MATH 347 HW5

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1 Q4.22

Verify that $f(x) = \frac{2x-1}{2x(1-x)}$ defines a bijection from the interval $(0,1)$ to $\mathbb{R}$. (Hint: In the proof that f is surjective, use the quadratic formula.)

Proof: proof by a lot of algebra, hang on!

First I will prove that the function is injective. A definition of injective is

$$\forall\, a, a' \in A \qquad f(a) = f(a') \to a = a'$$

In this case A is $(0,1)$.

$$f(a) = f(a') \qquad \frac{2a-1}{2a(1-a)} = \frac{2a'-1}{2a'(1-a')}$$

$$(2a-1)(a')(1-a') = (2a'-1)(a)(1-a) \qquad (2a-1)(a'-a'^2) = (2a'-1)(a-a^2)$$

$$2aa' - 2aa'^2 - a' + a'^2 = 2a'a - 2a'a^2 - a + a^2 \qquad -2aa'^2 - a' + a'^2 = -2a'a^2 - a + a^2$$

$$2a'a^2 - 2aa'^2 + a - a' = a^2 - a'^2 \qquad 2aa'(a-a') + (a-a') = (a-a')(a+a')$$

So now we have two possibilities:

$$(a - a') = 0 \qquad \text{or} \qquad 2aa' + 1 = a + a'$$

$$a = a' \qquad \text{or} \qquad 2aa' - a = a' - 1$$

$$a = \frac{(a'-1)}{2a'-1}$$

In this case, the $a = a'$ is valid because you could give me an $a' \in A$ and I can give you a valid $a \in A$ because a will be the same as a', so we know they are both in A.

For the other case, for $a \in A$ to be true, since $A = (0,1)$, that means $0 < a = \frac{(a'-1)}{2a'-1} < 1$ must also be true for all $a' \in A$. Let me show why this cannot be the case.

Note that $\frac{(a'-1)}{2a'-1}$ is not defined for $a' = .5$. Also note that for $0 < a' < .5, \quad (2a'-1) < 0$, and for $.5 < a' < 1, \quad 0 < 2a' - 1$

$$\text{for } 0 < a' < .5 \qquad \frac{(a'-1)}{2a'-1} < 1$$

$$a' - 1 > 2a' - 1 \qquad 0 > a'$$

But this is a contradiction because we assumed $0 < a'$

$$\text{for } .5 < a' < 1 \qquad\qquad 0 < \frac{(a'-1)}{2a'-1}$$

$$0 < a' - 1 \qquad\qquad 1 < a'$$

But this is a contradiction because we assumed $a' < 1$

Therefore for the case where $a = \frac{(a'-1)}{2a'-1}$, assuming $a' \in A$ leads to us concluding $a \notin A$. So this means there is only one case where $a, a' \in A \wedge f(a) = f(a')$, and that is only the case where $a = a'$ So $a, a' \in A, \quad f(a) = f(a') \to a = a'$ and f is an injection by definition of injective.

Now I am going to prove that f is surjective. A definition of surjective is

$$\forall\, b \in B,\ \exists\, a \in A \qquad f(a) = b$$

Going with the hint, we can do the following to get something like the inverse of f, which is not the inverse of f, but will still be helpful because it will be a non-function which takes in values for $b \in \mathbb{R}$ and spits out the "corresponding," (keyword) real $a \in \mathbb{R}$ values of such that $f(a) = b$.

$$b = f(a) = \frac{2a-1}{2a(1-a)}$$

$$2ab - 2a^2b = 2a - 1$$

$$0 = 2ba^2 + (-2b+2)a - 1$$

The two rules for using the quadratic formula (that the "x^2 coefficient" be non-zero and the discriminant be positive) can be met if we split this into two cases. We will first consider the case where $b \neq 0$ (because $2b$ is the "x^2 coefficient", so if b is zero then the quadratic formula fails). Also note we know the discriminant is always positive because

$$"b^2 - 4ac" = (2(-b+1))^2 - 4 * 2b * (-1) = 4b^2 - 8b + 4 + 8b = 4(b^2+1)$$

$4(b^2+1)$ is always positive, so the discriminant will be positive.

Plugging into the quadratic formula we get

$$a = \frac{b - 1 \pm \sqrt{b^2 - 1}}{2b} \qquad\qquad \text{where } b \neq 0$$

Now, since the resulting... equation... is not the inverse, we just have to show that if we tweak it by turning the $\pm$ into $+$, then it always give a result in A. Let's call this tweaked function g. I will do this by showing for all $b \in B - \{0\}$, $g(b) \in A$, or in other words $0 < g(b) < 1$

$$b > 0 \qquad\qquad 0 < g(b) = \frac{b - 1 + \sqrt{b^2+1}}{2b}$$

$$0 < b - 1 + \sqrt{b^2+1}$$

$$1 < b + \sqrt{b^2+1} \qquad\qquad \text{Squaring positive sides preserves inequality}$$

$$1 < b^2 + 2b\sqrt{b^2+1} + b^2 + 1$$

$$0 < 2b^2 + 2b\sqrt{b^2+1} \qquad\qquad \text{is true because } b > 0$$

$$b < 0 \qquad\qquad 0 < g(b) = \frac{x - 1 + \sqrt{b^2+1}}{2b}$$

$$0 < \frac{b - 1 + b\sqrt{1 + \frac{1}{b^2}}}{b}$$

$$0 < 1 - \frac{1}{b} + \sqrt{1 + \frac{1}{b^2}}$$

$$\frac{1}{b} < 1 + \sqrt{1 + \frac{1}{b^2}}$$

This will always be true because since $b < 0$ the left side will always be negative, meanwhile the right side will always be positive.

So as can be seen, for $b \neq 0$, $\frac{b-1+\sqrt{b^2-1}}{2b}$ is positive.

$$b > 0 \qquad\qquad g(b) = \frac{b - 1 + \sqrt{b^2+1}}{2b} < 1$$

$$b - 1 + \sqrt{b^2+1} < 2b$$

$$-1 + \sqrt{b^2+1} < b$$

$$\sqrt{b^2+1} < b + 1$$

$$b^2 + 1 < b^2 + 2b + 1 \qquad \text{Squaring positive sides preserves inequality}$$

$$0 < 2b \qquad \text{This will be true because } b > 0$$

$$b < 0 \qquad\qquad g(b) = \frac{b - 1 + \sqrt{b^2+1}}{2b} < 1$$

$$b - 1 + \sqrt{b^2+1} > 2b$$

$$-b + \sqrt{b^2+1} > 1$$

$$b^2 - 2b\sqrt{b^2+1} + b^2 + 1 > 1 \qquad \text{Squaring positive sides preserves inequality}$$

$$b^2 - b\sqrt{b^2+1} > 0$$

This will be true because the left side is strictly positive.

So now we have shown that for $b \neq 0$, $0 < \frac{b-1+\sqrt{b^2+1}}{2b} < 1$, or in other words for $b \in \mathbb{R} \quad g(b) \in A$

So, the argument goes, if g takes a any non-zero b and returns a valid a (because $g(b) \in A$) for which $f(a) = b$, **then** g demonstrates that $\forall\, b \in \mathbb{R} - \{0\} \quad \exists\, a \in A \quad f(a) = b$.

Alright. Next, we will prove $(b = 0) \in \mathbb{R} \;\; \exists\, a \in A \;\; f(a) = 0$

$$f(a) = 0 = \frac{2a - 1}{2a(1-a)}$$

$$0 = 2a - 1$$
$$a = \frac{1}{2}$$

So now we have proven $\forall\, b \in \mathbb{R} - \{0\} \;\; \exists\, a \in A \;\; f(a) = b$ and $(b = 0) \in \mathbb{R} \;\; \exists\, a \in A \;\; f(a) = 0$ which put together means we have proven $\forall\, b \in \mathbb{R} \;\; \exists\, a \in A \;\; f(a) = b$ which means f is surjective by definition of surjective.

In conclusion, f is both injective and surjective, and therefore it is a bijection. ■

2 Q4.31

4.31. (!)Let $f : A \to B$ be a bijection, where A and B are subsets of $\mathbb{R}$. Prove that if f is increasing on A, then f^{-1} is increasing on B.

Proof: direct method.

The book gives the definition of increasing as (well this is my interpretation)

$$x, x' \in X \qquad x < x' \wedge f(x) < f(x')$$

Let

$$a, a' \in A \qquad b, b' \in B \qquad f(a) = b \qquad f(a') = b' \qquad a < a' \wedge f(a) < f(a')$$

Then by the property of functions that $g^{-1}(g(x)) = x$

$$f(a) = b \qquad\qquad f(a') = b'$$

$$f^{-1}(f(a)) = f^{-1}(b) \qquad\qquad f^{-1}(f(a')) = f^{-1}(b')$$
$$a = f^{-1}(b) \qquad\qquad a' = f^{-1}(b')$$

And

$$b = f(a) < f(a') = b'$$
$$b < b'$$

And

$$f^{-1}(b) = a < a' = f^{-1}(b')$$
$$f^{-1}(b) < f^{-1}(b')$$

So

$$b, b' \in B \qquad b < b' \wedge f^{-1}(b) < f^{-1}(b')$$

Then, by definition of increasing, f^{-1} must be increasing on B. ■

3 Q4.35

(!) Consider $f : A \to B$ and $g : B \to A$. Answer each question below by providing a proof or a counterexample.
a) If $f(g(y)) = y$ for all $y \in B$, does it follow that f is a bijection?
b)If $g(f(x)) = x$ for all $x \in A$, does it follow that $f(g(y)) = y$ for all $y \in B$?

For a) the counterexample is very simple, let f and g be the identity function. Let $B = \{1\}$ and let $A = \{1, 2\}$. Then g and f are valid functions, and for all elements in B, they reappear in the image of $f \circ g$. However f is not a bijection, because there is an element in B which has no corresponding element in A. ■

For b) the counterexample is also very simple, let g be the identity function and let f be the function which maps 1 to 1 and 2 to 1, let $A = \{1\}$ and $B = \{1, 2\}$. Then

$$g(f(\{1\})) = g(\{1\}) = \{1\}$$

And

$$f(g(\{1, 2\})) = f(\{1, 2\}) = \{1\}$$

So there is an element in B which is not in the image of $f(g(B))$ so it is not the case for all elements of B. ■

4 Q4.37

Consider $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

Here is a definition for a function g to be injective.

$$\forall\, c, c' \in C \qquad\qquad g(c) = g(c') \to c = c'$$

Here is what it means for $f \circ f$ to be injective.

$$\forall\, a, a' \in A \qquad\qquad (f \circ f)(a) = (f \circ f)(a') \to a = a'$$

By the meaning of composition of function, this is equal to saying

$$\forall\, a, a' \in A \qquad\qquad f(f(a)) = f(f(a')) \to a = a'$$

Since f is a function, and functions map each point in its domain to one point in its target, we can apply the function to the equal elements a and a' and maintain equality.

$$\forall\, a, a' \in A \qquad\qquad f(f(a)) = f(f(a')) \to f(a) = f(a')$$

Now since $f : A \to A$ that means $f(a), f(a') \in A$

Let $b, b' \in (A)$ such that $b = f(a)$ and $b' = f(a')$ then

$$\forall\, a, a' \in A \qquad\qquad f(b) = f(b') \to b = b'$$

And out pops the fact that f matches the definition for injective. ■

5 Q4.43

Let B be a proper subset of a set A, and let f be a bijection from A to B. Prove that A is an infinite set. (Hint: Use Exercise 4.42.)

Proof: I will prove this by showing that there cannot be a bijection from A to B if A is finite, then use an example from the book to show that there exists an scenario where A is infinite and the bijection holds. I will then conclude that since it is possible that the bijection holds when A is infinite and impossible when A is finite, the bijection is only valid if A is infinite and therefore A is infinite.

From the book

"A set C is finite if there is a bijection from C to $[k]$ for some $k \in \mathbb{N} \cup \{0\}$."

If $A = \varnothing$ then it has no proper subset and is vacuously false because B cannot exist, so $A \neq \varnothing$.

If A is finite, then there is a bijection from A to $[k]$. If B is a proper subset of A, then B is finite since the subset of a finite set is finite, and B has at least 1 fewer elements than A. This means that there is a bijection from B to $[l]$ for some $l < k$.

Since the composition of bijections is a bijection, if there is a bijection from A to $[k]$, g, and a bijection from B to $[l]$,h, then there is a bijection from A to B if there is a bijection from $[k]$ to $[l]$, t. However, by the result promised by exercise 4.42, there is no bijection from $[k]$ to $[l]$ since $l < k$ and therefore $l \neq k$.

Therefore there is not bijection from A to B, and thus if f is a bijection, then A cannot be a finite set.

Here is an example of an infinite set having a bijection to it's proper subset.

4.9. Example. One-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$. We define a function from $\mathbb{N}$ to $\mathbb{Z}$ by letting $f(n) = \frac{-(n+l)}{2}$ if n is odd and $f(n) = n/2$ if n is even. Note that $f(n)$ is negative when n is odd and non-negative when n is even. Thus $f(n) = b$ for be $\mathbb{Z}$ has the unique solution $n = 2b$ when $b < 0$ and $n = -2b - 1$ when $b < 0$.

(Source: Mathematical Thinking, Problem-Solving and Proofs Second Edition by John P. D'angelo and Douglas B. West ISBN 0-13-014412-6)

The example describes a one-to-one correspondence, ie. bijection, from $\mathbb{Z}$ to $\mathbb{N}$, but $\mathbb{N}$ is a proper subset of $\mathbb{Z}$

Since a set can either be infinite or finite, and since it is impossible for there to be a bijection from a set to one of its proper subsets, if the set is finite, but it is possible, empirically, if the set is infinite, I conclude that if f is a bijection from A to it's proper subset B, then A is infinite.

Otherwise let A be some non-empty finite subset. Since $B \subset A$, B contains every element of A except at least one. Therefore there is no bijection from A to B for the following reason:

If you took the identity function from A into A, it would be a bijection. You could also make a new bijection, g, from A to A by taking an existing bijection, f, from A to A and changing two element mappings like so

$$a, a', a'', a''' \in A \qquad f(a) = a'' \quad f(a') = a'''$$

$$a, a', a'', a''' \in A \qquad g(a) = a''' \quad g(a') = a''$$

because this operation preserves one-to-one correspondence. Therefore by the lemma, every bijection from A to A is the identity function altered by a series of swap operations.

So if we took any bijection from A to A, it could be altered by a series of swaps back to the identity since swaps are invertible.

Lemma: Every bijection from the non-empty finite set A to itself consists of the identity function altered by a series of swap operations where two element mappings are swapped.

Proof: proof by induction.

If you want a particular a mapping from A to A, then you can make a $\mid A \mid \times 2$ matrix. In the first column of the matrix, you place the elements of A and in the second column, you place which element of A you would like the corresponding element of A in the first column to map to under your function.

Base case:
If $\mid A \mid= 1$, then no swaps are required. If $\mid A \mid= 2$ then the ordering is how you want it or you swap the two elements in the left column.
For instance:

$$ab$$

$$ba$$

Becomes

$$bb$$

$$aa$$

Inductive case:
If $\mid A \mid= n$ for $n \in \mathbb{N}$, then the target, t, is in the bottom row, right column. You find the row in the left column which holds t. Then you swap the element in the bottom row, left column, with the row that holds t in the left column.

You then treat the rest of the matrix except the bottom row as a $\mid A \mid -1 \times 2$ matrix and induct upon that.

By the principle of induction, you have now found a series of element mapping swaps that will get you from the identity to your desired mapping. ■

6 Q4.42

Let f be a bijection from $[m]$ to $[n]$. Prove that $m = n$. (Hint: Use induction.)

I am doing this one because it will help with Q 4.43. Arrow with tail means injection, double headed arrow means surjection, and double headed arrow with tail means bijection.

Proof: I will use induction to prove $f : [n] \rightarrowtail [m] \Rightarrow n \leq m$ and $f : [n] \twoheadrightarrow [m] \Rightarrow n \geq m$. Which combined give that

$$\begin{aligned} f : [n] \rightarrowtail [m] \wedge f : [n] \twoheadrightarrow [m] \quad &\Rightarrow \quad k \leq m \wedge n \geq m \\ f : [k] \rightarrowtail\!\!\!\!\twoheadrightarrow [m] \quad &\Rightarrow \quad n = m \end{aligned}$$

Proof for $f : [n] \rightarrowtail [m] \Rightarrow n \leq m$

Proof: proof by induction on n.

Base case: $n = 1$. Since there is only one element in $[n]$, it is vacuously true that f meets the definition of inductive

$$\forall\, a, a' \in A \qquad f(a) = f(a') \rightarrow a = a'$$

because a and a' will be the one element of $[n]$ so $a = a'$ because the one element is itself. And this is true for any $1 \geq m$.

Inductive case:

case 1: we assume

$$f : [k] \rightarrowtail [p] \text{ and } k < p$$

now we can define a function $\iota : [k] \rightarrowtail [k+1]$ where $\iota(x) = x$. We know ι is an injection because its input and output are the same, so if two outputs are the same, the two inputs that lead to the outputs must also be the same.

We can also define a function $g : [k+1] \rightarrowtail [p]$. We know g can be an injection because it only can't be an injection if $k + 1 > m$, but if $k < m$, then $k + 1 \leq m$. So g can be an injection.

Now we can let the composition $(g \circ i)$ be defined to be f which we can do because g and ι are injective, and the composition of two injective functions is injective, and also f and $(g \circ \iota)$ have the same function signature.

So now we have shown that for $f : [k] \rightarrowtail [p]$ and $k < p$ for every

$$g : [k+1] \rightarrowtail [p] \text{ it is the case that } k + 1 \leq p$$

case 2: we assume

$$f : [k] \rightarrowtail [p] \text{ and } k = p$$

Then we can make a function $\iota : [p] \rightarrowtail [p+1]$ such that $\iota(x) = x$ and for the same reasoning in the other case, ι is injective.

Then we can make another function $g : [k] \rightarrowtail [p+1]$ such that $g = (\iota \circ f)$. g is injective because it is the composition of two injective functions.

let define $l = (p + 1)$ and note that

$$k = p$$

$$k < p + 1 = l$$
$$k < l$$

so now we have

$$g : [k] \rightarrowtail [l] \text{ where } k < l$$

which means it now matches case 1, so we will use case 1 to induct on it.

Putting the two cases together: we have

$$f : [k] \rightarrowtail [m] \text{ and } k \leq m$$

implies that for all

$$g : [k+1] \to [p] \text{ it is the case that } k+1 \leq p$$