Gorenstein Rings

Notes for the course MA 842: Topics in Algebra II

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§1 Injective Modules

§§ Basic Properties of Injective Modules

Definition 1.1. Let R be a ring. An R-module E is said to be *injective* if for every inclusion of R-modules $N \hookrightarrow M$ and an R-linear map $N \to E$, there is an R-linear map $M \to E$ making

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[dl, dotted, "\exists"] \\ & E & \end{tikzcd}$$

commute.

An R-module M is said to be *divisible* if

$$\mu_a : M \longrightarrow M \qquad m \longmapsto am$$

is surjective for each non-zerodivisor $a \in R$.

Remark 1.2. It is easy to see that E is injective if and only if given any inclusion of R-modules $N \hookrightarrow M$, the induced map $\operatorname{Hom}_R(M, E) \to \operatorname{Hom}_R(N, E)$ is surjective. Further, since $\operatorname{Hom}_R(-, E)$ is always left-exact, we have:

An R-module E is injective if and only if $\operatorname{Hom}_R(-, E)$ is an exact functor.

Proposition 1.3. A direct product of injective modules is injective.

Proof. This follows from the natural isomorphism of functors

$$\operatorname{Hom}_R\left(-, \prod_{\lambda\in\Lambda} E_\lambda\right) \cong \prod_{\lambda\in\Lambda} \operatorname{Hom}_R(-, E_\lambda). \qquad \blacksquare$$

Proposition 1.4. Every injective R-module is divisible.

Proof. Let E be R-injective, $x \in E$, and $a \in R$ a non-zerodivisor. Let $\varphi : R \to E$ be the unique R-linear map sending $1 \mapsto x$. Since $R \xrightarrow{\mu_a^R} R$ is injective, there is a map $\widetilde{\varphi} : R \to E$ such that $\widetilde{\varphi} \circ \mu_a^R = \varphi$. In particular, $a\widetilde{\varphi}(1) = x$, whence $\mu_a^E : E \to E$ is surjective, as desired. $\blacksquare$

Theorem 1.5 (Baer's Criterion). Let R be a ring and E an R-module. Then E is injective if and only if for every ideal $I \trianglelefteqslant R$ and an R-linear map $f : I \to E$, there is an R-linear map $F : R \to E$ such that $F|_I = f$.

Proof. The forward implication is clear. We shall prove the converse. Let $0 \to N \to M$ be exact and $f : N \to E$ be an R-linear map. Consider the poset

$$\Omega = \{(P, g) \colon N \leqslant P \leqslant M \text{ and } g : P \to E \text{ is } R\text{-linear extending } f\},$$

where $(P, g) \leqq (P', g')$ if $P \leqslant P'$ and $g'|_P = g$. Using Zorn's lemma, choose a maximal element $(P, g) \in \Omega$. We claim that $P = M$. Suppose now and choose some $x \in M \setminus P$. Set $I = (P :_R x) \trianglelefteqslant R$ and consider the map

$$I \longrightarrow E \qquad a \mapsto g(ax).$$

This is well-defined and R-linear, whence it extends to an R-linear map $\varphi : R \to E$. Let $\alpha = \varphi(1)$ and define $F : P + Rx \to E$ by $F(p + ax) = g(p) + a\alpha$ for all $p \in P$ and $a \in R$. To see that this is well-defined, note that if $p_1 + a_1x = p_2 + a_2x$, then $a_1 - a_2 \in I$, so that

$$g(p_2) - g(p_1) = g((a_1 - a_2)x) = (a_1 - a_2)\alpha \implies g(p_1) + a_1\alpha = g(p_2) + a_2\alpha.$$

The map F is obviously R-linear and extends g, thereby contradicting the maximality of (P, g). Hence, $P = M$ and E is injective. $\blacksquare$

Corollary 1.6. An R-module E is injective if and only if $\operatorname{Ext}^1_R(R/I, E) = 0$ for all ideals $I \trianglelefteqslant R$. $\blacksquare$

Remark 1.7. We note that it is not sufficient to check the equivalent condition of Theorem 1.5 for finitely generated ideals. Indeed, let $R = \mathscr{O}(\mathbb{C})$ the ring of entire functions, or $R = \mathcal{O}_{\overline{\mathbb{Q}}}$ the ring of algebraic integers in $\mathbb{C}$. It is known that R is a non-Noetherian Bézout domain. As such, due to Interlude 1.14, there is a family of R-injectives $\{E_i\}_{i=1}^\infty$ such that $E = \bigoplus_i E_i$ is not injective.

Since each E_i is injective, it is divisible, consequently, E is a divisible R-module. Moreover, since R is a Bézout domain, every finitely generated ideal I in R is principal. It follows now that the equivalent condition of Theorem 1.5 holds for E but E is not injective.

Proposition 1.8. Let R be a PID. An R-module E is injective if and only if it is divisible.

Proof. The forward direction is clear from Proposition 1.4. Conversely, let E be a divisible R-module and let $f : I \to E$ be R-linear where $I \trianglelefteqslant R$ is an ideal. If $I = 0$, then $f = 0$ and the zero map $R \to E$ extends f. If $I \neq 0$, then there is some $0 \neq a \in R$ such that $I = (a)$. If $x = f(a)$, then choose $y \in E$ with $ay = x$ and let $F : R \to E$ be the unique R-linear map sending 1 to y. It is clear that R extends f and hence E is an injective R-module. $\blacksquare$

Proposition 1.9. Let R be an integral domain. A torsion-free and divisible R-module is injective.

Proof. Let E be a torsion-free and divisible R-module. We shall use Theorem 1.5 to show that E is injective. Let $0 \neq I \triangleleft R$ be a proper ideal and $f : I \to E$ be R-linear. Choose some $0 \neq a \in I$ and let $x \in E$ be the unique (since E is torsion-free) element such that $ax = f(a)$. Let $F : R \to E$ be the unique R-linear map sending $1 \mapsto x$. We contend that F extends f. Indeed, for $0 \neq b \in I$,

$$af(b) = bf(a) = abx \implies f(b) = bx = F(b),$$

as desired. $\blacksquare$

Lemma 1.10. Let S be an R-algebra and E an injective R-module. Then $\mathrm{Hom}_R(S, E)$ is an injective S-module.

Note. $\mathrm{Hom}_R(S, E)$ is naturally an S-module under the action

$$(s \cdot f)(s') = f(ss') \qquad \forall\, s, s' \in S,\ f \in \mathrm{Hom}_R(S, E).$$

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of S-modules. Using the Hom-Tensor adjunction, we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S\left(M'', \mathrm{Hom}_R(S,E)\right) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_S\left(M, \mathrm{Hom}_R(S,E)\right) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_S\left(M', \mathrm{Hom}_R(S,E)\right) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(M' \otimes_S S, E) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(M \otimes_S S, E) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(M'' \otimes_S S, E) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(M', E) \arrow[r] & \mathrm{Hom}_R(M, E) \arrow[r] & \mathrm{Hom}_R(M'', E) \arrow[r] & 0
\end{tikzcd}$$

The exactness of the bottom row is a consequence of the R-injectivity of E. Thus the top row is exact and we have our desideratum. ∎

Theorem 1.11. Every R-module can be embedded inside an R-injective.

Proof. First, we show this for $R = \mathbb{Z}$. Let M be a $\mathbb{Z}$-module, then $M \cong \bigoplus_I \mathbb{Z}/N$ for some submodule N of $\bigoplus_I \mathbb{Z}$. There is a natural inclusion of $\mathbb{Z}$-modules $\bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_I \mathbb{Q}$ which induces an inclusion

$$M \cong \frac{\bigoplus_I \mathbb{Z}}{N} \hookrightarrow \frac{\bigoplus_I \mathbb{Q}}{N} =: E$$

Being a quotient of a divisible module, E is divisible and hence $\mathbb{Z}$-injective.

Now, let R be any ring and M an R-module. Then M is naturally a $\mathbb{Z}$-module and admits a $\mathbb{Z}$-linear inclusion $\iota : M \hookrightarrow E$, where E is a $\mathbb{Z}$-injective. Consider the map

$$\varphi : M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, E) \qquad m \longmapsto \varphi_m,$$

where $\varphi_m : R \to E$ is given by $\varphi_m(r) = f(rm)$. The map φ is obviously R-linear and if $\varphi_m = 0$, then $f(m) = \varphi_m(1) = 0$, i.e., $m = 0$. As a result, φ is injective and we have embedded M inside an injective R-module. ∎

Corollary 1.12. Let E be an R-module. Then E is injective if and only if every R-linear inclusion $E \hookrightarrow M$ splits.

Proof. Suppose E is injective.

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & M \arrow[dl, dotted, "\exists"] \\
& E &
\end{tikzcd}$$

The above diagram constructs a splitting of $E \hookrightarrow M$.

Conversely, suppose every R-linear inclusion $E \hookrightarrow M$ splits. Due to Theorem 1.11, we may choose M to be injective, so that E is a direct summand of M, whence E is injective. ∎

Proposition 1.13. Let R be a Noetherian ring. A direct sum of injective R-modules is injective.

Proof. Let $\{E_\lambda\}_{\lambda \in \Lambda}$ be a collection of R-injectives and $E = \bigoplus_{\lambda \in \Lambda} E_\lambda$. Let $I \trianglelefteq R$ be a non-zero proper ideal and $f : I \to E$ an R-linear map. Since I is finitely generated, its image under f is finitely generated in E. Consequently, there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $f(I) \subseteq \bigoplus_{\lambda \in \Lambda_0} E_\lambda = E_0$. Being a finite direct sum of injectives, E_0 is injective and hence there is a map $F : R \to E_0$ extending $f : I \to E_0$. Composing F with the natural inclusion $E_0 \hookrightarrow E$, we obtain our desired extension of f. It now follows from Theorem 1.5 that E is an injective R-module. ∎

Interlude 1.14 (Bass-Papp Construction). Let R be a non-Noetherian ring. Choose a strictly increasing chain of proper non-zero ideals

$$0 \neq I_1 \subsetneq I_2 \subsetneq \cdots .$$

For each $n \geqslant 1$, choose an injective module E_n containing R/I_n, and set $E = \bigoplus_n E_n$. We contend that E is not R-injective.

Let $I = \bigcup_n I_n$. Since each I_n is proper, so is I. Let $f : I \to E$ be the map given by

$$f(x) = (x \bmod I_1, x \bmod I_2, \dots) .$$

If E were injective, then there must exist a map $F : R \to E$ extending f. Suppose $F(1) = (x_1, x_2, \dots)$. There is a positive integer N such that $x_n = 0$ for all $n \geqslant N$. Choose $x \in I_{N+1} \setminus I_N$. Since $x \in I$, we have

$$(xx_1, xx_2, \dots) = F(x) = f(x) = (x \bmod I_1, x \bmod I_2, \dots) .$$

In particular, $x \bmod I_N = xx_N = 0$, a contradiction. Thus E is not R-injective.

Proposition 1.15. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. If $E \neq 0$ is an finitely generated injective R-module, then R is Artinian.

Proof. We shall show that $\dim R = 0$. Suppose not; we contend that there is a prime $\mathfrak{p} \subsetneq \mathfrak{m}$ such that $\operatorname{Hom}_R(R/\mathfrak{p}, E) \neq 0$. Indeed, if there is a non-maximal prime $\mathfrak{p} \in \operatorname{Ass}_R(E)$, then $R/\mathfrak{p} \hookrightarrow E$, giving us the desideratum. On the other hand, if $\operatorname{Ass}_R(E) = \{\mathfrak{m}\}$, then the composition

$$R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{m} \hookrightarrow E$$

gives a non-zero map $R/\mathfrak{p} \to E$.

Choose $a \in \mathfrak{m} \setminus \mathfrak{p}$; this is a non-zerodivisor on $R/\mathfrak{p}$ and furnishes an exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{\cdot a} R/\mathfrak{p}.$$

Applying $\operatorname{Hom}_R(-, E)$, we get a surjection

$$\operatorname{Hom}_R(R/\mathfrak{p}, E) \xrightarrow{\cdot a} \operatorname{Hom}_R(R/\mathfrak{p}, E) \to 0.$$

Note that $\operatorname{Hom}_R(R/\mathfrak{p}, E) \cong (0 :_E \mathfrak{p}) \subseteq E$, is a finite R-module. Due to Nakayama's lemma, we must have that $\operatorname{Hom}_R(R/\mathfrak{p}, E) = 0$, a contradiction. Thus $\dim R = 0$, i.e. R is Artinian. ∎

Remark 1.16. One cannot drop the local condition in Proposition 1.15. This construction makes use of injective hulls. Let k be an algebraically closed field and

$$R = \frac{k[X, Y]}{(X - X^2, Y - XY)}.$$

Note that R is the coordinate ring of the disjoint union of the origin and the line $x = 1$ in $\mathbb{A}^2_k$. In particular, $\dim R = 1$, and R is not Artinian.

Let $\mathfrak{m} = (x, y)$ be the maximal ideal corresponding to the origin. Then $R_\mathfrak{m} \cong k$, since it is the local ring of an isolated point. Now,

$$E_R(k) \cong E_{R_\mathfrak{m}}(k) \cong E_k(k) = k,$$

so that k is a finitely generated injective R-module.

§§ Essential Extensions and Injective Hulls

Definition 1.17. A containment of R-modules $N \subseteq M$ is said to be *essential* if every non-zero submodule of M intersects N non-trivially.

An injective map $\iota : N \hookrightarrow M$ is said to be essential if $\iota(N) \subseteq M$ is essential.

Remark 1.18. Let $M \subseteq N$ be an essential extension of R-modules and $\varphi : M \hookrightarrow P$ be an R-linear injective map. If φ extends to an R-linear map $\widetilde{\varphi} : N \to P$, then $\widetilde{\varphi}$ is injective too. Indeed, if $K = \ker \widetilde{\varphi} \neq 0$, then $K \cap M \neq 0$, a contradiction.

Proposition 1.19. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Let M be an Artinian R-module. Then $\operatorname{Soc}_R(M) \subseteq M$ is an essential extension.

Proof. Let $0 \neq K \subseteq M$ be a submodule. Choose $0 \neq x \in K$. Since M is Artinian, the descending chain $Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2x \supseteq \cdots$ stabilizes. Let $n \geqslant 0$ be the least positive integer such that $\mathfrak{m}^n x = \mathfrak{m}^{n+1}x$. Due to Nakayama's lemma, $\mathfrak{m}^n x = 0$, whence $n \geqslant 1$. It follows that $0 \neq \mathfrak{m}^{n-1}x \subseteq \operatorname{Soc}_R(M) \cap K$, as desired. ■

Corollary 1.20. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring and M an Artinian R-module. If $\dim_k \operatorname{Soc}_R(M) = d$, then $E_R(M) \cong E^{\oplus d}$.

Proof. Since $\operatorname{Soc}_R(M) \cong k^{\oplus d}$, it is clear that $E_R(\operatorname{Soc}_R(M)) \cong E^{\oplus d}$. The inclusion $\operatorname{Soc}_R(M) \hookrightarrow E^{\oplus d}$ can be extended to M to obtain a commutative diagram:

$$\begin{tikzcd}
M \arrow[dr, hook] & \\
\operatorname{Soc}_R(M) \arrow[u, hook] \arrow[r, hook] & E_R(\operatorname{Soc}_R(M)) \cong E^{\oplus d}
\end{tikzcd}$$

where all maps are inclusion. It follows that $M \hookrightarrow E^{\oplus d}$ is an essential extension. Since $E^{\oplus d}$ is an injective module, we have that $E_R(M) \cong E^{\oplus d}$. ■

§2 Matlis Duality

Definition 2.1. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. For an R-module M, set $M^\vee = \operatorname{Hom}_R(M, E)$. This is known as the *Matlis dual* of a module.

Clearly $(-)^\vee$ is a contravariant exact functor on the category of R-modules. Note that if $I \subseteq \mathfrak{m}$ is an ideal, then as we have seen earlier,

$$E_{R/I}(k) = \operatorname{Hom}_R(R/I, E) = (R/I)^\vee .$$

In particular, taking $I = \mathfrak{m}$, we see that $k^\vee \cong k$ as R-modules.

Lemma 2.2. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then

(1) If $M \neq 0$, then $M^\vee \neq 0$.

(2) If $\lambda_R(M) < \infty$, then $\lambda_R(M^\vee) \neq 0$. Moreover, $\lambda_R(M) = \lambda_R(M^\vee)$.

Proof. (1) Let $0 \neq x \in M$. If $I = \operatorname{Ann}_R(x)$, then there is a natural inclusion $R/I \hookrightarrow M$ sending $\overline{1} \mapsto x$. Taking the Matlis dual, we have a surjection

$$M^\vee \twoheadrightarrow (R/I)^\vee = E_{R/I}(k) \neq 0,$$

consequently $M^\vee \neq 0$.

(2) We shall prove both statements by induction on $\lambda_R(M)$. If $\lambda_R(M) = 0$, then $M = 0$, so that $M^\vee = 0$ and we get $\lambda_R(M) = 0 = \lambda_R(M^\vee)$. Suppose now that $0 < \lambda_R(M) < \infty$. Then $\mathfrak{m} \in \operatorname{Ass}_R(M)$, and we have a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Since length is additive, $\lambda_R(N) = \lambda_R(M) - 1$; hence the induction hypothesis applies and $\lambda_R(N^\vee) = \lambda_R(N)$. Taking the Matlis dual of the above short exact sequence, we have

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow k^\vee \longrightarrow 0.$$

Since $k^\vee = 0$, we see that

$$\lambda_R(M^\vee) = \lambda_R(N^\vee) + 1 = \lambda_R(N) + 1 = \lambda_R(M),$$

as desired. ■

Theorem 2.3. Let $(R, \mathfrak{m}, k, E)$ be an Artinian local ring.

(1) E is a faithful finite R-module.

(2) The map

$$\mu : R \longrightarrow \operatorname{Hom}_R(E, E) \qquad a \longmapsto \mu_a$$

is an isomorphism of R-modules and rings.

(3) Given a finite R-module M, the natural map

$$\varphi_M : M \longrightarrow M^{\vee\vee} \qquad m \longmapsto \mathrm{ev}_m$$

is an isomorphism.

Proof. (1) Suppose $a \in R$ is such that $aE = 0$. Then

$$R^\vee = \operatorname{Hom}_R(R, E) = E = (E :_E a) \cong \operatorname{Hom}_R(R/aR, E) = (R/aR)^\vee .$$

Since R is Artinian, we then have

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R\left((R/aR)^\vee\right) = \lambda_R(R/aR) \implies \lambda_R(aR) = 0,$$

consequently, $a = 0$, i.e., E is a faithful R-module.

Next, since R is Artinian, $\mathfrak{m} \in \operatorname{Ass}_R(R)$, consequently, there is an injection $k = R/\mathfrak{m} \hookrightarrow R$. Due to Remark 1.18 extends to an inclusion $E \hookrightarrow R$, consequently, E is a finite R-module.

(2) First note that μ is injective due to (1). But note that

$$\infty > \lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = \lambda_R(E^\vee) = \lambda_R\left(\operatorname{Hom}_R(E, E)\right),$$

consequently μ is an isomorphism.

(3) It suffices to show that φ_M is injective since $\lambda_R(M) = \lambda_R(M^{\vee\vee})$. Suppose $0 \neq x \in M$ is such that $\varphi_M(x) = 0$, that is, for all $f \in \operatorname{Hom}_R(M, E)$, $f(x) = 0$. Let $I = \operatorname{Ann}_R(x)$. Now, there is a non-zero map

$$\psi : R/I \twoheadrightarrow R/\mathfrak{m} = k \hookrightarrow E,$$

which extends to a non-zero map $f : M \to E$ since $R/I \hookrightarrow M$ through $\overline{1} \mapsto x$. Thus, $f(x) = \psi(\overline{1}) \neq 0$, a contradiction. ∎

Porism 2.4. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite-length R-module. Then the "evaluation map" $\mathrm{ev} : M \to M^{\vee\vee}$ is an isomorphism of R-modules.

Proof. As in the preceding proof, ev is injective and due to Lemma 2.2, $\lambda_R(M) = \lambda_R(M^{\vee\vee})$, whence ev is an isomorphism. ∎

Interlude 2.5 (On $\widehat{R}$-modules). Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module such that $\Gamma_{\mathfrak{m}}(M) = M$. We contend that M is an $\widehat{R}$-module in a natural way. To this end, we need only define $\widehat{a} \cdot m$ for $\widehat{a} \in \widehat{R}$ and $m \in M$.

Let $\widehat{a} = (a_1, a_2, \dots)$, where we are using the isomorphism

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

Since $\Gamma_{\mathfrak{m}}(M) = M$, there is a positive integer $n \geqslant 1$ such that $\mathfrak{m}^n m = 0$. Hence, for $k \geqslant n$, we have $a_k \cdot m = a_n \cdot m$, as $a_k - a_n \in \mathfrak{m}^n$. In light of this, we define $\widehat{a} \cdot m = a_n \cdot m$. We must show that this makes M into an $\widehat{R}$-module.

Let $m_1, m_2 \in M$ and $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$. There are positive integers $n_1, n_2 \geqslant 1$ such that $\mathfrak{m}^{n_1} m_1 = 0 = \mathfrak{m}^{n_2} m_2$; then $\mathfrak{m}^n m_1 = 0 = \mathfrak{m}^n m_2$ for all $n \geqslant \max\{n_1, n_2\}$. Hence, for all such $n \geqslant 1$,

$$\widehat{a} \cdot (m_1 + m_2) = a_n \cdot (m_1 + m_2) = a_n \cdot m_1 + a_n \cdot m_2 = \widehat{a} \cdot m_1 + \widehat{a} \cdot m_2.$$

Next, let $\widehat{a}, \widehat{b} \in \widehat{R}$ and $m \in M$ with

$$\widehat{a} = (a_1, a_2, \dots) \qquad \text{and} \qquad \widehat{b} = (b_1, b_2, \dots).$$

There is a positive integer n such that $\mathfrak{m}^n m = 0$. Then

$$(\widehat{a} + \widehat{b}) \cdot m = (a_n + b_n) \cdot m = a_n \cdot m + b_n \cdot m = \widehat{a} \cdot m + \widehat{b} \cdot m.$$

Finally, note that $\widehat{b} \cdot m = b_n m$ and $\mathfrak{m}^n \left(\widehat{b} \cdot m\right) = 0$, so that

$$\widehat{a} \cdot \left(\widehat{b} \cdot m\right) = \widehat{a} \cdot (b_n \cdot m) = a_n \cdot (b_n \cdot m) = (a_n b_n) \cdot m = (\widehat{a}\widehat{b}) \cdot m.$$

This shows that M is indeed an $\widehat{R}$-module as described above. Further, since $R \to \widehat{R}$ is the diagonal map, it follows that the $\widehat{R}$-module structure on M agrees with the R-module struture through the diagonal map. In particular, this means that:

A subset of M is an R-submodule if and only if it is an $\widehat{R}$-submodule.

As a result, M is Noetherian (resp. Artinian) as an R-module if and only if it is so as an $\widehat{R}$-module.

Interlude 2.6 (On maps between $\mathfrak{m}$-power torsion modules). Again, let $(R, \mathfrak{m}, k)$ be a local ring and suppose M and N are R-modules such that $\Gamma_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(N)$. By Interlude 2.5, we know that they are $\widehat{R}$-modules in a natural way. Let $\varphi : M \to N$ be an R-linear map. We contend that φ is also $\widehat{R}$-linear. Indeed, let $m \in M$ and $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$. There is a positive integer $n \geqslant 1$ such that $\mathfrak{m}^n m = 0$, and hence, $\mathfrak{m}^n \varphi(m) = 0$. It follows that

$$\varphi(\widehat{a} \cdot m) = \varphi(a_n \cdot m) = a_n \cdot \varphi(m) = \widehat{a} \cdot \varphi(m),$$

as desired.

Theorem 2.7. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring.

(1) $\Gamma_{\mathfrak{m}}(E) = E$, and hence E is an $\widehat{R}$ module and for every R-module M.

(2) $E \cong E_{\widehat{R}}(k)$ as $\widehat{R}$-modules.

(3) $R^{\vee\vee} = \operatorname{Hom}_R(E, E) \cong \widehat{R}$ as R-algebras.

(4) E is an Artinian R-module.

Proof. (1) That E is an $\widehat{R}$-module follows immediately from Interlude 2.5.

(2) The containment $k \subseteq E$ is an essential extension of R-modules, both of which are $\mathfrak{m}$-power torsion. Due to Interlude 2.5, it follows that it is an essential extension of $\widehat{R}$-modules too. Now, due to Remark 1.18, there is a commutative diagram of inclusions

$$\begin{tikzcd} E \arrow[dr] & \\ k \arrow[u] \arrow[r] & E_{\widehat{R}}(k), \end{tikzcd}$$

where all maps are $\widehat{R}$-linear. It follows that $E \hookrightarrow E_{\widehat{R}}(k)$ is an essential extension of $\widehat{R}$-modules, and consequently, an essential extension of R-modules. Since E is R-injective, we must have that the inclusion is an isomorphism of R-modules. Finally, due to Interlude 2.6, this is an isomorphism of $\widehat{R}$-modules.

(3) For every positive integer $n \geqslant 1$, set $E_n = (0 :_E \mathfrak{m}^n)$. Note that $E_1 \subseteq E_2 \subseteq \cdots$, and $E = \bigcup_n E_n$. Define $\Phi : \widehat{R} \to \operatorname{End}_R(E)$ as follows: for $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$, let $\Phi(\widehat{a}) = f \in \operatorname{End}_R(E)$ where f is given by

$$f(x) = a_n x \quad \text{ if } x \in E_n.$$

First we must show that the above map is well-defined. Indeed, if $m < n$ and $x \in E_m \subseteq E_n$, then $a_m - a_n \in \mathfrak{m}^m$, whence $(a_m - a_n)x = 0$, i.e., $a_m x = a_n x$. That the map f is R-linear is clear from its definition.

That the map Φ is R-linear is also clear. We claim that Φ is a ring homomorphism. Let $\widehat{a} = (a_1, a_2, \dots)$, $\widehat{b} = (b_1, b_2, \dots) \in \widehat{R}$ and set $f = \Phi(\widehat{a})$, $g = \Phi(\widehat{b})$, and $h = \Phi(\widehat{a}\widehat{b})$. If $x \in E_n$, then

$$h(x) = (a_n b_n)x = f(g(x)) \implies h = f \circ g,$$

thus Φ is a ring homomorphism.

Finally, we show that Φ is bijective, so that it is an isomorphism of R-algebras. If $\widehat{a} \in \widehat{R}$ is such that $\Phi(\widehat{a}) = 0$, then $a_n \in \mathrm{Ann}_R(E_n)$ for every positive integer n. But recall that

$$E_n \cong \mathrm{Hom}_R(R/\mathfrak{m}^n, E) \cong E_{R/\mathfrak{m}^n}(k),$$

which is a faithful $R/\mathfrak{m}^n$-module due to Theorem 2.3. As a result, $\mathrm{Ann}_R(E_n) = \mathfrak{m}^n$, i.e., $a_n \in \mathfrak{m}^n$ for all $n \geqslant 1$; in other words, $\widehat{a} = 0$. This proves the injectivity of Φ.

Next, we must show surjectivity of Φ. Let $f \in \mathrm{End}_R(E)$, then f restricts to an R-linear endomorphism of $E_n \cong E_{R/\mathfrak{m}^n}(k)$. Due to Theorem 2.3, the restriction of f to each E_n is multiplication by some element $a_n \in R/\mathfrak{m}^n$. Further, it is clear that under the canonical surjection $R/\mathfrak{m}^n \twoheadrightarrow R/\mathfrak{m}^{n-1}$, a_n maps to a_{n-1}, so that $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$ and $\Phi(\widehat{a}) = f$. Thus Φ is surjective, as desired.

As a final subtle point, we must check that the R-algebra structure on $\widehat{R}$ is the canonical one. The natural map $R \to \mathrm{End}_R(E)$ is $a \mapsto \mu_a$, the "multiplication by a" map. From our definition of Φ, it is clear that $\Phi^{-1}(\mu_a) = (a, a, \dots)$, which is precisely the image of a under the canonical map $R \to \widehat{R}$.

(4) Let $M_1 \supseteq M_2 \supseteq \cdots$ be a chain of R-submodules in E. There are commutative diagrams

$$\begin{tikzcd} M_{j+1} \arrow[r, hook, "\iota_{j+1}"] \arrow[d, hook] & E \\ M_j \arrow[ur, hook, "\iota_j"'] & \end{tikzcd}$$

whose Matlis dual furnishes commutative diagrams

$$\begin{tikzcd} \widehat{R} = E^\vee \arrow[r, two heads, "\varphi_j"] \arrow[dr, two heads, "\varphi_{j+1}"'] & M_j^\vee \arrow[d, two heads] \\ & M_{j+1}^\vee \end{tikzcd} .$$

Note that all Matlis duals are $\mathfrak{m}$-power torsions and hence due to Interlude 2.6, the φ_j's are $\widehat{R}$-linear. Let $I_j = \ker \varphi_j \subseteq \widehat{R}$, which is an ideal. Due to the commutative diagram, it is clear that there is an ascending chain $I_j \subseteq I_{j+1}$. Since $\widehat{R}$ is Noetherian, this chain stabilizes, say $I_n = I_{n+1} = \dots$.

Then due to the first isomorphism theorem, $M_j^\vee \twoheadrightarrow M_{j+1}^\vee$ is an isomorphism for all $j \geqslant n$. Let $C_j = \mathrm{coker}\,(M_{j+1} \hookrightarrow M_j)$. The exactness of the Matlis dual gives $C_j^\vee = 0$, which, due to Lemma 2.2, implies that $C_j = 0$, that is, $M_{j+1} \hookrightarrow M_j$ is an isomorphism for all $j \geqslant n$, i.e., the descending chain stabilizes, as desired. ∎

Interlude 2.8 (The Matlis Dual is a module over $\widehat{R}$). Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring and M an R-module. The Matlis dual $M^\vee = \mathrm{Hom}_R(M, E)$ is naturally a $\widehat{R} = \mathrm{End}_R(E)$-module: for $f \in M^\vee$ and $\varphi \in \mathrm{End}_R(E)$, define $\varphi \cdot f = \varphi \circ f$. It is easy to check that this $\widehat{R}$-module structure on $M^\vee$ extends the R-module structure through the canonical map $R \to \mathrm{End}_R(E)$, $a \mapsto \mu_a$.

Now, if $f : M \to N$ is an R-linear map of R-modules, then $f^\vee : N^\vee \to M^\vee$ is $\widehat{R}$-linear. Indeed, for $\varphi \in N^\vee$, and $\psi \in \widehat{R} = \operatorname{End}_R(E)$, we have

$$f^\vee(\psi \cdot \varphi) = f^\vee(\psi \circ \varphi) = \psi \circ \varphi \circ f = \psi \cdot f^\vee(\varphi),$$

as desired.

Theorem 2.9 (Matlis Duality, version 1). Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{finitely generated} \\ \widehat{R}\text{-modules} \end{matrix} \right\} \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{Artinian } R\text{-modules} \end{matrix} \right\}.$$

Proof. Let M be an Artinian R-module and let $d = \dim_k \operatorname{Soc}_R(M)$. Due to Corollary 1.20, $E_R(M) \cong E^{\oplus d}$, so that there is an inclusion $M \hookrightarrow E^{\oplus d}$, which upon taking the Matlis dual furnishes an $\widehat{R}$-linear surjection $\widehat{R}^{\oplus d} \twoheadrightarrow M^\vee$. Thus $M^\vee$ is a finite $\widehat{R}$-module.

Conversely, suppose M is a finite $\widehat{R}$-module. Thus, there is a surjection $\widehat{R}^{\oplus n} \twoheadrightarrow M$. Taking the Matlis dual, we obtain an injection $M^\vee \hookrightarrow \left(\widehat{R}^\vee\right)^{\oplus n}$.

There is a natural "evaluation map" $\mathrm{ev} : M \to M^{\vee\vee}$, which we shall show is an isomorphism. That ev is injective follows in the same way as Theorem 2.3 (3). Next, since $\lambda_R(M) < \infty$, we have that $\lambda_R(M) = \lambda_R(M^\vee) = \lambda_R(M^{\vee\vee})$, whence ev is an isomorphism. ∎

Theorem 2.10. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then the following are equivalent:

(1) R is self-injective

(2) $R \cong E$ as R-modules.

(3) R is Artinian and $\dim_k \operatorname{Soc}_R(R) = 1$.

Proof. (1) $\implies$ (2) Due to Proposition 1.15, R must be an Artinian local ring, and hence, from Proposition 1.19, $\operatorname{Soc}_R(R) \subseteq R$ is an essential extension. It follows that R is the injective hull of $\operatorname{Soc}_R(R) \cong k^{\oplus d}$ for some positive integer d. Hence, $R \cong E^{\oplus d}$ as R-modules, and comparing lengths, we have

$$\lambda_R(R) = d\lambda_R(E) = d\lambda_R(R^\vee) = d\lambda_R(R),$$

whence $d = 1$ and $R \cong E$.

(2) $\implies$ (3) Due to Theorem 2.7 (4), R is Artinian. Using a length argument as above, we can show that $\dim_k \operatorname{Soc}_R(R) = 1$.

(3) $\implies$ (1) Again, since $k = \operatorname{Soc}_R(R) \subseteq R$ is essential, we have that $R \hookrightarrow E = E_R(k)$. Using a length argument, it follows that this inclusion must be an isomorphism, whence R is self-injective. ∎

Theorem 2.11 (Matlis Duality, version 2). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{ \begin{matrix} \mathfrak{m}\text{-primary ideals} \\ \text{in } R \end{matrix} \right\} \underset{(0:_R -)}{\overset{(0:_E -)}{\rightleftarrows}} \left\{ \begin{matrix} \text{finitely generated} \\ R\text{-submodules of } E \end{matrix} \right\}.$$

Proof. We must first show that the above maps are indeed defined between those sets. Let I be $\mathfrak{m}$-primary in R. Then

$$(0 :_E I) \cong \operatorname{Hom}_R(R/I, E) = (R/I)^\vee.$$

As a result, $\lambda_R((0 :_E I)) = \lambda_R(R/I) < \infty$, so that $(0 :_E I)$ is a finite R-module.

On the other hand, let $W \subseteq E$ be a finite R-submodule. Taking the Matlis dual of the exact sequence $0 \to W \to E$, one obtains an $\widehat{R}$-linear (due to Interlude 2.8) surjection $\varphi : \widehat{R} \twoheadrightarrow W^\vee$. Further, since $\lambda_R(W) < \infty$, we have $\lambda_R(W^\vee) = \lambda_R(W) < \infty$. Set $I = (0 :_R W)$ and $J = (0 :_R W^\vee)$; note that both I and J are $\mathfrak{m}$-primary. This shows that both the maps maps in the theorem are well-defined.

Claim. $I = J$

Since I annihilates W, it must also annihilate $W^\vee$, so that $I \subseteq J$. Now, since J annihilates $W^\vee$, it annihilates $W^{\vee\vee} \cong W$ (due to Porism 2.4), so that $J \subseteq I$; as a result, $I = J$. ♠

Finally, we show that the given maps are inverses to one another. Let $I \trianglelefteq R$ be $\mathfrak{m}$-primary. Then $(0 :_E I) \cong \operatorname{Hom}_R(R/I, E) \cong E_{R/I}(k)$, whence due to Theorem 2.3 (1), $(0 :_R (0 :_E I)) = I$. Next, let $W \subseteq E$ be a finite R-submodule. Clearly $W \subseteq (0 :_E (0 :_R W))$. Further, recall that $\widehat{R} \twoheadrightarrow W^\vee$ and $\lambda_{\widehat{R}}(W^\vee) = \lambda_R(W) < \infty$[1], the kernel of the surjection is $\widehat{\mathfrak{m}}$-primary, and hence, factors through $\widehat{R}/\widehat{\mathfrak{m}}^n$ for some positive integer n. But since $\widehat{R}/\widehat{\mathfrak{m}}^n \cong R/\mathfrak{m}^n$ as R-modules, it follows that $W^\vee$ is a cyclic R-module. In particular, $W^\vee \cong R/J = R/I$. In particular,

$$\lambda_R\left((0 :_E (0 :_R W))\right) = \lambda_R\left((R/I)^\vee\right) = \lambda_R(R/I) = \lambda_R(R/J) = \lambda_R(W^\vee) = \lambda_R(W),$$

whence $W = (0 :_E (0 :_R W))$, thereby completing the proof. ∎

§3 Injective Resolutions

§§ Bass's Lemma and ramifications

Definition 3.1. Let M be an R-module. An *injective resolution* for M is an exact complex

$$0 \to M \to E^0 \to E^1 \to E^2 \to \cdots,$$

where each E^n is an injective R-module. The resolution is often denoted succinctly as $0 \to M \to E^\bullet$.

We say that M has finite injective dimension if M has an injective resolution $0 \to M \to E^\bullet$ and an integer $N \geqslant 0$ such that $E^n = 0$ for $n \geqslant N$. We define

$$\operatorname{inj\,dim}_R M = \inf\left\{n \colon 0 \to M \to E^0 \to \cdots \to E^n \to 0 \text{ is an injective resolution of } M\right\}.$$

If M does not have finite injective dimension, then set $\operatorname{inj\,dim}_R M = \infty$.

Remark 3.2. It is possible to create a "canonical" injective resolution by successively taking injective hulls. Set $E^0 = E_R(M)$ and for $i \geqslant 0$, define

$$E^{i+1} = E_R\left(\operatorname{coker}\left(E^{i-1} \to E^i\right)\right),$$

with the convention that $E^{-1} = M$. We call this the *minimal injective resolution* of M.

Lemma 3.3. Let R be a Noetherian ring and $0 \to M \xrightarrow{\theta} E$ be an inclusion of R-modules with E injective. Then the inclusion is an injective hull of M if and only if

$$\operatorname{Hom}_R\left(R/\mathfrak{p}, M\right)_\mathfrak{p} \xrightarrow{\theta_\mathfrak{p}} \operatorname{Hom}_R\left(R/\mathfrak{p}, E\right)_\mathfrak{p}$$

is an isomorphism for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. Owing to the left exactness of $\operatorname{Hom}_R(R/\mathfrak{p}, -)$ and the exactness of localization, the map $\theta_\mathfrak{p}$ is injective for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Hence, it suffices to show that E is injective if and only if $\theta_\mathfrak{p}$ is surjective for each $\mathfrak{p} \in \operatorname{Spec}(R)$.

Recall that there are canonical isomorphisms

$$\operatorname{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \xrightarrow{\sim} \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \qquad \frac{\psi}{s} \longmapsto \left(\frac{a}{t} \mapsto \frac{\psi(a)}{st}\right),$$

where we are identifying $\kappa(\mathfrak{p})$ with the quotient field of $R/\mathfrak{p}$. Hence, surjectivity of $\theta_\mathfrak{p}$ is equivalent to the surjectivity of

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \to \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_\mathfrak{p}).$$

Henceforth, we shall identify M with a submodule of E, so that θ is simply the inclusion map.

[1] Since every $\widehat{R}$-submodule of $W^\vee$ is also an R-submodule, it follows that $W^\vee$ is both Noetherian and Artinian as an $\widehat{R}$-module.

Suppose first that $M \xrightarrow{\theta} E$ is an injective hull and let $0 \neq \varphi \in \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_\mathfrak{p})$. Using the above isomorphism, we can write $\varphi = \psi/s$ for some $\psi \in \operatorname{Hom}_R(R/\mathfrak{p}, E)$ and $s \in R \setminus \mathfrak{p}$. Let $\psi(\overline{1}) = z \in E$ and $a \in R$ such that $0 \neq az \in M$. Note that $a \in R \setminus \mathfrak{p}$, since $\mathfrak{p} \subseteq \operatorname{Ann}_R(z)$[2]. Define

$$\overline{\varphi} : R/\mathfrak{p} \longrightarrow M \qquad \overline{1} \longmapsto az.$$

This is well-defined, since $\mathfrak{p}$ annihilates $az \in M$. We claim that

$$\varphi = \frac{\overline{\varphi}}{as} \in \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E_\mathfrak{p}\right).$$

Indeed, for $x/t \in \kappa(\mathfrak{p})$ we have

$$\left(\frac{\overline{\varphi}}{as}\right)\left(\frac{x}{t}\right) = \frac{\overline{\varphi}(x)}{ast} = \frac{xaz}{ast} = \frac{xz}{st} = \left(\frac{\psi}{s}\right)\left(\frac{x}{t}\right) = \varphi\left(\frac{x}{t}\right),$$

as desired. This shows that $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \to \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_\mathfrak{p})$ is surjective.

Conversely, suppose the aforementioned map is surjective. We shall show that E is the injective hull of M. To this end, it suffices to show that the inclusion $M \subseteq E$ is essential. Let $0 \neq N \subseteq E$ be a submodule and $\mathfrak{p} \in \operatorname{Ass}_R(N)$. There is an injective map

$$0 \to R/\mathfrak{p} \longrightarrow N \qquad \overline{1} \longmapsto z.$$

Since $\mathfrak{p} = \operatorname{Ann}_R(z)$, it suffices to find $a \in R \setminus \mathfrak{p}$ such that $az \in M$. Consider the map

$$\varphi : \kappa(\mathfrak{p}) \longrightarrow E_\mathfrak{p} \qquad \overline{1} \longmapsto z/1.$$

The surjectivity of $\theta_\mathfrak{p}$ furnishes a $\psi : \kappa(\mathfrak{p}) \to M_\mathfrak{p}$ such that $\theta_\mathfrak{p}(\psi) = \varphi$. In particular, this means that

$$\frac{z}{1} = \varphi(\overline{1}) = \psi(\overline{1}) \in M_\mathfrak{p},$$

whence there is some $a \in R \setminus \mathfrak{p}$ such that $az \in M$, as desired. ∎

Corollary 3.4. Let R be a Noetherian ring and $0 \to M \to E^\bullet$ be an injective resolution of an R-module M. Then $E^\bullet$ is minimal if and only if the natural maps

$$\operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^{n+1}_\mathfrak{p}\right)$$

are identically zero for all $n \geqslant 0$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. Let $K^n = \ker\left(E^n \to E^{n+1}\right)$. Then there is an exact sequence $0 \to K^n \to E^n \to E^{n+1}$. Using Lemma 3.3, E^n is the injective hull of C^n if and only if

$$\Phi : \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), C^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \text{ is an isomorphism.}$$

But the left-exactness of Hom and exactness of localization implies that the sequence

$$0 \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), C^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^{n+1}_\mathfrak{p}\right)$$

is exact. Thus Φ is an isomorphism if and only if the map $\operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^{n+1}_\mathfrak{p}\right)$ is the zero map, as desired. ∎

Corollary 3.5. Let R be a Noetherian ring and M an R-module. Let $0 \to M \to E^\bullet$ be *the* minimal injective resolution of M. Then

$$E^j = \bigoplus_{\mathfrak{p}} E_R\left(R/\mathfrak{p}\right)^{a_j(\mathfrak{p})} \qquad \text{and} \qquad a_j(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), M_\mathfrak{p}\right).$$

In particular, if M is a finite R-module, $a_j(\mathfrak{p}) < \infty$ for all $j \geqslant 0$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.

[2]Note that $\mathfrak{p} = \operatorname{Ann}_R(z)$, for if not, then $\varphi = 0$.

Proof. ∎

Definition 3.6. Let R be a Noetherian ring and M a finite R-module. For $j \geqslant 0$ and $\mathfrak{p} \in \operatorname{Spec}(R)$, define the *j-th Bass number* as

$$\mu_j(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}).$$

Remark 3.7. We can now justify the name "minimal injective resolution". In particular, we shall show that the length of the minimal injective resolution is precisely the injective dimension of a module.

Let R be a Noetherian ring and M a finite R-module. Let $0 \to M \to E^\bullet$ be *the* minimal injective resolution in the sense of Remark 3.2. Let $0 \leqslant \ell \leqslant \infty$ denote the length of the resolution. Clearly $\operatorname{inj\,dim}_R M \leqslant \ell$. If $\operatorname{inj\,dim}_R M = \infty$, then $\ell \leqslant \operatorname{inj\,dim}_R M$ so that $\ell = \operatorname{inj\,dim}_R M$.

On the other hand, if $\operatorname{inj\,dim}_R M = n < \infty$, then using this injective resolution to compute the Ext's, we see that for $j > n$, and $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$\operatorname{Ext}^j_R(R/\mathfrak{p}, M) = 0 \implies \mu_j(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) = 0.$$

That is, $E^j = 0$ for all $j > n$ and hence, $\ell \leqslant n$. It follows that $\ell = \operatorname{inj\,dim}_R M$.

Lemma 3.8 (Bass). Let R be a Noetherian ring and M a finite R-module. Let $\mathfrak{p} \subsetneq \mathfrak{q}$ be primes in R such that $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = 1$. If for some $j \geqslant 0$, $\mu_j(\mathfrak{p}, M) \neq 0$, then $\mu_{j+1}(\mathfrak{q}, M) \neq 0$.

Proof. Localizing at $\mathfrak{q}$, we may assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring and $\operatorname{ht}(\mathfrak{m}/\mathfrak{p}) = 1$. If $a \in \mathfrak{m} \setminus \mathfrak{p}$, then $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$, and we have a short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{\cdot a} R/\mathfrak{p} \to R/(\mathfrak{p} + (a)) \to 0.$$

This gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \xrightarrow{\cdot a} \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \to \operatorname{Ext}^{j+1}_R(R/(\mathfrak{p} + (a)), M) \to \cdots,$$

for all $j \geqslant 0$.

$$\mu_j(\mathfrak{p}, M) \neq 0 \implies \operatorname{Ext}^j_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \neq 0 \implies \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \neq 0.$$

Since the Ext's are finite R-modules, Nakayama's lemma implies that $\operatorname{Ext}^{j+1}_R(R/(\mathfrak{p} + (a)), M) \neq 0$.

Since $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$, the R-module $R/(\mathfrak{p} + (a))$ is finite Artinian, so that it has a composition series with successive quotients isomorphic to $R/\mathfrak{m} = k$. Now, if $\operatorname{Ext}^{j+1}_R(k, M) \neq 0$, then through the short exact sequences induced by the composition series, it would follow that $\operatorname{Ext}^{j+1}_R(R/(\mathfrak{p} + (a)), M) = 0$, a contradiction. But since $R \setminus \mathfrak{m}$ consists of only units, we have that

$$0 \neq \operatorname{Ext}^{j+1}_R(k, M) = \operatorname{Ext}^{j+1}_{R_\mathfrak{m}}(\kappa(\mathfrak{m}), M_\mathfrak{m}),$$

and hence $\mu_{j+1}(\mathfrak{m}, M) \neq 0$. ∎

Remark 3.9. Let R be a Noetherian ring and M a finite R-module.

(i) If $\mu_i(\mathfrak{p}, M) \neq 0$, then for all primes $\mathfrak{q} \supseteq \mathfrak{p}$ with $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = h < \infty$, $\mu_{i+h}(\mathfrak{q}, M) \neq 0$.

(ii) Since $\mu_0(\mathfrak{p}, M) \neq 0$ if and only if $\mathfrak{p} \in \operatorname{Ass}_R(M)$, using (i) and Remark 3.7, we conclude that

$$\operatorname{inj\,dim}_R M \geqslant \sup\{\dim R/\mathfrak{p} \colon \mathfrak{p} \in \operatorname{Ass}_R(M)\} = \dim M.$$

(iii) If $(R, \mathfrak{m}, k, E)$ is a Noetherian local ring with $0 \to M \to E^\bullet$ as *the* minimal injective resolution. If $E^n \neq 0$ and $E^j = 0$ for all $j > n$, then we must have that

$$\mu_n(\mathfrak{p}, M) \neq 0 \iff \mathfrak{p} = \mathfrak{m}.$$

In particular, $E^n = E^{\mu_j(\mathfrak{m}, M)}$.

Corollary 3.10. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite R-module. Then

$$\operatorname{inj\,dim}_R M = \infty \iff \mu_j(\mathfrak{m}, M) \neq 0 \text{ for infinitely many } j \geqslant 0.$$

Proof. Let $0 \to M \to E^\bullet$ denote *the* minimal injective resolution. Since $\mu_j(\mathfrak{m}, M) = \dim_k \operatorname{Ext}^j_R(k, M)$, it is clear that if the supremum on the right hand side is infinite, then so is the length of the minimal injective resolution, which is the injective dimension of M.

Conversely, if $\operatorname{inj\,dim}_R M = \infty$, then $E^j \neq 0$ for infinitely many $j \geqslant 0$. We claim that for every integer $N \geqslant 0$, there is a $j \geqslant N$ with $\mu_j(\mathfrak{m}, M) \neq 0$. Indeed, there is an index $i \geqslant N$ with $E^i \neq 0$. Choose $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mu_i(\mathfrak{p}, M) \neq 0$. Using Lemma 3.8, setting $j = i + \operatorname{ht}(\mathfrak{m}/\mathfrak{p})$, we must have that $\mu_j(\mathfrak{m}, M) \neq 0$, as desired. ■

Theorem 3.11. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite R-module. Then

$$\operatorname{inj\,dim}_R M = \sup\left\{j \colon \operatorname{Ext}^j_R(k, M) \neq 0\right\}.$$

Proof. If $\operatorname{inj\,dim}_R M = \infty$, then due to Corollary 3.10, $\operatorname{Ext}^j_R(k, M) \neq 0$ for infinitely may $j \geqslant 0$, so that the supremum on the right hand side is infinite.

Suppose now wthat $\operatorname{inj\,dim}_R M = n < \infty$. Clearly, $\operatorname{Ext}^j_R(k, M) = 0$ for $j > n$ and hence,

$$\sup\left\{j \colon \operatorname{Ext}^j_R(k, M)\right\} \leqslant n = \operatorname{inj\,dim}_R M.$$

Let $0 \to M \to E^\bullet$ denote *the* minimal injective resolution. Due to Remark 3.9 (ii), we know that $\operatorname{Ext}^n_R(k, M) \neq 0$, and hence,

$$\sup\left\{j \colon \operatorname{Ext}^j_R(k, M)\right\} = n = \operatorname{inj\,dim}_R M,$$

as desired. ■

Corollary 3.12. Let $(R, \mathfrak{m}, k)$ be a regular local ring. If M is a finite R-module, then $\operatorname{inj\,dim}_R M < \infty$.

Proof. Since R is regular local, $\operatorname{proj\,dim}_R k < \infty$ and hence for any finite R-module M, $\operatorname{Ext}^j_R(k, M) = 0$ for $j \gg 0$. It follows from Theorem 3.11 that $\operatorname{inj\,dim}_R M < \infty$. ■

Corollary 3.13. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then $\operatorname{inj\,dim}_R k < \infty$ if and only if R is a regular local ring.

Proof. If $\operatorname{inj\,dim}_R k < \infty$, then $\operatorname{Ext}^j_R(k, k) = 0$ for $j \gg 0$. Hence, the Betti numbers $\beta_j(k) = \dim_k \operatorname{Ext}^j_R(k, k) = 0$ for $j \gg 0$, whence $\operatorname{proj\,dim}_R k < \infty$, that is, R is a regular local ring.

Conversely, if R is a regular local ring, then $\operatorname{proj\,dim}_R k < \infty$, so that $\operatorname{Ext}^j_R(k, k) = 0$ for $j \gg 0$, consequently, $\operatorname{inj\,dim}_R k < \infty$. ■

§4 Gorenstein Rings

§§ Modules of finite injective dimension

Definition 4.1. A Noetherian local ring $(R, \mathfrak{m}, k)$ is said to be a *Gorenstein local ring* if $\operatorname{inj\,dim}_R R < \infty$.

Proposition 4.2. If $(R, \mathfrak{m}, k)$ is a Gorenstein local ring and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_\mathfrak{p}$ is a Gorenstein local ring.

Proof. Since $\operatorname{inj\,dim}_R R < \infty$, the minimal injective resolution of R is finite, say of length n:

$$0 \to R \to E^0 \to \cdots \to E^n \to 0.$$

Localizing at $\mathfrak{p}$, one obtains a finite injective resolution of $R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module. Thus $R_\mathfrak{p}$ is a Gorenstein local ring. ■

This allows us to make the following

Definition 4.3. A Noetherian ring R is said to be *Gorenstein* if $R_\mathfrak{p}$ is a Gorenstein local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Due to Proposition 4.2, every Gorenstein local ring is a Gorenstein ring.

Proposition 4.4. A regular ring is Gorenstein.

Proof. It suffices to show this in the local case. Let $(R, \mathfrak{m}, k)$ be a regular local ring. Then $\operatorname{gl\,dim} R = \operatorname{proj\,dim}_R k < \infty$. This means that $\operatorname{Ext}^j_R(k, M) = 0$ for $j \gg 0$; which due to Theorem 3.11 implies $\operatorname{inj\,dim}_R M < \infty$ for each finite R-module M. In particular, $\operatorname{inj\,dim}_R R < \infty$, whence R is a Gorenstein local ring, as desired. ■

Remark 4.5. Note that if R is a Noetherian ring such that $\operatorname{inj\,dim}_R R < \infty$, then

$$\operatorname{inj\,dim}_{R_\mathfrak{p}} R_\mathfrak{p} \leqslant \operatorname{inj\,dim}_R R < \infty,$$

so that R is a Gorenstein ring. What about the converse?

Theorem 4.6 (Ischebeck's Formula). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M, N be finite R-modules. If $\operatorname{inj\,dim}_R N < \infty$, then

$$\operatorname{inj\,dim}_R N = \operatorname{depth} M + \sup\left\{i \colon \operatorname{Ext}^i_R(M, N) \neq 0\right\}.$$

Proof. Due to Theorem 3.11, we know that Ischebeck's formula is true for $M = k$. Next, we prove this by induction on $\operatorname{depth} M$.

Suppose first that $\operatorname{depth} M = 0$. Then $\mathfrak{m} \in \operatorname{Ass}_R(M)$, and hence there is a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow C \longrightarrow 0.$$

Let $t = \operatorname{inj\,dim}_R N$ and consider the long exact sequence induced:

$$\cdots \to \operatorname{Ext}^t_R(C, N) \to \operatorname{Ext}^t_R(M, N) \to \operatorname{Ext}^t_R(k, N) \to \operatorname{Ext}^{t+1}_R(C, N) = 0.$$

Due to Theorem 3.11, $\operatorname{Ext}^t_R(k, N) \neq 0$, and hence $\operatorname{Ext}^t_R(M, N) \neq 0$ since it surjects onto the former. It follows that $\sup\left\{i \colon \operatorname{Ext}^i_R(M, N) \neq 0\right\} = t = \operatorname{inj\,dim}_R N$. This shows that Ischebeck's formula holds when $\operatorname{depth} M = 0$.

Suppose now that $\operatorname{depth} M > 0$. Let $a \in \mathfrak{m}$ be a non-zerodivisor on M; this gives a short exact sequence

$$0 \to M \xrightarrow{\cdot a} M \to \overline{M} \to 0,$$

where $\overline{M} = M/aM$. Set $t = \operatorname{inj\,dim}_R N$ and $d = \operatorname{depth} M > 0$. Then $\operatorname{depth} \overline{M} = d - 1$. The induction hypothesis gives

$$\sup\left\{i \colon \operatorname{Ext}^i_R(\overline{M}, N) \neq 0\right\} = t - d + 1.$$

The short exact sequence above gives a long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(M, N) \xrightarrow{\cdot a} \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^{i+1}_R(\overline{M}, N) \to \operatorname{Ext}^{i+1}_R(M, N) \to \cdots.$$

If $i > t - d$, then $\operatorname{Ext}^{i+1}_R(\overline{M}, N) = 0$, and due to Nakayama's lemma, $\operatorname{Ext}^i_R(M, N) = 0$. On the other hand, for $i = t - d$, $\operatorname{Ext}^{i+1}_R(\overline{M}, N) \neq 0$ but $\operatorname{Ext}^{i+1}_R(M, N) = 0$. Thus $\operatorname{Ext}^i_R(M, N)$ surjects onto a non-zero module, whence it must be non-zero too. We have shown

$$\sup\left\{i \colon \operatorname{Ext}^i_R(M, N) \neq 0\right\} = t - d = \operatorname{inj\,dim}_R N - \operatorname{depth} M,$$

as desired. ■

Corollary 4.7. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite R-module. If $\operatorname{inj\,dim}_R M < \infty$, then $\operatorname{inj\,dim}_R M = \operatorname{depth} R$.

Proof. Using Ischebeck's formula,

$$\operatorname{inj\,dim}_R M = \operatorname{depth} R + \sup\left\{i \colon \operatorname{Ext}^i_R(R, M) \neq 0\right\} = \operatorname{depth} R,$$

as desired. ■

Corollary 4.8. A Gorenstein ring is Cohen-Macaulay.

Proof. It suffices to prove this in the local case $(R, \mathfrak{m}, k)$. Due to Corollary 4.7, $\operatorname{inj}\dim_R R = \operatorname{depth} R$. But due to Remark 3.9 (ii), $\operatorname{inj}\dim_R R \geqslant \dim R$. It follows that $\operatorname{depth} R = \dim R$ and hence R is Cohen-Macaulay. ∎

Corollary 4.9. A Gorenstein Artinian local ring is self-injective.

Proof. Due to Corollary 4.7, $\operatorname{inj}\dim_R R = \operatorname{depth} R = 0$. ∎

Definition 4.10. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, M a Cohen-Macaulay R-module, and $\underline{a} = a_1, \dots, a_s \in \mathfrak{m}$ a maximal M-sequence. Then $M/\underline{a}M$ is Artinian and we can define

$$\operatorname{type}(M) = \dim_k \left(\operatorname{Soc}_R \left(M/\underline{a}M\right)\right).$$

We must argue that this defintion is independent of the chosen maximal M-sequence. We begin with a

Lemma 4.11. Let R be a ring, M and N be R-modules, and $a \in \operatorname{Ann}_R(M)$ be a non-zerodivisor on N. Then

$$\operatorname{Ext}_R^{j+1}(M, N) \cong \operatorname{Ext}_R^{j}(M, N/aN) \qquad \forall\, j \geqslant 0.$$

Proof. Consider the short exact sequence

$$0 \to N \xrightarrow{\cdot a} N \to N/aN \to 0.$$

This gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}_R^{j}(M, N) \xrightarrow{\cdot a} \operatorname{Ext}_R^{j}(M, N) \to \operatorname{Ext}_R^{j}(M, N/aN) \to \operatorname{Ext}_R^{j+1}(M, N) \xrightarrow{\cdot a} \operatorname{Ext}_R^{j+1}(M, N) \to \cdots .$$

Since a annihilates M, both the above "multiplication by a" maps have zero image. In particular, this gives an exact sequence

$$0 \to \operatorname{Ext}_R^{j}(M, N/aN) \to \operatorname{Ext}_R^{j+1}(M, N) \to 0,$$

as desired. ∎

We return to the setup of Definition 4.10. Using the above Lemma, we have

$$\operatorname{Soc}_R(M/\underline{a}M) \cong \operatorname{Hom}_R(R/\mathfrak{m}, M/\underline{a}M) \cong \operatorname{Ext}_R^{0}(k, M) \cong \operatorname{Ext}_R^{s}(k, M).$$

This characterization is independent of the maximal regular sequence, as desired.

Interlude 4.12 (Constructing the minimal injective resolution of M/aM over R/aR).
Let R be a Noetherian ring, M a finite R-module, and $a \in R$ a non-zerodivisor on both M and R. Let $0 \to M \to E^\bullet$ be the minimal injective resolution of M over R. Set $\overline{R} = R/aR$ and $\overline{M} = M/aM$. Consider the short exact sequence

$$0 \to R \xrightarrow{\cdot a} R \to \overline{R} \to 0$$

of R-modules. Then $\operatorname{proj}\dim_R \overline{R} \leqslant 1$ so that $\operatorname{Ext}_R^{j}(\overline{R}, M) = 0$ for all $j > 1$. The above sequence also gives

$$0 \to \operatorname{Hom}_R(\overline{R}, M) \to \operatorname{Hom}_R(R, M) \xrightarrow{\cdot a} \operatorname{Hom}_R(R, M) \to \operatorname{Ext}_R^{1}(\overline{R}, M) \to 0.$$

It follows that $\operatorname{Ext}_R^{1}(\overline{R}, M) \cong \overline{M}$.

Now, consider the complex

$$0 \to \underbrace{\operatorname{Hom}_R(\overline{R}, M)}_{=0} \to \operatorname{Hom}_R(\overline{R}, E^0) \to \operatorname{Hom}_R(\overline{R}, E^1) \to \operatorname{Hom}_R(\overline{R}, E^2) \to \cdots .$$

Since $\operatorname{Ext}_R^{j}(\overline{R}, M) = 0$ for $j \geqslant 2$, the above complex is exact at $\operatorname{Hom}_R(\overline{R}, E^j)$ for $j \geqslant 2$. Further, since $\operatorname{Ass}_R(M) = \operatorname{Ass}_R(E^0)$, it follows that a is a non-zerodivisor on E^0, so that $\operatorname{Hom}_R(\overline{R}, E^0) = 0$. Therefore,

$$\ker\left(\operatorname{Hom}_R(\overline{R}, E^1) \to \operatorname{Hom}_R(\overline{R}, E^2)\right) \cong \operatorname{Ext}_R^{1}(\overline{R}, M) \cong \overline{M}.$$

Set $I^j = \operatorname{Hom}_R(\overline{R}, E^j)$. Then I^j is an injective $\overline{R}$-module and

$$0 \to \overline{M} \to I^1 \to I^2 \to \cdots$$

is an injective resolution of $\overline{M}$ over $\overline{R}$.

Finally, we claim that the above resolution is the minimal resolution of $\overline{M}$ over $\overline{R}$. Let $\overline{\mathfrak{p}}$ be a prime in $\overline{R}$. We must show that the map

$$\operatorname{Hom}_{\overline{R}_{\overline{\mathfrak{p}}}}\left(\kappa(\overline{\mathfrak{p}}), I^j_{\overline{\mathfrak{p}}}\right) \longrightarrow \operatorname{Hom}_{\overline{R}_{\overline{\mathfrak{p}}}}\left(\kappa(\overline{\mathfrak{p}}), I^{j+1}_{\overline{\mathfrak{p}}}\right)$$

is the zero map. But note that the above is the localization of the map

$$\operatorname{Hom}_{\overline{R}}\left(\overline{R}/\overline{\mathfrak{p}}, I^j\right) \longrightarrow \operatorname{Hom}_{\overline{R}}\left(\overline{R}/\overline{\mathfrak{p}}, I^{j+1}\right),$$

which, due to the Hom-Tensor adjunction is canonically isomorphic to

$$\operatorname{Hom}_R\left(\overline{R}/\overline{\mathfrak{p}} \otimes_{\overline{R}} \overline{R}, I^j\right) \longrightarrow \operatorname{Hom}_R\left(\overline{R}/\overline{\mathfrak{p}} \otimes_{\overline{R}} \overline{R}, I^{j+1}\right).$$

Finally, since $\overline{R}/\overline{\mathfrak{p}}$ is the same as $R/\mathfrak{p}$ as R-modules, the above map is the same as

$$\operatorname{Hom}_R\left(R/\mathfrak{p}, E^j\right) \to \operatorname{Hom}_R\left(R/\mathfrak{p}, E^{j+1}\right).$$

But it is known that this map is identically zero when localized at $\mathfrak{p}$, as desired.

Theorem 4.13. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring, M a finite R-module, and $a \in R$ a non-zerodivisor on both M and R. Set $\overline{M} = M/aM$ and $\overline{R} = R/aR$. Then

$$\operatorname{inj\,dim}_R M < \infty \iff \operatorname{inj\,dim}_{\overline{R}} \overline{M} < \infty.$$

In this case, $\operatorname{inj\,dim}_R M = \operatorname{inj\,dim}_{\overline{R}} \overline{M} + 1$.

Proof. Suppose first that $\operatorname{inj\,dim}_R M < \infty$. It is clear from Interlude 4.12 that $\operatorname{inj\,dim}_{\overline{R}} \overline{M} < \infty$ and $\operatorname{inj\,dim}_{\overline{R}} \overline{M} = \operatorname{inj\,dim}_R M - 1$.

On the other hand, if $\operatorname{inj\,dim}_R M = \infty$, then $\mu_j(\mathfrak{m}, M) \neq 0$ for infinitely many $j \geqslant 0$. But recall that $\operatorname{Hom}_R(\overline{R}, E) = E_{\overline{R}}(k)$. Hence, if $E \mid E^j$ for some $j \geqslant 0$, then $E_{\overline{R}}(k) \mid I^j$. That is, for $j \geqslant 1$,

$$\mu_j(\mathfrak{m}, M) \neq 0 \implies \mu_{j-1}(\overline{\mathfrak{m}}, \overline{M}) \neq 0.$$

Hence, $\operatorname{inj\,dim}_{\overline{R}} \overline{M} = \infty$. This completes the proof. ∎

Corollary 4.14. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $a \in R$ a non-zerodivisor. Then R is Gorenstein if and only if R/aR is Gorenstein. ∎

Proposition 4.15. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then the following are equivalent:

(1) R is Gorenstein

(2) R is Cohen-Macaulay and $\operatorname{type}(R) = 1$.

Proof. Let $\underline{a} = a_1, \dots, a_s \in \mathfrak{m}$ be a maximal R-sequence. Due to Corollary 4.14, it suffices to show the equivalence for $R/\underline{a}R$. So R is a depth zero Noetherian local ring. Clearly if R is Gorenstein, then it is self-injective and hence $\operatorname{type}(R) = 1$. Conversely, if R is Cohen-Macaulay, then $\dim R = \operatorname{depth} R = 0$, so that R is an Artinian local ring with $\operatorname{type}(R) = 1$, whence R is self-injective, in particular, Gorenstein. ∎

§§ A closer look at the Artinian case

Theorem 4.16. Let $(R, \mathfrak{m}, k, E)$ be an Artinian local ring. Then the following are equivalent:

(1) $\mathrm{idim}_R(R) < \infty$,

(2) R is self-injective,

(3) $R \cong E$ as R-modules,

(4) The ideal $(0) \lhd R$ is irreducible,

(5) $\dim_k (\mathrm{Soc}_R(R)) = 1$,

(6) for all ideals $I \subseteq R$, $(0 :_R (0 :_R I)) = I$.

Proof. (3) $\implies$ (2) $\implies$ (1) is clear. The implication (1) $\implies$ (3) follows from Corollary 4.7 so that R is self-injective, and hence, due to Theorem 2.10, $R \cong E$ as R-modules.

(3) $\implies$ (6) is a consequence of Theorem 2.11.

(6) $\implies$ (5) If $0 \neq a \in \mathrm{Soc}_R(R)$, then $\mathrm{Ann}_R(a) = \mathfrak{m}$. As a result,

$$\mathrm{Soc}_R(R) = (0 :_R \mathfrak{m}) = (0 :_R (0 :_R a)) = (a),$$

whence $\dim_k (\mathrm{Soc}_R(R)) = 1$.

(5) $\implies$ (3) is again a consequence of Theorem 2.10.

(5) $\implies$ (4) If $0 \neq I$ is any ideal of R, then $I \cap \mathrm{Soc}_R(R) \neq 0$, and hence, $\mathrm{Soc}_R(R) \subseteq I$, since the former is a simple R-module. In particular, this means that the intersection of two non-trivial ideals of R must contain the socle, and hence, must be non-zero; i.e., (0) is an irreducible ideal.

(4) $\implies$ (5) If $\dim_k (\mathrm{Soc}_R(R)) \neq 1$, then $\dim_k (\mathrm{Soc}_R(R)) \geqslant 2$[3]. Let $a, b \in \mathrm{Soc}_R(R)$ be linearly independent over k. Then $(a) = ka$ and $(b) = kb$. Thus $(a) \cap (b) = (0)$, i.e., (0) is not an irreducible ideal, a contradiction. ■

Proposition 4.17. The following are equivalent to the (equivalent) conditions of Theorem 4.16:

(7) R has a unique minimal non-zero ideal,

(8) $\mathrm{proj\,dim}_R E < \infty$,

(9) E is free,

(10) E is cyclic,

(11) Given any submodule $W \subseteq E$, $(0 :_R (0 :_R W)) \cong W$,

(12) E has a unique maximal proper submodule.

Proof. (7) $\implies$ (4) Let $\mathfrak{a} \trianglelefteq R$ be the unique non-zero minimal ideal. Let $I \trianglelefteq R$ be a non-zero ideal. Since R is Artinian, I contains a minimal non-zero ideal, say $\mathfrak{b}$, which, due to uniqueness, must be equal to $\mathfrak{a}$. Hence, every non-zero ideal of R contains $\mathfrak{a}$. It follows that (0) is an irreducible ideal.

(5) $\implies$ (7) It is clear that $\mathrm{Soc}_R(R)$ is a minimal ideal. Further, since $\mathrm{Soc}_R(R) \subseteq R$ is essential, the socle must be contained in every non-zero ideal as was argued in the preceding proof.

(8) $\implies$ (1) follows by taking a finite projective (and hence free) resolution of E and then taking its Matlis dual, which gives a finite injective resolution of R.

(1) $\implies$ (8) follows similarly by taking a finite injective resolution of R and then taking its Matlis dual.

(9) $\implies$ (3) Suppose $E \cong R^{\oplus d}$. Then

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = d\lambda_R(R) \implies d = 1.$$

Thus $R \cong E$. The implication (3) $\implies$ (9) is clear.

(10) $\implies$ (3) If $E \cong R/I$, then

$$\lambda_R(R) - \lambda_R(I) = \lambda_R(E) = \lambda_R(R^\vee) = \lambda(R),$$

whence $\lambda_R(I) = 0$, i.e., $I = 0$. Thus $R \cong E$. The converse (3) $\implies$ (10) is once again clear.

Through Theorem 2.11, the equivalence (7) $\iff$ (12) is clear, thereby completing the proof. ■

[3]Since $\mathfrak{m} \in \mathrm{Ass}_R(R)$.

§§ Fibres of a flat map

Theorem 4.18. Let $\varphi : (R, \mathfrak{m}_R, k) \to (S, \mathfrak{m}_S, \ell)$ be a flat map with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. Then

(1) $\dim R + \dim S/\mathfrak{m}_R S = \dim S$.

(2) if $\mathfrak{m}_R S = \mathfrak{m}_S$, then for any R-module M of finite length, $\lambda_R(M) = \lambda_S(M \otimes_R S)$.

(3) if $\underline{a} = a_1, \dots, a_n \in \mathfrak{m}_S$ is $S/\mathfrak{m}_R S$-regular, then $a_1, \dots, a_n$ is S-regular and $R \to S/(\underline{a})$ is flat.

(4) $\operatorname{depth} R + \operatorname{depth} S/\mathfrak{m}_R S = \operatorname{depth} S$.

(5) S is Cohen-Macaulay if and only if R and $S/\mathfrak{m}_R S$ are so.

(6) S is Gorenstein if and only if R and $S/\mathfrak{m}_R S$ are so.

Proof. (1) Induct on $\dim R$. If R is Artinian, then $\mathfrak{m}_R S$ is nilpotent and hence $\mathfrak{m}_R S \subseteq \mathfrak{N}(S)$, so that $\dim S/\mathfrak{m}_R S = \dim S$. This prove the assertion when $\dim R = 0$.

Suppose now that $\dim R > 0$ and let $\mathfrak{N} = \mathfrak{N}(R)$. The map $R/\mathfrak{N} \to S/\mathfrak{N}S$ is flat and $\mathfrak{N}S \subseteq \mathfrak{N}(S)$, so that

$$\dim R = \dim R/\mathfrak{N} \quad \text{and} \quad \dim S = \dim S/\mathfrak{N}S.$$

Replacing R and S by $R/\mathfrak{N}$ and $S/\mathfrak{N}S$ respectively, we may assume that R is reduced. In particular, this means that the zero ideal in R is the intersection of all its (finitely many) minimal primes. Since $\dim R > 0$, the maximal ideal is not minimal and using prime avoidance, choose a non-zerodivisor $a \in \mathfrak{m}_R$. The map $R/aR \to S/aS$ is flat, $\dim R/aR = \dim R - 1$ and the fibre of this map is still $S/\mathfrak{m}_R S$. The induction hypothesis implies

$$\dim S = \dim S/aS + 1 = \dim R/aR + \dim S/\mathfrak{m}_R S + 1 = \dim R + \dim S/\mathfrak{m}_R S.$$

(2) Let $n = \lambda_R(M)$. If $n = 0$, then $M = 0$, and there is nothing to prove. Suppose $n > 0$. Then, there is a composition series

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0,$$

giving rise to short exact sequences

$$0 \to M_{i+1} \to M_i \to k \to 0 \qquad 0 \leqslant i \leqslant n-1.$$

Applying the functor $- \otimes_R S$ and using the fact that $\mathfrak{m}_R S = \mathfrak{m}_S$ so that $k \otimes_R S = \ell$ (as S-modules), we obtain

$$0 \to M_{i+1} \otimes_R S \to M_i \otimes_R S \to \ell \to 0,$$

therefore, $\lambda_S(M \otimes_R S) = \lambda_R(M)$.

(3) It is clear that it suffices to prove the assertion for $n = 1$. Let $a \in \mathfrak{m}_S$ be $S/\mathfrak{m}_R S$-regular. We must show that a is S-regular and $R \to S/aS$ is flat.

Set $d_n = \dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$. There are short exact sequences

$$0 \to \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n \to 0.$$

Applying $- \otimes_R S$, we obtain short exact sequences

$$0 \to (S/\mathfrak{m}_R S)^{\oplus d_n} \to S/\mathfrak{m}_R^{n+1} S \to S/\mathfrak{m}_R^n S \to 0$$

of S-modules. Inducting on n, it is easy to show that a is a non-zerodivisor on $S/\mathfrak{m}_R^n S$ for all $n \geqslant 1$[4]. Suppose a is a zerodivisor on S, then there exists $0 \neq s \in S$ such that $as = 0$. By Krull's Intersection Theorem,

$$\bigcap_{n \geqslant 1} \mathfrak{m}_R^n S \subseteq \bigcap_{n \geqslant 1} \mathfrak{m}_S^n = 0,$$

[4] If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules and a is a non-zerodivisor on M' and M'', then it is a non-zerodivisor on M. This follows from the fact that $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(M') \cup \operatorname{Ass}_R(M'')$.

consequently, there is some $n \geqslant 1$ such that $s \notin \mathfrak{m}_R^n S$. In particular, $0 \neq \overline{s} \in S/\mathfrak{m}_R^n S$. But since a is a non-zerodivisor on S, we cannot have $a\overline{s} = 0$, a contradiction. Thus a is a non-zerodivisor on S.

It remains to show that $R \to S/aS$ is flat, i.e., we must show that

$$\operatorname{Tor}_1^R(M, S/aS) = 0 \quad \text{for all } R\text{-modules } M,$$

equivalently (using standard reduction techniques), since R is Noetherian, it suffices to show that

$$\operatorname{Tor}_1^R(R/\mathfrak{p}, S/aS) = 0 \quad \text{for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Set $\overline{R} = R/\mathfrak{p}$ and $\overline{S} = S/\mathfrak{p}S$ and note that $\overline{R} \to \overline{S}$ is flat and a morphism of local rings, further $\overline{S}/\mathfrak{m}_{\overline{R}}\overline{S} \cong S/\mathfrak{m}_R S$ as rings and S-modules. Now, a is a non-zerodivisor on $S/\mathfrak{m}_R S$, and hence is a non-zerodivisor on $\overline{S}/\mathfrak{m}_{\overline{R}}\overline{S}$. Thus, $\overline{s} \in \overline{S}$ is a non-zerodivisor on $\overline{S}/\mathfrak{m}_{\overline{R}}\overline{S}$ and because of what we have argued in the preceding paragraph, $\overline{a} \in \overline{S}$ is a non-zerodivisor.

Consider the short exact sequence

$$0 \to S \xrightarrow{\cdot a} S \to S/aS \to 0,$$

and applying $- \otimes_R R/\mathfrak{p}$, we obtain

$$0 = \operatorname{Tor}_1^R(S, R/\mathfrak{p}) \to \operatorname{Tor}_1^R(S/aS, R/\mathfrak{p}) \to \overline{S} \xrightarrow{\cdot a} \overline{S} \to S/aS \otimes_R R/\mathfrak{p} \to 0.$$

Thus $\operatorname{Tor}_1^R(S/aS, R/\mathfrak{p}) = 0$, as desired. This completes the proof of (3).

(4) Let $\underline{a} = a_1, \dots, a_s \in \mathfrak{m}_R$ be a maximal R-sequence. Since $R \to S$ is flat, $\underline{a}$ is an S-sequence, therefore, replacing R and S by $R/\underline{a}R$ and $S/\underline{a}S$, we may assume $\operatorname{depth}_R R = 0$. Note that the fibre of the map does not change during this reduction. Now, let $\underline{b} = b_1, \dots, b_r \in S$ be $S/\mathfrak{m}_R S$-regular. Using (3), we know that $\underline{b}$ is S-regular and the map $R \to S/\underline{b}S$ is flat. Let $\overline{S} = S/\underline{b}S$, then

$$\operatorname{depth} S - \operatorname{depth} S/\mathfrak{m}_R S = \operatorname{depth} \overline{S} - \operatorname{depth} \overline{S}/\mathfrak{m}_R \overline{S},$$

and hence, we may replace S by $S/\underline{b}S$ and assume that $\operatorname{depth}_R R = \operatorname{depth}_S S/\mathfrak{m}_R S = 0$. It remains to show that $\operatorname{depth}_S S = 0$ in this situation.

Since $\operatorname{depth}_R R = 0$, there is an injection $S/\mathfrak{m}_R \hookrightarrow R$, which upon tensoring with S and using flatness, gives an injection $S/\mathfrak{m}_R S \hookrightarrow S$ as S-modules. But $\operatorname{depth} S/\mathfrak{m}_R S = 0$ implies there is an injection $S/\mathfrak{m}_S \hookrightarrow S/\mathfrak{m}_R S$, and hence there is an injection $S/\mathfrak{m}_S \hookrightarrow S$. Thus $\mathfrak{m}_S \in \operatorname{Ass}_S(S)$, i.e., $\operatorname{depth}_S S = 0$, as desired.

(5) Immediate from (1) and (4).

(6) In light of (5), we can assume that S, R, and $S/\mathfrak{m}_R S$ are all Cohen-Macaulay. Let $\underline{a} \in R$ be a maximal R-sequence. Replacing R and S by $R/\underline{a}R$ and $S/\underline{a}S$, we may assume that R is Artinian (recall that depth $=$ dim for Cohen-Macaulay rings). Now let $\underline{b} = b_1, \dots, b_s \in \mathfrak{m}_S$ be $S/\mathfrak{m}_R S$-regular. Then due to (3), $\underline{b}$ is S-regular and $R \to S/\underline{b}S$. Again replacing S by $S/\underline{b}S$, we may assume that $S/\mathfrak{m}_R S$ is Artinian too. Due to (1), we conclude that S is Artinian too.

We shall show that

$$\operatorname{type}(S) = \operatorname{type}(R)\operatorname{type}(S/\mathfrak{m}_R S),$$

which implies the desideratum. Let $r = \operatorname{type}(R)$, then

$$k^{\oplus r} = \operatorname{Soc}_R(R) \cong \operatorname{Hom}_R(R/\mathfrak{m}_R/R).$$

Applying $- \otimes_R S$, we obtain[5]

$$(S/\mathfrak{m}_R S)^{\oplus r} \cong \operatorname{Hom}_S(S/\mathfrak{m}_R S, S) \cong (0 :_S \mathfrak{m}_R S).$$

[5]If M is a finitely presented R-module and N is any R-module, then for every flat algebra $R \to S$, there is a canonical isomorphism

$$\operatorname{Hom}_R(M, N) \otimes_R S \cong \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S).$$

Observe that $\mathrm{Soc}_S(S) \subseteq (0 :_S \mathfrak{m}_R S)$, and hence

$$\mathrm{Soc}_S(S) \subseteq \mathrm{Soc}_S\left((0 :_S \mathfrak{m}_R S)\right) \subseteq \mathrm{Soc}_S(S),$$

consequently,

$$\mathrm{Soc}_S\left(S/\mathfrak{m}_R S\right)^{\oplus r} \cong \mathrm{Soc}_S\left(\left(S/\mathfrak{m}_R S\right)^{\oplus r}\right) \cong \mathrm{Soc}_S(S)$$

therefore,

$$r\,\mathrm{type}\left(S/\mathfrak{m}_R S\right) = \mathrm{type}(S),$$

as desired. ■

Corollary 4.19. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then R is Gorenstein if and only if $\widehat{R}$ is so.

Proof. This follows immediately from Theorem 4.18 (6), since $\widehat{R}/\mathfrak{m}\widehat{R} \cong k$, which is always Gorenstein. ■

Corollary 4.20. If R is Gorenstein, then so is $R[X]$.

Proof. Let $\mathfrak{P} \in \mathrm{Spec}\left(R[X]\right)$ and set $\mathfrak{p} = R \cap \mathfrak{P}$. It is clear that $\left(R[X]\right)_{\mathfrak{P}} \cong \left(R_{\mathfrak{p}}[X]\right)_{\mathfrak{P}}$, and hence we can assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring and $\mathfrak{P} \cap R = \mathfrak{m}$. The map $R \to \left(R[X]\right)_{\mathfrak{P}}$ is a flat local homomorphism, since a composition of flat maps is flat. Thus, it suffices to show that

$$\frac{\left(R[X]\right)_{\mathfrak{P}}}{\mathfrak{m}\left(R[X]\right)_{\mathfrak{P}}}$$

is a Gorenstein ring. But the above is isomorphic to

$$\left(R[X]/\mathfrak{m}[X]\right)_{\overline{\mathfrak{P}}} \cong \left(k[X]\right)_{\overline{\mathfrak{P}}},$$

which is obviously Gorenstein. This completes the proof. ■

§§ Canonical Module

Lemma 4.21 (Depth Lemma). Let R be a Noetherian ring, $I \trianglelefteq R$ a proper ideal and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of finite R-modules such that $IM \neq M' \neq M'$, $IM \neq M$, and $IM'' \neq M''$. Let $m = \mathrm{depth}(I, M)$, $m' = \mathrm{depth}(I, M')$, and $m'' = \mathrm{depth}(I, M'')$.

(1) $m < m'' \implies m' = m$.

(2) $m > m'' \implies m' = m'' + 1$.

(3) $m = m'' \implies m' \geqslant m$.

Proof. Let T^i denote the functor $\mathrm{Ext}^i_R(R/I, -)$. We shall use Rees's characterization of depth as

$$\mathrm{depth}(I, N) = \sup\left\{i \colon \mathrm{Ext}^i_R(R/I, N) \neq 0\right\},$$

where N is a finite R-module. The given short exact sequence gives rise to a long exact sequence

$$\cdots \to T^{i-1}(M'') \to T^i(M') \to T^i(M) \to T^i(M'') \to T^{i+1}(M') \to \cdots,$$

with the convention that $T^i = 0$ for $i < 0$.

(1) Let $i < m$, then a part of the long exact sequence looks like $T^{i-1}(M'') \to T^i(M') \to T^i(M)$ where both $T^{i-1}(M'')$ and $T^i(M)$ are zero, so that $T^i(M') = 0$. If $i = m$, then we have an exact sequence

$$0 = T^{m-1}(M'') \to T^m(M') \to T^m(M) \to T^m(M'') = 0,$$

whence $T^m(M') \cong T^m(M) \neq 0$, i.e., $m' = m$.

(2) Let $i \leqslant m'' < m$. Then the exact sequence $0 = T^{i-1}(M'') \to T^i(M') \to T^i(M) = 0$ gives $T^i(M') = 0$. On the other hand, there is an exact sequence $0 = T^{m''}(M) \to T^{m''}(M'') \to T^{m''+1}(M')$ and since $T^{m''}(M'') \neq 0<$ we must have $T^{m''+1}(M') \neq 0$, i.e., $m' = m'' + 1$.

(3) Let $i < m = m''$, then the exact sequence $0 T^{i-1}(M'') \to T^i(M') \to T^i(M) = 0$ gives $T^i(M') = 0$, i.e., $m' \geqslant m$, thereby completing the proof ■

Theorem 4.22. Let $(S, \mathfrak{m}, k)$ be a Gorenstein local ring with $d = \dim S$. Let $\mathrm{CM}_S(i)$ denote the class of Cohen-Macaulay S-modules of dimension i. Then for $M \in \mathrm{CM}_S(i)$,

(1) $\mathrm{Ext}^j_S(M, S) = 0$ for $j \neq d - i$.

(2) $\mathrm{Ext}^{d-i}_S(M, S) \in \mathrm{CM}_S(i)$.

(3) $\mathrm{Ext}^{d-i}_S \left(\mathrm{Ext}^{d-i}_S(M, S), S\right) \cong M$ as S-modules.

Proof. Since $M \in \mathrm{CM}_S(i)$,

$$i = \dim\left(S/\operatorname{Ann}_S(M)\right) = \dim S - \operatorname{ht}\operatorname{Ann}_S(M) \implies \operatorname{ht}(\operatorname{Ann}_S(M)) = d - i,$$

where we used the fact that Gorenstein rings are Cohen-Macaulay. Again, since S is Cohen-Macaulay, there is an S-regular sequence $a_1, \dots, a_{d-i} \in \operatorname{Ann}_S(M)$. A standard "Ext-shifting" argument then gives that for $j < d - i$,

$$\mathrm{Ext}^j_S(M, S) \cong \operatorname{Hom}_S\left(M, S/(a_1, \dots, a_j)S\right) = 0,$$

since a_{j+1} is a non-zerodivisor on $S/(a_1, \dots, a_j)S$ but annihilates M. Further, we also have that

$$\mathrm{Ext}^{d-i}_S(M, S) \cong \operatorname{Hom}_{\overline{S}}(M, \overline{S}),$$

where $\overline{S} = S/(a_1, \dots, a_{d-i})S$.

Now, since $\operatorname{inj}\dim_S S = d < \infty$, using Theorem 4.6, we have

$$\operatorname{depth}_S M + \sup\left\{j \colon \mathrm{Ext}^j_S(M, S) \neq 0\right\} = d,$$

so that $\mathrm{Ext}^j_S(M, S) = 0$ for $j > d - i$. This completes the proof of (1).

Note that $\overline{S}$ is Gorenstein and M is a maximal Cohen-Macaulay $\overline{S}$-module. Replacing S by $\overline{S}$, we can assume that M is a maximal Cohen-Macaulay S-module, and we would like to show that

(2) $M^* := \operatorname{Hom}_S(M, S) \in \mathrm{CM}_S(d)$, and

(3) $M^{**} \cong M$ as S-modules.

If M is free, then both the above conclusions are trivial. Suppose now that M is not free; then we must have $\operatorname{proj}\dim_S M = \infty$, else, due to the Auslander-Buchsbaum Formula, $\operatorname{proj}\dim_S M = \operatorname{depth} S - \operatorname{depth} M = 0$, a contradiction to our assumption that M is not free.

Let $\cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$ be a free resolution of M and set $N = \ker \varphi_1$. This gives a short exact sequence

$$0 \to N \to F_0 \to M \to 0.$$

Due to Lemma 4.21, since $\operatorname{depth}_S M = \operatorname{depth}_S F_0$,

$$\operatorname{depth}_S N \geqslant \operatorname{depth}_S M = \dim S,$$

so that N is a maximal Cohen-Macaulay S-module. Due to (1), $\mathrm{Ext}^j_S(M, S) = 0$ for $j > 0$ and we get an exact complex

$$0 \to M^* \to F_0^* \xrightarrow{\varphi_1^*} F_1^* \to \cdots.$$

Breaking up the above long exact sequence into short exact sequences and making repeated use of Lemma 4.21, it is clear that $\operatorname{depth}_S M^* = \dim S$, and hence, M^* is a maximal Cohen-Macaulay S-module. ■