Gorenstein Rings

Notes for the course MA 842: Topics in Algebra II

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§1 INJECTIVE MODULES

§§ Essential Extensions

REMARK 1.1. Let $M \subseteq N$ be an essential extension of R-modules and $\varphi : M \hookrightarrow P$ be an R-linear injective map. If φ extends to an R-linear map $\widetilde{\varphi} : N \to P$, then $\widetilde{\varphi}$ is injective too. Indeed, if $K = \ker \widetilde{\varphi} \neq 0$, then $K \cap M \neq 0$, a contradiction.

PROPOSITION 1.2. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Let M be an Artinian R-module. Then $\operatorname{Soc}_R(M) \subseteq M$ is an essential extension.

Proof. Let $0 \neq K \subseteq M$ be a submodule. Choose $0 \neq x \in K$. Since M is Artinian, the descending chain $Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2 x \supseteq \cdots$ stabilizes. Let $n \geqslant 0$ be the least positive integer such that $\mathfrak{m}^n x = \mathfrak{m}^{n+1} x$. Due to Nakayama's lemma, $\mathfrak{m}^n x = 0$, whence $n \geqslant 1$. It follows that $0 \neq \mathfrak{m}^{n-1} x \subseteq \operatorname{Soc}_R(M) \cap K$, as desired. ■

COROLLARY 1.3. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring and M an Artinian R-module. If $\dim_k \operatorname{Soc}_R(M) = d$, then $E_R(M) \cong E^{\oplus d}$.

Proof. Since $\operatorname{Soc}_R(M) \cong k^{\oplus d}$, it is clear that $E_R(\operatorname{Soc}_R(M)) \cong E^{\oplus d}$. The inclusion $\operatorname{Soc}_R(M) \hookrightarrow E^{\oplus d}$ can be extended to M to obtain a commutative diagram:

$$\begin{tikzcd}
M \arrow[dr, hook] & \\
\operatorname{Soc}_R(M) \arrow[u, hook] \arrow[r, hook] & E_R(\operatorname{Soc}_R(M)) \cong E^{\oplus d}
\end{tikzcd}$$

where all maps are inclusion. It follows that $M \hookrightarrow E^{\oplus d}$ is an essential extension. Since $E^{\oplus d}$ is an injective module, we have that $E_R(M) \cong E^{\oplus d}$. ■

§2 MATLIS DUALITY

DEFINITION 2.1. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. For an R-module M, set $M^\vee = \operatorname{Hom}_R(M, E)$. This is known as the *Matlis dual* of a module.

Clearly $(-)^\vee$ is a contravariant exact functor on the category of R-modules. Note that if $I \subseteq \mathfrak{m}$ is an ideal, then as we have seen earlier,

$$E_{R/I}(k) = \operatorname{Hom}_R(R/I, E) = (R/I)^\vee.$$

In particular, taking $I = \mathfrak{m}$, we see that $k^\vee \cong k$ as R-modules.

LEMMA 2.2. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then

(1) If $M \neq 0$, then $M^\vee \neq 0$.

(2) If $\lambda_R(M) < \infty$, then $\lambda_R(M^\vee) \neq 0$. Moreover, $\lambda_R(M) = \lambda_R(M^\vee)$.

Proof. (1) Let $0 \neq x \in M$. If $I = \operatorname{Ann}_R(x)$, then there is a natural inclusion $R/I \hookrightarrow M$ sending $\overline{1} \mapsto x$. Taking the Matlis dual, we have a surjection

$$M^\vee \twoheadrightarrow (R/I)^\vee = E_{R/I}(k) \neq 0,$$

consequently $M^\vee \neq 0$.

(2) We shall prove both statements by induction on $\lambda_R(M)$. If $\lambda_R(M) = 0$, then $M = 0$, so that $M^\vee = 0$ and we get $\lambda_R(M) = 0 = \lambda_R(M^\vee)$. Suppose now that $0 < \lambda_R(M) < \infty$. Then $\mathfrak{m} \in \operatorname{Ass}_R(M)$, and we have a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Since length is additive, $\lambda_R(N) = \lambda_R(M) - 1$; hence the induction hypothesis applies and $\lambda_R(N^\vee) = \lambda_R(N)$. Taking the Matlis dual of the above short exact sequence, we have

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow k^\vee \longrightarrow 0.$$

Since $k^\vee = 0$, we see that

$$\lambda_R(M^\vee) = \lambda_R(N^\vee) + 1 = \lambda_R(N) + 1 = \lambda_R(M),$$

as desired. ∎

THEOREM 2.3. Let $(R, \mathfrak{m}, k, E)$ be an Artinian local ring.

(1) E is a faithful finite R-module.

(2) The map

$$\mu : R \longrightarrow \operatorname{Hom}_R(E, E) \qquad a \longmapsto \mu_a$$

is an isomorphism of R-modules and rings.

(3) Given a finite R-module M, the natural map

$$\varphi_M : M \longrightarrow M^{\vee\vee} \qquad m \longmapsto \mathrm{ev}_m$$

is an isomorphism.

Proof. (1) Suppose $a \in R$ is such that $aE = 0$. Then

$$R^\vee = \operatorname{Hom}_R(R, E) = E = (E :_E a) \cong \operatorname{Hom}_R(R/aR, E) = (R/aR)^\vee.$$

Since R is Artinian, we then have

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R\left((R/aR)^\vee\right) = \lambda_R(R/aR) \implies \lambda_R(aR) = 0,$$

consequently, $a = 0$, i.e., E is a faithful R-module.

Next, since R is Artinian, $\mathfrak{m} \in \operatorname{Ass}_R(R)$, consequently, there is an injection $k = R/\mathfrak{m} \hookrightarrow R$. Due to Remark 1.1 extends to an inclusion $E \hookrightarrow R$, consequently, E is a finite R-module.

(2) First note that μ is injective due to (1). But note that

$$\infty > \lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = \lambda_R(E^\vee) = \lambda_R\left(\operatorname{Hom}_R(E, E)\right),$$

consequently μ is an isomorphism.

(3) It suffices to show that φ_M is injective since $\lambda_R(M) = \lambda_R(M^{\vee\vee})$. Suppose $0 \neq x \in M$ is such that $\varphi_M(x) = 0$, that is, for all $f \in \operatorname{Hom}_R(M, E)$, $f(x) = 0$. Let $I = \operatorname{Ann}_R(x)$. Now, there is a non-zero map

$$\psi : R/I \twoheadrightarrow R/\mathfrak{m} = k \hookrightarrow E,$$

which extends to a non-zero map $f : M \to E$ since $R/I \hookrightarrow M$ through $\overline{1} \mapsto x$. Thus, $f(x) = \psi(\overline{1}) \neq 0$, a contradiction. ∎

INTERLUDE 2.4 (ON $\widehat{R}$-MODULES). Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module such that $\Gamma_{\mathfrak{m}}(M) = M$. We contend that M is an $\widehat{R}$-module in a natural way. To this end, we need only define $\widehat{a} \cdot m$ for $\widehat{a} \in \widehat{R}$ and $m \in M$.

Let $\widehat{a} = (a_1, a_2, \ldots)$, where we are using the isomorphism

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

Since $\Gamma_{\mathfrak{m}}(M) = M$, there is a positive integer $n \geqslant 1$ such that $\mathfrak{m}^n m = 0$. Hence, for $k \geqslant n$, we have $a_k \cdot m = a_n \cdot m$, as $a_k - a_n \in \mathfrak{m}^n$. In light of this, we define $\widehat{a} \cdot m = a_n \cdot m$. We must show that this makes M into an $\widehat{R}$-module.

Let $m_1, m_2 \in M$ and $\widehat{a} = (a_1, a_2, \ldots) \in \widehat{R}$. There are positive integers $n_1, n_2 \geqslant 1$ such that $\mathfrak{m}^{n_1} m_1 = 0 = \mathfrak{m}^{n_2} m_2$; then $\mathfrak{m}^n m_1 = 0 = \mathfrak{m}^n m_2$ for all $n \geqslant \max\{n_1, n_2\}$. Hence, for all such $n \geqslant 1$,

$$\widehat{a} \cdot (m_1 + m_2) = a_n \cdot (m_1 + m_2) = a_n \cdot m_1 + a_n \cdot m_2 = \widehat{a} \cdot m_1 + \widehat{a} \cdot m_2.$$

Next, let $\widehat{a}, \widehat{b} \in \widehat{R}$ and $m \in M$ with

$$\widehat{a} = (a_1, a_2, \ldots) \qquad \text{and} \qquad \widehat{b} = (b_1, b_2, \ldots).$$

There is a positive integer n such that $\mathfrak{m}^n m = 0$. Then

$$(\widehat{a} + \widehat{b}) \cdot m = (a_n + b_n) \cdot m = a_n \cdot m + b_n \cdot m = \widehat{a} \cdot m + \widehat{b} \cdot m.$$

Finally, note that $\widehat{b} \cdot m = b_n m$ and $\mathfrak{m}^n \left(\widehat{b} \cdot m\right) = 0$, so that

$$\widehat{a} \cdot \left(\widehat{b} \cdot m\right) = \widehat{a} \cdot (b_n \cdot m) = a_n \cdot (b_n \cdot m) = (a_n b_n) \cdot m = (\widehat{a}\widehat{b}) \cdot m.$$

This shows that M is indeed an $\widehat{R}$-module as described above. Further, since $R \to \widehat{R}$ is the diagonal map, it follows that the $\widehat{R}$-module structure on M agrees with the R-module struture through the diagonal map. In particular, this means that:

> A subset of M is an R-submodule if and only if it is an $\widehat{R}$-submodule.

As a result, M is Noetherian (resp. Artinian) as an R-module if and only if it is so as an $\widehat{R}$-module.

INTERLUDE 2.5 (ON MAPS BETWEEN $\mathfrak{m}$-POWER TORSION MODULES). Again, let $(R, \mathfrak{m}, k)$ be a local ring and suppose M and N are R-modules such that $\Gamma_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(N)$. By Interlude 2.4, we know that they are $\widehat{R}$-modules in a natural way. Let $\varphi : M \to N$ be an R-linear map. We contend that φ is also $\widehat{R}$-linear. Indeed, let $m \in M$ and $\widehat{a} = (a_1, a_2, \ldots) \in \widehat{R}$. There is a positive integer $n \geqslant 1$ such that $\mathfrak{m}^n m = 0$, and hence, $\mathfrak{m}^n \varphi(m) = 0$. It follows that

$$\varphi(\widehat{a} \cdot m) = \varphi(a_n \cdot m) = a_n \cdot \varphi(m) = \widehat{a} \cdot \varphi(m),$$

as desired.

THEOREM 2.6. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring.

(1) $\Gamma_{\mathfrak{m}}(E) = E$, and hence E is an $\widehat{R}$ module and for every R-module M, $M^\vee$ is $\mathfrak{m}$-power torsion.

(2) $E \cong E_{\widehat{R}}(k)$ as $\widehat{R}$-modules.

(3) $R^{\vee\vee} = \operatorname{Hom}_R(E, E) \cong \widehat{R}$ as R-modules.

(4) E is an Artinian R-module.

Proof. (1) That E is an $\widehat{R}$-module follows immediately from Interlude 2.4. Finally, $M^\vee = \operatorname{Hom}_R(M, E)$ is $\mathfrak{m}$-power torsion because E is so.

(2) The containment $k \subseteq E$ is an essential extension of R-modules, both of which are $\mathfrak{m}$-power torsion. Due to Interlude 2.4, it follows that it is an essential extension of $\widehat{R}$-modules too. Now, due to Remark 1.1, there is a commutative diagram of inclusions

$$\begin{tikzcd} E \arrow[dr] & \\ k \arrow[u] \arrow[r] & E_{\widehat{R}}(k), \end{tikzcd}$$

where all maps are $\widehat{R}$-linear. It follows that $E \hookrightarrow E_{\widehat{R}}(k)$ is an essential extension of $\widehat{R}$-modules, and consequently, an essential extension of R-modules. Since E is R-injective, we must have that the inclusion is an isomorphism of R-modules. Finally, due to Interlude 2.5, this is an isomorphism of $\widehat{R}$-modules.

(3) TODO: Write this out in gory detail.

(4) Let $M_1 \supseteq M_2 \supseteq \cdots$ be a chain of R-submodules in E. There are commutative diagrams

$$\begin{tikzcd} M_{j+1} \arrow[r, hook, "\iota_{j+1}"] \arrow[d, hook] & E \\ M_j \arrow[ur, hook, "\iota_j"'] & \end{tikzcd}$$

whose Matlis dual furnishes commutative diagrams

$$\begin{tikzcd} \widehat{R} = E^\vee \arrow[r, two heads, "\varphi_j"] \arrow[dr, two heads, "\varphi_{j+1}"'] & M_j^\vee \arrow[d, two heads] \\ & M_{j+1}^\vee \end{tikzcd} .$$

Note that all Matlis duals are $\mathfrak{m}$-power torsions and hence due to Interlude 2.5, the φ_j's are $\widehat{R}$-linear. Let $I_j = \ker \varphi_j \subseteq \widehat{R}$, which is an ideal. Due to the commutative diagram, it is clear that there is an ascending chain $I_j \subseteq I_{j+1}$. Since $\widehat{R}$ is Noetherian, this chain stabilizes, say $I_n = I_{n+1} = \ldots$.

Then due to the first isomorphism theorem, $M_j^\vee \twoheadrightarrow M_{j+1}^\vee$ is an isomorphism for all $j \geqslant n$. Let $C_j = \operatorname{coker}\left(M_{j+1} \hookrightarrow M_j\right)$. The exactness of the Matlis dual gives $C_j^\vee = 0$, which, due to Lemma 2.2, implies that $C_j = 0$, that is, $M_{j+1} \hookrightarrow M_j$ is an isomorphism for all $j \geqslant n$, i.e., the descending chain stabilizes, as desired. ■

THEOREM 2.7 (MATLIS DUALITY, VERSION 1). Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{\begin{matrix} \text{isomorphism classes of} \\ \text{finitely generated} \\ \widehat{R}\text{-modules} \end{matrix}\right\} \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \left\{\begin{matrix} \text{isomorphism classes of} \\ \text{Artinian } R\text{-modules} \end{matrix}\right\}.$$

Proof. Let M be an Artinian R-module and let $d = \dim_k \operatorname{Soc}_R(M)$. Due to Corollary 1.3, $E_R(M) \cong E^{\oplus d}$, so that there is an inclusion $M \hookrightarrow E^{\oplus d}$, which upon taking the Matlis dual furnishes an $\widehat{R}$-linear surjection $\widehat{R}^{\oplus d} \twoheadrightarrow M^\vee$. Thus $M^\vee$ is a finite $\widehat{R}$-module.

Conversely, suppose M is a finite $\widehat{R}$-module. Thus, there is a surjection $\widehat{R}^{\oplus n} \twoheadrightarrow M$. Taking the Matlis dual, we obtain an injection $M^\vee \hookrightarrow \left(\widehat{R}^\vee\right)^{\oplus n}$.

There is a natural "evaluation map" $\mathrm{ev} : M \to M^{\vee\vee}$, which we shall show is an isomorphism. That ev is injective follows in the same way as Theorem 2.3 (3). Next, since $\lambda_R(M) < \infty$, we have that $\lambda_R(M) = \lambda_R(M^\vee) = \lambda_R(M^{\vee\vee})$, whence ev is an isomorphism. ■