# MATH 413 HW4

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## 1 8

Prove $\frac{a}{b}$ ends in a repeating pattern using the pigeon hole principle.

Let $(r\%\text{y}) : r, y \in \mathbb{Z}$ come to mean the remainder of the divsion of r by y.

Let $a, b, k \in \mathbb{N} - \{0\}$ $\langle C \rangle \subseteq [k]$ where $b$ has $k$ digits.

We will prove this by showing there is an upper bound on the number of times number can go through conventional long division steps before it repeats, we will then use the pigeon hole principle to prove the statement about rational numbers.

$$\frac{a}{b} = \frac{a - a\%\text{b}}{b} + \frac{a\%\text{b}}{b}$$

Where the first term on the right, is the part of the quotient left of the decimal point for lack of a better phrase. We can ignore it because it doesn't interfere with the part of the quotient right of the decimal point.

$$0 \leq a\%\text{b} \leq b \qquad \text{by definition of remainder}$$

case one: $a\%\text{b} = 0$ means that the sequence has ended, because of the further division of 0 will result in more 0's

case two: $0 < a\%\text{b} < b$. In which case, by the rules of the algorithm of long division, we need to continue. To continue, we can multiply $a\%\text{b}$ by 10 until $b \leq a\%\text{b} * 10^{C_1} < 10b$. (shifting over one to the right by one in our final decimal quotient number every time we multiply by 10) By the definition of remainder

$$0 \leq ((a\%\text{b} * 10^{C_1})\%b) < b$$

So the division in the second step in the long division leaves that remainder to be further divided.

Analyzing this remainder, we see that case 1 still applies, if the remainder is 0, then we are done dividing because there would be nothing left to divide.

If the remainder isn't zero then we continue one with in the same manner, producing remainders of the form

$$(((a\%\text{b}10^{C_1})\%b)10^{C_2}\%b)10^{C_3}...$$

This is where I note that if any of these remainders terms are equal to any previous remainder terms, then we will get a repeating pattern because we know that dividing this remainder will eventually yield another identical remainder down the line. This in turn means that the same numbers will end up in the quotient, thus a repeating pattern will occur.

Now, since the remainder terms $\geq 0$ and $< b$, then we can set up our pigeon holes as the values 0 through $b-1$. And our pigeons will be the remainders from the first $b+1$ steps in the long division algorithm. By the pigeon hole principle, one of these remainder values will be repeated. Thus all divisions of two non-zero natural numbers will end in 0's or end in a repeating pattern.

Since this is true even if you flip the sign of the number, it extends to divisions of non-zero integers, and thus to the rational numbers.

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Suppose that the mn people of a marching band are standing in a rectangular formation of m rows and n columns in such a way that in each row each person is taller than the one to his or her left. Suppose that the leader rearranges the people in each column in increasing order of height from front to back. Show that the rows are still arranged in increasing order of height from left to right.

Proof: I will prove this by induction.

Base Case: we start with one row, all the tallest people in each column are at the bottom.

Inductive step: We assume the hypothesis that k rows were arranged, and the whole band is ordered from shortest to tallest in the vertical and horizontal directions of the rectangle. We introduce a new row of band mates.

If we can prove that the ordering in all four directions stayed the same for each band member from $k$ rows to $k+1$ rows, then we have proven the inductive step.

For each person in the original rectangle, there were $x$ people in their column who were taller than them, and therefore at least $x+1$ taller people than them in the row to their right, unless they were in the rightmost row. This is because in the row to the right of that person were $x$ people taller than the $x$ people taller than them from their own row, and 1 person to their immediate right who is taller than them.

If in a column, someone from the new row displaces someone already there, then since the person who displaced them was taller than them and someone taller than the new person came into the row to the right, there are now $x+1$ people taller than the original person in their column and at least $x+2$ people taller than the original person in the column to the right. Therefore the person to the right of the original person is either one of the previously known taller people or the new person who is taller than the taller person who displaced the original person.

Therefore for any person who got displaced has the gap to their right filled by someone taller than them. Since we know that all the gaps to the right have been filled by someone taller than that person, we also get for free that all the gaps to the left were filled by someone shorter. Thus the horizontal orderings were preserved.

Also the vertical orderings were preserved, because we simply inserted people into the appropriate spot in the order.Therefore the ordering of height in all directions of everyone in the band is preserved.

Thus k implies k+1, and this is true by the principle of induction.