

# Buildings

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## §2.4 Buildings

Let  $(G, B, N, R)$  be a Tits system with  $H = B \cap N$ . Suppose there is a reduced and irreducible root system  $\Sigma_0$  on a Euclidean space  $A$ , a chamber  $C$  of the associated root system  $\Sigma$ , and a surjective homomorphism  $\nu : N \twoheadrightarrow W$  such that

- (i)  $\ker \nu = H$ , so that we may identify the Weyl group  $N/H$  of the Tits system with the affine Weyl group  $W$  of  $\Sigma$ . We shall implicitly make this identification henceforth.
- (ii) under this identification, the distinguished generators of  $N/H$  are the reflections in the walls of the chamber  $C$ , i.e.,

$$R = \{w_\alpha : \alpha \in \Pi\}.$$

Following the notation of [Mac71], the conjugates of  $B$  in  $G$  are called the *Iwahori subgroups* of  $G$  and a *parahoric* subgroup of  $G$  is a *proper* subgroup containing an Iwahori subgroup. We have seen last time that every Iwahori subgroup of  $G$  is conjugate to a unique  $P_S := BW_S W$ , where  $S \subseteq R$ . In particular, each parahoric subgroup of  $G$  uniquely determines a subset  $S$  of  $R$ .

This sets up a bijective correspondence  $S \longleftrightarrow F$  between the subsets  $S$  of  $R$  and the facets  $F$  of the chamber  $C$ : to a facet  $F$  corresponds the set of all  $w_\alpha \in R$  which fix  $F$ . Under this correspondence,  $\emptyset \longleftrightarrow C$ , and  $R \longleftrightarrow \emptyset$ . If  $S \longleftrightarrow F$ , then we write  $P_F$  for  $P_S$ . Clearly, each parahoric subgroup  $P$  uniquely determines a facet  $F(P)$  of  $C$ : namely  $F(P) = F$  if and only if  $P$  is conjugate to  $P_F$ .

The *building* associated with the Tits system structure on  $G$  is the set

$$\mathcal{F} = \{(P, x) : x \in F(P)\}.$$

With each parahoric subgroup  $P$  associate

$$\mathcal{F}(P) = \{(P, x) : x \in F(P)\} \subseteq \mathcal{F}.$$

The set  $\mathcal{F}(P)$  is called a *facet* of  $\mathcal{F}$  of *type*  $F(P)$ . In particular, if  $P$  is an Iwahori subgroup,  $\mathcal{F}(P)$  is called a *chamber* of  $\mathcal{F}$ . We define the *closure of a facet* as

$$\overline{\mathcal{F}(P)} = \bigcup_{\substack{Q \supseteq P \\ Q \leq G}} \mathcal{F}(Q).$$

The group  $G$  acts on  $\mathcal{F}$  as

$$g \cdot (P, x) = (gPg^{-1}, x).$$

### §§ Apartments

Set

$$\mathcal{A}_0 := \bigcup_{w \in W} \overline{\mathcal{F}(wBw^{-1})} \subseteq \mathcal{F}.$$

Since  $\overline{\mathcal{F}(wBw^{-1})}$  is the union of  $\mathcal{F}(P)$ 's for all parahorics containing  $wBw^{-1}$  and conjugation by  $w$  is the same as conjugation by some  $n \in N$  for which  $\nu(n) = w$ , it follows that

$$\mathcal{A}_0 = \bigcup_{n \in N} n\mathcal{F}(P).$$

**Proposition 2.4.1.** There exists a *unique* bijection  $j : A \rightarrow \mathcal{A}_0$  such that

(1) for each facet  $F$  of  $C$  and each  $x \in F$ ,

$$j(x) = (P_F, x),$$

(2)  $j \circ w = w \circ j$  for all  $w \in W$ .

*Proof.* Let  $y \in A$ . Then there is a unique  $x \in \bar{C}$  such that there exists a  $w \in W$  such that  $y = wx$ . Let  $F$  be the facet of  $C$  containing  $x$ . Define  $j(y) = (wP_F w^{-1}, x) \in \mathcal{A}_0$ . We must check that  $j$  is well-defined. Suppose  $w' \in W$  is such that  $y = w'x$ . Then  $w^{-1}w'$  fixes  $x$  and hence, belongs to the subgroup of  $W$  generated by  $\{w_\alpha : \alpha \in \Pi, w_\alpha \text{ fixes } x\}$  ([Mac71, last line on pg. 16]). That is,  $w^{-1}w' \in W_S$ , where  $S \longleftrightarrow F$ . In particular,  $w^{-1}w' \in P_F = P_S$ , therefore,  $wP_F w^{-1} = w'P_F w'^{-1}$ . Hence,  $j$  is well-defined and clearly satisfies (1). As for (2), let  $w'' \in W$  and  $y \in Y$  as before. Then  $w''y$  is conjugate to  $x$  under  $w''w$ , therefore,  $j(w''y) = (w''wP_F w^{-1}w''^{-1}, x) = w''(P_F, x) = w''j(y)$ . The uniqueness is clear since the conjugates of  $\bar{C}$  cover  $A$ . ■

**Lemma 2.4.2.** If  $g\mathcal{A}_0 = \mathcal{A}_0$ , then  $j^{-1} \circ (g|_{\mathcal{A}_0}) \circ j \in W$ .

*Proof.* Let  $\mathcal{C}_0 \subseteq \mathcal{A}_0$  denote the chamber  $\mathcal{F}(B) = j(C)$  of  $\mathcal{J}$ . Note that  $g\mathcal{C}_0$  is another chamber of  $\mathcal{J}$  and is contained in  $\mathcal{A}_0 = \bigcup_{n \in N} \bigcup_{P \supseteq B} n\mathcal{F}(P)$ , therefore there exists  $n_0 \in N$  such that  $g\mathcal{C}_0 = n_0\mathcal{C}_0$ . Hence,  $g_0 = n_0^{-1}g$  normalizes  $B$ , and hence, lies in  $B$  as we have seen last time. Notice that  $g_0\mathcal{A}_0 = n_0^{-1}g\mathcal{A}_0 = n_0^{-1}\mathcal{A}_0 = \mathcal{A}_0$ , since  $\nu(n_0) \in W$ . It is also clear that  $g_0$  fixes  $\mathcal{C}_0$  and each of its facets. It is clear that the map  $j^{-1} \circ (g|_{\mathcal{A}_0}) \circ j$  is a bijection from  $A$  to  $A$  which fixes the chamber  $C$  and each of its facets. Now, since  $w \in W$ , and  $j$  commutes with the action of the affine Weyl group on  $A$ , we have

$$(j^{-1} \circ (g_0|_{\mathcal{A}_0}) \circ j)(wx) = w(j^{-1} \circ (g_0|_{\mathcal{A}_0}) \circ j)(x) = wx.$$

In particular,  $j^{-1} \circ (g_0|_{\mathcal{A}_0}) \circ j$  is the identity map. Hence,

$$j^{-1} \circ (g|_{\mathcal{A}_0}) \circ j = j^{-1} \circ (n_0|_{\mathcal{A}_0}) \circ j = \nu(n_0) \in W,$$

as desired. ■

The subsets  $g\mathcal{A}_0$  of  $\mathcal{J}$  for  $g \in G$  are called the *apartments* of the building  $\mathcal{J}$ . If  $\mathcal{A} = g\mathcal{A}_0$  is an apartment, transport the Euclidean structure of  $A$  onto  $\mathcal{A}$  via the bijection  $(g|_{\mathcal{A}_0}) \circ j : A \rightarrow \mathcal{A}$ . We must check that this structure is well-defined. Indeed, if  $\mathcal{A} = g'\mathcal{A}_0$ , then

$$[(g'|_{\mathcal{A}_0}) \circ j]^{-1} \circ [(g|_{\mathcal{A}_0}) \circ j] = j^{-1} \circ (g'^{-1}g|_{\mathcal{A}_0}) \circ j$$

is an element of the affine Weyl group, in particular, it is an affine transformation that preserves lengths. Therefore, there is a well-defined Euclidean structure on  $\mathcal{A}$ .

**Lemma 2.4.3.** Any two facets of  $\mathcal{J}$  are contained in a single apartment.

*Proof.* Consider two facets  $\mathcal{F}(P_1)$  and  $\mathcal{F}(P_2)$  where  $P_1, P_2$  are parahoric subgroups of  $G$ , say  $P_i = g_i P_{F_i} g_i^{-1}$  for  $i \in \{1, 2\}$ , where  $F_1, F_2$  are facets of the chamber  $C$  in  $A$ . Since  $G = BWB$ , we can write  $g_1^{-1}g_2 = b_1 n b_2$  for some  $b_1, b_2 \in B$  and  $n \in N$ . Setting  $g = g_1 b_1$ , then

$$P_1 = g P_{F_1} g^{-1} \quad \text{and} \quad P_2 = g (n P_{F_2} n^{-1}) g^{-1},$$

whence  $\mathcal{F}(P_1)$  and  $\mathcal{F}(P_2)$  are both contained in  $g\mathcal{A}_0$ . ■

**Lemma 2.4.4.**  $G$  acts transitively on the set

$$\{(\mathcal{C}, \mathcal{A}) : \mathcal{C} \text{ is a chamber in } \mathcal{A}\}.$$

*Proof.* Since  $\mathcal{C} = g\mathcal{C}_0$  where  $\mathcal{C}_0 = \mathcal{F}(B)$  for some  $g \in G$ , we may suppose without loss of generality that  $\mathcal{C} = \mathcal{C}_0$ . If  $\mathcal{A} = g\mathcal{A}_0$  contains  $\mathcal{C}_0$ , then  $g^{-1}\mathcal{C}_0 = n\mathcal{C}_0$  for some  $n \in N$ . Setting  $g_1 = gn$ , we see that  $\mathcal{A} = g_1\mathcal{A}_0$  and  $\mathcal{C}_0 = g_1\mathcal{C}_0$ . ■

**Proposition 2.4.5.** Let  $\mathcal{A}, \mathcal{A}'$  be two apartments and let  $\mathcal{C}$  be a chamber contained in  $\mathcal{A} \cap \mathcal{A}'$ . Then there exists a unique bijection  $\rho : \mathcal{A}' \rightarrow \mathcal{A}$  such that

(1) There exists  $g \in G$  such that  $\rho x = gx$  for all  $x \in \mathcal{A}'$ , and

- (2)  $\rho x = x$  for all  $x \in \ell$ .

Moreover,  $\rho x = x$  for all  $x \in \mathcal{A} \cap \mathcal{A}'$ , and  $d_{\mathcal{A}'}(x, y) = d_{\mathcal{A}}(\rho x, \rho y)$  for all  $x, y \in \mathcal{A}'$ .

*Proof.* Due to Lemma 4.4, there exists  $g \in G$  which sends the pair  $(\ell, \mathcal{A}')$  to the pair  $(\ell, \mathcal{A})$ . Note that  $g\ell = \ell$  and  $\ell = \mathcal{F}(B')$  for some Iwahori subgroup  $B'$  of  $G$ . This means that  $g$  normalizes  $B'$ , and hence,  $g \in B'$ . Thus, this map fixes every element of  $\ell$ , and hence, satisfies the desired conditions.

Next, we argue uniqueness. If  $\rho_1, \rho_2 : \mathcal{A}' \rightarrow \mathcal{A}$  are two such maps, then  $\rho_1 \circ \rho_2^{-1}$  is a bijection from  $\mathcal{A}$  to  $\mathcal{A}$  which fixes  $\ell$ . There exists  $h \in G$  such that  $h$  maps  $(\ell_0, \mathcal{A}_0)$  to the pair  $(\ell, \mathcal{A})$ . Therefore,  $h^{-1}gh\mathcal{A}_0 = \mathcal{A}_0$  and fixes  $\ell_0$ . Due to Lemma 4.2, it follows that  $h^{-1}gh$  is the identity on  $\mathcal{A}_0$ , whence  $g$  is the identity on  $\mathcal{A}$ . The assertion  $d_{\mathcal{A}}(\rho x, \rho y) = d_{\mathcal{A}'}(x, y)$  is clear from the definition of the metric.

It remains to show that  $\rho x = x$  for all  $x \in \mathcal{A} \cap \mathcal{A}'$ . Due to Lemma 4.4, we may assume  $\mathcal{A}' = \mathcal{A}_0$ ,  $\mathcal{A} = g\mathcal{A}_0$  and  $\ell = \ell_0 = \mathcal{F}(B)$ . Since  $g\ell_0 = \ell_0$ , it follows that  $b \in B$  as before. Now let  $\mathcal{F} = \mathcal{F}(P)$  be a facet contained in  $\mathcal{A} \cap \mathcal{A}'$ .

Since  $\mathcal{F}(P) \subseteq \mathcal{A}_0 \cap g\mathcal{A}_0$ , we have

$$P = n_1 P n_1^{-1} = g(n_2 P_F n_2^{-1})g^{-1}$$

for some facet  $F$  of  $C$  and  $n_1, n_2 \in N$ . The above equality implies  $n_1^{-1}g n_2$  normalizes  $P_F$  and hence lies in  $P_F$ , therefore,  $B n_1 P_F = B n_2 P_F$ . But due to [Mac71, 2.3.5],

$$B n_1 P_F = B n_1 W_F B = B n_2 W_F B = B n_2 P_F,$$

where  $W_F$  is the subgroup of  $W$  fixing  $F$ . Recall again ([Mac71, 2.3.1]) that there is a bijection between  $N/H$  and  $B \backslash G/B$ . Hence  $n_1 W_F = n_2 W_F$ , in other words,  $n_1 P_F n_1^{-1} = n_2 P_F n_2^{-1}$ , consequently,  $\mathcal{F}(P) = g\mathcal{F}(P) = \rho\mathcal{F}$ , as desired. ■

## §§ Retraction of the building onto an apartment

**Theorem 2.4.6.** Let  $\mathcal{A}$  be an apartment and  $\ell$  a chamber in  $\mathcal{A}$ . Then there exists a unique mapping  $\rho : \mathcal{F} \rightarrow \mathcal{A}$  such that for all apartments  $\mathcal{A}'$  containing  $\ell$ ,  $\rho|_{\mathcal{A}'}$  is the bijection  $\mathcal{A}' \rightarrow \mathcal{A}$  of Proposition 4.5.

*Proof.* Let  $x \in \mathcal{F}$ . By Lemma 4.3, there exists an apartment  $\mathcal{A}_1$  containing  $x$  and  $\ell$ . Let  $\rho_1 : \mathcal{A}_1 \rightarrow \mathcal{A}$  be the bijection of Proposition 4.5 and define  $\rho(x) := \rho_1(x)$ . We must show that this map is well-defined first. Indeed, suppose  $\mathcal{A}_2$  is another apartment of  $\mathcal{F}$  containing  $x$  and  $\ell$  and  $\rho_2 : \mathcal{A}_2 \rightarrow \mathcal{A}$  be the bijection of Proposition 4.5, then  $\rho_1^{-1} \circ \rho_2 : \mathcal{A}_2 \rightarrow \mathcal{A}_1$  is again the bijection of Proposition 4.5 for the apartments  $\mathcal{A}_2$ ,  $\mathcal{A}_1$ , and the chamber  $\ell$ . Thus,  $\rho_1^{-1} \circ \rho_2$  fixes  $x \in \mathcal{A}_1 \cap \mathcal{A}_2$ , i.e.,  $\rho_1(x) = \rho_2(x)$ . This shows the existence of a desired retraction.

To see uniqueness, again use the fact that for any  $x \in \mathcal{F}$ , there exists an apartment containing  $x$  and  $\ell$ . This completes the proof. ■

The mapping  $\rho$  defined above is called the *retraction of  $\mathcal{F}$  onto  $\mathcal{A}$  with centre  $\ell$* .

**Proposition 2.4.7.** Let  $\rho$  be the retraction of Theorem 4.6. Then

- (1)  $\rho x = x$  for all  $x \in \mathcal{A}$ .
- (2) For each facet  $\mathcal{F}$  in  $\mathcal{F}$ ,  $\rho|_{\overline{\mathcal{F}}}$  is a surjective affine isometry of  $\overline{\mathcal{F}} \rightarrow \overline{\rho\mathcal{F}}$ .
- (3) If  $x \in \overline{\ell}$ , then  $\rho^{-1}(x) = \{x\}$ .

*Proof.* (1) According to Theorem 4.6,  $\rho|_{\mathcal{A}}$  is the unique bijection of Proposition 4.5, which is just the identity map, and hence  $\rho x = x$  for all  $x \in \mathcal{A}$ .

- (2) Let  $\mathcal{A}'$  be an apartment containing  $\mathcal{F}$  and  $\ell$ , which exists due to Lemma 4.3. Note that  $\overline{\mathcal{F}} \subseteq \mathcal{A}'$ . Since  $\rho : \mathcal{A}' \rightarrow \mathcal{A}$  is an isometry due to Proposition 4.5, the assertion follows.

- (3) Let  $\mathcal{F}'$  be a facet of  $\mathcal{F}$  mapping to  $\mathcal{F}$  under  $\rho$ . Note that  $\rho : \mathcal{A}' \rightarrow \mathcal{A}$  is multiplication by some  $g \in G$  which leaves  $\ell$  fixed, therefore, must leave all its facets fixed too, after all the facets are those corresponding to the parahorics containing the Iwahori corresponding to  $\ell$ . ■

**Proposition 2.4.8.** (1) There exists a unique function  $d : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_+$  such that  $d|_{\mathcal{A} \times \mathcal{A}}$  is the metric  $d_{\mathcal{A}}$  for each apartment  $\mathcal{A}$  of  $\mathcal{F}$ .

- (2) If  $\rho$  is a retraction of  $\mathcal{F}$  onto an apartment  $\mathcal{A}$  as in Theorem 4.6, then  $d(\rho(x), \rho(y)) \leq d(x, y)$  for all  $x, y \in \mathcal{F}$ .
- (3)  $d$  is a  $G$ -invariant metric on  $\mathcal{F}$ .

*Proof.* (1) Let  $x, y \in \mathcal{F}$ , then due to Lemma 4.3, there is an apartment  $\mathcal{A}$  containing  $x$  and  $y$ . We define  $d(x, y) := d_{\mathcal{A}}(x, y)$ . Suppose  $\mathcal{A}'$  is another apartment containing  $x$  and  $y$ . We must show that  $d_{\mathcal{A}}(x, y) = d_{\mathcal{A}'}(x, y)$ .

Let  $\bar{\epsilon}$  be a chamber in  $\mathcal{A}$  such that  $x \in \bar{\epsilon}$ , this can be done, since every facet corresponds to a parahoric, which contains an Iwahori. Similarly, let  $\bar{\epsilon}'$  be a chamber in  $\mathcal{A}'$  such that  $y \in \bar{\epsilon}'$ . Again by Lemma 4.3, there is an apartment  $\mathcal{A}''$  containing  $\bar{\epsilon}$  and  $\bar{\epsilon}'$ . From Proposition 4.5, we have that  $d_{\mathcal{A}}(x, y) = d_{\mathcal{A}''}(x, y)$  because  $\mathcal{A}$  and  $\mathcal{A}''$  share the chamber  $\bar{\epsilon}$ . Analogously,  $d_{\mathcal{A}'}(x, y) = d_{\mathcal{A}''}(x, y)$ . Thus, the distance  $d$  is well-defined. That it is  $G$ -invariant follows from the definition of  $d_{\mathcal{A}}$  as  $(g|_{\mathcal{A}_0}) \circ j : A \rightarrow \mathcal{A}$ .

(2) This is cumbersome to write out formally but here's the main idea: Choose an apartment  $\mathcal{A}'$  in  $\mathcal{F}$  containing  $x$  and  $y$ . This apartment is in bijection with  $A$ , through which its metric is defined. The affine line joining  $x$  to  $y$  in  $A$  will intersect finitely many facets in the tessellation of  $A$ . Thus, this line segment can be broken into a union of smaller closed line segments, each lying in the closure of a facet. Under  $\rho$ , the image of each such line segment is a line segment of the same length. In particular, the image of  $[xy]$  under  $\rho$  is a polygonal line, whose "total length" is  $d_{\mathcal{A}'}(x, y)$ . The triangle inequality implies the desired conclusion.

(3) Let  $x, y, z \in \mathcal{F}$  and let  $\mathcal{A}$  be an apartment containing  $x$  and  $y$ . Let  $\rho$  be a retraction of  $\mathcal{F}$  onto  $\mathcal{A}$  as in Theorem 4.6. Then keeping in mind that  $\rho(x) = x$  and  $\rho(y) = y$ , we have

$$d(x, y) = d_{\mathcal{A}}(\rho(x), \rho(y)) \leq d(\rho(x), \rho(z)) + d(\rho(z), \rho(y)) \leq d(x, z) + d(z, y),$$

where the last equality follows from (2). ■

**Proposition 2.4.9.** Let  $x, y \in \mathcal{F}$ . Then there is a unique geodesic joining  $x$  to  $y$ .

*Proof.* ■

**Proposition 2.4.10.**  $\mathcal{F}$  is complete with respect to the metric  $d$ .

*Proof.* Let  $(x_n)_{n \geq 1}$  be a Cauchy sequence in  $\mathcal{F}$  with respect to the metric  $d$ . Let  $\rho$  be a retraction of  $\mathcal{F}$  onto an apartment  $\mathcal{A}_0$  as in Theorem 4.6. Then  $(\rho x_n)_{n \geq 1}$  is a Cauchy sequence in  $\mathcal{A}_0$ , and as such, converges to some  $x \in \mathcal{A}_0$ . Let  $x = (P, a) \in \mathcal{A}_0$  where  $a \in A$ . Then there is a  $\mu > 0$  such that  $d(x, wx) \geq \mu$  for all  $w \in W$ , the affine Weyl group. Let  $g \in G$  be such that  $x \neq gx$ . We claim that  $d(x, gx) \geq \mu$ . Indeed, there is an apartment  $\mathcal{A}' = h\mathcal{A}_0$  containing both  $x$  and  $gx$  for some  $h \in G$ . Then, from the  $G$ -invariance of  $d$ ,

$$d(x, gx) = d(h^{-1}x, h^{-1}gx) \geq \mu,$$

which is clear from the bijection  $A \leftrightarrow \mathcal{A}_0$ . Again, since  $d$  is  $G$ -invariant, it follows that  $d(gx, g'x) \geq \mu$  for all  $g, g' \in G$  such that  $gx \neq g'x$ .

Now, let  $N > 0$  be a positive integer such that for all  $m, n \geq N$ ,

$$d(\rho x_n, x) < \frac{1}{3}\mu \quad \text{and} \quad d(x_m, x_n) < \frac{1}{3}\mu.$$

By definition, each  $\rho x_n$  is of the form  $g_n x_n$  for some  $g_n \in G$ . Set  $y_n = g_n^{-1}x$ . Then for  $n \geq N$ , using the  $G$ -invariance of  $d$ , we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \\ &= d(x, \rho x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \\ &< \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{3}\mu < \mu. \end{aligned}$$

Hence,  $y_N = y_{N+1} = \dots =: y$ . Finally, for  $n \geq N$ , we have

$$d(x_n, y) = d(x_n, y_n) = d(g_n x_n, g_n y_n) = d(\rho x_n, x) \rightarrow 0,$$

as  $n \rightarrow \infty$ . ■

### Fixed point theorem

A subset  $X \subseteq \mathcal{F}$  is said to be *convex* if whenever  $x, y \in X$ ,  $[xy] \subseteq X$ .

**Lemma 2.4.11.** Let  $x, y, z \in \mathcal{F}$  and let  $m$  be the midpoint of  $[xy]$  Then

$$d(z, x)^2 + d(z, y)^2 \geq 2d(z, m)^2 + \frac{1}{2}d(x, y)^2.$$

**Theorem 2.4.12.** Let  $X$  be a bounded non-empty subset of  $\mathcal{F}$ . Then the group of isometries  $\gamma$  of  $\mathcal{F}$  such that  $\gamma(X) \subseteq X$  has a fixed point in the convex hull of  $X$ .

*Proof.* ■

A subset  $M \subseteq G$  is said to be *bounded* if  $MX$  is bounded for all bounded subsets  $X \subseteq \mathcal{F}$ .

### References

- [Mac71] I.G. Macdonald. *Spherical Functions on a Group of P-adic Type*. Publications of the Ramanujan Institute. Ramanujan Institute for Advanced Study in Mathematics, University of Madras, 1971.