Homework 6

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1 Introduction

1.1 question

6.6. Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We consider a game with two players. Each player has a deck of 40 cards. Each player shuffles their deck, then deals seven cards, called their hand. The rest of each player's deck is called their library. Assume that player one has 10 land cards in their deck and player two has 20. Write L1 for the number of lands in player one's hand and L2 for the number of lands in player two's hand. Write Lt for the number of lands in the top 10 cards of player one's library.
(a) Write $S = L1 + L2$.What is $P(S = 0)$?
(b) Write $D = L1L2$. What is $P(D = 0)$?
(c) What is the probability distribution for $L1$?
(d) Write out the probability distribution for $P(L1|Lt = 10)$.
(e) Write out the probability distribution $P(L1|Lt = 5)$.

1.2 answer

(a): A rephrase of the question in English: What is the probability that both players have 0 land cards in their hands?

$$S = L1 + L2$$

$$P(S = 0) = P(L1 + L2 = 0|L1 \geq 0, L2 \geq 0) = P(L1 = 0) \cap P(L2 = 0)$$

$$P(L1 = 0) = \frac{\binom{40-10}{7}}{\binom{40}{7}} \qquad P(L2 = 0) = \frac{\binom{40-20}{7}}{\binom{40}{7}}$$

$$P(L1 = 0) \cap P(L2 = 0) = \frac{\binom{40-10}{7}}{\binom{40}{7}} \frac{\binom{40-20}{7}}{\binom{40}{7}} = \frac{2610}{5748431} = 0.0004540369$$

(b): A rephrase of the question in English: What is the probability that both players have the same number of cards in their hands.

$$D = L1 - L2$$

$$P(D = 0) = P(L1 - L2 = 0|L1 \geq 0, L2 \geq 0) = P(L1 = L2)$$

$$P(L1 = L2) = \sum_{i=0}^{7} \frac{\binom{10}{i} * \binom{30}{7-i}}{\binom{40}{7}} \frac{\binom{20}{i} * \binom{20}{7-i}}{\binom{40}{7}} = \frac{592537}{4395859} = 0.1347943$$

(c): A rephrase of the question in English: Layout the probability for each of the values that L1 can take. The player can have $0, 1, 2, 3, 4, 5, 6$, or 7 land cards in their hand at one time.

L1 Value	Probability	Decimal
0	$\frac{2035800}{18643560}$	.10919
1	$\frac{5937750}{18643560}$	.31848
2	$\frac{6412770}{18643560}$	.34396
3	$\frac{3288600}{18643560}$	.17639
4	$\frac{852600}{18643560}$	.04573
5	$\frac{109620}{18643560}$	.00588
6	$\frac{6300}{18643560}$	.00033
7	$\frac{120}{18643560}$	.0000064

(d): So there were two interpretations of this question due to the fact that in the question there is a lack of temporal information for L_t the top ten cards in player 1's library. For my interpretation, I am defining the library as being defined after player one has drawn. So effectively the question becomes, layout the probability for each of the values that $L1$ can take given that after player one has drawn, the next 10 cards are all lands.

$$P(L1 = x | L_t = 10) = \frac{P(L1 = x) \cap (L_t = 10)}{P(L_t = 10)}$$

The following table describes those values.

L1 Value	Probability
0	1
1	0
2	0
3	0
4	0
5	0
6	0
7	0

(e): The probability distribution given that the within the next 10 cards, five of them are lands.

$$P(L1 = x|L_t = 5) = P(L1 = x|\text{deck-size} = 30, \text{player has 5 total land cards in their deck})$$

The following probability distribution shows the results of that equivalency.

L1 Value	Probability	Decimal
0	$\frac{480700}{2035800}$	0.23612339
1	$\frac{885500}{2035800}$	0.43496414
2	$\frac{531300}{2035800}$	0.26097848
3	$\frac{126500}{2035800}$	0.06213773
4	$\frac{11500}{2035800}$	0.00564888
5	$\frac{300}{2035800}$	0.00014736
6	0	0
7	0	0

2 6.8

2.1 question

There is some (small!) voltage over the terminals of a warm resistor caused by noise (electrons moving around in the heat and banging into one another). This is a good example of a continuous random variable, and we can assume there is some probability density function for it, say p(x). We assume that p(x) has the property that

$$\lim_{\epsilon \to 0} \int_{v-\epsilon}^{v+\epsilon} p(x)dx = 0$$

which is what you'd expect for any function you're likely to have dealt with. Now imagine I define a new random variable by the following procedure: I flip a coin; if it comes up heads, I report 0; if tails, I report the voltage over the resistor. This random variable, u, has a probability 1/2 of taking the value 0, and 1/2 of taking a value from p(x). Write this random variable's probability density function q(u). Show that

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} q(u)du = \frac{1}{2}$$

2.2 answer

Yea, so this question is really crazy because the integral from any point to the same point by definition is 0, and discrete variables don't really have density functions because discrete random variables only have raw probability functions. So I'm going to try my best, but forgive me for error because most of this is not very rigorous, but I had a lot of fun with this silly math.

$$Q(u) \begin{cases} \frac{1}{2} & 0 \\ \frac{1}{2} & p(x) \end{cases}$$

$$q(u)du = Q(u)$$

$$q(u) = \frac{Q(u)}{du} \begin{cases} \frac{1}{2du} & 0 \\ \frac{1}{2du} & p(x) \end{cases}$$

$$q(u) \begin{cases} \frac{1}{2du} & 0 \\ \frac{1}{2du} & p(x) \end{cases}$$

That should satisfy the first bizarre requirement. Now to prove the following

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} q(u)du = \frac{1}{2}$$

start

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} q(u)du = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} Q(u) \qquad \lim_{\epsilon \to 0} \epsilon = 0$$

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} Q(u) = \int_{0}^{0} Q(u) = \int_{0}^{0} Q(u = u)$$

So here is a lemma: The integral is the sum of areas under a curve. Normally the integral between any point and itself is 0, but why? It is because d(point), the change distance between a point and itself is 0, so you end up (in theory) multiplying the value of the function at that point with 0 and you get 0. In our integral however, we have eliminated the d(point) term. So really

$$\int_{0}^{0} Q(u = u) = Q(u = 0)$$

Or the value of the function at that point.

$$Q(u = 0)$$

According to out expansion near the top of the page, the probability that u =0 is $\frac{1}{2}$, the probability of getting heads, and also another one half probability of being 0 if $p(x) = 0$. However, $p(x)$ is a continuous random variable, and the probability that a continuous random variable ever takes the value of a single point is 0 for the same reason that "it never rains exactly two inches." -Salman Khan Therefore

$$Q(u = 0) = \frac{1}{2}$$

And so

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} q(u)du = \frac{1}{2}$$

3 6.10

3.1 question

A simple coin game is as follows: we have a box, which starts empty. $P1$ flips a fair coin. If it comes up heads, $P2$ gets the contents of the box, and the game ends. If it comes up tails, $P1$ puts a dollar in the box and they flip again; this repeats until it comes up heads
(a) With what probability will $P2$ win exactly 10 units?
(b) Write $S_\infty = \sum_0^\infty r^i$. Show that $(1-r)S_\infty = 1$, so that $S_\infty = \frac{1}{(1-r)}$
(c) Show that $\sum_0^\infty ir^i = \sum_0^\infty r^i + r\sum_0^\infty r^i + r^2\sum_0^\infty r^i + ...$ (look carefully at the limits of the sums!) and so show that $\sum_0^\infty ir^i = \frac{r}{(1-r)^2}$
(d) What is the expected value of the game? (you may find the results of the two previous subexercises helpful; they're not there just for show).
(e) How much should P2 pay to play, to make the game fair?

3.2 answer

(a): The formula for the probability of any senario is
(the probability that the game ended) $*$ (the probability it kept going)$^($units won$^)$

This comes out to

$$r * r^{10} = r^{11}$$

(b):

$$(1-r)\sum_0^\infty r^i = 1$$

$$(1-r)(1+r+r^2+r^3+...) = 1$$

$$(1-r)+(r-r^2)+(r^2-r^3)+(r^3-r^4)... = 1$$

$$1+(r-r)+(r^2-r^2)+(r^3-r^3)... = 1$$

$$1+0+0+0+... = 1$$

$$1 = 1 \text{ QED}$$

(c):

$$\sum_0^\infty ir^i = 0+r+2r^2+3r^3... = (r+r^2+r^3...)+(r^2+r^3+r^4...)+(r^3+r^4+r^5...)+...$$

$$= (r+r^2+r^3...)+r(r+r^2+r^3...)+r^2(r+r^2+r^3...)+... = \sum_1^\infty r^i + r\sum_1^\infty r^i + r^2\sum_1^\infty r^i + ...$$

QED for part one of part c

$$= (1+r+r^2+r^3...) * (r+r^2+r^3+r^4...) = \sum_0^\infty r^i * \sum_1^\infty r^i$$

$$= \sum_0^\infty r^i * \sum_1^\infty r^i * \frac{1}{r} * r = (\sum_0^\infty r^i)^2 * r = \frac{r}{(1-r)^2}$$

QED for part two of part c

(d): The expected value of the game is given by $\sum_0^\infty ir^i$ which we know evaluates to $\frac{r}{(1-r)^2}$ Therefore because the game is evaluated on the flipping of a fair coin, we know $r = \frac{1}{2}$

$$\frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

(e): To make the game fair, player 2 should pay the same expected return of 2 for each time they play the game.

4 6.13

4.1 question

An airline company runs a flight that has 10 seats. Each passenger who buys a ticket has a probability p of turning up for the flight. The gender of the passengers is not known until they turn up for a flight, and women buy tickets with the same frequency that men do. The pilot is eccentric, and will not fly unless at least two women turn up.
(a) How many tickets should the airline sell to ensure that the expected num- ber of passengers that turn up is greater than 10?
(b) The airline sells 10 tickets. What is the expected number of passengers on the aircraft, given that it flies? (i.e. that at least two women turn up). Estimate this value with a simulation.

4.2 maths answer

(a): Because expectations are linear, I am going to calculate the expectation for one seat and then infer from that for the whole ten seats. t is the number of tickets per seat, T is the total number of tickets.

$$\mathbb{E}(\text{seat is occupied}) = \sum_\Omega xP(x) = 1 * t * (p) + 0 * t * (1-p) = pt$$
$$\mathbb{E}(\text{seat is occupied}) = 1 = pt$$
$$t = \frac{1}{p}$$

That is the number of tickets you sell per seat. There are ten seats.

$$t * 10 = T = 10 * \frac{1}{p}$$

(b): The number of seats occupied will be called SO, SO_F if the number of seats occupied by females, SO_M is the number of seats occupied by males.

$$SO = SO_F + SO_M$$
$$\mathbb{E}(SO) = \mathbb{E}(SO_F + SO_M) = \mathbb{E}(SO_F) + \mathbb{E}(SO_M)$$
$$\mathbb{E}(SO|\text{the plane flies}) = \mathbb{E}(SO_F|SO_F \geq 2) + \mathbb{E}(SO_M)$$
$$\mathbb{E}(SO_M) = E(\text{Gender of ticket holder in binary}) * E(SO) = \frac{1}{2} * pT = \frac{1}{2} * 0.95 * 10 = 4.75$$
$$\mathbb{E}(SO_F|SO_F \geq 2) = \sum_{x=0}^{10} x \frac{P(SO_F = x|SO_F \geq 2)}{P(SO_F \geq 2)} = \sum_{x=0}^{10} x \frac{P(SO_F = x|SO_F \geq 2)}{1 - P(SO_F < 2)}$$

Because Minje set the probability that the passenger would show up to such a high probability, the chances of only one woman showing up are actually quite small, so small in fact that they are not significant, and the expected value (formally realized in the sum) comes out approximately to the same 4.75 when you plug it into R

$$\mathbb{E}(SO) = \mathbb{E}(SO_F) + \mathbb{E}(SO_M) = 4.75 + 4.75 \approx 9.5$$

4.3 simulation code

```
    #Obviously you have to set the number of trials
trials <- 100000

#Here are some variables that you can set
numSeats <- 10
numWomenRequired <- 2
numTickets <- 10

#Just in case you want to have a laugh at a
#universe where women buy plane tickets
#four times as often as men.
womanToManTicketRatio <- 1
womanTicketFreq <- womanToManTicketRatio/(womanToManTicketRatio+1)
manTicketFreq <- 1/(womanToManTicketRatio+1)

#The probability that each gender shows up
turnUpProbMen <- .95
turnUpProbWomen <- .95


ticketGoesToManShowsUp <- turnUpProbMen*manTicketFreq
ticketGoesToWomanShowsUp <- turnUpProbWomen*womanTicketFreq
emptySeatProb <- 1-(ticketGoesToManShowsUp+ticketGoesToWomanShowsUp)

genderSampleSpace <- matrix(nrow=trials,ncol=10)

#This assigns 1 to women, 0 to men and 11 to empty
#I chose 11 for empty because you can take the modulus of
#the sum of the row to get the number of women that way.
for(i in 1:trials){
```

```
genderSampleSpace[i,] <-sample(c(1,0,11),10,replace=TRUE,prob=c(ticketGoesToWomanShowsUp,ticketGoesToM


resultsList <- rep(0,trials)
#This records the trials where the criteria was not met
#and the plane did not take off
for(i in 1:trials){
if((sum(genderSampleSpace[i,])%%11)>(numWomenRequired-1)){resultsList[i] <- i}}

newSampleSpace <- genderSampleSpace[resultsList,]

#This gets the total number of people who showed up
positives<- nrow(newSampleSpace)*ncol(newSampleSpace)-length(grep("11",newSampleSpace[,]))

#The number of occupied seats per plane ride is what we
#were looking for
result<- positives/nrow(newSampleSpace)
result
```

4.4 simulation result

For 100000 trials, I got the number of seats occupied on average was 9.504043647 when the plane took off and the number of tickets sold was 10.

5 6.17

5.1 question

The random variable X takes the values $-2, -1, 0, 1, 2$, but has an unknown probability distribution. You know that $\mathbb{E}[|X|] = 0.2$. Use Markov's inequality to give a lower bound on P(X = 0). Hint: Notice that $P(X = 0) = 1 - P(|X| = 1) - P(|X| = 2)$.

5.2 answer

$$P(|X| = 0) = 1 - P(|X| > 0) - P(|X| < 0) = 1 - P(|X| > 0) = 1 - P(|X| \geq 1)$$

Markov's inequality states:

$$P(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}$$

It follows that

$$-P(|X| \geq a) \geq -\frac{\mathbb{E}(|X|)}{a}$$

$$1 - P(|X| \geq a) \geq 1 - \frac{\mathbb{E}(|X|)}{a}$$

$$P(|X| = 0) = 1 - P(|X| \geq 1) \geq 1 - \frac{\mathbb{E}(|X|)}{1} = 1 - .2 = .8$$

$$P(|X| = 0) \geq .8$$

6 6.19

6.1 question

You have a biased random number generator. This generator produces a ran- dom number with mean value -1, and standard deviation 0.5. Write A for the event that the number generator produces a non-negative number. Use Chebyshevs inequality to bound $P(A)$.

6.2 answer

Standard deviation is the root of the variance.

$$P(A) = P(X \geq 0)$$

$$1 = P(X \geq 0) + P(0 > X > -2) + P(X \leq -2)$$
$$P(X \geq 0) = 1 - P(0 > x > -2) - P(x \leq -2)$$
$$1 - P(0 > x > -2) = P(|X - \mathbb{E}(X)| \geq 1)$$
$$P(X \geq 0) + P(X \leq -2) = P(|X - \mathbb{E}(X)| \geq 1)$$

Chebyshev's Inequality states

$$P(|X - \mathbb{E}(X)| \geq a) \leq \frac{var(X)}{a^2}$$

It follows that

$$P(X \geq 0) + P(X \leq -2) \leq \frac{var(X)}{1^2}$$

Because random number generators follow some sort of usually smooth distribution, $P(x \leq -2) \neq 0$

$$P(X \geq 0) < \frac{var(X)}{1^2}$$
$$P(A) < \frac{var(X)}{1^2}$$
$$P(A) < \frac{\text{standard deviation}^2}{1^2} = \frac{.5^2}{1} = 0.25$$

$$P(A) < .25$$