

MA 824: ASSIGNMENT 1

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1. PROBLEM 1

Some Preliminary Estimates. First, suppose $1 \leq p < q < \infty$. Then

$$\|x\|_q^q = \sum_{i=1}^d |x_i|^q = \sum_{i=1}^d |x_i|^p \cdot |x_i|^{q-p} \leq \|x\|_q^{q-p} \sum_{i=1}^d |x_i|^p = \|x\|_q^{q-p} \|x\|_p^p,$$

where the first inequality follows from the fact that $|x_i| \leq \|x\|_q$ for $1 \leq i \leq d$. The above shows that $\|x\|_q^p \leq \|x\|_p^p$, that is, $\|x\|_q \leq \|x\|_p$.

On the other hand, using Hölder's inequality on the measure space $\{1, \dots, d\}$ equipped with the counting measure, we have

$$\|x^p\|_1 \leq \|x^p\|_{\frac{q}{p}} \|\mathbb{1}\|_{\frac{q}{q-p}},$$

where $x^p = (|x_1|^p, \dots, |x_d|^p) \in \mathbb{C}^d$. Now note that $\|\mathbb{1}\|_{\frac{q}{q-p}} = d^{\frac{q-p}{q}}$, and

$$\|x^p\|_{\frac{q}{p}} = \left(\sum_{i=1}^d |x_i|^q \right)^{\frac{p}{q}} = \|x\|_q^p.$$

This gives us

$$\|x\|_p^p \leq \|x\|_q^p d^{\frac{q-p}{q}} \implies \|x\|_p \leq d^{\frac{1}{p} - \frac{1}{q}} \|x\|_q,$$

that is,

$$\|x\|_q \leq \|x\|_p \leq d^{\frac{1}{p} - \frac{1}{q}} \|x\|_q.$$

Note that both inequalities are tight. Indeed, take $x = (1, 0, \dots, 0) \in \mathbb{C}^d$, then $\|x\|_q = \|x\|_p = 1$. On the other hand, taking $x = (1, 1, \dots, 1) \in \mathbb{C}^d$, we see that $\|x\|_p = d^{\frac{1}{p}}$ and $\|x\|_q = d^{\frac{1}{q}}$, whence $\|x\|_p = d^{\frac{1}{p} - \frac{1}{q}} \|x\|_q$.

Next, suppose $q = \infty$ and $p < q$. For $x = (x_1, \dots, x_d) \in \mathbb{C}^d$, there is an index $1 \leq i_0 \leq d$ such that $|x_{i_0}| = \|x\|_\infty$. Hence, we have

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{\frac{1}{p}} \geq |x_{i_0}| = \|x\|_\infty,$$

and

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^d \|x\|_\infty^p \right)^{\frac{1}{p}} = d^{\frac{1}{p}} \|x\|_\infty.$$

The inequalities

$$\|x\|_\infty \leq \|x\|_p \leq d^{\frac{1}{p}} \|x\|_\infty$$

are tight. Taking $x = (1, 0, \dots, 0) \in \mathbb{C}^d$, we have $\|x\|_p = \|x\|_\infty = 1$. And taking $x = (1, 1, \dots, 1) \in \mathbb{C}^d$, we have $\|x\|_p = d^{\frac{1}{p}} = d^{\frac{1}{p}} \|x\|_\infty$, as desired.

Conclusion. First, if $p = q$, then take $c = C = 1$. Henceforth, we assume that $p \neq q$. Next, suppose $1 \leq p, q < \infty$. Then using our estimates from the previous (sub)section,

$$\begin{cases} d^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \|x\|_p & p < q \\ \|x\|_p & p > q \end{cases} \leq \|x\|_q \leq \begin{cases} \|x\|_p & p < q \\ d^{\left(\frac{1}{q}-\frac{1}{p}\right)} \|x\|_p & p > q. \end{cases}$$

Finally, if $q = \infty$, then $p < \infty$ and we have

$$d^{-\frac{1}{p}} \leq \|x\|_q \leq \|x\|_p,$$

and if $p = \infty$ so that $q < \infty$, then

$$\|x\|_p \leq \|x\|_q \leq d^{\frac{1}{q}} \|x\|_p.$$

As we have seen in the preceding (sub)section, all the above estimates are tight, that is, the constants are the best possible.

2. PROBLEM 2

(a) There is a $0 < \lambda < 1$ such that $p = \lambda r + (1 - \lambda)s$. Hölder's inequality gives

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\lambda r + (1-\lambda)s}\|_1 \leq \|f^{\lambda r}\|_{\frac{1}{\lambda}} \|f^{(1-\lambda)s}\|_{\frac{1}{1-\lambda}}.$$

Note that

$$\|f^{\lambda r}\|_{\frac{1}{\lambda}} = \left(\int_X (|f|^{\lambda r})^{\frac{1}{\lambda}} d\mu \right)^{\lambda} = \left(\int_X |f|^r d\mu \right)^{\lambda} = \|f\|_r^{\lambda r} \leq \max\{\|f\|_r, \|f\|_s\}^{\lambda r}$$

and

$$\|f^{(1-\lambda)s}\|_{\frac{1}{1-\lambda}} = \left(\int_X (|f|^{(1-\lambda)s})^{\frac{1}{1-\lambda}} d\mu \right)^{1-\lambda} = \|f\|_s^{(1-\lambda)s} \leq \max\{\|f\|_r, \|f\|_s\}^{(1-\lambda)s}.$$

Thus

$$\|f\|_p^p \leq \max\{\|f\|_r, \|f\|_s\}^{\lambda r + (1-\lambda)s} \implies \|f\|_p \leq \max\{\|f\|_r, \|f\|_s\},$$

as desired.

(b) Suppose first that $\|f\|_{\infty} = \infty$. As a result, for every $C > 0$,

$$\mu\{x \in X: |f(x)| \geq C\} > 0.$$

Hence,

$$\|f\|_p^p = \int_X |f|^p d\mu \geq \int_{\{x: |f(x)| \geq C\}} |f(x)|^p d\mu \geq \mu\{x \in X: |f(x)| \geq C\} C^p,$$

and hence,

$$\|f\|_p \geq C \mu\{x \in X: |f(x)| \geq C\}^{\frac{1}{p}}.$$

Thus, as $p \rightarrow \infty$, using the fact that $\mu\{x \in X: |f(x)| \geq C\} > 0$, we have

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq C \liminf_{p \rightarrow \infty} \mu\{x \in X: |f(x)| \geq C\}^{\frac{1}{p}} = \infty.$$

It follows immediately that $\lim_{n \rightarrow \infty} \|f\|_p = \infty$.

Next, suppose $\|f\|_{\infty} < \infty$. For $p > r$, we have

$$\|f\|_p^p = \int_X |f|^p d\mu = \int_X |f|^{p-r} |f|^r d\mu \leq \|f\|_{\infty}^{p-r} \int_X |f|^r d\mu.$$

Thus,

$$\|f\|_p \leq \|f\|_{\infty}^{1-\frac{r}{p}} \|f\|_r^{\frac{r}{p}}.$$

Since $\|f\|_{\infty} > 0$, we have $\|f\|_r > 0$, consequently,

$$\limsup_{p \rightarrow \infty} \|f\|_p \leq \|f\|_{\infty} \lim_{p \rightarrow \infty} \|f\|_{\infty}^{-\frac{r}{p}} \|f\|_r^{\frac{r}{p}} = \|f\|_{\infty}.$$

On the other hand, let $0 < C < \|f\|_\infty$, so that

$$\mu \{x \in X: |f(x)| \geq C\} > 0.$$

An obvious estimate gives us

$$\|f\|_p^p = \int_X |f|^p d\mu \geq \int_{\{x \in X: |f(x)| \geq C\}} |f|^p d\mu \geq C^p \mu \{x \in X: |f(x)| \geq C\},$$

consequently,

$$\|f\|_p \geq C \mu \{x \in X: |f(x)| \geq C\}^{\frac{1}{p}}.$$

Finally, we claim that $0 < \mu \{x \in X: |f(x)| \geq C\} < \infty$. Indeed, since $\|f\|_\infty > 0$, it is obvious that the above measure is positive. Further, since $\|f\|_r < \infty$, we have

$$C \mu \{x \in X: |f(x)| \geq C\}^{\frac{1}{r}} \leq \left(\int_X |f|^r d\mu \right)^{\frac{1}{r}} = \|f\|_r < \infty,$$

whence the measure is finite. It follows that

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq C \liminf_{p \rightarrow \infty} \mu \{x \in X: |f(x)| \geq C\}^{\frac{1}{p}} = C.$$

Since the above inequality holds for all $0 < C < \|f\|_\infty$; taking a supremum over all such C , we get that

$$\|f\|_\infty \geq \limsup_{p \rightarrow \infty} \|f\|_p \geq \liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty,$$

therefore,

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty,$$

as desired.

- (c) Since $r < s$, we have $\frac{1}{r} > \frac{1}{s}$. Choose a $p \geq 1$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s} \implies 1 = \frac{r}{p} + \frac{r}{s}.$$

Hölder's inequality then gives us

$$\|f^r\|_1 \leq \|f^r\|_{\frac{s}{r}} \|\mathbb{1}\|_{\frac{p}{r}},$$

where $\mathbb{1}$ denotes the constant function 1. Now note that

$$\|f^r\|_{\frac{s}{r}} = \left(\int_X |f|^s d\mu \right)^{\frac{r}{s}} = \|f\|_s^r,$$

$\|\mathbb{1}\|_{\frac{p}{r}} = 1$, since $\mu(X) = 1$; and

$$\|f^r\|_1 = \int_X |f|^r d\mu = \|f\|_r^r.$$

Hence, we have shown that $\|f\|_r^r \leq \|f\|_s^r$. If $\|f\|_s = \infty$, then the inequality $\|f\|_r \leq \|f\|_s$ is trivial. If $\|f\|_s < \infty$, then taking r -th roots we get $\|f\|_r \leq \|f\|_s$, as desired.

3. PROBLEM 3

First, suppose $1 \leq p < \infty$. For $i \geq 1$, define the "standard basis vectors" $e_i \in \ell^p$ by

$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathbb{K} denote the base field over which ℓ^p is defined. If $\mathbb{K} = \mathbb{R}$, set $Q = \mathbb{Q}$ and if $\mathbb{K} = \mathbb{C}$, then set $Q = \mathbb{Q} + \mathbb{Q}i$. Note that in either case, Q is dense in \mathbb{K} .

Set

$$D = \bigcup_{n=1}^{\infty} \{q_1 e_1 + \cdots + q_n e_n : q_1, \dots, q_n \in Q\}.$$

Being a countable union of countable sets, D is itself countable. We contend that D is dense in ℓ^p . Let $x = (x_n) \in \ell^p$ and $\varepsilon > 0$. Since the sum $\sum_{n=1}^{\infty} |x_n|^p$ converges, there is a positive integer N such that the tail sum

$$\sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Let $y = (y_n) \in \ell^p$ be given by

$$y_n = \begin{cases} x_n & n \leq N \\ 0 & n > N. \end{cases}$$

Then,

$$\|x - y\|^p = \sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Therefore, $\|x - y\| < \frac{\varepsilon}{2}$. Next, using the density of Q in \mathbb{K} , for each $1 \leq n \leq N$, we can find a $z_n \in Q$ such that

$$|y_n - z_n|^p < \frac{1}{N} \left(\frac{\varepsilon}{2}\right)^p.$$

Setting $z = z_1 e_1 + \cdots + z_N e_N \in D \subseteq \ell^p$,

$$\|y - z\|^p = \sum_{n=1}^N |y_n - z_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

It follows from the triangle inequality that

$$\|x - z\| \leq \|x - y\| + \|y - z\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have shown that for all $x \in \ell^p$ and $\varepsilon > 0$, there is a $z \in D$ with $\|x - z\| < \varepsilon$, and hence, D is dense in ℓ^p .

Finally, we show that ℓ^∞ is not separable. For this part of the proof, we write an element $x \in \ell^\infty$ as $(x(n))_{n \geq 1}$. Suppose not and there were a countable dense subset $D \subseteq \ell^\infty$. For each subset $S \subseteq \mathbb{N} = \{1, 2, \dots\}$, define $x_S \in \ell^\infty$ as

$$x_S(n) = \begin{cases} 1 & n \in S \\ 0 & \text{otherwise,} \end{cases}$$

and set $U_S = B\left(x_S, \frac{1}{2}\right)$. We claim that the U_S 's are pairwise disjoint. Indeed, suppose $S, T \subseteq \mathbb{N}$ are distinct subsets and $y \in U_S \cap U_T$. It follows that

$$\|x_S - x_T\| = \|(x_S - y) - (x_T - y)\| \leq \|x_S - y\| + \|x_T - y\| < 1.$$

But for $n \in S \Delta T$, we have that $|x_S(n) - x_T(n)| = 1$, whence $\|x_S - x_T\| \geq 1$, a contradiction. Thus, the U_S 's are pairwise disjoint. For each $S \subseteq \mathbb{N}$, there is a $z_S \in D$ such that $z_S \in U_S \cap D$, owing to the density of D . Since the U_S 's are pairwise disjoint, the z_S 's are distinct. But \mathbb{N} has uncountably many subsets (due to Cantor), and D is countable, a contradiction. Hence ℓ^∞ is not separable.

4. PROBLEM 4

Let X be a locally compact normed linear space; then there is a neighborhood V of the origin in X such that \bar{V} is compact. Since V is open, there is an $r > 0$ such that $B(0, r) \subseteq V$, consequently, $\bar{B}(0, r) \subseteq \bar{V}$. Since the latter is compact, so is the former. Further, the map $x \mapsto rx$ is a homeomorphism $X \rightarrow X$ with inverse given by $x \mapsto r^{-1}x$. Under the former map, the closed unit ball $\bar{B}(0, 1)$ maps to $\bar{B}(0, r)$. Since the latter is compact, the former must be too, that is, $\bar{B}(0, 1)$ is compact. Hence, we may suppose without loss of generality that $V = B(0, 1)$.

The following method is due to André Weil and generalizes well to topological vector spaces over complete valued fields. Note that \bar{V} is compact and

$$\bar{V} \subseteq \bigcup_{x \in \bar{V}} \left(x + \frac{1}{2}V\right).$$

Since the latter is an open cover, it contains a finite subcover of \bar{V} . That is, there are $x_1, \dots, x_n \in \bar{V}$ such that

$$\bar{V} \subseteq \bigcup_{i=1}^n \left(x_i + \frac{1}{2}V \right).$$

Let Y denote the span of $\{x_1, \dots, x_n\}$. Being a finite-dimensional subspace of X , Y is closed in X as we have seen in class. The above containment implies

$$V \subseteq \bar{V} \subseteq \bigcup_{i=1}^n \left(x_i + \frac{1}{2}V \right) \subseteq Y + \frac{1}{2}V.$$

But then

$$Y + \frac{1}{2}V \subseteq Y + \frac{1}{2} \left(Y + \frac{1}{2}V \right) = Y + \frac{1}{2}Y + \frac{1}{4}V = Y + \frac{1}{4}V,$$

where the last equality follows from the fact that Y is a vector space and hence, $\frac{1}{2}Y = Y$ and $Y + Y = Y$. Inductively, suppose we have shown that

$$V \subseteq Y + \frac{1}{2^m}V$$

for some positive integer m . Then,

$$V \subseteq Y + \frac{1}{2^m} \left(Y + \frac{1}{2}V \right) = Y + \frac{1}{2^m}Y + \frac{1}{2^{m+1}}V = Y + \frac{1}{2^{m+1}}V,$$

where the last equality follows from the fact that Y is a vector space. Consequently, we have

$$V \subseteq \bigcap_{m=1}^{\infty} \left(Y + \frac{1}{2^m}V \right).$$

CLAIM.

$$\bar{Y} = \bigcap_{m=1}^{\infty} \left(Y + \frac{1}{2^m}V \right)$$

Proof. Suppose $y \in \bar{Y}$ and m a positive integer. Then, by definition, there is some $x \in Y$ such that $\|x - y\| < 2^{-m}$, that is, $y - x \in 2^{-m}V$, consequently, $y \in Y + 2^{-m}V$. It follows that $y \in \bigcap_{m=1}^{\infty} (Y + 2^{-m}V)$.

Conversely, if $y \in \bigcap_{m=1}^{\infty} (Y + 2^{-m}V)$ and $r > 0$. Choose a positive integer m such that $2^{-m} < r$. Since $y \in Y + 2^{-m}V$, there is some $x \in Y$ such that $y \in x + 2^{-m}V$, equivalently, $\|y - x\| < 2^{-m} < r$. Hence, $B(y, r) \cap Y \neq \emptyset$. It follows that $y \in \bar{Y}$. ■

Using the above claim, we have

$$V \subseteq \bar{Y} = Y,$$

since Y is closed in X , owing to it being finite-dimensional. Since Y is a vector space, we see that $\text{Span}(V) \subseteq Y$. Now, for any $0 \neq x \in X$, we have $\frac{x}{2\|x\|} \in V$, since it has norm $\frac{1}{2}$. Hence, $x \in \text{Span}(V)$, in particular, $\text{Span}(V) = X$. This shows that $X \subseteq Y$, that is, $X = Y$, hence X is finite-dimensional, as desired.

5. PROBLEM 5

Before we begin, let $x = (x(i)) \in X$ and $j \in I$. Then, for $1 \leq p < \infty$,

$$\|x(i)\|^p \leq \sum_{i \in I} \|x(i)\|^p = \|x\|^p \implies \|x(i)\| \leq \|x\|.$$

And if $p = \infty$, then obviously $\|x(i)\| \leq \|x\|$. All integrals over I henceforth are with respect to the counting measure $(I, \mathcal{P}(I), \mu)$ on I .

Throughout this solution, let $\|\cdot\|_p$ denote the $L^p(I, \mathcal{P}(I), \mu)$ norm with $1 \leq p \leq \infty$. We also identify sequences indexed by I with measurable functions $I \rightarrow \mathbb{K}$, so that there is no difference between a sum indexed by I and an integral over I .

- (a) Let $1 \leq p \leq \infty$, $x, y \in \bigoplus_p X_i$, and $\alpha, \beta \in \mathbb{K}$. We shall show that $\alpha x + \beta y \in \bigoplus_p X_i$ and $\|\alpha x + \beta y\| \leq |\alpha|\|x\| + |\beta|\|y\|$. Indeed, note that $\|\alpha x + \beta y\|$ is the L^p -norm of the function $f : I \rightarrow \mathbb{K}$ given by $f(i) = \|\alpha x(i) + \beta y(i)\|$. The triangle inequality gives us $f(i) \leq |\alpha|\|x(i)\| + |\beta|\|y(i)\|$ for all $i \in I$. Let $g, h : I \rightarrow \mathbb{K}$ be given by $g(i) = \|x(i)\|$ and $h(i) = \|y(i)\|$. Then $f = |\alpha|g + |\beta|h$. Since the L^p -spaces form a normed linear space, we have

$$\|\alpha x + \beta y\| \leq \|f\|_p \leq |\alpha|\|g\|_p + |\beta|\|h\|_p = |\alpha|\|x\| + |\beta|\|y\| < \infty.$$

Hence, $\alpha x + \beta y \in \bigoplus_p X_i$ for all $\alpha, \beta \in \mathbb{K}$. This shows that $\bigoplus_p X_i$ is a vector space and the inequality proved above shows that $\|\cdot\|$ is indeed a norm on $\bigoplus_p X_i$.

Finally, we must show that $P_i : X \rightarrow X_i$ has norm ≤ 1 . That P_i is a linear map is obvious. For any $x \in X$, as we had observed in the paragraph preceding the solution of part (a), $\|x(i)\| \leq \|x\|$, and hence, $\|P_i\| \leq 1$.

- (b) Suppose first that each X_i is a Banach space and let $(x_n)_{n \geq 1}$ be a Cauchy sequence in X . That is, given any $\varepsilon > 0$, there is a positive integer $N > 0$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$. Hence, for $i \in I$, we have $\|x_m(i) - x_n(i)\| < \varepsilon$ for all $m, n \geq N$. This shows that the sequence $(x_n(i))_{n \geq 1}$ is Cauchy in X_i , therefore converges to some $x(i) \in X_i$ since X_i is Banach. Since the norm is a continuous function on each X_i , it follows that $\|x_n(i)\| \rightarrow \|x(i)\|$. Set $x = (x(i))$. We now treat the cases $p < \infty$ and $p = \infty$ separately.

Let $p < \infty$. First, we show that $x \in X$. Indeed, by Fatou's lemma, we have

$$\|x\|^p = \int_I \|x(i)\|^p d\mu(i) = \int_I \liminf_{n \rightarrow \infty} \|x_n(i)\|^p d\mu(i) \leq \liminf_{n \rightarrow \infty} \int_I \|x_n(i)\|^p d\mu = \liminf_{n \rightarrow \infty} \|x_n\|^p < \infty,$$

since the sequence (x_n) is bounded in X , owing to it being Cauchy (this is a standard fact from metric spaces). Next, we must show that $x_n \rightarrow x$ in X . For $\varepsilon > 0$, there is a positive integer N such that $\|x_m - x_n\| < \varepsilon$ whenever $m, n \geq N$. Then, by Fatou's Lemma, for $n \geq N$, we have

$$\|x - x_n\|^p = \int_I \liminf_{m \rightarrow \infty} \|x_m(i) - x_n(i)\|^p d\mu(i) \leq \liminf_{m \rightarrow \infty} \int_I \|x_m(i) - x_n(i)\|^p d\mu = \liminf_{m \rightarrow \infty} \|x_m - x_n\|^p \leq \varepsilon^p,$$

that is, $\|x - x_n\| \leq \varepsilon$. This shows that $x_n \rightarrow x$ in X , and hence X is a Banach space for $1 \leq p < \infty$.

Next, let $p = \infty$. First, we show that $x \in X$. Indeed, there is a positive integer N such that $\|x_m - x_n\| < 1$ for all $m, n \geq N$. In particular, $\|x_n - x_N\| < 1$ for all $n \geq N$, whence $\|x_n(i) - x_N(i)\| < 1$ for all $i \in I$. Consequently, $\|x_n(i)\| < \|x_N(i)\| + 1 \leq \|x_N\| + 1$ for all $i \in I$. Since $x_n(i) \rightarrow x(i)$ in X_i , we have that

$$\|x(i)\| = \lim_{n \rightarrow \infty} \|x_n(i)\| \leq \|x_N\| + 1.$$

Taking a supremum over $i \in I$, we get that $\|x\| \leq \|x_N\| + 1$, that is, $x \in X$. Finally, we show that $x_n \rightarrow x$ in X . Let $\varepsilon > 0$. Then there is a positive integer N such that $\|x_m - x_n\| < \varepsilon$ whenever $m, n \geq N$. Since $x_m(i) \rightarrow x(i)$ in X_i , we see that for $n \geq N$,

$$\|x(i) - x_n(i)\| = \lim_{m \rightarrow \infty} \|x_m(i) - x_n(i)\| \leq \varepsilon.$$

Taking a supremum over $i \in I$, we have $\|x - x_n\| \leq \varepsilon$ for all $n \geq N$, whence $x_n \rightarrow x$ in X . This shows that X is a Banach space when $p = \infty$.

Conversely, suppose X is Banach; we shall show that each X_i is Banach. Let (x_n) be a Cauchy sequence in X_i . Define a sequence (y_n) in X by

$$y_n(j) = \begin{cases} x_n & j = i \\ 0 & \text{otherwise.} \end{cases}$$

We claim that (y_n) is a Cauchy sequence in X . Indeed, for $p < \infty$, and positive integers m, n , we have

$$\|y_m - y_n\| = \left(\int_I \|y_m(j) - y_n(j)\|^p d\mu(j) \right)^{1/p} = \|x_m - x_n\|,$$

and for $p = \infty$,

$$\|y_m - y_n\| = \sup_{j \in I} \|y_m(j) - y_n(j)\| = \|x_m - x_n\|.$$

Thus, (y_n) is a Cauchy sequence in X , since (x_n) is a Cauchy sequence in X_i . Thus, there is some $y = (y(j))$ such that $y_n \rightarrow y$ in X . Then, as we have observed at the beginning, $\|y_n(i) - y(i)\| \leq \|y_n - y\|$, whence $y_n(i) \rightarrow y(i)$ in X_i , that is, $x_n \rightarrow y(i)$ in X_i . This shows that X_i is a Banach space, thereby completing the proof.

- (c) We first show that the image of a ball $B_X(0, r)$ centered at 0 in X is open under $P_i : X \rightarrow X_i$. In fact, we claim that the image of this ball is $B_{X_i}(0, r)$. Indeed, if $x = (x(i)) \in B_X(0, r)$, then $\|x(i)\| \leq \|x\| < r$, whence the image of $B_X(0, r)$ under P_i is contained in $B_{X_i}(0, r)$. Conversely, if $x_i \in X_i$ with $\|x_i\| < r$, then setting $y = (y(j)) \in X$ where

$$y(j) = \begin{cases} x_i & j = i \\ 0 & \text{otherwise,} \end{cases}$$

we note that $\|y\| = \|x_i\|$ in both cases $p < \infty$ and $p = \infty$. Therefore, $y \in B_X(0, r)$. It follows that $B_{X_i}(0, r)$ is contained in the image of $B_X(0, r)$ under P_i , whence $P(B_X(0, r)) = B_{X_i}(0, r)$.

Now, obviously $P_i : X \rightarrow X_i$ is a linear map, for if $c \in \mathbb{C}$ and $x = (x(j)) \in X$, then $P_i(cx) = P_i((cx(j))) = cx(j)$ and if $y = (y(j)) \in X$, then

$$P_i(x + y) = P_i((x(j)) + (y(j))) = P_i((x(j) + y(j))) = P_i(x) + P_i(y).$$

Let $U \subseteq X$ be an open set. Then, for each $x = (x(i)) \in U$, there is an $r_x > 0$ such that $B_X(x, r_x) \subseteq U$, whence $U = \bigcup_{x \in U} B_X(x, r_x)$. Note that for any $y \in X$ and $r > 0$,

$$P_i(B_X(y, r)) = P_i(y + B_X(0, r)) = P_i(y) + P_i(B_X(0, r)) = y(i) + B_{X_i}(0, r) = B_{X_i}(y(i), r).$$

Consequently,

$$P_i(U) = P_i\left(\bigcup_{x \in U} B_X(x, r_x)\right) = \bigcup_{x \in U} P_i(B_X(x, r_x)) = \bigcup_{x \in U} B_{X_i}(x(i), r_x),$$

which is an open subset of X_i , as desired.

6. PROBLEM 6

That the dual space of c_0 is isometrically isomorphic to ℓ^1 has been argued in class and I shall not reproduce that argument. We show that the dual space of c isometrically isomorphic to ℓ^1 . In this case, we denote an element $x \in \ell^1$ by a sequence indexed by $n \geq 0$. This is in contrast to the standard indexing of $n \geq 1$. It will be clear why this is done.

Define a map $T : \ell^1 \rightarrow c^*$ given by $a = (a(n))_{n \geq 0} \mapsto T_a$ where

$$T_a(x) = a(0)x_\infty + \sum_{n=1}^{\infty} a(n)x(n),$$

where $x_\infty = \lim_{n \rightarrow \infty} x(n)$. Obviously, we must have $|x_\infty| \leq \|x\|$. We must show that this sum converges, for which it suffices to show absolute convergence. Indeed, every partial sum (of the absolute value sum) is bounded as

$$|a(0)x_\infty| + \sum_{n=1}^N |a(n)x(n)| \leq \|x\| \left(\sum_{n=0}^N |a(n)| \right) \leq \|a\| \|x\|,$$

and hence, must converge. It follows that T_a is a well-defined function. That T_a is linear is clear from the definition. To see that it is bounded, we again have that

$$\begin{aligned} |T_a(x)| &= \left| a(0)x_\infty + \sum_{n=1}^{\infty} a(n)x(n) \right| \\ &\leq |a(0)||x_\infty| + \sum_{n=1}^{\infty} |a(n)||x(n)| \\ &\leq \|x\| \left(\sum_{n=0}^{\infty} |a(n)| \right) = \|a\| \|x\|. \end{aligned}$$

Thus, $T_a \in c^*$ and $\|T_a\| \leq \|a\|$. We claim that the map T is linear. Indeed, if $a, b \in \ell^1$ and $\alpha \in \mathbb{K}$, then

$$\begin{aligned} T_{a+\alpha b}(x) &= (a(0) + \alpha b(0)) x_\infty + \sum_{n=1}^{\infty} (a(n) + \alpha b(n)) x(n) \\ &= a_0 x_\infty + \sum_{n=1}^{\infty} a(n) x(n) + \alpha \left(b_0 x_\infty + \sum_{n=1}^{\infty} b(n) x(n) \right) \\ &= T_a(x) + \alpha T_b(x) \end{aligned}$$

for all $x \in c$. This shows that $T(a + \alpha b) = T(a) + \alpha T(b)$, whence T is linear. Further, since $\|T_a\| \leq \|a\|$, the map $T : \ell^1 \rightarrow c^*$ is a bounded linear functional.

Next, we show that T is an isometry. Let $a \in \ell^1$. If $a = 0$, then it is clear that $T_a = 0$, whence $\|T_a\| = 0$. Suppose now that $a \neq 0$. For every $j \geq 0$, let

$$z_j = \begin{cases} \frac{\overline{a(j)}}{a(j)} & a(j) \neq 0 \\ 0 & a(j) = 0, \end{cases}$$

whereby $|z_j| \leq 1$. Note that the z_j 's are chosen so that $z_j a(j) = |a(j)|$. For every positive integer N , let $x_N \in c$ be given by

$$x_N(n) = \begin{cases} z_n & n \leq N \\ z_0 & n > N. \end{cases}$$

Since the sequence $x_N(n)$ eventually stabilizes, it lies in c and $\|x_N\| \leq 1$. According to our definition,

$$\begin{aligned} |T_a(x_N)| &= \left| a(0) + \sum_{j=1}^N |a(j)| + \sum_{j=N+1}^{\infty} a(j) z_0 \right| \\ &\geq |a(0)| + \sum_{j=1}^N |a(j)| - |z_0| \left| \sum_{j=N+1}^{\infty} a(j) \right| \\ &\geq |a(j)| + \sum_{j=1}^N |a(j)| - \left| \sum_{j=N+1}^{\infty} a(j) \right|, \end{aligned}$$

since $|z_0| \leq 1$. But since $\|x_N\| \leq 1$, we have

$$\|T_a\| \geq |T_a(x_N)| \geq |a(0)| + \sum_{j=1}^N |a(j)| - \left| \sum_{j=N+1}^{\infty} a(j) \right| \geq |a(0)| + \sum_{j=1}^N |a(j)| - \sum_{j=N+1}^{\infty} |a(j)| = \|a\| - 2 \sum_{j=N+1}^{\infty} |a(j)|.$$

Since the sum $\sum_{j=0}^{\infty} |a(j)|$ converges, the tail sum goes to 0. In particular, taking $N \rightarrow \infty$ in the above inequality, we get

$$\|T_a\| \geq \|a\| \implies \|T_a\| = \|a\|,$$

that is, T is an isometry, whence T is injective, as the kernel is trivial; for if $T(x) = 0$, then $\|x\| = \|T(x)\| = 0$, i.e., $x = 0$.

Finally, to show that T is an isometric isomorphism, we must show that T is surjective. Indeed, let $\Lambda \in c^*$ and let the e_i 's denote the "standard basis vectors" for c , that is,

$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Set $a(n) = \Lambda(e_n)$ and $a(0) = T(\xi)$, where $\xi(j) = 1$ for all $j \geq 1$. We contend that $a \in \ell^1$ and $\Lambda = T_a$. Indeed, for $x = (x(n))_{n \geq 1} \in c$, set $x_\infty = \lim_{n \rightarrow \infty} x(n)$ and $y = x - x_\infty \xi \in c$. Obviously, we have that

$$\lim_{n \rightarrow \infty} y(n) = 0.$$

Let $y_N \in c$ be given by

$$y_N = y(1)e_1 + \cdots + y(N)e_N \in c.$$

Then

$$y - y_N = \left(\underbrace{0, \dots, 0}_{N \text{ times}}, y(N+1), y(N+2), \dots \right).$$

Note that

$$\|y - y_N\| = \sup_{n \geq N+1} |y(n)| \rightarrow 0$$

as $N \rightarrow \infty$, since $\lim_{n \rightarrow \infty} y(n) = 0$. Since Λ is continuous, we have

$$\Lambda y = \lim_{N \rightarrow \infty} \Lambda y_N = \lim_{N \rightarrow \infty} \sum_{i=1}^N y(i) \Lambda(e_i) = \lim_{N \rightarrow \infty} \sum_{i=1}^N a(i) y(i) = \sum_{n=1}^{\infty} a(n) y(n).$$

Hence,

$$\Lambda x = \Lambda y + x_{\infty} \Lambda \xi = a(0) x_{\infty} + \sum_{n=1}^{\infty} a(n) x(n).$$

Finally, we must show that $a \in \ell^1$. Again, for $n \geq 1$, set

$$z_n = \begin{cases} \frac{\bar{a}(n)}{|a(n)|} & a(n) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$w_N = z_1 e_1 + \dots + z_N e_N \in c.$$

Since $\lim_{n \rightarrow \infty} w_N(n) = 0$, we see that

$$|\Lambda w_N| = \left| \sum_{n=1}^N z_n a(n) \right| = \sum_{n=1}^N |a(n)|.$$

Since each $|z_n| \leq 1$, we see that $\|w_N\| \leq 1$, consequently,

$$\sum_{n=1}^N |a(n)| = |\Lambda w_N| \leq \|\Lambda\| \|w_N\| \leq \|\Lambda\|.$$

Since the sum is bounded independent of N and the left hand side is a monotonically increasing sequence, it must converge, that is,

$$\sum_{n=1}^{\infty} |a(n)| < \infty \implies \sum_{n=0}^{\infty} |a(n)| < \infty,$$

equivalently, $a \in \ell^1$, as desired. Thus, we have shown that $\Lambda = T_a$ for some $a \in \ell^1$. This proves surjectivity of T and establishes the isometric isomorphism.

Now, we show that c_0 and c are not isometrically isomorphic. Suppose there was an isometric isomorphism $T : c \rightarrow c_0$. Set $\xi = (1, 1, \dots) \in c$. Since T is an isometry, $x = T(\xi)$ must have norm 1. But since $\lim_{n \rightarrow \infty} x(n) = 0$, there is a positive integer N such that for all $n \geq N$, $|x(n)| < \frac{1}{2}$. But since

$$\sup_{n \in \mathbb{N}} |x(n)| = 1,$$

there is an $n_0 < N$ with $|x(n_0)| = 1$. Define y, z as

$$y(n) = \begin{cases} x(n) & n < N \\ x(n) + \frac{1}{4n} & n \geq N \end{cases} \quad \text{and} \quad z(n) = \begin{cases} x(n) & n < N \\ x(n) - \frac{1}{4n} & n \geq N. \end{cases}$$

Note that $y(n_0) = z(n_0) = 1$ since $n_0 < N$. Further, for $n \geq N$, we have

$$|y(n)| \leq |x(n)| + \frac{1}{4n} < \frac{1}{2} + \frac{1}{4} < 1 \quad |z(n)| \leq |x(n)| + \frac{1}{4n} < \frac{1}{2} + \frac{1}{4} < 1,$$

and

$$\lim_{n \rightarrow \infty} y(n) = \lim_{n \rightarrow \infty} x(n) = 0 = \lim_{n \rightarrow \infty} z(n).$$

It follows that $y, z \in c_0$, $\|y\| = \|z\| = 1$, and $x = \frac{1}{2}(y + z)$. Since T is an isometric isomorphism, there exist $\zeta, \eta \in c$ such that $T(\zeta) = y$ and $T(\eta) = z$ and $\|\zeta\| = \|\eta\| = 1$. We also have that

$$T(\xi) = x = \frac{1}{2}(y + z) = \frac{1}{2}(T(\zeta) + T(\eta)) = T\left(\frac{\zeta + \eta}{2}\right),$$

consequently, $\xi = \frac{1}{2}(\zeta + \eta)$, in other words,

$$\zeta(n) + \eta(n) = 2 \quad \forall n \in \mathbb{N}.$$

But since $|\zeta(n)|, |\eta(n)| \leq 1$, we have that $\zeta(n) = \eta(n) = 1$ for all $n \in \mathbb{N}$, i.e., $\zeta = \eta = \xi$, a contradiction, since $x \neq y$ and $x \neq z$. It follows that c and c_0 are not isometric, thereby completing the proof.

7. PROBLEM 7

First, consider the case when μ is a positive measure. Then,

$$\mu([0, 1]) = \int_0^1 d\mu = 0,$$

and hence, $\mu(E) = 0$ for all Borel sets $E \subseteq [0, 1]$, that is, $\mu = 0$.

Next, suppose μ is a complex measure. We claim that μ is a regular Borel measure. To this end, we use the following theorem:

THEOREM 7.1. Let X be a locally compact Hausdorff space in which every open set is σ -compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K . Then λ is regular.

Proof. See [Rud87, Theorem 2.17] ■

Obviously every open set in $[0, 1]$ is σ -compact and for every compact set $K \subseteq [0, 1]$, $|\mu|(K) < \infty$, since $|\mu|([0, 1]) < \infty$, where $|\mu|$ is the total variation measure. It follows that $|\mu|$ is a positive regular Borel measure on $[0, 1]$, and hence, μ is a regular complex Borel measure on $[0, 1]$. Thus, the map $T_\mu : C[0, 1] \rightarrow \mathbb{C}$ given by

$$T_\mu f = \int_0^1 f d\mu$$

is a bounded linear functional on $[0, 1]$, since the dual space of $C[0, 1]$ is identified with the Banach space of all complex regular Borel measures on $[0, 1]$. Further, we know that $T_\mu(x^n) = 0$ for all $n \geq 0$, and hence by taking finite linear combinations, $T_\mu(p(x)) = 0$ for all polynomials $p(x) \in \mathbb{C}[x]$. Due to Weierstrass' Theorem, the space $\mathbb{C}[x]$ is dense in $C[0, 1]$ with respect to the sup-norm. Thus, T_μ is identically zero on a dense subspace of $C[0, 1]$, consequently, T_μ must be identically 0. That is, $T_\mu f = 0$ for all $f \in C[0, 1]$.

Recall that there is an isometric isomorphism (in particular, a bijection) $\mathcal{M}([0, 1]) \rightarrow (C[0, 1])^*$ given by $\lambda \mapsto T_\lambda$, where T_λ is as defined above and $\mathcal{M}([0, 1])$ is the space of all regular complex Borel measures on $[0, 1]$ equipped with the total variation norm. Since $\mu \in \mathcal{M}([0, 1])$ maps to $0 \in (C[0, 1])^*$, we see that $\mu = 0$, as desired.

8. PROBLEM 8

We have seen in class and it is a standard fact from real analysis that the space $Y = C[0, 1]$ is complete with respect to the sup-norm. We claim that X is not complete. Note that $X = C^1[0, 1]$ is a subspace of Y with the same norm. Thus to show that X is not complete, it suffices to exhibit a sequence in X converging to an element of $Y \setminus X$. Take $f \in Y \setminus X$ given by

$$f(x) = \left| x - \frac{1}{2} \right| \quad 0 \leq x \leq 1.$$

This is obviously not an element of X since f is not differentiable at $\frac{1}{2}$. Due to a theorem of Weierstraß, we know that there is a sequence of polynomials $p_n \in Y$, which converge uniformly to f on $[0, 1]$, that is, $p_n \rightarrow f$ in Y . Since polynomials are infinitely differentiable, they are elements of X . Thus, we have found a sequence of elements of X which converges to an element of $Y \setminus X$. Since every convergent sequence is Cauchy, the sequence $\{p_n\}$ is Cauchy in Y , and hence in X (since the norm on X is the restriction of the norm on Y). But p_n cannot converge to some $g \in X$ since that would imply $f = g$ due to the uniqueness of limits in Y ; indeed, since convergence in X is the same as convergence in Y . This argument shows that X is not complete.

Now, we show that the map $A : X \rightarrow Y$ given by $Af = f'$ has a closed graph but is not continuous. We shall need the following result which is usually covered in a first course on real analysis:

LEMMA 8.1. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f , and

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) \quad a \leq x \leq b.$$

Proof. See [Rud53, Theorem 7.17]. ■

First, we show that the graph of A is closed in $X \times Y$. Since $X \times Y$ is a metric space, it suffices to show that the graph of A is sequentially closed. To this end, let (f_n, Af_n) be a sequence in $\text{Graph}(A)$ converging to some $(f, g) \in X \times Y$, that is, $f \in C^1[0, 1]$ and $g \in C[0, 1]$. Let $Af_n = g_n \in Y$. Since $(f_n, g_n) \rightarrow (f, g)$, we have that $f_n \rightarrow f$ in X and $g_n \rightarrow g$ in Y (this is a standard fact about the product topology on metric spaces). Since both X and Y are equipped with the sup-norm on $[0, 1]$, we have that $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly on $[0, 1]$. Hence, Lemma 8.1 applies and we get that

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) = \lim_{n \rightarrow \infty} g_n(x) = g(x).$$

That is, $g = Af$, equivalently, $(f, g) \in \text{Graph}(A)$. This shows that $\text{Graph}(A)$ is closed.

We show that A is not continuous. Clearly A is linear (since taking the derivative is a linear operation). Consider, for positive integers $n \geq 1$, the functions $f_n(x) = x^n$. Then $f_n \in X = C^1[0, 1]$ and $g_n = Af_n \in Y$ are given by $g_n(x) = nx^{n-1}$. Obviously, $\|f_n\| = 1$ and $\|Af_n\| = n$, whence

$$\|A\| = \sup_{\|f\| \leq 1} \|Af\| \geq \sup_{n \in \mathbb{N}} n = \infty,$$

i.e., A is not bounded and hence not continuous.

9. PROBLEM 9

Define the map $T : V \rightarrow C(E)$ given by $Tf = f|_E$. Obviously, T is a linear map, for if $f, g \in V$ and $c \in \mathbb{K}$, then

$$T(f + cg) = (f + cg)|_E = f|_E + cg|_E = Tf + cTg.$$

Further, for any $g \in C(E)$, according to the hypothesis of the question, there is an $f \in V$ such that $g = f|_E$, whence $T : V \rightarrow C(E)$ is a surjective linear map. Finally, for any $f \in V$,

$$\|Tf\| = \|f|_E\| \leq \|f\|,$$

consequently, T is a bounded linear functional, that is, T is continuous.

LEMMA 9.1. Let $T : X \rightarrow Y$ be a surjective linear map between Banach spaces. Then there is a constant $c > 0$ such that for every $y \in Y$, there is an $x \in X$ with $\|x\| \leq c\|y\|$ such that $Tx = y$.

Proof. Due to the open mapping theorem, T is an open map, consequently, $T(B_X(0, 1))$ is an open set in Y containing 0. Thus, there is an $r > 0$ such that $B_Y(0, r) \subseteq T(B_X(0, 1))$. Let $y \in Y$. If $y = 0$, then set $x = 0$. If $y \neq 0$, then consider $z = \frac{ry}{2\|y\|}$, where $\|z\| < r$. Hence, there is a $w \in B_X(0, 1)$ such that $Tw = z$. Set $x = \frac{2\|y\|}{r}w \in X$ and note that

$$Tx = \frac{2\|y\|}{r}Tw = y,$$

and

$$\|x\| = \frac{2\|y\|}{r}\|w\| \leq \frac{2}{r}\|y\|.$$

Pick $c = \frac{2}{r}$. We have shown that for every $y \in Y$, there is an $x \in X$ with $\|x\| \leq c\|y\|$. ■

Since both V and $C(E)$ are Banach spaces, the conclusion follows immediately from Lemma 9.1.

10. PROBLEM 10

Let $J : X \rightarrow X^{**}$ denote the canonical isometry given by $x \mapsto \text{ev}_x$, the evaluation map at x . That this is indeed an isometry has been argued in class. Let $x^{**} \in X^{**}$ denote the image of $x \in X$ under the map J .

Let $f \in X^*$, then for any $x \in S$, we have

$$|x^{**}(f)| = |f(x)| \leq M_f$$

for some constant $M_f > 0$, since $f(S)$ is a bounded subset of \mathbb{C} according to the hypothesis. Due to the *Uniform Boundedness Principle* (or Banach-Steinhaus Theorem), there is a constant $M > 0$ such that $\|x^{**}\| \leq M$ for all $x \in S$. Note that the theorem applies since X^* is a Banach space as we have argued in class. Finally, since J is an isometry, for every $x \in S$, we have

$$\|x\| = \|J(x)\| = \|x^{**}\| \leq M,$$

that is,

$$\sup \{\|x\| : x \in S\} \leq M < \infty,$$

as desired.

11. PROBLEM 10

For each $x \in X$, since the sequence $(T_n(x))$ converges, it is bounded in Y , that is, there is an $M_x > 0$ such that $\|T_n x\| \leq M_x$ for all $n \geq 1$. Since X is Banach space, the *Uniform Boundedness Principle* applies and there is an $M > 0$ such that $\|T_n\| \leq M$ for all $n \geq 1$. In particular, this means that $\|T_n(x)\| \leq M\|x\|$ for all $x \in X$. As a result, for all $x \in X$,

$$\|T(x)\| = \left\| \lim_{n \rightarrow \infty} T_n(x) \right\| = \lim_{n \rightarrow \infty} \|T_n(x)\| \leq M\|x\|.$$

Finally, we show that T is a linear functional. Indeed, if $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$, then

$$T(\alpha x + \beta y) = \lim_{n \rightarrow \infty} T_n(\alpha x + \beta y) = \lim_{n \rightarrow \infty} \alpha T_n(x) + \beta T_n(y) = \alpha T(x) + \beta T(y).$$

This shows that $T : X \rightarrow Y$ is a bounded linear functional, as desired.

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