Buildings

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§2.4 Buildings

Let (G, B, N, R) be a Tits system with $H = B \cap N$. Suppose there is a reduced and irreducible root system Σ_0 on a Euclidean space A, a chamber C of the associated root system Σ, and a surjective homomorphism $\nu : N \twoheadrightarrow W$ such that

(i) $\ker \nu = H$, so that we may identify the Weyl group N/H of the Tits system with the affine Weyl group W of Σ. We shall implicitly make this identification henceforth.

(ii) under this identification, the distinguished generators of N/H are the reflections in the walls of the cmaber C, i.e.,

$$R = \{w_\alpha \colon \alpha \in \Pi\}.$$

Following the notation of [Mac71], the conjugats of B in G are called the *Iwahori subgroups* of G and a *parahoric* subgroup of G is a *proper* subgroup containing an Iwahori subgroup. We have seen last time that that every Iwahori subgroup of G is conjugate to a unique $P_S := BW_S W$, where $S \subseteq R$. In particular, each parahoric subgroup of G uniquely determines a subset S of R.

This sets up a bijective correspondence $S \longleftrightarrow F$ between the subsets S of R and the facets F of the chamber C: to a facet F corresponds the set of all $w_\alpha \in R$ which fix F. Under this correspondence, $\emptyset \longleftrightarrow C$, and $R \longleftrightarrow \emptyset$. If $S \longleftrightarrow F$, then we write P_F for P_S. Clearly, each parahoric subgroup P uniquely determines a facet $F(P)$ of C: namely $F(P) = F$ if and only if P is conjugate to P_F.

The *building* associated with the Tits system structure on G is the set

$$\mathscr{I} = \{(P, x) \colon x \in F(P)\}.$$

With each parahoric subgroup P associate

$$\mathscr{F}(P) = \{(P, x) \colon x \in F(P)\} \subseteq \mathscr{I}.$$

The set $\mathscr{F}(P)$ is called a *facet* of $\mathscr{I}$ of *type* $F(P)$. In particular, if P is an Iwahori subgroup, $\mathscr{F}(P)$ is called a *chamber* of $\mathscr{I}$. We define the *closure of a facet* as

$$\overline{\mathscr{F}(P)} = \bigcup_{\substack{Q \supseteq P \\ Q \lneq G}} \mathscr{F}(Q).$$

The group G acts on $\mathscr{I}$ as

$$g \cdot (P, x) = \left(gPg^{-1}, x\right).$$

§§ Apartments

Set

$$\mathscr{A}_0 := \bigcup_{w \in W} \overline{\mathscr{F}(wBw^{-1})} \subseteq \mathscr{I}.$$

Since $\overline{\mathscr{F}(wBw^{-1})}$ is the union of $\mathscr{F}(P)$'s for all parahorics containing wBw^{-1} and conjugation by w is the same as conjugation by some $n \in N$ for which $\nu(n) = w$, it follows that

$$\mathscr{A}_0 = \bigcup_{n \in N} n\mathscr{F}(P).$$

Proposition 2.4.1. There exists a *unique* bijection $j : A \to \mathscr{A}_0$ such that

(1) for each facet F of C and each $x \in F$,
$$j(x) = (P_F, x),$$

(2) $j \circ w = w \circ j$ for all $w \in W$.

Proof. Let $y \in A$. Then there is a unique $x \in \overline{C}$ such that there exists a $w \in W$ such that $y = wx$. Let F be the facet of C containing x. Define $j(y) = (wP_F w^{-1}, x) \in \mathscr{A}_0$. We must check that j is well-defined. Suppose $w' \in W$ is such that $y = w'x$. Then $w^{-1}w'$ fixes x and hence, belongs to the subgroup of W generated by $\{w_\alpha : \alpha \in \Pi,\ w_\alpha \text{ fixes } x\}$ ([Mac71, last line on pg. 16]). That is, $w^{-1}w' \in W_S$, where $S \longleftrightarrow F$. In particular, $w^{-1}w' \in P_F = P_S$, therefore, $wP_F w^{-1} = w' P_F w'^{-1}$. Hence, j is well-defined and clearly satisfies (1). As for (2), let $w'' \in W$ and $y \in Y$ as before. Then $w''y$ is conjugate to x under $w''w$, therefore, $j(w''y) = \left(w''wP_F w^{-1}w''^{-1}, x\right) = w''\left(P_F, x\right) = w''j(y)$. The uniqueness is clear since the conjugates of $\overline{C}$ cover A. ∎

Lemma 2.4.2. If $g\mathscr{A}_0 = \mathscr{A}_0$, then $j^{-1} \circ (g|_{\mathscr{A}_0}) \circ j \in W$.

Proof. Let $\mathscr{C}_0 \subseteq \mathscr{A}_0$ denote the chamber $\mathscr{F}(B) = j(C)$ of $\mathscr{I}$. Note that $g\mathscr{C}_0$ is another chamber of $\mathscr{I}$ and is contained in $\mathscr{A}_0 = \bigcup_{n \in N} \bigcup_{P \supseteq B} n\mathscr{F}(P)$, therefore there exists $n_0 \in N$ such that $g\mathscr{C}_0 = n_0\mathscr{C}_0$. Hence, $g_0 = n_0^{-1}g$ normalizes B, and hence, lies in B as we have seen last time. Notice that $g_0\mathscr{A}_0 = n_0^{-1}g\mathscr{A}_0 = n_0^{-1}\mathscr{A}_0 = \mathscr{A}_0$, since $\nu(n_0) \in W$. It is also clear that g_0 fixes $\mathscr{C}_0$ and each of its facets. It is clear that the map $j^{-1} \circ (g|_{\mathscr{A}_0}) \circ j$ is a bijection from A to A which fixes the chamber C and each of its facets. Now, since $w \in W$, and j commutes with the action fo the affine Weyl group on A, we have
$$\left(j^{-1} \circ (g_0|_{\mathscr{A}_0}) \circ j\right)(wx) = w\left(j^{-1} \circ (g_0|_{\mathscr{A}_0}) \circ j\right)(x) = wx.$$
In particular, $j^{-1} \circ (g_0|_{\mathscr{A}_0}) \circ j$ is the identity map. Hence, idk why
$$j^{-1} \circ (g|_{\mathscr{A}_0}) \circ j = j^{-1} \circ (n_0|_{\mathscr{A}_0}) \circ j = \nu(n_0) \in W,$$
as desired. ∎

The subsets $g\mathscr{A}_0$ of $\mathscr{I}$ for $g \in G$ are called the *apartments* of the building $\mathscr{I}$. If $\mathscr{A} = g\mathscr{A}_0$ is an apartment, transport the Euclidean structure of A onto $\mathscr{A}$ via the bijection $(g|_{\mathscr{A}_0}) \circ j : A \to \mathscr{A}$. We must check that this structure is well-defined. Indeed, if $\mathscr{A} = g'\mathscr{A}_0$, then
$$\left[(g'|_{\mathscr{A}_0}) \circ j\right]^{-1} \circ \left[(g|_{\mathscr{A}_0} \circ j)\right] = j^{-1} \circ \left(g'^{-1}g|_{\mathscr{A}_0}\right) \circ j$$
is an element of the affine Weyl group, in particular, it is an affine transformation that preserves lengths. Therefore, there is a well-defined Euclidean structure on $\mathscr{A}$.

Lemma 2.4.3. Any two facets of $\mathscr{I}$ are contained in a single apartment.

Proof. Consider two facets $\mathscr{F}(P_1)$ and $\mathscr{F}(P_2)$ where P_1, P_2 are parahoric subgroups of G, say $P_i - g_i P_{F_i} g_i^{-1}$ for $i \in \{1, 2\}$, where F_1, F_2 are facets of the chamber C in A. Since $G = BWB$, we can write $g_1^{-1}g_2 = b_1 n b_2$ for some $b_1, b_2 \in B$ and $n \in N$. Setting $g = g_1 b_1$, then
$$P_1 = gP_{F_1}g^{-1} \quad \text{and} \quad P_2 = g\left(nP_{F_2}n^{-1}\right)g^{-1},$$
whence $\mathscr{F}(P_1)$ and $\mathscr{F}(P_2)$ are both contained in $g\mathscr{A}_0$. ∎

Lemma 2.4.4. G acts transitively on the set
$$\{(\mathscr{C}, \mathscr{A}) : \mathscr{C} \text{ is a chamber in } \mathscr{A}\}.$$

Proof. Since $\mathscr{C} = g\mathscr{C}_0$ where $\mathscr{C}_0 = \mathscr{F}(B)$ for some $g \in G$, we may suppose without loss of generality that $\mathscr{C} = \mathscr{C}_0$. If $\mathscr{A} = g\mathscr{A}_0$ contains $\mathscr{C}_0$, then $g^{-1}\mathscr{C}_0 = n\mathscr{C}_0$ for some $n \in N$. Setting $g_1 = gn$, we see that $A = g_1\mathscr{A}_0$ and $\mathscr{C}_0 = g_1\mathscr{C}_0$. ∎

Proposition 2.4.5. Let $\mathscr{A}$, $\mathscr{A}'$ be two apartments and let $\mathscr{C}$ be a chamber contained in $\mathscr{A} \cap \mathscr{A}'$. Then there exists a unique bijection $\rho : \mathscr{A}' \to \mathscr{A}$ such that

(1) There exists $g \in G$ such that $\rho x = gx$ for all $x \in \mathscr{A}'$, and

(2) $\rho x = x$ for all $x \in \mathscr{C}$.

Moreover, $\rho x = x$ for all $x \in \mathscr{A} \cap \mathscr{A}'$, and $d_{\mathscr{A}'}(x, y) = d_{\mathscr{A}}(\rho x, \rho y)$ for all $x, y \in \mathscr{A}'$.

Proof. Due to Lemma 4.4, there exists $g \in G$ which sends the pair $(\mathscr{C}, \mathscr{A}')$ to the pair $(\mathscr{C}, \mathscr{A})$. Note that $g\mathscr{C} = \mathscr{C}$ and $\mathscr{C} = \mathscr{F}(B')$ for some Iwahori subgroup B' of G. This means that g normalizes B', and hence, $g \in B'$. Thus, this map fixes every element of $\mathscr{C}$, and hence, satisfies the desired conditions.

Next, we argue uniqueness. If $\rho_1, \rho_2 : \mathscr{A}' \to \mathscr{A}$ are two such maps, then $\rho_1 \circ \rho_2^{-1}$ is a bijection from $\mathscr{A}$ to $\mathscr{A}$ which fixes $\mathscr{C}$. There exists $h \in G$ such that h maps $(\mathscr{C}_0, \mathscr{A}_0)$ to the pair $(\mathscr{C}, \mathscr{A})$. Therefore, $h^{-1}gh\mathscr{A}_0 = \mathscr{A}_0$ and fixes $\mathscr{C}_0$. Due to Lemma 4.2, it follows that $h^{-1}gh$ is the identity on $\mathscr{A}_0$, whence g is the identity on $\mathscr{A}$. The assertion $d_{\mathscr{A}}(\rho x, \rho y) = d_{\mathscr{A}'}(x, y)$ is clear from the definition of the metric.

It remains to show that $\rho x = x$ for all $x \in \mathscr{A} \cap \mathscr{A}'$. Due to Lemma 4.4, we may assume $\mathscr{A}' = \mathscr{A}_0$, $\mathscr{A} = g\mathscr{A}_0$ and $\mathscr{C} = \mathscr{C}_0 = \mathscr{F}(B)$. Since $g\mathscr{C}_0 = \mathscr{C}_0$, it follows that $b \in B$ as before. Now let $\mathscr{F} = \mathscr{F}(P)$ be a facet contained in $\mathscr{A} \cap \mathscr{A}'$.

Since $\mathscr{F}(P) \subseteq \mathscr{A}_0 \cap g\mathscr{A}_0$, we have

$$P = n_1 P n_1^{-1} = g\left(n_2 P_F n_2^{-1}\right) g^{-1}$$

for some facet F of C and $n_1, n_2 \in N$. The above equality implies $n_1^{-1} g n_2$ normalizes P_F and hence lies in P_F, therefore, $Bn_1 P_F = Bn_2 P_F$. But due to [Mac71, 2.3.5],

$$Bn_1 P_F = Bn_1 W_F B = Bn_2 W_F B = Bn_2 P_F,$$

where W_F is the subgroup of W fixing F. Recall again ([Mac71, 2.3.1]) that there is a bijection between N/H and $B\backslash G/B$. Hence $n_1 W_F = n_2 W_F$, in other words, $n_1 P_F n_1^{-1} = n_2 P_F n_2^{-1}$, consequently, $\mathscr{F}(P) = g\mathscr{F}(P) = \rho\mathscr{F}$, as desired. ■

§§ Retraction of the building onto an apartment

Theorem 2.4.6. Let $\mathscr{A}$ be an apartment and $\mathscr{C}$ a chamber in $\mathscr{A}$. Then there exists a unique mapping $\rho : \mathscr{I} \to \mathscr{A}$ such that for all apartments $\mathscr{A}'$ containing $\mathscr{C}$, $\rho|_{\mathscr{A}'}$ is the bijection $\mathscr{A}' \to \mathscr{A}$ of Proposition 4.5.

Proof. Let $x \in \mathscr{I}$. By Lemma 4.3, there exists an apartment $\mathscr{A}_1$ containing x and $\mathscr{C}$. Let $\rho_1 : \mathscr{A}_1 \to \mathscr{A}$ be the bijection of Proposition 4.5 and define $\rho(x) := \rho_1(x)$. We must show that this map is well-defined first. Indeed, suppose $\mathscr{A}_2$ is another apartment of $\mathscr{I}$ containing x and $\mathscr{C}$ and $\rho_2 : \mathscr{A}_2 \to \mathscr{A}$ be the bijectio nof Proposition 4.5, then $\rho_1^{-1} \circ \rho_2 : \mathscr{A}_2 \to \mathscr{A}_1$ is again the bijection of Proposition 4.5 for the apartments $\mathscr{A}_2$, $\mathscr{A}_1$, and the chamber $\mathscr{C}$. Thus, $\rho_1^{-1} \circ \rho_2$ fixes $x \in \mathscr{A}_1 \cap \mathscr{A}_2$, i.e., $\rho_1(x) = \rho_2(x)$. This shows the existence of a desired retraction.

To see uniqueness, again use the fact that for any $x \in \mathscr{I}$, there exists an apartment containing x and $\mathscr{C}$. This completes the proof. ■

The mapping ρ defined above is called the *retraction of $\mathscr{I}$ onto $\mathscr{A}$ with centre $\mathscr{C}$*.

Proposition 2.4.7. Let ρ be the retraction of Theorem 4.6. Then

(1) $\rho x = x$ for all $x \in \mathscr{A}$.

(2) For each facet $\mathscr{F}$ in $\mathscr{I}$, $\rho|_{\overline{\mathscr{F}}}$ is a surjective affine isometry of $\overline{\mathscr{F}} \twoheadrightarrow \overline{\rho\mathscr{F}}$.

(3) If $x \in \overline{\mathscr{C}}$, then $\rho^{-1}(x) = \{x\}$.

Proof. (1) According to Theorem 4.6, $\rho|_{\mathscr{A}}$ is the unique bijection of Proposition 4.5, which is just the identity map, and hence $\rho x = x$ for all $x \in \mathscr{A}$.

(2) Let $\mathscr{A}'$ be an apartment containing $\mathscr{F}$ and $\mathscr{C}$, which exists due to Lemma 4.3. Note that $\overline{\mathscr{F}} \subseteq \mathscr{A}'$. Since $\rho : \mathscr{A}' \to \mathscr{A}$ is an isometry due to Proposition 4.5, the assertion follows.

(3) Let $\mathscr{F}'$ be a facet of $\mathscr{I}$ mapping to $\mathscr{F}$ under ρ. Note that $\rho : \mathscr{A}' \to \mathscr{A}$ is multiplication by some $g \in G$ which leaves $\mathscr{C}$ fixed, therefore, must leave all its facets fixed too, after all the facets are those corresponding to the parahorics containing the Iwahori corresponding to $\mathscr{C}$. ■

Proposition 2.4.8. (1) There exists a unique function $d : \mathscr{I} \times \mathscr{I} \to \mathbb{R}_+$ such that $d|_{\mathscr{A} \times \mathscr{A}}$ is the metric $d_{\mathscr{A}}$ for each apartment $\mathscr{A}$ of $\mathscr{I}$.

(2) If ρ is a retraction of $\mathscr{I}$ onto an apartment $\mathscr{A}$ as in Theorem 4.6, then $d(\rho(x), \rho(y)) \leqslant d(x, y)$ for all $x, y \in \mathscr{I}$.

(3) d is a G-invariant metric on $\mathscr{I}$.

Proof. (1) Let $x, y \in \mathscr{I}$, then due to Lemma 4.3, there is an apartment $\mathscr{A}$ containing x and y. We define $d(x, y) := d_{\mathscr{A}}(x, y)$. Suppose $\mathscr{A}'$ is another apartment containing x and y. We must show that $d_{\mathscr{A}}(x, y) = d_{\mathscr{A}'}(x, y)$. Let $\mathscr{C}$ be a chamber in $\mathscr{A}$ such that $x \in \overline{\mathscr{C}}$, this can be done, since every facet corresponds to a parahoric, which contains an Iwahori. Similarly, let $\mathscr{C}'$ be a chamber in $\mathscr{A}'$ such that $y \in \overline{\mathscr{C}}'$. Again by Lemma 4.3, there is an apartment $\mathscr{A}''$ containing $\mathscr{C}$ and $\mathscr{C}'$. From Proposition 4.5, we have that $d_{\mathscr{A}}(x, y) = d_{\mathscr{A}''}(x, y)$ because $\mathscr{A}$ and $\mathscr{A}''$ share the chamber $\mathscr{C}$. Analogously, $d_{\mathscr{A}'}(x, y) = d_{\mathscr{A}''}(x, y)$. Thus, the distance d is well-defined. That it is G-invariant follows from the definition of $d_{\mathscr{A}}$ as $(g|_{\mathscr{A}_0}) \circ j : A \to \mathscr{A}$.

(2) This is cumbersome to write out formally but here's the main idea: Choose an apartment $\mathscr{A}'$ in $\mathscr{I}$ containing x and y. This apartment is in bijection with A, through which its metric is defined. The affine line joining x to y in A will intersect finitely many facets in the tessellation of A. Thus, this line segment can be broken into a union of smaller closed line segments, each lying in the closure of a facet. Under ρ, the image of each such line segment is a line segment of the same length. In particular, the image of $[xy]$ under ρ is a polygonal line, whose "total length" is $d_{\mathscr{A}'}(x, y)$. The triangle inequality implies the desired conclusion.

(3) Let $x, y, z \in \mathscr{I}$ and let $\mathscr{A}$ be an apartment containing x and y. Let ρ be a retraction of $\mathscr{I}$ onto $\mathscr{A}$ as in Theorem 4.6. Then keeping in mind that $\rho(x) = x$ and $\rho(y) = y$, we have

$$d(x, y) = d_{\mathscr{A}}(\rho(x), \rho(y)) \leqslant d(\rho(x), \rho(z)) + d(\rho(z), \rho(y)) \leqslant d(x, z) + d(z, y),$$

where the last equality follows from (2). ■

Proposition 2.4.9. Let $x, y \in \mathscr{I}$. Then there is a unique geodesic joining x to y.

Proof. ■

Proposition 2.4.10. $\mathscr{I}$ is complete with respect to the metric d.

Proof. Let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in $\mathscr{I}$ with respect to the metric d. Let ρ be a retraction of $\mathscr{I}$ onto an apartment $\mathscr{A}_0$ as in Theorem 4.6. Then $(\rho x_n)_{n\geqslant 1}$ is a Cauchy sequence in $\mathscr{A}_0$, and as such, converges to some $x \in \mathscr{A}_0$. Let $x = (P, a) \in \mathscr{A}_0$ where $a \in A$. Then there is a $\mu > 0$ such that $d(x, wx) \geqslant \mu$ for all $w \in W$, the affine Weyl group. Let $g \in G$ be such that $x \neq gx$. We claim that $d(x, gx) \geqslant \mu$. Indeed, there is an apartment $\mathscr{A}' = h\mathscr{A}_0$ containing both x and gx for some $h \in G$. Then, from the G-invariance of d,

$$d(x, gx) = d(h^{-1}x, h^{-1}gx) \geqslant \mu,$$

which is clear from the bijection $A \leftrightarrow \mathscr{A}_0$. Again, since d is G-invariant, it follows that $d(gx, g'x) \geqslant \mu$ for all $g, g' \in G$ such that $gx \neq g'x$.

Now, let $N > 0$ be a positive integer such that for all $m, n \geqslant N$,

$$d(\rho x_n, x) < \frac{1}{3}\mu \quad \text{and} \quad d(x_m, x_n) < \frac{1}{3}\mu.$$

By definition, each ρx_n is of the form $g_n x_n$ for some $g_n \in G$. Set $y_n = g_n^{-1}x$. Then for $n \geqslant N$, using the G-invariance of d, we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leqslant d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \\ &= d(x, \rho x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \\ &< \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{3}\mu < \mu. \end{aligned}$$

Hence, $y_N = y_{N+1} = \cdots =: y$. Finally, for $n \geqslant N$, we have

$$d(x_n, y) = d(x_n, y_n) = d(g_n x_n, g_n y_n) = d(\rho x_n, x) \to 0,$$

as $n \to \infty$. ■

Fixed point theorem

A subset $X \subseteq \mathscr{I}$ is said to be *convex* if whenever $x, y \in X$, $[xy] \subseteq X$.

Lemma 2.4.11. Let $x, y, z \in \mathscr{I}$ and let m be the midpoint of $[xy]$ Then

$$d(z,x)^2 + d(z,y)^2 \geqslant 2d(z,m)^2 + \frac{1}{2}d(x,y)^2.$$

Proof. If x, y, z lie in the same apartment, then upon moving to the Euclidean space A, this is just a restatement of the well-known Apollonius' theorem. In the general case, let $\mathscr{A}$ be an apartment containing x and y and choose a chamber $\mathscr{C}$ in $\mathscr{A}$ such that $m \in \overline{\mathscr{C}}$. Let $\rho : \mathscr{I} \to \mathscr{A}$ be the retraction with centre $\mathscr{C}$ as in Theorem 4.6. Note that due to Lemma 4.3, we can choose an apartment $\mathscr{A}'$ containing $\mathscr{C}$ and z. Then, using Proposition 4.5, it is clear that $d(\rho(z), m) = d(z, m)$. Hence, we have

$$\begin{aligned} d(z,x)^2 + d(z,y)^2 &\geqslant d(\rho(z),x)^2 + d(\rho(z),y)^2 \\ &= 2d(\rho(z),m)^2 + \frac{1}{2}d(x,y)^2 \\ &= 2d(z,m)^2 + \frac{1}{2}d(x,y)^2, \end{aligned}$$

as desired. ■

Theorem 2.4.12. Let X be a bounded non-empty subset of $\mathscr{I}$. Then the group of isometries γ of $\mathscr{I}$ such that $\gamma(X) \subseteq X$ has a fixed point in the convex hull of X.

Proof. ■

A subset $M \subseteq G$ is said to be *bounded* if MX is bounded for all bounded subsets $X \subseteq \mathscr{I}$.

References

[Mac71] I.G. Macdonald. *Spherical Functions on a Group of P-adic Type*. Publications of the Ramanujan Institute. Ramanujan Institute for Advanced Study in Mathematics, University of Madras, 1971.