MATH 413 HW8

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1 Q below

"Question A. A population grows as follows: in the current generation, each man chooses one woman for the next generation, and each woman chooses one man and one woman for the next generation. Generation 0 consists of one man. What is the size of generation n?"

Premise: First prove a Fibonacci relationship, Then solve closed form Fibonacci for closed form formula.

First label men a, and women b. So that we can have

$$\begin{tikzcd} a \arrow[r] & b \end{tikzcd}$$

$$\begin{tikzcd} b \arrow[r] \arrow[dr] & a \\ & b \end{tikzcd}$$

From the diagram it can be seen that the number of a's, A, in the next generation is the number of b's, B in the current generation. And B in the next generation is the same as the A plus B in the current generation. Or...

$$A_{n+1} = B_n$$
$$B_{n+1} = B_n + A_n$$

So

$$A_{n+2} + B_{n+2} = A_{n+1} + B_{n+1} + B_{n+1} = A_{n+1} + B_{n+1} + A_n + B_n$$

Since a generation is made of only a's and b's the size of a generation, G is

$$G_{n+2} = G_{n+1} + G_n$$

Now we solve the characteristic equation to get the closed form for this Fibonacci relation where $G_0 = 1$ and $G_1 = 1$ since there is only one man in generation 0 and 1 woman in generation 1.

$$f(n) = f(n-1) + f(n-2)$$
$$f(n) - f(n-1) - f(n-2) = 0$$

let $f(n) = x^n$

$$x^n - x^{n-1} - x^{n-2} = 0$$
$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad\qquad r_2 = \frac{1-\sqrt{5}}{2}$$

So

$$f(n) = \alpha r_1^n + \beta r_2^n$$

Then

$$f(0) = 1 = \alpha + \beta$$
$$\alpha = 1 - \beta$$

And

$$f(1) = 1 = \alpha r_1 + \beta r_2$$
$$1 = (1-\beta) r_1 + \beta r_2$$
$$1 = r_1 - \beta r_1 + \beta r_2$$
$$1 - r_1 = \beta(r_2 - r_1)$$
$$\beta = \frac{1 - r_1}{r_2 - r_1}$$

And

$$\alpha = \frac{r_2 - r_1}{r_2 - r_1} - \frac{1 - r_1}{r_2 - r_1} = \frac{r_2 - 1}{r_2 - r_1}$$

$$\alpha = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad\qquad \beta = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

So the size of generation n is

$$G(n) = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

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2 Q 9

Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

Premise: prove relationship, use characteristic equation to solve for closed formula. Let r mean red, b blue, and w white.

As with last time, we will consider the problem as a tree problem. We know that $f(1) = 3$ because there are obviously three ways to color a 1×1 chess board with 3 colors. Then when we add a new square, if the previous square was red, then the new square can't be red, so we get the following tree below.

The tree has the following formulas for the number of r -R- b, -B- and w -W- for each expansion.

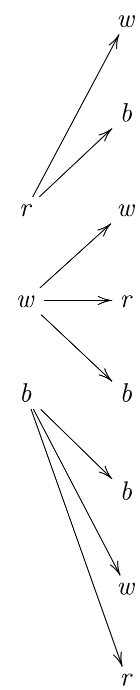
$$R(n) = W(n-1) + B(n-1)$$

$$W(n) = R(n-1) + W(n-1) + B(n-1)$$
$$B(n) = R(n-1) + W(n-1) + B(n-1)$$

So

$$B(n) + R(n) + W(n) = 2\Big(R(n-1) + W(n-1) + B(n-1)\Big) + W(n-1) + B(n-1)$$

$$= 2\Big(R(n-1) + W(n-1) + B(n-1)\Big) + 2\Big(R(n-2) + W(n-2) + B(n-2)\Big)$$



Since the size of an expansion is $R + B + W$, then $h_n = 2h_{n-1} + 2h_{n-2}$.

This shows there is a recurrence relation for h_n.

Now we derive the characteristic equation.

$$h_n - 2h_{n-1} - 2h_{n-2} = 0$$

Let $h_n = x^n$.

$$x^n - 2x^{n-1} - 2x^{n-2} = 0$$

$$x^2 - 2x - 2 = 0$$

By the quadratic formula

$$x = 1 \pm \sqrt{3}$$

$$r_1 = 1 + \sqrt{3} \qquad\qquad r_2 = 1 - \sqrt{3}$$

So

$$h_n = \alpha r_1^n + \beta r_2^n$$

Since we can see from the tree that $f(1) = 3$ and $f(2) = 8$, we let

$$h_1 = 3 = \alpha r_1 + \beta r_2$$

$$\alpha = \frac{3 - \beta r_2}{r_1}$$

And

$$h_2 = 8 = \alpha r_1^2 + \beta r_2^2$$

$$8 = \frac{3 - \beta r_2}{r_1} r_1^2 + \beta r_2^2$$

$$8 = 3r_1 - \beta r_2 r_1 + \beta r_2^2$$

$$\beta = \frac{8 - 3r_1}{r_2^2 - r_1 r_2}$$

So

$$\beta = \frac{3 - 2\sqrt{3}}{6} \qquad \alpha = \frac{3 + 2\sqrt{3}}{6}$$

Plugging in we get

$$h_n = \frac{3 + 2\sqrt{3}}{6}\left(1 + \sqrt{3}\right)^n + \frac{3 - 2\sqrt{3}}{6}\left(1 - \sqrt{3}\right)^n$$

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3 Q 11

The Lucas numbers $l_0, l_1, l_2, ..., l_n...$ are defined using the same recurrence relation defining the Fibonacci numbers, but with different initial conditions:

$$l_n = l_{n-1} + l_{n-2}, (n \geq 2), l_0 = 2, l_1 = 1.$$

Prove that

(a) $l_n = f_{n-1} + f_{n+l}$ for $n \geq 1$

(b) $l_0^2 + l_1^2 + ... + l_n^2 = l_n l_{n+l} + 2$ for $n \geq 0$

(a) Premise: Exploit the fact that Fibonacci sequences are linear.

The Lucas numbers start with the seed $l_0 = 2, l_1 = 1$, so then the lucas numbers are a linear combination of Fibonacci sequences. One with the starting seed $g_0 = 1, g_1 = 0$ and one with starting seed $f_0 = 0, f_1 = 1$. They look like so

$$\begin{aligned} g(n) =& 1, 0, 1, 1, 2, 3, 5, 8, 13, ... \\ f(n) =& 0, 1, 1, 2, 3, 5, 8, 13, 21, ... \end{aligned}$$

So you can see that for $n > 0$, $g(n) = f(n-1)$ and since $l(n) = f(n) + 2g(n)$ then

$$l(n) = f(n) + f(n-1) + f(n-1) = f(n+1) + f(n-1)$$

(b) Premise: Geometric interpretation and induction. (More interesting than regular induction.)

For any Fibonacci sequence f. We are going to show that a rectangle of size $f(n) \times f(n+1)$ with a "spillover rectangle" will have the same area as all $f(i) \times f(i)$ squares where $0 \leq i \leq n$.

We make a "spillover rectangle" of area $f(0)^2 - f(0) \times f(1)$.

For the first addition to the main rectangle, we start with a $f(0) \times f(1)$ rectangle.

We will consider this a base case. The combined area of the "spillover rectangle" and main rectangle is $f(0)^2 - f(0) \times f(1) + f(0) \times f(1) = f(0)^2$ which is how we planned it. So the equation is valid for $n = 0$.

Next another base case. We take the side in our rectangle of size $f(0)$ and elongate it to size $f(2)$ by adding $f(1)$ length. Since the other side was already length $f(1)$ and we elongated this side by $f(1)$, we effectively added $f(1)^2$ area to the main rectangle. Also now our main rectangle has dimension

$f(1) \times f(2)$ and, together with the "spillover rectangle," holds the square area of $f(1)$ and $f(0)$. Which implies $f(n+1) \cdot f(n) + f(0)^2 - f(1) \cdot f(0) = \sum_{i=0}^{n} f(i)^2$

The inductive step is basically the same. Take the shorter side of the main rectangle and elongate by the length of the longer side, effectively adding the square area of the longer side and leaving a rectangle of area $f(n) \times f(n+1)$. With a "spillover rectangle" of area $f(0)^2 - f(1) \cdot f(0)$. Which, together, hold the area of the square of each element in the sequence up to n. Implying $f(n+1) \cdot f(n) + f(0)^2 - f(1) \cdot f(0) = \sum_{i=0}^{n} f(i)^2$.

By the principle of induction, this must be the case.

Since we know this works for any Fibonacci sequence, we can apply it to Lucas numbers. We get

$$\sum_{i=0}^{n} l(i) = \big(l(i) \cdot l(i+1)\big) + l(0)^2 - l(1) \cdot l(0)$$
$$= l(i) \cdot l(i+1) + 2$$

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4 Q 14

Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$. Determine the generating function for the sequence $h_0, h_1, h_2, ..., h_n, ...$, where h_n is the number of n-combinations of S with the following added restrictions:
(a) Each e_i occurs an odd number of times.
(b) Each e_i occurs a multiple-of-3 number of times.
(c) The element e_1 does not occur, and e_2 occurs at most once.
(d) The element e_1 occurs 1, 3, or 11 times, and the element e_2 occurs 2, 4, or 5 times.
(e) Each e_i occurs at least 10 times.

(a) premise: only allow them to appear an odd number of times.

For the four of them only to appear an odd number of times is to consider the coefficients of this infinite series

$$(x + x^3 + x^5 + x^7 ...)^4$$

To get the generating function, we use dirty trickery

$$\begin{aligned}
S &= x + x^3 + x^5 + x^7 ... \\
S - x &= x^3 + x^5 + x^7 + x^9 ... \\
\frac{S - x}{x^2} &= x + x^3 + x^5 + x^7 ... \\
\frac{S - x}{x^2} &= S \\
S(1 - x^2) &= x \\
S &= \frac{x}{1 - x^2}
\end{aligned}$$

So the generating function we are looking for is

$$\left(\frac{x}{1 - x^2}\right)^4$$

(b) premise: only allow them to appear a multiple of 3 number of times.

For the four of them only to appear a multiple of 3 number of times, and since mathematically 0 is a multiple of 3, is to consider the coefficients of this infinite series

$$(1 + x^3 + x^6 + x^9 + x^{12}...)^4$$

To get the generating function, we use dirty trickery

$$\begin{aligned}
&S = 1 + x^3 + x^6 + x^9 + x^{12}...\\
&S - 1 = x^3 + x^6 + x^9 + x^{12} + x^{15}...\\
&\frac{S-1}{x^3} = 1 + x^3 + x^6 + x^9 + x^{12}...\\
&\frac{S-1}{x^3} = S\\
&S(1 - x^3) = 1\\
&S = \frac{1}{1 - x^3}
\end{aligned}$$

So the generating function we are looking for is

$$\left(\frac{1}{1 - x^3}\right)^4$$

(c) premise: only allow e_1 to appear 0 times, allow e_2 to appear 0 or 1 times, and allow e_3 and e_4 to appear any number of times. Since we know the generating series of allowing them any number of times from class, and finite cases are trivial, I can just say the generating function we are looking for is.

$$(1 + x)\left(\frac{1}{1 - x}\right)^2$$

(d) premise: only allow e_1 to appear 1,3, or 11 times, allow e_2 to appear 2,4 or 5 times, and allow e_3 and e_4 to appear any number of times. Since we know the generating series of allowing them any number of times from class, and finite cases are trivial, I can just say the generating function we are looking for is.

$$(x + x^3 + x^{11})(x^2 + x^4 + x^5)\left(\frac{1}{1 - x}\right)^2$$

(e) premise: only allow them to appear at least 10 times.

For the four of them only to appear an odd number of times is to consider the coefficients of this infinite series

$$(x^{10} + x^{11} + x^{12} + x^{13}...)^4$$

To get the generating function, we use dirty trickery

$$\begin{aligned}
&S = x^{10} + x^{11} + x^{12} + x^{13}...\\
&S - x^{10} = x^{11} + x^{12} + x^{13} + x^{14}...\\
&\frac{S - x^{10}}{x} = x^{10} + x^{11} + x^{12} + x^{13}...\\
&\frac{S - x^{10}}{x} = S\\
&S(1 - x) = x^{10}\\
&S = \frac{x^{10}}{1 - x}
\end{aligned}$$

So the generating function we are looking for is

$$\left(\frac{x^{10}}{1-x}\right)^4$$

5 Q 40

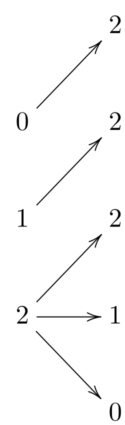
Let a_n equal the number of ternary strings of length n made up of Os,1s, and 2s, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$a_n = a_{n-1} + 2a_{n-2}, (n \geq 2),$$

with $a_0 = 1$ and $a_1 = 3$. Then find a formula for a_n.

Premise: prove relationship, use characteristic equation to solve for closed formula. Let Z mean the number of 0's in the previous iteration, T the number of 2's in the previous iteration, and N the number of 1's in the previous iteration.

We can consider the strings to be paths through a tree with the following grammar. This is correct because the tree will produce every ternary string that doesn't have the restricted substrings.



The tree has the following formulas

$$\begin{aligned} T(n) &= T(n-1) + N(n-1) + Z(n-1) \\ N(n) &= T(n-1) \\ Z(n) &= T(n-1) \end{aligned}$$

So

$$\begin{aligned} N(n) &= T(n-1) = T(n-2) + N(n-2) + Z(n-2) \\ Z(n) &= T(n-1) = T(n-2) + N(n-2) + Z(n-2) \end{aligned}$$

Since $a_n = N(n) + T(n) + Z(n)$, then

$$a_n = N(n)+T(n)+Z(n) = T(n-1)+N(n-1)+Z(n-1)+2\left(T(n-2) + N(n-2) + Z(n-2)\right) = a_{n-1}+2a_{n-2}$$

So now we can fine the characteristic equation to derive a closed form formula.

$$a_n = a_{n-1} + 2a_{n-2}$$

$$a_n - a_{n-1} - 2a_{n-2} = 0$$

let $a_n = x^n$

$$x^n - x^{n-1} - 2x^{n-2} = 0$$

$$x^2 - x - 2x = 0$$

by the quadratic formula

$$x = \frac{1 \pm 3}{2}$$

$$r_1 = -1 \qquad\qquad r_2 = 2$$

So

$$a_n = \alpha(-1)^n + \beta 2^n$$

Since $a_0 = 1$

$$a_0 = 1 = \alpha + \beta$$

$$\alpha = 1 - \beta$$

Since $a_1 = 3$

$$a_1 = 3 = \alpha(-1) + \beta(2)$$

$$3 = (1 - \beta)(-1) + \beta(2)$$

$$3 = 3\beta - 1$$

$$\beta = 4/3 \qquad\qquad \alpha = -1/3$$

So

$$a_n = \frac{-1}{3}(-1)^n + \frac{4}{3}(2)^n$$

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