Galois Categories

Swayam Chube

April 29, 2025

§1 Preliminaries on Profinite Groups

Definition 1.1. A profinite group is an inverse limit of finite discrete topological groups. Note that a profinite group is always compact, Hausdorff, and totally disconnected.

Proposition 1.2. Let π be a profinite group acting on a set E. Then

(1) The action is continuous if and only if for each $e \in E$, $\mathrm{Stab}_\pi(e)$ is open in π.

(2) If E is finite, the action is continuous if and only if its kernel $\{\sigma \in \pi \colon \sigma e = e \ \forall\ e \in E\}$ is open in π.

(3) Any finite transitive π-set is isomorphic to π/π' for a certain open subgroup π' of π.

Proof. (1) If the action is continuous, then the function $\pi \to E$ given by $\sigma \mapsto \sigma e$ is continuous and the preimage of e, which is precisely the stabilizer of e in π, is open.

Conversely, suppose every stabilizer is open. Let $A : \pi \times E \to E$ denote the action. Since E is discrete, it suffices to show that $A^{-1}(e)$ is open for each $e \in E$. Let $e' \in \pi \cdot e$ and suppose $\tau_{e'} \in \pi$ is such that $\tau_{e'} e = e'$. Then

$$\{\sigma \colon \sigma e' = e\} = \tau_{e'}^{-1} \mathrm{Stab}_\pi(e'),$$

which is an open subset of π. Consequently,

$$A^{-1}(e) = \bigcup_{e' \in \pi \cdot e} \{(\sigma, e') \colon \sigma e' = e\} = \bigcup_{e' \in \pi \cdot e} \tau_{e'}^{-1} \mathrm{Stab}_\pi(e') \times \{e'\}$$

is an open subset of $\pi \times E$, as desired.

(2) The kernel of the action (denoted π') is the intersection of all the stabilizers. If E is finite, then since the stabilizers are open, the kernel is also open. Conversely, any open subgroup of π must have finite index, i.e., π' has finite index in π. Let $\tau_1, \dots, \tau_n$ be a collection of left coset representatives of π' in π, and supopse that for $1 \leqslant i \leqslant m$, we have $\tau_i e = e$, which implies

$$\pi_e = \bigcup_{i=1}^{m} \tau_i \pi',$$

and so the stabilizers of each $e \in E$ are open.

(3) This is trivial from (a) and (b). ■

§2 Galois Categories

§§ Statement of the Main Theorem

Definition 2.1. Let $\mathscr{C}$ be a category, X an object of $\mathscr{C}$, and G a subgroup of $\mathrm{Aut}_{\mathscr{C}}(X)$. The *quotient* of X by G is an object X/G of $\mathscr{C}$ together with a morphism $p : X \to X/G$ satisfying

(i) $p = p \circ \sigma$ for all $\sigma \in G$,

(ii) if $X \xrightarrow{f} Y$ is a morphism in $\mathscr{C}$ such that $f = f \circ \sigma$ for all $\sigma \in G$, then there is a unique morphism $X/G \xrightarrow{g} Y$ making

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/G \arrow[ur, "g"'] & \end{tikzcd}$$

commute.

The quotient of an object by a group need not exist in a category, but when it does, it must be unique up to a unique isomorphism.

Definition 2.2. Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to \mathbf{FinSets}$ a (covariant) functor from $\mathscr{C}$ to the category of finite sets. We say that the pair $(\mathscr{C}, F)$ is a *Galois category*, or that $\mathscr{C}$ is a Galois category with *fundamental functor* F, if the following axioms are satisfied:

(G1) There is a terminal object and $\mathscr{C}$ admits all fibred products.

(G2) An initial object exists in $\mathscr{C}$, finite coproducts exist in $\mathscr{C}$, and for any object in $\mathscr{C}$, the quotient by a finite group of automorphisms exists.

(G3) Any morphism u in $\mathscr{C}$ factors as $u = u' \circ u''$ where u' is a monomorphism and u'' is an epimorphism. Every monomorphism $X \xrightarrow{f} Y$ in $\mathscr{C}$ is an isomorphism of X with a direct summand of Y; i.e., there is an object $Z \xrightarrow{g} Y$ such that

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

is a coproduct diagram.

(G4) The functor F sends terminal objects to terminal objects and commutes with fibred products.

(G5) The functor F sends initial objects to initial objects, commutes with finite coproducts, sends epimorphisms to epimorphisms, and commutes with passage to the quotient by a finite group of automorphisms.

(G6) If u is a morphism in $\mathscr{C}$ such that $F(u)$ is an isomorphism, then u is an isomorphism.

Proposition 2.3. Let $(\mathscr{C}, F)$ be a small Galois category and set $\mathscr{D} = [\mathscr{C}, \mathbf{FinSets}]$, the functor category between $\mathscr{C}$ and the category of finite sets. Then $\mathrm{Aut}_{\mathscr{D}}(F)$ is a profinite group acting continuously on $F(X)$ for every $X \in \mathscr{C}$.

Proof. An element of $\mathrm{Aut}_{\mathscr{D}}(F)$ is a natural isomorphism $\eta : F \Rightarrow F$, i.e, each $\eta_X : F(X) \to F(X)$ is an isomorphism. Hence, we can identify $\mathrm{Aut}_{\mathscr{D}}(F)$ with a subgroup of $\prod_{X \in \mathscr{C}} \mathfrak{S}_{F(X)}$, where $\mathfrak{S}_{F(X)}$ is the group of permutations of $F(X)$. In particular,

$$\mathrm{Aut}_{\mathscr{D}}(F) = \left\{ (\eta_X)_X \in \prod_{X \in \mathscr{C}} \mathfrak{S}_{F(X)} : \text{ for each } Y \xrightarrow{f} Z \text{ in } \mathscr{C},\ \eta_Z \circ F(f) = F(f) \circ \eta_Y \right\}.$$

Let $Y \xrightarrow{f} Z$ be a morphism in $\mathscr{C}$. Then the set

$$\mathfrak{A}_f = \left\{ (\eta_X)_X \in \prod_{X \in \mathscr{C}} \mathfrak{S}_{F(X)} : \eta_Z \circ F(f) = F(f) \circ \eta_Y \right\}.$$

is closed, as it is the finite union of the closed sets

$$\prod_{\substack{X \in \mathscr{C} \\ X \neq Y, Z}} \mathfrak{S}_{F(X)} \times \{\eta_Y\} \times \{\eta_Z\},$$

where $\eta_Y \in \mathfrak{S}_{F(Y)}$ and $\eta_Z \in \mathfrak{S}_{F(Z)}$ satisfy $\eta_Z \circ F(f) = F(f) \circ \eta_Y$. Now, since

$$\mathrm{Aut}_{\mathscr{D}}(F) = \bigcap_{\substack{Y \xrightarrow{f} Z \\ \text{in } \mathscr{C}}} \mathfrak{A}_f,$$

it is a closed subgroup of $\prod_{X \in \mathscr{C}} \mathfrak{S}_{F(X)}$, so that it is a profinite group.

Finally, the map $\mathrm{Aut}_{\mathscr{D}}(F) \times F(X) \to F(X)$ given by $((\eta_X)_{X \in \mathscr{C}}, a) \longmapsto \eta_X(a)$ defines an action of $\mathrm{Aut}_{\mathscr{D}}(F)$ on $F(X)$. The stabilizer of each $a \in F(X)$ is precisely

$$\mathrm{Aut}_{\mathscr{D}}(F) \times \left(\prod_{\substack{Y \in \mathscr{C} \\ Y \neq X}} \mathfrak{S}_{F(Y)} \times \mathrm{Stab}_{\mathfrak{S}_{F(X)}}(a) \right),$$

which is an open subgroup of $\mathrm{Aut}_{\mathscr{D}}(F)$. Due to Proposition 1.2, this action is continuous. ■

Interlude 2.4 (Construction of the Main Functor). Let $(\mathscr{C}, F)$ be a small Galois category. Define the functor $H : \mathscr{C} \to \mathrm{Aut}(F)$-**sets** sending each $X \in \mathscr{C}$ to $F(X)$ with the $\mathrm{Aut}(F)$-action as defined in the proof of Proposition 2.3. If $Y \xrightarrow{f} Z$ is a morphism in $\mathscr{C}$, then the induced morphism $F(f) : F(Y) \to F(Z)$ is $\mathrm{Aut}(F)$-linear: indeed, if $\eta = (\eta_X)_X \in \mathrm{Aut}(F)$, then for $y \in Y$,

$$F(f)(\eta y) = F(f)(\eta_Y y) = \eta_Z(F(f)(z)) = \eta F(f)(z).$$

Theorem 2.5 (Fundamental Theorem of Galois Categories). Let $(\mathscr{C}, F)$ be an essentially small Galois category. Then

(1) The functor $H : \mathscr{C} \to \mathrm{Aut}(F)$-**sets** is an equivalence of categories.

(2) If π is a profinite group such that the categories $\mathscr{C}$ and π-**sets** are equivalent by an equivalence, that when composed with the forgetful functor π-**sets** $\to$ **FinSets** yields the funtor F, then π is canonically isomorphic to $\mathrm{Aut}(F)$.

(3) If F' is a second fundamental functor on $\mathscr{C}$, then F and F' are naturally isomorphic.

(4) If π is a profinite group such that the categories $\mathscr{C}$ and π-**sets** are equivalent, then there is an isomorphism of profinite groups $\pi \cong \mathrm{Aut}(F)$ that is canonically determined up to an inner automorophism of $\mathrm{Aut}(F)$.

Henceforth, let $(\mathscr{C}, F)$ be a small Galois category.

§§ Subobjects and connected objects

Definition 2.6. Lt $X \in \mathscr{C}$. Consider the set $\{Y \to X \text{ a monomorphism}\} / \sim$ where

$$Y \xrightarrow{f} X \sim Y' \xrightarrow{f'} X$$

if and only if there is an isomorphism $Y \xrightarrow{\cong} Y'$ making

$$\begin{tikzcd} Y \arrow[r, "\cong"] \arrow[d, "f"'] & Y' \arrow[dl, "f'"] \\ X & \end{tikzcd}$$

commute. Every equivalence class in the above is called a *subobject* of X.

Lemma 2.7. f is a monomorphism if and only if $F(f)$ is injective.

Proof. Let $Y \xrightarrow{f} X$. We first show that f is a monomorphism if and only if the canonical map $p_1 : Y \times_X Y \to Y$ is an isomorphism. If f is a monomorphism, then it is clear that $\begin{tikzcd} Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$ is a coproduct diagram, so that $p_1 : Y \times_X Y \to Y$ is an isomorphism.

Conversely, suppose $p_1 : Y \times_X Y \to Y$ is an isomorphism and consider the commutative diagram

$$\begin{tikzcd} Y \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "\mathbf{id}_Y"] \arrow[ddr, bend right, "\mathbf{id}_Y"'] & & \\ & Y \times_X Y \arrow[r, "p_1"] \arrow[d, "p_2"] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd}$$

Since p_1 is an isomorphism, it follows that $\theta = p_1^{-1}$ is an isomorphism. Further, since $p_2 \circ \theta = \mathbf{id}_Y$, we must have that $p_1 = p_2$.

Now, suppose $h_1, h_2 : Z \to Y$ are morphisms in $\mathscr{C}$ satisfying $f \circ h_1 = f \circ h_2$, then there is a morphism $\varphi : Z \to Y \times_X Y$ making the required diagram commute. But then

$$h_1 = p_1 \circ \varphi = p_2 \circ \varphi = h_2,$$

so that f is a monomorphism.

Coming back to the proof of the Lemma, we have

$$\begin{aligned} F(f) \text{ is injective } &\iff F(f) \text{ is a monomorphism} \\ &\iff F(p_1) \text{ is an isomorphism} \\ &\iff p_1 \text{ is an isomorphism} \\ &\iff f \text{ is a monomorphism,} \end{aligned}$$

where the first equivalence follows from the classification of monomorphisms in **FinSets**, the second and last equivalences follow from what we just proved and **(G4)**, and the third isomorphism follows from **(G6)**. ■

Lemma 2.8. Two monomorphisms $Y \xrightarrow{f} X$ and $Y' \xrightarrow{f'} X$ are representative of the same subobject of X if and only if $F(f)(F(Y)) = F(f')(F(Y'))$ as subsets of $F(X)$.

Proof. Suppose the two objects represent the same subobject of X. Then there is an isomorphism $\theta : Y \xrightarrow{\sim} Y'$ such that $f = f' \circ \theta$. Then, $F(f)(F(Y)) = F(f') \circ F(\theta)(F(Y))$ but $F(\theta)$ is an isomorphism, so is surjective and hence $F(f)(F(Y)) = F(f')(F(Y'))$.

Conversely, suppose $F(f)(F(Y)) = F(f')(F(Y'))$. As F commutes with fibred products, we have the following pullback squares

$$\begin{tikzcd} Y \times_X Y' \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "f"] \\ Y' \arrow[r, "f'"'] & X \end{tikzcd} \qquad \begin{tikzcd} F(Y \times_X Y') \arrow[r, "F(p_1)"] \arrow[d, "F(p_2)"'] & Y \arrow[d, "F(f)"] \\ Y' \arrow[r, "F(f')"'] & X \end{tikzcd}$$

Since the latter is a pullback square, we have

$$F\left(Y \times_X Y'\right) = \{(y, y') \in F(Y) \times F(Y') \colon F(f)(y) = F(f')(y')\}.$$

As $F(f)$ and $F(f')$ are injective with the same image in X, it is clear that both $F(p_1)$ and $F(p_2)$ must be bijections, consequently, due to **(G6)**, both p_1 and p_2 must be isomorphisms isomorphisms in $\mathscr{C}$. Finally, this gives $f = f' \circ (p_2 \circ p_1^{-1})$, as desired. ∎

Definition 2.9. An object $X \in \mathscr{C}$ is said to be *connected* if it has exactly two subobjects, $0 \to X$ and $\mathbf{id}_X : X \to X$.

Proposition 2.10. Every object in $\mathscr{C} \neq 0$ is the coproduct of its connected subobjects.

Proof. Let X be a non-initial object in $\mathscr{C}$. We shall argue by induction on $\#F(X)$. If $\#F(X) = 1$, then X is connected, for if $Y \xrightarrow{f} X$ is a subobject, then $F(Y) \xrightarrow{F(f)} F(X)$ is injective, so that $F(Y) = \emptyset$ or $F(Y) = F(X)$. In the latter case, $F(f)$ is an isomorphism and hence, so is f; on the other hand, if $F(Y) = \emptyset$, then Y must be the initial object of $\mathscr{C}$[1]. Suppose now that $\#F(X) \geqslant 2$; since there is nothing to prove when X is connected, we may suppose that X is not connected. Then there is a subobject $Y \xrightarrow{q_1} X$ of X which is neither initial, nor an isomorphism. Due to **(G3)**, there is a morphism $Z \xrightarrow{q_2} X$ such that $X = Y \coprod Z$. This coproduct diagram transforms into a coproduct diagram in **FinSets**, so that $F(q_2)$ is injective, consequently due to Lemma 2.7, q_2 is a monomorphism. It follows that $Z \xrightarrow{q_2} X$ is another subobject of X. The inductive hypothesis applies and we can write X coproduct of *some* of its connected components. Since $\#F(X)$ is finite, it is clear that this is a finite coproduct.

It remains to show that X is the disjoint union of *each* of its connected subobjects. Suppose $X = \coprod_{i=1}^n X_i$ and Y a connected subobject of X. I shall treat $F(Y)$ and $F(X_i)$ as subsets of $F(X)$ for ease of notation. Since $F(X) = \coprod_i F(X_i)$, there is some index j such that $F(Y) \times_{F(X)} F(X_j) = F(Y) \cap F(X_j) \neq \emptyset$. As a result, $Y \times_X X_j$ is not the initial object of $\mathscr{C}$. Since $F(Y \times_X X_j) \to F(X_j)$ and $F(Y \times_X X_j) \to F(Y)$ are injective, due to Lemma 2.7, the maps $Y \times_X X_j \to X_j$ and $Y \times_X X_j \to Y$ must be monomorphisms, and hence, must be isomorphisms. It follows that X_j and Y are the same subobject of X. ∎

Lemma 2.11. $\mathscr{C}$ admits all equalizers.

Proof. Let $f, g : X \to Y$ be morphisms in $\mathscr{C}$. There are two fibred product diagrams

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} (X \times_Y X) \times_{X \times X} X \arrow[r] \arrow[d] & X \arrow[d, "\mathbf{id}_X \times \mathbf{id}_X"] \\ X \times_Y X \arrow[r, "p_1 \times p_2"'] & X \times X \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \\ X & X \arrow[l, "\mathbf{id}_X"] \arrow[u, "\mathbf{id}_X"'] \arrow[ul, dotted] \end{tikzcd}$$

We claim that $W = (X \times_Y X) \times_{X \times X} X \to X$ is the equalizer of f and g. Clearly, we have the following equality of compositions:

$$\begin{aligned} W \to X \xrightarrow{f} Y &= W \to X \xrightarrow{\mathbf{id}_X} X \xrightarrow{f} Y \\ &= W \to X \to X \times X \xrightarrow{\pi_1} X \xrightarrow{f} Y \\ &= W \to X \times_Y X \to X \times X \xrightarrow{\pi_1} X \xrightarrow{f} Y \\ &= W \to X \times_Y X \xrightarrow{p_1} X \xrightarrow{f} Y \\ &= W \to X \times_Y X \xrightarrow{p_2} X \xrightarrow{g} Y \\ &= W \to X \times_Y X \to X \times X \xrightarrow{\pi_2} X \xrightarrow{g} Y \\ &= W \to X \to X \times X \xrightarrow{\pi_2} X \xrightarrow{g} Y \\ &= W \to X \xrightarrow{\mathbf{id}_X} X \xrightarrow{g} Y \\ &= W \to X \xrightarrow{g} Y. \end{aligned}$$

[1]Indeed, if 0 is "the" initial object of $\mathscr{C}$, then there is a unique morphism $0 \xrightarrow{u} Y$ in $\mathscr{C}$. But since $F(u)$ is an isomorphism in **FinSets**, it follows from **(G6)** that u is an isomorphism.

If $h : Z \to X$ is such that $f \circ h = g \circ h$, then there is a unique map $\theta : Z \to X \times_Y X$ induced by $Z \xrightarrow{h} X$, which then induces a unique map $\phi : Z \to W$, as desired. ■

Proposition 2.12. Let A be a connected object in $\mathscr{C}$ and $a \in F(A)$. Then for every $X \in \mathscr{C}$, the map

$$\mathscr{C}(A, X) \longrightarrow F(X) \qquad f \longmapsto F(f)(a)$$

is injective.

Proof. Let $f, g \in \mathscr{C}(A, X)$ be such that $F(f)(a) = F(g)(a)$, and let (C, θ) be the equalizer of f, g, which is known to exist due to Lemma 2.11. Since F commutes with fibred products, it must commute with equalizers too, hence $(F(C), F(\theta))$ is an equalizer of $F(f), F(g) : F(A) \to F(X)$. In particular, $F(\theta)$ is injective, so that θ is a monomorphism due to Lemma 2.7. Moreover,

$$a \in F(C) = \{b \in F(A) \colon F(f)(b) = F(g)(b)\} \neq \emptyset,$$

and hence C is not the initial object of $\mathscr{C}$, whence $\theta : C \to A$ is an isomorphism, which implies $f = g$. ■

Interlude 2.13. Consider the set $I = \{(A, a) \colon A \text{ connected}, \ a \in F(A)\} / \sim$ where $\sim$ is the equivalence relation:

$$(A, a) \sim (B, b) \iff \exists\ f : A \to B \text{ an isomorphism such that } F(f)(a) = b.$$

We can define a partial order on I by

$$(A, a) \geqq (B, b) \iff \exists\ f : A \to B \text{ a morphism such that } F(f)(a) = b.$$

Note that due to Proposition 2.12 the above map f, if it exists, is unique. We claim that $(I, \geqq)$ is a directed set under this order relation:

Reflexivity: Taking $f = \mathbf{id}_A$, we have $F(\mathbf{id}_A)(a) = a$, so $(A, a) \geqq (A, a)$.

Anti-symmetry: If $(A, a) \geqq (B, b)$ and $(B, b) \geqq (A, a)$, then there are morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that $F(f)(a) = b$ and $F(g)(b) = a$. Consequently, $F(g \circ f)(a) = a$ and $F(f \circ g)(b) = b$. Using Proposition 2.12, it follows that $g \circ f = \mathbf{id}_A$ and $f \circ g = \mathbf{id}_B$, that is, $(A, a) = (B, b)$.

Transitivity: If $(A, a) \geqq (B, b) \geqq (C, c)$ and $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are the corresponding maps, then $g \circ f : A \to C$ is such that

$$F(g \circ f)(a) = F(g) \circ F(f)(a) = F(g)(b) = c.$$

Directedness: Let $(A, a), (B, b) \in I$. Choose a connected subobject $C \to A \times B$ such that the image of $F(C)$ in $F(A \times B) = F(A) \times F(B)$ contains $a \times b$; further, let $c \in C$ be the unique element in $F(C)$ mapping to $a \times b$. Compose the monomorphism $C \to A \times B$ with the canonical projections $A \times B \xrightarrow{p_1} A$ and $A \times B \xrightarrow{p_2} B$ to obtain maps f_1 and f_2. Then it is clear that $F(f_1)(c) = a$ and $F(f_2)(c) = b$, so that $(C, c) \geqq (A, a), (B, b)$.

We shall write $(A, a) \geqq_f (B, b)$ if we want to specify the morphism $A \xrightarrow{f} B$ satisfying $F(f)(a) = b$.

If $(A, a) \geqq_f (B, b)$, then the morphism $f : A \to B$ induces a natural transformation of functors $\mathscr{C}(B, -) \xrightarrow{-\circ f} \mathscr{C}(A, -)$. This gives us a projective system of functors in the functor category $[\mathscr{C}, \mathbf{FinSets}]$.

Theorem 2.14. There is an isomorphism of functors

$$\varinjlim_{(A,a) \in I} \mathscr{C}(A, -) \longrightarrow F(-) \qquad f \longmapsto F(f)(a)$$

Proof. Consider the maps $\phi_{(A,a)} : \mathscr{C}(A, X) \to F(X)$ given by $f \mapsto F(f)(a)$. If $(A, a) \geqq_\psi (B, b)$, then it is clear that the diagram

$$\begin{tikzcd}
\mathscr{C}(A, X) \arrow[dr, "\phi_{(A,a)}"'] & & \mathscr{C}(B, X) \arrow[ll, "-\circ\psi"'] \arrow[dl, "\phi_{(B,b)}"] \\
& X &
\end{tikzcd}$$

commutes. This clearly induces a map $\phi : \varinjlim_{(A,a)\in I} \mathscr{C}(A,X) \to F(X)$ given by

$$\phi(f) = \phi_{(A,a)}(f) \quad \text{if } f \in \mathscr{C}(A,X).$$

It suffices to show that this map is a bijection of sets, since then it would follow that ϕ is an isomorphism of functors.

First, we show injectivity. Suppose $F(f)(a) = F(g)(b)$ for some $(A,a),(B,b) \in I$ and $f \in \mathscr{C}(A,X)$ and $g \in \mathscr{C}(B,X)$. Let $C \to A \times B$ be a connected subobject such that $(a,b) \in f(C)$, and let p_1', p_2' be the compositions of the projection maps $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ with the monomorphism $C \to A \times B$. It is then clear that $(C,c) \geqq (A,a)$ and $(C,c) \geqq (B,b)$.

Under the map $\mathscr{C}(A,X) \to \mathscr{C}(C,X)$, the morphism f maps to $f \circ p_1'$ and under the map $\mathscr{C}(B,X) \to \mathscr{C}(C,X)$, the morphism g maps to $g \circ p_2'$. We contend that these two maps are the same. Indeed, since $F(fp_1')(c) = F(gp_2')(c)$, due to Proposition 2.12, $fp_1' = gp_2'$. This shows that f and g are equal in $\varinjlim_{(A,a)\in I} \mathscr{C}(A,X)$.

Finally, to see surjectivity, take $x \in F(X)$ and consider $f : A \to X$, the connected component of X such that $x \in F(A)$. Then $(A,x) \in I$ and $F(f)(x) = x$. This completes the proof. ■

§§ Galois Objects

If A is a connected object, then we have the inequalities:

$$\# \operatorname{Aut}_{\mathscr{C}}(A) \leqslant \# \mathscr{C}(A,A) \leqslant \# F(A),$$

where the second inequality follows from Proposition 2.12. In particular, the set of automorphisms of A is finite, and therefore, it makes sense to talk about the quotient of a connected object by its group of automorphisms.

Definition 2.15. An object $A \in \mathscr{C}$ is called a *Galois object* if $A/\operatorname{Aut}_{\mathscr{C}}(A)$ is a terminal object.

Proposition 2.16. Let $X \in \mathscr{C}$. There exists $(A,a) \in I$ with A Galois such that the map $\mathscr{C}(A,X) \to F(X)$ given by $f \mapsto F(f)(a)$ is bijective.

Proof. Let $Y = X^{\#F(X)}$ be the product of $\#F(X)$ copies of X. As F commutes with products, we have $F(Y) = F(X)^{\#F(X)}$. Let us index the coordinates of Y by the elements of $F(X)$, and let $a \in F(Y)$ be the element having in the x-th coordinate the element $x \in F(X)$. Let A be the connected component of Y suc tthat $a \in F(A)$ and $f_x : A \to Y \to X$ be the composition of the monomorphism $A \to Y$ and the projectio on the x-th coordinate $p_x : Y \to X$. Then $f_x \in \mathscr{C}(A,X)$ and $F(f_x)(a) = x$. As a has all the elements of $F(X)$ in its coordinates, then as f_x varies, we obtain all the elements $x \in F(X)$, and so the map is bijective (since we already know about injectivity from Proposition 2.12).

Moreover, we have also obtained that the only morphisms in $\mathscr{C}(A,X)$ are the ones of the form f_x for a certain $x \in f(X)$. We contend that A is Galois. Let $a' \in F(A)$, $a' \neq a$. The map $\mathscr{C}(A,X) \to F(X)$ given by $f \mapsto F(f)(a')$ is bijective as it is injective and we have just seen that the two sets cardinality. As for all $g \in \mathscr{C}(A,X)$, $g = f_x$ for a certain x, this proves that a' has all the elements of $F(X)$ in its coordinates.

We shall show that there is an automorphism of Y sending a to a'. Let $a = (a_x)_{x\in F(X)}$ and $a' = (a_{\sigma(x)})_{x\in F(X)}$ where σ is a permutation of the set $F(X)$. Note that $\mathscr{C}(Y,Y) = \prod_{x\in F(X)} \mathscr{C}(Y,X)$ and consider the map $f = \prod_{x\in F(X)} p_{\sigma(x)}$. Then $F(f)(a) = \prod_{x\in F(X)} F(p_{\sigma(x)})(a) = (a_{\sigma(x)})_{x\in F(X)} = a'$. Taking the inverse permutation to σ we see that f is an isomorphism. Then the map $A \to Y \xrightarrow{\sigma} Y$ is a monomorphism, which induces an automorphism $A \xrightarrow{\tau} A'$ from A to another connected component A' of Y. Moreover, as $a' \in F(A) \cap F(A')$, and A, A' are connected, we must have $F(A) = F(A')$, so that $A = A'$ and therefore τ is an automorphism of A which sends a to a'. In conclusion $\operatorname{Aut}_{\mathscr{C}}(A)$ acts transitively on $F(A)$, and therefore A is Galois. ■

Remark 2.17. The above result shows that the subset $J \subseteq I$ corresponding to connected Galois objects is a cofinal subset of I, so

$$\varinjlim_{J} \mathscr{C}(A,-) \cong \varinjlim_{I} \mathscr{C}(A,-) \cong F.$$

§§ Construction of the Equivalence

Lemma 2.18. Let A be a connected Galois object, and B a connected object such that $\mathscr{C}(A, B) \neq \emptyset$. Then, the action

$$\mathrm{Aut}_{\mathscr{C}}(A) \times \mathscr{C}(, B) \to \mathscr{C}(A, B) \qquad (\sigma, f) \mapsto f \circ \sigma$$

is transitive.

Proof. Let $f \in \mathscr{C}(A, B)$, then we can factor $f = gh$ where h is an epimorphism and g is a monomorphism. Since B is connected, g must be an isomorphism since both A and B are connected. In particular, this means that $F(f)$ is an isomorphism. Thus, given $f' : A \to B$, there exists an $a' \in F(A)$ such that $F(f)(a') = F(f')(a)$. Since A is Galois, there exists a unique $\sigma \in \mathrm{Aut}_{\mathscr{C}}(A)$ such that $F(\sigma)(a) = a'$. Then $F(f\sigma)(a) = F(f')(a)$, and due to Proposition 2.12, we have that $f \circ \sigma = f'$. ■

Lemma 2.19. Let $(A, a), (B, b) \in J$, $(A, a) \geqq_f (B, b)$. Given $\sigma \in \mathrm{Aut}_{\mathscr{C}}(A)$, there exists a unique $\tau \in \mathrm{Aut}_{\mathscr{C}}(B)$ such that $\tau \circ f = f \circ \sigma$ and the mapping $\sigma \mapsto \tau$ is a surjective group homomorphism $\mathrm{Aut}_{\mathscr{C}}(A) \to \mathrm{Aut}_{\mathscr{C}}(B)$.

Proof. Let $a' := F(\sigma)(a)$ and $b' := F(f)(a')$. Then, since B is Galois, there exists a unique $\tau \in \mathrm{Aut}_{\mathscr{C}}(B)$ such that $F(\tau)(b) = b'$ due to Proposition 2.16. So, we have

$$F(f\sigma)(a) = b' = F(\tau f)(a) \implies f \circ \sigma = \tau \circ f$$

due to Proposition 2.12. It remains to show that such a $\tau \in \mathrm{Aut}_{\mathscr{C}}(B)$ is unique. Indeed, if there were two automorphisms $\tau_1, \tau_2 \in \mathrm{Aut}_{\mathscr{C}}(B)$ satisfying the property, i.e., $\tau_1 \circ f = f \circ \sigma = \tau_2 \circ f$, then $F(\tau_1)(b) = F(\tau_2)(b)$. Due to Proposition 2.12, it follows that $\tau_1 = \tau_2$.

Finally, we must show that the association $\sigma \mapsto \tau$ is a surjective group homomorphism $\mathrm{Aut}_{\mathscr{C}}(A) \to \mathrm{Aut}_{\mathscr{C}}(B)$. Indeed, if $\sigma_1 \mapsto \tau_1$ and $\sigma_2 \mapsto \tau_2$, then we have

$$f\sigma_1\sigma_2 = \tau_1 f \sigma_2 = \tau_1\tau_2 f,$$

and so $\sigma_1\sigma_2 \mapsto \tau_1\tau_2$. This proves that the association $\sigma \mapsto \tau$ is a group homomorphism. Further, due to Lemma 2.18, the action of $\mathrm{Aut}_{\mathscr{C}}(A)$ on $\mathscr{C}(A, B)$ is transitive, and hence, given $\tau \in \mathrm{Aut}_{\mathscr{C}}(B)$, there exists a $\sigma \in \mathrm{Aut}_{\mathscr{C}}(A)$ such that $\tau \circ f = f \circ \sigma$, whence the association $\sigma \mapsto \tau$ is surjective, thereby completing the proof. ■

Note that the above result gives rise to an inverse system indexed by J. Set

$$\pi := \varprojlim_J \mathrm{Aut}_{\mathscr{C}}(A) \subseteq \prod_J \mathrm{Aut}_{\mathscr{C}}(A).$$

Proposition 2.20. For all $X \in \mathscr{C}$, the action

$$\varprojlim_J \mathrm{Aut}_{\mathscr{C}}(A) \times \varinjlim_J \mathscr{C}(A, X) \longrightarrow \varinjlim_J \mathscr{C}(A, X) \qquad ((\sigma_A)_{A\in J}, f) \longmapsto f \circ \sigma^{-1}$$

defines a functor $H' : \mathscr{C} \to \pi$-**sets**.

Proof. First, we must check that this action is well-defined. Let $f_A \in \mathscr{C}(A, X)$ and $f_B \in \mathscr{C}(B, X)$ be representatives of the same element in $\varinjlim_J \mathscr{C}(A, X)$. This means that there exists a $(C, c) \in J$ such that $(C, c) \geqq_{f_1} (A, a)$ and $(C, c) \geqq_{f_2} (B, b)$ and $f_B \circ f_2 = f_C = f_A \circ f_1$. Let $(\sigma_A)_{A\in J} \in \varprojlim_J \mathrm{Aut}_{\mathscr{C}}(A)$. Then we have

$$f_A f_1 \sigma_C^{-1} = f_C \sigma_C^{-1} = f_B f_2 \sigma_C^{-1}.$$

But $\sigma_A^{-1} f_1 = f_1 \sigma_C^{-1}$ and $\sigma_B^{-1} f_2 = f_2 \sigma_C^{-1}$. Therefore,

$$f_C \sigma_C^{-1} = f_A \sigma_A^{-1} f_1 = f_B \sigma_B^{-1} f_2,$$

whence $f_A \sigma_A^{-1} = f_B \sigma_B^{-1}$ in $\varinjlim_B \mathscr{C}(A, X)$.

It is easy to check that the above action is continuous. We shall show functoriality. Let $f : X \to Y$ be a morphism in $\mathscr{C}$, then $H'(f)$ maps $(f_A)_{A\in J}$ to $(f \circ f_A)_{A\in J}$. Then it is clear that H' preserves compositions and maps the identity to the identity. ■

Remark 2.21. We have defined a functor $H' : \mathscr{C} \to \pi$-**sets** by endowing the functor $\varinjlim_J \mathscr{C}(A, -) \cong F(-)$. This isomorphism of functors induces a π-action on $F(X)$ for each $X \in \mathscr{C}$, which induces a functor $\mathscr{C} \to \pi$-**sets**, which, upon composing with the forgetful functor π-**sets** $\to$ **Sets** recovers F. All that remains is to show that H' is an equivalence of categories.

Proposition 2.22. Let B be a connected object in $\mathscr{C}$. Then $B \cong A/G$ for some Galois object A and G a finite subgroup of $\mathrm{Aut}_{\mathscr{C}}(A)$.

Proof. Due to Proposition 2.16 there exists a Galois object A and $a \in F(A)$ such that the map

$$\mathscr{C}(A, B) \longrightarrow F(B) \qquad f \longmapsto F(f)(a)$$

is bijective. But since $F(B) \cong \varinjlim_J \mathscr{C}(A, B)$ and $\mathrm{Aut}_{\mathscr{C}}(A)$ acts transitively on $\mathscr{C}(A, B)$, we have that $\mathrm{Aut}_{\mathscr{C}}(A)$ acts transitively on $H'(B)$, and therefore, $H'(B) \cong \mathrm{Aut}_{\mathscr{C}}(A)/G$, where G is the stabilizer of a certain element $f \in H'(B)$.

In particular, we have $\#F(B) = \#\,\mathrm{Aut}_{\mathscr{C}}(A)/G$. Further, for each $\sigma \in G$, we have $f \circ \sigma = \sigma$, consequently, there is a morphism $g : A/G \to B$ induced by f. Since $F(f)$ is surjective, so is $F(g)$. Moreover, $\#F(A/G) = \#F(A)/G = \#\,\mathrm{Aut}_{\mathscr{C}}(A)/G$, so that $F(g)$ is an isomorphism. It follows from **(G6)** that $B \cong A/G$, as desired. ∎

Proposition 2.23. The functor H' maps connected objects to connected objects.

Proof. For every connected object B, choose a Galois object A such that $\mathscr{C}(A, B) \cong F(B)$. Then $\mathrm{Aut}_{\mathscr{C}}(A)$ acts transitively on $\mathscr{C}(A, B)$. Further, since the maps in the inverse system $(\mathrm{Aut}_{\mathscr{C}}(A))_{A \in J}$ are surjective, it follows that π acts transitively on $H'(B)$, so that $H'(B)$ is a connected object in the category π-**sets**. ∎

Lemma 2.24. Let $f : X \to Z$ be an epimorphism in $\mathscr{C}$ and $g : W \to Z$ a non-trivial subobject. Then $W \times_Z X \to Z$ is a non-trivial subobject. In particular, if $X \to Z$ is an epimorphism and X is connected, then Z is connected.

Proof. First, we show that the map $p_2 : W \times_Z X \to X$ is a monomorphism. Indeed, let $s, t : Y \to W \times_Z X$ satisfy $p_2 s = p_2 t$.

$$\begin{tikzcd}
Y \arrow[dr, dotted] \arrow[drr, bend left, "p_1 s = p_1 t"] \arrow[ddr, bend right, "p_2 s = p_2 t"'] & & \\
& W \times_Z X \arrow[r, "p_1"] \arrow[d, "p_2"] & W \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Composing with f, we have $fp_2 s = fp_2 t$, and hence $gp_1 s = gp_2 t$. As g is a monomorphism, this implies that $p_1 s = p_1 t$. Thus $s, t : Y \to W \times_Z X$ makes the above diagram commute. Hence, $s = t$ by uniqueness. This proves that p_2 is a monomorphism.

It remains to check that it is a non-trivial subobject. For this, it is enough to check that $F(W \times_Z X)$ is neither the empty set, nor all of $F(X)$. Note that

$$F(W \times_Z X) = F(W) \times_{F(Z)} F(X) = \{(a, b) \in F(W) \times F(X) \colon F(f)(b) = F(g)(a)\}.$$

- If $F(W \times_Z X) = F(X)$, then since f is an epimorphism, this means that for all $z \in F(Z)$, there exists a pair $(a, b) \in F(W) \times F(X)$ such that $F(f)(b) = z$, and $F(g)(a) = z$, so that $F(g)$ is surjective. Since $F(g)$ is also injective, it must be an isomorphism, whence, due to **(G6)**, so is g, i.e., $W \to Z$ is the full subobject, a contradiction.
- If $F(W \times_Z X) = 0$, then there is no pair $(a, b) \in F(W) \times F(X)$ satisfying $F(g)(a) = F(f)(b)$. But as f is an epimorphism, we must have that $F(W) = 0$, so that W is the initial object in $\mathscr{C}$, a contradiction. ∎

Lemma 2.25. If $f, g : X \to Y$ are two morphisms in $\mathscr{C}$ satisfying $F(f) = F(g)$, then $f = g$.

Proof. Let $E \to X$ denote the equalizer of (f, g). As we have seen earlier, $F(E) \to F(X)$ is the equalizer of $(F(f), F(g))$. But since $F(f) = F(g)$, we must have that $F(E) \to F(X)$ is an isomorphism, whence due to **(G6)**, $E \to X$ is an isomorphism, so that $f = g$, as desired. ∎

Theorem 2.26. The functor $H' : \mathscr{C} \to \pi$-**sets** is an equivalence of categories.

Proof. It suffices to show that H' is fully-faithful and essentially surjective. Any π-set is a disjoint union of transitive orbits, so it suffices to show that every transitive π-set is of the form $H'(X)$ for some $X \in \mathscr{C}$ (this is because H' preserves coproducts).

Note that every trasitiv eπ-set is of the form $\mathrm{Aut}_{\mathscr{C}}(A)/G$ for some $G \subseteq \mathrm{Aut}_{\mathscr{C}}(A)$ and A connected Galois. Note that the map

$$\mathrm{Aut}_{\mathscr{C}}(A) \longrightarrow H'(A) \qquad f \longmapsto F(f)(a)$$

is bijective. Therefore the map

$$H'(A) \longrightarrow \mathrm{Aut}_{\mathscr{C}}(A) \qquad F(f)(a) \longmapsto f^{-1}$$

is a bijection, and $F(f\sigma^{-1}) \longmapsto \sigma f^{-1}$, so the map respects the π-action, and it is therefore an isomorphism of π-sets. Thus

$$H'(A/G) \cong H'(A)/G \cong \mathrm{Aut}_{\mathscr{C}}(A)/G,$$

thereby proving essential surjectivity.

As for fully-faithfulness, we already know that $\mathscr{C}(X,Y) \to \pi$-**sets** $(H'(X), H'(Y))$ is injective due to Proposition 2.12. Therefore it would suffice to show that the sets have the same cardinality. First, we reduce this to the case of connected objects.

- For all $X \in \mathscr{C}$, wecan write a decomposition $X = \coprod_{i=1}^n X_i$, and due to the universal property of coproducts, we have

$$\mathscr{C}(X,Y) \cong \prod_{i=1}^n \mathscr{C}(X_i, Y).$$

  As H' commutes with finite coproducts, we also have that

$$\pi\text{-}\mathbf{sets}(H'(X), H'(Y)) \cong \prod_{i=1}^n \pi\text{-}\mathbf{sets}\left(H'(X_i), H'(Y)\right),$$

  whence we can reduce to the case that X is connected.

- Let $X \to Y$ be a morphism. Using **(G3)**, we can factor it as $X \xrightarrow{\text{epi}} Z \xrightarrow{\text{mono}} Y$. If X is connected, due to Lemma 2.24, we know that Z is connected too, and hence $Z \to Y$ is a connected component of Y. This shows that any morphism $X \to Y$ factors through connected components of Y, so that

$$\mathscr{C}(X,Y) \cong \coprod_{i=1}^n \mathscr{C}(X, Y_i)$$

  for X connected. Using that H' maps connected components to connected components, we also have that

$$\pi\text{-}\mathbf{sets}\left(H'(X), H'(Y)\right) \cong \coprod_{i=1}^n \pi\text{-}\mathbf{sets}\left(H'(X), H'(Y_i)\right).$$

Now choose $A \in \mathscr{C}$ so that $X \cong A/G_1$ and $Y \cong A/G_2$. This can always be done: For example, one can take A a connected component of $X^{\#F(X)} \times Y^{\#F(Y)}$ and repeat the same proof of Proposition 2.16, and then use Proposition 2.22.

Then we have that $H'(X) \cong \mathrm{Aut}_{\mathscr{C}}(A)/G_1$ and $H'(Y) \cong \mathrm{Aut}_{\mathscr{C}}(A)/G_2$. Consider a morphism of π-sets, $f : \mathrm{Aut}_{\mathscr{C}}(A)/G_1 \to \mathrm{Aut}_{\mathscr{C}}(A)/G_2$. Then, $f(\tau G_1) = \tau\sigma G_2$, for a certain σ that completely characterizes f. The morphism is well-defined $\iff$ two representatives of the same class are mapped to the same element $\iff$ for all $g \in G_1$, $gG_1 \mapsto \sigma G_2 \iff$ for all $g \in G_1$, $g\sigma G_2 = \sigma G_2 \iff$ for all $g \in G_1$, $g\sigma \in \sigma G_2 \iff G_1\sigma \subseteq \sigma G_2$. Then

$$\#\pi\text{-}\mathbf{sets}\left(H'(X), H'(Y)\right) = \#\left\{\sigma G_2 \colon G_1\sigma \subseteq \sigma G_2\right\}.$$

On the other hand, the choice of A implies that $\mathrm{Aut}_{\mathscr{C}}(A)$ acts transitively on both $\mathscr{C}(A, X)$ and $\mathscr{C}(A, Y)$. Then, cosnider the projection morphisms $A \xrightarrow{h_1} A/G_1$ and $A \xrightarrow{h_2} A/G_2$. Given $f : X \to Y$, there exists a $\sigma \in \mathrm{Aut}_{\mathscr{C}}(A)$ such that $h_2\sigma = fh_1$.

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "\sigma"'] & A/G_1 = X \arrow[d, "f"] \\ A \arrow[r, "h_2"'] & A/G_2 = Y \end{tikzcd}$$

Note that $h_2\sigma = h_2\sigma' \iff \sigma'\sigma^{-1} \in G_2 \iff G_2 sigma = G_2\sigma'$, so f uniquely determines the coset $G_2\sigma$. Reciprocially, an element $\sigma \in \mathrm{Aut}_{\mathscr{C}}(A)$ gives rise to a morphism $f : X \to Y$ ifa nd only if $h_2\sigma$ factors through A/G_1, that is, if and only if $h_2\sigma\tau = h_2\sigma$, for all $\tau \in G_1$ if and only if $\sigma G_2 \subseteq G_2\sigma$. This proves that

$$\#\mathscr{C}(X, Y) = \#\{G_2\sigma \colon \sigma G_2 \subseteq G_2\sigma\}.$$

In conclusion, $\#\mathscr{C}(X, Y) = \#\pi\text{-}\mathbf{sets}(H'(X), H'(Y))$, thereby completing the proof. ∎

§§ Proof of the Main Theorem

Lemma 2.27. Let π be a profinite group, $F : \pi\text{-}\mathbf{sets} \to \mathbf{Sets}$ the forgetful functor. Then $\mathrm{Aut}(F) \cong \pi$.

Proof. Note that given $\theta \in \mathrm{Aut}(F)$, the action of θ on every π-set is determined by its action on the transitive π-sets, and as every transitive π-set is isomorphic to one of the form π/π', with π' an open subgroup of π, the action of θ is totally determined by the action on the sets of this form.

Moreover, we know that in a compact totally disconnected group, every neighborhood of 1 contains an open normal subgroup. Therefore, there exists π'' an open normal subgroup of π such that $\pi'' \subseteq \pi'$. Consider the natural morphism of π-sets $f : \pi/\pi'' \twoheadrightarrow \pi/\pi'$. The automorphism θ of F has to commute with f. Let $\sigma \in \pi$ be such that $\theta_{\pi/\pi''}(\tau\pi'') = \tau\sigma\pi''$. Then we have $f \circ \theta_{\pi/\pi''}(\tau\pi'') = \tau\sigma\pi'$, and so $\theta_{\pi/\pi'} \circ f(\tau\pi') = \tau\sigma\pi'$. As $f(\tau\pi'') = \tau\pi'$, we have then $\sigma_{\pi/\pi'}(\tau\pi') = \tau\sigma\pi'$. Thus, the action of $\theta \in \mathrm{Aut}(F)$ is totally determined by the morphisms $\theta_{\pi/\pi'}$ where π' runs over open (and hence, finite index) normal subgroups of π.

Let π' be an open normal subgroup of π. Note that $\pi\text{-}\mathbf{sets}(\pi/\pi') \cong \pi/\pi'$, with the following isomorphism

$$\mathrm{Aut}_{\pi\text{-}\mathbf{sets}}(\pi/\pi') \to \pi/\pi' \qquad f \mapsto \tau^{-1}\pi' \text{ if } f(\pi') = \tau\pi'.$$

Now let $f : \pi/\pi' \to \pi/\pi$ be a set theoretic map commuting with all π-set automorphisms. Then $f(\tau\pi')\sigma = f(\tau\pi'\sigma)$ if and only if $f(\pi'\tau)\sigma = f(\pi'\tau\sigma)$. Let $f(\pi') = a\pi'$. Then $f(\pi'\sigma) = f(\sigma\pi') = f(\pi')\sigma = a\pi'\sigma$, so f is given by left multiplication by an element of π/π'. Therefore, we can define a map $\psi : \pi \to \mathrm{Aut}(F)$ given by

$$\psi(\sigma)_{\pi/\pi'}(\pi') = \sigma\pi'$$

for every open normal subgroup of π. We shall now show that this is an isomorphism of groups.

Well-defined: To see that $\psi(\sigma) \in \mathrm{Aut}(F)$, it is enough to check that it commutes with every morphism of π-sets, and this can clearly be reduced to proving that it commutes with every morphism $\pi/\pi' \to \pi/\pi''$, where π' and π'' are open normal subgroups of π. Let $f : \pi/\pi' \to \pi/\pi''$ be given by $f(\pi') = a\pi''$. Let $x \in \pi' \setminus \pi''$. Then $xa\pi'' f(x\pi') = f(\pi') = a\pi''$ and hence $x\pi''a = \pi''a$ for all $x \in \pi'$. This implies $\pi'' \supseteq \pi'$, and it is clear that $\psi(\sigma)$ commutes with f, so ψ is well-defined.

Injectivity: This is clear because an element $\theta \in \mathrm{Aut}(F)$ is totally characterized by the coordinates $\theta_{\pi/\pi'} \in \pi/\pi'$, and

$$\pi \cong \varprojlim_{\pi' \text{ open normal}} \pi/\pi'.$$

Surjective: The fact that every morphism $\pi/\pi' \to \pi/\pi'$ commuting with π-set automorphisms is given by left product by an element of π/π' implies that every element of $\mathrm{Aut}(F)$ ha to be defined by left product by an element of $\varprojlim \pi/\pi' \cong \pi$.

This completes the proof. ■

Proof of Theorem 2.5. (b) Let π be a profinite group, and $H : \mathscr{C} \to \pi$-**sets** an equivalence that composed with the forgetful functor $F_1 : \pi$-**sets** $\to$ **Sets** yields F. Then we have $\mathrm{Aut}(F_1) \cong \pi$ by Lemma 2.27. Therefore, it would be enough to check that $\mathrm{Aut}(F) \cong \mathrm{Aut}(F_1)$.

Note that an automorphism $\varepsilon \in \mathrm{Aut}(F_1)$ induces naturally an automorphism of F, $\psi(\varepsilon) = \left(\varepsilon_{H(X)}\right)_{X \in \mathscr{C}}$. Indeed, for $A, B \in \pi$-**sets** and $f : A \to B$, there is a commutative diagram

$$\begin{tikzcd} F_1(A) \arrow[r, "F_1(g)"] \arrow[d, "\varepsilon_A"'] & F_1(B) \arrow[d, "\varepsilon_B"] \\ F_1(A) \arrow[r, "F_1(g)"'] & F_1(B) \end{tikzcd}$$

Given $Y, Z \in \mathscr{C}$ and $f : Y \to X$, we can take $A = H(X)$, $B = H(Y)$, and $g = H(f)$ and substituting into the diagram above, taking into account that $F_1 \circ H = F$, it yields

$$\begin{tikzcd} F(Y) \arrow[r, "F(f)"] \arrow[d, "\varepsilon_{H(Y)}"'] & F(Z) \arrow[d, "\varepsilon_{H(Z)}"] \\ F(Y) \arrow[r, "F(f)"'] & F(Z) \end{tikzcd}$$

Reciprocially, we shall show that every automorphism of F will induce an automorphism of F_1. As H is an equivalence of categories, we have that there exists a functor $G : \pi$-**sets** $\to \mathscr{C}$, and an isomorphism of functors $\theta : \mathbf{id} \Rightarrow HG$:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\theta_A"'] & B \arrow[d, "\theta_B"] \\ HG(A) \arrow[r, "HG(g)"'] & HG(B) \end{tikzcd}$$

Then let $\sigma \in \mathrm{Aut}(F)$, and take $Y = G(A)$, $Z = G(B)$, $f = G(g)$. We have commutative diagrams:

$$\begin{tikzcd} F(Y) \arrow[r, "F(f)"] \arrow[d, "\sigma_Y"'] & F(Z) \\ F(Y) \arrow[r, "F(f)"'] & F(Z) \end{tikzcd} \qquad \begin{tikzcd} F_1HG(A) \arrow[r, "F_1HG(g)"] \arrow[d, "\theta_{G(A)}"'] & F_1HG(B) \arrow[d, "\theta_{G(B)}"] \\ F_1HG(A) \arrow[r, "F_1HG(g)"'] & F_1HG(B) \end{tikzcd}$$

Then, we can define $\varphi(\sigma) := (\varphi(\sigma)_A)_{A \in \pi\text{-}\mathbf{sets}} = \left(F_1(\theta_A^{-1})\sigma_{G(A)}F_1(\theta_A)\right)_A$. We contend that $\varphi(\sigma)$ is an automorphism of functors. Indeed, $F_1(g) = F_1(\theta_B^{-1} \circ HG(g) \circ \theta_A)$ by the diagram of the equivalence of categories. Then

$$F_1(g) \circ \varphi(\sigma)_A = F_1\left(\theta_B^{-1} \circ HG(g) \circ \theta_A\right) F_1(\theta_A^{-1})\sigma_{G(A)}F_1(\theta_A) = F_1(\theta_B^{-1}) \circ F_1(HG(g)) \circ \sigma_{G(A)} \circ F_1(\theta_A),$$

and similarly,

$$\varphi(\sigma)_B F_1(g) = F_1(\theta_B^{-1}) \circ \sigma_{G(B)} \circ F_1(HG(g)) \circ F_1(\theta_A).$$

Using that σ is a natural transformation, we have that $\sigma_{G(B)}F_1HG(g) = F_1HG(g)\sigma_{G(A)}$, and so $F_1(g) \circ \varphi(\sigma)_A = \varphi(\sigma)_B \circ F_1(g)$ and hence $\varphi(\sigma)$ is a well-defined automorphism of the functor F_1.

It remains to show that $\varphi\psi$ and $\psi\varphi$ are identities. Let $\sigma \in \mathrm{Aut}(F)$, then

$$\psi\varphi(\sigma) = (\psi\varphi(\sigma)_X)_X = \left(F_1(\theta_{H(X)}^{-1})\sigma_{GH(X)}F_1(\theta_{H(X)})\right).$$

As σ is a natural isomorphism, it commutes with the morphism $\theta_{H(X)}$, and hence $\sigma_{GH(X)}F_1(\theta_{H(X)}) = F_1(\theta_{H(X)})\sigma_X$, and therefore $\psi\varphi(\sigma)_X = \sigma_X$, i.e., $\psi\varphi(\sigma) = \mathbf{id}_{\mathrm{Aut}(F)}$. Similarly, one can show that $\varphi\psi = \mathbf{id}_{\mathrm{Aut}(F_1)}$. This completes the proof of (b).

(a) Applying (b) to the profinite group $\pi = \varprojlim_{(A,a)\in J} \mathrm{Aut}_{\mathscr{C}}(A)$ and recall the functor H' constructed earlier which we have shown is an equivalence of categories in Theorem 2.26 which when composed with the forgetful functor F_1 yields F. Then $\pi \cong \mathrm{Aut}(F)$ and via this isomorphism we can identify H' and the previously defined $H : \mathscr{C} \to \mathrm{Aut}(F)$-**sets**. Therefore, H is an equivalence of categories.

(c) Let $F' : \mathscr{C} \to$ **Sets** be a second fundamental functor. Then we have $\varinjlim_J \mathscr{C}(A, -) \cong F$, $\varinjlim_{J'} \mathscr{C}(A, -) \cong F'$. Note that all the pairs $(A, a) \in J$ with the same A are isomorphic so we can replace J by $J_0 \subseteq J$ with exactly one pair (A, a) for each A Galois; similarly, we replace J' by $J_0' \subseteq J$ with exactly one pair (A, a) for each A Galois. Note here that the notion of Galois objects is independent of the fundamental functor.

Now given $(A, a), (B, b) \in J_0$ and $g : A \to B$ a morphism, there exists a unique $\beta \in \mathrm{Aut}_{\mathscr{C}}(B)$ such that $F(\beta)(F(g)(a)) = b$. Then $f := \beta g$ satisfies $F(f)(a) = b$, so $(A, a) \geqq_f (B, b)$ in J_0, and this happens if and only if $(A, a') \geqq_{f'} (B, b')$ in J_0' but the morphisms $f, f' : A \to B$ are not necessarily the same.

But it is true that for all $\alpha \in \mathrm{Aut}_{\mathscr{C}}(A)$, there exists a $\gamma \in \mathrm{Aut}_{\mathscr{C}}(B)$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\gamma"] \\ A \arrow[r, "f'"'] & B \end{tikzcd}$$

Now mapping $\alpha \mapsto \gamma$ we obtain a system of morphisms between the finite non-empty groups $\mathrm{Aut}_{\mathscr{C}}(A)$ giving rise to a projective sstem. Thsi limit is non-empty. This implies that we can make a simultaneous choice $(\alpha_A)_{(A,a)\in J_0}$ such that all the diagrams commute. This induces an isomorphism

$$\varinjlim_{J_0} \mathscr{C}(A, -) \cong \varinjlim_{J_0'} \mathscr{C}(A, -),$$

so that $F \cong F'$.

(d) Let $H' : \mathscr{C} \to \pi$-**sets** be an equivalence, and F' the composite of H' with the forgetful functor. Then $\pi \cong \mathrm{Aut}(F')$ by (b) and $F' \cong F$ by (c). The isomorphism between the functors F and F' induces an isomorphism $\sigma : \mathrm{Aut}(F) \to \mathrm{Aut}(F')$ by letting $\varepsilon' \in \mathrm{Aut}(F')$ correspond to $\varepsilon := \sigma\varepsilon'\sigma^{-1}$. In conclusion, $\pi \cong \mathrm{Aut}(F)$ canonically, thereby completing the proof. ■