Product Developments

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§1 The Space of Holomorphic Functions

Theorem 1.1. If $\Omega \subseteq \mathbb{C}$ is open, then there is a sequence $(K_n)_{n\geqslant 1}$ of compact subsets of Ω such that $\Omega = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:

(i) $K_n \subseteq K_{n+1}^{\circ}$.

(ii) If $K \subseteq \Omega$ is compact, then $K \subseteq K_n$ for some $n \geqslant 1$.

(iii) For every $n \geqslant 1$, each component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$.

Proof. ■

Let $\Omega \subseteq \mathbb{C}$ be an open set, and (X, d) be a complete metric space. Let $C(\Omega, X)$ denote the set of all continuous functions from Ω to X. Our first goal will be to define a complete metric on this space. In particular, when $X = \mathbb{C}$, $C(\Omega, X)$ will be a Fréchet space (not that we shall ever use this fact seriously).

Begin with an exhaustion $(K_n)_{n\geqslant 1}$ of Ω. That is,

$$\Omega = \bigcup_{n=1}^{\infty} K_n \quad \text{and} \quad K_n \subseteq K_{n+1}^{\circ} \quad \forall\, n \geqslant 1.$$

We may further assume that $K_n \neq \emptyset$ for all $n \geqslant 1$. For functions $f, g \in C(\Omega, X)$, define

$$\rho_n(f,g) = \sup\{d(f(z), g(z)) \colon z \in K_n\}.$$

Further, define

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)}. \tag{$\clubsuit$}$$

Clearly the right hand side converges for all $f, g \in C(\Omega, X)$. We shall show that ρ is a metric on $C(\Omega, X)$.

Lemma 1.2. If (S, d) is a metric space then

$$\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$$

is a metric on S inducing the same topology. Further, a sequence in S is Cauchy for d if and only if it is Cauchy for μ.

Proof. ■

Proposition 1.3. $\big(C(\Omega, X), \rho\big)$ is a metric space.

Proof. It is clear from the definition that $\rho(f,g) = \rho(g,f)$ for all $f, g \in C(\Omega, X)$. Further, due to Lemma 1.2, each factor in the infinite sum satisfies the triangle inequality, and so ρ also satisfies the triangle inequality. Finally, suppose $\rho(f,g) = 0$. Since the infinite sum is a sum of positive terms, they must all be zero, consequently, $\rho_n(f,g) = 0$ for all $n \geqslant 1$. That is, $f(z) = g(z)$ for all $z \in K_n$ for all $n \geqslant 1$. But $\Omega = \bigcup_{n=1}^{\infty} K_n$, and hence $f = g$ on Ω. ■

Lemma 1.4. Let ρ be the metric as in ($\clubsuit$).

(1) If $\varepsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subseteq \Omega$ such that for $f, g \in C(\Omega, X)$,

$$\sup\{d(f(z), g(z)) \colon z \in K\} < \delta \implies \rho(f,g) < \varepsilon.$$

(2) If $\delta > 0$ and a compact set K are given, then there is an $\varepsilon > 0$ such that for $f, g \in C(\Omega, X)$,

$$\rho(f,g) < \varepsilon \implies \sup\{d(f(z), g(z)) \colon z \in K\} < \delta.$$

Proof. (1) Since the sum $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, there is a positive integer N such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

Set $K = K_N$ and choose $\delta > 0$ such that

$$\frac{\delta}{1+\delta} < \frac{\varepsilon}{2}.$$

If $f, g \in C(\Omega, X)$ are such that $\sup\{d(f(z), g(z)) \colon z \in K\} < \delta$, then

$$\rho(f,g) = \sum_{n=1}^{N} \frac{1}{2^n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \frac{\varepsilon}{2} \sum_{n=1}^{N} \frac{1}{2^n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon.$$

(2) Choose a positive integer N such that $K \subseteq K_N$. If $\rho(f,g) < \varepsilon$, then

$$\frac{1}{2^N} \frac{\rho_N(f,g)}{1+\rho_N(f,g)} \leqslant \rho(f,g) < \varepsilon.$$

Set $\varepsilon = \frac{1}{2^N} \frac{\delta}{1+\delta}$. Then

$$\frac{\rho_N(f,g)}{1+\rho_N(f,g)} < \frac{\delta}{1+\delta}.$$

Since the function $t \mapsto \frac{t}{1+t}$ is an increasing function, we have that $\rho_N(f,g) < \delta$, and hence

$$\sup\{d(f(z),g(z)) \colon z \in K\} \leqslant \rho_N(f,g) < \delta,$$

as desired. ■

Proposition 1.5. (1) A set $\mathscr{U} \subseteq C(\Omega,X)$ is open if and only if for each $f \in \mathscr{U}$ there is a compact set $K \subseteq \Omega$ and a $\delta > 0$ such that

$$\{g \in C(\Omega,X) \colon d(f(z),g(z)) < \delta,\ \forall\ z \in K\} \subseteq \mathscr{U}.$$

(2) A sequence $(f_n)_{n\geqslant 1}$ in $C(\Omega,X)$ converges to $f \in C(\Omega,X)$ if and only if $(f_n)_{n\geqslant 1}$ converges to f uniformly on all compact subsets of Ω.

Proof. (1) Suppose $\mathscr{U}$ is open. Then there is an $\varepsilon > 0$ such that whenever $\rho(f,g) < \varepsilon$, $g \in \mathscr{U}$. Using Lemma 1.4, there is a compact set $K \subseteq \Omega$ and a $\delta > 0$ such that

$$\sup\{d(f(z),g(z)) \colon z \in K\} < \delta \implies \rho(f,g) < \varepsilon \implies g \in \mathscr{U}.$$

Conversely, suppose for every $f \in \mathscr{U}$, there is a compact set $K \subseteq \Omega$ and a $\delta > 0$ such that

$$\{g \in C(\Omega,X) \colon d(f(z),g(z)) < \delta,\ \forall\ z \in K\} \subseteq \mathscr{U}.$$

Again, using Lemma 1.4, there is an $\varepsilon > 0$ such that

$$\rho(f,g) < \varepsilon \implies \sup\{d(f(z),g(z)) \colon z \in K\} < \delta \implies g \in \mathscr{U}.$$

(2) Suppose $(f_n)_{n\geqslant 1}$ converges to f in $C(\Omega,X)$ and let $K \subseteq \Omega$ be a compact set. For any $\delta > 0$, there exists an $\varepsilon > 0$ such that

$$\rho(f,g) < \varepsilon \implies \sup\{d(f(z),g(z)) \colon z \in K\} < \delta.$$

But since $f_n \to f$ in $C(\Omega,X)$, there exists a positive integer N such that $\rho(f_n,f) < \varepsilon$ for all $n \geqslant N$. As a result, $\sup\{d(f_n(z),f(z)) \colon z \in K\} < \delta$ for all $n \geqslant N$. Hence $(f_n)_{n\geqslant 1}$ converges to f uniformly on compact subsets of Ω.

Conversely, suppose $(f_n)_{n\geqslant 1}$ converges to f uniformly on compact subsets of Ω and let $\varepsilon > 0$. Then there is a compact set $K \subseteq \Omega$ and $\delta > 0$ such that

$$\sup\{d(f(z),g(z)) \colon z \in K\} < \delta \implies \rho(f,g) < \varepsilon.$$

Since $(f_n)_{n\geqslant 1}$ converges to f uniformly on K, there is a positive integer N such that

$$\sup\{d(f_n(z),f(z)) \colon z \in K\} < \delta$$

for all $n \geqslant N$. As a result, $\rho(f_n,f) < \varepsilon$ for all $n \geqslant N$, i.e., $(f_n)_{n\geqslant 1}$ converges to f in $C(\Omega,X)$, thereby completing the proof. ■

An upshot of the above result is that the topology on $C(\Omega, X)$ is independent of the chosen exhaustion of Ω. That is, if

$$G = \bigcup_{n=1}^{\infty} K_n' \quad \text{and} \quad K_n' \subseteq \left(K_{n+1}'\right)^{\circ},$$

and this induces the metric ρ' on $C(\Omega, X)$, then the topology induced by ρ is the same as the topology induced by ρ'. This is clear because the characterization of open sets in Proposition 1.5 is independent of the chosen exhaustion. This "canonical" topology on $C(\Omega, X)$ is called the *compact-open topology*.

THEOREM 1.6. $\left(C(\Omega, X), \rho\right)$ is a complete metric space.

Proof. Let $(f_n)_{n\geqslant 1}$ be a Cauchy sequence in $C(\Omega, X)$. First, we shall show that there is a function $f : \Omega \to X$ with the property that

$$\lim_{n\to\infty} f_n(z) = f(z) \qquad \forall\ z \in \Omega.$$

Indeed, for any $\delta > 0$ and $z \in \Omega$, the set $K = \{z\}$ is compact, so in view of Lemma 1.4, there is an $\varepsilon > 0$ such that

$$\rho(f, g) < \varepsilon \implies d(f(z), g(z)) < \delta.$$

Since $(f_n)_{n\geqslant 1}$ is Cauchy, the above implies that $(f_n(z))_{n\geqslant 1}$ is also Cauchy. Since (X, d) is complete, there exists $f(z) \in X$ such that

$$\lim_{n\to\infty} f_n(z) = f(z).$$

This defines a function $f : \Omega \to X$ with the required property. It remains to show that f is continuous and $f_n \to f$ in $C(\Omega, X)$.

Note that $\Omega = \bigcup_{n=1}^{\infty} K_n^{\circ}$, and just as we argued earlier using Lemma 1.4 and $K = K_N$ for some $N \geqslant 1$, the sequence $(f_n)_{n\geqslant 1}$ is uniformly Cauchy on each K_N. Thus, $(f_n)_{n\geqslant 1}$ converges uniformly to f on K_N, and hence on K_N°. In particular, this means that f is continuous on K_N°. Since the K_N°'s cover Ω, it follows that f is continuous on Ω.

Next, we shall show that $f_n \to f$ in $C(\Omega, X)$. By Proposition 1.5, $f_n \to f$ in $C(\Omega, X)$ if and only if $(f_n)_{n\geqslant 1}$ converges to f uniformly on compact subsets Ω. But since every compact subset of Ω is contained in some K_N, it follows from the preceding paragraph that $f_n \to f$ in $C(\Omega, X)$, thereby completing the proof. ■

§§ The Arzelà-Ascoli Theorem and Normal Families

DEFINITION 1.7. Let (S, μ) be a metric space. A subset $\mathscr{F} \subseteq S$ is said to be *normal* if each sequence in $\mathscr{F}$ has a subsequence that converges in S.

PROPOSITION 1.8. Let (S, μ) be a metric space. A subset $\mathscr{F} \subseteq S$ is normal if and only if $\overline{\mathscr{F}}$ is compact in S.

Proof. Recall that a metric space is compact if and only if it is sequentially compact, that is, every sequence has a convergent subsequence. So if $\overline{\mathscr{F}}$ were compact, then every sequence in $\mathscr{F}$ would have a convergent subsequence in $\overline{\mathscr{F}} \subseteq S$.

Conversely, suppose every sequence in $\mathscr{F}$ has a convergent subsequence in S. Let $(y_n)_{n\geqslant 1}$ be a sequence in $\overline{\mathscr{F}}$. There is a sequence $(x_n)_{n\geqslant 1}$ in $\mathscr{F}$ such that $\mu(x_n, y_n) < \frac{1}{n}$. According to our assumption, there exists an $x \in S$ and a subsequence $(x_{n_k})_{k\geqslant 1}$ such that $x_{n_k} \to x$ in S. Clearly $x \in \overline{\mathscr{F}}$ and

$$d(y_{n_k}, x) \leqslant d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + d(x_{n_k}, x)$$

for all $k \geqslant 1$. Taking $k \to \infty$, we get that $y_{n_k} \to x$, whence $\overline{\mathscr{F}}$ is sequentially compact and hence compact. ∎

Lemma 1.9. Let (S, μ) be a metric space. A subset $\mathscr{F} \subseteq S$ is totally bounded if and only if $\overline{\mathscr{F}}$ is so.

Proof. Suppose $\mathscr{F}$ is totally bounded and let $\varepsilon > 0$. There exist $x_1, \dots, x_n \in \mathscr{F}$ such that

$$\mathscr{F} \subseteq \bigcup_{k=1}^{n} B_S\left(x_k, \frac{\varepsilon}{2}\right) \subseteq \bigcup_{k=1}^{n} \overline{B}_S\left(x_k, \frac{\varepsilon}{2}\right).$$

Since the latter union is closed, we have that

$$\overline{\mathscr{F}} \subseteq \bigcup_{k=1}^{n} \overline{B}_S\left(x_k, \frac{\varepsilon}{2}\right) \subseteq \bigcup_{k=1}^{n} B_S(x_k, \varepsilon).$$

Thus $\overline{\mathscr{F}}$ is totally bounded.

Conversely, suppose $\overline{\mathscr{F}}$ is totally bounded and let $\varepsilon > 0$. There exist $y_1, \dots, y_n \in \overline{\mathscr{F}}$ such that

$$\overline{\mathscr{F}} \subseteq \bigcup_{k=1}^{n} B_S\left(y_k, \frac{\varepsilon}{2}\right).$$

For each $1 \leqslant k \leqslant n$, there is some $x_k \in B_S\left(y_k, \frac{\varepsilon}{2}\right) \cap \mathscr{F}$, and hence

$$\overline{\mathscr{F}} \subseteq \bigcup_{k=1}^{n} B_S\left(y_k, \frac{\varepsilon}{2}\right) \subseteq \bigcup_{k=1}^{n} B_S(x_k, \varepsilon),$$

so that $\mathscr{F}$ is totally bounded, thereby completing the proof. ∎

Proposition 1.10. A set $\mathscr{F} \subseteq C(\Omega, X)$ is normal if and only if for each compact set $K \subseteq \Omega$ and $\delta > 0$, there are functions $f_1, \dots, f_n \in \mathscr{F}$ such that for any $f \in \mathscr{F}$, there is an index $1 \leqslant k \leqslant n$ with

$$\sup\{d(f(z), f_k(z)) \colon z \in K\} < \delta.$$

Proof. Recall that a metric space is compact if and only if it is complete and totally bounded. In view of Theorem 1.6, Proposition 1.8, and Lemma 1.9, $\mathscr{F}$ is normal if and only if it is totally bounded.

Suppose $\mathscr{F}$ is normal, then it is totally bounded. Let $K \subseteq \Omega$ be a compact set and $\delta > 0$. By Lemma 1.4 there is a $\varepsilon > 0$ such that

$$\rho(f, g) < \varepsilon \implies \sup\{d(f(z), g(z)) \colon z \in K\} < \delta.$$

There are $f_1, \dots, f_n \in \mathscr{F}$ such that

$$\mathscr{F} \subseteq \bigcup_{k=1}^{n} B_\rho(f_k, \varepsilon).$$

Now, for any $f \in \mathscr{F}$, there is an index $1 \leqslant k \leqslant n$ with $\rho(f, f_k) < \varepsilon$, and hence

$$\sup\{d(f(z), f_k(z)) \colon z \in K\} < \delta.$$

Conversely, suppose the given condition holds and let $\varepsilon > 0$. Then by Lemma 1.4 there is a compact set $K \subseteq \Omega$ and a $\delta > 0$ such that

$$\sup\{d(f(z), g(z)) \colon z \in K\} < \delta \implies \rho(f, g) < \varepsilon.$$

We claim that $\mathscr{F} \subseteq \bigcup_{k=1}^{n} B_\rho(f_k, \varepsilon)$. Indeed, if $f \in \mathscr{F}$, then there exists an index $1 \leqslant k \leqslant n$ such that

$$\sup\{d(f(z), f_k(z)) \colon z \in K\} < \delta \implies \rho(f, f_k) < \varepsilon,$$

that is, $f \in B_\rho(f_k, \varepsilon)$. This completes the proof. ∎

Our next goal is to prove the Arzelà-Ascoli theorem, which we shall adapt to normal families of holomorphic functions in order to prove Montel's theorem.

Definition 1.11. A set $\mathscr{F} \subseteq C(\Omega, X)$ is *equicontinuous at a point* $z_0 \in \Omega$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \varepsilon$ for all $f \in \mathscr{F}$.

The set $\mathscr{F}$ is said to be *equicontinuous over a set* $E \subseteq \Omega$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all pairs $z, z' \in E$ with $|z - z'| < \delta$, $d(f(z), f(z')) < \varepsilon$ for all $f \in \mathscr{F}$.

Clearly every finite collection of functions in $C(\Omega, X)$ is equicontinuous. Furthermore, if a family $\mathscr{F}$ is equicontinuous over $E \subseteq \Omega$, then every element of the family is uiformly continuous on E.

Proposition 1.12. If $\mathscr{F} \subseteq C(\Omega, X)$ is equicontinuous at each point of Ω, then it is equicontinuous over every compact set of Ω.

Proof. Let $K \subseteq \Omega$ be a compact set and let $\varepsilon > 0$. For each point $z \in K$, there is a $\delta_z > 0$ such that whenever $|\zeta - z| < \delta_z$, $d(f(\zeta), f(z)) < \frac{\varepsilon}{2}$ for all $f \in \mathscr{F}$. Note that the open balls $\{B(z, \delta_z) \colon z \in K\}$ form an open cover of K and hence, has a corresponding Lebesgue number, say $\lambda > 0$. Thus if $z, z' \in K$ are such that $|z - z'| < \lambda$, then there is a $z_0 \in K$ such that $z, z' \in B(z_0, \delta_{z_0})$. As a result, for any $f \in \mathscr{F}$.

$$d(f(z), f(z')) \leqslant d(f(z), f(z_0)) + d(f(z'), f(z_0)) < \varepsilon,$$

whence $\mathscr{F}$ is equicontinuous over K. ∎

Theorem 1.13 (Arzelà-Ascoli). A set $\mathscr{F} \subseteq C(\Omega, X)$ is normal if and only if

(1) for each $z \in \Omega$, $\{f(z) \colon f \in \mathscr{F}\}$ has compact closure in X, and

(2) $\mathscr{F}$ is equicontinuous at each point of Ω.

Proof. ∎

§§ Convergence of Holomorphic Functions and Montel's Theorem

§2 The Riemann Mapping Theorem

Theorem 2.1 (Riemann). Let $\Omega \subsetneq \mathbb{C}$ be a proper simply connected region and let $a \in \Omega$. Then there is a unique holomorphic function $f \in \mathscr{O}(\Omega)$ with the properties:

(i) $f(a) = 0$ and $f'(a) > 0$.

(ii) f is injective.

(iii) The image of f is the unit disk $\mathbb{D}$.

§3 Product Developments

§§ Generalities

Definition 3.1. If $(z_n)_{n \geqslant 1}$ is a sequence of complex numbers, then $z \in \mathbb{C}$ is said to be the *infinite product* of the sequence $(z_n)_{n \geqslant 1}$ if

$$z = \lim_{n \to \infty} \prod_{k=1}^{n} z_k.$$

Suppose $z_n \neq 0$ for all $n \geqslant 1$ and $z \neq 0$. Then, setting

$$p_n = \prod_{k=1}^{n} z_k,$$

we have, by definition that $p_n \to z \neq 0$ in $\mathbb{C}$. But since $z_n = p_n/p_{n-1}$ with the convention that $p_0 = 1$, we see that $z_n \to 1$ as $n \to \infty$.

PROPOSITION 3.2. Let $(z_n)_{n\geqslant 1}$ be a sequence of complex numbers with $\operatorname{Re} z_n > 0$ for all $n \geqslant 1$. Then $\prod_{n=1}^{\infty} z_n$ converges to a *non-zero* complex number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proof. ■

DEFINITION 3.3. If $(z_n)_{n\geqslant 1}$ is a sequence of complex numbers with $\operatorname{Re} z_n > 0$ for all n, then the infinite product $\prod_{n=1}^{\infty} z_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely.

LEMMA 3.4. If $|z| < \frac{1}{2}$, then

$$\frac{1}{2}|z| \leqslant |\log(1+z)| \leqslant \frac{3}{2}|z|.$$

Proof. Using the power series expansion of $\log(1+z)$ about $z = 0$, we get

$$\left|1 - \frac{\log(1+z)}{z}\right| = \left|\frac{1}{2}z - \frac{1}{3}z^2 + \cdots\right| \leqslant \frac{1}{2}\left(|z| + |z|^2 + \cdots\right) = \frac{1}{2}\frac{|z|}{1-|z|} < \frac{1}{2},$$

whence the conclusion follows. ■

PROPOSITION 3.5. Let $(z_n)_{n\geqslant 1}$ be a sequence of complex numbers with $\operatorname{Re} z_n > -1$ for all $n \geqslant 1$. Then the series $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n$ converges absolutely.

Proof. ■

COROLLARY 3.6. If $(z_n)_{n\geqslant 1}$ is a sequence of complex numbers with $\operatorname{Re} z_n > 0$ for all $n \geqslant 1$, then the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely.

Proof. ■

PROPOSITION 3.7. Let X be a set, and $(f_n)_{n\geqslant 1}$ be a sequence of complex-valued functions on X converging uniformly to $f : X \to \mathbb{C}$. Suppose there exists $a \in \mathbb{R}$ such that $\operatorname{Re} f_n(x) \leqslant a$ for all $x \in X$ and $n \geqslant 1$, then the sequence of functions $(\exp(f_n))_{n\geqslant 1}$ converges uniformly to $\exp(f)$.

Proof. ■

LEMMA 3.8. Let X be a compact topological space and $(g_n)_{n\geqslant 1}$ a sequence of complex-valued continuous functions on X such that $\sum_{n=1}^{\infty} |g_n(x)|$ converges uniformly on X. Then the product

$$f(x) = \prod_{n=1}^{\infty} (1 + g_n(x))$$

converges uniformly for all $x \in X$. Further there is an integer $n_0 \geqslant 1$ such that $f(x) = 0$ if and only if $g_n(x) = -1$ for some $1 \leqslant n \leqslant n_0$.

Proof. Since $\sum_{n=1}^{\infty} |g_n(x)|$ converges uniformly on X, there is a positive integer $n_0 \geqslant 1$ such that $|g_n(x)| < \frac{1}{2}$ for all $x \in X$ and $n > n_0$. Thus $\mathrm{Re}(1+g_n(x)) > 0$ for all $x \in X$ and $n > n_0$, and hence due to Lemma 3.4

$$|\log(1+g_n(x))| \leqslant \frac{3}{2}|g_n(x)| \qquad \forall\, x \in X,\ \forall n > n_0.$$

Thus, the sum

$$h(x) := \sum_{n=n_0}^{\infty} \log(1+g_n(x))$$

converges uniformly on X so that h is a continuous function. Since X is compact, there is an $a \in \mathbb{R}$ such that $\mathrm{Re}\, h(x) \leqslant a$ for all $x \in X$. In view of Proposition 3.7,

$$\exp h(x) = \prod_{n=n_0}^{\infty} (1+g_n(x))$$

converges uniformly on X. In particular, the product on the right is non-zero for all $x \in X$.

Finally, since

$$f(x) = (1+g_1(x))\cdots(1+g_{n_0}(x))\exp h(x),$$

it follows that if $f(x) = 0$, then $g_n(x) = -1$ for some $1 \leqslant n \leqslant n_0$. ■

THEOREM 3.9. Let $\Omega \subseteq \mathbb{C}$ be a region and let $(f_n)_{n\geqslant 1}$ be a sequence of holomorphic functions such that no f_n is identically zero. If $\sum_{n=1}^{\infty} |f_n(z) - 1|$ converges uniformly on compact subsets of Ω, then $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets of Ω to a holomorphic function $f(z)$.

If $a \in \Omega$ is a zero of f, then a is a zero of only a finite number of functions f_n , and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.

§§ Jensen's Formula

THEOREM 3.10 (JENSEN). Let $\Omega \subseteq \mathbb{C}$ be a region containing a closed disk $\overline{B}(0,R)$ for some $R > 0$. Let $f \in \mathscr{O}(\Omega)$ be a holomorphic function such that

(i) $f(0) \neq 0$, and

(ii) f has no zeros on the circle $\{z : |z| = R\}$.

If $a_1,\ldots,a_n$ are the zeros of f in $B(0,R)$ repeated according to multiplicity, then

$$|f(0)| \prod_{k=1}^{n} \frac{R}{|a_k|} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta\right).$$

Proof. Define $g \in \mathscr{O}(\Omega)$ as

$$g(z) = \frac{f(z)}{(z-a_1)\cdots(z-a_n)}.$$

Then g is a holomorphic function with no zeros in the closed ball $\overline{B}(0,R)$. To prove Jensen's formula for f, we shall prove it for g and for functions of the form $z \mapsto z-a$ for some $a \in B(0,R)$. The conclusion would then follow because if f_1 and f_2 are two holomorphic functions for which Jensen's formula holds, then it must hold for $f_1 f_2$.

Since g does not vanish in a neighborhood of the compact set $\overline{B}(0,R)$, the function $z \mapsto \log|g(z)|$ is a harmonic function and as such, has the mean value property, that is,

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\, d\theta.$$

Exponentiating both sides, g satisfies Jensen's formula.

Next, we claim that

$$\int_0^{2\pi} \log|e^{i\theta} - a|\, d\theta = 0$$

whenever $|a| < 1$. Making the change of variables $\theta \mapsto -\theta$, this is equivalent to proving

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|\, d\theta = 0$$

whenever $|a| < 1$. Consider the function $h(z) = 1 - az$, which does not vanish in a neighborhood of closed unit disk $\overline{\mathbb{D}}$. Again, using the mean value property for the harmonic function $z \mapsto |h(z)|$ and integrating over the unit disk, we have

$$0 = \log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|1 - ae^{i\theta}|\, d\theta,$$

as desired.

Finally, we must show that the function $F\colon z \mapsto z - a$ satisfies Jensen's formula when $a \in B(0,R)$. That is, we must show that

$$\log|F(0)| + \log R - \log|a| = \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - a|\, d\theta.$$

Note that $F(0) = -a$, and hence, the above is equivalent to showing that

$$\int_0^{2\pi} \log\left|e^{i\theta} - \frac{a}{R}\right|\, d\theta = 0,$$

which has already been established. ■

THEOREM 3.11. Suppose f is a bounded holomorphic function on $\mathbb{D}$ which is not identically zero, and $a_1, a_2, \ldots$ are the zeros of f, repeated according to multiplicity and $|a_n| \leqslant |a_{n+1}|$ for all $n \geqslant 1$. Then

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

Proof. Replacing $f(z)$ by $f(z)/z^m$ if necessary, we may suppose without loss of generality that $f(0) \neq 0$. Since f has only countably many zeros, there are uncountably many $0 < r < 1$ such that $|a_n| \neq r$ for any $n \geqslant 1$. Extract an increasing subsequence $(r_n)_{n\geqslant 1}$ from these values of r such that $r_n \to 1^-$ as $n \to \infty$. For $0 < r < 1$, let $\mathfrak{n}(r)$ denote the number of zeros of f contained in the closed ball $\overline{B}(0,r)$.

Let $k > 0$ be a positive integer and let $N \geqslant 1$ be such that $\mathfrak{n}(r_n) \geqslant k$ for all $n \geqslant N$. Then, due to Theorem 3.10,

$$|f(0)| \prod_{j=1}^{k} \frac{r_n}{|a_j|} \leqslant |f(0)| \prod_{j=1}^{\mathfrak{n}(r_n)} \frac{r_n}{|a_j|} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(r_n e^{i\theta})|\, d\theta\right).$$

Since f is bounded on $\mathbb{D}$, there is a constant $C > 0$ such that the right hand side of the above expression is bounded above by C for every $n \geqslant 1$. Thus

$$\prod_{j=1}^{k} |a_j| \geqslant C^{-1}|f(0)|r_n^k$$

for all $n \geqslant N$. Taking $n \to \infty$, we obtain

$$\prod_{j=1}^{k} |a_j| \geqslant C^{-1}|f(0)| > 0.$$

Note that the partial products of $\prod_{j=1}^{\infty} |a_j|$ form a decreasing sequence, and hence must converge. The above property implies that the product converges to a non-zero quantity. Finally, note that

$$C^{-1}|f(0)| \leqslant \prod_{j=1}^{k} |a_j| \leqslant \exp\left(-\sum_{j=1}^{k}(1-|a_j|)\right),$$

so that

$$\sum_{j=1}^{k}(1-|a_j|) \leqslant -\log\left(C^{-1}|f(0)|\right),$$

and hence, the sum $\sum_{j=1}^{k}(1-|a_j|)$ converges. ■

§§ The Muntz-Szasz Theorem

Let I denote the unit interval $[0,1]$.

THEOREM 3.12 (MUNTZ-SZASZ). Let $0 < \lambda_1 < \lambda_2 < \cdots$ be a sequence of positive real numbers and let X be the closure in $C(I)$ of the span of $\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$.

(1) If $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty$, then $X = C(I)$.

(2) If $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, and if $\lambda \notin (\lambda_n)_{n\geqslant 1}$, $\lambda \neq 0$, then X does not contain the function t^{λ}.

Proof. Consider the case (1) first. If X were not dense in $C(I)$, then there would exist a non-zero bounded linear functional $\Lambda\colon C(I) \to \mathbb{C}$ which vanishes on X. Due to the Riesz Representation Theorem, there exists a complex Borel measure μ on I such that

$$\Lambda f = \int_I f \, d\mu.$$

By our hypothesis,

$$\int_I t^{\lambda_n} \, d\mu = 0$$

for all $n \geqslant 1$. Define the function $f : \{z : \operatorname{Re} z > 0\} \to \mathbb{C}$ by

$$f(z) = \int_{(0,1]} t^z \, d\mu(t) = \int_I t^z \, d\mu(t).$$

The continuity of f can be verified using the Dominated Convergence Theorem[1]. Further, due to Morera's theorem, the integral of t^z over any triangle contained in the right half plane is zero, whence, due to Fubini's theorem, the integral of $f(z)$ over any triagle contained in the right half plane is zero. Thus f is holomorphic on the right half plane. For any $z = x + \iota y$ with $x > 0$, note that $|t^z| = t^x \leqslant 1$ for any $t \in (0,1]$, consequently f is bounded on the right half plane.

Suppose f is not identically zero. Define $g\colon \mathbb{D} \to \mathbb{C}$ by

$$g(z) = f\left(\frac{1+z}{1-z}\right).$$

This is a bounded holomorphic function on $\mathbb{D}$ with zeros at $\dfrac{\lambda_n - 1}{\lambda_n + 1}$. But it is easy to see that the sum

$$\sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n - 1}{\lambda_n + 1}\right) = +\infty,$$

and hence, in light of Theorem 3.11, f must be identically zero, that is,

$$\int_I t^\lambda \, d\mu = 0$$

for each $\lambda > 0$. But since the polynomials are dense in $C(I)$, we see that $\Lambda = 0$, a contradiction. Thus X is dense in $C(I)$. ■

§4 Runge's Theorem

Theorem 4.1 (Runge). Let $K \subseteq \mathbb{C}$ be a compact set and let E be a subset of $\mathbb{C}_\infty \setminus K$ meeting each connected component of $\mathbb{C}_\infty \setminus K$. If f is a function holomorphic in an open set $\Omega \supseteq K$ and $\varepsilon > 0$, then there exists a rational function $R(z)$ whose only poles lie in E such that

$$|f(z) - R(z)| < \varepsilon$$

for all $z \in K$.

Let $C(K)$ denote the Banach space of all complex-valued continuous functions on K equipped with the supremum norm on K, that is,

$$\|f\|_\infty := \sup\{|f(z)| : z \in K\} \qquad \forall f \in C(K).$$

Let $B(E) \subseteq C(K)$ denote the set of all functions $f \in C(K)$ such that for every $\varepsilon > 0$, there is a rational function $R(z)$ with poles only in E such that

$$\|f - R\|_\infty < \varepsilon.$$

Theorem 4.1 essentially states that $f|_K \in B(E)$ for every holomorphic function in a neighborhood of K.

Lemma 4.2. $B(E)$ is a closed $\mathbb{C}$-subalgebra of $C(K)$ containing every rational function with all poles in E.

[1]Recall that $\mu = h d|\mu|$ for any complex Borel measure μ, where $|\mu|$ is the total variation measure.

Proof. The latter part of the assertion is clear. To see that $B(E)$ is a subalgebra, suppose $f, g \in B(E)$ and $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$ and choose rational functions $R(z), S(z)$ such that

$$\|f - R\|_\infty < \frac{\varepsilon}{|\alpha| + |\beta| + 1} \quad \text{and} \quad |g - S| < \frac{\varepsilon}{|\alpha| + |\beta| + 1}.$$

Then

$$\left\|(\alpha f + \beta g) - (\alpha R + \beta S)\right\|_\infty < \frac{|\alpha| + |\beta|}{|\alpha| + |\beta| + 1}\varepsilon < \varepsilon,$$

whence $\alpha f + \beta g \in B(E)$. Next, we shall show that $fg \in B(E)$. Indeed, let $\varepsilon > 0$, and choose positive real numbers $M_1, M_2 > 0$ such that $\|f\|_\infty < M_1$ and $\|g\|_\infty < M_2$. Choose rational functions $R(z), S(z)$ such that

$$\|f - R\|_\infty < \frac{\varepsilon}{M_1 + M_2} \quad \text{and} \quad \|g - S\|_\infty < \frac{\varepsilon}{M_1 + M_2}.$$

Then $R(z)S(z)$ is a rational function with poles only in E, and

$$\|fg - RS\|_\infty \leqslant \|g(f - R) + R(g - S)\|_\infty \leqslant M_2\|f - R\|_\infty + M_1\|g - S\|_\infty < \varepsilon,$$

as desired. Thus $B(E)$ is a subalgebra of $C(K)$.

It remains to show that $B(E)$ is closed in the topology of $C(K)$. Indeed, let $f_n \to f$ in $C(K)$ and $\varepsilon > 0$. There is a positive integer N such that $\|f - f_N\|_\infty < \frac{\varepsilon}{2}$, and further, a rational function $R(z)$ with poles only in E such that $\|f_N - R\|_\infty < \frac{\varepsilon}{2}$. Thus

$$\|f - R\|_\infty < \|f - f_N\|_\infty + \|f_N - R\|_\infty < \varepsilon,$$

whence $f \in B(E)$, thereby completing the proof. ■

The outline of the rest of the proof is as follows:

- First, we show that $\dfrac{1}{z - a} \in B(E)$ for each $a \in \mathbb{C} \setminus K$.
- Since $B(E)$ is an algebra containing all polynomials, using partial fractions, we conclude that every rational function with poles only in $\mathbb{C} \setminus K$ belongs to $B(E)$.
- Finally, using Cauchy's integral formula, we show that every holomorphic function can be approximated arbitrarily well by rational functions with poles only in $\mathbb{C} \setminus K$.

Lemma 4.3. Let V and U be open subsets of $\mathbb{C}$ with $V \subseteq U$ and $\partial V \cap U = \emptyset$. If H is a component of U with $H \cap V \neq \emptyset$, then $H \subseteq V$.

Proof. Let $a \in H \cap V$ and let G be the connected component of V containing a; then $H \cup G$ is connected and contained in U. But since H is a connected component, $H \cup G = H$, that is, $G \subseteq H$. Note that $\partial G \subseteq \partial V$[2] and so $\partial G \cap H = \emptyset$, whence

$$H \setminus G = H \cap (\mathbb{C} \setminus G) = H \cap \left[(\mathbb{C} \setminus \overline{G}) \cup \partial G\right] = H \cap (\mathbb{C} \setminus \overline{G}),$$

whence $H \setminus G$ is open in H. But since G is open, $H \setminus G$ is both closed and open in H, and since H is connected and $G \neq \emptyset$, it follows that $H = G \subseteq V$, as desired. ■

Proposition 4.4. Let $a \in \mathbb{C} \setminus K$. Then $\dfrac{1}{z - a} \in B(E)$.

[2] This is because $\mathbb{C}$ is locally connected.

Proof. We split our analysis into two cases.
CASE 1. $\infty \notin E$. Let $U = \mathbb{C} \setminus K$ and let

$$V = \left\{ a \in \mathbb{C} : \frac{1}{z-a} \in B(E) \right\},$$

so that $E \subseteq V \subseteq U$. We first claim that V is open. Indeed, suppose $a \in V$ and $|b-a| < d(a,K)$. Then there exists $0 < r < 1$ such that $|b-a| < r|z-a|$ for all $z \in K$. But

$$\frac{1}{z-b} = \frac{1}{z-a} \frac{1}{1 - \frac{b-a}{z-a}},$$

and since $|(b-a)/(z-a)| < r < 1$ for all $z \in K$, we note that the series

$$\frac{1}{1 - \frac{b-a}{z-a}} = \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a} \right)^n$$

converges uniformly on K due to the Weierstraß M-test. Set

$$Q_n(z) = \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a} \right)^n,$$

then $\dfrac{1}{z-a} Q_n(z) \in B(E)$ since $a \in V$ and $B(E)$ is an algebra. Since $B(E)$ is closed, the uniform convergence of $\dfrac{1}{z-a} Q_n(z)$ to $\dfrac{1}{z-b}$ yields that the latter lies in $B(E)$, so that V is open.

Now suppose $b \in \overline{V} \setminus V = \partial V$ and let $(a_n)_{n \geq 1}$ be a sequence in V converging to b. We have that $|b - a_n| \geq d(a_n, K)$ and taking $n \to \infty$ and using the continuity of $d(\cdot, K)$, one obtains $d(b,K) = 0$, that is, $b \in K$. Thus $\partial V \cap U = \emptyset$. If H is a component of U, then $H \cap E \neq \emptyset$, so $H \cap V \neq \emptyset$. By Lemma 4.3, $H \subseteq V$. But since H was arbitrary, we have that $U \subseteq V$, i.e., $U = V$.
CASE 2. $\infty \in E$. Let d_∞ denote the metric on $\mathbb{C}_\infty$. Choose a_0 in the unbounded component of $\mathbb{C} \setminus K$ (i.e., the component containing ∞) such that $d_\infty(a_0, \infty) \leq \frac{1}{2} d_\infty(\infty, K)$ and $|a_0| > 2 \max\{|z| : z \in K\}$. Let $E_0 = (E \setminus \{\infty\}) \cup \{a_0\}$. Then E_0 meets each component of $\mathbb{C}_\infty \setminus K$, and $\infty \notin E_0$.

If $a \in \mathbb{C} \setminus K$, then due to CASE 1, $\dfrac{1}{z-a} \in B(E_0)$. We shall show that $\dfrac{1}{z-a_0} \in B(E_0)$. Once this is shown, we could approximate rational functions with poles only in E_0 by rational functions with poles only in E, since $E_0 \setminus E = \{a_0\}$. This would then immediately give us that $\dfrac{1}{z-a} \in B(E_0) \subseteq B(E)$, as desired.

Note that for all $z \in K$, $|z/a_0| \leq \frac{1}{2}$ and so

$$\frac{1}{z-a_0} = -\frac{1}{a_0} \frac{1}{1 - \frac{z}{a_0}} = -\frac{1}{a_0} \sum_{n=0}^{\infty} \left(\frac{z}{a_0} \right)^n$$

converges uniformly on K due to the Weierstraß M-test. Set

$$Q_n(z) = -\frac{1}{a_0} \sum_{k=0}^{n} \left(\frac{z}{a_0} \right)^k,$$

which is a sequence of polynomials converging uniformly to $\dfrac{1}{z-a_0}$ on K. Since $Q_n \in B(E)$ for each $n \geq 1$, we have shown that $\dfrac{1}{z-a_0} \in B(E)$, thereby completing the proof. ∎

LEMMA 4.5. Let Ω be a region contianing K. Then there are straight line segments $\gamma_1,\ldots,\gamma_n$ in $\Omega\setminus K$ such that for every holomorphic function f on Ω,

$$f(z)=\frac{1}{2\pi\iota}\sum_{k=1}^{n}\int_{\gamma_k}\frac{f(w)}{w-z}\,dw$$

for all $z\in K$. The line segments form a finite number of closed polygons in Ω.

Proof. Covering K by finitely many compact disks (contained in Ω), we can replace K with the union of these disks and suppose that $K=\overline{K^\circ}$. Let $0<\delta<\frac{1}{2}d(K,\mathbb{C}\setminus\Omega)$ and place a "grid" of horizontal and vertical lines in the plane with consecutive lines less than a distance δ apart. Let $R_1,\ldots,R_m$ be the resulting rectangles intersecting K. These rectangles are finite in number because K is compact. Consider ∂R_j, the boundary of R_j as a polygon oriented in the counter-clockwise direction.

If $z\in R_j$ for some $1\leqslant j\leqslant m$, then $d(z,K)\leqslant\operatorname{diam}R_j=\sqrt{2}\delta$, and hence $z\in\Omega$. This shows that every R_j is contained in Ω. Next, suppose R_j and R_j intersect in an edge σ. With respect to the two rectangles, σ will have opposite orientations, and hence, for any continuous function φ on σ, the sum of the integrals will cancel out.

Let $\gamma_1,\ldots,\gamma_n$ be those directed line segments that constitute an edge of exactly one of the R_j's. Then

$$\sum_{k=1}^{n}\int_{\gamma_k}\varphi=\sum_{j=1}^{m}\int_{\partial R_j}\varphi \tag{1}$$

for any continuous function φ on $\bigcup_{j=1}^{m}\partial R_j$.

We contend that each γ_k lies in $\Omega\setminus K$. Indeed, if one of the γ_k intersects K, then there are two rectangles in the grid with γ_k as a side, both of which intersect K, whence both of these rectangles must lie in the set $\{R_1,\ldots,R_m\}$, which is absurd, since γ_k is a side of exactly one of those rectangles.

Now, if $z\in K\setminus\bigcup_{j=1}^{m}\partial R_j$, then for any holomorphic function f on Ω,

$$\varphi(w)=\frac{1}{2\pi\iota}\frac{f(w)}{w-z}$$

is continuous on $\bigcup_{j=1}^{m}\partial R_j$. From (1), it follows that

$$\sum_{j=1}^{m}\frac{1}{2\pi\iota}\int_{\partial R_j}\frac{f(w)}{w-z}\,dw=\sum_{k=1}^{n}\frac{1}{2\pi\iota}\int_{\gamma_k}\frac{f(w)}{w-z}\,dw.$$

But z belongs to the interior of exactly one of the R_j's whence the sum on the left is precisely $f(z)$ whenever $z\in K\setminus\bigcup_{j=1}^{m}\partial R_j$. But both sides are continuous functions on K (since $f(z)$ is clearly continuous and every γ_k misses K) and because $K=\overline{K^\circ}$, the set $K\setminus\bigcup_{j=1}^{m}\partial R_j$ is dense in K; it follows that both sides must be equal for all $z\in K$, as desired. ■

Now that we have an integral representation of $f(z)$, we shall approximate it using rational functions having poles on the $\{\gamma_k\}$'s.

LEMMA 4.6. Let γ be a rectifiable curve and K a compact set such that $K\cap\{\gamma\}=\emptyset$. If f is a continuous function on $\{\gamma\}$, and $\varepsilon>0$, then there is a rational function $R(z)$ having all its poles on $\{\gamma\}$ such that

$$\left|\int_{\gamma}\frac{f(w)}{w-z}\,dw-R(z)\right|<\varepsilon$$

for all $z\in K$.

Proof. We may assume that $\gamma\colon [0,1] \to \mathbb{C}$. First, since K and $\{\gamma\}$ are disjoint, there is a real number $0 < r < d(\{\gamma\},K)$. For $0 \leqslant s < t \leqslant 1$ and $z \in K$,

$$\begin{aligned}
\left|\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z}\right| &= \left|\frac{\gamma(s)f(\gamma(t)) - \gamma(t)f(\gamma(s)) - z\big(f(\gamma(t)) - f(\gamma(s))\big)}{(\gamma(t)-z)(\gamma(s)-z)}\right| \\
&\leqslant \frac{1}{r^2}\left|\gamma(s)f(\gamma(t)) - \gamma(t)f(\gamma(s)) - z\big(f(\gamma(t)) - f(\gamma(s))\big)\right| \\
&\leqslant \frac{1}{r^2}\left|f(\gamma(t))\big(\gamma(s)-\gamma(t)\big) + \gamma(t)\big(f(\gamma(t)) - f(\gamma(s))\big) - z\big(f(\gamma(t)) - f(\gamma(s))\big)\right| \\
&\leqslant \frac{1}{r^2}\left|f(\gamma(t))\right|\left|\gamma(s)-\gamma(t)\right| + \frac{1}{r^2}\left|\gamma(t)-z\right|\left|f(\gamma(t)) - f(\gamma(s))\right|.
\end{aligned}$$

Using the compactness of $\{\gamma\}$ and K, there is a constant $C > 0$ such that $d(x,z) \leqslant C$ for all $x \in \{\gamma\}$ and $z \in K$, and $f(x) \leqslant C$ for all $x \in \{\gamma\}$. Thus

$$\left|\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z}\right| \leqslant \frac{C}{r^2}\left(\left|\gamma(s)-\gamma(t)\right| + \left|f(\gamma(t)) - f(\gamma(s))\right|\right).$$

Finally, using the uniform continuity of the functions $\gamma, f \circ \gamma\colon [0,1] \to \mathbb{C}$, there is a $\delta > 0$ such that whenever $|s-t| < \delta$,

$$\left|\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z}\right| < \frac{\varepsilon}{2V(\gamma)}$$

for all $z \in K$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that $|t_j - t_{j-1}| < \delta$ for $1 \leqslant j \leqslant n$. Set

$$R(z) = \sum_{i=1}^{n} \frac{f(\gamma(t_{j-1}))\big(\gamma(t_j) - \gamma(t_{j-1})\big)}{\gamma(t_{j-1}) - z}.$$

Now, there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $[0,1]$ such that

$$\left|\int_\gamma \frac{f(w)}{w-z}\,dw - \sum_{j=1}^{m} \frac{f(\gamma(s_j))}{\gamma(s_j) - \gamma(s_{j-1})}\right| < \frac{\varepsilon}{2}.$$

Thus

$$\left|\int_\gamma \frac{f(w)}{w-z}\,dw - R(z)\right| \leqslant \left|\int_\gamma \frac{f(w)}{w-z}\,dw - \sum_{j=1}^{m} \frac{f(\gamma(s_j))}{\gamma(s_j) - \gamma(s_{j-1})}\right| + \left|\sum_{j=1}^{m} \frac{f(\gamma(s_j))}{\gamma(s_j) - \gamma(s_{j-1})} - \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))\big(\gamma(t_j) - \gamma(t_{j-1})\big)}{\gamma(t_{j-1}) - z}\right|.$$

Taking a union of both partitions $\underline{s}$ and $\underline{t}$ and using the triangle inequality, it is clear that both terms are smaller than $\varepsilon/2$, therefore,

$$\left|\int_\gamma \frac{f(w)}{w-z}\,dw - R(z)\right| < \varepsilon,$$

for all $z \in K$. ■

Proof of Theorem 4.1. Due to Proposition 4.4 and the fact that $B(E)$ contains all polynomials, using partial fractions it follows that $B(E)$ contains all rational functions with all poles in $\mathbb{C} \setminus K$. Finally, using Lemma 4.5 and Lemma 4.6, it follows that $f \in B(E)$, as desired. ■

§§ Simply connected regions

THEOREM 4.7. Let $\Omega \subseteq \mathbb{C}$ be a region. Then the following are equivalent:

(1) Ω is simply connected.

(2) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in Ω and every point $a \in \mathbb{C} \setminus \Omega$.

(3) $\mathbb{C}_\infty \setminus \Omega$ is connected.

(4) For any $f \in \mathscr{O}(\Omega)$, there is a sequence of polynomials that converges to f in $\mathscr{O}(\Omega)$.

(5) For any $f \in \mathscr{O}(\Omega)$ and any closed rectifiable curve γ in Ω, $\int_\gamma f = 0$.

(6) Every function $f \in \mathscr{O}(\Omega)$ has a primitive.

(7) For any nowhere-vanishing function $f \in \mathscr{O}(\Omega)$, there is a $g \in \mathscr{O}(\Omega)$ such that $f = \exp g$.

(8) For any nowhere-vanishing function $f \in \mathscr{O}(\Omega)$, there is a $g \in \mathscr{O}(\Omega)$ such that $f = g^2$.

(9) Ω is homeomorphic to the unit disk.

(10) If $u \colon \Omega \to \mathbb{R}$ is harmonic, then there is a harmonic function $v \colon \Omega \to \mathbb{R}$ such that $f = u + \imath v$ is holomorphic on Ω.

§§ Mittag-Leffler's Theorem

THEOREM 4.8 (MITTAG-LEFFLER). Let $\Omega \subseteq \mathbb{C}$ be a region and $(a_n)_{n\geqslant 1}$ a sequence of distinct points in Ω with no limit point in Ω. Let $(S_n(z))_{n\geqslant 1}$ be a sequence of rational functions of the form

$$S_n(z) = \sum_{j=1}^{m_n} \frac{c_{nj}}{(z - a_n)^j},$$

where m_n is a positive integer and $c_{nj} \in \mathbb{C}$ for all $n \geqslant 1$ and $1 \leqslant j \leqslant m_n$. Then there exists a meromorphic function f on Ω which is holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$ and whose singular part at each a_n is given by $S_n(z)$.

Proof. Choose an exhaustion $(K_n)_{n\geqslant 1}$ of Ω as in Theorem 1.1 and as such, every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$. Next, since each K_n is compact, and $(a_k)_{k\geqslant 1}$ has no limit point in Ω, only finitely many of the a_k's can lie in each K_n. Define

$$I_n := \{k : a_k \in K_n \setminus K_{n-1}\}$$

with the convention that $K_0 = \emptyset$. Define the functions

$$f_n(z) = \sum_{k \in I_n} S_k(z).$$

This is clearly a meromorphic function on Ω with all its poles in $K_n \setminus K_{n-1}$. Using Theorem 4.1 with $E = \mathbb{C}_\infty \setminus \Omega$, there exists a rational function $R_n(z)$ with all its poles in $\mathbb{C}_\infty \setminus \Omega$ such that

$$|f_n(z) - R_n(z)| < \frac{1}{2^n}$$

for all $z \in K_{n-1}$ and $n \geqslant 2$. For $n = 1$, we set $R_1 = 0$. Define

$$f(z) = \sum_{n=1}^{\infty} (f_n(z) - R_n(z)).$$

We contend that this is our desired meromorphic function. We must first show that f is holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$ and then show that its singular part at each a_k is $S_k(z)$.

Indeed, let $K \subseteq \Omega \setminus \{a_1, a_2, \dots\}$ be a compact set. Then there is a positive integer $N \geqslant 1$ such that $K \subseteq K_N$. For all $n \geqslant N+1$, and $z \in K_N$, we have that

$$|f_n(z) - R_n(z)| < \frac{1}{2^n}.$$

Due to the Weierstraß M-test, the sum converges uniformly on K, whence the limiting function f is a holomorphic function on $\Omega \setminus \{a_1, a_2, \dots\}$.

Let $k \geqslant 1$. Since the sequence $(a_n)_{n \geqslant 1}$ has no limit point, there is an $r > 0$ such that $|a_j - a_k| > r$ for all $j \neq k$. Then, the sum for $f(z) - S_k(z)$ converges uniformly on $\overline{B}(a_k, r)$ to a holomorphic function there, again due to the Weierstraß M-test. As a result, $f(z)$ has singular part $S_k(z)$ at a_k. This completes the proof. ■

PROPOSITION 4.9. Let $\Omega \subseteq \mathbb{C}$ be a region. If $(a_n)_{n \geqslant 1}$ is a sequence of distinct points in Ω with no limit point in Ω, and $(c_n)_{n \geqslant 1}$ is a sequence of complex numbers, then there is a holomorphic function $f \in \mathscr{O}(\Omega)$ such that $f(a_n) = c_n$ for all $n \geqslant 1$.

Proof. Let $g \in \mathscr{O}(\Omega)$ be a holomorphic function with simple zeros at only the a_n's. Then we can write $g(z) = (z - a_n) g_n(z)$ for some holomorphic function $g_n \in \mathscr{O}(\Omega)$ with $g_n(a_n) \neq 0$. Using Theorem 4.8 let h be a meromorphic function on Ω, holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$, and having singular part

$$\frac{c_n}{g_n(a_n)} \frac{1}{z - a_n}$$

at a_n for each $n \geqslant 1$. Clearly $f(z) = g(z)h(z)$ has removable singularities at each a_n and $f(a_n) = c_n$. ■

A significantly more general statement is true; instead of just specifying values of a function at countably many points, we can specify the tail of its power series representation at those points:

THEOREM 4.10. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $(a_n)_{n \geqslant 1}$ be a sequence of distinct points in Ω with no limit point in Ω. For each $n \geqslant 1$, associate a non-negative integer $m_n \geqslant 0$, and complex numbers w_{nj} for $0 \leqslant j \leqslant m_n$. Then there exists a holomorphic function $f \in \mathscr{O}(\Omega)$ such that

$$f^{(j)}(a_n) = j! w_{nj}$$

for all $n \geqslant 1$ and $0 \leqslant j \leqslant m_n$[3].

Proof. Let $g \in \mathscr{O}(\Omega)$ have zeros at only the a_n's with multiplicity $m_n + 1$ respectively. We shall use Theorem 4.8 to find a meromorphic function h on Ω, which is holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$ and has singular part

$$S_n(z) = \frac{b_{n1}}{z-a} + \frac{b_{n2}}{(z-a)^2} + \cdots + \frac{b_{n,m_n+1}}{(z-a)^{m_n+1}}$$

[3]That is, the power series representation of f about a_n is of the form

$$f(z) = w_{n0} + w_{n1}(z - a_n) + \dots.$$

at each a_n, where $b_{nj} \in \mathbb{C}$ are complex numbers to be chosen later. Conisder the power series expansion of $g(z)$ about $z - a_n$:

$$g(z) = (z-a_n)^{m_n+1}\left(c_{n0} + c_{n1}(z-a_n) + c_{n2}(z-a_n)^2 + \dots\right),$$

for some complex numbers c_{nj}, $j \geqslant 0$. Note that $c_{n0} \neq 0$. Then

$$g(z)S_n(z) = \left(b_{n,m_n+1} + b_{n,m_n}(z-a) + \dots + b_{n1}(z-a)^{m_n}\right)(c_{n0} + c_{n1}(z-a_n) + \dots).$$

We would like to choose $b_{n1}, \dots, b_{n,m_n+1}$ such that the above product expands to

$$w_{n0} + w_{n1}(z-a_n) + w_{n2}(z-a_n)^2 + \dots.$$

The b_{nj}'s can be chosen inductively since $c_{n0} \neq 0$, so that we begin by setting $b_{n,m_n+1} = w_{n0}c_{n0}^{-1}$. And at each stage, one obtains a linear equation in b_{nj} with coefficient c_{n0}, which is again non-zero, and so that equation has a (unique) solution.

Finally, using Theorem 4.8 to choose a meromorphic function h on Ω having poles at precisely the a_n's with singular parts $S_n(z)$ respectively, it is clear that $f(z) = g(z)h(z)$ has the desired power series expansion at each a_n, thereby completing the proof. ■

THEOREM 4.11. Let $\Omega \subseteq \mathbb{C}$ be a region. Then $\mathscr{O}(\Omega)$ is a Bézout domain, that is, every finitely generated ideal in $\mathscr{O}(\Omega)$ is principal.

Proof. Inductively, it suffices to show that (f,g) is a principal ideal for $f, g \in \mathscr{O}(\Omega)$. First, we shall show that if f and g have no common zeros, then $(f,g) = (1)$. Let $a_1, a_2, \dots$ be the distict zeros of f with multiplicities $m_1, m_2, \dots$ respectively (note that these zeros can be finite in number). We contend that there exists $\varphi \in \mathscr{O}(\Omega)$ such that $1 - \varphi g$ has zeros $a_1, a_2, \dots$ with multiplicities $m'_1, m'_2, \dots$ respectively such that $m'_j \geqslant m_j$ for all $j \geqslant 1$.

Let $k \geqslant 1$ and consider the power series representation of g about a_k:

$$g(z) = b_{k0} + b_{k1}(z-a_k) + b_{k2}(z-a_k)^2 + \dots,$$

where $b_{k0} \neq 0$ since f and g do not share a zero. We want the power series representation of φ about a_k

$$\varphi(z) = w_{k0} + w_{k1}(z-a_k) + w_{k2}(z-a_k)^2 + \dots$$

to be such that

$$\varphi(z)g(z) = 1 + c_{m_k}(z-a_k)^{m_k} + \dots$$

for some $c_{m_k} \in \mathbb{C}$. This can clearly be done inductively just as in the proof of Theorem 4.10 since $b_{k0} \neq 0$. Further, the existence of such a $\varphi \in \mathscr{O}(\Omega)$ is guaranteed by Theorem 4.10. By construction, it is clear that there exists a holomorphic function $h \in \mathscr{O}(\Omega)$ such that $h(z)f(z) = 1 - \varphi(z)g(z)$, i.e., $1 \in (f,g)$, as desired.

Finally, suppose f and g are arbitrary holomorphic functions in $\mathscr{O}(\Omega)$. Let $a_1, a_2, \dots$ be the common zeros of f and g with

$$m_n = \min\{m(f;a_n), m(g;a_n)\} \geqslant 1,$$

for all $n \geqslant 1$. Let $\varphi \in \mathscr{O}(\Omega)$ be a holomorphic function with zeros at precisely the a_n's with multiplicities m_n respectively. Then there exist holomorphic functions $\widetilde{f}, \widetilde{g} \in \mathscr{O}(\Omega)$ such that $f = \varphi\widetilde{f}$ and $g = \varphi\widetilde{g}$; further f and g do not have common zeros. As a result,

$$(f,g) = (\varphi\widetilde{f}, \varphi\widetilde{g}) = (\varphi)(\widetilde{f}, \widetilde{g}) = (\varphi),$$

thereby completing the proof. ■