Homework 5

Charles Swarts
Swarts2

February 2016

1 Note to Graders

For the purposes of accuracy, I will will be aiming for within 1% error. This seems arbitrarily fair.

2 5.13

2.1 question

You shuffle a standard deck of cards, then draw four cards.
(b) What is the probability all four are red?
(c) What is the probability each has a different suit?

2.2 maths answer

(b): For this question, there are 26 cards per color, 2 and 52 in the deck, so the equation is simply.

$$\frac{\binom{26}{4} * 2!}{\binom{52}{4}} = 92/833 = .110444$$

(c): For this question, there are 13 per suit, and the first card is guaranteed to be of a different suit. But, because the other suits have not been drawn upon, the probability of the next suit being one of the other unique suits is never less than 13/52. But then there are 3! to arrange the new uniquely suited cards. Here is the equation.

$$\frac{(13)^3(51-3)! * 3!}{(51)!} = 2197/20825 = .105498$$

2.3 raw simulation code

```
#You manually enter the number of trials you would like to simulate.
simTimes <- 1000000
#The sample space includes the 52 cards
sampleSpace <- 1:52
#Because in each trial we always draw 4 cards, I
#choose this matrix to keep track of them.
outcomes <- matrix(ncol=4,nrow=simTimes)
#This fills the outcomes matrix with outcomes.
for(i in 1:simTimes){
outcomes[i,] <- sample(sampleSpace,4,replace=FALSE)}
#results is a binary hit/miss tracker.
```

```
#Did we get that card or not?
results <- rep(0,simTimes)
#This loop fills records the results.
for(i in 1:simTimes){
if(outcomes[i,1]>26&outcomes[i,2]>26&outcomes[i,3]>26&outcomes[i,4]>26)
{results[i] <- 1}
if(outcomes[i,1]<27&outcomes[i,2]<27&outcomes[i,3]<27&outcomes[i,4]<27)
{results[i] <- 1}}
#Add up all the hits
positives<- sum(results)
#Divide by the total number of trials
ratio<- positives/simTimes
ratio


#This second simulator does the same thing mostly.
simTimes <- 1000000
sampleSpace <- 1:52
outcomes <- matrix(ncol=4,nrow=simTimes)
for(i in 1:simTimes){
  outcomes[i,] <- sample(sampleSpace,4,replace=FALSE)}
#This double loop compresses the outcomes down to suit.
for(i in 1:simTimes){
for(j in 1:4){
  if(outcomes[i,j]<14){outcomes[i,j] <- 1}
  if(outcomes[i,j]>13&outcomes[i,j]<27){outcomes[i,j] <- 2}
  if(outcomes[i,j]>26&outcomes[i,j]<40){outcomes[i,j] <- 3}
  if(outcomes[i,j]>39){outcomes[i,j] <- 4}}}
results <- rep(0,simTimes)
#I thought a clever way to check if there
#were four different suits in the trial
#was to check if the intersection between
#the trial and having all different suits
#would just be having all different suits
for(i in 1:simTimes){
  if(length(intersect(outcomes[i,],c(1,2,3,4)))==4){
    results[i] <-1 }}
positives<- sum(results)
ratio <- positives/simTimes
ratio
```

2.4 Simulation Results

For part b with a million trials, I got 0.110228 for the probability.

$$\frac{0.110228 - 0.110444}{0.110444} = 0.1956\%$$

For part c with a million trials, I got 0.105292 for the probability

$$\frac{0.105292 - 0.105498}{0.105498} = 0.1952\%$$

3 5.27

3.1 question

Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We consider a game with two players. Each player has a deck of 40 cards. Each player shuffles their deck, then deals seven cards, called their hand.
(b) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will player one have two lands and player two have three lands in hand?
(c) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will player two have more lands in hand than player one?

3.2 maths answer

(b): As Minje explained in class, the probability of one player getting a certain hand is the choice that they get the right number of land cards times the choice they get the right number of other cards over the choice they pick a hand of cards. Two of these independent events can be multiplied so that you can get the probability of their joint occurence. The math solution is as follows:

$$(\frac{\binom{10}{2} * \binom{30}{5}}{\binom{40}{7}}) * (\frac{\binom{20}{3} * \binom{20}{4}}{\binom{40}{7}}) = 246645/2420392 = 0.101902$$

(c): This was a much harder question in my opinion, but my classmates explained it was very simple. Just this combination of loops or in maths terms: "sums". The equation should have been as follows:

$$\sum_{i=0}^{6} \left(\sum_{k=i+1}^{7} \left(\frac{\binom{10}{i}\binom{30}{7-i}\binom{20}{k}\binom{20}{7-k}}{\binom{40}{7}^2} \right) \right) = \frac{272620541412000}{347582329473600} = 0.78433372$$

3.3 raw simulation code

```
    #Here is a section where you can set all the various constants.
trials <- 1000000
deckSize <- 40
handSize <- 7
landP1 <- 10
landP2 <- 20
otherP1 <- deckSize-landP1
otherP2 <- deckSize-landP2
#Here is where I am creating the sample space from which I sample.
P1sampleSpace <- c(rep(1,landP1),rep(0,otherP1))
P2sampleSpace <- c(rep(1,landP2),rep(0,otherP2))

#Matricies to store outcomes.
P1Hands <- matrix(ncol=handSize,nrow=trials)
P2Hands <- matrix(ncol=handSize,nrow=trials)
#Sample the space.
for(i in 1:trials){
  P1Hands[i,] <- sample(P1sampleSpace,7,replace=FALSE)
  P2Hands[i,] <-sample(P2sampleSpace,7,replace=FALSE)}
#Reset the results list.
results <- rep(0,trials)
#Record results
```

```
for(i in 1:trials){
  if(sum(P1Hands[i,])==2&sum(P2Hands[i,])==3){
    results[i] <- 1}}
#Report results.
positives <- sum(results)
ratio <- positives/trials
ratio

#Reset results.
results <- rep(0,trials)
#Record results for problem c.
for(i in 1:trials){
  if(sum(P1Hands[i,])<sum(P2Hands[i,])){
    results[i] <- 1}}
#Report results.
positives2 <- sum(results)
ratio2 <- positives2/trials
ratio2
```

3.4 Simulation Results

For part b with a million trials, I got 0.101606 for the probability

$$\frac{0.101902 - 0.101606}{0.101902} = 0.2905\%$$

For part c with a million trials, I got 0.784052 for the probability.

$$\frac{0.784334 - 0.784052}{0.784334} = 0.0359\%$$

4 5.32

4.1 question

You take a standard deck of cards, shuffle it, and remove one card. You then draw a card.

(b) What is the conditional probability that the card you draw is a red king, conditioned on the removed card being a red king?
(c) What is the conditional probability that the card you draw is a red king, conditioned on the removed card being a black ace?

4.2 maths answer

(b): Let $P(RK) = P(\text{Red King})$ $\neg RK = RK^C$

$$P(RK) = P(RK|\neg RK) * P(\neg RK) + P(RK|RK) * P(RK)$$
$$P(RK|RK) = \frac{P(RK) - P(RK|\neg RK) * P(\neg RK)}{P(RK)}$$
$$P(RK|RK) = \frac{\frac{2}{52} - \frac{2}{51} * \frac{50}{52}}{\frac{2}{52}}$$
$$P(RK|RK) = \frac{1}{51}$$

(c): Let $P(BA) = P(\text{Black Ace})$, $P(RK) = P(\text{Red King})$ $\neg BA = BA^C$

$$P(RK) = P(RK|\neg BA) * P(\neg BA) + P(RK|BA) * P(BA)$$
$$P(RK|BA) = \frac{P(BA) - P(RK|\neg BA) * P(\neg BA)}{P(BA)}$$
$$P(RK|BA) = \frac{P(BA) - P(\neg BA|RK) * P(RK)}{P(BA)}$$
$$P(RK|BA) = 1 - \frac{P(\neg BA|RK) * P(RK)}{P(BA)}$$
$$P(RK|BA) = 1 - \frac{\frac{49}{51} * \frac{2}{52}}{\frac{2}{52}}$$
$$P(RK|BA) = \frac{2}{51}$$

4.3 raw simulation code

```
#For these simulations, here are what the
#numbers symbolize for the cards
#1 is the king of hearts
#2 is the king of diamonds
#52 is the black ace

trials <- 10000000
#sampleSpace1 simulates if the king of hearts
#was the red king that was removed.
```

```
sampleSpace1 <- 2:52
#sampleSpace2 simulates is the king of diamons
#was the red king that was removed.
sampleSpace2 <- c(1,3:52)
#sampleSpace3 simulates if the ace of spaces
#was removed.
sampleSpace3 <- c(1:51)
#This makes the removing of a red king random.
controller <- 0:1
results <- rep(0,trials)
#Sample the sample space and store the outcome directly
#in results.
for(i in 1:trials){
  switch <- sample(controller,1)
  if(switch==0)
    {results[i] <- sample(sampleSpace1,1)}
  if(switch==1)
    {results[i] <- sample(sampleSpace2,1)}}
#This counts the positive outcomes.
positives <- 0
for(i in 1:trials){
  if(results[i]==1|results[i]==2){
    positives <- positives+1}}
positives
#This reports the ratios.
ratio <- positives/trials
ratio


#This is the whole simulation for condition that the
#ace of spades is missing. It's simple and works
#on the principles of the previous simulation
results <- rep(0,trials)
for(i in 1:trials){
  results[i] <- sample(sampleSpace3,1)}
positives <- 0
for(i in 1:trials){
  if(results[i]==1|results[i]==2){
    positives <- positives+1}}
ratio2 <- positives/trials
ratio2
```

4.4 Simulation Results

For part a with ten million trials I got 0.019577 for the probability.

$$\frac{0.019608 - 0.019577}{0.019608} = 0.1581\%$$

For part b with ten million trials I got 0.039202 for the probability.

$$\frac{0.039215 - 0.039202}{0.039215} = 0.0349\%$$

4.5 question

Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We will consider a deck of 40 cards, containing 10 land cards and 30 other cards. A player shuffles that deck, and draws seven cards but does not look at them. The player then chooses three of these cards at random; each of these three is a land.
(a) What is the conditional probability that the original hand of seven cards is all lands?
(b) What is the conditional probability that the original hand of seven cards contains only three lands?

4.6 maths answer

(a): because of the phrasing "conditional probability, I interpret that "original hand" means the hand described after we know that three of the cards are land cards.
$AL =$All land cards $TL =$First 3 are land cards
$RL =$ Remaining 4 are land cards $\neg RL =$ Remaining are "other" cards
$OT =$Only 3 land cards

$$P(AL|TL) = \frac{P(TL \cap RL)}{P(TL)}$$

$$P(TL \cap RL) = P(AL) = \frac{\frac{10!}{3!}}{\frac{40!}{33!}} * \frac{7!}{7!} = \frac{\binom{10}{7}}{\binom{40}{7}}$$

$$P(TL) = \frac{\binom{10}{3}}{\binom{40}{3}}$$

$$P(AL|TL) = \frac{\frac{\binom{10}{7}}{\binom{40}{7}}}{\frac{\binom{10}{3}}{\binom{40}{3}}} = \frac{1}{1887} = .0005299$$

(b):

$$P(OT|TL) = \frac{P(TL \cap OT)}{P(TL)} = \frac{P(TL \cap \neg RL)}{P(TL)}$$

$$P(OT|TL) = \frac{P(TL) * P(OT)}{P(TL)} = P(OT)$$

$$P(OT) = \frac{\binom{30}{4}}{\binom{40-3}{4}} = \frac{261}{629} = 0.414944$$

4.7 raw simulation code

```
    #Define the number of trials.
trials <- 10000000
#Define key variables.
land <- 10
other <- 30
#This simulates the first 3 cards being land cards.
land <- land-3
#This creates a sample space where lands are represented
#by ones and others by zero.
sampleSpace <- c(rep(1,land),rep(0,other))
#This matrix keeps track of outcomes.
```

```
outcomes <- matrix(nrow=trials,ncol=4)
#Sample the sample space.
for(i in 1:trials){
  outcomes[i,]<- sample(sampleSpace,4,replace=FALSE)}
#This list keeps track of results.
results <- rep(0,trials)
#This list fills the results list with the results.
for(i in 1:trials)
{
  if(sum(outcomes[i,])==4){results[i] <- 1}
}
#These lines convert the results to a probability.
positives<- sum(results)
ratio<- positives/trials
ratio

#And while we have a perfectly good outcome list,
#we may as well see how many have only three lands.
results <- rep(0,trials)
#This list fills the results list with the results.
for(i in 1:trials)
{
  if(sum(outcomes[i,])==0){results[i] <- 1}
}
#These lines convert the results to a probability.
positives<- sum(results)
ratio2<- positives/trials
ratio2
```

4.8 Simulation Results

For part a with ten million trials I got 0.019577 for the probability.

$$\frac{0.0005329 - 0.0005299}{0.0005299} = 0.5661\%$$

For part b with ten million trials I got 0.039202 for the probability.

$$\frac{0.415077 - 0.414944}{0.414944} = 0.03205\%$$

5 5.41

5.1 question

Rule 4: choose from the doors with goats behind them uniformly and at random.
Monty Hall, Rule 4: If the host uses rule 4, and shows you a goat behind door 2, what is $P(C1|G2, r4)$? Do this by computing conditional probabilities.

5.2 answer

According to the equation in the book:

$$P(C_1|G_m, r_n) = \frac{P(G_m|C_1, Rn)P(C_1)}{P(G_m|C_1, r_n)P(C_1) + P(G_m|C_2, r_n)P(C_2) + P(G_m|C_3, r_n)P(C_3)}$$

To be clear here G_m means that the host reveals a goat behind that door, it does not mean there is goat behind that door.

Also it should be obvious that $P(C_1) = P(C_3) = P(C_3) = 1/3$ because the car is put behind one of the doors uniformly and at random.

I am also going to use the framework used in the book for solving the equation. Working it out it looks like this:

To work this out, we need to know $P(G_2|C_1, r_2)$, $P(G_2|C_2, r_2)$ and $P(G_2|C_3, r_2)$. Now $P(G_2|C_2, r_2) = 0$, because the host chooses from doors with goats behind them and if the door has a car behind it, it can't also have a goat. $P(G_2|C_1, r_2) = 1/2$, because the host chooses uniformly and at random from doors with goats behind them that are not door one; if the car is behind door one, there are two such doors. $P(G_2|C_3, r_2) = 1/2$, because the host chooses uniformly and at random from doors with goats behind them that are not door one; if the car is behind door three, there are two such doors. Now Plug into the formula.

$$P(C_1|G_m, r_n) = \frac{P(G_m|C_1, Rn)(1/3)}{P(G_m|C_1, r_n)(1/3) + P(G_m|C_2, r_n)(1/3) + P(G_m|C_3, r_n)(1/3)}$$

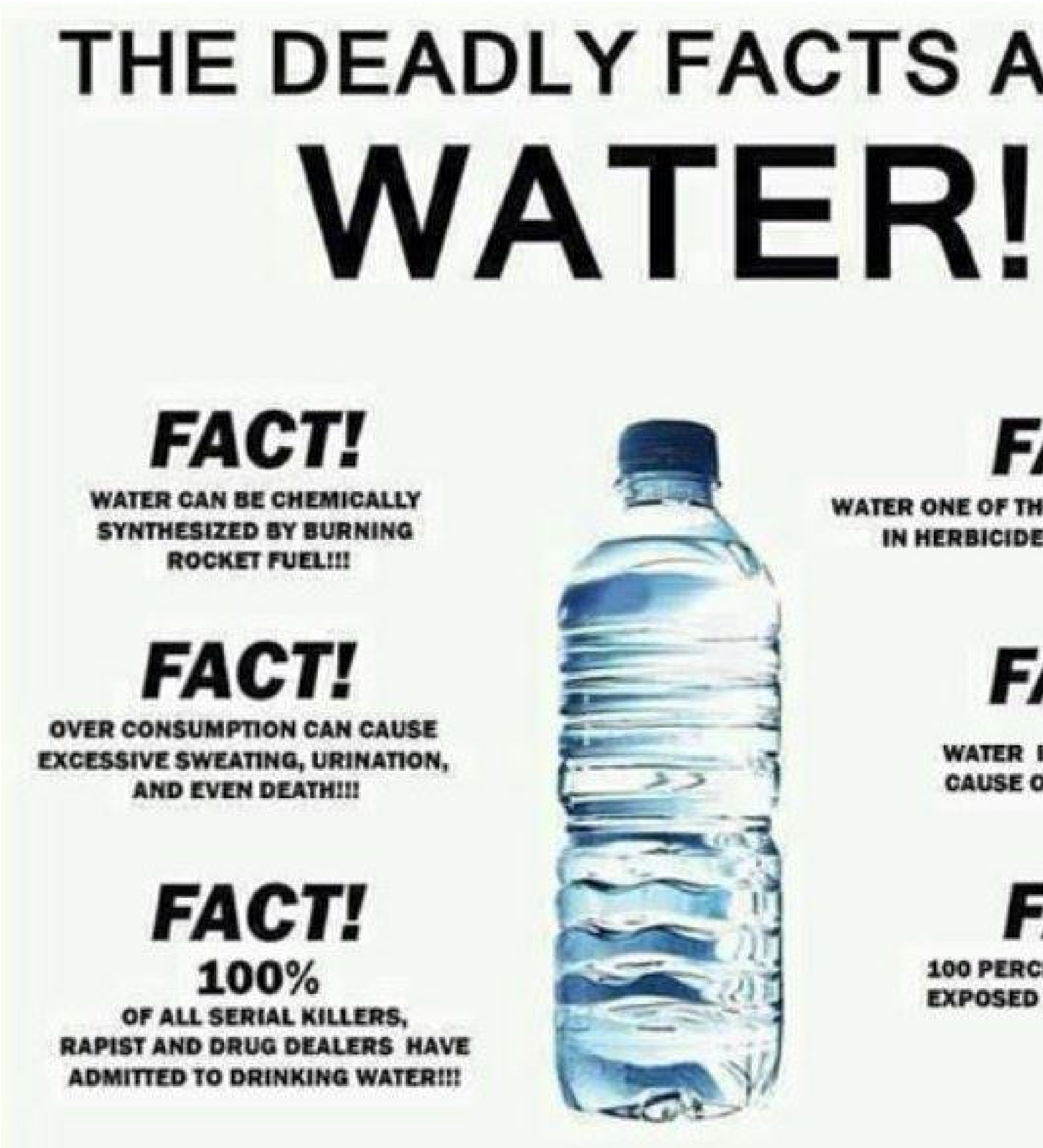
$$P(C_1|G_m, r_n) = \frac{P(G_m|C_1, Rn)}{P(G_m|C_1, r_n) + P(G_m|C_2, r_n) + P(G_m|C_3, r_n)}$$

$$P(C_1|G_m, r_n) = \frac{(1/2)}{(1/2) + 0 + (1/2)}$$

$$P(C_1|G_m, r_n) = \frac{1}{2}$$

So the implications are that if the host is using rule 4, you have a 50/50 chance that the car is behind your door.

6 Extra Credit



6.1 question

I saw the attached picture today. Extra homework credit to anyone who also submits the equations that show why the lower left statement is silly (hint: conditional probabilities)

6.2 answer

Let K = the killer/rapist/drug dealer admits drinking water.
Let W = the killer/rapist/drug dealer drinks water.

Then:

$$P(K) = P(K|W)P(W) + P(K|\neg W)P(\neg W)$$

Logically though, $P(\neg W) = 0$ because if the killer/rapist/drug-dealer never drank water then they most likely died shortly after birth. In the event the child killed the mother, the child has no ill thoughts and so can't be a killer. The child may have caused the mother to want drugs, but has not made money and so is not a real drug dealer. And the child is not a rapist because babies are too cute (and also prepubescent).

Because the probability of anthing and its compliment is 1, the killer/rapist/drug-dealer must have drank water so $P(W) = 1$

This reduces the original formula to:

$$P(K) = P(K|W) * 1 + P(K|\neg W) * 0$$

$$P(K) = P(K|W)$$

So because the fact they drink water can be deduced from being a killer, some would say their mere existence is a form of admitting they drink water. So given a killer or rapist, or a drug dealer, $P(K|W) = 1$ and therefore $P(K) = 1$ Thus the original proposition on the poster can be interpreted as being correct. But it is silly because it is a tautology as it does not matter what kind of human you choose, if they survived long enough to do something, they admit to drinking water.