Higher Homotopy Groups

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§1 Function Spaces

We begin with some preliminaries about function spaces.

Definition 1. Let X and Y be topological spaces. We use the shorthand X^Y to denote the set of continuous functions from Y to X. Endow this set with the *compact open topology*, that is, the topology generated by the subbasis elements

$$(K;U) := \left\{ f \in X^Y : f(K) \subseteq U \right\},$$

for all compact sets $K \subseteq Y$ and open sets $U \subseteq X$.

For each continuous function $F \colon Z \times Y \to X$, there is the *associate* $F^\sharp \colon Z \to X^Y$ defined by

$$F^\sharp(z) = (y \longmapsto F(z,y)).$$

Further, there is the *evaluation map* $\mathrm{ev} \colon X^Y \times Y \to X$ given by $\mathrm{ev}(f,y) = f(y)$.

Theorem 2. Let X and Z be topological spaces, Y a locally compact Hausdorff space, and equip X^Y with the compact-open topology.

(1) The evaluation map $\mathrm{ev} \colon X^Y \times Y \to X$ is continuous.

(2) A function $F \colon Z \times Y \to X$ is continuous if and only if its associate $F^\sharp \colon Z \to X^Y$ is continuous.

§2 Group and Cogroup Objects

In a category $\mathscr{C}$, if the product $X \times Y$ exists, then given any object Z and morphisms $f \colon Z \to X$ and $g \colon Z \to Y$, there is a unique map $(f,g) \colon Z \to X \times Y$ making the diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \\
Y & Z \arrow[l, "g"] \arrow[u, "f"'] \arrow[ul, dotted, "(f,g)"]
\end{tikzcd}$$

commute.

Similarly, if the coproduct $X \coprod Y$ exists, then given any object Z and morphisms $f \colon X \to Z$ and $g \colon Y \to Z$, there is a unique map $(f,g) \colon X \coprod Y \to Z$ making

$$\begin{tikzcd}
Z & X \arrow[l, "f"'] \arrow[d, "\iota_1"] \\
Y \arrow[u, "g"] \arrow[r, "\iota_2"'] & X \coprod Y \arrow[ul, dotted, "(f,g)"]
\end{tikzcd}$$

Definition 3. Let $\mathscr{C}$ be a category admitting finite products and a terminal object Z. A *group object* in $\mathscr{C}$ is an object G together with morphisms

$$\mu\colon G\times G\to G,\quad \eta\colon G\to G,\quad \text{and}\quad \varepsilon\colon Z\to G$$

such that the following diagrams commute:

Associativity

$$\begin{tikzcd}
G\times G\times G \arrow[r, "\mathbb{1}\times\mu"] \arrow[d, "\mu\times\mathbb{1}"'] & G\times G \arrow[d, "\mu"] \\
G\times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

Identity

$$\begin{tikzcd}
G\times Z \arrow[r, "\mathbb{1}\times\varepsilon"] \arrow[dr, "\mathrm{pr}_1"'] & G\times G \arrow[d, "\mu"'] & Z\times G \arrow[l, "\varepsilon\times\mathbb{1}"'] \arrow[dl, "\mathrm{pr}_2"] \\
& G &
\end{tikzcd}$$

Note that the projections pr_1 and pr_2 are isomorphisms in the category.

Inverse

$$\begin{tikzcd}
G \arrow[r, "(\mathbb{1},\eta)"] \arrow[d] & G\times G \arrow[d, "\mu"'] & G \arrow[l, "(\eta,\mathbb{1})"'] \arrow[d] \\
Z \arrow[r, "\varepsilon"'] & G & Z \arrow[l, "\varepsilon"]
\end{tikzcd}$$

The maps μ, η, and μ are called *multiplication*, *inversion*, and *unit* respectively.

Definition 4. Let $\mathscr{C}$ be a category admitting finite coproducts and an initial object A. A *cogroup object* in $\mathscr{C}$ is an object C together with morphisms

$$m\colon C\to C\coprod C,\quad h\colon C\to C,\quad \text{and}\quad e\colon C\to A$$

such that the following diagrams commute

Co-associativity

$$\begin{tikzcd}
C \arrow[r, "m"] \arrow[d, "m"'] & C\coprod C \arrow[d, "\mathbb{1}\coprod m"] \\
C\coprod C \arrow[r, "m\coprod\mathbb{1}"'] & C\coprod C\coprod C
\end{tikzcd}$$

Co-identity

$$\begin{tikzcd}
C\coprod A & C\coprod C \arrow[l, "\mathbb{1}\coprod e"'] \arrow[r, "e\coprod\mathbb{1}"] & A\coprod C \\
& C \arrow[ul, "\iota_1"] \arrow[u, "m"] \arrow[ur, "\iota_2"'] &
\end{tikzcd}$$

Co-inverse

$$\begin{tikzcd}
C & C\coprod C \arrow[l, "(1,h)"'] \arrow[r, "(h,1)"] & C \\
A \arrow[u] & C \arrow[l, "e"] \arrow[r, "e"'] \arrow[u, "m"] & A \arrow[u]
\end{tikzcd}$$

THEOREM 5. Let $\mathscr{C}$ be a category admitting finite coproducts and a terminal object. An object G in $\mathscr{C}$ is a group object if and only if $\mathrm{Hom}_{\mathscr{C}}(X,G)$ has the structure of a group for every object X of G.

Proof. Omitted. ∎

COROLLARY 6. Every abelian group is a group object in **Grp** and every topological group (with identity as basepoint) is a group object in $\mathbf{Top}_*$.

PROPOSITION 7. A group object in **Grp** is an abelian group.

Proof. Suppose (G,μ,η,ε) is a group object in **Grp**. Clearly, this gives an alternate group structure on G, which we denote by $(G,\otimes,a\mapsto\eta(a))$. Since $\mu\colon G\times G\to G$ is a group homomorphism with respect to the original group structure of G, we have

$$(ac)\otimes(bd)=(a\otimes b)(c\otimes d).$$

Due to Eckmann-Hilton, both group structures on G must agree and must be commutative, as desired. ∎

COROLLARY 8. The fundamental group of a topological group (with identity as basepoint) is abelian.

Proof. $\pi_1\colon \mathbf{Top}_*\to\mathbf{Grp}$ is a functor preserving finite products, and sending terminal objects to terminal objects, therefore, π_1 sends group objects to group objects. Since a topological group with identity as its basepoint is a group object in $\mathbf{Top}_*$, it follows that π_1 sends it to an abelian group due to Proposition 7. ∎