MATH 413 HW7

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1 Q 17

Determine the number of permutations of the multiset $S = \{3 \cdot a, 4 \cdot b, 2 \cdot c\}$, where, for each type of letter, the letters of the same type do not appear consecutively. (Thus *abbbbcaca* is not allowed, but *abbbacacb* is.)

The way to do this problem is to make "super-characters" out of each subset of letters of the same type. Then use PIE to cancel out over and under counting. Thus:

$$S = | P(\{a, a, a, b, b, b, b, c, c\}) | = \binom{9}{3, 4, 2}$$

$$r_a = | P(\{\text{“}aaa\text{”}, b, b, b, b, c, c\}) | = \binom{7}{1, 4, 2}$$

$$r_b = | P(\{a, a, a, \text{“}bbbb\text{”}, c, c\}) | = \binom{6}{3, 1, 2}$$

$$r_c = | P(\{a, a, a, b, b, b, b, \text{“}cc\text{”}\}) | = \binom{8}{3, 4, 1}$$

$$r_{ab} = | P(\{\text{“}aaa\text{”}, \text{“}bbbb\text{”}, c, c\}) | = \binom{4}{1, 1, 2}$$

$$r_{ac} = | P(\{\text{“}aaa\text{”}, b, b, b, b, \text{“}cc\text{”}\}) | = \binom{6}{1, 4, 1}$$

$$r_{bc} = | P(\{a, a, a, \text{“}bbbb\text{”}, \text{“}cc\text{”}\}) | = \binom{5}{3, 1, 1}$$

$$r_{abc} = | P(\{\text{“}aaa\text{”}, \text{“}bbbb\text{”}, \text{“}cc\text{”}\}) | = \binom{3}{1, 1, 1}$$

complementary PIE give us that the cardinality of the set we are looking for is

$$S - r_a - r_b - r_c + r_{ab} + r_{ac} + r_{bc} - r_{abc}$$

■

2 Q 24(c)

What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?

x	x				
	x	x			
		x			
				x	x
					x

The way to do this one is to label each row and column 1-6. We then use complementary PIE cheat formula to exclude all the cases where rooks are placed on a forbidden square. To use the formula, we need to count by hand the "r" terms.

$r_1 = \mid \{\{(1,1)\},\{(1,2)\}\} \mid =$ 2
$r_2 = \mid \{\{(2,2)\},\{(2,3)\}\} \mid =$ 2
$r_3 = \mid \{\{(3,3)\}\} \mid =$ 1
$r_4 = \mid \{\{(4,5)\},\{(4,6)\}\} \mid =$ 2
$r_5 = \mid \{\{(5,6)\}\} \mid =$ 1

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$r_{12} = \mid \{\{(1,1),(2,2)\},\{(1,1),(2,3)\},\{(1,2),(2,3)\}\} \mid =$ 3
$r_{13} = \mid \{\{(1,1),(3,3)\},\{(1,2),(3,3)\}\} \mid =$ 2
$r_{14} = \mid \{\{(1,1),(4,5)\},\{(1,1),(4,6)\},\{(1,2),(4,5)\},\{(1,2),(4,6)\}\} \mid =$ 4
$r_{15} = \mid \{\{(1,1),(5,6)\},\{(1,2),(5,6)\}\} \mid =$ 2
$r_{23} = \mid \{\{(2,2),(3,3)\}\} \mid =$ 1
$r_{24} = \mid \{\{(2,2),(4,5)\},\{(2,2),(4,6)\},\{(2,3),(4,5)\},\{(2,3),(4,6)\}\} \mid =$ 4
$r_{25} = \mid \{\{(2,2),(5,6)\},\{(2,3),(5,6)\}\} \mid =$ 2
$r_{34} = \mid \{\{(3,3),(4,5)\},\{(3,3),(4,6)\}\} \mid =$ 2
$r_{35} = \mid \{\{(3,3),(5,6)\}\} \mid =$ 1
$r_{45} = \mid \{\{(4,5),(5,6)\}\} \mid =$ 1

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$r_{123} = \mid \{\{(1,1),(2,2),(3,3)\}\} \mid =$ 1
$r_{124} = \mid \{\{(1,1),(2,2),(4,5)\},\{(1,1),(2,2),(4,6)\},$
$\{(1,1),(2,3),(4,5)\},\{(1,1),(2,3),(4,6)\},\{(1,2),(2,3),(4,5)\},\{(1,2),(2,3),(4,6)\}\} \mid =$ 6
$r_{125} = \mid \{\{(1,1)\,(2,2),(5,6)\},\{(1,1),(2,3),(5,6)\},\{(1,2),(2,3),(5,6)\}\} \mid =$ 3
$r_{134} = \mid \{\{(1,1)\,(3,3),(4,5)\},\{(1,1)\,(3,3),(4,6)\},$
$\{(1,2)\,(3,3),(4,5)\},\{(1,2)\,(3,3),(4,6)\}\} \mid =$ 4
$r_{135} = \mid \{\{(1,1),(3,3),(5,6)\},\{(1,2),(3,3),(5,6)\}\} \mid =$ 2
$r_{145} = \mid \{\{(1,1),(4,6),(5,6)\},\{(1,2),(4,5),(5,6)\}\} \mid =$ 2
$r_{234} = \mid \{\{(2,2),(3,3),(4,5)\},\{(2,2),(3,3),(4,6)\}\} \mid =$ 2
$r_{235} = \mid \{\{(2,2),(3,3),(5,6)\}\} = \mid$ 1
$r_{245} = \mid \{\{(2,2),(4,5),(5,6)\},\{(2,3),(4,5),(5,6)\}\} \mid =$ 2
$r_{345} = \mid \{\{(3,3),(4,5),(5,6)\}\} \mid =$ 1

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$r_{1234} = \mid \{\{(1,1),(2,2),(3,3),(4,5)\},\{\{(1,1),(2,2),(3,3),(4,6)\}\} =$ 2
$r_{1235} = \mid \{\{(1,1),(2,2),(3,3),(5,6)\}\} =$ 1

$r_{1245} =| \{\{(1,1),(2,2),(4,5),(5,6)\},\{(1,1),(2,3),(4,5),(5,6)\},\{(1,2),(2,3),(4,5),(5,6)\}\} = \quad 3$

$r_{1345} =| \{\{(1,1),(3,3),(4,5),(5,6)\},\{(1,2),(3,3),(4,5),(5,6)\}\} = \quad 3$

$r_{2345} =| \{\{(2,2),(3,3),(4,5),(5,6)\}\} = \quad 1$

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$r_{12345} =| \{\{(1,1),(2,2),(3,3),(4,5),(5,6)\}\} = \quad 1$

So according to the cheat formula, there are

$$6! - r_x(5!) + r_{xx}(4!) - r_{xxx}(3!) + r_{xxxx}(2!) - r_{xxxxx}(1!)$$

$$6! - 8(5!) + 22(4!) - 24(3!) + 10(2!) - 1(1!) = 163$$

ways to place the rooks so they aren't attacking each other. ■

3 Q25

Count the permutations $i_1 i_2 i_3 i_4 i_5 i_6$ of $\{1,2,3,4,5,6\}$, where $i_1 \neq 1,5$; $i_3 \neq 2,3,5$; $i_4 \neq 4$; and $i_6 \neq 5,6$.

The way to solve this is to convert this question to a forbidden rook placement problem. To do this, we convert the question to a 6×6 with the forbidden placements corresponding to the values the various i's can't take on.

x				x	
	x	x		x	
			x		
				x	x

We then use complementary PIE cheat formula to exclude all the cases where rooks are placed on a forbidden square. To use the formula, we need to count by hand the "r" terms.

$r_1 =| \{\{(1,1)\},\{(1,5)\}\} |= \quad 2$

$r_3 =| \{\{(3,2)\},\{(3,3)\},\{(3,5)\}\} |= \quad 3$

$r_4 =| \{\{(4,4)\}\} |= \quad 1$

$r_6 =| \{\{(6,5)\},\{(6,6)\}\} |= \quad 2$

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$r_1 3 =| \{\{(1,1),(3,2)\},\{\{(1,1),(3,3)\},\{\{(1,1),(3,5)\},\{(1,5),(3,2)\},\{(1,5),(3,3)\}\} |= \quad 5$

$r_{14} =| \{\{(1,1),(4,4)\},\{\{(1,5),(4,4)\}\} |= \quad 2$

$r_{16} =| \{\{(1,1),(6,5)\},\{(1,1),(6,6)\},\{\{(1,5),(6,6)\}\} |= \quad 3$

$r_{34} =| \{\{(3,2),(4,4)\},\{(3,3),(4,4)\},\{\{(3,5),(4,4)\}\} |= \quad 3$

$r_{36} =| \{\{(3,2),(6,5)\},\{(3,2),(6,6)\},\{(3,3),(6,5)\},\{(3,3),(6,6)\},\{\{(3,5),(6,6)\}\} |= \quad 5$

$r_{46} =| \{\{(4,4),(6,5)\},\{\{(4,4),(6,6)\}\} |= \quad 2$

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$r_{134} =| \{\{(1,1),(3,2),(4,4)\},\{(1,1),(3,3),(4,4)\},\{(1,1),(3,5),(4,4)\},$
$\{(1,5),(3,2),(4,4)\},\{(1,5),(3,3),(4,4)\}\} |= \quad 5$

$r_{136} =| \{\{(1,1),(3,2),(6,5)\},\{(1,1),(3,2),(6,6)\},\{(1,1),(3,3),(6,5)\},\{(1,1),(3,3),(6,6)\},$

$\{(1,1),(3,5),(6,6)\},\{(1,5),(3,2),(6,6)\},\{(1,5),(3,3),(6,6)\}\}\mid=$ 7

$r_{146} = \mid \{\{(1,1),(4,4),(6,5)\},\{(1,1),(4,4),(6,6)\},\{(1,5),(4,4),(6,6)\}\}\mid=$ 3

$r_{346} = \mid \{\{(3,2),(4,4),(6,5)\},\{(3,2),(4,4),(6,6)\},\{(3,3),(4,4),(6,5)\},$
$\{(3,3),(4,4),(6,6)\},\{(3,5),(4,4),(6,6)\}\}\mid=$ 5

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$r_{1346} = \mid \{\{(1,1),(3,2),(4,4),(6,5)\},\{(1,1),(3,2),(4,4),(6,6)\},\{(1,1),(3,3),(4,4),(6,5)\},$
$\{(1,1),(3,3),(4,4),(6,6)\},\{(1,1),(3,5),(4,4),(6,6)\},\{(1,5),(3,2),(4,4),(6,6)\},$
$\{(1,5),(3,3),(4,4),(6,6)\}\}\mid=$ 7

So according to the cheat formula, there are

$$6! - r_x(5!) + r_{xx}(4!) - r_{xxx}(3!) + r_{xxxx}(2!)$$

$$6! - 8(5!) + 20(4!) - 20(3!) + 7(2!) = 134$$

ways to assign values to the i's such that they meet the restrictions. ■

4 Q 28

A carousel has eight seats, each representing a different animal. Eight boys are seated on the carousel but facing inward, so that each boy faces another (each boy looks at another boy's front). In how many ways can the boys change seats so that each faces a different boy? How does the problem change if all the seats are identical?

This question is full of interpretation decisions. I will quickly list my interpretation.

- The question asks about two scenarios: when the seats are unique and identical. This means that the first boy who sits down in the new order is a "reference" for the position of the other boys, and in the case of the unique seats there are 8 ways to place the reference, whereas in the identical seat case, there is 1 way to place the reference boy in a seat.
- The boys are stated to be facing inward. I am understanding this to mean that each boy is looking straight through the middle of the carousel, (physically impossible due to the giant pillar, but ok), and, at the end, they will again be looking at the other end. So that there are three boys to the left and right of who each boy is looking at.

We can thing of each pair of boys staring at each other as a pair. We label four boys, one from each pair, a-d. We also label the boy each is staring at as their complement, a^c-d^c. We arrange them like so.

$$a - - - - a^c$$

$$b - - - - b^c$$

$$c - - - - c^c$$

$$d - - - - d^c$$

We would like to scramble the pairs so that no pair is the same as before. We treat the original 4 as positions which the other 4 could be assigned to. According to complementary PIE, if we consider every possibility and subtract the union of sets where each complement boy ends in a pair with his original boy, then we get the cardinality of the set in which no boy gets matched up with his original boy.

$$4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0! = 9$$

That equation states there are $\binom{4}{x}$ ways to pick how many points get stuck in their original position and $(4-x)!$ factorial ways to scramble the rest. The structure of the equation is the form of complementary PIE.

So now all we have to do is put the pairs back in place on the carousel. For the unique seats arrangement, there are 8 ways to place a and his new complement boy. After him there are 6 ways to place b and his new complement boy, 4 for c and new complement, and 2 for d and new complement. So that's

$$8 \cdot 6 \cdot 4 \cdot 2$$

Ways to put the boys back if they look straight across from each other.

So in conclusion we are left with

$$9 \cdot 8 \cdot 6 \cdot 4 \cdot 2$$

ways to rearrange the boys on the carousel to not look at the boy they were previously looking at.

If the seats are non-unique, then it doesn't matter where the first pair gets put back in, so it's only

$$9 \cdot 6 \cdot 4 \cdot 2$$

ways. ■

5 Q 32

Let n be a positive integer and let $p_1, p_2, ..., p_k$ be all the different prime numbers that divide n. Consider the Euler function ϕ defined by $\phi(n) =| \{k : 1 \leq k \leq n, GCD\{k, n\} = 1$. Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{i=1}^{k} (1 - \frac{1}{p_i})$$

Right, easy enough. Complementary PIE says that if we make a series where we alternate subtracting and adding the cardinality of sets where a criterion, or a union of criterion have been met, from the cardinality of the set where any criteria may or may not have been met, then we can find the cardinality of the set in which no criterion has been met. In this case, if we use complementary PIE and have our base sets be the numbers in 1-n which are divisible by one prime factor of n, then we will get out the set of number in 1-n in which no element shares any common factor prime factor with n. Since 1 is non-prime, and the identity factor, the GCD of n and every number in this set will be 1.

Let us use the theorem that there are $\lfloor \frac{n}{p} \rfloor$ numbers between 1 and n divisible by p. So if $p \mid n$, then n/p is an integer, so we can take out the floor function in our use.

Following the complementary PIE formula for our base criteria.

$$n - \sum_{i=1}^{k} \frac{n}{p_i} + \sum_{i=1, i<j \leq k}^{k} \frac{n}{p_i \cdot p_j} + ... \pm \frac{n}{p_1 ... p_k}$$

$$= n\Big(1 - \sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i=1, i<j \leq k}^{k} \frac{1}{p_i \cdot p_j} + ... \pm \frac{1}{p_1 ... p_k}\Big)$$

$$= n \prod_{i=1}^{k} (1 - \frac{1}{p_i})$$

■