Valuation Rings and Dedekind Domains

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§1 General Valuation Rings

Definition 1.1. An integral domain R with fraction field K is said to be a *valuation ring* if for each $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$.

We remark that if R is a valuation ring with fraction field K, then any subring of K containing R is also a valuation ring.

Proposition 1.2. The ideals in a valuation ring R are totally ordered by inclusion.

Proof. Let I and J be ideals in a valuation ring. If either I or J is zero or $I = J$, then there is nothing to prove. Assume hence that both are non-zero and $I \neq J$. Without loss of generality, suppose $x \in I \setminus J$. Then for any $0 \neq y \in J$, $\frac{x}{y} \notin R$ lest $x \in J$. But since R is a valuation ring, $\frac{y}{x} \in R$, consequently, $y \in I$, so that $J \subseteq I$. ■

Corollary 1.3. A valuation ring is a local domain.

Proof. Since the ideals are totally ordered, there must be a unique maximal ideal. ■

Corollary 1.4. A finitely generated ideal in a valuation ring is principal, i.e., a valuation ring is a Bézout domain.

Proof. Suppose $I = (a_1, \dots, a_n) \trianglelefteq R$, a valuation ring. In view of Proposition 1.2 there exists an index i such that $(a_j) \subseteq (a_i)$ for all $1 \leqslant j \leqslant n$, and hence $I = (a_i)$ is principal. ■

Definition 1.5. A ring is said to be Bézout if every finitely generated ideal is principal.

Proposition 1.6. Let R be a ring. Then R is a valuation ring if and only if it is a local Bézout domain.

Proof. We have shown above that every valuation ring is a local Bézout domain. Conversely, suppose $(R, \mathfrak{m})$ is a local Bézout domain and let $0 \neq x \in K$, the fraction field of R. Then there exist $f, g \in R \setminus \{0\}$ such that $x = \frac{f}{g}$. Since R is a Bézout domain, there exists $h \in R$ such that $(f, g) = (h)$. Let $a, b \in R$ be such that $f = ah$ and $g = bh$. Then $(a, b) = (1)$, and hence, at least one of a or b must be a unit. In any case, either $\frac{f}{g}$ or $\frac{g}{h}$ is an element of R, that is, R is a valuation ring. ■

Proposition 1.7. A valuation ring is integrally closed in its field of fractions.

Proof. Let $(R, \mathfrak{m})$ be a valuation ring with field of fractions K. Suppose R is not integrally closed in K, then there exists $0 \neq x \in R$ such that $x^{-1} \in K \setminus R$ is integral over R, and hence, satisfies an equation of the form

$$x^{-n} + a_1 x^{-n+1} + \cdots + a_n = 0,$$

where $a_i \in R$ for $1 \leqslant i \leqslant n$. Further, since K is a field, we may assume that $a_n \neq 0$. Multiplying by x^n, we obtain

$$a_n x^n + \cdots + a_1 x + 1 = 0.$$

Since x is not a unit in R, $x \in \mathfrak{m}$, but the above equation would then imply that $1 \in \mathfrak{m}$, a contradiction. Thus R is integrally closed in K. ■

There is a very simple characterization of flat modules over valuation rings which we include here, although it will never be used throughout this article.

THEOREM 1.8. A module over a Bézout domain is flat if and only if it is torsion-free. In particular, this is true for valuation rings.

Proof. Let R be a Bézout domain. It is well-known that a flat module over an integral domain is torsion-free; this follows by considering, for each $0 \neq a \in R$, the injective map $0 \to R \xrightarrow{\cdot a} R$ and tensoring it with M.

Conversely, let M be a torsion-free R-module. It suffices to show that $\operatorname{Tor}_1^R(R/\mathfrak{a}, M) = 0$ for every finitely generated ideal $\mathfrak{a}$ of R. Disregarding the trivial case, we may assume that $\mathfrak{a} \neq 0$. Since R is a Bézout domain, $\mathfrak{a} = (a)$ for some $0 \neq a \in R$. Tensoring the short exact sequence

$$0 \to R \xrightarrow{\cdot a} R \to R/aR \to 0$$

with M and taking the induced long exact sequence, we get

$$\cdots \to 0 = \operatorname{Tor}_R^1(R, M) \to \operatorname{Tor}_R^1(R/aR, M) \to R \otimes_R M \xrightarrow{\cdot a} R \otimes_R M \to R/aR \otimes_R M \to 0.$$

But since $R \otimes_R M$ is canonically isomorphic to M, and M is torsion-free, we have that $\operatorname{Tor}_R^1(R/aR, M) = 0$, whence M is a flat R-module. ■

THEOREM 1.9. Let R be a valuation ring with fraction field K, and let R' be another subring of K properly containing R. Let $\mathfrak{m}$ denote the maximal ideal of R and $\mathfrak{p}$ the maximal ideal of R'. Then

(1) $\mathfrak{p} \subsetneq \mathfrak{m} \subseteq R \subseteq R'$.

(2) $\mathfrak{p}$ is a prime ideal in R, and $R' = R_{\mathfrak{p}}$.

(3) $R/\mathfrak{p}$ is a valuation ring of the field $R'/\mathfrak{p}$.

(4) Given any valuation ring $\overline{S}$ of the field $R/\mathfrak{m}$, let S be its inverse image in R. Then S is a valuation ring having the same fraction field K as R.

Proof. (1) Let $x \in \mathfrak{p}$ so that x is not a unit in R', i.e., $x^{-1} \notin R'$. Thus $x^{-1} \notin R$, equivalently, $x \in \mathfrak{m}$. Next, to see that the inclusion $\mathfrak{p} \subseteq \mathfrak{m}$ is strict, choose some $y \in R' \setminus R$. Then $y^{-1} \in R$ and is not a unit in R, whence $y^{-1} \in \mathfrak{m}$, but $y^{-1} \notin \mathfrak{p}$, else $y \notin R'$. Thus the inclusion $\mathfrak{p} \subseteq \mathfrak{m}$ is strict.

(2) Since $\mathfrak{p} = \mathfrak{p} \cap R$, it is a prime ideal in R. Clearly every element in $R \setminus \mathfrak{p}$ is invertible in R', so that $R \subseteq R_{\mathfrak{p}} \subseteq R'$. But by construction, the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ is contained in the maximal ideal $\mathfrak{p}$ of R'; in view of (1), this means $R_{\mathfrak{p}} = R'$.

(3) Let $\pi\colon R' \to R'/\mathfrak{p}$ denote the natural surjection. Let $0 \neq \overline{x} \in R'/\mathfrak{p}$ and choose some $x \in R' \setminus p$ such that $\overline{x} = \pi(x)$. If $x \in R$, then $\overline{x} \in R/\mathfrak{p}$, else $x^{-1} \in R$ and $\overline{x}^{-1} = \pi(x^{-1}) \in R/\mathfrak{p}$, as desired.

(4) Note that S is a subring of R containing $\mathfrak{m}$ and $S/\mathfrak{m} = \overline{S} \subseteq R/\mathfrak{m}$. Let $0 \neq x \in K$. Since R is a valuation ring, either $x \in R$ or $x^{-1} \in R$. We may suppose without loss of generality that $x \in R$. Let $\overline{x} \in R/\mathfrak{m}$ denote its image. If $\overline{x} = 0$, then $x \in \mathfrak{m} \subseteq S$. Otherwise, either $\overline{x} \in \overline{S}$ or $\overline{x}^{-1} \in \overline{S}$. Hence, either $x \in S$ or $x^{-1} \in S$, i.e., S is a valuation ring with fraction field K. ∎

THEOREM 1.10. Let K be a field, $A \subseteq K$ a subring, and $\mathfrak{p}$ a prime ideal of A. Then there exists a valuation ring $(R, \mathfrak{m})$ of K satisfying

$$A \subseteq R \quad \text{and} \quad \mathfrak{m} \cap A = \mathfrak{p}.$$

Proof. Replacing A by $A_{\mathfrak{p}}$, we may assume that A is a local ring with $\mathfrak{p} = \mathfrak{m}_A$ the maximal ideal of A. This can be done because $A \cap \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$. Next, let $\mathscr{F}$ denote the subrings of K containing A such that $1 \notin \mathfrak{p}B$; and order $\mathscr{F}$ by inclusion of rings. Clearly every chain in $\mathscr{F}$ has an upper bound given by union of rings constituting the chain. In view of Zorn's lemma, $\mathscr{F}$ admits a maximal element, say R. Since $\mathfrak{p}R \subsetneq R$, there exists a maximal ideal $\mathfrak{m}$ of R containing $\mathfrak{p}R$. Then $R_{\mathfrak{m}} \in \mathscr{F}$, since the extension of $\mathfrak{m}$ to $R_{\mathfrak{m}}$ is $\mathfrak{m}R_{\mathfrak{m}}$, which is a proper ideal. Thus $R_{\mathfrak{m}} \in \mathscr{F}$ and by maximality of R, we have $R = R_{\mathfrak{m}}$, that is, $(R, \mathfrak{m})$ is a local ring.

Since $\mathfrak{m} \cap A \supseteq \mathfrak{p}$ and $\mathfrak{p}$ is a maximal ideal, we have $\mathfrak{m} \cap A = \mathfrak{p}$. It remains to show that R is a valuation ring. Let $0 \neq x \in K$ and suppose that $x, x^{-1} \notin R$. Since $x \notin R$, $R \subsetneq R[x]$, so that $R[x] \notin \mathscr{F}$ and hence $\mathfrak{p}$ generates the unit ideal in $R[x]$. Thus there exists a relation

$$1 = a_0 + a_1 x + \cdots + a_n x^n$$

for some positive integer n and $a_i \in \mathfrak{p}R$ for $0 \leqslant i \leqslant n$. Note that $n \geqslant 1$ since $1 \notin \mathfrak{p}R$. Since $1 - a_0$ is a unit in R, we can multiply by its inverse to get a relation of the form

$$1 = b_1 x + \cdots + b_n x^n \quad \text{where } b_i \in \mathfrak{m} \text{ for } 1 \leqslant i \leqslant n. \tag{$\star$}$$

Choose such a relation that minimizes $n \geqslant 1$. Similarly, since $x^{-1} \notin R$, we can find another relation

$$1 = c_1 x^{-1} + \cdots + c_m x^{-m} \quad \text{where } c_i \in \mathfrak{m} \text{ for } 1 \leqslant i \leqslant m. \tag{$\star\star$}$$

Again, choose such a relation that minimizes $n \geqslant 1$. If $n \geqslant m$, multiply ($\star\star$) by $b_n x^n$ and substitute in ($\star$) to obtain a relation of smaller x-degree, a contradiction. On the other hand, if $n < m$, then we obtain a similar contradiction by interchanging the roles of x and x^{-1}. This completes the proof. ∎

THEOREM 1.11. Let K be a field, $A \subseteq K$ a subring, and B the integral closure of A in K. Then B is the intersection of all the valuation rings of K containing A.

Proof. Let B' denote the intersection of all valuation rings of K containing A. Due to Proposition 1.7, every such valuation ring contains B, that is, $B \subseteq B'$. Let $x \in K \setminus B$, that is, x is not integral over A. It suffices to find a valuation ring of K containing A but not x. Set $y = x^{-1}$, and consider the ideal $yA[y]$ of the ring $A[y]$. Note that this ideal is proper, else, there would exist a relation

$$1 = a_1 y + \cdots + a_n y^n$$

for some $a_1, \ldots, a_n \in A$; which upon multiplying by x^n forces x to be integral over A, a contradiction. Let $\mathfrak{p}$ be a maximal ideal of $A[y]$ containing $yA[y]$. In view of Theorem 1.10, there exists a valuation ring $(V, \mathfrak{m})$ of K containing $A[y]$ such that $\mathfrak{m} \cap A[y] = \mathfrak{p}$, in particular, $y \in \mathfrak{m}$, and hence $x = y^{-1} \notin V$, as desired. ∎

DEFINITION 1.12. An abelian group $(H,+)$ together with a total ordering $(H,\leqq)$ is said to be an *ordered group* if

$$\forall\ x,y,z,w \in H \quad x \geqq y \text{ and } z \geqq w \implies x+z \geqq y+w.$$

Note that if $x>0$ and $y \geqq 0$ in H, then

$$x+y \geqq x > 0,$$

and if $x \geqq y$ in H, then adding $-(x+y)$ to both sides of the inequality, we obtain: $-y \geqq -x$.

Given an ordered abliean group $(H,\leqq)$, we can extend the ordering to the set $H \cup \{\infty\}$ by setting $\infty \geqq x$ for all $x \in H$, $\infty + x = \infty$ for all $x \in H$, and $\infty+\infty=\infty$.

DEFINITION 1.13. A(n) (additive) *valuation* of a field K is a map $v: K \to H \cup \{\infty\}$ where $(H,\leqq)$ is an ordered abelian group such that for all $x,y \in K$,

(i) $v(xy) = v(x)+v(y)$,

(ii) $v(x+y) \geqq \min\{v(x),v(y)\}$, and

(iii) $v(x)=\infty$ if and only if $x=0$.

Clearly, the restriction $v: K^\times \to H$ defines a group homomorphism. Set

$$R_v = \{x \in K : v(x) \geqslant 0\} \quad \text{ and } \quad \mathfrak{m}_v = \{x \in K : v(x) > 0\}.$$

It is easy to see that $(R_v, \mathfrak{m}_v)$ is a valuation ring of the field K. The image of the group homomorphism $v: K^\times \to H$ is called the *value group* of the valuation v. Note that we may restrict the codomain of v to its value group without changing the valuation ring.

Now, let $(R,\mathfrak{m})$ be a valuation ring with fraction field K. Let G denote the set of non-zero principal R-submodules of K, that is,

$$G = \left\{xR : x \in K^\times\right\}.$$

Note that G is clearly an abelian group under the "multiplication" defined by

$$xR \cdot yR = xyR.$$

The identity element is R and the inverse of xR is given by $x^{-1}R$. The canonical map $v: K^\times \to G$ given by $x \mapsto xR$ is a surjective group homomorphism with kernel $R^\times$. Thus $G \cong K^\times/R^\times$ as abelian groups. Note that the submodules in G are totally ordered, indeed, if $x,y \in K^\times$, either $\frac{x}{y}$ or $\frac{y}{x} \in R$, and thus, one of xR and yR must be contained in the other. Define the relation

$$xR \leqq yR \iff xR \supseteq yR$$

on G. Clearly G forms an ordered abelian group under this relation. Extend the map $v: K^\times \to G \subseteq G \cup \{\infty\}$ by setting $v(0)=\infty$. We contend that v is a valuation. To this end, it suffices to verify axiom (ii); for this, we may assume $x,y \in K^\times$. Now,

$$v(x+y) = xR + yR = \min\{v(x),v(y)\},$$

since the submodules in G are totally ordered. Thus v is a valuation, and the corresponding valuation ring is $(R,\mathfrak{m})$ by construction. Call this the *canonical valuation* of the valuation ring $(R,\mathfrak{m})$. In essence, we have shown that there's no substantial difference between "abstract" valuation rings and those valuation rings that come from (additive) valuations of a field.

PROPOSITION 1.14. Let v and v' be two (additive) valuations of the field K with value groups H and H' respectively. If both v and v' give rise to the same valuation ring, then there is an order isomorphism $\varphi\colon H \to H'$ such that $v' = \varphi \circ v$.

Thus, in some sense, the value group of a valuation ring is determined up to order-isomorphism, and in particular, is isomorphic to $K^\times / R^\times$.

Proof. Note that it sufficess to assume v' is the canonical valuation with value group $G = \{xR : x \in K^\times\}$. Since $\ker v = R^\times$ and $\ker v' = R^\times$, there is an injective map $\varphi\colon G \to H$ such that $\varphi \circ v' = v$. Since v is surjective, so is φ. That is, v is an isomorphism. Finally, suppose $v'(x) \leqq v'(y)$, that is, $xR \supseteq yR$, equivalently, $\frac{y}{x} \in R$, whence $v\left(\frac{x}{y}\right) \geqq 0$, equivalently $v(x) \geqq v(y)$, as desired. ∎

§2 Discrete Valuation Rings and Dedekind Domains

DEFINITION 2.1. A valuation ring whose value group is isomorphic to $\mathbb{Z}$ is called a *discrete valuation ring (DVR)*.

THEOREM 2.2. Let R be a valuation ring. The following are equivalent:

(1) R is a DVR.

(2) R is a PID.

(3) R is Noetherian.

Proof. ∎

THEOREM 2.3. Let R be a ring. The following are equivalent:

(1) R is a DVR.

(2) R is a local PID which is not a field.

(3) R is a Noetherian local ring of positive Krull dimension with principal maximal ideal.

(4) R is a one-dimensional normal Noetherian local domain.

Proof. ∎

§§ Fractional Ideals and Dedekind Domains

DEFINITION 2.4. Let R be an integral domain with fraction field K. A *fractional ideal* of R is an R-submodule I of K such that there exists $0 \neq \alpha \in R$ such that $\alpha I \subseteq R$.

Just like ordinary ideals of R, we can take the sum and product of R-submodules of K:

$$I + J = \{x + y : x \in I,\ y \in J\} \quad \text{ and } I \cdot J = \{xy : x \in I,\ y \in J\}R.$$

Note that the sum and product of fractional ideals is again a fractional ideal. Indeed, suppose $\alpha, \beta \in R \setminus \{0\}$ such that $\alpha I, \beta J \subseteq R$. Then it is clear that $\alpha\beta(I + J) \subseteq R$ and $\alpha\beta(I \cdot J) \subseteq R$.

Next, we take a look at localization. Let $S \subseteq R$ be a multiplicative subset. Then

$$S^{-1}I = \left\{\frac{x}{s} : x \in I,\ s \in S\right\}$$

is an $S^{-1}R$ submodule of K such that $\alpha(S^{-1}I) \subseteq S^{-1}R$, so that $S^{-1}I$ is a fractional ideal of $S^{-1}R$. The usual properties of localization for ideals carries over to the case of fractional ideals. Indeed, if I and J are R-submodules of K, then:

(i) $S^{-1}I \cdot S^{-1}J = S^{-1}(I \cdot J)$

(ii) $S^{-1}I :_{S^{-1}R} S^{-1}J = S^{-1}(I :_R J)$.

The first one is clear. For the second one, the inclusion $S^{-1}(I :_R J) \subseteq S^{-1}I :_{S^{-1}R} S^{-1}J$ is also clear. Conversely, if $\frac{\alpha}{s} \in S^{-1}I :_{S^{-1}R} S^{-1}J$, then for any $\frac{y}{t} \in S^{-1}J$, we have $\frac{\alpha y}{st} \in S^{-1}I$, that is, $\alpha y \in I$, whence $\alpha \in I :_R J$. This establishes the equality.

DEFINITION 2.5. An R-submodule I of K is said to be *invertible* if there exists an R-submodule J of K such that $I \cdot J = R$.

Clearly, if I is an invertible R-submodule of K, then it must be a fractional ideal. Further, if $I \cdot J = R$, then

$$J = \{\alpha \in K : \alpha I \subseteq R\} = R :_R I.$$

Indeed, we have the inclusion $J \subseteq R :_R I = (R :_R I) \cdot I \cdot J \subseteq J$, and hence, equality holds everywhere.

THEOREM 2.6. Let R be an integral domain and I a fractional ideal of R. The following are equivalent:

(1) I is invertible.

(2) I is a projective R-module.

(3) I is finitely generated, and for every maximal ideal $\mathfrak{m}$ of R, the fractional ideal $I_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is principal.

Proof. ■