Projective, Injective, and Flat Modules

Swayam Chube

December 10, 2024

§1 Projective Modules

Definition 1.1. An A-module M is said to be *projective* if the functor $\operatorname{Hom}_A(M, -) : \mathfrak{Mod}_A \to \mathfrak{Mod}_A$ is exact.

§§ Kaplansky's Theorem

Theorem 1.2. Let $(A, \mathfrak{m}, k)$ be a local ring. If M is a projective A-module, then M is free.

§2 Flat Modules

Definition 2.1. An A-module M is said to be *flat* if the functor $- \otimes_A M : \mathfrak{Mod}_A \to \mathfrak{Mod}_A$ is exact.

Definition 2.2. Let M be an A-module and $\sum_{i=1}^n f_i x_i = 0$ be a relation in M for $f_i \in A$ and $x_i \in M$. We say that the relation is *trivial* if there exists an integer $m \geqslant 0$, elements $y_j \in M$ for $1 \leqslant j \leqslant m$ and $a_{ij} \in A$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that

$$x_i = \sum_{j=1}^m a_{ij} y_j \quad \forall\, 1 \leqslant i \leqslant n \quad \text{and} \quad 0 = \sum_{i=1}^n a_{ij} f_i \quad \forall\, 1 \leqslant j \leqslant m.$$

Lemma 2.3 (Equational criterion of flatness). An A-module M is flat if and only if every relation in M is trivial.

Proof. Suppose M is flat and $\sum_{i=1}^n f_i x_i = 0$ is a relation in M. Let $\mathfrak{a} = (f_1, \dots, f_n) \subseteq A$ and consider the A-linear surjection $A^n = \bigoplus_{i=1}^n A e_i \to I$ given by $e_i \mapsto f_i$ whose kernel is $K \subseteq A^n$. That is, $0 \to K \to A^n \to \mathfrak{a} \to 0$. Since M is flat, tensoring with M preserves exactness and we have an exact sequence

$$0 \longrightarrow K \otimes_A M \longrightarrow A^n \otimes_A M \longrightarrow \mathfrak{a} \otimes_A M \longrightarrow 0.$$

Note that the natural map $\mathfrak{a} \otimes_A M \to R \otimes_A M$ is injective due to the flatness of M. Consequently, $\sum_{i=1}^n f_i \otimes x_i$ maps to 0 in $R \otimes_A M$ and hence, must be zero in $\mathfrak{a} \otimes_A M$. The exactness of the above sequence furnishes an element $\sum_{j=1}^m k_j \otimes y_j \in K \otimes_A M$ that maps to 0 in $A^n \otimes_A M$.

Each k_j can be written in the form

$$\sum_{i=1}^n a_{ij} e_i \quad \forall\, 1 \leqslant j \leqslant m,$$

and hence, the image of $\sum_{j=1}^m k_j \otimes y_j$ in $A^n \otimes_A M$ is

$$\sum_{j=1}^m \sum_{i=1}^m a_{ij} e_i \otimes y_j = \sum_{i=1}^n e_i \otimes \left(\sum_{j=1}^m a_{ij} y_j \right) = 0,$$

and the conclusion follows.

Conversely, suppose every relation in M is trivial and let $\mathfrak{a}$ be a finitely generated ideal of A. It suffices to show that $\operatorname{Tor}_1^A(A/\mathfrak{a}, M) = 0$, which is equivalent (from the Tor long exact sequence) to showing that the map $\mathfrak{a} \otimes_A M \to A \otimes_A M$ is injective.

Suppose $\sum_{i=1}^n f_i \otimes x_i \in \mathfrak{a} \otimes_A M$ maps to 0 in $A \otimes_A M$. Then, $\sum_{i=1}^n f_i x_i = 0$ in M, consequently, there is an $m \geqslant 0$, $y_j \in M$, $a_{ij} \in M$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that

$$x_i = \sum_{j=1}^m a_{ij} y_j \quad \forall\, 1 \leqslant i \leqslant n \quad \text{and} \quad 0 = \sum_{i=1}^n a_{ij} f_i \quad \forall\, 1 \leqslant j \leqslant m.$$

Consequently, in $\mathfrak{a} \otimes_A M$,

$$\sum_{i=1}^n f_i \otimes x_i = \sum_{i=1}^n f_i \otimes \left(\sum_{j=1}^m a_{ij} y_j \right) = \left(\sum_{i=1}^n a_{ij} f_i \right) \otimes y_j = 0.$$

This proves injectivity, thereby completing the proof. ■

Lemma 2.4. Let $(A, \mathfrak{m}, k)$ be a local ring and M a flat A-module. If $x_1, \dots, x_n \in M$ are such that their images $\overline{x}_1, \dots, \overline{x}_n \in M/\mathfrak{m}M$ are linearly independent over k, then $x_1, \dots, x_n$ are linearly independent over A.

Proof. We prove this statement by induction on n. If $n = 1$, then $a \in A$ is such that $ax_1 = 0$ and $\overline{x}_1 \neq 0$. From Lemma 2.3, there are $b_1, \dots, b_m \in A$ and $y_1, \dots, y_m \in M$ such that

$$x_1 = \sum_{j=1}^m b_j y_j \quad \text{and} \quad ab_j = 0 \quad \forall\, 1 \leqslant j \leqslant m.$$

Since $x_1 \notin \mathfrak{m}M$, it follows that at least one of the b_j's must be a unit, whence $a = 0$.

Now, suppose $n > 1$ and there is a relation $\sum_{i=1}^n a_i x_i = 0$ in M. From Lemma 2.3, there is an $m \geqslant 0$, $y_j \in M$, and $b_{ij} \in A$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that

$$x_i = \sum_{j=1}^m b_{ij} y_j \quad \forall\, 1 \leqslant i \leqslant n \quad \text{and} \quad 0 = \sum_{i=1}^n b_{ij} a_i \quad \forall\, 1 \leqslant j \leqslant m.$$

Since $x_n \notin \mathfrak{m}M$, at least one of the b_{nj}'s must be a unit, whence we can write

$$a_n = \sum_{i=1}^{n-1} c_i a_i,$$

for some $c_i \in A$ for $1 \leqslant i \leqslant n-1$. Therefore, we have

$$0 = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n-1} a_i (x_i + c_i x_n).$$

Since $\overline{x}_1, \ldots, \overline{x}_{n-1}$ are k-linearly independent in $M/\mathfrak{m}M$, we see that $\overline{x}_1 + \overline{c}_1\overline{x}_n, \ldots, \overline{x}_{n-1} + \overline{c}_{n-1}\overline{x}_n$ must also be k-linearly independent. Due to the induction hypothesis, $a_1 = \cdots = a_{n-1} = 0$ and hence, $a_n = 0$. This completes the proof. ■

THEOREM 2.5. Let $(A, \mathfrak{m}, k)$ be a local ring. If M is a finitely generated flat A-module, then M is free.

Proof. Let $x_1, \ldots, x_n \in M$ be a minimal generating set, that is, $\overline{x}_1, \ldots, \overline{x}_n$ are k-linearly independent in $M/\mathfrak{m}M$. Due to the preceding lemma, $x_1, \ldots, x_n$ are linearly independent over A, and hence, M is a free A-module. ■

§§ Cartier's Theorem

THEOREM 2.6 (CARTIER). Let M be a finitely generated module over an integral domain A. If for every $\mathfrak{m} \in \operatorname{MaxSpec}(A)$, $M_\mathfrak{m}$ is free as an $A_\mathfrak{m}$-module, then M is a projective A-module.

Proof. First show that M is a torsion-free A-module. Suppose $am = 0$ for some $0 \neq a \in A$ and $m \in M$. Let $\mathfrak{a}$ be the annihilator of m in A and $\mathfrak{m}$ a maximal ideal containing A. Note that $\frac{a}{1}\frac{m}{1} = 0$ in $M_\mathfrak{m}$, which is free over $A_\mathfrak{m}$, an integral domain, whence, is torsion free. That is, $\frac{m}{1} = 0$, whence, there is some $s \in A \setminus \mathfrak{m}$ such that $sm = 0$, which is absurd, since $\mathfrak{a} \subseteq \mathfrak{m}$. This shows that M is torsion-free.

Now, choose a set of generators $\{m_i \colon 1 \leqslant i \leqslant n\}$ for M over A. Let $\mathscr{P}$ be the collection of A-endomorphisms of M which are of the form

$$m \longmapsto \sum_{i=1}^{n} f_i(m) m_i,$$

where $f_1, \ldots, f_n : M \to A$ are A-module homomorphisms. Note that $\mathscr{P}$ is an A-submodule of $\operatorname{End}_A(M)$. We shall show that $\mathbf{id}_M \in \mathscr{P}$.

Let $\mathfrak{m}$ be a maximal ideal of A. We know that $M_\mathfrak{m}$ is free as an $A_\mathfrak{m}$-module and hence, there are $A_\mathfrak{m}$-module homomorphisms $f_i : M_\mathfrak{m} \to A_\mathfrak{m}$ such that

$$m' = \sum_{i=1}^{n} f_i'(m') \frac{m_i}{1} \quad \forall m' \in M_\mathfrak{m}.$$

To see that this is possible, first consider an $A_{\mathfrak{m}}$-basis $\{e_i \colon 1 \leqslant i \leqslant N\}$ for $M_{\mathfrak{m}}$. We can write

$$e_i = \sum_{j=1}^{n} a_{ij} \frac{m_j}{1} \quad \forall\, 1 \leqslant i \leqslant N.$$

Further, there are $A_{\mathfrak{m}}$-linear maps $f_i : M_{\mathfrak{m}} \to A_{\mathfrak{m}}$ such that

$$m' = \sum_{j=1}^{N} f_j(m') e_j.$$

Set

$$f_j'(m') = \sum_{i=1}^{N} a_{ij} f_i(m') \quad \forall\, m' \in M_{\mathfrak{m}}.$$

Then,

$$\sum_{j=1}^{n} f_j'(m') \frac{m_j}{1} = \sum_{i=1}^{N} \sum_{j=1}^{n} a_{ij} f_i(m') \frac{m_j}{1} = \sum_{i=1}^{N} f_i(m') e_i = m'.$$

Coming back, since M is torsion-free, the canonical map $M \to M_{\mathfrak{m}}$ is an injective map of A-modules. Further, we can find an $s \in A \setminus \mathfrak{m}$ such that $s f_i' \left(\frac{m_j}{1} \right) \in A$ for $1 \leqslant i, j \leqslant n$.

Note that $m' \mapsto s f_i'(m')$ is $A_{\mathfrak{m}}$-linear as a map $M_{\mathfrak{m}} \to A_{\mathfrak{m}}$, and hence, is A-linear. The restriction of this map to $M \subseteq M_{\mathfrak{m}}$ takes values in A. Thus, we can identify $s f_i'$ with an A-linear map $M \to A$. Further, for every $m \in M$, we have

$$sm = \sum_{i=1}^{n} s f_i'(m) m_i.$$

That is, $s \cdot \mathbf{id}_M \in \mathscr{P}$. Now, let $\mathfrak{a}$ be the collection of all $a \in A$ such that $a \cdot \mathbf{id}_M \in \mathscr{P}$. Then $\mathfrak{a}$ is an ideal of A. If $\mathfrak{a}$ were a proper ideal, it would be contained in a maximal ideal $\mathfrak{m}$. But from our preceding conclusion, there is some $s \in A \setminus \mathfrak{m}$ such that $s \cdot \mathbf{id}_M \in \mathscr{P}$, a contradiction. Thus, $\mathfrak{a} = A$, in particular, $\mathbf{id}_M \in \mathscr{P}$.

Finally, we show that M is projective. We have shown that there are A-linear maps $f_i : M \to A$ such that

$$m = \sum_{i=1}^{n} f_i(m) m_i \quad \forall\, m \in M.$$

Let F be the free module $\bigoplus_{i=1}^{n} A e_i$ and let $g : F \to M$ be given by $e_i \mapsto m_i$ and $f : M \to F$ given by

$$f(m) = \sum_{i=1}^{n} f_i(m) e_i.$$

By our construction, $g \circ f = \mathbf{id}_M$, and hence M is a direct summand of F, i.e. M is projective. ∎

Corollary. A finitely generated flat module over an integral domain is projective.

Proof. Follows from Theorem 2.6 and Theorem 2.5. ∎

§§ Finitely Presented Modules and Flatness

THEOREM 2.7. Let M be a finitely presented A-module and N be any A-module. If B is a flat A-algebra, then there is a natural isomorphism

$$\mathrm{Hom}_A(M, N) \otimes_A B \cong \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$

Proof. Fixing N and B, there are contravariant functors $\mathscr{F}, \mathscr{G} : \mathfrak{Mod}_A^{op} \to \mathfrak{Mod}_B$ given by

$$\mathscr{F}(M) = \mathrm{Hom}_A(M, N) \otimes_A B \qquad \mathscr{G}(M) = \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$

Define the natural transformation $\lambda : \mathscr{F} \implies \mathscr{G}$ given by

$$\lambda_M(f \otimes b) = b \cdot (f \otimes \mathbf{id}_B).$$

We first show that this is natural in M. Indeed, suppose $\varphi : M' \to M$ is A-linear, we wish to show that

$$\begin{tikzcd} \mathscr{F}(M) \arrow[r] \arrow[d, "\lambda_M"'] & \mathscr{F}(M') \arrow[d, "\lambda_{M'}"] \\ \mathscr{G}(M) \arrow[r] & \mathscr{G}(M') \end{tikzcd}$$

commutes. Consider $f \otimes b \in \mathscr{F}(M)$, which maps to $f \circ \varphi \otimes b \in \mathscr{F}(M')$, which maps to $b \cdot (f \circ \varphi \otimes \mathbf{id}_B) \in \mathscr{G}(M')$. On the other hand, under λ_M, $f \otimes b$ maps to $b \cdot (f \otimes \mathbf{id}_B) \in \mathscr{G}(M)$, which maps to $b \cdot (f \circ \varphi \otimes \mathbf{id}_B)$, which shows commutativity.

Next, suppose $M = A^n$ were free of finite rank. In this case, there is a sequence of isomorphisms

$$\mathrm{Hom}_A(A^n, N) \otimes_A B \cong N^n \otimes_A B \cong (N \otimes_A B)^n \cong \mathrm{Hom}_B(B^n, N \otimes_A B) \cong \mathrm{Hom}_B(A^n \otimes_A B, N \otimes_A B).$$

Under the above isomorphism, $f \otimes b$ first maps to $(f(e_1), \ldots, f(e_n))^\top \otimes b$ in $N^n \otimes_A B$. Under the second map, it goes to $(f(e_1) \otimes b, \ldots, f(e_n) \otimes b)^\top$ in $(N \otimes_A B)^n$. Under the third map it goes to the unique morpism $g : B^n \to N \otimes_A B$ that sends $e_i \mapsto f(e_i) \otimes b$.

Consider the map $b \cdot (f \otimes \mathbf{id}_B) \in \mathrm{Hom}_B(A^n \otimes_A B, N \otimes_A B)$. Under this map, $e_i \in B^n$ is the same as $e_i \otimes 1 \in A^n \otimes B$, which maps to $b \cdot (f(e_i) \otimes 1) = f(e_i) \otimes b \in N \otimes_A B$. It follows that this is the same as the aforementioned g. Thus, λ_M is an isomorphism in this case.

Finally, there is an exact sequence $A^m \to A^n \to M \to 0$ since M is finitely presented. This fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \mathscr{F}(M) \arrow[r] \arrow[d, "\lambda"] & \mathscr{F}(A^n) \arrow[r] \arrow[d, "\lambda"] & \mathscr{F}(A^m) \arrow[d, "\lambda"] \\ 0 \arrow[r] & \mathscr{G}(M) \arrow[r] & \mathscr{G}(A^n) \arrow[r] & \mathscr{G}(A^m) \end{tikzcd}$$

where the last two λ's are isomorphisms. Due to the Five Lemma (after adding another column of zeros to the left), we see that $\lambda_M : \mathscr{F}(M) \to \mathscr{G}(M)$ must be an isomorphism, thereby completing the proof. ∎

COROLLARY. Let M be a finitely presented A-module and N be any A-module. Then for every $\mathfrak{p} \in \operatorname{Spec}(A)$,

$$\operatorname{Hom}_A(M, N)_{\mathfrak{p}} \cong \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})\,.$$

Proof. Note that the localization functor at $\mathfrak{p} \in \operatorname{Spec}(A)$ is naturally isomorphic to $- \otimes_A A_{\mathfrak{p}}$. ∎

THEOREM 2.8. Let M be a finitely presented A-module. Then the following are equivalent

(a) M is projective.

(b) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

(c) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \operatorname{MaxSpec}(A)$.

Proof. That $(a) \implies (b) \implies (c)$ is obvious. It suffices to show that $(c) \implies (a)$. To this end, we shall show that $\operatorname{Hom}_A(M, -)$ is an exact functor. We know that $\operatorname{Hom}_A(M, -)$ is left exact so let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Upon application of the above functor, note that we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(M, N') \longrightarrow \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, N'') \to K \to 0,$$

where K is the cokernel. Localizing the above sequence at a maximal ideal $\mathfrak{m}$ and using the exactness of localization and the preceding result, we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N'_{\mathfrak{m}}) \longrightarrow \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \longrightarrow \operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N''_{\mathfrak{m}}) \to K_{\mathfrak{m}} \to 0.$$

But since $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module, the functor $\operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, -)$ is exact, whence $K_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \operatorname{MaxSpec}(A)$. This shows that $K = 0$, that is, M is projective. ∎

§3 INJECTIVE MODULES

DEFINITION 3.1. An A-module M is said to be *injective* if the (contravariant) functor $\operatorname{Hom}_A(-, M) : \mathfrak{Mod}_A^{op} \to \mathfrak{Mod}_A$ is exact.

§§ Injective Hulls

DEFINITION 3.2. Let $M \leqslant E$ be A-modules. Then E is said to be an *essential extension* of M if every non-zero submodule of E intersects M non-trivially. We denote this by $M \leqslant_e E$.

REMARK 3.3. The above is equivalent to requiring that for every $x \in E \setminus \{0\}$, there is an $a \in A \setminus \{0\}$ such that $ax \in M \setminus \{0\}$.

We note some trivial properties of essential extensions before proceeding.

PROPOSITION 3.4. Let $L \leqslant M \leqslant N$ be A-modules. Then

$$L \leqslant_e M \text{ and } M \leqslant_e N \iff L \leqslant_e N.$$

Proof. Straightforward. ■

PROPOSITION 3.5. Let $M \leqslant E$ be A-modules. Consider the set

$$\mathcal{E} = \{N \leqslant E \colon M \leqslant_e N\}.$$

Then $\mathcal{E}$ has a maximal element.

Proof. Standard application of Zorn's lemma. ■

PROPOSITION 3.6. If $N_1 \leqslant_e M_1$ and $N_2 \leqslant_e M_2$, then $N_1 \oplus N_2 \leqslant_e M_1 \oplus M_2$.

Proof. Trivial. ■

§4 UNCATEGORIZED

§§ Eakin-Nagata Theorem

THEOREM 4.1 (FORMANEK). Let A be a ring, and B a finitely generated faithful A-module. Suppose the set of A-submodules $\Sigma = \{\mathfrak{a}B \colon \mathfrak{a} \trianglelefteq A\}$ has the ascending chain condition, then A is noetherian.

Proof. It suffices to show that B is a noetherian A-module since it is finitely generated and faithful. Suppose not. Then consider the collection

$$\Gamma = \{\mathfrak{a}B \colon \mathfrak{a} \trianglelefteq A,\ B/\mathfrak{a}B \text{ is a non-noetherian } A\text{-module}\},$$

which contains (0) and hence is non-empty. Since Σ has the ascending chain condition, so does Γ, whence, it contains a maximal element $\mathfrak{a}B$.

Replacing B by $B/\mathfrak{a}B$, we see that B is a non-noetherian A-module. This may not be faithful and hence, replace A by $A/\operatorname{Ann}_A(B)$. Then, B is a finite, non-noetherian, faithful A-module such that for every ideal $0 \neq \mathfrak{a} \triangleleft A$, $B/\mathfrak{a}B$ is a noetherian A-module.

Next, set

$$\mathfrak{M} = \{N \leqslant B \colon B/N \text{ is a faithful } A\text{-module}\},$$

which is non-empty, since $\{0\} \in \mathfrak{M}$. Suppose B is generated as an A-module by $b_1, \dots, b_n$. It is not hard to argue that

$$N \in \mathfrak{M} \iff \forall\, a \in A \setminus \{0\},\ \{ab_1, \dots, ab_n\} \not\subseteq N.$$

It follows that every chain in $\mathfrak{M}$ has a maximal element and hence Zorn's Lemma applies to furnish a maximal element $N_0 \in \Gamma$.

If B/N_0 is a noetherian A-module, then A is noetherian since B/N_0 is faithful and finite. If not, replace B with B/N_0, which is still a finite faithful A-module and satisfies:

(1) B is a non-noetherian A-module.

(2) for any ideal $0 \neq \mathfrak{a} \trianglelefteq A$, $B/\mathfrak{a}B$ is a noetherian A-module.

(3) for any submodule $0 \neq N$ of B, B/N is not faithful as an A-module.

Now, let N be a non-zero submodule of B. Due to (3), there is a $0 \neq a \in A$ such that $aB \subseteq N$. Due to (2), B/aB is a noetherian A-module with N/aB as a submodule. Thus, N/aB is a noetherian, in particular, a finite A-module. Since aB is also finite as an A-module, we have that N is a finite A-module. Hence, B is a noetherian A-module, which is absurd. This completes the proof. ∎

THEOREM 4.2 (EAKIN-NAGATA). Let $A \subseteq B$ be an extension of rings such that B is a finite A-module. If B is a noetherian ring, then so is A.

Proof. Note that B is a finite, faithful A-module, since $1 \in B$. The conclusion follows from Theorem 4.1. ∎