MA 534: HOMEWORK 1

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Throughout this article, we fix a sequence of mollifiers $\rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ for $\varepsilon > 0$ given by

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right) \qquad \forall\, x \in \mathbb{R}^n,$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\rho(x) = \begin{cases} C\exp\left(-\frac{1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geqslant 1, \end{cases}$$

with the constant $C > 0$ chosen such that $\int_{\mathbb{R}^n} \rho = 1$, and consequently, $\int_{\mathbb{R}^n} \rho_\varepsilon = 1$ for all $\varepsilon > 0$.

For an open set $\Omega \subseteq \mathbb{R}^n$, define

$$\Omega_\varepsilon = \{x \in \Omega \colon \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > \varepsilon\},$$

which is an open subset of Ω.

Lemma 0.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in C(\Omega)$. Set $u_\varepsilon = u * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$. Then, the sequence of smooth functions $\{u_\varepsilon\}$ converges uniformly on compact subsets of Ω to u.

That is, given a compact set $K \subseteq \Omega$ and a $\delta > 0$, there is an $\eta > 0$ such that for all $\varepsilon < \eta$, $K \subseteq \Omega_\varepsilon$, and $\|u_\varepsilon - u\|_K < \delta$.

Proof. For $\varepsilon < \operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$, we know that $K \subseteq \Omega_\varepsilon$. For such ε, we have for $x \in K$, that

$$\begin{aligned} |u_\varepsilon(x) - u(x)| &= \left| \int_{B(0,\varepsilon)} u(x-y)\rho_\varepsilon(y)\, dy - u(x) \right| \\ &= \left| \int_{B(0,\varepsilon)} \left(u(x-y) - u(x)\right)\rho_\varepsilon(y)\, dy \right| \\ &\leqslant \int_{B(0,\varepsilon)} |u(x-y) - u(x)|\rho_\varepsilon(y)\, dy \end{aligned}$$

Let $K_\varepsilon = \bigcup_{x \in K} B(x, \varepsilon)$, which is a bounded open set containing K, and is contained in Ω. Thus, for sufficiently small ε, we know that K_ε is relatively compact in Ω (since its closure would be contained in Ω and its closure is compact). Fix an $\alpha > 0$ such that $\overline{K}_\alpha$ is contained in Ω and hence, is compactly contained in the latter.

Since u is continuous, it is uniformly continuous on $\overline{K}_\alpha$, and hence, there is an $\eta > 0$ such that whenever $|x - y| < \eta$, and $x, y \in \overline{K}_\alpha$, $|u(x) - u(y)| < \delta$. Using the above equation, with $\varepsilon < \min\{\alpha, \eta\}$ so that $K \subseteq \Omega_\varepsilon$ and $B(x, \varepsilon) \subseteq K_\alpha$ for all $x \in K$, we have

$$|u_\varepsilon(x) - u(x)| \leqslant \int_{B(0,\varepsilon)} |u(x-y) - u(x)|\rho_\varepsilon(y)\, dy \leqslant \delta,$$

as desired. This completes the proof. ∎

Theorem 0.2 (Green's Second Identity). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $u, v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then

$$\int_\Omega u(x)\Delta v(x) - v(x)\Delta u(x)\, dx = \int_{\partial\Omega} u(x)\frac{\partial v}{\partial n}(x) - v(x)\frac{\partial u}{\partial n}(x)\, ds(x).$$

Throughout this article, let ω_n denote the surface area of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. In particular, this means that the surface area of a sphere of radius R is $\omega_n R^{n-1}$.

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1. Problem 1

Fix an exhaustion $\{K_n\}$ of Ω, that is, $\Omega = \bigcup_{n=1}^{\infty} K_n$ and $K_i \subseteq K_{i+1}^{\circ}$. Set $\omega_i = K_i^{\circ}$. We may suppose without loss of generality that $\omega_1 \neq \emptyset$. Note that each ω_i is open and relatively compact in Ω. Therefore, $f \in L^1(\omega_i)$ for all $i \in \mathbb{N}$. Further, we know that $\int_{\omega_i} f\varphi = 0$ for all $\varphi \in C_c^{\infty}(\omega_i)$ and all $i \in \mathbb{N}$. We shall show that $f = 0$ a.e. on ω_i for all $i \in \mathbb{N}$, whence it would follow that $f = 0$ a.e. on Ω, since Ω is a countable union of the ω_i's. Henceforth, we shall replace ω_i by Ω, so that we may assume $f \in L^1(\Omega)$ and $\int f\varphi = 0$ for all $\varphi \in C_c^{\infty}(\Omega)$. In particular, we have assumed Ω to be open and bounded, whence it is a finite measure space.

Claim. $\displaystyle\int_{\Omega} f\varphi = 0$ for all $\varphi \in C_c(\Omega)$.

Proof. Let $\varphi \in C_c(\Omega)$ and $K = \operatorname{Supp}\varphi$, which is a compact subset of Ω. Fix a $\delta > 0$ such that $2\delta < \operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$, so that the set

$$Q_\varepsilon = \{x \in \Omega \colon \operatorname{dist}(x, K) \leqslant \varepsilon\}$$

is contained in Ω for all $\varepsilon < \delta$. Further, since the function $\operatorname{dist}(\cdot, K)$ is continuous, the above set is closed (in $\mathbb{R}^n$) and also bounded, whence is compact in Ω. In particular, note that $Q_\varepsilon \subseteq Q_\delta$ for all $\varepsilon < \delta$. Note that for $x \in \Omega \setminus Q_\varepsilon$ with $\varepsilon < \delta$, we have that

$$\varphi_\varepsilon(x) = (\varphi * \rho_\varepsilon)(x) = \int_{B(0,\varepsilon)} \varphi(x-y)\rho(y)\, dy = 0,$$

since $x - y \notin K$ for all $|y| < \varepsilon$. It follows that $\operatorname{Supp}\varphi_\varepsilon \subseteq Q_\varepsilon \subseteq Q_\delta \subseteq \Omega$, in particular, is compact. Further, due to Lemma 0.1, we know that φ_ε converges to φ uniformly on compact subsets of Ω, and thus, converges uniformly on Q_δ (for $\varepsilon < \delta$). We then have for $\varepsilon < \delta$,

$$\begin{aligned}\left|\int_{\Omega} f(x)\varphi(x)\, dx\right| &= \left|\int_{Q_\delta} f(x)\varphi(x)\, dx\right| \\ &= \left|\int_{Q_\delta} f(x)\varphi(x) - f(x)\varphi_\varepsilon(x)\, dx\right| \\ &\leqslant \|\varphi - \varphi_\varepsilon\|_{Q_\delta}\|f\|_{L^1(Q_\delta)} \\ &\leqslant \|\varphi - \varphi_\varepsilon\|_{Q_\delta}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Due to uniform convergence, the right hand side goes to 0 as $\varepsilon \to 0$. It follows that $\int_{\Omega} f\varphi = 0$, as desired. ∎

Lemma 1.1. Let X be a locally compact Hausdorff space with a Radon measure μ, $A \subseteq X$ have finite μ-measure, and f a complex measurable function on X such that $f(x) = 0$ whenever $x \notin A$. Further, suppose that $|f| \leqslant 1$ on X. Then there is a sequence $\{g_n\}$ such that $g_n \in C_c(X)$, $|g_n| \leqslant 1$, and

$$f(x) = \lim_{n\to\infty} g_n(x) \quad \text{a.e. on } X.$$

Proof. See [Rud87, Corollary to Theorem 2.24]. ∎

We shall now show that $f = 0$ a.e. on Ω. Since Ω is a finite measure space and the Lebesgue measure is Radon, the above furnishes a sequence $\{g_n\}$ in $C_c(\Omega)$ with $|g_n| \leqslant 1$ on Ω such that

$$\lim_{n\to\infty} g_n(x) = \frac{\overline{f}(x)}{1 + |f(x)|} \quad \text{a.e. on } \Omega.$$

Thus, $|fg_n| \leqslant |f|$, and hence, the Lebesgue Dominated Convergence Theorem applies to get

$$\int_{\Omega} \frac{|f|^2}{1+|f|} = \int_{\Omega} fg_n = 0.$$

Since the integrand $|f|^2/(1+|f|)$ is non-negative measurable, we see that $|f|^2/(1+|f|) = 0$ a.e. on Ω, in other words, $f = 0$ a.e. on Ω, as desired.

Recall now that our Ω was in fact ω_i on which f is L^1, but on Ω, f is just L^1_{loc}. We have shown that there is a measure zero set $E_i \subseteq \omega_i$ such that $f = 0$ on $\omega_i \setminus E_i$, therefore, $f = 0$ on $\Omega \setminus \bigcup_{i\in\mathbb{N}} E_i$, and since $\bigcup_{i\in\mathbb{N}} E_i$ is also measure zero, we see that $f = 0$ a.e. on Ω.

2. Problem 2

For any distribution u, note that

$$(\Delta u, \varphi) = \left(\sum_{i=1}^{n} \partial_i^2 u, \varphi\right) = \sum_{i=1}^{n} \left(u, \partial_i^2 \varphi\right) = (u, \Delta\varphi).$$

So, according to the hypothesis of the question,

$$\int_\Omega u\Delta\varphi = 0 \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Lemma 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. If $\{u_\varepsilon\}_{\varepsilon>0}$ is a sequence of harmonic functions on Ω that converges uniformly on compacta to $u \in C(\Omega)$, then u is harmonic, and in particular, u is smooth on Ω

Proof. It suffices to show that u has the mean value property. Indeed, for some point $x_0 \in \Omega$, there is an $R > 0$ such that $\overline{B}(x_0, R) \subseteq \Omega$. Consequently, for $0 < r < R$, we have

$$0 = \lim_{\varepsilon\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u_\varepsilon(x)\,dx = \frac{1}{B(x_0,r)} \int_{B(x_0,r)} \lim_{\varepsilon\to 0} u_\varepsilon(x)\,dx = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(x)\,dx,$$

where we can interchange the limit with the integral since the convergence $u_\varepsilon \to u$ is uniform on $B(x,r) \subsetneq \overline{B}(x,R)$, since the latter is compact and contained in Ω. This shows that u has the mean value property on Ω, consequently, is harmonic on Ω. ∎

Coming back to the problem at hand, let $u_\varepsilon = u * \rho_\varepsilon$, defined and smooth on Ω_ε. Fix a point $p \in \Omega$ and choose a relatively compact ball $\omega \Subset \Omega$ centered at p.

There is a $\delta > 0$ such that $\omega \subseteq \Omega_\delta$, and hence, for all $\varepsilon < \delta$, we have that $\omega \subseteq \Omega_\varepsilon$. For $\varphi \in C_c^\infty(\omega) \subseteq C_c^\infty(\Omega)$, using Green's second identity, we have (since the boundary terms corresponding to φ vanish, owing to it having compact support in ω)

$$\begin{aligned}\int_\omega \Delta u_\varepsilon(x)\varphi(x)\,dx &= \int_\omega u_\varepsilon(x)\Delta\varphi(x)\,dx \\ &= \int_\omega \Delta\varphi(x) \int_{B(0,\varepsilon)} u(x-y)\rho(y)\,dy\,dx \\ &= \int_{B(0,\varepsilon)} \int_\omega u(x-y)\Delta\varphi(x)\,dx\,dy.\end{aligned}$$

Perform the substitution $z = x - y$, that is, $x = z + y$, then the inner integral transforms into

$$\int_{\omega-y} u(z)\Delta\varphi(z+y)\,dz = \int_{\omega-y} u(z)\Delta_z(\varphi(z+y))\,dz.$$

Since $|y| < \varepsilon$ and $\omega \subseteq \Omega_\varepsilon$, we know that $\omega - y \subseteq \Omega$. Further, $\operatorname{Supp}_z \varphi(z+y) = \operatorname{Supp}\varphi - y \subseteq \omega - y \subseteq \Omega$ is still compactly supported in Ω. Thus, according to our hypothesis,

$$\int_\omega u(x-y)\Delta\varphi(x)\,dx = \int_{\omega-y} u(z)\Delta\varphi(z+y)\,dz = \int_\Omega u(z)\Delta_z\left(\varphi(z+y)\right)\,dz = 0,$$

since $\Delta\varphi(z+y)$ vanishes outside $\omega - y$, the integral can be taken to be over all of Ω. It follows that

$$\int_\omega \Delta u_\varepsilon(x)\varphi(x)\,dx = 0$$

for al $\varphi \in C_c^\infty(\omega)$, consequently, $\Delta u_\varepsilon = 0$ in ω for all $\varepsilon < \delta$. Finally, due to Lemma 0.1, we know that $\{u_\varepsilon\}_{\varepsilon<\delta}$ converges uniformly on compacta to u on ω. Due to Lemma 2.1, we see that u is harmonic on ω, whence is smooth on ω.

We have shown that every point in Ω has a neighborhood on which u is smooth and harmonic (in the classical sense). Thus, u is smooth and $\Delta u = 0$ on Ω, since both properties of being smooth and harmonic are local properties.

3. Problem 3

Let $K \subseteq \mathbb{R}^n$ be a compact subset. Fix a $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\rho \equiv 1$ on V, an open subset of $\mathbb{R}^n$ containing K, and $\rho \geqslant 0$ everywhere. Let $\varphi \in C_c^\infty(K)$. Since φ is $\mathbb{C}$-valued, we can write $\varphi = \phi + i\psi$ where $\phi, \psi \in C_c^\infty(K)$ are real-valued.

Let $M = \max\left\{\sup_{x\in K}|\phi(x)|, \sup_{x\in K}|\psi(x)|\right\}$. Then,

$$|(u,\varphi)| = |(u,\phi) + i(u,\psi)| \leqslant |(u,\phi)| + |(u,\psi)|.$$

Note that $M\rho - \phi \geqslant 0$, since on K, $M\rho(x) = M \geqslant \phi(x)$ and outside K, $\phi \equiv 0$. Hence, $(u,\phi) \leqslant M(u,\rho)$. Similarly, $M\rho + \phi \geqslant 0$, since on K, $\phi(x) \geqslant -M$ and outside K, $\phi \equiv 0$. It follows that $(u,\phi) \geqslant -M(u,\rho)$. Note that $(u,\rho) \geqslant 0$ since $\rho \geqslant 0$, and hence

$$|(u,\phi)| \leqslant M(u,\rho).$$

Similarly, one can show that $|(u,\psi)| \leqslant M(u,\rho)$. Finally, note that for all $x \in K$, we have

$$|\varphi(x)| = \sqrt{\phi(x)^2 + \psi(x)^2} \geqslant |\phi(x)| \implies \sup_{x\in K}|\varphi(x)| \geqslant \sup_{x\in K}|\phi(x)|,$$

and similarly, for ψ. Thus,

$$\sup_{x\in K}|\varphi(x)| \geqslant \max\left\{\sup_{x\in K}|\phi(x)|, \sup_{x\in K}|\psi(x)|\right\} = M.$$

Hence,

$$|(u,\varphi)| \leqslant |(u,\phi)| + |(u,\psi)| \leqslant 2M(u,\rho) \leqslant 2(u,\rho)\sup_{x\in K}|\varphi(x)|.$$

Since (u,ρ) depends only on K and is independent of φ, we see that u has order 0.

From a result we have seen in class, u can be extended to a linear functional on $C_c(\mathbb{R}^n)$. Recall that this extension was defined to be

$$(u,\varphi) = \lim_{\varepsilon\to 0^+}(u, \varphi * \rho_\varepsilon) \qquad \forall\, \varphi \in C_c(\mathbb{R}^n),$$

where the ρ_ε are the standard mollifiers discussed in the introduction. If $\varphi \geqslant 0$, then obviously $\varphi * \rho_\varepsilon \geqslant 0$, and hence $(u,\varphi) \geqslant 0$, that is, u is a positive linear functional on $C_c(\mathbb{R}^n)$. Due to the Riesz Representation Theorem ([Rud87, Theorem 2.14]), there is a positive Borel measure μ on $\mathbb{R}^n$ such that

$$(u,\varphi) = \int_{\mathbb{R}^n} \varphi\, d\mu \qquad \forall\, \varphi \in C_c(\mathbb{R}^n),$$

thereby completing the proof.

4. Problem 4

We have seen in class that the distribution $\mathrm{p.v.}\left(\frac{1}{x}\right)$ has order at most 1. Suppose, for the sake of contradiction that $\mathrm{p.v.}\left(\frac{1}{x}\right)$ has order 0, then for every compact set $K \subseteq \mathbb{R}$, there is a constant $C > 0$ such that

$$\left|\left(\mathrm{p.v.}\left(\frac{1}{x}\right), \varphi\right)\right| \leqslant C\|\varphi\|_K \qquad \forall\, \varphi \in C_c^\infty(K).$$

Choose $K = [0,1]$, then there is a constant $C > 0$ such that the above inequality is satisfied. For $n \geqslant 3$, let $\varphi \in C_c^\infty([0,1])$ be such that it is identically equal to 1 on the interval $\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. Then, $\|\varphi\|_K = 1$, and

$$\begin{aligned}\left(\mathrm{p.\,v.}\left(\frac{1}{x}\right), \varphi\right) &= \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x}\, dx \\ &= \lim_{\substack{\varepsilon \to 0 \\ \varepsilon < \frac{1}{n}}} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x}\, dx \\ &\geqslant \int_{\frac{1}{n}}^{1 - \frac{1}{n}} \frac{1}{x}\, dx \\ &= \log(n-1).\end{aligned}$$

This shows that $\log(n-1) \leqslant C$ for every positive integer $n \geqslant 3$, which is absurd. Thus, p. v. $\left(\frac{1}{x}\right)$ cannot have order 0 as a distribution and hence, must have order 1.

5. Problem 5

First, we show that u is indeed a distribution on $(0, \infty)$. Let $K \subseteq (0, \infty)$ be a compact set. Since $0 \notin K$, there is a $\delta > 0$ such that $(-\delta, \delta) \cap K = \varnothing$. Thus, there is a positive integer $M > 0$ such that for all $n \geqslant M$, $\frac{1}{n} \notin K$. Now, let N be the largest positive integer such that $\frac{1}{N} \in K$. If there is no such N, then for all $\varphi \in C_c^\infty(K)$,

$$|(u, \varphi)| = \left|\sum_{k=1}^{\infty} \partial^k \varphi\left(\frac{1}{k}\right)\right| = 0 \leqslant \sup_{\substack{|\alpha| \leqslant 0 \\ x \in K}} |\partial^k \varphi(x)|,$$

since for all positive integers k, $\partial^k \varphi\left(\frac{1}{k}\right) = 0$.

On the other hand, if such an N exists, then for all $\varphi \in C_c^\infty(K)$,

$$|(u, \varphi)| = \left|\sum_{k=1}^{\infty} \partial^k \varphi\left(\frac{1}{k}\right)\right| = \left|\sum_{\substack{\frac{1}{k} \in K \\ k \in \mathbb{N}}} \partial^k \varphi\left(\frac{1}{k}\right)\right| \leqslant \sum_{\substack{\frac{1}{k} \in K \\ k \in \mathbb{N}}} \left|\partial^k \varphi\left(\frac{1}{k}\right)\right|.$$

Note that the last sum is finite, since for $k > N$, $\frac{1}{k} \notin K$. As a result, for all $k \in \mathbb{N}$ such that $\frac{1}{k} \in K$, we have

$$\left|\partial^k \varphi\left(\frac{1}{k}\right)\right| \leqslant \sup_{\substack{|\alpha| \leqslant N \\ x \in K}} |\partial^\alpha \varphi(x)|.$$

Hence,

$$|(u, \varphi)| \leqslant \sum_{\substack{\frac{1}{k} \in K \\ k \in \mathbb{N}}} \sup_{\substack{|\alpha| \leqslant N \\ x \in K}} |\partial^\alpha \varphi(x)| \leqslant N \sup_{\substack{|\alpha| \leqslant N \\ x \in K}} |\partial^\alpha \varphi(x)|.$$

This shows that u is a distribution on $(0, \infty)$.

Now, we deal with the second part of the problem. Suppose there is a distribution $\Lambda \in \mathscr{D}'(\mathbb{R})$ which restricts to u on $(0, \infty)$. Let $K = [0,1]$. Then there is a constant $C > 0$ and a non-negative integer m such that

$$|(\Lambda, \varphi)| \leqslant C \sup_{\substack{|\alpha| \leqslant m \\ x \in K}} |\partial^\alpha \varphi(x)|.$$

Choose N to be a very large positive integer, say $N \geqslant m + 100 \geqslant 100$ such that $4 \mid N$, and let $\delta > 0$ be such that

$$Q = \left[\frac{1}{N} - \frac{1}{\delta}, \frac{1}{N} + \frac{1}{\delta}\right] \subseteq \left(\frac{1}{N+1}, \frac{1}{N-1}\right) \subseteq (0,1).$$

Choose $\eta \in C_c^\infty(Q)$ such that $\eta \geqslant 0$ and $\eta\left(\frac{1}{N}\right) > 0$. For $\lambda > 1$, define $\varphi_\lambda \in C_c^\infty(Q) \subseteq C_c^\infty(K)$ by

$$\varphi_\lambda(x) = \eta(x)\cos\left(\lambda\left(x - \frac{1}{N}\right)\right) \qquad \forall\, x \in \mathbb{R}.$$

Then, for $k \leqslant m$, we have

$$\partial^k \varphi_\lambda(x) = \sum_{r=0}^{k} \binom{k}{r} \eta^{(k-r)}(x) \cos^{(r)}\left(\lambda\left(x - \frac{1}{N}\right)\right)\lambda^r.$$

Let $C' > 0$ be such that

$$\sup_{\substack{r \leqslant m \\ x \in Q}} |\eta^{(r)}(x)| < C',$$

and $M > 0$ be such that

$$\binom{k}{r} < M' \qquad \forall\, 0 \leqslant r \leqslant k \leqslant m.$$

Then, forall $0 \leqslant k \leqslant m$ and $x \in Q$,

$$|\partial^k \varphi_\lambda(x)| \leqslant \sum_{r=0}^{k} MC'\lambda^r \leqslant (k+1)MC'\lambda^k \leqslant (m+1)MC'\lambda^m \qquad \forall\, x \in Q,$$

where the last two inequalities follow from the fact that $\lambda > 1$.

On the other hand, since $\varphi_\lambda \in C_c^\infty(Q) \subseteq C_c^\infty((0,1))$, we see that

$$(\Lambda, \varphi_\lambda) = (u, \varphi_\lambda) = \sum_{k=1}^{\infty} \partial^k \varphi_\lambda\left(\frac{1}{k}\right) = \partial^N \varphi_\lambda\left(\frac{1}{N}\right).$$

We have

$$\partial^N \varphi_\lambda\left(\frac{1}{N}\right) = \sum_{k=0}^{N} \binom{N}{k} \eta^{(N-k)}\left(\frac{1}{N}\right)\cos^{(k)}(0)\lambda^k,$$

which is a polynomial in λ, say $p(\lambda) \in \mathbb{R}[\lambda]$, with leading coefficient

$$\eta\left(\frac{1}{N}\right)\cos^{(N)}(0) = \eta\left(\frac{1}{N}\right) \neq 0,$$

where the first equality follows from the fact that $4 \mid N$ and hence $\cos^{(N)}(x) = \cos x$. The seminorm estimate then gives us

$$|p(\lambda)| \leqslant (m+1)MCC'\lambda^m.$$

Dividing throughout by λ^N and taking $\lambda \to \infty$, the left hand side goes to $|\eta\left(\frac{1}{N}\right)| > 0$ while the right hand side goes to 0, since $N > m$. This is an immediate contradiction, and hence, there is no such $\Lambda \in \mathscr{D}'(\mathbb{R})$.

6. Problem 6

We shall make use of Problem 11. Define the distribution $v \in \mathscr{D}'(\mathbb{R})$ by $v = u - \text{p.v.}\left(\frac{1}{x}\right)$. Then, for $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
(xv, \varphi) &= \left(u - \text{p.v.}\left(\frac{1}{x}\right), x\varphi\right) \\
&= (u, x\varphi) - \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{1}{x} \cdot x\varphi(x)\, dx \\
&= (xu, \varphi) - \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \varphi(x)\, dx.
\end{aligned}$$

The integral above can be written as

$$\int_{\mathbb{R}^n} \chi_{|x|>\varepsilon}\varphi,$$

where the integrand is pointwise bounded by $|\varphi|$, since

$$|\chi_{|x|>\varepsilon}(y)\varphi(y)| \leqslant |\varphi(y)|,$$

and the latter is compactly supported and continuous, whence integrable. Thus, the Dominated Convergence Theorem applies and we have

$$\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\varphi(x)\,dx=\int_{\mathbb{R}^n}\lim_{\varepsilon\to 0}\chi_{|x|>\varepsilon}(y)\varphi(y)\,dy=\int_{\mathbb{R}^n}\chi_{\mathbb{R}^n\setminus\{0\}}(y)\varphi(y)\,dy=\int_{\mathbb{R}^n}\varphi.$$

But we also have that $xu=1$, and hence,

$$(xv,\varphi)=\int_{\mathbb{R}^n}\varphi-\int_{\mathbb{R}^n}\varphi=0.$$

Thus, $xv=0$. Using the result of Problem 11, we know that $v=c\delta$ for some $c\in\mathbb{C}$, where δ is the Dirac delta distribution centered at 0. Hence,

$$u=\mathrm{p.\,v.}\left(\frac{1}{x}\right)+c\delta\qquad\text{for some } c\in\mathbb{C}.$$

Conversely, if u is of the above form, then for $\varphi\in C_c^\infty(\mathbb{R})$, we can write

$$\begin{aligned}(xu,\varphi)&=(u,x\varphi)\\&=\left(\mathrm{p.\,v.}\left(\frac{1}{x}\right),x\varphi\right)+(\delta,x\varphi)\\&=\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{1}{x}\cdot x\varphi(x)\,dx\\&=\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\varphi(x)\,dx=\int_{\mathbb{R}^n}\varphi,\end{aligned}$$

where the last equality follows in the same way using the Dominated Convergence Theorem as we have argued in the earlier paragraphs. It follows that $xu=1$. This completes the characterization of u.

7. Problem 7

For $\varphi\in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n}f_\varepsilon(x)\varphi(x)\,dx\overset{x\mapsto\varepsilon y}{=\!=}\int_{\mathbb{R}^n}f(y)\varphi(\varepsilon y)\,dy,$$

and hence,

$$(f_\varepsilon,\varphi)-(\delta,\varphi)=\int_{\mathbb{R}^n}f(y)\left(\varphi(\varepsilon y)-\varphi(0)\right)\,dy,$$

since $\int_{\mathbb{R}^n}f=1$. Further, since φ is compactly supported, there is an $M>0$ such that $|\varphi(x)|\leqslant M$ for all $x\in\mathbb{R}^n$, consequently,

$$|\varphi(\varepsilon y)-\varepsilon(0)|\leqslant|\varphi(\varepsilon y)|+|\varphi(0)|\leqslant 2M$$

due to the triangle inequality. In particular, $|f(y)(\varphi(\varepsilon y)-\varphi(0))|\leqslant 2M|f(y)|$, which is an integrable function on $\mathbb{R}^n$. It follows from the Dominated Convergence Theorem that

$$\begin{aligned}\lim_{\varepsilon\to 0}|(f_\varepsilon,\varphi)-(\delta,\varphi)|&=\lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^n}f(y)(\varphi(\varepsilon y)-\varphi(0))\,dy\right|\\&\leqslant\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}|f(y)(\varphi(\varepsilon y)-\varphi(0))|\,dy\\&=\int_{\mathbb{R}^n}\lim_{\varepsilon\to 0}|f(y)(\varphi(\varepsilon y)-\varphi(0))|\,dy=0,\end{aligned}$$

since $\lim\limits_{\varepsilon\to 0}f(y)(\varphi(\varepsilon y)-\varphi(0))=0$ for all $y\in\mathbb{R}^n$ due to continuity of φ. This shows that $f_\varepsilon\to\delta$ in $\mathscr{D}'(\mathbb{R}^n)$.

8. Problem 8

We make use of Problem 7. Let $f \in L^1(\mathbb{R})$ be given by

$$f(x) = \frac{1}{\pi(x^2+1)} \qquad \forall x \in \mathbb{R}.$$

Then, following in the notation of Problem 7,

$$f_\varepsilon(x) = \frac{1}{\varepsilon} \frac{1}{\pi\left(\frac{x^2}{\varepsilon^2}+1\right)} = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}.$$

Thus, f_ε converges to δ in $\mathscr{D}'(\mathbb{R})$.

9. Problem 9

This is called the *Sokhotski-Plemelj formula*. For $\varphi \in C_c^\infty(\mathbb{R})$, we can write

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \frac{1}{x+i\varepsilon}\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{x-i\varepsilon}{x^2+\varepsilon^2}\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx - i\int_{\mathbb{R}} \frac{\varepsilon}{x^2+\varepsilon^2}\varphi(x)\,dx.$$

From the conclusion of Problem 8, we note immediately that the second term in the above limit converges to $-i\pi\varphi(0)$. Thus, we have

$$\lim_{\varepsilon\to 0} \frac{1}{x+i\varepsilon}\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx - i\pi(\delta,\varphi),$$

where δ is the Dirac delta distribution centered at 0. For $0 < \delta \leqslant 1$, we can break the first integral as

$$\int_{|x|>\delta} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx + \int_{|x|\leqslant\delta} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx.$$

The second integral above can be written as

$$\int_{-\delta}^{0} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx + \int_{0}^{\delta} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx.$$

Performing the substitution $x \mapsto -y$ in the first integral, we obtain

$$-\int_0^\delta \frac{y^2}{y^2+\varepsilon^2}\frac{\varphi(-y)}{y}\,dy + \int_0^\delta \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx = \int_0^\delta \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

Since $|x| \leqslant \delta \leqslant 1$, using the mean value property, for $x > 0$, there is some $c_x \in (-x, x) \subseteq [-1, 1]$ such that $\varphi(x) - \varphi(-x) = 2x\varphi'(c_x)$. Since φ' is a continuous function, it is bounded on $[-1, 1]$ in absolute value by some $M > 0$. Thus, the integrand is equal to

$$\frac{x^2}{x^2+\varepsilon^2}\cdot 2\varphi'(c_x),$$

and hence, is bounded in absolute value by $2M$. Since the constant function $2M$ is integrable on $[0,\delta]$, the Dominated Convergence Theorem applies and we can write

$$\lim_{\varepsilon\to 0} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int_0^\delta \lim_{\varepsilon\to 0}\frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int_0^\delta \frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

Next, we take care of the first integral,

$$\int_{|x|>\delta} \frac{x^2}{x^2+\varepsilon^2}\frac{\varphi(x)}{x}\,dx.$$

First, note that φ has compact support, and hence, there is an $R > 0$ such that $\operatorname{Supp}\varphi \subseteq (-R, R)$. In particular, the above integral is over a bounded measure space, $\delta < |x| < R$. Since the closure of this domain in $\mathbb{R}$, namely $\delta \leqslant |x| \leqslant R$ is compact, and $\frac{\varphi(x)}{x}$ is a continuous function on it, it is bounded above by some $\widetilde{M} > 0$ in absolute value. It follows that the integrand above is bounded in absolute value by $\widetilde{M}$, which is an integrable function on the measure space $\delta < |x| < R$. Hence, the Dominated Convergence Theorem applies and we can write

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_{|x|>\delta} \frac{x^2}{x^2+\varepsilon^2} \frac{\varepsilon(x)}{x}\, dx &= \lim_{\varepsilon \to 0} \int_{\delta<|x|<R} \frac{x^2}{x^2+\varepsilon^2} \frac{\varphi(x)}{x}\, dx \\
&= \int_{\delta<|x|<R} \lim_{\varepsilon \to 0} \frac{x^2}{x^2+\varepsilon^2} \frac{\varphi(x)}{x}\, dx \\
&= \int_{\delta<|x|<R} \frac{\varphi(x)}{x}\, dx \\
&= \int_{|x|>\delta} \frac{\varphi(x)}{x}\, dx.
\end{aligned}$$

We have shown that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{x^2}{x^2+\varepsilon^2} \frac{\varphi(x)}{x}\, dx = \int_{|x|>\delta} \frac{\varphi(x)}{x}\, dx + \int_0^\delta \frac{\varphi(x)-\varphi(-x)}{x}\, dx,$$

for all $0 < \delta \leqslant 1$. Thus, the equality holds in the limit $\delta \to 0^+$. In this limit, we shall show that the second integral goes to 0. Indeed, using the mean value theorem, we have

$$\left| \int_0^\delta \frac{\varphi(x)-\varphi(-x)}{x}\, dx \right| = \left| \int_0^\delta 2\varphi'(c_x)\, dx \right| \leqslant \int_0^\delta 2|\varphi'(c_x)|\, dx \leqslant 2M\delta,$$

where M is the same constant introduced earlier. It follows now that the second integral goes to 0 as $\delta \to 0^+$. This leaves us with

$$\lim_{\varepsilon \to 0} \frac{x^2}{x^2+\varepsilon^2} \frac{\varphi(x)}{x}\, dx = \lim_{\delta \to 0} \int_{|x|>\delta} \frac{\varphi(x)}{x}\, dx = \left(\text{p. v.} \left(\frac{1}{x} \right), \varphi \right).$$

Combining this with our simplification of the secon term in the beginning, we have

$$\lim_{\varepsilon \to 0} \left(\frac{1}{x+i\varepsilon}, \varphi \right) = \left(\text{p. v.} \left(\frac{1}{x} \right) - i\pi\delta, \varphi \right) \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}),$$

as desired.

10. Problem 10

We shall make (light) use of the Fourier transform to solve this. Consider the function $\chi(x)$, the indicator function of the interval $\left[-\frac{n}{2\pi}, \frac{n}{2\pi}\right]$. The Fourier transform of this is given by

$$\begin{aligned}
\widehat{\chi}(\xi) &= \int_{\mathbb{R}} \chi(x) e^{-2\pi i \xi}\, dx \\
&= \int_{-\frac{n}{2\pi}}^{\frac{n}{2\pi}} e^{-2\pi i x \xi}\, dx \\
&= \frac{1}{2\pi i \xi} \left(e^{in\xi} - e^{-in\xi} \right) \\
&= \frac{\sin n\xi}{\pi \xi}.
\end{aligned}$$

Let $\varphi \in C_c^\infty(\mathbb{R})$. Then there is an $R > 0$ such that the support of φ is contained in the open interval $(-R, R)$. We can write

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\sin nx}{\pi x}\varphi(x)\,dx &= \int_{-R}^{R} \frac{\sin nx}{\pi x}\varphi(x)\,dx \\
&= \int_{-R}^{R}\left(\int_{-\frac{n}{2\pi}}^{\frac{n}{2\pi}} e^{-2\pi i x y}\,dy\right)\varphi(x)\,dx \\
&= \int_{-\frac{n}{2\pi}}^{\frac{n}{2\pi}}\int_{-R}^{R} \varphi(x)e^{-2\pi i x y}\,dx\,dy \\
&= \int_{-\frac{n}{2\pi}}^{\frac{n}{2\pi}} \widehat{\varphi}(y)\,dy.
\end{aligned}$$

Note that we can make use of Fubini's theorem because the integrand $\varphi(x)e^{-2\pi i x y}$ is a continuous function on $\mathbb{R}\times\mathbb{R}$, which contains the domain of integration. As a result,

$$\lim_{n\to\infty}\int_{\mathbb{R}} \frac{\sin nx}{\pi x}\varphi(x)\,dx = \lim_{n\to\infty}\int_{-\frac{n}{2\pi}}^{\frac{n}{2\pi}} \widehat{\varphi}(y)\,dy = \int_{\mathbb{R}}\widehat{\varphi}(y)\,dy = \varphi(0).$$

where the last equality follows from the Fourier inversion formula on the Schwartz class

$$\int_{\mathbb{R}}\widehat{\varphi}(y)e^{2\pi i x y}\,dy = \varphi(x) \quad \text{for all } x \in \mathbb{R},$$

evaluated at $x = 0$. We have shown that

$$\left(\frac{\sin nx}{\pi x}, \varphi\right) \to (\delta, \varphi)$$

for every $\varphi \in C_c^\infty(\Omega)$, where δ is the Dirac delta distribution centered at 0. This completes the proof.

11. Problem 11

First, we show that $\operatorname{Supp} u \subseteq \{0\}$. To this end, let $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{0\}$, then there is some $a_i \neq 0$ for $1 \leqslant i \leqslant n$. Then, consider the open ball

$$U = \{x \in \mathbb{R}^n \colon |x - a| < |a_i|\}.$$

For any $x = (x_1, \dots, x_n) \in U$, we have that $|x_i - a_i| \leqslant |x - a| < |a_i|$, and hence, $x_i \neq 0$. It follows that the function $x \mapsto \frac{1}{x_i}$ is a well-defined smooth function on U. Now, for any $\varphi \in C_c^\infty(U)$, we have

$$(u, \varphi) = \left(u, x_i \cdot \frac{1}{x_i}\varphi\right) = \left(x_i u, \frac{1}{x_i}\varphi\right) = 0,$$

since $\frac{1}{x_i}\varphi$ is a compactly supported smooth function on U, and thus, a compactly supported smooth function on all of $\mathbb{R}^n$ (simply extend by 0 to all of $\mathbb{R}^n$). Thus, we have shown that $a \notin \operatorname{Supp}\varphi$, consequently, $\operatorname{Supp}\varphi \subseteq \{0\}$. We have seen in class that for such distributions, there is a positive integer N and constants $c_\alpha \in \mathbb{C}$ such that

$$u = \sum_{|\alpha| \leqslant N} c_\alpha \partial^\alpha \delta,$$

where δ is the Dirac delta distribution centered at 0. We contend that $c_\alpha = 0$ for $1 \leqslant |\alpha| \leqslant N$. Indeed, suppose $\beta \neq \alpha$ with $|\beta| \leqslant N$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. If $\beta_i > \alpha_i$ for any $1 \leqslant i \leqslant n$, then $\partial^\beta x^\alpha = 0$ identically, and hence,

$$\left(\partial^\beta \delta, x^\alpha\right) = (-1)^{|\beta|}\left(\delta, \partial^\beta x^\alpha\right) = 0.$$

On the other hand, if $\beta_i \leqslant \alpha_i$ for $1 \leqslant i \leqslant n$, then

$$\partial^\beta x^\alpha = \prod_{i=1}^{n} \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}} x_i^{\alpha_i} = \prod_{i=1}^{n} \frac{\alpha_i!}{(\alpha_i - \beta_i)!} x_i^{\alpha_i - \beta_i},$$

and since $\beta \neq \alpha$, there is an index j such that $\beta_j < \alpha_j$, consequently,

$$\left(\partial^\beta \delta, x^\alpha\right) = (-1)^{|\beta|} \left(\delta, \partial^\beta x^\alpha\right) = 0.$$

Hence,

$$(u, x^\alpha) = \sum_{|\beta| \leqslant N} c_\beta \left(\partial^\beta \delta, x^\alpha\right) = c_\alpha (-1)^{|\alpha|} (\delta, \partial^\alpha x^\alpha) = (-1)^{|\alpha|} c_\alpha \prod_{i=1}^n \alpha_i!.$$

Now, there is an index k such that $\alpha_k > 0$. Set $\gamma = (\gamma_1, \dots, \gamma_n)$, with $\gamma_i = \alpha_i$ for $i \neq k$ and $\gamma_k = \alpha_k - 1$. We can write

$$(u, x^\alpha) = (u, x_i x^\gamma) = (x_i u, x^\gamma) = 0.$$

It follows that $c_\alpha = 0$ whenever $1 \leqslant |\alpha| \leqslant N$. Therefore, $u = c_0 \delta$, as desired.

12. Problem 12

Let $\rho = \rho_1 \in C_c^\infty(\mathbb{R})$ be the standard mollifier as defined in the introduction, so that $\int_{\mathbb{R}} \rho = 1$. For $\varphi \in C_c^\infty(\mathbb{R})$, let $c = \int_{\mathbb{R}} \varphi$ and set $\psi = \varphi - c\rho$. This is a compactly supported smooth function on $\mathbb{R}$. Indeed, let N be a positive integer such that both ρ and φ have supports contained in the compact interval $[-N, N]$, then ψ must be supported inside $[-N, N]$ too, as a consequence, ψ is compactly supported. Define

$$\Phi(t) = \int_{-N}^t \psi(t) \, dt.$$

Note that for $t \geqslant N$, we have

$$\Phi(t) = \int_{-N}^N \psi(t) \, dt + \int_N^t \psi(t) \, dt = \int_{-N}^N \varphi(t) \, dt - c \int_{-N}^N \rho(t) \, dt = 0,$$

since $\int_{-N}^N \varphi = \int_{\mathbb{R}} \varphi = c$ and $\int_{-N}^N \rho = \int_{\mathbb{R}} \rho = 1$. On the other hand, for $t \leqslant -N$, we have that

$$\Phi(t) = \int_{-N}^t \psi(t) \, dt = -\int_t^{-N} \psi(t) \, dt = 0,$$

since ψ is identically zero on the interval $[t, -N]$. Thus, Φ is compactly supported in $[-N, N]$ and $\Phi'(t) = \psi(t)$. We have

$$0 = (u', \Phi) = -(u, \Phi') = -(u, \psi),$$

and hence,

$$(u, \varphi) = c(u, \rho) = (u, \rho) \int_{\mathbb{R}} \varphi.$$

Hence, $u = (u, \rho)$ is a constant, as desired.

13. Problem 13

We claim that the answer is $0 < a < n$. First, let $a < n$. We shall show that $\frac{1}{|x|^a}$ is locally integrable. Let $K \subseteq \mathbb{R}^n$ be compact. Then, there is an $R > 0$ such that $K \subseteq B(0, R)$. Note that

$$\int_K \frac{1}{|x|^a} \, dx \leqslant \int_{B(0,R)} \frac{1}{|x|^a} \, dx = \int_{\mathbb{R}^n} \chi_{B(0,R) \setminus \{0\}} \frac{1}{|x|^a} \, dx.$$

Consider the sequence of functions

$$\chi_{B(0,R) \setminus B(0,\varepsilon)} \frac{1}{|x|^a},$$

which are positive, measurable, pointwise increasing (with respect to ε), and converge pointwise to

$$\chi_{B(0,R) \setminus \{0\}} \frac{1}{|x|^a}.$$

Thus, the Monotone Convergence Theorem applies and

$$\int_{B(0,R) \setminus \{0\}} \frac{1}{|x|^a} \, dx = \lim_{\varepsilon \to 0^+} \int_{B(0,R) \setminus B(0,\varepsilon)} \frac{1}{|x|^a} \, dx.$$

The integral on the right can be computed using polar coordinates as

$$\int_{r=\varepsilon}^{R} \int_{\partial B(0,r)} \frac{1}{r^a} \, d\sigma \, dr = \int_{r=\varepsilon}^{R} \omega_n r^{n-1-a} \, dr = \frac{\omega_n}{n-a} \left(R^{n-a} - \varepsilon^{n-a} \right),$$

which converges as $\varepsilon \to 0^+$, since $a < n$. This shows that $\frac{1}{|x|^a}$ is locally integrable for $a < n$.

Suppose now that $a \geqslant n$. We shall show that the function is not locally integrable. Indeed, let $K = \overline{B}(0, R)$ for some $R > 0$ and consider the integral

$$\int_{B(0,R)} \frac{1}{|x|^a} \, dx.$$

Due to the above arguments, we can write this (using the Monotone Convergence Theorem) as

$$\lim_{\varepsilon \to 0^+} \int_{\overline{B}(0,R) \setminus B(0,\varepsilon)} \frac{1}{|x|^a} \, dx = \lim_{\varepsilon \to 0^+} \int_{r=\varepsilon}^{R} \int_{\partial B(0,r)} \frac{1}{r^a} \, d\sigma \, dr = \lim_{\varepsilon \to 0^+} \int_{r=\varepsilon}^{R} \omega_n r^{n-1-a} \, dr.$$

If $a = n$, then the limit on the right is $\omega_n \log \left(\frac{R}{\varepsilon} \right)$, which diverges as $\varepsilon \to 0^+$. On the other hand, if $a > n$, then the integral is

$$\frac{\omega_n}{a-n} \left(\varepsilon^{n-a} - R^{n-a} \right),$$

which diverges as $\varepsilon \to 0^+$ since $n - a < 0$. Thus, $\frac{1}{|x|^a}$ is not locally integrable for $a \geqslant n$, thereby completing the proof.

14. Problem 14

That $u \in \mathscr{D}'(\mathbb{R}^n)$ follows from the preceding problem. We shall show that $\Delta u = \delta$. First note that $\Delta u = \sum_{i=1}^n \partial_i^2 u$, and

$$(\partial_i^2 u, \varphi) = (-1)^2 (u, \partial_i^2 \varphi) \implies (\Delta u, \varphi) = (u, \Delta \varphi).$$

Hence, it sufices to show that $(u, \Delta\varphi) = \varphi(0)$ for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Let K denote the support of φ, which is compact, and hence, is contained in an open ball of the form $B(0, R)$ for some $R > 0$. Note that the support of every partial derivative of φ is also contained in this open ball. In particular, this means that $\Delta\varphi$ is compactly supported in $B(0, R)$.

We wish to compute

$$\int_{\mathbb{R}^n} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx = \int_{B(0,R)} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx.$$

First, we must argue that the latter is indeed integrable. This is easy to see, since $\Delta\varphi$ is compactly supported and hence, bounded by some $M > 0$ on $\mathbb{R}^n$. It follows that $\left| \frac{1}{|x|^{n-2}} \Delta\varphi(x) \right| \leqslant \frac{M}{|x|^{n-2}}$, which is locally integrable as argued in the preceding problem. In particular, it is integrable over $\overline{B}(0, R)$, and hence on $B(0, R)$. Note that this also implies integrability over all of $\mathbb{R}^n$, since $\Delta\varphi$ is compactly supported inside $B(0, R)$.

Next, we show that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < R} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx = \int_{B(0,R) \setminus \{0\}} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx = \int_{B(0,R)} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx$$

The second equality is obvious. To see the first, consider the sequence of functions

$$\chi_{B(0,R) \setminus \overline{B}(0,\varepsilon)}(x) \frac{1}{|x|^{n-2}} \Delta\varphi(x).$$

These are bounded in absolute value by $\frac{1}{|x|^{n-2}} |\Delta\varphi(x)|$, which we have argued to be integrable on $\mathbb{R}^n$ in the preceding paragraph. It follows that the dominated convergence theorem applies. The above sequence of functions converges pointwise to the function

$$\chi_{B(0,R) \setminus \{0\}}(x) \frac{1}{|x|^{n-2}} \Delta\varphi(x).$$

Hence, we have shown that

$$\lim_{\varepsilon \to 0} \int_{B(0,R) \setminus \overline{B}(0,\varepsilon)} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx = \int_{B(0,R) \setminus \{0\}} \frac{1}{|x|^{n-2}} \Delta\varphi(x) \, dx,$$

as desired.

Finally, we evaluate the above limit on the left hand side. Using Green's second identity on the domain $\Omega = B(0,R) \setminus \overline{B}(0,\varepsilon)$ with the notation $u(x) = \frac{1}{|x|^{n-2}}$, we have

$$\int_\Omega u(x)\Delta\varphi(x) - \varphi\Delta u(x) = \int_{\partial\Omega} u(x)\frac{\partial\varphi}{\partial n}(x) - \varphi(x)\frac{\partial u}{\partial n}(x)\, ds(x).$$

The boundary $\partial\Omega$ consists of two pieces, one the outer sphere $|x| = R$, which we shall denote by C_1 and the inner sphere $|x| = \varepsilon$, which we shall denote by C_2. Note that C_1 has the outward pointing normal and C_2 the inward pointing normal. For each $x \in C_1$, there is a neighborhood on which $\Delta\varphi = 0$, since φ is compactly supported within $B(0,R)$. It follows that both $\varphi(x)$ and $\frac{\partial\varphi}{\partial n}(x)$ are identically 0 on C_1. This leaves us with the terms corresponding to C_2. Note that the inward normal on C_2 is precisely $\frac{-x}{\varepsilon}$ for all $x \in C_2$. This gives us:

$$\begin{aligned}
\int_{|x|=\varepsilon} \varphi(x)\frac{\partial u}{\partial n}(x)\, ds(x) &= \int_{|x|=\varepsilon} \varphi(x)\nabla u(x)\cdot\frac{-x}{\varepsilon}\, ds(x) \\
&= \int_{|x|=\varepsilon} \varphi(x)\frac{-(n-2)x}{|x|^n}\cdot\frac{-x}{\varepsilon}\, ds(x) \\
&= \frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} \varphi(x)\, ds(x) \\
&= \frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} \varphi(x) - \varphi(0)\, ds(x) + \underbrace{\frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} \varphi(0)\, ds(x)}_{(n-2)\omega_n\varphi(0)}
\end{aligned}$$

We claim that the first term vanishes in the limit $\varepsilon \to 0$. Indeed, note that

$$\left|\frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} \varphi(x) - \varphi(0)\, ds(x)\right| \leqslant \frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} |\varphi(x) - \varphi(0)|\, ds(x) \leqslant (n-2)\omega_n \sup_{|x|=\varepsilon} |\varphi(x) - \varphi(0)|.$$

Given a $\delta > 0$, there is a corresponding $\eta > 0$ such that

$$|\varphi(x) - \varphi(0)| < \frac{\delta}{(n-2)\omega_n} \qquad \forall\ |x| < \eta.$$

Hence, for all $\varepsilon < \eta$, we have that

$$\left|\frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} \varphi(x) - \varphi(0)\, ds(x)\right| < \delta.$$

It follows that

$$\lim \frac{n-2}{\varepsilon^{n-1}}\int_{|x|=\varepsilon} \varphi(x) - \varphi(0)\, ds(x) = 0,$$

so that

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \varphi(x)\frac{\partial u}{\partial n}(x)\, ds(x) = (n-2)\omega_n\varphi(0).$$

Next, we show that

$$\lim_{\varepsilon\to 0}\int_{C_2} u(x)\frac{\partial\varphi}{\partial n}(x)\, ds(x) = 0.$$

Indeed, the above integral in absolute value is equal to

$$\begin{aligned}
\left|\int_{|x|=\varepsilon} u(x)\nabla\varphi(x)\cdot\frac{-x}{\varepsilon}\, ds(x)\right| &\leqslant \int_{|x|=\varepsilon} |u(x)| \left|\nabla\varphi(x)\cdot\frac{x}{\varepsilon}\right|\, ds(x) \\
&\leqslant \int_{|x|=\varepsilon} |u(x)| \|\nabla\varphi(x)\| \left\|\frac{x}{\varepsilon}\right\|\, ds(x) \\
&= \int_{|x|=\varepsilon} |u(x)| \|\nabla\varphi(x)\|\, ds(x),
\end{aligned}$$

where the second inequality follows from the Cauchy-Schwarz inequality. We may suppose that $\varepsilon \leqslant 1$. Since u and $\|\nabla\varphi(x)\|$ are continuous functions on the compact ball $\overline{B}(0,1)$, both are bounded there, in the sense that there are constants $M_1, M_2 > 0$ such that $|u(x)| \leqslant M_1$ and $\|\nabla\varphi(x)\| \leqslant M_2$ for all $x \in \overline{B}(0,1)$. Thus,

$$\left|\int_{|x|=\varepsilon} u(x)\nabla\varphi(x)\cdot\frac{-x}{\varepsilon}\,ds(x)\right| \leqslant \int_{|x|=\varepsilon} M_1M_2\,ds(x) = M_1M_2\omega_n\varepsilon^n.$$

As $\varepsilon \to 0$, the right hand side goes to 0, consequently,

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} u(x)\nabla\varphi(x)\cdot\frac{-x}{\varepsilon}\,ds(x) = 0.$$

In conclusion, this gives us

$$\lim_{\varepsilon\to 0}\int_{B(0,R)\setminus\overline{B}(0,\varepsilon)} u(x)\Delta\varphi(x) - \varphi(x)\Delta u(x) = -(n-2)\omega_n\varphi(0).$$

Note that u is a harmonic function on $\mathbb{R}^n \setminus \{0\}$, consequently, $\Delta u = 0$ on $\mathbb{R}^n \setminus \{0\}$. It follows that

$$(\Delta u, \varphi) = \lim_{\varepsilon\to 0}\int_{B(0,R)\setminus\overline{B}(0,\varepsilon)} u(x)\Delta\varphi(x) = -(n-2)\omega_n\varphi(0).$$

This gives us that $\Delta u = -(n-2)\omega_n\delta$, where δ is the Dirac delta distribution centered at 0.

15. Problem 15

Let

$$u_n(x) = n\int_{\frac{1}{n}}^{n} \phi\left(n(x-t)\right)\,dt.$$

We shall show that $u_n \to H$ where H is the Heaviside function. Let $\psi \in C_c^\infty(\mathbb{R})$. Consequently, there is an $R > 0$ such that $\operatorname{Supp}\psi \subseteq [-R, R]$. Then

$$(u_n, \psi) = n\int_{\mathbb{R}}\int_{\frac{1}{n}}^{n} \phi\left(n(x-t)\right)\psi(x)\,dt\,dx = n\int_{\frac{1}{n}}^{n}\int_{\mathbb{R}} \phi\left(n(x-t)\right)\psi(x)\,dx\,dt.$$

Performing the substitution $x = y + t$ to get

$$\begin{aligned}
(u_n, \psi) &= n\int_{\frac{1}{n}}^{n}\int_{\mathbb{R}} \phi(ny)\psi(y+t)\,dy\,dt \\
&= n\int_{\mathbb{R}}\int_{\frac{1}{n}}^{n} \phi(ny)\psi(y+t)\,dt\,dy \\
&= n\int_{\mathbb{R}} \phi(ny)\int_{\frac{1}{n}}^{n} \psi(y+t)\,dt\,dy \\
&= n\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(ny)\int_{\frac{1}{n}}^{n} \psi(y+t)\,dt\,dy \\
&= n\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(ny)\int_{\frac{1}{n}+y}^{n+y} \psi(z)\,dz\,dy,
\end{aligned}$$

where we have performed the substitution $z = y + t$. Let $N > 0$ be a positive integer such that $N - \frac{1}{N} > R$. Then, for all $n \geqslant N$, $n + y > R$ whenever $-\frac{1}{n} \leqslant y \leqslant \frac{1}{n}$. Thus

$$\begin{aligned}
(u_n, \psi) &= n\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(ny)\int_{\frac{1}{n}+y}^{\infty} \psi(z)\,dz\,dy \\
&= n\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(ny)\left(\int_0^\infty \psi - \int_0^{\frac{1}{n}+y} \psi(z)\,dz\right)\,dy \\
&= \int_0^\infty \psi - n\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(ny)\int_0^{\frac{1}{n}+y} \psi(z)\,dz\,dy.
\end{aligned}$$

Now, we show that the second quantity goes to 0 as $n \to \infty$. First, make the substitution $w = ny$ and rewrite the integral as

$$\int_{-1}^{1} \phi(w) \int_{0}^{\frac{1+y}{n}} \psi(z)\, dz\, dy,$$

which is bounded in absolute value by

$$\left| \int_{-1}^{1} \phi(w) \int_{0}^{\frac{1+y}{n}} \psi(z)\, dz\, dy \right| \leqslant \int_{-1}^{1} |\phi(w)| \int_{0}^{\frac{1+y}{n}} |\psi(z)|\, dz\, dy.$$

Since ψ is compactly supported, it is bounded on $\mathbb{R}$, similarly, so is ϕ. Let $M > 0$ be such that $|\phi(x)| \leqslant M$ and $|\psi(x)| \leqslant M$ for all $x \in \mathbb{R}$. Then, the above quantity is bounded by

$$\int_{-1}^{1} M \int_{0}^{\frac{1+y}{n}} M\, dz\, dy = M^2 \int_{-1}^{1} \frac{1+y}{n}\, dy = \frac{2M^2}{n},$$

which goes to 0 as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int_{-1}^{1} \phi(w) \int_{0}^{\frac{1+y}{n}} \psi(z)\, dz\, dy = 0,$$

which gives

$$\lim_{n\to\infty} (u_n, \psi) = \int_{0}^{\infty} \psi = (H, \psi),$$

as desired.

16. Problem 16

This is quite straightforward. For $\varphi \in C_c^\infty(\mathbb{R})$, we have

$$(u', \varphi) = -(u, \varphi') = -\int_0^1 \varphi'(x)\, dx = \varphi(0) - \varphi(1).$$

Thus, $u' = \delta_0 - \delta_1$, where δ_c denotes the Dirac delta distribution centered at $c \in \mathbb{R}$.

References

[Rud87] W. Rudin. *Real and Complex Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1987.