Homological methods in Commutative Algebra

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§1 The Koszul Complex

Definition 1.1. Let A be a ring and $x_1, \ldots, x_n \in A$. Set $K_0 = A$ and for $1 \leqslant p \leqslant n$, let

$$K_p = \bigoplus_{1 \leqslant i_1 < \cdots < i_p \leqslant n} Ae_{i_1} \wedge \cdots \wedge e_{i_p},$$

which is a free module of rank $\binom{n}{p}$.

Define $d : K_p \to K_{p-1}$ as

$$d\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_p},$$

and extending linearly. This is called the *Koszul complex*. For an A-module M, we set $K_\bullet(\underline{x}, M) = K_\bullet(\underline{x}) \otimes_A M$. The homologies of this complex $H_p\left(K_\bullet(\underline{x}, M)\right)$ are denoted by $H_p(\underline{x}, M)$. For a complex $C_\bullet$ of A-modules, we set $C_\bullet(\underline{x}) = C_\bullet \otimes_A K_\bullet(\underline{x})$.[1]

Proposition 1.2. For $p \geqslant 2$, $d \circ d = 0$ as a map $K_p \to K_{p-2}$.

Proof. In the expression for $(d \circ d)\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right)$, we find the coefficient of

$$e_{i_1} \wedge \cdots \wedge \widehat{e_{i_a}} \wedge \cdots \wedge \widehat{e_{i_b}} \wedge \cdots \wedge e_{i_p},$$

where $1 \leqslant a < b \leqslant p$. The coefficient is equal to the coefficient in

$$(-1)^{a-1} x_{i_a} d\left(e_{i_1} \wedge \ldots \widehat{e}_{i_a} \wedge \cdots \wedge e_{i_p}\right) + (-1)^{b-1} x_{i_b} d\left(e_{i_1} \wedge \cdots \wedge e_{i_b} \wedge \cdots \wedge e_{i_p}\right),$$

which is equal to

$$(-1)^{a-1} x_{i_a} \cdot (-1)^{b-2} x_{i_b} + (-1)^{b-1} x_{i_b} \cdot (-1)^{a-1} x_{i_a} = 0. \qquad \blacksquare$$

[1] Recall that the tensor product of two complexes is obtained by taking the total complex corresponding to the tensor double complex.

THEOREM 1.3. Let $C_\bullet$ be a complex of A-modules and $x \in A$. There is an exact sequence of complexes

$$0 \longrightarrow C_\bullet \longrightarrow C_\bullet(x) \longrightarrow C_\bullet[-1] \longrightarrow 0.$$

This furnishes an exact sequence

$$\cdots \to H_p(C_\bullet) \to H_p(C_\bullet(x)) \to H_{p-1}(C_\bullet) \xrightarrow{(-1)^{p-1}x} H_{p-1}(C_\bullet) \to \cdots.$$

Further, we have $x \cdot H_p(C_\bullet(x)) = 0$ for all p.

COROLLARY. Let M be an A-module and $x_1, \ldots, x_n \in A$. Then, $(\underline{x})$ annihilates $H_p(\underline{x}, M)$ for all p.

Proof. Induct on n. The inductive step follows from the fact that

$$K_\bullet(x_1, \ldots, x_n, M) \cong K_\bullet(x_n) \otimes_A K(x_1, \ldots, x_{n-1}, M).$$

We know that $(x_1, \ldots, x_{n-1})$ annihilates the homology groups of the latter and hence, they annihilate the homology groups of $K_\bullet(x_1, \ldots, x_n, M)$. Further, due to the preceding theorem, x_n annihilates the homologies of the above tensor product. This completes the proof. ∎

THEOREM 1.4. Let M be an A-module and $x_1, \ldots, x_n \in A$ an M-sequence. Then

$$H_p(\underline{x}, M) = 0 \quad \text{for } p > 0 \qquad H_0(\underline{x}, M) = M/(\underline{x})M.$$

Proof. ∎

THEOREM 1.5. Let A be a Noethering ring, M a finite A-odule, and I an ideal of A; suppose that $IM \neq M$. For a positive integer n, the following conditions are equivalent:

(a) $\operatorname{Ext}^i_A(N, M) = 0$ for $0 \leqslant i < n$ and for any finite A-module N with $\operatorname{Supp}(N) \subseteq V(I)$.

(b) $\operatorname{Ext}^i_A(A/I, M) = 0$ for $0 \leqslant i < n$.

(c) $\operatorname{Ext}^i_A(N, M) = 0$ for $0 \leqslant i < n$ and *some* finite A-module N with $\operatorname{Supp}(N) = V(I)$.

(d) there exists an M-sequence of length n contained in I.

Proof. ∎

COROLLARY. Let A be a Noetherian ring, I an ideal of A, and M a finite A-module such that $M \neq IM$; then the length of a maximal M-sequence in I is determined by

$$\operatorname{Ext}^i_A(A/I, M) = 0 \quad \text{for } i < n \quad \text{and} \quad \operatorname{Ext}^n_A(A/I, M) \neq 0.$$

This integer is denoted by $\operatorname{depth}(I, M)$ and is called the *I-depth* of M. In other words,

$$\operatorname{depth}(I, M) = \inf\left\{i \colon \operatorname{Ext}^i_A(A/I, M) \neq 0\right\}.$$

For a Noetherian local ring $(A, \mathfrak{m}, k)$, we write $\operatorname{depth}_A M$ for $\operatorname{depth}(\mathfrak{m}, M)$.

COROLLARY. Let A be a Noetherian ring, I, I' ideals of A, and M a finite A-module such that $M \neq IM$ and $M \neq I'M$. Then, $\operatorname{depth}(I, M) = \operatorname{depth}(I', M)$.

§2 Regular Rings

Definition 2.1. Let $(A, \mathfrak{m}, k)$ be a local ring, M a finite A-module. A *minimal (free) resolution* is an exact sequence $L_\bullet \to M \xrightarrow{\varepsilon} 0$ such that

(a)

Lemma 2.2. Let $(A, \mathfrak{m}, k)$ be a local ring, and M a finite A-module. Suppose $L_\bullet \to M$ is a minimal resolution of M; then

(a) $\dim_k \operatorname{Tor}_i^A(M, k) = \operatorname{rank} L_i$ for all i,

(b) $\operatorname{proj\,dim} M = \sup \left\{ i \colon \operatorname{Tor}_i^A(M, k) \neq 0 \right\} \leqslant \operatorname{proj\,dim}_A k$,

(c) if $M \neq 0$ and $\operatorname{proj\,dim} M = r < \infty$, then for any finite A-module $N \neq 0$ we have $\operatorname{Ext}_A^r(M, N) \neq 0$.