HIGHER MATHEMATICS

Lectures

Part Four

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Preface

At German universities, lectures on higher mathematics are an integral part of the curriculum in natural and engineering sciences. These lectures aim to provide students with the mathematical foundations for their respective subject areas, typically in the first four semesters. This was also the case for me when a good forty years ago at the beginning of my physics studies I first entered the lecture hall of the Technische Universität München (TUM), the place where Prof. Dr. Armin Leutbecher taught Higher Mathematics. I realize of course that not everyone can or wants to share the same enthusiasm for mathematics. However, I hope that those who are reading these lines will understand what I mean when I say that to me as a student those mathematics courses have been a real source of happiness. Happiness in the sense that back then I always looked forward to each and every one of these lectures. This certainly did not only have to do with the content of the lectures, but at least as much with the way they were delivered by Prof. Leutbecher. Of course, one always expects clarity from a mathematician. But the clarity with which professional mathematicians generally conduct their discussions does not necessarily carry over to how a mathematician might then impart his knowledge to students. Prof. Leutbecher's clarity and style of delivery made his Higher Mathematics lectures an intellectual delight. In addition, I also had the good fortune that the exercises for Prof. Leutbecher lectures were given by Dr. Peter Vachenauer. At the beginning of the 1990s the first edition of a two-volume textbook on Higher Mathematics co-authored by Dr. Vachenauer was published. The exemplary methodology and care with which the lecture materials were studied during my time at TUM in Dr. Vachenauer's tutorial exercises is reflected in this textbook.

A little over a year ago, while tidying up, I stumbled across my transcripts of the Higher Mathematics lectures from the years 1981-1983 and the corresponding exercises. At first I was surprised that these forty-year-old documents were not lost during various moves over four decades, some of them between continents. When I then curiously began to leaf through my rediscovered lecture notes I all of a sudden experienced the same kind of joy which I once felt when I was sitting in the lecture hall, listening spellbound to Prof. Leutbecher's lectures some forty years ago. Although these notes, my transcript of Prof. Leutbecher's lectures, cannot replace a textbook, they do convey the essential content of Higher Mathematics with a vividness that I believe should make them a reading pleasure for students or anyone else seriously interested in mathematics. All too often such lecture notes are riddled with errors, and this was no different here. After reviewing and correcting my notes several times, hopefully the vast majority of them have been corrected. Preserving the clarity and style of Prof. Leutbecher's lectures, as I captured them in my notes more than forty years ago, was something I attached great importance to when revising my notes. Translating those notes from their original German into English added of course another challenge. Quite likely some of the elegance of the German language lectures may have been lost in translation. However, I do hope that the English language version still conveys the essence of the lectures original style and clarity. This volume, **HIGHER MATHEMATICS - Lectures Part Four**, contains the material of the Higher Mathematics IV lectures as given by Prof. Leutbecher in the summer semester 1983 at the TUM.

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1. Elementary Solution Methods for Ordinary Differential Equations

Let f be real and continuous on the interval I and let g be real and continuous on the interval J. With that

$$\boxed{y' \;=\; f(x)g(y) \qquad (1)}$$

declares the differential equation with separate variables.

Ansatz: $\quad \dfrac{y'}{g(y)} \;=\; f(x)$

Integrating both sides over x yields $\; G(y) \;=\; F(x)$. Solve for y.

<u>Theorem 1:</u>

Suppose g is continuous and without zeros on J. For initial conditions $(x_0, y_0) \in D = I \times J$ consider

$$F(x) \;=\; \int\limits_{x_0}^{x} f(t)\,\mathrm{d}t \quad , \quad G(y) \;=\; \int\limits_{y_0}^{y} \frac{\mathrm{d}t}{g(t)} \qquad x \in I\,,\;\; y \in J$$

Let I' be a subinterval of I with $F(I') \subset G(J)\,,\; x_0 \in I\,,'$. Then there exists exactly one solution φ on I' with $\varphi(x_0) = y_0$ and for that

$$G(\varphi(x)) \;=\; F(x)$$

Proof of Theorem 1:

1) $F(x) \;=\; \displaystyle\int\limits_{x_0}^{x} f(\xi)\,\mathrm{d}\xi \qquad G(y) \;=\; \int\limits_{y_0}^{y} \frac{\mathrm{d}\eta}{g(\eta)}$

Since $g(\eta)$ has a fixed sign on J, $G(y)$ becomes strictly monotonic with a differentiable inverse function

$$H \; : \; G(J) \;\longrightarrow\; J$$

with derivative: $\quad H'(w) \;=\; \dfrac{1}{G'(H(w))}$

2) Let $I' \subset I$ with $x_0 \in I'$, $F(I') \subset G(J)$. Furthermore $\varphi \,:\, I' \mapsto \mathbb{R}$ solution of differential equation (1) with $\varphi(x_0) = y_0$. Then

$$\frac{\varphi'(\xi)}{g(\varphi(\xi))} \;=\; f(\xi)$$

Integration over ξ from x_0 to x

$$F(x) \;=\; \int_{x_0}^{x} f(\xi)\,\mathrm{d}\xi \;=\; \int_{x_0}^{x} \frac{\varphi'(\xi)}{g(\varphi(\xi))}\,\mathrm{d}\xi \qquad \text{substitution: } \eta = \varphi(\xi)$$

$$= \int_{\varphi(x_0)=y_0}^{\varphi(x)} \frac{\mathrm{d}\eta}{g(\eta)} \;=\; G(\varphi(x)) \;-\; G(y_0)$$

According to the definition of $G(y)$, $G(y_0) = 0$. That means

$$G(\varphi(x)) \;=\; F(x)$$

Apply H

$$\varphi(x) \;=\; HF(x) \qquad (*)$$

$F(I') \subset G(J)$ is required here. Thus, on the one hand, uniqueness is proven, on the other hand, $(*)$ defines a differentiable function φ with value

$$\varphi(x_0) \;=\; H(F(x_0)) \;=\; H(0) \;=\; y_0$$

and with the derivative

$$\varphi'(x) \;=\; H'(F(x))F'(x) \;=\; \frac{1}{G'(H(F(x)))}\,f(x) \;=\; g(\varphi(x))f(x)$$

□

Remarks:

(1) In the formula $G(\varphi(x)) \;=\; F(x)$, φ is given implicitly.

(2) Uniqueness could be proven here without the Lipschitz condition.

Example:

(1) $y' \;=\; y^2 \;,\quad f(x) \;=\; 1 \qquad \forall\, x \in I = \mathbb{R}$

$g(y) \;=\; y^2 \qquad J = \,]0,\infty[\;$ or $\;J = \,]-\infty, 0[$

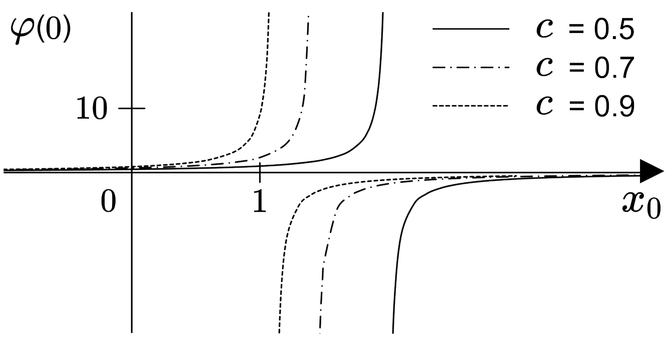
The initial condition $\varphi(0) = 0$ does not fall under Theorem 1. Nevertheless, exactly one solution exists with $\varphi(0) = 0$, namely the zero function. If $y_0 = c \neq 0$ then

$$G(y) = \int_c^y \frac{\mathrm{d}\eta}{\eta^2} = \frac{1}{c} - \frac{1}{y}$$

$$G(J) = \begin{cases} \left]-\infty, \frac{1}{c}\right[&, \text{ if } c > 0 \\ \left]\frac{1}{c}, +\infty\right[&, \text{ if } c < 0 \end{cases}$$

$$\frac{1}{c} - \frac{1}{\varphi(x)} = G(\varphi(x)) = x - x_0$$

$$\varphi(x) = \frac{1}{\frac{1}{c} - (x - x_0)} = \frac{c}{1 - c(x - x_0)}$$



$x = 0$:

$$\varphi(0) = \frac{c}{1 + cx_0}$$

$$\boxed{y' = f\left({}^{x}/_{y}\right) \qquad (2)}$$

(so-called homogeneous differential equation)

Not to be confused with the homogeneous linear differential equation.

f is declared and continuous on an interval J

$J =]0, \infty[$ or $J =]-\infty, 0[$

$D = \left\{(x, y)\,;\, x > 0\,,\, \frac{x}{y} \in J\right\}$ or $D = \left\{(x, y)\,;\, x < 0\,,\, \frac{x}{y} \in J\right\}$

Theorem 2:

Let f be continuous on J and $(x_0, y_0) \in D$. The differentiable function φ declared on the interval I is the solution of the differential equation (2) with initial condition $\varphi(x_0) = y_0$ if and only if there the function

$$\psi(x) = \frac{\varphi(x)}{x}$$

is solution of the differential equation

$$\boxed{z' = \frac{f(z) - z}{x} \qquad (3)} \qquad \text{with} \quad \psi(x_0) = \frac{y_0}{x_0}$$

Remark:

(3) Theorem 2 reduces the homogeneous differential equation (2) to a special differential equation with separate variables.

Proof sketch:

Let φ be the solution of (2) with $\varphi(x_0) = y_0$.

$$\psi(x) = \frac{\varphi(x)}{x}$$

is differentiable on I with derivative

$$\psi'(x) = \frac{x\varphi'(x) - \varphi(x)}{x^2} = \frac{xf\left(\frac{\varphi(x)}{x}\right) - \varphi(x)}{x^2} = \frac{f(\psi(x)) - \psi(x)}{x}$$

Analogously for the reverse direction.

$\square$

Examples:

(2) $y' = \dfrac{y}{x} + \dfrac{y^2}{x}$; $f(t) = t + t^2$ declared on $J = \mathbb{R}$

$$z' = \frac{z^2}{x} \qquad (3) \qquad \text{separate variables}$$

Ansatz $f(x) = \dfrac{1}{x}$ und $g(z) = z^2$. With that

$$G(z) = \int_{z_0}^{z} \frac{\mathrm{d}w}{w^2} = \frac{1}{z_0} - \frac{1}{z}$$

$$F(x) = \int_{x_0}^{x} \frac{\mathrm{d}t}{t} = \ln x_0 - \ln x$$

Because of $G(\psi(x)) = F(x)$ (Theorem 1)

$$\frac{1}{z_0} - \frac{1}{\psi(x)} = \ln \frac{x}{x_0}$$

(3) $y' = \dfrac{y}{x} + \sqrt{1 - \dfrac{y^2}{x^2}} \qquad x < 0 \quad \text{or respectively} \quad x > 0$

$$D_\pm = \{(x,y)\,;\, \operatorname{sgn} x = \pm 1\,,\ y/x \in\,]-1,1[\}$$

Solution ansatz: $y = xz$

$$y' = xz' + z = z + \sqrt{1-z^2}$$

$$z' = \frac{\sqrt{1-z^2}}{x} \qquad \text{separate variables}$$

$$\frac{z'}{\sqrt{1-z^2}} = \frac{1}{x} \qquad \text{integration gives}$$

$$\arcsin(z) - \arcsin(z_0) = \ln\left|\frac{x}{x_0}\right|$$

$$\arcsin(z) = \ln|x| + \text{const}$$

$$z = \sin(\ln|x| + \text{const})$$

$$y(x) = x\sin(\ln|x| + \text{const})$$

Solution for $y = \pm x$ is missing (not captured by the standard method).

1.1 Euler's Multiplier

Recall the description of the level lines of a continuously differentiable function $U(x,y)$ by implicit functions, that means curves $U(x,y) = c$.

If $U(x_0,y_0) = c$ and ${}^{\partial U}\!/_{\partial y} \neq 0$, then there exists a function $\psi(x)$ defined in a neighborhood of x_0 with

$$\psi(x_0) = 0 \qquad \text{and} \qquad U(x,\psi(x)) = c\,.$$

It is (if continuous) even differentiable with derivative

$$\psi'(x) = -\frac{\partial U}{\partial x}\Big/\frac{\partial U}{\partial y}$$

(Compare theorem about implicit functions.[1])

Analogously exists, if ${}^{\partial U}\!/_{\partial x} \neq 0$, a function $\varphi(y)$ with $\varphi(y_0) = x_0$ and $U(\varphi(y),y) = c$. Again φ (if continuous) is even differentiable with derivative

[1] See chapter 9, section 1 in HIGHER MATHEMATICS Lectures Part Two.

$$\varphi'(y) \;=\; -\frac{\partial U}{\partial y}\Big/\frac{\partial U}{\partial x}$$

Both functions are solutions of the differential equation

$$\boxed{A\mathrm{d}x + B\mathrm{d}y = 0 \qquad \text{(E)}} \qquad \text{with} \quad A = \frac{\partial U}{\partial x}\ , \quad B = \frac{\partial U}{\partial y}$$

The question of when the vector field $\big(A(x,y), B(x,y),\big)$ is the gradient of a scalar-valued function U is answered locally in the integrability criterion.

$$\frac{\partial A}{\partial y} \;=\; \frac{\partial B}{\partial x}$$

If this is the case, then $A\mathrm{d}x + B\mathrm{d}y = 0$ is called "exact". If (E) is not exact, one seeks to achieve with the help of a zero-free function $\mu(x,y)$ that μA, μB becomes exact.

The integrability criterion:

$$\frac{\partial}{\partial y}(\mu A) \;=\; \frac{\partial \mu}{\partial y}A + \mu\frac{\partial A}{\partial y}$$

$$\frac{\partial}{\partial x}(\mu B) \;=\; \frac{\partial \mu}{\partial x}B + \mu\frac{\partial B}{\partial x}$$

hence μ becomes an Euler multiplier (integrating factor) if and only if

$$\Big(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\Big)\mu \;=\; B\frac{\partial \mu}{\partial x} - A\frac{\partial \mu}{\partial y}$$

Simple cases: existence of a multiplier dependent only on x if

$$\Big(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\Big)\cdot\frac{1}{B} \quad \text{independent of } y$$

<u>Examples:</u>

(4) $$\big(\underbrace{a(x)y + b(x)}_{A}\big)\,\mathrm{d}x \;\underbrace{-\ \mathrm{d}y}_{-1=B} \;=\; 0$$

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \;=\; a(x)$$

Differential equation for the multiplier that depends only on x

$$\mu'(x) \;=\; \frac{\dfrac{\partial A}{\partial y} - \dfrac{\partial B}{\partial x}}{B}\,\mu \;=\; -a(x)\mu(x)$$

hence

$$\mu(x) \;=\; \exp\Big(-\int_{x_0}^{x} a(t)\,\mathrm{d}t\Big)$$

(5) The differential equation with separate variables

$$\underbrace{f(x)}_{A}\,\mathrm{d}x - \underbrace{\frac{\mathrm{d}y}{g(y)}}_{B} \;=\; 0 \qquad \text{satisfies the integrability criterion} \;\Rightarrow\; \text{exact.}$$

(6) $$(\underbrace{2x^2y + 2xy^3 + y}_{A})\,\mathrm{d}x \;+\; (\underbrace{3y^2 + x}_{B}) \;=\; 0$$

$$\frac{\partial A}{\partial y} \;=\; 2x^2 \;+\; 6xy^2 \;+\; 1 \quad ; \quad \frac{\partial B}{\partial x} \;=\; 1 \qquad \text{not exact}$$

$$\Big(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\Big)\cdot\frac{1}{B} \;=\; \frac{2x^2 + 6xy^2}{3y^2 + x} \;=\; 2x$$

Therefore a multiplier μ exists that only depends on x with

$$\mu'(x) \;=\; 2x\mu(x) \quad ; \quad \mu(x) \;=\; \mathrm{e}^{x^2}$$

Sought after is the scalar-valued function U with gradient $\mu A, \mu B$

$$\frac{\partial U}{\partial x} \;=\; (2x^2y + 2xy^3 + y)\mathrm{e}^{x^2} \quad , \quad \frac{\partial U}{\partial y} \;=\; (3y^2 + x)\mathrm{e}^{x^2}$$

Integration of the second equation

$$U(x,y) \;=\; (y^3 + xy)\mathrm{e}^{x^2} \;+\; V(x)$$

Substitute into the first equation

$$(\underbrace{y + (y^3 + xy)2x}_{2x^2y + 2xy^3 + y})\mathrm{e}^{x^2} \;+\; V'(x) \;=\; (2x^2y + 2xy^3 + y)\mathrm{e}^{x^2}$$

Hence $V(x)$ yields a U,

$$U(x,y) \;=\; (y^3 + xy)\mathrm{e}^{x^2}$$

<u>Result:</u> the level lines $U(x,y) = \text{const}$ are solutions of $A\mathrm{d}x + B\mathrm{d}y \;=\; 0$ (implicitly).

If in general the continuous vector field A, B is proportional to a gradient of a continuously differentiable potential function $U(x,y)$

$$\operatorname{grad} U(x,y) \;=\; \mu(A,B)$$

with continuous zero-free $\mu(x,y)$ and $x = \varphi(t)$, $y = \psi(t)$ is a parameterized level line of $U(x,y)$, i.e. $U(\varphi(t),\psi(t)) = \text{const}$, then the chain rule yields (for differentiable φ, ψ)

$$U\big(\varphi(t),\psi(t)\big)\cdot\big(\dot\varphi(t),\dot\psi(t)\big) \;=\; 0$$

Therefore

$$A\big(\varphi(t),\psi(t)\big)\cdot\dot\varphi(t) \,+\, B\big(\varphi(t),\psi(t)\big)\cdot\dot\psi(t) \;=\; 0$$

hence

$$x = \varphi(t) \;\;,\;\; y = \psi(t) \quad \text{solves} \quad A\dot x + B\dot y \;=\; 0$$

1.2 Autonomous Systems

are called systems

$$\begin{array}{l} \dot y_1 \;=\; f_1(y_1,y_2) \\ \dot y_2 \;=\; f_2(y_1,y_2) \end{array} \quad (\text{A}) \;, \qquad \text{vectorial:} \quad \dot{\vec y} \;=\; \vec f(\vec y)$$

in which the independent variable t (frequently the time) does not appear explicitly.

1.2.1 Class of Examples

The linear homogeneous systems with constant coefficients! If in $\vec y_0$ not at the same time $f_1(\vec y_0) = f_2(\vec y_0) = 0$ then at least one of the coordinate functions $y_i(t)$ is reversible at $\vec y_0$. Time then can be written locally as a function of y_1 (or respectively of y_2), the variable time can thus be eliminated from (A):

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1} \;=\; \frac{f_2(y_1,y_2)}{f_1(y_1,y_2)} \qquad \text{or respectively} \qquad \frac{\mathrm{d}y_1}{\mathrm{d}y_2} \;=\; \frac{f_1(y_1,y_2)}{f_2(y_1,y_2)}$$

Together:

$$f_2(y_1,y_2)\,\mathrm{d}y_1 \,-\, f_1(y_1,y_2)\,\mathrm{d}y_2 \;=\; 0$$

Remark:

(4) If $\vec\varphi(t)$ is declared in a neighborhood of t_0 and solution of the differential equation

$$\dot{\vec y} \;=\; \vec f(\vec y)$$

then

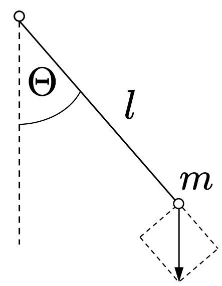
$$\vec{\psi}(t) \;=\; \vec{\varphi}(t-t_0)$$

becomes a solution declared in a zero neighborhood (of the time axis). Normalization option.

In the special points y_1, y_2 with $f_1(y_1,y_2)=0,\ f_2(y_1,y_2)=0$ lie the rest positions of the system, that means the constant functions $y_1(t)=y_1\,,\ y_2(t)=y_2$ solve the differential equation.

Example:

(1) The mathematical pendulum (without friction)



Pendulum arm of length l (massless)

$$\ddot{\Theta} \;=\; -\alpha^2 \sin\Theta \quad ; \quad \alpha^2 \;=\; \frac{g}{l}$$

$$\frac{1}{2}\dot{\Theta} \;=\; \alpha^2\cos\Theta \,+\, c$$

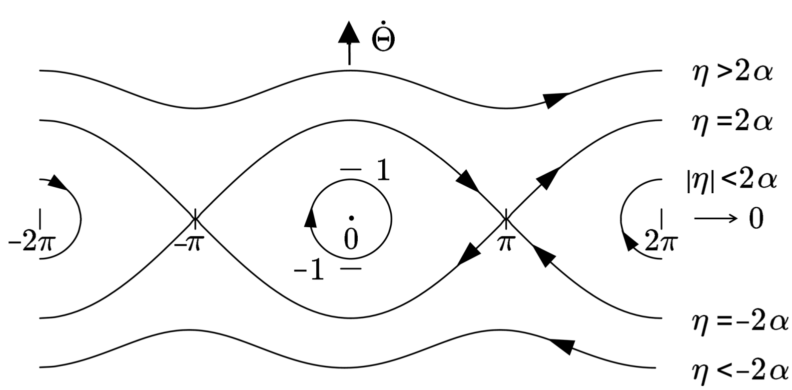
Ansatz: $y_1 \,=\, \Theta\,;\quad y_2 \,=\, \dot{\Theta}\,.$ (y_1,y_2)-plane is called "phase space" or state space. The differential equation of the phase curves

$$\dot{\Theta} \;=\; \pm\sqrt{2c+2\alpha^2\cos\Theta}\ ; \qquad \cos\Theta \;=\; 1-2\sin^2\frac{\Theta}{2}$$

$$\dot{\Theta} \;=\; \pm\sqrt{\eta^2-4\alpha^2\sin^2\frac{\Theta}{2}}$$

$\eta \;=\;$ velocity of Θ in the zero crossing.

Rest positions: $\quad \Theta \,=\, \pi m \quad ; \quad \dot{\Theta} \,=\, 0 \qquad (m\in\mathbb{R})$



At $2\alpha = \pm\eta$ separatrix: $\quad \dot{\Theta} \;=\; \pm 2\alpha\cos\dfrac{\Theta}{2}$

If not $\Theta \equiv 0$ and $\Theta = \pi m$ with odd m (unstable rest position) and $\Theta(t)$ a solution of the pendulum equation, then Θ will go through the zero position. Normalization $t_0 = 0\,,\ \Theta(t_0) = \Theta(0) = 0\,.$

Integration of the pendulum equation over time

Fall 1: $\eta > 2\alpha$ (orbiting pendulum)

Fall 2: $\eta = 2\alpha$ (movement on the separatrix)

Fall 3: $0 < \eta < 2\alpha$ (swinging pendulum)

Case 1: $\dot{\Theta} \;=\; \pm\sqrt{\eta^2 - 4\alpha^2\sin^2\dfrac{\Theta}{2}}$

Differential equation with separate variables

$$\int\limits_0^{\Theta(t)} \frac{\mathrm{d}\varphi}{\sqrt{\eta^2 - 4\alpha^2\sin^2\dfrac{\varphi}{2}}} \;=\; t$$

$$\frac{T}{2} \;=\; \int\limits_0^{\pi} \frac{\mathrm{d}\varphi}{\sqrt{\eta^2 - 4\alpha^2\sin^2\dfrac{\varphi}{2}}}\;, \qquad T \;=\; \text{orbital period}$$

$$\Theta(t+T) \;=\; \Theta(t) \,+\, 2\pi$$

Case 2: $\dot{\Theta} \;=\; 2\alpha\sqrt{1 - \sin^2\dfrac{\Theta}{2}}$

$$\frac{1}{2\alpha} \;=\; \int\limits_0^{\Theta(t)} \frac{\mathrm{d}\varphi}{\sqrt{1 - \sin^2\dfrac{\varphi}{2}}} \;=\; t \qquad \text{substitution:}\quad s = \sin\frac{\varphi}{2}\;,\quad \mathrm{d}s \;=\; \frac{1}{2}\cos\frac{\varphi}{2}\,\mathrm{d}\varphi$$

$$2\alpha t \;=\; 2\int\limits_0^{\sin\left(\Theta(t)/2\right)} \frac{\mathrm{d}s}{1-s^2} \;=\; \ln\frac{1+s}{1-s}\bigg|_0^{\sin\left(\Theta(t)/2\right)}$$

$$\mathrm{e}^{2\alpha t} \;=\; \frac{1+\sin\left(\Theta(t)/2\right)}{1-\sin\left(\Theta(t)/2\right)}\quad;\quad \sin\left(\frac{\Theta(t)}{2}\right) \;=\; \frac{\mathrm{e}^{\alpha t} - \mathrm{e}^{-\alpha t}}{\mathrm{e}^{\alpha t} + \mathrm{e}^{-\alpha t}} \;=\; \tanh(\alpha t)$$

$$\Theta(t) \;=\; \arcsin(\tanh(\alpha t))$$

In particular, for all times t: $-\pi \,<\, \Theta(t) \,<\, \pi$

Case 3: $0 < \eta < 2\alpha$

$$\delta \;:=\; 2\arcsin\left({}^{\eta}\!/_{2\alpha}\right) \quad , \quad \frac{T}{4} \;:=\; \int\limits_0^{\delta} \frac{\mathrm{d}\varphi}{\sqrt{\eta^2 - 4\alpha^2 \sin^2\dfrac{\varphi}{2}}}$$

$$\int\limits_0^{\Theta(t)} \frac{\mathrm{d}\varphi}{\sqrt{\eta^2 - 4\alpha^2 \sin^2\dfrac{\varphi}{2}}} \;=\; t \qquad -\frac{\pi}{4} \le t \le \frac{\pi}{4}$$

Useful continuation: $\Theta(t + T/2) \;=\; -\Theta(t)$

T is the period of oscillation of the swinging pendulum.

Connection with the elliptic functions of Jacobi $(0 \le k < 1)$

Complete elliptic integral of the first kind:

$$K(k) \;:=\; \int\limits_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$\int\limits_0^y \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad , \quad y \;=\; \operatorname{sn}(x,k) \quad , \quad -K \le x < K$$

sn , the so-called *sinus amplitudinis*, is the elliptical sine function.

$$\operatorname{sn}(x+2K,k) \;=\; \operatorname{sn}(x,k) \;\;, \quad \text{periodic with period } 4K$$

$$\operatorname{sn}(x,0) \;=\; \sin x \quad ; \qquad \lim_{k\to 1} \operatorname{sn}(x,k) \;=\; \tanh x$$

$\operatorname{sn}(x,k)$ is the solution of the differential equation

$$y'^{\,2} \;=\; (1+y^2)(1-k^2y^2)$$

with initial condition $y(0) \;=\; 0$

Example:

(2) Case 3: $0 < \eta < 2\alpha \quad ; \quad k \;=\; \dfrac{1}{2\alpha} \quad ; \quad \delta = 2\arcsin\left({}^{\eta}\!/_{2\alpha}\right)$

$$T \;=\; \int\limits_0^{\delta} \frac{\mathrm{d}\varphi}{\sqrt{\eta^2 - 4\alpha^2 \sin^2\dfrac{\varphi}{2}}} \qquad \text{subsitution: } s \;=\; \frac{2\alpha}{\eta}\sin\frac{\varphi}{2}$$

$$=\; \frac{4}{\alpha}\int\limits_0^1 \frac{\mathrm{d}s}{\sqrt{(1-s^2)(1-k^2s^2)}} \;>\; \frac{4}{\alpha}\int\limits_0^1 \frac{\mathrm{d}s}{\sqrt{1-s^2}} \;=\; \frac{2\pi}{\alpha}$$

1.2.2 State Space of a System of Two Homogeneous Real 1st Order Differential Equations

$$\boxed{\dot{\vec{x}} \;=\; A\vec{x}} \qquad A \in M_2(\mathbb{R})$$

The real normal form $S^{-1}AS$ is in the similarity class of A. The characteristic polynomial only depends on the similarity class

$$P_A(x) \;=\; P_{S^{-1}AS}(x)$$

Generally for n instead of 2

$$P_A(x) \;=\; \det(1_n X - A) \;=\; \det S^{-1} \det(1_n X - A) \det S$$

Product theorem for determinates[2]

$$\begin{aligned} P_A(x) &= \det(S^{-1}(1_n X - A)S) \\ &= \det(1_n X - S^{-1}AS) \;=\; P_{S^{-1}AS}(x) \end{aligned}$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the characteristic polynomial becomes

$$P_A(x) \;=\; x^2 - (a+d)x + (ad - bc)$$

Zeros

$$\lambda_{1,2} \;=\; \frac{a+d}{2} \pm \frac{1}{2}\sqrt{\Delta}$$

$$\Delta \;=\; (a+d)^2 - 4(ad-bc) \;=\; (a-d)^2 + 4bc \qquad \text{discriminant}$$

<u>Three types of normal form</u>

I. A real diagonalizable if $\Delta > 0$ or $\Delta = 0$ and $\operatorname{rk}(A - \lambda 1_n) = 0$.

II. A has a double real eigenvalue, is not diagonalizable, that means $\Delta = 0$ and $\operatorname{rk}(A - \lambda 1_n) = 1$.

III. A has a pair of non-real conjugate complex eigenvalues $\lambda_{1,2} = \mu \pm j\omega$ $(\omega > 0)$, if $\Delta < 0$.

The normal form $S^{-1}AS$ becomes in these cases

[2]See chapter 5, section 2 in HIGHER MATHEMATICS Lectures Part Three

$$\textbf{I.} \qquad S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\textbf{II.} \qquad S^{-1}AS = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

$$\textbf{III.} \qquad S^{-1}AS = \begin{pmatrix} \mu & \omega \\ -\omega & \mu \end{pmatrix} \qquad \mu \text{ and } \omega \text{ real}$$

Construction of the real normal form in the case **III**

$$0 > \Delta = (a-d)^2 + 4bc \Rightarrow bc \neq 0$$

$$S = \begin{pmatrix} 0 & -2b \\ \sqrt{|\Delta|} & a-d \end{pmatrix} \qquad \text{yields}$$

$$S^{-1}AS = \begin{pmatrix} \mu & \omega \\ -\omega & \mu \end{pmatrix}$$

<u>The phase diagram of the differential equation</u>

$$\boxed{\dot{\vec{x}} = A\vec{x}}$$

New coordinates in state space:

$$\dot{\vec{x}} = S\dot{\vec{y}} \quad ; \quad \vec{x} = S\vec{y} \quad ; \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

hence

$$S\dot{\vec{y}} = AS\vec{y} \quad ; \quad \dot{\vec{y}} = S^{-1}AS\vec{y}$$

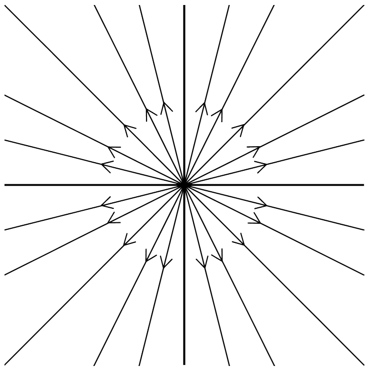
<u>Case **I**</u>:

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= \lambda_2 y_2 \end{aligned} \qquad \text{abbreviation} \qquad \begin{aligned} x &= y_1 \\ y &= y_2 \end{aligned}$$

The differential equation of the phase curves

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\lambda_2}{\lambda_1}\frac{y}{x} \qquad (\lambda_1 \neq 0)$$

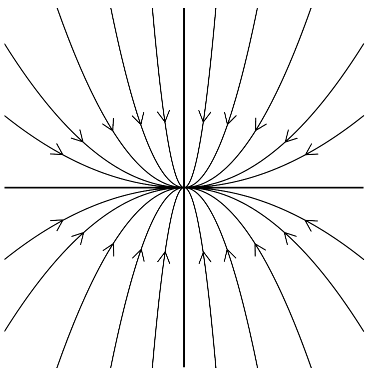
Equation of the phase curves: $\qquad y = C|x|^{\lambda_2/\lambda_1}$



$\lambda_1 = \lambda_2 \neq 0$

(the case when $\Delta = 0$ and $\operatorname{rk}(A - \lambda 1_n) = 0$)

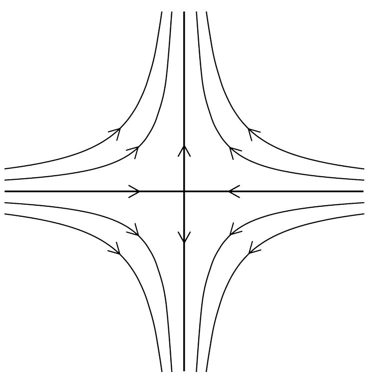
Source point, if $\lambda 1 > 0$ node of the 1st kind.



$\frac{\lambda_1}{\lambda_2} = 2 > 0$

$\lambda_1 < 0$

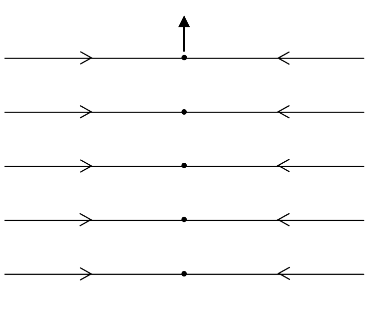
Node of the 2nd kind, stable!



$\lambda_1 < 0$

$\lambda_2 > 0$

Saddle point, unstable.



$\lambda_1 < 0$

$\lambda_2 = 0$

Special case, weakly unstable.

The phase curves do not change if the sign of the eigenvalues changes at the same time. The run-through direction of $(t \to \infty)$ becomes reversed everywhere.

Case **II**:

$$\begin{aligned} \dot{y}_1 &= \lambda y_1 \\ \dot{y}_2 &= y_1 + \lambda y_2 \end{aligned} \qquad \text{abbreviation} \qquad \begin{aligned} x &= y_1 \\ y &= y_2 \end{aligned}$$

The differential equation of the phase curves

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\lambda} + \frac{y}{x} \qquad \text{homogeneous differential equation}$$

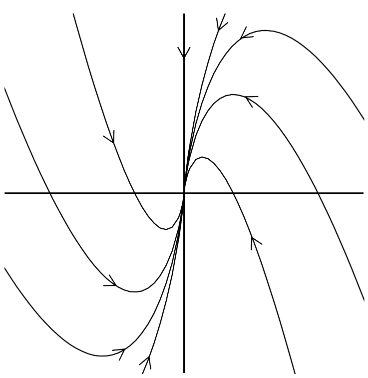
Ansatz: $y = xz$

$$y' = z + xz' = \frac{1}{\lambda} + z$$

$$z' = \frac{1}{\lambda x} \quad ; \quad z = \ln|\lambda x| + C$$

Equation of the phase curves

$$y = x\ln|\lambda x| + Cx$$



$\lambda < 0$

Node of the 3rd kind.

Case **III**:

$$\begin{aligned} \dot{y}_1 &= \mu y_1 + \omega y_2 \\ \dot{y}_2 &= -\omega y_1 + \mu y_2 \end{aligned} \qquad \text{abbreviation} \qquad \begin{aligned} x &= y_1 \\ y &= y_2 \end{aligned}$$

The differential equation of the phase curves

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\omega x - \mu y}{\mu x + \omega y}$$

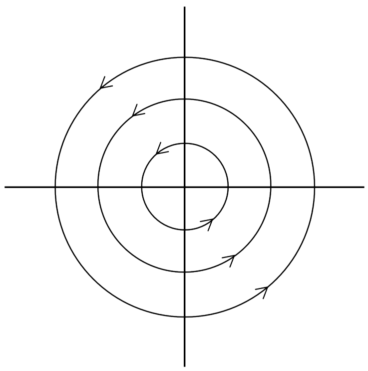
Ansatz: parameterization

$$\left.\begin{aligned} x &= r(\varphi)\cos\varphi \\ y &= r(\varphi)\sin\varphi \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x' &= r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi \\ y' &= r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi \end{aligned}$$

Insertion into the differential equation of the phase curves and comparison

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{\mu}{\omega}\sin\varphi + \cos\varphi}{-\frac{\mu}{\omega}\cos\varphi - \sin\varphi} = \frac{\frac{r\,'}{r}\sin\varphi + \cos\varphi}{\frac{r\,'}{r}\cos\varphi - \sin\varphi} = \frac{x\,'}{y\,'}$$

$$\Rightarrow \quad \frac{r'(\varphi)}{r(\varphi)} = -\frac{\mu}{\omega} \quad ; \quad r(\varphi) = C\exp\Big(-\frac{\omega}{\mu}\varphi\Big)$$



$\mu = 0$

Center, vortex, weakly stable.

1.3 On the Stability of Equilibrium Positions

$$\dot{\vec{y}} = \vec{f}(\vec{y}) \qquad \text{(A)} \quad ; \quad \vec{f} \text{ continuously differentiable}$$

Definitions (regarding stability):

(i) Let $\vec{\varphi}$ be a solution function of (A) (declared on $[0,\infty[$). $\vec{\varphi}$ is stable (in the sense of Lyapunov) if a $\delta_0 > 0$ exists such that firstly every further solution $\vec{\psi}$ of (A) with

$$\left\|\vec{\varphi}(0) - \vec{\psi}(0)\right\| \leq \delta_0$$

can be continued as a solution on all of $[0,\infty[$ and secondly, that for every $\epsilon > 0$ there exists a bound $\delta = \delta(\epsilon) > 0$ such that to every (on all of $[0,\infty[$) continued solution $\vec{\psi}$ of (A) with

$$\left\|\vec{\varphi}(0) - \vec{\psi}(0)\right\| \leq \delta$$

applies for all future times $t \geq 0$

$$\left\|\vec{\varphi}(t) - \vec{\psi}(t)\right\| \leq \epsilon$$

(ii) If $\vec{\varphi}$ on $[0,\infty[$ is a stable solution, then $\vec{\varphi}$ is said to be asymptotically stable if for all solutions $\vec{\psi}$ of (A) continued on $[0,\infty[$ with

$$\left\|\vec{\varphi}(0) - \vec{\psi}(0)\right\| \leq \delta \quad \text{applies} \quad \lim_{t\to\infty}\left\|\vec{\varphi}(t) - \vec{\psi}(t)\right\| = 0$$

Main example: The linear autonomous systems with constant coefficients

$$\dot{\vec{y}} = A\vec{y} \qquad \text{(H)} \qquad ; \qquad A \in M_n(\mathbb{R})$$

Positions of rest (equilibria) $\vec{y}_0 \in \mathbb{R}^n$ with $A\vec{y}_0 = \vec{0}$ (for invertible A only $\vec{y}_0 = \vec{0}$).

If all eigenvalues $\lambda \in \mathbb{C}$ of A have negative real parts, then $\beta > 0$ exists with[3]

$$\operatorname{Re}\lambda < -\beta \qquad \forall\, \lambda$$

Therefore

$$\left\|\mathrm{e}^{At}\right\| \leq \gamma \mathrm{e}^{-\beta t} \qquad \forall\, t \geq 0$$

if $\gamma > 0$ and large enough. Then for every solution of (H) one has

$$\vec{\varphi}(t) = \mathrm{e}^{At}\vec{c}$$

hence

$$\|\vec{\varphi}(t)\|_{\mathrm{e}} = \left\|\mathrm{e}^{At}\vec{c}\right\|_{\mathrm{e}} \leq \left\|\mathrm{e}^{At}\right\|_{\mathrm{e}}\|\vec{c}\|_{\mathrm{e}} \leq \gamma\|\vec{c}\|_{\mathrm{e}}\mathrm{e}^{-\beta t}$$

Result: In the case $\operatorname{Re}\lambda < 0$ for all eigenvalues $\lambda \in \mathbb{C}$ of A, the zero solution $\vec{0}$ of A is asymptotically stable.

On the other hand, if there is an eigenvalue λ with $\operatorname{Re}\lambda > 0$, then the zero solution is unstable.

$$\dot{\vec{y}} = A\vec{y} + \vec{b} \qquad \text{(L)}$$

is only autonomous if the inhomogeneity $\vec{b}$ is time-independent, i.e. constant.

Positions of rest: $\quad \vec{y}_0 \quad$ with $\quad A\vec{y} + \vec{b} = \vec{0}$

If A is invertible, then there exists exactly one rest position, namely $\vec{y}_0 = -A^{-1}\vec{b}$.

Because the difference between any two solutions of (L) is a solution of (H), the stability questions regarding the rest position carry over.

Remark about the stability of the remaining solutions from (L) :

(5) The difference $\vec{\varphi}(t) - \vec{\psi}(t)$ of two solutions of (L) is the solution of the homogeneous system, i.e

$$\vec{\varphi}(t) - \vec{\psi}(t) = \mathrm{e}^{At}(\vec{\varphi}(0) - \vec{\psi}(0))$$

Estimation of the norm

[3] Compare with chapter 7, section 8 in HIGHER MATHEMATICS Lectures Part Three.

$$\left\|\vec{\varphi}(t) - \vec{\psi}(t)\right\|_{\mathrm{e}} \;\le\; \left\|\mathrm{e}^{At}\right\|_{\mathrm{e}} \left\|\vec{\varphi}(0) - \vec{\psi}(0)\right\|_{\mathrm{e}}$$

So if $\left\|\mathrm{e}^{At}\right\|_{\mathrm{e}}$ on $[0,\infty[$ is bounded, then every solution of (L) is stable.

For the general autonomous system

$$\dot{\vec{y}} = \vec{f}(\vec{y}) \qquad \text{(A)} \qquad ; \qquad \vec{f} \text{ continuously differentiable}$$

examine rest positions $\vec{y}_0$ with $\vec{f}(\vec{y}_0) = \vec{0}$ for stability by means of linearization.

Because $\vec{f}$ is continuously differentiable, for zeros $\vec{y}_0$ of $\vec{f}$

$$\vec{f}(\vec{y}) \;=\; A(\vec{y} - \vec{y}_0) + \vec{g}(\vec{y}) \qquad \text{with} \qquad \lim_{\vec{y}\to\vec{y}_0} \frac{\vec{g}(\vec{y})}{\|\vec{y} - \vec{y}_0\|} \;=\; 0$$

Without restriction $\vec{y}_0 = \vec{0}$ (otherwise $\vec{z} = \vec{y} - \vec{y}_0$).

Stability theorem:

$$\dot{\vec{y}} \;=\; \vec{f}(\vec{y}) \qquad \text{(A)}$$

with continuously differentiable $\vec{f}$ shall be an autonomous system with equilibrium at $\vec{y}_0 = \vec{0}$. Let A be the Jacobian matrix of $\vec{f}$ at the point $\vec{y} = \vec{y}_0$. Then the rest position $\vec{y}_0 = \vec{0}$ is asymptotically stable if all eigenvalues λ of A have negative real parts. However, if an eigenvalue λ has a real part > 0, then the rest position is unstable.

Proof sketch for the first part:

Let $\beta > 0$ with

$$\left\|\mathrm{e}^{At}\right\| \;\le\; \gamma \mathrm{e}^{-\beta t} \quad \forall\, t \ge 0 \qquad (*)$$

as per the assumption about A. If $\|\vec{y}\| \le \delta$ it follows from the differentiability of $\vec{f}$

$$\left\|\vec{f}(\vec{y}) - A\vec{y}\right\| \;\le\; \frac{\beta}{2\gamma}\|\vec{y}\,\| \qquad (**)$$

$\vec{f}(\vec{y}) - A\vec{y} = \vec{g}(\vec{y})$ is considered as a “disturbance” of linearity at $\vec{y}_0$.

Conclusion from $(*)$ and $(**)$: For all solutions $\vec{\psi}$ of (A) with

$$\|\underbrace{\vec{\psi}(0)}_{\vec{\eta}_0}\| \;\le\; \frac{\delta}{\gamma} \qquad \text{holds} \qquad \left\|\vec{\psi}(t)\right\| \;\le\; \gamma\|\vec{\eta}_0\|\mathrm{e}^{-\beta t/2}$$

First step: Integral representation of $\vec{\psi}(t)$

$$\vec{\psi}(t) \;=\; \mathrm{e}^{At}\vec{\eta}_0 + \int_0^t \mathrm{e}^{A(t-s)}\vec{g}\big(\vec{\psi}(s)\big)\,\mathrm{d}s$$

Second step: Standard integral estimation with $(*)\,,(**)$

$$\|\vec{\psi}(t)\| \;\leq\; \gamma\|\vec{\eta}_0\|\mathrm{e}^{-\beta t} + \int\limits_0^t \gamma\mathrm{e}^{-\beta(t-s)}\frac{\beta}{2\gamma}\|\vec{\psi}(s)|\|\,\mathrm{d}s$$

Auxiliary function:

$$\phi(t) \;=\; \mathrm{e}^{\beta t}\|\vec{\psi}(t)\| \;\leq\; \gamma\|\vec{\eta}_0\| + \int\limits_0^t \frac{\beta}{2}\phi(s)\,\mathrm{d}s$$

Third step: Gronwall's lemma[4]

$$\phi(t) \;\leq\; \gamma\|\vec{\eta}_0\|\mathrm{e}^{\,\beta t/2}$$

□

[4]See chapter 8, section 1 in HIGHER MATHEMATICS Lectures Part Three.

2. Boundary Value Problems of Ordinary Differential Equations

Example:

(1) Let on $0 \leq x \leq l$ be given

$$y'' + \lambda y = 0 \qquad \text{(H)}$$

We are looking for a non-trivial ($\neq 0$) solution $\varphi(x)$ with the boundary conditions $\varphi(0) = 0\,,\ \varphi(l) = 0$ where $\lambda > 0\,.$

The totality of all solutions of (H)

$$\varphi(x) = c_1 \cos \omega x + c_2 \sin \omega x \qquad (\omega = \sqrt{\lambda}\,,\ \ c_1, c_2 \in \mathbb{R})$$

Boundary values: $\varphi(0) = c_1 = 0$ then $\varphi(l) = c_2 \sin \omega l = 0$

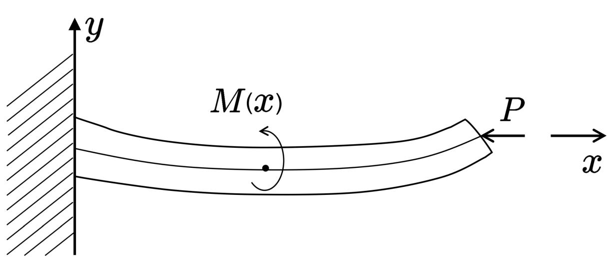
Result: Only then does the homogeneous boundary value problem have a solution $\varphi(x) \neq 0$ if ωl is a zero of the sine, i.e. for

$$\omega l = \sqrt{\lambda} l = n\pi \qquad (n = 1, 2, \ldots)$$

hence if

$$\lambda = \frac{n^2 \pi^2}{l^2}$$

Application in mechanics (buckling beam)



Curvature of the neutral fiber:

$$\frac{y''}{(1 + y')^{3/2}} = \frac{M(x)}{EI} \ ;$$

$$Q = E \cdot I$$

Precondition: $|y'| \ll 1 \;\rightarrow\;$ then linearization

$$y'' = \frac{M(x)}{Q} \qquad , \qquad M(x) = -y(x)P$$

$$y'' + \lambda y = 0 \qquad \text{with} \qquad \lambda = \frac{P}{Q}$$

Boundary condition $y(0) = y(l) = 0$. If

$$0 < P < \frac{\pi^2 EI}{l^2} =: P^*$$

then only the trivial solution $y = 0$ exists. But for $P = P^*$ a non-trivial solution exists for the first time. $P^* =$ "Euler buckling load".

2.1 The General Linear Boundary Value Problem

Let a compact interval $I = [a, b]$ be given and there three continuous functions p, q, r

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{(L)}$$

with two linear boundary conditions:

$$R_i(y) = \alpha_{i1}y(a) + \alpha_{i2}y'(a) + \beta_{i1}y(b) + \beta_{i2}y'(b) = \eta_i$$

with real α_{ik}, β_{ik}, η_i ; furthermore $i = 1, 2$

In the event of

$\eta_1 = \eta_2 = 0$ "semi-homogeneous boundary value problem"

$\eta_1 = \eta_2 = 0$; $r(x) = 0$ "fully homogeneous boundary value problem"

General case: "inhomogeneous boundary value problem".

Example:

(2) $y'' - y = 0$ (H) $0 \le x \le l$

Boundary conditions:

$$y(0) + y(l) = \eta_1$$
$$y(0) - y'(l) = \eta_2$$

General solution from a fundamental system, such as e^x, e^{-x}

$$\varphi(x) = c_1 e^x + c_2 e^{-x}$$

Boundary conditions:

$$\varphi(0) + \varphi(l) = c_1 + c_2 + c_1 e^l + c_2 e^{-l} = \eta_1$$
$$\varphi(0) - \varphi'(l) = c_1 + c_2 - c_1 e^l + c_2 e^{-l} = \eta_2$$

System of linear equations in c_1 , c_2 ; $u_1(x) = \mathrm{e}^x$, $u_2(x) = \mathrm{e}^{-x}$

$$c_1 R_1(u_1) + c_2 R_1(u_2) = \eta_1$$

$$c_1 R_2(u_1) + c_2 R_2(u_2) = \eta_2$$

It is uniquely solvable if and only if the coefficient matrix

$$\begin{pmatrix} R_1(u_1) & R_1(u_2) \\ R_2(u_1) & R_2(u_2) \end{pmatrix}$$

is invertible.

$$\begin{pmatrix} 1+\mathrm{e}^l & 1+\mathrm{e}^{-l} \\ 1-\mathrm{e}^l & 1+\mathrm{e}^{-l} \end{pmatrix} \text{ is invertible.}$$

2.1.1 Uniqueness Criterion

Let

$$L(y) = y'' + p(x)y' + q(x)y$$

denote the left side of the general linear boundary value problem. Let the functions u_1, u_2 be a basis of the solution space of

$$L(y) = 0 \qquad \text{(H)}$$

The semi-homogeneous boundary value problem is solvable if and only if

$$\det\begin{pmatrix} R_1(u_1) & R_1(u_2) \\ R_2(u_1) & R_2(u_2) \end{pmatrix} \neq 0$$

Proof:

Let $v(x)$ be a particular solution of

$$L(y) = r(x) \qquad \text{(L)}$$

From the theory: general solution

$$\varphi(x) = v(x) + c_1 u_1(x) + c_2 u_2(x)$$

Boundary conditions

$$R_i(v) + c_1 R_i(u_1) + c_2 R_i(u_2) = 0 \quad ; \quad i = 1, 2$$

System of linear equations for c_1, c_2 . Uniquely solvable if and only if

$$\det\big(R_i(u_k)\big)_{i,k=1,2} \neq 0$$

□

Remark:

(1) In the case $\det(R_i(u_k)) = 0$, the boundary value problem can be unsolvable, or it has several solutions. In particular, the fully homogeneous boundary value problem then has a non-trivial solution!

Example:

(3) $y'' = r(x) \qquad 0 \le x \le l$

Boundary conditions: $\quad y(0) = 0 \;, \quad y(l) = 0$

Basis of the solution space of $\; y'' = 0$

$$u_1(x) = 1 \quad , \quad u_2(x) = x$$

$$\det\begin{pmatrix} R_1(u_1) & R_1(u_2) \\ R_2(u_1) & R_2(u_2) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 1 & l \end{pmatrix} = l \neq 0$$

$$y'(x) = \int_0^x r(\xi)\,\mathrm{d}\xi + c_1$$

$$y(x) = \int_0^x \underset{\downarrow \atop u'}{1} \underbrace{\Big(\int_0^\xi r(\eta)\,\mathrm{d}\eta \Big)}_{v} \mathrm{d}\xi + c_1 x + c_2$$

$$= \Big[\xi \int_0^\xi r(\eta)\,\mathrm{d}\eta \Big]_0^x - \int_0^x \xi r(\xi)\,\mathrm{d}\xi + c_1 x + c_2$$

$$= \int_0^x (x-\xi) r(\xi) \mathrm{d}\xi + c_1 x + c_2$$

is the general solution of $y'' = r(x)$.

1. Boundary condition: $\; y(0) = 0 \;\Rightarrow\; c_2 = 0$

2. Boundary condition: $\; y(l) = 0 \;\Rightarrow\; c_1 l = \int_0^l (\xi - l) r(\xi)\,\mathrm{d}\xi$

$$
\begin{aligned}
y(x) \;&=\; \int_0^x (x-\xi) r(\xi)\,\mathrm{d}\xi \;+\; \frac{x}{l} \underbrace{\int_0^l (\xi - l) r(\xi)\,\mathrm{d}\xi}_{\int_0^x \;+\; \int_x^l} \\
&=\; \int_0^x \frac{\xi}{l}(x-l) r(\xi)\,\mathrm{d}\xi \;+\; \int_x^l \frac{x}{l}(\xi - l) r(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

<u>Result:</u> The general solution of the semi-homogeneous boundary value problem is

$$
y(x) \;=\; \int_0^l G(x,\xi) r(\xi)\,\mathrm{d}\xi
$$

with “Green’s function”

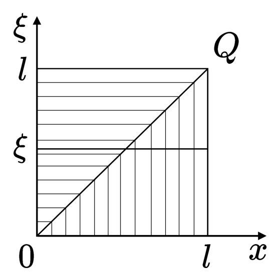
$$
G(x,\xi) \;=\; \begin{cases} \dfrac{\xi}{l}(x-l) & \text{if} \;\; 0 \le \xi \le x \le l \\[2mm] \dfrac{x}{l}(\xi - l) & \text{if} \;\; 0 \le x \le \xi \le l \end{cases}
$$

<u>Properties of $G(x,\xi)$</u>

a) Continuity:

$G(x,\xi)$ is continuous on the square

$$
Q \;=\; \big\{(x,\xi)\,;\;\; 0 \le x \le l\,,\, 0 \le \xi \le l\big\}
$$



b) Differentiability: For each ξ, $x \mapsto G(x,\xi)$ is twice continuously differentiable on the subintervals $0 \le x \le \xi\,,\;\; \xi \le x \le l$ and satisfies the associated homogeneous linear differential equation $y'' = 0$.

c) $x \mapsto G(x,\xi)$ satisfies the boundary conditions of the semi-homogeneous boundary value problem.

d) $\displaystyle \lim_{x \searrow \xi} \frac{\partial G}{\partial x}(x,\xi) \;-\; \lim_{x \nearrow \xi} \frac{\partial G}{\partial x}(x,\xi) \;=\; \frac{\xi}{l} - \frac{\xi - l}{l} \;=\; 1$ (jump condition)

2.1.2 Green's Function for the General Semi-homogeneous (Linear) Boundary Value Problem

Let there be given three continuous functions p, q, and r on $J = [a, b]$, further the linear differential equation

$$L(y) \;=\; y'' + p(x)y' + q(x)y \;=\; r(x) \qquad \text{(L)}$$

and the boundary conditions $(i = 1, 2)$:

$$R_i(y) \;=\; \alpha_{i1}y(a) + \alpha_{i2}y'(a) + \beta_{i1}y(b) + \beta_{i2}y'(b) \;=\; 0$$

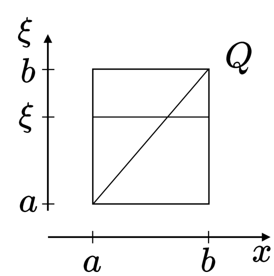
Finally, the uniqueness criterion shall apply

$$\det\big(R_i(u_k)\big)_{1 \le i,k \le 2} \;\neq\; 0$$

for a basis u_1, u_2 of the solution space $L(y) = 0$. Then the unique solution $y(x)$ of the semi-homogeneous boundary value problem is

$$y(x) \;=\; \int_a^b G(x,\xi)r(\xi)\,\mathrm{d}\xi$$

Therein "Green's function" $G(x, \xi)$ is determined by



a) Continuity:

$G(x, \xi)$ is continuous on the square

$$Q \;=\; \big\{(x,\xi)\,;\;\; a \le x \le b\,,\, a \le \xi \le b\big\}$$

b) Differentiability: For all ξ, $x \mapsto G(x, \xi)$ is twice continuously differentiable on each subinterval $a \le x \le \xi$, $\xi \le x\, leqb$ and is solution of $L(y) = 0$.

c) For each ξ, $G(x, \xi)$ satisfies the boundary conditions

$$R_i\big(G(-,\xi)\big) \;=\; 0 \qquad i = 1, 2$$

d) $$\lim_{x \searrow \xi} \frac{\partial G}{\partial x}(x,\xi) - \lim_{x \nearrow \xi} \frac{\partial G}{\partial x}(x,\xi) \;=\; 1 \qquad \text{(jump condition)}$$

Proof:

1) Construction of Green's function.

Every solution of $L(y) = 0$ is a linear combination of u_1 and u_2. Therefore ansatz

$$G(x,\xi) \;=\; \begin{cases} (a_1 - b_1)u_1(x) \,+\, (a_2 - b_2)u_2(x) \quad , \quad (a \le x \le \xi) \\ (a_1 + b_1)u_1(x) \,+\, (a_2 + b_2)u_2(x) \quad , \quad (\xi \le x \le b) \end{cases}$$

with coefficients a_i, b_i, which depend on ξ. Continuity on the diagonal and jump condition

$$(a_1 - b_1)(\xi)u_1(\xi) + (a_2 - b_2)(\xi)u_2(\xi) \;=\; (a_1 + b_1)(\xi)u_1(\xi) + (a_2 + b_2)(\xi)u_2(\xi)$$

hence

$$\left.\begin{aligned} 2b_1(\xi)u_1(\xi) + 2b_2(\xi)u_2(\xi) &= 0 \qquad \text{(diagonal condition)} \\ 2b_1(\xi)u_1'(\xi) + 2b_2(\xi)u_2'(\xi) &= 1 \qquad \text{(jump condition)} \end{aligned}\right\} \quad (*)$$

Coefficient matrix $2\begin{pmatrix} u_1(\xi) & u_2(\xi) \\ u_1'(\xi) & u_2'(\xi) \end{pmatrix}$ is invertible (Wronski)

$(*)$ is therefore uniquely solvable for b_1 and b_2.

Boundary conditions for G

$$\begin{aligned} R_i\big(G(-,\xi)\big) \;&=\; \alpha_{i1}G(a,\xi) + \alpha_{i2}\frac{\partial}{\partial x}G(a,\xi) \\ &\quad + \beta_{i1}G(b,\xi) + \beta_{i2}\frac{\partial}{\partial x}G(b,\xi) \;=\; 0 \end{aligned}$$

$$\begin{aligned} 0 \;=\; & \alpha_{i1}(a_1 - b_1)u_1(a) + \alpha_{i1}(a_2 - b_2))u_2(a) \\ & + \alpha_{i2}(a_1 - b_1)u_1'(a) + \alpha_{i2}(a_2 - b_2))u_2'(a) \\ & + \beta_{i1}(a_1 + b_1)u_1(b) + \beta_{i1}(a_2 + b_2))u_2(b) \\ & + \beta_{i2}(a_1 + b_1)u_1'(b) + \beta_{i2}(a_2 + b_2))u_2'(b) \qquad ; \quad (i = 1,2) \end{aligned}$$

Expressed for a_1, a_2

$$a_1R_i(u_1) + a_2R_i(u_2) \;=\; f_i(b_1,b_2) \qquad (**) \quad (i = 1,2)$$

System of linear equations for a_1 and a_2 with invertible coefficient matrix, thus uniquely solvable.

2) $y(x) \;=\; \displaystyle\int_a^b G(x,\xi)r(\xi)\,\mathrm{d}\xi$

is solution of the semi-homogeneous boundary value problem. By G's construction:

$$\begin{aligned}\frac{\partial}{\partial x}G(x-0,x) &= \frac{\partial}{\partial x}G(x,x+0)\\ \frac{\partial}{\partial x}G(x+0,x) &= \frac{\partial}{\partial x}G(x,x-0)\end{aligned}$$

Differentiation of $\quad y(x) = \int\limits_a^b G(x,\xi)r(\xi)\,\mathrm{d}\xi$

according to the rule for integrals with parameters

$$\begin{aligned}y'(x) &= \int\limits_a^b \frac{\partial}{\partial x}G(x,\xi)r(\xi)\,\mathrm{d}\xi \;=\; \int\limits_a^x \ldots\ldots \;+\; \int\limits_x^b \ldots\ldots\\ y''(x) &= \int\limits_a^b \frac{\partial^2}{\partial x^2}G(x,\xi)r(\xi)\,\mathrm{d}\xi + \frac{\partial}{\partial x}G(x,x-0)r(x)\\ &= \int\limits_a^b \frac{\partial^2}{\partial x^2}G(x,\xi)r(\xi)\,\mathrm{d}\xi - \frac{\partial}{\partial x}G(x,x+0)r(x)\\ y''(x) &= \int\limits_a^b \frac{\partial^2}{\partial x^2}G(x,\xi)r(\xi)\,\mathrm{d}\xi + r(x)\\ L(y) &= \int\limits_a^b \underbrace{\Big[\frac{\partial^2 G(x,\xi)}{\partial x^2} + p(x)\frac{\partial G(x,\xi)}{\partial x} + q(x)G(x,\xi)\Big]}_{=0} r(\xi)\,\mathrm{d}\xi + r(x) \implies\\ L(y) &= r(x) \qquad (L(y)=0 \text{ is wrong})\\ R_i(y) &= \int\limits_a^b \underbrace{R_i\big(G(-,\xi)\big)}_{=0} r(\xi)\,\mathrm{d}\xi = 0 \qquad (i=1,2)\end{aligned}$$

□

Remark:

(2) In the case of the inhomogeneous linear boundary value problem

$$R_i(y) = \eta_i \qquad (i=1,2)$$

one can easily procure a twice continuously differentiable function ϕ with

$$R_i(\phi) = \eta_i$$

Instead of y, consider the difference $z = y - \phi$. Conditions on z:

$$L(z) = L(y) - L(\phi) = r(x) - L(\phi)(x)$$

$$R_i(z) = R_i - R_i(\phi) = 0 \qquad (i = 1, 2)$$

The modified problem (in the case $\det\big(R_i(u_k)\big) \neq 0$) can be solved using Green's function.

2.2 Vibrations of a String

With l denoting the string length and t the time ($0 \leq t$) the differential equation for a vibrating string on $0 \leq x \leq l$ is

$$\frac{\partial^2}{\partial x^2}\, y - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\, y = 0$$

Boundary conditions for the sought after function $y(x,t)$

$$\left.\begin{aligned} y(x,0) &= p(x) \\ \frac{\partial y}{\partial t}(x,0) &= q(x) \end{aligned}\right\} \qquad \left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right\} \qquad \forall\, t$$

2.2.1 d'Alembert's Solution Ansatz

New variable $\xi = x + ct$; $\eta = x - ct$ that means $x = \frac{1}{2}(\xi + \eta)$ and $t = \frac{1}{2c}$

$$z(\xi,\eta) = y\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big)$$

$$\begin{aligned} \frac{\partial^2}{\partial\xi\partial\eta}\, z(\xi,\eta) &= \frac{\partial}{\partial\xi}\left[\frac{1}{2}\frac{\partial y}{\partial x}\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big) - \frac{1}{2c}\frac{\partial y}{\partial t}\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big)\right] \\ &= \frac{1}{4}\Bigg[\underbrace{\frac{\partial^2 y}{\partial x^2}\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big) - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big)}_{=\,0 \text{ for the solution } y}\Bigg] \end{aligned}$$

We are looking for a twice continuously differentiable function z with $\frac{\partial^2}{\partial\xi\partial\eta}\, z = 0$

That means $\frac{\partial z}{\partial\eta}$ is independent of ξ, hence ansatz

$$\frac{\partial z}{\partial\eta} = \psi'(\eta) \qquad z = \phi(\xi) + \psi(\eta)$$

with twice continuously differentiable functions ϕ, ψ.

Hence for the original function

$$y(x,t) \;=\; \phi(x+ct) \,+\, \psi(x-ct)$$

Interpretation: superposition of two waves, one progressing to the left, the other to the right.

Thus the boundary conditions for $t=0$ (that means the first pair of boundary conditions) become

$$\phi(x) \,+\, \psi(x) \;=\; p(x) \qquad \Rightarrow \qquad \phi' \,+\, \psi' \;=\; p'$$

$$c\phi'(x) \,-\, c\psi'(x) \;=\; q(x)$$

It follows

$$\phi'(x) \;=\; \frac{1}{2}p'(x) \,+\, \frac{1}{2c}\,q(x)$$

$$\psi'(x) \;=\; \frac{1}{2}\,p'(x) \,-\, \frac{1}{2c}\,q(x)$$

Integration

$$\phi(x) \;=\; \frac{1}{2}\,p(x) \,+\, \frac{1}{2c}\int_0^x q(\xi)\,\mathrm{d}\xi \,+\, c_1$$

$$\psi(x) \;=\; \frac{1}{2}\,p(x) \,-\, \frac{1}{2c}\int_0^x q(\xi)\,\mathrm{d}\xi \,+\, c_2$$

Second pair of boundary conditions

$$\phi(ct) \,+\, \psi(ct) \;=\; 0 \qquad \forall\, t \,\geq\, 0$$

$$\phi(l+ct) \,+\, \psi(l-ct) \;=\; 0 \qquad \forall\, t \,\geq\, 0$$

Result: continuation of ϕ and ψ onto all of $\mathbb{R}$

$$\phi(ct) \;=\; -\psi(-ct)$$

$$\phi(l+ct) \;=\; -\psi(l-ct) \;=\; \phi(-l+ct)$$

ϕ, ψ become periodic with period $2l$.

p, q become automatically continued to $2l$-periodic functions.

2.2.2 Fourier's Solution Method

With two solutions of the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

every linear combination is also a solution.

Furthermore: If the initial solutions do satisfy the second pair of boundary conditions $y(0,t) = y(l,t) = 0$, then every linear combination of them also satisfies these boundary conditions.

Solution ansatz as product: Let

$$y(x,t) = U(x)V(t)$$

be a non-trivial solution with $U(x_0) \neq 0 \neq V(t_0)$.

Wave equation

$$U''(x)V(t) - \frac{1}{c^2}U(x)V''(t) = 0$$

$$U''(x) + \lambda U(x) = 0 \qquad \text{with} \qquad \lambda = -\frac{V''(t_0)}{c^2 V(t_0)} = -\frac{U''(x_0)}{U(x_0)}$$

$$V''(t) + c^2\lambda V(t) = 0$$

Under the boundary conditions $U(0) = U(l) = 0$, the first differential equation (see example 1) has a non-trivial solution only for the special eigenvalues $\lambda = \dfrac{n^2\pi^2}{l^2}$, $n = 1, 2, 3\ldots,$ viz

$$U_n(x) = \sin\frac{n\pi x}{l}$$

$$V_n(x) = A_n \cos\frac{\pi nct}{l} + B_n \sin\frac{\pi nct}{l}$$

and therefore

$$y_n(x,t) = \sin\frac{n\pi x}{l}\left(A_n \cos\frac{\pi nct}{l} + B_n \sin\frac{\pi nct}{l}\right)$$

Thus all finite sums become solutions.

$$y_n(x,t) = \sum_{n=1}^{N} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{\pi nct}{l} + B_n \sin\frac{\pi nct}{l}\right)$$

is thus the solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

with boundary conditions $y(0,t) = y(l,t) = 0$.

Final question: When do solutions arise for $N \to \infty$?

3. Inversive Geometry and Number Sphere

Consider $\quad z \;\mapsto\; \dfrac{1}{z} \qquad (\text{for } z \in \mathbb{C}^*)$

This mapping is "circle-preserving". By adding ∞ (as image and preimage / inverse image of 0 at $z \mapsto \frac{1}{z}$) the number sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ (Riemann sphere) is created on $\mathbb{C}$.

Translations are also circle-preserving ($z \;\mapsto\; z + c$, c constant).

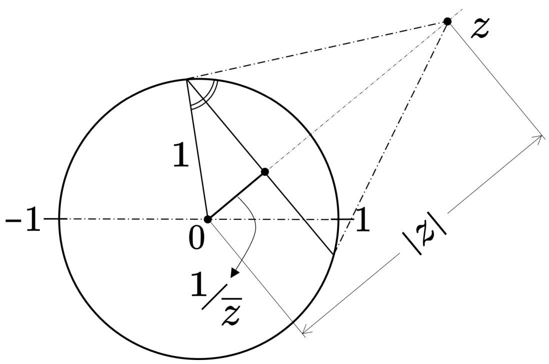
By combining these mapping types, the group of circle-preserving "Möbius transformations" arises

$$z \;\mapsto\; \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

$z \mapsto \dfrac{1}{z}$ is composed of $z \mapsto \bar{z}$ and $z \mapsto \dfrac{1}{\bar{z}} = \dfrac{1}{|z|^2}\, z$

<u>Geometric interpretation:</u>

If $z \in \mathbb{C}^*$, then z and $\dfrac{1}{\bar{z}}$ lie on the same ray through 0 and have the absolute product 1.



$z \;\mapsto\; \dfrac{1}{\bar{z}}$:

"Reflection on the unit circle"

or

"Mapping with reciprocal radii"

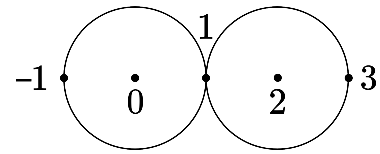
$z \;\mapsto\; \dfrac{1}{z}$ therefore means reflection on the unit circle with subsequent reflection on the real axis.

Circle preservation

Examples:

(1) The circle $|z-2| = 1$

turns at $w = \frac{1}{z}$ into the set of all w with

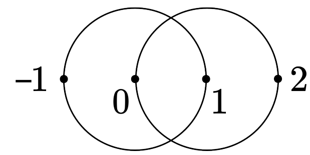


$$\left|\frac{1}{w} - 2\right| = 1$$

$$\Leftrightarrow \quad \left(\frac{1}{w} - 2\right)\left(\frac{1}{\bar{w}} - 2\right) = 1$$

$$\Leftrightarrow \quad (1-2\bar{w})(1-2w) = |w|^2$$

$$\Leftrightarrow \quad 3|w|^2 - 2(w+\bar{w}) + 1 = 0$$



and thus $\left|w - \frac{2}{3}\right|^2 = \left(\frac{1}{3}\right)^2$

(2) $|z-1| = 1$

turns into the set of all w where either $w = \infty$ or

$$\left(\frac{1}{w} - 1\right)\left(\frac{1}{\bar{w}} - 1\right) = 1$$

$$\Leftrightarrow \quad (1-\bar{w})(1-w) = |w|^2$$

$$\Leftrightarrow \quad 1 - (w+\bar{w}) = 0 \quad \Longleftrightarrow \quad \operatorname{Re} w = \frac{1}{2}$$

Definition:
A "circle" on the number sphere $\mathbb{P}$ denotes the ordinary circles in $\mathbb{C}$ and the straight lines supplemented by ∞ in $\mathbb{C}$.

Equation of the circle representation in $\mathbb{P}$

$$\varrho|z|^2 - \bar{\alpha}z - \alpha\bar{z} + \sigma = 0 \qquad \text{with real } \varrho,\ \sigma \text{ and } \alpha\bar{\alpha} - \varrho\sigma = 1$$

Rationale for the circle preservation of $z \mapsto w = \frac{1}{z}$:

The z of the equation $\varrho|z|^2 - \bar{\alpha}z - \alpha\bar{z} + \sigma = 0$ turn into the w of the equation

$$\varrho - \bar{\alpha}\bar{w} - \alpha w + \sigma|w|^2 = 0$$

Example:

(3) ϱ, α, σ for unit circle $|z|^2 = 1$

$$\varrho = 1 \ ; \quad \alpha = 0 \ ; \quad \sigma = -1$$

for the imaginary axis $\operatorname{Re} z = 0$

$$\varrho = 0 \ ; \quad \alpha = 1 \ ; \quad \sigma = 0$$

for the real axis $\operatorname{Im} z = 0$

$$\varrho = 0 \ ; \quad \alpha = j \ ; \quad \sigma = 0$$

Proper circles $\iff$ $\varrho \neq 0$

Improper circles $\iff$ $\varrho = 0$

Proper circles through 0 $\iff$ $\varrho \neq 0\,,\ \sigma = 0$

Straight lines not through 0 $\iff$ $\varrho = 0\,,\ \sigma \neq 0$

Calculation rule 1 for ∞

For every translation $z \mapsto z + c \quad (c \in \mathbb{C})$ ∞ is a fixed point.

Möbius transformations are called the mappings of the number sphere resulting from iteration of inverse formation and translations. They are all circle-preserving. The general form

$$z \mapsto w = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

$$z = 0 \quad \text{turns into} \quad \begin{cases} w = \dfrac{b}{d} & \text{if} \quad d \neq 0 \\ w = \infty & \text{if} \quad d = 0 \end{cases}$$

$$z = \infty \quad \text{turns into} \quad \begin{cases} w = \dfrac{a}{c} & \text{if} \quad c \neq 0 \\ w = \infty & \text{if} \quad c = 0 \end{cases}$$

If $c \neq 0$, then $\quad w = \dfrac{az+b}{cz+d} = \dfrac{a}{c}\dfrac{ad-bc}{c(cz+d)}$

Regarding the statement that all the mappings mentioned are composed of inversions and translations, it remains to add:

$$z \mapsto hz \qquad h \in \mathbb{C}^*$$

is a Möbius transformation.

$$a + \frac{1}{-a^{-1} + \frac{1}{a + \frac{1}{z}}} \;=\; a + \frac{1}{-a^{-1} + \frac{z}{az+1}}$$

$$= \; a + \frac{az+1}{a^{-1}} \;=\; -a^2 z \quad , \qquad (a \neq 0)$$

(h always has the form $-a^2$)

<u>Calculation rule 2 for ∞</u>

For every rotational stretch $z \mapsto hz$ $(h \in \mathbb{C}^*)$ ∞ is a fixed point.

3.1 The Cross-ratio

For every three pairwise distinct points $z_0,\ z_1,\ z_\infty \in \mathbb{P}$ there is exactly one Möbius transformation T with $Tz_0 = 0\,,\ Tz_1 = 1\,,\ Tz_{\ infty} = \infty\,,$ namely

$$\frac{z - z_0}{z - z_\infty} \cdot \frac{z_1 - z_\infty}{z_1 - z_0}$$

If one of these points $z_k = \infty$, divide the numerator and denominator formally by z_k.

For example, if $z_0 = \infty$ then

$$\frac{z - z_0}{z - z_\infty} \cdot \frac{z_1 - z_\infty}{z_1 - z_0} \;=\; \frac{z/z_0 - 1}{z - z_\infty} \cdot \frac{z_1 - z_\infty}{z_1/z_0 - 1} \;=\; \frac{z_1 - z_\infty}{z - z_\infty}$$

This Möbius transformation T is also called double ratio or cross-ratio of $z,\ z_0,\ z_1,\ z_\infty$, written $D(z, z_0, z_1, z_\infty)$. The cross-ratio is invariant under Möbius transformations S

$$D(Sz, Sz_0, Sz_1, Sz_\infty) \;=\; D(z, z_0, z_1, z_\infty)$$

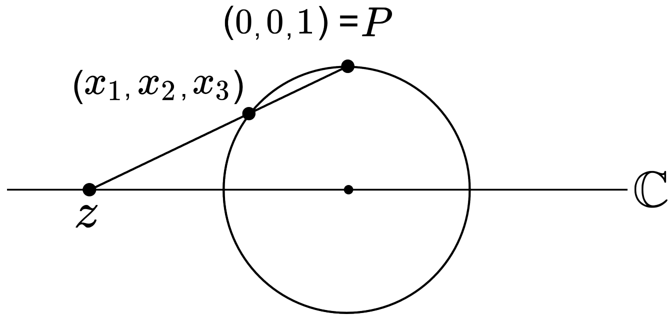
Immediately clear for translations. Check: $\quad Sz \,=\, \dfrac{1}{z}$

$$D\Big(\frac{1}{z}, \frac{1}{z_0}, \frac{1}{z_1}, \frac{1}{z_\infty}\Big) \;=\; \frac{1/z - 1/z_0}{1/z - 1/z_\infty} \cdot \frac{1/z_1 - 1/z_\infty}{1/z_1 - 1/z_0} \qquad \text{expanding}$$

$$= \; \frac{z_0 - z}{z_\infty - z} \cdot \frac{z_\infty - z_1}{z_0 - z_1} \;=\; D(z, z_0, z_1, z_\infty)$$

$$S^2 = \left\{ \vec{x} \in \mathbb{R}^3 , \ \ \|\vec{x}\| = 1 \right\}$$

$$z = \frac{x_1 + j x_2}{1 - x_3}$$



Starting from the 1-sphere in $\mathbb{R}^3$, each additional point (x_1, x_2, x_3) is projected straight from the north pole $P = (0,0,1)$ onto the equatorial plane $\mathbb{C}$

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1 - x_3^2}{(1-x_3)^2} = \frac{1 + x_3}{1 - x_3}$$

$$|z|^2 + 1 = \frac{2}{1 - x_3} \ , \qquad |z|^2 - 1 = \frac{2x_3}{1 - x_3}$$

The stereographic projection is circle-preserving.

As proof: The circles on S^2 are the non-trivial plane intersections $a_1x_1 + a_2x_2 + a_3x_3 = a_0$, where $a_k \in \mathbb{R}, \ \ a_1^2 + a_2^2 + a_3^2 = 1 , \ \ -1 < a_0 < 1$

Multiplying the plane intersection equation with $\dfrac{2}{1 - x_3}$ gives for z the image points

$$a_1(z + \bar{z}) - a_2 j (z - \bar{z}) + a_3(|z|^2 - 1) = a_0(|z|^2 + 1)$$

$$\underbrace{(a_0 - a_3)}_{\varrho} |z|^2 - \underbrace{(a_1 - j a_2)}_{\bar{\alpha}} z - \underbrace{(a_1 + j a_2)}_{\alpha} \bar{z} + \underbrace{(a_0 + a_3)}_{\sigma} = 0$$

$$\alpha\bar{\alpha} - \varrho\sigma = a_1^2 + a^2 - (a_0^2 - a_3^2) = 1 - a_0^2 \qquad \text{(possibly now normalization)}$$

Interpretation of the inversion $\ z \mapsto \dfrac{1}{z}\ $ on $\ S^2$

$$z = \frac{x_1 + j x_2}{1 - x_3} \ ; \quad \frac{1}{z} = \frac{x_1 - j x_2}{1 + x_3} \qquad \text{(the product} = 1)$$

hence $\quad (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$

that means a rotation of 180° around the x_1-axis.

3.1.1 The Rational Functions as Mapping of the Number Sphere $\mathbb{P}$ in Itself

$$f(z) \;=\; \frac{P(z)}{Q(z)} \;,$$

where $P(z)\,,\,Q(z)$ are polynomials in z without a common root, Q is not the zero polynomial, possibly normalized. The zeros of the denominator Q are assigned the value $f(z_0) = \infty$. The (possibly improper) limit $\lim\limits_{z\to\infty}\big[P(z)\,/\,Q(z)\big]$ is assigned to the point $z = \infty$. The zeros z_0 of Q are called poles of f.

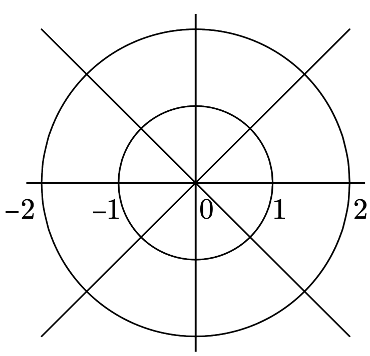
The Joukowsky mapping

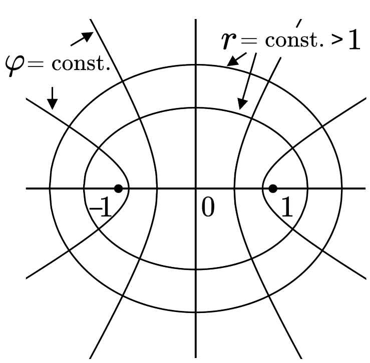
$$w \;=\; f(z) \;=\; \frac{1}{2}\big(z + \frac{1}{z}\big)$$

Each point gets the same image as the reciprocal point

$$0 \mapsto \infty \quad ; \quad \infty \mapsto \infty \quad ; \quad 1 \mapsto 1 \quad ; \quad -1 \mapsto -1$$

Every point w different from ± 1 has two preimages.





Polar notation:

$$z \;=\; r\mathrm{e}^{i\phi}$$

$$w \;=\; \frac{1}{2}\Big(r\mathrm{e}^{i\varphi} + \frac{1}{r}\mathrm{e}^{-i\varphi}\Big)$$

$$u \;=\; \operatorname{Re} w \;=\; \frac{1}{2}\Big(r + \frac{1}{r}\Big)\cos\varphi$$

$$v \;=\; \operatorname{Im} w \;=\; \frac{1}{2}\Big(r - \frac{1}{r}\Big)\sin\varphi$$

$$a = \frac{1}{2}\Big(r + \frac{1}{r}\Big) \quad , \quad a = \frac{1}{2}\Big(r - \frac{1}{r}\Big)$$

$$\Rightarrow \quad \frac{u^2}{a^2} + \frac{v^2}{b^2} \;=\; 1$$

$$\varphi \;=\; \text{const} \;\neq\; 0,\, \pm\pi,\, \pm\frac{\pi}{2}$$

$$\frac{u^2}{\cos^2\varphi} - \frac{v^2}{\sin^2\varphi} \;=\; 1$$

3.2 A Class $\mathscr{H}$ of Rational Functions

$\mathscr{H}$ contains all rational functions $h(z) = \dfrac{P(z)}{Q(z)}$ having the following three properties

$$\begin{aligned}
\operatorname{Re} h(z) \; &> \; 0\,, \quad \text{if} \quad \operatorname{Re} z \; > \; 0 \\
\operatorname{Re} h(z) \; &< \; 0\,, \quad \text{if} \quad \operatorname{Re} z \; < \; 0 \\
h\big(i\mathbb{R} \cup \{\infty\}\big) \; &\subset \; i\mathbb{R} \cup \{\infty\}
\end{aligned}$$

3.2.1 Elementary Properties of $\mathscr{H}$

Let h, h_1 be functions in $\mathscr{H}$ and $a > 0$, b be purely imaginary. Then, likewise in $\mathscr{H}$ are

$$h + h_1 \quad , \quad \frac{1}{h} \quad , \quad ah + b \quad , \quad z \mapsto h(az + b) \quad , \quad z \mapsto h\Big(\frac{1}{z}\Big)$$

<u>Rationale:</u> Each of the three sets $\operatorname{Re}(z) > 0$, $\operatorname{Re}(z) < 0$, $\operatorname{Re}(z) = 0$ contain with each two points their sum, with each point the reciprocal point, and with each point positive multiples and purely imaginary translations. For example the Joukowsky mapping

$$z \mapsto \frac{1}{2}\Big(z + \frac{1}{z}\Big)$$

is an element of $\mathscr{H}$.

Functions $h \in \mathscr{H}$ have their zeros on the imaginary axis and all their zeros are simple. Obviously, zeros z_0 of h have real part 0. Because of the fourth elementary property, it is sufficient to consider $z_0 = 0$.

$$\begin{aligned}
h(z) \; &= \; \frac{P(z)}{Q(z)} \; , \qquad P\,,\, Q \text{ without common zeros} \\
P(z) \; &= \; z^m P_1(z) \qquad \text{with} \qquad P_1(0) \neq 0 \neq Q(0) \\
m \geq 1 \quad &, \quad z \; = \; r\mathrm{e}^{i\varphi} \\
h(z) \; &= \; \mathrm{e}^{im\varphi}\, r^m \, \frac{P_1(z)}{Q(z)}
\end{aligned}$$

If $m > 1$, then $\operatorname{Re} h(z)$ would change its sign for z in the right half plane, that means for $-\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{2}$ (contradiction).

3.2.2 Partial Fraction Decomposition of the Function $h(z) \in \mathscr{H}$

$$h(z) \;=\; c_0 z \,+\, d_0 \,+\, \sum_{k=1}^{n} \frac{a_k}{z - i\beta_k}$$

where $n \geq 0$, all $a_k > 0$, the β_k pairwise distinct and real, $c_0 \geq 0$,, $\mathrm{Re}\,(d_0) = 0$. Moreover, if $n = 0$, then $c_0 > 0$.

Proof:

All summands are in $\mathscr{H}$ because of the elementary properties, therefore $\sum$ is in $\mathscr{H}$ (if $\neq 0$). So all those partial fractions belong to $\mathscr{H}$.
If $h(z) = \dfrac{P(z)}{Q(z)}$ in $\mathscr{H}$, then $\dfrac{1}{h} \in \mathscr{H}$, therefore $Q(z)$ has nothing but simple roots $i\beta_k$ $(1 \leq k \leq n)$. As to the derivation of the partial fraction expansion with

$$Q(z) \;=\; \prod_{k=1}^{n} (z - i\beta_k)$$

Auxiliary polynomials

$$Q_k(z) \;=\; \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - i\beta_j) \qquad \text{yield} \qquad 1 \;=\; \sum_{k=1}^{n} q_k(x) Q_k(x)$$

Multiplication with $P(z)$

$$\begin{aligned} P(z) \;&=\; \sum_{k=1}^{n} P_k(z) Q_k(z) \\ &=\; P_0(z)Q(z) \,+\, \sum_{k=1}^{n} \underset{\substack{\downarrow \\ \text{constant}}}{a_k}\, Q_k(z) \end{aligned}$$

Division by $Q(z) \;\Rightarrow$

$$h(z) \;=\; \frac{P(z)}{Q(z)} \;=\; P_0(z) \,+\, \sum_{k=1}^{n} \frac{a_k}{z - i\beta_k}$$

Special consideration in the vicinity of each pole $i\beta_k$ yields $a_k > 0$ (because of $h(z) \in \mathscr{H}$).

Left to examine: $z \to \infty$. Result: $\deg P_0 \leq 1$, since otherwise $\mathrm{Re}\, h(z)$ would change sign in $\mathrm{Re}\, z > 0$. Finally $P_0(z) \;=\; c_0 z + d_0$ with $c_0 \geq 0$, Re , $(d_0) = 0$.

$\square$

3.2.3 Division Algorithm for Functions in $\mathscr{H}$

Let P_0, P_1 be polynomials, $P_1 \neq 0$ and $\deg P_0 > \deg P_1$ as well as P_0, P_1 without common roots. Then $\dfrac{P_0}{P_1}(z) = h(z)$ is a function in $\mathscr{H}$ if and only if the division algorithm for P_0 and P_1

$$P_{k-1}(z) \;=\; q_k P_k(z) \,+\, P_{k+1}(z) \quad , \qquad \deg P_{k+1} < \deg P_k$$

has the form

$$q_k(z) \;=\; c_k z \,+\, d_k \quad \text{with} \quad c_k > 0\,,\ \operatorname{Re} d_k = 0$$

until the algorithm terminates.

Proof:

Numerator degree > denominator degree means $\lim\limits_{z\to\infty} h(z) = \infty$

1) Let $h(z) \in \mathscr{H}$. Partial fraction decomposition $\Rightarrow$

$$P_0(z) \;=\; (c_0 z + d_0)P_1(z) \,+\, P_2(z)$$

where $c_0 > 0$ because of $\lim\limits_{z\to\infty} h(z) = \infty$

If P_2 is not the zero polynomial then

$$\frac{P_2(z)}{P_1(z)} \;=\; \sum \frac{a_k}{z - i\beta_k} \;\in\; \mathscr{H}$$

This means that also $\quad h_1(z) \;=\; \dfrac{P_1(z)}{P_2(z)} \;\in\; \mathscr{H}$

and the same conclusion is thus carried forward until termination.

2) If the division algorithm has the special form, then for the first time $P_{n+1} = 0$

$$\frac{P_{n-1}(z)}{P_n(z)} \;=\; (c_n z + d_n) \,+\, 0 \;\in\; \mathscr{H}$$

$$\frac{P_{k-1}(z)}{P_k(z)} \;=\; \underbrace{(c_k z + d_k)}_{\in\,\mathscr{H}} \,+\, \underbrace{\frac{P_{k+1}(z)}{P_k(z)}}_{\substack{\in\,\mathscr{H} \\ \text{induction} \\ \text{premise}}}$$

$$\Rightarrow \quad \frac{P_{k-1}}{P_k} \in \mathscr{H} \quad \text{therefore also} \quad \frac{P_0}{P_1} \;=\; h(z) \;\in\; \mathscr{H}$$

$\square$

3.2.4 Application: The Stability Criterion

Let $f(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0$ be a normalized polynomial. With the auxiliary polynomials

$$\begin{aligned} P_0(z) &= z^n + \big(i\,\mathrm{Im}\,a_{n-1}\big)z^{n-1} + \big(\mathrm{Re}\,a_{n-2}\big)z^{n-2} + \ldots \\ P_1(z) &= \big(\mathrm{Re}\,a_{n-1}\big)z^{n-1} + \big(i\,\mathrm{Im}\,a_{n-2}\big)z^{n-2} + \ldots \end{aligned}$$

the following applies: f is stable (i.e. all zeros have real parts < 0) if and only if in the division algorithm

$$P_{k-1}(z) = q_k(z)P_k(z) + P_{k+1}(z) \qquad (\deg P_{k+1} < \deg P_k)$$

firstly the length up to termination is exactly $n = \deg f$ and second, the factors are

$$q_k(z) = c_k z + d_k$$

with $c_k > 0$ and $\mathrm{Re}\,d_k = 0$.

Proof sketch:

The Möbius transformation $w \mapsto \dfrac{-w+1}{w+1}$ maps the unit circle onto the imaginary axis $i\mathbb{R} \cup \{\infty\}$ and the interior of the unit circle is mapped to the half-plane $\mathrm{Re}\,w > 0$. Out of $f(z)$ form the function $f^*(z) = \overline{f(-\bar{z})}$

$$h(z) = \frac{f(z) - f^*(z)}{f(z) + f^*(z)} = \frac{-\dfrac{f^*(z)}{f(z)} + 1}{\dfrac{f^*(z)}{f(z)} + 1}$$

Therefore $\mathrm{Re}\,h(z) > 0$ is equivalent to $\dfrac{f^*(z)}{f(z)}$ lying in the unit circle! In addition, $h^*(z) = -h(z)$ is used in our special case. From this combined with the knowledge of $\mathscr{H}$ follows the assertion!

Examples:

(4) $f(z) = z^3 + (1+2i)z^2 + (3+i)z + 3 + 2i$

$$\begin{aligned} P_0(z) &= z^3 + 2iz^2 + 3z + 2i \\ P_1(z) &= z^2 + iz + 3 \\ P_0(z) &= (z+i)P_1(z) + \underbrace{(z-i)}_{P_2(z)} \end{aligned}$$

$$P_1(z) = (z+2i)P_2(z) + \underset{\displaystyle P_3(z)}{\overset{\displaystyle 1}{\downarrow}}$$

$$P_2(z) = (z-i)P_3(z) + \underset{\displaystyle P_4(z)}{\overset{\displaystyle 0}{\downarrow}}$$

Length $= 3 =$ grad f , all factors good $\Rightarrow$ f stable.

(5) $f(z) = z^3 + (1+i)z^2 + 2z + 2 + 2i$

$$P_0(z) = z^3 + iz^2 + 2z + 2i$$

$$P_1(z) = z + 2$$

$$P_0(z) = (z+i)P_1(z) + 0$$

Length $= 1 <$ grad f $\Rightarrow$ f not stable.

$$\frac{P_0(z)}{P_1(z)} = h(z) \in \mathscr{H}$$

$$f(z) = P_0(z) + P_1(z) = (z+i+1)(z^2+2)$$

Zeros are: $\pm i\sqrt{2}$; $-1-i$

4. Holomorphic or Analytic Functions

These are complex-valued functions $f(z)$, which are defined on a region $D \subset \mathbb{C}$ and for which the following limit exists in every point $z_0 \in D$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

It is called the (complex) derivative of f at the point z_0. Abbreviations:

$$f'(z_0)\,, \quad \Big(\frac{\mathrm{d}}{\mathrm{d}z} f\Big)(z_0) \quad \ldots$$

<u>Examples:</u>

(1) All constant functions are analytic everywhere and have the derivative 0.

(2) $f(z) \;=\; z^n \;\; (n \in \mathbb{N})$ is holomorphic everywhere with derivative

$$f'(z) \;=\; n z^{n-1}$$

Polar notation

$$r\mathrm{e}^{i\varphi} \;=\; z \qquad , \qquad f(z) \;=\; r^n \mathrm{e}^{ni\varphi}$$

(3) If f and g are both holomorphic (in the same region), then $f + g$ and $f \cdot g$ are also holomorphic there with the derivatives

$$(f+g)'(z) \;=\; f'(z) + g'(z)$$

$$(f\,g)'(z) \;=\; f'(z)g(z) + f(z)g'(z)$$

In particular, all polynomial functions

$$f(z) \;=\; \sum_{n=0}^{N} a_n z^n$$

are everywhere in $\mathbb{C}$ analytic with the derivative

$$f'(z) \;=\; \sum_{n=1}^{N} a_n n z^{n-1}$$

(4) If f and g are both holomorphic (in the same region), then (with the exception of the zeros of the denominator) the quotient $\dfrac{f(z)}{g(z)}$ is holomorphic and the quotient rule applies

$$\left(\frac{f}{g}\right)'(z) \;=\; \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$$

In particular, the rational functions $f(z) = \dfrac{P(z)}{Q(z)}$ (with polynomials P, Q) are holomorphic outside the zero set of Q.

(5) If f and g are analytic and the values $g(z)$ are in the domain of f, then $f(g(z))$ is also analytic and the $\mathbb{C}$-derivative is calculated according to the chain rule

$$(f \circ g)'(z) \;=\; f'(g(z))g'(z)$$

4.1 Geometric Interpretation of Holomorphy

Example:

The square function $q(z) = z^2$ has the derivative $q'(z) = 2z$. This derivative has exactly one zero, namely at $z = 0$. Polar notation $z = r\mathrm{e}^{i\varphi}$ results in $q(z) = r^2\mathrm{e}^{2i\varphi}$; hence z becomes mapped by doubling the argument and squaring the amount. The rays through 0 ($\varphi = \text{const}$) are mapped into rays through 0 and the circles around 0 are mapped into circles around 0 with radius r^2 ($r = \text{const}$). Mapping of the coordinate lines $y = \operatorname{Im} z = \text{const}$ or respectively $x = \operatorname{Re} z = \text{const}$!

$$u \;=\; \operatorname{Re} q(x+iy) \;=\; x^2 - y^2 \quad ; \quad v \;=\; \operatorname{Im} q(x+iy) \;=\; 2xy$$

with $\quad y = c \;\Rightarrow$

$$u \;=\; x^2 - c^2 \quad ; \quad v \;=\; 2cx \;=\; \pm 2c\sqrt{u + c^2} \qquad \text{(parabolas with focus 0)}$$

The importance of $\mathbb{C}$-differentiability for real functions

$$u \;=\; \operatorname{Re} q(x+iy) \quad , \quad v \;=\; \operatorname{Im} q(x+iy)$$

$$\frac{\partial u}{\partial x} = u_x = 2x \quad , \quad \frac{\partial v}{\partial x} = v_x = 2y$$

$$\frac{\partial u}{\partial y} = u_y = -2y \quad , \quad \frac{\partial v}{\partial y} = v_y = 2y$$

So the Cauchy-Riemann differential equations apply

$$u_x = v_y \quad ; \quad u_y = -v_x$$

The Jacobian matrix of the real mapping

$$(x,y) \longmapsto \big(u(x,y), v(x,y)\big)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = 2\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

The column vectors are orthogonal of equal length and the functional determinant > 0 (if not $x = y = 0$). This mapping is therefore angle preserving.

Definition of angle preservation for complex-valued functions f at the point z_0.

For each φ with a constant $\zeta \neq 0$ holds

$$\lim_{r\searrow 0} \frac{f(z_0 + r\mathrm{e}^{i\varphi}) - f(z_0)}{|f(z_0 + r\mathrm{e}^{i\varphi}) - f(z_0)|} = \zeta \mathrm{e}^{i\varphi}$$

Every function f holomorphic on D is angle-preserving in all points $z_0 \in D$ where $f'(z_0) \neq 0$,. Because there

$$\frac{f(z_0 + r\mathrm{e}^{i\varphi}) - f(z_0)}{r\mathrm{e}^{i\varphi}} \to f'(z_0) \quad \text{if} \quad r \searrow 0$$

hence

$$\lim_{r\searrow 0} \frac{f(z_0 + r\mathrm{e}^{i\varphi}) - f(z_0)}{|f(z_0 + r\mathrm{e}^{i\varphi}) - f(z_0)|} = \mathrm{e}^{i\varphi} \frac{f'(z_0)}{|f'(z_0)|}$$

4.2 Holomorphy and Cauchy-Riemann Differential Equations

Let f be holomorphic on D, $f(x+iy) = u(x,y) + iv(x,y)$

By specific choice of $h = t$ or $h = it$ $(t \in \mathbb{R})$ the derivative becomes

$$f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = u_x + iv_x$$

$$f'(z) = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it} = \frac{u_y}{i} + v_y = v_y - iu_y$$

Therefore (real or respectively imaginary part)

$$u_x = v_y \quad , \quad v_x = -u_y$$

Conversely, if u, v are continuously differentiable solutions of the Cauchy-Riemann differential equations on D, then:

Definition:

$$f(x+iy) \;:=\; u(x,y) + iv(x,y) \quad \text{a holomorphic function on } D$$

<u>As to the derivative of $f(z)$:</u>

$$f(z+h) - f(z) \;=\; \operatorname{grad} u \cdot (\operatorname{Re} h, \operatorname{Im} h) + i \cdot \operatorname{grad} v \cdot (\operatorname{Re} h, \operatorname{Im} h) + \operatorname{Rest}(h)$$

where $\lim\limits_{h \to 0} \dfrac{\operatorname{Rest}(h)}{|h|} = 0$; With that:

$$\begin{aligned} f(z+h) - f(z) &= (u_x, u_y) \cdot (\operatorname{Re} h, \operatorname{Im} h) + i \cdot (-u_y, u_x) \cdot (\operatorname{Re} h, \operatorname{Im} h) + \operatorname{Rest}(h) \\ &= u_x \cdot \underbrace{(\operatorname{Re} h + i \cdot \operatorname{Im} h)}_{h} - i u_y \cdot \underbrace{(\operatorname{Re} h + i \cdot \operatorname{Im} h)}_{h} + \operatorname{Rest}(h) \\ &= u_x h - i u_y h + \operatorname{Rest}(h) \end{aligned}$$

From this: f is complex differentiable in z with derivative

$$f'(z) \;=\; u_x - i u_y$$

4.2.1 Harmonic Functions

It turns out: Every function $f = u + iv$ that is holomorphic on D is differentiable there any number of times. The vector field on D

$$\operatorname{grad} u = (u_x, u_y)$$

is therefore not just a potential field (gravitational field), i.e

$$\operatorname{rot}(u_x, u_y) = 0$$

but the Cauchy-Riemann differential equations yield in addition

$$\operatorname{div}(u_x, u_y) \;=\; u_{xx} + u_{yy} \;=\; v_{yx} - v_{xy} \;=\; 0$$

Hence u is a solution of the Laplace equation in the plane

$$\Delta u \;=\; u_{xx} + u_{yy} \;=\; 0$$

The vector field under consideration is therefore both vortex-free and source-free! Conversely, if $u(x, y)$ is a twice continuously differentiable solution of the Laplace equation $\Delta u = 0$ on D, then u is on every disk $K \ subsetD$ (even on every simply connected subregion D' of D) the real part of a function f holomorphic on K (or respectively D').

Proof:

Consider the vector field

$$(v_x, v_y) \coloneqq -(u_y, -u_x)$$

It satisfies the integrability condition

$$\frac{\partial v_x}{\partial y} = -u_{yy} = u_{xx} = \frac{\partial v_y}{\partial x}$$

Therefore, the new vector field is actually (as the notation already suggests) the gradient of a continuously differentiable function v on D'. Therefore it follows just as above that

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic.

$\square$

Example:

A harmonic function u that has no global harmonic conjugate.

$$\begin{aligned} u(x, y) &= \ln |x + iy| \qquad (x^2 + y^2 \neq 0) \\ &= \frac{1}{2} \ln(x^2 + y^2) \end{aligned}$$

$$u_x = \frac{x}{x^2 + y^2} \quad , \quad u_y = \frac{y}{x^2 + y^2}$$

$$u_{xx} = (x^2 + y^2)^{-2}(x^2 + y^2 - 2x^2)$$

$$u_{yy} = (x^2 + y^2)^{-2}(x^2 + y^2 - 2y^2)$$

$$u_{xx} + u_{yy} = 0$$

A significance of the chain rule for holomorphic functions: Transplantation of harmonic functions.

4.2.2 Transplantation of Harmonic Functions

Let $U(x,y)$ be a real harmonic function on D, and let $D : \tilde{D} \mapsto D$ be holomorphic. Then

$$V(x,y) \;=\; U\big(\operatorname{Re} f(x+iy)\,,\, \operatorname{Im} f(x+iy)\big)$$

is a harmonic function on $\tilde{D}$.

<u>Rationale:</u> Let $z_0 \in D$ and $k \subset D$ be a disk around z_0. Then U on D is the real part of a function F which is holomorphic there. Thus V becomes the real part of the function $F(f(z))$. $F \circ f$ is differentiable according to the chain rule, i.e. $\operatorname{Re}(F \circ f) = V$ is harmonic.

<u>Application:</u> For every holomorphic function $f(z)$ outside the zero set of f,

$$V(x,y) \;=\; \ln|f(x+iy)|$$

is a harmonic function.

<u>Remark:</u>

(1) The imaginary parts of holomorphic functions are also harmonic, because with $f(z)$, $g(z) = -if(z)$ is also holomorphic.

Definition:

A bijective mapping $f : \tilde{D} \mapsto D$ between two regions $\tilde{D}$, D of the plane is called “conformal” if f including its inverse mapping is holomorphic in all points of the corresponding region. (Then f has everywhere the derivative $f'(x) \neq 0$, i.e. it is angle-preserving at every point $z_0 \in D$.)

<u>Example:</u>

$$f(z) \;=\; \frac{-z^2 - 2z + 1}{z^2 - 2z - 1}$$

provides a conformal mapping of the bisected disk

$$\tilde{D} \;=\; \big\{z \in \mathbb{C};\, |z| < 1,\, \operatorname{Re} z > 0\big\}$$

onto the circular disk

$$E \;=\; \big\{z \in \mathbb{C},\, |z| < 1\big\}$$

Execution of the construction:

i) The denominator

$$z^2 - 2z - 1 = (z - 1 + \sqrt{2})(z - 1 - \sqrt{2})$$

has only two zeros, outside of these zeros f is holomorphic.

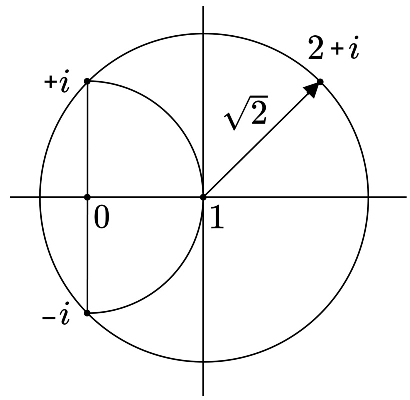
ii) The Möbius transformation

$$T(z) = \frac{z+i}{iz+1}$$

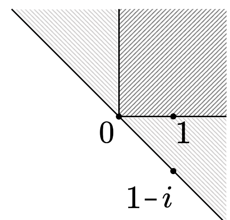
$$T(0) = i \quad ; \quad T(1) = 1 \quad ; \quad T(\infty) = -i$$

$$T(i) = \infty \quad ; \quad T(-i) = 0$$

$$T(2+i) = \frac{2+2i}{2i} = i - 1$$



T maps the circle of radius $\sqrt{2}$ around 1 onto the straight line $\operatorname{Re} z + \operatorname{Im} z = 0$, the interior is mapped bijectively and holomorphically onto the half-plane $\operatorname{Re} z + \operatorname{Im} z > 0$.



Furthermore, $\tilde{D}$ is mapped onto the positive quadrant Q.

iii) The square mapping $q(z) = z^2$ maps the half-plane $\operatorname{Re} z + \operatorname{Im} z > 0$ onto the slit plane $\mathbb{C} \setminus \{-t\,i\,;\, t \geq 0\}$ while it bijectively and holomophorically maps Q onto the upper half-plane $\operatorname{Im} z > 0$.

iv) Because T maps the circular disk E to the upper half-plane, T^{-1} maps the upper half-plane to E in a conformal manner. The mapping we are looking for is:

$$T^{-1} \circ q \circ T(z)$$

Here

$$T^{-1}(z) = \frac{z-i}{-iz+1}$$

With $\quad q \circ T(z) = (T(z))^2 = \dfrac{(z+i)^2}{(iz+1)^2} \quad$ follows

$$T^{-1} \circ q \circ T(z) = \frac{(T(z))^2 - i}{-i(T(z))^2 + 1} = \frac{(z+i)^2 - i(iz+1)^2}{-i(z+i)^2 + (iz+1)^2}$$

$$= \frac{(1+i)\left[z^2 + 2z - 1\right]}{-(1+i)\left[z^2 - 2z - 1\right]} = f(z)$$

4.3 Power Series, the Basic Example of Holomorphic Functions

To each power series

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with real or complex coefficients a_n belongs its radius of convergence

$$\varrho := \left\{ r \geq 0 \, ; \; \left(a_n r^n\right)_{n \geq 0} \text{ is bounded} \right\}$$

Primary characteristic:

If $|z - z_0| < \varrho$, then $S(z)$ even converges absolutely (compare with the geometric series), if $|z - z_0| > \varrho$ then $S(z)$ is divergent.

$S(z)$ always has the same radius of convergence as its derivative $S'(z)$ (which arises from $S(z)$ by termwise differentiation according to z).

Proof:

Let ϱ (or respectively ϱ') be the radius of convergence

$$S'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1} \; ; \quad \varrho' \leq \varrho \text{ is almost immediately evident.}$$

Now one shows: If $r < \varrho$, then $r \leq \varrho'$, therefore also $\varrho' \geq \varrho$.

Let $r > 0$ without restriction. Choose $r_1 = r(1 + \vartheta) < \varrho$ with $\vartheta \in \,]0, 1[$

$$\begin{aligned} n\left(\frac{r}{r_1}\right)^n &= \frac{n}{(1+\vartheta)^n} \qquad \text{(Bernoulli)} \\ &\leq \frac{n}{1 + n\vartheta} \leq \frac{1}{\vartheta} \end{aligned}$$

Moreover, because of $r_1 < \varrho$, $a_n r_1^n$ is bounded. Hence

$$\left| n \, a_n r^{n-1} \right| = \left| a_n r_1^n \right| \frac{1}{r} \, n \left(\frac{r}{r_1}\right)^n$$

also bounded.

$\square$

4.3.1 Theorem on the Holomorphy of Power Series

$$S(z) \;=\; \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

shall have positive convergence radius ϱ. Then for z with $|z-z_0| < \varrho$ always

$$\lim_{h\to 0} \frac{S(z+h)-S(z)}{h} \;=\; S'(z)$$

Proof:

Without restriction $z_0 = 0$; Choose r with $|z| < r < \varrho$

Remember: $\quad a^n - b^n \;=\; (a-b)\sum_{m=0}^{n-1} a^m b^{n-m-1}$

$$p_n(z,h) \;=\; \frac{(z+h)^n - z^n}{h} \;=\; \sum_{m=0}^{n-1} (z+h)^m z^{n-m-1}$$

is a polynomial in h with $\quad p_n(z,0) \;=\; nz^{n-1}$.

In addition, for $|z+h| \;\le\; r$ holds

$$|p_n(z,h)| \;\le\; \sum_{m=0}^{n-1} |z+h|^m |z|^{n-m-1} \;\le\; n\cdot r^{n-1}$$

For $\epsilon > 0$ select N with $\quad \sum_{n=N+1}^{\infty} |n\,a_n| r^{n-1} \;<\; \frac{\epsilon}{4}$

With that $\quad \left|\frac{S(z+h)-S(z)}{h} - S'(z)\right|$

$$= \left|\sum_{n=1}^{N} a_n\big[p_n(z,h) - nz^{n-1}\big] + \sum_{n=N+1}^{\infty} a_n\big[p_n(z,h) - nz^{n-1}\big]\right|$$

$$\le \left|\sum_{n=1}^{N} a_n\big[p_n(z,h) - nz^{n-1}\big]\right| + \frac{\epsilon}{2}$$

Because of $\lim_{h\to 0} p_n(z,h) \;=\; nz^{n-1}$, follows

$$\left|\sum_{n=1}^{N} a_n\big[p_n(z,h) - nz^{n-1}\big]\right| \;<\; \frac{\epsilon}{2} \qquad \text{if} \quad |h| < \delta$$

$\square$

Quotient criterion for the convergence radius

If for the coefficients a_n of the series

$$S(z) \;=\; \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

it holds that

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \;=\; R$$

then $S(z)$ has the convergence radius $\varrho = R$.

Examples:

(1) $\displaystyle\sum_{n=0}^{\infty} 2^n z^n$ has the radius of convergence $\varrho = \dfrac{1}{2}$

(2) For all $k \in \mathbb{N}$, $\displaystyle\sum_{n=0}^{\infty} n^k z^n$ has the radius of convergence $\displaystyle\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k \;=\; 1$

5. Integration of Complex-valued Functions

A line integral for complex-valued functions $f(z) = u(x,y) + iv(x,y)$ on regions $D \subset \mathbb{C}$ and for (piecewise continuously differentiable) curves

$$\gamma : [a,b] \longmapsto D \qquad \text{is being declared,}$$

as soon as f is continuous (that means $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ both continuous on D)[1] by the formula

$$\int_\gamma f(w)\,\mathrm{d}w \;:=\; \int_a^b \big[u(\gamma_1(t),\gamma_2(t)) + iv(\gamma_1(t),\gamma_2(t))\big]\big[\dot\gamma_1(t) + i\dot\gamma_2(t)\big]\,\mathrm{d}t$$

wherein $\qquad \gamma_1(t) = \operatorname{Re}\gamma(t) \quad ; \quad \gamma_2(t) = \operatorname{Im}\gamma(t)$

Separation into real and imaginary parts

$$\int_\gamma f(w)\,\mathrm{d}w = \int_a^b \big(u(\gamma(t))\dot\gamma_1(t) - v(\gamma(t))\dot\gamma_2(t)\big)\,\mathrm{d}t$$

$$+\, i\int_a^b \big(u(\gamma(t))\dot\gamma_2(t) + v(\gamma(t))\dot\gamma_1(t)\big)\,\mathrm{d}t$$

$$\boxed{\int_\gamma f(w)\,\mathrm{d}w \;:=\; \int_a^b f(\gamma(t))\dot\gamma(t)\,\mathrm{d}t}$$

Here $[a,b]$ is the definition interval of the curve γ, which runs in the domain of definition D of f. Decomposing $f = u + iv$ into real and imaginary parts of f results in

[1] Compare chapter 3 in HIGHER MATHEMATICS Lectures Part Three.

$$\int_\gamma f(w)\,\mathrm{d}w \;=\; \int_\gamma (u,-v)\,\mathrm{d}\vec{x} \;+\; i\int_\gamma (v,u)\,\mathrm{d}\vec{x}$$

5.1 Integral and Primitive Function

Let f be the (continuous) derivative of the holomorphic function F on D. Then the following formula applies to every curve $\gamma:\ [a,b] \mapsto D$ running in D

$$\int_\gamma f(w)\,\mathrm{d}w \;=\; F\big(\gamma(b)\big) \;-\; F\big(\gamma(a)\big)$$

In particular, then for every closed curve $\delta:\ [a,b]$ (that means $\delta(a) = \delta(b)$)

$$\oint_\delta f(w)\,\mathrm{d}w \;=\; 0$$

<u>Rationale:</u>

$$F \;=\; U \;+\; iV \qquad \Rightarrow$$

$$f(x+iy) \;=\; F'(x+iy) \;=\; U_x \;+\; iV_x$$

$$\;=\; u+iv \;=\; V_y \;-\; iU_y$$

Hence $(u,-v) \;=\; \operatorname{grad} U$ and $(v,u) \;=\; \operatorname{grad} V$

With that[2]

$$\int_\gamma (u,-v)\,\mathrm{d}\vec{x} \;=\; U\big(\gamma(b)\big) \;-\; U\big(\gamma(a)\big)$$

$$\int_\gamma (v,u)\,\mathrm{d}\vec{x} \;=\; V\big(\gamma(b)\big) \;-\; V\big(\gamma(a)\big)$$

$$\int_\gamma f(w)\,\mathrm{d}w \;=\; U\big(\gamma(b)\big) \;-\; U\big(\gamma(a)\big) + i\big[V\big(\gamma(b)\big) \;-\; V\big(\gamma(a)\big)\big]$$

$$\;=\; F\big(\gamma(b)\big) \;-\; F\big(\gamma(a)\big)$$

The second assertion about δ is a special case.

□

[2]See HIGHER MATHEMATICS Lectures Part Three.

Remark:

(1) For any power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

inside its circle of convergence

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

is a primitive function of f.

Warning example: Non-existence of a global holomorphic logarithm.

The function $f(z) = \dfrac{1}{z}$ holomorphic in $D = \mathbb{C} \setminus \{0\}$ has no primitive function there.

As to the rationale:

Because suppose F is a primitive function of f in D. Then one would have

$$\int_\delta f(w)\,\mathrm{d}w = 0$$

on (for example) the boundary curve $\delta = e^{it}$, $t \in [0, 2\pi]$ of the unit circle.

Thereby wrong! Direct calculation of the integral

$$\int_\delta \frac{\mathrm{d}w}{w} = \int_0^{2\pi} \frac{\dot{\delta}(t)\,\mathrm{d}t}{\delta(t)} = \int_0^{2\pi} \frac{ie^{it}}{\mathrm{e}^{it}}\,\mathrm{d}t = 2\pi i \neq 0$$

□

Three elementary properties of the line integral $\int_\gamma f(w)\,\mathrm{d}w$

(P1) It is $\mathbb{C}$-linear with respect to the integrand:

$$\int_\gamma (f+g)(w)\,\mathrm{d}w = \int_\gamma f(w)\,\mathrm{d}w + \int_\gamma g(w)\,\mathrm{d}w$$

$$\int_\gamma \lambda f(w)\,\mathrm{d}w = \lambda \int_\gamma f(w)\,\mathrm{d}w \qquad ; \qquad \lambda \in \mathbb{C}$$

(P2) It is invariant under parameter transformation. If $\varphi : [c,d] \mapsto [a,b]$ is a (including the inverse mapping) continuously differentiable transformation with $\varphi(c) = a$, $\varphi(d) = b$, then for the transformed curve $\delta(t) = \gamma(\varphi(t))$:

$$\int_{\delta} f(w)\,\mathrm{d}w \quad = \quad \int_{\gamma} f(w)\,\mathrm{d}w$$

On the other hand, if $\gamma^*(t) = \gamma((a+b)-t)$ is the opposite curve to γ, then

$$\int_{\gamma^*} f(w)\,\mathrm{d}w \quad = \quad -\int_{\gamma} f(w)\,\mathrm{d}w$$

(P3) The line integral is additive with respect to composition of curves.

Let $\gamma_1 : [a,b] \mapsto D$, $\gamma_2 : [b,c] \mapsto D$ with $\gamma_1(b) = \gamma_2(b)$

and $\gamma(t) = \begin{cases} \gamma_1(t) , & a \le t \le b \\ \gamma_2(t) , & b \le t \le c \end{cases}$

then

$$\int_{\gamma = \gamma_1 \cup \gamma_2} f(w)\,\mathrm{d}w \quad = \quad \int_{\gamma_1} f(w)\,\mathrm{d}w + \int_{\gamma_2} f(w)\,\mathrm{d}w$$

5.2 Cauchy's Integral Formula

Let f be holomorphic on D. For each $z_0 \in D$ and each $r > 0$, such that the closed disk $|z - z_0| \le r$ is included in D, applies

$$\boxed{f(z) \quad = \quad \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z}\,\mathrm{d}w}$$

Here $|z - z_0| < r$ and $\gamma_r = z_0 + r\mathrm{e}^{it}$, $t \in [0, 2\pi]$.

5.2.1 Theorem on Taylor Expansion of Holomorphic Functions

i) A function $f(z)$ that is holomorphic in the region $D \subset \mathbb{C}$ can be expanded into a power series around every point $z_0 \in D$.

$$f(z) \quad = \quad \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

ii) Its radius of convergence is at least as large as the radius of the largest open circular disk around z_0 in D.

iii) For the Taylor coefficients the following formulas apply

$$a_n \quad = \quad \frac{f^{(n)}(z_0)}{n!} \quad = \quad \frac{1}{2\pi i} \oint\limits_{\gamma_r} \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w$$

Here $\gamma_r \;=\; z_0 + r\mathrm{e}^{it}\,,\ t \in [0, 2\pi]\,,$ and $r > 0$ is arbitrary under the constraint that the circular disk $|z - z_0| \leq r$ lies in D.

Proof:

From Cauchy's formula it follows for $|z - z_0| < r$

$$f(z) \quad = \quad \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(w)}{w - z} \, \mathrm{d}w \quad = \quad \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{it})\dot{\gamma}_r}{r\mathrm{e}^{it} - (z - z_0)} \, \mathrm{d}t$$

$$= \quad \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{it})i}{1 - \left(\dfrac{z - z_0}{r\mathrm{e}^{it}}\right)} \, \mathrm{d}t$$

rewrite using the geometric series:

$$f(z) \quad = \quad \frac{1}{2\pi i} \int\limits_0^{2\pi} f(z_0 + r\mathrm{e}^{it})i \sum_{n=0}^{\infty} \left(\frac{z - z_0}{r\mathrm{e}^{it}}\right)^n \mathrm{d}t$$

since uniformly convergent in t it follows $(w = z_0 + r\mathrm{e}^{it})$

$$f(z) \quad = \quad \sum_{n=0}^{\infty} (z - z_0)^n \cdot \underbrace{\frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{it})\dot{\gamma}_r(t)}{\left(r\mathrm{e}^{it}\right)^{n+1}} \, \mathrm{d}t}_{a_n \;=\; \frac{1}{2\pi i} \oint\limits_{\gamma_r} \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w}$$

According to the above deduction, this series converges as soon as $|z - z_0| < r$ and the circular disk $|z - z_0| \leq r$ lies in D. The (second) integral formula for the a_n is proven (for fixed r). The formula $a_n = f^{(n)}(z_0)/n!$ is now a direct consequence of the connection between the power series and the derivative. This formula shows that the coefficients are independent of the auxiliary number r.

$\square$

By using the standard integral estimate

$$\left| \int_\gamma g(w)\,\mathrm{d}w \right| \;\leq\; l(\gamma) \max_{z \,\text{along}\, \gamma} |g(z)|$$

with line length

$$l(\gamma) \;=\; \int_a^b |\dot\gamma(t)|\,\mathrm{d}t \;=\; \int_a^b \sqrt{\dot\gamma_1^2(t) + \dot\gamma_2^2(t)}\,\mathrm{d}t$$

one obtains the Cauchy inequalities for the Taylor coefficients:

Let $M(r) \;=\; \max_{z - z_0 = r} |f(z)|$

Then: $|a_n| \;\leq\; M(r) r^{-n}$

Definition:

Every function that is holomorphic on $\mathbb{C}$ is called an "entire function".

Examples:

All polynomials, also e^z, e^{it}, and linear combinations such as for example

$$\cos z \;=\; \frac{1}{2}\left(\mathrm{e}^{iz} + \mathrm{e}^{-iz}\right) \qquad \cosh z \;=\; \frac{1}{2}\left(\mathrm{e}^{z} + \mathrm{e}^{-z}\right)$$

$$\sin z \;=\; \frac{1}{2i}\left(\mathrm{e}^{iz} - \mathrm{e}^{-iz}\right) \qquad \sinh z \;=\; \frac{1}{2}\left(\mathrm{e}^{z} - \mathrm{e}^{-z}\right)$$

Remark:

(2) The Taylor expansion of an entire function has the radius of convergence $\varrho = \infty$ around every point z_0.

5.2.2 Liouville's Theorem

Every bounded entire function f is constant.

Proof from the Cauchy inequality:

By assumption (for $z_0 = 0$) $M(r)$ is bounded, e.g. $\leq M$. Taylor expansion

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n \qquad |a_n| \;\leq\; M r^{-n} \quad \text{for every } r > 0$$

For $n > 0$, the limit transition $r \to \infty$ gives $|a_n| = 0$, i.e. $a_n = 0 \quad \forall n > 0$.

$\square$

Application:

New proof for the fundamental theorem of algebra over $\mathbb{C}$. Every non-constant polynomial P has at least one root in $\mathbb{C}$.

5.3 Identity Theorem for Analytic Functions

Let $f(z)$ and $g(z)$ be analytic in the region D. For $A = \{z \in D\,;\, f(z) = g(z)\}$ there are only two options. Either $A = D$ or every point $a \in A$ has a neighborhood U with $f(z) \neq g(z)$ for all $z \in U$, $z \neq a$. Equivalent formulation for the difference $h(z) = f(z) - g(z)$.

5.3.1 Theorem about the Zeros of Analytic Functions

The set of zeros N of an analytic and non-constant function $h(z)$ in the region D consists only of isolated points in D. (On the boundary of D could possibly lie cluster points of zeros.)

The **proof** consists of a local and a global perspective.

1) The local perspective:

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be the non-constant Taylor series of h around z_0

$$k := \min_{a_n \neq 0} n$$

$$h(z) = a_k (z - z_0)^k \left[1 + \sum_{n=k+1}^{\infty} \frac{a_n}{a_k} (z - z_0)^{n-k}\right]$$

The angular bracket is continuous with value 1 at the point $z = z_0$. Therefore there is a $r > 0$ such that it has no roots on $|z - z_0| \leq r$.

Therefore $h(z)$ on $|z - z_0| \leq r$ also has at most the one zero $z = z_0$. Expressed differently: In a cluster point z_0 of zeros of the analytic function $h(z)$, its Taylor series has nothing but coefficients $a_n = 0$.

2) The global perspective:

Assumption, $z_0 \in D$ be a cluster point of zeros of $h(z)$. Then according to **1)**, on a (closed) disk K_0 around z_0 contained in D, $h(z)$ is identical 0. Let z_1 be another point in D. Join z_0 with z_1 by a curve γ in D. Circular disk covering of the line $\gamma\big([a,b]\big)$ with finitely many closed circles K_n in D such that the center of K_{n+1} falls into K_N.

Local perspective: If $h(z)$ vanishes identically on K_n, then the center of K_{n+1} is a cluster point of zeros, therefore $h(z)$ is identical to 0 also on K_{n+1} .

□

5.3.2 Cauchy's Integral Formula for $z = z_0$

Mean value property of analytic functions

For each circular disk $|z - z_0| \le r$ in the holomorphic region D of the functions f the following formula applies

$$f(z_0) \;=\; \frac{1}{2\pi}\int\limits_0^{2\pi} f(z_0 + r\mathrm{e}^{it})\,\mathrm{d}t$$

The value of f at the center is equal to the integral value over the boundary of the circle!

Transfer to solutions $u(x,y)$ of the Lapalace equation, i.e. to harmonic functions $u(x,y)$. On circular disks in the domain of definition of u, u is known to be the real part of an analytical function $f(z)$.

Separate real and imaginary parts in the above integral:

$$u(x_0, y_0) \;=\; \frac{1}{2\pi}\int\limits_0^{2\pi} u(x_0 + r\cos t, y_0 + r\sin t)\,\mathrm{d}t$$

This mean value formula for harmonic functions $u(x,y)$ applies to every circular disk contained entirely in the domain of u:

$$(x - x_0)^2 \,+\, (y - y_0)^2 \;\le\; r^2$$

5.3.3 Principle of Maximum and Minimum for Analytic Functions

Every harmonic function $u(x,y)$ in the region D which has a maximum or a minimum at the point $(x_0, y_0) \in D$ is already constant on D.

Proof:

It is sufficient to discuss maxima, since with $u(x,y)$ also $-u(x,y)$ solves the Laplace equation. Furthermore, $u(x,y) > 0$ can be assumed without restriction, because with $u(x,y)$, $u(x,y) + \text{const}$ is also harmonic.

In addition to a local consideration, the rest of the proof contains a global consideration, which is carried out using the method of covering circular disks.

The local perspective: Integral estimate from the mean value formula.

$M(r)$ shall denote the maximum of $u(x,y)$ on

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

where $|z - z_0|^2 \leq r^2$ lies in D. Since $u(x_0, y_0)$ maximal, it follows

$$M(r) \leq u(x_0, y_0)$$

Further, on the circle

$$u(x_0 + r\cos t, y_0 + r\sin t) \leq M(r) \qquad \forall\, t$$

Mean value formula:

$$\begin{aligned} 0 &\leq u(x_0, y_0) - u(r) \\ &= \frac{1}{2\pi} \int\limits_0^{2\pi} \big[u(x_0 + r\cos t, y_0 + r\sin t) - u(r)\big]\, \mathrm{d}t \leq 0 \end{aligned}$$

The chain of inequalities is only valid if simultaneously

$$u(x_0, y_0) = M(r) \qquad \text{and}$$

$$M(r) = u(x_0 + r\cos t, y_0 + r\sin t) \qquad \forall\, t$$

Therefore, for every permissible $r > 0$, $u(x,y) = u(x_0, y_0)$ on the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$. That means u is constant on those circular disks around (x_0, y_0).

□

The maximum principle for analytic functions

If for the on D holomorphic function $f(z)$ the absolute value function $|f(z)|$ has a maximum at the point $z_0 \in D$, then $f(z)$ is constant.

Proof:

In the case $f(z_0) = 0$ there is nothing to show. Therefore without restriction $f(z_0) \neq 0$. Even $f(z_0) = 1$ can be considered (otherwise consider $g(z) = f(z)/f(z_0)$). The real part $u(x,y)$ of $f(x+iy)$ is harmonic, furthermore

$$u(x,y) \;\leq\; |u(x,y)| \;\leq\; |f(x+iy)|$$

and

$$u(x_0,y_0) \;=\; f(x_0+iy_0) \;=\; 1$$

Hence (x_0,y_0) is a maximum of $u(x,y)$. Therefore, according to the last theorem, $u(x,y) =$ const.

Cauchy-Riemann differential equations:

$$f'(z) \;=\; u_x(x,y) + iv_x(x,y) \;=\; u_x(x,y) - iv_y(x,y) \;=\; 0$$

$f'(z) = 0$ implies $f(z) = \text{const}$ on the region D.

$\square$

Application:

Harmonic functions have their maxima and minima on the edge.

Let a bounded region D be given. Let $u(x,y)$ be harmonic on D and on $\bar{D} = D \cup \text{RdD}$ (RdD = set of all boundary points on D). Then u takes on its maxima and minima on RdD.

Proof:

$\bar{D}$ is compact, so the continuous real function u on $\bar{D}$ takes on both maxima and minima. If u is not constant, then according to the maximum principle, the maxima and minima do not occur in D.

$\square$

Consequence:
The Dirichlet problem for regions which are bounded has at most one solution.

Rationale:

A continuous function $u(x,y)$ is given on the boundary of a bounded region. Then there exists at most one continuation of $u(x,y)$ that is continuous on $\bar{D}$ and harmonic on D. Because suppose $U(x,y)$, $V(x,y)$ are two solutions of the problem. Then $U(x,y) - V(x,y)$ is continuous on $\bar{D}$ and harmonic on D. Moreover, this difference has the value 0 everywhere on the boundary RdD. Hence for $W(x,y) = U(x,y) - V(x,y)$ the maximum = minimum = 0, i.e. $U = V$.

5.4 The Poisson Integral Formula

Let $f(z)$ be holomorphic in a neighborhood of $|z| \leq 1$. Then for $|z| < 1$ holds

$$\boxed{f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1-|z|^2}{|\mathrm{e}^{it}-z|^2}\,\mathrm{d}t}$$

Proof:

In the case $z = 0$ this is the mean value formula. For $0 < |z| < 1$, $\frac{1}{\bar{z}}$ is outside the unit circle. The function

$$g(w) \;=\; \frac{f(w)}{w - \frac{1}{\bar{z}}}$$

is then holomorphic in a circle containing $|z| \leq 1$ in its interior. It therefore has a primitive function $G(w)$ on this circle. Therefore, for $\gamma(t) = \mathrm{e}^{it}$, $t \in [0, 2\pi]$

$$0 \;=\; \int_\gamma g(w)\,\mathrm{d}w \;=\; \int_0^{2\pi} \frac{f(e^{it})\dot{\gamma}(t)}{\mathrm{e}^{it} - \frac{1}{\bar{z}}}\,\mathrm{d}t \;=\; \int_\gamma \frac{f(w)}{w - \frac{1}{\bar{z}}}\,\mathrm{d}w \qquad (*)$$

because $\frac{1}{\bar{z}}$ outside γ! On $|w| = 1$ applies $\bar{w} = w^{-1}$. With that

$$\frac{1}{w-z} - \frac{1}{w-\frac{1}{\bar{z}}} \;=\; \frac{z - \frac{1}{\bar{z}}}{(w-z)(w-\frac{1}{\bar{z}})} \qquad \text{(expand with } \bar{z}\bar{w}\text{)}$$

$$\frac{1}{w-z} - \frac{1}{w-\frac{1}{\bar{z}}} \;=\; \frac{(1-|z|^2)\bar{w}}{|w-z|^2} \qquad (**)$$

Subtract zero from Cauchy's integral formula by combing it with $(*)$

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(w)}{w-z}\,\mathrm{d}w - \underbrace{\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(w)}{w-\frac{1}{\bar z}}\,\mathrm{d}w}_{=0}$$

$$= \frac{1}{2\pi i}\int\limits_{\gamma} f(w)\left[\frac{1}{w-z}-\frac{1}{w-\frac{1}{\bar z}}\right]\mathrm{d}w$$

and then use $(**)$ for the expression in square brackets under the integral:

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(w)\bar w(1-|z|^2)}{|w-z|^2}\,\mathrm{d}w \qquad ; \quad w = \mathrm{e}^{it}$$

$$= \frac{1}{2\pi}\int\limits_{0}^{2\pi} f(\mathrm{e}^{it})\frac{1-|z|^2}{|\mathrm{e}^{it}-z|^2}\,\mathrm{d}t$$

$\square$

<u>Remark:</u>

(3) For the constant function $f(z) \equiv 1$ follows the relation:

$$\int\limits_{0}^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}\,\mathrm{d}t = 2\pi$$

The solution to Dirichlet's boundary value problem for the circular disk can be derived from it.

5.4.1 Solution of Dirichlet's Boundary Value Problem for a Circular Disk

Let there be given a continuous real function $\hat u$ on the boundary S^1 of the unit circle. Then

$$u(z) = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\hat u(\mathrm{e}^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,\mathrm{d}t$$

defines a continuous extension of $\hat u$ in $|z| \le 1$. For $|z| < 1$, $u(z)$ is the real part of the in $|z| < 1$ holomorphic function

$$f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \hat{u}(e^{it})\frac{e^{it}+z}{\mathrm{e}^{it}-z}\,\mathrm{d}t$$

In particular, $u(z)$ is a harmonic continuation of $\hat{u}$ in $|z|<1$.

Remark:

(4) Difficult in the proof is only the continuity of the continuation in the points of S^1.

6. Extending the Theory of Analytic Functions

6.1 The Holomorphic Logarithm

$$\oint_{|w|=1} \frac{\mathrm{d}w}{w} \;=\; 2\pi i \qquad \text{is not equal to} \quad 0$$

Therefore $z \mapsto \dfrac{1}{z}$ has no primitive function in the whole domain of definition $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. The search for a holomorphic primitive function

$$F(z) \;=\; U(x,y) \,+\, iV(x,y)$$

leads to the vector fields

$$\big(u(x,y), -v(x,y)\big) \quad \text{or respectively} \quad \big(v(x,y), u(x,y)\big)$$

where $\quad u \;=\; \mathrm{Re}\,\dfrac{1}{z} \;=\; \dfrac{x}{x^2+y^2} \quad ; \quad v \;=\; \mathrm{Im}\,\dfrac{1}{z} \;=\; \dfrac{-y}{x^2+y^2}$

$$(u,-v) \;=\; \Big(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\Big)$$

is the gradient of $\;U(x,y) \;=\; \ln|z|\;$ and

$$(v,u) \;=\; \Big(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\Big)$$

is the orthogonal vector field.

Polar coordinates in the plane:

$$r(x,y) \;=\; \sqrt{x^2+y^2} \qquad , \qquad \varphi(x,y) \quad \text{with} \quad \begin{cases} x \;=\; r\cos\varphi \\ y \;=\; r\sin\varphi \end{cases}$$

From the second (or respectively first) equation implicit differentiation yields

$$\varphi_x \;=\; \frac{-y}{x^2+y^2} \quad , \quad \varphi_y \;=\; \frac{x}{x^2+y^2}$$

Possible solutions for $V(x,y)$ are the continuous fixations of $\varphi(x,y) = \arg z$, of the angle between the position vector of z and the positive real axis. The obstacle to the existence of a continuous global argument is also the obstacle to the existence of a global analytic logarithm.

On $D_+ = \mathbb{C} \setminus \big\{x \in \mathbb{R}\,;\; x \le 0\big\}$ the principal value (branch) of the complex logarithm becomes

$$\mathit{Log}\, z \;=\; \ln|z| \,+\, i\varphi(x,y) \qquad -\pi \,<\, \varphi(x,y) \,<\, \pi$$

Side branches are

$$\mathit{Log}\, z \;=\; \ln|z| \,+\, i\varphi(x,y) \,\pm\, 2\pi i k \quad (k = \pm 1, \pm 2, \ldots)$$

$$D_+ = \mathbb{C} \setminus \big\{x \in \mathbb{R}\,;\; x \le 0\big\} \quad \mathit{Log}\, z \;=\; \ln|z| + i\varphi(x,y) \quad (-\pi \,<\, \varphi(x,y) \,<\, \pi)$$

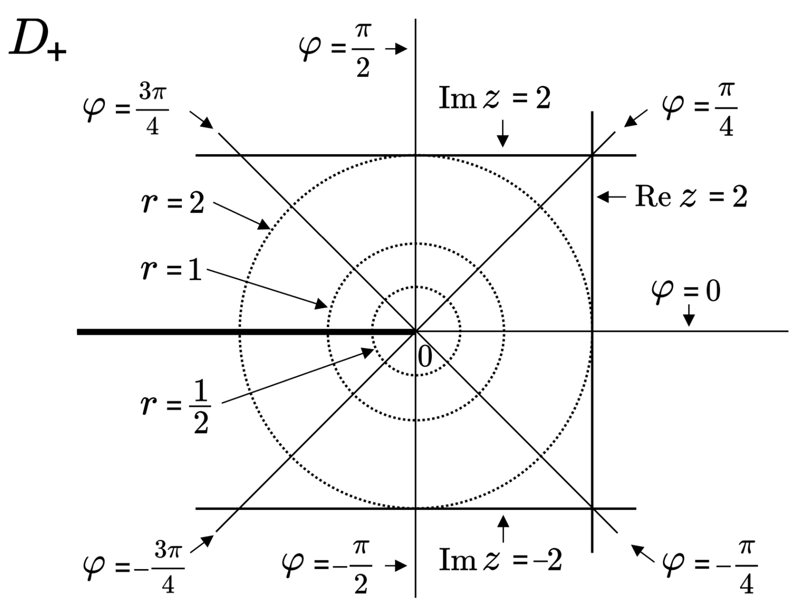
$$D_- = \mathbb{C} \setminus \big\{x \in \mathbb{R}\,;\; x \ge 0\big\} \quad \mathit{Log}\, z \;=\; \ln|z| \,+\, i\varphi(x,y) \quad (0 \,<\, \varphi(x,y) \,<\, 2\pi)$$

Definition of the general power z^α by using the holomorphic logarithm

$$z^\alpha \;:=\; \exp\big(\alpha \mathit{Log}\, z\big)$$

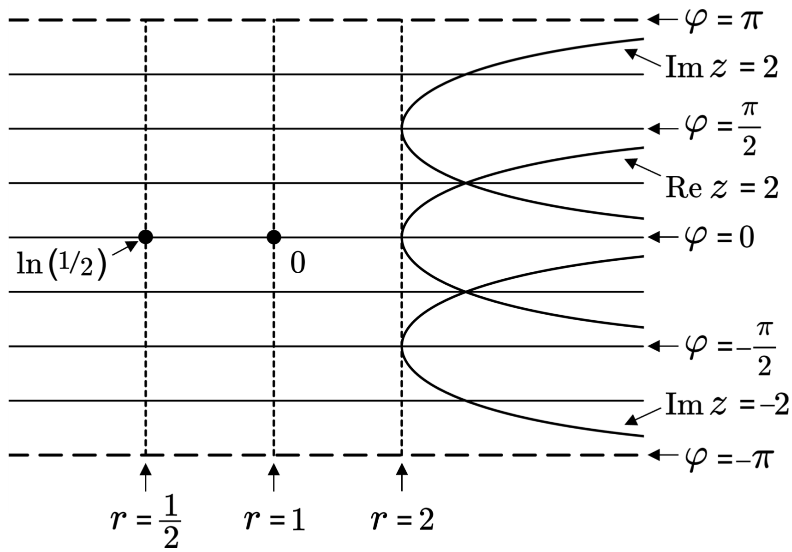
To keep in mind: the exact definition requires a cut from ∞ to 0 in a plane, which makes the definition of the holomorphic logarithm possible in the first place. It also includes the fixation of one of the main or side branches.

Mapping properties of the holomorphic logarithm:



w-plane:

$$Log\, z \;=\; \ln|z| \,+\, i\varphi$$



- The rays

$$z \;=\; r\mathrm{e}^{i\varphi} \quad ; \quad r \in]0, \infty[$$

are mapped to straight lines

$$\{w \in \mathbb{C}\,;\ \operatorname{Im} w = \varphi\}$$

- The dotted circles

$$\{z = r\mathrm{e}^{i\varphi}\,,\ \ -\pi \,<\, \varphi \,<\, \pi\}$$

are mapped onto straight line segments

$$\{w \in \mathbb{C}\,,\ \ -\pi \,<\, \operatorname{Im} w \,<\, \pi\,, \operatorname{Re} w = \ln r\}$$

- The straight lines $\operatorname{Re} z \;=\; c \;>\; 0\,;\quad z \;=\; c + it$ are mapped onto the $t \in \mathbb{R}$ curves

$$w \;=\; \ln|c + it| \,+\, i \arctan\left(\frac{t}{c}\right)$$

- To rotations in the z-plane by an angle α correspond purely imaginary translations in the w-plane
- The straight lines $\operatorname{Im} z = \pm 2$ in the z-plane are rotations of $\operatorname{Re} z = 2$ by $\pm\frac{\pi}{2}$
- In the w-plane, these straight lines therefore correspond to translations of $\operatorname{Re} z = 2$ by $\pm\frac{\pi}{2}$

6.2 The Winding Number

is declared for closed curves $\gamma : [a,b] \mapsto \mathbb{C}$ and for points $z \in \mathbb{C} \setminus \gamma\big([a,b]\big)$, from the complement of the curve image

$$n_\gamma(z) \;=\; \oint\limits_\gamma \frac{\mathrm{d}w}{w-z}$$

This number (consider the local primitive function of $w \mapsto \dfrac{1}{w-z}$) measures (normalized by the factor $\dfrac{1}{2\pi}$) the increase in angle of $\gamma(t) - z$ as the curve is being traversed.

Because γ is closed, $n_\gamma(z)$ is always an integer. $n_\gamma(z)$ is continuous as a function of z. Therefore $n_\gamma(z)$ is always constant on subregions of $\mathbb{C} \setminus \gamma\big([a,b]\big)$. On the unbounded subregion of $\mathbb{C} \setminus \gamma\big([a,b]\big)$, the winding number is $n_\gamma(z) = 0$.

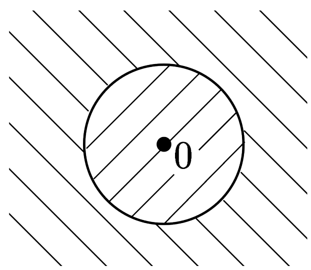
Examples:

(1)

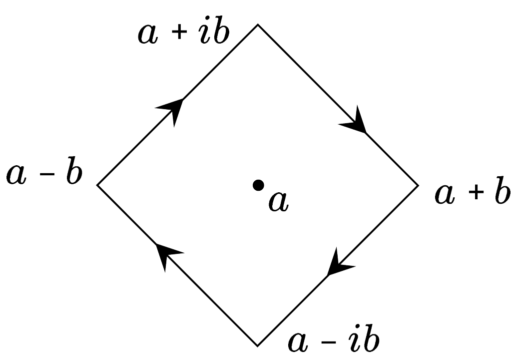
$$\gamma(t) \;=\; re^{it} \qquad t \in [0, 2\pi]$$

$$n_\gamma(z) \;=\; 1 \quad , \quad \text{if} \quad |z| < r$$

$$n_\gamma(z) \;=\; 0 \quad , \quad \text{if} \quad |z| > r$$



(2)



$b \neq 0$; γ = edge curve of the square with the corners

$$a+b \quad ; \quad a-ib \quad ; \quad a-b \quad ; \quad a+ib$$

$n_\gamma(z) \;=\; 0$ outside of the square

$n_\gamma(z) \;=\; -1$ inside the square

The actual calculation (as a warning)

$$\gamma \;=\; \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$$

where each γ_k is a connection route from $a + i^k b$ to $a + i^{k-1} b$.

$$\gamma_k(t) \;=\; a + i^k b \;+\; (t-k)(i^{k-1}b - i^k b) \qquad k \;\leq\; t \;\leq\; k+1$$

$$\dot\gamma_k(t) \;=\; -i^k b(1+i)$$

$$2\pi i n_\gamma(a) = \sum_{k=0}^{3} \int\limits_{k}^{k+1} \frac{\dot\gamma_k(t)}{\gamma_k(t)-a}\,\mathrm{d}t$$

$$= \sum_{k=0}^{3} \int\limits_{k}^{k+1} \frac{-bi^k(1+i)}{bi^k\big(1-(t-k)(1+i)\big)}\,\mathrm{d}t$$

$$= -4\int\limits_0^1 \frac{(1+i)}{1-t((i+1)}\,\mathrm{d}t = -4\int\limits_0^1 \frac{(1+i)(1-t+it)}{(1-t)^2+t^2}\,\mathrm{d}t$$

With that

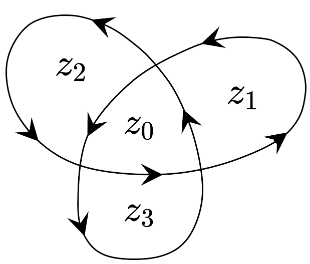
$$2\pi i n_\gamma(a) = -4\int\limits_0^1 \frac{1-2t+i}{(1-t)^2+t^2}\,\mathrm{d}t$$

The left side (hence also the integral) is purely imaginary. Therefore

$$2\pi i n_\gamma(a) = -4i\int\limits_0^1 \frac{\mathrm{d}t}{(1-t)^2+t^2}\mathrm{d}t \qquad (x = 2t-1)$$

$$= -4i\int\limits_{-1}^{+1} \frac{\mathrm{d}x}{x^2+1} = -4i\arctan x\Big|_{-1}^{+1} = -2\pi i$$

(3)



$$n_\gamma(z_0) = 2$$

$$n_\gamma(z_1) = n_\gamma(z_2) = n_\gamma(z_3) = 1$$

A principle for identification of the winding number in the case of curves with self-intersections: Decomposition into several simply closed partial curves.

6.2.1 General Version of Cauchy's Integral Formula

Let f be holomorphic in the region $D \subset \mathbb{C}$ and $\gamma : I \mapsto D$ be a closed curve such that for all $z_0 \in \mathbb{C} \setminus D$ the winding number $n_\gamma(z_0) = 0$ (any holes present in D should not be enclosed by γ). Then

$$\boxed{n_\gamma(z) f(z) \;=\; \frac{1}{2\pi i} \oint\limits_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w} \qquad z \in D \setminus \gamma(I)$$

A **proof** from Liouville's theorem after Dixon (1971):

$$E := \left\{ z \in \mathbb{C} \setminus \gamma(I)\,;\; n_\gamma(z) = 0 \right\}$$

According to premise $\mathbb{C} = D \cup E$

1) Auxiliary function for $(z, w) \in D \times D$

$$g(z,w) \;=\; \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if} \quad w \neq z \\[2ex] f\,'(w) & \text{if} \quad w = z \end{cases}$$

g is continuous in $D \times D$ and for each w holomorphic in z.

2) By integrating one obtains a second auxiliary function

$$G(z) \;=\; \begin{cases} \displaystyle\int\limits_\gamma g(z,w)\,\mathrm{d}w & \text{if} \quad z \in D \\[2ex] \displaystyle\int\limits_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w & \text{if} \quad z \in E \end{cases}$$

On $z \in D \cap E$, both expressions for $G(z)$ are identical because there the winding number n_γ is 0:

$$\begin{aligned} \int\limits_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \;&=\; \int\limits_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \;-\; 2\pi i n_\gamma(z) f(z) \\ &=\; \int\limits_\gamma \frac{f(w)-f(z)}{w-z}\,\mathrm{d}w \;=\; \int\limits_\gamma g(w,z)\,\mathrm{d}w \end{aligned}$$

Hence $G(z)$ is an entire function. From the explanation for $z \in E$ comes $\lim\limits_{z\to\infty} G(z) = 0$. Hence $G(z)$ is a bounded entire function and thus by Liouville's theorem constant with $G(z) = 0$ for all z.

3) The first formula applies for the $z \in D \setminus \gamma(I)$. Result:

$$0 \;=\; G(z) \;=\; \int\limits_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w - \int\limits_\gamma \frac{f(z)}{w-z}\,\mathrm{d}w$$

that means

$$n_\gamma(z)f(z) = \frac{1}{2\pi i}\int\limits_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w$$

□

6.3 General Version of Cauchy's Integral Theorem

Let f be holomorphic on the region $D \subset \mathbb{C}$ and $\gamma : I \mapsto D$ be a closed curve with winding number $n_\gamma(z_0) = 0 \;\forall\, z_0 \in \mathbb{C} \setminus D$. Then

$$\boxed{\oint\limits_\gamma f(w)\,\mathrm{d}w = 0}$$

Proof:

By inserting the two functions $f(z)$, $zf(z)$ into the integral formula.

For $z \in D$:

$$\frac{1}{2\pi i}\int\limits_\gamma f(w)\,\mathrm{d}w = \underbrace{\frac{1}{2\pi i}\int\limits_\gamma \frac{wf(w)}{w-z}\,\mathrm{d}w}_{n_\gamma(z)\,(z\,f(z))} - \underbrace{\frac{1}{2\pi i}\int\limits_\gamma \frac{zf(w)}{w-z}\,\mathrm{d}w}_{z\cdot n_\gamma(z)f(z)} = 0$$

□

Definition of "simply connected":

A region $D \subset \mathbb{C}$ is called "simply connected" if for every closed curve $\gamma : I \mapsto D$ and all $z_0 \in \mathbb{C} \setminus D$ the winding number $n_\gamma(z_0) = 0$.

Remark:

(1) For functions f which are holomorphic on simply connected regions D the following applies to every closed curve $\gamma : I \mapsto D$

$$\oint_\gamma f(w)\,\mathrm{d}w \;=\; 0$$

6.3.1 Existence of Global Primitive Functions

Let $f(z)$ be holomorphic on the simply connected region D. Then on D there exists a holomorphic function $F(z)$ with

$$F'(z) \;=\; f(z)$$

Proof:

Let $z_0 \in D$ be constant. Two connecting curves $\gamma\,,\delta:\; I \mapsto D$ with the same starting point z_0 and the same end point $z \in D$ yield

$$\int_\gamma f(w)\,\mathrm{d}w \;=\; \int_\delta f(w)\,\mathrm{d}w$$

since according to Cauchy

$$\int_{\gamma\,\cup\,\delta^*} f(w)\,\mathrm{d}w \;=\; 0$$

In other words, every curve γ_z connecting z_0 with $z \in D$ yields the same integral

$$F(z) \;=\; \int_{\gamma_z} f(w)\,\mathrm{d}w$$

Elementary integral estimate yields

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \;=\; 0$$

□

Application: Existence of holomorphic logarithms

Let $f(z)$ be holomorphic and zero-free on the simply connected region D. Then on D there exists a holomorphic function $g(z)$ with

$$f(z) \;=\; \mathrm{e}^{g(z)}$$

For the **proof:**

one integrates the logarithmic derivative $\dfrac{f'(z)}{f(z)}$ of the function $f(z)$

$$g(z) \;=\; \int\limits_{\gamma} \frac{f'(w)}{f(w)}\,\mathrm{d}w \;+\; \text{const}$$

with $\quad \mathrm{e}^{\,\text{const}} = f(z_0)$

□

6.3.2 Isolated Singularities

If $f(z)$ is holomorphic in the punctured circular disk $0 < |z - z_0| < \varrho$ then z_0 is called an "isolated singularity" of f. If this is the case, then $f(z)$ in $0 < |z - z_0| < \varrho$ has a so-called Laurent expansion

$$\boxed{f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n}$$

It is absolutely convergent in $0 < |z - z_0| < \varrho$ and the coefficients (uniquely determined by f) have the integral form

$$\boxed{a_n \;=\; \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(w)}{(w - z_0)^{n+1}}\,\mathrm{d}w}$$

with $\gamma_r \;= z_0 + r\mathrm{e}^{it}$, $t \in [0, 2\pi]$, and any r in $]0, \varrho[$.

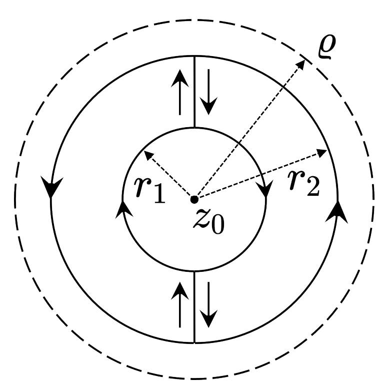
Proof sketch:

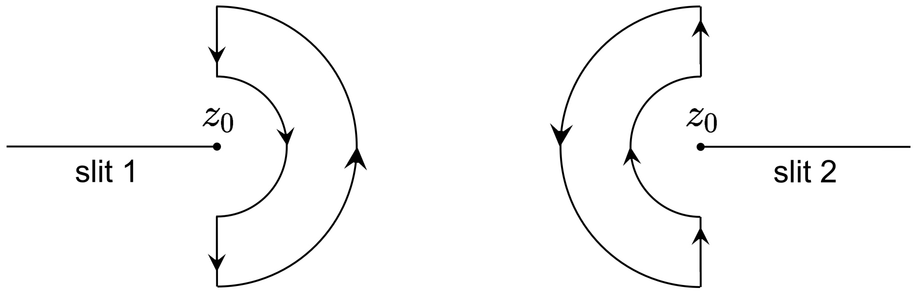
1) The integrals are independent of r

$$0 \;<\; r_1 \;<\; r_2 \;<\; \varrho$$

Decomposition of the paths of integration into two simply closed curves in a simply connected holomorphic region of

$$\frac{f(w)}{(w - z_0)^{n+1}}$$





2) Validity of the Laurent expansion by integrating the auxiliary function

$$g(w) \quad = \quad \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if} \quad w \neq z \\ \\ f\,'(w) & \text{if} \quad w = z \end{cases}$$

Important remark:

(2) Exactly then possesses the Laurent series

$$f(z) \quad = \quad \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad \text{in} \qquad 0 < |z-z_0| < \varrho$$

a primitive function, namely

$$\sum_{n=-\infty}^{\infty} \frac{a_n (z-z_0)^{n+1}}{n+1} \qquad , \quad \text{if} \quad a_{-1} \; = \; 0\,.$$

Definition:

$$a_{-1} \quad = \quad \frac{1}{2\pi i} \int\limits_{\gamma_r} f(w)\,\mathrm{d}w \quad = \quad \operatorname{Res}(f, z_0)$$

is called the residue of f in the isolated singularity z_0.

Natural occurrence of isolated singularities

with quotients $f(z) = \dfrac{g(z)}{h(z)}$ of two functions $g\,,\,h \neq 0$ which are analytic in a region D.

h has only isolated zeros (compare Identity Theorem) and therefore in the quotient $f(z) = \dfrac{g(z)}{h(z)}$ the zeros of h become isolated singularities. If z_0 is a zero of h, then there exists a largest number n with

$$f^{(k)} = 0 \qquad (0 \leq k < n)$$

in other words

$$n = \min_{c_k \neq 0} k \ ,$$

where the c_k come from the Taylor series $\sum\limits_{k=0}^{\infty} c_k(z-z_0)^k$ of h.

n is called "multiplicity" or "order" of the zero of h.

6.3.3 A Formula for Residues

Let z_0 be a simple zero of $h(z)$; then it holds

$$\operatorname{Res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}$$

Proof:

Because z_0 is of the 1st order, $h(z) = (z-z_0)h_1(z)$ where $h_1(z)$ is holomorphic and zero-free in a neighborhood of z_0.

$$f_1(z) = \frac{g(z)}{h_1(z)}$$

is therefore holomorphic in a neighborhood of z_0. Because of

$$f(z) = \frac{g(z)}{h(z)} = \frac{1}{z-z_0}\frac{g(z)}{h_1(z)}$$

the Laurent series of $f(z)$ around z_0 results from the Taylor series of $f_1(z)$ around z_0 by division with $z-z_0$

$$\begin{aligned}\operatorname{Res}(f, z_0) &= f_1(z_0) \\ &= \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z)-h(z_0)}{z-z_0}} = \frac{g(z_0)}{h'(z_0)}\end{aligned}$$

□

Example:

(1) $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

$z_0 = k\pi$, $\sin z$ has a single zero there

$$\operatorname{Res}(\cot z, k\pi) = \frac{\cos k\pi}{\sin' k\pi} = 1$$

6.3.4 Logarithmic Derivatives

$\dfrac{h'(z)}{h(z)}$ for holomorphic functions $h \neq$ const have isolated singularities in the zeros z_0 (n times) of $h(z)$.

Taylor series of $h(z)$

$$\begin{aligned} h(z) &= \sum_{k=n}^{\infty} c_k (z-z_0)^k \qquad c_n \neq 0 \\ h(z) &= c_n(z-z0)^n \big[1 + \text{higher terms in } (z-z_0)\big] \\ h'(z) &= nc_n(z-z0)^{n-1} \big[1 + \text{higher terms in } (z-z_0)\big] \qquad \Longrightarrow \\ \operatorname{Res}\left(\frac{h'(z)}{h(z)}\,;\, z_0\right) &= n \end{aligned}$$

In words: The residue of the logarithmic derivative $\dfrac{h'(z)}{h(z)}$ in a zero of the non-constant holomorphic function h is equal to the order of the zero.

Decomposition of the Laurent series into principal part and regular part

Let z_0 be an isolated singularity of the holomorphic function $f(z)$. Therefore, f is holomorphic in $0 < |z-z_0| < \varrho$ and it has there the unique, absolutely convergent Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Here one calls

$$f_-(z) = \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

or respectively

$$f_+(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

the “principal part” or respectively the “regular part” of f at the point z_0. The principal part series converges (absolute value estimate) if $z \neq z_0$, it thus represents a function series that is holomorphic in $\mathbb{C} \setminus \{z_0\}$. In contrast, the regular part series is holomorphic at least in $|z-z_0| < \varrho$ (including the midpoint).

6.4 Residue Theorem

Let there be given an analytic function $f(z)$ in the region $D \subset \mathbb{C}$. Furthermore, let $\gamma : I \mapsto D$ be a closed curve whose winding number $n_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus D$ apart from finitely many exceptions $z_1, z_2, \ldots, z_N$. Then:

$$\boxed{\frac{1}{2\pi i} \oint_\gamma f(w)\,\mathrm{d}w \;=\; \sum_{k=1}^{N} n_\gamma(z_k)\,\mathrm{Res}\,(f, z_k)}$$

Proof:

$z_1, z_2, \ldots, z_N$ are isolated singularities, therefore in a punctured circular disk $0 < |z - z_k| < \varrho$

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n^{(k)} (z - z_k)^n$$

Let

$$f_k(z) \;=\; \sum_{n=1}^{\infty} a_{-n}^{(k)} (z - z_k)^{-n}$$

be the principal part of f at z_k. Then

$$g(z) \;=\; f(z) \,-\, f_1(z) \,-\, \ldots \,-\, f_N(z)$$

is even in $D' = D \cup \{z_1, \ldots, z_N\}$ holomorphic (note: $f_k(z)$ is holomorphic in $\mathbb{C} \setminus \{z_k\}$). Then by Cauchy's integral theorem

$$\int_\gamma g(w)\,\mathrm{d}w \;=\; 0$$

that is

$$\int_\gamma f(w)\,\mathrm{d}w \;=\; \sum_{k=1}^{N} \int_\gamma f_k(w)\,\mathrm{d}w$$

In

$$f_k(z) \;=\; \frac{a_{-1}^{(k)}}{(z - z_k)} \;+\; \sum_{k=2}^{\infty} a_{-n}^{(k)} (z - z_k)^{-n}$$

the second summand has a primitive function in $\mathbb{C} \setminus \{z_k\}$, that means

$$\int\limits_\gamma f_k(w)\,\mathrm{d}w \quad = \quad \int\limits_\gamma \frac{a_{-1}^{(k)}}{w - z_k}\,\mathrm{d}w \quad = \quad \operatorname{Res}(f, z_k) \cdot 2\pi i n_\gamma(z_k)$$

$\square$

Example:

(1) Let h be holomorphic in a neighborhood of the circular disk $|z - z_0| \leq r$ and without a root on $|z - z_0| = r$,. Then the value of

$$\frac{1}{2\pi i} \oint\limits_{|z-z_0|=r} \frac{h'(z)}{h(z)}\,\mathrm{d}z$$

is equal to the number of zeros of h counted with multiplicity in $|z - z_0| < r$.

6.4.1 Classification of Isolated Singularities

Let f be holomorphic on $0 < |z - z_0| \leq \varrho$, that means z_0 is an isolated singularity of f. It is called

removable: if $f(z)$ can be continued holomorphically to z_0.

non-essential: if $(z - z_0)^n f(z)$ can be continued holomorphically to z_0 for a $n \in \mathbb{N}$.

essential: if $(z - z_0)^n f(z)$ cannot be continued holomorphically to z_0 for any $n \in \mathbb{N}$.

Non-essential singularities of z_0 are also called poles; in this case there exists a smallest natural number n_0, such that $(z - z_0)^{n_0} f(z)$ can be holomorphically continued to z_0; n_0 is called the “order of the pole”.

Examples:

(2) $\dfrac{\sin z}{z}$ has a removable singularity in $z_0 = 0$, the holomorphic continuation is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, z^{2k}$$

(3) $\dfrac{\mathrm{e}^z}{z^n}$ has a pole of n-th order in $z_0 = 0$.

(4) $\mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ has an essential singularity in $z_0 = 0$.

From the Laurent expansion of f around z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n ,$$

or respectively from its principal part

$$f_-(z) = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

the type of singularity can be read: If

$f_-(z)$ is identical to zero $\iff$ z_0 removable.

$f_-(z) = \sum_{n=1}^{N} a_{-n} (z - z_0)^{-n}$, $a_{-N} \neq 0$ $\iff$ z_0 is a pole of N-th order.

$f_-(z) = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$ with infinitely many $a_{-n} \neq 0$ $\iff$ z_0 is an essential singularity.

h shall have a pole of n-th order in z_0. Then for the residue of the logarithmic derivative $\frac{h'(z)}{h(z)}$ of h applies

$$\operatorname{Res}\left(\frac{h'(z)}{h(z)}, z_0\right) = -n$$

Because

$$h(z) = (z - z_0)^{-n} h_1(z) , \quad h_1(z) \neq 0 \text{ and holomorphic}$$

$$h'(z) = -n(z - z_0)^{-n-1} h_1(z) + (z - z_0)^n h_1'(z)$$

$$\frac{h'(z)}{h(z)} = \frac{-n}{(z - z_0)} + \frac{h_1'(z)}{h_1(z)}$$

Hence: $\frac{h_1'(z)}{h_1(z)}$ as a Taylor series

$$\operatorname{Res}\left(\frac{h'(z)}{h(z)}, z_0\right) = -n , \quad \text{since} \quad a_{-1} = -n$$

6.4.2 The Zeros- and Poles-counting Integral

Let $\gamma : I \mapsto \mathbb{C}$ be a closed curve whose complement $\mathbb{C} \setminus \gamma(I)$ splits into exactly two regions, a bounded D and an unbounded one. Furthermore let $n_\gamma(z) = 1$ for all $z \in D$. If then, except for the poles, $h(z)$ is holomorphic (i.e. meromorphic) in a neighborhood $\overline{D} = D \cup \gamma(I)$, as well as without zeros and poles on $\gamma(I)$, then

$$\oint_\gamma \frac{h'(w)}{h(w)}\,\mathrm{d}w \quad = \quad N - P$$

where N denotes the number of zeros and P the number of poles of h, counted according to their multiplicity and order.

Remark:

(3) $\Gamma(t) \quad = \quad h(\gamma(t)) \quad , \qquad t \in I = [a, b]$

$$\frac{1}{2\pi i}\oint_\gamma \frac{h'(w)}{h(w)}\,\mathrm{d}w \quad = \quad \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}w}{w}$$

$$= \quad \frac{1}{2\pi i}\int_a^b \frac{h'(\gamma(t)) \cdot \gamma'(t)}{h(\gamma(t))}\,\mathrm{d}t \quad = \quad n_\Gamma(0)$$

6.4.3 Ronché's Theorem

Again let $\gamma : I \mapsto \mathbb{C}$ be a closed curve whose complement $\mathbb{C} \setminus \gamma(I)$ splits into a bounded component D and an unbounded one; $n_\gamma(z) = 1 \ \forall\, z \in D$. If then $f(z)$ and $g(z)$ are analytic in a neighborhood of $\overline{D} = D \cup \gamma(I)$ and $|g(z)| < |f(z)|$ holds for all $z \in \gamma(I)$, then f and $f+g$ in D have the same number of roots!

Proof:

f and $f+g$ are without zeros on $\gamma(I)$; N_{f+g} shall denote the number of roots of $f+g$ in D :

$$N_{f+g} \quad = \quad \frac{1}{2\pi i}\oint_\gamma \frac{f' + g'}{f+g}\,\mathrm{d}w$$

$$\frac{f' + g'}{f+g} - \frac{f'}{f} \quad = \quad \frac{g'f - f'g}{f(f+g)}$$

$$
\begin{aligned}
&= \frac{g'f - f'g}{f^2} \cdot \frac{1}{1 + {}^{g}\!/_{f}} \\
&= \left(1 + {}^{g}\!/_{f}\right)' \cdot \frac{1}{1 + {}^{g}\!/_{f}}
\end{aligned}
$$

Hence

$$
N_{f+g} \;=\; \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f'}{f}\,\mathrm{d}w \;+\; \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{\left(1 + {}^{g}\!/_{f}\right)'}{1 + {}^{g}\!/_{f}}\,\mathrm{d}w \;=\; N_f + n_\Gamma(0)
$$

where

$$
\Gamma(t) \;=\; 1 + \frac{g(\gamma(t))}{f(\gamma(t))}
$$

and $\Gamma(t)$ lies in $|z-1| < 1$ for all $t \in I$, i.e. $n_\Gamma(0) = 0$.

$\square$

7. Applications of the Residue Theorem

7.1 Applications I

Evaluation of integrals of the form

$$\int_0^{2\pi} R(e^{it})\,\mathrm{d}t$$

with rational $R(z)$ without poles on the boundary S^1 of the unit circle.

With $f(z) = \dfrac{R(z)}{iz}$ applies

$$\boxed{\int_0^{2\pi} R(\mathrm{e}^{it})\,\mathrm{d}t \;=\; 2\pi i \sum_{|\zeta|<1} \operatorname{Res}(f,\zeta)} \qquad \text{(I)}$$

$$2\pi i \sum_{|\zeta|<1} \operatorname{Res}(f,\zeta) \;=\; \oint_{|z|=1} f(w)\,\mathrm{d}w \;\underset{\gamma(t)=\mathrm{e}^{it}}{\underset{\downarrow}{=}}\; \int_0^{2\pi} f(\mathrm{e}^{it}) i\mathrm{e}^{it}\,\mathrm{d}t \;=\; \int_0^{2\pi} R(\mathrm{e}^{it})\,\mathrm{d}t$$

<u>Examples:</u>

(1) $(|a| \neq 1)$

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1-2a\cos t + a^2} = \begin{cases} \dfrac{2\pi}{1-a^2} & \text{if } |a| < 1 \\[2ex] \dfrac{2\pi}{a^2-1} & \text{if } |a| > 1 \end{cases}$$

(2)
$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}t}{\cos t+z} \;=\; \frac{1}{\sqrt{z^2-1}} \qquad z\in\underbrace{\mathbb{C}\setminus[-1,1]}_{D}$$

where for $z=x>1$ the holomorphic root branch is normalized by $\sqrt{x^2-1}>0$.

Remarkable:

Although D is not simply connected, the root branch can be defined on D.

Proof:

Both sides are holomorphic, for the left side one uses the theorem about parameter dependent integrals $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Thereafter it is then twice continuously differentiable and satisfies the Cauchy-Riemann differential equations. According to the identity theorem, the proof for $z=x>1$ suffices. Use (I)

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}t}{\cos t+x} \;=\; \frac{1}{2\pi}\int_0^{2\pi}\underbrace{\frac{1}{i\mathrm{e}^{it}}\frac{1}{\frac{1}{2}\left(\mathrm{e}^{it}+\mathrm{e}^{-it}\right)+x}}_{f(\mathrm{e}^{it})}\,i\mathrm{e}^{it}\,\mathrm{d}t$$

$$\overset{\substack{w=\mathrm{e}^{it}\\ \downarrow}}{=} \; i\operatorname{Res}\left(\frac{i}{iw\left({}^{w}/_{2}+{}^{w^{-1}}/_{2}+x\right)}\,,\,x_0\right)$$

$$= \;\operatorname{Res}\left(\frac{2}{w^2+2wx+1}\,,\,x_0\right)$$

where x_0 is, by absolute value, the smaller root of the denominator polynomial

$$x_0 \;=\; -x+\sqrt{x^2-1}$$

Because x_0 is a simple zero of the denominator polynomial, it follows

$$\operatorname{Res}\left(\frac{2}{w^2+2xw+1}\,,\,w=x_0\right) \;=\; \frac{1}{x_0+x}$$
$$=\;\frac{1}{\sqrt{x^2-1}}$$

□

Remark:

(1) For $z = x < -1$ the integrand is always negative and hence so is the value of the function $g(z)$ represented by the integral.

It follows: The analytic continuation of the root function $g(z)$ defined by the integral is odd: $g(-z) = -g(z)$

From the standard estimate for complex line integrals:

Lemma 1:

Let $B = \{\mathrm{e}^{it}\,;\, \alpha \le t \le \beta\}$ be a partial arc of the unit circle S^1 with $0 < \beta - \alpha < 2\pi$; $R > 0$.

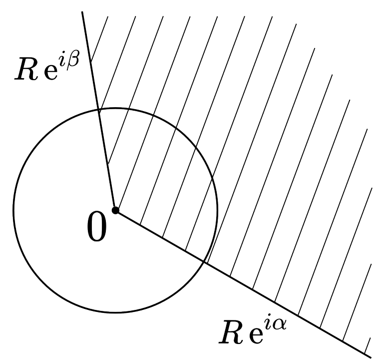
Let $f(z)$ be continuous in angular space $|z| \ge R\,,\ \dfrac{z}{|z|} \in B$.

i) If

$$\lim_{\substack{z\to\infty\\ z/|z|\in B}} |zf(z)| \;=\; 0$$

then with $\gamma_r(t) = r\mathrm{e}^{it}\,,\ t \in [\alpha,\beta]$

$$\lim_{r\to\infty} \int_{\gamma_r} f(w)\,\mathrm{d}w \;=\; 0$$

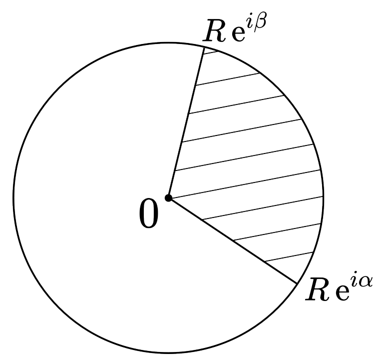


ii) If

$$\lim_{\substack{z\to 0\\ z/|z|\in B}} |zf(z)| \;=\; 0$$

then

$$\lim_{r\searrow 0} \int_{\gamma_r} f(w)\,\mathrm{d}w \;=\; 0$$



Polynomial estimation:

Let $P(z)$ be a real or complex polynomial of degree $n > 0$ with leading coefficient a_0. Then there exists a $R > 1$ such that

$$\frac{1}{2}|a_0||z|^n \;\le\; |P(z)| \;\le\; 2|a_0||z|^n\;; \qquad |z| \;\ge\; R$$

Proof:

follows from the polynomial representation

$$\begin{aligned} P(z) &= a_0 z^n + a_1 z^{n-1} + \ldots + a_n \\ &= a_0 z^n \Big(1 + \frac{a_1}{a_0}\frac{1}{z} + \frac{a_2}{a_0}\frac{1}{z^2} + \ldots + \frac{a_n}{a_0}\frac{1}{z^n} \Big) \end{aligned}$$

For $z \to \infty$, the round bracket has the limit value 1, so if $|z|$ is large enough, its absolute value is $\geq \frac{1}{2}$ and ≤ 2.

□

7.2 Applications II

Let $P(z)$, $Q(z)$ be polynomials, where $Q(z)$ has no real roots. Furthermore shall hold for the straight line

$$\deg Q \;\geq\; 2 + \deg P$$

finally let $\alpha > 0$. Then

$$\boxed{\int_{-\infty}^{+\infty} \frac{P(t)}{Q(t)} \mathrm{e}^{i\alpha t}\,\mathrm{d}t \;=\; 2\pi i \sum_{\substack{\operatorname{Im} a > 0 \\ Q(a) = 0}} \operatorname{Res}\left(\frac{P(z)}{Q(z)} \mathrm{e}^{i\alpha z}, a \right)} \qquad \text{(II)}$$

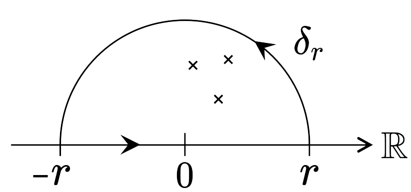
Moreover, the integral is absolutely convergent.

Proof sketch:

$\dfrac{P(z)}{Q(z)} \mathrm{e}^{-i\alpha z}$ has infinitely many singularities in $\mathbb{C}$.

If $\gamma_r = r\mathrm{e}^{it}$ $(0 \leq t \leq \pi)$ and if r larger than the absolute value of all zeros of Q, then the residual theorem for the (closed) path composed of $[-r, r]$ and $\gamma_r(t)$ δ_r yields

$$\int_{\delta_r} \frac{P(z)}{Q(z)} \mathrm{e}^{i\alpha z}\,\mathrm{d}z \;=\; 2\pi i \sum_{\substack{\operatorname{Im} a > 0 \\ Q(a) = 0}} \operatorname{Res}\left(\frac{P(z)}{Q(z)} \mathrm{e}^{i\alpha z}, a \right)$$



According to the assumption $\deg Q \geq 2 + \deg P$ there exists $R > 1$, $M > 1$ with

$$\left|\frac{P(z)}{Q(z)}\right| \;\leq\; \frac{M}{|z|^2}\;; \qquad |z| \,\geq\, R$$

Moreover, because of $\alpha \geq 0$ in $\operatorname{Im} z \geq 0$

$$|\mathrm{e}^{i\alpha z}| \;=\; \mathrm{e}^{-\operatorname{Im}\alpha z} \;\leq\; 1$$

The assumptions of Lemma 1 are satisfied for the angular space $\operatorname{Im} z \geq 0\,,\; |z| \geq R$. Therefore for $r \to \infty$

$$\lim_{r\to\infty}\int\limits_{-r}^{r}\frac{P(t)}{Q(t)}\mathrm{e}^{i\alpha t}\,\mathrm{d}t \;=\; 2\pi i\sum_{\operatorname{Im} a>0}\operatorname{Res}\left(\frac{P(z)}{Q(z)}\mathrm{e}^{i\alpha z}, a\right)$$

□

Remark:

(2) In the case $\alpha < 0$ one can work analogously with an auxiliary path in the lower half-plane.

Examples:

(1)
$$\int\limits_{-\infty}^{+\infty}\frac{t^{2k}}{1+t^{2n}}\,\mathrm{d}t \;=\; \frac{\pi/n}{\sin\left(\dfrac{2k+1}{2n}\pi\right)}$$

if k, n are natural numbers and $k < n$.

The roots of $1+z^{2n}$ are the odd powers of $\mathrm{e}^{2\pi i/4n} \;=\; \mathrm{e}^{\pi i/2n}$. Among these fall into the upper half-plane

$$a_m \;=\; \exp\frac{\pi i(2m+1)}{2n}\;; \qquad 0 \leq m < n$$

The derivative of $\;Q(z) \,=\, 1+z^{2n}$

$$Q'(z) \;=\; 2nz^{2n-1}$$

has no roots in a_m, therefore

$$\operatorname{Res}\left(\frac{z^{2k}}{1+z^{2n}}\,,\,a_m\right) \;=\; \frac{1}{2n}\,a_m^{2k-2n+1} \;=\; -\frac{1}{2n}\,a_m^{2k+1}$$

$$\sum_{m=0}^{n-1} a_m^{2k+1} \;=\; \sum_{m=0}^{n-1}\exp\left(\frac{\pi i}{2n}(2k+1)(2m+1)\right)$$

$$= \exp\frac{\pi i(2k+1)}{2n}\sum_{m=0}^{n-1}\exp\Big(\frac{\pi i}{n}(2k+1)m\Big)$$

Geometric series $\Rightarrow$

$$\sum_{m=0}^{n-1} a_m^{2k+1} = \exp\frac{\pi i(2k+1)}{2n}\,\frac{1-\exp\big(\pi i(2k+1)\big)}{1-\exp\Big(\dfrac{\pi i(2k+1)}{n}\Big)}$$

$$= \frac{2}{\exp\Big(\dfrac{-\pi i(2k+1)}{2n}\Big)-\exp\Big(\dfrac{\pi i(2k+1)}{2n}\Big)}$$

$$= \frac{i}{\sin\Big(\dfrac{2k+1}{2n}\pi\Big)}$$

With (II) it follows

$$\int_{-\infty}^{+\infty}\frac{P(t)}{Q(t)}\mathrm{e}^{i\alpha t}\,\mathrm{d}t = 2\pi i\sum \mathrm{Res} = -\frac{\pi i}{n}\,\frac{i}{\sin\Big(\dfrac{2k+1}{2n}\pi\Big)}$$

$$= \frac{\pi/n}{\sin\Big(\dfrac{2k+1}{2n}\pi\Big)}$$

With analytic continuation (identity theorem) follows

$$\int_0^{\infty}\frac{s^{z-1}}{1+s}\,\mathrm{d}s = \frac{\pi}{\sin(\pi z)}\,, \qquad 0 < \mathrm{Re}\,z < 1$$

The formula was proven for infinitely many rational $z = \dfrac{2k+1}{2n}$.

(2)
$$\int_0^{+\infty}\frac{\cos t}{w^2+t^2}\,\mathrm{d}t = \frac{\pi}{2w}\mathrm{e}^{-w}\ ; \quad \mathrm{Re}\,w>0$$

Since $\cos t$ is even $\Rightarrow$

$$\int_0^{+\infty}\frac{\cos t}{w^2+t^2}\,\mathrm{d}t = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos t}{w^2+t^2}\,\mathrm{d}t$$

$$= \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{it}}{w^2+t^2}\,\mathrm{d}t \qquad \text{(II) with } \alpha = 1 \text{ is applicable}$$

Roots of $Q(z) = w^2 + z^2$ are $\pm iw$, where iw lies in $\mathrm{Im}\, z > 0$.

$$\int_0^{+\infty} \frac{\cos t}{w^2+t^2}\,\mathrm{d}t = \frac{1}{2}\,\mathrm{Re}\left(2\pi i\,\mathrm{Res}\left(\frac{\mathrm{e}^{iz}}{w^2+z^2}\,,\, iw\right)\right)$$

$$= \frac{1}{2}\,\mathrm{Re}\, 2\pi i\,\frac{e^{-w}}{2iw} = \frac{\pi}{2w}\mathrm{e}^{-w}\ , \quad \text{if } w \text{ real and } w > 0$$

Because in the asserted formula both sides (in $\mathrm{Re}\, w > 0$) are analytic, the general formula follows from the identity theorem.

Lemma 2:

$$\pi\cot(\pi z) = \frac{\pi\cos(\pi z)}{\sin(\pi z)}$$

is holomorphic in $\mathbb{C}$ except for simple poles in $m \in \mathbb{Z}$, the residue is

$$\mathrm{Res}\big(\pi\cot(\pi z)\,,\, m\big) = 1$$

For the $z \in \mathbb{C}\setminus\mathbb{R}$ the following estimate applies

$$|\cot(\pi z)| \le \frac{1+\exp(-2\pi|\mathrm{Im}\, z|)}{1-\exp(-2\pi|\mathrm{Im}\, z|)}$$

Proof:

The first half follows from $\dfrac{\sin'(\pi z)}{\sin(\pi z)} = \pi\cot(\pi z)$

$\cot(\pi z)$ is odd, hence estimate only for z with $\mathrm{Im}\, z > 0$

$$|\cot\pi z| = \left|\frac{\exp(\pi i z)+\exp(-\pi i z)}{\exp(\pi i z)-\exp(-\pi i z)}\right|$$

$$= \left|\frac{1+\exp(-2\pi i z)}{1-\exp(-2\pi i z)}\right| \le \frac{1+|\exp(-2\pi i z)|}{1-|\exp(-2\pi i z)|}$$

□

7.3 Applications III

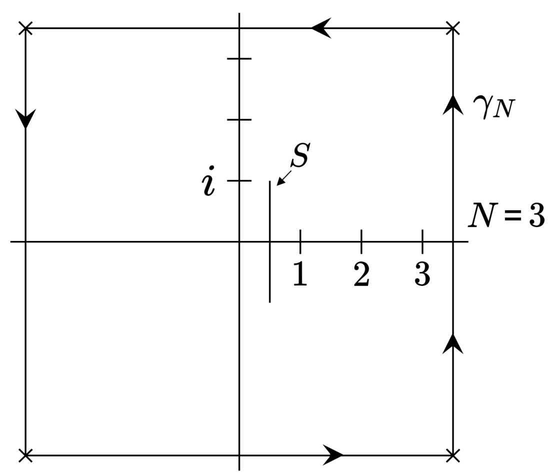
Let $R(z) = \dfrac{P(z)}{Q(z)}$ be the quotient of the polynomials $P(z)$, $Q(z)$ with $\deg Q \geq 2 + \deg P$. Then

$$\boxed{\sum_{\substack{m=-\infty \\ Q(m)\neq 0}}^{\infty} R(m) \;=\; -\sum_{\substack{Q\in C \\ Q(a)=0}} \operatorname{Res}\big(R(z)\,\pi\cot(\pi z)\,,\,a\big)} \qquad \text{(III)}$$

The series on the left is absolutely convergent.

Proof:

1) Let N be a (large) natural number. γ_N shall denote the positive oriented boundary curve of the square with the four corners $(\pm 1, \pm i)\big(N + \frac{1}{2}\big)$.



Let $N > |a|$ for all roots a of $Q(z)$. Then one obtains with the residue theorem

$$\frac{1}{2\pi i}\int_{\gamma_N} R(w)\pi\cot(\pi w)\,\mathrm{d}w \qquad (*)$$

$$= \sum_{\substack{Q\in C \\ Q(a)=0}} \operatorname{Res}\big(R(z)\,\pi\cot(\pi z)\,,\,a\big) + \sum_{\substack{m=-N \\ Q(m)\neq 0}}^{N} \operatorname{Res}\big(R(z)\,\pi\cot(\pi z)\,,\,z=m\big)$$

In the second summand one has with Lemma 2

$$\operatorname{Res}\big(R(z)\,\pi\cot(\pi z)\,,\,m\big) \;=\; R(m)\cdot 1 \qquad (**)$$

According to the polynomial estimation (after Lemma 1) there exists an $R \geq 1$ and a $M_1 > 0$ with

$$|R(z)| \;\leq\; \frac{M_1}{|z|^2} \qquad \text{for} \quad |z| \;\geq\; R$$

Comparison with the convergent series $\displaystyle\sum_{m=1}^{\infty} \frac{2M_1}{m^2}$ yields absolute convergence of the series in question.

2) $\pi \cot(\pi z)$ is 1-periodic and for $|\mathrm{Im}\, z| \geq 1$ holds with Lemma 2

$$|\pi \cot \pi z| \;\leq\; \pi \, \frac{1 + |\exp(-2\pi|\mathrm{Im}\, z|)|}{1 - |\exp(-2\pi|\mathrm{Im}\, z|)|} \;\leq\; \pi \, \frac{1 + \exp(-2\pi)}{1 - \exp(-2\pi)}$$

On $S \;=\; \left\{ \frac{1}{2} + it\,;\; -1 \leq t \leq 1 \right\}$ $\pi \cot(\pi z)$ is bounded because it is a continuous function. Thus it is also bounded on the region traced out by γ_N with the same bound. Therewith applies with a constant $M_2 > 0$

$$|R(z)\pi \cot(\pi z)| \;\leq\; \frac{M_2}{|z|^2} \;, \qquad \text{if} \quad |z| > R \;\text{ and }\; z \text{ on a } \gamma_N$$

On γ_N there is $|z| \geq N$ everywhere, hence standard integral estimation yields

$$\left| \int_{\gamma_N} R(w)\pi \cot(\pi z)\, \mathrm{d}w \right| \;\leq\; l(\gamma_N)\, \frac{M_2}{N^2}$$

$$\leq\; \frac{M_2\, 4(2N+1)}{N^2} \;\to\; 0$$

With $(*)$ and inserting $(**)$ follows

$$\sum_{\substack{m=-\infty \\ Q(m) \neq 0}}^{\infty} R(m) \;=\; -\sum_{\substack{Q \in C \\ Q(a) = 0}} \mathrm{Res}\left(R(z)\, \pi \cot(\pi z)\,,\, a\right)$$

Example:

(1) Partial fraction decomposition of the cotangent

$$\pi \cot(\pi z) \;=\; \frac{1}{z} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left(\frac{1}{z-m} + \frac{1}{m} \right) \qquad z \in \mathbb{C} \setminus \mathbb{Z}$$

Proof:

By applying the summation principle III to the series on the right (z is a fixed parameter):

$$R(w) \;=\; \frac{1}{z-w} + \frac{1}{w} \;=\; \frac{z}{zw - w^2}$$

The denominator polynomial

$$Q(w) \;=\; zw \,-\, w^2 \qquad \text{of degree } 2$$

with two simple zeros at $w = 0$ and $w = z$. The numerator polynomial $P(w) = z$ is constant.

According to (III)

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{z-m} + \frac{1}{m} \right) \;=\;$$

$$-\operatorname{Res}\left(\frac{z}{zw-w^2} \pi \cot(\pi z)\,,\; w = 0 \right) \,-\, \operatorname{Res}\left(\frac{z}{zw-w^2} \pi \cot(\pi z)\,,\; w = z \right)$$

$$\operatorname{Res}\left(\frac{z}{zw-w^2} \pi \cot(\pi z)\,,\; w = z \right) \;=\; -\pi \cot(\pi z)$$

$$\operatorname{Res}\left(\frac{z}{zw-w^2} \pi \cot(\pi z)\,,\; w = 0 \right) \;=\;$$

$$\underbrace{\operatorname{Res}\left(\frac{1}{z-w} \pi \cot(\pi z)\,,\; w = 0 \right)}_{\frac{1}{z}} \;+\; \underbrace{\operatorname{Res}\left(\frac{\pi \cot(\pi z)}{w}\,,\; w = 0 \right)}_{=\; 0^{(*)}}$$

$(*)$: Because $\dfrac{\cot(\pi w)}{w}$ is an even function and therefore the leading coefficients a_{2k+1} are all $= 0$. With that follows

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{z-m} + \underset{\substack{\downarrow \\ \text{convergence generating summands}}}{\frac{1}{m}} \right) \;=\; -\frac{1}{z} \,+\, \pi \cot(\pi z)$$

□

Assertion: For each $R > 0$ the series (subseries)

$$\sum_{|m| \geq 2R} \Big(\frac{1}{z-m} + \frac{1}{m} \Big)$$

converges absolutely uniform on $|z| \leq R$.

Rationale:

$$\left| \frac{1}{z-m} + \frac{1}{m} \right| \;=\; \left| \frac{z}{(z-m)m} \right| \;\leq\; \frac{R}{|m|(|m|-R)} \;\leq\; \frac{2R}{m^2}$$

A comparison with the z-independent convergent series $\displaystyle\sum_{n \geq 2R} \frac{4R}{n^2}$ then yields the assertion.

Application: Product decomposition of the sine

$$\sin(\pi z) \;=\; \pi z \prod_{\substack{m=\infty \\ m \neq 0}}^{\infty} \Big(1 - \frac{z}{m}\Big) \mathrm{e}^{\,z/m} \;=\; \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big)$$

Proof:

$f(z) \;=\; \dfrac{\sin(\pi z)}{\pi z}$ is holomorphic and zero-free on

$$D \;=\; \mathbb{C} \setminus \big\{ x \in \mathbb{R}\,;\; |x| \;\geq\; 1 \big\}$$

once one has by means of

$$f(0) \;=\; \lim_{z \to 0} f(z) \;=\; 1$$

removed the removable singularity in $z = 0$.

As a star-shaped region with respect to 0, D is simply connected. Hence $f(z)$ has a holomorphic logarithm on D.

$$l(z) \;:\;\; f(z) \;=\; \exp\big(l(z)\big)$$

$l(z)$ is a suitable primitive function of the logarithmic derivative $f\,'(z)/f(z)$, normalized by $l(0) \;=\; 0$.

Straight-line integration from 0 to z

$$\frac{f\,'(z)}{f(z)} \;=\; -\,\frac{1}{z} \;+\; \pi \cot(\pi z)$$

$$= \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \Big(\frac{1}{z-m} + \frac{1}{m}\Big)$$

$\dfrac{1}{z-m} + \dfrac{1}{m}$ is the logarithmic derivative of $\Big(1 - \dfrac{z}{m}\Big)\mathrm{e}^{z/m}$

Termwise integration (permissible because of locally uniform convergence)

$$l(z) \;=\; \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \underbrace{\log\Big(\Big(1-\frac{z}{m}\Big)\mathrm{e}^{z/m}\Big)}_{\substack{\downarrow \\ \text{normalized by 0 at the origin.}}}$$

Continuity and functional equation of exp

$$\begin{aligned} \frac{\sin(\pi z)}{\pi z} \;&=\; f(z) \;=\; \exp\big(l(z)\big) \\ &= \lim_{N\to\infty} \prod_{\substack{m=-N \\ m\neq 0}}^{N} \Big(1-\frac{z}{m}\Big)\mathrm{e}^{z/m} \\ &= \lim_{N\to\infty} \prod_{n=1}^{N} \Big(1-\frac{z^2}{n^2}\Big) \end{aligned}$$

□

The significance of locally uniform convergence for sequences of holomorphic functions on a region $D \subset \mathbb{C}$:

Weierstraß Theorem

If $\big(f_n(z)\big)_{n\geq 1}$ is a uniformly convergent sequence of holomorphic functions on the common region D, then the limit function $f(z)$ itself is holomorphic. Moreover, the sequence $f_n'(z)$ converges locally uniformly on D to the derivative $f'(z)$ of $f(z)$.

Example:

(2) $D \;=\; \big\{z \in \mathbb{C}\,;\, \mathbb{R}\, z > 0\big\}$

$$f_n(z) \;=\; \int_{1/n}^{n} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

Limit function:

$$\Gamma_n(z) \;=\; \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

By partial integration of $\Gamma(z+1)$ one quickly finds

$$\Gamma(z+1) \;=\; z \cdot \Gamma(z)$$

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