MATH 347 HW1

Charles Swarts
swarts2@illinois.edu

August 2016

1 1.7

1.1

Disprove the following statement.

$$x, y \in \mathbb{R} - \{0\}, \;\; x > y \implies -1/x > -1/y$$

I will disprove this through counter-example.

Take the counter-example: x is the real number 1 which is greater than y which takes the value -1.

$$\frac{-1}{x} = \frac{-1}{1} = -1 \not> 1 = \frac{-1}{-1} = \frac{-1}{y}$$

So a value of x greater than a value of y does not imply the inequality and thus the statement is false.

1.2

Add a hypothesis to make the previous statement true
Adding the hypothesis that $y > 0$ makes the following statement

$$x, y \in \mathbb{R} - \{0\}, \;\; x > y > 0 \implies -1/x > -1/y$$

Proof: I will prove this by showing that the implication is the hypothesis.

I will take any positive real value for y and a positive real value for x which is higher than y.

Multiplying both sides of the implication by -1 will flip the inequality and leave

$$\frac{1}{x} < \frac{1}{y}$$

Since x and y are positive and multiplying by a positive real number preserves inequalities, I will multiply both sides by x and then y.

This makes the statement proposition $x > y > 0 \implies x > y$ which is always true $\square$

2 1.13

$$A = \{2k + 1 : k \in \mathbb{Z}\}, \;\; B = \{2k - 1 : k \in \mathbb{Z}\}$$

Prove that $A = B$

I will prove this by finding a way to make the definition of A and B equal. Let's restate B and redefine A to have a new definition by substituting $(k-1)$ for k.

$$B = \{2k - 1 : k \in \mathbb{Z}\} \quad A = \{2(k-1) + 1 : (k-1) \in \mathbb{Z}\}$$

If $(k-1)$ is an integer, by completeness of the integers, k is an integer, and if $(k-1)$ is any integer, k is any integer $+1$ which is still any integer. This means we can rewrite the definition of A again to

$$A = \{2(k-1) + 1 : k \in \mathbb{Z}\} = \{2k - 2 + 1 : k \in \mathbb{Z}\} = \{2k - 1 : k \in \mathbb{Z}\}$$

So now

$$A = \{2k - 1 : k \in \mathbb{Z}\} = B \;\; A = B \;\; \square$$

3 1.32

Assuming only arithmetic, prove that

$$\{x \in \mathbb{R} : x^2 - 2x - 3 < 0\} = \{x \in \mathbb{R} : -1 < x < 3\}$$

Proof: I will prove this by showing that the condition of both sets constrain the real numbers to the same interval of the real numbers.

By the distributive law,

$$x^2 - 2x + 3 = (x-3)(x+1)$$

In order for $(x-3)(x+1) < 0$, one of the two terms needs to be negative while the other one is positive. Here are some observations:

$$x > 3, \quad \Leftrightarrow \quad (x-3) > 0 \text{ and } (x+1) > 0$$

$$x < -1, \quad \Leftrightarrow \quad (x+1) < 0 \text{ and } (x-3) < 0$$

$$x = -1 \quad \Leftrightarrow \quad (x+1) = 0 \qquad x = 3 \;\; (x-3) = 0$$

$$-1 < x < 3, \quad \Leftrightarrow \quad (x+1) > 0 \text{ and } (x-3) < 0$$

Ad oculos, the only subset of the real numbers that x can be an element of for which one term is negative while the other is positive is the interval $(-1, 3)$.

For the other term, the only subset of the real numbers where $-1 < x < 3$ is when x is an element of the interval $(-1, 3)$.

Thus both conditions constrain the real numbers down to the same interval of the real numbers. Since an interval is equal to itself, these sets are equal. $\square$

4 1.36

4.1

Let $S = [3] \times [3]$. Let T be the set of ordered pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq 3x + y - 4 \leq 8$.Prove that $S \subseteq T$. Does equality hold?

Prove that $S \subseteq T$:

Proof: I will prove this by redefining T in terms of a function, f(x,y), and show that each element in the image f(S) complies with the condition for being in T.

Let T be redefined as the set of ordered pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq f(x, y) \leq 8$ where f :

$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},$ and $f(x, y) = 3x + y - 4$.

Since $f(S)$ is a finite subset of the integers, and thus has the reflexive, antisymmetric, transitive, and total ordering properties, $f(S)$ will have both a lowest and highest valued element i.e. all elements in $f(S)$ will be less than or equal to the highest valued element in $f(S)$ and greater than or equal to the lowest valued element in $f(S)$.

The lowest valued element of $f(S)$ is $f(1,1) = 3(1) + 1 - 4 = 0$, and the highest valued element of $f(S)$ is $f(3,3) = 3(3) + 3 - 4 = 8$. Therefore $\forall\ s \in f(S),\ 0 \leq s \leq 8$

Therefore the ordered pairs in S are all elements in the set of ordered pairs T where $\forall\ t \in T\ 0 \leq f(t) \leq 8$. Therefore $S \subseteq T$. $\square$

4.2

Does equality hold? No.

Proof: I will disprove this through counter example.

(note: let $\mathbb{Z}^+$ denote the set of positive integers)

Take a counter example: the ordered pair $(0, 5)$. For $n \in \mathbb{Z}^+$, $[n]$ never includes 0 as part of the set of natural numbers 1 through n. Therefore $(0, 5) \notin S$ because $(0, 5)$ could not be an ordered pair in $[3] \times [3]$. However...

$$f(0, 5) = 3(0) + 5 - 4 = 1$$
$$0 \leq f(0, 5) = 1 \leq 8$$

By the second definition of T, $(0, 5) \in T$.
Since there is an element of T which is not an element of S, $T \nsubseteq S$
Since $T = S$ requires $T \subseteq S$, $T \neq S$. $\square$

5 1.47

5.1

Let $J : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by $J(a, b) = (a + 1)(a + 2b)/2$.
a) Show that the image of J is contained in N.
By the distributive law...

$$\begin{aligned}(a+1)(a+2b)/2 &= \frac{a(a+1)}{2} + \frac{2b(a+1)}{2} \\ &= \frac{a(a+1)}{2} + b(a+1)\end{aligned}$$

Lemma 1: $\frac{a(a+1)}{2}$ is a natural number.

One property of natural numbers is that each is either odd or even. Even natural numbers have the property that they can be written as 2*c where c is a natural number which might be odd or even. Odd natural numbers have a property where adding one to them makes them an even number.

Case 1: a is an odd number.

If a is an odd number, then $a + 1$ is an even number and can be substituted with 2*c. This leaves

$$\frac{a(a+1)}{2} = \frac{a(2c)}{2} = \frac{2ac}{2} = ac$$

$a * c$ is a natural number times a natural number, which, by completeness, is a natural number.

Case 2: a is an even number.

Since a is an even number, then it can be replaced with $2c$ this leaves

$$\frac{a(a+1)}{2} = \frac{(2c)(a+1)}{2} = c(a+1)$$

c(a+1) is a natural number because any combination of multiplication and addition between natural numbers yields a natural number. $\square$

By Lemma 1, $\frac{a(a+1)}{2}$ is a natural number, let's call it d. $d + b(a+1)$ is a natural number because by completeness, any combination of addition and multiplication between natural numbers yields a natural number.

So for any two numbers a and b, $J(a, b)$ is a natural number. Since by definition $\mathbb{N} \times \mathbb{N}$ is an ordered pair of two natural numbers, $J(\mathbb{N} \times \mathbb{N})$ will be a set of natural numbers. Since any set comprised of only natural numbers is a subset of the natural numbers, the image $J(\mathbb{N} \times \mathbb{N}) \subseteq \mathbb{N}$

5.2

b) (+) Determine exactly which natural numbers are in the image of J. (Hint: Formulate a hypothesis by trying values.)

Let me restate the function adjusted with the suggestion in part a.
Let $J : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $J(a, b) = (a + 1)(a + 2b)/2$.

Hypothesis: The image of J includes every natural number except 1 and any number 2^n where $n \in \mathbb{N}$

Proof: I will prove this by showing what elements of the natural numbers cannot be in the image of the function J over $\mathbb{N} \times \mathbb{N}$, and then I will describe a method to use the function J to get any natural number except the natural numbers that are not in the image.

Let me start with a parity analysis of the function J. In J, the $(a + 1)$ term will always have the opposite parity to the $(a + 2b)$ term. This is because the $2b$ portion of $(a + 2b)$ will always be even by definition of even. Since an even number added to an even number is an even number, and an odd number plus an even number is an odd number, this $2b$ will not affect the parity of the term. Therefore the $(a + 2b)$ term has a logically equivalent parity with a. Also a must have the opposite parity as $a + 1$; this is because adding one to a natural number flips its parity.

Now I will explain the exceptions. Because the lowest value of the natural numbers is 1, the lowest value of the function over the domain is $J(1, 1) = (1 + 1)(1 + 2(1))/2 = 3$. So right away we need to remove 1 and 2 from the image.

Now let us rewrite J a little so it becomes.

$$2 * J(x, y) = (a + 1)(a + 2b) = 2x \text{ where } x \in X \subseteq \mathbb{N} - \{1, 2\} \text{ would be the normal answer}$$

By definition, $2x$ is an even number. By the fundamental theorem of algebra, a natural number can be "prime factorized" where it is broken down into the multiplication of a series of prime numbers. We can group these prime numbers into two categories by parity so that every even natural number ($2x$) becomes a product of a series of even numbers multiplied by a series of odd numbers. Because we know $2x$ is even, we know that there is at least one 2 in the even parity series of $2x$. Because an even times an even is an even and an odd times an odd is an odd, this means that the series of odd numbers can be represented by a single odd number and the series of even numbers can be represented by a single even number. Thus...

$$2x = \text{some unique odd natural number} * \text{some unique even natural number}$$

However, there is a group of even natural numbers whose odd factor is 1. These are the set of numbers defined by 2^n, $n \in \mathbb{N}$ because their prime factorization is all 2's and one 1. Because of the terms $(a+1)$ and $(a+2b)$ have the opposite parity, and both a and b must be greater than 1, the terms cannot match the even and odd components of a 2^n, $n \in \mathbb{N}$ and thus these values are impossible to obtain using the function J over an ordered pair of natural numbers.

Finally, now that the exceptions are described, I can describe a method to get any of the other even natural number. To do this, you see which is bigger, the odd component or the even component. You then solve the following system of equations.

$$a + 1 = \text{The smaller component}$$

$$a + 2b = \text{The larger component}$$

So that

$$a = \text{The smalleer component} - 1$$

$$b = \frac{\text{The larger component} - \text{The smaller component} + 1}{2}$$

This will work because all we said is, "Every even natural number, (other than the exceptions) has an odd component and an even component which are both greater than 1, if we set $(a+1)$ equal to one component and $(a+2b)$ (which we know to have the opposite cardinality) equal to the other component, they should multiply back to get our original number." All that's left to prove is that a and b are both natural numbers.

For a, the smallest component, even or odd, is 2. $2 - 1 = 1$, so even with the smallest component a is still a natural number. Increasing the size of the smaller component will still make a a natural number. Therefore a is always a natural number.

If a natural number is larger than a natural number of the opposite parity, then it is greater by an odd number of at least 1. So

$$\text{The larger component} - \text{The smaller component} = \text{an odd number} \geq 1$$

Adding 1 to all sides turns the odd number into an even number.

$$\text{The larger component} - \text{The smaller component} + 1 = \text{an even number} \geq 2$$

Therefore b is an even number greater than 2 divided by 2, which is always a natual number.

In conclusion, our function J, given an ordered pair of natural numbers, can produce any natural number except 1 and any 2^n, $n \in \mathbb{N}$ therefore $J(\mathbb{N} \times \mathbb{N}) = \mathbb{N} - \{1\} \cup \{2^n : n \in \mathbb{N}\}$ $\square$

6 1.50

(!) For S in the domain of a function f, let $f(S) = \{f(x) : x \in S\}$. Let C and D be subsets of the domain of f.

6.1

a)Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

Proof:

For all elements $f(a)$ in $f(C \cap D)$, there is an element a in $C \cap D$ for which it corresponds to. This implies a is an element in C and a is an element in D. It follows that $f(a)$ is an element in $f(C)$ and $f(a)$ is an element in $f(D)$. Thus $f(a)$ is an element in $f(C) \cap f(D)$. Since all elements in $f(C \cap D)$ are elements in $f(c) \cap f(D)$, $f(C \cap D) \subseteq f(C) \cap f(D)$. $\square$

6.2

b) Give an example where equality does not hold in part (a).

Example:

Suppose $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$. Again, suppose $C = \{-1\}$ and $D = \{1\}$. Then...

$$C \cap D = \emptyset \qquad f(C \cap D) = \emptyset$$

$$f(C) = \{1\} \qquad f(D) = \{1\}$$
$$f(C) \cap f(D) = \{1\}$$

For equality to hold, $f(C) \cap f(D) \subseteq f(C \cap D)$. However, there is an element of $f(C) \cap f(D)$ which is not an element of $f(C \cap D)$, thus $f(C) \cap f(D) \nsubseteq f(C \cap D)$ thus $f(C) \cap f(D) \neq f(C \cap D)$. $\square$