F} \cdot \boldsymbol{v} \tag{4.2}$$

A comparison of eq. (4.1) and eq. (4.2) shows that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \;=\; -\frac{\mathrm{d}U}{\mathrm{d}t} \qquad \Rightarrow \qquad \frac{\mathrm{d}\big(T+U\big)}{\mathrm{d}t} \;=\; 0$$

This means that the sum of kinetic and potential energy of a point mass is constant. In other words: E_0, the sum of the kinetic and potential energy of a point mass at the location $\boldsymbol{r}_0$ at the time t_0 is the same as E_1, the total energy of the point mass at the location $\boldsymbol{r}_1$ at the time t_1

$$T + U \;=\; E \qquad ; \qquad E \;=\; E_0 \;=\; E_1 \;=\; \mathit{const} \tag{4.3}$$

For example, if a point mass moves parallel to the x-axis, then with eq. (4.3) one has

$$\frac{m}{2}\dot{\boldsymbol{x}}(t)^2 + U\big(x(t)\big) \;=\; E \;=\; \mathit{const}$$

Therefore

$$\dot{\boldsymbol{x}}(t) = \sqrt{\frac{2}{m}\big[E - U\big(x(t)\big)\big]} \qquad \text{and} \qquad \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\big[E - U(x)\big]}} = \mathrm{d}t$$

Integration results in

$$t - t_0 = \int\limits_{x_0}^{x} \frac{\mathrm{d}\xi}{\sqrt{\frac{2}{m}\big[E - U(\xi)\big]}} \qquad \rightarrow \qquad t = t(x) \;\; \Rightarrow \;\; x = x(t)$$

Example 4.1 The harmonic oscillator

$$U = c \cdot x^2 \qquad \text{implies} \qquad F_x = -2 \cdot c \cdot x = -k \cdot x$$

where

$$U = \frac{k}{2} \cdot x^2 = \frac{m}{2}\omega^2 x^2$$

This gives for the equation of motion

$$m\ddot{x}(t) = -k \cdot x(t)$$

With the approach $x(t) = a\cos\omega t + b\sin\omega t$ follows

$$\ddot{x}(t) = -\omega^2 x(t) \qquad \rightarrow \qquad \omega^2 = \frac{k}{m} = \left(\frac{2\pi}{T_P}\right)^2$$

and one obtains the general solution in the form

$$x(t) = A\cos\big[\omega(t - t_0)\big] \qquad \rightarrow \qquad t - t_0 = \frac{1}{\omega}\arccos\frac{x(t)}{A}$$

4.2 Angular Momentum

The following applies to the torque of a point mass at a distance $\boldsymbol{r}$ from the coordinate origin on which the force $\boldsymbol{F}$ acts

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times \dot{\boldsymbol{p}} = m\big[\underbrace{(\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}})}_{=0} + (\boldsymbol{r} \times \ddot{\boldsymbol{r}})\big] = \frac{\mathrm{d}(\boldsymbol{r} \times \boldsymbol{p})}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \tag{4.4}$$

The torque $\boldsymbol{M}$ is the time derivative of the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \tag{4.5}$$

Definition

Central force: A force whose line of action is always directed towards a center of force.

Hence, for central forces $\boldsymbol{F}(\boldsymbol{r}) = F(r)\cdot\frac{\boldsymbol{r}}{r}$ applies. From this follows with eq. (4.4)

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{r}\times\boldsymbol{F}(\boldsymbol{r}) = \frac{F(r)}{r}\underbrace{(\boldsymbol{r}\times\boldsymbol{r})}_{=0} = 0$$

This means that for central forces $\boldsymbol{L}$ is a conserved quantity

$$\boldsymbol{L} = m\left(\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = \boldsymbol{L}_0 = const$$

The areal velocity, i.e., the area A which the position vector $\boldsymbol{r}$ of a point mass sweeps per unit time, is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left|\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|$$

For central forces, the following therefore applies

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{|\boldsymbol{L}|}{2m} = const \tag{4.6}$$

This is Johannes Kepler's second law: The radius vector between the Sun and a planet sweeps in identical time intervals areas of the same size.

Definition

Conservative central force: A central force is conservative if its strength only depends on the magnitude of the distance $|\boldsymbol{r}|$ from the center of the force.

The following then applies to the potential of conservative central forces

$$U(\boldsymbol{r}) = V(|\boldsymbol{r}|) = V(r) \tag{4.7}$$

and to conservative central forces themselves applies

$$\boldsymbol{F} = -\mathrm{grad}\,V(r) = -\frac{\partial V}{\partial r}\cdot\mathrm{grad}\,r \tag{4.8}$$

In Cartesian coordinates with

$$r = \sqrt{x^2+y^2+z^2}$$

one obtains e.g., for the derivative with respect to x

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

and analogously for the derivative with respect to y and z. With that follows

$$-\frac{\partial V}{\partial r}\cdot\operatorname{grad} r \;=\; -\frac{\partial V}{\partial r}\cdot\frac{\boldsymbol{r}}{r}$$

For a point mass in a potential $V(r)$ it follows with eq. (4.3) that

$$\frac{m}{2}\dot{\boldsymbol{r}} + V(r) \;=\; E \tag{4.9}$$

Rather than measuring the velocity of a point mass with respect to a fixed coordinate system, it is in many cases advantageous to measure it with respect to the trajectory itself. Thus, one has

$$\frac{m}{2}\dot{s}^2 + V(r) \;=\; E$$

where $\mathrm{d}s$ is the infinitesimal arc segment of the trajectory sketched in fig. 4.1 through which the point mass runs in the period $\mathrm{d}t$. From fig. 4.1 one can see that

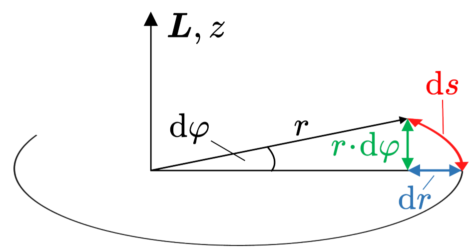

Fig. 4.1

$$\mathrm{d}s \;=\; \sqrt{\mathrm{d}r^2 + (r\mathrm{d}\varphi)^2} \tag{4.10}$$

and therefore

$$\frac{m}{2}\left(\frac{\sqrt{\mathrm{d}r^2 + (r\mathrm{d}\varphi)^2}}{\mathrm{d}t}\right)^2 + V \;=\; \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V \;=\; E \tag{4.11}$$

It must hold that (conservation of angular momentum)

$$|\boldsymbol{L}| \;=\; l \;=\; mr(r\dot{\varphi}) \;=\; mr^2\dot{\varphi} \;=\; const$$

hence, the angular velocity is

$$\dot{\varphi} = \frac{l}{mr^2} \tag{4.12}$$

With that, eq. (4.11) becomes

$$\frac{m}{2}\dot{r}^2 + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{U(r) = V_{Cf}(r) + V(r)} \;=\; E \tag{4.13}$$

where

$$V_{Cf} \;=\; \frac{l^2}{2mr^2} \;=\; \frac{m}{2}\left(\dot{\boldsymbol{\varphi}} \times \boldsymbol{r}\right)^2 \tag{4.14}$$

is the so-called centrifugal potential. The associated force in radial direction, the central force F_Z, is given by

$$F_C \;=\; -\frac{\partial V_{Cf}}{\partial r} \;=\; \frac{l^2}{mr^3} \;=\; m\dot{\varphi}^2 r \tag{4.15}$$

Using $U(r) = V_{Cf}(r) + V(r)$ one can rewrite eq. (4.13) with the result

$$\frac{m\dot{r}^2}{2} \;=\; E - U(r)$$

and therefore

$$\dot{r} \;=\; \frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \pm\sqrt{\frac{2}{m}\big[E - U(\boldsymbol{r})\big]} \tag{4.16}$$

With $\quad \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} \;=\; \dfrac{l}{mr^2} \quad$ and $\quad r(t) \;=\; r\big(\varphi(t)\big) \quad$ follows

$$\frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} \;=\; \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{l}{mr^2} \;=\; r'\cdot\frac{l}{mr^2} \tag{4.17}$$

and therefore

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} \;=\; r' \;=\; \pm\frac{mr^2}{l}\sqrt{\frac{2}{m}\big[E - U(\boldsymbol{r})\big]} \tag{4.18}$$

4.3 Kepler Motion

The term Kepler motion refers to the motion of a point mass under the influence of the gravitational force. That means that for the force acting between a point mass m and a mass M, Newton's law of gravitation applies

$$\boldsymbol{F} \;=\; -G\frac{M\cdot m}{r^3}\boldsymbol{r}$$

The corresponding gravitational potential of Kepler motion is

$$V(r) \;=\; -G\frac{M\cdot m}{r}$$

Thus the potential $U(r) = V_{Cf}(r) + V(r)$ in eq. (4.13) becomes

$$U(r) = \frac{l^2}{2mr^2} - G\frac{M\cdot m}{r}$$

and, by using $\mathrm{d}r/\mathrm{d}t$ from eq. (4.17), eq. (4.13) therefore becomes

$$\frac{m}{2}\Big(\frac{l}{mr^2}r'\Big)^2 + \frac{l^2}{2mr^2} - G\frac{M\cdot m}{r} \;=\; E$$

Solution approach with substitution:

$$\xi = \frac{1}{r} \qquad ; \qquad \xi' = \frac{\mathrm{d}\xi}{\mathrm{d}\varphi} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{r^2}\cdot r'$$

Inserting in the above equation yields

$$\frac{l^2}{2m}\cdot\xi'^2 + \frac{l^2}{2m}\cdot\xi^2 - GM\cdot m\cdot\xi = E$$

Now one completes the square

$$\xi'^2 + \left(\xi - G\frac{M\cdot m^2}{l^2}\right)^2 = \frac{2mE}{l^2} + \left(G\frac{M\cdot m^2}{l^2}\right)^2 \tag{4.19}$$

With the abbreviations

$$\frac{1}{p} = G\frac{M\cdot m^2}{l^2} \qquad \text{und} \qquad \frac{1}{p}\cdot\epsilon = \frac{1}{p}\cdot\sqrt{\frac{2mE}{l^2}p^2 + 1} \tag{4.20}$$

the significance of which will become apparent in the following, eq. (4.19) reads

$$\xi'^2 + \underbrace{\left(\xi - \frac{1}{p}\right)}_{\eta}^2 = \frac{\epsilon^2}{p^2}$$

This differential equation has the form

$$\eta'^2 + \eta^2 = \lambda^2 \qquad \text{and the solution} \qquad \eta = \eta_0\cos(\varphi - \varphi_0).$$

where $\eta_0 = \pm(\epsilon/p)$ and with a suitable choice of the initial conditions $\varphi_0 = 0$. With that one obtains as the solution of eq. (4.19)

$$\xi = \frac{1}{r} = \frac{1}{p}(1 \pm \epsilon\cos\varphi)$$

or respectively the trajectory curve

$$r = \frac{p}{1 \pm \epsilon\cos\varphi} \tag{4.21}$$

This equation describes the trajectory curve of a conic section and especially for $\epsilon < 1$ it is the trajectory curve of an ellipse in polar coordinate representation. Hence, this equation states nothing different than Kepler's first law: The planets move in elliptical orbits with the Sun sitting in one focal point. The minus sign means that the coordinate origin lies in the left focal point (as shown in fig. 4.2b) and the plus sign that it lies in the right focal point.

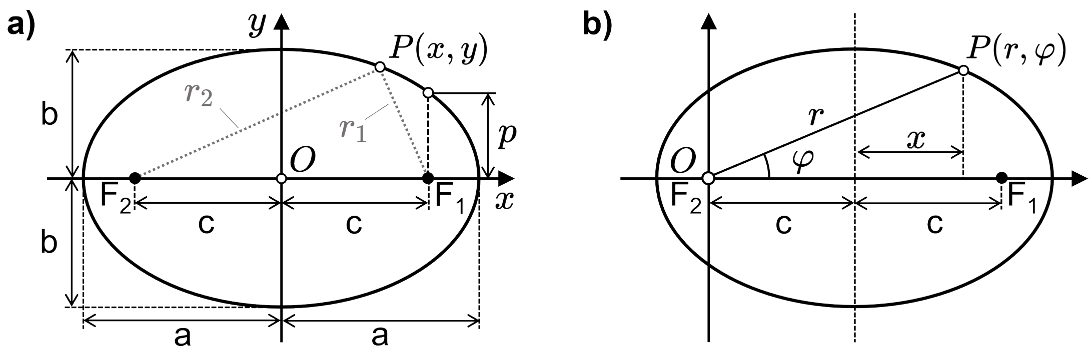

Fig. 4.2: Ellipse in Cartesian (a) and polar coordinates (b). In polar coordinates, one has the choice of placing the origin as shown here in the left focal point - minus sign in eq. (4.21) - or in the right focal point - plus sign in eq. (4.21).

Trajectory curve of the ellipse

An ellipse is defined as the set of all points for which the sum of the distances to two given points, the so-called focal points F_1 and F_2 separated by a distance $2c$ in fig. 4.2, is constant:

$$r_1 + r_2 \;=\; 2a \tag{4.22}$$

where a is the semi-major axis of the ellipse. For the relationship between the semi-major and the semi-minor axis, one can read from fig. 4.2 that

$$a^2 \;=\; c^2 + b^2$$

For r_1 and r_2 one can also read from fig. 4.2a in Cartesian coordinates

$$r_1^2 = (x-c)^2 + y^2 \qquad \text{und} \qquad r_2^2 = (x+c)^2 + y^2 \tag{4.23}$$

If one inserts this into eq. (4.22), one obtains after some calculation the equation describing an ellipse in Cartesian coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1 \tag{4.24}$$

With eq. (4.22) also applies

$$r_2^2 - r_1^2 \;=\; (r_2 + r_1)(r_2 - r_1) \;=\; 2a(r_2 - r_1)$$

But it also follows from eq. (4.23) that

$$r_2^2 - r_1^2 \;=\; 4cx \;=\; 2a\frac{2cx}{a}$$

A comparison of the right-hand sides of the two equations yields

$$r_2 - r_1 \;=\; \frac{2cx}{a}$$

and with eq. (4.22) it therefore holds that

$$r_1 \;=\; a - \frac{cx}{a} \qquad \text{and} \qquad r_2 \;=\; a + \frac{cx}{a}$$

By setting $r = r_2$ for the polar coordinate representation, one transfers the coordinate origin into the left focal point of the ellipse (fig. 4.2b). With $x = r\cos\varphi - c$, the corresponding transfer of the x-coordinate of an ellipse point from fig. 4.2a into fig. 4.2b, one obtains with

$$r \;=\; a + \frac{cx}{a} \;=\; a + \frac{c}{a}(r\cos\varphi - c) \;=\; \frac{a^2 - c^2}{a} + \frac{c}{a} r\cos\varphi \;=\; \frac{b^2}{a} + \frac{c}{a} r\cos\varphi$$

the polar coordinate equation of the ellipse with the origin in the left focal point

$$r \;=\; \frac{p}{1 - \epsilon\cos\varphi} \tag{4.25}$$

with the ellipse parameters

$$p = \frac{b^2}{a} \qquad \text{and} \qquad \epsilon = \frac{c}{a} \quad \text{(eccentricity)} \tag{4.26}$$

The semi-major and semi-minor axes of the ellipse, a and b, as well as the distance c of the focal points from the center of the ellipse can be expressed by p and ϵ:

$$a \;=\; \frac{p}{1 - \epsilon^2} \qquad ; \qquad c \;=\; \frac{p\epsilon}{1 - \epsilon^2} \qquad ; \qquad b \;=\; \frac{p^2}{1 - \epsilon^2}$$

By inserting p and ϵ from eq. (4.20) it follows from these equations that

$$\frac{1}{a - c} \;=\; \frac{1 + \epsilon}{p} \;=\; G\frac{M\cdot m^2}{l^2} + \sqrt{\frac{2mE}{l^2} + \left(G\frac{M\cdot m^2}{l^2}\right)^2} \tag{4.27a}$$

and

$$\frac{1}{a + c} \;=\; \frac{1 - \epsilon}{p} \;=\; G\frac{M\cdot m^2}{l^2} - \sqrt{\frac{2mE}{l^2} + \left(G\frac{M\cdot m^2}{l^2}\right)^2} \tag{4.27b}$$

Multiplying eq. (4.27a) by eq. (4.27b) yields

$$\frac{1}{a^2 - c^2} = \frac{1}{b^2} = \frac{1}{p^2} - \frac{\epsilon^2}{p^2} \;=\; \frac{2mE}{l^2} \tag{4.28}$$

and addition of eq. (4.27a) and eq. (4.27b) yields

$$\frac{2a}{a^2 - c^2} = \frac{2a}{b^2} = \frac{2}{p} \;=\; 2\gamma\frac{M\cdot m^2}{l^2} \tag{4.29}$$

Division of eq. (4.28) by eq. (4.29) shows that

$$E \;=\; G\frac{M\cdot m}{2} \cdot \frac{1 - \epsilon^2}{p} \;=\; G\frac{M\cdot m}{2a} \tag{4.30}$$

With eq. (4.6) and the area of the ellipse, that is

$$l = \frac{2mA}{\tau} \qquad \text{and} \qquad A = \pi \cdot a \cdot b$$

as well as with eq. (4.29), it follows that

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{G \cdot M} \tag{4.31}$$

Here, τ is the time required for a complete orbit of the point mass on the elliptical trajectory curve. Eq. (4.31) states that the ratio of the square of the orbital period to the third power of the semi-major axis depends only on the mass M and not on the mass m of the point mass in orbit. This ratio is therefore identical for all point masses m on any arbitrary elliptical trajectory curve with the mass M sitting in one of the focal points. Obviously, this is nothing but a restatement of Kepler's third law: The square of the orbital period divided by the third power of the semi-major axis is a constant for all planets orbiting the Sun. This relationship between the orbital period and orbital radius is not only valid in our solar system but it is valid for all planets that move on elliptical orbits around their star.

As eq. (4.30) shows, the trajectory curve of the ellipse has the property that the total energy can be expressed through the semi-major axis a of the ellipse. Eq. (4.30) can be formulated more generally as

$$E = -\frac{\alpha}{2a} \tag{4.32}$$

Eq. (4.32) states that for an elliptical trajectory curve the energy only depends on the semi-major axis. Each motion tracing an elliptical trajectory curve with the same a corresponds to the same total energy, independent of the eccentricity ϵ of the respective elliptical trajectory curve. Here specifically, the case of the gravitational potential was considered for which

$$V(r) = -\frac{\alpha}{r} = -G\frac{M \cdot m}{r}$$

applies with $\alpha = G \cdot M \cdot m$. But eq. (4.32) also applies, for example, to the Coulomb potential between two electric charges q_1 and q_2

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1 \cdot q_2}{r} \qquad \text{with} \qquad \alpha = -\frac{1}{4\pi\epsilon_0} q_1 q_2$$

For the Coulomb potential α can be positive or negative. For example, if the charges q_1 and q_2 are both positive or both negative, then $\alpha < 0$. But if q_1 is positive and q_2 is negative or vice versa, then $\alpha > 0$.

In both cases, the gravitational potential and the Coulomb potential, the total energy can be determined using the simple formula in eq. (4.32). If $\alpha > 0$, then there is an attractive force and $E < 0$. With that $\epsilon < 1$ and the trajectory curve is an ellipse. The trajectory curve remains an ellipse as long as $0 < \epsilon < 1$ and at $\epsilon = 0$ the ellipse becomes a circle. If $\alpha = 0$ then $E = 0$ and thus $\epsilon = 1$ and the trajectory curve is thus a parabola. If, on the other hand, $\alpha < 0$, then one is dealing with a repulsive force and because of $E > 0$ it follows that $\epsilon > 1$ and the trajectory is therefore a hyperbola.

4.3.1 Bounded and Unbounded Motion

For bounded motion, the total energy of the relative motion is $E < 0$ and the trajectory curve is an ellipse or a circle. For unbounded motion $E \geq 0$ and the trajectory curve is a parabola or a hyperbola. First, the bounded motion on an elliptical trajectory curve will be considered as sketched in fig. 4.2b. However, now Earth shall sit at the origin of the coordinate system, i.e., the left focal point of the ellipse, and a satellite shall orbit it on the respective trajectory curve. The trajectory curve is therefore given by

$$r(\varphi) = \frac{p}{1 - \epsilon \cos \varphi}$$

and the total energy by

$$E = \frac{m_S}{2}\dot{r}^2 + U(r) = \frac{m_S}{2}\dot{r}^2 + V_{Cf}(r) + V(r)$$

with

$$V_{Cf}(r) = \frac{l^2}{2m_S r^2} \qquad \text{and} \qquad V(r) = -G\frac{M_E \cdot m_S}{r}$$

where M_E is the mass of the Earth and m_S is the mass of the satellite. Fig. 4.3 shows the shape of the potential curve $U(r)$, the sum of the attracting potential $V(r)$ and the repelling potential $V_{Cf}(r)$. If the angular momentum $l = |L|$ is too small, then the repelling force caused by it will be too small for a body to be able to move in an orbit around the Earth. The minimum distance r_{min} that a body has on an elliptical orbit around the Earth can be read from fig. 4.2b with eq. (4.20)

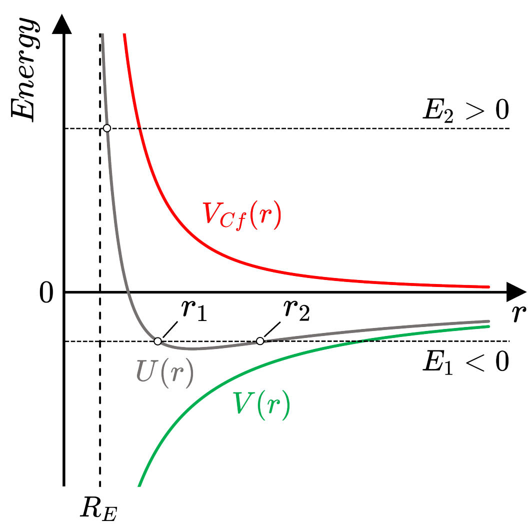

Fig. 4.3

$$r_{min} = r(\pi) = \frac{p}{1+\epsilon} = \frac{l^2}{GM_E m_S(1+\epsilon)}$$

So that a body does not plunge back onto Earth $r_{min} > R_E$ must apply. Therefore, the required minimum angular momentum must be

$$l > \sqrt{GM_E m_S(1+\epsilon)R_E} \tag{4.33}$$

The smallest possible value of l results for $\epsilon = 0$, i.e., for a circular orbit. For motion in a circular orbit with radius r, the following applies to the angular momentum $l = |L|$ and the tangential component of the velocity of the orbiting body v_φ

$$l = mr^2\dot{\varphi} \qquad \text{and} \qquad v_\varphi = r\cdot\dot{\varphi} \tag{4.34}$$

Inserting these relationships for the limiting case $r = R_E$ and $m = m_S$ in eq. (4.33) yields for v_φ

$$v_\varphi > \sqrt{\frac{GM_E}{R_E}} = 7.9\,\mathrm{km\cdot s^{-1}} = v^{(1)} \tag{4.35}$$

$v_\varphi = 7.9\,\mathrm{km\cdot s^{-1}}$ is the so-called first cosmic velocity $v^{(1)}$. If Earth had no atmosphere and its spherical surface were perfectly flat, one could throw a pebble just above Earth's surface in such a way as to impart it only with the tangential velocity component $v^{(1)}$ and thereby launch it into a stable orbit just above Earth's surface. However, because of the Earth's atmosphere, satellites must orbit our planet in a circular orbit at a much greater distance from the Earth's surface. The angular momentum for the respective desired circular orbit with radius R can be obtained analogously to eq. (4.33) and correspondingly one obtains the necessary tangential velocity. For a circular orbit, the distances r_1 and r_2 coincide and one is presented with the situation sketched in fig. 4.4.

Of particular interest is the so-called geostationary orbit, i.e., an orbit for which a satellite stays always above the same point on the Earth's surface. From eq. (4.33) with R_{geo} instead of R_E and eq. (4.35) with $\dot{\varphi} = 2\pi/1\,\text{day}$ one can calculate for the geostationary orbit

$$R_{geo} = \frac{G\cdot M_E\cdot 1\,\text{day}}{2\pi} = 42.2\,\text{km}$$

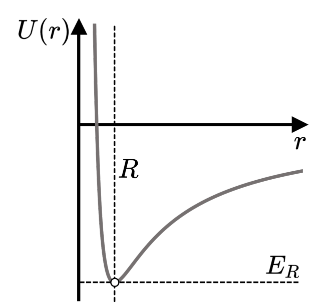

Fig. 4.4

The geostationary orbit is therefore located at a height of approximately 35.9 km above the Earth's surface. The energy required to launch a satellite into such a geostationary bound orbit is considerable. This is of course due to the fact that with the satellite one must also accelerate the much heavier launch vehicle, most of its weight being the fuel it carries, to a correspondingly high speed. To understand what it takes to completely escape Earth's gravitational field, one must consider the energy required to bring a body from the Earth's surface into a specific orbit of radius R. The required energy is given by

$$\Delta E = G \cdot M_E \cdot m \cdot \left(\frac{1}{R_E} - \frac{1}{R} \right)$$

To completely escape Earth's gravitational field, i.e., $R \to \infty$, requires at least the kinetic energy

$$\Delta E = \frac{m}{2} \cdot v_{2_E} = G \cdot M_E \cdot m \cdot \frac{1}{R_E} \tag{4.36}$$

Hence, the escape velocity from the Earth's gravitational field, also known as the second cosmic velocity $v^{(2)}$, is given by

$$v_{2_E} = \sqrt{\frac{2\gamma M_E}{R_E}} = 11.2\,\mathrm{km \cdot s^{-1}} = v^{(2)} \tag{4.37}$$

In relation to the elliptical trajectory curve in fig. 4.2b, $R \to \infty$ in eq. (4.36) means that the point on the trajectory curve with the greatest distance from the focal point moves further and further away and finally vanishes into infinity. For eq. (4.25) must therefore apply

$$r(0) = \frac{p}{1 - \epsilon} \to \infty \qquad \Rightarrow \qquad \epsilon \to 1$$

So, the elliptical orbit breaks up and becomes a parabola. Just as there is an escape velocity for escaping Earth's gravity, there is a corresponding escape velocity for leaving our solar system. In order to calculate the escape velocity from the Sun at the location of the Earth, one replaces the mass of the Earth M_E and the radius of the Earth R_E in eq. (4.37) with the mass of the Sun M_S and the mean distance between the Sun and the Earth $R_{\overline{SE}}$. With that one calculates the escape velocity from the Sun in Earth's orbital trajectory to be $v_{2_S} = 42.1\,\mathrm{km \cdot s^{-1}}$. The Earth orbits the Sun with a speed of $v_E \approx 29.8\,\mathrm{km \cdot s^{-1}}$. This intrinsic speed of Earth reduces of course the escape velocity for leaving the solar system and therefore must be subtracted from v_{2_S}. Finally, one also has to consider the effect of Earth's gravitation, that is v_{2_E}. With all of this taken into account, one obtains for the third cosmic velocity $v^{(3)}$, the speed that one has to reach in order to leave the solar system from the position of Earth

$$v^{(3)} = \sqrt{(v_{2_S} - v_E)^2 + v_{2_E}^2} \approx 16.6\,\mathrm{km \cdot s^{-1}} \tag{4.38}$$

Trajectory curve of the hyperbola

As with the ellipse, with the hyperbola one also has the choice of placing the origin of the coordinate system either in the left or the right focal point. Here, as sketched in fig. 4.5, the left focal point is chosen again to describe the right branch of the hyperbola. For every point on the right branch of the hyperbola applies $r_2 - r_1 = 2a$ and for the left branch of the hyperbola $r_2 - r_1 = -2a$ applies. For all points on the hyperbola applies in Cartesian coordinates (fig. 4.5a)

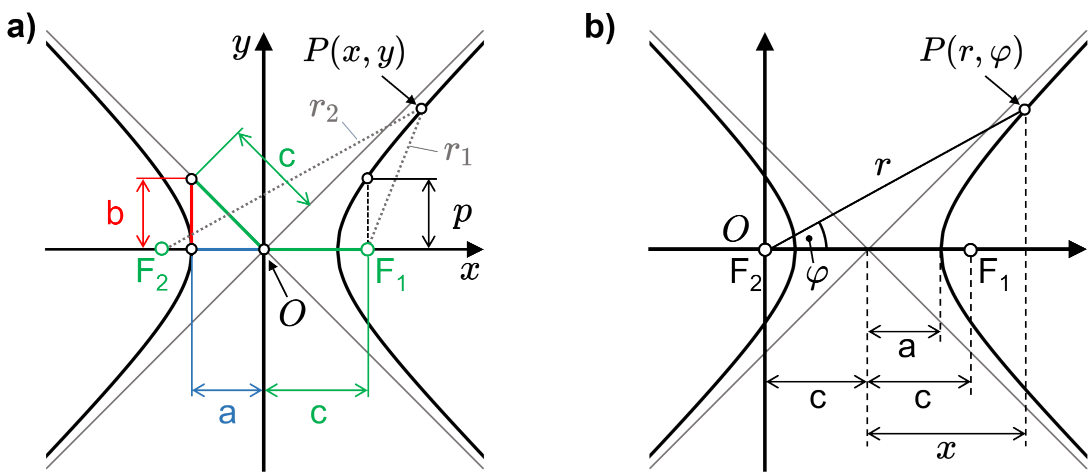

Fig. 4.5: Hyperbola in Cartesian (a) and polar coordinates (b). In polar coordinates, one has the choice of placing the origin as shown here in the left focal point - minus sign in eq. (4.21) - or in the right focal point - plus sign in eq. (4.21).

$$r_2 - r_1 \;=\; \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} \;=\; \pm 2a \tag{4.39}$$

With that and with $c^2 \;=\; a^2 + b^2$, the equation of the hyperbola can be derived in Cartesian coordinates

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \;=\; 1 \tag{4.40}$$

Analogously to the procedure for the ellipse equation, by comparing the right-hand sides of

$$r_2^2 - r_1^2 = (r_2 - r_1)(r_2 + r_1) = \pm 2a(r_2 + r_1)$$

and

$$r_2^2 - r_1^2 \;=\; 4cx \;=\; 2a\frac{2cx}{a}$$

it follows that

$$r_2 + r_1 \;=\; \mp\frac{2cx}{a}$$

This gives with eq. (4.39) for the right branch of the hyperbola

$$r_1 = -a + \frac{cx}{a} \qquad \text{and} \qquad r_2 = a + \frac{cx}{a} \qquad \text{with} \qquad x \geq a$$

and for the left branch of the hyperbola

$$r_1 = a - \frac{cx}{a} \qquad \text{and} \qquad r_2 = -a - \frac{cx}{a} \qquad \text{with} \qquad x \leq -a$$

For the representation in polar coordinates with the origin in the left focal point as sketched in fig. 4.5b, $r = r_2$ is set again and one obtains with the transferred x-coordinate $x = r\cos\varphi - c$ of the right branch

$$r \;=\; a + \frac{cx}{a} \;=\; a + \frac{c}{a}(r\cos\varphi - c) \;=\; \frac{a^2 - c^2}{a} + \frac{c}{a} r\cos\varphi \;=\; -\frac{b^2}{a} + \frac{c}{a} r\cos\varphi$$

the equation for the right branch of the hyperbola in polar coordinates with the origin at the left focal point

$$r \;=\; \frac{p}{1 - \epsilon\cos\varphi} \tag{4.41}$$

with the hyperbola parameters

$$p \;=\; -\frac{b^2}{a} \;=\; -(\epsilon^2 - 1)a \qquad \text{und} \qquad \epsilon \;=\; \frac{c}{a} \;>\; 1 \tag{4.42}$$

To the left branch of the hyperbola eq. (4.41) also applies, but with the hyperbola parameters

$$p \;=\; \frac{b^2}{a} \;=\; (\epsilon^2 - 1)a \qquad \text{und} \qquad \epsilon \;=\; \frac{c}{a} \;>\; 1 \tag{4.43}$$

4.4 Scattering off a Central Potential

Scattering off a central potential is closely related to Kepler motion. In the simplest case, the scattering process describes the relative movement of two non-bound particles (i.e., $E > 0$) which approach each other coming from different directions, then at a small relative distance change their trajectories due to potential scattering, and after that move away from each other in different directions. Incidentally, the trajectory curve of this scattering process corresponds to a hyperbola branch of the Kepler motion.

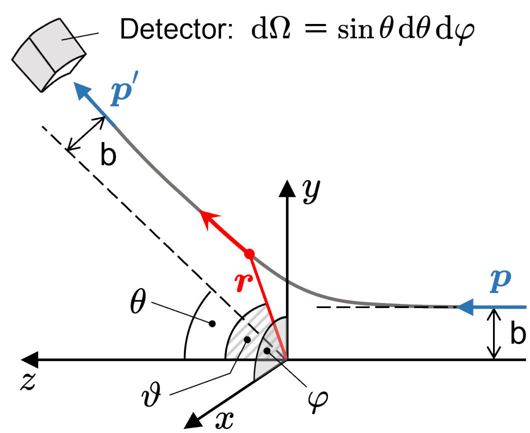

Fig. 4.6

In the following, a particle beam is considered in which all particles (point masses) have the same momentum $\boldsymbol{p}$. This particle beam shall be scattered at a central potential that has its center at the coordinate origin (fig. 4.6). A particle that is incident at a distance b from the z-axis with momentum $\boldsymbol{p}$ is scattered by the angle θ and after scattering has the momentum $\boldsymbol{p}'$. The coordinate system in fig. 4.6 is rotated around the z-axis in such a way that the plane spanned by the two momentum vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ coincides with the yz-plane. The number of particles that fly through a cross-sectional area perpendicular to the xy-plane, i.e., the intensity I of the particle flux, shall be spatially constant. As can be seen from fig. 4.6, scattering off a central potential is rotationally symmetric. The number of particles $\mathrm{d}N$ entering the solid angle element $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ (detector) around the scattering angle θ does not depend on φ; scattering takes place respectively in planes for which $\varphi = const.$ It is therefore sufficient to consider the problem in the scattering plane, in fig. 4.6 the yz-plane. The trajectory curve of a particle in this plane is only a function of the impact parameter b and the magnitude of the initial momentum $p = |\boldsymbol{p}|$, i.e., $\theta = \theta(b, p)$. With considerations like those made in fig. 4.1 and the use of conservation of momentum and energy conservation, one can derive the equations for $\dot{\boldsymbol{r}}$ and $\dot{\vartheta}$; with the difference that the corresponding angle here is ϑ and not φ. Since the angle ϑ for the incident particle is larger than for the scattered particle one has in addition $\dot{\vartheta} < 0$ with

$$\dot{\vartheta} = -\frac{l}{mr^2} \tag{4.44}$$

and

$$\dot{r}^2 = \pm\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - V(r)\right) \tag{4.45}$$

Because of $\dot{\vartheta} < 0$, the sign in the equation equivalent to eq. (4.18) is reversed (i.e., from $\pm$ to $\mp$)

$$\frac{\mathrm{d}r}{\mathrm{d}\vartheta} = \mp\frac{\sqrt{2m}\cdot r^2}{l}\left(E - \frac{l^2}{2mr^2} - V(r)\right)^{-1/2} \tag{4.46}$$

From this follows the equation for the scattering angle

$$\mathrm{d}\vartheta = \mp\frac{l}{\sqrt{2m}}\left(E - \frac{l^2}{2mr^2} - V(r)\right)^{-1/2}\cdot r^{-2}\cdot\mathrm{d}r \tag{4.47}$$

Integration from ϑ_0 to ϑ and r_0 to r yields

$$\vartheta - \vartheta_0 = \mp\frac{l}{\sqrt{2m}}\cdot\int_{r_0}^{r}\left(E - \frac{l^2}{2m\xi^2} - V(\xi)\right)^{-1/2}\cdot\xi^{-2}\cdot\mathrm{d}\xi \tag{4.48}$$

When choosing ϑ_0 and r_0, one uses the fact that the equation of motion is invariant with respect to time reversal $t \to -t$ and, theoretically, the scattering process can therefore run "backwards", that is, the scattered particle becomes the incident particle and vice versa. This is reflected in the mirror symmetry of the trajectory curve (fig. 4.7) with respect to the straight line through the scattering center and the shortest distance r_0 of the scattered particle from the scattering center, the perihelion. Fig. 4.7 shows that $\alpha = \vartheta_0 - \theta$ and $\theta + 2\alpha = \pi$. Thus, for the angle ϑ_0 at which the particle is at the position of the perihelion the following applies:

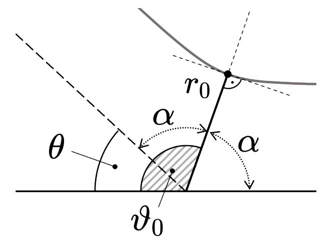

Fig. 4.7

$$\vartheta_0 = \frac{\theta}{2} + \frac{\pi}{2}$$

Now one considers the incident and the scattered branch separately. In eq. (4.45) $\dot{r}$ is negative for the incident branch and positive for the scattered branch. The following applies to the values of ϑ and r:

$$\left.\begin{array}{l} r_0 \le r \le \infty \\ \vartheta_0 \le \vartheta \le \pi \end{array}\right\} \begin{array}{c}\text{incident branch} \\ (\dot{r} < 0)\end{array} \qquad \left.\begin{array}{l} r_0 \le r \le \infty \\ \theta \le \vartheta \le \vartheta_0 \end{array}\right\} \begin{array}{c}\text{scattered branch} \\ (\dot{r} > 0)\end{array}$$

For the incident branch in eq. (4.48) applies

$$\pi - \left(\frac{\theta}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{\theta}{2} = \frac{l}{\sqrt{2m}} \cdot \int_{r_0}^{\infty} \left(E - \frac{l^2}{2m\xi^2} - V(\xi)\right)^{-1/2} \cdot \xi^{-2} \cdot \mathrm{d}\xi \qquad (4.49)$$

and correspondingly for the scattered branch in eq. (4.48)

$$\theta - \left(\frac{\theta}{2} + \frac{\pi}{2}\right) = \frac{\theta}{2} - \frac{\pi}{2} = -\frac{l}{\sqrt{2m}} \cdot \int_{r_0}^{\infty} \left(E - \frac{l^2}{2m\xi^2} - V(\xi)\right)^{-1/2} \cdot \xi^{-2} \cdot \mathrm{d}\xi \qquad (4.50)$$

With the magnitude of the angular momentum

$$l = |\boldsymbol{L}| = |\boldsymbol{r} \times \boldsymbol{p}| = \mathrm{b} \cdot p = const$$

and the energy

$$E = \frac{p^2}{2m} = const$$

the equation for the scattered branch becomes

$$\frac{\theta}{2} - \frac{\pi}{2} = -\mathrm{b}\cdot p\cdot \int_{r_0}^{\infty} \left(p^2 \left(1 - \frac{\mathrm{b}^2}{\xi^2} \right) - 2mV(\xi) \right)^{-1/2} \cdot \xi^{-2} \cdot \mathrm{d}\xi \tag{4.51}$$

With the substitution

$$u = \frac{\mathrm{b}}{\xi} \quad , \quad \mathrm{d}u = -\frac{\mathrm{b}}{\xi^2}\mathrm{d}\xi$$

one obtains after transformation ($p^2 = 2mE$)

$$\theta = \pi - 2\cdot \int_0^{\mathrm{b}/r_0} \left(1 - u^2 - \frac{V(\mathrm{b}/u)}{E} \right)^{-1/2} \cdot \mathrm{d}u \tag{4.52}$$

In order to be able to calculate the scattering angle from eq. (4.52) with a known impact parameter and known energy, r_0 must now be determined. At the perihelion $\dot{r} = 0$ must apply in eq. (4.45), i.e., (with $l^2 = \mathrm{b}^2 p^2 = \mathrm{b}^2 2mE$)

$$E - \frac{l^2}{2mr_0^2} - V(r_0) = E\left(1 - \frac{\mathrm{b}^2}{r_0^2} \right) - V(r_0) = 0 \tag{4.53}$$

With the help of eq. (4.52) and eq. (4.53) the scattering angle θ can be calculated as a function of the impact parameter and the energy E of the incident particles for a given central potential.

Example 4.2 Rutherford scattering

The term Rutherford scattering refers to the scattering of charged particles off a charged scattering center. The electrical potential (Coulomb potential) acting on a test charge q has the general form

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q\cdot q}{r} = -\frac{\alpha}{r} \tag{4.54}$$

where the constant prefactor is the Coulomb constant and Q is the electrical charge located at the scattering center. With Rutherford scattering, the electrical charges have the same sign, i.e., $V(r)$ is positive and $\alpha < 0$ (repelling force). Inserted into eq. (4.53) to determine r_0 this gives

$$E\left(1 - \frac{\mathrm{b}^2}{r_0^2} \right) + \frac{\alpha}{r_0} = 0$$

Multiplication with $\dfrac{r_0^2}{E\mathrm{b}^2}$ gives

$$\frac{r_0^2}{\mathrm{b}^2} - 1 + \frac{r_0}{\mathrm{b}}\frac{\alpha}{E\mathrm{b}} \;=\; 0$$

With the dimensionless quantities

$$\rho \;=\; \frac{r_0}{\mathrm{b}} \qquad \text{and} \qquad \Lambda \;=\; \frac{\alpha}{2E\mathrm{b}}$$

one can, using completion of the square, rewrite this equation as

$$\rho^2 + 2\rho\Lambda + \Lambda^2 \;=\; 1 + \Lambda^2$$

and hence

$$\rho \;=\; -\Lambda \pm \sqrt{1+\Lambda^2}$$

Since r_0 must be positive, the solution with the positive sign must be chosen, i.e.,

$$\rho \;=\; -\Lambda + \sqrt{1+\Lambda^2} \tag{4.55}$$

With that, eq. (4.52) for the calculation of the scattering angle θ becomes

$$\theta \;=\; \pi - 2\cdot\int_0^{1/\rho} \left(1 - u^2 + 2u\Lambda\right)^{-1/2} \mathrm{d}u \tag{4.56}$$

The expression in brackets under the integral can be transformed into

$$1 - u^2 - 2u\Lambda \;=\; (1+\Lambda^2)\left[1 - \left(\frac{u-\Lambda}{\sqrt{1+\Lambda^2}}\right)^2\right]$$

With the substitution

$$x \;=\; \frac{u-\Lambda}{\sqrt{1+\Lambda^2}} \quad , \quad \mathrm{d}x \;=\; \frac{\mathrm{d}u}{\sqrt{1+\Lambda^2}}$$

the integral in eq. (4.56) becomes

$$\int \left(1 - u^2 + 2u\Lambda\right)^{-1/2} \mathrm{d}u \;=\; \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} \;=\; \arcsin x \;=\; \arcsin \frac{u-\Lambda}{\sqrt{1+\Lambda^2}}$$

One therefore obtains for the scattering angle θ in eq. (4.56)

$$\theta \;=\; \pi - 2\arcsin \frac{u-\Lambda}{\sqrt{1+\Lambda^2}}\Bigg|_0^{1/\rho} \;=\; \pi - 2\left(\arcsin\frac{1/\rho - \Lambda}{\sqrt{1+\Lambda^2}} - \arcsin\frac{-\Lambda}{\sqrt{1+\Lambda^2}}\right)$$

Inserting ρ from eq. (4.55) into this expression gives

$$\theta = \pi - 2\left(\arcsin \underbrace{\left[\frac{1}{\sqrt{1+\Lambda^2}} \left(\frac{1+\Lambda^2 - \Lambda\sqrt{1+\Lambda^2}}{\sqrt{1+\Lambda^2} - \Lambda} \right) \right]}_{=1} - \arcsin \frac{-\Lambda}{\sqrt{1+\Lambda^2}} \right)$$

$$= \pi - 2\left(\underbrace{\arcsin 1}_{\frac{\pi}{2}} - \arcsin \frac{-\Lambda}{\sqrt{1+\Lambda^2}} \right) = 2 \arcsin \frac{-\Lambda}{\sqrt{1+\Lambda^2}}$$

With that and using $\Lambda = \alpha/2E\mathrm{b}$ one obtains for the Rutherford scattering angle θ ($\alpha < 0$, $E > 0$ and thus $-\Lambda > 0$)

$$\theta = 2 \arcsin \frac{1}{\sqrt{1 + \frac{4E^2\mathrm{b}^2}{\alpha^2}}} \tag{4.57}$$

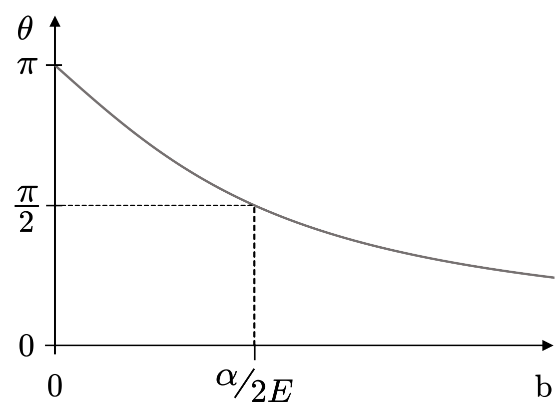

Fig. 4.8: Rutherford scattering: Scattering angle θ as a function of the impact parameter b.

Fig. 4.8 shows the dependence of the scattering angle θ in eq. (4.57) on the impact parameter b in graphic form. In the case of a central impact (b $=$ 0) one has direct backscattering ($\theta = \pi$) and the scattered branch is congruent with the incident branch. Eq. (4.57) shows that the argument of the arcsin function scales like b^{-1}. Because $\arcsin(x) \approx x$ applies for small x, $\theta(b)$ also scales for large impact parameters b like b^{-1}. The reason for this is the long range of potentials of the type $V(r) = \alpha/r$. Since the scattering angle θ in eq. (4.57) depends on the impact parameter b and the energy E in the same way, one obtains the same curve shape as for $\theta(\mathrm{b})$ in fig. 4.8 if one plots θ as a function of E.

In order to better illustrate the connection between Kepler motion and the trajectory curve of Rutherford scattering, the latter can be derived directly from Kepler motion. To do this one considers the plane trajectory curve of an incoming particle in polar coordinates. The particle shall come in from an “infinite” distance and shall be scattered at a central potential centered in the origin of the coordinate system. The radius vector from the origin of the central potential (origin of the coordinate system) to the incident particle shall be

$$\boldsymbol{r} = r \cdot \hat{\boldsymbol{u}} \tag{4.58}$$

where $\hat{\boldsymbol{u}}$ is the unit vector pointing in the direction of the particle. For $\hat{\boldsymbol{u}}$ holds

$$\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}} \;=\; 1 \quad \text{and therefore} \quad \frac{\partial \hat{\boldsymbol{u}}^2}{\partial t} \;=\; 2\hat{\boldsymbol{u}}\cdot\dot{\hat{\boldsymbol{u}}} \;=\; 0 \tag{4.59}$$

For the velocity of the incoming particle, it follows from eq. (4.58)

$$\dot{\boldsymbol{r}} \;=\; \dot{r}\cdot\hat{\boldsymbol{u}} + r\cdot\dot{\hat{\boldsymbol{u}}}$$

and its angular momentum is thus

$$\boldsymbol{L} \;=\; \boldsymbol{r}\times\boldsymbol{p} \;=\; m(\boldsymbol{r}\times\dot{\boldsymbol{r}}) \;=\; m\big(r\hat{\boldsymbol{u}}\times(\dot{r}\cdot\hat{\boldsymbol{u}} + r\cdot\dot{\hat{\boldsymbol{u}}})\big) \;=\; mr^2(\hat{\boldsymbol{u}}\times\dot{\hat{\boldsymbol{u}}})$$

For the central potential, the force always acts along the radius vector, i.e.,

$$m\ddot{\boldsymbol{r}} \;=\; -\boldsymbol{\nabla}V(r) = -\boldsymbol{\nabla}\Big(-\frac{\alpha}{r}\Big) \;=\; -\frac{\alpha}{r^2}\hat{\boldsymbol{u}} \qquad (\alpha < 0)$$

This directly implies angular momentum conservation in central potentials since

$$\frac{\partial \boldsymbol{L}}{\partial t} \;=\; m\frac{\partial}{\partial t}(\boldsymbol{r}\times\dot{\boldsymbol{r}}) \;=\; m\big(\underbrace{\dot{\boldsymbol{r}}\times\dot{\boldsymbol{r}}}_{=0} + \boldsymbol{r}\times\ddot{\boldsymbol{r}}\big) \;=\; -\frac{m\alpha}{r}\underbrace{(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{u}})}_{=0} = 0$$

Now one uses the cross product of acceleration and angular momentum

$$\ddot{\boldsymbol{r}}\times\boldsymbol{L} = -\frac{\alpha}{mr^2}\hat{\boldsymbol{u}}\times mr^2\big(\hat{\boldsymbol{u}}\times\dot{\hat{\boldsymbol{u}}}\big) \;=\; -\alpha\big(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{u}}\times\dot{\hat{\boldsymbol{u}}}\big)$$

which can be rewritten using the Graßmann identity and eq. (4.59) to give

$$\ddot{\boldsymbol{r}}\times\boldsymbol{L} = -\alpha\big(\hat{\boldsymbol{u}}\underbrace{(\hat{\boldsymbol{u}}\cdot\dot{\hat{\boldsymbol{u}}})}_{=0} - \dot{\hat{\boldsymbol{u}}}\underbrace{(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}})}_{=1}\big) \;=\; \alpha\dot{\hat{\boldsymbol{u}}} \tag{4.60}$$

Eq. (4.60) can be integrated directly, since $\boldsymbol{L} = const$ and one obtains

$$\dot{\boldsymbol{r}}\times\boldsymbol{L} \;=\; \alpha\hat{\boldsymbol{u}} + \boldsymbol{c} \tag{4.61}$$

where the vector $\boldsymbol{c}$ is a constant of integration. The vector equation eq. (4.61) can be converted into a scalar equation by multiplying from the left with $\boldsymbol{r} \;=\; r\hat{\boldsymbol{u}}$. Because the scalar triple product is commutative, one obtains

$$\boldsymbol{r}\big(\dot{\boldsymbol{r}}\times\boldsymbol{L}\big) \;=\; \big(\boldsymbol{r}\times\dot{\boldsymbol{r}}\big)\boldsymbol{L} \;=\; \frac{1}{m}\boldsymbol{L}\boldsymbol{L} \;=\; \alpha r\hat{\boldsymbol{u}}\hat{\boldsymbol{u}} + \boldsymbol{r}\boldsymbol{c}$$

If φ is the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{c}$ and $l = |\boldsymbol{L}|$, resolving this equation for r results in (with $\alpha < 0$ replaced by $-|\alpha|$)

$$r \;=\; \frac{p}{1-\epsilon\cos\varphi} \tag{4.62}$$

with the parameters

$$p \;=\; -\frac{l^2}{m|\alpha|} \qquad \text{and} \qquad \epsilon \;=\; \frac{c}{|\alpha|} \tag{4.63}$$

Eq. (4.62) is the trajectory curve of a conic section. The fact that the conic section parameter p in eq. (4.63) is negative already suggests that one is dealing with a hyperbola. In order to verify that $\epsilon > 1$ holds, one has to determine c. One obtains the value of c by considering the initial conditions. If these are set as sketched in fig. 4.6, then the incoming particle comes from the $+z$-direction with the velocity v_∞ on a straight line which runs parallel to the z-axis at a distance b; the trajectory curve lies in the yz-plane. At very large distances, i.e., when the incident particle is still "infinitely" far away from the scattering center, the following applies to the total energy of the relative motion:

$$E = \frac{m}{2} \cdot v_\infty^2 \tag{4.64}$$

Without the influence of the scattering center, the incoming particle with momentum mv_∞ would intersect the y-axis on a straight-line path at b. With the unit vectors of the coordinate system $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ the particle has thus the angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = b \cdot \boldsymbol{e}_y \times mv_\infty \cdot \boldsymbol{e}_z = mbv_\infty \cdot \boldsymbol{e}_x \tag{4.65}$$

The angular momentum of the system is a conserved quantity and thus eq. (4.65) also applies if the incoming particle is still "infinitely" far away from the scattering center. For this case, one now evaluates eq. (4.61) with eq. (4.65) and thus obtains the initial condition

$$\dot{\boldsymbol{r}} \times \boldsymbol{L} = \dot{r} \cdot \boldsymbol{e}_z \times mbv_\infty \cdot \boldsymbol{e}_x = mbv_\infty^2 \cdot \boldsymbol{e}_y = \alpha \cdot \boldsymbol{e}_z + \boldsymbol{c}$$

The unit vectors $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are perpendicular to each other and therefore the absolute value of $\boldsymbol{c}$ is

$$c = \sqrt{\alpha^2 + (mbv_\infty^2)^2}$$

For ϵ in eq. (4.63) this results in

$$\epsilon = \frac{c}{|\alpha|} = \sqrt{1 + \left(\frac{mbv_\infty^2}{\alpha}\right)^2} > 1$$

Hence, the trajectory curve of Rutherford scattering (eq. (4.62)) is indeed that of a hyperbola. With the incident particle at a large distance from the scattering center, the total energy is given by eq. (4.64) and thus ϵ becomes

$$\epsilon = \sqrt{1 + \frac{4E^2 b^2}{\alpha^2}} \tag{4.66}$$

A comparison with eq. (4.57) shows that for the scattering angle θ holds

$$\theta = 2 \arcsin \frac{1}{\epsilon} \tag{4.67}$$

From eq. (4.66) one can now derive the relationship between the impact parameter b and the hyperbola parameters. Solving eq. (4.66) for b^2 gives

$$b^2 = \frac{\alpha^2}{4E^2}(\epsilon^2 - 1) \tag{4.68}$$

and from eq. (4.63) it follows with the help of eq. (4.64) and eq. (4.65) that

$$p = -\frac{l^2}{m|\alpha|} = -\frac{mv_\infty^2 b^2}{|\alpha|} = -\frac{2Eb^2}{|\alpha|} \qquad \Rightarrow \qquad \frac{\alpha^2}{4E^2} = \frac{b^4}{p^2}$$

The latter inserted in eq. (4.68) for b^2 gives for the connection between the impact parameter b and the hyperbolic parameters p and ϵ

$$b = \frac{p}{\sqrt{\epsilon^2 - 1}} \tag{4.69}$$

4.4.1 The Effective Cross Section

Generally speaking, the effective cross section σ is a measure for the probability of an interaction between two particles. In the simplest case, σ is identical with the geometric cross section. A simple example of the latter is an umbrella. If one holds an umbrella above one's head when it rains, assuming the raindrops fall vertically, no raindrops will hit the ground in a small circle around oneself. The area of this circle around one's feet represents the geometric cross section of the umbrella. However, if the umbrella has holes, then it no longer interacts with all the raindrops aimed at it from above and some pass through it unhindered - the effective cross section of the umbrella has therefore become smaller.

In order to determine the cross-section for Rutherford scattering, one considers again a particle stream which runs parallel to the z-axis towards a scattering center (fig. 4.9). Through the interaction with the scattering center, particles whose impact parameter is between b and $b + \mathrm{d}b$ are being scattered into an angle element between θ and $\theta + \mathrm{d}\theta$. Because the scattering potential (e.g., the Coulomb potential) does not depend on the angle ϕ, particles whose impact parameters fall within the annulus with radius b and thickness $\mathrm{d}b$ will all be scattered into the solid angle element $\mathrm{d}\Omega$ integrated over ϕ.

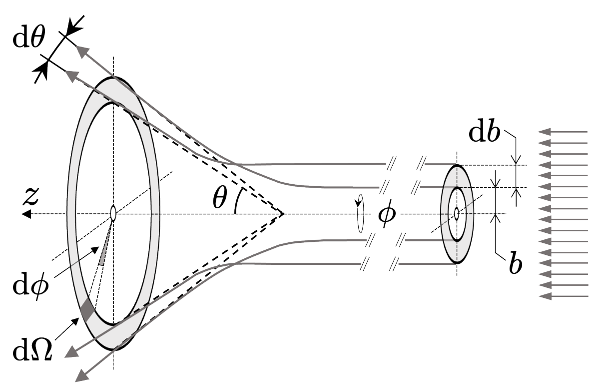

Fig. 4.9

The annulus with radius b and thickness $\mathrm{d}b$ has the area

$$\mathrm{d}\sigma = 2\pi b \mathrm{d}b$$

and the solid angle integrated over ϕ is

$$\mathrm{d}\Omega = 2\pi \sin\theta \mathrm{d}\theta$$

With that, the differential effective cross section becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{4.70}$$

Since the cross section must always be positive (it has the measure of an area) and $\mathrm{d}b/\mathrm{d}\theta$ can also assume negative values, eq. (4.70) uses its absolute magnitude (b is always positive and with $0 \leq \theta \leq \pi$ it always holds that $\sin\theta \geq 0$). From eq. (4.57) one obtains for b

$$b = \frac{\alpha}{2E}\cot\frac{\theta}{2} \tag{4.71}$$

and with that

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{\alpha}{4E}\frac{1}{\sin^2 \theta/2} \tag{4.72}$$

Inserting eq. (4.71) and eq. (4.72) into eq. (4.70) yields the differential effective cross section as a function of the scattering angle

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4 \theta/2} \tag{4.73}$$

What is remarkable about eq. (4.73) is that the differential effective cross section depends on α^2 and not on α. Because of that, in the case of Coulomb scattering eq. (4.73) is independent of the sign of the electrical charges. The total effective cross section σ_{tot} is obtained by integration over the solid angle $\mathrm{d}\Omega = 2\pi\sin\theta\mathrm{d}\theta$, i.e.,

$$\sigma_{tot} = 2\pi\left(\frac{\alpha}{4E}\right)^2 \int_0^\pi \frac{\sin\theta}{\sin^4 \theta/2}\mathrm{d}\theta \tag{4.74}$$

The integrand in eq. (4.74) diverges for $\theta \to 0$. For small θ holds $\sin\theta \approx \theta$ and $\sin \theta/2 \approx \theta/2$, the result of which is that the integrand diverges for $\theta \to 0$ like θ^{-3} and that the respective primitive function $\sigma_{tot}(\theta)$ therefore diverges like θ^{-2}. This divergence at small scattering angles or large impact parameters is caused by the long range of central potentials of the type $V(r) = -\alpha/r$.

5. Dynamics of Rigid Bodies

5.1 The Tensor of Inertia

As sketched in fig. 5.1, let $\boldsymbol{r}_P$ be the vector to the origin of the body-fixed reference system at P and $\boldsymbol{r}_K$ be the vector of a point mass at position K within the rigid body with respect to the coordinate origin O. Then

$$\boldsymbol{r}_K = \boldsymbol{r}_P + \boldsymbol{r}_{PK} \tag{5.1a}$$

$$\boldsymbol{v}_K = \boldsymbol{v}_P + \boldsymbol{v}_{PK} \tag{5.1b}$$

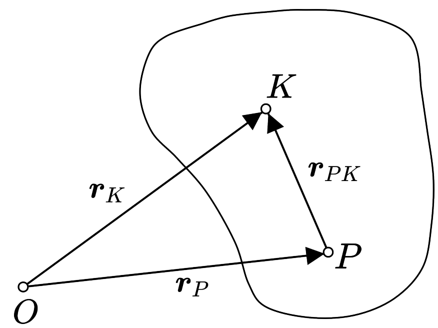

Fig. 5.1

The distances between any two point masses in the rigid body itself do not change (that is the definition of a rigid body) and thus

$$\left(\boldsymbol{r}_{PK}\right)^2 = \boldsymbol{r}_{PK}\cdot\boldsymbol{r}_{PK} = const$$

With the time derivative of this expression it follows

$$0 = 2\cdot\frac{\mathrm{d}\boldsymbol{r}_{PK}}{\mathrm{d}t}\cdot\boldsymbol{r}_{PK} \qquad \Rightarrow \qquad \boldsymbol{v}_{PK} \perp \boldsymbol{r}_{PK}$$

This allows eq. (5.1b) to be rewritten as

$$\boldsymbol{v}_K = \boldsymbol{v}_P + \underbrace{\boldsymbol{\omega} \times \boldsymbol{r}_{PK}}_{\boldsymbol{v}_{PK}} \tag{5.2}$$

The total angular momentum of the rigid body is given by the sum of the angular momenta of all point masses of the body

$$\boldsymbol{L} = \sum_m m_K\left(\boldsymbol{r}_K \times \boldsymbol{v}_K\right) \quad \to \quad \int \left(\boldsymbol{r}_K \times \boldsymbol{v}_K\right) \underbrace{\mathrm{d}m}_{\rho\cdot\mathrm{d}V}$$

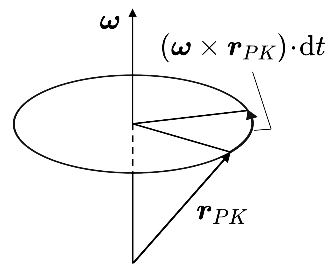

Fig. 5.2

Specifically, if the origin of the body-fixed reference system is the vector to the center of mass, i.e.,

$$\boldsymbol{r}_P = \boldsymbol{r}_S \quad \text{und} \quad \boldsymbol{r}_{PK} = \boldsymbol{r}_{SK}$$

the total angular momentum of the rigid body becomes

$$\boldsymbol{L} = \int (\boldsymbol{r}_S + \boldsymbol{r}_{SK}) \times (\boldsymbol{v}_S + \boldsymbol{\omega} \times \boldsymbol{r}_{SK})\mathrm{d}m$$

where

$$\mathrm{d}m = \rho\cdot\mathrm{d}x_{SK}\cdot\mathrm{d}y_{SK}\cdot\mathrm{d}z_{SK}$$

With the mass $M = \int \mathrm{d}m$ of the rigid body, this becomes

$$\boldsymbol{L} = (\boldsymbol{r}_S \times \boldsymbol{v}_S)\cdot M + \int (\boldsymbol{r}_S \times \boldsymbol{\omega} \times \boldsymbol{r}_{SK})\mathrm{d}m + \int \boldsymbol{r}_{SK} \times (\boldsymbol{v}_S + \boldsymbol{\omega} \times \boldsymbol{r}_{SK})\mathrm{d}m$$

As the center of mass coincides with the origin of the body-fixed reference system

$$\int \boldsymbol{r}_{SK}\mathrm{d}m = M\boldsymbol{R}_{SK} \equiv 0$$

and thus, integrals with integrands linear in $\boldsymbol{r}_{SK}$ vanish and one obtains

$$\boldsymbol{L} = \underbrace{(\boldsymbol{r}_S \times \boldsymbol{v}_S)\cdot M}_{\boldsymbol{L}_S} + \underbrace{\int \boldsymbol{r}_{SK} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{SK})\mathrm{d}m}_{\boldsymbol{L}_{rot}} \tag{5.3}$$

Here $\boldsymbol{L}_S$ is the angular momentum due to the motion of the body's center of mass around the coordinate origin and $\boldsymbol{L}_{rot}$ is the angular momentum due to the body's own rotation. This self-rotation of rigid bodies is also referred to as spin.

With the help of (Graßmann identity)

$$\boldsymbol{r}_{SK} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{SK}) = \boldsymbol{r}_{SK}^2\cdot\boldsymbol{\omega} - (\boldsymbol{r}_{SK}\cdot\boldsymbol{\omega})\cdot\boldsymbol{r}_{SK}$$

one obtains for the components of $\boldsymbol{L}_{rot}$

$$L_{j_{rot}} = \sum_{l=1}^{3} \int \Big(\sum_{i=1}^{3} x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}}\Big)\mathrm{d}m\cdot\omega_l = \sum_{l=1}^{3} J_{jl}\cdot\omega_l$$

where

$$J_{jl} = \int \Big(\sum_{i=1}^{3} x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}}\Big)\mathrm{d}m$$

depends only on the mass distribution of the body and the choice of reference system, i.e., the choice of $\boldsymbol{r}_{SK}$. The J_{jl} are the components of the symmetric tensor of inertia $\underline{\underline{\boldsymbol{J}}}$

$$\underline{\underline{\boldsymbol{J}}} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \tag{5.4}$$

and with it applies for $\boldsymbol{L}_{rot}$

$$\boldsymbol{L}_{rot} = \underline{\underline{\boldsymbol{J}}}\cdot\boldsymbol{\omega} \qquad \text{or respectively} \qquad \boldsymbol{L}_{rot} = \underline{\underline{\boldsymbol{J}}}^{(d)}\cdot\boldsymbol{\omega} \tag{5.5}$$

where the index d indicates that the notation $\underline{\underline{\boldsymbol{J}}}_d$ refers to the tensor of inertia of intrinsic rotation, i.e., to the body's spin ("Drall" in German). Hence eq. (5.3) becomes

$$\boldsymbol{L} = \boldsymbol{L}_S + \underline{\underline{\boldsymbol{J}}}\cdot\boldsymbol{\omega} = \boldsymbol{L}_S + \underline{\underline{\boldsymbol{J}}}^{(d)}\cdot\boldsymbol{\omega} \tag{5.6}$$

The diagonal components of $\underline{\underline{\boldsymbol{J}}}$, i.e., the J_{jj}, are called mass moments of inertia and the J_{jl} with $j \neq l$ are called products of inertia. If all J_{jl} with $j \neq l$ are equal to zero, then $\boldsymbol{L}_{rot}$ points in the same direction as $\boldsymbol{\omega}$, otherwise, which is the more general case, $\boldsymbol{L}_{rot}$ and $\boldsymbol{\omega}$ point in different directions.

On the calculation of the J_{jl} :

$$J_{11} = \int \left(\boldsymbol{r}_{SK}^2 - x_{1_{SK}}^2\right)\mathrm{d}m = \int \left(y_{SK}^2 + z_{SK}^2\right)\mathrm{d}m$$

J_{22} and J_{33} are obtained by cyclic permutation.

$$J_{12} = -\int x_{1_{SK}} x_{2_{SK}}\mathrm{d}m = -\int x_{SK} y_{SK}\mathrm{d}m = J_{21}$$

The remainder of the J_{jl} with $j \neq l$ is obtained again by cyclic permutation.

In summary, one obtains:

$$\begin{aligned} J_{11} &= \int \left(y_{SK}^2 + z_{SK}^2\right)\mathrm{d}m & J_{12} = J_{21} &= -\int x_{SK} y_{SK}\mathrm{d}m \\ J_{22} &= \int \left(x_{SK}^2 + z_{SK}^2\right)\mathrm{d}m & J_{13} = J_{31} &= -\int x_{SK} z_{SK}\mathrm{d}m \\ J_{33} &= \int \left(x_{SK}^2 + y_{SK}^2\right)\mathrm{d}m & J_{23} = J_{32} &= -\int y_{SK} z_{SK}\mathrm{d}m \end{aligned} \tag{5.7}$$

Adding the mass moments of inertia in eq. (5.7) results in

$$J_{11} + J_{22} + J_{33} \;=\; 2\cdot\!\int (x_{SK}^2 + y_{SK}^2 + z_{SK}^2)\mathrm{d}m \;=\; 2\cdot\!\int \boldsymbol{r}_{SK}^2 \mathrm{d}m \tag{5.8}$$

The integral depends only on the reference point and not on the choice of the reference system, that is, the coordinate origin. Therefore, the integral remains unchanged when the coordinate system is rotated.

Mass moment of inertia conservation
The sum of the three mass moments of inertia of a rigid body is a conserved quantity under rotation of the coordinate system.

The following triangle inequalities can also be read from eq. (5.7):

$$J_{11} + J_{22} \;\geq\; J_{33} \quad , \quad J_{22} + J_{33} \;\geq\; J_{11} \quad , \quad J_{33} + J_{11} \;\geq\; J_{22} \tag{5.9}$$

Mass moment of inertia triangle inequality
The sum of two mass moments of inertia of a rigid body is always greater than or at least equal to the third mass moment of inertia.

Huygens-Steiner theorem
The tensor of inertia $\underline{\underline{\boldsymbol{J}}}$ of a body does depend on the choice of the reference point relative to the center of mass of the body. As sketched in fig. 5.3, the reference point P in the rigid body shall be shifted with respect to the body's center of mass S, but the reference system, i.e., the coordinate system, shall remain unchanged:

$$\boldsymbol{r}_{PK} \;=\; \boldsymbol{r}_{PS} + \boldsymbol{r}_{SK} \qquad (*)$$

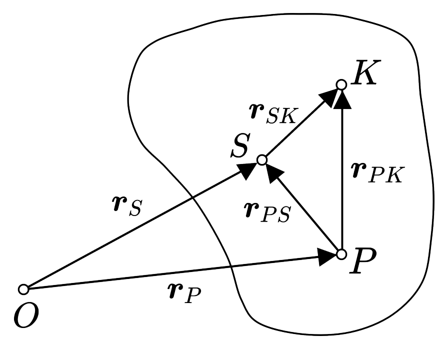

Fig. 5.3

If an axis of rotation running through the center of mass is shifted parallel to run through a new reference point P, then the following applies to the components of $\underline{\underline{\boldsymbol{J}}}$ with respect to this shifted axis of rotation:

$$J_{P_{jl}} \;=\; \int \underbrace{\Big(\sum_{i=1}^{3} x_{i_{PK}}^2 \delta_{jl} - x_{j_{PK}} x_{l_{PK}}\Big)}_{Int} \mathrm{d}m \tag{5.10}$$

In the integrand "Int" we replace $\boldsymbol{r}_{PK}$ by the relation $(*)$ and get

$$Int \;=\; \sum_{i=1}^{3} (x_{i_{PS}} + x_{i_{SK}})^2 \delta_{jl} - (x_{j_{PK}} + x_{j_{SK}})(x_{l_{PK}} + x_{l_{SK}})$$

After multiplying and rearranging this becomes

$$
\begin{aligned}
Int \;=\; & \sum_{i=1}^{3} x_{i_{PS}}^2 \delta_{jl} - x_{j_{PS}} x_{l_{PS}} + \sum_{i=1}^{3} x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}} \\
& + \; 2 \sum_{i=1}^{3} x_{i_{PS}} x_{i_{SK}} \delta_{jl} - x_{j_{SK}} x_{l_{PS}} - x_{j_{PS}} x_{l_{SK}}
\end{aligned}
$$

With this reshaped integrand eq. (5.10) becomes

$$
\begin{aligned}
J_{P_{jl}} \;=\; & \underbrace{\int \Big(\sum_{i=1}^{3} x_{i_{PS}}^2 \delta_{jl} - x_{j_{PS}} x_{l_{PS}} \Big) \mathrm{d}m}_{J_{PS}} \; + \; \underbrace{\int \Big(\sum_{i=1}^{3} x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}} \Big) \mathrm{d}m}_{J_{S_{jl}}} \\
& + \; 2\delta_{jl} \sum_{i=1}^{3} x_{i_{PS}} \underbrace{\int x_{i_{SK}} \mathrm{d}m}_{=0} \; - \; x_{l_{PS}} \underbrace{\int x_{j_{SK}} \mathrm{d}m}_{=0} \; - \; x_{j_{PS}} \underbrace{\int x_{l_{SK}} \mathrm{d}m}_{=0}
\end{aligned}
$$

The last three integrals on the left side of this equation disappear because for integration in the center of mass system, i.e., the origin of the coordinate system lies in the body's center of mass, of course it applies again

$$
\int \boldsymbol{r}_{SK} \mathrm{d}m \;\equiv\; 0
$$

Thus, the components of the tensor of inertia for an axis of rotation through a point P, which runs at a distance of $\boldsymbol{r}_{PS}$ parallel to the axis of rotation through the center of mass S, are then given by

$$
J_{P_{jl}} \;=\; J_{PS} + J_{S_{jl}} \qquad \text{(Steiner theorem)} \tag{5.11}
$$

In this equation the

$$
J_{S_{jl}} \;=\; M \cdot \Big(\sum_{i=1}^{3} x_{i_{SK}}^2 \delta_{jl} - x_{j_{SK}} x_{l_{SK}} \Big)
$$

are the components of the tensor of inertia for the parallel axis of rotation running through the center of mass and

$$
J_{PS} \;=\; M \cdot \Big(\sum_{i=1}^{3} x_{i_{PS}}^2 \delta_{jl} - x_{j_{PS}} x_{l_{PS}} \Big)
$$

tells us how much larger the components $J_{P_{jl}}$ of the tensor of inertia are for the rotation axis running through the point P as compared to the parallel axis of rotation running through the center of mass.

In summary, one obtains for the components through the new axis of rotation the relationships known under the label Huygens-Steiner theorem, named after Christiaan Huygens and Jakob Steiner:

$$
\begin{aligned}
J_{P_{11}} &= M(y_{PS}^2 + z_{PS}^2) + J_{11} \quad , \quad J_{P_{12}/P_{21}} = M\cdot x_{PS} y_{PS} + J_{12/21} \\
J_{P_{22}} &= M(x_{PS}^2 + z_{PS}^2) + J_{22} \quad , \quad J_{P_{13}/P_{31}} = M\cdot x_{PS} z_{PS} + J_{13/31} \\
J_{P_{33}} &= M(x_{PS}^2 + y_{PS}^2) + J_{22} \quad , \quad J_{P_{23}/P_{32}} = M\cdot y_{PS} z_{PS} + J_{23/32}
\end{aligned}
\tag{5.12}
$$

In particular, the tensor of inertia $\underline{\underline{\boldsymbol{J}}}$ for a rotation axis that does not run through the center of mass is always larger than for a rotation axis that runs parallel to it through the body's center of mass. This is a direct consequence of eq. (5.11) since $J_{S_{ij}}$ is always positive.

Huygens-Steiner theorem
The mass moments of inertia of a rigid body for rotational axes running through its center of mass are always smaller than those for parallel rotational axes that do not run through the center of mass.

The Huygens-Steiner theorem states that the tensor of inertia $\underline{\underline{\boldsymbol{J}}}$ of a rigid body will change when the body's reference point is displaced from the body's center of mass. However, the angular velocity remains unchanged since it describes the rotation of the rigid body with respect to the coordinate system and the latter is independent of the choice of the body reference point.

5.2 Euler's Equations

With eq. (5.2) the kinetic energy of a rigid body is given by

$$T = \frac{1}{2}\int \Big[\boldsymbol{v}_P + (\boldsymbol{\omega}\times\boldsymbol{r}_{PK})\Big]^2 \mathrm{d}m \qquad \text{with} \qquad \mathrm{d}m = \rho\cdot \mathrm{d}x_{SK}\mathrm{d}y_{SK}\mathrm{d}z_{SK}$$

If one calculates from the center of mass, that is $P = S$, then

$$\boldsymbol{r}_{PK} \to \boldsymbol{r}_{SK} \quad \text{and with} \quad \int \boldsymbol{r}_{SK}\mathrm{d}m = 0$$

one obtains for the kinetic energy

$$T = \frac{1}{2}\Big[M\cdot\boldsymbol{v}_S^2 + 2\cdot\underbrace{\int \boldsymbol{v}_S\cdot(\boldsymbol{\omega}\times\boldsymbol{r}_{SK})\mathrm{d}m}_{=0} + \int (\underbrace{\boldsymbol{\omega}\times\boldsymbol{r}_{SK}}_{\boldsymbol{a}})(\boldsymbol{\omega}\times\boldsymbol{r}_{SK})\mathrm{d}m\Big]$$

The scalar triple product $\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}_{SK})$ under the integral in the last term can be rewritten as

$$\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}_{SK}) \;=\; \boldsymbol{r}_{SK}\cdot(\boldsymbol{a}\times\boldsymbol{\omega}) \;=\; \boldsymbol{\omega}\cdot(\boldsymbol{r}_{SK}\times\boldsymbol{a})$$

With this and with the help of eq. (5.3) and eq. (5.5) one gets for the kinetic energy of a rigid body:

$$T \;=\; \frac{1}{2}M\boldsymbol{v}_S^2+\frac{1}{2}\boldsymbol{\omega}\boldsymbol{L}_{rot} \;=\; \frac{1}{2}M\boldsymbol{v}_S^2+\frac{1}{2}\boldsymbol{\omega}\underline{\underline{\boldsymbol{J}}}\boldsymbol{\omega} \tag{5.13}$$

The total kinetic energy of a rigid body is the sum of two terms, a translation term T_T and a rotation term T_R. If a rigid body rotates around an axis D whose direction is given by the unit vector $\hat{\boldsymbol{n}}_D$, then $\boldsymbol{\omega} = \omega\hat{\boldsymbol{n}}_D$ and therefore one obtains for the rotation term T_R in eq. (5.13)

$$T_R \;=\; \frac{1}{2}\hat{\boldsymbol{n}}_D\underline{\underline{\boldsymbol{J}}}\hat{\boldsymbol{n}}_D\cdot\omega^2 \;=\; \frac{1}{2}J_D\cdot\omega^2$$

where J_D is the moment of inertia for rotation around the axis of rotation D. A comparison with T_T shows that the tensor of inertia plays the same role for T_R as the inertial mass M does for T_T.

A symmetric tensor ($A_{ij} = A_{ji}$) like the tensor of inertia $\underline{\underline{\boldsymbol{J}}}$ has three real eigenvalues, the so-called principal moments of inertia of a body, with mutually perpendicular eigenvectors, the so-called principal axes of inertia of the body.

Example 5.1 **Rotating cuboid**

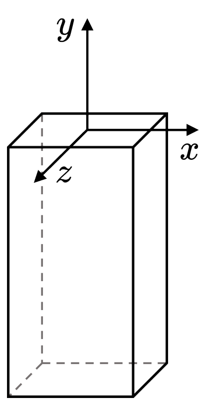

Fig. 5.4

In relation to a body-fixed principal axis system, such as for the cuboid shown in fig. 5.4, the off-diagonal elements of $\underline{\underline{\boldsymbol{J}}}$ vanish. The diagonal elements, i.e., the mass moments of inertia of the cuboid, then correspond to its principal moments of inertia. With

$$J_{xx} = A \quad , \quad J_{yy} = B \quad , \quad J_{zz} = A$$

$\boldsymbol{\omega}\underline{\underline{\boldsymbol{J}}}\boldsymbol{\omega}$ in eq. (5.13) becomes

$$\boldsymbol{\omega}\underline{\underline{\boldsymbol{J}}}\boldsymbol{\omega} \;=\; \begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}\begin{bmatrix}A & & \\ & B & \\ & & A\end{bmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} \;=\; A(\omega_x^2+\omega_z^2)+B\omega_y^2$$

In the case $\boldsymbol{v}_S = 0$ the total kinetic energy of the cuboid is rotational energy, i.e.,

$$T \;=\; T_R \;=\; \frac{1}{2}A\big[(\omega_x^2+\omega_z^2)+B\omega_y^2\big]$$

In the following, one considers a spatially fixed reference system x, y, z with origin O and a second reference system x', y', z' with origin O' moving relative to the first one. The position vectors of any point in the respective coordinate system are linked by

$$\boldsymbol{r} = \boldsymbol{r}_{O'} + \boldsymbol{r}'$$

The motion of the O'-system relative to the O-system can be a translation or a rotation, or both together.

Now let $\boldsymbol{A}$ be a vector quantity in the space-fixed coordinate system with origin O

$$\boldsymbol{A} = \sum_{i=1}^{3} A_i \hat{\boldsymbol{e}}_i$$

The time derivative of $\boldsymbol{A}$ is

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \sum_{i=1}^{3} \frac{\mathrm{d}A_i}{\mathrm{d}t} \hat{\boldsymbol{e}}_i + \sum_{i=1}^{3} A_i \frac{\mathrm{d}\hat{\boldsymbol{e}}_i}{\mathrm{d}t} \tag{5.14}$$

With $\dfrac{\mathrm{d}\hat{\boldsymbol{e}}_i}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_i$ this becomes

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \underbrace{\sum_{i=1}^{3} \dot{A}_i \hat{\boldsymbol{e}}_i}_{\frac{\mathrm{d}'\boldsymbol{A}}{\mathrm{d}t}} + \boldsymbol{\omega} \times \boldsymbol{A} \tag{5.15}$$

Here $\mathrm{d}'\boldsymbol{A}/\mathrm{d}t$ is the rate of change of the vector $\boldsymbol{A}$ as observed from the reference system in motion, i.e.,

$$\mathrm{d}'\boldsymbol{A}/\mathrm{d}t = \begin{pmatrix} \dot{A}_{x'} \\ \dot{A}_{y'} \\ \dot{A}_{z'} \end{pmatrix}$$

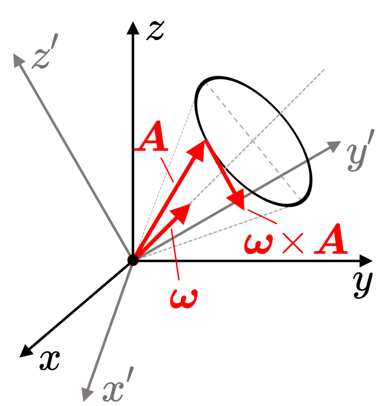

Fig. 5.5

That this is actually the case becomes understandable when, as shown in fig. 5.5, the motion of the O'-system relative to the O-system is limited to a rotation and at the same time $O' = O$. If then e.g., $\boldsymbol{A}$ is constant, i.e., $\mathrm{d}'\boldsymbol{A}/\mathrm{d}t = 0$, it follows that $\boldsymbol{A}$ will move in the O system on a circle whose plane is perpendicular to $\boldsymbol{\omega}$, the vector of the rotational speed of the O' system. In that case $\boldsymbol{A}$ moves in the O system with the velocity $\boldsymbol{\omega} \times \boldsymbol{A}$.

If one now transfers this consideration to the angular momentum vector $\boldsymbol{L}$ with center of mass velocity $\boldsymbol{v}_S = 0$, then the situation is the same as in fig. 5.5. That means

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}'\boldsymbol{L}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{L} \tag{5.16}$$

With eq. (5.5) one thereby obtains the Euler equations in vector form (sometimes also referred to as Euler's gyroscope equations) for the torque $\boldsymbol{M} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$

$$\boldsymbol{M} = \underline{\underline{\boldsymbol{J}}} \cdot \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega} \times \left(\underline{\underline{\boldsymbol{J}}}\boldsymbol{\omega}\right) \tag{5.17}$$

For example, in coordinate form the Euler equations for the cuboid from fig. 5.4 are given by

$$\begin{aligned} M_x &= A\dot{\omega}_x - (B - A)\omega_y\omega_z \\ M_y &= B\dot{\omega}_y \\ M_z &= A\dot{\omega}_z - (A - B)\omega_x\omega_y \end{aligned} \tag{5.18}$$

Tab. 5.1: Important relationships in rigid body dynamics (S = center of mass).

Angular momentum:	$\boldsymbol{L} = \boldsymbol{L}_S + \boldsymbol{L}_{rot} = m \cdot \boldsymbol{r}_S \times \boldsymbol{v}_S + \underline{\underline{\boldsymbol{J}}} \cdot \boldsymbol{\omega}$
Tensor of inertia:	$J_{ij} = \int \left(\boldsymbol{r}^2\delta_{ij} - x_i x_j\right)\mathrm{d}m$
Huygens-Steiner theorem:	$J_{P_{ij}} = m\left(\boldsymbol{r}_{PS}^2\delta_{ij} - x_{PS_i}x_{PS_j}\right) + J_{S_{ij}}$
Kinetic energy:	$T = \frac{1}{2}m\boldsymbol{v}_S^2 + \frac{1}{2}\boldsymbol{\omega}\underline{\underline{\boldsymbol{J}}}\boldsymbol{\omega}$
Rate of change of vector quantities:	$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\mathrm{d}'A}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{A}\ , \quad \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}'\boldsymbol{\omega}}{\mathrm{d}t}$
Rotating part of $\boldsymbol{L}$:	$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}'L}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{L}$ $= \underline{\underline{\boldsymbol{J}}}\frac{\mathrm{d}'\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{L}$ $= \underline{\underline{\boldsymbol{J}}}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega} \times \left(\underline{\underline{\boldsymbol{J}}} \cdot \boldsymbol{\omega}\right)$ $= \underline{\underline{\boldsymbol{J}}}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{\underline{\boldsymbol{J}}}}{\mathrm{d}t} \cdot \boldsymbol{\omega} = \boldsymbol{M}$

5.2.1 Euler Angles

Euler's angles have already been briefly mentioned in chapter 2 when discussing the degrees of freedom of a rigid body. Here they will now be examined in more detail. Fig. 5.6 shows the rotation of a body-fixed coordinate system O' with the axes x_1', x_2' and x_3' with respect to a space-fixed coordinate system O with the axes x_1, x_2 and x_3. The line of intersection of the x_1x_2-plane with the $x_1'x_2'$-plane is the so-called line of nodes, in fig. 5.6 the straight line K. The Euler angles α, β and γ in fig. 5.6 are defined as follows:

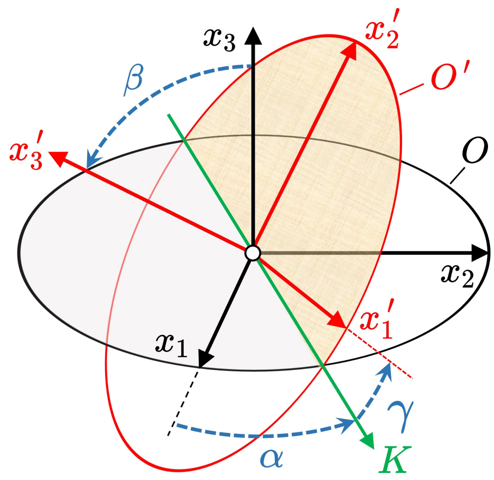

Fig. 5.6: Euler Angles

α: The angle between the x_1-axis and the line of nodes K.

β: The angle between the x_3- and the x_3'-axis.

γ: The angle between the line of nodes K and the x_1'-axis.

The importance of the Euler angles derives from the fact that with their help a spatially fixed coordinate system O can be transformed into a body-fixed coordinate system O' by carrying out the three successive rotations $\boldsymbol{R}^\alpha$, $\boldsymbol{R}^\beta$ and $\boldsymbol{R}^\gamma$ in fixed (none-interchangeable) order.

- In the first step, the system O is rotated by the angle α around the x_3-axis thereby aligning the x_1-axis to be coincident with the line of nodes K. The rotation matrix for this operation is

$$\boldsymbol{R}^\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- In the second step, this rotated system O is then rotated again by the angle β around the line of nodes K resulting in the x_3-axis becoming coincident with the x_3'-axis. The rotation matrix for this operation is

$$\boldsymbol{R}^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}$$

- In the third step, this new O system is then rotated by the angle γ around the x_3'-axis, thus making the x_1-axis, which through step one is already coincident with the line of nodes, coincident with the x_1'-axis. The rotation matrix for this operation is

$$\boldsymbol{R}^\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

After the execution of the third step, i.e., after the operation

$$\boldsymbol{R}^{\alpha\beta\gamma} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or respectively with the abbreviations $C_\alpha = \cos\alpha$, $S_\alpha = \sin\alpha$, $C_\beta = \cos\beta$, $S_\beta = \sin\beta$, $C_\gamma = \cos\gamma$ and $S_\gamma = \sin\gamma$

$$\boldsymbol{R}^{\alpha\beta\gamma} = \begin{pmatrix} C_\alpha C_\gamma - S_\alpha C_\beta S_\gamma & -C_\alpha S_\gamma - S_\alpha C_\beta C_\gamma & S_\alpha S_\beta \\ S_\alpha C_\gamma + C_\alpha C_\beta S_\gamma & -S_\alpha S_\gamma + C_\alpha C_\beta C_\gamma & -C_\alpha S_\beta \\ S_\beta S_\gamma & S_\beta C_\gamma & C_\beta \end{pmatrix} \tag{5.19}$$

O is now congruent with O'. The operation (notation from now on: $\boldsymbol{R} = \boldsymbol{R}^{\alpha\beta\gamma}$)

$$\hat{\mathbf{e}}_i' = \boldsymbol{R}\hat{\mathbf{e}}_i \qquad i = 1, 2, 3 \tag{5.20}$$

converts the base vectors $\hat{\mathbf{e}}_i$ of the space-fixed coordinate system O into the base vectors $\hat{\mathbf{e}}_i'$ of the body-fixed coordinate system O'. For the components R_{ij} of the rotation matrix $\boldsymbol{R}$ applies

$$R_{ij} = \hat{\mathbf{e}}_i^T \boldsymbol{R}\hat{\mathbf{e}}_j \tag{5.21}$$

which can be rewritten with eq. (5.20) as

$$R_{ij} = \hat{\mathbf{e}}_i^T \hat{\mathbf{e}}_j' \tag{5.22}$$

The components R_{ij} are each respectively the cosine of the angle between the x_i-axis in the spatially fixed coordinate system O and the x_j'-axis in the body-fixed coordinate system O'. For the component representation of a vector $\boldsymbol{b}$ in the coordinate systems O and O' applies

$$\boldsymbol{b} = b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + b_3\hat{\mathbf{e}}_3 = b_1'\hat{\mathbf{e}}_1' + b_2'\hat{\mathbf{e}}_2' + b_3'\hat{\mathbf{e}}_3'$$

For the components b'_j in the body-fixed coordinate system O' one has with eq. (5.22)

$$b'_j = \sum_{i=1}^{3} b_i \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}'_j = \sum_{i=1}^{3} b_i R_{ij}$$

In matrix notation

$$\begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \text{that means} \qquad \boldsymbol{b}' = \boldsymbol{R}^T \boldsymbol{b} \tag{5.23}$$

where $\boldsymbol{R}^T$ is the transposed matrix of $\boldsymbol{R}$. Eq. (5.20) shows that $\boldsymbol{R}$ is the rotation matrix of the unit vectors from O to O' and eq. (5.23) shows that $\boldsymbol{R}^T$ is the rotation matrix of the vector components from O to O'.

$$\boldsymbol{R}^T = \begin{pmatrix} C_\alpha C_\gamma - S_\alpha C_\beta S_\gamma & S_\alpha C_\gamma + C_\alpha C_\beta S_\gamma & S_\beta S_\gamma \\ -C_\alpha S_\gamma - S_\alpha C_\beta C_\gamma & -S_\alpha S_\gamma + C_\alpha C_\beta C_\gamma & S_\beta C_\gamma \\ S_\alpha S_\beta & -C_\alpha S_\beta & C_\beta \end{pmatrix}$$

As can be easily checked, det $\boldsymbol{R} = 0$. Therefore, an inverse matrix $\boldsymbol{R}^{-1}$ exists. Since the column vectors of $\boldsymbol{R}$ are pairwise orthogonal, $\boldsymbol{R}^{-1} \equiv \boldsymbol{R}^T$ holds true. This means that by multiplying eq. (5.23) with $\boldsymbol{R}$ it follows that

$$\boldsymbol{R}\boldsymbol{b}' = \boldsymbol{R}\boldsymbol{R}^T\boldsymbol{b} = \boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{b} \qquad \text{and hence} \qquad \boldsymbol{b} = \boldsymbol{R}\boldsymbol{b}'$$

In order to be able to calculate the position of a body from its rotational velocity, one must know the relationship between the rotational velocities ω' in the body-fixed coordinate system O' and the rates of change of the Euler angles $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$. In what follows, this relationship will be established by determining the respective axes of rotation of the three individual rotations in the coordinate system O': around the x_3-axis, the line of nodes K, and the x'_3-axis. One then obtains the rotation vector components for ω' by multiplying the corresponding unit vectors $\hat{\boldsymbol{e}}_3$, $\hat{\boldsymbol{e}}_K$ and $\hat{\boldsymbol{e}}'_3$ with the associated angular velocities $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$.
With the unit vectors $\hat{\boldsymbol{e}}'_1$, $\hat{\boldsymbol{e}}'_2$ and $\hat{\boldsymbol{e}}'_3$ in O', $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ in O, as well as the unit vector $\hat{\boldsymbol{e}}_K$ in the direction of the line of nodes, the Euler angles are given by (directional cosines)

$$\cos\alpha = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_K \quad ; \quad \cos\beta = \hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}'_3 \quad ; \quad \cos\gamma = \hat{\boldsymbol{e}}'_1\hat{\boldsymbol{e}}_K \tag{5.24}$$

For rotations around the x_3-axis (first step) in the coordinate system O', one must express $\hat{\boldsymbol{e}}_3$ as a function of $\hat{\boldsymbol{e}}'_1$, $\hat{\boldsymbol{e}}'_2$ and $\hat{\boldsymbol{e}}'_3$. From fig. 5.6 it is immediately apparent that the projection of $\hat{\boldsymbol{e}}_3$ onto the x_3'-axis, i.e., the $\hat{\boldsymbol{e}}_3'$ component of $\hat{\boldsymbol{e}}_3$, is equal to $\cos\beta$. However, one can also get this result simply from eq. (5.24) by multiplying the expression for $\cos\beta$ with $\hat{\boldsymbol{e}}_3'$. The $\hat{\boldsymbol{e}}_1'$ and $\hat{\boldsymbol{e}}_2'$ components of $\hat{\boldsymbol{e}}_3$ follow from the projection of $\hat{\boldsymbol{e}}_3$ onto the $x_1'x_2'$-plane. The absolute value of this projection is $\cos(90^\circ - \beta)$, i.e., $\sin\beta$. If one were to rotate the x_1-axis around the x_3-axis until the x_1-axis becomes coincident with the line of nodes K, then one would find that the projection of the unit vector $\hat{\boldsymbol{e}}_3$ onto the $x_1'x_2'$-plane lies in the x_2x_3-plane, i.e., this projected vector and x_2' enclose the angle γ. Thus, the components of $\hat{\boldsymbol{e}}_3$ in O' are

$$\hat{\boldsymbol{e}}_3 \;=\; \sin\beta\sin\gamma\cdot\hat{\boldsymbol{e}}_1' + \sin\beta\cos\gamma\cdot\hat{\boldsymbol{e}}_2' + \cos\beta\cdot\hat{\boldsymbol{e}}_3' \tag{5.25}$$

For rotations around the line of nodes K (second step) in the coordinate system O', $\hat{\boldsymbol{e}}_K$ must be expressed as a function of $\hat{\boldsymbol{e}}_1'$, $\hat{\boldsymbol{e}}_2'$ and $\hat{\boldsymbol{e}}_3'$. The unit vector $\hat{\boldsymbol{e}}_K$ in the direction of the line of nodes can be expressed in the coordinates of O as well as the coordinates of O'. As can easily be seen from fig. 5.6

$$\hat{\boldsymbol{e}}_K \;=\; \cos\alpha\cdot\hat{\boldsymbol{e}}_1 - \sin\alpha\cdot\hat{\boldsymbol{e}}_2 \;=\; \cos\gamma\cdot\hat{\boldsymbol{e}}_1' - \sin\gamma\cdot\hat{\boldsymbol{e}}_2' \tag{5.26}$$

The axis relevant for the rotation about $\hat{\boldsymbol{e}}_3'$ in O' (third step) is of course $\hat{\boldsymbol{e}}_3'$ itself. With that and with the axes of rotation $\hat{\boldsymbol{e}}_3$ and $\hat{\boldsymbol{e}}_K$ as determined for O' through eq. (5.25) and eq. (5.26) one now obtains for the relationship between the rotational velocities ω' in the body-fixed coordinate system O' and the rate of change of the Euler angles $\dot\alpha$, $\dot\beta$ and $\dot\gamma$

$$\begin{aligned}
\omega' \;&=\; \dot\alpha\cdot\hat{\boldsymbol{e}}_3 + \dot\beta\cdot\hat{\boldsymbol{e}}_K + \dot\gamma\cdot\hat{\boldsymbol{e}}_3' \\
&=\; \dot\alpha\big[\sin\beta\sin\gamma\cdot\hat{\boldsymbol{e}}_1' + \sin\beta\cos\gamma\cdot\hat{\boldsymbol{e}}_2' + \cos\beta\cdot\hat{\boldsymbol{e}}_3'\big] + \dot\beta\big[\cos\gamma\cdot\hat{\boldsymbol{e}}_1' - \sin\gamma\cdot\hat{\boldsymbol{e}}_2'\big] + \dot\gamma\cdot\hat{\boldsymbol{e}}_3' \\
&=\; \big[\dot\alpha\sin\beta\sin\gamma + \dot\beta\cos\gamma\big]\cdot\hat{\boldsymbol{e}}_1' + \big[\dot\alpha\sin\beta\cos\gamma - \dot\beta\sin\gamma\big]\cdot\hat{\boldsymbol{e}}_2' + \big[\dot\alpha\cos\beta + \dot\gamma\big]\cdot\hat{\boldsymbol{e}}_3'
\end{aligned}$$

or respectively in matrix notation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{pmatrix} = \begin{pmatrix} \sin\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta\cos\gamma & -\sin\gamma & 0 \\ \cos\beta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{pmatrix} \tag{5.27}$$

These equations are frequently referred to as kinematic Euler equations. By integrating eq. (5.27) one can in principle calculate the position of a body from its known angular velocities.

5.3 Gyroscope, Nutation, Precession

Example 5.2 Unbalanced wheel

As sketched in fig. 5.7, due to an imbalance of the wheel, the wheel axis (axis 3) is no longer parallel to the axis of rotation $\boldsymbol{\omega}$ but tilted with respect to it by an angle α. The axis of the running direction of the wheel (axis 2) no longer points straight ahead but sideways. The third coordinate axis (axis 1) points out of the image plane. The mass moments of inertia of the wheel in the directions of axis 1 and 2 are identical and denoted here by A; the mass moment of inertia in the direction of the wheel axis, axis 3 shall be C. Thus, the tensor of inertia in the principal axis system is:

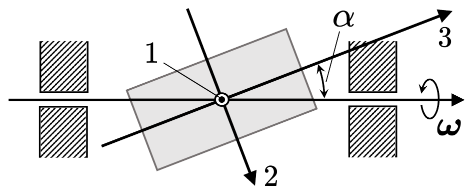

Fig. 5.7

$$\underline{\underline{\boldsymbol{J}}}^{(d)} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}$$

According to eq. (5.17), one thus obtains for the components of the torque

$$\begin{aligned} M_1 &= A\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \omega_2\omega_3(C - A) \\ M_2 &= A\frac{\mathrm{d}\omega_2}{\mathrm{d}t} + \omega_1\omega_3(A - C) \\ M_3 &= A\frac{\mathrm{d}\omega_3}{\mathrm{d}t} + \omega_1\omega_2(A - A) \end{aligned}$$

From fig. 5.7 one reads:

$$\omega_1 = 0 \quad ; \quad \omega_2 = \omega\sin\alpha \quad ; \quad \omega_3 = \omega\cos\alpha$$

That inserted into the components of the torque one gets:

$$\begin{aligned} M_1 &= (C - A)\omega^2\sin\alpha\cos\alpha = \frac{(C-A)}{2}\sin 2\alpha \\ M_2 &= A\dot{\omega}\sin\alpha \\ M_3 &= A\dot{\omega}\cos\alpha \end{aligned}$$

For uniform rotation $\dot{\omega} = 0$, which means that only the M_1 component of the torque remains. The torque exerted by the rotating wheel via the wheel axle on the wheel bearing, the so-called gyroscopic moment $\boldsymbol{M}^K$, points in the opposite direction of the torque M_1, hence

$$M_1^K = -M_1 = \frac{(A-C)}{2}\sin 2\alpha$$

If $A > C$ then M_1^K increases the imbalance of the wheel, i.e., the angle α increases. Conversely, in the case $A < C$, as should be the case with tires, M_1^K counteracts the imbalance and works to reduce the angle α.

Example 5.3 Torque-free motion

One speaks of a torque-free motion when the resulting total moment, i.e., the sum of all moments caused by all external forces acting on a rigid body, disappears. To examine torque-free motion one considers the ellipsoid of inertia (fig. 5.8) in the principal axis system. Of the kinetic energy of the rigid body (eq. (5.13)), only the kinetic energy due to its own rotation remains in the case of torque-free motion and without losses (e.g., due to friction), it remains constant. Hence:

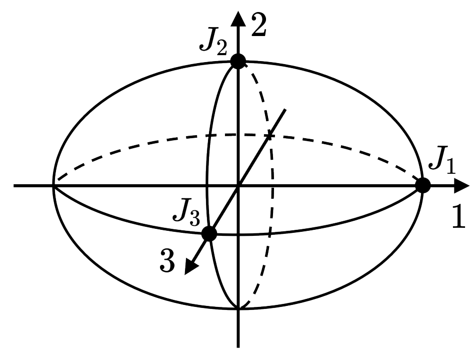

Fig. 5.8

$$T(\boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega}\big(\underline{\underline{\boldsymbol{J}}}\cdot\boldsymbol{\omega}\big) = \frac{1}{2}\sum_{i,j}^{3}\omega_i J_{ij}\omega_j = C \tag{5.28}$$

For the moments of inertia of the ellipsoid in fig. 5.8 applies

$$J_1 > J_2 > J_3$$

That inserted gives for the kinetic energy due to self-rotation

$$T(\boldsymbol{\omega}) = \frac{1}{2}\big(\omega_1^2 J_1 + \omega_2^2 J_2 + \omega_3^2 J_3\big) \tag{5.29}$$

With $L_i = J_i\omega_i\,,\ \ i = 1,2,3$ it follows

$$T(\boldsymbol{\omega}) = \frac{1}{2}\Big(\frac{L_1^2}{J_1} + \frac{L_2^2}{J_2} + \frac{L_3^2}{J_3}\Big) = const \tag{5.30}$$

that means for torque-free motion it must hold that

$$L^2 = L_1^2 + L_2^2 + L_3^2 = const \tag{5.31}$$

With $\big|\boldsymbol{L}^2\big| = const$ it follows

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 0 \tag{5.32}$$

The square of the angular momentum, i.e., its magnitude, is a conserved quantity in the body-fixed system.

In the following, the case of two-axis symmetry will be considered. In this specific case, it holds for the principal moments of inertia $A = J_{11}$ along the ξ_1 axis, $B = J_{22}$ along the ξ_2-axis and $C = J_{33}$ along the ξ_3-axis that:

$$A = B \neq C$$

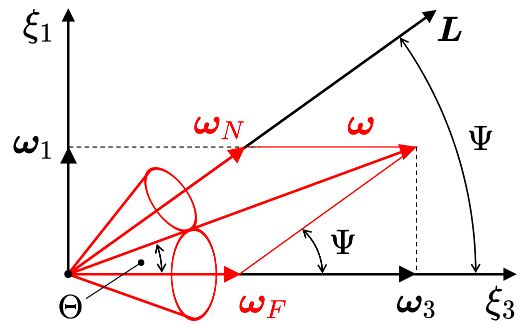

Fig. 5.9

As sketched in fig. 5.9, the figure axis is identical with the coordinate axis ξ_3. $\boldsymbol{\omega}_F$ denotes the rotation around the figure axis and $\boldsymbol{\omega}_N$ the rotation around the axis of nutation. Between the vectors of the angular velocities $\boldsymbol{\omega}$, $\boldsymbol{\omega}_F$ and $\boldsymbol{\omega}_N$ and the components of the angular velocity along the coordinate axes ξ_1 and ξ_3 the relationship

$$\boldsymbol{\omega} = \boldsymbol{\omega}_N + \boldsymbol{\omega}_F = \omega_1 \hat{\boldsymbol{\xi}}_1 + \omega_3 \hat{\boldsymbol{\xi}}_3$$

holds. As fig. 5.9 shows, for the angle Θ applies

$$\left.\begin{aligned} \omega_1 &= \omega \sin\Theta \\ \omega_3 &= \omega \cos\Theta \end{aligned}\right\} \quad \tan\Theta = \frac{\omega_1}{\omega_3} \tag{5.33}$$

The following relationships for the angle Ψ can be read from fig. 5.9

$$\tan\Psi = \frac{L_1}{L_3} = \frac{A\omega_1}{C\omega_3} = \frac{\omega_1}{\omega_3 - \omega_F} \tag{5.34}$$

and therefore

$$\omega_F = \frac{A - C}{A} \cdot \omega_3 \tag{5.35}$$

For example, the following applies to the Earth: $\dfrac{A-C}{A} \approx \dfrac{1}{305}$

From the Euler equations eq. (5.18) for torque-free motion follows:

$$\begin{aligned} A\dot{\omega}_1 + \omega_2\omega_3(C - A) &= M_1 = 0 \\ A\dot{\omega}_2 + \omega_1\omega_3(A - C) &= M_2 = 0 \\ C\dot{\omega}_3 + \omega_1\omega_2(A - A) &= M_3 = 0 \end{aligned}$$

From the third equation follows immediately

$$\omega_3 = \omega_3^0 = const \tag{5.36}$$

This means that the projection of $\boldsymbol{\omega}$ onto the figure axis is a time invariant. If one now makes use of the relationship from eq. (5.35) in the first two of Euler's equations, one obtains

$$\dot{\omega}_1 = \omega_2\omega_F \qquad \text{und} \qquad \dot{\omega}_2 = -\omega_1\omega_F \tag{5.37}$$

Taking the time derivatives in eq. (5.37) and inserting the respective original equations yields two differential equations, one for ω_1 and one for ω_2

$$\ddot{\omega}_1 + \omega_F^2\omega_1 = 0 \qquad \text{und} \qquad \ddot{\omega}_2 + \omega_F^2\omega_2 = 0 \tag{5.38}$$

These differential equations are equations of motion for harmonic oscillators. The general solutions of eq. (5.37) are thus

$$\omega_1(t) = \alpha\sin(\omega_F t + \beta) \qquad \text{und} \qquad \omega_2(t) = \alpha\cos(\omega_F t + \beta) \tag{5.39}$$

It follows from eq. (5.36) that the absolute value of $\boldsymbol{\omega}$ must be constant. That means it must hold that (see fig. 5.10)

$$|\boldsymbol{\omega}| = \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\alpha^2 + \omega_3^{0\,2}} = const \tag{5.40}$$

The projection of $\boldsymbol{\omega}(t)$ onto the $\xi_1\xi_2$-plane, the plane perpendicular to the figure axis ξ_3, therefore describes a circle in the $\xi_1\xi_2$-plane with radius α. The circular cone which $\boldsymbol{\omega}$ describes is the so-called pole cone (fig. 5.10). The opening angle of this pole cone is given by eq. (5.33) and the angular velocity with which this rotation of $\boldsymbol{\omega}$ occurs is ω_F from eq. (5.35) with $\omega_3 = \omega_3^0$. This circular movement of the axis of rotation of a rigid body around its figure axis is the so-called nutation motion. In the case of Earth, one obtains for the period of this nutation motion, referred to as Chandler's period after its discoverer, the value

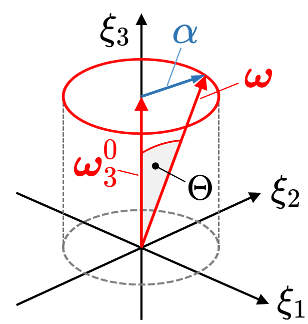

Fig. 5.10

$$T_F = \frac{2\pi}{\omega_F} = \frac{A}{A-C}\underbrace{\frac{2\pi}{\omega_3^0}}_{1\,\text{day}} \approx 305\,\text{days}$$

However, in reality T_F is not $\approx$ 305 days long, but $T_F \approx$ 433 days. The reason for this deviation lies in the earlier assumption that the Earth is a rigid body. But that is not the case, because the Earth is an elastic body and in parts it is liquid.

The coordinate system (ξ_1, ξ_2, ξ_3) in fig. 5.9 is the Earth body's principal axis system. With the help of eq. (5.27) one can determine the Euler angles, denoted here by ϕ, ϑ and ψ, with respect to the spatially fixed coordinate system (x_1, x_2, x_3):

$$\boldsymbol{\omega} \;=\; \begin{pmatrix} \alpha\sin(\omega_F t+\beta) \\ \alpha\cos(\omega_F t+\beta) \\ \omega_3^0 \end{pmatrix} \;=\; \begin{pmatrix} \sin\vartheta\sin\psi & \cos\psi & 0 \\ \sin\vartheta\cos\psi & -\sin\psi & 0 \\ \cos\vartheta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot\phi \\ \dot\vartheta \\ \dot\psi \end{pmatrix} \tag{5.41}$$

It is best to choose the space-fixed coordinate system (x_1, x_2, x_3) such that $\hat{\boldsymbol{x}}_3$ coincides with the direction of the angular momentum vector $\boldsymbol{L}$, i.e., $\hat{\boldsymbol{x}}_3 \parallel \boldsymbol{L}$. With eq. (5.25), the representation of $\hat{x}_3$ in the body's own coordinate system becomes

$$\hat{\boldsymbol{x}}_3 \;=\; \sin\vartheta\sin\psi\cdot\hat{\boldsymbol{\xi}}_1 + \sin\vartheta\cos\psi\cdot\hat{\boldsymbol{\xi}}_2 + \cos\vartheta\cdot\hat{\boldsymbol{\xi}}_3 \tag{5.42}$$

For torque-free motion, the magnitude of the angular momentum is a constant of motion (eq. (5.32)). For the components of $\boldsymbol{L}$ in the principal axis system therefore applies:

$$\boldsymbol{L} \;=\; L\cdot\big(\sin\vartheta\sin\psi\,,\;\sin\vartheta\cos\psi\,,\;\cos\vartheta\big) \tag{5.43}$$

With eq. (5.41) and eq. (5.43) one obtains for $\boldsymbol{L} \;=\; \underline{\underline{\boldsymbol{J}}}\cdot\boldsymbol{\omega}$ in the principal axis system the system of equations

$$\begin{aligned} L\sin\vartheta\sin\psi \;&=\; A\dot\phi\sin\vartheta\sin\psi + A\dot\vartheta\cos\psi \\ L\sin\vartheta\cos\psi \;&=\; A\dot\phi\sin\vartheta\cos\psi - A\dot\vartheta\sin\psi \\ L\cos\vartheta \;&=\; C\dot\phi\cos\vartheta + C\dot\psi \end{aligned} \tag{5.44}$$

By multiplying the first of these three equations by $\cos\psi$ and subtracting from it the second equation multiplied by $\sin\psi$, one can eliminate $\dot\phi$ and the angle ψ, thereby obtaining the relationship

$$0 \;=\; A\dot\vartheta\big(cos^2\psi + \sin^2\psi\big) \;=\; A\dot\vartheta$$

From that it follows

$$\vartheta \;=\; const \;=\; \vartheta_0 \tag{5.45}$$

and for the components of $\boldsymbol{\omega}$ from eq. (5.41) one therefore gets

$$\begin{aligned} \omega_1 \;&=\; \alpha\sin(\omega_F t+\beta) \;=\; \dot\phi\sin\vartheta_0\sin\psi \\ \omega_2 \;&=\; \alpha\cos(\omega_F t+\beta) \;=\; \dot\phi\sin\vartheta_0\cos\psi \\ \omega_3 \;&=\; \omega_3^0 \;=\; \dot\phi\cos\vartheta_0 + \dot\psi \end{aligned} \tag{5.46}$$

A possible approach to determine $\phi(t)$ and $\psi(t)$ follows from the consideration of

$$\omega_1\cos\psi - \omega_2\sin\psi \;=\; 0$$

Thus, according to eq. (5.46), it must also apply

$$\alpha \sin(\omega_F t + \beta) \cos\psi - \alpha \cos(\omega_F t + \beta) \sin\psi \;=\; \alpha \sin(\omega_F t + \beta - \psi) \;=\; 0$$

This condition is fulfilled for two values of $\psi(t)$

$$\psi(t) \;=\; \begin{cases} \omega_F t + \beta \\ \omega_F t + \beta + \pi \end{cases} \tag{5.47}$$

Inserting eq. (5.47) into the third equation of eq. (5.46) yields

$$\omega_3^0 \;=\; \dot{\phi} \cos\vartheta_0 + \dot{\psi} \;=\; \dot{\phi} \cos\vartheta_0 + \omega_F$$

With eq. (5.35) it follows for $\dot{\phi}$

$$\dot{\phi} \;=\; \frac{\omega_3^0 - \omega_F}{\cos\vartheta_0} \;=\; \frac{C}{A} \frac{\omega_3^0}{\cos\vartheta_0} \;=\; const$$

and hence

$$\phi(t) \;=\; \frac{C}{A} \frac{\omega_3^0}{\cos\vartheta_0} \cdot t + \phi_0 \tag{5.48}$$

The value of α can now be determined by setting $\omega_F t + \beta = 0$ for ω_2 in eq. (5.46). Then, according to eq. (5.47) $\psi = 0$ or $\psi = \pi$, i.e., $\cos\psi = \pm 1$, and one obtains for α the two solutions

$$\alpha \;=\; \pm\dot{\phi} \sin\vartheta_0 \;=\; \pm\frac{C}{A} \omega_3^0 \tan\vartheta_0 \tag{5.49}$$

Of course, given the geometric interpretation of α (see eq. (5.40)) only the positive sign makes sense. There is an alternative way to determine $\phi(t)$ and $\psi(t)$ by multiplying the first equation in eq. (5.44) by $\sin\psi$ and then adding the second equation in eq. (5.44) multiplied by $\cos\psi$ to it. Therewith, one can eliminate $\dot{\vartheta}$ and the angles ϑ and ψ and thus obtains

$$A\dot{\phi} \;=\; L$$

From that it follows

$$\phi(t) \;=\; \frac{L}{A} \cdot t + \phi_0 \tag{5.50}$$

One can obtain $\dot{\psi}$, for example, with the help of eq. (5.35) and eq. (5.45) by inserting eq. (5.50) into the third equation of eq. (5.44):

$$\dot{\psi} \;=\; \left(\frac{L}{C} - \frac{L}{A}\right)\cos\vartheta_0 \;=\; \frac{L}{C}\frac{A-C}{A}\cos\vartheta_0 \;=\; \frac{L}{C}\frac{\omega_F}{\omega_3^0}\cos\vartheta_0 \tag{5.51}$$

However, $\dot{\psi}$ can also be obtained from the equation for ω_3 in eq. (5.46)

$$\dot{\psi} \;=\; \omega_3^0 - \dot{\phi}\cos\vartheta_0 \;=\; \omega_3^0 - \frac{L}{A}\cos\vartheta_0 \tag{5.52}$$

A comparison of eq. (5.52) with eq. (5.51) shows that it must be true that

$$\omega_3^0 \;=\; \frac{L}{C}\cos\vartheta_0 \tag{5.53}$$

Inserting this into eq. (5.51) yields $\dot{\psi} = \omega_F$ and therefore

$$\psi(t) \;=\; \omega_F t + \psi_0 \tag{5.54}$$

Now one can also determine the constants ϑ_0 and ψ_0. Inserting eq. (5.50) into the first two equations of eq. (5.46) yields

$$\alpha\sin(\omega_F t + \beta) \;=\; \frac{L}{A}\sin\vartheta_0\sin\psi$$

$$\alpha\cos(\omega_F t + \beta) \;=\; \frac{L}{A}\sin\vartheta_0\cos\psi$$

Dividing these two equations and a comparison with eq. (5.54) gives

$$\tan\psi \;=\; \tan(\omega_F t + \beta) \qquad \Rightarrow \qquad \psi_0 \;=\; \beta \tag{5.55}$$

As expected, $\psi_0 = \beta$ corresponds to the result of eq. (5.47). If one inserts this solution into the first equation of eq. (5.46) one finds that

$$\alpha\sin(\omega_F t + \beta) \;=\; \alpha\sin\psi \;=\; \frac{L}{A}\sin\vartheta_0\sin\psi$$

With that applies

$$\sin\vartheta_0 \;=\; \frac{A}{L}\alpha$$

If one divides this equation by eq. (5.53), one eventually gets

$$\tan\vartheta_0 \;=\; \frac{A}{C}\frac{\alpha}{\omega_3^0} \tag{5.56}$$

This corresponds to the result of eq. (5.49) for α. In summary, one has for the Euler angles and the associated angular velocities

$$\begin{aligned} \phi(t) &= \frac{C}{A}\frac{\omega_3^0}{\cos\vartheta_0}\cdot t + \phi_0 &&;\qquad \dot\phi = \frac{C}{A}\frac{\omega_3^0}{\cos\vartheta_0} \\ \vartheta(t) &= \vartheta_0 = \arctan\left(\frac{A}{C}\frac{\alpha}{\omega_3^0}\right) &&;\qquad \dot\vartheta = 0 \\ \psi(t) &= \frac{A-C}{A}\cdot\omega_3^0 t + \beta &&;\qquad \dot\psi = \frac{A-C}{A}\cdot\omega_3^0 \end{aligned} \tag{5.57}$$

In these relationships one still finds five of the original six variables ϕ_0, ϑ_0, ψ_0, α, β and ω_3^0 present, but these are no longer all independently freely selectable. Through eq. (5.49) or eq. (5.56) the parameters α, ϑ_0 and ω_3^0 are connected. Because of eq. (5.40), however, α and ω_3^0 cannot be chosen independently of each other. With the choice of $\hat{\boldsymbol{x}}_3 \parallel \boldsymbol{L}$ made here, ϑ_0 became fixed and therefore α and ω_3^0 are no longer freely selectable. According to eq. (5.56) $\psi_0 = \beta$ applies and therefore only one of these two variables can be chosen freely; in eq. (5.57) this is β. From the original six variables ϕ_0, ϑ_0, ψ_0, α, β and ω_3^0 only ϕ_0 and β remain therefore as free variables.

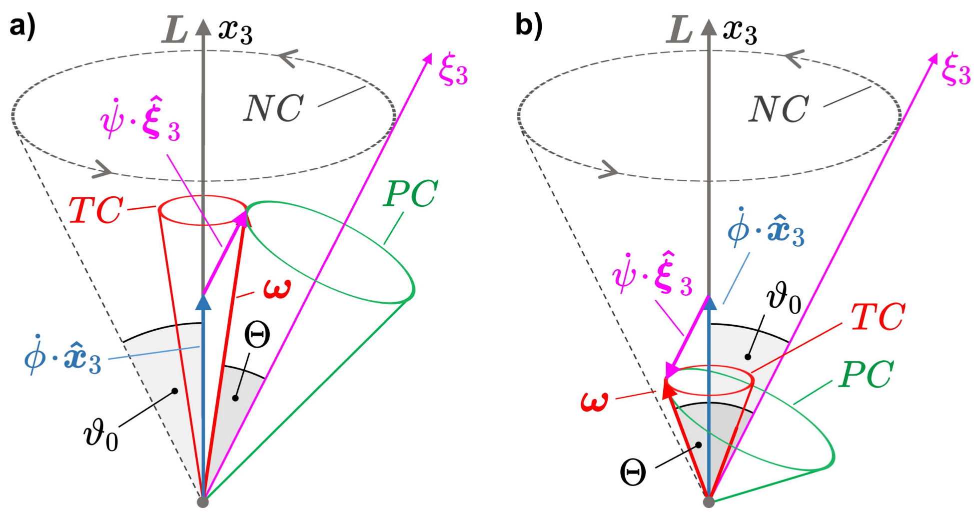

Fig. 5.11: Position of the angles, rotation vectors, nutation cone (NC), tracking cone (TK) and pole cone (PC) for the case $A > C$ (a) and the case $A < C$ (b).

In Fig. 5.11 the positions of the angles, the rotation vectors, the nutation cone, the tracking cone and the pole cone are sketched for two cases: In fig. 5.11a for $A > C$, such as is the case with the Earth; and in fig. 5.11b for the case $A < C$. With $\hat{\boldsymbol{x}}_3 \parallel \boldsymbol{L}$, ϑ_0 is the angle between the figure axis ξ_3 and $\boldsymbol{L}$ or respectively ξ_3 and x_3 (the angle ϑ_0 in fig. 5.11 is therefore identical to the angle Ψ in fig. 5.9). The figure axis rotates with the angular velocity $\dot\phi$ around the x_3-axis and thereby traces out the nutation cone of the so-called free nutation. The associated rotation vector $\dot\phi\cdot\hat{\boldsymbol{x}}_3$ is identical to $\boldsymbol{\omega}_N$ in fig. 5.9. $\dot\psi$ is the angular velocity with which the rigid body under consideration (in fig. 5.11a the Earth) rotates around the figure axis. The corresponding rotation vector $\dot{\psi}\cdot\hat{\boldsymbol{\xi}}_3$ is either parallel or antiparallel to the ξ_3-axis, depending on whether $A > C$ as in fig. 5.11a or $A < C$ as in fig. 5.11b. The rotation vector $\dot{\psi}\cdot\hat{\boldsymbol{\xi}}_3$ is identical to $\boldsymbol{\omega}_F$ in fig. 5.9. The total rotation vector $\boldsymbol{\omega}$ in fig. 5.11 is thus

$$\boldsymbol{\omega} \;=\; \dot{\phi}\cdot\hat{\boldsymbol{x}}_3 + \dot{\psi}\cdot\hat{\boldsymbol{\xi}}_3 \;=\; \boldsymbol{\omega}_N + \boldsymbol{\omega}_F$$

The rotation vector $\boldsymbol{\omega}$ always lies in the $x_3\xi_3$-plane and rotates with the figure axis around the direction of angular momentum $\boldsymbol{L}$ or respectively around the x_3-axis. In the process, $\boldsymbol{\omega}$ traces out the so-called tracking cone around $\boldsymbol{L}$ or respectively around the x_3 axis. The fixed angle between $\boldsymbol{\omega}$ and the figure axis ξ_3 is the angle Θ from fig. 5.9. Θ is the opening angle of the so-called pole cone. While the figure axis performs its nutation motion with ω_N around $\boldsymbol{L}$ or respectively the x_3-axis, this pole cone rolls off on the tracking cone with ω_N. In the case $A > C$ (fig. 5.11a) the pole cone rolls off with its outer surface on the outer surface of the tracking cone. In the case $A < C$ (fig. 5.11b), however, the pole cone rolls off with its inner surface on the outer surface of the tracking cone.

Example 5.4 **Motion with non-vanishing torque**

A simple example of the behavior of a rotating body when there is a non-vanishing torque is sketched in fig. 5.12. The figure shows a rotating wheel suspended on one side from a rope. The angular momentum vector $\boldsymbol{L}$ of the rotating wheel lies in the wheel axis whose length is l_A. At one end, the wheel axle is held by the rope force, at the other end an equal but opposite weight force acts on the center of mass of the wheel.

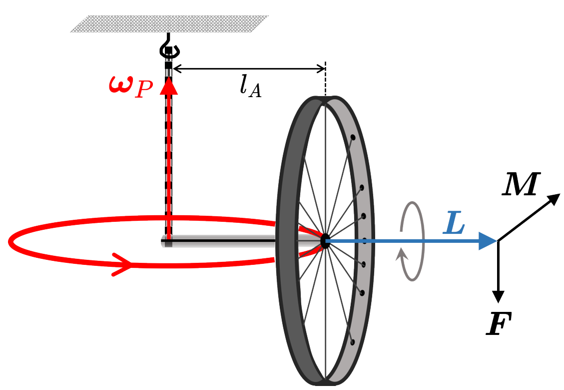

Fig. 5.12

The resulting torque $\boldsymbol{M}$ is perpendicular to the plane spanned by the wheel axis and the weight force. If the wheel does not rotate, then it naturally will tilt down. However, if the wheel rotates, the torque $\boldsymbol{M}$ perpendicular to $\boldsymbol{L}$ causes a change in direction of the wheel axis, which is called precession. If the angular velocity ω with which the wheel rotates is large enough, then the wheel will rotate or respectively precess with the wheel axis in horizontal direction around the rope suspension. With the Euler equations in eq. (5.17) applies

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \;=\; \underline{\underline{\boldsymbol{J}}}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega}\times\left(\underline{\underline{\boldsymbol{J}}}\cdot\boldsymbol{\omega}\right) \;=\; \boldsymbol{M}$$

As the direction of the rotation axis of the wheel changes, the rate of change of the angular momentum $\mathrm{d}\boldsymbol{L}/\mathrm{d}t$ remains perpendicular to the angular momentum vector $\boldsymbol{L}$ and perpendicular to the angular momentum vector $\boldsymbol{\omega}_P$ of precession. Therefore:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{\omega}_P \times \boldsymbol{L} = \boldsymbol{M}$$

Without friction losses, the absolute value of $\boldsymbol{L}$ will remain constant. With $|\boldsymbol{L}|$ and $|\boldsymbol{M}|$ each being constant, the magnitude of the angular velocity of the precession

$$|\boldsymbol{\omega}_P| = \frac{|\boldsymbol{M}|}{|\boldsymbol{L}|}$$

is also constant. In this case one speaks of a so-called regular precession. As can easily be seen from fig. 5.13, the angular velocity of the precession will remain constant even when the wheel axis should move out of the horizontal position. For the torque applies

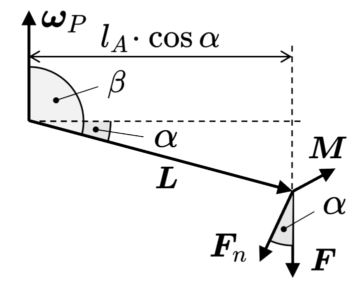

Fig. 5.13

$$|\boldsymbol{M}| = l_A{\cdot}\cos\alpha{\cdot}|\boldsymbol{F}|$$

and for the rate of change of the angular momentum applies

$$\left|\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right| = |\boldsymbol{\omega}_P \times \boldsymbol{L}| = |\boldsymbol{\omega}_P||\boldsymbol{L}|\sin\beta = |\boldsymbol{\omega}_P||\boldsymbol{L}|\cos\alpha$$

The left-hand sides of the two equations are identical since $\boldsymbol{M} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$ and thus the corresponding right-hand sides of the two equations are also the same:

$$|\boldsymbol{\omega}_P||\boldsymbol{L}|\cos\alpha = l_A{\cdot}\cos\alpha{\cdot}|\boldsymbol{F}|$$

The $\cos\alpha$ term cancels out and with $\boldsymbol{L} = \underline{\underline{\boldsymbol{J}}}{\cdot}\boldsymbol{\omega}$ and $|\boldsymbol{F}| = m{\cdot}g$ one obtains

$$\omega_P = \frac{l_A{\cdot}m{\cdot}g}{I_A{\cdot}\omega}$$

where m is the mass of the wheel and I_A is the moment of inertia for rotation about the wheel axle. This expression is independent of α and given the length of the wheel axis (l_A), mass of the wheel (m) and mass distribution with respect to the wheel axis (I_A), the angular velocity of precession ω_P depends only on the rotational velocity of the wheel ω. However, the expression derived here for ω_P only applies if $\omega \gg \omega_P$.

In general, for a body which is subject to a non-zero moment of force (torque), the components of the torque in the body-fixed coordinate system will depend on the orientation of the body in the space-fixed coordinate system. This however means that one cannot any longer solve the Euler equations independently from the respective equations for the

Euler angles ϕ, ϑ and ψ. To address this problem a different approach is required such as the Lagrange formalism which we will discuss at length in the next chapter. There, equipped with the necessary tools, we will revisit gyroscopic motion again and learn how it can be solved (see ex. 6.10 in section 6.3.2).

To summarize the above discussion, gyroscopic motion is composed of the superposition of three different motions (compare fig. 5.11):

$\psi(t)$: Rotation of the gyroscope around the body-fixed figure axis with rotation vector $\dot{\psi}(t)\cdot\hat{\boldsymbol{\xi}}_3$.

$\phi(t)$: Rotation of the figure axis around the space-fixed coordinate x_3 with rotation vector $\dot{\phi}(t)\cdot\hat{\boldsymbol{x}}_3$. This motion is called precession.

$\vartheta(t)$: Periodic change of the angle $\vartheta(t)$ between the figure axis ξ_3 and the spatially fixed coordinate x_3 when there is a non-vanishing torque. This motion is then called nutation. In the case of torque-free motion, $\vartheta(t)$ does not vary but is constant, i.e., $\vartheta(t) = \vartheta_0$; in this case one speaks of free nutation.

Fig. 5.14 illustrates the precession and nutation motions using the Earth as an example. In spherical coordinates the direction of the figure axis in the spatially fixed coordinate system is

$$\hat{\boldsymbol{\xi}}_3 = \sin\phi\sin\vartheta\cdot\hat{\boldsymbol{x}}_1 - \cos\phi\sin\vartheta\cdot\hat{\boldsymbol{x}}_2 + \cos\vartheta\cdot\hat{\boldsymbol{x}}_3$$

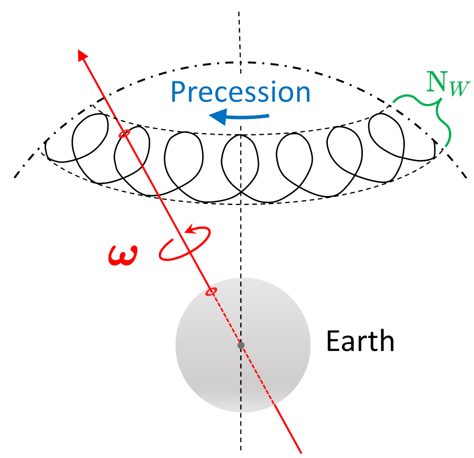

Fig. 5.14

If one now follows the motion of the figure axis on a spherical surface as a function of time, i.e., $\xi_3(t)$, then one can represent nutation and precession as depicted in fig. 5.14. In fig. 5.14, N_W is the range between which $\vartheta(t)$ oscillates. For Earth, it takes about 26 000 years, 25 772 to be exact, for one full cycle of its precession to complete. Largely, this precession of Earth's rotational axis is caused by the combined gravitational pull of Sun and Moon exerted on Earth's equatorial bulge; gravitational interactions with other planets matter as well but to a much lesser degree. As for Earth's nutation, it takes about 41 000 years for the tilt of our planets axis to complete a cycle, i.e., for $\vartheta(t)$ to vary from its maximum value to its minimum value and back again to its maximum value. In the past, over such a 41 000-year cycle, $\vartheta(t)$ has varied between a minimum value of $\vartheta = 22.1°$ and a maximum value $\vartheta = 24.5°$.

5.3.1 Moving Frames of Reference

$$\boldsymbol{r}(t) = \boldsymbol{R}(t) + \boldsymbol{r}'(t)$$

$$\boldsymbol{r}'(t) = \sum_{i=1}^{3} x_i' \hat{\boldsymbol{e}}_i' \qquad \text{all time dependent}$$

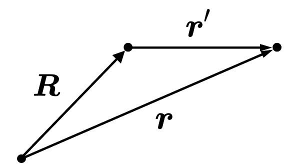

Fig. 5.15

$$\dot{\boldsymbol{r}}(t) = \dot{\boldsymbol{R}}(t) + \sum_{i=1}^{3} \dot{x}_i'(t) \hat{\boldsymbol{e}}_i'(t) + (\boldsymbol{\omega} \times \boldsymbol{r}')$$

$$\ddot{\boldsymbol{r}}(t) = \ddot{\boldsymbol{R}}(t) + \sum_{i=1}^{3} \ddot{x}_i' \hat{\boldsymbol{e}}_i' + 2 \sum_{i=1}^{3} \dot{x}_i' \dot{\hat{\boldsymbol{e}}}_i' + (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}') + \left[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')\right]$$

If $\boldsymbol{a}$, $\boldsymbol{A}$, and $\boldsymbol{a}'$ are the acceleration vectors in the corresponding coordinate directions, then

$$\ddot{\boldsymbol{r}}(t) = \boldsymbol{a} = \boldsymbol{A} + \boldsymbol{a}' + \underbrace{2(\boldsymbol{\omega} \times \boldsymbol{v}')}_{\substack{\text{Coriolis-} \\ \text{acceleration}}} + (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}') + \Big[\underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')}_{\substack{\text{Centrifugal} \\ \text{acceleration}}}\Big] \tag{5.58}$$

With $\boldsymbol{F} = m\boldsymbol{a}$ being the force acting on a mass m in a reference system at rest, the force $\boldsymbol{F}' = m\boldsymbol{a}'$ observed by someone in the moving system is given by

$$\boldsymbol{F}' = \boldsymbol{F} - \underbrace{m\Big\{\boldsymbol{A} + 2(\boldsymbol{\omega} \times \boldsymbol{v}') + (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}') + \big[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')\big]\Big\}}_{\text{Inertial forces}} \tag{5.59}$$

The inertial forces in eq. (5.59) can only be made to disappear point by point because the gravitational force has a different $\boldsymbol{r}$-dependence. The Coriolis force and the centrifugal force are so-called apparent forces. The Coriolis force only acts on bodies moving in a rotating system, for a body at rest in a moving system, i.e., $\boldsymbol{v} = 0$, it does not exist. The centrifugal force exists of course for every body in a rotating system and disappears when the rotation stops, i.e., $\boldsymbol{\omega} = 0$. Hence, the Coriolis force and the centrifugal force are only experienced by bodies in a rotating system. In that respect they are similar to the inertial force which only bodies in an accelerated system experience. Apparent forces such as the Coriolis force, the centrifugal force or the inertial force depend on the reference frame. They do not exist for an observer outside the reference frame, that is for an observer who is not part of the respective rotating or linearly accelerated system being considered. Just consider a standing passenger in a bus moving at constant speed who is not holding on to a handrail. As long as there is no acceleration or deceleration nothing undue will happen as the bus and the passenger both will continue to move with the same constant speed and hence no forces are at play. However, this changes when the bus suddenly accelerates or decelerates with the result that the passenger may either fly out through the back or the front window of the bus. When this happens, as far as the passenger is concerned, nothing has changed for she or he is still moving with the same constant velocity. Not so from the perspective of fellow travelers on the bus who see the passenger thrown out the back or the front window by an invisible force. Of course, something very similar will happen if the bus does not change speed at all but takes a sharp turn left or right; in that case the passenger will just fly out through a window on the left or the right side of the bus depending on the direction of the turn the bus takes. Here, an observer on the sidewalk will just see the passenger continuing in linear motion at constant speed as if no forces were acting on her or him.

Let's look at the Coriolis force and the centrifugal force associated with the rotation of a system O' with respect to O which is the system at rest. To do that we will pick a system that rotates at constant angular velocity $\boldsymbol{\omega}$, i.e, we are looking at a case of uniform circular motion of O'. For a point mass at rest in the rotating system $\boldsymbol{v}' = 0$ and hence, the Coriolis force vanishes but the centrifugal force is still there. The total force $\boldsymbol{F}'$ on a point mass that is at rest in the rotating system O' must be zero, hence for $\boldsymbol{v}' = 0$ eq. (5.59) becomes

$$0 \;=\; \boldsymbol{F}' \;=\; \boldsymbol{F} - m\cdot\big[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')\big] \;=\; \boldsymbol{F} - mr'_{\perp}\omega^2\hat{\boldsymbol{r}}'_{\perp}$$

Here, we have used the fact that $\boldsymbol{\omega}$ points in the direction of the rotational axis running through O' and therefore only $\boldsymbol{r}_{\perp}$, the component of $\boldsymbol{r}$ perpendicular to the axis of rotation contributes. Eq. (5.59) therefore reduces to

$$\boldsymbol{F} \;=\; mr'_{\perp}\omega^2\hat{\boldsymbol{r}}'_{\perp}$$

This tells us that someone in O, the system at rest, sees a force of the magnitude $mr'_{\perp}\omega^2$ at work which is just the centripetal force associated with a uniform circular motion. From the perspective of an observer in O, it is this force which keeps the point mass in the rotating system O' at rest.

Considering again a system O' rotating at constant angular velocity $\boldsymbol{\omega}$ where a point mass now moves in a radial direction from O' outwards, i.e., its velocity $\boldsymbol{v}'$ is parallel to its position vector $\boldsymbol{r}'$. The Coriolis force $\boldsymbol{F}_C$ in $\boldsymbol{F}'$ acting on this moving point mass, is perpendicular to $\boldsymbol{\omega}$ and $\boldsymbol{v}'$, i.e., it is perpendicular to $\boldsymbol{r}'$. For an outside observer this looks like the point mass is subject to a momentum

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{r}' \times \boldsymbol{F}_C = 2m\boldsymbol{r}' \times (\boldsymbol{\omega} \times \boldsymbol{v}') \\ &= 2m\big(\boldsymbol{\omega}(\underbrace{\boldsymbol{r}'\cdot\boldsymbol{v}'}_{r'\cdot v'}) - \boldsymbol{v}'(\underbrace{\boldsymbol{r}'\cdot\boldsymbol{\omega}}_{0})\big) = 2mr'v'\boldsymbol{\omega}\end{aligned}$$

A moment of force, i.e., a torque is always the result of a change in angular momentum $\boldsymbol{L}$. The angular momentum associated with the rotation of the radially outward moving point mass as observed from O must therefore be given by

$$\boldsymbol{L} = m\boldsymbol{r}'^2\boldsymbol{\omega}$$

so that its rate of change becomes

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = 2mr'v'\boldsymbol{\omega} = \boldsymbol{M}$$

Example 5.5 The weather map

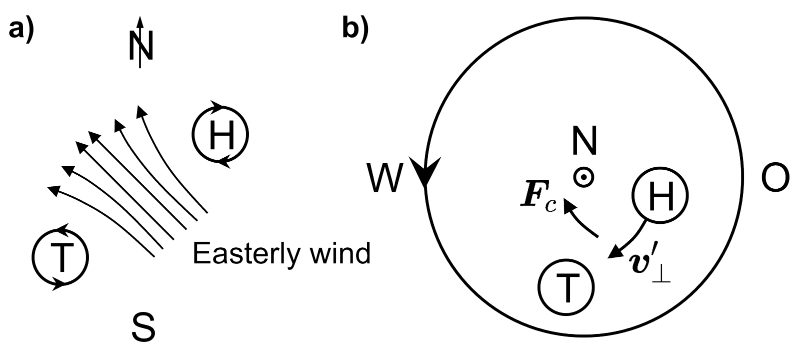

Fig. 5.16: (a) Formation of easterly winds in the northern hemisphere. (b) Coriolis force $\boldsymbol{F}_C$ in the northern hemisphere. The relative motion $\boldsymbol{v}'_\perp$ occurs perpendicular to $\boldsymbol{F}_C$.

A high-pressure area in the atmosphere is characterized by the fact that air masses flow from higher air layers towards the Earth's surface and lead to an increase in pressure there. The air masses escaping from this high-pressure area above the Earth's surface are deflected by the Coriolis force to the right in the northern hemisphere and to the left in the southern hemisphere. Thus, high-pressure areas are surrounded by right-hand vortices in the northern hemisphere and left-hand vortices in the southern hemisphere. For low-pressure areas, exactly the opposite is the case. Low-pressure areas are characterized by the fact that near-surface air masses flow into an area with lower air pressure and rise there. So the velocity vector of these air masses points upwards while in high-pressure areas it points downwards. Therefore, low-pressure areas are surrounded by left-hand vortices in the northern hemisphere and by right-hand vortices in the southern hemisphere. Fig. 5.16a illustrates how an easterly wind can be created by adjacent high- and low-pressure areas in the northern hemisphere.

Example 5.6 **East deviation in free fall**

At small angular velocities $\boldsymbol{\omega}$, terms proportional to $\boldsymbol{\omega}^2$, i.e., the centrifugal force, can be neglected in eq. (5.59). Under this condition (in a first approximation), the equation of motion for free fall reads:

$$\ddot{\boldsymbol{r}}' = -g\hat{\boldsymbol{e}}_{z'} - 2(\boldsymbol{\omega} \times \boldsymbol{v}')$$

Fig. 5.17 shows the situation for free fall in the northern hemisphere at latitude φ. The y'-coordinate points out of the image plane in fig. 5.17.

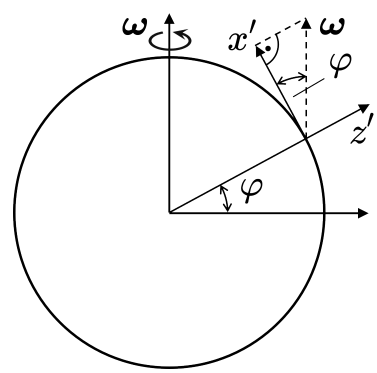

Fig. 5.17

For the rotation vector $\boldsymbol{\omega}$ one can see that:

$$\boldsymbol{\omega} = |\boldsymbol{\omega}| \cdot \begin{pmatrix} \cos\varphi \\ 0 \\ \sin\varphi \end{pmatrix}$$

The equations of motion are thus in a first approximation ($\boldsymbol{\omega}$ small):

$$\begin{pmatrix} \ddot{x}' \\ \ddot{y}' \\ \ddot{z}' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - 2 \cdot \begin{pmatrix} \omega\cos\varphi \\ 0 \\ \omega\sin\varphi \end{pmatrix} \times \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix}$$

or respectively

$$\left.\begin{aligned} \ddot{x}' &= 2\omega\sin\varphi \cdot \dot{y}' \\ \ddot{y}' &= -2\omega\sin\varphi \cdot \dot{x}' + 2\omega\cos\varphi \cdot \dot{z}' \\ \ddot{z}' &= -g + 2\omega\cos\varphi \cdot \dot{y}' \end{aligned}\right\} \tag{5.60}$$

In the zeroth-order approximation $\omega \approx 0$, i.e., the Coriolis force falls away and one thus has

$$\dot{z}_0' = -g \cdot t \quad ; \quad z_0' = h - \frac{g}{2} \cdot t^2 \quad ; \quad \dot{x}_0' = \dot{y}_0' = 0$$

If one inserts the result from the zeroth-order approximation into eq. (5.60) - the first-order approximation - then only the equation for the y'-coordinate remains.

$$\ddot{y}' = -2\omega\cos\varphi \cdot g \cdot t$$

The y'-coordinate in fig. 5.17 runs from west to east along the constant latitude φ. A value for $y' \neq 0$ means that there will be a deviation along the direction of the latitude during free fall due to the Coriolis force. Integrating once yields

$$\dot{y}' = -\omega \cos\varphi \cdot g \cdot t^2$$

and integrating one more time gives for the y'-coordinate

$$y' = -\omega \cos\varphi \cdot g \cdot \frac{t^3}{3}$$

From the zeroth-order approximation one can determine the fall time from the initial height h to the Earth's surface $z'_0 = 0$. Inserting this gives for the deviation in y'-direction during free-fall in the northern hemisphere

$$y' = -\frac{2\sqrt{2}}{3} \cdot h \cdot \sqrt{\frac{h}{g}} \cdot \omega \cos\varphi$$

The minus sign means that the direction of the deviation points into the image plane of fig. 5.17, i.e., in the direction of the negative y'-axis. Hence there is an easterly deviation for free falling objects in the northern hemisphere. To illustrate: With a drop-height of 500 m at a latitude of $\varphi = 45°$, this easterly deviation is ≈ -0.17 m.

6. Systems of Point Masses

6.1 Momentum Theorem

Every point mass in a system of point masses is subject to Newton's second law:

$$\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \boldsymbol{F}_i \tag{6.1}$$

Here $\boldsymbol{p}_i$ is the momentum of the point mass m_i and $\boldsymbol{F}_i$ is the force acting on it. With respect to the forces acting on a point mass, a distinction is made between internal and external forces.

Definition
Internal forces: These arise from the interaction between points masses which only include two-body interactions between any two given point masses.

$$\boldsymbol{F}_i = \sum_{j \neq i} \boldsymbol{F}_{ij}$$

With respect to the interaction between any two point masses Newton's third law applies (actio = reactio):

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} \tag{6.2}$$

Definition
External forces: These depend only on the coordinates of a point mass or on its velocity or are a function of time; but they do not depend on other point masses. Examples of this are the motion of a point mass in an electromagnetic field or in the gravitational field.

$$\boldsymbol{F}_i \neq \sum \boldsymbol{F}_{ij}$$

Without the influence of external forces, the total momentum of a system of point masses is constant, which means:

$$\sum_i \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\sum_i \boldsymbol{p}_i}_{\boldsymbol{P}} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \sum_{j \neq i} \boldsymbol{F}_{ij} = -\sum_{j \neq i} \boldsymbol{F}_{ji} \equiv 0 \tag{6.3}$$

Theorem
The rate of change observed in the total momentum of a system of point masses is only caused by external forces. The total momentum of such a system cannot be changed by internal forces.

Central impact:

If the velocities of the masses m and M are v_0 and V_0 before and v and V after the central impact (fig. 6.1), the momentum theorem states

$$mv_0 + MV_0 = mv + MV$$

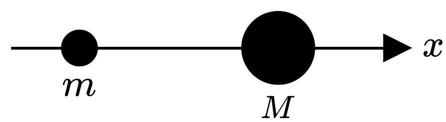

Fig. 6.1

Systems of rigid bodies: (analytical mechanics)

f shall be the number of degrees of freedom of a system. Two examples:

Example 6.1 A rotatable system

The system in fig. 6.2 consists of three rods, each with 3 degrees of freedom, connected by 4 bearings each of valence two. The number of remaining degrees of freedom of the system is thus

$$f = 9 - 8 = 1$$

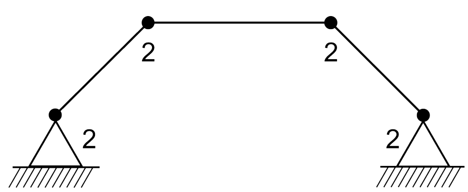

Fig. 6.2

Example 6.2 Spatial thread pendulum

The motion of the pendulum mass in fig. 6.3 by itself, i.e., without the thread suspension, would have 3 degrees of freedom. However, the pendulum thread of length a restricts the motions of the pendulum mass to a spherical surface and therefore

$$f = 3 - 1 = 2$$

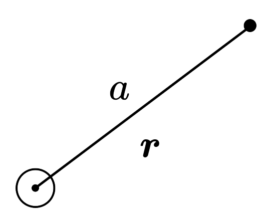

Fig. 6.3

6.2 Auxilliary Conditions and Constraint Forces

In general, all systems of rigid bodies have limitations in their motion as illustrated in ex. 6.1 and ex. 6.2 because the respective masses a system is made of cannot move independently from each other. For the number of these restrictions, the so-called auxiliary or side conditions for the coordinates of a system of n point masses, it holds that

$$\varphi_\mu(x_1, x_2, \dots x_{3n}) = 0 \qquad \mu = 1, 2, \dots 3n - f \tag{6.4}$$

where f is the number of degrees of freedom of the considered system. Auxiliary conditions φ_μ, which are only a function of the coordinates, are called holonomic. If the φ_μ also do depend on the rates of change of coordinates, i.e., on velocities, then they are called non-holonomic.

For example, in ex. 6.2 with $n = 1$ and $f = 2$ there is only one auxiliary condition, namely that the pendulum mass must move on a spherical surface with radius a. Hence it must apply

$$\varphi = x_1^2 + x_2^2 + x_3^2 - a^2 = 0 \tag{6.5}$$

Auxiliary conditions are also referred to as constraints and the corresponding forces are called constraint forces to distinguish them from the so-called imprinted forces. For the equation of motion this means for a given coordinate

$$m_i \ddot{x}_i = F_i + Z_i \tag{6.6}$$

where F_i are the imprinted forces and Z_i are the constraint forces. A potential as sketched in fig. 6.4 results in a constraint force. For example, if the potential has the general form

$$V = V_0 \varphi^2$$

it follows that

$$Z_i = -\frac{\partial V}{\partial x_i} = -\underbrace{2V_0\varphi}_{\lambda}\frac{\partial \varphi(x_1 \dots x_{3n})}{\partial x_i}$$

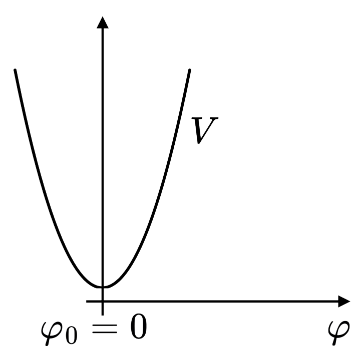

Fig. 6.4

Here λ is a multiplier that will be discussed below. For ex. 6.2 with the auxiliary conditions from eq. (6.5) holds

$$\frac{\partial \varphi}{\partial x_i} = 2x_i \quad \text{and the constraint force is} \quad Z_i = 2\lambda x_i$$

This approach of using constraint forces leads to the so-called Lagrange equations of the first kind. However, before these can be introduced, a brief digression is required to discuss the so-called D'Alembert principle of virtual work.

6.2.1 D'Alembert Principle - Virtual Work

The D'Alembert principle of virtual displacements, named after Jean-Baptiste le Rond D'Alembert, is derived from the principle of virtual work. The latter states that in an equilibrium state the virtual work resulting from virtual displacements $\delta \boldsymbol{r}_i$ of a system's point masses m_i caused by forces $\boldsymbol{F}_i$ acting on the system is zero:

$$\delta A = \sum_i \boldsymbol{F}_i \delta \boldsymbol{r}_i = 0 \qquad \text{with} \qquad \boldsymbol{F}_i = \boldsymbol{F}_i^{(i)} + \boldsymbol{F}_i^{(e)} \tag{6.7}$$

where $\boldsymbol{F}_i^{(i)}$ are the internal forces and $\boldsymbol{F}_i^{(e)}$ are the imprinted or respectively external forces. The internal forces cannot do any work on the system because otherwise a perpetuum mobile of the first kind would be possible and thus applies

$$\delta A = \sum_i \boldsymbol{F}_i^{(e)} \delta \boldsymbol{r}_i = 0 \tag{6.8}$$

D'Alembert extended the principle of virtual work to constraint forces. Contrary to today's common usage, the word "virtual" has here the meaning of "possible" in the sense of all possible displacements compatible with the constraints imposed by auxiliary conditions on the system. What this means exactly is best illustrated with a simple example.

Example 6.3 **The inclined plane**

There are only two possible, i.e., virtual, directions of motion for the ball in fig. 6.5 under the constraints of the inclined plane, namely along the directions of $\delta \boldsymbol{x}$. $\boldsymbol{Z}_{IP}$ is the constraint force which corresponds to the auxiliary condition of the inclined plane. The imprinted force is $\boldsymbol{F}^{(e)} = \boldsymbol{F}_G$.

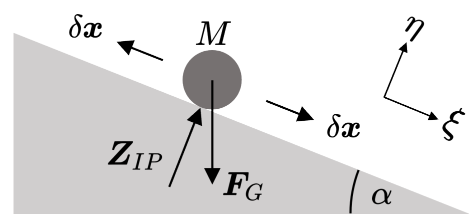

Fig. 6.5

With that, Newton's equation of motion for a ball of mass M on the inclined plane thus becomes

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}^{(e)} + \boldsymbol{Z}_{IP} \tag{6.9}$$

According to D'Alembert, the constraint forces do not perform any work under virtual displacements, i.e., $\boldsymbol{Z}_{IP}\delta\boldsymbol{x} = 0$. Hence, if one multiplies eq. (6.9) by $\delta\boldsymbol{x}$ one obtains

$$\left(\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} - \boldsymbol{F}^{(e)}\right)\cdot\delta\boldsymbol{x} \;=\; 0 \tag{6.10}$$

Evidently, to solve the equation of motion for the ball on the inclined plane one does not need to know the constraint force $\boldsymbol{Z}_{IP}$. If one now chooses the coordinate system as sketched in fig. 6.5 with the coordinate axes ξ and η then one has

$$\delta\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{F}^{(e)} = \begin{pmatrix} Mg\sin\alpha \\ -Mg\cos\alpha \end{pmatrix}$$

and thus for the equations of motion

$$\left[M\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_\xi \\ v_\eta \end{pmatrix} - \begin{pmatrix} Mg\sin\alpha \\ -Mg\cos\alpha \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; 0 \tag{6.11}$$

From this follows the well-known equation for the inclined plane

$$\frac{\mathrm{d}v_\xi}{\mathrm{d}t} \;=\; g\sin\alpha \tag{6.12}$$

In general case, with

$$\frac{\partial\varphi_\mu}{\partial x_i} \;=\; \varphi_{\mu_i}$$

the equivalence principle or the D'Alembert principle states that to all displacements δx_i applies

$$\sum_i \varphi_{\mu_i}\delta x_i \;=\; 0$$

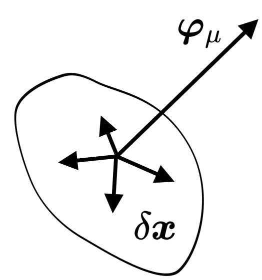

Fig. 6.6

The displacements δx_i are, as sketched in fig. 6.6, each orthogonal to φ_{μ_i}. From this it follows for the equation of motion

$$\sum_i \big(F_i - m_i\ddot{x}_i\big)\delta x_i \;=\; 0 \tag{6.13}$$

One therefore does not need to know the constraint forces to solve the equation of motion. However, sometimes one really wants to determine the constraint forces. This becomes possible with the help of the Lagrange equations of the first kind.

6.3 Lagrange Equations

6.3.1 Lagrange Equations of the First Kind

With eq. (6.4), the following applies to the auxiliary conditions:

$$\sum_i \varphi_{\mu_i} \dot{x}_i + \varphi_t \ = \ 0 \tag{6.14}$$

If these auxiliary conditions φ_μ only depend on the coordinates x_i and either explicitly or not explicitly on the time t, then they are called holonomic auxiliary conditions. Written out, this means that

$$\left.\begin{aligned} \varphi_{\mu_i} &= \frac{\partial \varphi_u}{\partial x_i} \quad \text{and} \\ \varphi_t &= \frac{\partial \varphi_\mu}{\partial t} \quad \text{or} \quad \varphi_t \ = \ 0 \end{aligned}\right\} \quad \text{with} \quad \mu = 1, \ldots, 3n - f \qquad \text{(holonomic)}$$

If the auxiliary conditions are not only a function of the coordinates x_i but also of the velocities $\dot{x}_i$, then they are called non-holonomic, regardless of whether they explicitly or not explicitly depend on the time t

$$\left.\begin{aligned} \varphi_{\mu_i} &\neq \frac{\partial \varphi_u}{\partial x_i} \quad \text{and} \\ \varphi_t &= \frac{\partial \varphi_\mu}{\partial t} \quad \text{or} \quad \varphi_t \ = \ 0 \end{aligned}\right\} \quad \text{with} \quad \mu = 1, \ldots, 3n - f \qquad \text{(non-holonomic)}$$

In addition to the distinction between holonomic and non-holonomic auxiliary conditions, there is also a differentiation regarding the explicit time dependency. Auxiliary conditions that do not explicitly depend on time ($\varphi_t = 0$) are called scleronomic auxiliary conditions and those that are an explicit function of time ($\varphi_t \neq 0$) are called rheonomic auxiliary conditions.

The general formulation of the equations of motion including constraint forces with holonomic and scleronomic auxiliary conditions ($\varphi_t = 0$) is given by

$$m_i \ddot{x}_i \ = \ F_i + \sum_{\mu=1}^{r} \lambda_\mu \varphi_{\mu_i} \qquad \varphi_\mu(x_i \ldots x_{3n}) \ = \ 0 \qquad \text{Lagrange I} \tag{6.15}$$

Equations of motion of the type of eq. (6.15) are so-called Lagrange equations of the first kind, named after Joseph-Louis Lagrange. Lagrange equations of the first kind are equations in $3n+r$ unknowns, the $3n$ degrees of freedom of the system and the r Lagrange multipliers of the constraint conditions. To determine those unknowns one has the $3n$ coordinate equations and the r equations for the auxiliary conditions.

Equations of motion of the type of eq. (6.15) with non-holonomic auxiliary conditions cannot be integrated. However, even in such cases it is often possible to find analytic solutions and in those instances, where that does not work, one can always identify solutions numerically. One example of that is driving a car, where the steering wheel is used to specify a direction and not an area and this direction can be changed again by steering. This means, among other things, that even if a parking space may offer sufficient space for a car of a given size, it may be impossible to park the vehicle in the space. Modern cars must take this into account when parking automatically and need to decide whether the problem can be solved in a reasonable time or not.

Lagrange equations of the first kind are helpful in determining the constraint forces. Using a simple example (ex. 6.2), the three-dimensional thread pendulum in fig. 6.7, this is easy to illustrate. With the auxiliary condition from eq. (6.5), the pendulum mass m and the resulting imprinted force $F = (0, 0, -mg)$, the equations of motion for ex. 6.2 according to eq. (6.15) are

$$m\ddot{x}_i \;=\; F_i + \lambda\frac{\partial\varphi}{\partial x_i} \qquad i = 1, 2, 3$$

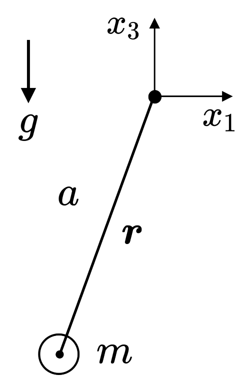

Fig. 6.7

The auxiliary condition is

$$\varphi(x_1, x_2, x_3) \;=\; x_1^2 + x_2^2 + x_3^2 - a^2 \;=\; 0$$

With that one has

$$\frac{\partial\varphi}{\partial x_i} \;=\; 2x_i \qquad \text{and thus} \qquad m\ddot{x}_i \;=\; F_i + 2\lambda x_i \qquad (*)$$

No explicit time dependency of the auxiliary condition, hence

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0 \quad \rightarrow \quad 2\sum_i x_i\dot{x}_i \;=\; 0$$

and with

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = 0 \quad \rightarrow \quad 2\sum_i \left[\dot{x}_i^2 + x_i\ddot{x}_i\right] \;=\; 0 \quad \rightarrow \quad v^2 \;=\; -\sum_i x_i\ddot{x}_i \qquad (**)$$

Multiplication of $(*)$ by $\sum_i x_i$ results in

$$\sum_i m x_i\ddot{x}_i \;=\; \boldsymbol{F}\cdot\boldsymbol{r} + 2\lambda\cdot\boldsymbol{r}^2$$

By exploiting (∗∗) one gets

$$-mv^2 \;=\; \boldsymbol{F}\cdot\boldsymbol{r} + 2\lambda\cdot a^2$$

With that one obtains for λ

$$\lambda \;=\; \frac{-\boldsymbol{F}\cdot\boldsymbol{r} - mv^2}{2a^2}$$

If one inserts the gravitational force $\mathbf{F}_G \;=\; (0, 0, -mg)$ for $\mathbf{F}$, then it follows for λ

$$\lambda \;=\; m\cdot\frac{gx_3 - v^2}{2a^2}$$

Thus, one obtains for the constraint force $\boldsymbol{F}_Z$ restricting the motion of the thread pendulum, i.e., for the force exerted by the thread on the pendulum mass m

$$\boldsymbol{F}_Z \;=\; 2\lambda\boldsymbol{x} \;=\; m\cdot\frac{gx_3 - v^2}{a^2}\cdot\boldsymbol{x} \tag{6.16}$$

If one rewrites this expression using the unit vector in the direction of the pendulum thread $\boldsymbol{e}_a = \boldsymbol{x}/a$ as

$$\boldsymbol{F}_Z \;=\; m\cdot\Big(\frac{gx_3}{a} - \frac{v^2}{a}\Big)\cdot\boldsymbol{e}_a \tag{6.17}$$

then one sees that the first component of $\boldsymbol{F}_Z$ corresponds to the weight force in the direction of the thread and the second component is the centrifugal force acting in the direction of the thread.

At the reversal points of the pendulum, one has $v = 0$ and for small deflections of the pendulum $x_3 \approx -a$. With that, λ becomes

$$\lambda \;=\; \frac{mgx_3}{a^2} \;=\; -\frac{mg}{a}$$

The equation of motion for x_1 is thus

$$m\ddot{x}_1 \;=\; 2\lambda x_1 \;=\; -\frac{mg}{a}x_1 \tag{6.18}$$

or respectively

$$\ddot{x}_1 \;=\; -\omega^2 x_1 \qquad \text{with} \qquad \omega \;=\; \sqrt{\frac{g}{a}}$$

and has the solution

$$x_1(t) \;=\; x_0 \sin\omega(t - t_0)$$

Example 6.4 **The yo-yo toy**

The yo-yo toy (fig. 6.8) consists of two disks of radius R which are connected by a central rod of radius r. The axis of rotation of the yo-yo runs perpendicularly through the center of the disks and the central rod. A thread is connected to the central rod and wound around it. Holding the upper end of the thread, the yo-yo is released while the thread is taut and sinks as the thread unwinds. When the thread has run out, it then winds up again in the opposite sense and the yo-yo rises. As this plays out, the length of the unwound or respectively rewound thread $r \cdot \Theta$ is always equal to the change in the position of the yo-yo on the z-axis. With the two degrees of freedom z and Θ one has two auxiliary conditions, one for the sinking and one for the rising of the yo-yo:

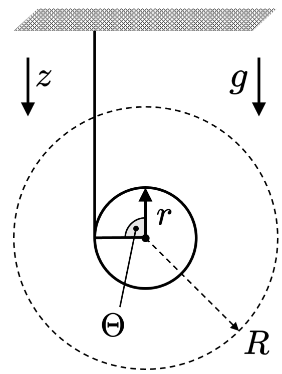

Fig. 6.8

$$\varphi_1 = r \cdot \Theta - z = 0 \quad \text{(sinking)} \quad ; \quad \varphi_2 = r \cdot \Theta + z = 0 \quad \text{(rising)}$$

The process of sinking will be considered first. The two equations of motion are obtained from eq. (6.15) where I is the moment of inertia of the yo-yo about its axis of rotation. Multiplying these equations of motion accordingly and subtracting them from each other yields an equation for the multiplier λ_1.

$$\left.\begin{aligned} m\ddot{z} &= mg - \lambda_1 \qquad / \cdot ({}^1/_m) \\ I\ddot{\Theta} &= \lambda_1 r \qquad\quad / \cdot ({}^r/_I) \end{aligned}\right\} \Rightarrow \quad r\ddot{\Theta} - \ddot{z} = 0 = \frac{\lambda_1 r^2}{I} - g + \frac{\lambda_1}{m}$$

This gives for λ_1 and the constraint force F_{Z_1} during the process of sinking

$$\lambda_1 = g\frac{Im}{mr^2 + I} \qquad \text{and} \qquad F_{Z_1} = \lambda_1 \frac{\partial \varphi_1}{\partial z} = -g\frac{Im}{mr^2 + I}$$

Analog now for the equations of motion during the process of rising. Here the two equations are added after multiplication to get an equation for λ_2.

$$\left.\begin{aligned} m\ddot{z} &= mg + \lambda_2 \qquad / \cdot ({}^1/_m) \\ I\ddot{\Theta} &= \lambda_2 r \qquad\quad / \cdot ({}^r/_I) \end{aligned}\right\} \Rightarrow \quad r\ddot{\Theta} + \ddot{z} = 0 = \frac{\lambda_2 r^2}{I} + g + \frac{\lambda_2}{m}$$

This gives for λ_2 and the constraint force F_{Z_2} during the process of rising

$$\lambda_2 = -g\frac{Im}{mr^2 + I} \qquad \text{and} \qquad F_{Z_2} = \lambda_2 \frac{\partial \varphi_1}{\partial z} = -g\frac{Im}{mr^2 + I} = F_{Z_1}$$

The constraint forces exerted on the thread are therefore identical for the sinking and the rising movement of the yo-yo.

Example 6.5 **Atwood's falling machine**

A rope runs over a friction-free rotating cylinder and to begin with two equally heavy weights of mass M shall be attached to the respective ends of the rope (see fig. 6.9). If one now adds a smaller additional weight of mass m to one of these two weights, this respective weight will move downwards with uniform acceleration. However, while in the process the total mass $2M + m$ has to be accelerated, only the weight mg of the smaller mass m acts as the accelerating force. The occurring acceleration is (with the assumption of a negligible moment of inertia of the cylinder) only a fraction of the gravitational acceleration g. It is intuitively clear that $F_{Z_1} = F_{Z_2} = F_Z$ must apply to the constraint forces acting on the rope as otherwise the rope would tear. The equations of motion can be easily determined as ($m_1 = M$, $m_2 = M + m$)

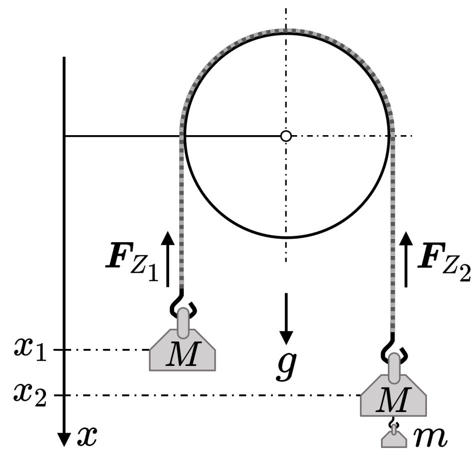

Fig. 6.9

$$m_1\ddot{x}_1 = m_1 g - F_Z \qquad \text{and} \qquad m_2\ddot{x}_2 = m_2 g - F_Z$$

By subtracting and adding these equations, one obtains

$$m_1\ddot{x}_1 - m_2\ddot{x}_2 = (m_1 - m_2)g \qquad \text{and} \qquad m_1\ddot{x}_1 + m_2\ddot{x}_2 = (m_1 + m_2)g - 2F_Z$$

The length of the rope is fixed and so with $x_1 = q$ it holds that $x_2 = l - q$, where l is the length of the freely hanging parts of the rope. Inserting this gives

$$\ddot{q} = \frac{m_1 - m_2}{m_1 + m_2} g \qquad \text{and} \qquad \ddot{q} = \frac{m_1 + m_2}{m_1 - m_2} g - \frac{2F_Z}{m_1 - m_2} \tag{6.19}$$

Since the left sides of these equations are identical, so must be their right sides. Hence,

$$F_Z = 2g\frac{m_1 m_2}{m_1 + m_2} = 2g\frac{M(M + m)}{2M + m}$$

For $m = 0$, $F_Z = Mg$. Inserting this into the equations of motion shows that in this case Atwood's falling machine is in equilibrium. If $m > 0$, then it follows from the equations of motion

$$M\ddot{x}_1 = -M\frac{m}{2M + m} g \qquad \text{and} \qquad (M + m)\ddot{x}_2 = (M + m)\frac{m}{2M + m} g$$

Hence, the acting acceleration is reduced by the factor $\dfrac{m}{2M + m}$ compared to the gravitational acceleration g.

6.3.2 Lagrange Equations of the Second Kind

According to d'Alembert's principle, for all virtual displacements δx_i of a system with constraint forces proportional to φ_{μ_i} it holds that

$$\sum_i \varphi_{\mu_i} \delta x_i = 0$$

From this follows with eq. (6.15)

$$\sum_i \big(F_i - m_i \ddot{x}_i\big) \delta x_i = 0$$

In eq. (6.15) the constraint forces are

$$\sum_\mu \lambda_\mu \varphi_{\mu_i} = Z_i \qquad \text{and thus applies} \qquad \sum_i Z_i \delta x_i = 0$$

As already discussed (fig. 6.3), the movement of the pendulum mass of the 3-dimensional thread pendulum is limited to motion on a spherical surface with radius a around the pendulum suspension. This motion has two degrees of freedom; therefore, two coordinates are required to describe it. A good choice for describing motion on a spherical surface are angular coordinates, i.e., azimuth φ and elevation ϑ. For this, x_1, x_2 and x_3 must be expressed as functions of φ and ϑ, i.e., $x_1 = x_1(\varphi, \vartheta)$, $x_2 = x_2(\varphi, \vartheta)$ and $x_3 = x_3(\varphi, \vartheta)$. In the general case, for the description of motion on any arbitrary surface, this means that the respective x_i must be expressed by the corresponding coordinates of the surface. Importantly, this applies not only to motion on two-dimensional surfaces but also more generally to motion on higher-dimensional surfaces.

The surface which must be considered for motion according to eq. (6.15) possesses the dimension f of the remaining degrees of freedom f of the system. If $f = 2$, as in ex. 6.2, two so-called generalized coordinates, for example q_1 and q_2 as sketched in fig. 6.10, are required to describe point mass motion on this surface. In the general case of f remaining degrees of freedom, an f-dimensional surface must be considered with the generalized coordinates $q_1, \ldots, q_f$ and the parametrization

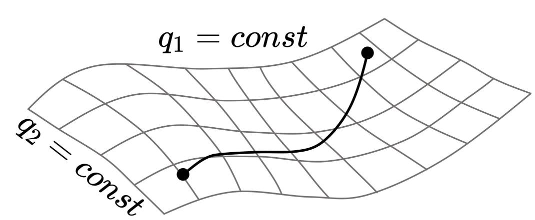

Fig. 6.10

$$x_i = x_i(q_1, \ldots, q_f) = x_i(q_K) \qquad i = 1, \ldots, 3n \tag{6.20}$$

The following then applies to the auxiliary conditions

$$0 \equiv \varphi_\mu\big(x_1(q_K), x_2(q_K), \dots, x_{3n}(q_K)\big) \tag{6.21}$$

and

$$\frac{\partial \varphi_\mu}{\partial q_K} \equiv 0 \equiv \sum_i \frac{\partial \varphi_\mu}{\partial x_i}\frac{\partial x_i}{\partial q_K} \tag{6.22}$$

For the motion of a body on the f-dimensional surface, the q_r $(r = 1, \dots, f)$ as a function of time t must be calculated

$$\sum_i \left(m_i \ddot{x}_i = F_i + \sum_\mu \lambda_\mu \frac{\partial \varphi_\mu}{\partial x_i}\right) \cdot \frac{\partial x_i}{\partial q_r} \qquad \text{(holonomic auxiliary conditions)} \tag{6.23}$$

According to d'Alembert

$$\sum_\mu \lambda_\mu \frac{\partial \varphi_\mu}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_r} = 0$$

and therefore

$$\sum_i m_i \ddot{x}_i \cdot \frac{\partial x_i}{\partial q_r} = \underbrace{\sum_i F_i \cdot \frac{\partial x_i}{\partial q_r}}_{F_r} \qquad \text{with} \qquad r = 1, \dots, f \tag{6.24}$$

The F_r with $r = 1, \dots, f$ designate the so-called generalized forces. The left side of this equation can be rewritten as

$$\sum_i m_i \ddot{x}_i \cdot \frac{\partial x_i}{\partial q_r} = \sum_i \left[m_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_i \cdot \frac{\partial x_i}{\partial q_r}\right) - m_i \dot{x}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x_i}{\partial q_r}\right] \tag{6.25}$$

Using the relationships

$$m_i \dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x_i}{\partial q_r} = m_i \dot{x}_i \cdot \frac{\partial \dot{x}_i}{\partial q_r} = \frac{m_i}{2}\frac{\partial \dot{x}_i^2}{\partial q_r}$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \dot{x}_i = \sum_{r=1}^{f} \frac{\partial x_i}{\partial q_r}\dot{q}_r = \frac{\partial x_i}{\partial q_K} \cdot \dot{q}_K \quad (\Delta) \quad \Rightarrow \quad \frac{\partial \dot{x}_i}{\partial \dot{q}_K} = \frac{\partial x_i}{\partial q_K}$$

one can further rewrite the left side of eq. (6.24) as

$$\sum_i m_i \ddot{x}_i \cdot \frac{\partial x_i}{\partial q_r} = \sum_i \left[\frac{m_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{x}_i^2}{\partial \dot{q}_r} - \frac{m_i}{2}\frac{\partial \dot{x}_i^2}{\partial q_r}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} \tag{6.26}$$

where

$$T = \sum_i \frac{m_i}{2} \dot{x}_i^2$$

From (Δ) follows

$$T = \sum_i \frac{m_i}{2} \frac{\partial x_i}{\partial q_K} \frac{\partial x_i}{\partial q_K} \cdot \dot{q}_K \cdot \dot{q}_K = \frac{1}{2} M_K(q_1 \ldots q_f) \cdot \dot{q}_K^2 \tag{6.27}$$

If one now puts the right side of eq. (6.26), i.e., the transformed left side of eq. (6.24), back into eq. (6.24), then one obtains the so-called Lagrangian equations of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = F_r \qquad r = 1, \ldots, f \qquad \text{Lagrange II} \tag{6.28}$$

For conservative forces applies

$$F_i = -\frac{\partial V}{\partial x_i}$$

and therefore

$$F_r = -\sum_i \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_r} = -\frac{\partial V}{\partial q_r} \tag{6.29}$$

Definition

Lagrange function: The Lagrange function L, also known as kinetic potential, is the difference between kinetic energy T and potential energy V:

$$\mathrm{L} = T - V$$

V does not depend on the velocity. The general equation of motion for L follows with eq. (6.28) and eq. (6.29) from the Lagrange equations of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathrm{L}}{\partial \dot{q}_r} - \frac{\partial \mathrm{L}}{\partial q_r} = 0 \tag{6.30}$$

The advantage of the Lagrange equations of the second kind consists above all in their great generality and that one can formulate the equations of motion without having to explicitly write down the forces or constraint force as is required for the solution of the Lagrange equations of the first kind. Another advantage is that one can freely choose the generalized coordinates, which often simplifies the solution of problems. Since the number of these generalized coordinates is always equal to the number of degrees of freedom of a system, the generalized coordinates are always independent of each other.

Another advantage of the Lagrange mechanism that must not be underestimated is that it is usually much easier to determine the kinetic and potential energy, i.e., T and V for the Lagrange function than to formulate Newton's equations of motion or respectively Euler's equations for any given problem. A disadvantage is, however, that with the Lagrange mechanism physical clarity is lost and because of that the physical meaning of solutions obtained with the Lagrange mechanism, i.e., their physics interpretation, must frequently be supplied later.

Example 6.6 **Back to the spatial thread pendulum**

For a practical illustration of the equation of motion eq. (6.30) we turn again to ex. 6.2, the three-dimensional thread pendulum. In section 6.3.1 the constraint force for this problem was already determined with the help of the Lagrange equations of the first kind and the solution was identified. Here the problem of the planar pendulum is tackled using the Lagrange equations of motion with the generalized coordinates.

$$\left.\begin{aligned} x_1 &= a\sin\vartheta \\ x_3 &= -a\cos\vartheta \end{aligned}\right\} \qquad x_i = x_i(q_K)$$

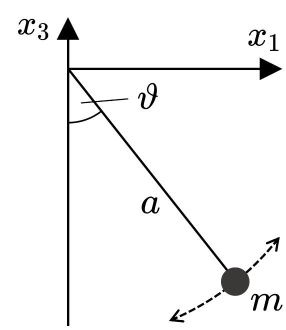

Fig. 6.11

Auxiliary condition

$$\varphi = x_1^2 + x_3^2 - a^2 \equiv a(\sin^2\vartheta + cos^2\vartheta - 1) = 0$$

For the kinetic and the potential energy, one gets

$$\left.\begin{aligned} \dot{x}_1 &= a\cos\vartheta\dot{\vartheta} \\ \dot{x}_3 &= a\sin\vartheta\dot{\vartheta} \end{aligned}\right\} \qquad \frac{m}{2}v^2 = \frac{a^2m}{2}\dot{\vartheta}^2 = T$$

or respectively

$$V = mgx_3 = -mga\cos\vartheta$$

Thus, one has for the Lagrange function

$$\mathrm{L} = T - V = \frac{a^2m}{2}\dot{\vartheta}^2 + mga\cos\vartheta \tag{6.31}$$

Calculating the derivatives of the Lagrange function

$$\frac{\partial\mathrm{L}}{\partial\dot{\vartheta}} = ma^2\dot{\vartheta} \quad ; \quad \frac{\partial\mathrm{L}}{\partial\vartheta} = -mga\sin\vartheta$$

and inserting into eq. (6.30) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(ma^2\dot{\vartheta}\right) + mga\sin\vartheta = 0 \tag{6.32}$$

From this follows the equation of motion for the ϑ-coordinate

$$\ddot{\vartheta} = -\frac{g}{a}\sin\vartheta \tag{6.33}$$

For small deflections $\vartheta \ll 1$, $\sin\vartheta \approx \vartheta$ applies and one obtains the equation of motion

$$\ddot{\vartheta} = -\frac{g}{a}\vartheta = \omega^2\vartheta \tag{6.34}$$

As expected, this corresponds to the result from the treatment of the problem with the Lagrange equations of the first kind (see section 6.3.1). According to eq. (6.24), the generalized force F_ϑ can be determined from $\boldsymbol{F} = (0, 0, -mg)$

$$F_\vartheta = F_1\frac{\partial x_1}{\partial\vartheta} + F_3\frac{\partial x_3}{\partial\vartheta} = -mga\sin\vartheta \tag{6.35}$$

Example 6.7 **Atwood's falling machine once more**

In ex. 6.5 Atwood's falling machine (fig. 6.9) was already treated with the Lagrange equations of the first kind in order to determine the constraint forces acting on the rope. The auxiliary condition for the Lagrange equations of the second kind results from the constancy of the rope length and the fact that the piece of rope running over the cylinder always has the same length. If l denotes the rope section that does not run over the cylinder, the auxiliary constraint is:

$$x_1 + x_2 = l$$

So, the system has only one degree of freedom for which $x_1 = q$ is chosen as in ex. 6.5. With the generalized coordinate q it then follows that $x_2 = l - q$. With $\dot{x}_1 = \dot{q}$ and $\dot{x}_2 = -\dot{q}$ one obtains (with a negligible moment of inertia of the cylinder) for the kinetic energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{q}^2$$

where for m_1 and m_2 with fig. 6.9 again $m_1 = M$ and $m_2 = M + m$. For the potential energy of Atwood's falling machine, one obtains

$$V = -m_1 g x_1 - m_2 g x_2 = -g\big[m_1 q - m_2(l - q)\big] = -g(m_1 - m_2)q - g m_2 l$$

Thus, one has for the Lagrangian function $\mathrm{L} = T - V$

$$\mathrm{L} = \frac{1}{2}(m_1 + m_2)\dot{q}^2 + g(m_1 - m_2)q + g m_2 l$$

Setting up the Lagrange equation for the generalized coordinate q

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{q}} - \frac{\partial \mathrm{L}}{\partial q} \;=\; (m_1+m_2)\ddot{q} - g(m_1-m_2) \;=\; 0$$

gives the equation of motion

$$\ddot{q} \;=\; \frac{m_1-m_2}{m_1+m_2}g \;=\; \frac{m}{2M+m}g \tag{6.36}$$

This corresponds to the result from eq. (6.19), which was based on the intuitive assumption that the constraint forces acting on the rope in fig. 6.9 must be identical. Eq. (6.36) confirms the correctness of this assumption. As already stated in ex. 6.5, eq. (6.36), the equation of motion of Atwood's falling machine, corresponds to that of a free fall with reduced acceleration.

Example 6.8 The rolling cylinder

The two coordinates x and ϑ describe the motion of the rolling cylinder in fig. 6.12. However, x and ϑ are not independent of each other and the system therefore has only one degree of freedom. The rolling velocity of the cylinder with radius a is

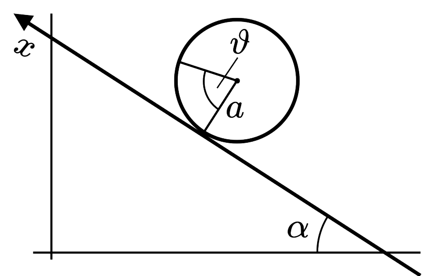

Fig. 6.12

$$v_a = a\cdot\omega = a\cdot\dot{\vartheta}$$

This velocity corresponds to the velocity with which the center of the cylinder moves along the x coordinate, i.e.,

$$a\dot{\vartheta} - \dot{x} \;=\; 0 \qquad (*)$$

The auxiliary condition is here that the distance covered by rolling along the x-coordinate must be equal to the part of the cylinder's circumference that is unrolled in the process. This means that the cylinder only moves on the inclined plane by rolling and not by sliding

$$\varphi_\mu(\vartheta, x) \;=\; a\vartheta - x \;=\; 0$$

This condition does not explicitly depend on time. Hence, with eq. (6.14) applies

$$\frac{\partial \varphi_\mu}{\partial \vartheta}\dot{\vartheta} + \frac{\partial \varphi_\mu}{\partial x}\dot{x} \;=\; a\dot{\vartheta} - \dot{x} \;=\; 0$$

But that is exactly the statement of $(*)$ and thus eq. (6.14) is fulfilled here. For the kinetic energy of the rolling cylinder holds

$$T \;=\; \frac{m}{2}\dot{x}^2 + \frac{I}{2}\dot{\vartheta}^2 \tag{6.37}$$

where I is the moment of inertia of the cylinder. If one defines an effective moment of inertia mass m^* with $m^* = I/a^2$, one can rewrite T as

$$T \;=\; \frac{m}{2}\dot{x}^2 + \frac{I}{2}\dot{\vartheta}^2 \;=\; \frac{m+m^*}{2}\dot{x}^2 \tag{6.38}$$

With the potential energy of the cylinder $V = m{\cdot}g{\cdot}x{\cdot}\sin\alpha$ one therefore obtains for the Lagrange function L

$$\mathrm{L} \;=\; T - V \;=\; \frac{m+m^*}{2}\dot{x}^2 - m{\cdot}g{\cdot}x{\cdot}\sin\alpha \tag{6.39}$$

Calculating the derivatives of the Lagrange function

$$\frac{\partial \mathrm{L}}{\partial \dot{x}} = (m+m^*)\dot{x} \qquad ; \qquad \frac{\partial \mathrm{L}}{\partial x} = -m{\cdot}g{\cdot}\sin\alpha$$

and inserting into eq. (6.30) gives the equation of motion

$$(m+m^*)\ddot{x} + m{\cdot}g{\cdot}\sin\alpha \;=\; 0 \tag{6.40}$$

After two integrations one gets for the motion of the rolling cylinder along the x-coordinate in fig. 6.12

$$x(t) \;=\; x_0 + v_0 t - \frac{mg}{m+m^*}\sin\alpha{\cdot}\frac{t^2}{2} \tag{6.41}$$

If one treats the problem of the rolling cylinder with the help of the Lagrange equations of the first kind, then ϑ is not eliminated, but one obtains the constraint forces. The constraint forces Z_x and Z_ϑ follow from the auxiliary condition $\varphi_\mu = a\vartheta - x$

$$\big(Z_x, Z_\vartheta\big) \;=\; \lambda\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial\vartheta}\right) \;=\; \lambda(-1, a) \tag{6.42}$$

and the Lagrange equations of the first kind for the force in the direction of the x-coordinate $F = m\ddot{x}$ and for the torque of the cylinder due to the angular acceleration $M = I\ddot{\vartheta}$ are

$$\begin{aligned} m\ddot{x} \;&=\; -mg\sin\alpha - \lambda \\ m^* a^2\ddot{\vartheta} \;&=\; \lambda{\cdot}a \quad \rightarrow \quad \lambda \;=\; m^* a\ddot{\vartheta} \end{aligned} \tag{6.43}$$

Differentiating the auxiliary condition twice yields $a\ddot{\vartheta} \;=\; \ddot{x}$. Inserting this into the second equation and using for $\ddot{x}$ the first equation gives

$$\lambda \;=\; m^* a\ddot{\vartheta} \;=\; m^*\ddot{x} \;=\; -m^* g\sin\alpha - \frac{m^*}{m}\lambda \tag{6.44}$$

Equating the first and the last part of this equation yields for λ

$$\lambda \;=\; -\frac{m{\cdot}m^*}{m+m^*}g\sin\alpha \tag{6.45}$$

Example 6.9 **Cylinders rolling on each other**

In this example one consider a cylinder with radius R_2 and mass m which, as sketched in fig. 6.13, shall roll off on a second cylinder with radius R_1 and mass M. The axis of rotation of the large cylinder is firmly anchored. At the time $t = 0$, the smaller cylinder (R_2) is placed on the larger cylinder (R_1) at an angle Θ_{10} against the vertical and then let go. The assumptions one makes here are that the two cylinders then roll on top of each other without sliding, and that the lower large cylinder shall be able to rotate freely around its fixed axis without friction. The question then is: At which angle Θ_1 will the small cylinder jump off the lower larger cylinder. The jump condition is easy to find: The upper cylinder will jump off when the centrifugal force for the smaller cylinder moving in an orbit with radius r becomes equal to the component of gravity acting along r that presses the small cylinder onto the large cylinder. Therefore, the jump condition is given by

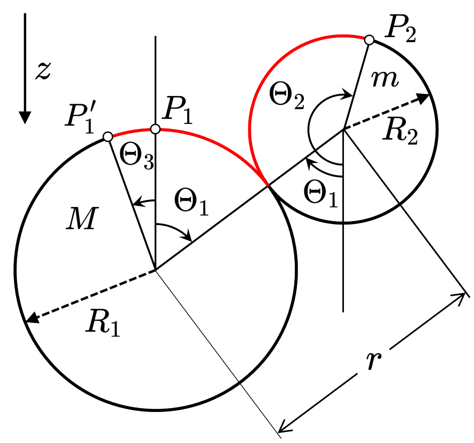

Fig. 6.13

$$mg\cos\Theta_1 + mr\dot{\Theta}_1^2 \;=\; 0 \tag{6.46}$$

Hence, it is sufficient to know $\dot{\Theta}_1$ to determine the take-off angle Θ_1. For the corresponding auxiliary conditions, one must differentiate between two cases:

A The lower larger cylinder can rotate freely without friction. In this case, the lower larger cylinder rotates to the left by the angle Θ_3 while the small cylinder rolls off on it, so the point P_1 moves to the location P_1'. With that, the two auxiliary conditions read

$$r \;=\; R_1 + R_2 \qquad \text{and} \qquad R_1\Theta_3 + R_1\Theta_1 \;=\; R_2(\Theta_2 - \Theta_1)$$

B The lower larger cylinder is fixed. In this case the point P_1 does not move and Θ_3 is always zero. With that, the two auxiliary conditions read

$$r \;=\; R_1 + R_2 \qquad \text{and} \qquad R_1\Theta_1 \;=\; R_2(\Theta_2 - \Theta_1)$$

Case **A**: The lower larger cylinder can rotate freely without friction

The kinetic energy of the small cylinder is the sum of the kinetic energy of the motion of its center of mass and the kinetic energy of its own rotation. For the large cylinder one only has the kinetic energy of its own rotation. The total kinetic energy of the system of two cylinders is thus

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\Theta}_1^2) + \frac{I_m}{2}\dot{\Theta}_2^2 + \frac{I_M}{2}\dot{\Theta}_3^2$$

Since the vertical position of the lower cylinder does not change, its potential energy is constant and can be set to zero, so the point $z = 0$ is in the center of the large cylinder. With this choice of coordinates, the entire potential energy of the system is that of the smaller cylinder, i.e., $V = -mgr\cos\Theta_1$. With that, the Lagrange function $\mathrm{L} = T - V$ becomes

$$\mathrm{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\Theta}_1^2) + \frac{I_m}{2}\dot{\Theta}_2^2 + \frac{I_M}{2}\dot{\Theta}_3^2 + mgr\cos\Theta_1$$

From the auxiliary conditions for the case **A** it follows with $r = R_1 + R_2 = const$

$$\dot{r} = 0 \qquad \text{and} \qquad \dot{\Theta}_3 = \frac{R_2\dot{\Theta}_2 - \dot{\Theta}_1}{R_1} - \dot{\Theta}_1$$

With this one can rewrite L as a function of the generalized coordinates Θ_1, Θ_2

$$\mathrm{L} = \frac{m}{2}r^2\dot{\Theta}_1^2 + \frac{I_m}{2}\dot{\Theta}_2^2 + \frac{I_M}{2}\left(\frac{R_2\dot{\Theta}_2 - \dot{\Theta}_1}{R_1} - \dot{\Theta}_1\right)^2 + mgr\cos\Theta_1$$

For the Lagrange equations of the second kind one then obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial\dot{\Theta}_1} + \frac{\partial \mathrm{L}}{\partial\Theta_1} = \frac{\mathrm{d}}{\mathrm{d}t}\left[mr^2\dot{\Theta}_1 - I_M\left(\frac{R_2}{R_1}\dot{\Theta}_2 - \frac{r}{R_1}\dot{\Theta}_1\right)\frac{r}{R_1}\right] + mgr\sin\Theta_1 = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial\dot{\Theta}_2} + \frac{\partial \mathrm{L}}{\partial\Theta_2} = \frac{\mathrm{d}}{\mathrm{d}t}\left[I_m\dot{\Theta}_2 + I_M\left(\frac{R_2(\dot{\Theta}_2 - \dot{\Theta}_2)}{R_1} - \dot{\Theta}_1\right)\frac{R_2}{R_1}\right] = 0$$

From these Lagrange equations follow the two equations of motion

$$\left(m + \frac{I_m}{R_1^2}\right)r^2\ddot{\Theta}_1 - I_M\frac{R_2 r}{R_1^2}\ddot{\Theta}_2 + mgr\sin\Theta_1 = 0 \tag{6.47}$$

$$\left(I_m + I_M\frac{R_2^2}{R_1^2}\right)\ddot{\Theta}_2 - I_M\frac{R_2^2 + R_2R_1}{R_1^2}\ddot{\Theta}_1 = 0 \tag{6.48}$$

Now one inserts the expression for $\ddot{\Theta}_2$ from eq. (6.48) in eq. (6.47) and solves for $\ddot{\Theta}_1$. After a little calculation one obtains the differential equation for $\ddot{\Theta}_1$

$$\left(mr^2 + \frac{I_mI_Mr^2}{I_mR_1^2 + I_MR_2^2}\right)\ddot{\Theta}_1 + mgr\sin\Theta_1 = 0 \tag{6.49}$$

The two cylinders shall be solid cylinders. With that applies

$$I_m = \frac{m}{2}R_2^2 \qquad \text{and} \qquad I_M = \frac{M}{2}R_1^2$$

Inserting this into eq. (6.49) yields after transformation

$$\ddot{\Theta}_1 = -\frac{g}{r}\frac{2(m+M)}{2m+3M}\sin\Theta_1 = -c{\cdot}\sin\Theta_1$$

A differential equation of the form $\ddot{\Theta}_1 = -c{\cdot}\sin\Theta_1$ can be solved by multiplying it with $2\dot{\Theta}_1$, i.e.,

$$2\dot{\Theta}_1\ddot{\Theta}_1 = -2c{\cdot}\sin\Theta_1\dot{\Theta}_1$$

and then integrating it, i.e.,

$$\dot{\Theta}_1^2 = 2c\cos\Theta_1 + d$$

At the time $t = 0$, $\dot{\Theta}_1 = 0$ and $\Theta_1 = \Theta_{10}$ apply. With that the constant of integration becomes $d = -2c\cos\Theta_{10}$ and one obtains

$$\dot{\Theta}_1^2 = 2\frac{g}{r}\frac{2(m+M)}{2m+3M}(\cos\Theta_1 - \cos\Theta_{10}) \tag{6.50}$$

With this, the take-off angle in the case **A** can now be determined. Inserting eq. (6.50) into eq. (6.46) gives the equation for the take-off angle Θ_1

$$mg\cos\Theta_1 + mr{\cdot}2\frac{g}{r}\frac{2(m+M)}{2m+3M}(\cos\Theta_1 - \cos\Theta_{10}) = 0$$

Solving for Θ_1, one obtains for the take-off angle in case **A**

$$\Theta_1^A = \arccos\left[\frac{4(m+M)}{6m+7M}\cos\Theta_{10}\right]$$

Case **B**: The lower larger cylinder is fixed

In order to obtain the Lagrange function for this case, one only needs to set $\Theta_3 = 0$ in the Lagrange function for the case **A**, i.e.,

$$\mathrm{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\Theta}_1^2) + \frac{I_m}{2}\dot{\Theta}_2^2 + mgr\cos\Theta_1$$

In this case, it follows from the auxiliary conditions

$$\dot{r} = 0 \qquad \text{and} \qquad \dot{\Theta}_2 = \frac{r}{R_2}\dot{\Theta}_1$$

Inserting this into the Lagrange function yields

$$\mathrm{L} = \frac{m}{2}r^2\dot{\Theta}_1^2 + \frac{I_m}{2}\left(\frac{r}{R_2}\right)^2\dot{\Theta}_1^2 + mgr\cos\Theta_1$$

From this follows the Lagrange equation of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{\Theta}_1} + \frac{\partial \mathrm{L}}{\partial \Theta_1} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left[mr^2\dot{\Theta}_1 + I_m\frac{r^2}{R_2^2}\dot{\Theta}_1\right] + mgr\sin\Theta_1 \;=\; 0$$

and the equation of motion

$$\left(mr^2 + I_m\frac{r^2}{R_2^2}\right)\ddot{\Theta}_1 + mgr\sin\Theta_1 \;=\; 0$$

Now assuming solid cylinders again and solving for $\ddot{\Theta}_1$ yields

$$\ddot{\Theta}_1 \;=\; -\frac{2}{3}\frac{g}{r}\sin\Theta_1 = -c\sin\Theta_1$$

This differential equation is again treated as in case **A** and one thus obtains

$$\dot{\Theta}_1^2 \;=\; \frac{4}{3}\frac{g}{r}(\cos\Theta_1 - \cos\Theta_{10}) \tag{6.51}$$

Inserting this in eq. (6.46) yields for the take-off angle Θ_1 in case **B**

$$\Theta_1^B \;=\; \arccos\left[\frac{4}{7}\cos\Theta_{10}\right]$$

It is no surprise that when the large cylinder is stationary, the take-off angle of the small cylinder is independent of the respective cylinder masses.

Example 6.10 The nutational motion of the gyroscope

In section 5.3 the nutational motion of the gyroscope was briefly discussed with the remark that when a torque acts on the gyroscope, the Euler equations can no longer be solved independently from the equations for the Euler angles ϕ, ϑ and ψ. It is there where the approach of the Lagrange formalism with ϕ, ϑ and ψ as the generalized coordinates becomes helpful. For the kinetic energy of the symmetrical gyroscope considered here, such as the spinning top sketched in fig. 6.14, the following applies with eq. (5.26) and the components of the tensor of inertia in the principal axis system $J_{11} = J_{22} = A$ and $J_{33} = C$:

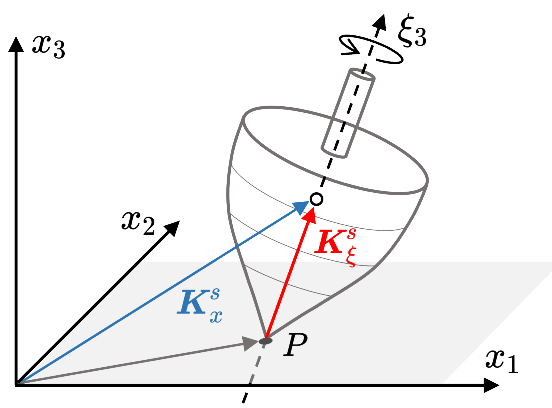

Fig. 6.14

$$T(\boldsymbol{\omega}) \;=\; \frac{1}{2}\boldsymbol{\omega}\big(\underline{\underline{\boldsymbol{J}}}\cdot\boldsymbol{\omega}\big) \;=\; \frac{1}{2}\sum_{i,j}^{3}\omega_i J_{ij}\omega_j \;=\; \frac{1}{2}\big(\omega_1^2 A + \omega_2^2 A + \omega_3^2 C\big)$$

With the components ω_1, ω_2 and ω_3 from eq. (5.36) and eq. (5.39) one thus obtains for T the expression

$$T = \frac{A}{2}\left(\dot\phi^2 \sin^2\vartheta + \dot\vartheta^2\right) + \frac{C}{2}\left(\dot\phi\cos\vartheta + \dot\psi\right)^2$$

If x_1^s, x_2^s, x_3^s are the coordinates of the gyroscope's center of mass $\boldsymbol{K}_x^s$ in the space-fixed coordinate system, then the potential energy of a spinning top of mass m is given by

$$V(\phi,\vartheta,\psi) = m\cdot g\cdot x_3^s$$

Transformation of the space-fixed center of mass vector $\boldsymbol{K}_x^s$ into the body-fixed center of mass vector $\boldsymbol{K}_\xi^s$ using the rotation matrix $\boldsymbol{R}^{\phi\vartheta\psi}$ (see section 5.2.1)

$$\boldsymbol{K}_x^s = \boldsymbol{R}^{\phi\vartheta\psi}\boldsymbol{K}_\xi^s$$

returns for x_3^s in the body-fixed coordinate system

$$x_3^s = \sin\vartheta\sin\psi\cdot\xi_1^s + \sin\vartheta\cos\psi\cdot\xi_2^s + \cos\vartheta\cdot\xi_3^s$$

For the symmetrical spinning top, the center of mass lies on the axis of rotation which is identical with the ξ_3-axis and thus $\xi_1^s = \xi_2^s = 0$. If the gyroscope's center of mass $\boldsymbol{K}_\xi^s$ lies now on the x_3-axis at a distance d from the reference point P, i.e., $\xi_3^s = d$, then one obtains for the potential energy in the body-fixed coordinate system

$$V(\phi,\vartheta,\psi) = m\cdot g\cdot d\cdot\cos\vartheta$$

With that, the total energy of the gyroscope $E = T + V$ becomes

$$E = \frac{A}{2}\left(\dot\phi^2 \sin^2\vartheta + \dot\vartheta^2\right) + \frac{C}{2}\left(\dot\phi\cos\vartheta + \dot\psi\right)^2 + m\cdot g\cdot d\cdot\cos\vartheta \tag{6.52}$$

and for the Lagrange function $\mathrm{L} = T - V$ of the gyroscope one obtains

$$\mathrm{L} = \frac{A}{2}\left(\dot\phi^2 \sin^2\vartheta + \dot\vartheta^2\right) + \frac{C}{2}\left(\dot\phi\cos\vartheta + \dot\psi\right)^2 - m\cdot g\cdot d\cdot\cos\vartheta \tag{6.53}$$

L only depends on $\dot\phi$, $\dot\vartheta$, $\dot\psi$, and ϑ but not on ϕ and ψ. Generalized coordinates q for which $\partial\mathrm{L}/\partial q = 0$ holds true are referred to as cyclic coordinates. These cyclic coordinates have the special property that one can eliminate them from the Lagrange equations. ϕ and ψ are two such cyclic coordinates and for them the Lagrange equations of the second kind are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathrm{L}}{\partial\dot\phi} + \frac{\partial\mathrm{L}}{\partial\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left[A\dot\phi\sin^2\vartheta + C\left(\dot\phi\cos\vartheta + \dot\psi\right)\cos\vartheta\right] = 0 \tag{6.54}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{\psi}} + \frac{\partial \mathrm{L}}{\partial \psi} = \frac{\mathrm{d}}{\mathrm{d}t}\Big[C\big(\dot{\phi}\cos\vartheta + \dot{\psi}\big)\Big] = 0 \tag{6.55}$$

From eq. (6.54) and eq. (6.55) one obtains the equations

$$A\dot{\phi}\sin^2\vartheta + C\big(\dot{\phi}\cos\vartheta + \dot{\psi}\big)\cos\vartheta = c_1$$

$$C\big(\dot{\phi}\cos\vartheta + \dot{\psi}\big) = c_2$$

with the constants c_1 and c_2. Solving for $\dot{\phi}$ and $\dot{\psi}$ one gets

$$\dot{\phi} = \frac{c_1 - c_2\cos\vartheta}{A\sin^2\vartheta} \tag{6.56}$$

and

$$\dot{\psi} = \frac{c_2}{C} - \frac{c_1 - c_2\cos\vartheta}{A\sin^2\vartheta}\cos\vartheta \tag{6.57}$$

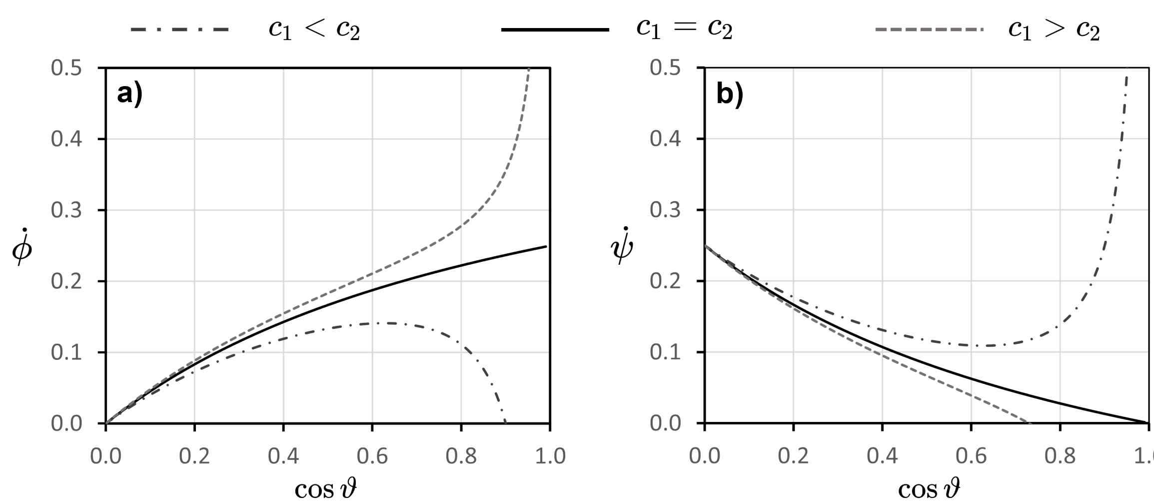

Fig. 6.15: The angular velocities $\dot{\phi}$ (a) and $\dot{\psi}$ (b) from eq. (6.56) and eq. (6.57) as functions of $\cos\vartheta$ for $c_1 = 0.9{\cdot}c_2$, $c_1 = c_2$ and $c_2 = 1.1{\cdot}c_2$ ($A = 1$ and $C = 2A$).

Fig. 6.15 illustrates the dependence of both $\dot{\phi}$ and $\dot{\psi}$ on $\cos\vartheta$ in normalized form for three selected relative values of c_1 and c_2 respectively; with A set to 1, in both cases a C:A-ratio of 2:1 was selected. For $c_1 < c_2$, $\dot{\phi}$ first increases to a maximum, then decreases again and finally becomes zero for $\cos\vartheta = c_1/c_2$. For $c_1 = c_2$, $\dot{\phi}$ increases continuously, as well as for $c_1 > c_2$, but in this case it has a turning point, after which $\dot{\phi}$ increases significantly stronger. For $c_1 < c_2$, $\dot{\psi}$ falls to a minimum and then rises steeply. For $c_1 = c_2$, $\dot{\psi}$ falls steadily until it vanishes at $\cos\vartheta = 1$. The behavior of $\dot{\psi}$ for $c_1 > c_2$ is similar, with the difference that the decrease is somewhat faster and $\dot{\psi}$ already disappears at a value of $\cos\vartheta < 1$.

By inserting the formulas for $\dot{\phi}$ and $\dot{\psi}$ from eq. (6.56) and eq. (6.57) into eq. (6.52) one obtains for the total energy E

$$E = \frac{c_2}{2C} + A\left[\dot{\vartheta}^2 + \frac{(c_1 - c_2\cos\vartheta)^2}{A^2\sin^2\vartheta}\right] + m\cdot g\cdot d\cdot\cos\vartheta$$

Solving this equation for $\dot{\vartheta}$ and separating the variables yields

$$\dot{\vartheta} = \sqrt{\frac{2}{A}\left(E - \frac{c_2^2}{2C} - mgd\cos\vartheta\right) - \frac{(c_1 - c_2\cos\vartheta)^2}{A^2\sin^2\vartheta}} \tag{6.58}$$

and

$$\mathrm{d}t = \frac{\mathrm{d}\vartheta}{\sqrt{\frac{2}{A}\left(E - \frac{c_2^2}{2C} - mgd\cos\vartheta\right) - \frac{(c_1 - c_2\cos\vartheta)^2}{A^2\sin^2\vartheta}}}$$

With integration and substitution $u = \cos\vartheta$ one gets

$$t - t_0 = -A\int_{\cos\vartheta_0}^{\cos\vartheta}\left(2A(1-u^2)\big(E - \frac{c_2^2}{2C} - mgdu\big) - (c_1 - c_2u)^2\right)^{-1/2}\cdot\mathrm{d}u \tag{6.59}$$

This elliptic integral has no simple analytic solution and must be evaluated either using approximations or must be solved numerically. In both cases one obtains the time dependence of the coordinate ϑ. If one inserts the thus obtained $\vartheta(t)$ into eq. (6.56) and eq. (6.57), then one gets $\phi(t)$ and $\psi(t)$ through a further integration step. With that, the motion of the gyroscope as a function of $\phi(t)$, $\vartheta(t)$ and $\psi(t)$ is completely determined.

The solution path is therefore clear. However, one can already get a better physics understanding of the gyroscopic through a more detailed consideration of the expression R under the root in eq. (6.59):

$$R = 2A(1-u^2)\big(E - \frac{c_2^2}{2C} - mgdu\big) - (c_1 - c_2u)^2$$

As a third-degree polynomial in u, R has at most three zeros. One can immediately see that R is negative for $u = 1$ ($\vartheta = 0$) and for $u = -1$ ($\vartheta = \pi$) and that therefore the integral in eq. (6.59) has no real solution (i.e., it only has imaginary solutions). Somewhere between $0 < \vartheta < \pi$, however, a real solution for $\dot{\vartheta}$ in eq. (6.58) must exist. This means that between $-1 < u < 1$, the value of R must be positive and then negative again: In other words, R must have two zeros u_1 and u_2 in the range $-1 < u < 1$ and therefore, two values ϑ_1 and ϑ_2 must exist for each of which $\dot{\vartheta}$ vanishes in eq. (6.58). As can easily be checked, for $R \to +\infty$ applies $u \to +\infty$, that means u_3, the third zero of R, must be greater than 1 and therefore lies outside the domain of arccos; hence, the corresponding angle ϑ_3 does not exist.

In the two limiting cases $c_1 = c_2$ and $c_1 = -c_2$, the factor $(1-u)$ or respectively the factor $(1+u)$ can be split off from R:

- In the case of $c_1 = c_2$, $u = 1$ is a zero of R with the associated angle $\vartheta_1 = 0$. This means that for $c_1 = c_2$ the nutational motion $\vartheta(t)$ forces the figure axis of the spinning top into vertical position until it lies parallel to the coordinate x_3.
- In the of case $c_1 = -c_2$, $u = -1$ is a zero of R and the associated angle is $\vartheta_2 = \pi$. This means that for $c_1 = -c_2$ the nutational motion $\vartheta(t)$ vertically inverts the figure axis of the spinning top. However, this only works for spinning tops that are mounted accordingly (e.g., for magnetically levitated spinning tops) because the floor of a normal tabletop spinning top prevents the figure axis from becoming antiparallel to the coordinate x_3.

According to eq. (6.56), two precession velocities, i.e., $\dot\phi_1$ and $\dot\phi_2$, between which $\dot\phi$ varies, are associated with the two zeros ϑ_1 and ϑ_2 of R

$$\dot\phi_1 = \frac{c_1 - c_2\cos\vartheta_1}{A\sin^2\vartheta_1} \tag{6.60}$$

and

$$\dot\phi_2 = \frac{c_1 - c_2\cos\vartheta_2}{A\sin^2\vartheta_2} \tag{6.61}$$

If ϑ_1 denotes the smaller of the two critical nutation angles, then three cases can be distinguished:

1) $c_1 > c_2\cos\vartheta_1$:
The precession is a monotonous motion with a velocity which varies periodically (fig. 6.16a).

2) $c_1 = c_2\cos\vartheta_1$:
The precession stops periodically when the nutation angle ϑ_1 is reached because $\dot\phi_1 = 0$ (fig. 6.16b).

3) $c_1 < c_2\cos\vartheta_1$:
The precession becomes retrograde, i.e., $\dot\phi_1 < 0$ for a certain nutation range $\vartheta_1 \leq \vartheta < \vartheta_2$ (fig. 6.16c).

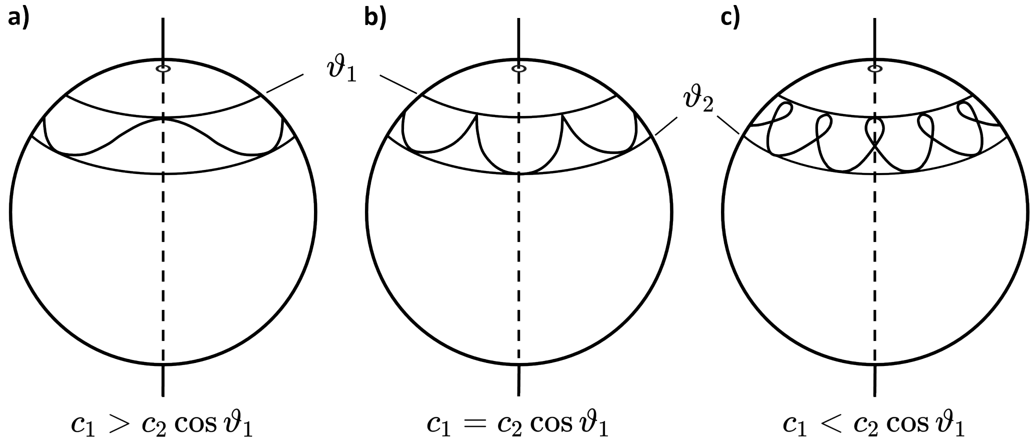

Fig. 6.16: Trajectory curve of the top of a gyroscope traced on a spherical surface: (a) $\dot\phi$ varies monotonically between $\dot\phi_1$ and $\dot\phi_2$; (b) $\dot\phi$ periodically becomes zero because $\dot\phi_1 = 0$; and (c) $\dot\phi$ becomes periodically retrograde when $\dot\phi_1 < 0$.

For each of the three cases, fig. 6.16 shows the trajectory curve which a point lying on the figure axis above the top of the gyroscope would describe on a spherical surface between the "latitudes" defined by the nutation angles ϑ_1 and ϑ_2. With eq. (6.57), the extreme values $\dot\psi_1$ and $\dot\psi_2$ for rotation around the figure axis associated with ϑ_1 and ϑ_2 are

$$\dot\psi_1 = \frac{c_2}{C} - \frac{c_1 - c_2\cos\vartheta_1}{A\sin^2\vartheta_1}\cos\vartheta_1 \tag{6.62}$$

and

$$\dot\psi_2 = \frac{c_2}{C} - \frac{c_1 - c_2\cos\vartheta_2}{A\sin^2\vartheta_2}\cos\vartheta_2 \tag{6.63}$$

Just as for $\dot\phi$, the rotational velocity around the figure axis $\dot\psi$ fluctuates periodically between these extreme values. In the limiting case $\vartheta_1 = \vartheta_2$, the two "latitudes" in fig. 6.16 coincide, which means ϑ becomes a constant and with that nutational motion stops completely. The angular velocities $\dot\phi$ and $\dot\psi$ are then constant and therefore the gyroscope motion is that of a regular precession as it has already been discussed in the context of torque-free motion (see section 5.3).

6.3.3 Non-holonomic Systems

In a non-holonomic system at least one of the constraints (auxiliary conditions) is not only a function of the generalized coordinates q_i but also depends on the velocities $\dot q_i$. Hence, the constraints (auxiliary conditions) have the form

$$\varphi_\mu\big(q_i, \dot{q}_k, t\big) \;=\; 0 \qquad \mu \;=\; 1, 2, \ldots 3n - f \qquad i, k \;=\; 1, 2, \ldots n \tag{6.64}$$

This means that the generalized coordinates q_i of the auxiliary conditions can no longer be varied independently of each other and that the auxiliary conditions can no longer be integrated. In addition to the auxiliary conditions for a non-holonomic system with, for example, S degrees of freedom q_j, further boundary conditions are required in the form

$$\sum_{j=1}^{S} a_{ij}\mathrm{d}q_j + b_i\mathrm{d}t \;=\; 0 \qquad \text{with} \qquad 1 \,\leq\, i \,\leq\, p \tag{6.65}$$

For the p additional constraints, the following applies (with $\delta t = 0$)

$$\sum_{j=1}^{S} a_{ij}\delta q_j \;=\; 0$$

In the Lagrange formalism, the additional constraints are taken into account by multiplying them by a Lagrange parameter and adding them as constraint forces to the Lagrange function eq. (6.30)

$$\sum_{j=1}^{S}\left\{\frac{\partial \mathrm{L}}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{q}_j} + \sum_{i=1}^{p}\lambda_i a_{ij}\right\}\delta q_j \;=\; 0 \tag{6.66}$$

The generalized coordinates are then chosen in such a way that for the q_j applies:

$$q_j \;=\; \begin{cases} \text{independent if} & 1 \leq j \leq S - p \\ \text{dependent if} & S - p + 1 \leq j \leq S \end{cases}$$

Now, one chooses the p Lagrange parameters in eq. (6.66) in such a way that the expression between the curly brackets becomes equal to zero for the dependent variables. This means that the independent variables can now be varied separately. Consequently, one then only has to deal with the independent variables and therefore the outer sum in eq. (6.66) disappears and it remains

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{q}_j} - \frac{\partial \mathrm{L}}{\partial q_j} \;=\; \sum_{i=1}^{p}\lambda_i a_{ij} \;=\; Q_j \tag{6.67}$$

where the Q_j are the constraint forces resulting from the additional constraints. A comparison with eq. (6.15) shows that eq. (6.67) corresponds to a Lagrange equation of the first kind. With eq. (6.67) one has S equations (there are S independent variables) in $S + p$ unknowns. The missing p equations to solve eq. (6.67) are provided by the p additional constraints (boundary conditions) from eq. (6.65).

Example 6.11 **The case of the non-holonomic roll condition (car)**

The wheel in fig. 6.17 rolls with a velocity $\boldsymbol{v}$ in the xy-plane with α being the angle between the wheel axis and the x-coordinate. For $\alpha = 0$, the wheel axis is parallel to the x-axis and the wheel rolls in the negative y-direction, i.e., out of the image plane. $\alpha = 90°$ means the wheel axis is perpendicular to the x-axis and the wheel rolls parallel to the positive x-direction. The motion of the wheel rolling in the xy-plane is completely described by the four coordinates x, y, ϑ and α. The following relationships can be read from fig. 6.17

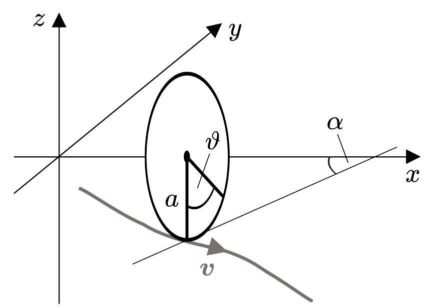

Fig. 6.17

$$v = a\dot{\vartheta} \quad ; \quad \dot{x} = v \sin\alpha \quad ; \quad \dot{y} = -v\cos\alpha$$

Both the x- and y-coordinate are a function of the wheel's velocity; hence, this is a non-holonomic system. From the equations for $\dot{x}$ and $\dot{y}$ follows

$$\dot{x}\cos\alpha + \dot{y}\sin\alpha \;=\; 0 \qquad \text{or} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \;=\; 0 \tag{6.68}$$

This non-holonomic auxiliary condition states that the velocity vector $\boldsymbol{v}$ is always perpendicular to the wheel axis. Although the wheel can assume any orientation x, y, ϑ and α in the plane, due to this non-holonomic auxiliary condition, from moment to moment, there are only two degrees of freedom, the motion in the momentary forward direction $\delta\hat{\boldsymbol{v}}$ and the change in angle $\delta\vartheta$. Two differential boundary conditions follow from the above equations

$$\left.\begin{aligned} &\dot{x} - a\cdot\dot{\vartheta}\cdot\sin\alpha \;=\; 0 \qquad (*) \\ &\dot{y} + a\cdot\dot{\vartheta}\cdot\cos\alpha \;=\; 0 \end{aligned}\right\} \tag{6.69}$$

No integration solution can be found for these differentials, so these two differential boundary conditions are indeed non-holonomic. That this is really the case can also be shown without attempting integration by assuming that there exists a holonomic auxiliary condition $\varphi_\mu(x, y, \alpha, \vartheta) = 0$ from which, for example, the first of these two differential boundary conditions could be derived. In that case it must hold

$$\varphi_{\mu x}\dot{x} + \varphi_{\mu y}\dot{y} + \varphi_{\mu\alpha}\dot{\alpha} + \varphi_{\mu_\vartheta}\dot{\vartheta} \;=\; 0 \qquad (**)$$

Since $(*)$ only has terms with $\dot{x}$ and $\dot{\vartheta}$, the two middle terms in $(**)$ must vanish, which means that $\varphi_{\mu_y} = 0$ and $\varphi_{\mu_\alpha} = 0$ and thus $\varphi_{\mu_{\alpha\vartheta}} = \varphi_{\mu_{\alpha x}} = 0$ also applies. With that the boundary condition $(**)$ becomes

$$\varphi_{\mu_x}\dot{x} + \varphi_{\mu_\vartheta}\dot{\vartheta} \;=\; 0$$

With $(*)$ applies

$$\frac{\varphi_{\mu_\vartheta}}{\varphi_{\mu_x}} \;=\; -a\sin\alpha \qquad \text{or} \qquad \varphi_{\mu_\vartheta} \;=\; -a\sin\alpha\cdot\varphi_{\mu_x}$$

With that and because of $\varphi_{\mu_{\alpha\vartheta}} = \varphi_{\mu_{\vartheta\alpha}} = 0$ follows

$$0 \;=\; \varphi_{\mu_{\vartheta\alpha}} \;=\; \frac{\partial}{\partial\alpha}\varphi_{\mu_\vartheta} \;=\; -a\cos\alpha\cdot\varphi_{\mu_x} - a\sin\alpha\cdot\underbrace{\varphi_{\mu_{x\alpha}}}_{=\varphi_{\mu_{\alpha x}}=0}$$

Thus $\varphi_{\mu_x} = 0$ must apply and therefore because of $(*)$ also $\varphi_{\mu_\vartheta} = 0$. That, however, means that φ_μ must be constant! Hence, contrary to the assumption, one finds that $\varphi_\mu \neq 0$ and thus the auxiliary conditions cannot be holonomic. To solve the problem, one first returns to the Lagrange equations of the first kind in order to determine the constraint forces (right side of eq. (6.67)) and then constructs the equations of motion using the Lagrange function (left side of eq. (6.67)) on.

The four generalized coordinates of the problem are the coordinates of the support point of the wheel, i.e., x and y, as well as the two angles ϑ and α:

$$q_1 = x \quad ; \quad q_2 = y \quad ; \quad q_3 = \vartheta \quad ; \quad q_4 = \alpha$$

With these coordinates and after multiplication by $\mathrm{d}t$, the differential boundary conditions from eq. (6.69) thus become

$$\mathrm{d}q_1 - a\sin q_4\cdot\mathrm{d}q_3 \;=\; 0$$

$$\mathrm{d}q_2 + a\cos q_4\cdot\mathrm{d}q_3 \;=\; 0$$

With eq. (6.65), however, to these two differential boundary conditions must also apply:

$$a_{11}\mathrm{d}q_1 + a_{12}\mathrm{d}q_2 + a_{13}\mathrm{d}q_3 + a_{14}\mathrm{d}q_4 + b_1\mathrm{d}t \;=\; 0$$

$$a_{21}\mathrm{d}q_1 + a_{22}\mathrm{d}q_2 + a_{23}\mathrm{d}q_3 + a_{24}\mathrm{d}q_4 + b_2\mathrm{d}t \;=\; 0$$

Therefore, the following must hold:

$$a_{11} = 1 \quad ; \quad a_{12} = 0 \quad ; \quad a_{13} = -a\sin q_4 \quad ; \quad a_{14} = 0 \quad ; \quad b_1 = 0$$

$$a_{21} = 0 \quad ; \quad a_{22} = 1 \quad ; \quad a_{23} = a\cos q_4 \quad ; \quad a_{24} = 0 \quad ; \quad b_2 = 0$$

With these coefficients one can now determine the constraint forces according to eq. (6.67). Two Lagrange multipliers λ_1 and λ_2 are required for the two constraint conditions from eq. (6.69), with the help of which the four constraint forces follow from eq. (6.67):

$$Q_1 \;=\; \lambda_1 a_{11} + \lambda_2 a_{21} \;=\; \lambda_1$$

$$Q_2 \;=\; \lambda_1 a_{12} + \lambda_2 a_{22} \;=\; \lambda_2$$

$$Q_3 \;=\; \lambda_1 a_{13} + \lambda_2 a_{23} \;=\; -\lambda_1 a \sin q_4 + \lambda_2 a \cos q_4$$

$$Q_4 \;=\; \lambda_1 a_{14} + \lambda_2 a_{24} \;=\; 0$$

Now one determines the potential and the kinetic energy of the rolling wheel. Since the wheel rolls in the plane, its potential energy is constant and one can therefore set $V = 0$. The kinetic energy of the wheel is the sum of its translation energy T_T and its rotation energy T_R. The rotation energy of the wheel comes from its own rotation, i.e., self-rotation, ($T_R^{(1)}$) and from the steering contribution ($T_R^{(2)}$). Self-rotation is independent of the choice of the coordinate system, but it is best to use the main axis system with the main moment of inertia $J_{11} = m^* a^2$ in the direction of the wheel axis, where m^* again denotes the effective mass of the moment of inertia. This gives for $T_R^{(1)}$:

$$T_R^{(1)} \;=\; \frac{1}{2} m^* a^2 \dot{\vartheta}^2 \;=\; \frac{1}{2} m^* a^2 \dot{q}_3^2$$

The two moments of inertia normal to the wheel axle are identical, i.e., $J_{22} = J_{33} = I$. Steering occurs through rotation around the axis that goes through the wheel support point and the wheel hub with the rotational speed being $\dot{\alpha}$. Its contribution to the rotational energy is thus

$$T_R^{(2)} \;=\; \frac{I}{2}\dot{\alpha}^2 \;=\; \frac{I}{2}\dot{q}_4^2$$

With that one has for the Lagrangian function $\mathrm{L} = T - V$

$$\mathrm{L} \;=\; T_T + T_R^{(1)} + T_R^{(2)} - V \;=\; \frac{1}{2} M(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} m^* a^2 \dot{q}_3^2 + \frac{I}{2}\dot{q}_4^2$$

or respectively

$$\mathrm{L} \;=\; \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m^* a^2 \dot{\vartheta}^2 + \frac{I}{2}\dot{\alpha}^2$$

where M is the mass of the wheel. With the help of eq. (6.67) and the constraint forces already determined, one can now write down the Lagrangian equations of the first kind:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{x}} - \frac{\partial \mathrm{L}}{\partial x} &= M\ddot{x} = F_x + \lambda_1 \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{y}} - \frac{\partial \mathrm{L}}{\partial y} &= M\ddot{y} = F_y + \lambda_2 \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{\vartheta}} - \frac{\partial \mathrm{L}}{\partial \vartheta} &= m^* a^2 \ddot{\vartheta} = M_\vartheta - \lambda_1 a \sin\alpha + \lambda_2 a \cos\alpha \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{\alpha}} - \frac{\partial \mathrm{L}}{\partial \alpha} &= I\ddot{\alpha} = M_\alpha + 0
\end{aligned} \tag{6.70}$$

where the external forces F_x and F_y added here stand for gravity, frictional forces or air resistance whereas the added moments M_ϑ and M_α stand for motor and braking torques and for the moment caused by steering, respectively. The last of the equations is based on the control of the wheel, i.e., the steering force.

With the two boundary conditions from eq. (6.69) and the four Lagrange equations of the first kind from eq. (6.70) one now has six equations for six unknowns: λ_1, λ_2, x, y, ϑ and α. For the sake of simplicity, eq. (6.70) shall be considered here without external forces / moments, i.e., $F_x = 0$, $F_y = 0$, $M_\vartheta = 0$ and $M_\alpha = 0$. Direct integration of the fourth equation in eq. (6.70) with $M_\alpha = 0$ then yields

$$\dot{\alpha} = const = \omega_\alpha \qquad \text{and} \qquad \alpha = \omega_\alpha t + \alpha_0$$

For $\omega_\alpha = 0$ the wheel rolls straight and for $\omega_\alpha > 0$ it runs through a curve with constant angular velocity. In the next step, one inserts $\alpha = \omega_\alpha t + \alpha_0$ into the constraint conditions in eq. (6.69) and takes the time derivative of the equations thus obtained. The results for $\ddot{x}$ and $\ddot{y}$ are then inserted into the first two equations of eq. (6.70) (with $F_x = 0$ and $F_y = 0$). This yields for λ_1 and λ_2

$$\begin{aligned}
\lambda_1 &= Ma\ddot{\vartheta}\sin(\omega_\alpha t + \alpha_0) + Ma\dot{\vartheta}\omega_\alpha \cos(\omega_\alpha t + \alpha_0) \\
\lambda_2 &= -Ma\ddot{\vartheta}\cos(\omega_\alpha t + \alpha_0) + Ma\dot{\vartheta}\omega_\alpha \sin(\omega_\alpha t + \alpha_0)
\end{aligned}$$

Now one inserts λ_1 and λ_2 into equation three of eq. (6.70) (with $M_\vartheta = 0$). This gives

$$\begin{aligned}
m^* a^2 \ddot{\vartheta} = {} & -a\sin(\omega_\alpha t + \alpha_0)\big(Ma\ddot{\vartheta}\sin(\omega_\alpha t + \alpha_0) + Ma\dot{\vartheta}\omega_\alpha\cos(\omega_\alpha t + \alpha_0)\big) \\
& + a\cos(\omega_\alpha t + \alpha_0)\big(-Ma\ddot{\vartheta}\cos(\omega_\alpha t + \alpha_0) + Ma\dot{\vartheta}\omega_\alpha\sin(\omega_\alpha t + \alpha_0)\big)
\end{aligned}$$

That can be simplified to

$$\underbrace{m^* a^2}_{>0}\ddot{\vartheta} = \underbrace{-Ma^2}_{<0}\ddot{\vartheta} \qquad \Rightarrow \qquad \ddot{\vartheta} = 0$$

So, it must apply

$$\dot{\vartheta} = const = \Omega \qquad \text{and therefore} \qquad \vartheta = \Omega t + \vartheta_0$$

For the generalized constraint forces according to eq. (6.67) one then obtains

$$Q_1 = M a \omega_\alpha \Omega \cos(\omega_\alpha t + \alpha_0)$$

$$Q_2 = M a \omega_\alpha \Omega \sin(\omega_\alpha t + \alpha_0)$$

$$Q_3 = Q_4 = 0$$

The constraint forces Q_1 and Q_2 only exist if the wheel does not roll straight ahead, i.e., they only exist when $\omega_\alpha \neq 0$. The constraint forces Q_1 and Q_2 ensure that the roll condition in eq. (6.68) is always satisfied.

6.4 Small Oscillations

Oscillations of small amplitude for which the restoring force can be assumed to be proportional to the deflection and for which the system under consideration executes linear oscillations around an equilibrium position are referred to as small oscillations. Small oscillations are an important area of application for the Lagrange approach, especially when the number of degrees of freedom f of the system under consideration is large. The kinetic energy T for a system of N masses m_i $(i = 1 \dots N)$ with f degrees of freedom is according to eq. (6.27) given by

$$T = \frac{1}{2} \sum_{r,s=1}^{f} M_{rs}(q_1 \dots q_f) \dot{q}_r \dot{q}_s \qquad \text{with} \qquad M_{rs} = \sum_i m_i \frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_s}$$

and the potential energy of the system is given by

$$V = V(q_1 \dots q_f)$$

Let $\boldsymbol{q^0} = (q_1^0 \dots q_f^0)$ be the rest position of the system in which its potential energy $V(\boldsymbol{q^0})$ is minimal. For small deviations $\boldsymbol{q}$ from this equilibrium position

$$\hat{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}^0$$

the respective motion of the system is then an oscillation. The first terms of the Taylor expansion of the potential V around the rest position $\boldsymbol{q^0}$ are

$$V(\hat{q}_1 \dots \hat{q}_f) = V(q_1^0 \dots q_f^0) + \sum_{r=1}^{f} \left(\frac{\partial V}{\partial q_r} \right)_0 \cdot \hat{q}_r + \frac{1}{2} \sum_{r,s=1}^{f} \left(\frac{\partial^2 V}{\partial q_r \partial q_s} \right)_0 \cdot \hat{q}_r \hat{q}_s + \dots$$

The second term of this expansion corresponds to the virtual work performed on the system by the acting forces $F_r = \partial V/\partial q_r$ and is therefore equal to zero (see section 6.2.1). For the quadratic term applies with $\boldsymbol{q}^0 = \mathbf{0}$

$$\left(\frac{\partial^2 V}{\partial q_r \partial q_s}\right)_0 \;=\; \frac{\partial^2 V}{\partial q_r \partial q_s} \;=\; V_{rs} \tag{6.71}$$

and the Lagrange function $\mathrm{L} = T - V$ becomes

$$\mathrm{L} \;=\; \frac{1}{2}\sum_{r,s=1}^{f} M_{rs}\dot{q}_r\dot{q}_s - V_0 - \frac{1}{2}\sum_{r,s=1}^{f} V_{rs}q_r q_s \tag{6.72}$$

From the general equation of motion according to Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{q}_r} - \frac{\partial \mathrm{L}}{\partial q_r} \;=\; 0$$

then follows the matrix equation

$$\sum_s M_{rs}\ddot{q}_s + \sum_s V_{rs}q_s \;=\; 0 \tag{6.73}$$

The following applies to the matrices M_{rs} and V_{rs}:

$$M_{rs} \;=\; M_{sr} \quad \text{and} \quad V_{rs} \;=\; V_{sr}$$

A brute force approach to solving eq. (6.73) is

$$\begin{aligned} q_r(t) \;&=\; a_r \mathrm{e}^{i\omega t} \;\;,\;\; a_r \cos\omega t \;\;,\;\; a_r \sin\omega t \\ \dot{q}_r(t) \;&=\; i\omega a_r \mathrm{e}^{i\omega t} \\ \ddot{q}_r(t) \;&=\; -\omega^2 a_r \mathrm{e}^{i\omega t} \;=\; -\omega^2 q_r \end{aligned}$$

With that eq. (6.73) becomes

$$\sum_s^{f}\big(-M_{rs}\omega^2 + V_{rs}\big)q_s \;=\; 0 \tag{6.74}$$

This equation only has a solution if the corresponding determinant is zero, which means it must hold

$$\det\big|V_{rs} - M_{rs}\omega^2\big| \;=\; 0 \tag{6.75}$$

This leads to an equation of f-th degree for ω^2 and f solutions $\pm\omega_\lambda$ with $\lambda = 1 \ldots f$.

Correspondingly, there are f vectors $Q_{l\lambda}$ with $l, \lambda = 1 \ldots f$ such that

$$\sum_{l=1}^{f} \big(V_{kl} - \omega_\lambda^2 M_{kl}\big) Q_{l\lambda} \;=\; 0 \tag{6.76}$$

Eq. (6.76) has the general solution

$$q_k(t) \;=\; \mathrm{Re}\left(\sum_{\lambda=1}^{f} Q_{k\lambda}\eta_\lambda \mathrm{e}^{i\omega_\lambda t}\right) \tag{6.77}$$

Small oscillations with several degrees of freedom are mostly coupled oscillations. In the following, three examples of coupled oscillations will be considered in detail: The double pendulum, two pendula coupled with a spring and a model for the stretching vibration of a linear diatomic molecule.

Example 6.12 **The double pendulum**

For the coordinates of the double pendulum in fig. 6.18 one can see that

$$y_1 \;=\; -a_1 \cos\varphi_1$$

$$x_1 \;=\; a_1 \sin\varphi_1$$

$$y_2 \;=\; -a_1 \cos\varphi_1 - a_2 \cos\varphi_2$$

$$x_2 \;=\; a_1 \sin\varphi_1 + a_2 \sin\varphi_2$$

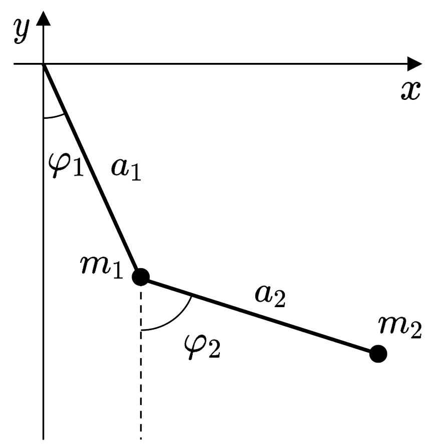

Fig. 6.18

With that one gets for the velocities

$$\dot{y}_1 \;=\; a_1 \sin\varphi_1 \cdot \dot{\varphi}_1$$

$$\dot{x}_1 \;=\; a_1 \cos\varphi_1 \cdot \dot{\varphi}_1$$

$$\dot{y}_2 \;=\; \dot{y}_1 + a_2 \sin\varphi_2 \cdot \dot{\varphi}_2$$

$$\dot{x}_2 \;=\; \dot{x}_1 + a_2 \cos\varphi_2 \cdot \dot{\varphi}_2$$

The kinetic energy of the double pendulum is the sum of the kinetic energies of the two masses m_1 and m_2

$$T \;=\; T_1 + T_2 \;=\; \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

After inserting the velocities, one therefore obtains for the kinetic energy of the double pendulum

$$T = \frac{m_1+m_2}{2} a_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} a_2^2 \dot{\varphi}_2^2 + m_2 a_1 a_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$

The potential energy of the double pendulum is the sum of the potential energies of the masses m_1 and m_2

$$V = V_1 + V_2 = -m_1 g y_1 - m_2 g y_2 = -(m_1+m_2) g a_1 \cos\varphi_1 - m_2 g a_2 \cos\varphi_2$$

Here, the approximation for small deflections is the Taylor expansion of the cosine

$$\cos\varphi_{1,2} = 1 - \frac{\varphi_{1,2}^2}{2}$$

If one plugs this approximation for small deflections into the equations for T and V and keeps only terms that are at most quadratic in $\varphi_{1,2}$ and its derivatives, then one obtains for the kinetic energy T

$$T = \frac{m_1+m_2}{2} a_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} a_2^2 \dot{\varphi}_2^2 + m_2 a_1 a_2 \dot{\varphi}_1 \dot{\varphi}_2$$

and for the potential energy V

$$V = V_0 + \frac{m_1+m_2}{2} g a_1 \varphi_1^2 + \frac{m_2 g a_2}{2} \varphi_2^2$$

where

$$V_0 = -(m_1+m_2) g a_1 - m g a_2$$

With that the Lagrange function $\mathrm{L} = T - V$ becomes

$$\mathrm{L} = \frac{m_1+m_2}{2} a_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} a_2^2 \dot{\varphi}_2^2 + m_2 a_1 a_2 \dot{\varphi}_1 \dot{\varphi}_2 - V_0 - \frac{m_1+m_2}{2} g a_1 \varphi_1^2 - \frac{m_2 g a_2}{2} \varphi_2^2$$

A comparison of this equation with eq. (6.72) shows that here the matrices M_{rs} and V_{rs} can be read off directly

$$M_{rs} = \begin{bmatrix} (m_1+m_2) a_1^2 & m_2 a_1 a_2 \\ m_2 a_1 a_2 & m_2 a_2^2 \end{bmatrix}$$

and

$$V_{rs} = \begin{bmatrix} (m_1+m_2) g a_1 & 0 \\ 0 & m_2 g a_2 \end{bmatrix}$$

From the solution condition $\det |V_{rs} - \omega^2 M_{rs}| = 0$ therefore follows

$$\begin{bmatrix} (m_1+m_2) g a_1 - \omega^2 (m_1+m_2) a_1^2 & -m_2 a_1 a_2 \omega^2 \\ -m_2 a_1 a_2 \omega^2 & m_2 g a_2 - m_2 a_2^2 \end{bmatrix} = 0$$

Hence

$$(m_1+m_2)a_1(g-a_1\omega^2)\cdot m_2a_2(g-a_2\omega^2)-m_2a_1a_2\omega^2\cdot m_2a_1a_2\omega^2 \;=\; 0$$

or respectively simplified

$$(m_1+m_2)(g-a_1\omega^2)(g-a_2\omega^2)-m_2a_1a_2\omega^4 \;=\; 0 \qquad (*)$$

To solve $(*)$, a double pendulum with $a_1 = a_2$ shall be considered here. With the abbreviations

$$\omega_0^2 \;=\; \frac{g}{a_1} \;=\; \frac{g}{a_2} \qquad \text{and} \qquad \mu \;=\; \frac{m_1}{m_2}$$

$(*)$ becomes

$$\left(1+\frac{1}{\mu}\right)(\omega^2-\omega_0^2)-\omega^4 \;=\; 0$$

from which follows for ω^2

$$\omega^2 \;=\; \left(1+\mu\pm\mu\sqrt{1+\frac{1}{\mu}}\right)\omega_0^2 \tag{6.78}$$

For large μ the positive branch of this solution obviously becomes $\omega^2=(1+2\mu)\omega_0^2$. The asymptotic value $\omega_0^2/2$ for large μ in the negative branch can be found by replacing the square root in eq. (6.78) by the first two terms of its Taylor expansion. Fig. 6.19 shows the graphical representation of the two branches of eq. (6.78). The lower branch, the solution with the minus sign in eq. (6.78) corresponds to an in-phase motion ($\omega^2 < \omega_0^2$) and the upper branch, the solution with the plus sign in eq. (6.78), to an anti-phase ($\omega^2 > \omega_0^2$) motion of the double pendulum. The solution for $\omega^2 = \omega_0^2$ is not defined.

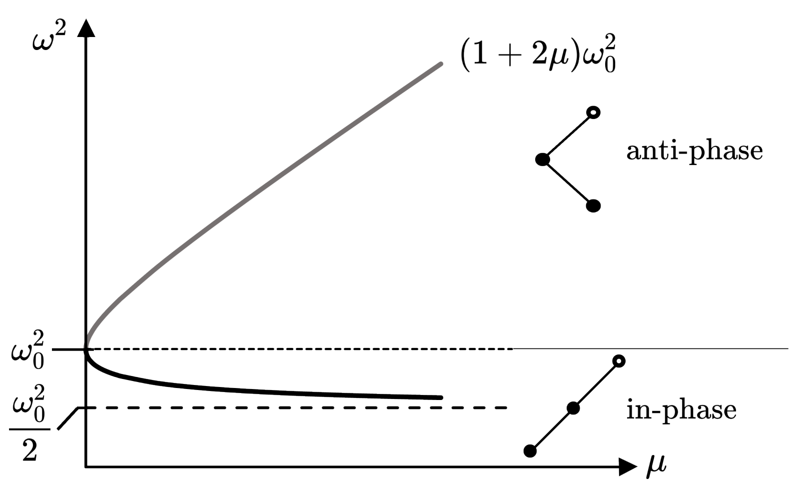

Fig. 6.19: Graphic representation of the solution for the double pendulum motion. The lower branch, the in-phase motion, corresponds to the minus sign and the upper branch, the anti-phase motion, corresponds to the plus sign in eq. (6.78).

The solution from eq. (6.78) for the equation of motion of the double pendulum is only valid for small deflections. Without the simplifications of T and V that follow from this assumption, the equation of motion for the Lagrange function $\mathrm{L} = T - V$ cannot be solved analytically. It is not for nothing that the motion of the double pendulum (if the deflections are not small) is a prime example of chaotic motion.

Example 6.13 **Two spring-coupled pendula**

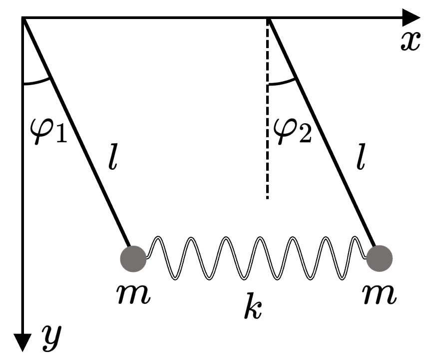

Fig. 6.20

As sketched in fig. 6.20, two identical pendula, each with a pendulum arm l and a pendulum mass m, shall be coupled through a spring with the spring constant k. For identical deflection, i.e., $\varphi_1 = \varphi_2$, the spring is relaxed. The kinetic energy of the system is the sum of the kinetic energies of the two pendula

$$T = \frac{m}{2}\dot{\varphi}_1^2 l^2 + \frac{m}{2}\dot{\varphi}_2^2 l^2 = \frac{m}{2} l^2 \big(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\big)$$

The potential energy of the system is the sum of the potential energies of the two pendulum masses and the potential energy of the spring. For the potential energies of the pendulum masses, one reads from fig. 6.20

$$V_{m+m} = mg\big(l - l\cos\varphi_1\big) + mg\big(l - l\cos\varphi_2\big) = 2mgl - mgl\big(\cos\varphi_1 + \cos\varphi_2\big)$$

The potential energy of the spring is

$$V_{Spring} = \frac{k}{2} l^2 \big(\sin\varphi_1 - \sin\varphi_2\big)^2$$

Thus, one has for the Lagrange function $\mathrm{L} = T - V$

$$\mathrm{L} = \frac{m}{2} l^2 \big(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\big) - \frac{k}{2} l^2 \big(\sin\varphi_1 - \sin\varphi_2\big)^2 - 2mgl + mgl\big(\cos\varphi_1 + \cos\varphi_2\big)$$

Now one uses the approximation for small deflections (i.e., small oscillations)

$$\sin\varphi_{1,2} \approx \varphi_{1,2} \qquad \text{and} \qquad \cos\varphi_{1,2} \approx 1 - \frac{\varphi_{1,2}^2}{2}$$

This simplifies the Lagrange function to

$$\mathrm{L} = \frac{m}{2} l^2 \big(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\big) - \frac{k}{2} l^2 \big(\varphi_1 - \varphi_2\big)^2 - mgl\Big(\frac{\varphi_1^2}{2} + \frac{\varphi_2^2}{2}\Big)$$

Multiplying out the quadratic terms and rewriting gives

$$\mathrm{L} = \frac{m}{2} l^2 \big(\dot{\varphi}_1^2 + \dot{\varphi}_1^2\big) + \frac{1}{2}\Big(\big[kl^2 + mgl\big]\varphi_1^2 - 2kl^2\varphi_1\varphi_2 + \big[kl^2 + mgl\big]\varphi_2^2\Big)$$

A comparison with the Lagrange function in the form of eq. (6.72)

$$\mathrm{L} \;=\; \frac{1}{2}\sum_{r,s=1}^{f} M_{rs}\dot{q}_r\dot{q}_s - V_0 - \frac{1}{2}\sum_{r,s=1}^{f} V_{rs}q_rq_s$$

shows that M_{rs} and V_{rs} are given by

$$M_{rs} \;=\; \begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \end{bmatrix} \qquad ; \qquad V_{rs} \;=\; \begin{bmatrix} kl^2 + mgl & -kl^2 \\ -kl^2 & kl^2 + mgl \end{bmatrix}$$

From the solution condition $\det|V_{rs} - \omega^2 M_{rs}|$ follows

$$\begin{bmatrix} (kl^2 + mgl) - ml^2\omega^2 & -kl^2 \\ -kl^2 & (kl^2 + mgl) - ml^2\omega^2 \end{bmatrix} \;=\; 0$$

So, it must apply

$$\left[(kl^2 + mgl) - ml^2\omega^2\right]^2 - k^2l^4 \;=\; 0$$

Solving for ω yields

$$(kl^2 + mgl) - ml^2\omega^2 \;=\; \pm kl^2$$

$$\omega^2 \;=\; \mp\frac{k}{m} + \frac{k}{m} - \frac{g}{l}$$

Hence the two solutions are

$$\omega_1^2 \;=\; \frac{g}{l} \qquad \text{and} \qquad \omega_2^2 \;=\; \frac{2k}{m} + \frac{g}{l}$$

or respectively

$$\omega_1 \;=\; \sqrt{\frac{g}{l}} \qquad \text{and} \qquad \omega_2 \;=\; \sqrt{\frac{2k}{m} + \frac{g}{l}} \tag{6.79}$$

From the Lagrange equations of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial\dot{\varphi}_{1,2}} - \frac{\partial \mathrm{L}}{\partial\varphi_{1,2}} \;=\; 0$$

one obtains the equations of motion for φ_1 and φ_2:

$$\ddot{\varphi}_1 + \frac{k}{m}(\varphi_1 - \varphi_2) + \frac{g}{l}\varphi_1 \;=\; 0$$

and

$$\ddot{\varphi}_2 + \frac{k}{m}(\varphi_2 - \varphi_1) + \frac{g}{l}\varphi_2 \;=\; 0$$

or respectively rearranged

$$\ddot{\varphi}_1 + \left(\frac{k}{m} + \frac{g}{l}\right)\varphi_1 - \frac{k}{m}\varphi_2 \;=\; 0 \tag{6.80}$$

and

$$\ddot{\varphi}_2 + \left(\frac{k}{m} + \frac{g}{l}\right)\varphi_2 - \frac{k}{m}\varphi_1 \;=\; 0 \tag{6.81}$$

So, there are two solutions, one with $\omega = \omega_1$ and one with $\omega = \omega_2$:

$$\varphi_1 \;=\; A_1 \mathrm{e}^{i\omega_1 t} \quad ; \quad \varphi_2 \;=\; A_2 \mathrm{e}^{i\omega_1 t}$$

and

$$\varphi_1 \;=\; A_1 \mathrm{e}^{i\omega_2 t} \quad ; \quad \varphi_2 \;=\; A_2 \mathrm{e}^{i\omega_2 t}$$

Inserting the solution approach $\omega = \omega_{1,2}$ into the equation of motion for φ_1 (eq. (6.80)) results in

$$-\omega^2 A_1 \mathrm{e}^{i\omega t} + \left(\frac{k}{m} + \frac{g}{l}\right) A_1 \mathrm{e}^{i\omega t} - \frac{k}{m} A_2 \mathrm{e}^{i\omega t} \;=\; 0$$

It therefore must apply

$$\left(-\omega^2 + \frac{k}{m} + \frac{g}{l}\right) A_1 \;=\; \frac{k}{m} A_2$$

With the two solutions $\omega = \omega_1$ and $\omega = \omega_2$ (eq. (6.79)) it follows for A_1 and A_2

$$\omega^2 \;=\; \begin{cases} \omega_1^2 \;=\; \dfrac{g}{l} & \Rightarrow \quad A_1 \;=\; A_2 \\[2ex] \omega_2^2 \;=\; \dfrac{2k}{m} + \dfrac{g}{l} & \Rightarrow \quad A_1 \;=\; -A_2 \end{cases}$$

As can easily be checked, the very same result follows if one alternatively inserts the solution approach into the equation of motion for φ_2 (eq. (6.81)). The result is in both cases that two solutions exist, one in-phase (fig. 6.21a) and one anti-phase solution (fig. 6.21b).

$$\varphi_1 \;=\; A\mathrm{e}^{i\omega_1 t} \quad ; \quad \varphi_2 \;=\; A\mathrm{e}^{i\omega_1 t} \qquad \text{in-phase solution}$$

$$\varphi_1 \;=\; A\mathrm{e}^{i\omega_2 t} \quad ; \quad \varphi_2 \;=\; -A\mathrm{e}^{i\omega_2 t} \qquad \text{anti-phase solution}$$

For further treatment of the problem one introduces the normal coordinates λ_a and λ_b

$$\lambda_a \;=\; \varphi_1 - \varphi_2 \qquad \text{and} \qquad \lambda_b \;=\; \varphi_1 + \varphi_2$$

For φ_1 and φ_2 expressed in these normal coordinates one has

$$\varphi_1 \;=\; \frac{\lambda_a + \lambda_b}{2} \qquad \text{and} \qquad \varphi_2 \;=\; \frac{\lambda_b - \lambda_a}{2}$$

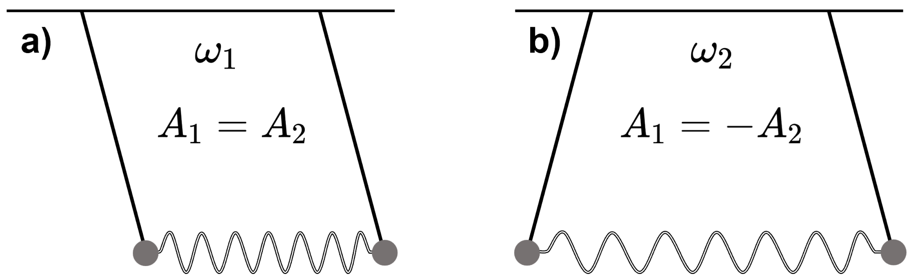

Fig. 6.21: In-phase (a) and anti-phase (b) solution of the equation of motion for two identical pendula coupled by a mechanical spring.

Inserting these two expressions into the equations of motion for φ_1 (eq. (6.80)) and φ_2 (eq. (6.81)) results in

$$\frac{1}{2}\ddot{\lambda}_a + \frac{1}{2}\ddot{\lambda}_b + \left(\frac{k}{m} + \frac{g}{l}\right)\left(\frac{\lambda_a + \lambda_b}{2}\right) - \frac{k}{m}\left(\frac{\lambda_b - \lambda_a}{2}\right) = 0 \tag{6.82}$$

and

$$\frac{1}{2}\ddot{\lambda}_b - \frac{1}{2}\ddot{\lambda}_a + \left(\frac{k}{m} + \frac{g}{l}\right)\left(\frac{\lambda_b - \lambda_a}{2}\right) - \frac{k}{m}\left(\frac{\lambda_a + \lambda_b}{2}\right) = 0 \tag{6.83}$$

If one subtracts eq. (6.83) from eq. (6.82) one obtains

$$\ddot{\lambda}_a + \frac{k}{m}(\lambda_a + \lambda_b) - \frac{k}{m}(\lambda_b - \lambda_a) + \frac{g}{l}\left(\frac{\lambda_a + \lambda_b}{2} - \frac{\lambda_b - \lambda_a}{2}\right) = 0$$

The terms with λ_b cancel out and it remains

$$\ddot{\lambda}_a + \left(\frac{2k}{m} + \frac{g}{l}\right)\lambda_a = 0 \tag{6.84}$$

If one adds eq. (6.82) and eq. (6.83) one obtains

$$\ddot{\lambda}_b + \frac{g}{l}\left(\frac{\lambda_a + \lambda_b}{2} + \frac{\lambda_b - \lambda_a}{2}\right) = 0$$

Here the terms with λ_a cancel out and it remains

$$\ddot{\lambda}_b + \frac{g}{l}\lambda_b = 0 \tag{6.85}$$

Eq. (6.84) and eq. (6.85) are equations for harmonic oscillations in the normal coordinates λ_a and λ_b with the oscillation frequencies $\omega = \omega_1$ or respectively $\omega = \omega_2$. Solution approach:

$$\lambda_b = A_b \mathrm{e}^{i\omega_1 t} \qquad \text{and} \qquad \lambda_a = A_a \mathrm{e}^{i\omega_2 t}$$

hence

$$\varphi_1 = \frac{1}{2}A_a e^{i\omega_2 t} + \frac{1}{2}A_b e^{i\omega_1 t} \tag{6.86}$$

and

$$\varphi_2 = \frac{1}{2}A_b e^{i\omega_1 t} - \frac{1}{2}A_a e^{i\omega_2 t} \tag{6.87}$$

Now one compares these equations with the already identified solutions

$$\varphi_1 = A_1 e^{i\omega_1 t} \quad ; \quad \varphi_2 = A_2 e^{i\omega_1 t}$$

and

$$\varphi_1 = A_1 e^{i\omega_2 t} \quad ; \quad \varphi_2 = A_2 e^{i\omega_2 t}$$

For $t = 0$ must apply

$$A_a = A_1 - A_2 \qquad \text{and} \qquad A_b = A_1 + A_2$$

Inserting into eq. (6.86) and eq. (6.87) yields

$$\varphi_1 = \frac{1}{2}\left[A_1 e^{i\omega_2 t} - A_2 e^{i\omega_2 t} + A_1 e^{i\omega_1 t} + A_2 e^{i\omega_1 t}\right]$$

and

$$\varphi_2 = \frac{1}{2}\left[A_1 e^{i\omega_1 t} + A_2 e^{i\omega_1 t} - A_1 e^{i\omega_2 t} + A_2 e^{i\omega_2 t}\right]$$

These equations can be rearranged into

$$\begin{aligned}\varphi_1 = {} & \frac{1}{2}A_1 e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_2-\omega_1}{2}\right)t} + e^{-i\left(\frac{\omega_2-\omega_1}{2}\right)t}\right] - \\ & \frac{1}{2}A_2 e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_2-\omega_1}{2}\right)t} - e^{-i\left(\frac{\omega_2-\omega_1}{2}\right)t}\right]\end{aligned}$$

and

$$\begin{aligned}\varphi_2 = {} & \frac{1}{2}A_1 e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{-i\left(\frac{\omega_2-\omega_1}{2}\right)t} - e^{i\left(\frac{\omega_2-\omega_1}{2}\right)t}\right] - \\ & \frac{1}{2}A_2 e^{i\left(\frac{\omega_1+\omega_2}{2}\right)t}\left[e^{i\left(\frac{\omega_2-\omega_1}{2}\right)t} + e^{-i\left(\frac{\omega_2-\omega_1}{2}\right)t}\right]\end{aligned}$$

If one now defines

$$\frac{\omega_1 + \omega_2}{2} = \bar{\omega} \qquad \text{and} \qquad \omega_2 - \omega_1 = \Delta\omega$$

then the equations for φ_1 and φ_2 become

$$\varphi_1 \;=\; A_1 \mathrm{e}^{i\bar{\omega}t} \cos\left(\frac{\Delta\omega}{2}t\right) - iA_2 \mathrm{e}^{i\bar{\omega}t} \sin\left(\frac{\Delta\omega}{2}t\right)$$

and

$$\varphi_2 \;=\; A_2 \mathrm{e}^{i\bar{\omega}t} \cos\left(\frac{\Delta\omega}{2}t\right) - iA_1 \mathrm{e}^{i\bar{\omega}t} \sin\left(\frac{\Delta\omega}{2}t\right)$$

The amplitudes A_1 and A_2 follow from the choice of initial conditions for $t = 0$:

$$\varphi_1(0) \;=\; \dot{\varphi}_1(0) \;=\; \dot{\varphi}_2(0) \;=\; 0 \quad \text{and} \quad \varphi_2(0) \;=\; \alpha$$

This means that pendulum number 1 is kept still, pendulum number 2 is moved to a deflection angle α and then both pendula are released at $t = 0$. From these initial conditions follows $A_1 = 0$ and $A_2 = \alpha$. The equations for φ_1 and φ_1 are thus

$$\varphi_1 = -i\alpha \sin\left(\frac{\Delta\omega}{2}t\right)\left[\cos\left(\bar{\omega}t\right) + i \sin\left(\bar{\omega}t\right)\right]$$

and

$$\varphi_2 = \alpha \cos\left(\frac{\Delta\omega}{2}t\right)\left[\cos\left(\bar{\omega}t\right) + i \sin\left(\bar{\omega}t\right)\right]$$

For the real part of the general solution, one thus obtains

$$\varphi_1 = \left[\alpha \sin\left(\frac{\Delta\omega}{2}t\right)\right] \sin\left(\bar{\omega}t\right) \tag{6.88}$$

and

$$\varphi_2 = \left[\alpha \cos\left(\frac{\Delta\omega}{2}t\right)\right] \cos\left(\bar{\omega}t\right) \tag{6.89}$$

with

$$\frac{\Delta\omega}{2} \;=\; \frac{\omega_2 - \omega_1}{2} \;=\; \frac{1}{2}\left(\sqrt{\frac{2k}{m} + \frac{g}{l}} - \sqrt{\frac{g}{l}}\right)$$

and

$$\bar{\omega} \;=\; \frac{\omega_1 + \omega_2}{2} \;=\; \frac{1}{2}\left(\sqrt{\frac{2k}{m} + \frac{g}{l}} + \sqrt{\frac{g}{l}}\right)$$

The amplitude of the oscillations described by eq. (6.88) and eq. (6.89) is time-dependent with the frequency $\Delta\omega/2$ and both pendula oscillate with the frequency $\bar{\omega}$, however, out of phase by 90°. Because one amplitude oscillates with the sine (i.e., φ_1) while the other oscillates with the cosine (i.e., φ_2), one will be zero when the other one will reach its maximum value and vice versa. As a result of the coupling, the pendulum (φ_2) deflected by α at the time $t = 0$ transfers its energy to the other pendulum (φ_1) and comes to rest when the latter has reached its maximum amplitude; then the process is repeated in the reverse direction.

Example 6.14 **Stretching vibration of a molecule**

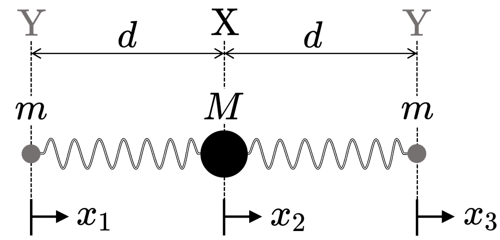

Fig. 6.22

To model the stretching vibration of a linear tri-atomic molecule XY_2 one considers the situation sketched in fig. 6.22. Three point masses with the respective atomic masses M for the middle atom X and m for each of the two outer atoms Y shall be connected by two identical springs where each has the same spring constant f. The equilibrium distance for the identical springs, i.e., the respective molecular distance when the springs are relaxed, shall be d. The natural choice of the generalized coordinates for this problem are the linear displacements (compression or extension of the springs) x_1, x_2 and x_3.

Using these generalized coordinates, the kinetic energy T of this three-atom system is given by

$$T = \frac{m}{2}\dot{x}_1^2 + \frac{M}{2}\dot{x}_2^2 + \frac{m}{2}\dot{x}_3^2$$

and for the potential energy of the springs one obtains

$$V = f\frac{(x_2 - x_1)^2}{2} + f\frac{(x_3 - x_2)^2}{2}$$

The Lagrange function $\mathrm{L} = T - V$ of the molecular model in fig. 6.22 therefore is

$$\mathrm{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2}\dot{x}_2^2 - \frac{f}{2}\left[(x_3 - x_2)^2 + (x_2 - x_1)^2\right]$$

From this Lagrange function one obtains with

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{L}}{\partial \dot{x}_r} - \frac{\partial \mathrm{L}}{\partial x_r} = 0 \qquad r = 1, 2, 3$$

the three equations

$$m\ddot{x}_1 - f(x_2 - x_1) = 0$$

$$M\ddot{x}_2 - f(x_1 - 2x_2 + x_3) = 0$$

$$m\ddot{x}_3 + f(x_2 - x_3) = 0$$

or respectively in matrix notation

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}}_{M_{rs}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \underbrace{\begin{bmatrix} f & -f & 0 \\ -f & 2f & -f \\ 0 & -f & f \end{bmatrix}}_{V_{rs}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution approach for this system of equations: $\quad x_i = A_i \mathrm{e}^{i\omega t}$

With that one gets the matrix equation

$$\sum_s \left(-M_{rs}\omega^2 + V_{rs}\right) x_s = 0 \tag{6.90}$$

From the solution condition $\det |V_{rs} - \omega^2 M_{rs}| = 0$ follows

$$\begin{aligned} 0 &= (f-\omega^2 m)(2f-\omega^2 M)(f-\omega^2 m) - (f-\omega^2 m)f^2 - (f-\omega^2 m)f^2 \\ &= (f-\omega^2 m)\omega^2\left[\omega^2 mM - f(2m+M)\right] \end{aligned}$$

and one obtains the solutions (positive roots only)

$$\omega_1 = \sqrt{\frac{f}{m}} \quad ; \quad \omega_2 = 0 \quad ; \quad \omega_3 = \sqrt{f\left(\frac{2}{M} + \frac{1}{m}\right)}$$

To determine the eigenvectors and vibration modes of the molecule, one now inserts these eigenvalues into eq. (6.90) and applies the Gauß-algorithm. With $\mu = M/m$ one gets for ω_1

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 0 & -f & 0 & 0 \\ -f & f(2-\mu) & -f & 0 \\ 0 & -f & 0 & 0 \end{array} \quad \Rightarrow \quad \begin{array}{ccc|cl} x_1 & x_2 & x_3 & & \\ \hline f & f(\mu-2) & f & 0 & ;\ -1 \times 2.\,\text{row} \\ 0 & f & 0 & 0 & ;\ -1 \times 1.\,\text{row} \\ 0 & 0 & 0 & 0 & ;\ 1.\,\text{row} - 3.\,\text{row} \end{array}$$

With backwards substitution one obtains for the eigenvalue

$$\omega_1 = \sqrt{\frac{f}{m}}$$

the eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

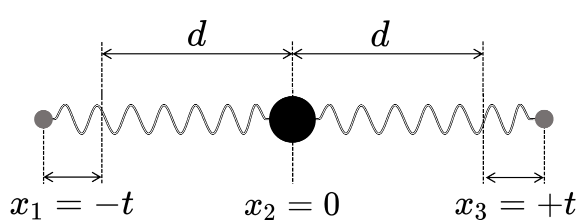

Fig. 6.23

As sketched in fig. 6.23, the two outer atoms Y oscillate in anti-phase, while the central atom X remains in its rest position. Since the mass of the central atom remains in its rest position, the motion of the outer atoms is independent of the mass ratio μ.

Now one inserts the eigenvalue ω_2 into eq. (6.90) and solves the equation system again by using the Gauß-algorithm:

$$
\begin{array}{ccc|c}
x_1 & x_2 & x_3 & \\
\hline
f & -f & 0 & 0 \\
-f & 2f & -f & 0 \\
0 & -f & f & 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc|cl}
x_1 & x_2 & x_3 & & \\
\hline
f & -2f & f & 0 & ;\ -1 \times 2.\,\text{row} \\
0 & f & -f & 0 & ;\ 2.\,\text{row} + 1.\,\text{row} \\
0 & -f & f & 0 &
\end{array}
$$

$$
\Rightarrow \quad
\begin{array}{ccc|cl}
x_1 & x_2 & x_3 & & \\
\hline
f & -2f & f & 0 & \\
0 & f & -f & 0 & \\
0 & 0 & 0 & 0 & ;\ 2.\,\text{row} + 3.\,\text{row}
\end{array}
$$

With backwards substitution one obtains for the eigenvalue $\omega_2 = 0$ the eigenvector

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
$$

The fact that all three masses experience the same deflection t suggests that the molecule moves as a whole without any oscillation occurring, and it does so independent of the mass ratio μ. Of course, in this case, t can also simply be zero, that means nothing moves at all.

Finally, one inserts the eigenvalue ω_3 into eq. (6.90) and solves the equation system again by using the Gauß-algorithm:

$$
\begin{array}{ccc|c}
x_1 & x_2 & x_3 & \\
\hline
-\frac{2f}{\mu} & -f & 0 & 0 \\
-f & -f\mu & -f & 0 \\
0 & -f & -\frac{2f}{\mu} & 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc|cl}
x_1 & x_2 & x_3 & & \\
\hline
f & f\mu & f & 0 & ;\ -1 \times 2.\,\text{row} \\
0 & f & \frac{2f}{\mu} & 0 & ;\ 1.\,\text{row} - \frac{2}{\mu} \times 2.\,\text{row} \\
0 & -f & -\frac{2f}{\mu} & 0 &
\end{array}
$$

$$\Rightarrow \quad \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline f & f\mu & f & 0 \\ 0 & f & \dfrac{2f}{\mu} & 0 \\ 0 & 0 & 0 & 0 \end{array} \qquad ; \;\; 2.\,\text{row} + 3.\,\text{row}$$

With backwards substitution one obtains for the eigenvalue

$$\omega_3 = \sqrt{f\left(\frac{2}{M} + \frac{1}{m}\right)}$$

the eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ -2/\mu \\ 1 \end{pmatrix}$$

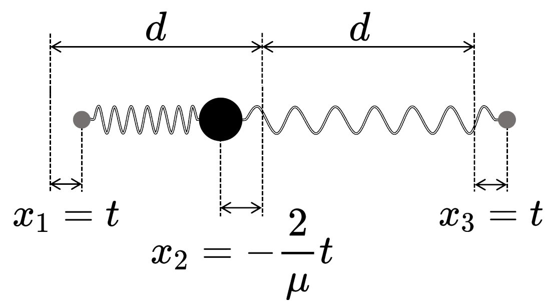

Fig. 6.24

In this case, as sketched in fig. 6.24, the two outer atoms Y oscillate in phase with each other and in anti-phase (out of phase by 90°) to the middle atom X. Here, the amplitude of the deflection of the central atom depends on the mass ratio μ. For example, if $\mu = 1$ then the middle atom oscillates with twice the amplitude of the outer atoms. If $\mu = 2$, the X atom has twice the mass of a Y atom; in which case all three atoms oscillate with the same amplitude however the middle atom oscillates in anti-phase to the outer atoms.

7. Continuum Mechanics

7.1 Elastodynamics

The field of elastostatics (see chapter 3) deals with elastic bodies in situations of force equilibrium. In an equilibrium of forces situation, the respective forces acting on such a body cancel each other out, the result of which is that the body does not move. Elastodynamics considers situations in which this balance of forces is disturbed, and the elastic body is therefore subject to acceleration by a resulting force. The stress tensor (eq. (3.5)), already familiar from elastostatics, is

$$\underline{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{7.1}$$

Displacements or distortions in the direction of the spatially fixed coordinate axes (x, y, z) associated with the effect of an acting force are represented by the point mass coordinates (ξ_1, ξ_2, ξ_3). In order to determine the resultant force in a situation in which the forces acting on an elastic body are not in equilibrium, one considers the situation of the cuboid in fig. 7.1 (also compare with fig. 3.4). Here, the forces acting on the cuboid shall only lead to a non-equilibrium of the stress and shear components in the x-direction, while the stress and shear components in the y- and z-directions shall be unaffected and continue to be in balance. For the resulting force F_x in the x-direction one reads from fig. 7.1

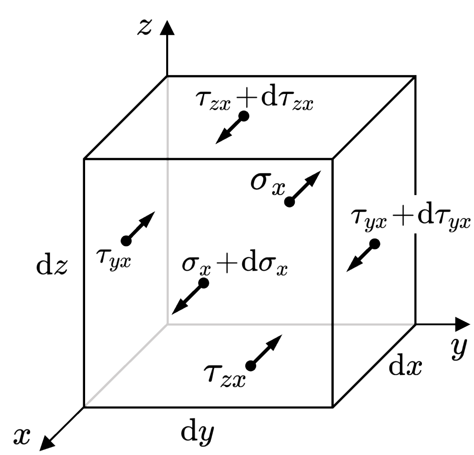

Fig. 7.1

$$F_x = \mathrm{d}\sigma_x \mathrm{d}y\mathrm{d}z + \mathrm{d}\tau_{yx}\mathrm{d}x\mathrm{d}z + \mathrm{d}\tau_{zx}\mathrm{d}x\mathrm{d}y + f_x\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{7.2}$$

In addition to the components of the stress tensor that act on the respective cuboid surfaces, the volume forces already known from eq. (3.4) must also be taken into account. This happens in eq. (7.2) through the factor $f_x\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where f_x is the force density of the elastic body in x-direction. To begin with, the case of force equilibrium shall be considered once more.

The equilibrium case

In the case of equilibrium, the forces that are transferred according to eq. (7.2) from the volume element $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ in x-direction disappear, i.e., $F_x = 0$. With $\tau_{xy} = \tau_{yx}$, it hence follows from eq. (7.2)

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + f_x = 0 \tag{7.3}$$

With x, y, z fixed in space and the point mass coordinates (ξ_1, ξ_2, ξ_3), this can be written in a generalized way as

$$\sum_k \frac{\partial\sigma_{ik}}{\partial\xi_k} + f_i = 0 \quad \text{with} \quad i = 1, 2, 3 \tag{7.4}$$

For the coordinates, $\xi_i = \xi_i(x, y, z, t)$ applies here and for the volume forces $f_i = f_i(x, y, z, t)$. With the discussion of Hooke's law in section 3.3, the components of the stress tensor and of the strain tensor are according to eq. (3.32) linked via the relationship

$$\sigma_{ik} = 2\mathrm{G}\Big[\epsilon_{ik} + \frac{\mu}{1-2\mu}\delta_{ik}\sum_{l=1}^{3}\epsilon_{ll}\Big] \tag{7.5}$$

where for the strain tensor with eq. (3.20) applies

$$\epsilon_{ik} = \frac{1}{2}\Big(\frac{\partial\xi_i}{\partial x_k} + \frac{\partial\xi_k}{\partial x_i}\Big) \tag{7.6}$$

The derivatives of the components of the stress tensor in eq. (7.4) are determined using eq. (7.5) and eq. (7.6) as well as the following relationships

$$\sum_{l=1}^{3}\epsilon_{ll} = \sum_{l=1}^{3}\frac{\partial\xi_l}{\partial x_l} = \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \mathrm{div}\,\boldsymbol{\xi}$$

$$\sum_k \frac{\partial\epsilon_{ik}}{\partial x_k} = \frac{1}{2}\Big(\sum_k \frac{\partial^2\xi i}{\partial x_k^2} + \frac{\partial}{\partial x_i}\mathrm{div}\,\boldsymbol{\xi}\Big) = \frac{1}{2}\Big(\Delta\xi_i + \frac{\partial}{\partial x_i}\mathrm{div}\,\boldsymbol{\xi}\Big)$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial x_k} \delta_{ik} \big(\mathrm{div}\, \boldsymbol{\xi}\big) \;=\; \frac{\partial}{\partial x_i} \big(\mathrm{div}\, \boldsymbol{\xi}\big)$$

Inserted into eq. (7.4) one obtains

$$G\Big[\Delta\xi_i + \frac{1}{1-2\mu}\frac{\partial}{\partial x_i}\big(\mathrm{div}\, \boldsymbol{\xi}\big)\Big] + f_i \;=\; 0 \quad \text{with} \quad i = 1,2,3 \tag{7.7}$$

It is easy to see that

$$\mathrm{div}\boldsymbol{\xi} \;=\; \frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_3}{\partial x_3} \;=\; \epsilon_V$$

describes the (negative or positive) volume expansion of the elastic body. Insertion into eq. (7.7) gives the basic equations of the linear theory of elasticity

$$\Delta\xi_i + \frac{1}{1-2\mu}\frac{\partial \epsilon_V}{\partial x_i} + \frac{1}{G} f_i \;=\; 0 \quad \text{with} \quad i = 1,2,3 \tag{7.8}$$

As the name already suggests, these three equations are linear in the displacement coordinates of the point masses, but they are also linear in the volume forces f_i. Consequently, the principle of superposition applies both to the displacements and to the volume forces. This is of course of great advantage for complicated problems if it is possible to represent them as linear superpositions of problems which are easier to solve. The next step is to consider the case in which there is no equilibrium of forces.

The non-equilibrium case

In the above example (fig. 7.1) this means $F_x \neq 0$ in eq. (7.2). More generally applies for $F_i \neq 0$ $(i = 1,2,3)$ that according to Newton each F_i must equal the accelerated mass times the acceleration component of the strain or respectively compression in the corresponding coordinate direction. That means it must apply

$$F_i \;=\; \rho \mathrm{d}x \mathrm{d}y \mathrm{d}z \frac{\partial^2 \xi_i}{\partial t^2} \tag{7.9}$$

Analogous to the equilibrium case in eq. (7.4) with $F_i = 0$ $(i = 1,2,3)$ this results in the equations:

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} \;=\; f_i + \sum_k \frac{\partial \sigma_{ik}}{\partial x_k} \quad \text{with} \quad i = 1,2,3 \tag{7.10}$$

Hence, instead of the equation eq. (7.7) one obtains

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} \;=\; f_i + G\Big[\Delta\xi_i + \frac{1}{1-2\mu}\frac{\partial}{\partial x_i}\big(\mathrm{div}\, \boldsymbol{\xi}\big)\Big] \quad \text{with} \quad i = 1,2,3 \tag{7.11}$$

7.1.1 Elastic Waves

Here, plane elastic waves in particular shall be considered, such as for example a plane wave traveling in the x-direction with the amplitude $\boldsymbol{\xi}^0 = (\xi_1^0, \xi_2^0, \xi_3^0)$ as given by

$$\boldsymbol{\xi}(x,y,z,t) \;=\; \big(\xi_1, \xi_2, \xi_3\big) \;=\; \mathrm{Re}\Big[\boldsymbol{\xi}^0 \cdot \mathrm{e}^{i(\boldsymbol{k}\boldsymbol{r}-\omega t)}\Big] \tag{7.12}$$

The expression in the exponential function is constant if

$$kx - \omega t = \varphi_0 = const$$

that means

$$x \;=\; \frac{\varphi_0 + \omega t}{k} \;=\; \frac{\varphi_0}{k} + \underbrace{\frac{\omega t}{k}}_{c_p}$$

The following relationships apply to the phase velocity c_p, wavelength λ and time period T of the plane wave:

$$\omega \;=\; c_p k \quad ; \quad \lambda \;=\; \frac{2\pi}{k} \quad ; \quad T \;=\; \frac{2\pi}{\omega} \;=\; \frac{1}{\nu} \;=\; \frac{1}{f}$$

For a sine wave

$$\sin\underbrace{(kx - \omega t)}_{\varphi}$$

with wavelength $\lambda = 2\pi/k$ (fig. 7.2) applies accordingly

$$x \;=\; \frac{\omega}{k}\,t \quad ; \quad \varphi \;=\; const \quad ; \quad \omega \;=\; c_p k$$

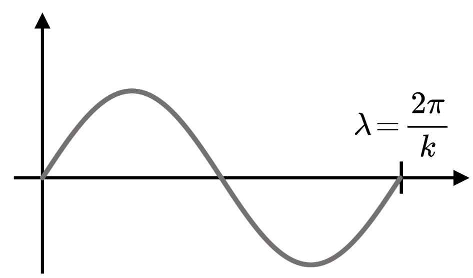

Fig. 7.2

For transverse waves the oscillation $\boldsymbol{\xi}$ is perpendicular to the direction of propagation, i.e., $\boldsymbol{k} \perp \boldsymbol{\xi}$. With a wave propagating in x-direction and the deflection occurring for example in z-direction, i.e., $\boldsymbol{\xi}^0 = (0, 0, \xi_3^0)$, it follows that

$$\mathrm{div}\,\boldsymbol{\xi} \;=\; \frac{\partial \xi_3}{\partial z} \;=\; \frac{\partial}{\partial y}\big(\xi_3^0 \cdot \mathrm{e}^{i(kx-\omega t)}\big) \;=\; 0$$

Eq. (7.12) with these parameters inserted into eq. (7.11) returns

$$-\rho\omega^2\xi_3^0 \;=\; f_i - Gk^2\xi_3^0$$

In the case of $f_i = 0$ it follows

$$\omega^2 \;=\; \frac{G}{\rho} \cdot k^2 \;=\; c_t^2 \cdot k^2$$

where

$$c_t = \sqrt{c_{p,t}^2} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2\rho(\mu+1)}} \tag{7.13}$$

is the phase velocity of the transverse wave and for the last step the relationships from eq. (3.29) were used. For longitudinal waves the oscillation $\boldsymbol{\xi}$ is parallel to the direction of propagation, i.e., $\boldsymbol{k} \parallel \boldsymbol{\xi}$. For a longitudinal wave traveling in x-direction we have $\boldsymbol{\xi}^0 = (\xi_1^0, 0, 0)$ and eq. (7.12) with this parameter inserted into eq. (7.11) returns

$$-\rho\omega^2\xi_1^0 = f_i - 2G\frac{1-\mu}{1-2\mu}k^2\xi_1^0$$

In the case of $f_i = 0$ it follows

$$\omega^2 = \frac{2G}{\rho}\frac{1-\mu}{1-2\mu}k^2 = c_l^2 \cdot k^2$$

where

$$c_l = \sqrt{c_{p,l}^2} = \sqrt{\frac{2G}{\rho}\frac{1-\mu}{1-2\mu}} = \sqrt{\frac{E}{\rho}\frac{1-\mu}{(1+\mu)(1-2\mu)}} \tag{7.14}$$

is the phase velocity of the longitudinal wave. For the last step, the relationships from eq. (3.29) were used again.

The considerations made here with respect to the propagation of transverse and longitudinal waves in elastic bodies assume that the wave propagation takes place in an infinite medium. In practice, however, this is not the case. If the wavelength of a longitudinal wave is comparable to or greater than the transverse dimension of the medium, for example a thin rod, then something happens at the edge of the rod in which the longitudinal wave propagates. The lateral limitation of the rod causes a transverse contraction which is not prevented by surrounding matter, as is the case with an infinite medium. The density modulation caused by a longitudinal wave correlates with the stretching / compression in the x-direction.

$$\epsilon_x = \frac{1}{E}\sigma_x = \frac{\partial\xi_x}{\partial x}$$

From that, analogously to the previous treatment for an infinitely extended medium, the equation of motion follows

$$\rho\frac{\partial^2\xi_x}{\partial t^2} = \frac{\partial\sigma_x}{\partial x} = E\frac{\partial^2\xi_x}{\partial x^2}$$

Insertion of a longitudinal wave

$$\xi_x = \xi_x^0 \mathrm{e}^{i(kx-\omega t)}$$

into the equation of motion gives for the phase velocity c_l^{rod} in the rod:

$$c_l^{rod} = \sqrt{\frac{E}{\rho}}$$

From the derivation of Poisson's number μ (see eq. (3.28)) one can see that μ can assume values between 0 and 0.5. This means that the phase velocity of the longitudinal wave in a thin rod is lower than the corresponding phase velocity in the infinite medium from eq. (7.14).

7.2 Hydrodynamics

In hydrodynamics, one can approach the description of flow processes in two possible ways by either using the method of Lagrange or Euler's method. With the Lagrange method, one describes the trajectory curves of point masses by their position vectors, i.e., $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t)$ where $\boldsymbol{r}_0$ is the known position vector of a point mass at time t_0. Velocities and accelerations of point masses are then obtained as usual by taking the time derivative once or twice, respectively. In the case of flow processes, however, it is often not so much of interest where a point mass is located, but what the velocity field looks like in which the point mass is moving.

Euler description

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t) \quad ; \quad \rho = \rho(\boldsymbol{x}, t)$$

Here $\boldsymbol{v}$ is the velocity field, ρ is the local density and $\boldsymbol{x}$ is a space-fixed coordinate. In the Euler description, physical quantities $a(\boldsymbol{x}, t)$, such as $\rho(\boldsymbol{x}, t)$, are each considered at a fixed point in space: The respective point in space is

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\xi}(\boldsymbol{x}_0, t)$$

where the $\boldsymbol{x}_0$ are Lagrangian coordinates (as in the elastic equations). A point mass that moves along a trajectory curve $\boldsymbol{x}$ is at every point in space both under the influence of the physical quantity $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$ at the respective point in space, i.e., it registers the local strength or weakness of $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$, but also experiences the rate of change of $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$ at the respective location. The rate of change of a physical quantity $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$ is therefore given by the total derivative with respect to time, often also called the substantial derivative[1]:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \underset{\substack{\downarrow \\ \text{local} \\ \text{part}}}{\frac{\partial a}{\partial t}} + \underbrace{\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial a}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial a}{\partial z}\cdot\frac{\partial z}{\partial t}}_{\substack{\text{convective part} \\ \text{(with fixed } \boldsymbol{x}_0)}}$$

This means that for every $a(\boldsymbol{x}, t)$ applies the relationship

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\partial a}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \frac{\partial a}{\partial t} + \boldsymbol{v}\cdot\operatorname{grad} a \tag{7.15}$$

The substantial derivative of a physical quantity $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$ consists of two parts, the local change of this physical quantity at a given point in space and a so-called convective part of $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$. The convective part can be imagined as a set of markers distributed all over the elastic body, which at each point in space $\boldsymbol{x}_0$ indicate the magnitude of $a\big(\boldsymbol{x}(\boldsymbol{x}_0, t), t\big)$.

Sometimes a different notation of the substantive derivative is useful which makes use of the relationship

$$\big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\big)\boldsymbol{a} = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{a}^2 - \big[\boldsymbol{a}\times\big(\boldsymbol{\nabla}\times\boldsymbol{a}\big)\big]$$

(as to the proof of this so-called Weber transform see Appendix). With that the substantial derivative becomes

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\partial a}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} a = \frac{\partial a}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{a}^2 - \big[\boldsymbol{a}\times\big(\boldsymbol{\nabla}\times\boldsymbol{a}\big)\big] \tag{7.16}$$

7.2.1 Equation of Continuity

I) **Derivation according to Lagrange:**

The mass of a fluid element is $\Delta m = \rho\Delta V$. From the conservation of mass follows

$$\frac{\mathrm{d}\Delta m}{\mathrm{d}t} = 0 = \frac{\mathrm{d}\rho}{\mathrm{d}t}\Delta V + \rho\frac{\mathrm{d}\Delta V}{\mathrm{d}t} \tag{7.17}$$

Replacing in eq. (3.21) V with ΔV and ξ_i with $\mathrm{d}\xi_i$ shows that

$$\frac{\mathrm{d}\Delta V}{\Delta V} = \operatorname{div}(\mathrm{d}\boldsymbol{\xi})$$

[1]The term substantial derivative has a purely physical meaning. It expresses the fact that the substantial derivative describes the change in a physical quantity along the motion of the substance itself, i.e., the moving fluid or gas.

Using this relationship, one can rewrite eq. (7.17) to become

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\Delta V + \rho\Delta V \mathrm{div}\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) \; = \; 0$$

From this follows with $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t = \boldsymbol{v}$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\cdot\mathrm{div}\,\boldsymbol{v} \; = \; 0$$

Using the relationship from eq. (7.15), one can rewrite this as

$$\frac{\partial\rho}{\partial t} + \big(\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho + \rho\cdot\boldsymbol{\nabla}\boldsymbol{v}\big) \; = \; 0$$

and thus obtain the continuity equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}(\rho\boldsymbol{v}) \; = \; 0 \tag{7.18}$$

II) **Derivation without Lagrange using the theorem of Gauß:**

As sketched in fig. 7.3, the considered fluid mass shall be given by

$$m \; = \; \int\limits_V \rho \mathrm{d}V$$

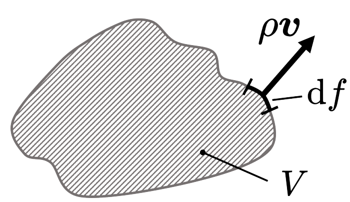

Fig. 7.3

For the mass loss or respectively mass gain of the volume V then applies

$$\frac{\partial m}{\partial t} \; = \; \int\limits_V \frac{\partial\rho}{\partial t}\mathrm{d}V \tag{7.19}$$

To calculate the mass loss or respectively the mass gain per unit of time through the area A_V enclosing the volume V, one only requires the normal component of the flow $\rho\boldsymbol{v}$, i.e., $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} \; = \; v_n$:

$$\frac{\partial m}{\partial t} - \oiint\limits_{A_V} \rho\cdot v_n\cdot\mathrm{d}f \; = \; 0 \tag{7.20}$$

According to the theorem of Gauß, for the surface integral in eq. (7.20) applies

$$\oiint\limits_{A_V} \rho\cdot v_n\cdot\mathrm{d}f \; = \; \int\limits_V \mathrm{div}\big(\rho\boldsymbol{v}\big)\mathrm{d}V$$

and with that also applies

$$\frac{\partial m}{\partial t} - \int\limits_V \operatorname{div}\big(\rho \boldsymbol{v}\big)\mathrm{d}V \;=\; 0 \tag{7.21}$$

A comparison of eq. (7.19) and eq. (7.21) shows that it must hold

$$\int\limits_V \left(\frac{\partial \rho}{\partial t} + \operatorname{div}\big(\rho \boldsymbol{v}\big)\right)\mathrm{d}V \;=\; 0$$

This can only be true if the integrand vanishes, which means nothing other than that the continuity equation eq. (7.17)

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\big(\rho \boldsymbol{v}\big) \;=\; 0$$

must apply.

7.2.2 Streamlines and Stream Function

To describe flow processes, hydrodynamics distinguishes between streamlines, pathlines (i.e., trajectory lines or curves), streaklines and timelines.

Definition
Pathlines (trajectories): These describe the path of a point mass or respectively of a liquid particle in a flow.

Pathlines can easily be made visible by adding, for example, small cork particles to the flowing liquid.

Definition
Streamlines: Designate curves whose tangents at each curve point are identical to the respective velocity vectors $\boldsymbol{v}$ of the flow.

If one takes a snapshot of the flowing liquid at a point in time $t = t_0$ then the streamlines give an image of the velocity field of the flow.

Definition
Streaklines: Designate curves of point masses or respectively liquid particles that all run through a common point in space.

A simple example for streaklines is the deployment of buoys at a fixed location in a liquid flow. If one connects the positions of the successively deployed buoys, one gets the streakline.

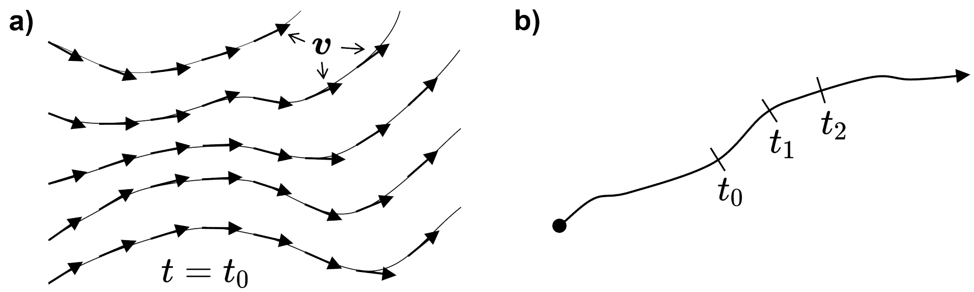

Fig. 7.4: (a) Streamlines provide a snapshot of the velocity field. The length of the $\boldsymbol{v}$-vectors at each tangent point (here drawn with the same length) depends on the local flow. (b) A pathline (trajectory) describes the path of a single particle as a function of time.

Definition

Timelines: Denote the connecting line between the current locations of particles whose locations were already known at an earlier point in time.

To illustrate timelines, in the simplest case, for example, light-weight sample particles are placed in the flow at different positions at the same time t_0. At each later point in time t, the connecting line between the current positions of the sample particles then represents the timeline at the point in time t.

In the present context, the streamlines are of particular interest. Pathlines represent the Lagrange description of a flow (fig. 7.4a) while streamlines correspond to the Euler description of a flow (fig. 7.4b). In the case of a stationary flow, i.e., $\dot{\boldsymbol{v}} = 0$, the flow velocity is a time invariant (i.e., it is constant) at every location and thus streamline and pathline are identical.

In other words, a flow is always stationary if a sample particle, which is put into the liquid flow at different times but always at the same position of the flow, will move on the same pathline each time. If this is not the case, then one speaks of a so-called non-stationary flow. In the case of a non-stationary flow, it is no longer possible to determine from a given pathline the flow velocity present at a specific point in time. That, however, means nothing more than that for non-stationary flows, streamline and pathline are no longer identical.

In mathematical terms, the streamline definition given above states that the change in the position vector of the streamline curve at any point on the curve is always parallel to the velocity vector of the flow at that point. Hence, in every point of the streamline the vector product of the velocity field $\boldsymbol{v}$ and the infinitesimal change $\mathrm{d}\boldsymbol{r}$ of the position vector $\boldsymbol{r}$ of the streamline curve must vanish:

$$\boldsymbol{v} \times \mathrm{d}\boldsymbol{r} = 0 \tag{7.22}$$

To understand what, in the case of incompressible liquids and planar stationary flows, distinguishes two adjacent streamlines from one another, one considers the so-called stream function Ψ, also referred to as Stokes' stream function. If, for example, the coordinate system is chosen in such a way that the unit vector $\hat{\boldsymbol{e}}_z$ in z-direction is at the same time the normal vector of the plane through which the flow occurs (flow in the xy-plane), then Ψ is defined by the condition

$$\boldsymbol{v} = \boldsymbol{\nabla} \times \Psi\hat{\boldsymbol{e}}_z = \boldsymbol{\nabla}\Psi \times \hat{\boldsymbol{e}}_z \tag{7.23}$$

$$= \left(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}, \frac{\partial\Psi}{\partial z}\right) \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}, 0\right) = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix}$$

Hence, the velocity field $\boldsymbol{v}$ can be calculated from the stream function Ψ. It immediately follows from the definition of Ψ that

$$\boldsymbol{\nabla}\Psi \cdot \boldsymbol{v} = \boldsymbol{\nabla}\Psi \cdot \big(\boldsymbol{\nabla}\Psi \times \hat{\boldsymbol{e}}_z\big) = 0 \tag{7.24}$$

In other words: Along lines whose tangential vectors are parallel to the respective velocity vectors of the velocity field at every point, the stream function is always constant. This also follows directly by inserting eq. (7.23) into the definition of the streamline given in eq. (7.22). For a flow in the xy- plane with $\mathrm{d}\boldsymbol{r} = (\mathrm{d}x, \mathrm{d}y, 0)$ this results in (Graßmann identity)

$$\boldsymbol{v} \times \mathrm{d}\boldsymbol{r} = 0 = \big(\boldsymbol{\nabla}\Psi \times \hat{\boldsymbol{e}}_z\big) \times \mathrm{d}\boldsymbol{r} = \big(\boldsymbol{\nabla}\Psi \cdot \mathrm{d}\boldsymbol{r}\big) \cdot \hat{\boldsymbol{e}}_z - \big(\underbrace{\hat{\boldsymbol{e}}_z \cdot \mathrm{d}\boldsymbol{r}}_{=\,0}\big) \cdot \boldsymbol{\nabla}\Psi$$

Therefore, to streamlines must apply

$$\boldsymbol{\nabla}\Psi \cdot \mathrm{d}\boldsymbol{r} = \frac{\partial\Psi}{\partial x}\mathrm{d}x + \frac{\partial\Psi}{\partial y}\mathrm{d}y = \mathrm{d}\Psi = 0$$

The stream function is constant along a streamline and streamlines are thus the contour lines (equipotential lines) of the stream function. That means of course that each streamline corresponds to a specific value of the stream function Ψ. For a better understanding of Ψ, the so-called specific discharge q shall be considered in the two-dimensional case (planar flow).

The specific discharge q is defined as the “liquid area” which per unit of time flows through the normal area in fig. 7.5 between the two streamlines $\Psi = \Psi_A$ and $\Psi = \Psi_B$. Hence

$$q = \int_A^B \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s$$

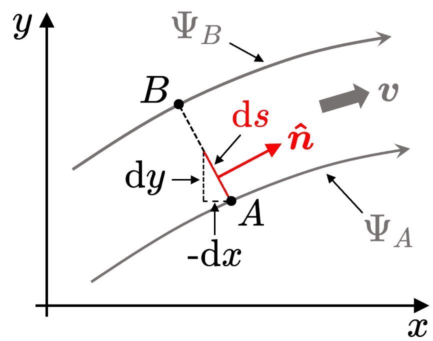

Fig. 7.5

where $\mathrm{d}s$ is the infinitesimal line element along the connecting segment from A to B and $\hat{\boldsymbol{n}}$ is the normal unit vector on this line element. In fig. 7.5, for the sake of simplicity, this connecting line is chosen as a straight line, but it can have any arbitrary shape. From fig. 7.5 one can read that

$$\hat{\boldsymbol{n}}\mathrm{d}s = \left(\mathrm{d}y\,, -\mathrm{d}x\right)$$

Using eq. (7.23), this gives for q

$$q = \int_A^B \left(v_x\mathrm{d}y - v_y\mathrm{d}x\right) = \int_A^B \mathrm{d}\Psi = \Psi_B - \Psi_A \tag{7.25}$$

With $\Psi_B = \mathit{const}$ and $\Psi_A = \mathit{const}$, the difference q is also constant. For incompressible liquids in a stationary flow the specific discharge is therefore constant. From that follows: If the distance between the streamlines $\Psi = \Psi_A$ and $\Psi = \Psi_B$ decreases / increases, then the flow velocity between the two streamlines increases / decreases. Another consequence of eq. (7.23) is

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\Psi\hat{\boldsymbol{e}}_z\right) = 0 \tag{7.26}$$

From this it follows that the liquid flow contains neither sources nor sinks. This means that streamlines can neither begin nor end inside the liquid under consideration but must be closed curves or run along the edge of the flow. Furthermore, it can be read from eq. (7.25) that the specific discharge vanishes through an area whose boundary consists of a streamline and a connecting line that begins and ends on the streamline ($\Psi_B = \Psi_A$). This applies in particular to a “connecting line” that runs along the streamline itself. That means, the specific discharge through a streamline is always zero. Streamlines act like impenetrable walls, which means that since there is no flow through a streamline itself, the flow rate between any two streamlines must vary with the distance between those streamlines.

7.2.3 Momentum Balance

An important concept of hydrodynamics is the flux tube, also known as streamline bundle (fig. 7.6). Here one first considers a so-called streamline surface, which is the surface that is spanned by the sum of all streamlines that go through a stationary line. If this stationary line happens to be a closed curve, then the respective streamline surface forms the mantle of a tube, i.e., of the flux tube. The surface normal vector of the entry or respectively exit area of the flux tube is

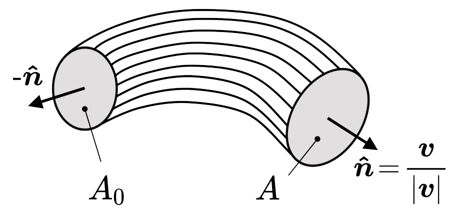

Fig. 7.6

$$\hat{\boldsymbol{n}} = \pm\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

In a stationary flow, the position and shape of a flux tube does not change with time. The following continuity condition applies

$$\rho_0 \cdot v_0 \cdot A_0 = \rho \cdot v \cdot A \tag{7.27}$$

Hence, the same mass must flow through each cross-sectional area of such a flux tube per unit of time. If the course of the flux tube is parameterized in such a way that the line element ds runs through the centroid of each cross-sectional area of the flux tube, then eq. (7.27) becomes

$$\rho_0 \cdot v \cdot A = \rho \cdot A \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = \rho \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t} = const$$

To derive the continuity condition in the non-stationary flow case, one uses the conservation of mass and Newton's second law for incremental changes Δm and $\Delta \boldsymbol{p}$ of mass and momentum. It must apply:

$$\frac{\mathrm{d}(\Delta m)}{\mathrm{d}t} = 0 \qquad \text{(mass)}$$

$$\frac{\mathrm{d}(\Delta \boldsymbol{p})}{\mathrm{d}t} = \Delta \boldsymbol{F} \qquad \text{(momentum)}$$

Therefore, with eq. (7.9) and eq. (7.10), it applies component-wise

$$\frac{\mathrm{d}(\Delta p_i)}{\mathrm{d}t} = \Delta F_i = f_i \Delta V + \left(\sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k}\right)\Delta V$$

$$= \frac{\mathrm{d}(\Delta m)}{\mathrm{d}t} v_i + \Delta m \frac{\mathrm{d}v_i}{\mathrm{d}t}$$

It follows with $\rho = \Delta m/\Delta V$

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} \;=\; f_i + \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} \tag{7.28}$$

But with eq. (7.15) it must also apply

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} \;=\; \rho \left(\frac{\partial v_i}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} v_i \right) \tag{7.29}$$

Since the left sides of eq. (7.28) and eq. (7.29) are identical, the right sides must also be identical, i.e.,

$$f_i + \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} \;=\; \rho \frac{\partial v_i}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} v_i \tag{7.30}$$

Now one rewrites the continuity equation, i.e., eq. (7.18), to read

$$0 \;=\; \frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial (\rho v_k)}{\partial x_k}$$

If one multiplies this equation by v_i and then adds the right side of it as a zero to the right side of eq. (7.30), then one obtains

$$f_i + \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} \;=\; \frac{\partial (\rho v_i)}{\partial t} + \sum_{k=1}^{3} \frac{\partial (\rho v_i v_k)}{\partial x_k}$$

or respectively rewritten

$$f_i \;=\; \underset{\substack{\downarrow \\ \text{momentum} \\ \text{density}}}{\frac{\partial (\rho v_i)}{\partial t}} \;+\; \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \underbrace{\Big(\rho v_i v_k - \sigma_{ik} \Big)}_{\substack{\text{momentum flux} \\ \text{density} \;=\; \Pi_{ik}}} \tag{7.31}$$

A) Microscopic examination of pressure

$$\sigma_{ik} \;=\; -P \delta_{ik} \qquad ; \qquad \sigma_{ik} \;=\; - \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

Assertion: For the momentum flux density applies

$$\Pi_{ik} \;=\; \rho\langle(v_i + u_i)(v_k + u_k)\rangle$$

$\langle a\rangle$ is the mean value of a over the thermal motion of an ensemble of atoms or molecules and u_i is the velocity component associated with this thermal motion for which applies $\langle u_i\rangle = 0$. Furthermore, if $i \neq k$, then

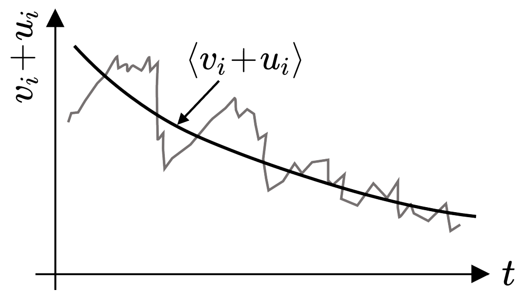

Fig. 7.7

$$\langle u_i u_k\rangle \;=\; \langle u_i\rangle\langle u_k\rangle$$

and therefore $\langle u_i u_k\rangle = 0$ if $i \neq k$. In addition, the so-called equipartition theorem applies

$$\frac{m}{2}\langle u_i^2\rangle \;=\; \frac{k_B T}{2}$$

This means that for Π_{ik}

$$\Pi_{ik} \;=\; \rho v_i v_k + \rho\langle u_i u_k\rangle \;=\; \rho v_i v_k + \rho\frac{k_B T}{m}\delta_{ik}$$

From the equation of state of the ideal gas one obtains

$$P\cdot V \;=\; R\cdot T \quad ; \quad k_B \;=\; \frac{R}{N_A} \quad \rightarrow \quad P \;=\; \frac{N_A}{V}\frac{m}{m}k_B T \;=\; \frac{\rho}{m}k_B T$$

and hence

$$\Pi_{ik} \;=\; \rho v_i v_k + P\delta_{ik} \tag{7.32}$$

B) Force impact on pipes

The stationary case is considered, i.e., in eq. (7.31)

$$\frac{\partial\rho v_i}{\partial t} \;=\; \frac{\partial\rho}{\partial t} \;=\; 0$$

and $f_i = 0$. With that must hold

$$\sum_{k=1}^{3}\frac{\partial\Pi_{ik}}{\partial x_k} \;=\; 0 \qquad (*)$$

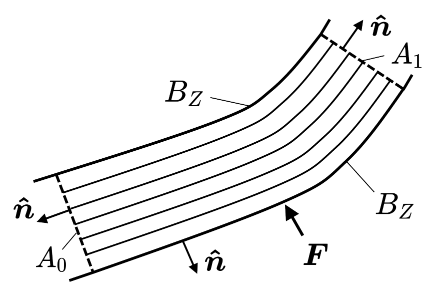

Fig. 7.8

The force on the cylindrical pipe wall is B_Z

$$\boldsymbol{F} \;=\; \iint\limits_{B_Z} \underline{\underline{\boldsymbol{\sigma}}}\,\hat{\boldsymbol{n}}\,\mathrm{d}A$$

where $\hat{\boldsymbol{n}}$ is the normal unit vector of the pipe wall and $\mathrm{d}A$ is the corresponding differential surface element. For an indirect calculation of $\boldsymbol{F}$ one considers the momentum flux density for the pipe volume. With the theorem of Gauß applies in general

$$\iiint \mathrm{div}_k \Pi_{i(k)} \mathrm{d}V \;=\; \oiint \sum_{k=1}^{3} \Pi_{i(k)} n_k \mathrm{d}A$$

However, according to the presumption in ($*$)

$$\iiint \sum_{k=1}^{3} \frac{\partial \Pi_{ik}}{\partial x_k} \mathrm{d}V \;=\; 0 \;=\; \iiint \mathrm{div}_k \Pi_{i(k)} \mathrm{d}V$$

and with that applies

$$\oiint \sum_{k=1}^{3} \Pi_{i(k)} n_k \mathrm{d}A \;=\; 0$$

For the total force on the walls of the considered pipe volume, i.e., including the forces on the two cross-sectional areas that limit the volume in the pipe at its two respective ends, one has

$$\iint\limits_{B_Z} + \iint\limits_{B_{A_0}} + \iint\limits_{B_{A_1}} \;=\; \boldsymbol{F} + \iint\limits_{B_{A_0}} + \iint\limits_{B_{A_1}} \;=\; \oiint \sum_{k=1}^{3} \Pi_{i(k)} n_k \mathrm{d}A \;=\; 0$$

From this it follows for the force impact on the pipe in fig. 7.8

$$\boldsymbol{F_i} = (\rho v_i v + P)_1 \cdot A_1 - (\rho v_i v + P)_0 \cdot A_0 \tag{7.33}$$

7.2.4 Bernoulli Equation

From the relationships

$$\sigma_{ik} \;=\; -P\delta_{ik} \qquad \text{and} \qquad \sum_k \frac{\partial \sigma_{ik}}{\partial x_k} \;=\; -\frac{\partial P}{\partial x_i} \;=\; -(\mathrm{grad}\, P)_i$$

follows with eq. (7.31) the Euler equation of hydrodynamics in the form

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;=\; \boldsymbol{f} - \mathrm{grad} P \;=\; \boldsymbol{f} - \boldsymbol{\nabla} P \tag{7.34}$$

or respectively rewritten also in the forms

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;=\; \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big) \boldsymbol{v} \;=\; \boldsymbol{f} - \boldsymbol{\nabla} P \tag{7.35}$$

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;=\; \rho \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\rho}{2} \boldsymbol{\nabla} \boldsymbol{v}^2 - \rho \big[\boldsymbol{v} \times \big(\boldsymbol{\nabla} \times \boldsymbol{v}\big)\big] \;=\; \boldsymbol{f} - \boldsymbol{\nabla} P \tag{7.36}$$

As can be seen from the derivation of the Euler equation, frictional forces, which often play a significant role in practical cases, are not taken into account. The Euler equation only considers the flow of ideal incompressible liquids. However, without viscosity and without thermal conductivity, there are no energy losses in ideal liquids. The motion of an ideal liquid is therefore everywhere adiabatic. That means that any change in state of such a liquid takes place without heat transfer. This in turn means nothing other than that the entropy S of each liquid element is a constant of motion. With s as the entropy per unit mass, it therefore applies

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \qquad \text{where} \qquad s = \frac{S}{M}$$

From this follows with eq. (7.15) the adiabatic equation, describing the entropy change of volume elements in moving ideal liquids or respectively ideal gases

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}s = 0 \tag{7.37}$$

With eq. (7.18) the continuity equation for entropy follows from eq. (7.37)

$$\frac{\partial(\rho s)}{\partial t} + \boldsymbol{\nabla}\cdot\big(\rho s\boldsymbol{v}\big) = 0 \tag{7.38}$$

Eq. (7.37) refers to moving volume elements of a liquid or a gas. That means that between any two arbitrary chosen volume elements the entropy can certainly be different. However, if this is not the case and if the entropy S in all volume elements of the liquid or respectively the gas under consideration has the same value at a point in time t_0, then the entropy S of the liquid or the gas will not change for times $t > t_0$. Hence it must apply, if

$$S(\boldsymbol{r}, t_0) = \mathit{const} \quad \Rightarrow \quad S(\boldsymbol{r}, t > t_0) = \mathit{const} \quad \forall\, t > t_0$$

In the case of $S(\boldsymbol{r}, t_0) = \mathit{const}$ one then speaks of an isentropic motion. The Euler equations (eq. (7.34) or respectively eq. (7.35) or eq. (7.36)) then can be simplified by making use of the enthalpy H. To achieve that one uses the differential form of the enthalpy equation in the natural variables entropy S and pressure P

$$\mathrm{d}H(S,P) = T\cdot\mathrm{d}S + V\cdot\mathrm{d}P$$

In relation to a unit mass, i.e., with $h = H/M$ and $s = S/M$, this equation reads

$$\mathrm{d}h = T\cdot\mathrm{d}s + \frac{V}{M}\cdot\mathrm{d}P = T\cdot\mathrm{d}s + \frac{1}{\rho}\cdot\mathrm{d}P$$

In the isentropic case $\mathrm{d}s = 0$ and the equation simplifies to the relationship

$$\mathrm{d}h = \frac{1}{\rho}\cdot \mathrm{d}P$$

and hence

$$\boldsymbol{\nabla} h = \frac{1}{\rho}\cdot \boldsymbol{\nabla} P \tag{7.39}$$

Inserting this into the Euler equation eq. (7.34) results in the following new form of the Euler equation

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{f} - \rho\boldsymbol{\nabla} h \tag{7.40}$$

The same applies to the forms of the Euler equation given in eq. (7.35) or respectively eq. (7.36). If one takes the curl of eq. (7.36), then one obtains the so-called Euler equation for isentropic motion. The simplified form of eq. (7.36) is

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} - \rho\big[\boldsymbol{v}\times\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\big] = \boldsymbol{f} - \rho\boldsymbol{\nabla}\left(h - \frac{\boldsymbol{v}^2}{2}\right) \tag{7.41}$$

Eq. (7.41) multiplied by $\boldsymbol{\nabla}\times$ from the left yields

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{v}) - \boldsymbol{\nabla}\times\big[\boldsymbol{v}\times\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\big] = \frac{1}{\rho}\boldsymbol{\nabla}\times\boldsymbol{f} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\left(h - \frac{\boldsymbol{v}^2}{2}\right)$$

In the case that $\boldsymbol{f}$ is either the gradient of a potential, e.g., as with the force of gravity and the gravitational potential, or respectively if there are no external forces at all ($\boldsymbol{f} = 0$) it follows because of $\boldsymbol{\nabla}\times\boldsymbol{\nabla} = 0$ that

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{v}) = \boldsymbol{\nabla}\times\big[\boldsymbol{v}\times\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\big] \tag{7.42}$$

This is Euler's equation for isentropic motion in which the velocity field $\boldsymbol{v}$ is the only remaining quantity. If we now consider a stationary flow, i.e., $\dot{\boldsymbol{v}} = 0$, then with eq. (7.36) and eq. (7.39)

$$\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{v}^2 - \big[\boldsymbol{v}\times\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\big] = \frac{1}{\rho}\boldsymbol{f} - \boldsymbol{\nabla} h \tag{7.43}$$

If one substitutes gravity for the external force $\boldsymbol{F}$, then with the respective gravitational potential $U = gz$ applies

$$\boldsymbol{F} = -m\cdot\boldsymbol{\nabla} U = -m\cdot\boldsymbol{\nabla}(gz)$$

or respectively for the force density of the elastic body, i.e., for the liquid volume $\widetilde{V}$ under consideration follows

$$\boldsymbol{f} = \frac{\boldsymbol{F}}{\widetilde{V}} = -\frac{m}{\widetilde{V}} \cdot \boldsymbol{\nabla} U = -\rho \boldsymbol{\nabla} U$$

With that, eq. (7.43) becomes

$$\boldsymbol{\nabla}\left(\frac{\boldsymbol{v}^2}{2} + h + U\right) = \boldsymbol{v} \times \left(\boldsymbol{\nabla} \times \boldsymbol{v}\right) \tag{7.44}$$

If one now considers a specific streamline, then $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$ applies to this streamline and therefore

$$\frac{\boldsymbol{v}^2}{2} + h + U = const \qquad \text{(Bernoulli equation)} \tag{7.45}$$

It should be noted here that the value of the constant is different for different streamlines. The Bernoulli equation is therefore always only valid for an individual streamline. With eq. (7.39) and $U = g \cdot z$ one obtains the Bernoulli equation in the more familiar form

$$\frac{\rho \boldsymbol{v}^2}{2} + P + \rho g z = const \tag{7.46}$$

In this equation, the first term is the so-called dynamic pressure and P is the static pressure.

7.2.5 Energy Balance

The force connected to the gravitational potential $U = gz$ is

$$\boldsymbol{F} = -m \cdot \boldsymbol{\nabla} U = -m \cdot \boldsymbol{\nabla}(gz)$$

From this follows for the force density of the elastic body, i.e., the volume of liquid under consideration $\widetilde{V}$

$$\boldsymbol{f} = \frac{\boldsymbol{F}}{\widetilde{V}} = -\frac{m}{\widetilde{V}} \cdot \boldsymbol{\nabla} U$$

Substituting into Eq. (7.34), the Euler equation of hydrodynamics, one obtains

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\rho \cdot \boldsymbol{\nabla} U - \boldsymbol{\nabla} P \tag{7.47}$$

and by multiplying this equation from the left by $\boldsymbol{v}$ and using

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}^2 \;=\; 2\boldsymbol{v}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$$

eq. (7.47) becomes

$$\frac{\rho}{2}\frac{\mathrm{d}\boldsymbol{v}^2}{\mathrm{d}t} \;=\; -(\rho\boldsymbol{v})\boldsymbol{\nabla}U - \boldsymbol{v}\cdot\boldsymbol{\nabla}P \tag{7.48}$$

Now, in the next step, one first considers the left side (denoted by LS) and the right side (denoted by RS) of eq. (7.48) separately. With the help of eq. (7.15) one can rewrite the left side of eq. (7.48) as

$$\mathrm{LS} \;=\; \frac{\rho}{2}\left(\frac{\partial\boldsymbol{v}^2}{\partial t} + \boldsymbol{v}\boldsymbol{\nabla}\boldsymbol{v}^2\right) \tag{7.49}$$

Now one uses the relationship

$$\boldsymbol{\nabla}\big(\rho\boldsymbol{v}\boldsymbol{v}^2\big) \;=\; \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v})\boldsymbol{v}^2 + \rho\boldsymbol{v}\boldsymbol{\nabla}\boldsymbol{v}^2$$

which can be rewritten with the help of the continuity equation

$$\boldsymbol{\nabla}\big(\rho\boldsymbol{v}\boldsymbol{v}^2\big) \;=\; -\frac{\partial\rho}{\partial t}\boldsymbol{v}^2 + \rho\boldsymbol{v}\boldsymbol{\nabla}\boldsymbol{v}^2$$

Hence one has

$$\rho\boldsymbol{v}\boldsymbol{\nabla}\boldsymbol{v}^2 \;=\; \frac{\partial\rho}{\partial t}\boldsymbol{v}^2 + \boldsymbol{\nabla}\big(\rho\boldsymbol{v}\boldsymbol{v}^2\big)$$

If one inserts this in eq. (7.48) one obtains for the left side of eq. (7.48)

$$\mathrm{LS} \;=\; \frac{\rho}{2}\frac{\partial\boldsymbol{v}^2}{\partial t} + \frac{1}{2}\frac{\partial\rho}{\partial t}\boldsymbol{v}^2 + \frac{1}{2}\boldsymbol{\nabla}\big(\rho\boldsymbol{v}\boldsymbol{v}^2\big)$$

which ultimately results for the left side of eq. (7.48) to become

$$\mathrm{LS} \;=\; \frac{\partial}{\partial t}\Big(\frac{\rho}{2}\boldsymbol{v}^2\Big) + \boldsymbol{\nabla}\big(\frac{\rho}{2}\boldsymbol{v}\boldsymbol{v}^2\big)$$

Next one considers the right side (RS) of eq. (7.48). With the relationships

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\rho\boldsymbol{v}U\big) \;=\; (\rho\boldsymbol{v})\cdot\boldsymbol{\nabla}U + U\cdot\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v}) \;=\; (\rho\boldsymbol{v})\cdot\boldsymbol{\nabla}U - \frac{\partial\rho}{\partial t}\cdot U$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\big(P\boldsymbol{v}\big) \;=\; \boldsymbol{v}\cdot\boldsymbol{\nabla}P + P\cdot\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}$$

the right side of eq. (7.48) can be transformed into

$$\mathrm{RS} \;=\; -\left(\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\rho\boldsymbol{v}U\big) + \frac{\partial\rho}{\partial t}\cdot U\right) - \big(\boldsymbol{\nabla}\boldsymbol{\cdot}\big(P\boldsymbol{v}\big) - P\cdot\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}\big)$$

Inserting the transformed right and left sides of eq. (7.48) into the original equation eq. (7.48) yields

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{2}\boldsymbol{v}^2\Big) + \boldsymbol{\nabla}\big(\frac{\rho}{2}\boldsymbol{v}\boldsymbol{v}^2\big) + \bigg(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}U) + \frac{\partial\rho}{\partial t}\cdot U\bigg) + \big(\boldsymbol{\nabla}\cdot(P\boldsymbol{v}) - P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v}\big) \;=\; 0$$

Now one pulls together the respective time derivatives and spatial derivatives (U not a function of time)

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho U\Big) + \boldsymbol{\nabla}\cdot\Big[\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho U + P\Big)\boldsymbol{v}\Big] - P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; 0 \tag{7.50}$$

The last term in eq. (7.50), $P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v}$, can be reshaped into

$$P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; \frac{P}{\rho}\big(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) - \boldsymbol{v}\boldsymbol{\nabla}\rho\big) \;=\; \frac{P}{\rho}\bigg[-\frac{\partial\rho}{\partial t} - \bigg(\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\partial\rho}{\partial t}\bigg)\bigg]$$

Hence

$$P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; -\frac{P}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \;=\; \rho\cdot P\cdot\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{\rho}\bigg) \tag{7.51}$$

To better understand the right-hand side in this last expression, one considers an adiabatic state change of a system as in fig. 7.9, where the change of state takes place in such a way that no heat is exchanged with the environment during compression or expansion of the volume $\widetilde{V}$. With

$$\frac{1}{\rho} \;=\; \frac{\widetilde{V}}{M}$$

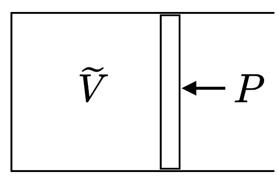

Fig. 7.9

one has

$$P\cdot\mathrm{d}\bigg(\frac{1}{\rho}\bigg) \;=\; P\cdot\frac{\mathrm{d}\widetilde{V}}{M}$$

For the adiabatic work done with the change in volume applies per unit mass

$$\frac{\mathrm{d}A}{M} \;=\; -\frac{P\cdot\mathrm{d}\widetilde{V}}{M} \;=\; -\frac{\mathrm{d}E}{M} - \underset{=0}{\underset{\downarrow}{\mathrm{d}Q}} \qquad \text{(adiabaticity)}$$

and therefore

$$-P\cdot\frac{\mathrm{d}(1/\rho)}{\mathrm{d}t} \;=\; \frac{\mathrm{d}e}{\mathrm{d}t} \;=\; \frac{\mathrm{d}(E/M)}{\mathrm{d}t}$$

If one put this in eq. (7.51), one gets for the expression $P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v}$

$$P\cdot\boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; \rho\cdot P\cdot\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{\rho}\bigg) \;=\; -\rho\cdot\frac{\mathrm{d}e}{\mathrm{d}t}$$

Application of eq. (7.51) and the continuity equation yields

$$\rho\cdot\frac{\mathrm{d}e}{\mathrm{d}t} \;=\; \rho\bigg(\frac{\partial e}{\partial t} + \boldsymbol{v}\boldsymbol{\nabla}e\bigg) \;=\; \rho\frac{\partial e}{\partial t} + \boldsymbol{\nabla}\cdot(\rho e\boldsymbol{v}) - e\cdot\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})$$

$$= \rho\frac{\partial e}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho e\boldsymbol{v}) + e\frac{\partial \rho}{\partial t} \;=\; \frac{\partial(\rho e)}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho e\boldsymbol{v})$$

And with that

$$P\cdot\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} \;=\; \frac{\partial(\rho e)}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho e\boldsymbol{v})$$

Inserting this into eq. (7.50) yields the so-called energy flux equation

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho U + \rho e\Big) + \boldsymbol{\nabla}\boldsymbol{\cdot}\Big[\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho U + \rho e + P\Big)\cdot\boldsymbol{v}\Big] \;=\; 0 \tag{7.52}$$

Important relationships for ideal liquid and ideal gas

$$\Delta\big(\rho v A\big) \;=\; \rho v A - \rho_0 v_0 A_0 \;=\; 0 \qquad \text{Flux tube}$$

$$\Delta\Big[(\rho\boldsymbol{v}^2 + P)\frac{\boldsymbol{v}}{v}A\Big] \;=\; \boldsymbol{F} \qquad \text{Force on pipe}$$

$$\Delta\Big[\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho U + \rho e + P\Big)vA\Big] \;=\; 0 \qquad \text{Energy flux}$$

$$\Delta\Big[\frac{\rho}{2}\boldsymbol{v}^2 + \rho U + \rho e + P\Big] \;=\; 0 \qquad \text{Bernoulli equation}$$

Example 7.1 Throttled expansion

In the case of throttled expansion, a liquid in a volume $\widetilde{V}$ under high pressure is being pressed from this volume through a porous wall (or a so-called throttle valve) into a volume with lower pressure. The respective pressures ($P_0 > P$) on the left or right of the porous wall in fig. 7.10 are kept constant. Depending on the Joule-Thomson coefficient of the liquid used in the process, the liquid will cool or heat up.

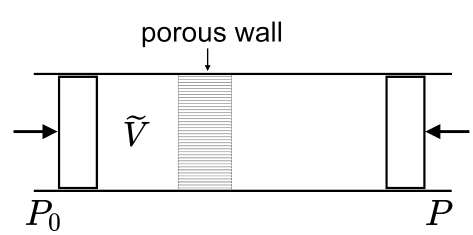

Fig. 7.10

So, it applies

$$(E + P\widetilde{V})_0 \;=\; (E + PV)$$

with the energy flux density

$$\frac{E}{\widetilde{V}} \;=\; e\cdot\rho \;=\; \frac{I}{\widetilde{V}}(P,T)$$

where I is the flux strength or enthalpy.

Discharge from pressure vessels

The following applies to pressure P, flow velocity v and filling level h at level **a** and level **b** in fig. 7.11

	Level $\boldsymbol{a}$	Level $\boldsymbol{b}$
$P:$	$P_0 + \Delta P$	P_0
$v:$	≈ 0	v_0
$h:$	h	0

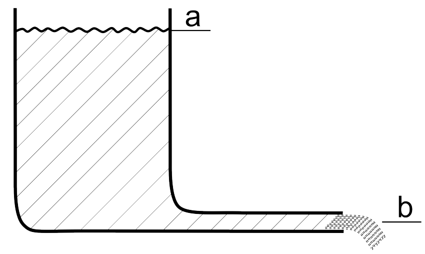

Fig. 7.11

Incompressible liquid in the gravitational field

With the Bernoulli equation (eq. (7.46)) applies

$$g{\cdot}h + \frac{P_0 + \Delta P}{\rho} = \frac{v_0^2}{2} + \frac{P_0}{\rho}$$

The air pressure difference ΔP is negligible in most cases and therefore

$$\rho{\cdot}g{\cdot}h + \underbrace{\Delta P}_{\approx 0} = \rho{\cdot}\frac{v_0^2}{2}$$

Gas flow into a vacuum

For the change of the energy flux density e one can apply $\Delta e \approx 0$ (weak cooling). In addition, $\rho{\cdot}g{\cdot}h \approx 0$ is a good approximation and thus one obtains

$$v_0^2 = \frac{2\Delta P}{\rho} = 2\frac{P}{\rho}$$

where P is the pressure inside the vessel for which applies (ideal gas!)

$$\frac{P}{\rho} = \frac{k_B T}{m} \qquad \text{and} \qquad P = \frac{R{\cdot}T}{\widetilde{V}}$$

Example 7.2 Discharge from a pressure vessel

One considers a closed cylindrical pressure vessel with cross-section A that is partially filled with water. Below the water level, the pressure vessel has a small opening with cross-section a for which $a \ll A$ shall apply. The external pressure at the small opening is P_0 and the air pressure P above the water level in the pressure vessel shall be kept constant. The quantities to be determined for this pressure vessel are the velocity v_2 with which the water flows out of the pressure vessel, the recoil experienced by the vessel, and the change in the liquid level in the vessel as a function of time for $P = P_0$.

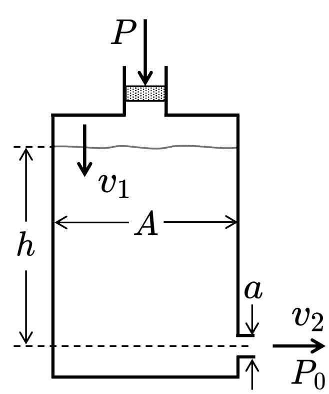

Fig. 7.12

With Bernoulli applies

$$\frac{1}{2}\rho v_1^2 + \rho g h + P \;=\; \frac{1}{2}\rho v_2^2 + P_0$$

This results in the discharge velocity v_2

$$v_2 \;=\; \sqrt{v_1^2 + 2gh + \frac{2}{\rho}(P - P_0)}$$

So, for $P = P_0$ and $v_1 \approx 0$ one has

$$v_2 \;=\; \sqrt{2gh}$$

For the recoil force one uses the approach

$$F_R \;=\; \frac{\mathrm{d}p}{\mathrm{d}t} \;=\; \frac{\mathrm{d}mv_2}{\mathrm{d}t} \;=\; \frac{\mathrm{d}m}{\mathrm{d}t}v_2 + m\frac{\mathrm{d}v_2}{\mathrm{d}t}$$

The mass flowing through the opening a per unit time is equal to $\rho a v_2$. With $v_2 = const$ the result for the recoil force is

$$F_R \;=\; \frac{\mathrm{d}m}{\mathrm{d}t}v_2 \;=\; \rho a v_2^2$$

With the continuity condition (see eq. (7.27)) it holds

$$\rho\cdot a\cdot v_2 = \rho\cdot A\cdot v_1$$

For $p = p_0$ this results in the equation

$$a\sqrt{v_1^2 + 2gh} \;=\; Av_1$$

Solving for v_1 returns

$$\left(1 - \frac{a^2}{A^2}\right)v_1^2 \;=\; 2g\frac{a^2}{A^2}h$$

Because of $v_1 = \dfrac{\mathrm{d}h}{\mathrm{d}t}$ it follows

$$\left(1 - \frac{a^2}{A^2}\right)\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 \;=\; 2g\frac{a^2}{A^2}h$$

Separation of the variables gives

$$\frac{\mathrm{d}h}{\sqrt{h}} \;=\; \pm\frac{a}{A}\frac{\sqrt{2g}}{\sqrt{1 - \frac{a^2}{A^2}}}\mathrm{d}t \;\approx\; \pm\frac{a}{A}\sqrt{2g}\mathrm{d}t$$

where $a \ll A$ was used in the second step. The integration

$$\int\limits_{h_0}^{h}\frac{\mathrm{d}h}{\sqrt{h}} \;=\; \pm\frac{a}{A}\sqrt{2g}\int\limits_{0}^{t}\mathrm{d}t$$

yields

$$2\sqrt{h} - 2\sqrt{h_0} = \pm\frac{a}{A}\sqrt{2g}$$

Since $h_0 \geq h$, the minus sign must be chosen on the right-hand side and one thus obtains

$$h(t) = \left(\sqrt{h_0} - \frac{a\sqrt{2g}}{2A}\cdot t\right)^2$$

At the time $t = t_A$ the expression in the brackets shall be zero and thus $h(t_A) = 0$. This is the case for

$$t_A = \frac{2A}{a}\sqrt{\frac{h_0}{2g}}$$

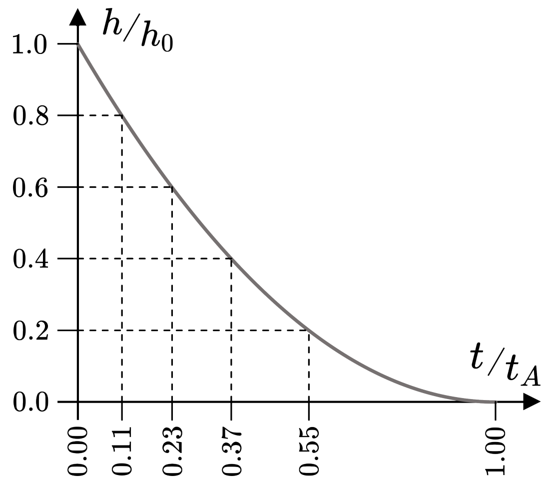

Fig. 7.13

Fig. 7.13 shows the normalized filling level of the container with respect to h_0 as a function of time in units of t_A. For the ratio $A:a$, $1000:1$ was chosen here. Up to a level of approx. $0.5 \cdot h_0$, the water level in the container falls almost linearly, but then it falls more and more slowly. After just a little more than half of t_A, the filling level is only $\approx 20\%$ of h_0. On closer inspection, one finds that to discharge the last $\approx 25\%$ of the water takes just as long as it took the first $\approx 75\%$ of water to discharge.

Example 7.3 **Bernoulli "paradox"**

The effect called Bernoulli paradox is due to a negative pressure which is created by the bending of streamlines. In fig. 7.14 the lower disk is suspended from the upper disk so that it can move in the z direction. If an air flow is blown in from above, the air escapes between the two disks. If P_0 is the ambient pressure, then with Bernoulli the pressure between the two disks is given by

$$P_S = P_0 - \frac{\rho}{2}v^2$$

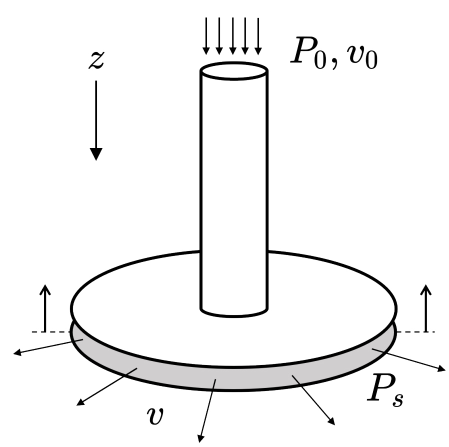

Fig. 7.14

where v is the velocity at which the air escapes from between the disks. Since P_0 is constant, this means that the pressure between the two disks decreases. The resulting negative pressure pulls the lower disk upwards. If A is the area of the lower disk, then it levitates if

$$P_S\cdot A > m\cdot g$$

If one completely immerses the apparatus from fig. 7.14 in a liquid and uses a liquid jet, i.e., liquid pressure, instead of an air jet, i.e., air pressure, to operate it, then the very same effect can be observed.

Summary of conservation theorems

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \;\rightarrow\; \Delta(\rho v A) = 0 \qquad \textbf{\textit{Mass}}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\Pi_{ik} = f_i \;\rightarrow\; \Delta\Big[(\rho v^2 + P)\frac{\boldsymbol{v}}{v}A\Big] = 0 \qquad \textbf{\textit{Momentum}}$$

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho e + \rho U\Big) + \mathrm{div}\Big[\Big(\frac{\rho}{2}\boldsymbol{v}^2 + \rho e + \rho U + P\Big)\cdot \boldsymbol{v}\Big] = 0 \qquad \textbf{\textit{Energy}}$$

$$\Delta\Big(\frac{\boldsymbol{v}^2}{2} + U + e + \frac{P}{\rho}\Big) = 0 \qquad \textbf{\textit{Bernoulli}}$$

Venturi tube

In a so-called Venturi tube (fig. 7.15), named after Giovanni Battista Venturi, the dynamic pressure P_0 is at a maximum at the narrowest point of the Venturi tube and the hydrostatic pressure is minimal. This leads to a negative pressure in the narrower tube where $P < P_0$ and $v > v_0$; there the air flows with higher velocity. The respective pressure difference is indicated by the different levels of liquid in the two arms of the U-tube. One can use this, for example, to measure the flow velocities of gases or liquids.

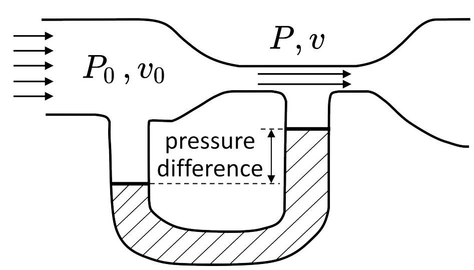

Fig. 7.15

7.2.6 Circulation

The line integral over the velocity field $\boldsymbol{v}$ between two points A and B

$$\int_A^B \boldsymbol{v}\mathrm{d}\boldsymbol{s} = \int_A^B (v_x \mathrm{d}x + v_y \mathrm{d}y + v_z \mathrm{d}z) \tag{7.53}$$

along a curve C with the line element d$\boldsymbol{s}$ that runs in the liquid under consideration is called flow. In the case that one looks at the flow along a closed curve ,i.e., the integral turns into a closed curve integral, then one speaks of a circulation:

$$\Gamma = \oint_C \boldsymbol{v} \mathrm{d}\boldsymbol{s} \tag{7.54}$$

Γ is a measure of the vortices present in any given velocity field. If for example $\Gamma = 0$, then the velocity field within the considered closed curve is vortex-free, otherwise vortices exist there.

In general, the flow between two points A and B (eq. (7.53)) depends on the respective path considered. However, if $\Gamma = 0$ applies to all areas between A and B, then the flow only depends on the locations of the points A and B. If $A = A(s)$ is the area enclosed by the path, then it follows from the application of Stoke's theorem and the mean value theorem that A must have a point P where

$$\oint_C \boldsymbol{v} \mathrm{d}\boldsymbol{s} = \iint_A (\boldsymbol{\nabla} \times \boldsymbol{v}) \mathrm{d}A = A\hat{\boldsymbol{n}}(\boldsymbol{\nabla} \times \boldsymbol{v}) \tag{7.55}$$

applies. Here $\hat{\boldsymbol{n}}$ is the normal vector of the surface A in point P. If one now contracts the curve C onto the point P, it follows with

$$\hat{\boldsymbol{n}}\Big(\boldsymbol{\nabla} \times \boldsymbol{v}\Big) = \lim_{A \to 0} \frac{1}{A} \oint_C \boldsymbol{v} \mathrm{d}\boldsymbol{s} = 0 \tag{7.56}$$

that a flow without circulation is always vortex-free, i.e., $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$. As will be shown in section 7.2.10, then one always deals with a so-called potential flow. However, the opposite conclusion, that a potential flow is always vortex-free, is only valid for simply connected areas.

Definition

A curve is called free of double points if it runs through each point it contains only once.
A domain is called simply connected if every closed and double-point-free curve in the domain is continuous and can be contracted onto a point P of the domain. A domain that is not simply connected is a multiply connected domain.

With $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$, according to Stoke's theorem, the circulation vanishes for a simply connected domain.

$$\Gamma = \oint_C \boldsymbol{v} \mathrm{d}\boldsymbol{s} = \iint_A (\boldsymbol{\nabla} \times \boldsymbol{v}) \mathrm{d}A = 0$$

For domains which include regions that do not belong to the liquid itself, i.e., they are not simply connected domains, the integration path must be chosen appropriately for the application of Stoke's theorem. This is done by cutting open the domain that is not simply connected.

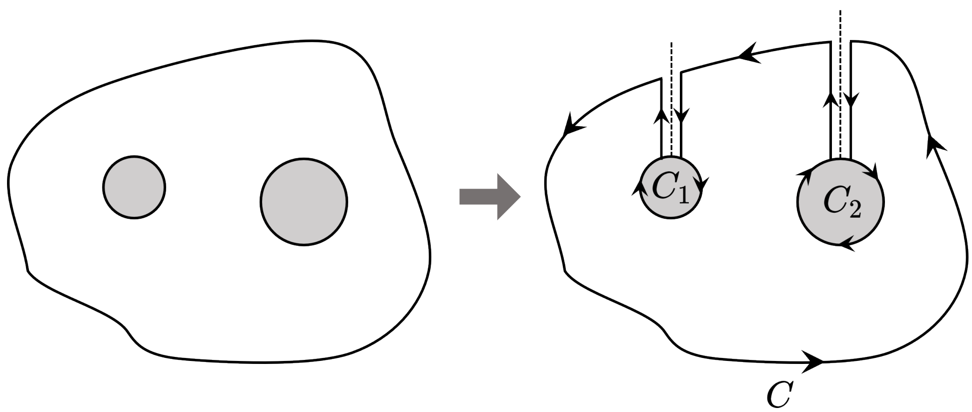

Fig. 7.16: Left: A multiply (3-fold) connected domain. Right: Slicing transforms the multiply connected domain into a simply connected domain for which Stoke's theorem can be applied along the path of integration outlined.

The left side of Fig. 7.16 shows a domain with two islands that do not belong to the domain, i.e., a multiply (3-fold) connected domain. On the right-hand side of fig. 7.16 the path of integration is sketched, which one obtains by cutting open the domain. The cuts are thought to be infinitesimally thin and the paths of integration along the cuts themselves cancel each other out. For $\boldsymbol{\nabla}\times\boldsymbol{v}=0$, Stoke's theorem for the 3-fold connected domain in Fig. 7.16 along the outlined path of integration yields

$$\oint_C \boldsymbol{v}\mathrm{d}\boldsymbol{s}+\oint_{C_1} \boldsymbol{v}\mathrm{d}\boldsymbol{s}+\oint_{C_2} \boldsymbol{v}\mathrm{d}\boldsymbol{s} = \iint_A (\boldsymbol{\nabla}\times\boldsymbol{v})\mathrm{d}A = 0$$

To the circulation along the path C which borders the 3-fold connected domain thus applies

$$\Gamma_C = \oint_C \boldsymbol{v}\mathrm{d}\boldsymbol{s} = -\oint_{C_1} \boldsymbol{v}\mathrm{d}\boldsymbol{s} - \oint_{C_2} \boldsymbol{v}\mathrm{d}\boldsymbol{s} = \Gamma_{C_1}+\Gamma_{C_2} \tag{7.57}$$

Hence, for a domain that is not simply connected, the circulation is therefore equal to the sum of the circulations around the regions of the domain which do not belong to the liquid. Next, the time dependence of the circulation shall be considered. The change of Γ along a curve moving with the liquid particles of the curve is given by the substantial derivative

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\oint\limits_C \boldsymbol{v}\mathrm{d}\boldsymbol{s} = \oint\limits_C \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\mathrm{d}\boldsymbol{s} + \oint\limits_C \boldsymbol{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s} \tag{7.58}$$

With eq. (7.46) the equation of motion reads

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\boldsymbol{\nabla}\left(U - \frac{P}{\rho}\right)$$

and thus (due to $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\Omega = 0$) the first term on the right side of eq. (7.58) vanishes

$$\oint\limits_C \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\mathrm{d}\boldsymbol{s} = -\oint\limits_C \boldsymbol{\nabla}\left(U - \frac{P}{\rho}\right)\cdot\mathrm{d}\boldsymbol{s} = -\iint\limits_A \left[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\left(U - \frac{P}{\rho}\right)\right]\mathrm{d}A = 0$$

For the line element between two infinitesimally adjacent points X and X' with velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$ applies

$$\boldsymbol{v}' = \boldsymbol{v} + \big(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v}$$

After the time $\mathrm{d}t$ has elapsed, the line element $\mathrm{d}\boldsymbol{s}$ becomes the line element

$$\mathrm{d}\boldsymbol{s}' = \mathrm{d}\boldsymbol{s} + (\boldsymbol{v}' - \boldsymbol{v})\mathrm{d}t = \mathrm{d}\boldsymbol{s} + \big(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v}\mathrm{d}t$$

and thus one obtains for the rate of change of the line element $\mathrm{d}\boldsymbol{s}$

$$\frac{\mathrm{d}\boldsymbol{s}' - \mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\boldsymbol{s} = \big(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v}$$

Inserting this into the second term on the right side of eq. (7.58)

$$\oint\limits_C \boldsymbol{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s} = \oint\limits_C \boldsymbol{v}\cdot\big(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v} = \oint\limits_C \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\nabla}\frac{v^2}{2} = \iint\limits_A \left[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\frac{v^2}{2}\right]\mathrm{d}A = 0$$

shows that this term also vanishes (because of $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\Omega = 0$). Hence, it follows

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \tag{7.59}$$

This is the conservation of circulation theorem, also known as Thomson's theorem, named after William Thomson, the later Lord Kelvin. This theorem states that in the case of isentropic flow and conservative forces (which were assumed in eq. (7.46) above) the circulation along a closed curve is an invariant, i.e., it is constant. If one considers an infinitesimal curve δC enclosing the area δA, then the following applies

$$\oint\limits_{\delta C} \boldsymbol{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s} = \iint\limits_{\delta A} (\boldsymbol{\nabla}\times\boldsymbol{v})\mathrm{d}A \approx \big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\cdot\delta A = const$$

In this context, the expression $\boldsymbol{\nabla} \times \boldsymbol{v}$ is referred to as the vortex of the flow. The respective interpretation for the fact that $\big(\boldsymbol{\nabla} \times \boldsymbol{v}\big) \cdot \delta A = const$ is then that the vortex moves with the liquid flow.

7.2.7 Helmholtz Vortex Theorems

In what follows, the vortex field of an incompressible ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) ideal fluid is considered, which is characterized by the vortex vector $\boldsymbol{w}$:

$$\boldsymbol{w} = \frac{1}{2}(\boldsymbol{\nabla} \times \boldsymbol{v}) \tag{7.60}$$

Analogous to the streamlines that illustrate a velocity field $\boldsymbol{v}$, vortex lines fulfill the same purpose for the vortex field. Just as in every point of a streamline curve the tangent vector is given by the respective local velocity vector, so the tangent in every point of a vortex line points in the direction of the local vortex vector. The density of the vortex lines, which is proportional to the magnitude of the rotational velocity $|\boldsymbol{w}|$, serves to further characterize a vortex field. The greater the density of vortex lines through a surface normal to $\boldsymbol{w}$, the greater $|\boldsymbol{w}|$. Because for the vortex field applies

$$\boldsymbol{\nabla} \cdot \boldsymbol{w} = \frac{1}{2}\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = 0$$

vortex lines, just as is the case for streamlines (cf. eq. (7.26)), can neither begin nor end inside a liquid, but instead must either be closed curves or must run along the edges of the flow. Fig. 7.17 shows a section of a so-called vortex tube. A vortex tube is the tube formed by all vortex lines that enter through the surface A_1 and exit through the surface A_2. If O denotes the surface of the depicted vortex tube section (mantle and side surfaces A_1 and A_2) and if V denotes the respective volume enclosed by it, then with theorem of Gauß and $\boldsymbol{\nabla} \cdot \boldsymbol{w} = 0$ applies

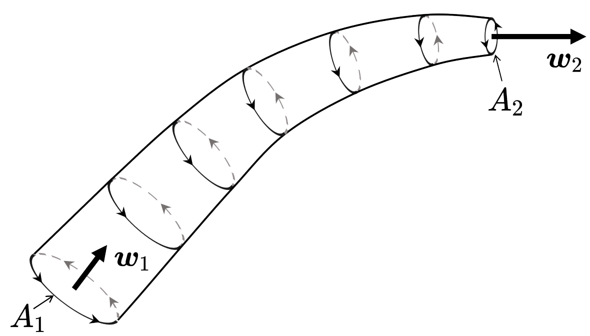

Fig. 7.17

$$\int\limits_O \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{A} = \int\limits_V (\boldsymbol{\nabla} \cdot \boldsymbol{w})\mathrm{d}V = 0 \tag{7.61}$$

The normal unit vector of the tube mantle surface is everywhere perpendicular to $\boldsymbol{w}$ and thus only the entry- and the exit surfaces, i.e., A_1 and A_2, contribute to the surface integral. That means it must apply

$$\int_{A_1} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{A} + \int_{A_2} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{A} = 0 \tag{7.62}$$

The normal unit vectors of the vortex tube entry and exit surfaces A_1 and A_2, each point outwards, i.e., for A_1 the normal unit vector is antiparallel to $\boldsymbol{w}_1$ and for A_2 it is parallel to $\boldsymbol{w}_2$. This means, the vortex flux is the same in all cross-sectional areas of a vortex tube, i.e.,

$$\int_{A} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{A} \;=\; const \tag{7.63}$$

If the cross-section of a vortex tube is so small that $\boldsymbol{w}$ is constant over the cross-section, in which case one speaks of a vortex filament, then it holds

$$|\boldsymbol{w}_1|\cdot A_1 \;=\; |\boldsymbol{w}_2|\cdot A_2 \;=\; |\boldsymbol{w}|\cdot A \;=\; const \tag{7.64}$$

Hence the rotation velocity of a vortex filament is inversely proportional to the respective cross-section at every point. The so-called vortex strength, defined as

$$2\cdot|\boldsymbol{w}|\cdot A \;=\; A\cdot|\boldsymbol{\nabla}\times\boldsymbol{v}| \tag{7.65}$$

is also constant along a vortex filament. Another constant is the circulation around a vortex tube or a vortex filament, since with the definition of $\boldsymbol{v}$ it follows from Stoke's theorem

$$\int_{A} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{A} \;=\; \frac{1}{2}\int_{A} (\boldsymbol{\nabla}\times\boldsymbol{v})\mathrm{d}\boldsymbol{A} \;=\; \frac{1}{2}\oint_{\partial A} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s} \;=\; \frac{\Gamma}{2} \tag{7.66}$$

where ∂A is the boundary (boundary curve) of the cross-sectional area A on the mantle of the vortex tube or respectively the vortex filament. According to eq. (7.66), the circulation around a vortex tube or a vortex filament is the same at all points, meaning, it is spatially constant.

The relationship between the vortex vector $\boldsymbol{w}$ and Γ expressed in eq. (7.66) can also be derived by means of a simple consideration. Fig. 7.18 shows an infinitesimal square surface in the xz-plane with the lower left corner at the origin $(0, 0)$, the upper right corner at $(\mathrm{d}x, \mathrm{d}z)$ and the area $\mathrm{d}A = \mathrm{d}x\cdot\mathrm{d}z$. At the origin, the velocity field shall have the value v_x in x-direction and v_z in z-direction. Now one considers the infinitesimal changes in the velocities v_x and v_z along the square sides s_1 and s_3 in v_x-direction and along the square sides s_2 and s_4 in v_z-direction.

In v_x-direction applies to s_1:

$$v_1 = v_x + \frac{\partial v_x}{\partial x}\mathrm{d}x$$

and to s_3:

$$v_3 = v_x + \frac{\partial v_x}{\partial x}\mathrm{d}x + \frac{\partial v_x}{\partial z}\mathrm{d}z$$

In v_z-direction applies to s_2 and s_4

$$v_2 = v_z + \frac{\partial v_z}{\partial z}\mathrm{d}z \qquad \text{and}$$

$$v_4 = v_z + \frac{\partial v_z}{\partial z}\mathrm{d}z + \frac{\partial v_z}{\partial x}\mathrm{d}x$$

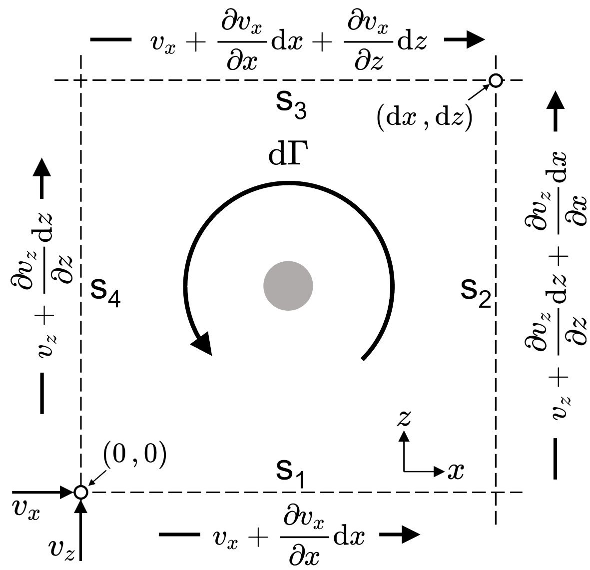

Fig. 7.18

For $\mathrm{d}\Gamma$ in fig. 7.18 then applies

$$\mathrm{d}\Gamma = \sum_{i=1}^{4} v_i \mathrm{d}s_i$$

Adding all terms along the direction of rotation of $\mathrm{d}\Gamma$ with the correct signs gives

$$\mathrm{d}\Gamma = \left(v_x + \frac{\partial v_x}{\partial x}\mathrm{d}x\right)\mathrm{d}x - \left(v_x + \frac{\partial v_x}{\partial x}\mathrm{d}x + \frac{\partial v_x}{\partial z}\mathrm{d}z\right)\mathrm{d}x$$
$$- \left(v_z + \frac{\partial v_z}{\partial z}\mathrm{d}z)\right)\mathrm{d}z - \left(v_z + \frac{\partial v_z}{\partial z}\mathrm{d}z + \frac{\partial v_z}{\partial x}\mathrm{d}x\right)\mathrm{d}z$$

hence

$$\mathrm{d}\Gamma = \frac{\partial v_x}{\partial z}\mathrm{d}z\mathrm{d}x - \frac{\partial v_z}{\partial x}\mathrm{d}x\mathrm{d}z = \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathrm{d}A = 2\cdot w_y \cdot \mathrm{d}A$$

or respectively with the unit vector $\hat{\boldsymbol{n}}_y$ in y-direction

$$\Gamma = 2\cdot\int_A w_y \cdot \mathrm{d}A = 2\cdot\int_A \boldsymbol{w}\cdot\hat{\boldsymbol{n}}_y\cdot\mathrm{d}A = 2\cdot\int_A \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{A}$$

But that is exactly what eq. (7.66) says. Next, the rate of change of vortices shall be examined. The starting point is eq. (7.41), the Euler equation for isentropic motion with conservative forces ($\boldsymbol{f} = -\rho\boldsymbol{\nabla}U$):

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} - \rho\big[\boldsymbol{v}\times\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\big] = -\boldsymbol{\nabla}(\rho\cdot U + P) \tag{7.67}$$

Multiplying eq. (7.67) from the left by $\boldsymbol{\nabla}\times$ gives with eq. (7.60) and $\boldsymbol{\nabla}\times\boldsymbol{\nabla} = 0$

$$\frac{\partial \boldsymbol{w}}{\partial t} - \boldsymbol{\nabla}\times\big(\boldsymbol{v}\times\boldsymbol{w}\big) = 0 \tag{7.68}$$

The second term can be transformed (see eq. (A.1)) and with $\boldsymbol{\nabla} \cdot \boldsymbol{w} = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ (incompressible liquid) eq. (7.68) becomes

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{w} \;=\; (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{7.69}$$

On the left side of eq. (7.69) stands the substantial derivative of $\boldsymbol{w}$

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} \;=\; (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{7.70}$$

From eq. (7.70) it immediately follows that a vortex-free liquid, that means $\boldsymbol{w} = 0$ on the right side of eq. (7.70), will always remain vortex-free because the substantial derivative of $\boldsymbol{w}$ is equal to zero. Therefore, in an ideal incompressible liquid, vortices can neither arise nor disappear.

In the next step, two liquid particles will be considered which at time t shall lie within infinitesimal proximity to each other at the respective points P_1 and P_2 of a vortex line. Because the corresponding line element $\mathrm{d}\boldsymbol{s}$ between the points P_1 and P_2 on the vortex line (by definition) always points in the direction of the vortex vector $\boldsymbol{w}$, it applies with a scalar parameter $\mathrm{d}\xi$ that

$$\mathrm{d}\boldsymbol{s} \;=\; \boldsymbol{w} \cdot \mathrm{d}\xi$$

If $\boldsymbol{r}$ is the position vector of P_1 and $\boldsymbol{v}$ is the speed with which P_1 moves in the liquid then the position vector of P_2 is

$$\boldsymbol{r} + \mathrm{d}\boldsymbol{s} \;=\; \boldsymbol{r} + \boldsymbol{w} \cdot \mathrm{d}\xi$$

and P_2 moves in a first approximation with the velocity

$$\boldsymbol{v} + \big(\mathrm{d}\boldsymbol{s} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v} \;=\; \boldsymbol{v} + \mathrm{d}\xi\big(\mathrm{d}\boldsymbol{w} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v}$$

At the time $t + \mathrm{d}t$, P_1 and P_2 have moved to P_1' and P_2' and $\mathrm{d}\boldsymbol{s}$ has become

$$\mathrm{d}\boldsymbol{s}' \;=\; \mathrm{d}\boldsymbol{s} + \big(\mathrm{d}\boldsymbol{s} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v}\mathrm{d}t \;=\; \mathrm{d}\xi\big[\boldsymbol{w} + \mathrm{d}t\big(\boldsymbol{w} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v}\big]$$

Now, however, with eq. (7.70) holds

$$\boldsymbol{w} + \mathrm{d}t\big(\boldsymbol{w} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v} \;=\; \boldsymbol{w} + \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}\mathrm{d}t \;=\; \boldsymbol{w}'$$

where $\boldsymbol{w}'$ is the vortex vector at time $t + \mathrm{d}t$. However, this means that at the time $t + \mathrm{d}t$ it holds again

$$\mathrm{d}\boldsymbol{s}' \;=\; \boldsymbol{w}' \cdot \mathrm{d}\xi$$

But that means nothing other than that a vortex line moves along with the liquid particles that constitute it. Therefore, liquid particles lying on a vortex line will always remain on this vortex line and hence vortex lines are so-called material lines (i.e., made out of particles). A vortex tube or respectively a vortex filament thus will always be constituted by the very same liquid particles. From

$$\left|\frac{\mathrm{d}\boldsymbol{s}'}{\mathrm{d}\boldsymbol{s}}\right| = \left|\frac{\boldsymbol{w}'}{\boldsymbol{w}}\right|$$

it also follows that the volume of a vortex tube is a constant.

Summary of the vortex theorems

First Helmholtz vortex theorem:
Vortex-free flow regions remain (in the absence of vortex-enhancing external forces) vortex-free.

Second Helmholtz vortex theorem:
Fluid elements lying on a vortex line remain on that vortex line. Vortex lines are therefore material lines.

Third Helmholtz vortex theorem:
The circulation along a vortex tube is constant. A vortex line can therefore neither arise nor end in the liquid. Just like streamlines, vortex lines must be closed or run along the edge.

Since vortex-free flows remain vortex-free, the question arises as to how vortices arise in the first place. A prerequisite for the derivation of eq. (7.70) was the assumption of conservative forces in eq. (7.67), i.e., forces of the form $\boldsymbol{F} = -k\boldsymbol{\nabla}\phi(\boldsymbol{r})$ for which the curl vanishes. The gravitational force is such a force, but not, for example, the Coriolis force $\boldsymbol{F}_C$ for which $\boldsymbol{\nabla} \times \boldsymbol{F}_C \neq 0$ applies. So, if one take into account the Coriolis force in eq. (7.67) then eq. (7.70) becomes

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \big(\boldsymbol{\nabla} \times f_c\big) \tag{7.71}$$

The significance of this equation is that even if no vortices should exist at a time t, meaning $\boldsymbol{w} = 0$ at that time, then the right-hand side of this equation, the substantial derivative of $\boldsymbol{w}$, does no longer vanish and therefore $\boldsymbol{w} \neq 0$ at later point in time $t + \mathrm{d}t$; hence, vortices are being formed.

However, the Coriolis force can be neglected for most considerations of flow processes, since it only becomes noticeable in very large extensive systems. An example of the latter are atmospheric vortices for which the Coriolis force is decisive in their formation at higher latitudes. For much smaller systems, such as the flow of liquids, vortices arise at the boundary surfaces.

If the vortex field of a liquid is known, the velocity field $\boldsymbol{v}$ can be determined from it. For an incompressible liquid $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ applies and because of $\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{A}\big) = 0$ the velocity field $\boldsymbol{v}$ can be derived from a vector potential $\boldsymbol{A}$. This means that $\boldsymbol{v} = \boldsymbol{\nabla}\times\boldsymbol{A}$ must apply. With that follows from eq. (7.60)

$$\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\boldsymbol{A}\big) = \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{A}\big) - \boldsymbol{\nabla}^2\boldsymbol{A} = 2\boldsymbol{w} \tag{7.72}$$

Since it is possible to choose $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ (see the discussion in the Appendix), this expression reduces to the well-known form of the Laplace equation

$$\Delta\boldsymbol{A} = -2\boldsymbol{w} \tag{7.73}$$

As one knows for example from electrodynamics, this equation has the solution

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2\pi}\int\frac{\boldsymbol{w}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}' \tag{7.74}$$

and with that one obtains for the velocity field $\boldsymbol{v}(\boldsymbol{r})$

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{2\pi}\int\frac{\boldsymbol{w}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\boldsymbol{r}' \tag{7.75}$$

The equivalent of $\boldsymbol{A}(\boldsymbol{r})$ and $\boldsymbol{v}(\boldsymbol{r})$ in electrodynamics are the vector field and the magnetic field strength of a current-carrying wire. This analogy with a current-carrying wire goes even further and relationships analogous to the Bio-Savart law of electrodynamics can be derived for a vortex filament:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\Gamma}{4\pi}\int\frac{\mathrm{d}\boldsymbol{s}}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{7.76}$$

and

$$\boldsymbol{v}(\boldsymbol{r}) = -\frac{\Gamma}{4\pi}\int\frac{(\boldsymbol{r}-\boldsymbol{r}')\times\mathrm{d}\boldsymbol{s}}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{7.77}$$

where Γ is the circulation and $\mathrm{d}\boldsymbol{s}$ is the line element of the vortex filament along which the integration takes place.

Example 7.4 Buoyancy (Kutta-Joukowski equation)

One considers the air circulation Γ around the wing of an airplane (fig. 7.19) in transverse direction. If $\Gamma \neq 0$ applies, vortices are present. To calculate the lift due to the airflow around the wing, only velocity components parallel to the transverse profile (i.e., the cross-section) of the wing are being considered; velocity components in the longitudinal direction of the airplane wing are not taken into account. As sketched in fig. 7.20, the airplane wing is situated in an air flow $\boldsymbol{u}$ that shall only have a component in the x-direction. Far in front of and far behind the airplane wing, the airflow has the flow velocity $\boldsymbol{u}$. Around the wing, however, the airflow has additional components w_x and w_z. Hence, to the velocity of the air flow around the wing applies

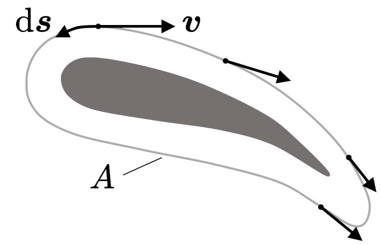

Fig. 7.19

$$v_x = u + w_x \quad ; \quad v_z = w_z$$

From eq. (7.32) one obtains for the momentum flux density in x- and in z-direction

$$\Pi_{xx} + \Pi_{xy} + \Pi_{xz} = \rho v_x^2 + P + \rho v_x v_z$$

$$\Pi_{zx} + \Pi_{zy} + \Pi_{zz} = \rho v_z v_x + P + \rho v_z^2$$

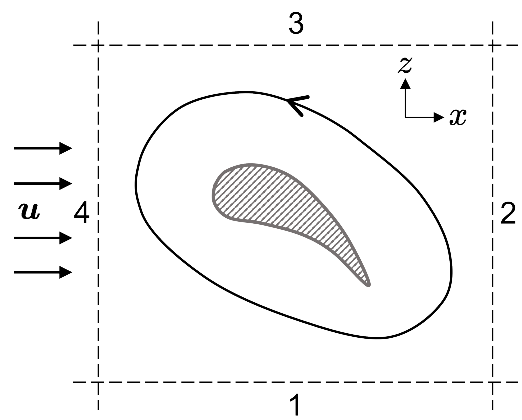

Fig. 7.20

For a given volume V enclosed by the surface A, one obtains the respective force components by integration over the momentum density (see eq. (7.31)), i.e.,

$$\boldsymbol{F}_i = \frac{\partial}{\partial t}\int_V \rho v_i \mathrm{d}V = -\int_A \Pi_{ik}\mathrm{d}A_i + \int_V \boldsymbol{f}_i \mathrm{d}V \tag{7.78}$$

Without external forces one thus obtains for the force in the x-direction

$$-F_x = \int_1 \Pi_{xk}\mathrm{d}A_k + \int_2 \Pi_{xk}\mathrm{d}A_k + \int_3 \Pi_{xk}\mathrm{d}A_k + \int_4 \Pi_{xk}\mathrm{d}A_k \tag{7.79}$$

and for the force in the z-direction, i.e., the buoyancy

$$-F_z = \int_1 \Pi_{zk}\mathrm{d}A_k + \int_2 \Pi_{zk}\mathrm{d}A_k + \int_3 \Pi_{zk}\mathrm{d}A_k + \int_4 \Pi_{zk}\mathrm{d}A_k \tag{7.80}$$

The orientation of the surface element $\mathrm{d}A_k$ is determined by the normal vector in the direction of the coordinate with the index k. First F_z^{24}, the contribution of the surfaces 2 and 4 shall be considered. With constant pressure P, the corresponding terms drop out and insertion of the Π_{zk} with consideration for the normal vectors of the surfaces pointing in opposite directions results in

$$F_z^{24} = \int_2 \left[\rho \cdot v_z \cdot v_x + \rho \cdot v_z^2\right] \mathrm{d}y\mathrm{d}z - \int_4 \left[\rho \cdot v_z \cdot v_x + \rho \cdot v_z^2\right] \mathrm{d}y\mathrm{d}z$$

By inserting $v_x = u + w_x$, $v_z = w_z$ and neglecting terms nonlinear in w_x and w_z one obtains

$$F_z^{24} = \int_2 \rho \cdot w_z \cdot (u + w_x) \mathrm{d}y\mathrm{d}z - \int_4 \rho \cdot w_z \cdot (u + w_x) \mathrm{d}y\mathrm{d}z$$

If one shifts the surfaces 2 and 4 towards ∞, then w_x disappears and F_z^{24} becomes

$$F_z^{24} = \rho \cdot u \cdot \int_2 w_z \mathrm{d}y\mathrm{d}z - \rho \cdot u \cdot \int_4 w_z \mathrm{d}y\mathrm{d}z$$

For the contribution of the surfaces 1 and 3 one obtains

$$F_z^{13} = \int_1 \left[\rho \cdot v_z \cdot v_x - \rho \cdot v_z^2\right] \mathrm{d}x\mathrm{d}y - \int_3 \left[\rho \cdot v_z \cdot v_x + \rho \cdot v_z^2\right] \mathrm{d}x\mathrm{d}y$$

By inserting $v_x = u + w_x$, $v_z = w_z$ and neglecting terms nonlinear in w_x and w_z one obtains

$$F_z^{13} = \int_1 \rho \cdot w_z \cdot (u + w_x) \mathrm{d}x\mathrm{d}y - \int_3 \rho \cdot w_z \cdot (u + w_x) \mathrm{d}x\mathrm{d}y$$

If one pushes the surfaces 1 and 3 towards each other, then ultimately $w_z = u$ and F_z^{13} will become

$$F_z^{13} = \rho \cdot u \cdot \int_1 w_x \mathrm{d}x\mathrm{d}y - \rho \cdot u \cdot \int_3 w_x \mathrm{d}x\mathrm{d}y$$

After rearranging the respective terms along the path of integration 1-2-3-4, this gives for $-F_z = F_z^{24} + F_z^{13}$

$$-F_z = \rho \cdot u \cdot \left\{ \int_1 w_x \mathrm{d}x + \int_2 w_z \mathrm{d}z - \int_3 w_x \mathrm{d}x - \int_4 w_z \mathrm{d}z \right\} \cdot \int \mathrm{d}y$$

hence

$$-F_z = \rho \cdot u \cdot \underbrace{\oint \boldsymbol{w} \mathrm{d}\boldsymbol{s}}_{=\Gamma} \cdot \int \mathrm{d}y \qquad \text{(Kutta-Joukowski equation)} \tag{7.81}$$

If v_u and v_o are the mean velocities in the x-direction along the lower (1) and upper (3) horizontal lines in fig. 7.20, respectively, then one can determine a mean u and estimate the circulation Γ with

$$u = \frac{v_o + v_u}{2} \qquad \text{and} \qquad \Gamma \approx B(v_o - v_u)$$

If L is the length of the wing and B is its width, then one obtains the following approximation for the Kutta-Joukowski formula:

$$-F_z \approx \rho \cdot u \cdot \Gamma \cdot L \approx \frac{\rho}{2} \cdot B \cdot \left(v_o^2 - v_u^2\right) \cdot L \tag{7.82}$$

With the Bernoulli equation (eq. (7.46)) applies

$$\Delta P = P_o - P_u = \frac{\rho}{2}(v_u^2 - v_o^2)$$

and with that the approximation for the Kutta-Joukowski formula for the lift of an airplane wing becomes

$$F_z \approx B \cdot \Delta P \cdot L \tag{7.83}$$

Convection terms that are important for calculating the force in the z-direction (eq. (7.80)), i.e., the buoyancy, can be neglected when calculating the force in the x-direction. This simplifies eq. (7.79) to

$$-F_x = \int_2 \left[P + \rho v_x^2\right] \mathrm{d}y \mathrm{d}z - \int_4 \left[P + \rho v_x^2\right] \mathrm{d}y \mathrm{d}z$$

If one replaces P using the Bernoulli equation eq. (7.47) without the gravitational force then (integrations over constants cancel each other out)

$$-F_x = \frac{1}{2} \int_2 \rho v_x^2 \mathrm{d}y \mathrm{d}z - \frac{1}{2} \int_4 \rho v_x^2 \mathrm{d}y \mathrm{d}z$$

Now one retains from

$$v_x^2 = (u + w_x)^2 = u^2 + 2uw_x + w_x^2$$

only terms linear in w_x. With this and because $u = \mathit{const}$ it follows for F_x

$$-F_x = \rho \cdot u \cdot \int_2 w_x \mathrm{d}y \mathrm{d}z - \rho \cdot u \cdot \int_4 w_x \mathrm{d}y \mathrm{d}z$$

If L is again the length of the wing in x-direction, then

$$-F_x = \rho \cdot u \cdot \underbrace{\left\{ \int_2 w_x \mathrm{d}z - \int_4 w_x \mathrm{d}z \right\}}_{=0} \cdot L \;=\; 0 \tag{7.84}$$

The expression between the brackets is obviously zero and that means nothing else than that the continuity equation is fulfilled in the x-direction.

As illustrated in fig. 7.21, vortex trails are unavoidable. The vortex energy E_W is proportional to the kinetic energy arising from the velocity difference $\Delta v = v_o - v_u$. Hence, one can estimate

$$\frac{E_W}{l} \;\approx\; \frac{\rho}{2}(v_o - v_u)^2 \cdot B \;=\; F_x^{\mathrm{ind}}$$

F_x^{ind} is a force induced by the vortex energy in the x-direction. In differential form

$$\mathrm{d}E_W \;=\; F_x^{\mathrm{ind}} \cdot \mathrm{d}l$$

where for F_x^{ind} with eq. (7.82) applies

$$F_x^{\mathrm{ind}} \;\approx\; \frac{F_z^2}{2 \cdot \rho \cdot u^2 \cdot B \cdot L^2}$$

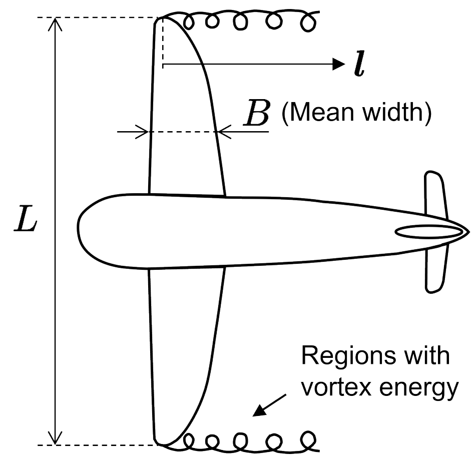

Fig. 7.21

Frictional forces (viscosity) at the wing surfaces also generate further vortices.

7.2.8 Sound Waves

Mechanical waves which as sketched in fig. 7.22 propagate as pressure or respectively density fluctuations in an elastic body such as a gas, a liquid or in a solid body, are known as sound waves. The Euler equation without internal forces, i.e., $\boldsymbol{f} = 0$ in eq. (7.34), applies

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;=\; -\boldsymbol{\nabla} P$$

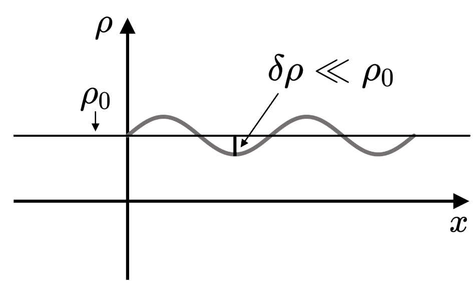

Fig. 7.22

and so does the continuity equation eq. (7.18):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \;=\; 0$$

The substantial derivative of the velocity for a sound wave propagating in an elastic medium in the x-direction (fig. 7.22) is given by

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\partial v_x}{\partial t} + \boldsymbol{v}\boldsymbol{\nabla}v_x$$

The change in density caused by a sound wave is small compared to the density of the elastic body ($\delta\rho \ll \rho_0$). In addition, the rate of change and spatial changes in the velocity, $\partial\boldsymbol{v}/\partial t$ and $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ respectively, are small compared to $\boldsymbol{v}$. This enables the linearization of the Euler equation and the continuity equation in $\delta\rho$ and $\boldsymbol{v}$. It further applies:

$$P = P(\rho,T) = P(\rho, \underset{\substack{\downarrow \\ \text{adiabatic}}}{T(\rho)}) = \underset{\substack{\downarrow \\ \text{adiabatic}}}{P(\rho)}$$

The gradient of P can therefore be rewritten as (κ = adiabatic compressibility)

$$\boldsymbol{\nabla}P = \left(\frac{\partial P}{\partial\rho}\right)_{\text{ad.}\,\rho=\rho_0}\cdot\boldsymbol{\nabla}(\delta\rho) = \frac{1}{\kappa\rho_0}\boldsymbol{\nabla}(\delta\rho) \tag{7.85}$$

With that the continuity equation becomes

$$\frac{\partial(\delta\rho)}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \tag{7.86}$$

and the Euler equation becomes

$$\rho_0\frac{\partial\boldsymbol{v}}{\partial t} = -\frac{1}{\kappa\rho_0}\boldsymbol{\nabla}(\delta\rho) \tag{7.87}$$

From these two equations follows with $\boldsymbol{\nabla}(\boldsymbol{\nabla}(\delta\rho)) = \Delta(\delta\rho)$:

$$\frac{\partial^2(\delta\rho)}{\partial t^2} = -\rho_0\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{1}{\kappa\rho_0}\Delta(\delta\rho) = c_s^2\Delta(\delta\rho) \tag{7.88}$$

where c_s^2 is the speed of sound. This equation is a so-called wave equation which has the general form

$$\boxed{\frac{\partial^2\rho}{\partial t^2} = c_s^2\Delta\rho} \qquad \text{Wave equation} \tag{7.89}$$

Solution approach: $\quad \rho = \rho_0 + \delta\rho = \rho_0 + (\delta\rho)_0\mathrm{e}^{i(\boldsymbol{kr}-\omega t)}$

Calculating the respective derivatives

$$\frac{\partial(\delta\rho)}{\partial t} = (\delta\rho)_0(-i\omega)\mathrm{e}^{i(\boldsymbol{kr}-\omega t)} = -i\omega\delta\rho \quad ; \quad \frac{\partial(\delta\rho)}{\partial x} = ik_x\delta\rho$$

$$\frac{\partial^2(\delta\rho)}{\partial t^2} = (i\omega)^2\delta\rho \quad ; \quad \frac{\partial^2(\delta\rho)}{\partial x^2} = -k_x^2\delta\rho$$

and substituting into eq. (7.89) yields

$$\omega^2 \;=\; c_s^2 \boldsymbol{k}^2$$

Wave number ν, wavelength λ and speed of sound c_s are linked via the relationship $\nu\lambda = c_s$. Hence, for the angular frequency ω and the wave vector $\boldsymbol{k}$ applies

$$\omega \;=\; 2\pi\nu \qquad \text{and} \qquad |\boldsymbol{k}| \;=\; \frac{2\pi}{\lambda}$$

Assuming the propagation direction $\boldsymbol{v}$ of the sound waves is perpendicular to the wave vector $\boldsymbol{k}$, i.e., $\boldsymbol{v} \perp \boldsymbol{k}$, it then follows from

$$\boldsymbol{v} \;=\; \boldsymbol{v}_{0_\perp} \mathrm{e}^{i(\boldsymbol{kr}-\omega t)} \qquad \text{that} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; i(\boldsymbol{k}\cdot\boldsymbol{v}) \;=\; 0$$

However, since according to the above derivation of the wave equation (eq. (7.86) and eq. (7.87) as well as the solution approach) it holds that

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; -\frac{1}{\rho_0}\frac{\partial(\delta\rho)}{\partial t} \;=\; i\omega\frac{\delta\rho}{\rho_0} \;\neq\; 0$$

and

$$\boldsymbol{\nabla}(\delta\rho) \;=\; i\boldsymbol{k}\delta\rho \;=\; -\kappa\rho_0^2\frac{\partial\boldsymbol{v}}{\partial t} \;=\; \kappa\rho_0^2\cdot i\omega\boldsymbol{v}$$

$\boldsymbol{v} \perp \boldsymbol{k}$ cannot be correct, but instead $\boldsymbol{v} \parallel \boldsymbol{k}$ must apply. Therefore, sound waves are longitudinal waves. Observation from a "moving train":

$$\boldsymbol{r} \to \boldsymbol{r} - \boldsymbol{v}_0 t \quad \text{and} \quad \mathrm{e}^{i(\boldsymbol{kr}-\omega t)} \to \mathrm{e}^{i(\boldsymbol{kr}-\omega' t)} \quad \text{with} \quad \omega' \;=\; \omega + \boldsymbol{k}\boldsymbol{v}_0$$

Hence, one has a source of sound which moves with the relative velocity $\boldsymbol{v}_0$. From fig. 7.23 one can read

$$v_0 \;=\; \frac{\Delta x}{\tau} \quad \text{and} \quad c_s \;=\; \frac{\Delta x}{\Delta\tau}$$

$$\Rightarrow \qquad \frac{\Delta\tau}{\tau} \;=\; \frac{v_0}{c_s}$$

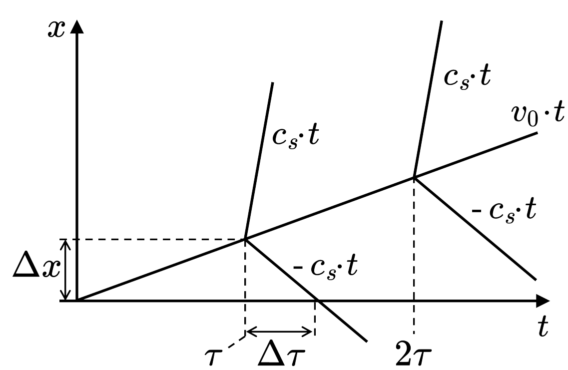

Fig. 7.23

With

$$\tau' \;=\; \tau \pm \Delta\tau \;=\; \tau\left(1 \pm \frac{v_0}{c_s}\right)$$

follows accordingly

$$\omega' \;=\; \frac{\omega}{1 \pm {}^{v_0}/_{c_s}} \;\approx\; \omega\left(1 \mp \frac{v_0}{c_s}\right)$$

where the approximation in the last step applies for $v_0 \ll c_s$.

As sketched in fig. 7.24, if $v_0 > c_s$, as for example with airplanes that fly faster than the speed of sound, a so-called Mach cone, named after Ernst Mach, forms. The cone angle is

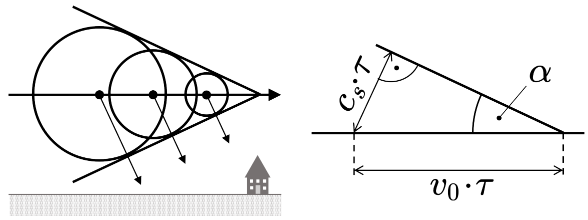

Fig. 7.24

$$\sin\alpha = \frac{c_s}{v_0}$$

7.2.9 Viscous Liquids

A viscous liquid, unlike an ideal fluid, offers frictional resistance to any change in its shape. A fluid's viscosity is a measure of how much it resists such change. As previously, it applies

$$\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} = f_i + \sum_{k=1}^{3}\frac{\partial\sigma_{ik}}{\partial x_k} \quad ; \quad \sigma_{ik} = -P\delta_{ik}$$

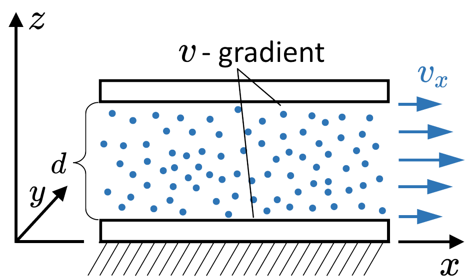

Fig. 7.25

Because of frictional resistance, a velocity gradient exists in a pipe (fig. 7.25) which is perpendicular to the flow direction. The flow velocity v_x is lower at the pipe wall than in the middle. For a given v_x, the velocity gradient is the larger, the smaller the diameter d of the tube.

$$\frac{\partial v_x}{\partial z} = \frac{v_x}{d}$$

With the transverse viscosity (toughness) η, also referred to as shear viscosity, first viscosity or dynamic viscosity, applies

$$\sigma_{xz} = \eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

The so-called dynamic viscosity η of a substance is the ratio of shear stress σ_{xz} to the spatial velocity gradient $\boldsymbol{\nabla v}$. The reciprocal of it is referred to as the fluidity of a substance. The spatial velocity gradient describes the spatial rate of change of a fluid's velocity, and it is a second order tensor, also referred to as the velocity gradient tensor. In three dimensions it reads

$$\underline{\underline{l}} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{7.90}$$

With its help, one can for example reformulate the expression for the substantial derivative of the velocity $\boldsymbol{v}$ as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;=\; \frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v} \;=\; \frac{\partial \boldsymbol{v}}{\partial t} + \underline{\underline{\mathbf{1}}}\boldsymbol{v} \tag{7.91}$$

With eq. (7.16) it also applies to $\underline{\underline{\mathbf{1}}}\boldsymbol{v}$

$$\underline{\underline{\mathbf{1}}}\boldsymbol{v} \;=\; \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{v}^2 - \big[\boldsymbol{v}\times\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)\big] \tag{7.92}$$

Including the volume viscosity λ, which takes into account the viscosity of fluids when their volume changes and which is also referred to as the second viscosity, the following applies in general to the stress tensor or respectively the distortion tensor

$$\sigma_{ik} \;=\; -P\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \lambda\delta_{ik}\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{7.93}$$

and therefore

$$\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} \;=\; f_i - \frac{\partial P}{\partial x_i} + \eta\Delta v_i + (\eta+\lambda)\frac{\partial}{\partial x_i}\big(\boldsymbol{\nabla}\cdot\boldsymbol{v}\big) \tag{7.94}$$

or respectively in vector form

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \;=\; \boldsymbol{f} - \boldsymbol{\nabla}P + \eta\Delta\boldsymbol{v} + (\eta+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \tag{7.95}$$

The transition to the Euler description for a physical vector quantity $\boldsymbol{a}$ according to eq. (7.15) and insertion into eq. (7.95) finally yields the Navier-Stokes equation

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v}\right] \;=\; \boldsymbol{f} - \boldsymbol{\nabla}P + \eta\Delta\boldsymbol{v} + (\eta+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \tag{7.96}$$

Example 7.5 Damped sound waves

If one neglects gravity, i.e., $\boldsymbol{f} = 0$, uses eq. (7.85) to replace $\boldsymbol{\nabla}P$ with $\boldsymbol{\nabla}\rho$, and keeps only terms linear in $\boldsymbol{v}$, then eq. (7.96) becomes

$$\rho_0\frac{\partial \boldsymbol{v}}{\partial t} \;=\; -\frac{1}{\kappa\rho_0}\boldsymbol{\nabla}\rho + \eta\Delta\boldsymbol{v} + (\eta+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \tag{7.97}$$

Of this equation one then takes the divergence, i.e., one multiplies the equation from the left by $\boldsymbol{\nabla}$

$$\frac{\partial}{\partial t}\big(\rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v}\big) \;=\; -\frac{1}{\kappa\rho_0}\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\,\rho\big) + \eta\cdot\underbrace{\boldsymbol{\nabla}\cdot\big(\Delta\boldsymbol{v}\big)}_{\substack{\parallel \\ \Delta(\boldsymbol{\nabla}\cdot\boldsymbol{v})}} + (\eta+\lambda)\boldsymbol{\nabla}\cdot\Big(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\Big) \tag{7.98}$$

Now the continuity equation

$$\frac{\partial\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v} \;=\; 0$$

is used to replace the $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ terms in eq. (7.98) with the partial time derivative of ρ. One thus obtains the result

$$-\frac{\partial^2\rho}{\partial t^2} \;=\; -\frac{1}{\kappa\rho_0}\Delta\rho - \frac{\eta}{\rho_0}\cdot\Delta\left(\frac{\partial\rho}{\partial t}\right) - \frac{\eta+\lambda}{\rho_0}\Delta\left(\frac{\partial\rho}{\partial t}\right) \tag{7.99}$$

By rearranging and inserting the speed of sound c_s

$$c_s^2 \;=\; \frac{1}{\kappa\rho_0}$$

one obtains the wave equation for damped sound waves

$$\frac{\partial^2\rho}{\partial t^2} \;=\; c_s^2\Delta\rho + \frac{2\eta+\lambda}{\rho_0}\Delta\left(\frac{\partial\rho}{\partial t}\right) \tag{7.100}$$

Solution approach: plane wave in x-direction

$$\rho \;=\; \rho_0 + \delta\rho \;=\; \rho_0 + (\delta\rho)_0\cdot\exp[i(kx-\omega t)] \tag{7.101}$$

By calculating the derivatives and inserting them into eq. (7.100) it follows

$$\omega^2 \;=\; c_s^2k^2 - i\frac{2\eta+\lambda}{\rho_0}k^2\omega \tag{7.102}$$

One first solves eq. (7.102) for ω. By completing the square one gets (only the positive sign of the square root is of physical relevance)

$$\omega + i\frac{2\eta+\lambda}{2\rho_0}k^2 \;=\; c_sk\sqrt{1-\left(\frac{2\eta+\lambda}{2\rho_0}\cdot\frac{k}{c_s}\right)^2} \tag{7.103}$$

The square root can be replaced by an approximation for small k (see eq. (A.10)). With that one obtains for ω

$$\omega \;=\; c_sk - i\frac{2\eta+\lambda}{2\rho_0}k^2 \tag{7.104}$$

Inserted into eq. (7.101) this gives for the time dependence of the plane wave

$$\exp(-i\omega t) \;=\; \exp(-ic_s kt)\cdot\exp\left(-\frac{2\eta+\lambda}{2\rho_0}k^2 t\right) \tag{7.105}$$

Now one solves eq. (7.102) for k

$$k \;=\; \frac{\omega}{c_s}\frac{1}{\sqrt{1-i\dfrac{2\eta+\lambda}{\rho_0 c_s}\cdot\dfrac{\omega}{c_s}}} \tag{7.106}$$

With an approximation for small ω (see eq. (A.11)) one obtains for k

$$k \;=\; \frac{\omega}{c_s}+i\frac{2\eta+\lambda}{2\rho_0 c_s}\left(\frac{\omega}{c_s}\right)^2 \tag{7.107}$$

Inserted into eq. (7.101) this gives for the spatial dependence of the plane wave

$$\exp(-ikx) \;=\; \exp\left(-i\frac{\omega}{c_s}\cdot x\right)\cdot\exp\left[-\underbrace{\frac{2\eta+\lambda}{2\rho_0 c_S}\left(\frac{\omega}{c_s}\right)^2}_{=\,\alpha/2}\cdot x\right] \tag{7.108}$$

Eq. (7.105) and eq. (7.108) show that sound waves propagating in a body with viscosity η and volume viscosity λ are subject to a temporal damping as well as a spatial damping (illustrated in fig. 7.26). The first viscosity or respectively dynamic viscosity η of a material is linked to its kinematic viscosity ν by the relationship $\eta = \nu{\cdot}\rho$. Tab. 7.1 gives the corresponding values for three well-known substances.

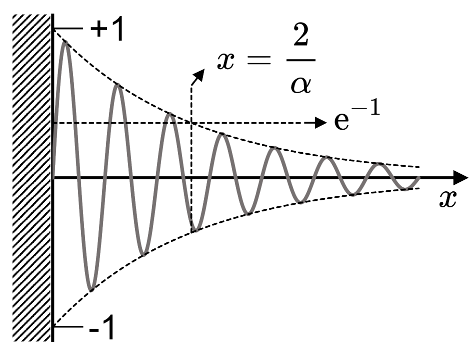

Fig. 7.26

Tab. 7.1: Density, dynamic and kinematic viscosity of air, water and glycerol at 20 °C.

Body	ρ [g·cm^{-3}]	η [g·cm^{-1}·s^{-1}]	ν [cm^2·s^{-1}]
Air	$1.20\cdot10^{-3}$	$1.83\cdot10^{-4}$	0.15
H_2O	1.00	0.01	0.01
Glycerol	1.26	14.12	11.2

In the second example in this subsection on viscous fluids, the flow of a liquid with dynamic viscosity η in a cylindrical tube shall now be considered for the stationary case.

Example 7.6 Hagen-Poiseuille flow (stationary)

Stationary flow means

$$\frac{\partial \rho}{\partial t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = 0$$

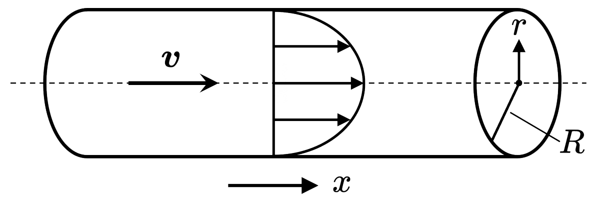

Fig. 7.27

With the continuity equation (eq. (7.18)), this also means $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ and thus the fluid under consideration is incompressible. It follows from eq. (7.95) with $\boldsymbol{f} = 0$ and $\lambda = 0$

$$0 = -\frac{1}{\rho}\boldsymbol{\nabla} P + \frac{\eta}{\rho}\Delta\boldsymbol{v}$$

For the flow situation sketched in fig. 7.27 only the x-component of the equation is required:

$$\eta \Delta v_x(r) = \frac{\partial P}{\partial x} = \frac{P_l - P_0}{l} \tag{7.109}$$

The pressure gradient in the x direction is given here by the pressure difference between the pressure $P_l = P(x = l)$ at the end of a piece of pipe of length l and the pressure $P_0 = P(x = 0)$ at the beginning of this piece of pipe divided by the length l of the piece of pipe. The problem of pipe flow is best treated in polar coordinates. Transforming Cartesian coordinates (x, y) into polar coordinates (r, φ) means for the Laplace operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \to \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \varphi} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}$$

Eq. (7.109) only depends on r and thus eq. (7.109) becomes in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x(r)}{\partial r}\right) = \frac{1}{\eta}\frac{P_l - P_0}{l}$$

Integrating this equation once gives

$$r\frac{\partial v_x(r)}{\partial r} = \frac{r^2}{2\eta}\frac{P_l - P_0}{l} + C_1$$

For $r = 0$ the left side vanishes. So, the right-hand side must also disappear and with that follows $C_1 = 0$. Integrating one more time gives

$$v_x(r) \;=\; \frac{r^2}{4\eta}\frac{P_l - P_0}{l} + C_2$$

At the edge of the tube $r = R$ must apply

$$v_x(R) \;=\; 0 \;=\; \frac{R^2}{4\eta}\frac{P_l - P_0}{l} + C_2$$

and therefore the value of C_2 becomes

$$C_2 \;=\; -\frac{R^2}{4\eta}\frac{P_l - P_0}{l}$$

With that, the result obtained for the radial dependence of the flow velocity in the pipe from fig. 7.27 is ($\Delta P = P_l - P_0$)

$$v_x(r) \;=\; \frac{r^2 - R^2}{4\eta}\cdot\frac{\Delta P}{l} \tag{7.110}$$

With the help of eq. (7.110) one can now calculate the flow rate $\mathrm{d}V/\mathrm{d}t \;=\; \dot{V}$, i.e., the volume of liquid flowing through the pipe with radius R and length L per unit of time, also referred to as the volume flow, by simple integration:

$$\begin{aligned}\dot{V} \;&=\; \int_0^R 2\pi r v_x(r)\mathrm{d}r \;=\; \int_0^R 2\pi r\frac{r^2 - R^2}{4\eta}\cdot\frac{\Delta P}{l}\mathrm{d}r \\ &=\; \frac{\pi}{2}\cdot\frac{\Delta P}{l}\left[\frac{r^2R^2}{2} - \frac{r^4}{4}\right]_0^R \;=\; \frac{\pi}{8\eta}\cdot\frac{\Delta P}{l}\cdot R^4\end{aligned}$$

The equation

$$\dot{V} \;=\; \frac{\pi}{8\eta}\cdot\frac{\Delta P}{l}\cdot R^4 \qquad \text{(Hagen-Poiseuille law)} \tag{7.111}$$

named after Gotthilf Heinrich Ludwig Hagen and Jean Léonard Marie Poiseuille shows that the volume flow in a pipe with radius R varies linearly with the pressure difference over the pipe length and with the fourth power of the radius R. Prerequisite for the applicability of the Hagen-Poiseuille law is the existence of a stationary flow. Further boundary conditions are that the boundary layers on the pipe wall are negligible, i.e., the radius R must be large enough for this to apply, and the pipe must also be long enough for the pressure difference to be stable. The Hagen-Poiseuille law is not only useful for considering pipe flow in the case of liquids, but also for gas flow in pipes as long as the density of the gas is high enough. The latter is the case when the mean free path of the gas particles is much shorter than the radius of the pipe.

7.2.10 Potential Flow

If one considers the rotation of the velocity field in the case of a planar stationary flow of an incompressible liquid, then according to the section on streamlines and stream functions with eq. (7.23) applies

$$\left.\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{v} &= \boldsymbol{\nabla}\times\boldsymbol{\nabla}\Psi\times\hat{\boldsymbol{e}}_z\\ &= \underbrace{\boldsymbol{\nabla}\Psi\big(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{e}}_z\big)}_{=0} - \hat{\boldsymbol{e}}_z\big(\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Psi)\big) + \underbrace{\big(\hat{\boldsymbol{e}}_z\boldsymbol{\nabla}\big)\boldsymbol{\nabla}\Psi}_{=0} - \underbrace{\big(\boldsymbol{\nabla}\Psi\boldsymbol{\nabla}\big)\hat{\boldsymbol{e}}_z}_{=0}\\ &= \big[-\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Psi)\big]\cdot\hat{\boldsymbol{e}}_z = -\Delta\Psi\cdot\hat{\boldsymbol{e}}_z\end{aligned}\right\} \tag{7.112}$$

The rotation of $\boldsymbol{v}$ only has a component perpendicular to the flow plane, here the xy-plane. In this case, the above equation written out for $\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)_z$ reads

$$\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)_z = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\cdot\hat{\boldsymbol{e}}_z = \left(-\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2}\right)\cdot\hat{\boldsymbol{e}}_z = -\Delta\Psi\cdot\hat{\boldsymbol{e}}_z \tag{7.113}$$

For vortex-free flows, which means $\big(\boldsymbol{\nabla}\times\boldsymbol{v}\big)_z = 0$, the stream function therefore satisfies a Laplace equation

$$\Delta\Psi\cdot\hat{\boldsymbol{e}}_z = 0 \tag{7.114}$$

If for the vector field $\boldsymbol{v}$ of the velocities of a flow also applies

$$\boldsymbol{v} = \boldsymbol{\nabla}\Phi \tag{7.115}$$

then one speaks of a potential flow (or ideal flow) where Φ is the so-called velocity potential. Since the rotation of a gradient field vanishes, i.e., $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Phi) = 0$, $\boldsymbol{\nabla}\times\boldsymbol{v} = 0$ automatically applies to a potential flow; hence a potential flow is always vortex free. If one further requires that the potential flow shall be incompressible, then $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, which means that the continuity equation applies. In summary, the following applies for an incompressible potential flow (or ideal flow):

$$\boldsymbol{\nabla}\times\boldsymbol{v} = 0 \quad ; \quad \boldsymbol{v} = \boldsymbol{\nabla}\Phi \quad ; \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$$

It follows directly from these relations that the velocity potential Φ also satisfies a Lapalace equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Phi) = \Delta\Phi \tag{7.116}$$

The stream function automatically fulfills the continuity equation, but it is not automatically vortex-free, the latter must be explicitly required. By definition, the velocity potential is vortex-free and for incompressible liquids it also automatically satisfies the continuity equation. The following relationships apply between the velocity field $\boldsymbol{v}$ and the stream function Ψ or respectively the velocity potential Φ

$$\boldsymbol{v} = \left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right) = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right) \tag{7.117}$$

So, for Ψ and Φ applies

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \qquad \text{and} \qquad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x} \tag{7.118}$$

It follows immediately from these equations that

$$(\boldsymbol{\nabla}\Phi)\cdot(\boldsymbol{\nabla}\Psi) = \frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial y} = 0 \tag{7.119}$$

This means that the equipotential lines of the velocity potential $\Phi(x,y) = const$ are orthogonal to the equipotential lines of the stream function $\Psi(x,y) = const$, i.e., orthogonal to the streamlines. The differential equations for the variables Ψ and Φ in eq. (7.118) correspond to the Cauchy-Riemann differential equations which follow from the complex differentiability of the holomorphic function

$$w(z) = \Phi(x,y) + i\cdot\Psi(x,y) \qquad \text{with} \qquad z = x + i\cdot y \tag{7.120}$$

or respectively guarantee complex differentiability. $w(z)$ is the so-called complex velocity potential. Since Ψ and Φ each satisfy a Laplace equation, $w(z)$ also satisfies a Laplace equation.

$$\Delta w(z) = \Delta\Phi(x,y) + i\cdot\Delta\Psi(x,y) \tag{7.121}$$

The complex derivative of $w(z)$ is then the complex velocity for which with eq. (7.117) applies

$$\frac{\mathrm{d}w(z)}{\mathrm{d}z} = \frac{\partial\Phi}{\partial x} + i\cdot\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial y} - i\cdot\frac{\partial\Phi}{\partial y} = v_x - i\cdot v_y = v\mathrm{e}^{i\theta} \tag{7.122}$$

where θ is the angle between the velocity vector $\boldsymbol{v}$ and the x-axis. At the surface of a body immersed in a flow, the normal component of Φ vanishes, so the velocity is always tangential to the surface of the body immersed in the flow, and the surface of the body is therefore automatically a streamline. Thus the boundary condition $\Psi = const$ applies to the surface of a body immersed in a flow. Zero can be chosen for the constant without any restriction, from which it follows that $w(z)$ on the surface of the body immersed in the flow can only assume real values. From eq. (7.119) it also follows that the equipotential lines of the velocity potential Φ must be perpendicular to the surface of the body immersed in the flow, since this surface is an equipotential line of the stream function. The boundary condition that must be met for the normal component of the velocity field to vanish at the surface of a body immersed in a flow is given by

$$\boldsymbol{v}_n \;=\; \boldsymbol{n}\cdot\boldsymbol{\nabla}\Phi \;=\; \frac{\partial\Phi}{\partial n} \;=\; 0 \tag{7.123}$$

(Compare with electrostatics where $\Phi = \Phi_0 = const$ applies to the boundary.)

Planar potential flow around an obstacle

To understand the flow around obstacles one first considers the complex velocity potential in the case of a planar flow around an obstacle that shall be located with its center at the coordinate origin. At large distances from the obstacle, the flow is constant. With that, according to eq. (7.122), the complex velocity at large distances from the obstacle, i.e., at infinity, is also constant, hence

$$w'(z\to\infty) \;=\; \left.\frac{\mathrm{d}w(z)}{\mathrm{d}z}\right|_{z\to\infty} \;=\; const$$

One can therefore expand the complex velocity in powers of $1/z$

$$w'(z) \;=\; \sum_{n=0}^{\infty}\frac{a_n}{z^n} \;=\; a_0 + \frac{a_1}{z} + \sum_{n=2}^{\infty}\frac{a_n}{z^n} \tag{7.124}$$

For $z\to\infty$ this series converges with eq. (7.122) to the constant value

$$w'(z\to\infty) \;=\; a_0 \;=\; v_x^\infty - i v_y^\infty \;=\; u$$

Therefore, a_0 is the asymptotic velocity v_∞ of the potential flow. To understand the significance of a_1, one considers the curve integral of the complex velocity along a closed path in the complex plane. The integration path is laid out in such a way that it runs along the surface of the obstacle located at the coordinate origin because the stream function Ψ is constant there.

$$\oint_C w'(z)\mathrm{d}z \;=\; \oint_C (v_x - i v_y)(\mathrm{d}x + i\mathrm{d}y)$$

Separation of real and imaginary parts gives

$$\oint_C w'(z)\mathrm{d}z \;=\; \oint_C (v_x\mathrm{d}x + v_y\mathrm{d}y) + i\oint_C (v_x\mathrm{d}y - v_y\mathrm{d}x)$$

A comparison with eq. (7.53) and eq. (7.54) shows that the real part is precisely the circulation:

$$\oint_C (v_x\mathrm{d}x + v_y\mathrm{d}y) \;=\; \oint_C (\boldsymbol{v}\mathrm{d}\boldsymbol{s}) \;=\; \Gamma$$

and because of the choice of the path of integration to which applies $\Psi = const$, the imaginary part vanishes

$$\oint_C (v_x\mathrm{d}y - v_y\mathrm{d}x) \;=\; \oint_C \left(\frac{\partial\Psi}{\partial y}\mathrm{d}y + \frac{\partial\Psi}{\partial x}\mathrm{d}x\right) \;=\; \oint_C \mathrm{d}\Psi \;=\; 0$$

Thus one obtains for the integral

$$\oint_C w'(z)\mathrm{d}z \;=\; \Gamma \tag{7.125}$$

The residue theorem of function theory offers another possible way for evaluating this integral. With that one gets

$$\frac{1}{2\pi i}\oint_C w'(z)\mathrm{d}z \;=\; a_1 \tag{7.126}$$

A comparison of eq. (7.125) and eq. (7.126) shows that $\Gamma = 2\pi i a_1$ and thus a_1 is a measure for the circulation. Inserting this value for a_1 and $a_0 = v_\infty$ into eq. (7.124) yields for the complex velocity $w'(z)$

$$w'(z) \;=\; v_\infty + \frac{\Gamma}{2\pi i}\frac{1}{z} + \sum_{n=2}^{\infty}\frac{a_n}{z^n} \tag{7.127}$$

With that one can, by integrating eq. (7.127), now determine the complex velocity potential. The result is

$$w(z) \;=\; v_\infty z + \frac{\Gamma}{2\pi i}\ln z - \sum_{n=2}^{\infty}\frac{a_n}{(n-1)z^{n-1}} \tag{7.128}$$

With $a_n = \alpha_n + i\beta_n$, splitting eq. (7.128) into its real and imaginary part gives for the velocity potential $\Phi = \mathrm{Re}\big(w(z)\big)$ and the stream function $\Psi = \mathrm{Im}\big(w(z)\big)$ in polar coordinates (r,φ)

$$\Phi(r,\varphi) \;=\; v_\infty r\cos\varphi + \frac{\Gamma}{2\pi}\varphi - \sum_{n=2}^{\infty}\left(\frac{\alpha_n\cos[(n-1)\varphi]}{(n-1)r^{n-1}} + \frac{\beta_n\sin[(n-1)\varphi]}{(n-1)r^{n-1}}\right) \tag{7.129}$$

$$\Psi(r,\varphi) \;=\; v_\infty r\sin\varphi - \frac{\Gamma}{2\pi}\ln r + \sum_{n=2}^{\infty}\left(\frac{\alpha_n\sin[(n-1)\varphi]}{(n-1)r^{n-1}} - \frac{\beta_n\cos[(n-1)\varphi]}{(n-1)r^{n-1}}\right) \tag{7.130}$$

In order to determine the coefficients a_n with $n > 1$ in eq. (7.127), one must now make assumptions about the obstacle itself. A cylinder immersed in a flow will be considered here.

Example 7.7 Potential for flow around a cylinder

Here, one considers, as sketched in fig. 7.28, a cylinder with radius R whose axis is perpendicular to the xz-plane. Onto the circular disk of the cylinder in fig. 7.28 flows coming from the left, parallel to the x-axis, the constant velocity field $\boldsymbol{u}$, meaning $v_\infty = \boldsymbol{u}_x = u$. Far away from the cylindrical disk, the velocity field $\boldsymbol{u}$ remains undisturbed, but not so around the cylinder itself. There, not only is the velocity field $\boldsymbol{u}$ disturbed, but there is also circulation. If v_φ and v_r denote the tangential and the perpendicular velocity components at the cylinder surface, then the circulation is given by

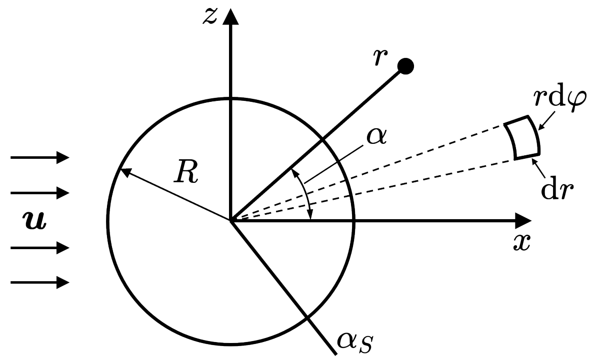

Fig. 7.28

$$\Gamma = \oint v_\alpha \mathrm{d}s = r \int_0^{2\pi} v_\varphi \mathrm{d}\varphi$$

where $\mathrm{d}s$ is the infinitesimal line element along the circle with radius R. In order to determine the coefficients with $n > 1$ in eq. (7.129) and eq. (7.130) one uses the fact that the stream function Ψ is constant on the cylinder surface. With that holds

$$\mathrm{d}\Psi\big|_{r=R} = 0$$

and thus $\Psi|_{r=R}$ must be independent of φ. From eq. (7.130) one obtains ($v_\infty = u$)

$$\begin{aligned}\Psi\big|_{r=R} = \; & uR\sin\varphi - \frac{\Gamma}{2\pi}\ln R + \frac{\alpha_2 \sin\varphi}{R} \\ & + \sum_{n=3}^{\infty} \frac{\alpha_n \sin[(n-1)\varphi]}{(n-1)R^{n-1}} - \sum_{n=2}^{\infty} \frac{\beta_n \cos[(n-1)\varphi]}{(n-1)R^{n-1}}\end{aligned} \tag{7.131}$$

In eq. (7.131), the term with $n = 2$ was extracted from the sum over the α_n, because one then immediately recognizes which coefficients α_n and β_n must disappear so that $\Psi|_{r=R}$ no longer depends on φ. From eq. (7.131) one can read that it must hold

$$\alpha_2 = -uR^2 \quad ; \quad \alpha_n = 0 \quad \text{for} \;\; n \geq 3 \quad ; \quad \beta_n = 0 \quad \text{for} \;\; n \geq 2$$

The velocity potential Φ and the stream function Ψ for the flow $\boldsymbol{u}$ around the cylinder in fig. 7.28 are obtained by inserting these coefficients in eq. (7.129) and eq. (7.130) with the result

$$\Phi(r,\varphi) \;=\; ur\cos\varphi\left(1+\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi}\varphi \tag{7.132}$$

$$\Psi(r,\varphi) \;=\; ur\sin\varphi\left(1-\frac{R^2}{r^2}\right)-\frac{\Gamma}{2\pi}\ln r \tag{7.133}$$

For the complex velocity potential $w(z)$ and the complex velocity $w'(z)$ of the cylinder immersed in the flow one obtains

$$w(z) \;=\; uz\left(1+\frac{R^2}{z^2}\right)+\frac{\Gamma}{2\pi i}\ln z \tag{7.134}$$

$$w'(z) \;=\; u\left(1-\frac{R^2}{z^2}\right)+\frac{\Gamma}{2\pi i}\frac{1}{z} \tag{7.135}$$

Now one can determine $\boldsymbol{v} \;=\; \boldsymbol{\nabla}\Phi$. In cylindrical coordinates one has

$$v_r \;=\; \frac{\partial\Phi}{\partial r} \;=\; u\cos\varphi\left(1-\frac{R^2}{r^2}\right) \tag{7.136}$$

$$v_\varphi \;=\; \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} \;=\; -u\sin\varphi\left(1+\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi r} \tag{7.137}$$

The boundary condition for the normal component v_n is fulfilled because

$$v_n \;=\; v_r\big|_{r=R} \;=\; u\cos\varphi\left(1-\frac{R^2}{R^2}\right)=0$$

The so-called stagnation angle $\varphi = \alpha_S$ in fig. 7.28 is determined by

$$v_\varphi(\alpha_S)\big|_{r=R} = -2u\sin\alpha_S + \frac{\Gamma}{2\pi R} \;=\; 0$$

For low circulation Γ and small α_S, i.e., $\sin\alpha_S \approx \alpha_S$, one can estimate the stagnation angle:

$$\alpha_S \;\approx\; \frac{\Gamma}{4\pi R u}$$

The stagnation angle can be determined exactly from the zero points, i.e., the roots of the complex velocity potential, the so-called stagnation points. The roots of eq. (7.135) are given by

$$z_{1,2} \;=\; i\frac{\Gamma}{4\pi u} \pm \sqrt{R^2-\left(\frac{\Gamma}{4\pi u}\right)^2} \tag{7.138}$$

The importance of the stagnation points lies in the fact that the velocity vanishes in them. In eq. (7.138) one must distinguish between two cases, $|\Gamma| \leq 4\pi u$ and $|\Gamma| > 4\pi u$.

Case I: $|\Gamma| \leq 4\pi u$

In this case the root is real and $|z_1| = |z_2| = R$. The stagnation points are therefore on the cylinder surface at $z_{1,2} = Re^{i\varphi_{1,2}}$. With $\sin\varphi = \mathrm{Im}(z)/|z|$ and $\cos\varphi = \mathrm{Re}(z)/|z|$ one gets for the angles $\varphi_{1,2}$:

$$\begin{aligned} \sin\varphi_1 &= \sin\varphi_2 = \frac{\Gamma}{4\pi u R} \\ \cos\varphi_1 &= -\cos\varphi_2 = \sqrt{1 - \left(\frac{\Gamma}{4\pi u R}\right)^2} \end{aligned}$$

$\Gamma = 0$: Without circulation, the two stagnation points fall on the x-axis at $x = \pm R$ and the flow around the cylinder is symmetrical (fig. 7.29a). The flow velocities above and below the cylinder are the same.

$\Gamma < 0$: With increasing $|\Gamma|$, the stagnation point that was at $-R$ moves counterclockwise and the stagnation point that was at R moves clockwise until both finally become coincident at $(x = 0, y = -R)$ for $|\Gamma| = 4\pi u R$ (the situations are illustrated by fig. 7.29b and fig. 7.29c). The flow velocity above the cylinder is greater than below.

$\Gamma > 0$: With increasing $|\Gamma|$, the stagnation point that was at $-R$ moves clockwise and the stagnation point that was at R moves counterclockwise until both finally become coincident at $(x = 0, y = R)$ for $|\Gamma| = 4\pi u R$ (corresponds to a reflection of the situation in fig. 7.29b and fig. 7.29c on the x-axis). The flow velocity below the cylinder is greater than above.

Case II: $|\Gamma| > 4\pi u$

In this case eq. (7.138) becomes

$$z_{1,2} = i\left(\frac{\Gamma}{4\pi u} \pm \sqrt{\left(\frac{\Gamma}{4\pi u}\right)^2 - R^2}\right)$$

and the stagnation points are no longer lying on the cylinder. If one sets $\Gamma = -4\pi u R(1 + \Delta)$ with $\Delta > 1$, then the magnitude of $z_{1,2}$ is

$$|z_{1,2}| = R \cdot \left|1 + \Delta \pm \Delta\sqrt{1 + \frac{2}{\Delta}}\right|$$

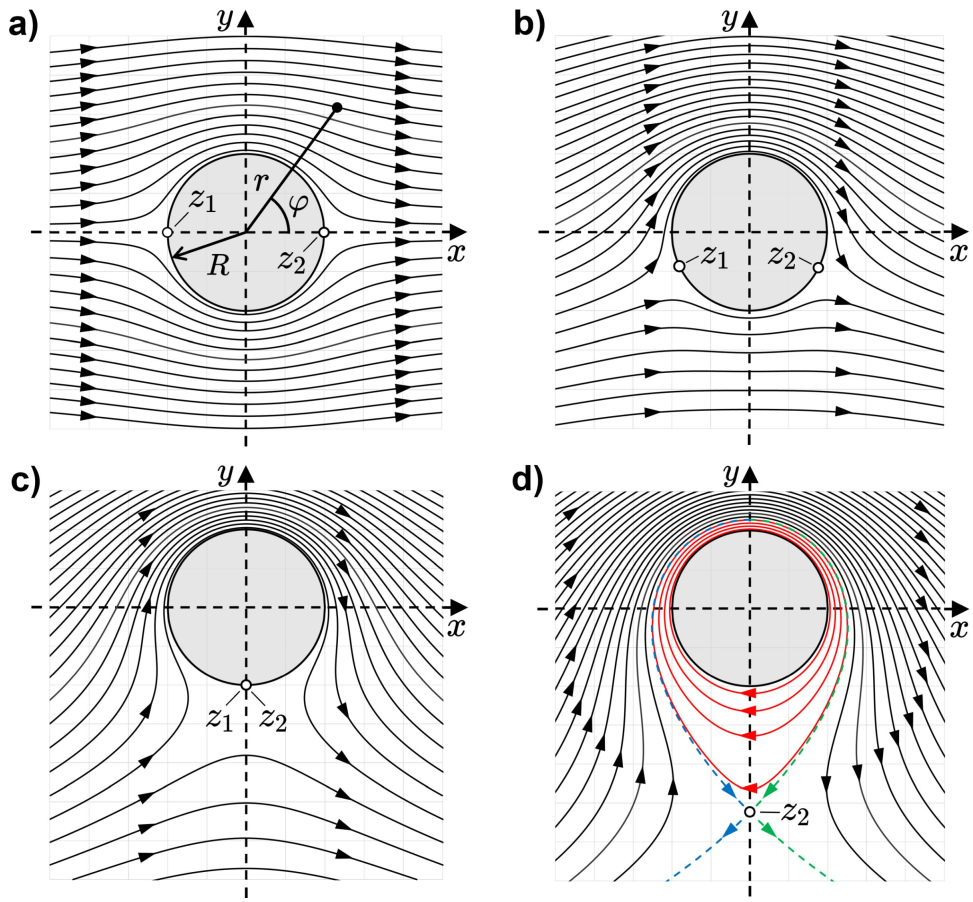

Fig. 7.29: Contour plot of the stream function $\Psi(r,\varphi)$ for the cylinder immersed in the flow from fig. 7.28. (a) no circulation, $\Gamma = 0$; (b) some circulation, $\Gamma = -2\pi uR$; (c) $\Gamma = -4\pi uR$, the stagnation points coincide; and (d) strong circulation, $\Gamma = -6\pi uR$ and there is only one stagnation point in the flow field. The difference between any two adjacent equipotential lines of the stream function is the same in all four plots.

The expression under the root is always > 1 and thus it must either hold that $|z_1| < R$ and $|z_2| > R$ or it must hold that $|z_1| > R$ and $|z_2| < R$. One stagnation point is always located inside the cylinder. Hence, there can only be one real stagnation point which however lies on the negative imaginary axis (fig. 7.29d), which means that it lies not any longer on the cylinder surface but is located in the flow field below the cylinder. The dashed streamline in fig. 7.29d, which goes through this stagnation point, divides the flow field around the cylinder into three areas. Above the stagnation point and within the streamline running through it, the flow circulates around the cylinder. To the right and left of the streamline running through the stagnation point, the streamlines run above the cylinder. Below the stagnation point and between the streamline passing through the stagnation point, all streamlines run below the cylinder. The flow velocity above the cylinder is in this case also greater than below.

Pressure field of the cylinder in a flow

With the help of Bernoulli's equation, one obtains the pressure field surrounding the cylinder in the flow directly from the velocity field. According to Bernoulli, with eq. (7.46) and the potential $U = 0$ applies

$$\frac{\rho v^2}{2} + P \;=\; const \;=\; \frac{\rho u^2}{2} + P_\infty \tag{7.139}$$

where v^2 at the location $\boldsymbol{r}$ is given by

$$v^2 \;=\; v_r^2 + v_\varphi^2$$

with the velocity components v_r and v_φ in cylindrical coordinates from eq. (7.136) and eq. (7.137). The constant u is again the velocity field far away from the cylinder where $v = u$ applies to the velocity and $P = P_\infty$ to the pressure. Inserting v_r and v_φ into eq. (7.139) gives the pressure P

$$P \;=\; \frac{\rho u^2}{2} + P_\infty - \frac{\rho}{2}\left\{u^2\cos^2\varphi\left(1 - \frac{R^2}{r^2}\right)^2 + \left[-u\sin\varphi\left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi r}\right]^2\right\} \tag{7.140}$$

After carrying out the multiplications and some rearranging one has

$$2\frac{P - P_\infty}{\rho u^2} \;=\; -\frac{R^4}{r^4} + \frac{2R^2}{r^2}\cos 2\varphi + 2\sin\varphi\left(1 + \frac{R^2}{r^2}\right)\frac{\Gamma}{2\pi u r} - \left(\frac{\Gamma}{2\pi u r}\right)^2 \tag{7.141}$$

At the cylinder surface this expression simplifies to

$$2\frac{P - P_\infty}{\rho u^2} \;=\; -1 + 2\cos 2\varphi + 4\sin\varphi\frac{\Gamma}{2\pi u R} - \left(\frac{\Gamma}{2\pi u R}\right)^2 \tag{7.142}$$

If there is no circulation, i.e., $\Gamma = 0$ as in fig. 7.29a, eq. (7.141) becomes

$$2\frac{P - P_\infty}{\rho u^2} \;=\; -1 + 2\cos 2\varphi$$

Taking the derivative with respect to φ, one finds that this expression takes on its extreme value where $\sin 2\varphi = 0$, i.e., at $2\varphi = 0, \pi, 2\pi, 3\pi$ or at $\varphi = 0, \pi/2, \pi, 3\pi/2$. Evaluating the second derivative at these points shows, as the pressure field in fig. 7.30a illustrates, that the pressure P has its maximum values at $\varphi = 0$ and $\varphi = \pi$ and its minimum values at $\varphi = \pi/2$ and $\varphi = 3\pi/2$. Hence, the pressure is maximal at the stagnation points (that's why they are called that way). With some circulation, $\Gamma = -2\pi u R$ in fig. 7.29b, eq. (7.141) becomes

$$2\frac{P - P_\infty}{\rho u^2} \;=\; -2 + 2\cos 2\varphi - 4\sin\varphi$$

and one finds the extreme values of the pressure at $\varphi = -\pi/6$ and $\varphi = -7\pi/6$. Both points are pressure maxima of the corresponding pressure field in fig. 7.30b and in turn coincide with the stagnation points in fig. 7.29b. For a circulation of $\Gamma = -4\pi uR$ as in fig. 7.29c one obtains from eq. (7.141)

$$2\frac{P - P_\infty}{\rho u^2} = -5 + 2\cos 2\varphi - 8\sin\varphi$$

Since the stagnation points coincide in this case, one finds, as expected, only one extreme value for the pressure P in the associated pressure field in fig. 7.30c, namely at $\varphi = 3\pi/2$ where the pressure in eq. (7.142) has its maximum value.

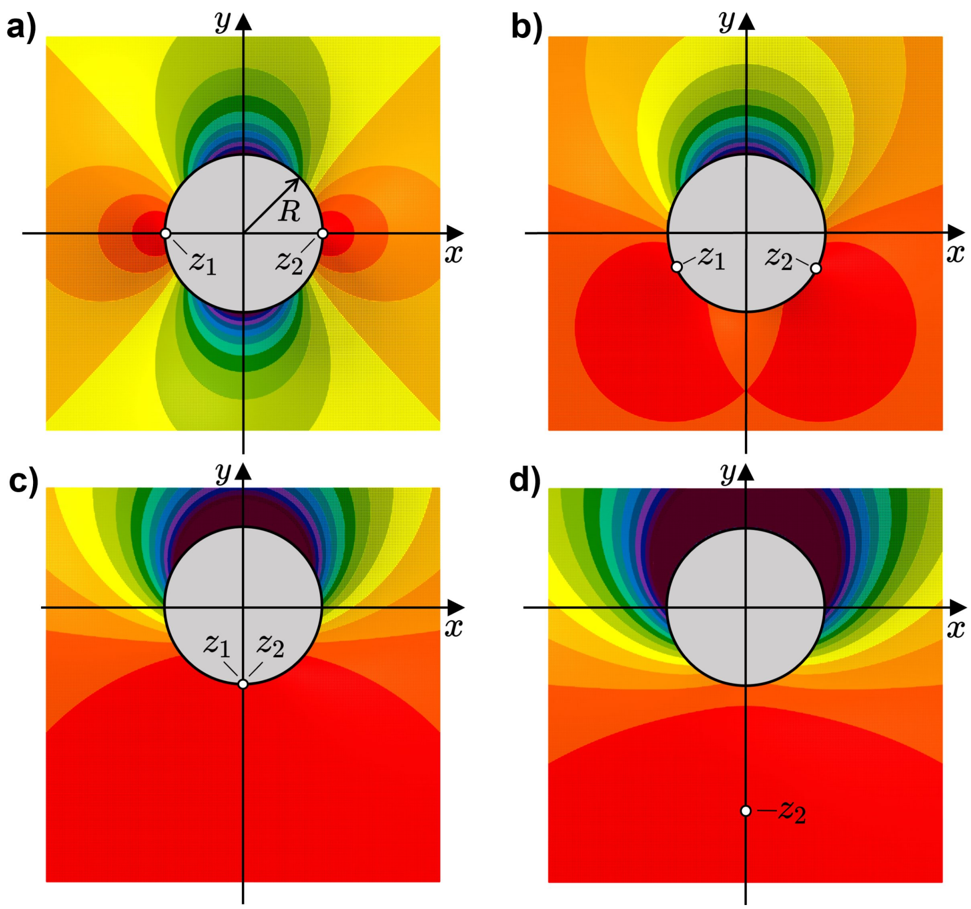

Fig. 7.30: Pressure field of the cylinder in the flow for the four cases from fig. 7.29 according to the normalized pressure difference from eq. (7.141). (a) no circulation, $\Gamma = 0$; (b) some circulation, $\Gamma = -2\pi uR$; (c) $\Gamma = -4\pi uR$; and (d) strong circulation with $\Gamma = -6\pi uR$. Each pressure field is divided into sixteen pressure levels (colors), from high pressure in red to low pressure in purple. The absolute pressure difference is not identical in the four cases but in (b) 2.25-times, in (c) 3.98-times and in (d) 6.23-times as large as in (a).

In the case of strong circulation, i.e., $\Gamma = -6\pi uR$ as in fig. 7.29d, the obvious assumption is that the location of maximum pressure again coincides with the stagnation point. The former is now no longer on the cylinder surface but on the negative y-axis in the flow field. In order to determine the exact location on the y-axis at which the pressure assumes an extreme value, one plugs into eq. (7.141) for the angle φ and the circulation Γ the values $\varphi = 3\pi/2$ and $\Gamma = -6\pi u R$ and measures r in units of R (i.e., $R = 1$); then one determines the extreme values as a function of r

$$2\frac{P - P_\infty}{\rho u^2} = -\frac{1}{r^4} - \frac{2}{r^2} - 6\sin\varphi\left(1 + \frac{1}{r^2}\right) - \left(\frac{6}{r}\right)^2$$

Differentiating with respect to r and setting the result equal to zero leads to the cubic equation

$$r^3 - \frac{11}{3}r^3 + 3r - \frac{2}{3} = 0$$

which can be solved with Cardano's method. After some calculation one gets for the absolute value of r

$$r = \frac{2}{9}\sqrt[3]{5 \cdot 16\sqrt{10}} \cdot \cos\left[\frac{1}{3}\arccos\left(\frac{5 \cdot 9}{32\sqrt{10}}\right)\right] + \frac{11}{3} = 2.618$$

That means $r = 2.618 \cdot R$. The location of the maximum pressure of the pressure field shown in fig. 7.30d is therefore congruent with the stagnation point of the flow field in fig. 7.29d which lies at $(x = 0, y = -2.618 \cdot R)$.

Force on a body in a flow

As can be seen from fig. 7.30, the pressure difference between the bottom ($\varphi = 3\pi/2$) and the top ($\varphi = \pi/2$) of the cylinder is zero without circulation and increases with increasing circulation. The force per unit length $\boldsymbol{f}$ acting on the cylinder due to this pressure difference is given by

$$\boldsymbol{f} = -\oint_C P \cdot \boldsymbol{n}\, \mathrm{d}s \tag{7.143}$$

where $\boldsymbol{n}$ is the normal unit vector on the line element $\mathrm{d}s$ along the cylinder boundary C. Eq. (7.143) applies to any arbitrary body in a flow as outlined in fig. 7.31. According to Bernoulli, with eq. (7.139) the pressure is given by

$$\frac{\rho v^2}{2} + P = const = P_0$$

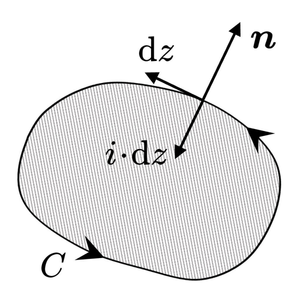

Fig. 7.31

Substituting into eq. (7.143) yields (P_0 is a constant)

$$\boldsymbol{f} = -\oint_C \left(P_0 - \frac{\rho v^2}{2}\right) \cdot \boldsymbol{n}\, \mathrm{d}s = \oint_C \frac{\rho v^2}{2} \cdot \boldsymbol{n}\, \mathrm{d}s$$

In complex representation, $\boldsymbol{n}\, \mathrm{d}s$ becomes $-i\mathrm{d}z$ with

$$-i\mathrm{d}z \;=\; -i(\mathrm{d}x + i\mathrm{d}y) \;=\; \mathrm{d}y - i\mathrm{d}x$$

and with $\boldsymbol{f} = (f_x, f_y)$ and $f = f_x + if_y$ one obtains for the force

$$f \;=\; -i\frac{\rho}{2}\oint\limits_C v^2\!\cdot\!\mathrm{d}z \tag{7.144}$$

On the boundary C, the boundary condition again applies that the normal component of the velocity $\boldsymbol{v}$ must vanish. With the velocity $\boldsymbol{v} = (v_x, v_y)$ and the line element $\mathrm{d}\boldsymbol{s} = (\mathrm{d}x, \mathrm{d}y)$ therefore holds

$$\boldsymbol{v}_\perp\cdot\mathrm{d}\boldsymbol{s} \;=\; \begin{pmatrix}-v_y\\ v_x\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} \;=\; v_x\mathrm{d}y - v_y\mathrm{d}x \;=\; 0 \tag{7.145}$$

With the complex velocity $w'(z) = v_x - iv_y$ one now considers the quantities $w'(z)\mathrm{d}z$ as well as $w'(z)^2\mathrm{d}z$. For $w'(z)\mathrm{d}z$ one obtains

$$\begin{aligned} w'(z)\mathrm{d}z \;&=\; (v_x - iv_y)(\mathrm{d}x + i\mathrm{d}y) \\ &=\; (v_x\mathrm{d}x + v_y\mathrm{d}y) + i(v_x\mathrm{d}y - v_y\mathrm{d}x) \\ &=\; \boldsymbol{v}\,\mathrm{d}\boldsymbol{s} + i\,\underbrace{\boldsymbol{v}_\perp\mathrm{d}\boldsymbol{s}}_{=\,0} \end{aligned}$$

where the last term vanishes because of eq. (7.145). Thus, one finds for $w'(z)^2\mathrm{d}z$, again using $v_x\mathrm{d}y = v_y\mathrm{d}x$ from eq. (7.145)

$$\begin{aligned} w'(z)^2\mathrm{d}z \;&=\; w'(z)\big(w'(z)\mathrm{d}z\big) \;=\; (v_x - iv_y)(v_x\mathrm{d}x + v_y\mathrm{d}y) \\ &=\; (v_x^2\mathrm{d}x + v_y^2\mathrm{d}x) - i(v_x^2\mathrm{d}y + v_y^2\mathrm{d}y) \;=\; v^2\overline{\mathrm{d}z} \end{aligned}$$

Thus applies $v^2\mathrm{d}z \;=\; \overline{w'(z)^2\mathrm{d}z}$ and eq. (7.144) can be rewritten as

$$f \;=\; -i\frac{\rho}{2}\oint\limits_C \overline{w'(z)^2\mathrm{d}z} \tag{7.146}$$

From eq. (7.127) one obtains for $w'(z)^2$

$$w'(z)^2 \;=\; v_\infty^2 + 2v_\infty\frac{\Gamma}{2\pi i}\frac{1}{z} + \mathscr{O}(z^{-2})$$

With $v_\infty \;=\; v_{x_\infty} - iv_{y_\infty}$ and the residue theorem, one then obtains for the complex conjugate of f in eq. (7.146)

$$\overline{f} \;=\; i\frac{\rho}{2}\oint\limits_C w'(z)^2\mathrm{d}z \;=\; i\frac{\rho}{2}\cdot\frac{2\Gamma}{2\pi i}(v_{x_\infty} - iv_{y_\infty})\cdot 2\pi i \;=\; \rho\Gamma(v_{y_\infty} + iv_{x_\infty})$$

and accordingly, for f

$$f = \rho\Gamma(v_{y\infty} - iv_{x\infty}) \qquad \text{or} \qquad \boldsymbol{f} = \rho\Gamma\begin{pmatrix} v_{y\infty} \\ -v_{x\infty}\end{pmatrix} = -\rho\Gamma\boldsymbol{v}_{\perp\infty} \tag{7.147}$$

Eq. (7.147) is the Kutta-Joukowski theorem. It states that f, the force acting on the body per unit length, is proportional to Γ and since Γ is real, this force is perpendicular to the complex velocity at infinity, i.e., to $v_\infty = v_{x\infty} - iv_{y\infty}$. According to equation eq. (7.147), a force only acts on the body if $|\Gamma| > 0$.

As for the force on the cylinder in a flow in fig. 7.28, with $\boldsymbol{v}_{\perp\infty} = (0, -u)$, one obtains from eq. (7.147) directly

$$\boldsymbol{f} = (0, -\rho\Gamma u) \tag{7.148}$$

One obtains the same result by directly integrating eq. (7.143). With eq. (7.140) the pressure on the cylinder surface is given by

$$P = P_\infty + \frac{\rho}{2}\left\{u^2 - \left(2u\sin\varphi + \frac{\Gamma}{2\pi R}\right)^2\right\} \tag{7.149}$$

With the normal unit vector $\boldsymbol{n} = (\cos\varphi, \sin\varphi)$ and the parameterization of the line element of the cylinder boundary $\mathrm{d}s = R\mathrm{d}\varphi$ one obtains for the force per unit length on the cylinder according to eq. (7.143)

$$\boldsymbol{f} = -\oint_C P\boldsymbol{n}\,\mathrm{d}s = -\int_0^{2\pi} P\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} R\mathrm{d}\varphi = \begin{pmatrix} 0 \\ -\rho\Gamma u\end{pmatrix} \tag{7.150}$$

As expected, this result corresponds to that of eq. (7.148), which was calculated directly from the theorem of Kutta-Joukowski in eq. (7.147). If one multiplies the force $\boldsymbol{f}$ acting on the circular cylinder per unit length in eq. (7.148) by the length of the cylinder, one gets the same solution which was already found in eq. (7.81) for the buoyancy force caused by the circulation around a body immersed in a flow.

Transfer of the results for the cylinder to arbitrary profiles

Conformal mapping is of great practical importance in determining the flow behavior of objects of arbitrary profile. On the one hand, the circulation is an invariant under conformal mapping. On the other hand, the angle preserving property of conformal mapping ensures that the boundary conditions on the surface of a body immersed in a flow will be correctly transformed. Through the combination of both properties, it becomes possible to determine from the known potential flow around a given profile the unknown potential flow around objects of different geometry.

Example 7.8 **Potential flow around a plate**

The potential flow around a plate is to be determined. For this purpose, the known potential flow around a circular cylinder is first converted into the potential flow around an ellipse. The corresponding conformal mapping is

$$z \longmapsto \widetilde{z} = f(z) = z + \frac{\Delta^2}{z} \tag{7.151}$$

where Δ is a scalable parameter. With $\Delta = 1$ and $f(z)$ provided with a prefactor $1/2$, eq. (7.151) is the so-called Joukowski function, which is helpful for many problems in fluid mechanics. Inserting $z = r\mathrm{e}^{i\varphi}$ into $f(z)$ and separating the real and imaginary parts gives for $\widetilde{z} = \widetilde{x} + i\widetilde{y}$

$$\widetilde{x} = \left(r + \frac{\Delta^2}{r}\right)\cos\varphi = \left(1 + \frac{\Delta^2}{r^2}\right)\cdot x$$

and

$$\widetilde{y} = \left(r - \frac{\Delta^2}{r}\right)\sin\varphi = \left(1 - \frac{\Delta^2}{r^2}\right)\cdot y$$

Substituting this into the circle equation $x^2 + y^2 = r^2$ results in

$$\frac{\widetilde{x}^2}{\left(1 + \frac{\Delta^2}{r^2}\right)^2} + \frac{\widetilde{y}^2}{\left(1 - \frac{\Delta^2}{r^2}\right)^2} = r^2$$

or respectively rewritten

$$\frac{\widetilde{x}^2}{\left(r + \frac{\Delta^2}{r}\right)^2} + \frac{\widetilde{y}^2}{\left(r - \frac{\Delta^2}{r}\right)^2} = \frac{\widetilde{x}^2}{a^2} + \frac{\widetilde{x}^2}{b^2} = 1 \tag{7.152}$$

Eq. (7.152) is the equation for an ellipse with the semi-axes

$$a = r + \frac{\Delta^2}{r} \qquad \text{and} \qquad b = r - \frac{\Delta^2}{r} \tag{7.153}$$

Now, let $r = R$ be the radius of the circular disk under consideration with the center at the origin. For $\Delta \to R$, the semi-major axis of the ellipse in eq. (7.153) goes towards $a = R$ and the semi-minor axis goes towards $b = 0$, so the ellipse turns into a plate. The transformation sketched in fig. 7.32 with two singularities at $\pm R$ maps the circular line uniquely into the boundary of the plate. The respective inverse mapping is given by

$$z = \frac{1}{2}\left(\widetilde{z} + \sqrt{\widetilde{z}^2 + 4R^2}\right) \tag{7.154}$$

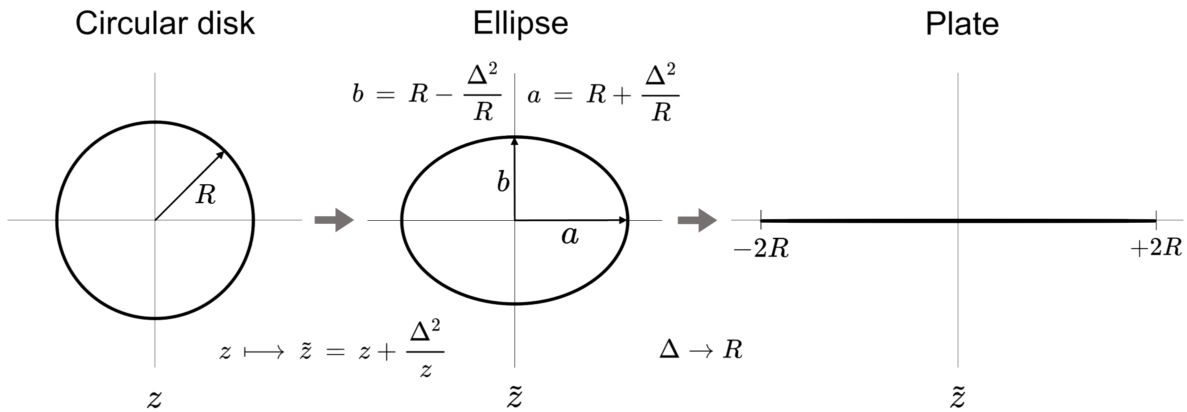

Fig. 7.32: Conformal mapping of a circular disk into a plate.

The complex velocity potential $w(z)$ is transformed according to the same rule as the circular line

$$w(z) \longmapsto \widetilde{w}(\widetilde{z}) \qquad \text{with} \qquad \widetilde{z} = z + \frac{R^2}{z}$$

If one looks at the complex velocity potential for the circular cylinder in a flow given by eq. (7.134), one recognizes that at infinity the flows around the circular cylinder and the plate are identical. So, it applies

$$w(z \to \infty) = u \cdot z \qquad \text{and} \qquad \widetilde{w}(\widetilde{z} \to \infty) = u \cdot \widetilde{z}$$

In addition, it is immediately clear that the transformation from circular cylinder to plate leaves the circulation unchanged. With eq. (7.150) it then also follows directly that the force on the plate and the force on the circular cylinder are identical. With the transformation of the streamlines, the stagnation points of the circular cylinder in the flow at $x = \pm R$ in fig. 7.29a are also converted into the corresponding points of the plate at $x = \pm 2R$. If one only considers the circulation term in eq. (7.134), i.e., $u = 0$ in eq. (7.134), then one gets

$$w(z) = \frac{\Gamma}{2\pi i} \ln z \qquad \text{and} \qquad \widetilde{w}(\widetilde{z}) = \frac{\Gamma}{2\pi i} \ln \widetilde{z}$$

for the vortex flow around the circular cylindrical disk and for the vortex flow around the plate. As for the respective stream functions, with eq. (7.133) it applies for the circular disk

$$\Psi(r,\varphi) = \Psi(r) = -\frac{\Gamma}{2\pi} \ln r \tag{7.155}$$

and accordingly, it applies for the plate

$$\widetilde{\Psi}(r,\varphi) = -\frac{\Gamma}{2\pi} \ln \left| z + \frac{R^2}{z} \right| \tag{7.156}$$

The stream function for the disk for $u = 0$ in eq. (7.155) does not depend on φ and only varies with the logarithm of the distance from the coordinate origin. For $u = 0$, the stream function of the circular cylinder in the flow consists of concentric circles. With respect to the corresponding stream function for the plate in eq. (7.156), one first considers the argument of the logarithm

$$\widetilde{z} = \widetilde{x} + i\widetilde{y} = r\mathrm{e}^{i\varphi} + \frac{R^2}{r}\mathrm{e}^{-i\varphi} = \left(r + \frac{R^2}{r}\right)\cos\varphi + i\left(r + \frac{R^2}{r}\right)\sin\varphi$$

It therefore applies (as already evident from the derivation of eq. (7.152) for $\Delta = R$)

$$\widetilde{x} = \left(r + \frac{R^2}{r}\right)\cos\varphi \qquad \text{and} \qquad \widetilde{y} = \left(r - \frac{R^2}{r}\right)\sin\varphi$$

The angle φ can be eliminated and one obtains with

$$\cos^2\varphi + \sin^2\varphi = 1 = \frac{\widetilde{x}^2}{\left(r + \dfrac{R^2}{r}\right)^2} + \frac{\widetilde{y}^2}{\left(r - \dfrac{R^2}{r}\right)^2}$$

the equation of an ellipse with the semi-axes

$$a = r + \frac{R^2}{r} \qquad \text{and} \qquad b = r - \frac{R^2}{r}$$

The focus points of the ellipse on the real axis are located at

$$c = \pm\sqrt{a^2 - b^2} = \pm 2R$$

and their positions are independent of r. This means that the concentric circles on which the stream function of the circular cylinder in the flow is constant are transformed into confocal ellipses on which the stream function for the plate in the flow is constant.

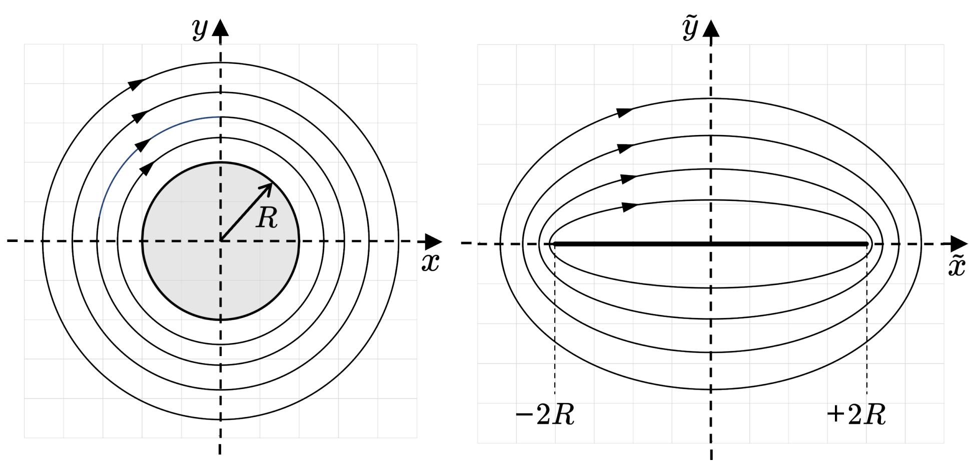

Fig. 7.33: Conversion of the circular flow around a circular cylinder (left) by means of conformal mapping into the corresponding flow around this circular cylinder transformed into a plate (right). The value selected for Γ corresponds to that from fig. 7.29b.

Fig. 7.33 shows the stream function for circulation around the circular cylinder and the transformed stream function for the corresponding circulation around the plate. The value for Γ chosen in fig. 7.33 is identical to the value from fig. 7.29b. However, in contrast to fig. 7.29b, in fig. 7.33 $u = 0$; the equipotential lines of the stream function in fig. 7.33 correspond to the equipotential lines 2, 3, 4 and 5 in fig. 7.29b (counted from the edge of the cylinder).

The example shown here illustrates how the known flow behavior of an object can be used to determine the unknown flow behavior of another object via conformal mapping. However, this is only possible with certain restrictions, such as the above-mentioned singularities at $\widetilde{x} = \pm 2R$, which in this case lead to unphysical solutions for the flow at these points. The understanding of the flow behavior directly at the surface of the plate requires a separate consideration at the ends of the plate. One possibility is for example to analyze the flow behavior of the object in a twisted flow, i.e., to rotate the object in the flow.

Appendix

I) From Vector Analysis

Summary of important equations

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \boldsymbol{A}\right) = 0 \tag{A.1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Lambda = 0 \tag{A.2}$$

$$\boldsymbol{\nabla}(\Lambda_1\Lambda_2) = (\boldsymbol{\nabla}\Lambda_1)\Lambda_2 + \Lambda_1(\boldsymbol{\nabla}\Lambda_2) \tag{A.3}$$

$$\boldsymbol{\nabla}(\Lambda\boldsymbol{A}) = (\boldsymbol{\nabla}\Lambda)\boldsymbol{A} + \Lambda(\boldsymbol{\nabla}\boldsymbol{A}) \tag{A.4}$$

$$\boldsymbol{\nabla} \times (\Lambda\boldsymbol{A}) = \Lambda(\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \times \boldsymbol{\nabla}\Lambda \tag{A.5}$$

$$\boldsymbol{\nabla}(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + (\boldsymbol{A}\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\boldsymbol{\nabla})\boldsymbol{A} \tag{A.6}$$

$$\boldsymbol{\nabla}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B}(\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A}(\boldsymbol{\nabla} \times \boldsymbol{B}) \tag{A.7}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\big(\boldsymbol{\nabla}\boldsymbol{B}\big) - \boldsymbol{B}\big(\boldsymbol{\nabla}\boldsymbol{A}\big) + \big(\boldsymbol{B}\boldsymbol{\nabla}\big)\boldsymbol{A} - \big(\boldsymbol{A}\boldsymbol{\nabla}\big)\boldsymbol{B} \tag{A.8}$$

When can $\boldsymbol{\nabla} \cdot \boldsymbol{A}$ be chosen freely?

Assertion:
For the vector potential $\boldsymbol{A}$ of a vector field $\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A}$, $\boldsymbol{\nabla} \cdot \boldsymbol{A}$ can be chosen appropriately, in particular $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ can be chosen.

Proof:
Suppose there are two vector potentials $\boldsymbol{A}$ and $\boldsymbol{A'}$ for which applies

$$\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A} \qquad \text{and} \qquad \boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A}'$$

Then for the quantity

$$\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{A}' \quad \text{must apply} \quad \boldsymbol{\nabla} \times \boldsymbol{B} = 0$$

However, this means that $\boldsymbol{B}$ must be the gradient of a scalar field Λ because according to eq. (A.2) the rotation of a gradient vanishes. Hence, it must apply

$$\boldsymbol{A} \;=\; \boldsymbol{A}' + \boldsymbol{\nabla}\Lambda$$

Except for the gradient of an arbitrary scalar field Λ, the vector potential $\boldsymbol{A}$ is uniquely determined. Because of

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} \;=\; \boldsymbol{\nabla} \cdot \boldsymbol{A}' + \boldsymbol{\nabla}^2\Lambda$$

$\boldsymbol{\nabla}^2\Lambda$ can therefore be chosen appropriately to give $\boldsymbol{\nabla} \cdot \boldsymbol{A}$ a desired value, one such value being of course $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.

The Weber Transform

The Weber transform reads

$$\big(\boldsymbol{F}\cdot\boldsymbol{\nabla}\big)\boldsymbol{F} \;=\; \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{F}^2 - \big[\boldsymbol{F} \times \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)\big] \tag{A.9}$$

For the proof of eq. (A.9) one writes out the right-hand side explicitly. First one writes curl $\boldsymbol{F}$ as

$$\boldsymbol{\nabla} \times \boldsymbol{F} \;=\; \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \;=\; \begin{pmatrix} \partial F_z/\partial y - \partial F_y/\partial z \\ \partial F_x/\partial z - \partial F_z/\partial x \\ \partial F_y/\partial x - \partial F_x/\partial y \end{pmatrix} \;=\; \begin{pmatrix} \big(\mathrm{rot}\boldsymbol{F}\big)_x \\ \big(\mathrm{rot}\boldsymbol{F}\big)_y \\ \big(\mathrm{rot}\boldsymbol{F}\big)_z \end{pmatrix}$$

With this one can write out the second term on the right side of eq. (A.9) as

$$\boldsymbol{F} \times \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \;=\; \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \times \begin{pmatrix} \big(\mathrm{rot}\boldsymbol{F}\big)_x \\ \big(\mathrm{rot}\boldsymbol{F}\big)_y \\ \big(\mathrm{rot}\boldsymbol{F}\big)_z \end{pmatrix} \;=\; \begin{pmatrix} F_y\big(\mathrm{rot}\boldsymbol{F}\big)_z - F_z\big(\mathrm{rot}\boldsymbol{F}\big)_y \\ F_z\big(\mathrm{rot}\boldsymbol{F}\big)_x - F_x\big(\mathrm{rot}\boldsymbol{F}\big)_z \\ F_x\big(\mathrm{rot}\boldsymbol{F}\big)_y - F_y\big(\mathrm{rot}\boldsymbol{F}\big)_x \end{pmatrix}$$

$$= \begin{pmatrix} F_y\left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right) - F_z\left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right) \\ F_z\left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right) - F_x\left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right) \\ F_x\left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right) - F_y\left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right) \end{pmatrix}$$

Now one considers the first term on the right side of eq. (A.9)

$$\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{F}^2 = \frac{1}{2}\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\cdot\left(F_x^2+F_y^2+F_z^2\right) = \begin{pmatrix}F_x\dfrac{\partial F_x}{\partial x}+F_y\dfrac{\partial F_y}{\partial x}+F_z\dfrac{\partial F_z}{\partial x}\\ F_x\dfrac{\partial F_x}{\partial y}+F_y\dfrac{\partial F_y}{\partial y}+F_z\dfrac{\partial F_z}{\partial y}\\ F_x\dfrac{\partial F_x}{\partial z}+F_y\dfrac{\partial F_y}{\partial z}+F_z\dfrac{\partial F_z}{\partial z}\end{pmatrix}$$

With these two written-out terms of the right-hand side of eq. (A.9) one gets, after back insertion and a little calculation, for the right-hand side of eq. (A.9)

$$\begin{aligned}\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{F}^2-\left[\boldsymbol{F}\times\left(\boldsymbol{\nabla}\times\boldsymbol{F}\right)\right] &= \begin{pmatrix}F_x\dfrac{\partial F_x}{\partial x}+F_y\dfrac{\partial F_x}{\partial y}+F_z\dfrac{\partial F_x}{\partial z}\\ F_x\dfrac{\partial F_y}{\partial x}+F_y\dfrac{\partial F_y}{\partial y}+F_z\dfrac{\partial F_y}{\partial z}\\ F_x\dfrac{\partial F_z}{\partial x}+F_y\dfrac{\partial F_z}{\partial y}+F_z\dfrac{\partial F_z}{\partial z}\end{pmatrix}\\ &= \left(F_x\frac{\partial}{\partial x}+F_y\frac{\partial}{\partial y}+F_z\frac{\partial}{\partial z}\right)\cdot\begin{pmatrix}F_x\\ F_y\\ F_z\end{pmatrix}\\ &= \left(\boldsymbol{F}\boldsymbol{\nabla}\right)\boldsymbol{F}\end{aligned}$$

But that is exactly the left side of eq. (A.9).

Graßmann Identity

For the vector product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ the Graßmann identity applies:

$$\boldsymbol{a}\times\left(\boldsymbol{b}\times\boldsymbol{c}\right) = \left(\boldsymbol{a}\cdot\boldsymbol{c}\right)\cdot\boldsymbol{b}-\left(\boldsymbol{a}\cdot\boldsymbol{b}\right)\cdot\boldsymbol{c}$$

or respectively

$$\left(\boldsymbol{a}\times\boldsymbol{b}\right)\times\boldsymbol{c} = \left(\boldsymbol{a}\cdot\boldsymbol{c}\right)\cdot\boldsymbol{b}-\left(\boldsymbol{b}\cdot\boldsymbol{c}\right)\cdot\boldsymbol{a}$$

II) From Function Theory

Complex Differentiability

In function theory, a complex-valued function is called holomorphic if it is complex differentiable at every point in its domain of definition D. Let

$$f(x+iy) \;=\; u(x,y) \,+\, iv(x,y)$$

be holomorphic on D. By special choice of $h = t$ or $h = it$ $(t \in \mathbb{R})$ the derivative becomes

$$f\,'(z) \;=\; \lim_{t\to 0} \frac{f(z+t)-f(z)}{t} \;=\; u_x \,+\, iv_x$$

$$f\,'(z) \;=\; \lim_{t\to 0} \frac{f(z+it)-f(z)}{it} \;=\; \frac{u_y}{i} + v_y \;=\; v_y - iu_y$$

Therefore, the following differential equations apply to the real or respectively imaginary part of $f\,'(z)$:

$$u_x = v_y \quad , \quad v_x = -u_y$$

These so-called Cauchy-Riemann differential equations follow directly from the differentiability of $f(z)$. Conversely, if u, v are continuously differentiable solutions of the Cauchy-Riemann differential equations on D, then $f(z)$ is a holomorphic function on D.

<u>On the derivative of $f(z)$:</u>

$$f(z+h) \,-\, f(z) \;=\; \operatorname{grad} u \cdot (\operatorname{Re} h, \operatorname{Im} h) \,+\, i \cdot \operatorname{grad} v \cdot (\operatorname{Re} h, \operatorname{Im} h) \,+\, \operatorname{rest}(h)$$

where $\displaystyle\lim_{h\to 0} \frac{\operatorname{rest}(h)}{|h|} = 0\,;$ With that:

$$\begin{aligned} f(z+h) \,-\, f(z) \;&=\; (u_x, u_y) \cdot (\operatorname{Re} h, \operatorname{Im} h) \,+\, i \cdot (-u_y, u_x) \cdot (\operatorname{Re} h, \operatorname{Im} h) \,+\, \operatorname{rest}(h) \\ &=\; u_x \cdot \underbrace{(\operatorname{Re} h + i \cdot \operatorname{Im} h)}_{h} \,-\, iu_y \cdot \underbrace{(\operatorname{Re} h + i \cdot \operatorname{Im} h)}_{h} \,+\, \operatorname{rest}(h) \\ &=\; u_x h \,-\, iu_y h \,+\, \operatorname{rest}(h) \end{aligned}$$

From this it follows: f is complex differentiable in z with derivative

$$f'(z) \;=\; u_x \,-\, iu_y$$

Cauchy's Integral Formula

Let f be holomorphic on D. For each $z_0 \in D$ and each $r > 0$, such that the closed disk $|z - z_0| \leq r$ is included in D, applies

$$\boxed{f(z) \;=\; \frac{1}{2\pi i}\int\limits_{\gamma_r} \frac{f(w)}{w-z}\,\mathrm{d}w}$$

Here $|z - z_0| \;<\; r$ and $\gamma_r \;=\; z_0 + r\mathrm{e}^{it}\,,\; t \in [0, 2\pi]\,.$

General Version of Cauchy's Integral Formula

Let f be holomorphic in the region $D \subset \mathbb{C}$ and $\gamma : I \mapsto D$ be a closed curve such that for all $z_0 \in \mathbb{C} \setminus D$ the winding number $n_\gamma(z_0) = 0$ (any holes present in D should not be enclosed by γ). Then

$$\boxed{n_\gamma(z) f(z) \;=\; \frac{1}{2\pi i}\oint\limits_{\gamma} \frac{f(w)}{w-z}\,\mathrm{d}w} \qquad z \in D \setminus \gamma(I)$$

A **proof** from Liouville's theorem after Dixon (1971):

$$E \;:=\; \left\{z \in \mathbb{C} \setminus \gamma(I)\,;\;\; n_\gamma(z) \;=\; 0\right\}$$

According to the above premise $\mathbb{C} = D \,\cup\, E$

1) Auxiliary function for $(z, w) \in D \times D$

$$g(z,w) \;=\; \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if} \quad w \neq z \\[2ex] f'(w) & \text{if} \quad w = z \end{cases}$$

g is continuous in $D \times D$ and for each w holomorphic in z.

2) By integrating one obtains a second auxiliary function

$$G(z) \;=\; \begin{cases} \displaystyle\int\limits_{\gamma} g(z,w)\,\mathrm{d}w & \text{if} \quad z \in D \\[2ex] \displaystyle\int\limits_{\gamma} \frac{f(w)}{w-z}\,\mathrm{d}w & \text{if} \quad z \in E \end{cases}$$

On $z \in D \,\cap\, E$, both expressions for $G(z)$ are identical because there the winding number n_γ is 0:

$$\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \;=\; \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \;-\; 2\pi i n_\gamma(z) f(z)$$

$$= \int_\gamma \frac{f(w)-f(z)}{w-z}\,\mathrm{d}w \;=\; \int_\gamma g(w,z)\,\mathrm{d}w$$

Hence $G(z)$ is an entire function. If $z \in E$, then it holds for $G(z)$ that $\lim\limits_{z\to\infty} G(z) = 0$. Hence $G(z)$ is a bounded entire function and thus by Liouville's theorem constant with $G(z) = 0$ for all z.

3) The first formula applies for the $z \in D \setminus \gamma(I)$. Result:

$$0 \;=\; G(z) \;=\; \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \;-\; \int_\gamma \frac{f(z)}{w-z}\,\mathrm{d}w$$

that means

$$n_\gamma(z) f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w$$

□

General Version of Cauchy's Integral Theorem

Let f be holomorphic on the region $D \subset \mathbb{C}$ and $\gamma : I \mapsto D$ be a closed curve with winding number $n_\gamma(z_0) = 0 \;\; \forall\, z_0 \in \mathbb{C} \setminus D$. Then

$$\boxed{\oint_\gamma f(w)\,\mathrm{d}w \;=\; 0}$$

Proof:

By inserting the two functions $f(z)$, $zf(z)$ into the integral formula.

For $z \in D$:

$$\frac{1}{2\pi i}\int_\gamma f(w)\,\mathrm{d}w \;=\; \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{w f(w)}{w-z}\,\mathrm{d}w}_{n_\gamma(z)\,(z\,f(z))}$$

$$- \underbrace{\frac{1}{2\pi i} \int\limits_{\gamma} \frac{z f(w)}{w - z} \, \mathrm{d}w}_{z \cdot n_\gamma(z) f(z)} \;=\; 0$$

$\square$

Definition of "simply connected":

A region $D \subset \mathbb{C}$ is called "simply connected" if for every closed curve $\gamma : \, I \mapsto D$ and all $z_0 \in \mathbb{C} \setminus D$ the winding number $n_\gamma(z_0) = 0$.

Remark:

(1) For functions f which are holomorphic on simply connected regions D the following applies to every closed curve $\gamma : \, I \mapsto D$

$$\oint\limits_{\gamma} f(w) \, \mathrm{d}w \;=\; 0$$

The Residue Theorem

Let there be given an analytic function $f(z)$ in the region $D \subset \mathbb{C}$. Furthermore, let $\gamma : \, I \mapsto D$ be a closed curve whose winding number $n_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus D$ apart from finitely many exceptions $z_1, z_2, \ldots, z_N$. Then:

$$\boxed{\frac{1}{2\pi i} \oint\limits_{\gamma} f(w) \, \mathrm{d}w \;=\; \sum_{k=1}^{N} n_\gamma(z_k) \, \mathrm{Res}\,(f, z_k)}$$

Proof:

$z_1, z_2, \ldots, z_N$ are isolated singularities in a punctured circular disk $0 < |z - z_k| < \varrho$ in which one can describe $f(z)$ by a Laurent series

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n^{(k)} (z - z_k)^n$$

Let

$$f_k(z) \;=\; \sum_{n=1}^{\infty} a_{-n}^{(k)} (z - z_k)^{-n}$$

be the principal part of f at z_k. Then

$$g(z) \;=\; f(z) \,-\, f_1(z) \,-\, \ldots \,-\, f_N(z)$$

is even in $D' = D \cup \{z_1, \ldots, z_N\}$ holomorphic (note: $f_k(z)$ is holomorphic in $\mathbb{C} \setminus \{z_k\}$). Then by Cauchy's integral theorem

$$\int\limits_\gamma g(w)\,\mathrm{d}w \;=\; 0$$

that is

$$\int\limits_\gamma f(w)\,\mathrm{d}w \;=\; \sum_{k=1}^{N} \int\limits_\gamma f_k(w)\,\mathrm{d}w$$

In

$$f_k(z) \;=\; \frac{a_{-1}^{(k)}}{(z-z_k)} + \sum_{k=2}^{\infty} a_{-n}^{(k)} (z-z_k)^{-n}$$

the second sum term has a primitive function in $\mathbb{C} \setminus \{z_k\}$, that means

$$\int\limits_\gamma f_k(w)\,\mathrm{d}w \;=\; \int\limits_\gamma \frac{a_{-1}^{(k)}}{w-z_k}\,\mathrm{d}w \;=\; \mathrm{Res}\,(f, z_k) \cdot 2\pi i n_\gamma(z_k)$$

□

z_0 is called a non-essential singularity or respectively a pole of the n-th order, if a $n \in \mathbb{N}$ exists such that $(z-z_0)^n f(z)$ can be holomorphically continued to z_0. For the residue then holds

$$\mathrm{Res}\,(f, z=z_0) \;=\; a_{-1}$$

and a_{-1} can be calculated without having to determine the Laurent series of $f(z)$. If $f(z)$ has a pole of n-th order in z_0, then it holds

$$a_{-1} \;=\; \frac{1}{(n-1)!} \lim_{z-z_0} \left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \big[(z-z_0)^n f(z) \big] \right)$$

III) Frequently Used Approximations

According to the Taylor expansion around $x = 0$, for small x apply the following approximations:

$$\sqrt{1+x} \;=\; 1 + \frac{x}{2} + \ldots \tag{A.10}$$

and

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \dots \tag{A.11}$$

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