Homework 4

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1 5.4

1.1 question

You read the book, so you know that King Lear had family problems. As a result, he decides to allocate two provinces to one daughter, one province to another daughter, and no provinces to the third. Because he's a bad problem solver, he does so at random. What is the space of outcomes?

1.2 answer

Outcome space for King Lear. (Note that this question is an outcome space question and not an event space question)

There are 3! ways to arrange the 3 provinces and 3! ways to pick daughters, so there are 36 outcomes in the outcome space. Let the provinces be represented by $\{P_1, P_2, P_3\}$ and Lear's daughters be represented by $\{G, R, C\}$.

Finally let a daughter being next to a province or two provinces enclosed by parenthesis mean she owns that/those province(s).

$$\{P_1, P_2, P_3\} \qquad \{G, R, C\}$$

$$\begin{array}{lll}
\{G(P_1P_2), R(P_3), C\} & \{G(P_2P_1), R(P_3), C\} & \{G(P_1P_3), R(P_2), C\} \\
\{G(P_3P_1), R(P_2), C\} & \{G(P_2P_3), R(P_1), C\} & \{G(P_3P_2), R(P_1), C\} \\
\{G(P_1P_2), C(P_3), R\} & \{G(P_2P_1), C(P_3), R\} & \{G(P_1P_3), C(P_2), R\} \\
\{G(P_3P_1), C(P_2), R\} & \{G(P_2P_3), C(P_1), R\} & \{G(P_3P_2), C(P_1), R\}
\end{array}$$

$$\begin{array}{lll}
\{C(P_1P_2), G(P_3), R\} & \{C(P_2P_1), G(P_3), R\} & \{C(P_1P_3), G(P_2), R\} \\
\{C(P_3P_1), G(P_2), R\} & \{C(P_2P_3), G(P_1), R\} & \{C(P_3P_2), G(P_1), R\} \\
\{C(P_1P_2), R(P_3), G\} & \{C(P_2P_1), R(P_3), G\} & \{C(P_1P_3), R(P_2), G\} \\
\{C(P_3P_1), R(P_2), G\} & \{C(P_2P_3), R(P_1), G\} & \{C(P_3P_2), R(P_1), G\}
\end{array}$$

$$\begin{array}{lll}
\{R(P_1P_2), G(P_3), C\} & \{R(P_2P_1), G(P_3), C\} & \{R(P_1P_3), G(P_2), C\} \\
\{R(P_3P_1), G(P_2), C\} & \{R(P_2P_3), G(P_1), C\} & \{R(P_3P_2), G(P_1), C\} \\
\{R(P_1P_2), C(P_3), G\} & \{R(P_2P_1), C(P_3), G\} & \{R(P_1P_3), C(P_2), G\} \\
\{R(P_3P_1), C(P_2), G\} & \{R(P_2P_3), C(P_1), G\} & \{R(P_3P_2), C(P_1), G\}
\end{array}$$

2 5.8

2.1 question

At a particular University, 1/2 of the students drink alcohol and 1/3 of the students smoke cigarettes. (a) What is the largest possible fraction of students who do neither? (b) It turns out that, in fact, 1/3 of the students do neither. What fraction of the students does both?

2.2 answer

(a): Let D represent the students who drink, and let S represent the students who smoke. And let their proportionality among the population be represented as a probability that the student does that activity, then:

$$P(D) = \frac{1}{2} \qquad P(S) = \frac{1}{3} \qquad P(\neg D \cup \neg P) = ?$$
$$(S \subset D) \rightarrow P(D \cup S) = P(D)$$
$$P(\neg D \cup \neg S) = 1 - P(D \cup S) = 1 - P(D) = 1 - \frac{1}{2} = \frac{1}{2}$$

Or, basically, if all the smokers drink, then the population of non smokers is equal to the population of non drinkers which in this case is $\frac{1}{2}$ of the population. (b): Same symbolism as before. Except now the teetotalers are set at $\frac{1}{3}$. Then using the rules of probability, we find the answer for the vice-loving students is:

$$P(D) = \frac{1}{2} \qquad P(S) = \frac{1}{3} \qquad P(\neg D \cup \neg P) = 1/3$$
$$\Omega = P(\neg D \cup \neg P) + P(D) + P(S) - P(D \cap S)$$
$$1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} - P(D \cap S)$$
$$\frac{1}{6} = P(D \cap S)$$

3 5.13

3.1 question

You shuffle a standard deck of cards, then draw four cards. (a) What is the probability all four are the same suit? (b) What is the probability all four are red? (c) What is the probability each has a different suit?

3.2 answer

(a): For this question, there are 13 cards per suit. But the first card is logically always in the suit as the last card. So really I am not picking 4, but 3 cards. So, the equation looks like this:

$$\frac{(12)!(51-3)!}{(12-3)!(51)!} = \frac{44}{4165} = 0.01056$$

(b): For this question, there are 26 cards per color. And the bit of logic from part a holds, the first card will have the same color as the zeroth card. Here is the equation:

$$\frac{(25)!(51-3)!}{(25-3)!(51)!} = \frac{92}{833} = 0.11044$$

(c): For this question, there are 13 per suit, and the first card is guaranteed to be of a different suit. But, because the other suits have not been drawn upon, the probability is never less than 13/52. Here is the equation.

$$\frac{(13)^3(51-3)!}{(51)!} = 2197/124950 = .01758$$

4 5.25

4.1 question

An airline sells T tickets for a flight with S seats, where $T > S$. Passengers turn up for the flight independently, and the probability that a passenger with a ticket will turn up for a flight is pt. The pilot is eccentric, and will fly only if precisely E passengers turn up, where $E < S$. Write an expression for the probability the pilot will fly.

4.2 answer

For this question, the probability of the first E passengers showing up is pt^E and then the probability for the rest of them to not show up is $(1-pt)^{(T-E)}$. This probability "chain" can be scrambled into $T!$ logically equivalent strings of probabilities. But since the passengers who do show up are otherwise homogeneous, you have to divide by $E!$ to unscramble the people who showed up. Then you have to divide by $(T-E)!$ to unscramble the people who didn't show up. The final formula is as follows:

$$\frac{(T)! * pt^E * (1-pt)^{(T-E)}}{(E)! * (T-E)!}$$

5 5.27 b,c

5.1 question

Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We consider a game with two players. Each player has a deck of 40 cards. Each player shuffles their deck, then deals seven cards, called their hand.
(b) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will player one have two lands and player two have three lands in hand?
(c) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will player two have more lands in hand than player one?

5.2 answer

(b): The probability "chain" for this one goes as follows, player one has $\frac{10!*(40-2)!}{40!*(10-2)!}$ ways of getting two land cards on the first two goes. That gets multiplied by the probability "chain" that the next 5 after that are "other" cards. That chain equals $\frac{30!*(40-2-5)!}{(30-5)!(40-2)!}$. Then there are 7! ways to scramble those chains, so I am going to multiply the whole thing by 7!. Finally I am going to unscramble the "other" cards because they are indistinguishable, and also unscramble the land cards for the same reason. To do this, I divide by $(5! * 2!)$. The formula is as follows:

$$\begin{aligned}
&\frac{7! * 10! * (40-2)! * 30! * (40-2-5)!}{2! * 5! * 40! * (10-2)! * (40-2)! * (30-5)!} \\
&= \frac{((40-7)!7!) * 10! * 30!}{40! * (2!(10-2)!) * (5!(30-5)!)} \\
&= \frac{\binom{10}{2} * \binom{30}{5}}{\binom{40}{7}} \\
&= 16443/47804 = .34396
\end{aligned}$$

(c): So after spending hours tying to do the math for this one, I found I was unable to because it was very hard. So I decided to use my God given right as a CS major to write a program that does all the work for me.

```
#The first step is to create a truly massive truth table
#This sets up a truth table that models "hits" aka getting a land card
#vs "misses" aka not getting a land card.

#Here you can set the number of cards in the hand.
nhands <- 7
deckSize <- 40

#These two are the number of land cards that each player has in their deck.
P1Land <- 20
P2Land <- 10
deckConstant <- 1
for(i in 0:(nhands-1))
{
  deckConstant <- deckConstant*(deckSize-i)^2
}
P1Other <- deckSize-P1Land
```

```
P2Other <- deckSize-P2Land
#Here is a simple matrix of the probabilities and their complements.
probMatrix <- matrix(ncol=4,nrow=1)
probMatrix[1,1] <- P1Land
probMatrix[1,2] <- P1Other
probMatrix[1,3] <- P2Land
probMatrix[1,4] <- P2Other
numcol <- nhands*2
numrow <- 4^nhands
ttable <- NULL
ttable <- matrix(nrow=numrow,ncol=numcol)


#This loop generates the truth table.
for(i in 1:numcol)
{
  curTruth <- TRUE
  for(j in 1:numrow)
  {
    ttable[j,i] <- curTruth
    if((j%%(numrow/(2^i)))==0){
      curTruth <- !(curTruth)}
  }
}
ttable

#Viable candidates are outcomes where the player with more cards
#in their deck has come out with more cards in their hand.
ViableCandidates <- c(0)

for(i in 1:numrow)
{
  Player1Count <- 0
  Player2Count <- 0
  for(j in 1:numcol)
  {
    if((j%%2)==1)
    {
      if(ttable[i,j]==TRUE){
      Player1Count <- Player1Count+1}
    }
    else
    {
      if(ttable[i,j]==TRUE){
      Player2Count <- Player2Count+1}
  }}
  if((Player1Count>Player2Count)){
  ViableCandidates <- c(ViableCandidates,i)}
}

ViableCandidates <- ViableCandidates[2:length(ViableCandidates)]
ViableCandidates
```

```
ttable <- ttable[ViableCandidates,1:numcol]
ttable

#This numeric table is the first step in getting to a more concise matrix
#for which we can plug values into a formula.
numericTable <- NULL
if(length(nrow(ttable))!=1)
{
special <- 1
numericTable <- matrix(nrow=1,ncol=4)
}
if(length(nrow(ttable))==1){
special <- nrow(ttable)
numericTable <- matrix(nrow=nrow(ttable),ncol=4)
}
numericTable
for(i in 1:special)
{
  for(j in 1:4)
  {
    numericTable[i,j] <- 0
  }
}
numericTable
ttable
#Ignore this, R has problems with lists that want to be vectors.
if(special==1)
{
  numericTable[1,1] <- 1
  numericTable[1,4] <- 1
}
if(special!=1){
#This is the translation loop from truth states to numbers.
for(i in 1:special)
{
    for(j in 1:numcol)
  {
    if((j%%2)==1)
    {
      if(ttable[i,j]==TRUE){
        numericTable[i,1] <- numericTable[i,1]+1}
      else{
        numericTable[i,2] <- numericTable[i,2]+1}
    }
    else
    {
      if(ttable[i,j]==TRUE){
        numericTable[i,3] <-numericTable[i,3]+1}
      else{
        numericTable[i,4] <-numericTable[i,4]+1}
}}}}
numericTable
```

```
colnames(numericTable) <- c("P1 Land","P1 Other","P2 Land","P2 Other")
TrackerTable <- matrix(nrow=(((nhands+1)*(nhands)/2)),ncol=5)
for(i in 1:((nhands+1)*(nhands)/2))
{
  for(j in 1:5)
  {
    TrackerTable[i,j] <- 0
  }}

value <- nhands
count <- nhands
#Tracker Table is another intermediary, It's job is to finalize
#The number of unique solutions with the same probability.
for(i in 1:((nhands+1)*(nhands)/2))
{
  TrackerTable[i,1] <- value
  count <- count -1

  TrackerTable[i,3] <- (count)

  if(count==0){count <- (value-1); value <- value-1}

  TrackerTable[i,2] <-nhands-TrackerTable[i,1]
  TrackerTable[i,4] <-nhands-TrackerTable[i,3]
}
TrackerTable
#More counting.
for(i in 1:nrow(TrackerTable))
{
  for(j in 1:nrow(numericTable))
  if((TrackerTable[i,1]==numericTable[j,1])&(TrackerTable[i,3]==numericTable[j,3]))
  {TrackerTable[i,5] <- TrackerTable[i,5]+1}
}
TrackerTable

FinalTable <- matrix(nrow=nrow(TrackerTable),ncol=5)
for(i in 1:nrow(FinalTable))
{for(j in 1:5)
{
 FinalTable[i,j] <- 1
}}
FinalTable
#Final Table holds the multiplication terms that need to be added together.
for(i in 1:nrow(FinalTable))
{for(j in 1:4)
{
  if(TrackerTable[i,j]!=0)
  {
  for(k in 0:(TrackerTable[i,j]-1))
  {
    FinalTable[i,j] <- FinalTable[i,j]*(probMatrix[1,j]-k)
  }}}}
```

```
for(i in 1:nrow(TrackerTable))
{
  for(j in 1:4)
  {FinalTable[i,5] <- FinalTable[i,5]*FinalTable[i,j]}
  FinalTable[i,5] <- FinalTable[i,5]*TrackerTable[i,5]
}
FinalTable
TrackerTable
#Sum all the outcomes weighted by their probabilities.
Answer <- sum(FinalTable[,5])
Answer <- Answer/deckConstant
Answer
```

And the final answer came out to be that the probability of player two having more lands in his hand than player one given that player two has 20 land cards while player one has only 10 is... drum roll please... 0.7843337

And I would just like to say that I believe that someone thought the answer to this question was obvious. That maybe player two was just twice as likely, but that isn't the case. It's actually mildly complex. You can test my code, I have manually tested it for 1 2 and 3 draws.

6 5.30

6.1 question

You take a standard deck of cards, shuffle it, and remove both red kings. You then draw a card.
(a) Is the event card is red independent of the event card is a queen?
(b) Is the event card is black independent of the event card is a king?

6.2 answer

(a): A and B are independent if $P(A) * P(B) = P(A \cap B)$

$P(A \cap B) = P(\text{red queen})$ $\quad P(A) = P(\text{queen})$ $\quad P(B) = P(\text{Red})$

$$P(A) * P(B) = P(A \cap B)$$
$$\frac{4}{50} * \frac{24}{50} = \frac{2}{50}$$
$$\frac{96}{50} \neq 2$$

So they are not independent.
(b): A and B are independent if $P(A) * P(B) = P(A \cap B)$
$P(A \cap B) = P(\text{black king})$ $\quad P(A) = P(\text{king})$ $\quad P(B) = P(\text{black})$

$$P(A) * P(B) = P(A \cap B)$$
$$\frac{2}{50} * \frac{26}{50} = \frac{2}{50}$$
$$\frac{26}{50} \neq 1$$

So they are not independent.

7 5.32

7.1 question

You take a standard deck of cards, shuffle it, and remove one card. You then draw a card.
(a) What is the conditional probability that the card you draw is a red king, conditioned on the removed card being a king?
(b) What is the conditional probability that the card you draw is a red king, conditioned on the removed card being a red king?
(c) What is the conditional probability that the card you draw is a red king, conditioned on the removed card being a black ace?

7.2 answer

(a): Let $P(K) = P(\text{King}),\ \ P(RK) = P(\text{Red King})\quad \neg K = K^C$

then by definition

$$P(RK) = P(RK|\neg K) * P(\neg K) + P(RK|K) * P(K)$$
$$P(RK|K) = \frac{P(RK) - P(RK|\neg K) * P(\neg K)}{P(K)}$$
$$P(RK|K) = \frac{\frac{2}{52} - \frac{2}{51} * \frac{48}{52}}{\frac{4}{52}}$$
$$P(RK|K) = \frac{1}{34}$$

(b): Let $P(RK) = P(\text{Red King})\ \ \neg RK = RK^C$

$$P(RK) = P(RK|\neg RK) * P(\neg RK) + P(RK|RK) * P(RK)$$
$$P(RK|RK) = \frac{P(RK) - P(RK|\neg RK) * P(\neg RK)}{P(RK)}$$
$$P(RK|RK) = \frac{\frac{2}{52} - \frac{2}{51} * \frac{50}{52}}{\frac{2}{52}}$$
$$P(RK|RK) = \frac{1}{51}$$

(c): Let $P(BA) = P(\text{Black Ace}),\ \ P(RK) = P(\text{Red King})\quad \neg BA = BA^C$

$$P(RK) = P(RK|\neg BA) * P(\neg BA) + P(RK|BA) * P(BA)$$
$$P(RK|BA) = \frac{P(BA) - P(RK|\neg BA) * P(\neg BA)}{P(BA)}$$
$$P(RK|BA) = \frac{P(BA) - P(\neg BA|RK) * P(RK)}{P(BA)}$$
$$P(RK|BA) = 1 - \frac{P(\neg BA|RK) * P(RK)}{P(BA)}$$
$$P(RK|BA) = 1 - \frac{\frac{49}{51} * \frac{2}{52}}{\frac{2}{52}}$$
$$P(RK|BA) = \frac{2}{51}$$

8 5.37

8.1 question

Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We will consider a deck of 40 cards, containing 10 land cards and 30 other cards. A player shuffles that deck, and draws seven cards but does not look at them. The player then chooses three of these cards at random; each of these three is a land.
(a) What is the conditional probability that the original hand of seven cards is all lands?
(b) What is the conditional probability that the original hand of seven cards contains only three lands?

8.2 answer

(a): because of the phrasing ”conditional probability, I interpret that ”original hand” means the hand described after we know that three of the cards are land cards.
AL =All land cards TL =First 3 are land cards
RL = Remaining 4 are land cards $\neg RL$ = Remaining are ”other” cards
OT =Only 3 land cards

$$P(AL|TL) = \frac{P(TL \cap RL)}{P(TL)}$$

$$P(TL \cap RL) = P(AL) = \frac{\frac{10!}{3!}}{\frac{40!}{33!}} * \frac{7!}{7!} = \frac{\binom{10}{7}}{\binom{40}{7}}$$

$$P(TL) = \frac{\binom{10}{3}}{\binom{40}{3}}$$

$$P(AL|TL) = \frac{\frac{\binom{10}{7}}{\binom{40}{7}}}{\frac{\binom{10}{3}}{\binom{40}{3}}} = \frac{1}{1887} = .0005299$$

(b):

$$P(OT|TL) = \frac{P(TL \cap OT)}{P(TL)} = \frac{P(TL \cap \neg RL)}{P(TL)}$$

$$P(OT|TL) = \frac{P(TL) * P(OT)}{P(TL)} = P(OT)$$

$$P(OT) = \frac{\binom{30}{4}}{\binom{40-3}{4}} = \frac{261}{629} = 0.414944$$

9 5.39

9.1 question

A student takes a multiple choice test. Each question has N answers. If the student knows the answer to a question, the student gives the right answer, and otherwise guesses uniformly and at random. The student knows the answer to 70% of the questions. Write K for the event a student knows the answer to a question and R for the event the student answers the question correctly.
(a) What is $P(K)$?
(b) What is $P(R|K)$?
(c) What is $P(K|R)$, as a function of N?
(d) What values of N will ensure that $P(K|R) > 99\%$?

9.2 answer

(a) If probabilities describe proportions of outcomes, then $P(K) = P($ Proportion of questions the student knows the answer to) which implies that $P(K) = 70\%$ aka $P(K) = .7$

(b):If the student answers correctly all the questions they know the answer to, then given that the student knows an answer to a question, the student will always get the question right. This implies that $P(R|K) = 100\% or P(R|K) = 1.0$

(c):Firstly, if we are given the probability that the student gets a problem right despite not knowing the answer, then that would mathematically be described as $P(R|\neg K)$ or in English as the random guess success ratio. This is generally given by $1/N$. Secondly $\neg K = K^C$.

$$P(K|R) = \frac{P(R|K) * P(K)}{P(R)}$$

$$P(R) = P(R|K) * P(K) + P(R|\neg K) * P(\neg K)$$

$$P(K|R) = \frac{P(R|K) * P(K)}{P(R|K) * P(K) + P(R|\neg K) * P(\neg K)}$$

$$P(K|R) = \frac{.7}{.7 + \frac{1}{N} * .3}$$

(d):

$$P(K|R) = 0.99 \leq \frac{.7}{.7 + \frac{1}{N} * .3}$$

$$0.99 \leq \frac{.7}{.7 + \frac{1}{N} * .3} \qquad 0.99 \leq \frac{.7 * N}{.7 * N + .3}$$

$$0.99 \leq \frac{N}{N + \frac{.3}{.7}} \qquad 0.99 * (N + \frac{3}{7}) \leq N$$

$$.99 * \frac{3}{7} \leq N(1 - .99) \qquad \frac{.99}{.01} * \frac{3}{7} \leq N$$

$$N \geq 99 * \frac{3}{7} = 42.42857 \qquad N \geq 43$$