

Galois Categories and the Étale Fundamental Group

or, what should be taught in MA 811

Swayam Chube

Indian Institute of Technology, Bombay

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Galois Theory à la Grothendieck

Let k be a field and fix its separable and algebraic closures $k_s \subseteq \bar{k}$.
Let $G_k := \text{Gal}(k_s | k)$, which is a profinite group through the isomorphism:

$$\text{Gal}(k_s | k) \cong \varprojlim_{\substack{k \subseteq K \subseteq k_s \\ [L:k] < \infty}} \text{Gal}(L | k).$$

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If L is a finite separable extension of k , then there is a natural action of G_k on $\text{Hom}_k(L, k_s)$ given by

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The stabilizer of $\varphi \in \text{Hom}_k(L, k_s)$ is $\text{Gal}(k_s | \varphi(L))$, i.e., an open subgroup of G_k , whence $\text{Hom}_k(L, k_s)$ is a continuous G_k -set.

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Étale Algebras and the Fundamental Theorem

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Theorem (Fundamental Theorem of Galois Theory)

The functor mapping a finite étale k -algebra A to the finite G_k -set $\operatorname{Hom}_k(A, k_s)$ gives an anti-equivalence between the category of finite étale k -algebras and the category of finite sets with a continuous G_k -action. Here separable extensions correspond to transitive G_k -sets.

Galois Theory of Covering Spaces

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Theorem (Classification of Covering Spaces)

The aforementioned fibre functor induces an equivalence between the category of finite-sheeted covers of X and the category of continuous finite sets with a continuous $\widehat{\pi_1(X, x_0)}$ -action. Here connected covers correspond to transitive $\widehat{\pi_1(X, x_0)}$ -sets.

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All that remains is to establish an equivalence of some suitable categories.

Galois Categories

A *Galois category* is a pair (\mathcal{C}, F) where \mathcal{C} is a category and $F : \mathcal{C} \rightarrow \mathbf{Sets}$ is a functor with $F(a)$ a finite set for each object $a \in \mathcal{C}$, satisfying the following axioms:

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- G3 Every morphism u in \mathcal{C} factors as $u = u' u''$ where u' is a monomorphism and u'' is an epimorphism. Every monomorphism $f : X \rightarrow Y$ in \mathcal{C} is an isomorphism of X with a direct summand of Y .

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In this case, the functor F is called a *fundamental functor*.

Examples of Galois Categories

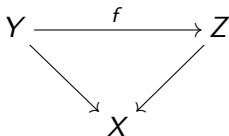
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- Let X be a connected scheme. Let \mathbf{FEt}_X denote the category of finite étale maps $Y \rightarrow X$ with morphisms $f : Y \rightarrow Z$ making



commute. Fix a geometric point $x_0 : \text{Spec } \Omega \rightarrow X$. This defines a fibre functor $\text{Fib}_{x_0} : \mathbf{FEt}_X \rightarrow \mathbf{Sets}$. The pair $(\mathbf{FEt}_X, \text{Fib}_{x_0})$ forms a Galois category.

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Further the tuple must be such that for each morphism $f : X \rightarrow Y$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

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The equivalence functor

Next, let $Y \in \mathcal{C}$. There is a natural action of $\text{Aut}_{\mathcal{C}}(F)$ on $F(Y)$ given by

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This gives a natural functor $H : \mathcal{C} \rightarrow \text{Aut}_{\mathcal{C}}(F)\text{-sets}$.

The Main Theorem

Theorem (Fundamental Theorem of Galois Categories)

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The Étale Fundamental Group

Definition

Let X be a connected scheme and $x_0 : \operatorname{Spec} \Omega \rightarrow X$ be a geometric point. The *étale fundamental group* $\pi_1^{\text{ét}}(X, x_0)$ to be $\operatorname{Aut}(\operatorname{Fib}_{x_0})$.

Recall that there is an anti-equivalence between the category of commutative rings and the category of affine schemes.

Therefore, there is an anti-equivalence between the category $\mathbf{F}\mathbf{Et}_{\operatorname{Spec} k}$ and the category of étale k -algebras.

Choosing a geometric point in $\operatorname{Spec} k$ is tantamount to fixing a separable closure k_s of k .

As we have seen earlier, $\mathbf{F}\mathbf{Et}_{\operatorname{Spec} k}$ is equivalent to the category G_k -**sets**. In particular, $\pi_1^{\text{ét}}(\operatorname{Spec} k, x_0) \cong \operatorname{Gal}(k_s | k)$.

Thank you for your attention!