MATH 413 HW5

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October 2016

1 Q11

Use combinatorial reasoning to prove the identity (In the given form)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

Story time! Imagine you are the big boss, the Don, of the mafia. You know in your mafia there are 3 snitches among the n mobsters you are in charge of. You also know there are $n-3$ loyal mobsters. You plan to whack k mobsters at random to get rid of a snitch. For probability purposes, you would like to know how many outcomes end in you killing at least one rat.

On the one hand, (the left hand side)

you could count the total number of ways you end up having k mobsters among n sleep with the fishes.

$$\binom{n}{k}$$

Then, by the subtraction principle, you could "uncount" the way in which you kill only loyal wise guys. Since there are $n-3$ loyal wise guys in your mob, there are

$$\binom{n-3}{k}$$

ways to only hit loyal mobsters. Hence

$$\binom{n}{k} - \binom{n-3}{k}$$

ways to kill at least one rat.

On the other hand, (the left hand side)

You could give each snitch a distinct letter so that we have snitch a, snitch b, and snitch c. Now you want to consider the powerset of $\{a, b, c\}$ and take out the empty set because it will represent not nabbing any squealers.

$$\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}, \{b, c\}, \{b\}, \{c\}$$

Let's say, after some thought, you come up with the following organization.

$$\{a\} \cup \{b, c\}$$
$$\{b\} \cup \{c\}$$
$$\{c\}$$

What it means is that you are going to consider three cases:
1: You for sure got rat a, and you may have gotten b, and you may have gotten c
2: You for sure got rat b, and you may have gotten c, but you DIDN'T nab rat a
3: You for sure got rat c, but you DIDN'T get a or b.

case 1 covers only the cases for $\{a,b,c\},\{a,b\},\{a,c\},\{a\}$, case 2 cover only the cases for $\{b,c\},\{b\}$, and case 3 covers only the case for $\{c\}$

So then if you enumerate cases 1,2, and 3, and sum them, you have also counted all the way to whack at least one rat in your mob.

To set aside a, is to say you know you got him, that leaves $n-1$ mobsters of which you take out a hit on $k-1$ of them, (because you already hit one and you want to whack k total) there are

$$\binom{n-1}{k-1}$$

ways carry out the remaining hits.
To set aside b, is to say you know you got him, and to set aside a also to say he's definitely dodged your wrath. That leaves $n-2$ mobsters left, of which you are going to whack $k-1$ of them to hit a total of k. There are

$$\binom{n-2}{k-1}$$

to choose who else sleeps with the fishes.
To set aside c, is to say you know you got him, and to set aside a and b is to say those rats live for now. That leaves $n-3$ mobsters left, of which you will still whack $k-1$. There are

$$\binom{n-3}{k-1}$$

way to do this, hence the right hand side:

$$\binom{n-1}{k-1}+\binom{n-2}{k-1}+\binom{n-3}{k-1}$$

Since both sides accurately describe the same scenario, they are equal.

2 Q12

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 == \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{if } n=2m. \end{cases}$$

Using the notation where sums from a higher number to a lower number is valid and means decrement the count each time...

For the case n is odd:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{k}^2 + \sum_{k=\frac{n+1}{2}}^{n}(-1)^k\binom{n}{k}^2$$

$$= \sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{k}^2 + \sum_{k=n}^{\frac{n+1}{2}}(-1)^k\binom{n}{k}^2$$

$$= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k}^2 + \sum_{k=n}^{\frac{n+1}{2}} (-1)^k \binom{n}{n-k}^2$$

$$= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k}^2 + \sum_{k=-n}^{\frac{-(n+1)}{2}} (-1)^{-k} \binom{n}{n+k}^2$$

$$= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k}^2 + \sum_{k=0}^{\frac{n-1}{2}} (-1)^{-k-n} \binom{n}{k}^2$$

$$= \sum_{k=0}^{\frac{n-1}{2}} \left((-1)^k \binom{n}{k}^2 + (-1)^{-k-n} \binom{n}{k}^2 \right)$$

Since n is odd, -1^k will have the opposite sign as -1^{-k-n} and since they multiply terms which have the same magnitude they are going to cancel out.

Therefore

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = 0$$

when n is odd.

When n is even:

When n is even, we will look at the special case provided by the hint.

We are given that $(1-x^2)^n = (1-x)^n(1+x)^n$ According to the binomial theorem, the middle most term in the expansion of $(1-x^2)^n$ is given by

$$\binom{n}{(n/2)} (-x^2)^{n/2} * (1)^{n/2}$$

$$= \binom{n}{\frac{n}{2}} * x^n * (-1)^{n/2}$$

We can also deduce that in the expansion of $(1-x)^n(1+x)^n$, the term which has the $(x^2)^{(n/2)}$, aka the x^n term, will be made up of the multiplication of terms from the expansion of $(1-x)^n$ and $(1+x)^n$. And what I really mean by that is the coefficients of the x^n term are made from the multiplication of terms from the expansion of $(1-x)^n$ and $(1+x)^n$.

Specifically, we know that the multiplications that matter are between terms whose powers add up to n. That's how we get this formula

$$x^n \text{ term of } (1-x^2)^{n/2} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} (-x)^k (x)^{n-k} (1)^k (1)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} (x)^n (-1)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 (x)^n (-1)^k$$

Great! So now we have two equations for the middle most term in the expansion of $(1-x^2)^n$

$$\binom{n}{\frac{n}{2}} * x^n * (-1)^{n/2} = \sum_{k=0}^{n} \binom{n}{k}^2 (x)^n (-1)^k$$

So now we let $x = 1$, substitute $2m$ for n on the left side, and out pops the proof.

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = \binom{2m}{m} * (-1)^m$$

3 Q16

By integrating the binomial expansion, prove that, for a positive integer n,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + ... + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

Fair enough, here's the binomial expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Integrating both sides from 0 to 1 yields

$$\int_0^1 (1+x)^n = \frac{(1+x)^{n+1}}{n+1}\Big|_0^1 = \frac{2^{n+1}-1}{n+1}$$

$$\int_0^1 \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1}\Big|_0^1 = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1}$$

And so

$$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = 1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + ... + \frac{1}{n+1}\binom{n}{n}$$

4 Q28

Let n and k be positive integers. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

Story time! There is particularly nasty tribe of humans in existence (jk) who have a strange set of ways. It so happens that every year $2n$ children are born, n male and n female. At age 17, the brood of humans go through the village's death-trials. Before the death-trials begin, they village chief draws a number 1 through n, called k out of a skull of a great ancestor. That number determines how many males are allowed to survive the death-trial, and also how many females are allowed to survive the death-trial. After the death-trial, the males are all thrown into a pit and forced to fight to the death! One of them will win, and that male now gets to take all k surviving females from the brood as his wives.

Now the left side of the equation has a summation which goes through every possible k that the chief could draw. It has $\binom{n}{k}^2$, which represents the ways the k males out of n could survive death-trial with the ways the k females out of n could survive death-trial. Finally, it has a k term which represents the number of ways one male could win the fight to the death.

The right side of the equation has an n term, this is the number of ways that one male can come out on top. It also has a $\binom{2n-1}{n-1}$ term, this term is complicated, let me explain it.

After death trial, $n - k$ females have perished. Also after death-trial, k males are sent to fight to the death! Since we know the eventual champion participates in the fight, we only don't know about $k - 1$ males. So really, after we pick a champion, (the n term in the equation), all we need to pick is $n - 1$ remaining tribe persons to go into the set

$$\text{(died in death-trial} \wedge \text{female)} \vee \text{(died in fight to the death} \wedge \text{male)}$$

Since besides the champion there are $2n - 1$ tribe persons and we can pick from any of them to go into the set, there are

$$\binom{2n-1}{n-1}$$

ways of choosing them.

So both the left and right side describe all the ways this particularly nasty tribe's extremely strange rituals could turn out, so both sides are equivalent.

5 Q37

Use the multinomial theorem to show that, for positive integers n and t,

$$t^n = \sum \binom{n}{n_1 n_2 ... n_t}$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, ..., n_t$ of $n_1 + n_2 + ... + n_t = n$

The multinomial theorem states that

$$(x_1 + x_2 + ... + x_t)^n = \sum \binom{n}{n_1 n_2 .. n_t} * x_1^{n_1} * x_2^{n_2} * ... * x_t^{n_t}$$

If we replace each x with 1, we get

$$(1_1 + 1_2 + ... + 1_t)^n = \sum \binom{n}{n_1 n_2 .. n_t} * 1_1^{n_1} * 1_2^{n_2} * ... * 1_t^{n_t}$$

$$t^n = \sum \binom{n}{n_1 n_2 .. n_t}$$