# Galois Categories and the Étale Fundamental Group

or, what should be taught in MA 811

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Let $k$ be a field and fix its separable and algebraic closures $k_s \subseteq \overline{k}$. Let $G_k := \mathrm{Gal}(k_s \mid k)$, which is a profinite group through the isomorphism:

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If $L$ is a finite separable extension of $k$, then there is a natural action of $G_k$ on $\mathrm{Hom}_k(L, k_s)$ given by

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The stabilizer of $\varphi \in \mathrm{Hom}_k(L, k_s)$ is $\mathrm{Gal}(k_s \mid \varphi(L))$, i.e., an open subgroup of $G_k$, whence $\mathrm{Hom}_k(L, k_s)$ is a continuous $G_k$-set.

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**Definition**

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# Étale Algebras and the Fundamental Theorem

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**Theorem (Fundamental Theorem of Galois Theory)**

*The functor mapping a finite étale $k$-algebra $A$ to the finite $G_k$-set $\mathrm{Hom}_k(A, k_s)$ gives an anti-equivalence between the category of finite étale $k$-algebras and the category of finte sets with a continuous $G_k$-action. Here separable extensions correspond to transitive $G_k$-sets.*

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## Galois Theory of Covering Spaces

Let $X$ be a path connected, locally path connected, semilocally simply connected topological space and fix a basepoint $x_0 \in X$. If $p : Y \to X$ is a finite-sheeted covering space, there is a natural action of $\pi_1(X, x_0)$ on the (finite) fibre $p^{-1}(x_0)$, known as the *monodromy action*.

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**Theorem (Classification of Covering Spaces)**

*The aforementioned fibre functor induces an equivalence between the category of finite-sheeted covers of $X$ and the category of continuous finite sets with a continuous $\widehat{\pi_1(X, x_0)}$-action. Here connected covers correspond to transitive $\widehat{\pi_1(X, x_0)}$-sets.*

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All that remains is to establish an equivalence of some suitable categories.

A *Galois category* is a pair $(\mathscr{C}, F)$ where $\mathscr{C}$ is a category and $F : \mathscr{C} \to \mathbf{Sets}$ is a functor with $F(a)$ a finite set for each object $a \in \mathscr{C}$, satisfying the following axioms:

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- G3 Every morphism $u$ in $\mathscr{C}$ factors as $u = u'u''$ where $u'$ is a monomorphism and $u''$ is an epimorphism. Every monomorphism $f : X \to Y$ in $\mathscr{C}$ is an isomorphism of $X$ with a direct summand of $Y$.

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In this case, the functor $F$ is called a *fundamental functor*.

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- Let $X$ be a connected scheme. Let $\textbf{FEt}_X$ denote the category of finite étale maps $Y \to X$ with morphisms $f : Y \to Z$ making

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

commute. Fix a geometric point $x_0 : \operatorname{Spec} \Omega \to X$. This defines a fibre functor $\mathrm{Fib}_{x_0} : \textbf{FEt}_X \to \textbf{Sets}$. The pair $(\textbf{FEt}_X, \mathrm{Fib}_{x_0})$ forms a Galois category.

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Further the tuple must be such that for each morphism $f : X \to Y$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

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$$C_f = \bigcup \prod_{\substack{X\in\mathscr{C} \\ X\neq Y,Z}} \mathfrak{S}_{F(X)} \times \{\eta_Y\} \times \{\eta_Z\},$$

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the conclusion follows.

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$$\mathrm{Aut}_{\mathscr{C}}(F) = \bigcap_{\substack{Y \xrightarrow{f} Z\\ \text{in } \mathscr{C}}} C_f,$$

the conclusion follows. In particular, $\mathrm{Aut}_{\mathscr{C}}(F)$ is a profinite group.

Next, let $Y \in \mathscr{C}$. There is a natural action of $\mathrm{Aut}_{\mathscr{C}}(F)$ on $F(Y)$ given by

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which is an open subgroup of $\mathrm{Aut}_{\mathscr{C}}(F)$. Thus, each $F(Y)$ is a finite $\mathrm{Aut}_{\mathscr{C}}(F)$-set with a continuous action.
This gives a natural functor $H : \mathscr{C} \to \mathrm{Aut}_{\mathscr{C}}(F)$-**sets**.

**Theorem (Fundamental Theorem of Galois Categories)**

*Let $(\mathscr{C}, F)$ be an essentially small Galois category. Then*

1. *The functor $H : \mathscr{C} \to \mathrm{Aut}_{\mathscr{C}}(F)$-**sets** is an equivalence of categories.*

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**Definition**

Let $X$ be a connected scheme and $x_0 : \operatorname{Spec}\Omega \to X$ be a geometric point. The *étale fundamental group* $\pi_1^{\acute{e}t}(X, x_0)$ to be $\operatorname{Aut}(\operatorname{Fib}_{x_0})$.

Recall that there is an anti-equivalence between the category of commutative rings and the category of affine schemes.

Therefore, there is an anti-equivalence between the category $\mathbf{FEt}_{\operatorname{Spec} k}$ and the category of étale $k$-algebras.

Choosing a geometric point in $\operatorname{Spec} k$ is tantamount to fixing a separable closure $k_s$ of $k$.

As we have seen earlier, $\mathbf{FEt}_{\operatorname{Spec} k}$ is equivalent to the category $G_k$-**sets**. In particular, $\pi_1^{\acute{e}t}(\operatorname{Spec} k, x_0) \cong \operatorname{Gal}(k_s \mid k)$.

**Thank you for your attention!**