# MA 824: ASSIGNMENT 2

SWAYAM CHUBE (200050141)

Throughout this document, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Most proofs should work over either of the two fields unless specified otherwise.

## 1. Problem 1

(a) Recall the following result from complex analysis:

**Theorem 1.1.** Let $\sum_{n\geqslant 0} a_n(z-a)^n$ be a power series with $a_n \in \mathbb{C}$. The radius of convergence of the above power series is the unique real number $R \geqslant 0$ such that

(i) if $|z-a| < R$, the series converges absolutely.

(ii) if $|z-a| > R$, the terms of the series become unbounded and so the series diverges.

*Proof.* See [Con73, Theorem III.1.3] ■

Returning back to our problem, let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Let $x \in \mathscr{H}$ denote the element $(1, |\lambda|, |\lambda|^2, \dots)$. That this is indeed an element of $\mathscr{H}$ is clear from the fact that the sum

$$\sum_{n\geqslant 0} |\lambda|^{2n} = \frac{1}{1-|\lambda|^2}$$

converges. Further, let $y \in \mathscr{H}$ denote the element $(|\alpha_0|, |\alpha_1|, \dots)$. That this is an element of $\mathscr{H}$ follows from the fact that the sum

$$\sum_{n\geqslant 0} |\alpha_n|^2$$

converges, since the element $(\alpha_0, \alpha_1, \dots)$ is an element of $\mathscr{H}$. The Cauchy-Schwarz inequality gives us:

$$\sum_{n\geqslant 0} |\alpha_n||\lambda|^n = \langle x, y\rangle \leqslant \|x\|\|y\| < \infty.$$

Thus, the sum $\sum_{n\geqslant 0} \alpha_n \lambda^n$ converges absolutely for $|\lambda| < 1$. Due to Theorem 1.1, it follows that the radius of convergence of the power series $\sum_{n\geqslant 0} \alpha_n z^n$ is $\geqslant 1$.

(b)

## 2. Problem 4

Clearly the bounded linear functional

$$\Lambda : y \longmapsto \left\langle y, \frac{x}{\|x\|} \right\rangle.$$

is such that $\Lambda x = \|x\|$ and $\|\Lambda\| = \left\| \frac{x}{\|x\|} \right\| = 1$. Suppose $\Phi : X \to \mathbb{C}$ is another bounded linear functional such that $\Phi x = \|x\|$ and $\|\Phi\| = 1$. By the Riesz Representation Theorem, there is some $z \in X$ such that $\Phi y = \langle y, z\rangle$ for all $y \in X$. Then, $\|z\| = \|\Phi\| = 1$. It follows that

$$\langle x, z\rangle = \|x\| \implies |\langle x, z\rangle| = \|x\| = \|x\|\|z\|.$$

Recall that the equality holds in the Cauchy-Schwarz inequality if and only if the vectors $x$ and $z$ are linearly dependent. Since both $x$ and $z$ are non-zero, this is tantamount to saying that there is some $\alpha \in \mathbb{C}$ such that $x = \alpha z$. In conclusion,

$$\alpha = \langle x, z\rangle = \|x\| \implies z = \frac{x}{\|x\|},$$

i.e., $\Phi = \Lambda$, as desired.

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## 3. Problem 5

Let $x, y \in \mathscr{H}$ and $\alpha \in \mathbb{C}$. According to our hypothesis, we have

$$\begin{aligned}0 &= \langle A(x+\alpha y), x+\alpha y\rangle \\ &= \langle Ax, x\rangle + \alpha\langle Ay, x\rangle + \overline{\alpha}\langle Ax, y\rangle + |\alpha|^2\langle Ay, y\rangle \\ &= \alpha\langle Ay, x\rangle + \overline{\alpha}\langle Ax, y\rangle.\end{aligned}$$

Set $\alpha = 1$ to get

$$\langle Ay, x\rangle + \langle Ax, y\rangle = 0$$

and set $\alpha = \iota$ to get

$$\iota(\langle Ay, x\rangle - \langle Ax, y\rangle) = 0 \implies \langle Ay, x\rangle - \langle Ax, y\rangle = 0.$$

Hence, $\langle Ax, y\rangle = 0 = \langle Ay, x\rangle$ for all $x, y \in \mathscr{H}$. In particular, for any $x \in \mathscr{H}$, setting $y = Ax$, we get

$$\langle Ax, Ax\rangle = \langle Ax, y\rangle = 0,$$

and hence $Ax = 0$, that is, $A = 0$.

## 4. Problem 9

**Exercise 12-1 (b) of [Lim14].** Suppose first that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $\lambda \in \sigma(A)$, we shall show that $p(\lambda) \in \sigma(p(A))$. The polynomial $p(X) - p(\lambda)$ has a root at $\lambda$, and hence, there is a polynomial $q(X) \in \mathbb{C}[X]$ such that $p(X) - p(\lambda) = (X - \lambda)q(X)$. Now, since $A$ commutes with the identity operator $I$, we get

$$p(A) - p(\lambda)I = (A - \lambda I)q(A).$$

Since $A - \lambda I$ is not invertible, the left hand side, $p(A) - p(\lambda)I$ is not invertible, i.e., $p(\lambda) \in \sigma(p(A))$. This shows that

$$p(\sigma(A)) := \{p(\lambda) \colon \lambda \in \sigma(A)\} \subseteq \sigma(p(A))$$

Suppose now that $\mathbb{K} = \mathbb{C}$. Further suppose that $p(X)$ is a non-constant polynomial. Let $\lambda \in \sigma(p(A))$. Then, due to the fundamental theorem of algebra, the polynomial $p(X) - \lambda$ factors as

$$p(X) - \lambda = a_n \prod_{i=1}^{n}(X - \alpha_i),$$

for some $\alpha_i \in \mathbb{C}$. Again, since $A$ commutes with the identity operator $I$, we have

$$p(A) - \lambda = \prod_{i=1}^{n}(A - \alpha_i I).$$

Since the left hand side is not invertible, at least one term on the right hand side must not be invertible, that is, there is an index $1 \leqslant j \leqslant n$ with $A - \alpha_j I$ not invertible. Thus $\alpha_j \in \sigma(A)$. It follows that $\sigma(p(A)) \subseteq p(\sigma(A))$. Thus $\sigma(p(A)) = p(\sigma(A))$.

Finally, if $p$ is a constant polynomial, say $p(X) \equiv c \in \mathbb{C}$, then $\sigma(p(A)) = \{c\}$, since $(c - \lambda)I$ is invertible if and only if $c \neq \lambda$. On the other hand, since $X$ is a Banach space over $\mathbb{C}$, due to Gelfand-Mazur, the spectrum $\sigma(A)$ is non-empty, whence $p(\sigma(A)) = \{c\}$, as desired.

**Exercise 17-5 of [Lim14].** Suppose $p(X) \in \mathbb{K}[X]$ is a polynomial such that $p(A)$ is a compact opeartor. If $p(X) \equiv c$ is a constant polynomial, then $p(A) = cI$, which is given to be compact. Thus, the closure of the image of the unit ball under $p(A)$ is compact, that is,

$$\overline{cB_X(0,1)} = \overline{B_X(0,c)}$$

is compact. If $c \neq 0$, using the fact that $\overline{B_X(0,c)}$ is homeomorphic to $\overline{B_X(0,1)}$ through the homeomorphism $x \mapsto c^{-1}x$ (as we have seen in class), it follows that the closed unit ball in $X$ is compact, which is absurd, since $X$ is infinite-dimensional. Thus $c = 0$. Clearly, if $c = 0$, then the operator $p(A) \equiv 0$ is compact.

Suppose now that $p(X) = a_n X^n + \cdots + a_0 \in \mathbb{K}[X]$ is a non-constant polynomial of degree $n > 0$, whence $a_n \neq 0$. According to the hypothesis, $p(A) = a_n A^n + \cdots + a_0 I$ is a compact operator. Recall that the compact operators form an ideal in the $\mathbb{K}$-algebra $\mathscr{B}(X)$ and $A$ is a compact operator, thus the operator

$$a_n A^n + \cdots + a_1 A$$

is a compact operator. It follows that

$$a_0 I = p(A) - \left(a_n A^n + \cdots + a_1 A\right)$$

is a compact operator. Because of what we just proved, we must have that $a_0 = 0$, that is, $p(0) = 0$.

Conversely, suppose $p(X) \in \mathbb{K}[X]$ is a polynomial such that $p(0) = 0$. If $p(X)$ is a constant polynomial, then it is identically zero, whence is trivially compact. Suppose now that $p(X)$ is non-constant. Then, we can write

$$p(X) = a_n X^n + \cdots + a_1 X$$

for $n > 0$ and $a_i \in \mathbb{K}$ with $a_n \neq 0$ where $n$ is the degree of the polynomial $p(X)$. Thus $p(A) = a_n A^n + \cdots + a_1 A$. As we argued earlier, since the compact operators form an ideal in $\mathscr{B}(X)$ and $A$ is a compact operator, it is clear that $p(A)$ is a compact operator, as desired.

## References

[Con73] J.B. Conway. *Functions of One Complex Variable*. Springer New York, 1973.

[Lim14] B.V. Limaye. *Functional Analysis*. New Age International (P) Limited, Publishers, 2014.