# MATH 413 HW8

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## 1 Q 1

Prove that the partition function $p(n)$(=number of partitions of $n$) satisfies $p(n+1) > p(n)$.

Since $p(0) = 1 = p(1)$, this isn't true. However for $n > 1$,

Proof: by induction on $k$

Base case:

$$p(1) = 1 < 2 = p(2)$$

Inductive case:

$$p(k) = \mid \text{partitions of k} \mid = \sum_{i=1}^{k} \mid \text{partitions of k of size i} \mid$$

For each partition of size $i$, we can add one more element, 1, into the partition-tuple to get a partition of size $i+1$ with $k+1$ elements.

This process of "adding one to the end" produces a valid and unique partition in $p(k+1)$.

Therefore we have an injection from elements of $p(k)$ to elements of $p(k+1)$. This implies that $p(k) \leq p(k+1)$

Finally we know there is one partition in $p(k)$ which is size 1 and contains $k$ as it's sole element. To this, we can "row append" 1 so that we get a partition of size 1 with $k+1$ as it's sole element which is a valid partition in $p(k+1)$ which cannot be produced by the previous process and is therefore unique.

So as an overview, the process outlined takes each element in $p(k)$ and give $p(k)+1$ valid unique partitions in $p(k+1)$. Therefore there must be strictly more partitions in $p(k+1)$ than in $p(k)$.

By the principle of induction, $p(n) < p(n+1)$. ■

## 2 2

For each integer $n > 2$ determine a self-conjugate partition of $n$ that has at least two parts.

Take $n$ and split it up into $1, \lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, (n - 2 \cdot \lfloor \frac{n-1}{2} \rfloor)$. Label these $\{1, a, b, c\}$ respectively.

It is obvious that $1 + \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + ((n-1) - 2 \cdot \lfloor \frac{n-1}{2} \rfloor) = n$

We can use these to make a Young diagram where we make one row of size $a+1$ then a row of size $1+c$ and then $b-1$ rows of size 1. Like so...

| | | | |
|---|---|---|---|
| 1 | $a$ | $a$ | $a$ |
| $b$ | $c$ | | |
| $b$ | | | |
| $b$ | | | |

Here are the base cases for what this will look like.

Case: $n = 3$

| | |
|---|---|
| 1 | $a$ |
| $b$ | |

Case: $n = 4$

| | |
|---|---|
| 1 | $a$ |
| $b$ | $c$ |

For cases where $n > 4$ if $n$ is odd, then $c$ will be 0 and there will be a self conjugate diagram for the following reasons.

We give the diagram coordinates where $(1, 1)$ is the top left. The first coordinate is the how far right, and the second is how far down. For the diagram to be self conjugate, there needs to be a square at $(x, y)$ for every square at $(y, x)$. There will be a square at $(1, 1)$ for the square at $(1, 1)$ and an "$a$" square at $(x, 1)$ for every "$b$" at $(1, x)$. So in these cases it will be self conjugate.

If $n$ is even, then $c$ will be 1 and there will be a self conjugate diagram for the previous reasons and for the fact that there will be a "$c$" square at $(2, 2)$ for the "$c$" square at $(2, 2)$. ■

# 3 Q 3

Prove that the number of partitions of $n$ in which no part appears exactly once is equal to the number of partitions of $n$ with no parts congruent to 1 or 5 (mod6).

let number refer to natural number.

Lemma: all numbers not congruent to 1 or 5 mod 6 are all numbers which are multiples of 2 or 3.

Proof: It is easily seen that we can generate all multiples of 2 by starting with
0 and adding some multiple of 6,
2 and adding some multiple of 6,
4 and adding some multiple of 6

It is also easily seen that we can generate all multiple of 3 by starting with
0 and adding some multiple of 6,
3 and adding some multiple of 6,

since 0,2,3,4 are all the numbers which a number can be congruent to mod 6 except for 1 or 5, all numbers not congruent to 1 or 5 mod 6 must be congruent to one of these and thus must be a multiple of 2 or 3. □

The generating series for the partitions for which no part appears exactly once is

$$\begin{aligned}
&1 + x^2 + x^3 + x^4 + x^5 + ...\\
&1 + x^4 + x^6 + x^8 + x^{10} + ...\\
&1 + x^6 + x^9 + x^{12} + x^{15} + ...\\
&\vdots
\end{aligned}$$

For the general case, we have a generating series

$$\begin{aligned}
S &= 1 + x^{2i} + x^{3i} + x^{4i}...\\
S - 1 &= x^{2i} + x^{3i} + x^{4i}...\\
\frac{S-1}{x^{2i}} &= 1 + x^i + x^{2i} + x^{3i} + x^{4i}...\\
\frac{S-1}{x^{2i}} &= \frac{1}{1-x^i}\\
S &= \frac{x^{2i} - x^i + 1}{1 - x^i}
\end{aligned}$$

So we have a produce of these generating series to get the generating series for the partions in which no part appears exactly once

$$\prod_{i=0}^{\infty} \frac{x^{2i} - x^i + 1}{1 - x^i}$$

The generating series for partitions which are congruent to 1 or 5 mod 6 is the same as the generating series for multiples of 2 or 3. Which is

$$\begin{aligned}
&1 + x^2 + x^4 + x^6 + x^8 + ...\\
&1 + x^3 + x^6 + x^9 + x^{12} + ...\\
&1 + x^4 + x^8 + x^{12} + x^{16} + ...\\
&1 + x^6 + x^{12} + x^{18} + x^{24} + ...\\
&\vdots
\end{aligned}$$

To get the formula for this generating series, we need to consider that it is made of all the ways to have multiples of 2 or 3. Since those overlap, we use PIE. We take the generating series for multiples of 2

$$\frac{1}{1 - x^{2i}}$$

and the generating series for the multiples of 3

$$\frac{1}{1 - x^{3i}}$$

and the generating series for $2 \cdot 3$, the intersection

$$\frac{1}{1 - x^{6i}}$$

. Put the two sets together and take away their intersection.

$$\prod_{i=0}^{\infty} \frac{1 - x^{6i}}{(1 - x^{3i})(1 - x^{2i})}$$

To prove the generating series are the same, we can see if each term is exactly equal in the product.

$$\frac{x^{2i}-x^i+1}{1-x^i} = \frac{1-x^{6i}}{(1-x^{3i})(1-x^{2i})}$$

$$\frac{(x^{2i}-x^i+1)(1+x^i)}{(1-x^i)(1+x^i)} = \frac{(1-x^{3i})(1+x^{3i})}{(1-x^{3i})(1-x^{2i})}$$

$$\frac{(x^{2i}-x^i+1)(1+x^i)}{(1-x^{2i})} = \frac{(1+x^{3i})}{(1-x^{2i})}$$

$$\frac{(1+x^{3i})}{(1-x^{2i})} = \frac{(1+x^{3i})}{(1-x^{2i})} \quad \checkmark$$

So they are the same generating series, so the number of partitions of $n$ must be the same under each condition. ■

# 4 Q 4

By considering partitions with distinct (that is, non-repeated) parts, prove that
$\prod_{k>=1}(1+x^k) = 1+\sum_{m>=1}(x^{m(m+1)/2}/(\prod_{k=1\ldots m}(1-x^k)))$ (Hint: look for a "maximal triangle" rather than a Durfee square.)

First we re-write the prompt as follows

$$\prod_{k=1}^{\infty}(1+x^k) = 1+\sum_{m=1}^{\infty}(x^{m(m+1)/2})\left(\prod_{k=1}^{m}\frac{1}{(1-x^k)}\right)$$

We start with a partition with only elements 1 through $m$ it looks like this:

| | | | | |
|---|---|---|---|---|
| $m$ | $m$ | $m$ | $m$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | ... |
| $m-1$ | $m-1$ | $m-1$ | ... | |
| $\vdots$ | $\vdots$ | $\vdots$ | ... | |
| 1 | | | | |
| $\vdots$ | | | | |
| 1 | | | | |

If you take the transpose of this, it looks like this

| | | | | | | |
|---|---|---|---|---|---|---|
| $m$ | ... | $m-1$ | ... | 1 | ... | 1 |
| $m$ | ... | $m-1$ | ... | | | |
| $m$ | ... | $m-1$ | ... | | | |
| $m$ | ... | ... | ... | | | |
| ... | .... | | | | | |

In this partition, there are $m$ rows. The relation between the rows is that

$$\mid \text{row}_1 \mid \geq \mid \text{row}_2 \mid \geq \mid \text{row}_3 \mid \geq ... \geq \mid \text{row}_m \mid$$

We can now add a maximal triangle into the diagram. It will add $m - i + 1$ elements to row $i$.

Our diagram now looks like this.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $m$ | $m$ | $m$ | ... | $m$ | ... | $m-1$ | ... | 1 | ... | 1 |
| $m-1$ | $m-1$ | $m-1$ | ... | $m$ | ... | $m-1$ | ... | | | | |
| $m-2$ | $m-2$ | ... | $m$ | ... | $m-1$ | ... | | | | | |
| $m-3$ | ... | $m$ | ... | ... | ... | | | | | | |
| ... | .... | | | | | | | | | | |

Before this step

$$\mid \text{row}_i \mid \geq \mid \text{row}_{i+1} \mid$$

So now

$$\mid \text{row} \mid_i + m - i + 1 > \mid \text{row}_{i+1} \mid + m - (i + 1) + 1$$

Meaning that the rule for each row after this step,

$$\mid \text{row}_i \mid > \mid \text{row}_{i+1} \mid$$

It is easily seen that since no rows are equally sized, all the rows have a distinct size.

Thus, since our process took an arbitrary element from the set of partitions with only elements 1 through $m$ and ended up with some partition with distinctly sized rows, our process is WELL DEFINED.

Lemma: this step of adding a maximal triangle to these kinds of partitions is an bijective function.

Proof:

Function is Injective:

Taking a partition with $m$ rows and adding a maximal triangle makes a new partition which has the same number of rows as the input partition and has rows of size $\mid \text{row}_i \mid + m - i + 1$. If the input had a different number of rows, then the output would have a different number of rows. If another input had even one row of a different size than this input, then the output would have a row of different size than this output and would be different. So adding a maximal triangle is injective.

Function is Surjective:

Now we prove we the image of the function is the target of the function.

We do this by showing only every combination of distinctly sized rows is part of the image.

We show this using the following inductive argument:

A partition is part of the image if the following rule is satisfied.

$$\mid \text{row}_i \mid > \mid \text{row}_{i+1} \mid$$

Base case: we have an empty partition and want to add a distinctly sized row. Since the empty partition can be thought to have no rows, it is vacuously true that this new row will have size less than it's previous row and greater than it's next row.

Inductive case: we have a non-empty partition of distinct elements and want to add a distinclty sized row. The distinctly sized row we want to add can be larger than every current row and become the first row or smaller than every current row and becomes the last row or not. If it isn't largest or smallest, since size(row) $\in \mathbb{Z}$, (and since $\mathbb{Z}$ is an ordered field), and since the new row's size is distinct, it will have size less than some existent row and size greater than some existent row. So the new row can fit into the partition.

Thus by the principle of induction, any combination of distinctly sized rows will make a partition that follows the rule. So all of these partitions will be in the image of the function.

The fact it produces every partition made of some combination of distinctly sized rows along with the fact stated earlier that it only produces partitions with distinctly sized rows implies that the image of this function, i.e. partitions of combinations of distinctly sized rows, is target of this function. $\square$

BIJECTION: We can consider our process to be a function. It is made of two other functions, a transpose/conjugation followed by adding a maximal triangle. Note that transposing/conjugation is a bijective function. We also proved adding a maximal triangle is bijective in this case. Therefore the whole process is bijection because it is made of the composition of bijective functions.

The above function is not weight preserving. But if the domain of the function is modified to be a tuple with a partition of only elements 1 through $m$ and it's inherent maximal triangle, it's still a well-defined bijection. The only difference is that it is now WEIGHT PRESERVING.

Now what's left is to show how this relates to the equation.

The LHS is the generating series for when there is or isn't a row of size $k$, but no other size. This makes it the generating series for a partition of distinct parts.

The RHS is the generating series for if the partition is empty, 1, or if there is a maximal triangle $x^{m(m+1)/2}$ of size $m(m+1)/2$ conjoined with a partition with parts of size 1 through $m$, $\prod_{k=1}^{m} \frac{1}{(1-x^k)}$.

Since we showed there is a weight preserving bijection from a partition with parts of size 1 through $m$ transposed/conjugated and conjoined with a maximal triangle of size $m(m+1)/2$, LHS, to a partition with only distinct parts, RHS, the two sides must be equal. $\blacksquare$

# 5 Q 5

5. Do part (c) of the last in class exercise of Lecture 19. That is, give a bijective proof that the number of partitions of N using distinct parts equals the number of partitions of N using only odd parts. (Hint: Consider the multiplicity mp of each part p occuring in partitions of the second type. Now look at the unique expansion of m as powers of 2 and distribute.)

Let $dp(n)$ be the number of partitions using distinct parts.

Let $op(n)$ be the number of partitions using odd parts.

BEGIN NUMBER THEORY

By the fundamental theorem of arithmetic, we know two integers are equal only if they have the same prime factorizerion, thus we know

$$1 \cdot 2^{x_1} \neq 3 \cdot 2^{x_2} \neq 5 \cdot 2^{x_3} \neq 7 \cdot 2^{x_4} ...$$

Because none of the terms can have the same prime factorization.

We also know that every integer can be broken down into the form

$$C \cdot 2^P$$

Where $C$ is the product of the odd prime factors and $2^P$ is the product of the even prime factors.

Therefore if you have 2 numbers who have different $C$ or $P$ when broken down into this form, then they are distinct numbers.

END NUMBER THEORY

For partitions using odd parts, they can be written in the form

$$\begin{gathered} a_1 \times 1 \\ a_3 \times 3 \\ a_5 \times 5 \\ a_7 \times 7 \\ a_9 \times 9 \\ \vdots \end{gathered}$$

Where $a_i$ is the number of times $i$ appears in the partition. ($i$ is not defined for even numbers).

We can break $a_i$ into its base 2 form which is a series of powers of 2. Then we can distribute this series across $i$. We are left with a series of the form

$$\sum_{n=0}^{\infty} ((a_i)_2)_{n+1} \cdot i \cdot 2^n$$

Where $((a_i)_2)_{n+1}$ is the $(n+1)^{th}$ bit in the base 2 expansion of $a_i$. Note that $\sum_{n=0}^{\infty} ((a_i)_2)_{n+1} \cdot i \cdot 2^n = a_i \cdot i$. Each term in that series has a unique combination of $C$ and $P$ when broken down into the $C \cdot 2^P$ form.

Thus if we turn the whole list representing the partition of odd parts into such a series, then every term in that series will be distinct, and the sum of the series will be the same as $n$ in $op(n)$ meaning we have preserved the weight. If we make this series of terms into a partition itself, then it will be a partition of distinct terms.

WELL DEFINED: This function, just described, is well defined because we took some arbitrary partition of odd parts and turned it into a partition of only distinct parts.

INJECTIVE: This function, just described, is injective because if even one $a_i$ is different in the list of odd parts, then the base 2 representation of that $a_i$ will be different. That means there will be some alternate series $\sum_{n=0}^{\infty}((a_i)_2)_{n+1}$ instead which will yield different distinct numbers who share the factor $i$. This will yield a different partition of distinct parts, so the output will be different.

So if the output of the function on $A$ matches the output of the function on $B$, then $A = B$. So the function is injective.

SURJECTIVE: This function is surjective because we can take an arbitrary partition with only distinct parts

$$
\begin{gathered}
b_1 \times 1 \\
b_2 \times 2 \\
b_3 \times 3 \\
b_4 \times 4 \\
b_5 \times 5 \\
\vdots
\end{gathered}
$$

Where $b_i$ is can have the value 1 or 0, and describes the number of times $i$ can appear in the partition.

Split each $i$ into the form $C \cdot 2^P$, and construct a partition of only odd parts by having $b_i \cdot 2^P$ parts of size $C$.

Since each $i$ is distinct, it has a unique combination of $C$ and $P$ from the $C \cdot 2^P$ breakdown. This means if we take away an $i$ which was already in the partition, or add an $i$ that wasn't, then in the partition of only odd parts, the $a_C$ for $C$ will have it's $P^{th}$ bit changed in its base 2 expansion. Since $i$ had the only combination of $C$ and $P$ to change this specific bit, this is an irrevocable change.

This means that this process outlined produces a unique partition of only odd parts.

When we apply the function to this partition of only odd parts, the result will be the original partition of only distinct parts.

Thus for each element in the target, there is an element in the domain which yields the element in the target after applying the function. So the function is surjective.

Since there exists a weight preserving bijection between the partition of $n$ with only odd parts and the partition of $n$ using only distinct parts, there must be an equal number of them. $\blacksquare$