MATH 347 HW4

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1 4.5

Only for sets with at least two distinct elements, there exists at least one non-identity self-bijective function.

Proof: imagine a set, S, with at least two distinct elements, we pick an element out of S, call it x. Then we continue picking elements out of the set until we find an element y, such that $x \neq y$. Now we define a function $f : S \to S$ such that f maps every element in S, except x or y, to itself, and maps x to y and y to x. f is a bijection because for every element of S, there is exactly one element of S that f maps to it. f is also not the identity because it didn't map every element to itself.

Since the notion of sets with a negative cardinality is not well defined, I will consider the empty set and sets of size one, thus covering sets of every known size.

Every function on the empty set is an identity, since the empty set has no elements and therefore it is vacuously true that for any function which maps the empty set to itself all elements of the empty set were mapped to themself. Since no non-identity function exists which maps the empty set to itself, it can't have a non-identity function, bijection or otherwise.

For sets of size/cardinality of 1, since there is only one element in such sets, if a function which maps the set to itself is a bijection, then it must map the one element to itself, thus being the identity function.

Therefore only for sets with at least two distinct elements, there exists at least one non-identity self-bijective function. $\square$

2 4.6

The problem asks us to consider the set

$$S = \{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday\}$$

Which gets mapped into the Natural numbers by a function, f, which maps each word to the natural number equal to the number of letters in the English word. The image $f(S$ is

$$\{6, 7, 9, 8\}$$

The definition of injective for a function, g, which maps A to B is

$$\forall\, x, x' \in A : \quad g(x) = g(x') \Rightarrow x = x'$$

But $\Big(f(Monday) = 6 = f(Friday) \Rightarrow Monday = Friday\Big)$ is false, so by definition, f, the function described by the problem, is not injective.

3 4.11

The function $f(x) = 2x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$, but not from $\mathbb{Z}$ to $\mathbb{Z}$

Let's point out that in both cases, the function is injective because of closure of the real numbers and integers under multiplication. By that I mean a real number times a real number is a real number, and an integer times an integer is an integer. Since 2 is both an integer and a real number, the image of multiplying the integers by two contains only integers, and the image of multiplying the real numbers by two contains only real numbers.

By Schroeder-Bernstein theorem, we can now prove a bijection by considering if the inverse of the function is also injective. The inverse of multiplication by 2 is multiplication by $1/2$. Since $1/2$ is a real number, this means that the inverse function will be injective when mapping the real numbers to the real numbers because of closure of real numbers under multiplication. However, because $1/2$ is not an integer, it is not guaranteed that the image of multiplying all integers by one half will yield a set of only integers. In fact, the integer 3 times $1/2$ is 1.5, which is not an integer, so the inverse is definitely not injective when mapping the integers to the integers.

So because the function, f, and it's inverse are both injective when mapping the real numbers to the real numbers, but not when mapping the integers to the integers, by Schroeder-Bernstein, the function is a bijection when mapping the real numbers into the real numbers, but the same cannot be said for the mapping of the integers to the integers.

4 4.21

Prove that the number of odd sized subsets of $[n]$ is the same as the number of even sized subsets.

Proof: To prove a bijection, I will first describe how the powerset of $[n]$ changes from one n to the next, and how that change ensures a bijection appears.

Let's show the subsets of [1]

$$\emptyset$$

$$\{1\}$$

And the subsets of [2]

$$\emptyset \;\; \{1,2\}$$

$$\{1\} \;\; \{2\}$$

Now let's see the subsets of [3]

$$\emptyset \;\; \{1,2\} \;\; \{1,3\} \;\; \{2,3\}$$

$$\{1\} \;\; \{2\} \;\; \{3\} \;\; \{1,2,3\}$$

Notice that I've arranged them so that the subsets with even size are on top and subsets with odd sizes are on the bottom.

Alright, so now lets describe how to get from the subsets of [2] to the subsets of [3] algorithmically. First we make two copies of the [2] like so

$$\emptyset \ \{1,2\} \ \emptyset \ \{1,2\}$$
$$\{1\} \ \{2\} \ \{1\} \ \{2\}$$

now to the second copy, we add the element 3 to each set,

$$\emptyset \ \{1,2\} \ \{3\} \ \{1,2,3\}$$
$$\{1\} \ \{2\} \ \{1,3\} \ \{2,3\}$$

now we flip the second copy

$$\emptyset \ \{1,2\} \ \{1,3\} \ \{2,3\}$$
$$\{1\} \ \{2\} \ \{3\} \ \{1,2,3\}$$

So we get to the subsets of [3] where the subsets of even size are in the top row and odd sized subsets are in the bottom row.

Let me give you a really concise formula. Let $P(x)$ mean power set of the set x. To get the subsets of $[n+1]$ from $[n]$ is:

$$P([n+1]) = P([n]) \cup \{x \cup \{n+1\} | x \in P([n])\}$$

This formula should work for the following reasons:

Since $[n] \subset [n+1]$, then $P([n]) \subset P([n+1])$ which is described by the formula
The size of the powerset of a set, S, doubles every time an element is added to S. And that occurs here since the two sets we are unioning are of equal size and disjoint.

Now, let's call the set of sets with $(n+1)$ as an element the "new sets." And those without $(n+1)$ "old sets." Each new set looks exactly like an old set but just with an extra element. Because it has an extra element, the parity is flipped from odd to even or even to odd. Therefore there is a bijection from each old element to it's corresponding new element, and they each have the opposite parity. Therefore for each odd paritied sized set, there is an even paritied sized set in the powerset of $[n+1]$.

So we can treat that as the inductive step in an inductive proof and use the subsets of [1] as a base case. $\square$

5 4.24

Let f and g be surjections from Z to Z, and let h = f g be their product (Definition 1.25). Must h also be surjective? Give a proof or a counterexample.

Counterexample: $f(x) = x$ and $g(x) = x$, $h(x) = x^2$ since there is no integer whose square is 2, there is an integer which is not in the image of $h(\mathbb{Z})$, therefore h is not a surjection.