Riemann-Roch for Riemann Surfaces

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§1 Čech Cohomology

Definition 1.1. Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. Define the *q-th cochain group* of $\mathscr{F}$ with respect to $\mathfrak{U}$ as

$$C^q(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

We denote elements of this cochain group by $\left(f_{i_0, \dots, i_q}\right)_{(i_0, \dots, i_q) \in I^{q+1}}$ and these are called q-cochains.

Next, define the *coboundary operators*

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}) \qquad \delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$$

as follows:

- For $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathscr{F})$, let $\delta\left((f_i)_{i \in I}\right) = (g_{ij})_{i,j \in I} \in C^2(\mathfrak{U}, \mathscr{F})$ where

$$g_{ij} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$

- For $(f_{ij})_{i,j \in I} \in C^1(\mathfrak{U}, \mathscr{F})$, let $\delta\left((f_{ij})_{i,j \in I}\right) = (g_{ijk})_{i,j,k \in I}$ where

$$g_{ijk} = f_{jk}|_{U_i \cap U_j \cap U_k} - f_{ik}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

The coboundary operators are group homomorphisms and let

$$Z^1(\mathfrak{U},\mathscr{F}) = \ker\left(C^1(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U},\mathscr{F})\right) \qquad B^1(\mathfrak{U},\mathscr{F}) = \operatorname{im}\left(C^0(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U},\mathscr{F})\right).$$

The elements of $Z^1(\mathfrak{U},\mathscr{F})$ are called 1-cocycles and the elements of $B^1(\mathfrak{U},\mathscr{F})$ are called 1-coboundaries. It is easy to see that $(f_{ij})_{i,j\in I}$ is a 1-cocycle if and only if

$$f_{ik}|_{U_i\cap U_j\cap U_k} = f_{jk}|_{U_i\cap U_j\cap U_k} + f_{ij}|_{U_i\cap U_j\cap U_k} \in \mathscr{F}(U_i\cap U_j\cap U_k).$$

The above relation is called the *cocycle relation*. Indeed, if $(f_{ij})_{i,j\in I}$ is a 1-cocycle, then taking $i=j$, we see that

$$f_{ii}|_{U_i\cap U_k} = 0 \qquad \forall k\in I.$$

Since the U_k's cover U_i, using the identity axiom, we have that $f_{ii} = 0 \in \mathscr{F}(U_i)$. As a consequence, we also see that

$$f_{ji}|_{U_i\cap U_j\cap U_k} + f_{ij}|_{U_i\cap U_j\cap U_k} = 0.$$

Again, using the same argument, we have that $f_{ij}+f_{ji} = 0 \in \mathscr{F}(U_i\cap U_j)$. It immediately follows from the above discussion that $\delta\circ\delta = 0$ as a map $C^0(\mathfrak{U},\mathscr{F}) \to C^2(\mathfrak{U},\mathscr{F})$.

Definition 1.2. The group

$$H^1(\mathfrak{U},\mathscr{F}) := \frac{Z^1(\mathfrak{U},\mathscr{F})}{B^1(\mathfrak{U},\mathscr{F})}$$

is called the *1-st cohomology group* with coefficients in $\mathscr{F}$ *with respect to the covering* $\mathfrak{U}$.

Definition 1.3. Let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_k)_{k\in K}$ be two open covers of X. We say that $\mathfrak{V}$ is *finer* than $\mathfrak{U}$ if every V_k is contained in some U_i.

Thus, there is a map $\tau : K \to I$ such that $V_k \subseteq U_{\tau(k)}$. This defines a mapping

$$t^{\mathfrak{U}}_{\mathfrak{V}} : Z^1(\mathfrak{U},\mathscr{F}) \to Z^1(\mathfrak{V},\mathscr{F})$$

as follows: for $(f_{ij}) \in Z^1(\mathfrak{U},\mathscr{F})$, let $t^{\mathfrak{U}}_{\mathfrak{V}}\left((f_{ij})\right) = (g_{kl})$, where

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k\cap V_l} \qquad \forall\, k,l\in K.$$

To see that this map is indeed well-defined, we need only check that (g_{kl}) is a 1-cocycle. To this end, we must check that the cocycle condition is satisfied. Indeed, for indices $k,l,m\in K$, we have

$$\begin{aligned}
g_{km}|_{V_k\cap V_l\cap V_m} &= f_{\tau(k)\tau(m)}|_{V_k\cap V_l\cap V_m} \\
&= \left(f_{\tau(k)\tau(l)}|_{U_{\tau(k)}\cap U_{\tau(l)}\cap U_{\tau(m)}} + f_{\tau(l)\tau(m)}|_{U_{\tau(k)}\cap U_{\tau(l)}\cap U_{\tau(m)}}\right)|_{V_k\cap V_l\cap V_m} \\
&= f_{\tau(k)\tau(l)}|_{V_k\cap V_l\cap V_m} + f_{\tau(k)\tau(l)}|_{V_k\cap V_l\cap V_m} \\
&= g_{kl}|_{V_k\cap V_l\cap V_m} + g_{lm}|_{V_k\cap V_l\cap V_m},
\end{aligned}$$

as desired. Further, we claim that the above map takes 1-coboundaries to 1-coboundaries. Indeed, suppose $(f_{ij})_{i,j\in I}$ is a 1-coboundary, that is, there is some $(f_i)_{i\in I}$ such that

$$f_{ij} = f_i|_{U_i\cap U_j} - f_j|_{U_i\cap U_j}.$$

Let $(g_k)_{k\in K}$ be such that $g_k = f_{\tau(k)}|_{V_k}$. Then $\delta\left((g_k)\right) = (g_{kl})$ where

$$\begin{aligned} g_{kl} &= g_k|_{V_k\cap V_l} - g_l|_{V_k\cap V_l} \\ &= f_{\tau(k)}|_{V_k\cap V_l} - f_{\tau(l)}|_{V_k\cap V_l} \\ &= f_{\tau(k)\tau(l)}|_{V_k\cap V_l}, \end{aligned}$$

that is, $(g_k l) = t^{\mathfrak{U}}_{\mathfrak{V}}\left((f_{ij})\right)$, that is, $t^{\mathfrak{U}}_{\mathfrak{V}}$ takes 1-coboundaries to 1-coboundaries. This induces a map

$$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F}).$$

Lemma 1.4. The map $t^{\mathfrak{U}}_{\mathfrak{V}}$ induced on cohomology is independent of the choice of $\tau : K \to I$.

Proof. Suppose $\widetilde{\tau} : K \to I$ is another such mapping. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and let

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k\cap V_l} \qquad \text{and} \qquad \widetilde{g}_{kl} = f_{\widetilde{\tau}(k)\widetilde{\tau}(l)}|_{V_k\cap V_l}.$$

We must show that the cocycles (g_{kl}) and $(\widetilde{g}_{kl})$ are cohomologous, that is, their difference lies in $B^1(\mathfrak{V}, \mathscr{F})$. Define

$$h_k = f_{\tau(k),\widetilde{\tau}(k)}|_{V_k} \in \mathscr{F}(V_k).$$

Then, on $V_k \cap V_l$, we have

$$\begin{aligned} g_{kl} - \widetilde{g}_{kl} &= f_{\tau(k)\tau(l)} - f_{\widetilde{\tau}(k)\widetilde{\tau}(l)} \\ &= f_{\tau(k)\tau(l)} + f_{\tau(l),\widetilde{\tau}(k)} - f_{\tau(l)\widetilde{\tau}(k)} - f_{\widetilde{\tau}(k)\widetilde{\tau}(l)} \\ &= f_{\tau(k)\widetilde{\tau}(k)} - f_{\tau(l)\widetilde{\tau}(l)} \\ &= h_k - h_l. \end{aligned}$$

Whence $(g_{kl} - \widetilde{g}_{kl})$ is a coboundary, as desired. ■

Lemma 1.5. The map $t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is injective.

Proof. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ is a 1-cocycle whose image in $Z^1(\mathfrak{V}, \mathscr{F})$ is a 1-coboundary. Then, there is some $(g_k) \in C^1(\mathfrak{U}, \mathscr{F})$ such that

$$f_{\tau(k)\tau(l)}|_{V_k\cap V_l} = g_k|_{V_k\cap V_l} - g_l|_{V_k\cap V_l}.$$

Then on $U_i \cap V_k \cap V_l$, we have

$$g_k - g_l = f_{\tau(k)\tau(l)} = f_{\tau(k)i} + f_{i\tau(l)} = f_{i\tau(l)} - f_{i\tau(k)}.$$

Hence $f_{i\tau(k)} + g_k = f_{i\tau(l)} + g_l$ on $U_i \cap V_k \cap V_l$. The gluability axiom applied to the open cover $\{U_i \cap V_k\}_{k\in K}$ furnishes a $h_i \in \mathscr{F}(U_i)$ such that

$$h_i = f_{i\tau(k)} + g_k \qquad \text{on } U_i \cap V_k \text{ for all } k \in K.$$

Then, on $U_i \cap U_j \cap V_k$ we have

$$f_{ij} = f_{i\tau(k)} + f_{\tau(k)j} = f_{i\tau(k)} + g_k - f_{j\tau(k)} - g_k = h_i - h_j.$$

Since $\{U_i \cap U_j \cap V_k\}$ forms an open cover of $U_i \cap U_j$, using the identity axiom, we have that $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. Thus, $(f_{ij}) \in B^1(\mathfrak{U}, \mathscr{F})$, thereby completing the proof. ■

DEFINITION 1.6. If $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$ are open covers of X, then it is easy to see that $t^{\mathfrak{V}}_{\mathfrak{W}} \circ t^{\mathfrak{U}}_{\mathfrak{V}} = t^{\mathfrak{U}}_{\mathfrak{W}}$. This gives us a directed system of cohomology groups, and we define

$$H^1(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}).$$

REMARK 1.7. Note that since the $t^{\mathfrak{U}}_{\mathfrak{V}}$ are all injective, the natural map $H^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ is injective. In particular, this means that $H^1(X, \mathscr{F}) = 0$ if and only if $H^1(\mathfrak{U}, \mathscr{F}) = 0$ for every open cover $\mathfrak{U}$ of X.

THEOREM 1.8 (LERAY). Let $\mathscr{F}$ be a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X such that $H^1(U_i, \mathscr{F}) = 0$ for every $i \in I$. Then for every open covering $\mathfrak{V} = (V_\alpha)_{\alpha \in A} < \mathfrak{U}$, the mapping

$$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is an isomorphism. The covering $\mathfrak{U}$ is called a *Leray covering* of X for the sheaf $\mathscr{F}$.

Proof. Let $\tau : A \to I$ be such that $V_\alpha \subseteq U_{\tau(\alpha)}$ for every $\alpha \in A$. Since we know that $t^{\mathfrak{U}}_{\mathfrak{V}}$ is injective, we would like to show that it is surjective. Let $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathscr{F})$. The family $(U_i \cap V_\alpha)_{\alpha \in A}$ is an open covering of U_i, which we denote by $U_i \cap \mathfrak{V}$. By assumption and Remark 1.7, we know that $H^1(U_i \cap \mathfrak{V}, \mathscr{F}) = 0$, that is, there exist $g_{i\alpha} \in \mathscr{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{on } U_i \cap V_\alpha \cap V_\beta.$$

On the intersection $U_i \cap U_j \cap V_\alpha \cap V_\beta$, we have

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}.$$

Using the gluability axiom on the open cover $\{U_i \cap U_j \cap V_\alpha\}_{\alpha \in A}$, there exist elements $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

We claim that f_{ij} satisfies the cocycle condition. Obviously, from the above description, on $U_i \cap U_j \cap U_k \cap V_\alpha$ we have that $F_{ik} = F_{ij} + F_{jk}$. Using the identity axiom, we see that this equality holds on $U_i \cap U_j \cap U_k$. Thus, $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. Let $h_\alpha = g_{\tau(\alpha)\alpha}|_{V_\alpha} \in \mathscr{F}(V_\alpha)$. The on $V_\alpha \cap V_\beta$, we have

$$\begin{aligned} F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} &= \left(g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}\right) - \left(g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta}\right) \\ &= g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} \\ &= h_\beta - h_\alpha, \end{aligned}$$

whence $(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta})$ is a coboundary, thereby completing the proof. ■

COROLLARY 1.9. If $\mathfrak{U}$ is a Leray covering of X, then $H^1(\mathfrak{U}, \mathscr{F}) \cong H^1(X, \mathscr{F})$.

§2 The Finiteness Theorem

§§ Laurent Schwartz's Theorem

Theorem 2.1 (Closed Range Theorem). Let $u : E \to F$ be a continuous linear map between Banach spaces. Then $u(E)$ is closed in F if and only if $u^*(F^*)$ is closed in E^*.

Proof. See [Rud91, Theorem 4.14]. ∎

Theorem 2.2 (Schauder). Let $u : E \to F$ be a continuous linear map between Banach spaces. Then u is compact if and only if u^* is.

Proof. See [Rud91, Theorem 4.19]. ∎

Lemma 2.3. Let E, F be Banach spaces and let $u : E \to F$ be a continuous linear map. Suppose that u is injective and that $u(E)$ is closed. Let $v : E \to F$ be a compact continuous linear map. Then $\ker(u+v)$ is finite-dimensional and $(u+v)(E)$ is closed in F.

Proof. Let $N = \ker(u+v)$. To see that this is finite-dimensional, it suffices to show that the closed unit ball in N is compact. To this end, let (x_n) be a sequence in the closed unit ball of N. Since v is compact, there is a subsequence (x_{n_k}) such that $(v(x_{n_k}))$ converges, as a result, $u(x_{n_k}) = -v(x_{n_k})$ also converges. Since $u(E)$ is closed in F, it is a Banach space and $u : E \to u(E)$ is a bijection, whence, due to the "bounded inverse theorem", there is a constant $c > 0$ such that $\|u(x)\| \geqslant c\|x\|$ for all $x \in E$, consequently, for $k, l \geqslant 1$,

$$\|x_{n_k} - x_{n_l}\| \leqslant \frac{1}{c}\|u(x_{n_k}) - u(x_{n_l})\|,$$

whence (x_{n_k}) is Cauchy, and thus converges. This shows that N is finite-dimensional.

Owing to N being finite-dimensional, there is a closed subspace N' of E such that $E = N \oplus N'$. Since $(u+v)(E) = (u+v)(N')$, it suffices to show that the latter is closed in F. Let (x_n) be a sequence in N' such that $(u+v)(x_n)$ converges in F; we show that the limit lies in $(u+v)(N')$. First, we claim that the sequence (x_n) is bounded. If not, then we can move to a subsequence and assume that $0 \neq x_n$ for all n and $\|x_n\| \to \infty$ as $n \to \infty$. Set $x'_n = x_n/\|x_n\|$. Then $\|x'_n\| = 1$ and $(u+v)(x'_n) \to 0$ as $n \to \infty$. Since v is compact, there is a subsequence (x'_{n_k}) such that $v(x'_{n_k})$ converges, consequently,

$$u(x'_{n_k}) = (u+v)(x'_{n_k}) - v(x'_{n_k})$$

also converges. As we argued in the preceding paragraph using the "bounded inverse theorem", this means that (x'_{n_k}) converges. It follows that there is some $x_0 \in N'$ with $\|x_0\| = 1$ and $(u+v)(x_0) = 0$, that is, $x_0 \in N \cap N' = \{0\}$, a contradiction. Hence, (x_n) is a bounded sequence in E.

Compactness of v implies that there is a subsequence (x_{n_k}) such that $v(x_{n_k})$ converges in F; and since $(u+v)(x_n)$ was assumed to be convergent, we see that $u(x_{n_k})$ is convergent too. Again, using the "bounded inverse theorem", we have that (x_{n_k}) is convergent to some $x_0 \in N'$. Hence,

$$\lim_{n\to\infty}(u+v)(x_n) = \lim_{k\to\infty}(u+v)(x_{n_k}) = (u+v)(x_0) \in (u+v)(E),$$

as desired. ∎

THEOREM 2.4 (L. SCHWARTZ). Let E, F be Banach space and let $u, v : E \to F$ be continuous linear maps. Suppose that u is surjective and that v is compact. Then $F' = (u+v)(E)$ is closed and F/F' is finite-dimensional.

Proof. Due to Theorem 2.1, it suffices to show that $(u^* + v^*)(F^*)$ is closed in E^*. Due to Theorem 2.2, we know that v^* is compact, and due to Theorem 2.1, we know that $u^*(F^*)$ is closed in E^*. Further, since u is surjective, it is easy to see that u^* must be injective. Thus, due to Lemma 2.3, we see that $(u^* + v^*)(F^*)$ is closed in E^*, as desired. We have shown that F' is closed in F.

To show that F/F' is finite-dimensional, we shall show that its closed unit ball is compact. Indeed, let (w_n) be a sequence in the closed unit ball of F/F', and choose preimages (x_n) in F satisfying $\|x_n\| \leqslant 2$. Since $u : E \to F$ is surjective, it is a consequence of the open mapping theorem, that there is a constant $M > 0$, independent of the sequence chosen, and a sequence (y_n) in E such that $\|y_n\| \leqslant M$ and $u(y_n) = x_n$. Since v is compact, there is a subsequence (x_{n_k}) such that $z_k = v(x_{n_k})$ converges in F to some $\widetilde{z}$. We can write

$$y_{n_k} = u(x_{n_k}) + v(x_{n_k}) - z_k = (u+v)(x_{n_k}) - z_k,$$

and hence, $-z_k$ maps to w_{n_k} in F/F'. Since the former converges, so does the latter. It follows that (w_n) admits a convergent subsequence, consequently, the closed unit ball in F/F' is compact, whence F/F' is finite-dimensional. This completes the proof. ■

§§ The Finiteness Theorem

REFERENCES

[Rud91] W. Rudin. *Functional Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1991.