

# Prouct Developments

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## §1 The Space of Holomorphic Functions

**THEOREM 1.1.** If  $\Omega \subseteq \mathbb{C}$  is open, then there is a sequence  $(K_n)_{n \geq 1}$  of compact subsets of  $\Omega$  such that  $\Omega = \bigcup_{n=1}^{\infty} K_n$ . Moreover, the sets  $K_n$  can be chosen to satisfy the following conditions:

- (i)  $K_n \subseteq K_{n+1}^\circ$ .
- (ii) If  $K \subseteq \Omega$  is compact, then  $K \subseteq K_n$  for some  $n \geq 1$ .
- (iii) For every  $n \geq 1$ , each component of  $\mathbb{C}_\infty \setminus K_n$  contains a component of  $\mathbb{C}_\infty \setminus \Omega$ .

## §2 Product Developments

### §§ Generalities

**DEFINITION 2.1.** If  $(z_n)_{n \geq 1}$  is a sequence of complex numbers, then  $z \in \mathbb{C}$  is said to be the *infinite product* of the sequence  $(z_n)_{n \geq 1}$  if

$$z = \lim_{n \rightarrow \infty} \prod_{k=1}^n z_k.$$

Suppose  $z_n \neq 0$  for all  $n \geq 1$  and  $z \neq 0$ . Then, setting

$$p_n = \prod_{k=1}^n z_k,$$

we have, by definition that  $p_n \rightarrow z \neq 0$  in  $\mathbb{C}$ . But since  $z_n = p_n/p_{n-1}$  with the convention that  $p_0 = 1$ , we see that  $z_n \rightarrow 1$  as  $n \rightarrow \infty$ .

**PROPOSITION 2.2.** Let  $(z_n)_{n \geq 1}$  be a sequence of complex numbers with  $\operatorname{Re} z_n > 0$  for all  $n \geq 1$ . Then  $\prod_{n=1}^{\infty} z_n$  converges to a *non-zero* complex number if and only if the series  $\sum_{n=1}^{\infty} \log z_n$  converges.

*Proof.* ■

**DEFINITION 2.3.** If  $(z_n)_{n \geq 1}$  is a sequence of complex numbers with  $\operatorname{Re} z_n > 0$  for all  $n$ , then the infinite product  $\prod_{n=1}^{\infty} z_n$  is said to *converge absolutely* if the series  $\sum_{n=1}^{\infty} \log z_n$  converges absolutely.

**LEMMA 2.4.** If  $|z| < \frac{1}{2}$ , then

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|.$$

*Proof.* Using the power series expansion of  $\log(1+z)$  about  $z=0$ , we get

$$\left|1 - \frac{\log(1+z)}{z}\right| = \left|\frac{1}{2}z - \frac{1}{3}z^2 + \dots\right| \leq \frac{1}{2}(|z| + |z|^2 + \dots) = \frac{1}{2} \frac{|z|}{1-|z|} < \frac{1}{2},$$

whence the conclusion follows. ■

**PROPOSITION 2.5.** Let  $(z_n)_{n \geq 1}$  be a sequence of complex numbers with  $\operatorname{Re} z_n > -1$  for all  $n \geq 1$ . Then the series  $\sum_{n=1}^{\infty} \log(1+z_n)$  converges absolutely if and only if the series  $\sum_{n=1}^{\infty} z_n$  converges absolutely.

*Proof.* ■

**COROLLARY 2.6.** If  $(z_n)_{n \geq 1}$  is a sequence of complex numbers with  $\operatorname{Re} z_n > 0$  for all  $n \geq 1$ , then the product  $\prod_{n=1}^{\infty} z_n$  converges absolutely if and only if the series  $\sum_{n=1}^{\infty} (z_n - 1)$  converges absolutely.

*Proof.* ■

**PROPOSITION 2.7.** Let  $X$  be a set, and  $(f_n)_{n \geq 1}$  be a sequence of complex-valued functions on  $X$  converging uniformly to  $f: X \rightarrow \mathbb{C}$ . Suppose there exists  $a \in \mathbb{R}$  such that  $\operatorname{Re} f_n(x) \leq a$  for all  $x \in X$  and  $n \geq 1$ , then the sequence of functions  $(\exp(f_n))_{n \geq 1}$  converges uniformly to  $\exp(f)$ .

*Proof.* ■

**LEMMA 2.8.** Let  $X$  be a compact topological space and  $(g_n)_{n \geq 1}$  a sequence of complex-valued continuous functions on  $X$  such that  $\sum_{n=1}^{\infty} |g_n(x)|$  converges uniformly on  $X$ . Then the product

$$f(x) = \prod_{n=1}^{\infty} (1 + g_n(x))$$

converges uniformly for all  $x \in X$ . Further there is an integer  $n_0 \geq 1$  such that  $f(x) = 0$  if and only if  $g_n(x) = -1$  for some  $1 \leq n \leq n_0$ .

*Proof.* Since  $\sum_{n=1}^{\infty} |g_n(x)|$  converges uniformly on  $X$ , there is a positive integer  $n_0 \geq 1$  such that  $|g_n(x)| < \frac{1}{2}$  for all  $x \in X$  and  $n > n_0$ . Thus  $\operatorname{Re}(1 + g_n(x)) > 0$  for all  $x \in X$  and  $n > n_0$ , and hence due to Lemma 2.4

$$|\log(1 + g_n(x))| \leq \frac{3}{2} |g_n(x)| \quad \forall x \in X, \forall n > n_0.$$

Thus, the sum

$$h(x) := \sum_{n=n_0}^{\infty} \log(1 + g_n(x))$$

converges uniformly on  $X$  so that  $h$  is a continuous function. Since  $X$  is compact, there is an  $a \in \mathbb{R}$  such that  $\operatorname{Re} h(x) \leq a$  for all  $x \in X$ . In view of Proposition 2.7,

$$\exp h(x) = \prod_{n=n_0}^{\infty} (1 + g_n(x))$$

converges uniformly on  $X$ . In particular, the product on the right is non-zero for all  $x \in X$ .

Finally, since

$$f(x) = (1 + g_1(x)) \cdots (1 + g_{n_0}(x)) \exp h(x),$$

it follows that if  $f(x) = 0$ , then  $g_n(x) = -1$  for some  $1 \leq n \leq n_0$ . ■

**THEOREM 2.9.** Let  $\Omega \subseteq \mathbb{C}$  be a region and let  $(f_n)_{n \geq 1}$  be a sequence of holomorphic functions such that no  $f_n$  is identically zero. If  $\sum_{n=1}^{\infty} |f_n(z) - 1|$  converges uniformly on compact subsets of  $\Omega$ , then  $\prod_{n=1}^{\infty} f_n(z)$  converges uniformly on compact subsets of  $\Omega$  to a holomorphic function  $f(z)$ .

If  $a \in \Omega$  is a zero of  $f$ , then  $a$  is a zero of only a finite number of functions  $f_n$ , and the multiplicity of the zero of  $f$  at  $a$  is the sum of the multiplicities of the zeros of the functions  $f_n$  at  $a$ .

### §3 Runge's Theorem

**THEOREM 3.1 (RUNGE).** Let  $K \subseteq \mathbb{C}$  be a compact set and let  $E$  be a subset of  $\mathbb{C}_{\infty} \setminus K$  meeting each connected component of  $\mathbb{C}_{\infty} \setminus K$ . If  $f$  is a function holomorphic in an open set  $\Omega \supseteq K$  and  $\varepsilon > 0$ , then there exists a rational function  $R(z)$  whose only poles lie in  $E$  such that

$$|f(z) - R(z)| < \varepsilon$$

for all  $z \in K$ .

Let  $C(K)$  denote the Banach space of all complex-valued continuous functions on  $K$  equipped with the supremum norm on  $K$ , that is,

$$\|f\|_{\infty} := \sup\{|f(z)| : z \in K\} \quad \forall f \in C(K).$$

Let  $B(E) \subseteq C(K)$  denote the set of all functions  $f \in C(K)$  such that for every  $\varepsilon > 0$ , there is a rational function  $R(z)$  with poles only in  $E$  such that

$$\|f - R\|_{\infty} < \varepsilon.$$

Theorem 3.1 essentially states that  $f|_K \in B(E)$  for every holomorphic function in a neighborhood of  $K$ .

**LEMMA 3.2.**  $B(E)$  is a closed  $\mathbb{C}$ -subalgebra of  $C(K)$  containing every rational function with all poles in  $E$ .

*Proof.* The latter part of the assertion is clear. To see that  $B(E)$  is a subalgebra, suppose  $f, g \in B(E)$  and  $\alpha, \beta \in \mathbb{C}$ . Let  $\varepsilon > 0$  and choose rational functions  $R(z), S(z)$  such that

$$\|f - R\|_{\infty} < \frac{\varepsilon}{|\alpha| + |\beta| + 1} \quad \text{and} \quad \|g - S\|_{\infty} < \frac{\varepsilon}{|\alpha| + |\beta| + 1}.$$

Then

$$\|(\alpha f + \beta g) - (\alpha R + \beta S)\|_{\infty} < \frac{|\alpha| + |\beta|}{|\alpha| + |\beta| + 1} \varepsilon < \varepsilon,$$

whence  $\alpha f + \beta g \in B(E)$ . Next, we shall show that  $fg \in B(E)$ . Indeed, let  $\varepsilon > 0$ , and choose positive real numbers  $M_1, M_2 > 0$  such that  $\|f\|_{\infty} < M_1$  and  $\|g\|_{\infty} < M_2$ . Choose rational functions  $R(z), S(z)$  such that

$$\|f - R\|_{\infty} < \frac{\varepsilon}{M_1 + M_2} \quad \text{and} \quad \|g - S\|_{\infty} < \frac{\varepsilon}{M_1 + M_2}.$$

Then  $R(z)S(z)$  is a rational function with poles only in  $E$ , and

$$\|fg - RS\|_{\infty} \leq \|g(f - R) + R(g - S)\|_{\infty} \leq M_2\|f - R\|_{\infty} + M_1\|g - S\|_{\infty} < \varepsilon,$$

as desired. Thus  $B(E)$  is a subalgebra of  $C(K)$ .

It remains to show that  $B(E)$  is closed in the topology of  $C(K)$ . Indeed, let  $f_n \rightarrow f$  in  $C(K)$  and  $\varepsilon > 0$ . There is a positive integer  $N$  such that  $\|f - f_N\|_\infty < \frac{\varepsilon}{2}$ , and further, a rational function  $R(z)$  with poles only in  $E$  such that  $\|f_N - R\|_\infty < \frac{\varepsilon}{2}$ . Thus

$$\|f - R\|_\infty < \|f - f_N\|_\infty + \|f_N - R\|_\infty < \varepsilon,$$

whence  $f \in B(E)$ , thereby completing the proof.  $\blacksquare$

The outline of the rest of the proof is as follows:

- First, we show that  $\frac{1}{z-a} \in B(E)$  for each  $a \in \mathbb{C} \setminus K$ .
- Since  $B(E)$  is an algebra containing all polynomials, using partial fractions, we conclude that every rational function with poles only in  $\mathbb{C} \setminus K$  belongs to  $B(E)$ .
- Finally, using Cauchy's integral formula, we show that every holomorphic function can be approximated arbitrarily well by rational functions with poles only in  $\mathbb{C} \setminus K$ .

**LEMMA 3.3.** Let  $V$  and  $U$  be open subsets of  $\mathbb{C}$  with  $V \subseteq U$  and  $\partial V \cap U = \emptyset$ . If  $H$  is a component of  $U$  with  $H \cap V \neq \emptyset$ , then  $H \subseteq V$ .

*Proof.* Let  $a \in H \cap V$  and let  $G$  be the connected component of  $V$  containing  $a$ ; then  $H \cup G$  is connected and contained in  $U$ . But since  $H$  is a connected component,  $H \cup G = H$ , that is,  $G \subseteq H$ . Note that  $\partial G \subseteq \partial V$ <sup>1</sup> and so  $\partial G \cap H = \emptyset$ , whence

$$H \setminus G = H \cap (\mathbb{C} \setminus G) = H \cap \left[ (\mathbb{C} \setminus \overline{G}) \cup \partial G \right] = H \cap (\mathbb{C} \setminus \overline{G}),$$

whence  $H \setminus G$  is open in  $H$ . But since  $G$  is open,  $H \setminus G$  is both closed and open in  $H$ , and since  $H$  is connected and  $G \neq \emptyset$ , it follows that  $H = G \subseteq V$ , as desired.  $\blacksquare$

**PROPOSITION 3.4.** Let  $a \in \mathbb{C} \setminus K$ . Then  $\frac{1}{z-a} \in B(E)$ .

*Proof.* We split our analysis into two cases.

**CASE 1.**  $\infty \notin E$ . Let  $U = \mathbb{C} \setminus K$  and let

$$V = \left\{ a \in \mathbb{C} : \frac{1}{z-a} \in B(E) \right\},$$

so that  $E \subseteq V \subseteq U$ . We first claim that  $V$  is open. Indeed, suppose  $a \in V$  and  $|b-a| < d(a, K)$ . Then there exists  $0 < r < 1$  such that  $|b-a| < r|z-a|$  for all  $z \in K$ . But

$$\frac{1}{z-b} = \frac{1}{z-a} \frac{1}{1 - \frac{b-a}{z-a}},$$

and since  $|(b-a)/(z-a)| < r < 1$  for all  $z \in K$ , we note that the series

$$\frac{1}{1 - \frac{b-a}{z-a}} = \sum_{n=0}^{\infty} \left( \frac{b-a}{z-a} \right)^n$$

converges uniformly on  $K$  due to the Weierstraß  $M$ -test. Set

$$Q_n(z) = \sum_{n=0}^{\infty} \left( \frac{b-a}{z-a} \right)^n,$$

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<sup>1</sup>This is because  $\mathbb{C}$  is locally connected.

then  $\frac{1}{z-a}Q_n(z) \in B(E)$  since  $a \in V$  and  $B(E)$  is an algebra. Since  $B(E)$  is closed, the uniform convergence of  $\frac{1}{z-a}Q_n(z)$  to  $\frac{1}{z-b}$  yields that the latter lies in  $B(E)$ , so that  $V$  is open.

Now suppose  $b \in \bar{V} \setminus V = \partial V$  and let  $(a_n)_{n \geq 1}$  be a sequence in  $V$  converging to  $b$ . We have that  $|b - a_n| \geq d(a_n, K)$  and taking  $n \rightarrow \infty$  and using the continuity of  $d(\cdot, K)$ , one obtains  $d(b, K) = 0$ , that is,  $b \in K$ . Thus  $\partial V \cap U = \emptyset$ . If  $H$  is a component of  $U$ , then  $H \cap E \neq \emptyset$ , so  $H \cap V \neq \emptyset$ . By Lemma 3.3,  $H \subseteq V$ . But since  $H$  was arbitrary, we have that  $U \subseteq V$ , i.e.,  $U = V$ .

**CASE 2.**  $\infty \in E$ . Let  $d_\infty$  denote the metric on  $\mathbb{C}_\infty$ . Choose  $a_0$  in the unbounded component of  $\mathbb{C} \setminus K$  (i.e., the component containing  $\infty$ ) such that  $d_\infty(a_0, \infty) \leq \frac{1}{2}d_\infty(\infty, K)$  and  $|a_0| > 2 \max\{|z| : z \in K\}$ . Let  $E_0 = (E \setminus \{\infty\}) \cup \{a_0\}$ . Then  $E_0$  meets each component of  $\mathbb{C}_\infty \setminus K$ , and  $\infty \notin E_0$ .

If  $a \in \mathbb{C} \setminus K$ , then due to CASE 1,  $\frac{1}{z-a} \in B(E_0)$ . We shall show that  $\frac{1}{z-a_0} \in B(E_0)$ . Once this is shown, we could approximate rational functions with poles only in  $E_0$  by rational functions with poles only in  $E$ , since  $E_0 \setminus E = \{a_0\}$ . This would then immediately give us that  $\frac{1}{z-a} \in B(E_0) \subseteq B(E)$ , as desired.

Note that for all  $z \in K$ ,  $|z/a_0| \leq \frac{1}{2}$  and so

$$\frac{1}{z-a_0} = -\frac{1}{a_0} \frac{1}{1-\frac{z}{a_0}} = -\frac{1}{a_0} \sum_{n=0}^{\infty} \left(\frac{z}{a_0}\right)^n$$

converges uniformly on  $K$  due to the Weierstraß  $M$ -test. Set

$$Q_n(z) = -\frac{1}{a_0} \sum_{k=0}^n \left(\frac{z}{a_0}\right)^k,$$

which is a sequence of polynomials converging uniformly to  $\frac{1}{z-a_0}$  on  $K$ . Since  $Q_n \in B(E)$  for each  $n \geq 1$ , we have shown that  $\frac{1}{z-a_0} \in B(E)$ , thereby completing the proof.  $\blacksquare$

**LEMMA 3.5.** Let  $\Omega$  be a region containing  $K$ . Then there are straight line segments  $\gamma_1, \dots, \gamma_n$  in  $\Omega \setminus K$  such that for every holomorphic function  $f$  on  $\Omega$ ,

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\gamma_k} \frac{f(w)}{w-z} dw$$

for all  $z \in K$ . The line segments form a finite number of closed polygons in  $\Omega$ .

*Proof.* Covering  $K$  by finitely many compact disks (contained in  $\Omega$ ), we can replace  $K$  with the union of these disks and suppose that  $K = \bar{K}^\circ$ . Let  $0 < \delta < \frac{1}{2}d(K, \mathbb{C} \setminus \Omega)$  and place a “grid” of horizontal and vertical lines in the plane with consecutive lines less than a distance  $\delta$  apart. Let  $R_1, \dots, R_m$  be the resulting rectangles intersecting  $K$ . These rectangles are finite in number because  $K$  is compact. Consider  $\partial R_j$ , the boundary of  $R_j$  as a polygon oriented in the counter-clockwise direction.

If  $z \in R_j$  for some  $1 \leq j \leq m$ , then  $d(z, K) \leq \text{diam } R_j = \sqrt{2}\delta$ , and hence  $z \in \Omega$ . This shows that every  $R_j$  is contained in  $\Omega$ . Next, suppose  $R_j$  and  $R_j$  intersect in an edge  $\sigma$ . With respect to the two rectangles,  $\sigma$  will have opposite orientations, and hence, for any continuous function  $\varphi$  on  $\sigma$ , the sum of the integrals will cancel out.

Let  $\gamma_1, \dots, \gamma_n$  be those directed line segments that constitute an edge of exactly one of the  $R_j$ 's. Then

$$\sum_{k=1}^n \int_{\gamma_k} \varphi = \sum_{j=1}^m \int_{\partial R_j} \varphi \tag{1}$$

for any continuous function  $\varphi$  on  $\bigcup_{j=1}^m \partial R_j$ .

We contend that each  $\gamma_k$  lies in  $\Omega \setminus K$ . Indeed, if one of the  $\gamma_k$  intersects  $K$ , then there are two rectangles in the grid with  $\gamma_k$  as a side, both of which intersect  $K$ , whence both of these rectangles must lie in the set  $\{R_1, \dots, R_m\}$ , which is absurd, since  $\gamma_k$  is a side of exactly one of those rectangles.

Now, if  $z \in K \setminus \bigcup_{j=1}^m \partial R_j$ , then for any holomorphic function  $f$  on  $\Omega$ ,

$$\varphi(w) = \frac{1}{2\pi i} \frac{f(w)}{w - z}$$

is continuous on  $\bigcup_{j=1}^m \partial R_j$ . From (1), it follows that

$$\sum_{j=1}^m \frac{1}{2\pi i} \int_{\partial R_j} \frac{f(w)}{w - z} dw = \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw.$$

But  $z$  belongs to the interior of exactly one of the  $R_j$ 's whence the sum on the left is precisely  $f(z)$  whenever  $z \in K \setminus \bigcup_{j=1}^m \partial R_j$ . But both sides are continuous functions on  $K$  (since  $f(z)$  is clearly continuous and every  $\gamma_k$  misses  $K$ ) and because  $K = \overline{K^\circ}$ , the set  $K \setminus \bigcup_{j=1}^m \partial R_j$  is dense in  $K$ ; it follows that both sides must be equal for all  $z \in K$ , as desired. ■

Now that we have an integral representation of  $f(z)$ , we shall approximate it using rational functions having poles on the  $\{\gamma_k\}$ 's.

**LEMMA 3.6.** Let  $\gamma$  be a rectifiable curve and  $K$  a compact set such that  $K \cap \{\gamma\} = \emptyset$ . If  $f$  is a continuous function on  $\{\gamma\}$ , and  $\varepsilon > 0$ , then there is a rational function  $R(z)$  having all its poles on  $\{\gamma\}$  such that

$$\left| \int_{\gamma} \frac{f(w)}{w - z} dw - R(z) \right| < \varepsilon$$

for all  $z \in K$ .

*Proof.* We may assume that  $\gamma: [0, 1] \rightarrow \mathbb{C}$ . First, since  $K$  and  $\{\gamma\}$  are disjoint, there is a real number  $0 < r < d(\{\gamma\}, K)$ . For  $0 \leq s < t \leq 1$  and  $z \in K$ ,

$$\begin{aligned} \left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right| &= \left| \frac{\gamma(s)f(\gamma(t)) - \gamma(t)f(\gamma(s)) - z(f(\gamma(t)) - f(\gamma(s)))}{(\gamma(t) - z)(\gamma(s) - z)} \right| \\ &\leq \frac{1}{r^2} |\gamma(s)f(\gamma(t)) - \gamma(t)f(\gamma(s)) - z(f(\gamma(t)) - f(\gamma(s)))| \\ &\leq \frac{1}{r^2} |f(\gamma(t))(\gamma(s) - \gamma(t)) + \gamma(t)(f(\gamma(t)) - f(\gamma(s))) - z(f(\gamma(t)) - f(\gamma(s)))| \\ &\leq \frac{1}{r^2} |f(\gamma(t))| |\gamma(s) - \gamma(t)| + \frac{1}{r^2} |\gamma(t) - z| |f(\gamma(t)) - f(\gamma(s))|. \end{aligned}$$

Using the compactness of  $\{\gamma\}$  and  $K$ , there is a constant  $C > 0$  such that  $d(x, z) \leq C$  for all  $x \in \{\gamma\}$  and  $z \in K$ , and  $f(x) \leq C$  for all  $x \in \{\gamma\}$ . Thus

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right| \leq \frac{C}{r^2} (|\gamma(s) - \gamma(t)| + |f(\gamma(t)) - f(\gamma(s))|).$$

Finally, using the uniform continuity of the functions  $\gamma, f \circ \gamma: [0, 1] \rightarrow \mathbb{C}$ , there is a  $\delta > 0$  such that whenever  $|s - t| < \delta$ ,

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right| < \frac{\varepsilon}{2V(\gamma)}$$

for all  $z \in K$ . Choose a partition  $0 = t_0 < t_1 < \dots < t_n = 1$  of  $[0, 1]$  such that  $|t_j - t_{j-1}| < \delta$  for  $1 \leq j \leq n$ . Set

$$R(z) = \sum_{i=1}^n \frac{f(\gamma(t_{j-1}))(\gamma(t_j) - \gamma(t_{j-1}))}{\gamma(t_{j-1}) - z}.$$

Now, there is a partition  $0 = s_0 < s_1 < \dots < s_m = 1$  of  $[0, 1]$  such that

$$\left| \int_{\gamma} \frac{f(w)}{w - z} dw - \sum_{j=1}^m \frac{f(\gamma(s_j))}{\gamma(s_j) - \gamma(s_{j-1}))} \right| < \frac{\varepsilon}{2}.$$

Thus

$$\left| \int_{\gamma} \frac{f(w)}{w - z} dw - R(z) \right| \leq \left| \int_{\gamma} \frac{f(w)}{w - z} dw - \sum_{j=1}^m \frac{f(\gamma(s_j))}{\gamma(s_j) - \gamma(s_{j-1}))} \right| + \left| \sum_{j=1}^m \frac{f(\gamma(s_j))}{\gamma(s_j) - \gamma(s_{j-1}))} - \sum_{j=1}^n \frac{f(\gamma(t_{j-1}))(\gamma(t_j) - \gamma(t_{j-1}))}{\gamma(t_{j-1}) - z} \right|.$$

Taking a union of both partitions  $\underline{s}$  and  $\underline{t}$  and using the triangle inequality, it is clear that both terms are smaller than  $\varepsilon/2$ , therefore,

$$\left| \int_{\gamma} \frac{f(w)}{w - z} dw - R(z) \right| < \varepsilon,$$

for all  $z \in K$ . ■

*Proof of Theorem 3.1.* Due to Proposition 3.4 and the fact that  $B(E)$  contains all polynomials, using partial fractions it follows that  $B(E)$  contains all rational functions with all poles in  $\mathbb{C} \setminus K$ . Finally, using Lemma 3.5 and Lemma 3.6, it follows that  $f \in B(E)$ , as desired. ■

## §§ Simply connected regions

**THEOREM 3.7.** Let  $\Omega \subseteq \mathbb{C}$  be a region. Then the following are equivalent:

- (1)  $\Omega$  is simply connected.
- (2)  $n(\gamma; a) = 0$  for every closed rectifiable curve  $\gamma$  in  $\Omega$  and every point  $a \in \mathbb{C} \setminus \Omega$ .
- (3)  $\mathbb{C}_{\infty} \setminus \Omega$  is connected.
- (4) For any  $f \in \mathcal{O}(\Omega)$ , there is a sequence of polynomials that converges to  $f$  in  $\mathcal{O}(\Omega)$ .
- (5) For any  $f \in \mathcal{O}(\Omega)$  and any closed rectifiable curve  $\gamma$  in  $\Omega$ ,  $\int_{\gamma} f = 0$ .
- (6) Every function  $f \in \mathcal{O}(\Omega)$  has a primitive.
- (7) For any nowhere-vanishing function  $f \in \mathcal{O}(\Omega)$ , there is a  $g \in \mathcal{O}(\Omega)$  such that  $f = \exp g$ .
- (8) For any nowhere-vanishing function  $f \in \mathcal{O}(\Omega)$ , there is a  $g \in \mathcal{O}(\Omega)$  such that  $f = g^2$ .
- (9)  $\Omega$  is homeomorphic to the unit disk.
- (10) If  $u: \Omega \rightarrow \mathbb{R}$  is harmonic, then there is a harmonic function  $v: \Omega \rightarrow \mathbb{R}$  such that  $f = u + iv$  is holomorphic on  $\Omega$ .

## §§ Mittag-Leffler's Theorem

**THEOREM 3.8 (MITTAG-LEFFLER).** Let  $\Omega \subseteq \mathbb{C}$  be a region and  $(a_n)_{n \geq 1}$  a sequence of distinct points in  $\Omega$  with no limit point in  $\Omega$ . Let  $(S_n(z))_{n \geq 1}$  be a sequence of rational functions of the form

$$S_n(z) = \sum_{j=1}^{m_n} \frac{c_{nj}}{(z - a_n)^j},$$

where  $m_n$  is a positive integer and  $c_{nj} \in \mathbb{C}$  for all  $n \geq 1$  and  $1 \leq j \leq m_n$ . Then there exists a meromorphic function  $f$  on  $\Omega$  which is holomorphic on  $\Omega \setminus \{a_1, a_2, \dots\}$  and whose singular part at each  $a_n$  is given by  $S_n(z)$ .

*Proof.* Choose an exhaustion  $(K_n)_{n \geq 1}$  of  $\Omega$  as in Theorem 1.1 and as such, every component of  $\mathbb{C}_\infty \setminus K_n$  contains a component of  $\mathbb{C}_\infty \setminus \Omega$ . Next, since each  $K_n$  is compact, and  $(a_k)_{k \geq 1}$  has no limit point in  $\Omega$ , only finitely many of the  $a_k$ 's can lie in each  $K_n$ . Define

$$I_n := \{k : a_k \in K_n \setminus K_{n-1}\}$$

with the convention that  $K_0 = \emptyset$ . Define the functions

$$f_n(z) = \sum_{k \in I_n} S_k(z).$$

This is clearly a meromorphic function on  $\Omega$  with all its poles in  $K_n \setminus K_{n-1}$ . Using Theorem 3.1 with  $E = \mathbb{C}_\infty \setminus \Omega$ , there exists a rational function  $R_n(z)$  with all its poles in  $\mathbb{C}_\infty \setminus \Omega$  such that

$$|f_n(z) - R_n(z)| < \frac{1}{2^n}$$

for all  $z \in K_{n-1}$  and  $n \geq 2$ . For  $n = 1$ , we set  $R_1 = 0$ . Define

$$f(z) = \sum_{n=1}^{\infty} (f_n(z) - R_n(z)).$$

We contend that this is our desired meromorphic function. We must first show that  $f$  is holomorphic on  $\Omega \setminus \{a_1, a_2, \dots\}$  and then show that its singular part at each  $a_k$  is  $S_k(z)$ .

Indeed, let  $K \subseteq \Omega \setminus \{a_1, a_2, \dots\}$  be a compact set. Then there is a positive integer  $N \geq 1$  such that  $K \subseteq K_N$ . For all  $n \geq N + 1$ , and  $z \in K_N$ , we have that

$$|f_n(z) - R_n(z)| < \frac{1}{2^n}.$$

Due to the Weierstraß  $M$ -test, the sum converges uniformly on  $K$ , whence the limiting function  $f$  is a holomorphic function on  $\Omega \setminus \{a_1, a_2, \dots\}$ .

Let  $k \geq 1$ . Since the sequence  $(a_n)_{n \geq 1}$  has no limit point, there is an  $r > 0$  such that  $|a_j - a_k| > r$  for all  $j \neq k$ . Then, the sum for  $f(z) - S_k(z)$  converges uniformly on  $\overline{B}(a_k, r)$  to a holomorphic function there, again due to the Weierstraß  $M$ -test. As a result,  $f(z)$  has singular part  $S_k(z)$  at  $a_k$ . This completes the proof. ■

**PROPOSITION 3.9.** Let  $\Omega \subseteq \mathbb{C}$  be a region. If  $(a_n)_{n \geq 1}$  is a sequence of distinct points in  $\Omega$  with no limit point in  $\Omega$ , and  $(c_n)_{n \geq 1}$  is a sequence of complex numbers, then there is a holomorphic function  $f \in \mathcal{O}(\Omega)$  such that  $f(a_n) = c_n$  for all  $n \geq 1$ .



*Proof.* Let  $g \in \mathcal{O}(\Omega)$  be a holomorphic function with simple zeros at only the  $a_n$ 's. Then we can write  $g(z) = (z - a_n)g_n(z)$  for some holomorphic function  $g_n \in \mathcal{O}(\Omega)$  with  $g_n(a_n) \neq 0$ . Using Theorem 3.8 let  $h$  be a meromorphic function on  $\Omega$ , holomorphic on  $\Omega \setminus \{a_1, a_2, \dots\}$ , and having singular part

$$\frac{c_n}{g_n(a_n)} \frac{1}{z - a_n}$$

at  $a_n$  for each  $n \geq 1$ . Clearly  $f(z) = g(z)h(z)$  has removable singularities at each  $a_n$  and  $f(a_n) = c_n$ . ■