

Commutative Algebra

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§1 Valuation Rings

DEFINITION 1.1. An integral domain R with field of fractions K is said to be a *valuation ring* if for every $x \in K^\times$, $x \in R$ or $x^{-1} \in R$. We also say that R is a valuation ring of K .

PROPOSITION 1.2. Let R be a valuation ring of K . The R -submodules of K are totally ordered with respect to inclusion.

Proof. Let I and J be distinct R -submodules of K . If $I \setminus J \neq \emptyset$, pick $x \in I \setminus J$. For any $0 \neq y \in J$, one of xy^{-1} or $x^{-1}y$ must belong to R . If $xy^{-1} \in R$, then $x = y \cdot (xy^{-1}) \in J$, which is absurd. Thus $x^{-1}y \in R$, so that $y = x \cdot (x^{-1}y) \in I$, and hence, $J \subseteq I$. Argue similarly for the case $J \setminus I \neq \emptyset$. ■

COROLLARY 1.3. A valuation ring is local.

PROPOSITION 1.4. The following are equivalent for a ring R :

- (1) R is a valuation ring.
- (2) R is a local Bézout domain.

Proof. (1) \implies (2) follows immediately from Proposition 1.2, since any finitely generated ideal in R can be written as $a_1R + \dots + a_nR$ for some $a_1, \dots, a_n \in R$.

(2) \implies (1): We must show that for every $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$. Choose $f, g \in R$ such that $x = \frac{f}{g}$, and let $(h) = (f, g)$ as ideals in R . We can find $a, b \in R$ such that $f = ah$ and $g = bh$, and can find $c, d \in R$ such that $h = cf + dg$. Since R is a domain and $h \neq 0$, it follows that $ac + bd = 1$. Therefore, either a or b must be a unit in R , so that either $x \in R$ or $x^{-1} \in R$, thereby completing the proof. ■

PROPOSITION 1.5. A valuation ring is integrally closed.

Proof. Let (R, \mathfrak{m}) be a valuation ring with fraction field K . If $x \in K \setminus R$ is integral over R , then there is a non-trivial relation of the form

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

with $a_i \in R$ for $0 \leq i \leq n-1$. Multiplying out by $x^{-n} \in \mathfrak{m}$, we have that $1 \in \mathfrak{m}$, a contradiction. Thus R is integrally closed. ■

REMARK 1.6. We note here that given a field K , a valuation ring of K is determined by its maximal ideal. Indeed, if (R, \mathfrak{m}) is a valuation ring of K , then we can write

$$K \setminus R = \{x^{-1} : x \in \mathfrak{m} \setminus \{0\}\}.$$

Further note that if R is a valuation ring of K , then any subring of K containing R is also a valuation ring of K .

THEOREM 1.7. Let R be a valuation ring of K and R' a subring of K containing R . Let \mathfrak{m} denote the maximal ideal of R , \mathfrak{p} the maximal ideal of R' , and suppose that $R \neq R'$. Then

- (1) $\mathfrak{p} \subsetneq \mathfrak{m} \subseteq R \subseteq R'$.

- (2) \mathfrak{p} is a prime ideal of R and $R' = R_{\mathfrak{p}}$.
- (3) R/\mathfrak{p} is a valuation ring of the field R'/\mathfrak{p} .
- (4) Given a valuation ring \bar{S} of the field R/\mathfrak{m} , let S denote its preimage in R . Then S is a valuation ring of K and is called the *composite* of R and \bar{S} .

Proof. (1) Let $0 \neq x \in \mathfrak{p}$. Then $x^{-1} \in K \setminus R' \subseteq K \setminus R$, so that $x \in \mathfrak{m}$. Thus $\mathfrak{p} \subseteq \mathfrak{m}$. In light of Remark 1.6, $\mathfrak{p} \neq \mathfrak{m}$.

- (2) Since $R/\mathfrak{p} \hookrightarrow R'/\mathfrak{p}$ as a subring, it is clear that \mathfrak{p} is a prime ideal in R . Further, since every element of $R \setminus \mathfrak{p}$ is invertible in R' , $R \subseteq R_{\mathfrak{p}} \subseteq R'$. Note that \mathfrak{p} is an ideal in $R_{\mathfrak{p}}$, and hence, $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}$, so that $R_{\mathfrak{p}} = R'$ due to (1).

(3) Straightforward.

- (4) Let $\pi: R \rightarrow R/\mathfrak{m}$ denote the projection map. Note that $\mathfrak{m} \subseteq S$ and $S/\mathfrak{m} = \bar{S}$. If $x \in R \setminus S$, then $\pi(x) \notin \bar{S}$, and therefore, $\pi(x^{-1}) = \pi(x)^{-1} \in S$. Hence $x^{-1} \in S$. On the other hand, if $x \in K \setminus R$, then $x^{-1} \in \mathfrak{m} \subseteq S$. This shows that S is a valuation ring of K . ■

THEOREM 1.8. Let K be a field, $A \subseteq K$ a subring, and \mathfrak{p} a prime ideal of A . Then there exists a valuation ring (R, \mathfrak{m}) of K such that $A \subseteq R$, and $\mathfrak{m} \cap A = \mathfrak{p}$.

Proof. First, replacing A by $A_{\mathfrak{p}}$, we may assume that A is a local ring with maximal ideal \mathfrak{p} . Let \mathcal{F} denote the set of all subrings B of K containing A such that $1 \notin \mathfrak{p}B$. Clearly, every ascending chain in \mathcal{F} has an upper bound given by the union of all elements of the chain. Using Zorn's lemma, choose a maximal element R in \mathcal{F} . First, we contend that R is a local ring. Indeed, since $1 \notin \mathfrak{p}R$, there is a maximal ideal \mathfrak{m} of R containing $\mathfrak{p}R$. Note that $R_{\mathfrak{m}} \in \mathcal{F}$, so that $R = R_{\mathfrak{m}}$ in view of the maximality of R . Next, $\mathfrak{m} \cap A$ is a proper ideal containing \mathfrak{p} , which, due to the maximality of \mathfrak{p} must be equal to \mathfrak{p} .

It remains to show that R is a valuation ring of K . Suppose $x \in K$ is such that $x, x^{-1} \notin R$. Then $R \subsetneq R[x]$, and hence $1 \in \mathfrak{p}R[x]$, i.e., there is a polynomial relation

$$1 = a_0 + a_1x + \cdots + a_nx^n$$

with $a_i \in \mathfrak{p}R \subseteq \mathfrak{m}$ for $0 \leq i \leq n$. Multiplying by $(1 - a_0)^{-1} \in R$, one obtains a relation of the form

$$1 = b_1x + \cdots + b_nx^n$$

with $b_i \in \mathfrak{p}R$ for $1 \leq i \leq n$. Choose one such relation with the smallest possible value of n . Arguing similarly for x^{-1} , choose a relation

$$1 = c_1x^{-1} + \cdots + c_mx^{-m}$$

with the smallest possible value of m . If $n \geq m$, then multiply the second equation by b_nx^n and subtract from the first to obtain a non-trivial relation of smaller degree than n , a contradiction. On the other hand, if $m > n$, then multiply the first relation by c_mx^{-m} and subtract from the second to obtain a non-trivial relation of smaller degree than m , a contradiction again. Thus $x \in R$ or $x^{-1} \in R$, i.e., R is a valuation ring of K . ■

THEOREM 1.9. Let K be a field, $A \subseteq K$ a subring, and B the integral closure of A in K . Then B is equal to the intersection of all valuation rings of K containing A .

Proof. Let C denote the intersection of all valuation rings of K containing A . In view of Proposition 1.5, B is contained in every such valuation ring, so that $B \subseteq C$. Now let $x \in K$ be non-integral over A and set $y = x^{-1}$. Note that $1 \notin yA[y]$, else x would be integral over A . Let \mathfrak{p} be a maximal ideal of $A[y]$ containing $yA[y]$, and using Theorem 1.8, choose a valuation ring (R, \mathfrak{m}) of K containing $A[y]$ such that $\mathfrak{m} \cap A[y] = \mathfrak{p}$. In particular, $y \in \mathfrak{m}$, so that $x \notin R$. Thus $x \notin C$, as desired. ■

§§ Valuations

DEFINITION 1.10. An abelian group Γ together with a total order relation \leq is said to be *ordered* if for all $x, y \in \Gamma$ with $x \leq y$, $x + z \leq y + z$ for all $z \in \Gamma$.

Let K be a field. A *valuation* on K is a map $v: K \rightarrow \Gamma \cup \{\infty\}$ satisfying:

- (i) $v(x) = \infty$ if and only if $x = 0$,
- (ii) $v(xy) = v(x) + v(y)$ ¹, and
- (iii) $v(x+y) \geq \min\{v(x), v(y)\}$

for all $x, y \in K$.

Corresponding to a valuation v on K , we can define

$$R_v = \{x \in K : v(x) \geq 0\} \quad \text{and} \quad \mathfrak{m}_v = \{x \in K : v(x) > 0\}.$$

It is not hard to see that (R_v, \mathfrak{m}_v) is a valuation ring of K . Conversely, we show that every valuation ring arises this way. Let

$$\Gamma = \{xR : x \in K^\times\}.$$

This is a group under the operation

$$(xR) \cdot (yR) = xyR$$

with neutral element R . Further, note that Γ is an ordered abelian group when equipped with the total ordering

$$xR \leq yR \iff xR \supseteq yR,$$

where we are implicitly invoking Proposition 1.2. Define $v: K \rightarrow \Gamma \cup \{\infty\}$ sending

$$v(x) = \begin{cases} xR & x \neq 0 \\ \infty & x = 0. \end{cases}$$

Note that $xR \geq R$ if and only if $xR \subseteq R$, that is, if $x \in R$. Hence, $R_v = R$ and $\mathfrak{m}_v = \mathfrak{m}$. This shows that every valuation ring arises from a valuation of K .

DEFINITION 1.11. The *value group* of a valuation ring (R, \mathfrak{m}) of K is $v(K^\times)$, where v is a valuation of K such that $(R_v, \mathfrak{m}_v) = (R, \mathfrak{m})$.

To see that the value group is well-defined:

PROPOSITION 1.12. If v and v' are two valuations of K corresponding to the same valuation ring R and having value groups H and H' respectively, then there is an order-isomorphism $\varphi: H \rightarrow H'$ such that $v' = \varphi \circ v$.

Proof. Let $v: K^\times \rightarrow \Gamma$ and $v': K^\times \rightarrow \Gamma'$ be the two valuations. Define $\varphi: H \rightarrow H'$ by $\varphi(v(x)) = v'(x)$ for all $x \in K^\times$. We must show that φ is well-defined. Indeed, if $x, y \in K^\times$ are such that $v(x) = v(y)$, then $x^{-1}y$ is a unit in R , so that $v'(x^{-1}y) = 0$, i.e., $v'(x) = v'(y)$. That φ is a group homomorphism is clear. The surjectivity of φ is immediate from the fact that $v' = \varphi \circ v$. As for injectivity, if $\varphi(v(x)) = 0$, then $v'(x) = 0$, and hence x is a unit in R , so that $v(x) = 0$. Finally, if $v(x) \leq v(y)$, then $v(x^{-1}y) \geq 0$, so that $x^{-1}y \in R$, i.e., $v'(x^{-1}y) \geq 0$, that is, $v'(y) \geq v'(x)$. Thus φ is an order-isomorphism. ■

§2 Discrete Valuation Rings and Dedekind Domains

DEFINITION 2.1. A valuation ring with value group order-isomorphic to \mathbb{Z} is called a *discrete valuation ring (DVR)*.

THEOREM 2.2. Let R be a valuation ring. Then the following are equivalent:

¹so that v is a group homomorphism when restricted to K^\times

- (1) R is a DVR.
- (2) R is a PID.
- (3) R is Noetherian.

Proof. Let K be the field of fractions of R and \mathfrak{m} its maximal ideal.

(1) \implies (2) Let $v: K^\times \rightarrow \mathbb{Z}$ be a surjective valuation corresponding to R . Let $t \in \mathfrak{m}$ be such that $v(t) = 1$. For $0 \neq x \in \mathfrak{m}$, $v(x) = n > 0$ for some positive integer n . Then $v(x/t^n) = 0$, i.e., $x = ut^n$ for some unit $u \in R^\times$. In particular, this shows that $\mathfrak{m} = tR$. Now let $0 \neq I$ be a proper ideal in R , and let

$$n = \min \{v(a) : 0 \neq a \in I\}.$$

Clearly n is a positive integer since I is proper. Let $x \in I$ with $v(x) = n$. Then $xR \subseteq I$, and for every $y \in I$, $v(y/x) \geq 0$, so that $y \in xR$. Thus $I = xR$, and R is a PID.

(2) \implies (3) is clear.

(3) \implies (2) A Noetherian Bézout domain is a PID.

(2) \implies (1) Let $t \in \mathfrak{m}$ be such that $\mathfrak{m} = tR$. Recall that a PID is a UFD, and let v denote the t -adic valuation on K . It is not hard to see that R is the valuation ring corresponding to v . ■

DEFINITION 2.3. If R is a DVR with maximal ideal \mathfrak{m} , then any element $t \in \mathfrak{m}$ such that $\mathfrak{m} = tR$ is said to be a *uniformizer* or a *uniformizing element* of R .

REMARK 2.4. We note here that a valuation ring whose maximal ideal is principal need not necessarily be a DVR. Indeed, let K be a field, (R, \mathfrak{m}_R) a DVR of K , set $k = R/\mathfrak{m}_R$, and suppose \mathfrak{N} is a DVR of k . Let S denote the composite of R and \mathfrak{N} . We contend that S is our desired counterexample.

Let $f \in \mathfrak{m}_R$ be a uniformizer of R , $g \in S$ such that $\bar{g} = g + \mathfrak{m}_R$ is a uniformizer of \mathfrak{N} . Then $\mathfrak{m}_S = \mathfrak{m}_R + gS$. Further, since $g \notin \mathfrak{m}_R$, it is a unit in R , so that $g^{-1} \in R$. Hence, for any element $h \in \mathfrak{m}_R$, we can write $h = g \cdot (g^{-1}h)$, so that $\mathfrak{m}_R \subseteq gS$. Hence $\mathfrak{m}_S = gS$ is principal.

Next, we show that S is not Noetherian. Indeed, consider the ascending chain of ideals in S :

$$(f) \subseteq (f, fg^{-1}) \subseteq (f, fg^{-1}, fg^{-2}) \subseteq \dots$$

We claim that all the above inclusions are proper. Indeed, if $fg^{-(n+1)} \in (f, fg^{-1}, \dots, fg^{-n})$ for some $n \geq 0$, then there exist $a_0, \dots, a_n \in S$ such that

$$fg^{-(n+1)} = a_0f + a_1fg^{-1} + \dots + a_nfg^{-n}.$$

Multiplying out by $f^{-1}g^{n+1}$, we obtain

$$a_0g^{n+1} + \dots + a_ng = 1,$$

which is absurd, since $1 \notin \mathfrak{m}_S = gS$, thereby completing the proof.

THEOREM 2.5. Let R be a ring. The following conditions are equivalent:

- (1) R is a DVR.
- (2) R is a local PID but not a field.
- (3) R is a Noetherian local ring, $\dim R > 0$, and the maximal ideal of R is principal.
- (4) R is a one-dimensional normal Noetherian local domain.

Proof. (1) \implies (2) \implies (3) is clear.

(3) \implies (1) Let $\mathfrak{m} = tR$ denote the maximal ideal of R . Due to Krull's intersection theorem,

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0).$$

Hence, for every $0 \neq x \in R$, there is a non-negative integer n such that $x \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$. Set $v(x) = n$. Now if $x, y \in R \setminus \{0\}$ are such that $v(x) = n$ and $v(y) = m$, then we can find units $u, v \in R^\times$ such that $x = t^n u$ and $y = t^m v$.

This shows that $xy = t^{m+n}uv \neq 0$, so that R is an integral domain and $v(xy) = v(x) + v(y)$. Let K denote the fraction field of R , and set

$$v\left(\frac{a}{b}\right) = v(a) - v(b)$$

for all $a, b \in R \setminus \{0\}$. This is clearly well-defined, and defines a valuation on K whose value group is \mathbb{Z} and corresponding valuation ring is R . Hence R is a DVR.

(1) \Rightarrow (4) Recall that in a DVR the only ideals are (0) and powers of the maximal ideal. Thus the only prime ideals are (0) and \mathfrak{m} , so that $\dim R = 1$. That R is Noetherian follows from it being a PID, and finally recall that every valuation ring is normal.

(4) \Rightarrow (3) Let \mathfrak{m} denote the maximal ideal of R . Due to Nakayama's lemma and the fact that $\dim R = 1$, $\mathfrak{m} \neq \mathfrak{m}^2$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We shall show that $\mathfrak{m} = xR$. Note that $\mathfrak{m} \in \text{Ass}_R(R/xR)$, so that there exists $y \in R \setminus xR$ such that $(xR : y) = \mathfrak{m}$. Set $a = yx^{-1} \in K := \text{Frac}(R)$. Note that $a \notin R$ lest $y \in xR$. Set

$$\mathfrak{m}^{-1} := \{\alpha \in K : \alpha\mathfrak{m} \subseteq R\}.$$

Note that \mathfrak{m}^{-1} is an R -submodule of K , and further, by construction,

$$\mathfrak{m}\mathfrak{m}^{-1} := \{R\text{-submodule of } K \text{ generated by } x_i y_i : x_i \in \mathfrak{m}, y_i \in \mathfrak{m}^{-1}\} \subseteq R \quad \text{and} \quad R \subseteq \mathfrak{m}^{-1}.$$

In particular, we have the inclusions $\mathfrak{m} \subseteq \mathfrak{m}\mathfrak{m}^{-1} \subseteq R$. Hence, $\mathfrak{m}\mathfrak{m}^{-1} \in \{\mathfrak{m}, R\}$. If $\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$, then $a\mathfrak{m} \subseteq \mathfrak{m}$. Since \mathfrak{m} is a finite R -module, due to Nakayama's lemma, a must be integral over R , but since R is normal, $a \in R$, a contradiction. Thus $\mathfrak{m}^{-1}\mathfrak{m} = R$. Next, note that $x\mathfrak{m}^{-1} \subseteq R$. If $x\mathfrak{m}^{-1} \subseteq \mathfrak{m}$, then

$$xR = x\mathfrak{m}^{-1}\mathfrak{m} \subseteq \mathfrak{m}^2,$$

a contradiction. Hence, $x\mathfrak{m}^{-1} = R$, so that

$$xR = x\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m},$$

thereby completing the proof. ■

DEFINITION 2.6. Let R be an integral domain with fraction field K . An R -submodule I of K is said to be a *fractional ideal* if there is a non-zero $\alpha \in R$ such that $\alpha I \subseteq R$.

The standard operations on ideals readily generalize to operations on fractional ideals. Indeed, if I and J are fractional ideals, define

$$\begin{aligned} I + J &= \{x + y : x \in I, y \in J\} \\ IJ &= \{R\text{-submodule generated by } x_i y_i : x_i \in I, y_i \in J\}. \end{aligned}$$

Clearly, both $I + J$ and IJ are fractional ideals of R . Further, if $S \subseteq R$ is a multiplicative set, then

$$S^{-1}I = \left\{ \frac{x}{s} : x \in I, s \in S \right\}$$

is a fractional ideal of $S^{-1}R$.

LEMMA 2.7. Let R be an integral domain, M and N R -submodules of $K = \text{Frac}(R)$. If N is finitely generated, then

$$S^{-1}(M : N) = (S^{-1}M : S^{-1}N).$$

Proof. Clearly $S^{-1}(M : N) \subseteq (S^{-1}M : S^{-1}N)$. Since N is finitely generated, we can write $N = a_1R + \dots + a_nR$ for some $a_1, \dots, a_n \in K$. If $x \in (S^{-1}M : S^{-1}N)$, then for all $1 \leq i \leq n$, $xa_i \in S^{-1}M$. Therefore, there exist $c_i \in S$ such that $c_i xa_i \in M$, whence $c_i x \in (M : N)$, so that $x \in S^{-1}(M : N)$. ■

DEFINITION 2.8. An R -submodule I of K is said to be *invertible* if there exists an R -submodule J of K such that $IJ = R$.

PROPOSITION 2.9. An invertible R -submodule of K must be a finite R -module.

Proof. Let I be an invertible fractional ideal of R and J an R -submodule of K such that $IJ = R$. Then we can find $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in J$ such that

$$1 = a_1 b_1 + \dots + a_n b_n.$$

For each $x \in I$, we can write

$$x = (xb_1)a_1 + \dots + (xb_n)a_n,$$

and note that $xb_i \in R$ for $1 \leq i \leq n$. It follows that $I = a_1R + \dots + a_nR$. ■

COROLLARY 2.10. Every invertible R -submodule of K is a fractional ideal.

REMARK 2.11. Suppose I is an invertible fractional ideal of R , and set

$$I^{-1} := \{\alpha \in K : \alpha I \subseteq R\}.$$

This is clearly an R -submodule of K . Further, note that $II^{-1} \subseteq R$, so that

$$I^{-1} = JI^{-1} \subseteq J.$$

But since $JI \subseteq R$, we clearly have $J \subseteq I^{-1}$. This shows that $J = I^{-1}$.

THEOREM 2.12. Let R be an integral domain and I an R -submodule of K . The following are equivalent:

- (1) I is an invertible fractional ideal.
- (2) I is a projective R -module.
- (3) I is a finite R -module, and for each maximal ideal \mathfrak{m} of R , the fractional ideal $I_{\mathfrak{m}} := IR_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is principal.

Proof. (1) \Rightarrow (2) Let $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in I^{-1}$ be such that $1 = a_1 b_1 + \dots + a_n b_n$. As we have seen in the proof of Proposition 2.9, $I = a_1R + \dots + a_nR$. Let $\pi: F := \bigoplus_{i=1}^n Re_i \rightarrow I$ be the map sending $e_i \mapsto a_i$. Define $\sigma: I \rightarrow F$ by

$$\sigma(x) = (xb_1)e_1 + \dots + (xb_n)e_n.$$

It is then clear that $\pi \circ \sigma = \text{id}_I$, so that I is projective.

(2) \Rightarrow (1) There is a free module $F = \bigoplus_i Re_i$ and an epimorphism $\pi: F \rightarrow I$ which splits through an R -linear map $\sigma: I \rightarrow F$. Let $a_i = \pi(e_i)$ and $\lambda_i = \pi_i \circ \sigma: I \rightarrow R$. Note that every R -linear map $I \rightarrow R$ is multiplication by some element of K . Say $\lambda_i(x) = b_i x$ for all $x \in I$. Note that $b_i \in I^{-1}$ for all i . Then

$$\pi(\sigma(x)) = \sum_i a_i b_i x \implies \sum_i a_i b_i = 1.$$

Set J denote the R -submodule of K generated by the b_i 's. Then $R \subseteq IJ \subseteq R$, and hence I is an invertible fractional ideal.

(1) \Rightarrow (3) That I is a finite R -module is the content of Proposition 2.9. Further,

$$R_{\mathfrak{m}} = (II^{-1})_{\mathfrak{m}} = I_{\mathfrak{m}}(I^{-1})_{\mathfrak{m}},$$

so that $I_{\mathfrak{m}}$ is an invertible fractional ideal of $R_{\mathfrak{m}}$. Due to (2), $I_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$ -module. But since any two elements of K are $R_{\mathfrak{m}}$ -linearly dependent, $I_{\mathfrak{m}}$ must be principal.

(3) \Rightarrow (1) Note that for every maximal ideal \mathfrak{m} of R ,

$$(I^{-1})_{\mathfrak{m}} = (R:I)_{\mathfrak{m}} = (R_{\mathfrak{m}}:I_{\mathfrak{m}}) = (I_{\mathfrak{m}})^{-1}.$$

If $II^{-1} \subsetneq R$, then there is a maximal ideal \mathfrak{m} containing II^{-1} , so that

$$\mathfrak{m}R_{\mathfrak{m}} \supseteq (II^{-1})_{\mathfrak{m}} = I_{\mathfrak{m}}(I^{-1})_{\mathfrak{m}} = I_{\mathfrak{m}}I_{\mathfrak{m}}^{-1} = R_{\mathfrak{m}},$$

a contradiction. This completes the proof. ■

THEOREM 2.13. Let R be a Noetherian domain, and \mathfrak{p} a non-zero prime ideal of R . If \mathfrak{p} is invertible, then $\text{ht}\mathfrak{p} = 1$ and $R_{\mathfrak{p}}$ is a DVR.

Proof. Since \mathfrak{p} is invertible, due to Theorem 2.12, $\mathfrak{p}R_{\mathfrak{p}}$ is a principal ideal. Since $\dim R_{\mathfrak{p}} = \text{ht}\mathfrak{p} > 0$, it follows from Theorem 2.5 that $R_{\mathfrak{p}}$ is a DVR, and hence $\text{ht}\mathfrak{p} = \dim R_{\mathfrak{p}} = 1$. ■

THEOREM 2.14. Let R be a normal Noetherian domain. Then

(1) all prime divisors of a non-zero principal ideal have height 1.

$$(2) R = \bigcap_{\text{ht}\mathfrak{p}=1} R_{\mathfrak{p}}.$$

Proof. (1) This follows from Krull's Hauptidealsatz.

(2) Suppose $\frac{a}{b} \in \bigcap_{\text{ht}\mathfrak{p}=1} R_{\mathfrak{p}}$ with $b \neq 0$. We shall show that $a \in bR$. If b is a unit in R , then there is nothing to prove. If b is not a unit in R , consider the primary decomposition

$$bR = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

with corresponding associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Due to (1), we know that $\text{ht}\mathfrak{p}_i = 1$ for all i , so that there are no embedded associated primes. Localising at \mathfrak{p}_i and contracting back to R , we have

$$\mathfrak{q}_i = bR_{\mathfrak{p}_i} \cap R \ni a.$$

Therefore,

$$a \in \bigcap_{i=1}^r \mathfrak{q}_i = bR,$$

as desired. ■

PORISM 2.15. Let R be a Noetherian domain. If all associated primes of a non-zero principal ideal have height 1, then

$$R = \bigcap_{\text{ht}\mathfrak{p}=1} R_{\mathfrak{p}}.$$

COROLLARY 2.16. Let R be a Noetherian domain. Then R is normal if and only if the following two conditions are satisfied:

- (i) for every height 1 prime \mathfrak{p} , $R_{\mathfrak{p}}$ is a DVR; and
- (ii) all associated primes of a non-zero principal ideal of R have height 1.

Proof. Necessity is the content of Theorem 2.14. For sufficiency, note that due to Porism 2.15, $R = \bigcap_{\text{ht}\mathfrak{p}=1} R_{\mathfrak{p}}$. But since each $R_{\mathfrak{p}}$ is a DVR, it is normal, so that R is normal. ■

DEFINITION 2.17. An integral domain for which every ideal is invertible is called a *Dedekind domain*.

THEOREM 2.18. For an integral domain R , the following conditions are equivalent:

- (1) R is a Dedekind domain.
- (2) R is either a field or a one-dimensional Noetherian normal domain.
- (3) every non-zero ideal of R can be written as a product of a finite number of prime ideals.

Proof. ■