Prouct Developments

Swayam Chube

Last Updated: May 23, 2025

§1 The Space of Holomorphic Functions

Theorem 1.1. If $\Omega \subseteq \mathbb{C}$ is open, then there is a sequence $(K_n)_{n\geqslant 1}$ of compact subsets of Ω such that $\Omega = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:

(i) $K_n \subseteq K_{n+1}^{\circ}$.

(ii) If $K \subseteq \Omega$ is compact, then $K \subseteq K_n$ for some $n \geqslant 1$.

(iii) For every $n \geqslant 1$, each component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$.

§2 Product Developments

§§ Generalities

Definition 2.1. If $(z_n)_{n\geqslant 1}$ is a sequence of complex numbers, then $z \in \mathbb{C}$ is said to be the *infinite product* of the sequence $(z_n)_{n\geqslant 1}$ if

$$z = \lim_{n\to\infty} \prod_{k=1}^{n} z_k.$$

Suppose $z_n \neq 0$ for all $n \geqslant 1$ and $z \neq 0$. Then, setting

$$p_n = \prod_{k=1}^{n} z_k,$$

we have, by definition that $p_n \to z \neq 0$ in $\mathbb{C}$. But since $z_n = p_n/p_{n-1}$ with the convention that $p_0 = 1$, we see that $z_n \to 1$ as $n \to \infty$.

Proposition 2.2. Let $(z_n)_{n\geqslant 1}$ be a sequence of complex numbers with $\operatorname{Re} z_n > 0$ for all $n \geqslant 1$. Then $\prod_{n=1}^{\infty} z_n$ converges to a *non-zero* complex number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proof. ■

Definition 2.3. If $(z_n)_{n\geqslant 1}$ is a sequence of complex numbers with $\operatorname{Re} z_n > 0$ for all n, then the infinite product $\prod_{n=1}^{\infty} z_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely.

Lemma 2.4. If $|z| < \frac{1}{2}$, then

$$\frac{1}{2}|z| \leqslant |\log(1+z)| \leqslant \frac{3}{2}|z|.$$

Proof. Using the power series expansion of $\log(1+z)$ about $z=0$, we get

$$\left|1-\frac{\log(1+z)}{z}\right| = \left|\frac{1}{2}z-\frac{1}{3}z^2+\cdots\right| \leqslant \frac{1}{2}\left(|z|+|z|^2+\cdots\right) = \frac{1}{2}\frac{|z|}{1-|z|} < \frac{1}{2},$$

whence the conclusion follows. ■

PROPOSITION 2.5. Let $(z_n)_{n\geqslant 1}$ be a sequence of complex numbers with $\operatorname{Re} z_n > -1$ for all $n \geqslant 1$. Then the series $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n$ converges absolutely.

Proof. ■

COROLLARY 2.6. If $(z_n)_{n\geqslant 1}$ is a sequence of complex numbers with $\operatorname{Re} z_n > 0$ for all $n \geqslant 1$, then the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty}(z_n-1)$ converges absolutely.

Proof. ■

PROPOSITION 2.7. Let X be a set, and $(f_n)_{n\geqslant 1}$ be a sequence of complex-valued functions on X converging uniformly to $f : X \to \mathbb{C}$. Suppose there exists $a \in \mathbb{R}$ such that $\operatorname{Re} f_n(x) \leqslant a$ for all $x \in X$ and $n \geqslant 1$, then the sequence of functions $(\exp(f_n))_{n\geqslant 1}$ converges uniformly to $\exp(f)$.

Proof. ■

LEMMA 2.8. Let X be a compact topological space and $(g_n)_{n\geqslant 1}$ a sequence of complex-valued continuous functions on X such that $\sum_{n=1}^{\infty} |g_n(x)|$ converges uniformly on X. Then the product

$$f(x) = \prod_{n=1}^{\infty}(1+g_n(x))$$

converges uniformly for all $x \in X$. Further there is an integer $n_0 \geqslant 1$ such that $f(x)=0$ if and only if $g_n(x) = -1$ for some $1 \leqslant n \leqslant n_0$.

Proof. Since $\sum_{n=1}^{\infty} |g_n(x)|$ converges uniformly on X, there is a positive integer $n_0 \geqslant 1$ such that $|g_n(x)| < \frac{1}{2}$ for all $x \in X$ and $n > n_0$. Thus $\operatorname{Re}(1+g_n(x)) > 0$ for all $x \in X$ and $n > n_0$, and hence due to Lemma 2.4

$$|\log(1+g_n(x))| \leqslant \frac{3}{2}|g_n(x)| \qquad \forall\ x \in X,\ \forall n > n_0.$$

Thus, the sum

$$h(x) := \sum_{n=n_0}^{\infty} \log(1+g_n(x))$$

converges uniformly on X so that h is a continuous function. Since X is compact, there is an $a \in \mathbb{R}$ such that $\operatorname{Re} h(x) \leqslant a$ for all $x \in X$. In view of Proposition 2.7,

$$\exp h(x) = \prod_{n=n_0}^{\infty}(1+g_n(x))$$

converges uniformly on X. In particular, the product on the right is non-zero for all $x \in X$.

Finally, since

$$f(x) = (1+g_1(x))\cdots(1+g_{n_0}(x))\exp h(x),$$

it follows that if $f(x)=0$, then $g_n(x) = -1$ for some $1 \leqslant n \leqslant n_0$. ■

THEOREM 2.9. Let $\Omega \subseteq \mathbb{C}$ be a region and let $(f_n)_{n\geqslant 1}$ be a sequence of holomorphic functions such that no f_n is identically zero. If $\sum_{n=1}^{\infty} |f_n(z)-1|$ converges uniformly on compact subsets of Ω, then $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets of Ω to a holomorphic function $f(z)$.

If $a \in \Omega$ is a zero of f, then a is a zero of only a finite number of functions f_n , and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.

§3 Runge's Theorem

THEOREM 3.1 (RUNGE). Let $K \subseteq \mathbb{C}$ be a compact set and let E be a subset of $\mathbb{C}_\infty \setminus K$ meeting each connected component of $\mathbb{C}_\infty \setminus K$. If f is a function holomorphic in an open set $\Omega \supseteq K$ and $\varepsilon > 0$, then there exists a rational function $R(z)$ whose only poles lie in E such that

$$|f(z) - R(z)| < \varepsilon$$

for all $z \in K$.

Let $C(K)$ denote the Banach space of all complex-valued continuous functions on K equipped with the supremum norm on K, that is,

$$\|f\|_\infty := \sup\{|f(z)| : z \in K\} \qquad \forall f \in C(K).$$

Let $B(E) \subseteq C(K)$ denote the set of all functions $f \in C(K)$ such that for every $\varepsilon > 0$, there is a rational function $R(z)$ with poles only in E such that

$$\|f - R\|_\infty < \varepsilon.$$

Theorem 3.1 essentially states that $f|_K \in B(E)$ for every holomorphic function in a neighborhood of K.

LEMMA 3.2. $B(E)$ is a closed $\mathbb{C}$-subalgebra of $C(K)$ containing every rational function with all poles in E.

Proof. The latter part of the assertion is clear. To see that $B(E)$ is a subalgebra, suppose $f, g \in B(E)$ and $\alpha, \beta \in \mathbb{C}$. Let $\varepsilon > 0$ and choose rational functions $R(z), S(z)$ such that

$$\|f - R\|_\infty < \frac{\varepsilon}{|\alpha| + |\beta| + 1} \quad \text{and} \quad |g - S| < \frac{\varepsilon}{|\alpha| + |\beta| + 1}.$$

Then

$$\left\|(\alpha f + \beta g) - (\alpha R + \beta S)\right\|_\infty < \frac{|\alpha| + |\beta|}{|\alpha| + |\beta| + 1}\varepsilon < \varepsilon,$$

whence $\alpha f + \beta g \in B(E)$. Next, we shall show that $fg \in B(E)$. Indeed, let $\varepsilon > 0$, and choose positive real numbers $M_1, M_2 > 0$ such that $\|f\|_\infty < M_1$ and $\|g\|_\infty < M_2$. Choose rational functions $R(z), S(z)$ such that

$$\|f - R\|_\infty < \frac{\varepsilon}{M_1 + M_2} \quad \text{and} \quad \|g - S\|_\infty < \frac{\varepsilon}{M_1 + M_2}.$$

Then $R(z)S(z)$ is a rational function with poles only in E, and

$$\|fg - RS\|_\infty \leqslant \|g(f - R) + R(g - S)\|_\infty \leqslant M_2\|f - R\|_\infty + M_1\|g - S\|_\infty < \varepsilon,$$

as desired. Thus $B(E)$ is a subalgebra of $C(K)$.

It remains to show that $B(E)$ is closed in the topology of $C(K)$. Indeed, let $f_n \to f$ in $C(K)$ and $\varepsilon > 0$. There is a positive integer N such that $\|f - f_N\|_\infty < \frac{\varepsilon}{2}$, and further, a rational function $R(z)$ with poles only in E such that $\|f_N - R\|_\infty < \frac{\varepsilon}{2}$. Thus

$$\|f - R\|_\infty < \|f - f_N\|_\infty + \|f_N - R\|_\infty < \varepsilon,$$

whence $f \in B(E)$, thereby completing the proof. ∎

The outline of the rest of the proof is as follows:

- First, we show that $\dfrac{1}{z-a} \in B(E)$ for each $a \in \mathbb{C} \setminus K$.
- Since $B(E)$ is an algebra containing all polynomials, using partial fractions, we conclude that every rational function with poles only in $\mathbb{C} \setminus K$ belongs to $B(E)$.
- Finally, using Cauchy's integral formula, we show that every holomorphic function can be approximated arbitrarily well by rational functions with poles only in $\mathbb{C} \setminus K$.

Lemma 3.3. Let V and U be open subsets of $\mathbb{C}$ with $V \subseteq U$ and $\partial V \cap U = \emptyset$. If H is a component of U with $H \cap V \neq \emptyset$, then $H \subseteq V$.

Proof. Let $a \in H \cap V$ and let G be the connected component of V containing a; then $H \cup G$ is connected and contained in U. But since H is a connected component, $H \cup G = H$, that is, $G \subseteq H$. Note that $\partial G \subseteq \partial V$[1] and so $\partial G \cap H = \emptyset$, whence

$$H \setminus G = H \cap (\mathbb{C} \setminus G) = H \cap \left[(\mathbb{C} \setminus \overline{G}) \cup \partial G\right] = H \cap (\mathbb{C} \setminus \overline{G}),$$

whence $H \setminus G$ is open in H. But since G is open, $H \setminus G$ is both closed and open in H, and since H is connected and $G \neq \emptyset$, it follows that $H = G \subseteq V$, as desired. ∎

Proposition 3.4. Let $a \in \mathbb{C} \setminus K$. Then $\dfrac{1}{z-a} \in B(E)$.

Proof. We split our analysis into two cases.
Case 1. $\infty \notin E$. Let $U = \mathbb{C} \setminus K$ and let

$$V = \left\{ a \in \mathbb{C} \colon \frac{1}{z-a} \in B(E) \right\},$$

so that $E \subseteq V \subseteq U$. We first claim that V is open. Indeed, suppose $a \in V$ and $|b-a| < d(a,K)$. Then there exists $0 < r < 1$ such that $|b-a| < r|z-a|$ for all $z \in K$. But

$$\frac{1}{z-b} = \frac{1}{z-a} \frac{1}{1 - \frac{b-a}{z-a}},$$

and since $|(b-a)/(z-a)| < r < 1$ for all $z \in K$, we note that the series

$$\frac{1}{1 - \frac{b-a}{z-a}} = \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a} \right)^n$$

converges uniformly on K due to the Weierstraß M-test. Set

$$Q_n(z) = \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a} \right)^n,$$

[1] This is because $\mathbb{C}$ is locally connected.

then $\frac{1}{z-a}Q_n(z) \in B(E)$ since $a \in V$ and $B(E)$ is an algebra. Since $B(E)$ is closed, the uniform convergence of $\frac{1}{z-a}Q_n(z)$ to $\frac{1}{z-b}$ yields that the latter lies in $B(E)$, so that V is open.

Now suppose $b \in \overline{V} \setminus V = \partial V$ and let $(a_n)_{n\geqslant 1}$ be a sequence in V converging to b. We have that $|b-a_n| \geqslant d(a_n,K)$ and taking $n \to \infty$ and using the continuity of $d(\cdot,K)$, one obtains $d(b,K)=0$, that is, $b \in K$. Thus $\partial V \cap U = \emptyset$. If H is a component of U, then $H \cap E \neq \emptyset$, so $H \cap V \neq \emptyset$. By Lemma 3.3, $H \subseteq V$. But since H was arbitrary, we have that $U \subseteq V$, i.e., $U = V$.

CASE 2. $\infty \in E$. Let d_∞ denote the metric on $\mathbb{C}_\infty$. Choose a_0 in the unbounded component of $\mathbb{C}\setminus K$ (i.e., the component containing ∞) such that $d_\infty(a_0,\infty) \leqslant \frac{1}{2}d_\infty(\infty,K)$ and $|a_0| > 2\max\{|z| : z \in K\}$. Let $E_0 = (E \setminus \{\infty\}) \cup \{a_0\}$. Then E_0 meets each component of $\mathbb{C}_\infty \setminus K$, and $\infty \notin E_0$.

If $a \in \mathbb{C}\setminus K$, then due to CASE 1, $\frac{1}{z-a} \in B(E_0)$. We shall show that $\frac{1}{z-a_0} \in B(E_0)$. Once this is shown, we could approximate rational functions with poles only in E_0 by rational functions with poles only in E, since $E_0 \setminus E = \{a_0\}$. This would then immediately give us that $\frac{1}{z-a} \in B(E_0) \subseteq B(E)$, as desired.

Note that for all $z \in K$, $|z/a_0| \leqslant \frac{1}{2}$ and so

$$\frac{1}{z-a_0} = -\frac{1}{a_0}\frac{1}{1-\frac{z}{a_0}} = -\frac{1}{a_0}\sum_{n=0}^{\infty}\left(\frac{z}{a_0}\right)^n$$

converges uniformly on K due to the Weierstraß M-test. Set

$$Q_n(z) = -\frac{1}{a_0}\sum_{k=0}^{n}\left(\frac{z}{a_0}\right)^k,$$

which is a sequence of polynomials converging uniformly to $\frac{1}{z-a_0}$ on K. Since $Q_n \in B(E)$ for each $n \geqslant 1$, we have shown that $\frac{1}{z-a_0} \in B(E)$, thereby completing the proof. ∎

LEMMA 3.5. Let Ω be a region contianing K. Then there are straight line segments $\gamma_1,\ldots,\gamma_n$ in $\Omega\setminus K$ such that for every holomorphic function f on Ω,

$$f(z) = \frac{1}{2\pi\iota}\sum_{k=1}^{n}\int_{\gamma_k}\frac{f(w)}{w-z}\,dw$$

for all $z \in K$. The line segments form a finite number of closed polygons in Ω.

Proof. Covering K by finitely many compact disks (contained in Ω), we can replace K with the union of these disks and suppose that $K = \overline{K^\circ}$. Let $0 < \delta < \frac{1}{2}d(K,\mathbb{C}\setminus\Omega)$ and place a "grid" of horizontal and vertical lines in the plane with consecutive lines less than a distance δ apart. Let $R_1,\ldots,R_m$ be the resulting rectangles intersecting K. These rectangles are finite in number because K is compact. Consider ∂R_j, the boundary of R_j as a polygon oriented in the counter-clockwise direction.

If $z \in R_j$ for some $1 \leqslant j \leqslant m$, then $d(z,K) \leqslant \operatorname{diam} R_j = \sqrt{2}\delta$, and hence $z \in \Omega$. This shows that every R_j is contained in Ω. Next, suppose R_j and R_j intersect in an edge σ. With respect to the two rectangles, σ will have opposite orientations, and hence, for any continuous function φ on σ, the sum of the integrals will cancel out.

Let $\gamma_1,\ldots,\gamma_n$ be those directed line segments that constitute an edge of exactly one of the R_j's. Then

$$\sum_{k=1}^{n}\int_{\gamma_k}\varphi = \sum_{j=1}^{m}\int_{\partial R_j}\varphi \tag{1}$$

for any continuous function φ on $\bigcup_{j=1}^{m} \partial R_j$.

We contend that each γ_k lies in $\Omega \setminus K$. Indeed, if one of the γ_k intersects K, then there are two rectangles in the grid with γ_k as a side, both of which intersect K, whence both of these rectangles must lie in the set $\{R_1, \ldots, R_m\}$, which is absurd, since γ_k is a side of exactly one of those rectangles.

Now, if $z \in K \setminus \bigcup_{j=1}^{m} \partial R_j$, then for any holomorphic function f on Ω,

$$\varphi(w) = \frac{1}{2\pi\iota} \frac{f(w)}{w-z}$$

is continuous on $\bigcup_{j=1}^{m} \partial R_j$. From (1), it follows that

$$\sum_{j=1}^{m} \frac{1}{2\pi\iota} \int_{\partial R_j} \frac{f(w)}{w-z}\, dw = \sum_{k=1}^{n} \frac{1}{2\pi\iota} \int_{\gamma_k} \frac{f(w)}{w-z}\, dw.$$

But z belongs to the interior of exactly one of the R_j's whence the sum on the left is precisely $f(z)$ whenever $z \in K \setminus \bigcup_{j=1}^{m} \partial R_j$. But both sides are continuous functions on K (since $f(z)$ is clearly continuous and every γ_k misses K) and because $K = \overline{K^\circ}$, the set $K \setminus \bigcup_{j=1}^{m} \partial R_j$ is dense in K; it follows that both sides must be equal for all $z \in K$, as desired. ■

Now that we have an integral representation of $f(z)$, we shall approximate it using rational functions having poles on the $\{\gamma_k\}$'s.

LEMMA 3.6. Let γ be a rectifiable curve and K a compact set such that $K \cap \{\gamma\} = \emptyset$. If f is a continuous function on $\{\gamma\}$, and $\varepsilon > 0$, then there is a rational function $R(z)$ having all its poles on $\{\gamma\}$ such that

$$\left| \int_{\gamma} \frac{f(w)}{w-z}\, dw - R(z) \right| < \varepsilon$$

for all $z \in K$.

Proof. We may assume that $\gamma\colon [0,1] \to \mathbb{C}$. First, since K and $\{\gamma\}$ are disjoint, there is a real number $0 < r < d(\{\gamma\}, K)$. For $0 \leqslant s < t \leqslant 1$ and $z \in K$,

$$\begin{aligned}
\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z} \right| &= \left| \frac{\gamma(s)f(\gamma(t)) - \gamma(t)f(\gamma(s)) - z\big(f(\gamma(t)) - f(\gamma(s))\big)}{(\gamma(t)-z)(\gamma(s)-z)} \right| \\
&\leqslant \frac{1}{r^2} \left| \gamma(s)f(\gamma(t)) - \gamma(t)f(\gamma(s)) - z\big(f(\gamma(t)) - f(\gamma(s))\big) \right| \\
&\leqslant \frac{1}{r^2} \left| f(\gamma(t))\big(\gamma(s)-\gamma(t)\big) + \gamma(t)\big(f(\gamma(t)) - f(\gamma(s))\big) - z\big(f(\gamma(t)) - f(\gamma(s))\big) \right| \\
&\leqslant \frac{1}{r^2} \left| f(\gamma(t)) \right| \left| \gamma(s) - \gamma(t) \right| + \frac{1}{r^2} \left| \gamma(t) - z \right| \left| f(\gamma(t)) - f(\gamma(s)) \right|.
\end{aligned}$$

Using the compactness of $\{\gamma\}$ and K, there is a constant $C > 0$ such that $d(x,z) \leqslant C$ for all $x \in \{\gamma\}$ and $z \in K$, and $f(x) \leqslant C$ for all $x \in \{\gamma\}$. Thus

$$\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z} \right| \leqslant \frac{C}{r^2} \left(\left| \gamma(s) - \gamma(t) \right| + \left| f(\gamma(t)) - f(\gamma(s)) \right| \right).$$

Finally, using the uniform continuity of the functions $\gamma, f\circ\gamma\colon [0,1]\to\mathbb{C}$, there is a $\delta > 0$ such that whenever $|s-t| < \delta$,

$$\left|\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z}\right| < \frac{\varepsilon}{2V(\gamma)}$$

for all $z \in K$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that $|t_j - t_{j-1}| < \delta$ for $1 \leqslant j \leqslant n$. Set

$$R(z) = \sum_{i=1}^{n} \frac{f(\gamma(t_{j-1}))\big(\gamma(t_j)-\gamma(t_{j-1})\big)}{\gamma(t_{j-1})-z}.$$

Now, there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $[0,1]$ such that

$$\left|\int_\gamma \frac{f(w)}{w-z}\,dw - \sum_{j=1}^{m} \frac{f(\gamma(s_j))}{\gamma(s_j)-\gamma(s_{j-1})}\right| < \frac{\varepsilon}{2}.$$

Thus

$$\left|\int_\gamma \frac{f(w)}{w-z}\,dw - R(z)\right| \leqslant \left|\int_\gamma \frac{f(w)}{w-z}\,dw - \sum_{j=1}^{m} \frac{f(\gamma(s_j))}{\gamma(s_j)-\gamma(s_{j-1})}\right| + \left|\sum_{j=1}^{m} \frac{f(\gamma(s_j))}{\gamma(s_j)-\gamma(s_{j-1})} - \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))\big(\gamma(t_j)-\gamma(t_{j-1})\big)}{\gamma(t_{j-1})-z}\right|.$$

Taking a union of both partitions $\underline{s}$ and $\underline{t}$ and using the triangle inequality, it is clear that both terms are smaller than $\varepsilon/2$, therefore,

$$\left|\int_\gamma \frac{f(w)}{w-z}\,dw - R(z)\right| < \varepsilon,$$

for all $z \in K$. ■

Proof of Theorem 3.1. Due to Proposition 3.4 and the fact that $B(E)$ contains all polynomials, using partial fractions it follows that $B(E)$ contains all rational functions with all poles in $\mathbb{C}\setminus K$. Finally, using Lemma 3.5 and Lemma 3.6, it follows that $f \in B(E)$, as desired. ■

§§ Simply connected regions

THEOREM 3.7. Let $\Omega \subseteq \mathbb{C}$ be a region. Then the following are equivalent:

(1) Ω is simply connected.

(2) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in Ω and every point $a \in \mathbb{C}\setminus\Omega$.

(3) $\mathbb{C}_\infty \setminus \Omega$ is connected.

(4) For any $f \in \mathscr{O}(\Omega)$, there is a sequence of polynomials that converges to f in $\mathscr{O}(\Omega)$.

(5) For any $f \in \mathscr{O}(\Omega)$ and any closed rectifiable curve γ in Ω, $\int_\gamma f = 0$.

(6) Every function $f \in \mathscr{O}(\Omega)$ has a primitive.

(7) For any nowhere-vanishing function $f \in \mathscr{O}(\Omega)$, there is a $g \in \mathscr{O}(\Omega)$ such that $f = \exp g$.

(8) For any nowhere-vanishing function $f \in \mathscr{O}(\Omega)$, there is a $g \in \mathscr{O}(\Omega)$ such that $f = g^2$.

(9) Ω is homeomorphic to the unit disk.

(10) If $u\colon \Omega \to \mathbb{R}$ is harmonic, then there is a harmonic function $v\colon \Omega\to\mathbb{R}$ such that $f = u + \imath v$ is holomorphic on Ω.

§§ Mittag-Leffler's Theorem

Theorem 3.8 (Mittag-Leffler). Let $\Omega \subseteq \mathbb{C}$ be a region and $(a_n)_{n\geqslant 1}$ a sequence of distinct points in Ω with no limit point in Ω. Let $(S_n(z))_{n\geqslant 1}$ be a sequence of rational functions of the form

$$S_n(z) = \sum_{j=1}^{m_n} \frac{c_{nj}}{(z-a_n)^j},$$

where m_n is a positive integer and $c_{nj} \in \mathbb{C}$ for all $n \geqslant 1$ and $1 \leqslant j \leqslant m_n$. Then there exists a meromorphic function f on Ω which is holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$ and whose singular part at each a_n is given by $S_n(z)$.

Proof. Choose an exhaustion $(K_n)_{n\geqslant 1}$ of Ω as in Theorem 1.1 and as such, every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$. Next, since each K_n is compact, and $(a_k)_{k\geqslant 1}$ has no limit point in Ω, only finitely many of the a_k's can lie in each K_n. Define

$$I_n := \{k : a_k \in K_n \setminus K_{n-1}\}$$

with the convention that $K_0 = \emptyset$. Define the functions

$$f_n(z) = \sum_{k \in I_n} S_k(z).$$

This is clearly a meromorphic function on Ω with all its poles in $K_n \setminus K_{n-1}$. Using Theorem 3.1 with $E = \mathbb{C}_\infty \setminus \Omega$, there exists a rational function $R_n(z)$ with all its poles in $\mathbb{C}_\infty \setminus \Omega$ such that

$$|f_n(z) - R_n(z)| < \frac{1}{2^n}$$

for all $z \in K_{n-1}$ and $n \geqslant 2$. For $n = 1$, we set $R_1 = 0$. Define

$$f(z) = \sum_{n=1}^{\infty} (f_n(z) - R_n(z)).$$

We contend that this is our desired meromorphic function. We must first show that f is holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$ and then show that its singular part at each a_k is $S_k(z)$.

Indeed, let $K \subseteq \Omega \setminus \{a_1, a_2, \dots\}$ be a compact set. Then there is a positive integer $N \geqslant 1$ such that $K \subseteq K_N$. For all $n \geqslant N+1$, and $z \in K_N$, we have that

$$|f_n(z) - R_n(z)| < \frac{1}{2^n}.$$

Due to the Weierstraß M-test, the sum converges uniformly on K, whence the limiting function f is a holomorphic function on $\Omega \setminus \{a_1, a_2, \dots\}$.

Let $k \geqslant 1$. Since the sequence $(a_n)_{n\geqslant 1}$ has no limit point, there is an $r > 0$ such that $|a_j - a_k| > r$ for all $j \neq k$. Then, the sum for $f(z) - S_k(z)$ converges uniformly on $\overline{B}(a_k, r)$ to a holomorphic function there, again due to the Weierstraß M-test. As a result, $f(z)$ has singular part $S_k(z)$ at a_k. This completes the proof. ∎

Proposition 3.9. Let $\Omega \subseteq \mathbb{C}$ be a region. If $(a_n)_{n\geqslant 1}$ is a sequence of distinct points in Ω with no limit point in Ω, and $(c_n)_{n\geqslant 1}$ is a sequence of complex numbers, then there is a holomorphic function $f \in \mathscr{O}(\Omega)$ such that $f(a_n) = c_n$ for all $n \geqslant 1$.

Proof. Let $g \in \mathscr{O}(\Omega)$ be a holomorphic function with simple zeros at only the a_n's. Then we can write $g(z) = (z - a_n)g_n(z)$ for some holomorphic function $g_n \in \mathscr{O}(\Omega)$ with $g_n(a_n) \neq 0$. Using Theorem 3.8 let h be a meromorphic function on Ω, holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$, and having singular part

$$\frac{c_n}{g_n(a_n)} \frac{1}{z - a_n}$$

at a_n for each $n \geqslant 1$. Clearly $f(z) = g(z)h(z)$ has removable singularities at each a_n and $f(a_n) = c_n$. ■

A significantly more general statement is true; instead of just specifying values of a function at countably many points, we can specify the tail of its power series representation at those points:

THEOREM 3.10. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $(a_n)_{n \geqslant 1}$ be a sequence of distinct points in Ω with no limit point in Ω. For each $n \geqslant 1$, associate a non-negative integer $m_n \geqslant 0$, and complex numbers w_{nj} for $0 \leqslant j \leqslant m_n$. Then there exists a holomorphic function $f \in \mathscr{O}(\Omega)$ such that

$$f^{(j)}(a_n) = j! w_{nj}$$

for all $n \geqslant 1$ and $0 \leqslant j \leqslant m_n$[2].

Proof. Let $g \in \mathscr{O}(\Omega)$ have zeros at only the a_n's with multiplicity $m_n + 1$ respectively. We shall use Theorem 3.8 to find a meromorphic function h on Ω, which is holomorphic on $\Omega \setminus \{a_1, a_2, \dots\}$ and has singular part

$$S_n(z) = \frac{b_{n1}}{z - a} + \frac{b_{n2}}{(z-a)^2} + \dots + \frac{b_{n,m_n+1}}{(z-a)^{m_n+1}}$$

at each a_n, where $b_{nj} \in \mathbb{C}$ are complex numbers to be chosen later. Conisder the power series expansion of $g(z)$ about $z - a_n$:

$$g(z) = (z - a_n)^{m_n+1} \left(c_{n0} + c_{n1}(z - a_n) + c_{n2}(z - a_n)^2 + \dots \right),$$

for some complex numbers c_{nj}, $j \geqslant 0$. Note that $c_{n0} \neq 0$. Then

$$g(z)S_n(z) = \left(b_{n,m_n+1} + b_{n,m_n}(z-a) + \dots + b_{n1}(z-a)^{m_n} \right) (c_{n0} + c_{n1}(z - a_n) + \dots).$$

We would like to choose $b_{n1}, \dots, b_{n,m_n+1}$ such that the above product expands to

$$w_{n0} + w_{n1}(z - a_n) + w_{n2}(z - a_n)^2 + \dots.$$

The b_{nj}'s can be chosen inductively since $c_{n0} \neq 0$, so that we begin by setting $b_{n,m_n+1} = w_{n0}c_{n0}^{-1}$. And at each stage, one obtains a linear equation in b_{nj} with coefficient c_{n0}, which is again non-zero, and so that equation has a (unique) solution.

Finally, using Theorem 3.8 to choose a meromorphic function h on Ω having poles at precisely the a_n's with singular parts $S_n(z)$ respectively, it is clear that $f(z) = g(z)h(z)$ has the desired power series expansion at each a_n, thereby completing the proof. ■

THEOREM 3.11. Let $\Omega \subseteq \mathbb{C}$ be a region. Then $\mathscr{O}(\Omega)$ is a Bézout domain, that is, every finitely generated ideal in $\mathscr{O}(\Omega)$ is principal.

[2] That is, the power series representation of f about a_n is of the form

$$f(z) = w_{n0} + w_{n1}(z - a_n) + \dots.$$

Proof. Inductively, it suffices to show that (f,g) is a principal ideal for $f,g \in \mathscr{O}(\Omega)$. First, we shall show that if f and g have no common zeros, then $(f,g) = (1)$. Let $a_1, a_2, \dots$ be the distict zeros of f with multiplicities $m_1, m_2, \dots$ respectively (note that these zeros can be finite in number). We contend that there exists $\varphi \in \mathscr{O}(\Omega)$ such that $1 - \varphi g$ has zeros $a_1, a_2, \dots$ with multiplicities $m_1', m_2', \dots$ respectively such that $m_j' \geqslant m_j$ for all $j \geqslant 1$.

Let $k \geqslant 1$ and consider the power series representation of g about a_k:

$$g(z) = b_{k0} + b_{k1}(z - a_k) + b_{k2}(z - a_k)^2 + \dots,$$

where $b_{k0} \neq 0$ since f and g do not share a zero. We want the power series representation of φ about a_k

$$\varphi(z) = w_{k0} + w_{k1}(z - a_k) + w_{k2}(z - a_k)^2 + \dots$$

to be such that

$$\varphi(z)g(z) = 1 + c_{m_k}(z - a_k)^{m_k} + \dots$$

for some $c_{m_k} \in \mathbb{C}$. This can clearly be done inductively just as in the proof of Theorem 3.10 since $b_{k0} \neq 0$. Further, the existence of such a $\varphi \in \mathscr{O}(\Omega)$ is guaranteed by Theorem 3.10. By construction, it is clear that there exists a holomorphic function $h \in \mathscr{O}(\Omega)$ such that $h(z)f(z) = 1 - \varphi(z)g(z)$, i.e., $1 \in (f,g)$, as desired.

Finally, suppose f and g are arbitrary holomorphic functions in $\mathscr{O}(\Omega)$. Let $a_1, a_2, \dots$ be the common zeros of f and g with

$$m_n = \min\{m(f; a_n), m(g; a_n)\} \geqslant 1,$$

for all $n \geqslant 1$. Let $\varphi \in \mathscr{O}(\Omega)$ be a holomorphic function with zeros at precisely the a_n's with multiplicities m_n respectively. Then there exist holomorphic functions $\widetilde{f}, \widetilde{g} \in \mathscr{O}(\Omega)$ such that $f = \varphi\widetilde{f}$ and $g = \varphi\widetilde{g}$; further f and g do not have common zeros. As a result,

$$(f,g) = (\varphi\widetilde{f}, \varphi\widetilde{g}) = (\varphi)(\widetilde{f}, \widetilde{g}) = (\varphi),$$

thereby completing the proof. ∎