

# Galois Categories and the Étale Fundamental Group

or, what should be taught in MA 811

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# Galois Theory à la Grothendieck

Let  $k$  be a field and fix its separable and algebraic closures  $k_s \subseteq \bar{k}$ .  
Let  $G_k := \text{Gal}(k_s | k)$ , which is a profinite group through the isomorphism:

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# Étale Algebras and the Fundamental Theorem

## Definition

A finite-dimensional  $k$ -algebra  $A$  is said to be *étale* over  $k$  if it is isomorphic to a finite direct product of separable extensions of  $k$ .

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## Theorem (Fundamental Theorem of Galois Theory)

*The functor mapping a finite étale  $k$ -algebra  $A$  to the finite  $G_k$ -set  $\operatorname{Hom}_k(A, k_s)$  gives an anti-equivalence between the category of finite étale  $k$ -algebras and the category of finite sets with a continuous  $G_k$ -action. Here separable extensions correspond to transitive  $G_k$ -sets.*



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## Theorem (Classification of Covering Spaces)

*The aforementioned fibre functor induces an equivalence between the category of finite-sheeted covers of  $X$  and the category of continuous finite sets with a continuous  $\widehat{\pi_1(X, x_0)}$ -action. Here connected covers correspond to transitive  $\widehat{\pi_1(X, x_0)}$ -sets.*

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All that remains is to establish an equivalence of some suitable categories.

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In this case, the functor  $F$  is called a *fundamental functor*.

# Examples of Galois Categories

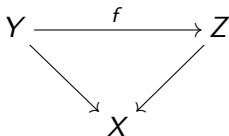
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- Let  $X$  be a connected scheme. Let  $\mathbf{FEt}_X$  denote the category of finite étale maps  $Y \rightarrow X$  with morphisms  $f : Y \rightarrow Z$  making



commute. Fix a geometric point  $x_0 : \text{Spec } \Omega \rightarrow X$ . This defines a fibre functor  $\text{Fib}_{x_0} : \mathbf{FEt}_X \rightarrow \mathbf{Sets}$ . The pair  $(\mathbf{FEt}_X, \text{Fib}_{x_0})$  forms a Galois category.

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This tuple is an element in the profinite group

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Further the tuple must be such that for each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

commutes.

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the conclusion follows. In particular,  $\text{Aut}_{\mathcal{C}}(F)$  is a profinite group.

# The equivalence functor

Next, let  $Y \in \mathcal{C}$ . There is a natural action of  $\text{Aut}_{\mathcal{C}}(F)$  on  $F(Y)$  given by

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The stabilizer of  $a \in F(Y)$  is

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This gives a natural functor  $H : \mathcal{C} \rightarrow \text{Aut}_{\mathcal{C}}(F)\text{-sets}$ .

# The Main Theorem

## Theorem (Fundamental Theorem of Galois Categories)

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# The Étale Fundamental Group

## Definition

Let  $X$  be a connected scheme and  $x_0 : \operatorname{Spec} \Omega \rightarrow X$  be a geometric point. The *étale fundamental group*  $\pi_1^{\text{ét}}(X, x_0)$  to be  $\operatorname{Aut}(\operatorname{Fib}_{x_0})$ .

Recall that there is an anti-equivalence between the category of commutative rings and the category of affine schemes.

Therefore, there is an anti-equivalence between the category  $\mathbf{F}\mathbf{Et}_{\operatorname{Spec} k}$  and the category of étale  $k$ -algebras.

Choosing a geometric point in  $\operatorname{Spec} k$  is tantamount to fixing a separable closure  $k_s$  of  $k$ .

As we have seen earlier,  $\mathbf{F}\mathbf{Et}_{\operatorname{Spec} k}$  is equivalent to the category  $G_k$ -**sets**. In particular,  $\pi_1^{\text{ét}}(\operatorname{Spec} k, x_0) \cong \operatorname{Gal}(k_s | k)$ .