MATH 347 HW7

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1 Q 14.5

$(-)$ Find a counterexample to the following false statement. "If $a_n < b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges."

Premise: counter-example.

Let $a_n = -1$ and $b_n = 0$ then $a_n < b_n$ for all n and $\sum b_n$ converges like so

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 0 = 0$$

But if we assume the series $\sum_{n=1}^{\infty} a_n$ converges to S

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} -1$$

$$S = \sum_{n=1}^{\infty} -1$$

$$S = -1 + \sum_{n=1}^{\infty} -1$$

$$S + 1 = \sum_{n=1}^{\infty} -1$$

$$S + 1 = S$$

$$1 = 0 \qquad ※$$

So $\sum a_n$ must be divergent.

Recap: we found an $\langle a \rangle$ and $\langle b \rangle$ where $a_n < b_n$ and $\sum b_n$ is convergent, but where $\sum a_n$ is divergent. ■

2 Q 14.12

If $a_n \to 0$ and $b_n \to 0$, then $\sum a_n b_n$. converges.

Premise: false due to counter-example.

Consider the sequence $c_n = \frac{1}{n}$. This sequence fails to produce a convergent series. Here is proof from the book

14.28. Example. The harmonic series. Consider $\sum_{k=1}^{\infty} 1/k$. To see that $\sum_{k=1}^{\infty} 1/k$ diverges even though $1/k \to 0$, we compare this with another divergent series whose terms approach 0. Let $\langle c \rangle = \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \cdots$; here there are 2^{j-1} copies of $1/2^j$ for each $j \geq 1$. Since the copies of $1/2^j$ for each fixed j sum to $1/2$, for each $M \in \mathbb{N}$ the partial $\sum_{k=1}^{n} c_k$ exceeds M for large enough n, and $\sum_{k=1}^{n} c_k$. The last copy of $1/2^j$ in $\langle c \rangle$ is the $2^j - 1th$ term. Thus we have $1/k > c_k$ for every k. For each n, summing n of these inequalities yields $\sum_{k=1}^{n} 1/k > \sum_{k=1}^{n} c_k$. Hence $\sum_{k=1}^{n} 1/k$ also diverges. ■

(Mathematical Thinking Problem Solving and Proofs, D'Angelo and West, Second Edition, ISBN 0-13-014412-6, page 282)

So then take $a_n = 1/\sqrt{n}$ and $b_n = 1/\sqrt{n}$

Lemma: $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$
Lemma Premise: definition of limit

The definition of a limit states that

$$\forall \epsilon \in \mathbb{R},\ \epsilon > 0,\ \ \exists N \in \mathbb{N},\ \ \forall n \in \mathbb{N},\ \ n \geq N \Rightarrow \mid a_n - L \mid < \epsilon$$

We need to show

$$\mid \frac{1}{\sqrt{n}} - 0 \mid < \epsilon$$

Since $1/\sqrt{n} > 0$, it is sufficient to show

$$\frac{1}{\sqrt{n}} < \epsilon$$

$$\frac{1}{\epsilon^2} < n$$

Therefore N exists. □

So according to the lemma, $a_n \to 0$ and $b_n \to 0$

But $a_n * b_n = c_n$, and we know $\langle c \rangle$ doesn't produce a converging series. ■

3 Q 14.13

Prove that if $\langle a \rangle$ converges, then every subsequence of $\langle a \rangle$ converges and has the same limit as $\langle a \rangle$.

Premise: Proof by definition of limit and subsequence.

By definition of subsequence, a subsequence, $\langle As \rangle$, is a subsequence of a sequence, $\langle A \rangle$, if $As_i{=}A_{f(i)}$ where $f : \mathbb{N} \to \mathbb{N}$ is an increasing function.

Lemma: $f(i) \geq i$
Lemma Premise: induction on k

Base Case: $k = 1$

$$f(k) = f(1)$$

$f(1)$ must be a natural number, so $f(1) \geq 1$. ✓

Inductive Step
Hypothesis: $f(k) \geq k$

By definition of increasing function

$$k + 1 > k \quad \Rightarrow \quad f(k + 1) > f(k)$$

$$f(k + 1) > f(k) \geq k$$

$$f(k + 1) > k$$

$$f(k + 1) \geq k + 1$$

So by the principle of induction, $f(i) \geq i$ $\square$

The definition of a sequence converging is the same as it having a limit. The definition of a limit, $L \in \mathbb{R}$, for an ordered sequence of real numbers, $\langle A \rangle$ is

$$\forall \epsilon \in \mathbb{R} \ \ , \epsilon > 0, \ \ \exists N \in \mathbb{N}, \ \ \forall n \in \mathbb{N}, \ \ n \geq N \Rightarrow \mid A_n - L \mid < \epsilon$$

So by the lemma,

$$f(N) \geq N$$

And by definition of increasing

$$n \geq N \quad \Rightarrow \quad f(n) \geq f(N) \geq N$$

So

$$n \geq N \Rightarrow f(n) \geq N \Rightarrow \mid A_{f(n)} - L \mid < \epsilon$$

$$n \geq N \Rightarrow f(n) \geq N \Rightarrow \mid As_n - L \mid < \epsilon$$

This means that any subsequence of $\langle A \rangle$ must also converge and have L as its limit. $\blacksquare$

4 Q 14.30

Let $\langle x \rangle$ be the sequence given by $x_1 = 1$ and $x_{n+1} = 1/(x_1 + \cdots + x_n)$ for $n \geq 1$. Prove that $\langle x \rangle$ converges, and obtain the limit.

Premise: Use MCT and infinity trick.

Lemma: First let us give a lower bound by showing that $\forall \, n, \ \ x_n > 0$

Lemma Premise: Proof by strong induction on k

Base Case: $k = 1 \quad x_k = x_1 = 1 \quad \checkmark$

Induction: Assume all x_1-x_k are positive, then their sum is positive.

$$x_{k+1} = \frac{1}{\sum_{i=1}^{k} x_i}$$

Since the the numerator and denominator on the RHS are positive, x_{k+1} is positive. $\square$

So we see there is a lower bound on x_n. Next we prove the x_n is decreasing $\forall\, n > 2$.

$$x_{n+1} < x_n$$

$$\frac{1}{\sum_{j=1}^{n} x_j} < \frac{1}{\sum_{i=1}^{n-1} x_i}$$

$$\frac{1}{\sum_{j=1}^{n-1} x_j + x_n} < \frac{1}{\sum_{i=1}^{n-1} x_i}$$

$$\sum_{i=1}^{n-1} x_i < \sum_{j=1}^{n-1} x_j + x_n$$

$$0 < x_n$$

Since the lemma showed all terms are positive, the last line is true. This implies the first line is also true. Since the next element in the sequence is always less than the current, x_n must be decreasing $\forall\, n > 2$.

By MCT, a decreasing sequence with a lower bound converges, so $\langle x \rangle$ converges.

Now we can use the infinity trick to find the limit.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n} x_i}$$

$$= \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n-1} x_i + x_n} = \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n} x_i + x_n}$$

So

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n} x_i + x_n}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\frac{1}{x_n} + x_n}$$

$$0 = \lim_{n\to\infty} \frac{1}{x_n(\frac{1}{x_n} + x_n)}$$

$$0 = \lim_{n\to\infty} \frac{1}{1 + x_n \cdot x_n}$$

$$\lim_{n\to\infty} 1 + x_n \cdot x_n = 1$$

$$\lim_{n\to\infty} x_n \cdot x_n = 0$$

$$\lim_{n\to\infty} x_n = 0$$

Therefore according to the MCT and the infinity trick, $\langle x \rangle$ converges to 0. $\blacksquare$

5 Q 14.44

Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Use this to obtain upper and lower bounds on $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Premise: Derive an easier partial sum formula. Then define a sequence from each of the given series and use them to give upper and lower bound on $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Lemma: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Lemma Premise: Proof by induction on k.

Base case: $k = 1$

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$
$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1}$$
$$\frac{1}{2} = \frac{1}{2} \qquad \checkmark$$

Inductive Step: Hypothesis $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Then by the principle of induction, the hypothesis of the lemma is true. $\square$

So using the lemma

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \lim_{n\to\infty} \frac{n}{n+1}$$
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} 1 - \frac{1}{n+1}$$
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 - \lim_{n\to\infty} \frac{1}{n+1}$$

We know from class that

$$0 < \frac{1}{n+1} < \frac{1}{n}$$
$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{1}{n+1} \le \lim_{n\to\infty} \frac{1}{n}$$
$$0 \le \lim_{n\to\infty} \frac{1}{n+1} \le 0$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

So

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Now that we know this, we can do a sequence analysis. We use the property of real numbers that

$$a < b \qquad c < d \qquad \Rightarrow \qquad a + c < b + d$$

So if we have two sequences, $\langle A \rangle$ and $\langle B \rangle$ where $\forall\, i \in \mathbb{N},\ A_i < B_i$ then the series $\sum_{i=0}^{\infty} A_i < \sum_{i=0}^{\infty} B_i$.

Let $\langle A \rangle$ be made of individual terms from $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, and let $\langle B \rangle$ be made of individual terms from $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\begin{aligned} \frac{1}{i(i+1)} &< \frac{1}{i^2} \\ \frac{1}{(i+1)} &< \frac{1}{i} \\ 0 &< 1 \qquad\qquad \checkmark \end{aligned}$$

So

$$\begin{aligned} \sum_{i=0}^{\infty} A_i &< \sum_{i=0}^{\infty} B_i \\ 1 &< \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

To get the other bound, we shift $\langle B \rangle$ one over.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \sum_{n=2}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

Let $\langle B \rangle$ now be the individual terms of the series on the RHS.

$$\begin{aligned} \frac{1}{(i+1)^2} &< \frac{1}{i(i+1)} \\ \frac{1}{(i+1)} &< \frac{1}{i} \\ 0 &< 1 \qquad\qquad \checkmark \end{aligned}$$

So

$$\begin{aligned} \sum_{i=0}^{\infty} B_i &< \sum_{i=0}^{\infty} A_i \\ \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} &< 1 \\ 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} &< 2 \\ \sum_{n=1}^{\infty} \frac{1}{(n)^2} &< 2 \end{aligned}$$

So using the computation for $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, and some facts about sequences, we derived an upper bound, 2, and a lower bound, 1, for $\sum_{n=1}^{\infty} \frac{1}{n^2}$. $\blacksquare$