MATH 413 HW6

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1 Q29

Find and prove a formula for

$$\sum_{\substack{r,s,t=\geq 0\\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$$

where the summation extends over all non-negative integers r,s and t with sum $r+s+t=n$.

Story time!

There is a town with 3 bars. Each bar has, m_1, m_2, and m_3 bar-stools respectively. There are also n people in the town.

We would like to find out all the ways that the bar-stools of the three bars can be filled if everybody in town goes to the bars and sits on a bar-stool.

The given formula takes the n people and splits them up into 3 groups, r, who go to bar 1, s who go to bar two, and t who go to bar three. Since negative people can't go to a bar, r,s, and t are positive. Also since it is the case that more people can't go into a bar, and sit, than there are bar-stools, the choose functions automatically cancel out scenarios where the size of the group is bigger than the capacity of the bar.

The formula sums over all the valid ways to split n people among the three bars. Within each summand term, there are

$$\binom{\text{bar-stools}}{\text{people in group}}$$

ways to permute the people among the bar-stools for each bar, and since each bar is independent, we multiply the choose terms.

Thus the given formula is valid for determining the ways that the n people in a town can sit in their $m_1+m_2+m_3$ bar-stools.

Another formula that calculates this is.

$$\binom{m_1+m_2+m_3}{n}$$

It literally calculates a scenario where we have $m_1+m_2+m_3$ bar-stools and n people, and it enumerates the $\binom{m_1+m_2+m_3}{n}$ ways to arrange the people to sit on barstools.

So by way of combinatorial reasoning, since both formulas describe the same situation, they are equal. ■

2 Q30

Prove that the only anti-chain of $S = \{1, 2, 3, 4\}$ of size 6 is the anti chain of all 2-subsets of S.

Proof: proof by contradiction.

I intend to prove that the largest anti-chain could only be made of 2-subsets of S or the anti-chain would be smaller than what Sperner's theorem predicts.

There are

$$\binom{4}{\lfloor \frac{4}{2} \rfloor} = \binom{4}{2} = 6$$

Elements in the longest anti-chain of S according to Sperner's theorem. There are

$$\binom{4}{2} = 6$$

2-subsets of S, none of which are a subset of any other, thus making them an anti-chain which is as long as the longest anti-chain.

Suppose not. Suppose the longest anti-chain included an n-subset where $n > 2$. Then that n-subset is a superset to $\binom{n}{2}$ 2-subsets. Since $n > 2$, $\binom{n}{2} > 1$, this means every n-subset in our anti-chain could be replaced with more than one 2-subsets and still be an anti-chain, so there will be no n-subsets in the longest anti-chain, because the longest anti-chain would have replaced the n-subsets with some 2-subsets to make a longer anti-chain.

Also suppose not. Suppose the longest anti-chain included a k-subset where $k < 2$. Then that k-subset is a subset to $\binom{4-k}{2-k} = \binom{4-k}{4-2} = \binom{4-k}{2}$ 2-subsets. Since $k < 2$, $\binom{4-k}{2} > 1$, this means every k-subset in our anti-chain could be replaced with more than one 2-subset and still be an anti-chain, so there will be no k-subsets in the longest anti-chain, because the longest anti-chain would have replaced the k-subsets with some 2-subsets to make a longer anti-chain.

Therefore, since the longest anti-chain is only made of 2-subsets and is length 6, the only anti-chain of length 6 must be made of 2-subsets since all other anti-chains are shorter than the longest anti-chain and are therefore shorter than length 6. ■

3 Q35

A talk show host has just bought 10 new jokes. Each night he tells some of the jokes. What is the largest number of nights on which you can tune in so that you never hear on one night at least all the jokes you heard on one of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1, 2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)

Well, if you aren't allowed to hear at least all of the joke you heard on any other night, that means you can't hear a subset of any other night's jokes tonight, and tonight's jokes can't be a subset of other night's jokes.

So what we need is a series of joke sets that form an anti-chain, and we want the longest one.

If there are ten jokes, according to Sperner's theorem, there are

$$\binom{10}{\lfloor \frac{10}{2} \rfloor} = \binom{10}{5}$$

joke subsets in the longest anti-chain of 10 jokes. Which means it's also the number of nights you could potentially go without any night's jokes being a subset of any other night's jokes. ■

4 Q40

What is the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expansion of $(x_1 - x_2 + 2x_3 - 2x_4)^9$?

According to the multinomial theorem, the coefficient of any specific monomial is that specific summand in the formula

$$(y_1 + y_2 + y_3 + ... + y_t)^n = \sum_{\substack{i_1, i_2, i_3 ... i_t > 0 \\ i_1 + i_2 + i_3 + ... + i_t = n}} \binom{n}{i_1, i_2, i_3, ..., i_n} y_1^{i_1} y_2^{i_2} y_3^{i_3} \cdots y_t^{i_t}$$

So if we just plug in we get

$$x_1^3 x_2^3 x_3 x_4^2 \text{termOf}((x_1 - x_2 + 2x_3 - 2x_4)^9) = \binom{9}{3, 3, 1, 2} x_1^3 \cdot (-x_2)^3 \cdot (2x_3)^1 \cdot (-2x_4)^2$$

So the coefficient is $-8\binom{9}{3,3,1,2}$. ■

5 Q45

Prove that

$$\sum_{n_1+n_2+n_3=n} \binom{n}{n_1 n_2 n_3} (-1)^{n_1 - n_2 + n_3} = (-3)^n$$

where the summation extends over all nonnegative integral solutions of $n_1 + n_2 + n_3 + n_4 = n$.

Restating the multinomial theorem.

$$(y_1 + y_2 + y_3 + ... + y_t)^n = \sum_{\substack{i_1, i_2, i_3 ... i_t > 0 \\ i_1 + i_2 + i_3 + ... + i_t = n}} \binom{n}{i_1, i_2, i_3, ..., i_n} y_1^{i_1} y_2^{i_2} y_3^{i_3} \cdots y_t^{i_t}$$

Then we could plug in -1 and -1^{-1} in some strategic places.

$$(-1 + -1^{-1} + -1)^n = \sum_{\substack{i_1, i_2, i_3 > 0 \\ i_1 + i_2 + i_3 = n}} \binom{n}{i_1, i_2, i_3} (-1)^{i_1} (-1)^{-i_2} (-1)^{i_3}$$

$$(-3)^n = \sum_{\substack{i_1, i_2, i_3 > 0 \\ i_1 + i_2 + i_3 = n}} \binom{n}{i_1, i_2, i_3} (-1)^{i_1 - i_2 + i_3}$$

So as can be seen, it's a consequence of the multinomial theorem. ■