Free, Projective, and Flat Modules

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§§ Cartier

THEOREM 0.1. Let M be a finitely generated module over an integral domain A. If for every $\mathfrak{m} \in \operatorname{MaxSpec}(A)$, $M_{\mathfrak{m}}$ is free as an $A_{\mathfrak{m}}$-module, then M is a projective A-module.

Proof. First show that M is a torsion-free A-module. Suppose $am = 0$ for some $0 \neq a \in A$ and $m \in M$. Let $\mathfrak{a}$ be the annihilator of m in A and $\mathfrak{m}$ a maximal ideal containing A. Note that $\frac{a}{1}\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$, which is free over $A_{\mathfrak{m}}$, an integral domain, whence, is torsion free. That is, $\frac{m}{1} = 0$, whence, there is some $s \in A \setminus \mathfrak{m}$ such that $sm = 0$, which is absurd, since $\mathfrak{a} \subseteq \mathfrak{m}$. This shows that M is torsion-free.

Now, choose a set of generators $\{m_i \colon 1 \leqslant i \leqslant n\}$ for M over A. Let $\mathscr{P}$ be the collection of A-endomorphisms of M which are of the form

$$m \longmapsto \sum_{i=1}^{n} f_i(m) m_i,$$

where $f_1, \dots, f_n : M \to A$ are A-module homomorphisms. Note that $\mathscr{P}$ is an A-submodule of $\operatorname{End}_A(M)$. We shall show that $\mathbf{id}_M \in \mathscr{P}$.

Let $\mathfrak{m}$ be a maximal ideal of A. We know that $M_{\mathfrak{m}}$ is free as an $A_{\mathfrak{m}}$-module and hence, there are $A_{\mathfrak{m}}$-module homomorphisms $f_i : M_{\mathfrak{m}} \to A_{\mathfrak{m}}$ such that

$$m' = \sum_{i=1}^{n} f_i'(m') \frac{m_i}{1} \quad \forall m' \in M_{\mathfrak{m}}.$$

To see that this is possible, first consider an $A_{\mathfrak{m}}$-basis $\{e_i \colon 1 \leqslant i \leqslant N\}$ for $M_{\mathfrak{m}}$. We can write

$$e_i = \sum_{j=1}^{n} a_{ij} \frac{m_j}{1} \quad \forall\, 1 \leqslant i \leqslant N.$$

Further, there are $A_{\mathfrak{m}}$-linear maps $f_i : M_{\mathfrak{m}} \to A_{\mathfrak{m}}$ such that

$$m' = \sum_{j=1}^{N} f_j(m') e_j.$$

Set

$$f'_j(m') = \sum_{i=1}^{N} a_{ij} f_i(m') \quad \forall\, m' \in M_{\mathfrak{m}}.$$

Then,

$$\sum_{j=1}^{n} f'_j(m') \frac{m_j}{1} = \sum_{i=1}^{N} \sum_{j=1}^{n} a_{ij} f_i(m') \frac{m_j}{1} = \sum_{i=1}^{N} f_i(m') e_i = m'.$$

Coming back, since M is torsion-free, the canonical map $M \to M_{\mathfrak{m}}$ is an injective map of A-modules. Further, we can find an $s \in A \setminus \mathfrak{m}$ such that $sf'_i\left(\frac{m_j}{1}\right) \in A$ for $1 \leqslant i, j \leqslant n$.

Note that $m' \mapsto sf'_i(m')$ is $A_{\mathfrak{m}}$-linear as a map $M_{\mathfrak{m}} \to A_{\mathfrak{m}}$, and hence, is A-linear. The restriction of this map to $M \subseteq M_{\mathfrak{m}}$ takes values in A. Thus, we can identify sf'_i with an A-linear map $M \to A$. Further, for every $m \in M$, we have

$$sm = \sum_{i=1}^{n} sf'_i(m) m_i.$$

That is, $s \cdot \mathbf{id}_M \in \mathscr{P}$. Now, let $\mathfrak{a}$ be the collection of all $a \in A$ such that $a \cdot \mathbf{id}_M \in \mathscr{P}$. Then $\mathfrak{a}$ is an ideal of A. If $\mathfrak{a}$ were a proper ideal, it would be contained in a maximal ideal $\mathfrak{m}$. But from our preceding conclusion, there is some $s \in A \setminus \mathfrak{m}$ such that $s \cdot \mathbf{id}_M \in \mathscr{P}$, a contradiction. Thus, $\mathfrak{a} = A$, in particular, $\mathbf{id}_M \in \mathscr{P}$.

Finally, we show that M is projective. We have shown that there are A-linear maps $f_i : M \to A$ such that

$$m = \sum_{i=1}^{n} f_i(m) m_i \quad \forall\, m \in M.$$

Let F be the free module $\bigoplus_{i=1}^{n} Ae_i$ and let $g : F \to M$ be given by $e_i \mapsto m_i$ and $f : M \to F$ given by

$$f(m) = \sum_{i=1}^{n} f_i(m) e_i.$$

By our construction, $g \circ f = \mathbf{id}_M$, and hence M is a direct summand of F, i.e. M is projective. ∎