Gorenstein Rings

Notes for the course MA 842: Topics in Algebra II

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CONTENTS

§1 INJECTIVE MODULES

§§ Basic Properties of Injective Modules

DEFINITION 1.1. Let R be a ring. An R-module E is said to be *injective* if for every inclusion of R-modules $N \hookrightarrow M$ and an R-linear map $N \to E$, there is an R-linear map $M \to E$ making

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[dl, dotted, "\exists"] \\ & E & \end{tikzcd}$$

commute.

An R-module M is said to be *divisible* if

$$\mu_a : M \longrightarrow M \qquad m \longmapsto am$$

is surjective for each non-zerodivisor $a \in R$.

REMARK 1.2. It is easy to see that E is injective if and only if given any inclusion of R-modules $N \hookrightarrow M$, the induced map $\operatorname{Hom}_R(M, E) \to \operatorname{Hom}_R(N, E)$ is surjective. Further, since $\operatorname{Hom}_R(-, E)$ is always left-exact, we have:

An R-module E is injective if and only if $\operatorname{Hom}_R(-, E)$ is an exact functor.

PROPOSITION 1.3. Every injective R-module is divisible.

Proof. Let E be R-injective, $x \in E$, and $a \in R$ a non-zerodivisor. Let $\varphi : R \to E$ be the unique R-linear map sending $1 \mapsto x$. Since $R \xrightarrow{\mu_a^R} R$ is injective, there is a map $\widetilde{\varphi} : R \to E$ such that $\widetilde{\varphi} \circ \mu_a^R = \varphi$. In particular, $a\widetilde{\varphi}(1) = x$, whence $\mu_a^E : E \to E$ is surjective, as desired. ■

THEOREM 1.4 (BAER'S CRITERION). Let R be a ring and E an R-module. Then E is injective if and only if for every ideal $I \trianglelefteqslant R$ and an R-linear map $f : I \to E$, there is an R-linear map $F : R \to E$ such that $F|_I = f$.

Proof. The forward implication is clear. We shall prove the converse. Let $0 \to N \to M$ be exact and $f : N \to E$ be an R-linear map. Consider the poset

$$\Omega = \{(P, g) \colon N \leqslant P \leqslant M \text{ and } g : P \to E \text{ is } R\text{-linear extending } f\},$$

where $(P, g) \leqq (P', g')$ if $P \leqslant P'$ and $g'|_P = g$. Using Zorn's lemma, choose a maximal element $(P, g) \in \Omega$. We claim that $P = M$. Suppose now and choose some $x \in M \setminus P$. Set $I = (P :_R x) \trianglelefteqslant R$ and consider the map

$$I \longrightarrow E \qquad a \mapsto g(ax).$$

This is well-defined and R-linear, whence it extends to an R-linear map $\varphi : R \to E$. Let $\alpha = \varphi(1)$ and define $F : P + Rx \to E$ by $F(p + ax) = g(p) + a\alpha$ for all $p \in P$ and $a \in R$. To see that this is well-defined, note that if $p_1 + a_1x = p_2 + a_2x$, then $a_1 - a_2 \in I$, so that

$$g(p_2) - g(p_1) = g((a_1 - a_2)x) = (a_1 - a_2)\alpha \implies g(p_1) + a_1\alpha = g(p_2) + a_2\alpha.$$

The map F is obviously R-linear and extends g, thereby contradicting the maximality of (P, g). Hence, $P = M$ and E is injective. ∎

COROLLARY 1.5. An R-module E is injective if and only if $\operatorname{Ext}^1_R(R/I, E) = 0$ for all ideals $I \trianglelefteqslant R$. ∎

REMARK 1.6. We note that it is not sufficient to check the equivalent condition of Theorem 1.4 for finitely generated ideals. Indeed, let $R = \mathscr{O}(\mathbb{C})$ the ring of entire functions, or $R = \mathcal{O}_{\overline{\mathbb{Q}}}$ the ring of algebraic integers in $\mathbb{C}$. It is known that R is a non-Noetherian Bézout domain. As such, due to Interlude 1.12, there is a family of R-injectives $\{E_i\}_{i=1}^{\infty}$ such that $E = \bigoplus_i E_i$ is not injective.

Since each E_i is injective, it is divisible, consequently, E is a divisible R-module. Moreover, since R is a Bézout domain, every finitely generated ideal I in R is principal. It follows now that the equivalent condition of Theorem 1.4 holds for E but E is not injective.

PROPOSITION 1.7. Let R be a PID. An R-module E is injective if and only if it is divisible.

Proof. ∎

LEMMA 1.8. Let S be an R-algebra and E an injective R-module. Then $\operatorname{Hom}_R(S, E)$ is an injective S-module.

Note. $\operatorname{Hom}_R(S, E)$ is naturally an S-module under the action

$$(s \cdot f)(s') = f(ss') \qquad \forall\, s, s' \in S,\ f \in \operatorname{Hom}_R(S, E).$$

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of S-modules. Using the Hom-Tensor adjunction, we have

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_S(M'', \operatorname{Hom}_R(S, E)) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_S(M, \operatorname{Hom}_R(S, E)) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_S(M', \operatorname{Hom}_R(S, E)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(M' \otimes_S S, E) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_R(M \otimes_S S, E) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_R(M'' \otimes_S S, E) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(M', E) \arrow[r] & \operatorname{Hom}_R(M, E) \arrow[r] & \operatorname{Hom}_R(M'', E) \arrow[r] & 0
\end{tikzcd}$$

The exactness of the bottom row is a consequence of the R-injectivity of E. Thus the top row is exact and we have our desideratum. ∎

THEOREM 1.9. Every R-module can be embedded inside an R-injective.

Proof. First, we show this for $R = \mathbb{Z}$. Let M be a $\mathbb{Z}$-module, then $M \cong \bigoplus_I \mathbb{Z}/N$ for some submodule N of $\bigoplus_I \mathbb{Z}$. There is a natural inclusion of $\mathbb{Z}$-modules $\bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_I \mathbb{Q}$ which induces an inclusion

$$M \cong \frac{\bigoplus_I \mathbb{Z}}{N} \hookrightarrow \frac{\bigoplus_I \mathbb{Q}}{N} =: E$$

Being a quotient of a divisible module, E is divisible and hence $\mathbb{Z}$-injective.

Now, let R be any ring and M an R-module. Then M is naturally a $\mathbb{Z}$-module and admits a $\mathbb{Z}$-linear inclusion $\iota : M \hookrightarrow E$, where E is a $\mathbb{Z}$-injective. Consider the map

$$\varphi : M \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R, E) \qquad m \longmapsto \varphi_m,$$

where $\varphi_m : R \to E$ is given by $\varphi_m(r) = f(rm)$. The map φ is obviously R-linear and if $\varphi_m = 0$, then $f(m) = \varphi_m(1) = 0$, i.e., $m = 0$. As a result, φ is injective and we have embedded M inside an injective R-module. ■

Corollary 1.10. Let E be an R-module. Then E is injective if and only if every R-linear inclusion $E \hookrightarrow M$ splits.

Proof. Suppose E is injective.

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, equal] & M \arrow[dl, dotted, "\exists"] \\ & E & \end{tikzcd}$$

The above diagram constructs a splitting of $E \hookrightarrow M$.

Conversely, suppose every R-linear inclusion $E \hookrightarrow M$ splits. Due to Theorem 1.9, we may choose M to be injective, so that E is a direct summand of M, whence E is injective. ■

Proposition 1.11. Let R be a Noetherian ring. A direct sum of injective R-modules is injective.

Proof. Let $\{E_\lambda\}_{\lambda \in \Lambda}$ be a collection of R-injectives and $E = \bigoplus_{\lambda \in \Lambda} E_\lambda$. Let $I \trianglelefteq R$ be a non-zero proper ideal and $f : I \to E$ an R-linear map. Since I is finitely generated, its image under f is finitely generated in E. Consequently, there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $f(I) \subseteq \bigoplus_{\lambda \in \Lambda_0} E_\lambda = E_0$. Being a finite direct sum of injectives, E_0 is injective and hence there is a map $F : R \to E_0$ extending $f : I \to E_0$. Composing F with the natural inclusion $E_0 \hookrightarrow E$, we obtain our desired extension of f. It now follows from Theorem 1.4 that E is an injective R-module. ■

Interlude 1.12 (Bass-Papp Construction). Let R be a non-Noetherian ring. Choose a strictly increasing chain of proper non-zero ideals

$$0 \neq I_1 \subsetneq I_2 \subsetneq \cdots.$$

For each $n \geqslant 1$, choose an injective module E_n containing R/I_n, and set $E = \bigoplus_n E_n$. We contend that E is not R-injective.

Let $I = \bigcup_n I_n$. Since each I_n is proper, so is I. Let $f : I \to E$ be the map given by

$$f(x) = (x \bmod I_1, x \bmod I_2, \dots).$$

If E were injective, then there must exist a map $F : R \to E$ extending f. Suppose $F(1) = (x_1, x_2, \dots)$. There is a positive integer N such that $x_n = 0$ for all $n \geqslant N$. Choose $x \in I_{N+1} \setminus I_N$. Since $x \in I$, we have

$$(xx_1, xx_2, \dots) = F(x) = f(x) = (x \bmod I_1, x \bmod I_2, \dots).$$

In particular, $x \bmod I_N = xx_N = 0$, a contradiction. Thus E is not R-injective.

Proposition 1.13. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. If $E \neq 0$ is an finitely generated injective R-module, then R is Artinian.

Proof. We shall show that $\dim R = 0$. Suppose not; we contend that there is a prime $\mathfrak{p} \subsetneq \mathfrak{m}$ such that $\operatorname{Hom}_R(R/\mathfrak{p}, E) \neq 0$. Indeed, if there is a non-maximal prime $\mathfrak{p} \in \operatorname{Ass}_R(E)$, then $R/\mathfrak{p} \hookrightarrow E$, giving us the desideratum. On the other hand, if $\operatorname{Ass}_R(E) = \{\mathfrak{m}\}$, then the composition

$$R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{m} \hookrightarrow E$$

gives a non-zero map $R/\mathfrak{p} \to E$.

Choose $a \in \mathfrak{m} \setminus \mathfrak{p}$; this is a non-zerodivisor on $R/\mathfrak{p}$ and furnishes an exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{\cdot a} R/\mathfrak{p}.$$

Applying $\operatorname{Hom}_R(-, E)$, we get a surjection

$$\operatorname{Hom}_R(R/\mathfrak{p}, E) \xrightarrow{\cdot a} \operatorname{Hom}_R(R/\mathfrak{p}, E) \to 0.$$

Note that $\operatorname{Hom}_R(R/\mathfrak{p}, E) \cong (0 :_E \mathfrak{p}) \subseteq E$, is a finite R-module. Due to Nakayama's lemma, we must have that $\operatorname{Hom}_R(R/\mathfrak{p}, E) = 0$, a contradiction. Thus $\dim R = 0$, i.e. R is Artinian. ∎

REMARK 1.14. One cannot drop the local condition in Proposition 1.13. This construction makes use of injective hulls. Let k be an algebraically closed field and

$$R = \frac{k[X, Y]}{(X - X^2, Y - XY)}.$$

Note that R is the coordinate ring of the disjoint union of the origin and the line $x = 1$ in $\mathbb{A}^2_k$. In particular, $\dim R = 1$, and R is not Artinian.

Let $\mathfrak{m} = (x, y)$ be the maximal ideal corresponding to the origin. Then $R_\mathfrak{m} \cong k$, since it is the local ring of an isolated point. Now,

$$E_R(k) \cong E_{R_\mathfrak{m}}(k) \cong E_k(k) = k,$$

so that k is a finitely generated injective R-module.

§§ Essential Extensions and Injective Hulls

DEFINITION 1.15. A containment of R-modules $N \subseteq M$ is said to be *essential* if every non-zero submodule of M intersects N non-trivially.

An injective map $\iota : N \hookrightarrow M$ is said to be essential if $\iota(N) \subseteq M$ is essential.

REMARK 1.16. Let $M \subseteq N$ be an essential extension of R-modules and $\varphi : M \hookrightarrow P$ be an R-linear injective map. If φ extends to an R-linear map $\widetilde{\varphi} : N \to P$, then $\widetilde{\varphi}$ is injective too. Indeed, if $K = \ker \widetilde{\varphi} \neq 0$, then $K \cap M \neq 0$, a contradiction.

PROPOSITION 1.17. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Let M be an Artinian R-module. Then $\operatorname{Soc}_R(M) \subseteq M$ is an essential extension.

Proof. Let $0 \neq K \subseteq M$ be a submodule. Choose $0 \neq x \in K$. Since M is Artinian, the descending chain $Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2x \supseteq \cdots$ stabilizes. Let $n \geqslant 0$ be the least positive integer such that $\mathfrak{m}^n x = \mathfrak{m}^{n+1}x$. Due to Nakayama's lemma, $\mathfrak{m}^n x = 0$, whence $n \geqslant 1$. It follows that $0 \neq \mathfrak{m}^{n-1}x \subseteq \operatorname{Soc}_R(M) \cap K$, as desired. ∎

COROLLARY 1.18. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring and M an Artinian R-module. If $\dim_k \operatorname{Soc}_R(M) = d$, then $E_R(M) \cong E^{\oplus d}$.

Proof. Since $\operatorname{Soc}_R(M) \cong k^{\oplus d}$, it is clear that $E_R(\operatorname{Soc}_R(M)) \cong E^{\oplus d}$. The inclusion $\operatorname{Soc}_R(M) \hookrightarrow E^{\oplus d}$ can be extended to M to obtain a commutative diagram:

$$\begin{tikzcd}
M \arrow[dr, hook] & \\
\operatorname{Soc}_R(M) \arrow[u, hook] \arrow[r, hook] & E_R(\operatorname{Soc}_R(M)) \cong E^{\oplus d}
\end{tikzcd}$$

where all maps are inclusion. It follows that $M \hookrightarrow E^{\oplus d}$ is an essential extension. Since $E^{\oplus d}$ is an injective module, we have that $E_R(M) \cong E^{\oplus d}$. ∎

§2 Matlis Duality

Definition 2.1. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. For an R-module M, set $M^\vee = \operatorname{Hom}_R(M, E)$. This is known as the *Matlis dual* of a module.

Clearly $(-)^\vee$ is a contravariant exact functor on the category of R-modules. Note that if $I \subseteq \mathfrak{m}$ is an ideal, then as we have seen earlier,

$$E_{R/I}(k) = \operatorname{Hom}_R(R/I, E) = (R/I)^\vee.$$

In particular, taking $I = \mathfrak{m}$, we see that $k^\vee \cong k$ as R-modules.

Lemma 2.2. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then

(1) If $M \neq 0$, then $M^\vee \neq 0$.

(2) If $\lambda_R(M) < \infty$, then $\lambda_R(M^\vee) \neq 0$. Moreover, $\lambda_R(M) = \lambda_R(M^\vee)$.

Proof. (1) Let $0 \neq x \in M$. If $I = \operatorname{Ann}_R(x)$, then there is a natural inclusion $R/I \hookrightarrow M$ sending $\overline{1} \mapsto x$. Taking the Matlis dual, we have a surjection

$$M^\vee \twoheadrightarrow (R/I)^\vee = E_{R/I}(k) \neq 0,$$

consequently $M^\vee \neq 0$.

(2) We shall prove both statements by induction on $\lambda_R(M)$. If $\lambda_R(M) = 0$, then $M = 0$, so that $M^\vee = 0$ and we get $\lambda_R(M) = 0 = \lambda_R(M^\vee)$. Suppose now that $0 < \lambda_R(M) < \infty$. Then $\mathfrak{m} \in \operatorname{Ass}_R(M)$, and we have a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Since length is additive, $\lambda_R(N) = \lambda_R(M) - 1$; hence the induction hypothesis applies and $\lambda_R(N^\vee) = \lambda_R(N)$. Taking the Matlis dual of the above short exact sequence, we have

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow k^\vee \longrightarrow 0.$$

Since $k^\vee = 0$, we see that

$$\lambda_R(M^\vee) = \lambda_R(N^\vee) + 1 = \lambda_R(N) + 1 = \lambda_R(M),$$

as desired. ∎

Theorem 2.3. Let $(R, \mathfrak{m}, k, E)$ be an Artinian local ring.

(1) E is a faithful finite R-module.

(2) The map

$$\mu : R \longrightarrow \operatorname{Hom}_R(E, E) \qquad a \longmapsto \mu_a$$

is an isomorphism of R-modules and rings.

(3) Given a finite R-module M, the natural map

$$\varphi_M : M \longrightarrow M^{\vee\vee} \qquad m \longmapsto \mathrm{ev}_m$$

is an isomorphism.

Proof. (1) Suppose $a \in R$ is such that $aE = 0$. Then

$$R^\vee = \operatorname{Hom}_R(R, E) = E = (E :_E a) \cong \operatorname{Hom}_R(R/aR, E) = (R/aR)^\vee.$$

Since R is Artinian, we then have

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R\left((R/aR)^\vee\right) = \lambda_R(R/aR) \implies \lambda_R(aR) = 0,$$

consequently, $a = 0$, i.e., E is a faithful R-module.

Next, since R is Artinian, $\mathfrak{m} \in \operatorname{Ass}_R(R)$, consequently, there is an injection $k = R/\mathfrak{m} \hookrightarrow R$. Due to Remark 1.16 extends to an inclusion $E \hookrightarrow R$, consequently, E is a finite R-module.

(2) First note that μ is injective due to (1). But note that

$$\infty > \lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = \lambda_R(E^\vee) = \lambda_R\left(\operatorname{Hom}_R(E,E)\right),$$

consequently μ is an isomorphism.

(3) It suffices to show that φ_M is injective since $\lambda_R(M) = \lambda_R(M^{\vee\vee})$. Suppose $0 \neq x \in M$ is such that $\varphi_M(x) = 0$, that is, for all $f \in \operatorname{Hom}_R(M,E)$, $f(x) = 0$. Let $I = \operatorname{Ann}_R(x)$. Now, there is a non-zero map

$$\psi : R/I \twoheadrightarrow R/\mathfrak{m} = k \hookrightarrow E,$$

which extends to a non-zero map $f : M \to E$ since $R/I \hookrightarrow M$ through $\overline{1} \mapsto x$. Thus, $f(x) = \psi(\overline{1}) \neq 0$, a contradiction. ■

INTERLUDE 2.4 (ON $\widehat{R}$-MODULES). Let $(R,\mathfrak{m},k)$ be a local ring and M an R-module such that $\Gamma_\mathfrak{m}(M) = M$. We contend that M is an $\widehat{R}$-module in a natural way. To this end, we need only define $\widehat{a} \cdot m$ for $\widehat{a} \in \widehat{R}$ and $m \in M$.

Let $\widehat{a} = (a_1, a_2, \ldots)$, where we are using the isomorphism

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

Since $\Gamma_\mathfrak{m}(M) = M$, there is a positive integer $n \geqslant 1$ such that $\mathfrak{m}^n m = 0$. Hence, for $k \geqslant n$, we have $a_k \cdot m = a_n \cdot m$, as $a_k - a_n \in \mathfrak{m}^n$. In light of this, we define $\widehat{a} \cdot m = a_n \cdot m$. We must show that this makes M into an $\widehat{R}$-module.

Let $m_1, m_2 \in M$ and $\widehat{a} = (a_1, a_2, \ldots) \in \widehat{R}$. There are positive integers $n_1, n_2 \geqslant 1$ such that $\mathfrak{m}^{n_1} m_1 = 0 = \mathfrak{m}^{n_2} m_2$; then $\mathfrak{m}^n m_1 = 0 = \mathfrak{m}^n m_2$ for all $n \geqslant \max\{n_1, n_2\}$. Hence, for all such $n \geqslant 1$,

$$\widehat{a} \cdot (m_1 + m_2) = a_n \cdot (m_1 + m_2) = a_n \cdot m_1 + a_n \cdot m_2 = \widehat{a} \cdot m_1 + \widehat{a} \cdot m_2.$$

Next, let $\widehat{a}, \widehat{b} \in \widehat{R}$ and $m \in M$ with

$$\widehat{a} = (a_1, a_2, \ldots) \qquad \text{and} \qquad \widehat{b} = (b_1, b_2, \ldots).$$

There is a positive integer n such that $\mathfrak{m}^n m = 0$. Then

$$(\widehat{a} + \widehat{b}) \cdot m = (a_n + b_n) \cdot m = a_n \cdot m + b_n \cdot m = \widehat{a} \cdot m + \widehat{b} \cdot m.$$

Finally, note that $\widehat{b} \cdot m = b_n m$ and $\mathfrak{m}^n \left(\widehat{b} \cdot m\right) = 0$, so that

$$\widehat{a} \cdot \left(\widehat{b} \cdot m\right) = \widehat{a} \cdot (b_n \cdot m) = a_n \cdot (b_n \cdot m) = (a_n b_n) \cdot m = (\widehat{ab}) \cdot m.$$

This shows that M is indeed an $\widehat{R}$-module as described above. Further, since $R \to \widehat{R}$ is the diagonal map, it follows that the $\widehat{R}$-module structure on M agrees with the R-module struture through the diagonal map. In particular, this means that:

> A subset of M is an R-submodule if and only if it is an $\widehat{R}$-submodule.

As a result, M is Noetherian (resp. Artinian) as an R-module if and only if it is so as an $\widehat{R}$-module.

INTERLUDE 2.5 (ON MAPS BETWEEN $\mathfrak{m}$-POWER TORSION MODULES). Again, let $(R,\mathfrak{m},k)$ be a local ring and suppose M and N are R-modules such that $\Gamma_\mathfrak{m}(M) = \Gamma_\mathfrak{m}(N)$. By Interlude 2.4, we know that they are $\widehat{R}$-modules in a natural way. Let $\varphi : M \to N$ be an R-linear map. We contend that φ is also $\widehat{R}$-linear. Indeed, let $m \in M$ and $\widehat{a} = (a_1, a_2, \ldots) \in \widehat{R}$. There is a positive integer $n \geqslant 1$ such that $\mathfrak{m}^n m = 0$, and hence, $\mathfrak{m}^n \varphi(m) = 0$. It follows that

$$\varphi(\widehat{a} \cdot m) = \varphi(a_n \cdot m) = a_n \cdot \varphi(m) = \widehat{a} \cdot \varphi(m),$$

as desired.

THEOREM 2.6. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring.

(1) $\Gamma_{\mathfrak{m}}(E) = E$, and hence E is an $\widehat{R}$ module and for every R-module M, $M^{\vee}$ is $\mathfrak{m}$-power torsion.

(2) $E \cong E_{\widehat{R}}(k)$ as $\widehat{R}$-modules.

(3) $R^{\vee\vee} = \operatorname{Hom}_R(E, E) \cong \widehat{R}$ as R-modules.

(4) E is an Artinian R-module.

Proof. (1) That E is an $\widehat{R}$-module follows immediately from Interlude 2.4. Finally, $M^{\vee} = \operatorname{Hom}_R(M, E)$ is $\mathfrak{m}$-power torsion because E is so.

(2) The containment $k \subseteq E$ is an essential extension of R-modules, both of which are $\mathfrak{m}$-power torsion. Due to Interlude 2.4, it follows that it is an essential extension of $\widehat{R}$-modules too. Now, due to Remark 1.16, there is a commutative diagram of inclusions

$$\begin{tikzcd} E \arrow[dr] & \\ k \arrow[u] \arrow[r] & E_{\widehat{R}}(k), \end{tikzcd}$$

where all maps are $\widehat{R}$-linear. It follows that $E \hookrightarrow E_{\widehat{R}}(k)$ is an essential extension of $\widehat{R}$-modules, and consequently, an essential extension of R-modules. Since E is R-injective, we must have that the inclusion is an isomorphism of R-modules. Finally, due to Interlude 2.5, this is an isomorphism of $\widehat{R}$-modules.

(3)

(4) Let $M_1 \supseteq M_2 \supseteq \cdots$ be a chain of R-submodules in E. There are commutative diagrams

$$\begin{tikzcd} M_{j+1} \arrow[r, hook, "\iota_{j+1}"] \arrow[d, hook] & E \\ M_j \arrow[ur, hook, "\iota_j"'] & \end{tikzcd}$$

whose Matlis dual furnishes commutative diagrams

$$\begin{tikzcd} \widehat{R} = E^{\vee} \arrow[r, two heads, "\varphi_j"] \arrow[dr, two heads, "\varphi_{j+1}"'] & M_j^{\vee} \arrow[d, two heads] \\ & M_{j+1}^{\vee} \end{tikzcd} .$$

Note that all Matlis duals are $\mathfrak{m}$-power torsions and hence due to Interlude 2.5, the φ_j's are $\widehat{R}$-linear. Let $I_j = \ker \varphi_j \subseteq \widehat{R}$, which is an ideal. Due to the commutative diagram, it is clear that there is an ascending chain $I_j \subseteq I_{j+1}$. Since $\widehat{R}$ is Noetherian, this chain stabilizes, say $I_n = I_{n+1} = \ldots$.

Then due to the first isomorphism theorem, $M_j^{\vee} \twoheadrightarrow M_{j+1}^{\vee}$ is an isomorphism for all $j \geqslant n$. Let $C_j = \operatorname{coker}(M_{j+1} \hookrightarrow M_j)$. The exactness of the Matlis dual gives $C_j^{\vee} = 0$, which, due to Lemma 2.2, implies that $C_j = 0$, that is, $M_{j+1} \hookrightarrow M_j$ is an isomorphism for all $j \geqslant n$, i.e., the descending chain stabilizes, as desired. ∎

THEOREM 2.7 (MATLIS DUALITY, VERSION 1). Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{finitely generated}\\ \widehat{R}\text{-modules}\end{matrix}\right\} \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{Artinian } R\text{-modules}\end{matrix}\right\}.$$

Proof. Let M be an Artinian R-module and let $d = \dim_k \operatorname{Soc}_R(M)$. Due to Corollary 1.18, $E_R(M) \cong E^{\oplus d}$, so that there is an inclusion $M \hookrightarrow E^{\oplus d}$, which upon taking the Matlis dual furnishes an $\widehat{R}$-linear surjection $\widehat{R}^{\oplus d} \twoheadrightarrow M^\vee$. Thus $M^\vee$ is a finite $\widehat{R}$-module.

Conversely, suppose M is a finite $\widehat{R}$-module. Thus, there is a surjection $\widehat{R}^{\oplus n} \twoheadrightarrow M$. Taking the Matlis dual, we obtain an injection $M^\vee \hookrightarrow \left(\widehat{R}^\vee\right)^{\oplus n}$.

There is a natural "evaluation map" $\mathrm{ev} : M \to M^{\vee\vee}$, which we shall show is an isomorphism. That ev is injective follows in the same way as Theorem 2.3 (3). Next, since $\lambda_R(M) < \infty$, we have that $\lambda_R(M) = \lambda_R(M^\vee) = \lambda_R(M^{\vee\vee})$, whence ev is an isomorphism. ■

THEOREM 2.8. Let $(R, \mathfrak{m}, k, E)$ be a Noetherian local ring. Then the following are equivalent:

(1) R is self-injective

(2) $R \cong E$ as R-modules.

(3) R is Artinian and $\dim_k \operatorname{Soc}_R(R) = 1$.

Proof. (1) $\implies$ (2) Due to Proposition 1.13, R must be an Artinian local ring, and hence, from Proposition 1.17, $\operatorname{Soc}_R(R) \subseteq R$ is an essential extension. It follows that R is the injective hull of $\operatorname{Soc}_R(R) \cong k^{\oplus d}$ for some positive integer d. Hence, $R \cong E^{\oplus d}$ as R-modules, and comparing lengths, we have

$$\lambda_R(R) = d\lambda_R(E) = d\lambda_R(R^\vee) = d\lambda_R(R),$$

whence $d = 1$ and $R \cong E$.

(2) $\implies$ (3) Due to Theorem 2.6 (4), R is Artinian. Using a length argument as above, we can show that $\dim_k \operatorname{Soc}_R(R) = 1$.

(3) $\implies$ (1) Again, since $k = \operatorname{Soc}_R(R) \subseteq R$ is essential, we have that $R \hookrightarrow E = E_R(k)$. Using a length argument, it follows that this inclusion must be an isomorphism, whence R is self-injective. ■

§3 INJECTIVE RESOLUTIONS

DEFINITION 3.1. Let M be an R-module. An *injective resolution* for M is an exact complex

$$0 \to M \to E^0 \to E^1 \to E^2 \to \cdots,$$

where each E^n is an injective R-module. The resolution is often denoted succinctly as $0 \to M \to E^\bullet$.

We say that M has finite injective dimension if M has an injective resolution $0 \to M \to E^\bullet$ and an integer $N \geqslant 0$ such that $E^n = 0$ for $n \geqslant N$. We define

$$\operatorname{inj\,dim}_R M = \inf\left\{n : 0 \to M \to E^0 \to \cdots \to E^n \to 0 \text{ is an injective resolution of } M\right\}.$$

If M does not have finite injective dimension, then set $\operatorname{inj\,dim}_R M = \infty$.

REMARK 3.2. It is possible to create a "canonical" injective resolution by successively taking injective hulls. Set $E^0 = E_R(M)$ and for $i \geqslant 0$, define

$$E^{i+1} = E_R\left(\operatorname{coker}\left(E^{i-1} \to E^i\right)\right),$$

with the convention that $E^{-1} = M$. We call this the *minimal injective resolution* of M.

LEMMA 3.3. Let R be a Noetherian ring and $0 \to M \xrightarrow{\theta} E$ be an inclusion of R-modules with E injective. Then the inclusion is an injective hull of M if and only if

$$\operatorname{Hom}_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \xrightarrow{\theta_{\mathfrak{p}}} \operatorname{Hom}_R(R/\mathfrak{p}, E)_{\mathfrak{p}}$$

is an isomorphism for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. Owing to the left exactness of $\operatorname{Hom}_R(R/\mathfrak{p}, -)$ and the exactness of localization, the map $\theta_{\mathfrak{p}}$ is injective for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Hence, it suffices to show that E is injective if and only if $\theta_{\mathfrak{p}}$ is surjective for each $\mathfrak{p} \in \operatorname{Spec}(R)$.

Recall that there are canonical isomorphisms

$$\operatorname{Hom}_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \xrightarrow{\sim} \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \qquad \frac{\psi}{s} \longmapsto \left(\frac{a}{t} \mapsto \frac{\psi(a)}{st}\right),$$

where we are identifying $\kappa(\mathfrak{p})$ with the quotient field of $R/\mathfrak{p}$. Hence, surjectivity of $\theta_{\mathfrak{p}}$ is equivalent to the surjectivity of

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \to \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}}).$$

Henceforth, we shall identify M with a submodule of E, so that θ is simply the inclusion map.

Suppose first that $M \xrightarrow{\theta} E$ is an injective hull and let $0 \neq \varphi \in \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}})$. Using the above isomorphism, we can write $\varphi = \psi/s$ for some $\psi \in \operatorname{Hom}_R(R/\mathfrak{p}, E)$ and $s \in R \setminus \mathfrak{p}$. Let $\psi(\overline{1}) = z \in E$ and $a \in R$ such that $0 \neq az \in M$. Note that $a \in R \setminus \mathfrak{p}$, since $\mathfrak{p} \subseteq \operatorname{Ann}_R(z)$[1]. Define

$$\overline{\varphi} : R/\mathfrak{p} \longrightarrow M \qquad \overline{1} \longmapsto az.$$

This is well-defined, since $\mathfrak{p}$ annihilates $az \in M$. We claim that

$$\varphi = \frac{\overline{\varphi}}{as} \in \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}}).$$

Indeed, for $x/t \in \kappa(\mathfrak{p})$ we have

$$\left(\frac{\overline{\varphi}}{as}\right)\left(\frac{x}{t}\right) = \frac{\overline{\varphi}(x)}{ast} = \frac{xaz}{ast} = \frac{xz}{st} = \left(\frac{\psi}{s}\right)\left(\frac{x}{t}\right) = \varphi\left(\frac{x}{t}\right),$$

as desired. This shows that $\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \to \operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}})$ is surjective.

Conversely, suppose the aforementioned map is surjective. We shall show that E is the injective hull of M. To this end, it suffices to show that the inclusion $M \subseteq E$ is essential. Let $0 \neq N \subseteq E$ be a submodule and $\mathfrak{p} \in \operatorname{Ass}_R(N)$. There is an injective map

$$0 \to R/\mathfrak{p} \longrightarrow N \qquad \overline{1} \longmapsto z.$$

Since $\mathfrak{p} = \operatorname{Ann}_R(z)$, it suffices to find $a \in R \setminus \mathfrak{p}$ such that $az \in M$. Consider the map

$$\varphi : \kappa(\mathfrak{p}) \longrightarrow E_{\mathfrak{p}} \qquad \overline{1} \longmapsto z/1.$$

The surjectivity of $\theta_{\mathfrak{p}}$ furnishes a $\psi : \kappa(\mathfrak{p}) \to M_{\mathfrak{p}}$ such that $\theta_{\mathfrak{p}}(\psi) = \varphi$. In particular, this means that

$$\frac{z}{1} = \varphi(\overline{1}) = \psi(\overline{1}) \in M_{\mathfrak{p}},$$

whence there is some $a \in R \setminus \mathfrak{p}$ such that $az \in M$, as desired. ∎

COROLLARY 3.4. Let R be a Noetherian ring and $0 \to M \to E^{\bullet}$ be an injective resolution of an R-module M. Then $E^{\bullet}$ is minimal if and only if the natural maps

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E^n_{\mathfrak{p}}) \longrightarrow \operatorname{Hom}_{R_{\mathfrak{p}}}\left(\kappa(\mathfrak{p}), E^{n+1}_{\mathfrak{p}}\right)$$

are identically zero for all $n \geqslant 0$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

[1] Note that $\mathfrak{p} = \operatorname{Ann}_R(z)$, for if not, then $\varphi = 0$.

Proof. Let $K^n = \ker\left(E^n \to E^{n+1}\right)$. Then there is an exact sequence $0 \to K^n \to E^n \to E^{n+1}$. Using Lemma 3.3, E^n is the injective hull of C^n if and only if

$$\Phi : \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), C^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \text{ is an isomorphism.}$$

But the left-exactness of Hom and exactness of localization implies that the sequence

$$0 \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), C^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^{n+1}_\mathfrak{p}\right)$$

is exact. Thus Φ is an isomorphism if and only if the map $\operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^n_\mathfrak{p}\right) \to \operatorname{Hom}_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), E^{n+1}_\mathfrak{p}\right)$ is the zero map, as desired. ■

Corollary 3.5. Let R be a Noetherian ring and M an R-module. Let $0 \to M \to E^\bullet$ be *the* minimal injective resolution of M. Then

$$E^j = \bigoplus_{\mathfrak{p}} E_R\left(R/\mathfrak{p}\right)^{a_j(\mathfrak{p})} \quad \text{and} \quad a_j(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), M_\mathfrak{p}\right).$$

In particular, if M is a finite R-module, $a_j(\mathfrak{p}) < \infty$ for all $j \geqslant 0$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. ■

Definition 3.6. Let R be a Noetherian ring and M a finite R-module. For $j \geqslant 0$ and $\mathfrak{p} \in \operatorname{Spec}(R)$, define the *j-th Bass number* as

$$\mu_j\left(\mathfrak{p}, M\right) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), M_\mathfrak{p}\right).$$

Lemma 3.7 (Bass). Let R be a Noetherian ring and M a finite R-module. Let $\mathfrak{p} \subsetneq \mathfrak{q}$ be primes in R such that $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = 1$. If for some $j \geqslant 0$, $\mu_j(\mathfrak{p}, M) \neq 0$, then $\mu_{j+1}(\mathfrak{q}, M) \neq 0$.

Proof. Localizing at $\mathfrak{q}$, we may assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring and $\operatorname{ht}(\mathfrak{m}/\mathfrak{p}) = 1$. If $a \in \mathfrak{m} \setminus \mathfrak{p}$, then $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$, and we have a short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{\cdot a} R/\mathfrak{p} \to R/(\mathfrak{p} + (a)) \to 0.$$

This gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \xrightarrow{\cdot a} \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \to \operatorname{Ext}^{j+1}_R(R/(\mathfrak{p} + (a)), M) \to \cdots,$$

for all $j \geqslant 0$.

$$\mu_j(\mathfrak{p}, M) \neq 0 \implies \operatorname{Ext}^j_{R_\mathfrak{p}}\left(\kappa(\mathfrak{p}), M_\mathfrak{p}\right) \neq 0 \implies \operatorname{Ext}^j_R(R/\mathfrak{p}, M) \neq 0.$$

Since the Ext's are finite R-modules, Nakayama's lemma implies that $\operatorname{Ext}^{j+1}_R(R/(\mathfrak{p} + (a)), M) \neq 0$.

Since $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$, the R-module $R/(\mathfrak{p} + (a))$ is finite Artinian, so that it has a composition series with successive quotients isomorphic to $R/\mathfrak{m} = k$. Now, if $\operatorname{Ext}^{j+1}_R(k, M) \neq 0$, then through the short exact sequences induced by the composition series, it would follow that $\operatorname{Ext}^{j+1}_R(R/(\mathfrak{p} + (a)), M) = 0$, a contradiction. But since $R \setminus \mathfrak{m}$ consists of only units, we have that

$$0 \neq \operatorname{Ext}^{j+1}_R(k, M) = \operatorname{Ext}^{j+1}_{R_\mathfrak{m}}(\kappa(\mathfrak{m}), M_\mathfrak{m}),$$

and hence $\mu_{j+1}(\mathfrak{m}, M) \neq 0$. ■