MATH 413 HW3

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1 Q33

$$S = \{A, A, A, B, B, B, B, C, C, C, C, C\}$$

First let's start with a worst case analysis of this set, given we have to choose ten elements. Trying to completely exclude any type of element will fail because there are not 10 of any element's compliment. In fact there are only 9 of A's complement, 8 of B's complement, and 7 of C's complement. Given we need 10 elements total, this means we are forced to have at least 1 A, 2 B's and 3 C's in our answer.

Rearranging the space to show predetermined elements on the left and TBD elements on the right.

$$D = \{A, B, B, C, C, C\} \qquad T = \{A, A, B, B, C, C\}$$

Now we can say that from the TBD elements, we must choose a unique combination of 4.

$$AABB \qquad AACC \qquad BBCC$$
$$AABC \qquad BBAC \qquad CCAB$$

Now all that's left is to calculate the sum of permutations. The permutations will be of $D \cup x \subset T$ where x is one of the six aforementioned valid possibilities.

The permutation is calculated with the following handy formula whose components are hopefully known by the reader.

$$\frac{n!}{(x_1! * x_2! * ... * x_m!)}$$

Plugging in we get

$$\frac{10!}{3! * 4! * 3!} + \frac{10!}{3! * 2! * 5!} + \frac{10!}{1! * 4! * 5!} + \frac{10!}{3! * 3! * 4!} + \frac{10!}{2! * 4! * 4!} + \frac{10!}{2! * 3! * 5!} = 17850$$

The following code in R gave the results.

```
sum <- 0
for(i in 0:2)
{
  for(j in 0:2)
  {
    if(i+j>=2)
    {
      temp <- factorial(10)/factorial(i+1)/factorial(j+2)/factorial(7-i-j)
      sum <- sum +temp
      print(paste("i:",i,"  j:", j,"  k:", 4-i-j, " partial:", temp,sep=""))
    }
  }
}
sum
```

2 Q57

We first pick a rank, then a suit for the two cards in our similar rank, given what ranks are left, we choose 3 and pick a suit for each of the three cards and to get the probability, we divide by all the possible way to choose 5 cards from a deck of 52.

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}} \approx .42256$$

3 Q60

Classic stars and bars problem. We assume that already there is one of each kind of bagel in our box. We then set up a stars and bars with the same number of bagel categories, but subtract the number of categories from the total number of bagels. We then plug into the stars and bars equation to get.

$$\binom{6+15-1-6}{6-1}$$

To get a probability, we divide by the total number of ways to put 15 of 6 kinds of bagels in a box, aka the normal stars and bars equation.

$$\frac{\binom{6+15-1-6}{6-1}}{\binom{6+15-1}{6-1}} = \frac{\binom{14}{5}}{\binom{20}{5}} \approx .1291$$

For the next one, since we define probability as the cardinality of a subset over the cardinality of the set, then what we need to do is divide the cardinality of the set of at least three sesame bagels over the cardinality of the set containing at least one sesame bagel.

This can be given by the stars and bars if with the assumption that three sesame bagels are already in the box, over the stars and bars equation with the assumption that one sesame bagel is already in the box.

$$\frac{\binom{6+15-1-3}{6-1}}{\binom{6+15-1-1}{6-1}} = \frac{\binom{17}{5}}{\binom{19}{5}} \approx .5321$$

4 Q64

So here's how I went at this problem. We will have three categories of integer: integers that repeat, integers that appear exactly once, and integers that don't appear at all.

First we will analyze the case with one repeating integer. Out of the n integers, there is only one that will repeat, so we choose it.

$$\binom{n}{1}$$

This leaves $n-1$ integers that don't repeat, and, since there will now be two integers that will not appear, we will choose those as well.

$$\binom{n-1}{2}$$

Now we need to choose an order. We have n integers in our ticket machine, and three of them are the same, so we will reserve space for the three of them.

$$\binom{n}{3}$$

Finally the rest of the numbers need to be scrambled so that we cover all cases like this, since they occupy $n-3$ spots, we will scramble them with an

$$(n-3)!$$

Finally leaving for the case with one repeating integer the numeration

$$\binom{n}{1} * \binom{n-1}{2} * \binom{n}{3} * (n-3)!$$

But now we have the case for 2 integers that repeat, this means we have to make some changes. Firstly we need to choose 2 integers and not one.

$$\binom{n}{2}$$

Next we removed 2 integers and not one from play, so we pick amongst n-1 integers now

$$\binom{n-2}{2}$$

We need space for the two tickets that belong to the first integer, and then we need space for the two tickets that belong to the second integer.

$$\binom{n}{2} * \binom{n-2}{2}$$

Finally, we now need to scramble $n-4$ integers

$$(n-4)!$$

This leaves the enumeration for the case with 2 repeating integers to be

$$\binom{n}{2} * \binom{n-2}{2} * \binom{n}{2} * \binom{n-2}{2} * (n-4)!$$

To get a probability, we add up the enumerations and divide by the total number of outcomes. Since each integer can be any integer 1 to n, and we have n of them, the total number of outcomes is n^n. Therefore the probability is

$$\frac{\binom{n}{1} * \binom{n-1}{2} * \binom{n}{3} * (n-3)! + \binom{n}{2} * \binom{n-2}{2} * \binom{n}{2} * \binom{n-2}{2} * (n-4)!}{n^n}$$

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So imagine every person, instead of standing in a line for a shop, they are all standing in one long massive line. Then we can use the shops as bars to split the people up into shorter queues. Finally, if we first scramble the lines and then divide the people up, we will have generated every situation. So the question is what two equations scramble a line and then split it up. I have decided to combine the stars and bars equation with the $n!$ to get

$$n! * \binom{n+q-1}{q-1}$$