Galois Categories and the Étale Fundamental Group

or, what should be taught in MA 811

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April 18, 2025

Galois Theory à la Grothendieck

Let k be a field and fix its separable and algebraic closures $k_s \subseteq \overline{k}$. Let $G_k := \mathrm{Gal}(k_s \mid k)$, which is a profinite group through the isomorphism:

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If L is a finite separable extension of k, then there is a natural action of G_k on $\mathrm{Hom}_k(L, k_s)$ given by

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The stabilizer of $\varphi \in \mathrm{Hom}_k(L, k_s)$ is $\mathrm{Gal}(k_s \mid \varphi(L))$, i.e., an open subgroup of G_k, whence $\mathrm{Hom}_k(L, k_s)$ is a continuous G_k-set.

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Étale Algebras and the Fundamental Theorem

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Theorem (Fundamental Theorem of Galois Theory)

The functor mapping a finite étale k-algebra A to the finite G_k-set $\mathrm{Hom}_k(A, k_s)$ gives an anti-equivalence between the category of finite étale k-algebras and the category of finte sets with a continuous G_k-action. Here separable extensions correspond to transitive G_k-sets.

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Galois Theory of Covering Spaces

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Since this action factors through a finite quotient of $\pi_1(X, x_0)$, it induces a natural continuous action of the profinite completion $\widehat{\pi_1(X, x_0)}$ on $p^{-1}(x_0)$.

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Theorem (Classification of Covering Spaces)

The aforementioned fibre functor induces an equivalence between the category of finite-sheeted covers of X and the category of continuous finite sets with a continuous $\widehat{\pi_1(X, x_0)}$-action. Here connected covers correspond to transitive $\widehat{\pi_1(X, x_0)}$-sets.

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All that remains is to establish an equivalence of some suitable categories.

Galois Categories

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- G3 Every morphism u in $\mathscr{C}$ factors as $u = u'u''$ where u' is a monomorphism and u'' is an epimorphism. Every monomorphism $f : X \to Y$ in $\mathscr{C}$ is an isomorphism of X with a direct summand of Y.

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In this case, the functor F is called a *fundamental functor*.

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- Let X be a connected scheme. Let $\textbf{FEt}_X$ denote the category of finite étale maps $Y \to X$ with morphisms $f : Y \to Z$ making

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

commute. Fix a geometric point $x_0 : \operatorname{Spec} \Omega \to X$. This defines a fibre functor $\mathrm{Fib}_{x_0} : \textbf{FEt}_X \to \textbf{Sets}$. The pair $(\textbf{FEt}_X, \mathrm{Fib}_{x_0})$ forms a Galois category.

The automorphism group of F

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Further the tuple must be such that for each morphism $f : X \to Y$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

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$$\mathrm{Aut}_{\mathscr{C}}(F) = \bigcap_{\substack{Y \xrightarrow{f} Z \\ \text{in } \mathscr{C}}} C_f,$$

the conclusion follows. In particular, $\mathrm{Aut}_{\mathscr{C}}(F)$ is a profinite group.

Next, let $Y \in \mathscr{C}$. There is a natural action of $\mathrm{Aut}_{\mathscr{C}}(F)$ on $F(Y)$ given by

$$(\eta_X)_{X \in \mathscr{C}} \cdot a = \eta_Y(a) \qquad \forall\ a \in F(Y).$$

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which is an open subgroup of $\mathrm{Aut}_{\mathscr{C}}(F)$. Thus, each $F(Y)$ is a finite $\mathrm{Aut}_{\mathscr{C}}(F)$-set with a continuous action.
This gives a natural functor $H : \mathscr{C} \to \mathrm{Aut}_{\mathscr{C}}(F)$-**sets**.

The Main Theorem

Theorem (Fundamental Theorem of Galois Categories)

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Definition

Let X be a connected scheme and $x_0 : \operatorname{Spec}\Omega \to X$ be a geometric point. The *étale fundamental group* $\pi_1^{\acute{e}t}(X, x_0)$ to be $\operatorname{Aut}(\operatorname{Fib}_{x_0})$.

Recall that there is an anti-equivalence between the category of commutative rings and the category of affine schemes.
Therefore, there is an anti-equivalence between the category $\mathbf{FEt}_{\operatorname{Spec} k}$ and the category of étale k-algebras.
Choosing a geometric point in $\operatorname{Spec} k$ is tantamount to fixing a separable closure k_s of k.
As we have seen earlier, $\mathbf{FEt}_{\operatorname{Spec} k}$ is equivalent to the category G_k-**sets**. In particular, $\pi_1^{\acute{e}t}(\operatorname{Spec} k, x_0) \cong \operatorname{Gal}(k_s \mid k)$.