

MA 534: HOMEWORK 2

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1. PROBLEM 1

For $\varepsilon > 0$, define the functions $F_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_\varepsilon(z) := \begin{cases} \sqrt{z^2 + \varepsilon^2} - \varepsilon & z > 0 \\ 0 & z \leq 0. \end{cases}$$

It is clear that F_ε is a continuously differentiable function on \mathbb{R} with

$$F'_\varepsilon(z) = \begin{cases} \frac{z}{\sqrt{z^2 + \varepsilon^2}} & z > 0 \\ 0 & z \leq 0. \end{cases}$$

Furthermore,

$$\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(z) = \begin{cases} z & z > 0 \\ 0 & z \leq 0, \end{cases}$$

and

$$\lim_{\varepsilon \rightarrow 0^+} F'_\varepsilon(z) = \begin{cases} 1 & z > 0 \\ 0 & z \leq 0. \end{cases}$$

Now, for any test function $\varphi \in C_c^\infty(\Omega)$, we have, using integration by parts with respect to the variable x_j with $1 \leq j \leq n$,

$$\int_{\Omega} F_\varepsilon(u) \frac{\partial \varphi}{\partial x_j} dx = - \int_{\Omega} F'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi dx$$

Note that $\frac{\partial \varphi}{\partial x_j}$ is of compact support in Ω , and since $|F_\varepsilon(z)| \leq |z|$, it is clear from the dominated convergence theorem that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} F_\varepsilon(u) \frac{\partial \varphi}{\partial x_j} dx = \int_{\Omega} \lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(u) \frac{\partial \varphi}{\partial x_j} dx = \int_{\Omega} u^+ \frac{\partial \varphi}{\partial x_j} dx.$$

Next, note that $|F'_\varepsilon(z)| \leq 1$ and hence, the dominated convergence theorem applies again to give

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} F'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi dx = \int_{\Omega} \lim_{\varepsilon \rightarrow 0^+} F'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi dx = \int_{\{u > 0\}} \frac{\partial u}{\partial x_j} \varphi dx.$$

In conclusion,

$$\int_{\Omega} u^+ \frac{\partial \varphi}{\partial x_j} dx = - \int_{u > 0} \frac{\partial u}{\partial x_j} \varphi dx.$$

Hence,

$$\frac{\partial u^+}{\partial x_j} = \begin{cases} \frac{\partial u}{\partial x_j} & u > 0 \\ 0 & u \leq 0, \end{cases}$$

almost everywhere. Similarly, using the fact that $u^- = (-u)^+$, we get

$$\frac{\partial u^-}{\partial x_j} = \begin{cases} -\frac{\partial u}{\partial x_j} & -u > 0 \\ 0 & -u \leq 0 \end{cases} = \begin{cases} -\frac{\partial u}{\partial x_j} & u < 0 \\ 0 & u \geq 0, \end{cases}$$

almost everywhere. Finally, note that

$$\left| \frac{\partial u^+}{\partial x_j} \right| \leq \left| \frac{\partial u}{\partial x_j} \right|,$$

almost everywhere, therefore,

$$\int_{\Omega} |u^+|^2 dx + \int_{\Omega} |Du^+|^2 dx \leq \int_{\Omega} |u|^2 dx + \int_{\Omega} |Du|^2 dx < \infty,$$

i.e., $u^+ \in H^1(\Omega)$. Similarly, $u^- \in H^1(\Omega)$; and since $|u| = u^+ + u^-$, it follows that $|u| \in H^1(\Omega)$.

2. PROBLEM 2

Suppose first that $p = \infty$, then $u \in L^\infty(\mathbb{R}^n)$, i.e., u is bounded on \mathbb{R}^n . It follows from Liouville's theorem that u must be constant. Conversely, note that every constant function on \mathbb{R}^n is trivially harmonic and in L^∞ . Thus, a harmonic function on \mathbb{R}^n is in $L^\infty(\mathbb{R}^n)$ if and only if it is constant.

Next, let $1 \leq p < \infty$. Let $x \in \mathbb{R}^n$. Then, using the mean value property of harmonic functions, we have

$$\begin{aligned} |u(x)| &= \frac{n}{\omega_n} \left| \int_{B(x,1)} u(y) dy \right| \\ &\leq \frac{n}{\omega_n} \int_{B(x,1)} |u(y)| dy. \end{aligned}$$

If $p = 1$, then the above inequality shows that

$$|u(x)| \leq \frac{n}{\omega_n} \int_{\mathbb{R}^n} |u(y)| dy = \frac{n}{\omega_n} \|u\|_{L^1(\mathbb{R}^n)}.$$

Thus, u is a bounded harmonic function on \mathbb{R}^n , whence, due to Liouville's theorem, u must be a constant function. But a constant function is in $L^1(\mathbb{R}^n)$ if and only if it is identically zero, so $u \equiv 0$. Clearly, if $u \equiv 0$, then u is a harmonic function in $L^1(\mathbb{R}^n)$.

Finally, suppose $1 < p < \infty$ and let q denote its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Then, using Hölder's inequality we have

$$\begin{aligned} |u(x)| &\leq \frac{n}{\omega_n} \int_{B(x,1)} |u(y)| dy \\ &\leq \frac{n}{\omega_n} \left(\int_{B(x,1)} |u(y)|^p dy \right)^{\frac{1}{p}} \left(\int_{B(x,1)} 1 dy \right)^{\frac{1}{q}} \\ &\leq \left(\frac{n}{\omega_n} \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u(y)|^p dy \right)^{\frac{1}{p}} \\ &= \left(\frac{n}{\omega_n} \right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Thus, u is a bounded harmonic function on \mathbb{R}^n , whence due to Liouville's theorem, u must be a constant function. But a constant function is in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if it is identically zero, so $u \equiv 0$. Clearly if $u \equiv 0$, then it is a harmonic function in $L^p(\mathbb{R}^n)$.

3. PROBLEM 3

I shall assume $\mathcal{L} := - \sum_{1 \leq i, j \leq n} a_{ij} \partial_i \partial_j$ is uniformly elliptic and that the derivatives of the coefficient functions a_{ij} are bounded on Ω . Note that $\mathcal{L}u = 0$, and hence

$$0 = D(\mathcal{L}u) = - \sum_{1 \leq i, j \leq n} u_{ij} D a_{ij} - \sum_{1 \leq i, j \leq n} a_{ij} D u_{ij}.$$

Set $v : \Omega \rightarrow \mathbb{R}$ to be

$$v(x) = |Du(x)|^2 + \lambda u(x)^2,$$

where $\lambda > 0$ will be fixed later. Then

$$\begin{aligned}
\mathcal{L}v &= - \sum_{1 \leq i, j \leq n} a_{ij} \partial_{ij} (Du \cdot Du + \lambda u^2) \\
&= - \sum_{1 \leq i, j \leq n} a_{ij} (2Du_{ij} \cdot Du + 2Du_i \cdot Du_j + 2\lambda u u_{ij} + 2\lambda u_i u_j) \\
&= -2 \sum_{1 \leq i, j \leq n} a_{ij} Du_{ij} \cdot Du - 2 \sum_{1 \leq i, j \leq n} a_{ij} Du_i \cdot Du_j - 2\lambda \underbrace{\sum_{1 \leq i, j \leq n} a_{ij} u_{ij}}_{=0} - 2\lambda \sum_{1 \leq i, j \leq n} a_{ij} u_i u_j \\
&= 2 \sum_{1 \leq i, j \leq n} u_{ij} D a_{ij} \cdot Du - 2 \sum_{1 \leq i, j \leq n} a_{ij} Du_i \cdot Du_j - 2\lambda \sum_{1 \leq i, j \leq n} a_{ij} u_i u_j.
\end{aligned}$$

Since the derivatives of a_{ij} , the Hessian of u , and the gradient of u are bounded, the sum of the first two terms in the above expression are bounded in absolute value. Finally, since \mathcal{L} is uniformly elliptic, the matrix (a_{ij}) is uniformly positive definite, in the sense that there is a $\theta > 0$ such that

$$\sum_{1 \leq i, j \leq n} a_{ij} u_i u_j \geq \theta |Du|^2.$$

In particular, this means that the last term is at most $-2\lambda\theta|Du|^2$. Thus, we can choose $\lambda \gg 0$ such that $\mathcal{L}u \leq 0$. Finally, we invoke the weak maximum principle to obtain

$$\begin{aligned}
\| |Du|^2 \|_{L^\infty(\Omega)} &\leq \| |Du|^2 + \lambda u^2 \|_{L^\infty(\Omega)} \\
&\leq \| v \|_{L^\infty(\Omega)}^2 \\
&= \| v \|_{L^\infty(\partial\Omega)}^2 \\
&= \| |Du|^2 + \lambda u^2 \|_{L^\infty(\partial\Omega)} \\
&\leq \| Du \|_{L^\infty(\partial\Omega)}^2 + \lambda \| u \|_{L^\infty(\partial\Omega)}^2 \\
&\leq C (\| Du \|_{L^\infty(\partial\Omega)} + \| u \|_{L^\infty(\partial\Omega)})^2,
\end{aligned}$$

where $C := \max\{1, \lambda\}$. This implies the desired conclusion.

4. PROBLEM 4

Define the quantities $H, D, N : (0, 1) \rightarrow (0, \infty)$ as

$$\begin{aligned}
H(r) &:= \int_{\partial B_r} |u(x)|^2 dx \\
D(r) &:= \int_{\partial B_r} u(x) \frac{\partial u}{\partial \nu}(x) ds(x) = \int_{B_r} |\nabla u(x)|^2 dx \\
N(r) &:= \frac{rD(r)}{H(r)}.
\end{aligned}$$

Note that the equality in the definition of $D(r)$ follows from Green's first identity, since u is harmonic, and thus $\Delta u = 0$.

CLAIM. $D'(r) = \frac{n-2}{r} D(r) + 2 \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x).$

Proof. Performing the substitution $x = ry$ and using

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \frac{x}{r},$$

we can write

$$D(r) = \int_{\partial B_1} u(ry) (\nabla u(ry) \cdot y) r^{n-1} ds(y).$$

Differentiating using the product rule, we obtain

$$\begin{aligned} D'(r) &= \int_{\partial B_1} (\nabla u(r y) \cdot y)^2 r^{n-1} + (n-1) \int_{\partial B_1} u(r y) (\nabla u(r y) \cdot y) r^{n-2} ds(y) + \int_{\partial B_1} u(r y) \frac{d}{dr} (\nabla u(r y) \cdot y) r^{n-1} ds(y) \\ &= \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) + \underbrace{\frac{n-1}{r} \int_{\partial B_r} u(x) \frac{\partial u}{\partial \nu}(x) ds(x)}_{D(r)} + \int_{\partial B_1} u(r y) \frac{d}{dr} (\nabla u(r y) \cdot y) r^{n-1} ds(y). \end{aligned}$$

We now simplify the last term on the right hand side. Indeed, we can write

$$\nabla u(r y) \cdot y = \sum_{i=1}^n \partial_i u(r y) y_i,$$

and thus,

$$\frac{d}{dr} (\nabla u(r y) \cdot y) = \frac{d}{dr} \sum_{i=1}^n \partial_i u(r y) y_i = \sum_{1 \leq i, j \leq n} y_i y_j \partial_i \partial_j u(r y).$$

Hence, we have that the last term is equal to

$$\int_{\partial B_1} u(r y) \left(\sum_{1 \leq i, j \leq n} y_i y_j \partial_i \partial_j u(r y) \right) r^{n-1} ds(y) = \frac{1}{r^2} \int_{\partial B_r} u(x) \left(\sum_{1 \leq i, j \leq n} x_i x_j u_{ij}(x) \right) ds(x),$$

where we shall use the shorthand u_{ij} to denote $\partial_i \partial_j u$ henceforth. Note that since u is harmonic, it is C^∞ , and hence, the order of differentiation does not really matter.

We can write the above quantity as follows, and then using Green's first identity, we obtain

$$\int_{\partial B_r} \sum_{i=1}^n \frac{x_i}{r} u(x) \frac{\partial u_i}{\partial \nu}(x) ds(x) = \sum_{i=1}^n \left(\int_{B_r} \nabla \left(\frac{x_i}{r} u(x) \right) \cdot \nabla u_i(x) dx - \int_{B_r} \frac{x_i}{r} u(x) \Delta u_i(x) dx \right).$$

Since each u_i is harmonic, the second term is zero and we are left with only the first term, which is equal to

$$\sum_{i=1}^n \int_{B_r} \sum_{j=1}^n \partial_j \left(\frac{x_i}{r} u(x) \right) \partial_j u_i(x) dx = \sum_{i=1}^n \int_{B_r} \sum_{j=1}^n \left(\frac{x_i}{r} u_j(x) + \frac{1}{r} \delta_{ij} u(x) \right) u_{ij}(x) dx,$$

where δ_{ij} denotes the Kronecker symbol. The above is equal to

$$\sum_{1 \leq i, j \leq n} \int_{B_r} \frac{x_i}{r} u_{ij}(x) u_j(x) dx + \sum_{i=1}^n \int_{B_r} \frac{1}{r} u_{ii}(x) u(x) dx.$$

The second integral is equal to $\frac{1}{r} \int_{B_r} \Delta u(x) u(x) dx = 0$, since u is harmonic. Hence, we are left with

$$(\clubsuit) \quad \sum_{1 \leq i, j \leq n} \int_{B_r} \frac{x_i}{r} u_{ij}(x) u_j(x) dx.$$

Consider the function

$$w(x) = \sum_{i=1}^n \frac{x_i}{r} u_i(x)$$

defined on the unit ball B_1 . Note that $w(x) = \frac{\partial u}{\partial \nu}(x)$ on ∂B_r . Thus, we can write

$$\begin{aligned}
 \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) &= \int_{\partial B_r} w(x) \frac{\partial u}{\partial \nu}(x) ds(x) \\
 &= \int_{B_r} \nabla w(x) \cdot \nabla u(x) - w(x) \Delta u(x) dx \\
 &= \int_{B_r} \nabla w(x) \cdot \nabla u(x) dx \\
 &= \int_{B_r} \sum_{j=1}^n \partial_j \left(\sum_{i=1}^n \frac{x_i}{r} u_i(x) \right) \cdot u_j(x) dx \\
 &= \int_{B_r} \sum_{j=1}^n \sum_{i=1}^n \left(\frac{1}{r} \delta_{ij} u_i(x) + \frac{x_i}{r} u_{ij}(x) \right) u_j(x) dx \\
 &= \sum_{1 \leq i, j \leq n} \int_{B_r} \frac{x_i}{r} u_{ij}(x) u_j(x) dx + \sum_{i=1}^n \frac{1}{r} \int_{B_r} (u_i(x))^2 dx \\
 &= \sum_{1 \leq i, j \leq n} \int_{B_r} \frac{x_i}{r} u_{ij}(x) u_j(x) dx + \sum_{i=1}^n \frac{1}{r} \int_{B_r} (u_i(x))^2 dx.
 \end{aligned}$$

Thus, the quantity in (♣) is equal to

$$\int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) - \frac{1}{r} \int_{B_r} |\nabla u(x)|^2 dx = \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) - \frac{1}{r} D(r).$$

Substituting this back into the expression for $D'(r)$, we get

$$D'(r) = \frac{n-2}{r} D(r) + 2 \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x),$$

as desired. ■

CLAIM. $H'(r) = \frac{n-1}{r} H(r) + 2D(r)$.

Proof. Performing the substitution $x = ry$, we have

$$H(r) = \int_{\partial B_1} |u(ry)|^2 r^{n-1} ds(y).$$

Differentiating, we obtain

$$\begin{aligned}
 H'(r) &= (n-1) \int_{\partial B_1} |u(ry)|^2 r^{n-2} ds(y) + 2 \int_{\partial B_1} r^{n-1} u(ry) (\nabla u(ry) \cdot y) ds(y) \\
 &= \frac{n-1}{r} \int_{\partial B_1} |u(x)|^2 ds(x) + 2 \int_{\partial B_r} u(x) \left(\nabla u(x) \cdot \frac{x}{r} \right) ds(x) \\
 &= \frac{n-1}{r} \int_{\partial B_1} |u(x)|^2 ds(x) + 2 \int_{\partial B_r} u(x) \frac{\partial u}{\partial \nu}(x) ds(x) \\
 &= \frac{n-1}{r} H(r) + 2D(r),
 \end{aligned}$$

as desired. ■

Coming back to the problem at hand, we would like to show that

$$N(r) = \frac{rD(r)}{H(r)}$$

is an increasing function of r . To this end, we shall show that $N'(r) \geq 0$ for $r \in (0, 1)$. Indeed,

$$\begin{aligned} N'(r) &= \frac{(D(r) + rD'(r))H(r) - rD(r)H'(r)}{H(r)^2} \\ &= \frac{\left((n-1)D(r) + 2r \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x)\right)H(r) - D(r)\left((n-1)H(r) + 2rD(r)\right)}{H(r)^2} \\ &= \frac{2rH(r) \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) - 2rD(r)^2}{H(r)^2}. \end{aligned}$$

Using the Cauchy Schwarz inequality, we have

$$H(r) \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) \geq \left(\int_{\partial B_r} \left| u(x) \frac{\partial u}{\partial \nu}(x) \right| ds(x) \right)^2 \geq D(r)^2,$$

computation — whence $N'(r) \geq 0$, thereby completing the proof.

5. PROBLEM 5

For $R > r$, let $A(r)$ denote the annulus

$$A(r) := \{x \in \mathbb{R}^n : r < |x| < R\}.$$

Note that

$$\partial A(r) = \{x \in \mathbb{R}^n : |x| = r\} \cup \{x \in \mathbb{R}^n : |x| = R\}.$$

The (weak) maximum principle gives us

$$\sup_{r \leq |x| \leq R} |u(x)| = \sup_{x \in \partial A(r)} |u(x)| = \sup_{|x|=R} |u(x)|,$$

where the last equality follows from the fact that $u \equiv 0$ on ∂B_r .

Let $x_0 \in \mathbb{R}^n \setminus \overline{B}_r$. We shall show that $u(x_0) = 0$. Indeed, for $R > |x_0|$, we have

$$|u(x_0)| \leq \sup_{r \leq |x| \leq R} |u(x)| = \sup_{|x|=R} |u(x)|.$$

But according to the hypothesis on u , we have

$$\lim_{R \rightarrow \infty} \sup_{|x|=R} |u(x)| = 0,$$

so that $0 \leq |u(x_0)| \leq 0$, that is, $u(x_0) = 0$. It follows that u is identically 0 on $\mathbb{R}^n \setminus B_r$, as desired.

6. PROBLEM 7

Define the function $\varphi : (0, \infty) \rightarrow \mathbb{R}$ by

$$\varphi(r) = \frac{1}{|\partial B(x_0, r)|} \int_{\partial B(x_0, r)} u(y) dy.$$

Performing the substitution $y = x_0 + rz$, we obtain

$$\varphi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,1)} u(x_0 + rz) r^{n-1} dz = \frac{1}{\omega_n} \int_{\partial B(0,1)} u(x_0 + rz) dz.$$

Differentiating the above,

$$\varphi'(r) = \frac{1}{\omega_n} \int_{\partial B(0,1)} \nabla u(x_0 + rz) \cdot z dz = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0, r)} \nabla u(y) \cdot \frac{y - x_0}{r} dy = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0, r)} \frac{\partial u}{\partial \nu}(y) ds(y).$$

Using Green's first identity, we have

$$\varphi'(r) = \frac{1}{\omega_n r^{n-1}} \int_{B(x_0, r)} \Delta u(y) dy = \frac{1}{\omega_n r^{n-1}} \int_{B(x_0, r)} 1 dy = \frac{r}{n}.$$

Solving this ordinary differential equation, we obtain

$$\varphi(r) = \frac{r^2}{2n} + C$$

for all $r \in (0, \infty)$, where $C \in \mathbb{R}$ is a constant. Note that $u \geq 0$, and hence $\varphi(r) \geq 0$ for all $r > 0$. Hence, $C = \lim_{r \rightarrow 0^+} \varphi(r) \geq 0$, in particular,

$$\varphi(r) \geq \frac{r^2}{2n} \quad \forall r > 0.$$

Let

$$M(r) := \sup_{|x-x_0|=r} u(x).$$

Then

$$M(r) - \varphi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0, r)} M(r) - u(y) \, ds(y) \geq 0,$$

so that $M(r) \geq \frac{r^2}{2n}$. Finally, using the weak maximum principle for subharmonic functions, and recalling that $u \geq 0$, we get

$$\sup_{\overline{B(x_0, r)}} u = \sup_{|x-x_0|=r} u(x) = M(r) \geq \frac{r^2}{2n},$$

as desired.

7. PROBLEM 8

Let $x^* \in \overline{\Omega}$ be a point of maxima, i.e.,

$$u(x^*) = \sup_{x \in \overline{\Omega}} u(x) \geq 0,$$

where the inequality follows since $u|_{\partial\Omega} \equiv 0$. Suppose $u(x^*) > 1$. Then $x^* \notin \partial\Omega$, since u vanishes identically there. Thus $x^* \in \Omega$, whence $\Delta u(x^*) \leq 0$. This forces

$$u(x^*)^3 - u(x^*) \leq 0 \implies u(x^*) \in (-\infty, -1] \cup [0, 1].$$

But since $u(x^*) \geq 0$, we must have that $u(x^*) \in [0, 1]$, in particular, $u(x^*) \leq 1$, a contradiction. Thus $u(x^*) \leq 1$.

Similarly, let $x_* \in \overline{\Omega}$ be a point of minima, i.e.,

$$u(x_*) = \inf_{x \in \overline{\Omega}} u(x) \leq 0,$$

where the inequality follows since $u|_{\partial\Omega} \equiv 0$. Suppose $u(x_*) < -1$. Then $x_* \notin \partial\Omega$, since u vanishes identically there. Thus $x_* \in \Omega$, whence $\Delta u(x_*) \geq 0$. This forces

$$u(x_*)^3 - u(x_*) \geq 0 \implies u(x_*) \in [-1, 0] \cup [1, \infty).$$

But since $u(x_*) \leq 0$, we must have that $u(x_*) \in [-1, 0]$, in particular, $u(x_*) \geq -1$, a contradiction. Thus $u(x_*) \geq -1$. In conclusion, we have that for any $x \in \Omega$,

$$-1 \leq u(x_*) \leq u(x) \leq u(x^*) \leq 1,$$

as desired.

Suppose now that there is some point $x_0 \in \Omega$ with $u(x_0) = 1$. This is clearly a point of maxima because $-1 \leq u \leq 1$ on Ω . Let

$$\omega := \{x \in \Omega : u(x) = 1\}.$$

Since u is a continuous function, ω is closed in Ω and is non-empty as it contains x_0 . We claim that ω is open in Ω .

last part

8. PROBLEM 9

- (a) This is an immediate consequence of the mean value property. Indeed, since u is harmonic, for $x \in \mathbb{R}^n$, we have

$$|u(x)| = \left| \frac{n}{\omega_n} \int_{B(x, 1)} u(y) \, dy \right| \leq \frac{n}{\omega_n} \int_{B(x, 1)} |u(y)| \, dy \leq \frac{nC}{\omega_n}.$$

That is, u is a bounded harmonic function on \mathbb{R}^n , and hence is constant due to Liouville's theorem.

(b) Suppose $u(x) = \phi(|x|)$ is indeed a harmonic function on \mathbb{R}^n , where $\phi : [0, \infty) \rightarrow \mathbb{R}$. Using the mean value property, for $r > 0$, we can write

$$u(0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,r)} u(y) ds(y) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,r)} \phi(r) ds(y) = \phi(r).$$

Thus, $\phi(r) = u(0)$ for all $r > 0$. Thus, u is constant on $\mathbb{R}^n \setminus \{0\}$. But since u is continuous, u must be constant on all of \mathbb{R}^n , as desired.

9. PROBLEM 10

Let

$$M = \sup_{\partial\Omega} |\varphi|.$$

Then $-M \leq \varphi(x) \leq M$ for all $x \in \partial\Omega$.

Let $x^* \in \bar{\Omega}$ be a point of maxima, i.e.,

$$u(x^*) = \sup_{x \in \bar{\Omega}} u(x).$$

If $x^* \in \Omega$, then it is an interior point, and hence $u(x^*)^3 = \Delta u(x^*) \leq 0$, that is, $u(x^*) \leq 0$. Else if $x^* \in \partial\Omega$, then we must have that

$$\frac{\partial u}{\partial \nu}(x^*) \geq 0 \implies a(x^*)u(x^*) \leq \varphi(x^*) \leq M \implies u(x^*) \leq \frac{M}{a(x^*)} \leq \frac{M}{a_0}.$$

In either case, the inequality

$$u(x^*) \leq \frac{M}{a_0}$$

holds true.

On the other hand, let $x_* \in \bar{\Omega}$ be a point of minima, i.e.,

$$u(x_*) = \inf_{x \in \bar{\Omega}} u(x).$$

If $x_* \in \Omega$, then it is an interior point, and hence $u(x_*)^3 = \Delta u(x_*) \geq 0$, that is, $u(x_*) \geq 0$. Else if $x_* \in \partial\Omega$, then we must have that

$$\frac{\partial u}{\partial \nu}(x_*) \leq 0 \implies a(x_*)u(x_*) \geq \varphi(x_*) \geq -M \implies u(x_*) \geq \frac{-M}{a(x_*)} \geq -\frac{M}{a_0}.$$

In either case, the inequality

$$u(x_*) \geq -\frac{M}{a_0}$$

holds true. Hence,

$$\sup_{\bar{\Omega}} |u| = \max \{|u(x^*)|, |u(x_*)|\} \leq \frac{M}{a_0} = \frac{1}{a_0} \sup_{\partial\Omega} |\varphi|,$$

thereby completing the proof.