# Homework 8

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# 1 9.5

## 1.1 question

9.5. Zeros on a Roulette Wheel: We now wish to make a more sophisticated estimate of the number of zeros on a roulette wheel without looking at the wheel. We will do so with Bayesian inference. Recall that when a ball on a roulette wheel falls into a non-zero slot, odd/even bets are paid; when it falls into a zero slot, they are not paid. There are 36 non-zero slots on the wheel. We assume that the number of zeros is one of $\{0, 1, 2, 3\}$. We assume that these cases have prior probability $\{0.1, 0.2, 0.4, 0.3\}$.

(a) Write $n$ for the event that, in a single spin of the wheel, an odd/even bet will not be paid (equivalently, the ball lands in one of the zeros). Write $z$ for the number of zeros in the wheel. What is $P(n|z)$ for each of the possible values of z (i.e. each of $\{0, 1, 2, 3\}$)?

(b) Under what circumstances is $P(z = 0|\text{observations})$ NOT 0?

(c) You observe 36 independent spins of the same wheel. A zero comes up in 2 of these spins. What is $P(z|observations)$?

## 1.2 answer

(a) where $n$ represents landing on 0 in one roll and $z$ represents the number of zeros in the wheel.

$$P(n|0) = \frac{0}{36} = 0$$
$$P(n|1) = \frac{1}{37} = 0.0270$$
$$P(n|2) = \frac{2}{38} = 0.0526$$
$$P(n|3) = \frac{3}{39} = 0.0769$$

(b) where $P(\text{observations})$ is a nonzero constant.

$$\neg(0 = P(z = 0|\text{observations}))$$

$$P(z = 0|\text{observations}) = \frac{P(\text{observations}|0) * P(0)}{P(\text{observations})} = 0$$

$$\neg\left(\frac{P(\text{observations}|0) * P(0)}{P(\text{observations})} = 0\right)$$

Since $P(0) \neq 0$ and $P(\text{observations})\neg 0$ the only way that the above statement can be false is if $P(\text{observations}|0) = 0$. Therefore the circumstances we are looking for occur when $P(\text{observations}|0) \neq 0$

This will continue to hold as long as WE OBSERVE NO ZEROS in the outcomes of the game.

(c) NZ's are nonzeros and Z's are zeros

$$P(z|34 \text{ NZ's}, 2 \text{ Z's}) = \frac{P(34 \text{ NZ's}, 2 \text{ Z's}|z)P(z)}{P(34 \text{ NZ's}, 2 \text{ Z's})}$$

First I will calculate $P(34 \text{ NZ's}, 2 \text{ Z's})$ because it is necessary.

$$P(34 \text{ NZ's}, 2 \text{ Z's}) = \sum_{i=0}^{3} P(34 \text{ NZ's}, 2 \text{ Z's}|i)P(i)$$

$$= \binom{36}{2}\left(0 * .1 + \left(\frac{1}{37}\right)^2\left(\frac{36}{37}\right)^{34} 0.2 + \left(\frac{2}{38}\right)^2\left(\frac{36}{38}\right)^{34} 0.3 + \left(\frac{3}{39}\right)^2\left(\frac{36}{39}\right)^{34} 0.4\right)$$

$$\approx 0.21763291194907557$$

Next I calculate the individual probabilities

$$P(34 \text{ NZ's}, 2 \text{ Z's}|0) * P(0) = 0$$

$$P(34 \text{ NZ's}, 2 \text{ Z's}|1) * P(1) = \binom{36}{2}((\frac{1}{37})^2\left(\frac{36}{37}\right)^{34} 0.2$$

$$= \frac{0.03625713676682445}{0.21763291194907557} = 0.16659767331196828 \approx 0.16659$$

$$P(34 \text{ NZ's}, 2 \text{ Z's}|2) * P(2) = \binom{36}{2}((\frac{2}{38})^2\left(\frac{36}{38}\right)^{34} 0.3$$

$$= \frac{0.08329085943831872}{0.21763291194907557} = 0.38271260854979572 \approx 0.38271$$

$$P(34 \text{ NZ's}, 2 \text{ Z's}|3) * P(3) = \binom{36}{2}((\frac{3}{39})^2\left(\frac{36}{39}\right)^{34} 0.4$$

$$= \frac{0.09808491574393239}{0.21763291194907557} = 0.45068971813823594 \approx 0.45068$$

Here is a nice easy to read map

$$P(z| \text{ NZ's}, 2 \text{ Z's}) \begin{cases} 0 & 0 \\ 0.16659 & 1 \\ 0.38271 & 2 \\ 0.45068 & 3 \end{cases}$$

# 2 9.6

## 2.1 question

9.6. A Normal Distribution: You are given a dataset of 3 numbers, $1, 0, 20$. You wish to model this dataset with a normal distribution with unknown mean  and standard deviation 1. You will make an MAP estimate of $\mu$. The prior on $\mu$ is normal, with mean 0 and standard deviation 10.
(a) What is the MAP estimate of $\mu$?
(b) A new datapoint, with value 1, arrives. What is the new MAP estimate of $\mu$?

### 2.2 answer

(a) $X$ will represent the dataset, $X_i$ will represent specific datapoints, D is for data, $\mu$ is for mean, $\sigma_D$ is the standard deviation for the dataset, and $\sigma_\mu$ is the standard deviation for the prior distribution.

$$P(\mu|D) = \prod_{i=1}^{i=3} \left( \frac{P(D_i|\mu)}{P(D_i)} \right) * P(\mu) \approx \prod_{i=1}^{i=3} \left( \frac{1}{\sqrt{2\pi}(\sigma_D)} e^{\frac{-(X_i-\mu)^2}{2(\sigma_D)}} \right) * \frac{-(\mu-0)^2}{2(\sigma_\mu)}$$

$$log(P(\mu|D)) = \sum_{i=1}^{3} \left( \frac{-(X_i-\mu)^2}{2(\sigma_D)} \right) + \frac{-(\mu-0)^2}{2(\sigma_\mu)}$$

$$= \frac{-(X_1-\mu)^2}{2(\sigma_D)} + \frac{-(X_2-\mu)^2}{2(\sigma_D)} + \frac{-(X_3-\mu)^2}{2(\sigma_D)} + \frac{-(\mu-0)^2}{2(\sigma_\mu)}$$

$$= \frac{X_1\mu}{(\sigma D)} + \frac{X_2\mu}{(\sigma D)} + \frac{X_3}{(\sigma D)} + 3\left( \frac{-\mu^2}{2(\sigma_\mu)} \right) - \frac{\mu}{2(\sigma_\mu)}$$

$$= \frac{(X_1+X_2+X_3)}{\sigma_R^2}\mu + \mu^2 \left( \frac{-3}{2(\sigma_D)^2} + \frac{-1}{2(\sigma_\mu^2)} \right)$$

$$\frac{d}{dx} log(P(\mu|D)) = \frac{(X_1+X_2+X_3)}{\sigma_R^2} + \mu \left( \frac{-3}{(\sigma_D)^2} + \frac{-1}{(\sigma_\mu^2)} \right)$$

Set the derivative equals to 0 for the last step in optimization.

$$0 = \frac{(X_1+X_2+X_3)}{\sigma_D^2} + \mu \left( \frac{-3}{(\sigma_D)^2} + \frac{-1}{(\sigma_\mu^2)} \right)$$

$$0 = \frac{(1+0+20)}{1^2} + \mu \left( \frac{-3}{(1)^2} + \frac{-1}{(10^2)} \right)$$

$$\mu = \frac{21}{3.01} = 6.976744$$

(b) Note to graders, I may have gotten the previous problem wrong, if so, please note that my technique for this question is rock solid because it's a formula question. Please be generous in your judgement.

According to the formula in the chapter

$$\mu_N = \frac{X_N * (\sigma_{N-1})^2 + \mu_{N-1}(\sigma_N)^2}{(\sigma_N)^2 + (\sigma_{N-1})^2}$$

inputting all the values using the following computer code:

```
data <- c(1,0,20,1)
sdNew <- 1
sdOld <- 10
meanOld <- 0

for(i in 1:4)
{
  meanOld <- (data[i]*sdOld^2+sdNew^2*meanOld)/(sdOld^2+sdNew^2)
  print(meanOld)
  sdOld <- sqrt(sdOld^2*sdNew^2/(sdOld^2+sdNew^2))
  print(sdOld)
}
```

```
meanOld
sdOld
```

I got that the answer to part (a) was actually 6.976744 and the previous $\sigma$ was 0.5763904

I get $\sigma$ to be 0.4993762

# 3 9.7

## 3.1 question

9.7. The Average Mouse: You wish to estimate the average weight of a mouse. You obtain 10 mice, sampled uniformly at random and with replacement from the mouse population. Their weights are $21, 23, 27, 19, 17, 18, 20, 15, 17, 22$ grams respectively.
(a) What is the best estimate for the average weight of a mouse, from this data?
(b) What is the standard error of this estimate?
(c) How many mice would you need to reduce the standard error to 0.1?

## 3.2 answer

(a) The chapter says the mean of the sample is a good estimate for the mean of the population so:

$$(21 + 23 + 27 + 19 + 17 + 18 + 20 + 15 + 17 + 22)/10 = 19.9$$

(b) Standard error of the estimate is the square root of the variance over the square root of the number of number of mice in the sample, so:

$$popsd(x) \approx sd(x) * \frac{\sqrt{k}}{\sqrt{k-1}}$$

To get the standard deviation of $\{X\}$ I just plugged this into R

```
data <- c(21,23,27,19,17,18,20,15,17,22)
sd(data)
```

I got 3.510302

$$popsd(x) = 3.510302 * \frac{\sqrt{10}}{\sqrt{10-1}} = 3.700183$$

$$std(x) = \frac{popsd(x)}{\sqrt{k}} = \frac{3.700183}{\sqrt{10}} = 1.170101$$

1.170101 is the standard error.
(c). You would need

$$\frac{popsd(x)}{\sqrt{k}} = 0.1$$

$$\frac{popsd(x)}{0.1} = 3.700183 * 10 = 37.00183 = \sqrt{k}$$

$$k = 37.00183^2 \approx 1370 \text{ mice}$$

# 4 9.8

## 4.1 question

9.8. Sample Variance and Standard Error: You encounter a deck of Martian playing cards. There are 87 cards in the deck. You cannot read Martian, and so the meaning of the cards is mysterious. However, you notice that some cards are blue, and others are yellow.

(a) You shuffle the deck, and draw one card. You repeat this exercise 10 times, replacing the card you drew each time before shuffling. You see 7 yellow and 3 blue cards in the deck. As you know, the maximum likelihood estimate of the fraction of blue cards in the deck is 0.3. What is the standard error of this estimate?
(b) How many times would you need to repeat the exercise to reduce the standard error to 0.05?

## 4.2 answer

(a) First I will give an encoding to each color. Blue is 0 and Yellow is 1.

Then I plug this into R

```
data <- c(1,1,1,1,1,1,1,0,0,0)
sd(data)
```

I got

$$sd(x) = 0.4830459$$

Then once again I multiply by $\frac{\sqrt{k}}{\sqrt{k-1}}$ to get 0.5091751 for $popsd(x)$
Standard error is just $\frac{popsd(x)}{\sqrt{k}} = \frac{0.5091751}{\sqrt{10}}$

The standard error is 0.1610153

(b)

You would need

$$\frac{popsd(x)}{\sqrt{k}} = 0.05$$

$$\frac{popsd(x)}{0.05} = 0.5091751 * 20 = 10.183502 = \sqrt{k}$$

$$k = 10.183502^2 \approx 104 \text{ card draws}$$

# 5 10.2

## 5.1 question

The Weight of Rats. You wish to estimate the average weight of a pet rat. You obtain 40 rats (easily and cheaply done; keep them, because they're excellent pets), sampled uniformly at random and with replacement from the pet rat population. The mean weight is 340 grams, with a standard deviation of 75 grams. (a) Give a 68% confidence interval for the weight of a pet rat, from this data (b) Give a 99% confidence interval for the weight of a pet rat, from this data

### 5.2 answer

(a) Firstly let me calculate the standard error.

$$\mathrm{STD_{ERR}} = \frac{popmeanx}{\sqrt{k}} = \frac{75}{\sqrt{40}} = 11.858541$$

For a 68% confidence interval is $mean(x) \mp \mathrm{STD_{ERR}}$

$$340 - 11.858541 \text{ to } 340 + 11.858541$$

$$328.141459 \text{ to } 351.858541$$

(b)
For a 99% confidence interval is $mean(x) \mp 3\mathrm{STD_{ERR}}$

$$340 - 3 * 11.858541 \text{ to } 340 + 3 * 11.858541$$

$$304.424377 \text{ to } 375.575623$$