Rings of Continuous Functions

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December 15, 2024

§1 $C(X)$

§§ Maximal Ideals

Theorem 1.1. Let X be a compact Hausdorff space. Every maximal ideal in $C(X)$ is of the form

$$\mathfrak{m}_x = \{f \in C(X) \colon f(x) = 0\}.$$

Proposition 1.2. Let X be a compact Hausdorff space. Every prime ideal in $C(X)$ is contained in a unique maximal idea.

Theorem 1.3 (Sury, ??). Every maximal ideal in $C[0,1]$ is uncountably generated.

§§ Krull Dimension

Throughout this (sub)section, X denotes a compact Hausdorff space. For every $x \in X$, there is a maximal ideal

$$\mathfrak{m}_x = \{f \in C(X) \colon f(x) = 0\}.$$

These are the only maximal ideals in $C(X)$. The goal of this (sub)section is to prove the following

Theorem 1.4. If there is a point $p \in X$ and an $f \in C(X)$ such that $f(p) = 0$ and there is no neighborhood of p on which f vanishes, then $C(X)$ has infinite Krull dimension.

Definition 1.5. A *partially ordered ring* is a pair $(A, \leqq)$ where $\leqq$ is a partial order on A such that

- $x \leqq y$ implies $x + z \leqq y + z$, and
- $0 \leqq x$ and $0 \leqq y$ implies $0 \leqq xy$,

for all $x, y, z \in A$. A *totally ordered ring* is a partially ordered ring $(A, \leqq)$ such that $\leqq$ is a total order.

The ring $C(X)$ has a canonical partial order, given by $f \leqq g$ if and only if $f(x) \leqslant g(x)$ for all $x \in X$.

Definition 1.6. An ideal $\mathfrak{a}$ of a partially ordered ring A is said to be *convex* if whenever $a, b \in A$ such that $0 \leqq a \leqq b$ and $b \in \mathfrak{a}$, then $a \in \mathfrak{a}$.

Proposition 1.7. If $(A, \leqq)$ is a partially ordered ring, and $\mathfrak{a} \trianglelefteq A$ is a convex ideal, then $A/\mathfrak{a}$ has a natural partial order given by:

$$(a+\mathfrak{a}) \leqq (b+\mathfrak{a}) \quad \text{if} \quad a \leqq b.$$

Proof. Standard. ∎

Proposition 1.8. Let $\mathfrak{P}$ be a prime ideal in $C(X)$. Then, $\mathfrak{P}$ is convex.

Proof. Suppsoe $0 \leqq f \leqq g$ in $C(X)$ and $g \in \mathfrak{P}$. The function $h : X \to \mathbb{R}$ given by

$$h(x) = \begin{cases} \frac{f(x)^2}{g(x)} & g(x) \neq 0 \\ 0 & g(x) = 0 \end{cases}$$

is a continuous function such that $f^2 = gh \in \mathfrak{P}$, whence $f \in \mathfrak{P}$. ∎

Proposition 1.9. The ring $A = C(X)/\mathfrak{P}$ is a totally ordered local domain. Further, the primes of A are totally ordered by inclusion.

Proof. That it is a local domain follows from the fact that there is a unique maximal ideal containing $\mathfrak{P}$. For any $f \in C(X)$, $f^2 \equiv |f|^2 \pmod{\mathfrak{P}}$, and hence, $f \equiv |f| \pmod{\mathfrak{P}}$ or $f \equiv -|f| \pmod{\mathfrak{P}}$. Consequently, $f + \mathfrak{P}$ is comparable with 0 in A, whence A is totally ordered.

Recall that the primes in A are of the form $\mathfrak{p} = \mathfrak{Q}/\mathfrak{P}$ for some prime $\mathfrak{Q} \supseteq \mathfrak{P}$. Since $\mathfrak{Q}$ is convex, so is $\mathfrak{p}$.

Finally, let $\mathfrak{p}$ and $\mathfrak{q}$ be two primes in A and suppose $a \in \mathfrak{q} \setminus \mathfrak{p}$. Then, for every $b \in \mathfrak{q}$, $b < a$, else $a \in \mathfrak{p}$. Hence, $b \in \mathfrak{p}$. This shows that $\mathfrak{p} \subseteq \mathfrak{q}$, whence the primes are totally ordered by inclusion. ∎

Let $p \in X$ be a point such that there is an $f \in C(X)$ such that $f(p) = 0$ but there is no neighborhood of p on which f is identically 0. Upon multiplying by a suitable real scalar, we may suppose that $0 \leqslant f(x) < e^{-2} < 1$ on X.

Proposition 1.10. The maximal ideal $\mathfrak{m}_p$ properly contains a prime ideal, say $\mathfrak{P}$.

Proof. Consider the local ring $C(X)_{\mathfrak{m}_p}$. If $\mathfrak{m}_p$ does not properly contain a prime ideal, then $C(X)_{\mathfrak{m}_p}$ is a local ring of dimension 0, whence the maximal ideal is the nilradical. But this ring is isomorphic to the ring of germs at p and the germ of f at p is not nilpotent since it does not vanish on any neighborhood of p. ∎

Let

$$I_p = \{g \in C(X) \colon g \text{ vanishes on a neighborhood of } p\} \subseteq \mathfrak{m}_p.$$

Proposition 1.11. $I_p \subseteq \mathfrak{P}$.

Proof. The localization map $C(X) \to C(X)_{\mathfrak{m}_p}$ is a surjective ring homomorphism whose kernel is I_p. Note that $\mathfrak{P}^{ec} = \mathfrak{P}$, since $\mathfrak{P}$ is prime. But upon contracting, we see that $\mathfrak{P}$ must contain the kernel. ■

Proof of Theorem 1.4. Let $A = C(X)/\mathfrak{P}$. We shall show that A has infinite Krull dimension. To this end, it suffices to show that we can find a prime ideal $\mathfrak{Q}$ such that $\mathfrak{P} \subsetneq \mathfrak{Q} \subsetneq \mathfrak{m}_p$, since this process can then be repeated ad infinitum.

Define the function $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{1}{|\log f(x)|} & f(x) \neq 0 \\ 0 & f(x) = 0. \end{cases}$$

This is a continuous function on X. Further, from basic calculus, it is evident that for every positive integer k, there is a neighborhood of 0 in $[0, \infty)$ on which $t|\log t|^k < 1$. Hence, for every positive integer k, there is a neighborhood U of p on which $g(x)^k \geqslant f(x)$.

Since $C(X)/I_p$ is ordered, we see that $g^k + I_p \geqq f + I_p$ for all positive integers k in $C(X)/I_p$. Since A is a quotient of $C(X)/I_p$, we have that $g^k + \mathfrak{P} \geqq f + \mathfrak{P}$ for all positive integers k in A.

Let $a, b \in A$ denote the images of f and g respectively. Then $a \leqslant b^k$ for all positive integers k. Note that by construction, $0 \leqslant g(x) < \frac{1}{2}$ on all of X. Suppose A has no prime ideals other than the maximal ideal and (0), then the radical of $(a) \trianglelefteq A$, which is the intersection of all primes containing (a) must be equal to the maximal ideal.

In particiular, there is a positive integer n such that $b^n \in (a)$, whence there is some $c \in A$ such that $b^n = ac$. Since $0 < a, b$, we have that $0 < c$. Therefore, we can find some $0 \leqq h \in C(X)$ such that c is the image of h in A. Since the supremum of g on X is smaller than $\frac{1}{2}$, and h is bounded on X (since X is compact), we have that for sufficiently large positive integers k, $0 < g^k h < 1$. That is, for sufficiently large k, $0 \leqq b^k c \leqq 1$.

Hence, for all sufficiently large k, we have

$$a \leqq b^{n+k} = a(b^k c) \leqq a \implies b^{n+k} = a.$$

Consequently, $b^N = b^{N+1}$ for sufficiently large N. Since A is a domain, this is possible if and only if $b \in \{0, 1\}$, neither of which is the case. This completes the proof. ■

PROPOSITION 1.12. Let X be a compact Hausdorff space such that $C(X)$ consists of only the locally constant functions on X. Then X is a finite set.

Proof. We first show that every G_δ-set in X is open. To this end, suppose $U_1, U_2, \ldots$ is a collection of open subsets of X containing a point $p \in X$. Urysohn's lemma furnishes continuous functions $f_n : X \to [0, 1]$ such that $f_n(p) = 1$ and f_n vanishes on $X \setminus U_n$. Define $f : X \to [0, 1]$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x) \qquad x \in X.$$

This series converges uniformly due to the Weierstrass M-test, whence f is continuous, i.e., locally constant. Thus, there is a neighborhood V of p in X on which f is identically 1.

Note that if $f(q) = 1$ then $q \in U_n$ for all $n \geqslant 1$. Thus,

$$V \subseteq \bigcap_{n=1}^{\infty} U_n \implies \bigcap_{n=1}^{\infty} U_n \text{ is open.}$$

Next, given disjoint points $a, b \in X$, there is a continuous function $f : X \to [0,1]$ such that $f(a) = 0$ and $f(1) = b$. Note that the zero set of f is open because f is locally constant. Whence, we have a disjoint union of clopen sets $U_{a,b} \sqcup U_{b,a}$ such that $a \in U_{a,b}$ and $b \in U_{b,a}$.

Suppose now that X is not finite and choose a countably infinite set $A \subseteq X$. Consider the collection $\mathscr{S}$ of sets S such that

- S is a collection of pairs (a,b) with $a, b \in A$ and $a \neq b$.
- For all $a \neq b$ in A, exactly one of (a,b) and (b,a) is in S.

Note that every $S \in \mathscr{S}$ is a countable set. Next, define

$$U_S = \bigcap_{(a,b) \in S} U_{a,b}.$$

Since every G_δ in X is open, every U_S is clopen. Further,

$$X = \bigsqcup_{S \in \mathscr{S}} U_S.$$

Finally, note that the elements of a lie in disjoint U_S's by construction. As a result, at least countably many of the U_S's are non-empty. Hence, we have expressed X as a disjoint union of at least countably many disjoint open sets, a contradiction to the compactness of X. This completes the proof. ■

To summarize, we have:

THEOREM 1.13. Let X be a compact Hausdorff space. Then,

- $\dim C(X) = 0$ if X is a finite set.
- $\dim C(X) = \infty$ in all other cases.

Proof. If X is finite, then $C(X) = \mathbb{R}^n$ as a ring, whence $\dim C(X) = 0$. The other cases have been handled above. ■