MATH 347 HW6

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October 2016

1 Q13.8

If S is a bounded set of real numbers, and S contains $sup(S)$ and $inf(S)$, then S is a closed interval.

Here is a counter-example:

Consider S is the set of real numbers $\{0, 1\}$. Here,

$$0 \leq 1$$

$$1 \leq 1$$

and so 1 must be the maximum of this set and hence $1 = sup(S)$

$$0 \geq 0$$

$$1 \geq 0$$

and so 0 must be the minimum of this set and hence $0 = inf(S)$

Also since S has an infimum and a supremum, it is bounded.

However, I can name a number, 1/2 which is not in $\{0, 1\}$, but which is less than 1 and greater than 0. So $\{0, 1\}$ is not a closed interval.

Hence the counter-example is a bounded set of real which contains a supremum and an infimum, but which is not a closed interval. ■

2 Q13.11

Suppose that $\langle a\rangle$ and $\langle b\rangle$ converge.
a) If $\lim a_n < \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < b_n$.
b) If $\lim a_n \leq \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq b_n$.

2.1 a)

Proof: direct method.

for $\langle a\rangle$ and $\langle b\rangle$ by definition of convergence

$$\forall \epsilon \in \mathbb{R},\ \epsilon > 0 \quad \exists N \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad n \geq N \rightarrow \mid a_n - \lim a_n \mid < \epsilon$$

$$\forall \epsilon \in \mathbb{R},\ \epsilon > 0 \quad \exists M \in \mathbb{N}, \quad \forall m \in \mathbb{N}, \quad m \geq M \rightarrow \mid a_m - \lim a_m \mid < \epsilon$$

So we can take the larger of N and M and call it O. So

$$\forall \epsilon \in \mathbb{R},\ \epsilon > 0 \quad \exists N, M, O \in \mathbb{N}, \quad \forall n \in \mathbb{N}$$

$$n \geq O \geq N \rightarrow \mid a_n - \lim a_n \mid < \epsilon \qquad\qquad n \geq O \geq M \rightarrow \mid a_m - \lim a_m \mid < \epsilon$$

So since $\lim a_n < \lim b_n$, then let $0 < \frac{\lim b_n - \lim a_n}{2} = \epsilon$

Then by convergence, there is some O such that

$$\lim a_n - \epsilon < a_n < \lim a_n + \epsilon \qquad \text{and} \qquad \lim b_n - \epsilon < b_n < \lim b_n + \epsilon$$

So

$$a_n < \lim a_n + \frac{\lim b_n - \lim a_n}{2} \qquad \text{and} \qquad \lim b_n - \frac{\lim b_n - \lim a_n}{2} < b_n$$

$$a_n < \frac{\lim b_n + \lim a_n}{2} \qquad \text{and} \qquad \frac{\lim b_n + \lim a_n}{2} < b_n$$

So

$$a_n < \frac{\lim b_n + \lim a_n}{2} < b$$

So

$$a_n < b_n$$

In conclusion, we supposed that $\langle a \rangle$ and $\langle b \rangle$ converged and that $\lim a_n < \lim b_n$, then there was a natural number, we called it O, for which all natural numbers $n > O$ yielded that $a_n < b_n$ ■

2.2 b)

Here is a counter-example, for $n \in \mathbb{N}$ take the sequence $\langle a \rangle$ whose values are gotten by the formula

$$a_n = \frac{1}{n}$$

and a sequence, $\langle b \rangle$ whose values are gotten by the formula

$$b_n = -\frac{1}{n}$$

As discussed in class, the $\frac{1}{n}$ sequence is a standard sequence whose limit as n becomes large is know to be 0. Since limits are linear, and since the n^{th} element of $\langle b \rangle$ is described by multiplying n^{th} element of $\langle a \rangle$ by -1, then the limit of $\langle b \rangle$ is the limit of $\langle a \rangle$ multiplied by -1. Therefore the limit as n becomes large of $\langle b \rangle$ is $-1 \cdot 0 = 0$. So then

$$\lim a_n = \lim b_n = 0$$

However,

$$a_n = \frac{1}{n} > -\frac{1}{n} = b_n$$

$$\frac{2}{n} > 0$$

$$2 > 0 \text{ this is always true}$$

there are no terms in b_n that are greater than a_n for any $n \in \mathbb{N}$.

So there is a case where $\lim a_n \leq \lim b_n$, but there does not exist $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq b_n$. ■

3 Q13.22

For each set S below, determine whether S is bounded, and determine $sup(S)$ and $inf(S)$, if they exist.
$a)S = \{x : x^2 < 5x\}$.
$b)S = \{x : 2x^2 < x^3 + x\}$.
c) $S = \{x : 4x^2 > x^3 + x\}$.

3.1 a)

$S = \{x : x^2 < 5x\}$

for $x > 0$

$$x : x < 5$$

5 is the supremum for this case because the inequality tells us all positive $x < 5$ are valid, so 5 is the least upper bound.

for $x < 0$

$$x : x > 5$$

But there are no negative numbers greater than 5, so there are no negative numbers in S.

for $x = 0$

$$x : x^2 = 5x$$

$$x : 0 \cdot 0 = 5 \cdot 0$$

$$0 \in x$$

So 0 must be the infimum because it is the minimum of the set.

Therefore $sup(S) = 5$ and $inf(S) = 0$

3.2 b)

$b)S = \{x : 2x^2 < x^3 + x\}$.

for $x > 0$

$$x : 2x^2 < x^3 + x$$

$$x : 2x < x^2 + 1$$

$$x : 0 < x^2 - 2x + 1$$

$$x : 0 < (x - 1)^2$$

This is true for every S except 1, so all the positive numbers except 1 are part of S. This means S has no supremum.

for $x < 0$

$$x : 2x^2 < x^3 + x$$

$$x : 2x > x^2 + 1$$

$$x : 0 > x^2 - 2x + 1$$

$$x : 0 > (x - 1)^2$$

This is never true, so no negative numbers are in S.

Since no negative numbers are in x, but all the positive numbers except 1 are in x, then 0 must be the infimum because the definition of positive is being greater than 0.

Therefore S has no supremum and $inf(S) = 0$

3.3 c)

$S = \{x : 4x^2 > x^3 + x\}$.

for $x > 0$

$$x : 4x^2 > x^3 + x$$
$$4x > x^2 + 1$$
$$3 > x^2 - 4x + 4$$
$$3 > (x-2)^2$$
$$\sqrt{3} > x - 2 \qquad\qquad \sqrt{3} > 2 - x$$
$$2 + \sqrt{3} > x$$

This must be the supremum because values immediately less than it are valid elements of S and it is the highest upper bound.

for $x < 0$

$$x : 4x^2 > x^3 + x$$
$$x : 4x < x^2 + 1$$
$$x : 3 < x^2 - 4x + 4$$
$$x : 3 < (x-2)^2$$
$$x : \sqrt{3} > x - 2$$

This will always be true because the right side is strictly positive, therefore the infimum doesn't exist because all negative $x \in S$.

Therefore $sup(S) = 2 + \sqrt{3}$ and there is no infimum.

4 Q13.25

Use the definition of limit to prove that $\lim \sqrt{1 + n^{-1}} = 1$.

If $\lim \sqrt{1 + n^{-1}} = 1$, then by definition of a limit

$$\epsilon \in \mathbb{R} \quad \epsilon > 0 \qquad \exists N \in \mathbb{R} \qquad \forall n \in \mathbb{R} \quad n > N \qquad \mid \sqrt{1 + n^{-1}} - 1 \mid < \epsilon$$

$$1 - \epsilon < \sqrt{1 + n^{-1}} < 1 + \epsilon$$
$$1 - 2\epsilon + \epsilon^2 < 1 + n^{-1} < 1 + 2\epsilon + \epsilon^2$$
$$-2\epsilon + \epsilon^2 < +n^{-1} < 2\epsilon + \epsilon^2$$

We will get back to this inequality shortly, but first let me render the left side of the inequality irrelevant.

While the definition of a limit says that ϵ must be positive and be able to get arbitrarily close to 0, it doesn't say it has to be able to be greater than any particular number. So let $\epsilon < 1$.

For $0 < \epsilon < 1$

$$\epsilon < 2$$
$$\epsilon^2 < 2\epsilon$$
$$-2\epsilon + \epsilon^2 < 0$$

So back to the original equation.

$$-2\epsilon + \epsilon^2 < n^{-1} < 2\epsilon + \epsilon^2$$

As can be seen, since n is a positive number, by the rules of multiplicative inverse, it's inverse will also be positive. Therefore the left side of the inequality will be met because $-2\epsilon + \epsilon^2 < 0$, while $0 < n^{-1}$, so

$$-2\epsilon + \epsilon^2 < 0 = 0 < n^{-1}$$
$$-2\epsilon + \epsilon^2 < n^{-1}$$

That leaves the right side of the inequality to deal with.

$$n^{-1} < 2\epsilon + \epsilon^2$$

Since $0 < \epsilon$, the right side will be positive. Therefore we can do this

$$\frac{1}{2\epsilon + \epsilon^2} < n$$

What this inequality tells us is that for any $0 < \epsilon < 1$, as long as $\frac{1}{2\epsilon+\epsilon^2} < n$, $\sqrt{1 + n^{-1}}$ will be within ϵ of 1. So we have proven the existence of N. ■

5 Q13.29

Let $X_n = (1 + n)/(1 + 2n)$. Prove that $\lim_{x\to\infty} x_n$ exists by using Monotone Convergence. Prove that $\lim_{x\to\infty} X_n = 1/2$ by using the definition of limit.

Proof: Part one using definition of MCT. Note that all numbers used are real numbers, and that my interpretation of the problem was for real numbers, but that since the results for real numbers are stronger, the case with natural numbers is also proved by the real case.

First I will prove that X_n is strictly decreasing on the interval $(1, \infty)$. The definition of increasing is

$$\forall\, x, x' \in A, \qquad x < x' \Rightarrow f(x) > f(x')$$

Where $A = (1, \infty)$, $x = n$, and $x' = n + h$ where $h > 0$

For $1 < n$, for $h > 0$

$$\frac{(1 + n)}{(1 + 2n)} > \frac{(1 + n + h)}{(1 + 2(n + h))}$$

Note that all terms are positive.

$$(1 + n)(1 + 2n + 2h) > (1 + n + h)(1 + 2n)$$
$$1 + 2n + 2h + n + 2n^2 + 2nh > 1 + n + h + 2n + 2n^2 + 2nh$$
$$2h + 2n^2 + 2nh > h + 2n^2 + 2nh$$

$$h > 0$$

Since this is true, it means X_n is strictly decreasing on A, this also means it is monotone since strictly decreasing falls into the category of non-increasing.

Since X_n is strictly decreasing, we now need to prove it is bounded from below. To do this, I will find any lower bound. 0 should do the trick.

$$\frac{(1+n)}{(1+2n)} > 0$$
$$(1+n) > 0$$
$$n > -1$$

This is true because we are assuming $n > 1$. Therefore X_n is bounded and monotone, so by monotone convergence theorem, it converges. ■

Proof: Part two, proof of limit by direct method.

If $\lim_{x\to\infty} X_n = 1/2$, then by definition of limit

$$\epsilon \in \mathbb{R} \quad \epsilon > 0 \qquad \exists\, N \in \mathbb{R} \qquad \forall\, n \in \mathbb{R} \quad n > N \qquad \mid \frac{1+n}{1+2n} - 1/2 \mid < \epsilon$$

$$1/2 - \epsilon < \frac{1+n}{1+2n} < 1/2 + \epsilon$$
$$(1+2n)(1/2-\epsilon) < 1+n < (1+2n)(1/2+\epsilon)$$
$$1/2 - \epsilon + n - 2n\epsilon < 1+n < 1/2 + \epsilon + n + 2n\epsilon$$
$$-\epsilon - 2n\epsilon < 1/2 < \epsilon + 2n\epsilon$$

Since n and ϵ are positive, the left side will be strictly negative, and any negative is less than 1/2, so only the right inequality is yet to be met.

$$1/2 < \epsilon + 2n\epsilon$$
$$-2n\epsilon < \epsilon - 1/2$$
$$-2n < 1 - \frac{1}{2\epsilon}$$
$$n > \frac{1-2\epsilon}{4\epsilon}$$

What this inequality tells us is that for any $0 < \epsilon$, as long as $n > \frac{1-2\epsilon}{4\epsilon}$, $\frac{(1+n)}{(1+2n)}$ will be within ϵ of 1/2. So we have proven the existence of N. ■

6 13.37

Explain why the technique of Theorem 13.27 does not prove that $\mathbb{Q}$ is uncountable. "Proceeding as in Theorem 13.27, we list the expansions of numbers in $\mathbb{Q}$ and create an expansion $\langle a \rangle$ for a number y that is not on our list. This contradicts the hypothesis that $\mathbb{Q}$ is countable."

Alright. I will describe this in English, because it is not so complicated. Let N denote random number $0-9$.

We need to take the list of all rational numbers, and create a new rational number not on the list by changing the n^{th} digit, right of the decimal place, of the n^{th} rational number in the list. Then storing that change in the n^{th} place in our new rational number. We it change to 1 if that sequence holds 0 in that place, and we change to 0 if that sequence holds one of 1,2,3,...,9 in that place.

It is a well known theorem that rational numbers end in repeating digits. In order for our new construction to end in repeating digits, it must have a base pattern, X which it repeats. This pattern will necessarily be made of 1's and 0's.

If it proves impossible to construct such a rational number through our process, then it has been shown that the technique does not work to prove $\mathbb{Q}$ is uncountable.

Here is why it is impossible to build a new rational number from all the rational numbers. Our construction has a base pattern, X, made of some finite length of some assortment of 1's and 0's. We start going through the list and get to the point in our construction where the repeating pattern is going to start emerging. The base pattern keeps repeating for a while, except that eventually we stumble onto a rational number in our list which has the same base pattern X in it's repeating section, and it is perfectly in sync with what our construction's repeating section should look like.

Here is a crude diagram.

Our construction so far:

$$00000...0000NNNNN.NNNNNNNNNNNNNNXXXXXXX$$

The number we run into:

$$00000...0000NNNNN.NNNNNNNNNNNXXXXXXXXX...$$

Since the number we stumbled upon is lined up with our construction at the point we are looking at, that means in the n^{th} place of the number we run into is the opposite of the number we want. It is going to mess up our repeating pattern.

So you say, "just move that number back in the list". The only problem is that it doesn't matter how far down the list you move it, it will always mess one digit in the pattern of our construction. AND there are an infinite number of rational number that end with this very same repeating base pattern! So if we want to go through the entire list of rational number, we are going to stumble onto an infinite number of rationals that mess up the base pattern of our construction.

Here comes the philosophical bit, "if you have a repeating pattern that gets messed up an infinite number of times, is it still a repeating pattern?" The answer is no.

RECAP: every time we try to construct a new rational number by changing every rational number at one place to the right of the decimal in its decimal expansion, we end up hitting an infinite amount of rational numbers on the list that force our base pattern to get interrupted. Hence our construction ends up being an irrational number and therefore not a new rational number. Hence the technique does not work to prove $\mathbb{Q}$ is uncountable. ■