Homework 7

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1 7.5

1.1 question

Write $\gamma = \sum_{i=0}^{\infty} r^i$. Show that

$$\sum_{i=0}^{\infty} i^2 r^i = (\gamma - 1) + 3r(\gamma - 1) + 5r^2(\gamma - 1) + ...$$

and so that

$$\sum_{i=0}^{\infty} i^2 r^i = \frac{r(1+r)}{(1-r)3}$$

1.2 answer

As previously proven, $\gamma = \frac{1}{(1-r)}$
it follows that

$$\gamma - 1 = \frac{1}{(1-r)} - 1 = \frac{1}{(1-r)} - \frac{(1-r)}{(1-r)} = \frac{r}{(1-r)}$$

it also follows that

$$2\gamma - 1 = \frac{2*1}{(1-r)} - 1 = \frac{2}{(1-r)} - \frac{(1-r)}{(1-r)} = \frac{(1+r)}{(1-r)}$$

It was also previously proven that $\sum_{i=0}^{\infty} ir^i = \frac{r}{(1-r)^2}$
it follows that since $\gamma = \frac{1}{(1-r)}$ and $\gamma = \frac{r}{(1-r)}$ that

$$\gamma * (\gamma - 1) = \frac{r}{(1-r)^2} = \sum_{i=0}^{\infty} ir^i$$

To be clear

$$\gamma = 1 + r + r^2 + r^3 + r^4 ... \qquad \gamma - 1 = r + r^2 + r^3 + r^4 + ...$$

Ok here goes

$$\sum_{i=0}^{\infty} i^2 r^i = 0 + r + r^2 + 9r^3 + 16r^4 ...$$

$$\sum_{i=0}^{\infty} i^2 r^i = \begin{cases} (r + r^2 + r^3 + r^4 + r^5 ...) \\ +3(r^2 + r^3 + r^4 + r^5 + r^6 ...) \\ +5(r^3 + r^4 + r^5 + r^6 + r^7 ...) \\ +7(r^4 + r^5 + r^6 + r^7 + r^8 ...) \\ + ... \end{cases}$$

$$\sum_{i=0}^{\infty} i^2 r^i = (\gamma - 1) + 3r(\gamma - 1) + 5r^2(\gamma - 1) + 7r^3(\gamma - 1) + ...)$$

$$= (\gamma - 1)(1 + 3r + 5r^2 + 7r^3 + 9r^4 + ...0$$

$$= (\gamma - 1)(1 + 3r + 5r^2 + 7r^3 + 9r^4 + ...0) = (\gamma - 1) * \begin{cases} 1 + r + r^2 + r^3 + r^4 + r^5 ... \\ +r + 2r^2 + 3r^3 + 4r^4 + 5r^5 + 6r^6 ... \\ +r + 2r^2 + 3r^3 + 4r^4 + 5r^5 + 6r^6 ... \end{cases}$$

$$= (\gamma - 1) * (\gamma + 2\sum_{i=0}^{\infty} ir^i) = (\gamma - 1) * (\gamma + \gamma * 2(\gamma - 1))$$

$$(\gamma - 1) * (\gamma + \gamma * 2(\gamma - 1)) = \gamma * (\gamma - 1)(1 + 2\gamma - 2) = \gamma * (\gamma - 1) * (2\gamma - 1)$$

$$\gamma * (\gamma - 1) * (2\gamma - 1) = \frac{1}{(1-r)} * \frac{r}{(1-r)} * \frac{1+r}{(1-r)} = \frac{r(r+1)}{(1-r)^3} \text{ QED}$$

2 7.7

2.1 question

7.7. Show that, for a geometric distribution with parameter p, the variance is $\frac{1-p}{p^2}$. To do this, note the variance is $\mathbb{E}[X^2] - \mathbb{E}[X]^2$. Now use the results of the previous exercises to show that

$$\mathbb{E}[X^2] = \sum_{i=0}^{\infty} i^2(1-p)^{(i-1)}p = \frac{p}{(1-p)}\frac{(1-p)(2-p)}{p^3}$$

then rearrange to get the expression for variance.

2.2 answer

From problem 7.5 we know that $\sum_{i=0}^{\infty} i^2 r^i = \frac{r(1+r)}{(1-r)^3}$

In the previous problem, problem 6.6, we are given the formula for this backwards geometric distribution is $\sum_{i=0}^{\infty} i(1-p)^{(i-1)}p$. The expected value of the square of the random variable is the sum of values squared times the probability of getting that value.

So as the problem suggests,

$$\mathbb{E}[X^2] = \sum_{i=0}^{\infty} i^2(1-p)^{(i-1)}p$$

$$\sum_{i=0}^{\infty} i^2(1-p)^{(i-1)}p = p\frac{(1-p)}{(1-p)}\sum_{i=0}^{\infty} i^2(1-p)^{(i-1)} = \frac{p}{(1-p)}\sum_{i=0}^{\infty} i^2(1-p)^i$$

Now, substituting $(1-p)$ for r into the pre-derived formula we get

$$\sum_{i=0}^{\infty} i^2(1-p)^i = \frac{(1-p)(1+(1-p))}{(1-(1-p))^3} = \frac{(1-p)(2-p)}{p^3}$$

Then substituting that in

$$\frac{p}{(1-p)}\sum_{i=0}^{\infty} i^2(1-p)^i = \frac{p}{(1-p)}\frac{(1-p)(2-p)}{p^3}$$

$$\mathbb{E}[X^2] = \frac{p}{(1-p)}\frac{(1-p)(2-p)}{p^3} \text{ QED}$$

From the previous problem, we are given that the mean of our geometric distribution is $\frac{1}{p}$ The mean is the same as the expected value in this case. So the square of the expected value is $(\frac{1}{p})^2 = \frac{1}{p^2}$
Now to get the variance we use the variance formula: $\mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$var(X) = \frac{p}{(1-p)}\frac{(1-p)(2-p)}{p^3} - \frac{1}{p^2} = \frac{(2-p)}{p^2} - \frac{1}{p^2} = \frac{(1-p)}{p^2}$$

3 7.11

3.1 question

Show that

$$P(w=i) = \sum_{j=0}^{N} P(h_t = j \cap h_{N-t} = i-j) = P(h_N = i)$$

3.2 answer

h_r = The number of heads in r flips. The coin has probability p of coming up heads.

This is the logical conclusion for the probability recurrence formula.

$$P_b(i|N,p) = p * P_b(i-1|N-1,p) + (1-p) * P_b(i|N-1,p)$$
$$= p^2 * P_b(i-2|N-2,p) + p*(1-p)*P_b(i-1|N-2,p) + (1-p)^2 * P_b(i|N-2,p)$$
$$= \sum_{j=0}^{n} \binom{n}{j} p^j * (1-p)^{n-j} * P_b(i-j|N-n,p)$$
$$= \sum_{j=0}^{n} P_b(j|n,p) * P_b(i-j|N-n,p)$$

It is logically equivalent to substitute N for n in the sum because the probability of more positive outcomes than trials is 0. I must mention the coin flips outcomes are described by the binomial distribution. Also, because the coin flips are independent, the multiplication of those two seprate events is the same as the probability of their union.

Therefore
$$\sum_{j=0}^{n} P_b(j|n,p) * P_b(i-j|N-n,p) = \sum_{j=0}^{n} P(h_t = j \cap h_{N-t} = i-j)$$

And
$$P(w=i) = \sum_{j=0}^{N} P(h_t = j \cap h_{N-t} = i-j) = P(h_N = i)$$

QED

4 7.15

4.1 question

An airline runs a regular flight with 10 seats on it. The probability that a passenger turns up for the flight is 0.95. What is the smallest number of seats the airline should sell to ensure that the probability the flight is full (i.e. 10 or more passengers turn up) is bigger than 0.99? (you'll probably need to use a calculator or write a program for this).

4.2 answer

t is the number of tickets sold.p is the probability they show up and is equals to 95% So the question wants to know when
$$\sum_{j=10}^{t} \binom{t}{j} * p^j * (1-p)^{t-j} \geq .99$$

I wrote this computer program to tell me when that was going to happen

```
probShow <- .95
probNoShow <- 1-probShow
numTickets <- 20
seats <- 10
prob <- 0
```

```
for(i in seats:numTickets){
  if(prob>(.99)){print(i-1); print(prob);}
  prob <- 0
  for(j in seats:i){
  prob <- prob+choose(i,j)*probShow^j*probNoShow^(i-j)
}  }
```

It told me the answer to the question is 13 tickets.