

Galois Categories and the Étale Fundamental Group

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§1 Preliminaries on Profinite Groups

Proposition 1.1. Let π be a profinite group acting on a set E . Then

- (1) The action is continuous if and only if for each $e \in E$, $\text{Stab}_\pi(e)$ is open in π .
- (2) If E is finite, the action is continuous if and only if its kernel $\{\sigma \in \pi : \sigma e = e \ \forall e \in E\}$ is open in π .
- (3) Any finite transitive π -set is isomorphic to π/π' for a certain open subgroup π' of π .

Proof. (1) If the action is continuous, then the function $\pi \rightarrow E$ given by $\sigma \mapsto \sigma e$ is continuous and the preimage of e , which is precisely the stabilizer of e in π , is open.

Conversely, suppose every stabilizer is open. Let $A : \pi \times E \rightarrow E$ denote the action. Since E is discrete, it suffices to show that $A^{-1}(e)$ is open for each $e \in E$. Let $e' \in \pi \cdot e$ and suppose $\tau_{e'} \in \pi$ is such that $\tau_{e'} e = e'$. Then

$$\{\sigma : \sigma e' = e\} = \tau_{e'}^{-1} \text{Stab}_\pi(e'),$$

which is an open subset of π . Consequently,

$$A^{-1}(e) = \bigcup_{e' \in \pi \cdot e} \{(\sigma, e') : \sigma e' = e\} = \bigcup_{e' \in \pi \cdot e} \tau_{e'}^{-1} \text{Stab}_\pi(e') \times \{e'\}$$

is an open subset of $\pi \times E$, as desired.

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§2 Galois Categories

§§ Statement of the Main Theorem

Definition 2.1. Let \mathcal{C} be a category, X an object of \mathcal{C} , and G a subgroup of $\text{Aut}_{\mathcal{C}}(X)$. The *quotient* of X by G is an object X/G of \mathcal{C} together with a morphism $p : X \rightarrow X/G$ satisfying

- (i) $p = p \circ \sigma$ for all $\sigma \in G$,
- (ii) if $X \xrightarrow{f} Y$ is a morphism in \mathcal{C} such that $f = f \circ \sigma$ for all $\sigma \in G$, then there is a unique morphism $X/G \xrightarrow{g} Y$ making

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p \downarrow & \nearrow g & \\ X/G & & \end{array}$$

commute.

The quotient of an object by a group need not exist in a category, but when it does, it must be unique up to a unique isomorphism.

Definition 2.2. Let \mathcal{C} be a category and $F : \mathcal{C} \rightarrow \mathbf{FinSets}$ a (covariant) functor from \mathcal{C} to the category of finite sets. We say that the pair (\mathcal{C}, F) is a *Galois category*, or that \mathcal{C} is a Galois category with *fundamental functor* F , if the following axioms are satisfied:

- (G1) There is a terminal object and \mathcal{C} admits all fibred products.
- (G2) An initial object exists in \mathcal{C} , finite coproducts exist in \mathcal{C} , and for any object in \mathcal{C} , the quotient by a finite group of automorphisms exists.
- (G3) Any morphism u in \mathcal{C} factors as $u = u' \circ u''$ where u' is a monomorphism and u'' is an epimorphism. Every monomorphism $X \xrightarrow{f} Y$ in \mathcal{C} is an isomorphism of X with a direct summand of Y ; i.e., there is an object $Z \xrightarrow{g} Y$ such that

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ Z & \xrightarrow{g} & Y \end{array}$$

is a coproduct diagram.

- (G4) The functor F sends terminal objects to terminal objects and commutes with fibred products.
- (G5) The functor F sends initial objects to initial objects, commutes with finite coproducts, sends epimorphisms to epimorphisms, and commutes with passage to the quotient by a finite group of automorphisms.
- (G6) If u is a morphism in \mathcal{C} such that $F(u)$ is an isomorphism, then u is an isomorphism.

Proposition 2.3. Let (\mathcal{C}, F) be a small Galois category and set $\mathcal{D} = [\mathcal{C}, \mathbf{FinSets}]$, the functor category between \mathcal{C} and the category of finite sets. Then $\text{Aut}_{\mathcal{D}}(F)$ is a profinite group acting continuously on $F(X)$ for every $X \in \mathcal{C}$.

Proof. An element of $\text{Aut}_{\mathcal{D}}(F)$ is a natural isomorphism $\eta : F \Rightarrow F$, i.e., each $\eta_X : F(X) \rightarrow F(X)$ is an isomorphism. Hence, we can identify $\text{Aut}_{\mathcal{D}}(F)$ with a subgroup of $\prod_{X \in \mathcal{C}} \mathfrak{S}_{F(X)}$, where $\mathfrak{S}_{F(X)}$ is the group of permutations of $F(X)$. In particular,

$$\text{Aut}_{\mathcal{D}}(F) = \left\{ (\eta_X)_X \in \prod_{X \in \mathcal{C}} \mathfrak{S}_{F(X)} : \text{for each } Y \xrightarrow{f} Z \text{ in } \mathcal{C}, \eta_Z \circ F(f) = F(f) \circ \eta_Y \right\}.$$

Let $Y \xrightarrow{f} Z$ be a morphism in \mathcal{C} . Then the set

$$\mathfrak{A}_f = \left\{ (\eta_X)_X \in \prod_{X \in \mathcal{C}} \mathfrak{S}_{F(X)} : \eta_Z \circ F(f) = F(f) \circ \eta_Y \right\}.$$

is closed, as it is the finite union of the closed sets

$$\prod_{\substack{X \in \mathcal{C} \\ X \neq Y, Z}} \mathfrak{S}_{F(X)} \times \{\eta_Y\} \times \{\eta_Z\},$$

where $\eta_Y \in \mathfrak{S}_{F(Y)}$ and $\eta_Z \in \mathfrak{S}_{F(Z)}$ satisfy $\eta_Z \circ F(f) = F(f) \circ \eta_Y$. Now, since

$$\text{Aut}_{\mathcal{D}}(F) = \bigcap_{\substack{Y \xrightarrow{f} Z \\ \text{in } \mathcal{C}}} \mathfrak{A}_f,$$

it is a closed subgroup of $\prod_{X \in \mathcal{C}} \mathfrak{S}_{F(X)}$, so that it is a profinite group.

Finally, the map $\text{Aut}_{\mathcal{D}}(F) \times F(X) \rightarrow F(X)$ given by $((\eta_X)_{X \in \mathcal{C}}, a) \mapsto \eta_X(a)$ defines an action of $\text{Aut}_{\mathcal{D}}(F)$ on $F(X)$. The stabilizer of each $a \in F(X)$ is precisely

$$\text{Aut}_{\mathcal{D}}(F) \times \left(\prod_{\substack{Y \in \mathcal{C} \\ Y \neq X}} \mathfrak{S}_{F(Y)} \times \text{Stab}_{\mathfrak{S}_{F(X)}}(a) \right),$$

which is an open subgroup of $\text{Aut}_{\mathcal{D}}(F)$. Due to Proposition 1.1, this action is continuous. ■

Interlude 2.4 (Construction of the Main Functor). Let (\mathcal{C}, F) be a small Galois category. Define the functor $H : \mathcal{C} \rightarrow \text{Aut}(F)\text{-sets}$ sending each $X \in \mathcal{C}$ to $F(X)$ with the $\text{Aut}(F)$ -action as defined in the proof of Proposition 2.3. If $Y \xrightarrow{f} Z$ is a morphism in \mathcal{C} , then the induced morphism $F(f) : F(Y) \rightarrow F(Z)$ is $\text{Aut}(F)$ -linear: indeed, if $\eta = (\eta_X)_X \in \text{Aut}(F)$, then for $y \in Y$,

$$F(f)(\eta y) = F(f)(\eta_Y y) = \eta_Z(F(f)(z)) = \eta F(f)(z).$$

Theorem 2.5 (Fundamental Theorem of Galois Categories). Let (\mathcal{C}, F) be an essentially small Galois category. Then

- (1) The functor $H : \mathcal{C} \rightarrow \text{Aut}(F)\text{-sets}$ is an equivalence of categories.
- (2) If π is a profinite group such that the categories \mathcal{C} and $\pi\text{-sets}$ are equivalent by an equivalence, that when composed with the forgetful functor $\pi\text{-sets} \rightarrow \mathbf{FinSets}$ yields the functor F , then π is canonically isomorphic to $\text{Aut}(F)$.
- (3) If F' is a second fundamental functor on \mathcal{C} , then F and F' are naturally isomorphic.
- (4) If π is a profinite group such that the categories \mathcal{C} and $\pi\text{-sets}$ are equivalent, then there is an isomorphism of profinite groups $\pi \cong \text{Aut}(F)$ that is canonically determined up to an inner automorphism of $\text{Aut}(F)$.

Henceforth, let (\mathcal{C}, F) be a small Galois category.

§§ Subobjects and connected objects

Definition 2.6. Let $X \in \mathcal{C}$. Consider the set $\{Y \rightarrow X \text{ a monomorphism}\} / \sim$ where

$$Y \xrightarrow{f} X \sim Y' \xrightarrow{f'} X$$

if and only if there is an isomorphism $Y \xrightarrow{\cong} Y'$ making

$$\begin{array}{ccc} Y & \xrightarrow{\cong} & Y' \\ f \downarrow & \swarrow f' & \\ X & & \end{array}$$

commute. Every equivalence class in the above is called a **subobject** of X .

Lemma 2.7. f is a monomorphism if and only if $F(f)$ is injective.

Proof. Let $Y \xrightarrow{f} X$. We first show that f is a monomorphism if and only if the canonical map $p_1 : Y \times_X Y \rightarrow Y$ is an isomorphism. If f is a monomorphism, then it is clear that $\begin{array}{ccc} Y & \xrightarrow{=} & Y \\ \parallel & & \downarrow f \\ Y & \xrightarrow{f} & X \end{array}$ is a coproduct diagram, so that

$$\begin{array}{ccc} Y & \xrightarrow{=} & Y \\ \parallel & & \downarrow f \\ Y & \xrightarrow{f} & X \end{array}$$

$p_1 : Y \times_X Y \rightarrow Y$ is an isomorphism.

Conversely, suppose $p_1 : Y \times_X Y \rightarrow Y$ is an isomorphism and consider the commutative diagram

$$\begin{array}{ccccc} Y & & & & \\ & \searrow \theta & & \searrow \text{id}_Y & \\ & & Y \times_X Y & \xrightarrow{p_1} & Y \\ & \swarrow \text{id}_Y & \downarrow p_2 & & \downarrow f \\ & & Y & \xrightarrow{f} & X \end{array}$$

Since p_1 is an isomorphism, it follows that $\theta = p_1^{-1}$ is an isomorphism. Further, since $p_2 \circ \theta = \text{id}_Y$, we must have that $p_1 = p_2$.

Now, suppose $h_1, h_2 : Z \rightarrow Y$ are morphisms in \mathcal{C} satisfying $f \circ h_1 = f \circ h_2$, then there is a morphism $\varphi : Z \rightarrow Y \times_X Y$ making the required diagram commute. But then

$$h_1 = p_1 \circ \varphi = p_2 \circ \varphi = h_2,$$

so that f is a monomorphism.

Coming back to the proof of the Lemma, we have

$$\begin{aligned} F(f) \text{ is injective} &\iff F(f) \text{ is a monomorphism} \\ &\iff F(p_1) \text{ is an isomorphism} \\ &\iff p_1 \text{ is an isomorphism} \\ &\iff f \text{ is a monomorphism,} \end{aligned}$$

where the first equivalence follows from the classification of monomorphisms in **FinSets**, the second and last equivalences follow from what we just proved and **(G4)**, and the third isomorphism follows from **(G6)**. ■

Lemma 2.8. Two monomorphisms $Y \xrightarrow{f} X$ and $Y' \xrightarrow{f'} X$ are representative of the same subobject of X if and only if $F(f)(F(Y)) = F(f')(F(Y'))$ as subsets of $F(X)$.

Proof. Suppose the two objects represent the same subobject of X . Then there is an isomorphism $\theta : Y \xrightarrow{\sim} Y'$ such that $f = f' \circ \theta$. Then, $F(f)(F(Y)) = F(f') \circ F(\theta)(F(Y))$ but $F(\theta)$ is an isomorphism, so is surjective, and hence $F(f)(F(Y)) = F(f')(F(Y'))$.

Conversely, suppose $F(f)(F(Y)) = F(f')(F(Y'))$. As F commutes with fibred products, we have the following pullback squares

$$\begin{array}{ccc} Y \times_X Y' & \xrightarrow{p_1} & Y \\ p_2 \downarrow & & \downarrow f \\ Y' & \xrightarrow{f'} & X \end{array} \quad \begin{array}{ccc} F(Y \times_X Y') & \xrightarrow{F(p_1)} & Y \\ F(p_2) \downarrow & & \downarrow F(f) \\ Y' & \xrightarrow{F(f')} & X \end{array}$$

Since the latter is a pullback square, we have

$$F(Y \times_X Y') = \{(y, y') \in F(Y) \times F(Y') : F(f)(y) = F(f')(y')\}.$$

As $F(f)$ and $F(f')$ are injective with the same image in X , it is clear that both $F(p_1)$ and $F(p_2)$ must be bijections, consequently, due to **(G6)**, both p_1 and p_2 must be isomorphisms in \mathcal{C} . Finally, this gives $f = f' \circ (p_2 \circ p_1^{-1})$, as desired. ■

Definition 2.9. An object $X \in \mathcal{C}$ is said to be **connected** if it has exactly two subobjects, $0 \rightarrow X$ and $\text{id}_X : X \rightarrow X$.

Proposition 2.10. Every object in $\mathcal{C} \neq 0$ is the coproduct of its connected subobjects.

Proof. Let X be a non-initial object in \mathcal{C} . We shall argue by induction on $\#F(X)$. If $\#F(X) = 1$, then X is connected, for if $Y \xrightarrow{f} X$ is a subobject, then $F(Y) \xrightarrow{F(f)} F(X)$ is injective, so that $F(Y) = \emptyset$ or $F(Y) = F(X)$. In the latter case, $F(f)$ is an isomorphism and hence, so is f ; on the other hand, if $F(Y) = \emptyset$, then Y must be the initial object of \mathcal{C} ¹. Suppose now that $\#F(X) \geq 2$; since there is nothing to prove when X is connected, we may suppose that X is not connected. Then there is a subobject $Y \xrightarrow{q_1} X$ of X which is neither initial, nor an isomorphism. Due to **(G3)**, there is a morphism $Z \xrightarrow{q_2} X$ such that $X = Y \amalg Z$. This coproduct diagram transforms into a coproduct diagram in **FinSets**, so that $F(q_2)$ is injective, consequently due to Lemma 2.7, q_2 is a monomorphism. It follows that $Z \xrightarrow{q_2} X$ is another subobject of X . The inductive hypothesis applies and we can write X coproduct of *some* of its connected components. Since $\#F(X)$ is finite, it is clear that this is a finite coproduct.

It remains to show that X is the disjoint union of *each* of its connected subobjects. Suppose $X = \coprod_{i=1}^n X_i$ and Y a connected subobject of X . I shall treat $F(Y)$ and $F(X_i)$ as subsets of $F(X)$ for ease of notation. Since $F(X) = \coprod_i F(X_i)$, there is some index j such that $F(Y) \times_{F(X)} F(X_j) = F(Y) \cap F(X_j) \neq \emptyset$. As a result, $Y \times_X X_j$ is not the initial object of \mathcal{C} . Since $F(Y \times_X X_j) \rightarrow F(X_j)$ and $F(Y \times_X X_j) \rightarrow F(Y)$ are injective, due to Lemma 2.7, the maps $Y \times_X X_j \rightarrow X_j$ and $Y \times_X X_j \rightarrow Y$ must be monomorphisms, and hence, must be isomorphisms. It follows that X_j and Y are the same subobject of X . ■

Lemma 2.11. \mathcal{C} admits all equalizers.

Proof. Let $f, g : X \rightarrow Y$ be morphisms in \mathcal{C} . There are two fibred product diagrams

$$\begin{array}{ccccc} X \times_Y X & \xrightarrow{p_1} & X & & (X \times_Y X) \times_{X \times X} X & \longrightarrow & X \\ p_2 \downarrow & & \downarrow f & & \downarrow & & \downarrow \text{id}_X \times \text{id}_X \\ X & \xrightarrow{g} & Y & & X \times_Y X & \xrightarrow{p_1 \times p_2} & X \times X \end{array} \quad \begin{array}{ccc} X \times X & \xrightarrow{\pi_1} & X \\ \pi_2 \downarrow & \nearrow f & \uparrow \text{id}_X \\ X & \xleftarrow{\text{id}_X} & X \end{array}$$

We claim that $W = (X \times_Y X) \times_{X \times X} X \rightarrow X$ is the equalizer of f and g . Clearly, we have the following equality of compositions:

$$\begin{aligned} W \rightarrow X &\xrightarrow{f} Y = W \rightarrow X \xrightarrow{\text{id}_X} X \xrightarrow{f} Y \\ &= W \rightarrow X \rightarrow X \times X \xrightarrow{\pi_1} X \xrightarrow{f} Y \\ &= W \rightarrow X \times_Y X \rightarrow X \times X \xrightarrow{\pi_1} X \xrightarrow{f} Y \\ &= W \rightarrow X \times_Y X \xrightarrow{p_1} X \xrightarrow{f} Y \\ &= W \rightarrow X \times_Y X \xrightarrow{p_2} X \xrightarrow{g} Y \\ &= W \rightarrow X \times_Y X \rightarrow X \times X \xrightarrow{\pi_2} X \xrightarrow{g} Y \\ &= W \rightarrow X \rightarrow X \times X \xrightarrow{\pi_2} X \xrightarrow{g} Y \\ &= W \rightarrow X \xrightarrow{\text{id}_X} X \xrightarrow{g} Y \\ &= W \rightarrow X \xrightarrow{g} Y. \end{aligned}$$

¹Indeed, if 0 is “the” initial object of \mathcal{C} , then there is a unique morphism $0 \xrightarrow{u} Y$ in \mathcal{C} . But since $F(u)$ is an isomorphism in **FinSets**, it follows from **(G6)** that u is an isomorphism.

If $h : Z \rightarrow X$ is such that $f \circ h = g \circ h$, then there is a unique map $\theta : Z \rightarrow X \times_Y X$ induced by $Z \xrightarrow{h} X$, which then induces a unique map $\phi : Z \rightarrow W$, as desired. ■

Proposition 2.12. Let A be a connected object in \mathcal{C} and $a \in F(A)$. Then for every $X \in \mathcal{C}$, the map

$$\mathcal{C}(A, X) \longrightarrow F(X) \quad f \longmapsto F(f)(a)$$

is injective.

Proof. Let $f, g \in \mathcal{C}(A, X)$ be such that $F(f)(a) = F(g)(a)$, and let (C, θ) be the equalizer of f, g , which is known to exist due to Lemma 2.11. Since F commutes with fibred products, it must commute with equalizers too, hence $(F(C), F(\theta))$ is an equalizer of $F(f), F(g) : F(A) \rightarrow F(X)$. In particular, $F(\theta)$ is injective, so that θ is a monomorphism due to Lemma 2.7. Moreover,

$$a \in F(C) = \{b \in F(A) : F(f)(b) = F(g)(b)\} \neq \emptyset,$$

and hence C is not the initial object of \mathcal{C} , whence $\theta : C \rightarrow A$ is an isomorphism, which implies $f = g$. ■

Interlude 2.13. Consider the set $I = \{(A, a) : A \text{ connected, } a \in F(A)\} / \sim$ where \sim is the equivalence relation:

$$(A, a) \sim (B, b) \iff \exists f : A \rightarrow B \text{ an isomorphism such that } F(f)(a) = b.$$

We can define a partial order on I by

$$(A, a) \geq (B, b) \iff \exists f : A \rightarrow B \text{ a morphism such that } F(f)(a) = b.$$

Note that due to Proposition 2.12 the above map f , if it exists, is unique. We claim that (I, \geq) is a directed set under this order relation:

Reflexivity: Taking $f = \text{id}_A$, we have $F(\text{id}_A)(a) = a$, so $(A, a) \geq (A, a)$.

Anti-symmetry: If $(A, a) \geq (B, b)$ and $(B, b) \geq (A, a)$, then there are morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that $F(f)(a) = b$ and $F(g)(b) = a$. Consequently, $F(g \circ f)(a) = a$ and $F(f \circ g)(b) = b$. Using Proposition 2.12, it follows that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$, that is, $(A, a) = (B, b)$.

Transitivity: If $(A, a) \geq (B, b) \geq (C, c)$ and $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are the corresponding maps, then $g \circ f : A \rightarrow C$ is such that

$$F(g \circ f)(a) = F(g) \circ F(f)(a) = F(g)(b) = c.$$

Directedness: Let $(A, a), (B, b) \in I$. Choose a connected subobject $C \rightarrow A \times B$ such that the image of $F(C)$ in $F(A \times B) = F(A) \times F(B)$ contains $a \times b$; further, let $c \in C$ be the unique element in $F(C)$ mapping to $a \times b$. Compose the monomorphism $C \rightarrow A \times B$ with the canonical projections $A \times B \xrightarrow{p_1} A$ and $A \times B \xrightarrow{p_2} B$ to obtain maps f_1 and f_2 . Then it is clear that $F(f_1)(c) = a$ and $F(f_2)(c) = b$, so that $(C, c) \geq (A, a), (B, b)$.

We shall write $(A, a) \geq_f (B, b)$ if we want to specify the morphism $A \xrightarrow{f} B$ satisfying $F(f)(a) = b$.

If $(A, a) \geq_f (B, b)$, then the morphism $f : A \rightarrow B$ induces a natural transformation of functors $\mathcal{C}(B, -) \xrightarrow{- \circ f} \mathcal{C}(A, -)$. This gives us a projective system of functors in the functor category $[\mathcal{C}, \mathbf{FinSets}]$.

Theorem 2.14. There is an isomorphism of functors

$$\lim_{(A, a) \in I} \mathcal{C}(A, -) \longrightarrow F(-) \quad f \longmapsto F(f)(a)$$

Proof. Consider the maps $\phi_{(A, a)} : \mathcal{C}(A, X) \rightarrow F(X)$ given by $f \mapsto F(f)(a)$. If $(A, a) \geq_\psi (B, b)$, then it is clear that the diagram

$$\begin{array}{ccc} \mathcal{C}(A, X) & \xleftarrow{- \circ \psi} & \mathcal{C}(B, X) \\ \phi_{(A, a)} \searrow & & \swarrow \phi_{(B, b)} \\ & X & \end{array}$$

commutes. This clearly induces a map $\phi : \varinjlim_{(A,a) \in I} \mathcal{C}(A, X) \rightarrow F(X)$ given by

$$\phi(f) = \phi_{(A,a)}(f) \quad \text{if } f \in \mathcal{C}(A, X).$$

It suffices to show that this map is a bijection of sets, since then it would follow that ϕ is an isomorphism of functors.

First, we show injectivity. Suppose $F(f)(a) = F(g)(b)$ for some $(A, a), (B, b) \in I$ and $f \in \mathcal{C}(A, X)$ and $g \in \mathcal{C}(B, X)$. Let $C \rightarrow A \times B$ be a connected subobject such that $(a, b) \in f(C)$, and let p'_1, p'_2 be the compositions of the projection maps $p_1 : A \times B \rightarrow A$ and $p_2 : A \times B \rightarrow B$ with the monomorphism $C \rightarrow A \times B$. It is then clear that $(C, c) \geq (A, a)$ and $(C, c) \geq (B, b)$.

Under the map $\mathcal{C}(A, X) \rightarrow \mathcal{C}(C, X)$, the morphism f maps to $f \circ p'_1$ and under the map $\mathcal{C}(B, X) \rightarrow \mathcal{C}(C, X)$, the morphism g maps to $g \circ p'_2$. We contend that these two maps are the same. Indeed, since $F(fp'_1)(c) = F(gp'_2)(c)$, due to Proposition 2.12, $fp'_1 = gp'_2$. This shows that f and g are equal in $\varinjlim_{(A,a) \in I} \mathcal{C}(A, X)$.

Finally, to see surjectivity, take $x \in F(X)$ and consider $f : A \rightarrow X$, the connected component of X such that $x \in F(A)$. Then $(A, x) \in I$ and $F(f)(x) = x$. This completes the proof. ■

§§ Galois Objects

If A is a connected object, then we have the inequalities:

$$\# \text{Aut}_{\mathcal{C}}(A) \leq \# \mathcal{C}(A, A) \leq \# F(A),$$

where the second inequality follows from Proposition 2.12. In particular, the set of automorphisms of A is finite, and therefore, it makes sense to talk about the quotient of a connected object by its group of automorphisms.

Definition 2.15. An object $A \in \mathcal{C}$ is called a *Galois object* if $A / \text{Aut}_{\mathcal{C}}(A)$ is a terminal object.

Proposition 2.16. Let $X \in \mathcal{C}$. There exists $(A, a) \in I$ with A Galois such that the map $\mathcal{C}(A, X) \rightarrow F(X)$ given by $f \mapsto F(f)(a)$ is bijective.

Proof. ■