Local Cohomology

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§1 The I-torsion functor

Definition 1.1. Let R be a ring, $I \trianglelefteq R$ an ideal, and M an R-module. Define

$$\Gamma_I(M) := \left\{x \in M : \text{there is a positive integer } n \in \mathbb{N} \text{ such that } I^n x = 0\right\} = \bigcup_{n \geqslant 1} (0 :_M I^n).$$

This is known as the *I-torsion functor*.

It is clear that $\Gamma_I(M)$ is a submodule of M and any R-linear map $\varphi : M \to N$ restricts to an R-linear map $\Gamma_I(\varphi) : \Gamma_I(M) \to \Gamma_I(N)$. Thus, $\Gamma_I : {}_R\mathfrak{Mod} \to {}_R\mathfrak{Mod}$ is a functor.

Lemma 1.2. The functor Γ_I is left-exact.

Proof. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be a short exact sequence of R-modules. ■

Definition 1.3. The right derived functors of $\Gamma_I : {}_R\mathfrak{Mod} \to {}_R\mathfrak{Mod}$ are called the *local cohomology functors with support in I*.

$\boxed{\text{Henceforth } R \text{ is a Noetherian ring unless specified otherwise.}}$

There are some properties of Γ_I which are trivial to verify:

- $\Gamma_I(M) = \Gamma_{\sqrt{I}}(M)$ as submodules of M.
- If $\Gamma_I(M) = 0$, then I cannot be contained in any associated prime of M. In particular, if M is a finite R-module, then $\operatorname{Ass}_R(M)$ is finite, and hence, using Prime Avoidance, there is an M-regular element in I, that is, $\operatorname{depth}(I, M) \geqslant 1$.
- Given a family of R-modules $\{M_\alpha\}_{\alpha\in\Lambda}$, $\Gamma_I\left(\bigoplus_{\alpha\in\Lambda} M_\alpha\right) = \bigoplus_{\alpha\in\Lambda} \Gamma_I(M_\alpha)$ as submodules of $\bigoplus_{\alpha\in\Lambda} M_\alpha$.
- If $S \subseteq R$ is a multiplicative subset, then $\Gamma_{S^{-1}I}(S^{-1}M) = S^{-1}\Gamma_I(M)$ as submodules of $S^{-1}M$.
- For $\mathfrak{p} \in \operatorname{Spec}(R)$,
$$\Gamma_I(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & I \subseteq \mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$
  In particular if E is an injective R-module, then $\Gamma_I(E)$ is an injective R-module, and is a direct summand of E.

Since all the above isomorphisms are natural, these extend to isomorphisms on local cohomology, that is, for $i \geqslant 0$:

- $H^i_I(M) = \Gamma_{\sqrt{I}}(M)$ as submodules of M.
- Given a family of R-modules $\{M_\alpha\}_{\alpha\in\Lambda}$, $H^i_I\left(\bigoplus_{\alpha\in\Lambda} M_\alpha\right) = \bigoplus_{\alpha\in\Lambda} \Gamma_I(M_\alpha)$ as submodules of $\bigoplus_{\alpha\in\Lambda} M_\alpha$.

- If $S \subseteq R$ is a multiplicative subset, then $H^i_{S^{-1}I}(S^{-1}M) = S^{-1}\Gamma_I(M)$ as submodules of $S^{-1}M$.

Lemma 1.4. Let M be an R-module. If $\Gamma_I(M) = M$, then $\Gamma_I(E_R(M)) = E_R(M)$.

Proof. Suppose not and choose some $x \in E_R(M) \setminus \Gamma_I(E_R(M))$. Since R is Noetherian, there is an associated prime $\mathfrak{p}$ of $E_R(M)$ containing $\operatorname{Ann}_R(x)$. But since there is no power of I annihilating x, we must have $I \not\subseteq \mathfrak{p}$.

On the other hand, since $\operatorname{Ass}_R(E_R(M)) = \operatorname{Ass}_R(M)$, it follows that there is some $y \in M$ with $\mathfrak{p} = \operatorname{Ann}_R(y)$. Further, since $\Gamma_I(M) = M$, there is a positive integer $n > 0$ such that $I^n y = 0$, i.e., $I^n \subseteq \mathfrak{p}$, and hence, $I \subseteq \mathfrak{p}$, a contradiction. ■

Corollary 1.5. Let M be an R-module. If $\Gamma_I(M) = M$, then $H^i(M) = 0$ for $i > 0$.

Proof. Let $0 \to M \to E^\bullet$ be a minimal injective resolution of M. Due to Lemma 1.4, it follows that $\Gamma_I(E^i) = E^i$ for $i \geqslant 0$, and the conclusion follows, since the resolution remains unchanged after applying Γ_I. ■

Proposition 1.6. Let M be an R-module, and set $N := M/\Gamma_I(M)$. Then $\Gamma_I(N) = 0$ and $H^i_I(N) \cong H^i_I(M)$ for $i > 0$.

Proof. Set $L := \Gamma_I(M)$. Then there is a short exact sequence $0 \to L \to M \to N \to 0$, and L is Γ_I-acyclic. It is then clear from the long exact sequence that $H^i_I(N) \cong H^i_I(M)$ for $i > 0$. Finally, since the induced map $\Gamma_I(L) \to \Gamma_I(M)$ is an isomorphism and $H^1_I(L) = 0$, it follows that $\Gamma_I(N) = 0$. ■

Theorem 1.7 (Grothendieck Vanishing Theorem). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $I \subseteq R$ an ideal, and M a finite R-module. Then $H^j_I(M) = 0$ for $j > \dim_R M$.

Proof. We argue by induction on $d := \dim_R(M)$. If $d = 0$, then M is Artinian, so that $\operatorname{Ass}_R(M) = \{\mathfrak{m}\}$. It follows that every element of M is annihilated by a power of $\mathfrak{m}$, and thus by a power of I. Consequently, $\Gamma_I(M) = M$. Due to Corollary 1.5, $H^i_I(M) = 0$ for $i > 0$, and this establishes the base case.

Suppose now that $d > 0$. Set $N := M/\Gamma_I(M)$. As we have seen in Proposition 1.6, $\Gamma_I(N) = 0$ and $H^i_I(N) \cong H^i_I(M)$ for $i > 0$. Further, $\dim_R N \leqslant d$. If this inequality is strict, then we are done due to the induction hypothesis. Hence we may assume that $\dim_R N = d$. As we remarked earlier, since $\Gamma_I(N) = 0$, and N is a finite R-module, $\operatorname{depth}(I, N) \geqslant 1$. Choose an N-regular element $a \in I$. Let $x \mapsto \overline{x}$ denote the natural surjection $M \twoheadrightarrow M/aM =: \overline{M}$ and $\mu_a : M \to M$ be multiplication by a. The short exact sequence $0 \to M \xrightarrow{\mu_a} M \to \overline{M} \to 0$ induces a long exact sequence:

$$\cdots \to H^{i-1}_I(\overline{M}) \to H^i_I(M) \xrightarrow{\mu_a} H^i_I(M) \to H^i_I(\overline{M}) \to \cdots.$$

For $i > d$, note that $i - 1 > d - 1 = \dim_R \overline{M}$, so that $H^{i-1}_I(\overline{M}) = H^i_I(\overline{M}) = 0$. This shows that $\mu_a : H^i_I(M) \to H^i_I(M)$ is an isomorphism of R-modules. Recall that $H^i_I(M)$ is I-torsion and $a \in I$. If $H^i_I(M) \neq 0$, then for $n \gg 0$, the composition μ_a^n would have non-trivial kernel, which is absurd since it is an isomorphism. This shows that $H^i_I(M) = 0$ for $i > d$, as desired. ■

Proposition 1.8. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with $d = \dim R$. Then

$$H^d_{\mathfrak{m}}(R) \cong E_R(k).$$

Proof. It is well-known that the minimal injective resolution of a Gorenstein local ring looks like:

$$0 \to R \to \bigoplus_{\operatorname{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p}) \to \bigoplus_{\operatorname{ht}\mathfrak{p}=1} E_R(R/\mathfrak{p}) \to \cdots \to E_R(k) \to 0.$$

Further, it is clear that

$$\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(k) & \mathfrak{p} = \mathfrak{m} \\ 0 & \text{otherwise,} \end{cases}$$

whence the conclusion follows. ■

It is also possible to characterize the depth of an ideal using the local cohomology modules:

Proposition 1.9. Let R be a Noetherian ring and $I \trianglelefteq R$ an ideal. If M is a finite R-module such that $IM \neq M$, then

$$\operatorname{depth}(I, M) = \inf\left\{i : H^i_I(M) \neq 0\right\}.$$

Proof. We induct on $d = \operatorname{depth}(I, M)$. If $d = 0$, then $I \subseteq \mathfrak{p}$ for some associated prime $\mathfrak{p}$ of M. ■

§§ The Mayer-Vietoris Sequences

Let M be an R-module and $I, J \trianglelefteq R$ be two ideals. It is easy to see that

$$0 \to \Gamma_{I+J}(M) \xrightarrow{x \mapsto (x,x)} \Gamma_I(M) \oplus \Gamma_J(M) \xrightarrow{(x,y) \mapsto x-y} \Gamma_{I \cap J}(M) \to 0$$

is exact.

Theorem 1.10 (Mayer-Vietoris, Version 1). Let R be a Noetherian ring, $I, J \trianglelefteq R$ be ideals, and M an R-module. Then there is a long exact sequence

$$0 \to \Gamma_{I+J}(M) \to \Gamma_I(M) \oplus \Gamma_J(M) \to \Gamma_{I \cap J}(M) \to H^1_{I+J}(M) \to H^1_I(M) \oplus H^1_J(M) \to H^1_{I \cap J}(M) \to H^2_{I+J}(M) \to \cdots.$$

Proof. Let $0 \to M \to E^\bullet$ be an R-injective resolution of M. In view of the above remark, there is a short exact sequence of complexes

$$0 \to \Gamma_{I+J}\left(E^\bullet\right) \to \Gamma_I\left(E^\bullet\right) \oplus \Gamma_J\left(E^\bullet\right) \to \Gamma_{I \cap J}\left(E^\bullet\right) \to 0.$$

Taking cohomologies, the conclusion follows. ∎

Theorem 1.11 (Mayer-Vietoris, Version 2). Let R be a Noetherian ring, $x \in R$, $I \trianglelefteq R$ an ideal, and M an R-module. Then there is a long exact sequence

$$0 \to \Gamma_{(I,x)}(M) \to \Gamma_I(M) \to \Gamma_{IR_x}(M_x) \to H^1_{(I,x)}(M) \to H^1_I(M) \to H^1_{IR_x}(M_x) \to H^2_{(I,x)}(M) \to \cdots.$$

Proof. ∎

§§ Set-theoretic Complete Intersections

Lemma 1.12. Let R be a Noetherian ring, $I \trianglelefteq R$ an ideal, and M an R-module. If I is generated by n elements, then $H^j_I(M) = 0$ for $j > n$.

Proof. We shall argue by induction on n. If $n = 0$, then $I = 0$, where it is clear that $H^j_I(M) = 0$ for $j > 0$. Suppose now that $n > 0$. Then there exists an ideal $J \subseteq I$ and $x \in I$ such that J is generated by $n-1$ elements and $I = (J, x)$. Using Theorem 1.11, for $j > n$, we have an exact sequence

$$H^{j-1}_{JR_x}(M_x) \to H^j_I(M) \to H^j_J(M).$$

Since $j - 1 > n - 1$, $H^j_J(M) = 0$ and since JR_x is generated by $n-1$ elements, $H^{j-1}_{JR_x}(M_x) = 0$. It follows that $H^j_I(M) = 0$ too. ∎

Definition 1.13. Let R be a Noetherian ring and $I \trianglelefteq R$ an ideal. We define the *arithmetic rank* of I to be

$$\operatorname{ara}(I) = \min\left\{n \in \mathbb{Z}_{\geqslant 0} : \text{there exist } a_1, \ldots, a_n \in R \text{ such that } \sqrt{I} = \sqrt{(a_1, \ldots, a_n)}\right\}.$$

§§ Connectedness of the Punctured Spectrum

Theorem 1.14 (Hartshorne). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring such that $\operatorname{depth} R \geqslant 2$. Then $\operatorname{Spec}^\circ(R) := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ is connected in the Zariski topology.

Proof. Suppose $\operatorname{Spec}^\circ(R)$ is not connected. Then there exist ideals I and J of R such that

$$\operatorname{Spec}^\circ(R) = (V(I) \setminus \{\mathfrak{m}\}) \sqcup (V(J) \setminus \{\mathfrak{m}\})$$

and $V(I) \setminus \{\mathfrak{m}\}$ is not empty, nor the entire $\operatorname{Spec}^\circ(\mathfrak{m})$. The latter condition is equivalent to the fact that I and J are neither $\mathfrak{m}$-primary or nilpotent. Further, the first condition is equivalent to $\sqrt{I+J} = \mathfrak{m}$ and $I \cap J$ being nilpotent.

With this setup, using Theorem 1.10, there is a long exact sequence

$$0 \to \Gamma_{I+J}(R) \to \Gamma_I(R) \oplus \Gamma_J(R) \to \Gamma_{I \cap J}(R) \to H^1_{I+J}(R) \to H^1_I(R) \oplus H^1_J(R) \to H^1_{I \cap J}(R) \to \cdots.$$

Since $I \cap J$ is nilpotent, it follows that $\Gamma_{I\cap J}(R) = R$ and $H^j_{I\cap J}(R) = 0$ for $j > 0$. Since $\sqrt{I+J} = \mathfrak{m}$, $H^j_{I+J}(R) = H^j_{\mathfrak{m}}(R)$ for $j \geqslant 0$. But due to Proposition 1.9, $H^j_{\mathfrak{m}}(R) = 0$ for $j = 0,1$. Hence, the above exact sequence gives

$$R \cong \Gamma_I(R) \oplus \Gamma_J(R),$$

But R being a direct sum of R-modules is equivalent to R being a product of rings, which is absurd since R is local. ■

§2 Čech Cohomology

Definition 2.1. Let R be a Noetherian ring, and $\underline{a} = a_1,\ldots,a_n \in R$. Let $\check{C}^\bullet(a_i)$ denote the cochain complex:

$$\cdots 0 \to R \to R_{a_i} \to 0 \to \cdots,$$

and define $\check{C}^\bullet(\underline{a})$ to be the cochain complex:

$$\check{C}^\bullet(a_1) \otimes \cdots \otimes \check{C}^\bullet(a_n).$$

Further, if M is an R-module, then define $\check{C}^\bullet(\underline{a}, M) := \check{C}^\bullet(\underline{a}) \otimes M$. This is known as the *Čech complex*. The cohomology modules of this cochain complex are known as the *Čech cohomology modules* and are denoted by $\check{H}^i_{\underline{a}}(M)$.

Remark 2.2. Since the tensor product of chain complexes is associative and commutative, the order of tensoring above doesn't matter.

Theorem 2.3. Let R be a Noetherian ring, $I \trianglelefteq R$ an ideal, and $\underline{a} := a_1,\ldots,a_n \in R$ such that $\sqrt{(a_1,\ldots,a_n)} = \sqrt{I}$. Then

$$\check{H}^j_{\underline{a}}(M) \cong H^j_I(M)$$

for every R-module M.

Proof. ■

Proposition 2.4. If $R \to S$ is a flat morphism of Noetherian rings, M an R-module, and I an ideal in R, then

$$H^j_I(M) \otimes_R S \cong H^j_{IS}(M \otimes_R S)$$

as S-modules.

Proposition 2.5. Let $R \to S$ be a homomorphism of Noetherian rings, $I \trianglelefteq R$ an ideal, and M an S-module. Then

$$H^j_I(M) \cong H^j_{IS}(M)$$

as S-modules.

Definition 2.6. Let R be a Noetherian ring and $I \trianglelefteq R$ an ideal. Define the *cohomological dimension* of I in R to be

$$\operatorname{cdim}(I,R) := \inf\left\{i : H^j_I(M) = 0 \text{ for all } R\text{-modules } M \text{ and all } j > i\right\}.$$

Proposition 2.7.

$$\operatorname{cdim}(I,R) = \inf\left\{i : H^j_I(R) = 0 \text{ for all } j > i\right\}.$$

Corollary 2.8. Let $(R,\mathfrak{m},k)$ be a Noetherian local ring. Then $\operatorname{cdim}(I,R) = \operatorname{cdim}(\widehat{I},\widehat{R})$.

Proof. This is immediate from Proposition 2.7 and the fact that $R \to \widehat{R}$ is faithfully flat. ■

References