Lie Algebras

Swayam Chube

December 19, 2024

Contents

§1 SEMISIMPLE LIE ALGEBRAS

Throughout this section, we shall assume that k is an algebraically closed field of characteristic 0.

The end goal of this section is to establish the root space decomposition and the corresponding Euclidean root system.

§§ Lie's Theorem

THEOREM 1.1. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$ where V is finite-dimensional. If $V \neq 0$, then V contains a common eigenvector for all the endomorphisms in $\mathfrak{g}$.

Proof. We induct on $\dim \mathfrak{g}$. The base case $\dim \mathfrak{g} = 0$ is trivial. Suppose now that $\dim \mathfrak{g} > 0$. Since $\mathfrak{g}$ properly contains $[\mathfrak{g}, \mathfrak{g}]$, and the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, we can choose an ideal $\mathfrak{h} \subsetneq \mathfrak{g}$ of codimension 1 by pulling back any subspace of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of codimension 1.

By the inductive hypothesis, there is a common eigenvector $v \in V$ for $\mathfrak{h}$, whence, there is a linear functional $\lambda : \mathfrak{h} \to k$ such that $x \cdot v = \lambda(x)v$ for every $x \in \mathfrak{h}$. Consider the subspace

$$W = \{w \in V \colon x \cdot w = \lambda(x)w, \ \forall\, x \in \mathfrak{h}\}.$$

Since $v \in W$, $W \neq 0$.

We contend that W is $\mathfrak{g}$-invariant. Let $w \in W$ and $x \in \mathfrak{g}$. To show that $x \cdot w \in W$, we must show that for every $y \in \mathfrak{h}$,

$$\lambda(y)x \cdot w = y \cdot (x \cdot w) = (xy) \cdot w - [x, y] \cdot w = \lambda(y)x \cdot w - \lambda([x, y])w.$$

That is, we must show that $\lambda([x, y]) = 0$ whenever $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$. Let $n > 0$ be the smallest integer for which $w, x \cdot w, \ldots, x^n \cdot w$ are linearly dependent. Let W_i be the subspace of V spanned by $w, x \cdot w, \ldots, x^{i-1} \cdot w$ with the convention that $W_0 = 0$, so that

$$W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n.$$

Obviously, y leaves W_0 and W_1 invariant. For $1 \leqslant i \leqslant n-1$, we have

$$y \cdot (x^i \cdot w) = x^i \cdot (y \cdot w) - [x^i, y] \cdot w = \lambda(y)x^i \cdot w - \lambda([x^i - y]) \cdot w.$$

Hence, y leaves every W_i invariant. Relative to the basis $\{w, x \cdot w, \ldots, x^{n-1} \cdot w\}$ of W_n, due to the above equation, y is represented by an upper triangular matrix with every diagonal entry equal to $\lambda(y)$. Hence, $\mathrm{Tr}_{W_n}(y) = n\lambda(y)$ and this equality holds for all $y \in \mathfrak{h}$.

But note that x stabilizes W_n (due to our choice of n), and hence, x is an endomorphism of W_n too. As a result, $\mathrm{Tr}_{W_n}([x, y]) = 0$, consequently, $n\lambda([x, y]) = 0$, that is, $\lambda([x, y]) = 0$ since we are in characteristic 0. Hence, we have shown that W is $\mathfrak{g}$-invariant.

Finally, write $\mathfrak{g} = \mathfrak{h} + \langle z \rangle$ as vector spaces. Since k is algebraically closed, there is an eigenvector $v_0 \in W$ for z. Then, v_0 is a common eigenvector for all of $\mathfrak{g}$. ■

COROLLARY 1.2 (LIE'S THEOREM). Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is a finite-dimensional k-vector space. Then $\mathfrak{g}$ stabilizes a complete flag in V.

Proof. Induct on $\dim V$. Choose a common eigenvector $v_1 \in V$ for $\mathfrak{g}$ and set $V_1 = \langle v_1 \rangle$. Note that V_1 is $\mathfrak{g}$-invariant and hence, $\mathfrak{g}$ acts naturally on V/V_1. The image of $\mathfrak{g}$ in $\mathfrak{gl}(V/V_1)$ is a solvable subalgebra, whence stabilizes a complete flag

$$\frac{V_2}{V_1} \subsetneq \cdots \frac{V_n}{V_1},$$

due to the inductive hypothesis. It follows that $\mathfrak{g}$ stabilizes the complete flag

$$V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n.$$

■

COROLLARY 1.3. Let $\mathfrak{g}$ be solvable. Then there exists a chain of ideals of $\mathfrak{g}$,

$$0 = \mathfrak{g}_0 \subsetneq \mathfrak{g}_1 \subsetneq \cdots \subsetneq \mathfrak{g}_n = \mathfrak{g},$$

such that $\dim \mathfrak{g}_i = i$.

Proof. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Due to the preceding result, there is a complete flag

$$0 = \mathfrak{g}_0 \subsetneq \mathfrak{g}_1 \subsetneq \cdots \subsetneq \mathfrak{g}_n = \mathfrak{g},$$

stabilized by $\mathfrak{g}$. That is, each $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$. This completes the proof. ■

COROLLARY 1.4. Let $\mathfrak{g}$ be solvable. Then for every $x \in [\mathfrak{g}, \mathfrak{g}]$, $\mathrm{ad}_{\mathfrak{g}} x$ is nilpotent. In particular, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. Due to the preceding result, there is a basis of $\mathfrak{g}$ with respect to which $\mathrm{ad}_{\mathfrak{g}} x$ is upper triangular for every $x \in \mathfrak{g}$. Consequently, $\mathrm{ad}_{\mathfrak{g}}[x, y] = [\mathrm{ad}_{\mathfrak{g}} x, \mathrm{ad}_{\mathfrak{g}} y]$ is strictly upper triangular, whence, is nilpotent. This proves the first assertion. Since $\mathrm{ad}_{\mathfrak{g}} x$ is nilpotent, so is $\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} x$. Due to Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. ■

§§ Jordan-Chevalley Decomposition

DEFINITION 1.5. Let V be a finite-dimensional k-vector space. An element $x \in \operatorname{End} V$ is called *semisimple* if its minimal polynomial over k is separable. Equivalently, since k is separable, x is semisimple if and only if it is diagonalizable.

REMARK 1.6. Two commuting semisimple endomorphisms of V are simultaneously diagonalizable. Further, if x is semisimple and stabilizes a subspace W of V, then the restriction of x to W is semisimple, since the minimal polynomial of x restricted to W divides the minimal polynomial of x.

THEOREM 1.7. Let V be a finite-dimensional vector space over k and $x \in \operatorname{End} V$.

(a) There exist unique $x_s, x_n \in \operatorname{End} V$ such that $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, and $x_s x_n = x_n x_s$.

(b) There exist polynomials $p(T), q(T) \in k[T]$ without constant term, such that $x_s = p(x)$ and $x_n = q(x_n)$.

(c) If $A \subseteq B \subseteq V$ are subspaces, and x maps B into A, then x_s and x_n also map B into A.

The decomposition $x = x_s + x_n$ is called the (additive) *Jordan-Chevalley decomposition* of x; x_s and x_n are called (respectively) the *semisimple part* and the *nilpotent part* of x.

Proof. Let $a_1, \ldots, a_r$ be the distinct eigenvalues of x with multiplicities $m_1, \ldots, m_r$, that is, the characteristic polynomial of x is $\prod_{i=1}^{r}(T - a_i)^{m_i}$. Set $V_i = \ker(x - a_i \cdot 1)^{m_i}$ (the generalized eigenspaces), and note that it is stable under the action x. It is not hard to argue that $V = V_1 + \cdots + V_r$. Further, since $\prod_{j \neq i}(x - a_j \cdot 1)$ annihilates every V_j for $j \neq i$ and acts on V_i by $\prod_{j \neq i}(a_j - a_i) \neq 0$, we see that the sum is direct, that is, $V = V_1 \oplus \cdots \oplus V_r$.

Note that the restriction of x to V_i has characteristic polynomial $(T - a_i)^{m_i}$. Using the Chinese Remainder Theorem, we can find a polynomial $p(T) \in k[T]$ satisfying

$$p(T) \equiv a_i \bmod (T - a_i)^{m_i} \quad \forall\, 1 \leqslant i \leqslant r, \qquad p(T) \equiv 0 \pmod{T}.$$

Set $q(T) = T - p(T)$, $x_s = p(x)$, and $x_n = q(x)$. Since they are polynomials in x, $x_s x_n = x_n x_s$, and they stabilize each V_i. Since $(T - a_i)^{m_i}$ divides $p(T) - a_i$, we note that the restriction of $x_s - a_i \cdot 1$ to V_i is zero, whence x_i acts by scalars on each V_i for $1 \leqslant i \leqslant r$. By definition, $x_n = x - x_s$, and hence, x_n acts on V_i by $(x - a_i \cdot 1)$. It follows that x_n is nilpotent on each V_i, and hence, on V.

It remains to establish the uniqueness of the decomposition in (a). Suppose $x = s + n$ is another such decomposition. Let $W_1 \oplus \cdots \oplus W_r$ be the eigenspace decomposition of W with respect to s (which exists because s is semisimple). Note that $x - a_i \cdot 1$ and $x - s = n$ restrict to the same endomorphism of W_i. Hence, $x - \lambda_i$ restricts to a nilpotent endomorphism of W_i. It follows that $W_i \subseteq V_i$. On the other hand, because $V = W_1 + \cdots + W_r$, we see that $W_i = V_i$. Since both x_s and s have the same eigenspaces (and are semisimple), they must be equal. It follows that $x_n = n$, thereby establishing uniqueness. ∎

PROPOSITION 1.8. Let V be a finite-dimensional k-vector space. If $x \in \mathfrak{gl}(V)$ is semisimple (resp. nilpotent), then $\operatorname{ad}_{\mathfrak{gl}(V)} x$ is semisimple (resp. nilpotent) as an element of $\mathfrak{gl}(\mathfrak{gl}(V))$.

Proof. Suppose x is semisimple. Choose a basis $v_1, \ldots, v_n$ of V with respect to which x is given by the matrix $\operatorname{diag}(a_1, \ldots, a_n)$. Let $\{e_{ij}\}$ denote the standard basis of $\mathfrak{gl}(V)$ with respect to this basis, that is, $e_{ij}(v_k) = \delta_{jk} v_i$. It is then easy to check that $\operatorname{ad} x(e_{ij}) = (a_i - a_j)e_{ij}$. Hence, $\operatorname{ad} x$ is semisimple as an element of $\mathfrak{gl}(\mathfrak{gl}(V))$.

Next, suppose x is nilpotent. We can write $\operatorname{ad} x = \lambda - \rho$, where $\lambda : y \mapsto xy$ and $\rho : y \mapsto yx$. Since λ and ρ are commuting nilpotent endomorphisms of $\mathfrak{gl}(V)$, we have that $\operatorname{ad} x = \lambda - \rho$ is a nilpotent endomorphism of $\mathfrak{gl}(\mathfrak{gl}(V))$. ∎

COROLLARY 1.9. Let V be a finite-dimensional k-vector space, $x \in \mathfrak{gl}(V)$, and $x = x_s + x_n$ be its Jordan decomposition. Then $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$ is the Jordan decomposition of $\operatorname{ad} x$ in $\mathfrak{gl}(\mathfrak{gl}(V))$.

Proof. Due to the preceding result, $\operatorname{ad} x_s$ is semisimple and $\operatorname{ad} x_n$ is nilpotent. Further, they commute because

$$[\operatorname{ad} x_s, \operatorname{ad} x_n] = \operatorname{ad}[x_s, x_n] = 0.$$

By uniqueness, $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$ is the Jordan decomposition in $\mathfrak{gl}(\mathfrak{gl}(V))$. ∎

LEMMA 1.10. Let $\mathfrak{A}$ be a k-algebra, $\delta \in \operatorname{Der}\mathfrak{A}$, $a, b \in k$, and $x, y \in \mathfrak{A}$. Then

$$(\delta - (a+b)\cdot 1)^n\,(xy) = \sum_{i=0}^{n} \binom{n}{i} \left((\delta - a\cdot 1)^{n-i} x\right) \cdot \left((\delta - b \cdot 1)^i y\right)$$

for all $n > 0$.

Proof. We prove this by induction on n. The base case with $n = 1$ is trivial. For $n > 1$, write

$$(\delta - (a+b)\cdot 1)^n (xy) = (\delta - (a+b)\cdot 1)^{n-1}\left((\delta - a\cdot 1)x \cdot y + x \cdot (\delta - b \cdot 1)y\right).$$

Now use the inductive hypothesis and the fact that

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}. \qquad \blacksquare$$

PROPOSITION 1.11. Let $\mathfrak{A}$ be a finite-dimensional k-algebra. Then $\operatorname{Der}\mathfrak{A}$ contains the semisimple and nilpotent parts (in $\operatorname{End}\mathfrak{A}$) of all its elements.

Proof. Let $\delta \in \operatorname{Der}\mathfrak{A}$, and let $\sigma, \nu \in \operatorname{End}\mathfrak{A}$ be its semisimple and nilpotent parts respectively. We shall show that $\sigma \in \operatorname{Der}\mathfrak{A}$. Let $\mathfrak{A}_a$ denote the generalized eigenspace of δ corresponding to $a \in k$, which is also the eigenspace corresponding to σ, by construction. Using the preceding proposition, it is easy to see that $\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}$ for all $a, b \in k$. For $x \in \mathfrak{A}_a$ and $y \in \mathfrak{A}_b$, we have

$$\sigma(xy) = (a+b)xy = \sigma(x)y + x\sigma(y).$$

Finally, since $\mathfrak{A} = \bigoplus \mathfrak{A}_a$, the above equality holds for all $x, y \in \mathfrak{A}$, whence σ is a derivation as desired. $\blacksquare$

§§ Cartan's Criterion

LEMMA 1.12. Let $A \subseteq B$ be two subspaces of $\mathfrak{gl}(V)$, $\dim V < \infty$. Let

$$M = \{x \in \mathfrak{gl}(V) \colon [x, B] \subseteq A\}.$$

Suppose $x \in M$ satisfies $\operatorname{Tr}(xy) = 0$ for all $y \in M$, then x is nilpotent.

Proof. Let s be the semisimple part of x. Fix a basis $v_1, \ldots, v_n$ of V relative to which s has matrix form $\operatorname{diag}(a_1, \ldots, a_n)$. Let E be the $\mathbb{Q}$-vector subspace of k spanned by $a_1, \ldots, a_n$. We shall show that $E = 0$, for which it would suffice to show that the dual space $E^* = 0$.

Let $f : E \to \mathbb{Q}$ be a linear transformation. Let $y \in \mathfrak{gl}(V)$ be such that the matrix representation of y with respect to the basis $v_1, \ldots, v_n$ is $\operatorname{diag}(f(a_1), \ldots, f(a_n))$. If $\{e_{ij}\}$ is the standard basis of $\mathfrak{gl}(V)$ with respect to the aforementioned basis, then $\operatorname{ad} s(e_{ij}) = (a_i - a_j)e_{ij}$ and $\operatorname{ad} y(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$.

Now, let $r(T) \in k[T]$ be a polynomial such that $r(a_i - a_j) = f(a_i) - f(a_j)$ and $r(0) = 0$. Note that this data is consistent, for if $a_i - a_j = a_k - a_l$, then due to the linearity of f, we have

$$f(a_i) - f(a_j) = f(a_i - a_j) = f(a_k - a_l) = f(a_k) - f(a_l).$$

It follows that $\operatorname{ad} y = f(\operatorname{ad} s)$ as a linear transformation $\mathfrak{gl}(V) \to \mathfrak{gl}(V)$.

Now, $\operatorname{ad} s$ is the semisimple part of $\operatorname{ad} x$, and hence it can be written as a polynomial in $\operatorname{ad} x$ without constant term. Therefore, $\operatorname{ad} y$ is also a polynomial in $\operatorname{ad} x$ without constant term (since $r(T)$ does not have a constant term). By the hypothesis, $\operatorname{ad} x$ maps B into A, consequently, $\operatorname{ad} y$ also maps B into A, consequently, $y \in M$. Thus,

$$0 = \operatorname{Tr}(xy) = \operatorname{Tr}(sy) + \operatorname{Tr}(x_n y) = \operatorname{Tr}(sy) = \sum_{i=1}^{n} a_i f(a_i). \qquad (??)$$

Applying f, we get $\sum_{i=1}^{n} f(a_i)^2 = 0$, that is, $f(a_i) = 0$ for $1 \leqslant i \leqslant n$, in particular, $f = 0$. This proves that $E = 0$, and consequently, each $a_i = 0$, whence $s = 0$ and $x = x_n$ is nilpotent. ■

THEOREM 1.13 (CARTAN'S CRITERION). Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$ where V is a finite-dimensional k-vector space. Suppose $\operatorname{Tr}(xy) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

Proof. It suffices to show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ being abelian, is solvable. Due to Engel's theorem, it suffices to show that $\operatorname{ad}_{[\mathfrak{g},\mathfrak{g}]} x$ is nilpotent, for which it suffices to show that $\operatorname{ad}_{\mathfrak{g}} x$ is nilpotent. We shall show that every $x \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent as an endomorphism of V, whence it would follow that $\operatorname{ad}_{\mathfrak{g}} x$ is nilpotent.

To this end, we would like to invoke the preceding result with $A = [\mathfrak{g}, \mathfrak{g}]$, and $B = \mathfrak{g}$ and

$$M = \{x \in \mathfrak{gl}(V) \colon [x, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]\} \supseteq \mathfrak{g}.$$

It remains to show that $\operatorname{Tr}(xy) = 0$ for every $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in M$.

Now, $[\mathfrak{g}, \mathfrak{g}]$ is generated by $[x, y]$ where $x, y \in \mathfrak{g}$. For any $z \in M$, we have

$$\operatorname{Tr}([x, y]z) = \operatorname{Tr}(x[y, z]) = \operatorname{Tr}([y, z]x).$$

By definition of M, $[y, z] \in [\mathfrak{g}, \mathfrak{g}]$, whence by our hypothesis, $\operatorname{Tr}([y, z]x) = 0$. The conclusion now follows. ■

COROLLARY 1.14. Let $\mathfrak{g}$ be a Lie algebra such that $\operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

Proof. Let $\mathfrak{h}$ denote the image of the adjoint representation of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$. Note that $[\mathfrak{h}, \mathfrak{h}] = \operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$, and hence, $\mathfrak{h}$ is solvable due to Cartan's criterion. Since $\mathfrak{h} \cong \mathfrak{g}/Z(\mathfrak{g})$ and $Z(\mathfrak{g})$ is solvable owing to it being abelian, we have that $\mathfrak{g}$ is solvable. ■

§§ Killing Form

DEFINITION 1.15. Let $\mathfrak{g}$ be a Lie algebra over k. Define $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$ by $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$, where the trace is taken as an element of $\mathfrak{gl}(\mathfrak{g})$. Then κ is a symmetric bilinear form and is called the *Killing form*.

REMARK 1.16. The Killing form is also *associative*, that is, $\kappa([x, y], z) = \kappa(x, [y, z])$. Indeed, we have

$$\kappa([x, y], z) = \operatorname{Tr}\left([\operatorname{ad} x, \operatorname{ad} y] \operatorname{ad} z\right) = \operatorname{Tr}(\operatorname{ad} x[\operatorname{ad} y, \operatorname{ad} z]) = \kappa(x, [y, z]).$$

LEMMA 1.17. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. If κ is the killing form of $\mathfrak{g}$, and $\kappa_{\mathfrak{a}}$ is the Killing form of $\mathfrak{a}$ (viewed as a Lie algebra), then $\kappa_{\mathfrak{a}} = \kappa|_{\mathfrak{a}\times\mathfrak{a}}$.

Proof. If $x, y \in \mathfrak{a}$, then $(\operatorname{ad}_{\mathfrak{g}} x)(\operatorname{ad}_{\mathfrak{g}} y)$ maps $\mathfrak{g}$ into $\mathfrak{a}$, so its trace as an endomorphism of $\mathfrak{g}$ is equal to the trace of the map viewed as an endomorphism of $\mathfrak{a}$. ■

REMARK 1.18. We have tacitly used the fact that if $T : V \to W \subseteq V$ is a linear transformation, then $\mathrm{Tr}_V(T) = \mathrm{Tr}_W(T)$. To see this, choose a basis of W and extend it to a basis of V. With respect to this basis, the diagonal elements corresponding to the basis elements of V not in V are 0. Hence, the trace can be computed over W.

DEFINITION 1.19. Let $\beta : V \times V \to k$ be a bilinear form. We define its *radical* to be

$$S = \{x \in V \colon \beta(x, y) = 0 \ \forall\, y \in V\},$$

which is obviously a subspace of V. We say that β is *nondegenerate* if $S = 0$.

REMARK 1.20. If $V = \mathfrak{g}$, a Lie algebra, and β is associative, then the radical $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Indeed, if $x \in \mathfrak{a}$, and $y \in \mathfrak{g}$, then for every $z \in \mathfrak{g}$, we have

$$\beta([x, y], z) = \beta(x, [y, z]) = 0.$$

THEOREM 1.21. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.

Proof. Let $\mathfrak{a}$ be the radical of κ. Suppose first that $\mathfrak{g}$ is semisimple, that is, $\operatorname{rad}\mathfrak{g} = 0$. By definition, we have $\mathrm{Tr}(\operatorname{ad}_{\mathfrak{g}} x \operatorname{ad}_{\mathfrak{g}} y) = 0$ for every $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$, in particular, for $y \in [\mathfrak{a}, \mathfrak{a}]$. Due to Cartan's criterion, it follows that $\operatorname{ad}_{\mathfrak{g}} \mathfrak{a}$ is solvable. The kernel of $\operatorname{ad}_{\mathfrak{g}}$ when restricted to $\mathfrak{a}$ is precisely $Z(\mathfrak{g}) \cap \mathfrak{a}$, which is abeilan, whence solvable. It follows that $\mathfrak{a}$ is solvable, but due to semisimplicity, $\mathfrak{a} = 0$.

Conversely, suppose the Killing form is nondegenerate and let $\mathfrak{a} \lhd \mathfrak{g}$ be an abelian ideal. Suppose $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. Then $(\operatorname{ad} x)(\operatorname{ad} y)$ maps $\mathfrak{g} \mapsto \mathfrak{g} \mapsto \mathfrak{a}$ and since $\mathfrak{a}$ is abelian, $(\operatorname{ad} x \operatorname{ad} y)^2 = 0$. In particular, $\kappa(x, y) = \mathrm{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$. That is, $\mathfrak{a} \subseteq \operatorname{rad}\kappa$. Hence, $\mathfrak{a} = 0$, and $\mathfrak{g}$ is semisimple since it contains no nontrivial abelian ideals. ■

DEFINITION 1.22. A Lie algebra $\mathfrak{g}$ is said to be the *direct sum* of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ provided $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$.

REMARK 1.23. Note that the above condition implicitly forces $[\mathfrak{a}_i, \mathfrak{a}_j] \subseteq \mathfrak{a}_i \cap \mathfrak{a}_j = 0$.

THEOREM 1.24. Let $\mathfrak{g}$ be semisimple. Then there exist ideal $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ of $\mathfrak{g}$, which are simple (as Lie algebras), such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$. Every simple ideal of $\mathfrak{g}$ is one of the $\mathfrak{g}_i$. Moreover the Killing form of $\mathfrak{g}_i$ is the restriction of κ to $\mathfrak{g}_i \times \mathfrak{g}_i$.

Proof. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and

$$\mathfrak{a}^{\perp} = \{x \in \mathfrak{g} \colon \kappa(x, y) = 0 \ \forall\, y \in \mathfrak{a}\},$$

then $\mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$. Further, due to Cartan's criterion, $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is solvable, whence 0 due to semisimplicity. Also, by a dimension argument, it is easy to see that $\dim \mathfrak{a}^{\perp} \geqslant \dim \mathfrak{g} - \dim \mathfrak{a}$ (choose a basis of $\mathfrak{a}$ and proceed in the obvious fashion), whence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$.

Next, we proceed by induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ has no nonzero proper ideal, then $\mathfrak{g}$ Is simple, and we are done. Otherwise, let $\mathfrak{g}_1$ be a minimal nonzero ideal of $\mathfrak{g}$. By the preceding paragraph, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^\perp$. Due to this decomposition, any ideal of $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$, consequently, $\mathfrak{g}_1$ must be semisimple (since an abelian ideal in $\mathfrak{g}_1$ is an abelian ideal in $\mathfrak{g}$), therefore, $\mathfrak{g}_1$ is simple. Similarly, any ideal of $\mathfrak{g}_1^\perp$ is an ideal of $\mathfrak{g}$, whence by the same reasoning, $\mathfrak{g}_1^\perp$ is also semisimple. By the induction hypothesis, $\mathfrak{g}_1^\perp$ splits into a direct sum of simple ideals; and since ideals of $\mathfrak{g}_1$ are ideals of $\mathfrak{g}$, we have proved the decomposition.

Next, we have to show that these simple ideals are unique. Suppose $\mathfrak{a}$ is a simple ideal of $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{a}]$ is an ideal of $\mathfrak{a}$ (obvious), and is nonzero, because $Z(\mathfrak{g}) = 0$. This forces $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$. On the other hand, we have

$$[\mathfrak{a}, \mathfrak{g}] = [\mathfrak{a}, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{a}, \mathfrak{g}_r],$$

so all but one summand must be 0. Say $[\mathfrak{a}, \mathfrak{g}_i] = \mathfrak{a}$. Then $\mathfrak{a} \subseteq \mathfrak{g}_i$, whence $\mathfrak{a} = \mathfrak{g}_i$ due to simplicity. This completes the proof. ∎

PORISM 1.25. Let $\mathfrak{g}$ be semisimple and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, where

$$\mathfrak{a}^\perp = \{x \in \mathfrak{g} \colon \kappa(x, y) = 0 \,\forall\, y \in \mathfrak{a}\}.$$

COROLLARY 1.26. If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and all ideals and homomorphic images of $\mathfrak{g}$ are semisimple. Moreover, each ideal of $\mathfrak{g}$ is a sum of certain simple ideals of $\mathfrak{g}$.

Proof. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. The porism above allows us to write $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. Again, every ideal of $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Hence, $\mathfrak{a}$ contains no abelian ideals, whence $\mathfrak{a}$ is semisimple, and so is $\mathfrak{a}^\perp$. It follows that $\mathfrak{a}$ is a direct sum of simple ideals of $\mathfrak{a}$, which are also simple ideals of $\mathfrak{g}$, and hence, are a subset of $\{\mathfrak{g}_1, \dots, \mathfrak{g}_r\}$. Finally, note that $\mathfrak{g}/\mathfrak{a} \cong \mathfrak{a}^\perp$ is semisimple. ∎

LEMMA 1.27. Let $\mathfrak{g}$ be a Lie algebra over k. Then $\operatorname{ad} \mathfrak{g}$ is an ideal in $\operatorname{Der} \mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$.

Proof. For $\delta \in \operatorname{Der} \mathfrak{g}$ and $x \in \mathfrak{g}$

$$\delta(\operatorname{ad} x(y)) - \operatorname{ad} x(\delta y) = [\delta x, y] = \operatorname{ad} \delta x(y).$$ ∎

THEOREM 1.28. If $\mathfrak{g}$ is semisimple, then $\operatorname{ad} \mathfrak{g} = \operatorname{Der} \mathfrak{g}$.

Proof. Let $\mathfrak{M} = \operatorname{ad} \mathfrak{g} \subseteq \mathfrak{D} = \operatorname{Der} \mathfrak{g}$, which is an ideal. Note that $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der} \mathfrak{g}$ is injective, since $Z(\mathfrak{g}) = 0$. Let $\mathfrak{I} = \mathfrak{M}^\perp$ with respect to the Killing form $\kappa_{\mathfrak{D}}$. If $\delta \in \mathfrak{M} \cap \mathfrak{I}$, then $\kappa_{\mathfrak{M}}(\delta, \sigma) = 0$ for every $\sigma \in \mathfrak{M}$. But since $\kappa_{\mathfrak{M}}$ is the restriction of $\kappa_{\mathfrak{D}}$ to $\mathfrak{M}$, and the former is nondegenerate, we see that $\delta = 0$. That is, $\mathfrak{M} \cap \mathfrak{I} = 0$.

Since $\mathfrak{M}$ and $\mathfrak{I}$ are both ideals, then $[\mathfrak{M}, \mathfrak{I}] \subseteq \mathfrak{M} \cap \mathfrak{I} = 0$. For any $x \in \mathfrak{g}$ and $\delta \in \mathfrak{I}$, we have $0 = [\delta, \operatorname{ad} x] = \operatorname{ad} \delta x$. Since ad is injective, $\delta x = 0$ for every $x \in \mathfrak{g}$ and $\delta \in \mathfrak{I}$. Hence, $\mathfrak{M}^\perp = \mathfrak{I} = 0$. In particular, this means that $\kappa_{\mathfrak{D}}$ is nondegenerate, and $\mathfrak{D}$ is semisimple, whence, $\mathfrak{D} = \mathfrak{M} \oplus \mathfrak{M}^\perp = \mathfrak{M}$, thereby completing the proof. ∎

We use the above to define the *abstract Jordan decomposition*. Let $\mathfrak{g}$ be a semisimple Lie algebra over k. The map $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der} \mathfrak{g}$ is an isomorphism. Thus, for any $x \in \mathfrak{g}$, $\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$ exists in $\operatorname{Der} \mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$. There are unique $x_s, x_n \in \mathfrak{g}$ such that $\operatorname{ad} x_s = (\operatorname{ad} x)_s$ and $\operatorname{ad} x_n = (\operatorname{ad} x)_n$. These are (respectively) the *semisimple part* and *nilpotent part* of x in $\mathfrak{g}$.

§§ Complete Reducibility of Representations

DEFINITION 1.29. Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra. A *$\mathfrak{g}$-module* is a vector space V, endowed with an operation $\mathfrak{g} \times V \to V$, denoted $(x, v) \mapsto x \cdot v$ satisfying the following:

(M1) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$.

(M2) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$.

(M3) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

A *homomorphism* of $\mathfrak{g}$-modules is a map $\phi : V \to W$ such that $\phi(x \cdot v) = x \cdot \phi(v)$. An $\mathfrak{g}$-module V is said to be *irreducible* if it has precisely two $\mathfrak{g}$-submodules (itself and 0). V is called *completely reducible* if V is a direct sum of irreducible $\mathfrak{g}$-submodules.

REMARK 1.30. It is evident from the above definition that we do not regard a zero-dimensional vector space as an irreducible $\mathfrak{g}$-module.

Let V and W be $\mathfrak{g}$-modules. We give $V \otimes_k W$ a $\mathfrak{g}$-module structure as follows:

$$x(v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w) \qquad \forall\, v \in V,\ w \in W.$$

Further, we also give $\mathrm{Hom}_k(V, W)$ the structure of a $\mathfrak{g}$-module by

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v) \qquad \forall f \in \mathrm{Hom}_k(V, W),\ v \in V.$$

Treating k as a trivial $\mathfrak{g}$-module, the above defines a natural $\mathfrak{g}$-module structure on V^* given by

$$(x \cdot f)(v) = -f(x \cdot v) \qquad \forall\, f \in V^*,\ v \in V.$$

PROPOSITION 1.31. The map $V^* \otimes_k W \to \mathrm{Hom}_k(V, W)$ given by

$$f \otimes w \longmapsto (v \mapsto f(v)w)$$

is an isomorphism of $\mathfrak{g}$-modules.

Proof. Call the map Φ. It is a standard fact from linear algebra that Φ is an isomorphism of vector spaces. It suffices to check that the map is $\mathfrak{g}$-linear. Indeed, for $x \in \mathfrak{g}$, $f \in V^*$, and $w \in W$, we have

$$\begin{aligned}
\Phi(x \cdot (f \otimes w))(v) &= \Phi\left((x \cdot f) \otimes w + f \otimes (x \cdot w)\right)(v) \\
&= (x \cdot f)(v)w + f(v)(x \cdot w) \\
&= -f(x \cdot v)w + f(v)(x \cdot w).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(x \cdot \Phi(f \otimes w))(v) &= x \cdot (\Phi(f \otimes w)(v)) - \Phi(f \otimes w)(x \cdot v) \\
&= x \cdot (f(v)w) - f(x \cdot v)w \\
&= f(v)(x \cdot w) - f(x \cdot v)w.
\end{aligned}$$

This completes the proof. ■

Next, we define the *Casimir element* of a representation.