Local Cohomology

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§1 The I-torsion functor

Definition 1.1. Let R be a ring, $I \trianglelefteq R$ an ideal, and M an R-module. Define

$$\Gamma_I(M) := \left\{x \in M : \text{there is a positive integer } n \in \mathbb{N} \text{ such that } I^n x = 0\right\} = \bigcup_{n \geqslant 1} (0 :_M I^n).$$

This is known as the *I-torsion functor*.

It is clear that $\Gamma_I(M)$ is a submodule of M and any R-linear map $\varphi : M \to N$ restricts to an R-linear map $\Gamma_I(\varphi) : \Gamma_I(M) \to \Gamma_I(N)$. Thus, $\Gamma_I : {}_R\mathfrak{Mod} \to {}_R\mathfrak{Mod}$ is a functor.

Lemma 1.2. The functor Γ_I is left-exact.

Proof. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be a short exact sequence of R-modules. ■

Definition 1.3. The right derived functors of $\Gamma_I : {}_R\mathfrak{Mod} \to {}_R\mathfrak{Mod}$ are called the *local cohomology functors with support in I*.

$\boxed{\text{Henceforth } R \text{ is a Noetherian ring unless specified otherwise.}}$

There are some properties of Γ_I which are trivial to verify:

- $\Gamma_I(M) = \Gamma_{\sqrt{I}}(M)$ as submodules of M.
- If $\Gamma_I(M) = 0$, then I cannot be contained in any associated prime of M. In particular, if M is a finite R-module, then $\operatorname{Ass}_R(M)$ is finite, and hence, using Prime Avoidance, there is an M-regular element in I, that is, $\operatorname{depth}(I, M) \geqslant 1$.
- Given a family of R-modules $\{M_\alpha\}_{\alpha \in \Lambda}$, $\Gamma_I\left(\bigoplus_{\alpha \in \Lambda} M_\alpha\right) = \bigoplus_{\alpha \in \Lambda} \Gamma_I(M_\alpha)$ as submodules of $\bigoplus_{\alpha \in \Lambda} M_\alpha$.
- If $S \subseteq R$ is a multiplicative subset, then $\Gamma_{S^{-1}I}(S^{-1}M) = S^{-1}\Gamma_I(M)$ as submodules of $S^{-1}M$.
- For $\mathfrak{p} \in \operatorname{Spec}(R)$,
$$\Gamma_I(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & I \subseteq \mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$
In particular if E is an injective R-module, then $\Gamma_I(E)$ is an injective R-module, and is a direct summand of E.

Since all the above isomorphisms are natural, these extend to isomorphisms on local cohomology, that is, for $i \geqslant 0$:

- $H^i_I(M) = \Gamma_{\sqrt{I}}(M)$ as submodules of M.
- Given a family of R-modules $\{M_\alpha\}_{\alpha \in \Lambda}$, $H^i_I\left(\bigoplus_{\alpha \in \Lambda} M_\alpha\right) = \bigoplus_{\alpha \in \Lambda} \Gamma_I(M_\alpha)$ as submodules of $\bigoplus_{\alpha \in \Lambda} M_\alpha$.

- If $S \subseteq R$ is a multiplicative subset, then $H^i_{S^{-1}I}(S^{-1}M) = S^{-1}\Gamma_I(M)$ as submodules of $S^{-1}M$.

LEMMA 1.4. Let M be an R-module. If $\Gamma_I(M) = M$, then $\Gamma_I(E_R(M)) = E_R(M)$.

Proof. Suppose not and choose some $x \in E_R(M) \setminus \Gamma_I(E_R(M))$. Since R is Noetherian, there is an associated prime $\mathfrak{p}$ of $E_R(M)$ containing $\operatorname{Ann}_R(x)$. But since there is no power of I annihilating x, we must have $I \not\subseteq \mathfrak{p}$.

On the other hand, since $\operatorname{Ass}_R(E_R(M)) = \operatorname{Ass}_R(M)$, it follows that there is some $y \in M$ with $\mathfrak{p} = \operatorname{Ann}_R(y)$. Further, since $\Gamma_I(M) = M$, there is a positive integer $n > 0$ such that $I^n y = 0$, i.e., $I^n \subseteq \mathfrak{p}$, and hence, $I \subseteq \mathfrak{p}$, a contradiction. ∎

COROLLARY 1.5. Let M be an R-module. If $\Gamma_I(M) = M$, then $H^i(M) = 0$ for $i > 0$.

Proof. Let $0 \to M \to E^\bullet$ be a minimal injective resolution of M. Due to Lemma 1.4, it follows that $\Gamma_I(E^i) = E^i$ for $i \geqslant 0$, and the conclusion follows, since the resolution remains unchanged after applying Γ_I. ∎

PROPOSITION 1.6. Let M be an R-module, and set $N := M/\Gamma_I(M)$. Then $\Gamma_I(N) = 0$ and $H^i_I(N) \cong H^i_I(M)$ for $i > 0$.

Proof. Set $L := \Gamma_I(M)$. Then there is a short exact sequence $0 \to L \to M \to N \to 0$, and L is Γ_I-acyclic. It is then clear from the long exact sequence that $H^i_I(N) \cong H^i_I(M)$ for $i > 0$. Finally, since the induced map $\Gamma_I(L) \to \Gamma_I(M)$ is an isomorphism and $H^1_I(L) = 0$, it follows that $\Gamma_I(N) = 0$. ∎

THEOREM 1.7 (GROTHENDIECK VANISHING THEOREM). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $I \subseteq R$ an ideal, and M a finite R-module. Then $H^j_I(M) = 0$ for $j > \dim_R M$.

Proof. We argue by induction on $d := \dim_R(M)$. If $d = 0$, then M is Artinian, so that $\operatorname{Ass}_R(M) = \{\mathfrak{m}\}$. It follows that every element of M is annihilated by a power of $\mathfrak{m}$, and thus by a power of I. Consequently, $\Gamma_I(M) = M$. Due to Corollary 1.5, $H^i_I(M) = 0$ for $i > 0$, and this establishes the base case.

Suppose now that $d > 0$. Set $N := M/\Gamma_I(M)$. As we have seen in Proposition 1.6, $\Gamma_I(N) = 0$ and $H^i_I(N) \cong H^i_I(M)$ for $i > 0$. Further, $\dim_R N \leqslant d$. If this inequality is strict, then we are done due to the induction hypothesis. Hence we may assume that $\dim_R N = d$. As we remarked earlier, since $\Gamma_I(N) = 0$, and N is a finite R-module, $\operatorname{depth}(I, N) \geqslant 1$. Choose an N-regular element $a \in I$. Let $x \mapsto \overline{x}$ denote the natural surjection $M \twoheadrightarrow M/aM =: \overline{M}$ and $\mu_a : M \to M$ be multiplication by a. The short exact sequence $0 \to M \xrightarrow{\mu_a} M \to \overline{M} \to 0$ induces a long exact sequence:

$$\cdots \to H^{i-1}_I(\overline{M}) \to H^i_I(M) \xrightarrow{\mu_a} H^i_I(M) \to H^i_I(\overline{M}) \to \cdots.$$

For $i > d$, note that $i - 1 > d - 1 = \dim_R \overline{M}$, so that $H^{i-1}_I(\overline{M}) = H^i_I(\overline{M}) = 0$. This shows that $\mu_a : H^i_I(M) \to H^i_I(M)$ is an isomorphism of R-modules. Recall that $H^i_I(M)$ is I-torsion and $a \in I$. If $H^i_I(M) \neq 0$, then for $n \gg 0$, the composition μ_a^n would have non-trivial kernel, which is absurd since it is an isomorphism. This shows that $H^i_I(M) = 0$ for $i > d$, as desired. ∎

PROPOSITION 1.8. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with $d = \dim R$. Then

$$H^d_{\mathfrak{m}}(R) \cong E_R(k).$$

Proof. It is well-known that the minimal injective resolution of a Gorenstein local ring looks like:

$$0 \to R \to \bigoplus_{\operatorname{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p}) \to \bigoplus_{\operatorname{ht}\mathfrak{p}=1} E_R(R/\mathfrak{p}) \to \cdots \to E_R(k) \to 0.$$

Further, it is clear that

$$\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(k) & \mathfrak{p} = \mathfrak{m} \\ 0 & \text{otherwise,} \end{cases}$$

whence the conclusion follows. ∎

It is also possible to characterize the depth of an ideal using the local cohomology modules:

PROPOSITION 1.9. Let R be a Noetherian ring and $I \trianglelefteq R$ an ideal. If M is a finite R-module such that $IM \neq M$, then

$$\operatorname{depth}(I, M) = \inf\left\{ i : H^i_I(M) \neq 0 \right\}.$$

Proof. We induct on $d = \operatorname{depth}(I, M)$. If $d = 0$, then $I \subseteq \mathfrak{p}$ for some associated prime $\mathfrak{p}$ of M. ∎

§2 Čech Cohomology

DEFINITION 2.1. Let R be a Noetherian ring, and $\underline{a} = a_1, \ldots, a_n \in R$. Let $\check{C}^\bullet(a_i)$ denote the cochain complex:

$$\cdots 0 \to R \to R_{a_i} \to 0 \to \cdots,$$

and define $\check{C}^\bullet(\underline{a})$ to be the cochain complex:

$$\check{C}^\bullet(a_1) \otimes \cdots \otimes \check{C}^\bullet(a_n).$$

Further, if M is an R-module, then define $\check{C}^\bullet(\underline{a}, M) := \check{C}^\bullet(\underline{a}) \otimes M$. This is known as the *Čech complex*. The cohomology modules of this cochain complex are known as the *Čech cohomology modules* and are denoted by $\check{H}^i_{\underline{a}}(M)$.

REMARK 2.2. Since the tensor product of chain complexes is associative and commutative, the order of tensoring above doesn't matter.

THEOREM 2.3. Let R be a Noetherian ring, $I \trianglelefteq R$ an ideal, and $\underline{a} := a_1, \ldots, a_n \in R$ such that $\sqrt{(a_1, \ldots, a_n)} = \sqrt{I}$. Then

$$\check{H}^j_{\underline{a}}(M) \cong H^j_I(M)$$

for every R-module M.

Proof. ■

PROPOSITION 2.4. If $R \to S$ is a flat morphism of Noetherian rings, M an R-module, and I an ideal in R, then

$$H^j_I(M) \otimes_R S \cong H^j_{IS}(M \otimes_R S)$$

as S-modules.

PROPOSITION 2.5. Let $R \to S$ be a homomorphism of Noetherian rings, $I \trianglelefteq R$ an ideal, and M an S-module. Then

$$H^j_I(M) \cong H^j_{IS}(M)$$

as S-modules.

DEFINITION 2.6. Let R be a Noetherian ring and $I \trianglelefteq R$ an ideal. Define the *cohomological dimension* of I in R to be

$$\operatorname{cdim}(I, R) := \inf\left\{i : H^j_I(M) = 0 \text{ for all } R\text{-modules } M \text{ and all } j > i\right\}.$$

PROPOSITION 2.7.

$$\operatorname{cdim}(I, R) = \inf\left\{i : H^j_I(R) = 0 \text{ for all } j > i\right\}.$$

COROLLARY 2.8. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then $\operatorname{cdim}(I, R) = \operatorname{cdim}(\widehat{I}, \widehat{R})$.

Proof. This is immediate from Proposition 2.7 and the fact that $R \to \widehat{R}$ is faithfully flat. ■

§3 Bass numbers of local cohomology modules

The goal of this section is to prove the following theorem from [HS93]:

THEOREM 3.1 (HUNEKE-SHARP, 1993). Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. If $I \trianglelefteq R$ is an ideal and $\mathfrak{p} \in \operatorname{Spec}(R)$, then

$$\mu^i_R\left(\mathfrak{p}, H^j_I(R)\right) \leqslant \mu^i_R\left(\mathfrak{p}, \operatorname{Ext}^j_A(A/I, A)\right) < \infty.$$

We begin with some preliminaries on the "Frobenius functor".

DEFINITION 3.2. Let R be a ring of characteristic $p > 0$. There is a ring homomorphism

$$\mathrm{Frob} : R \to R \qquad x \mapsto x^p.$$

Let $\widetilde{R}$ denote the (R, R)-bimodule where the left action of R is by the usual multiplication while the right action is through Frob. The *Frobenius functor* $\mathbf{F}_R : {}_R\mathfrak{Mod} \to {}_R\mathfrak{Mod}$ is defined to be the functor $\widetilde{R} \otimes_R -$.

PROPOSITION 3.3. With the setup of Definition 3.2,

(1) $\mathbf{F}(R) \cong R$

(2) $\mathbf{F}(R/I) \cong R/I^{[p]}$

(3) $S^{-1}\mathbf{F}(M) \cong \mathbf{F}_{S^{-1}R}(S^{-1}M)$,

as R-modules.

Proof. ∎

We have the following remarkable theorem of Kunz [Kun69], which we state without proof:

THEOREM 3.4 (KUNZ, 1969). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic $p > 0$. Then R is regular if and only if $\mathbf{F}$ is an exact functor.

LEMMA 3.5. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of characteristic $p > 0$. If $E = E_R(k)$, then $\mathbf{F}(E) \cong E$ as R-modules.

Proof. ∎

THEOREM 3.6. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of characteristic $p > 0$. If E is an injective R-module, then $\mathbf{F}(E) \cong E$ as R-modules.

Proof. ∎

LEMMA 3.7. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $(L_i)_{i \in \mathbb{N}}$ a direct system of R-modules such that there exists a non-negative integer $h \geqslant 0$ with $\dim_k(0 :_{L_i} \mathfrak{m}) \leqslant h$ for all $i \in \mathbb{N}$. If L denotes the direct limit of $(L_i)_{i \in \mathbb{N}}$, then $\dim_k(0 :_L \mathfrak{m}) \leqslant h$.

Proof. ∎

Proof of Theorem 3.1. ∎

References

[HS93] Craig L Huneke and Rodney Y Sharp. Bass numbers of local cohomology modules. *Transactions of the American Mathematical Society*, 339(2):765–779, 1993.

[Kun69] Ernst Kunz. Characterizations of regular local rings of characteristic p. *American Journal of Mathematics*, 91(3):772–784, 1969.