MATH 347 HW2

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1 1.21

Let $a, b, c, x, y \in \mathbb{R}$ where $a \neq 0$

Show that the following proof is false.

$$\begin{aligned} \text{step 1} \qquad & ax^2 + bx + c = 0 \\ & -(ay^2 + by + c) = 0 \\ & = (x-y)(a(x+y)+b) = 0 \\ \\ \text{step 2} \qquad & a(x+y)+b = 0 \end{aligned} \tag{1}$$

$$\text{step 3 let} \qquad x = y \qquad x = \frac{-b}{2a}$$

Note how going from step 1 to step 2 when we divide by $(x-y)$, that is assuming that $(x-y) \neq 0$, otherwise the division would not be allowed. The contradiction comes in step 3 when we then let $x = y$ which implies $(x-y) = 0$. Since it cannot be that $(x-y) = 0$ and $(x-y) \neq 0$, we have run into a contradiction, and the proof is a false proof. $\square$

2 2.4

2.4. Let A and B be sets of real numbers, let f be a function from $\mathbb{R}$ to $\mathbb{R}$, and let P be the set of positive real numbers. Without using words of negation, for each statement below write a sentence that expresses its negation.

a) For all $x \in A$, there is a $b \in B$ such that $b > x$.
a) There is an $a \in A$, such that for all $b \in B, b \leq x$.

b) There is an $x \in A$ such that, for all $b \in B$, $b > x$.
b) For all $a \in A$, there is a $b \in B$ such that $b \leq a$.

c) For all $x, y \in \mathbb{R}$. $f(x) = f(y) \to x = y$.
c) There exists $x, y \in \mathbb{R}$ such that $f(x) = f(y) \wedge x \neq y$.

d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$, such that $f(x) = b$.
d) For some $b \in \mathbb{R}$, for all $x \in \mathbb{R}, f(x) \neq b$.

e) For all $x, y \in \mathbb{R}$, and all $\epsilon \in P$, there is a $\delta \in P$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

e) For some $x, y \in \mathbb{R}$, and some $\epsilon \in P$, then for all $\delta \in P |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon$

f) For all $\epsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.
f) For some $\epsilon \in P$, then for all $\delta \in P$, there is some $x, y \in \mathbb{R}. \quad |x - y| < \delta \quad \wedge |f(x) - f(y)| \geq \epsilon$.

3 2.21

Consider the sentence "For every integer $n > 0$ there is some real number $x > 0$ such that $x < 1/n$". Without using words of negation, write a complete sentence that means the same as "It is false that for every integer $n > 0$ there is some real number $x > 0$ such that $x < 1/n$". Which sentence is true?

The first sentence in logic jargon: $\forall n \in \mathbb{Z}, n > 0 \;\; \exists x \in \mathbb{R}, x > 0, \quad x < 1/n$
The negation is $\exists n \in \mathbb{Z}, n > 0 \;\; \forall x \in \mathbb{R}, x > 0, \quad x \geq 1/n$
Back to Manglish.

There exists some integer n greater than 0 where, for any positive real number x, $x \geq 1/n$
Proof of the validity of statement 1:

$$\forall n \in \mathbb{Z}, \exists x \in \mathbb{R}, x < 1/n$$

Let x be a real number defined as $1/2n$ where n is a positive integer. Then because x and n are both positive, $x = \frac{1}{2n} < \frac{1}{n} \;\; \square$

4 2.28

(!) Consider the equation $x^4y + ay + x = 0$.

a) Show that the following statement is false. "For all $a, x \in \mathbb{R}$,there is a unique y such that $x^4y+ay+x = 0$."

If we show the negation is true, then the statement is false. The negation of the statement is:

$$\exists a, x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4y + ay + x \neq 0$$

Suppose x takes the value 1 and a takes the value -1, then the equation becomes $-1 \neq 0$ Therefore the negation is true, and the original statement is false. $\quad \square$

b) Find the set of real numbers a such that the following statement is true. "For all $x \in \mathbb{R}$,there is a unique y such that $x^4y + ay + x = 0$."

If we pull out the y from x^4 and a, then subtract x, the equation becomes

$$(x^4 + a)y = -x$$

Let us have the exception that $(x^4 + a) \neq 0$ or $a \neq -x^4$ Now dividing both sides by $(x^4 + a)$

$$y = -x/(x^4 + a)$$

Since the real numbers are closed under addition, subtraction, multiplication and division, x and a can be any value* and y will still be a valid real number which satisfies the equation. (* except the exceptions.)

Therefore $\forall a \in \mathbb{R} - \{x^4\}$ added as a proposition after $x \in \mathbb{R}$ will make the statement true.

5 2.31

Which of these statements are believable? (Hint: Consider Remark 2.34.)
a) "All of my 5-legged dogs can fly."
b) "I have no 5-legged dog that cannot fly."
c) "Some of my 5-legged dogs cannot fly."
d) "I have a 5-legged dog that cannot fly."

Assuming that 5-legged dogs do not exist in our universe:
a and b are believable because they both make universally quantified statements about an empty set, therefore they are vacuously true.
c and d are unbelievable because they make existentially quantified statements about an empty set and therefore are vacuously false.

6 2.34

(!) For each statement below about natural numbers, decide whether it is true or false, and prove your claim using only properties of the natural numbers.

a)If $n \in \mathbb{N}$ and $n^2 + (n+1)^2 = (n+2)^2$, then n =3.

TRUE
Expanding and simplifying, we get the roots of the equation.

$$n^2 + 2n + 1 = 4n + 4$$

$$n^2 - 2n - 3 = 0$$

$$(n-3)(n+1) = 0$$

This implies $n = 3$ or $n = -1$, since -1 is not a natural number, 3 is the only natural numbers solution to the problem.

b)For all $n \in \mathbb{N}$,it is false that $(n-1)^3 + n^3 = (n+1)^3$.

TRUE
the negation of this statement: for some $n \in \mathbb{N}$,$(n-1)^3 + n^3 = (n+1)^3$

expanding the negation equation, we get

$$n^3 = 6n^2 + 2$$

dividing both sides by n^2 yields

$$n = 6 + 2/n^2$$

This statement is false for $n = 1$. Above $n = 2$, the $2/n^2$ is a fraction. Since a natural number can't have a fraction component, there is no natural number above 1 which n can be to satisfy this equation. Therefore n cannot both satisfy the equation and be a natural number.

Since the negation is false, the original statement must be true.

7 2.35

Prove that if x and y are distinct real numbers, then $(x+1)^2 = (y+1)^2$ if and only if $x+y=-2$. How does the conclusion change if we allow $x=y$?

proof

$$\forall x, y \in \mathbb{R}, x \neq y, (x+1)^2 = (y+1)^2 \Leftrightarrow x+y=-2$$

$$(x+1)^2 = (y+1)^2$$
$$x^2 + 2(x-y) - y^2 = 0$$
$$(x-y)(x+y+2) = 0$$

Since $x \neq y$, $x - y \neq 0$ therefore we can divide both sides of the equation by $(x-y)$

$$(x+y+2) = 0$$
$$x+y=-2$$

Since we used a series of reversible steps to get from one conclusion to the other, the biconditional holds. $\square$

If we allow x=y, then we can substitute x for y and see that any value of y makes the equation true. Thus x=y is valid, and thus x-y=0 can be true/ x-y=-2 can be false while $(x+1)^2 = (y+1)^2$ is true. Since the biconditional holds only when both sides are true or both sides are false, but the left was true when the right was false, the biconditional is no longer true and the statement would become false.

8 2.41

A clerk returns n hats to n people who have checked them, but not necessarily in the right order. For which k is it possible that exactly k people get a wrong hat? Phrase your conclusion as a biconditional statement.

Let the natural numbers and the tuple of natural numbers 1 through n, aka $[n]$, include 0 for this problem.

Let $C : \mathbb{N} \to \mathbb{N}$ where $C(n)$ is a function which returns the number of people who got their own hat back from a club. Let r represent the number of people who got the right hat back, and let k be the number of people who got the wrong hat back.

$$\forall n \in \mathbb{N} \;\; r = C(n) \Rightarrow r \in [n] \Leftrightarrow k \in [n]$$

Each person gets one hat, so they cannot get both the wrong hat and the right hat, and they cannot get neither the wrong hat nor the right hat. As well, a "negative hat" is not a valid object, therefore the total number of rightly and wrongly placed hat are non-negative natural numbers. Also since the clerk cannot give out more than n hats, and each hat is or is not the hat of the person who accepts it, $r + k = n$. Since it could be that the first a, $a \in \mathbb{N}$, hats are in the correct order, up to n, and all the rest could be in the wrong order, we know that $r \in [n]$. By the lemma below, if $r \in [n]$, then $k = n - r \in [n]$. Also by the lemma, if $k \in [n] \Rightarrow r = n - k \in [n]$. $\square$

Lemma for $\forall p \in \mathbb{N}, \;\; \forall q \in [p], \;\; p - q \in [p]$ Proof: by definition of q being in the tuple of p ,$0 \leq q \leq p$. Subtracting p from all sides yields $-p \leq q - p \leq 0$. Multiplying by -1 and flipping the inequalities yields $p \geq p - q \geq 0$. Therefore $p - q \in [p]$