MA 824: ASSIGNMENT 1

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1. Problem 1

Some Preliminary Estimates. First, suppose $1 \le p < q < \infty$. Then

$$||x||_q^q = \sum_{i=1}^d |x_i|^q = \sum_{i=1}^d |x_i|^p \cdot |x_i|^{q-p} \le ||x||_q^{q-p} \sum_{i=1}^d |x_i|^p = ||x||_q^{q-p} ||x||_p^p,$$

where the first inequality follows from the fact that $|x_i| \leq ||x||_q$ for $1 \leq i \leq d$. The above shows that $||x||_q^p \le ||x||_p^p$, that is, $||x||_q \le ||x||_p$.

On the other hand, using Hölder's inequality on the measure space $\{1, \ldots, d\}$ equipped with the counting measure, we have

$$||x^p||_1 \leqslant ||x^p||_{\frac{q}{p}} ||1||_{\frac{q}{q-p}},$$

where $x^p=(|x_1|^p,\ldots,|x_d|^p)\in\mathbb{C}^d$. Now note that $\|\mathbb{1}\|_{\frac{q}{q-p}}=d^{\frac{q-p}{q}}$, and

$$\|x^p\|_{\frac{q}{p}} = \left(\sum_{i=1}^d |x_i|^q\right)^{\frac{p}{q}} = \|x\|_q^p.$$

This gives us

$$||x||_p^p \leqslant ||x||_q^p d^{\frac{q-p}{q}} \implies ||x||_p \leqslant d^{\frac{1}{p}-\frac{1}{q}} ||x||_q$$

that is,

$$||x||_q \le ||x||_p \le d^{\frac{1}{p} - \frac{1}{q}} ||x||_q.$$

Note that both inequalities are tight. Indeed, take $x = (1,0,\ldots,0) \in \mathbb{C}^d$, then $||x||_q = ||x||_p = 1$. On the other hand, taking $x = (1, 1, ..., 1) \in \mathbb{C}^d$, we see that $||x||_p = d^{\frac{1}{p}}$ and $||x||_q = d^{\frac{1}{q}}$, whence $||x||_p = d^{\frac{1}{p} - \frac{1}{q}}||x||_q$. Next, suppose $q = \infty$ and p < q. For $x = (x_1, ..., x_d) \in \mathbb{C}^d$, there is an index $1 \le i_0 \le d$ such that

 $|x_{i_0}| = ||x||_{\infty}$. Hence, we have

$$||x||_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}} \geqslant |x_{i_0}| = ||x||_{\infty},$$

and

$$||x||_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^d ||x||_{\infty}^p\right)^{\frac{1}{p}} = d^{\frac{1}{p}} ||x||_{\infty}.$$

The inequalities

$$||x||_{\infty} \leqslant ||x||_{p} \leqslant d^{\frac{1}{p}} ||x||_{\infty}$$

are tight. Taking $x = (1, 0, ..., 0) \in \mathbb{C}^d$, we have $||x||_p = ||x||_\infty = 1$. And taking $x = (1, 1, ..., 1) \in \mathbb{C}^d$, we have $||x||_p = d^{\frac{1}{p}} = d^{\frac{1}{p}} ||x||_{\infty}$, as desired.

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Conclusion. First, if p = q, then take c = C = 1. Henceforth, we assume that $p \neq q$. Next, suppose $1 \leq p, q < \infty$. Then using our estimates from the previous (sub)section,

$$\begin{cases} d^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \|x\|_p & p < q \\ \|x\|_p & p > q \end{cases} \le \|x\|_q \le \begin{cases} \|x\|_p & p < q \\ d^{\left(\frac{1}{q} - \frac{1}{p}\right)} \|x\|_p & p > q. \end{cases}$$

Finally, if $q = \infty$, then $p < \infty$ and we have

$$d^{-\frac{1}{p}} \leqslant ||x||_q \leqslant ||x||_p,$$

and if $p = \infty$ so that $q < \infty$, then

$$||x||_p \leqslant ||x||_q \leqslant d^{\frac{1}{q}} ||x||_p.$$

As we have seen in the preceding (sub)section, all the above estimates are tight, that is, the constants are the best possible.

2. Problem 2

(a) There is a $0 < \lambda < 1$ such that $p = \lambda r + (1 - \lambda)s$. Hölder's inequality gives

$$||f||_p^p = ||f^p||_1 = ||f^{\lambda r + (1-\lambda)s}||_1 \le ||f^{\lambda r}||_{\frac{1}{\lambda}} ||f^{(1-\lambda)s}||_{\frac{1}{1-\lambda}}$$

Note that

$$\|f^{\lambda r}\|_{\frac{1}{\lambda}} = \left(\int_X \left(|f|^{\lambda r}\right)^{\frac{1}{\lambda}} d\mu\right)^{\lambda} = \left(\int_X |f|^r d\mu\right)^{\lambda} = \|f\|_r^{\lambda r} \leqslant \max\{\|f\|_r, \|f\|_s\}^{\lambda r}$$

and

$$\|f^{(1-\lambda)s}\|_{\frac{1}{1-\lambda}} = \left(\int_X \left(|f|^{(1-\lambda)s}\right)^{\frac{1}{1-\lambda}} \ d\lambda\right)^{1-\lambda} = \|f\|_s^{(1-\lambda)s} \leqslant \max\{\|f\|_r, \|f\|_s\}^{(1-\lambda)s}.$$

Thus

$$||f||_p^p \le \max\{||f||_r, ||f||_s\}^{\lambda r + (1-\lambda)s} \implies ||f||_p \le \max\{||f||_r, ||f||_s\},$$

as desired.

(b) Suppose first that $||f||_{\infty} = \infty$. As a result, for every C > 0,

$$\mu \{x \in X : |f(x)| \ge C\} > 0.$$

Hence,

$$||f||_p^p = \int_X |f|^p \, d\mu \geqslant \int_{\{x: |f(x)| \geqslant C\}} |f(x)|^p \, d\mu \geqslant \mu \, \{x \in X: |f(x)| \geqslant C\} \, C^p,$$

and hence,

$$||f||_p \geqslant C\mu \{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{p}}.$$

Thus, as $p \to \infty$, using the fact that $\mu \{x \in X : |f(x)| \ge C\} > 0$, we have

$$\liminf_{p\to\infty} \|f\|_p \geqslant C \liminf_{p\to\infty} \mu \left\{ x \in X \colon |f(x)| \geqslant C \right\}^{\frac{1}{p}} = \infty.$$

It follows immediately that $\lim_{n\to\infty} ||f||_p = \infty$.

Next, suppose $||f||_{\infty} < \infty$. For p > r, we have

$$||f||_p^p = \int_X |f|^p d\mu = \int_X |f|^{p-r} |f|^r d\mu \leqslant ||f||_\infty^{p-r} \int_X |f|^r d\mu.$$

Thus,

$$||f||_p \le ||f||_{\infty}^{1-\frac{r}{p}} ||f||_r^{\frac{r}{p}}.$$

Since $||f||_{\infty} > 0$, we have $||f||_r > 0$, consequently,

$$\limsup_{n \to \infty} \|f\|_{p} \le \|f\|^{\infty} \lim_{p \to \infty} \|f\|_{\infty}^{-\frac{r}{p}} \|f\|_{r}^{\frac{r}{p}} = \|f\|_{\infty}.$$

On the other hand, let $0 < C < ||f||_{\infty}$, so that

$$\mu \{ x \in X \colon |f(x)| \geqslant C \} > 0.$$

An obvious estimate gives us

$$||f||_p^p = \int_X |f|^p d\mu \geqslant \int_{\{x \in X \colon |f(x)| \geqslant C\}} |f|^p d\mu \geqslant C^p \mu \{x \in X \colon |f(x)| \geqslant C\},$$

consequently,

$$||f||_p \geqslant C\mu \{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{p}}.$$

Finally, we claim that $0 < \mu \{x \in X \colon |f(x)| \geqslant C\} < \infty$. Indeed, since $||f||_{\infty} > 0$, it is obvious that the above measure is positive. Further, since $||f||_r < \infty$, we have

$$C\mu \{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{r}} \leqslant \left(\int_{X} |f|^{r} d\mu\right)^{\frac{1}{r}} = \|f\|_{r} < \infty,$$

whence the measure is finite. It follows that

$$\liminf_{p\to\infty} \|f\|_p \geqslant C \liminf_{p\to\infty} \mu \left\{x \in X \colon |f(x)| \geqslant C\right\}^{\frac{1}{p}} = C.$$

Since the above inequality holds for all $0 < C < \|f\|_{\infty}$; taking a supremum over all such C, we get that

$$||f||_{\infty} \geqslant \limsup_{p \to \infty} ||f||_{p} \geqslant \liminf_{p \to \infty} ||f||_{p} \geqslant ||f||_{\infty},$$

therefore,

$$\lim_{p\to\infty} \|f\|_p = \|f\|_{\infty},$$

as desired.

(c) Since r < s, we have $\frac{1}{r} > \frac{1}{s}$. Choose a $p \ge 1$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s} \implies 1 = \frac{r}{p} + \frac{r}{s}.$$

Hölder's inequality then gives us

$$||f^r||_1 \leq ||f^r||_{\frac{s}{r}} ||1||_{\frac{p}{r}}$$

where 1 denotes the constant function 1. Now note that

$$||f^r||_{\frac{s}{r}} = \left(\int_X |f|^s\right)^{\frac{t}{s}} = ||f||_s^r,$$

 $\|1\|_{\frac{p}{r}} = 1$, since $\mu(X) = 1$; and

$$||f^r||_1 = \int_X |f|^r d\mu = ||f||_r^r.$$

Hence, we have shown that $||f||_r^r \le ||f||_s^r$. If $||f||_s = \infty$, then the inequality $||f||_r \le ||f||_s$ is trivial. If $||f||_s < \infty$, then taking r-th roots we get $||f||_s < ||f||_s$, as desired.

3. Problem 3

First, suppose $1 \le p < \infty$. For $i \ge 1$, define the "standard basis vectors" $e_i \in \ell^p$ by

$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathbb{K} denote the base field over which ℓ^p is defined. If $\mathbb{K} = \mathbb{R}$, set $Q = \mathbb{Q}$ and if $\mathbb{K} = \mathbb{C}$, then set $Q = \mathbb{Q} + \mathbb{Q}i$. Note that in either case, Q is dense in \mathbb{K} .

Set

$$D = \bigcup_{n=1}^{\infty} \left\{ q_1 e_1 + \dots + q_n e_n \colon q_1, \dots, q_n \in Q \right\}.$$

Being a countable union of countable sets, D is itself countable. We contend that D is dense in ℓ^p . Let $x = (x_n) \in \ell^p$ and $\varepsilon > 0$. Since the sum $\sum_{n=1}^{\infty} |x_n|^p$ converges, there is a positive integer N such that the tail sum

$$\sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Let $y = (y_n) \in \ell^p$ be given by

$$y_n = \begin{cases} x_n & n \leqslant N \\ 0 & n > N. \end{cases}$$

Then,

$$||x-y||^p = \sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Therefore, $||x - y|| < \frac{\varepsilon}{2}$. Next, using the density of Q in \mathbb{K} , for each $1 \le n \le N$, we can find a $z_n \in Q$ such that

$$|y_n-z_n|^p<\frac{1}{N}\left(\frac{\varepsilon}{2}\right)^p.$$

Setting $z = z_1 e_1 + \cdots + z_N e_N \in D \subseteq \ell^p$,

$$||y-z||^p = \sum_{n=1}^N |y_n - z_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

It follows from the triangle inequality that

$$||x-z|| \le ||x-y|| + ||y-z|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have shown that for all $x \in \ell^p$ and $\varepsilon > 0$, there is a $z \in D$ with $||x - z|| < \varepsilon$, and hence, D is dense in ℓ^p .

Finally, we show that ℓ^{∞} is not separable. For this part of the proof, we write an element $x \in \ell^{\infty}$ as $(x(n))_{n\geqslant 1}$. Suppose not and there were a countable dense subset $D\subseteq \ell^{\infty}$. For each subset $S\subseteq \mathbb{N}=\{1,2,\ldots\}$, define $x_S\in \ell^{\infty}$ as

$$x_S(n) = \begin{cases} 1 & n \in S \\ 0 & \text{otherwise,} \end{cases}$$

and set $U_S = B\left(x_S, \frac{1}{2}\right)$. We claim that the U_S 's are pairwise disjoint. Indeed, suppose $S, T \subseteq \mathbb{N}$ are distinct subsets and $y \in U_S \cap U_T$. It follows that

$$||x_S - x_T|| = ||(x_S - y) - (x_T - y)|| \le ||x_S - y|| + ||x_T - y|| < 1.$$

But for $n \in S\Delta T$, we have that $|x_S(n) - x_T(n)| = 1$, whence $||x_S - x_T|| \geqslant 1$, a contradiction. Thus, the U_S 's are pairwise disjoint. For each $S \subseteq \mathbb{N}$, there is a $z_S \in D$ such that $z_S \in U_S \cap D$, owing to the density of D. Since the U_S 's are pairwise disjoint, the z_S 's are distinct. But \mathbb{N} has uncountably many subsets (due to Cantor), and D is countable, a contradiction. Hence ℓ^∞ is not separable.

4. Problem 4

Let X be a locally compact normed linear space; then there is a neighborhood V of the origin in X such that \overline{V} is compact. Since V is open, there is an r>0 such that $B(0,r)\subseteq V$, consequently, $\overline{B}(0,r)\subseteq \overline{V}$. Since the latter is compact, so is the former. Further, the map $x\mapsto rx$ is a homeomorphism $X\to X$ with inverse given by $x\mapsto r^{-1}x$. Under the former map, the closed unit ball $\overline{B}(0,1)$ maps to $\overline{B}(0,r)$. Since the latter is compact, the former must be too, that is, $\overline{B}(0,1)$ is compact. Hence, we may suppose without loss of generality that V=B(0,1).

The following method is due to André Weil and generalizes well to topological vector spaces over complete valued fields. Note that \overline{V} is compact and

$$\overline{V} \subseteq \bigcup_{x \in \overline{V}} \left(x + \frac{1}{2}V \right).$$

Since the latter is an open cover, it contains a finite subcover of \overline{V} . That is, there are $x_1, \ldots, x_n \in \overline{V}$ such that

$$\overline{V} \subseteq \bigcup_{i=1}^n \left(x_i + \frac{1}{2}V\right).$$

Let *Y* denote the span of $\{x_1, ..., x_n\}$. Being a finite-dimensional subspace of *X*, *Y* is closed in *X* as we have seen in class. The above containment implies

$$V \subseteq \overline{V} \subseteq \bigcup_{i=1}^{n} \left(x_i + \frac{1}{2}V\right) \subseteq Y + \frac{1}{2}V.$$

But then

$$Y + \frac{1}{2}V \subseteq Y + \frac{1}{2}\left(Y + \frac{1}{2}V\right) = Y + \frac{1}{2}Y + \frac{1}{4}V = Y + \frac{1}{4}V,$$

where the last equality follows from the fact that Y is a vector space and hence, $\frac{1}{2}Y = Y$ and Y + Y = Y. Inductively, suppose we have shown that

$$V\subseteq Y+\frac{1}{2^m}V$$

for some positive integer m. Then,

$$V \subseteq Y + \frac{1}{2^m} \left(Y + \frac{1}{2} V \right) = Y + \frac{1}{2^m} Y + \frac{1}{2^{m+1}} V = Y + \frac{1}{2^{m+1}} V,$$

where the last equality follows from the fact that Y is a vector space. Consequently, we have

$$V\subseteq\bigcap_{m=1}^{\infty}\left(Y+\frac{1}{2^{m}}V\right).$$

CLAIM.

$$\overline{Y} = \bigcap_{m=1}^{\infty} \left(Y + \frac{1}{2^m} V \right)$$

Proof. Suppose $y \in \overline{Y}$ and m a positive integer. Then, by definition, there is some $x \in Y$ such that $||x - y|| < 2^{-m}$, that is, $y - x \in 2^{-m}V$, consequently, $y \in Y + 2^{-m}V$. It follows that $y \in \bigcap_{m=1}^{\infty} (Y + 2^{-m}V)$.

Conversely, if $y \in \bigcap_{m=1}^{\infty} (Y + 2^{-m}V)$ and r > 0. Choose a positive integer m such that $2^{-m} < r$. Since $y \in Y + 2^{-m}V$, there is some $x \in Y$ such that $y \in x + 2^{-m}V$, equivalently, $||y - x|| < 2^{-m} < r$. Hence, $B(y,r) \cap Y \neq \emptyset$. It follows that $y \in \overline{Y}$.

Using the above claim, we have

$$V \subseteq \overline{Y} = Y$$

since *Y* is closed in *X*, owing to it being finite-dimensional. Since *Y* is a vector space, we see that $\text{Span}(V) \subseteq Y$. Now, for any $0 \neq x \in X$, we have $\frac{x}{2\|x\|} \in V$, since it has norm $\frac{1}{2}$. Hence, $x \in \text{Span}(V)$, in particular, Span(V) = X. This shows that $X \subseteq Y$, that is, X = Y, hence *X* is finite-dimensional, as desired.

5. Problem 5

Before we begin, let $x = (x(i)) \in X$ and $j \in I$. Then, for $1 \le p < \infty$,

$$||x(i)||^p \leqslant \sum_{i \in I} ||x(i)||^p = ||x||^p \implies ||x(i)|| \leqslant ||x||.$$

And if $p = \infty$, then obviously $||x(i)|| \le ||x||$. All integrals over I henceforth are with respect to the counting measure $(I, \mathcal{P}(I), \mu)$ on I.

Throughout this solution, let $\|\cdot\|_p$ denote the $L^p(I, \mathcal{P}(I), \mu)$ norm with $1 \leq p \leq \infty$. We also identify sequences indexed by I with measurable functions $I \to \mathbb{K}$, so that there is no difference between a sum indexed by I and an integral over I.

(a) Let $1 \le p \le \infty$, $x, y \in \bigoplus_p X_i$, and $\alpha, \beta \in \mathbb{K}$. We shall show that $\alpha x + \beta y \in \bigoplus_p X_i$ and $\|\alpha x + \beta y\| \le \|\alpha\| \|x\| + \|\beta\| \|y\|$. Indeed, note that $\|\alpha x + \beta y\|$ is the L^p -norm of the function $f: I \to \mathbb{K}$ given by $f(i) = \|\alpha x(i) + \beta y(i)\|$. The triangle inequality gives us $f(i) \le |\alpha| \|x(i)\| + |\beta| \|y(i)\|$ for all $i \in I$. Let $g, h: I \to \mathbb{K}$ be given by $g(i) = \|x(i)\|$ and $h(i) = \|y(i)\|$. Then $f = |\alpha|g + |\beta|h$. Since the L^p -spaces form a normed linear space, we have

$$\|\alpha x + \beta y\| \le \|f\|_p \le |\alpha| \|g\|_p + |\beta| \|h\|_p = |\alpha| \|x\| + |\beta| \|y\| < \infty.$$

Hence, $\alpha x + \beta y \in \bigoplus_p X_i$ for all $\alpha, \beta \in \mathbb{K}$. This shows that $\bigoplus_p X_i$ is a vector space and the inequality proved above shows that $\|\cdot\|$ is indeed a norm on $\bigoplus_p X_i$.

Finally, we must show that $P_i: X \to X_i$ has norm ≤ 1 . That P_i is a linear map is obvious. For any $x \in X$, as we had observed in the paragraph preceding the solution of part (a), $||x(i)|| \leq ||x||$, and hence, $||P_i|| \leq 1$.

(b) Suppose first that each X_i is a Banach space and let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in X. That is, given any $\varepsilon>0$, there is a positive integer N>0 such that $\|x_m-x_n\|<\varepsilon$ for all $m,n\geqslant N$. Hence, for $i\in I$, we have $\|x_m(i)-x_n(i)\|<\varepsilon$ for all $m,n\geqslant N$. This shows that the sequence $(x_n(i))_{n\geqslant 1}$ is Cauchy in X_i , therefore converges to some $x(i)\in X_i$ since X_i is Banach. Since the norm is a continuous function on each X_i , it follows that $\|x_n(i)\|\to \|x\|$. Set x=(x(i)). We now treat the cases $p<\infty$ and $p=\infty$ separately.

Let $p < \infty$. First, we show that $x \in X$. Indeed, by Fatou's lemma, we have

$$\|x\|^{p} = \int_{I} \|x(i)\|^{p} d\mu(i) = \int_{I} \liminf_{n \to \infty} \|x_{n}(i)\|^{p} d\mu(i) \leqslant \liminf_{n \to \infty} \int_{I} \|x_{n}(i)\|^{p} d\mu = \liminf_{n \to \infty} \|x_{n}\|^{p} < \infty,$$

since the sequence (x_n) is bounded in X, owing to it being Cauchy (this is a standard fact from metric spaces). Next, we must show that $x_n \to x$ in X. For $\varepsilon > 0$, there is a positive integer N such that $||x_m - x_n|| < \varepsilon$ whenever $m, n \ge N$. Then, by Fatou's Lemma, for $n \ge N$, we have

$$\|x-x_n\|^p = \int_I \liminf_{m\to\infty} \|x_m(i)-x_n(i)\|^p d\mu(i) \leqslant \liminf_{m\to\infty} \int_I \|x_m(i)-x_n(i)\| = \liminf_{m\to\infty} \|x_m-x_n\|^p \leqslant \varepsilon^p,$$

that is, $\|x - x_n\| \le \varepsilon$. This shows that $x_n \to x$ in X, and hence X is a Banach space for $1 \le p < \infty$. Next, let $p = \infty$. First, we show that $x \in X$. Indeed, there is a positive integer N such that $\|x_m - x_n\| < 1$ for all $m, n \ge N$. In particular, $\|x_n - x_N\| < 1$ for all $n \ge N$, whence $\|x_n(i) - x_N(i)\| < 1$ for all $i \in I$. Consequently, $\|x_n(i)\| < \|x_N(i)\| + 1 \le \|x_N\| + 1$ for all $i \in I$. Since $x_n(i) \to x(i)$ in X_i , we have that

$$||x(i)|| = \lim_{n \to \infty} ||x_n(i)|| \le ||x_N|| + 1.$$

Taking a supremum over $i \in I$, we get that $||x|| \le ||x_N|| + 1$, that is, $x \in X$. Finally, we show that $x_n \to x$ in X. Let $\varepsilon > 0$. Then there is a positive integer N such that $||x_m - x_n|| < \varepsilon$ whenever $m, n \ge N$. Since $x_m(i) \to x(i)$ in X_i , we see that for $n \ge N$,

$$||x(i)-x_n(i)|| = \lim_{m\to\infty} ||x_m(i)-x_n(i)|| \leqslant \varepsilon.$$

Taking a supremum over $i \in I$, we have $||x - x_n|| \le \varepsilon$ for all $n \ge N$, whence $x_n \to x$ in X. This shows that X is a Banach space when $p = \infty$.

Conversely, suppose X is Banach; we shall show that each X_i is Banach. Let (x_n) be a Cauchy sequence in X_i . Define a sequence (y_n) in X by

$$y_n(j) = \begin{cases} x_n & j = i \\ 0 & \text{otherwise.} \end{cases}$$

We claim that (y_n) is a Cauchy sequence in X. Indeed, for $p < \infty$, and positive integers m, n, we have

$$||y_m - y_n|| = \left(\int_I ||y_m(j) - y_n(j)||^p d\mu(j)\right)^{1/p} = ||x_m - x_n||,$$

and for $p = \infty$,

$$||y_m - y_n|| = \sup_{j \in I} ||y_m(j) - y_n(j)|| = ||x_m - x_n||.$$

Thus, (y_n) is a Cauchy sequence in X, since (x_n) is a Cauchy sequence in X_i . Thus, there is some y = (y(j)) such that $y_n \to y$ in X. Then, as we have observed at the beginning, $\|y_n(i) - y(i)\| \le \|y_n - y\|$, whence $y_n(i) \to y(i)$ in X_i , that is, $x_n \to y(i)$ in X_i . This shows that X_i is a Banach space, thereby completing the proof.

(c) We first show that the image of a ball $B_X(0,r)$ centered at 0 in X is open under $P_i: X \to X_i$. In fact, we claim that the image of this ball is $B_{X_i}(0,r)$. Indeed, if $x = (x(i)) \in B_X(0,r)$, then $||x(i)|| \le ||x|| < r$, whence the image of $B_X(0,r)$ under P_i is contained in $B_{X_i}(0,r)$. Conversely, if $x_i \in X_i$ with $||x_i|| < r$, then setting $y = y(i) \in X$ where

$$y(j) = \begin{cases} x_i & j = i \\ 0 & \text{otherwise,} \end{cases}$$

we note that $||y|| = ||x_i||$ in both cases $p < \infty$ and $p = \infty$. Therefore, $y \in B_X(0,r)$. It follows that $B_{X_i}(0,r)$ is contained in the image of $B_X(0,r)$ under P_i , whence $P(B_X(0,r)) = B_{X_i}(0,r)$.

Now, obviously $P_i: X \to X_i$ is a linear map, for if $c \in \mathbb{C}$ and $x = (x(j)) \in X$, then $P_i(cx) = P_i((cx(j))) = cx(j)$ and if $y = (y(j)) \in X$, then

$$P_i(x+y) = P_i((x(j)) + (y(j))) = P_i((x(j) + y(j))) = P_i(x) + P_i(y).$$

Let $U \subseteq X$ be an open set. Then, for each $x = (x(i)) \in U$, there is an $r_x > 0$ such that $B_X(x, r_x) \subseteq U$, whence $U = \bigcup_{x \in U} B_X(x, r_x)$. Note that for any $y \in X$ and r > 0,

$$P_i(B_X(y,r)) = P_i(y + B_X(0,r)) = P_i(y) + P_i(B_X(0,r)) = y(i) + B_{X_i}(0,r) = B_{X_i}(y(i),r).$$

Consequently,

$$P_i(U) = P_i\left(\bigcup_{x \in U} B_X(x, r_x)\right) = \bigcup_{x \in U} P_i\left(B_X(x, r_x)\right) = \bigcup_{x \in U} B_{X_i}(x(i), r_x),$$

which is an open subset of X_i , as desired.

6. Problem 6

That the dual space of c_0 is isometrically isomorphic to ℓ^1 has been argued in class and I shall not reproduce that argument. We show that the dual space of c isometrically isomorphic to ℓ^1 . In this case, we denote an element $x \in \ell^1$ by a sequence indexed by $n \ge 0$. This is in contrast to the standard indexing of $n \ge 1$. It will be clear why this is done.

Define a map $T: \ell^1 \to c^*$ given by $a = (a(n))_{n \geqslant 0} \longmapsto T_a$ where

$$T_a(x) = a(0)x_{\infty} + \sum_{n=1}^{\infty} a(n)x(n),$$

where $x_{\infty} = \lim_{n \to \infty} x(n)$. Obviously, we must have $|x_{\infty}| \leq ||x||$. We must show that this sum converges, for which it suffices to show absolute convergence. Indeed, every partial sum (of the absolute value sum) is bounded as

$$|a(0)x_{\infty}| + \sum_{n=1}^{N} |a(n)x(n)| \le ||x|| \left(\sum_{n=0}^{N} |a(n)|\right) \le ||a|| ||x||,$$

and hence, must converge. It follows that T_a is a well-defined function. That T_a is linear is clear from the definition. To see that it is bounded, we again have that

$$|T_{a}(x)| = \left| a(0)x_{\infty} + \sum_{n=1}^{\infty} a(n)x(n) \right|$$

$$\leq |a(0)||x_{\infty}| + \sum_{n=1}^{\infty} |a(n)||x(n)|$$

$$\leq ||x|| \left(\sum_{n=0}^{\infty} |a(n)| \right) = ||a|| ||x||.$$

Thus, $T_a \in c^*$ and $||T_a|| \le ||a||$. We claim that the map T is linear. Indeed, if $a, b \in \ell^1$ and $\alpha \in \mathbb{K}$, then

$$T_{a+\alpha b}(x) = (a(0) + \alpha b(0)) x_{\infty} + \sum_{n=1}^{\infty} (a(n) + \alpha b(n)) x(n)$$

$$= a_{0}x_{\infty} + \sum_{n=1}^{\infty} a(n)x(n) + \alpha \left(b_{0}x_{\infty} + \sum_{n=1}^{\infty} b(n)x(n) \right)$$

$$= T_{a}(x) + \alpha T_{b}(x)$$

for all $x \in c$. This shows that $T(a + \alpha b) = T(a) + \alpha T(b)$, whence T is linear. Further, since $||T_a|| \le ||a||$, the map $T : \ell^1 \to c^*$ is a bounded linear functional.

Next, we show that T is an isometry. Let $a \in \ell^1$. If a = 0, then it is clear that $T_a = 0$, whence $||T_a|| = 0$. Suppose now that $a \neq 0$. For every $j \geqslant 0$, let

$$z_j = \begin{cases} \frac{\overline{a(j)}}{a(j)} & a(j) \neq 0\\ 0 & a(j) = 0, \end{cases}$$

whereby $|z_j| \le 1$. Note that the z_j 's are chosen so that $z_j a(j) = |a(j)|$. For every positive integer N, let $x_N \in c$ be given by

$$x_N(n) = \begin{cases} z_n & n \leq N \\ z_0 & n > N. \end{cases}$$

Since the sequence $x_N(n)$ eventually stabilizes, it lies in c and $||x_N|| \le 1$. According to our definition,

$$|T_{a}(x_{N})| = \left| |a(0)| + \sum_{j=1}^{N} |a(j)| + \sum_{j=N+1}^{\infty} a(j)z_{0} \right|$$

$$\ge |a(0)| + \sum_{j=1}^{N} |a(j)| - |z_{0}| \left| \sum_{j=N+1}^{\infty} a(j) \right|$$

$$\ge |a(j)| + \sum_{j=1}^{N} |a(j)| - \left| \sum_{j=N+1}^{\infty} a(j) \right|,$$

since $|z_0| \le 1$. But since $||x_N|| \le 1$, we have

$$||T_a|| \ge |T_a(x_N)| \ge |a(0)| + \sum_{j=1}^N |a(j)| - \left|\sum_{j=N+1}^\infty a(j)\right| \ge |a(0)| + \sum_{j=1}^N |a(j)| - \sum_{j=N+1}^\infty |a(j)| = ||a|| - 2\sum_{j=N+1}^\infty |a(j)|.$$

Since the sum $\sum_{j=0}^{\infty} |a(j)|$ converges, the tail sum goes to 0. In particular, taking $N \to \infty$ in the above inequality, we get

$$||T_a|| \geqslant ||a|| \implies ||T_a|| = ||a||,$$

that is, T is an isometry, whence T is injective, as the kernel is trivial; for if T(x) = 0, then ||x|| = ||T(x)|| = 0, i.e., x = 0.

Finally, to show that T is an isometric isomorphism, we must show that T is surjective. Indeed, let $\Lambda \in c^*$ and let the e_i 's denote the "standard basis vectors" for c, that is,

$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Set $a(n) = \Lambda(e_n)$ and $a(0) = T(\xi)$, where $\xi(j) = 1$ for all $j \ge 1$. We contend that $a \in \ell^1$ and $\Lambda = T_a$. Indeed, for $x = (x(n))_{n \ge 1} \in c$, set $x_\infty = \lim_{n \to \infty} x(n)$ and $y = x - x_\infty \xi \in c$. Obviously, we have that

$$\lim_{n\to\infty}y(n)=0.$$

Let $y_N \in c$ be given by

$$y_N = y(1)e_1 + \cdots + y(N)e_N \in c.$$

Then

$$y-y_N = \left(\underbrace{0,\ldots,0}_{N \text{ times}}, y(N+1), y(N+2),\ldots\right).$$

Note that

$$||y - y_N|| = \sup_{n \geqslant N+1} |y(n)| \to 0$$

as $N \to \infty$, since $\lim_{n \to \infty} y(n) = 0$. Since Λ is continuous, we have

$$\Delta y = \lim_{N \to \infty} \Delta y_N = \lim_{N \to \infty} \sum_{i=1}^N y(i) \Delta(e_i) = \lim_{N \to \infty} \sum_{i=1}^N a(i) y(i) = \sum_{n=1}^\infty a(n) y(n).$$

Hence,

$$\Lambda x = \Lambda y + x_{\infty} \Lambda \xi = a(0)x_{\infty} + \sum_{n=1}^{\infty} a(n)x(n).$$

Finally, we must show that $a \in \ell^1$. Again, for $n \ge 1$, set

$$z_n = \begin{cases} \frac{\overline{a}(n)}{|a(n)|} & a(n) \neq 0\\ 0 & \text{otherwise}, \end{cases}$$

and let

$$w_N = z_1 e_1 + \dots + z_N e_N \in c.$$

Since $\lim_{n\to\infty} w_N(n) = 0$, we see that

$$|\Lambda w_N| = \left| \sum_{n=1}^N z_n a(n) \right| = \sum_{n=1}^N |a(n)|.$$

Since each $|z_n| \le 1$, we see that $||w_N|| \le 1$, consequently,

$$\sum_{n=1}^{N} |a(n)| = |\Lambda w_N| \leqslant ||\Lambda|| ||w_N|| \leqslant ||\Lambda||.$$

Since the sum is bounded independent of N and the left hand side is a monotonically increasing sequence, it must converge, that is,

$$\sum_{n=1}^{\infty} |a(n)| < \infty \implies \sum_{n=0}^{\infty} |a(n)| < \infty,$$

equivalently, $a \in \ell^1$, as desired. Thus, we have shown that $\Lambda = T_a$ for some $a \in \ell^1$. This proves surjectivity of T and establishes the isometric isomorphism.

Now, we show that c_0 and c are not isometrically isomorphic. Suppose there was an isometric isomorphism $T:c\to c_0$. Set $\xi=(1,1,\dots)\in c$. Since T is an isometry, $x=T(\xi)$ must have norm 1. But since $\lim_{n\to\infty}x(n)=0$, there is a positive integer N such that for all $n\geqslant N$, $|x(n)|<\frac{1}{2}$. But since

$$\sup_{n\in\mathbb{N}}|x(n)|=1,$$

there is an $n_0 < N$ with $|x(n_0)| = 1$. Define y, z as

$$y(n) = \begin{cases} x(n) & n < N \\ x(n) + \frac{1}{4n} & n \geqslant N \end{cases} \quad \text{and} \quad z(n) = \begin{cases} x(n) & n < N \\ x(n) - \frac{1}{4n} & n \geqslant N. \end{cases}$$

Note that $y(n_0) = z(n_0) = 1$ since $n_0 < N$. Further, for $n \ge N$, we have

$$|y(n)| \le |x(n)| + \frac{1}{4n} < \frac{1}{2} + \frac{1}{4} < 1$$
 $|z(n)| \le |x(n)| + \frac{1}{4n} < \frac{1}{2} + \frac{1}{4} < 1$,

and

$$\lim_{n \to \infty} y(n) = \lim_{n \to \infty} x(n) = 0 = \lim_{n \to \infty} z(n).$$

It follows that $y, z \in c_0$, ||y|| = ||z|| = 1, and $x = \frac{1}{2}(y+z)$. Since T is an isometric isomorphism, there exist $\zeta, \eta \in c$ such that $T(\zeta) = y$ and $T(\eta) = z$ and $||\zeta|| = ||\eta|| = 1$. We also have that

$$T(\xi) = x = \frac{1}{2}(y+z) = \frac{1}{2}(T(\zeta) + T(\eta)) = T(\frac{\zeta + \eta}{2}),$$

consequently, $\xi = \frac{1}{2} (\zeta + \eta)$, in other words,

$$\zeta(n) + \eta(n) = 2 \quad \forall n \in \mathbb{N}.$$

But since $|\zeta(n)|, |\eta(n)| \le 1$, we have that $\zeta(n) = \eta(n) = 1$ for all $n \in \mathbb{N}$, i.e., $\zeta = \eta = \xi$, a contradiction, since $x \ne y$ and $x \ne z$. It follows that c and c_0 are not isometric, thereby completing the proof.

7. Problem 7

First, consider the case when μ is a positive measure. Then,

$$\mu([0,1]) = \int_0^1 d\mu = 0,$$

and hence, $\mu(E) = 0$ for all Borel sets $E \subseteq [0, 1]$, that is, $\mu = 0$.

Next, suppose μ is a complex measure. We claim that μ is a regular Borel measure. To this end, we use the following theorem:

THEOREM 7.1. Let X be a locally compact Hausdorff space in which every open set is σ -compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then λ is regular.

Obviously every open set in [0,1] is σ -compact¹ and for every compact set $K \subseteq [0,1]$, $|\mu|(K) < \infty$, since $|\mu|([0,1]) < \infty$, where $|\mu|$ is the total variation measure. It follows that $|\mu|$ is a positive regular Borel measure on [0,1], and hence, μ is a regular complex Borel measure on [0,1]. Thus, the map $T_{\mu}: C[0,1] \to C$ given by

$$T_{\mu}f = \int_0^1 f \, d\mu$$

is a bounded linear functional on [0,1], since the dual space of C[0,1] is identified with the Banach space of all complex regular Borel measures on [0,1]. Further, we know that $T_{\mu}(x^n) = 0$ for all $n \ge 0$, and hence by taking finite linear combinations, $T_{\mu}(p(x)) = 0$ for all polynomials $p(x) \in \mathbb{C}[x]$. Due to Weierstrass' Theorem, the space $\mathbb{C}[x]$ is dense in C[0,1] with respect to the sup-norm. Thus, T_{μ} is identically zero on a dense subspace of C[0,1], consequently, T_{μ} must be identically 0. That is, $T_{\mu}f = 0$ for all $f \in C[0,1]$.

Recall that there is an isometric isomorphism (in particular, a bijection) $\mathcal{M}([0,1]) \to (C[0,1])^*$ given by $\lambda \mapsto T_{\lambda}$, where T_{λ} is as defined above and $\mathcal{M}([0,1])$ is the space of all regular complex Borel measures on [0,1] equipped with the total variation norm. Since $\mu \in \mathcal{M}([0,1])$ maps to $0 \in (C[0,1])^*$, we see that $\mu = 0$, as desired.

8. Problem 8

We have seen in class and it is a standard fact from real analysis that the space Y = C[0,1] is complete with respect to the sup-norm. We claim that X is not complete. Note that $X = C^1[0,1]$ is a subspace of Y with the same norm. Thus to show that X is not complete, it suffices to exhibit a sequence in X converging to an element of $Y \setminus X$. Take $f \in Y \setminus X$ given by

$$f(x) = \left| x - \frac{1}{2} \right| \qquad 0 \leqslant x \leqslant 1.$$

This is obviously not an element of X since f is not differentiable at $\frac{1}{2}$. Due to a theorem of Weierstraß, we know that there is a sequence of polynomials $p_n \in Y$, which converge uniformly to f on [0,1], that is, $p_n \to f$ in Y. Since polynomials are infinitely differentiable, they are elements of X. Thus, we have found a sequence of elements of X which converges to an element of $Y \setminus X$. Since every convergent sequence is Cauchy, the

¹This follows from the fact that every locally compact Hausdorff space admits an exhaustion; since the open subsets of a locally compact Hausdorff (in this case, compact Hausdorff) space are locally compact Hausdorff, we are done.

sequence $\{p_n\}$ is Cauchy in Y, and hence in X (since the norm on X is the restriction of the norm on Y). But p_n cannot converge to some $g \in X$ since that would imply f = g due to the uniqueness of limits in Y; indeed, since convergence in X is the same as convergence in Y. This argument shows that X is not complete.

Now, we show that the map $A: X \to Y$ given by Af = f' has a closed graph but is not continuous. We shall need the following result which is usually covered in a first course on real analysis:

LEMMA 8.1. Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a,b] and such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f'_n\}$ converges uniformly on [a,b], then $\{f_n\}$ converges uniformly on [a,b], to a function f, and

$$f'(x) = \lim_{n \to \infty} f'_n(x)$$
 $a \leqslant x \leqslant b$.

Proof. See [Rud53, Theorem 7.17].

First, we show that the graph of A is closed in $X \times Y$. Since $X \times Y$ is a metric space, it suffices to show that the graph of A is sequentially closed. To this end, let (f_n, Af_n) be a sequence in Graph(A) converging to some $(f,g) \in X \times Y$, that is, $f \in C^1[0,1]$ and $g \in C[0,1]$. Let $Af_n = g_n \in Y$. Since $(f_n,g_n) \to (f,g)$, we have that $f_n \to f$ in X and $g_n \to g$ in Y (this is a standard fact about the product topology on metric spaces). Since both X and Y are equipped with the sup-norm on [0,1], we have that $f_n \to f$ and $g_n \to g$ uniformly on [0,1]. Hence, Lemma 8.1 applies and we get that

$$f'(x) = \lim_{n \to \infty} f'_n(x) = \lim_{n \to \infty} g_n(x) = g(x).$$

That is, g = Af, equivalently, $(f, g) \in Graph(A)$. This shows that Graph(A) is closed.

We show that A is not continuous. Clearly A is linear (since taking the derivative is a linear operation). Consider, for positive integers $n \ge 1$, the functions $f_n(x) = x^n$. Then $f_n \in X = C^1[0,1]$ and $g_n = Af_n \in Y$ are given by $g_n(x) = nx^{n-1}$. Obviously, $||f_n|| = 1$ and $||Af_n|| = n$, whence

$$||A|| = \sup_{\|f\| \leqslant 1} ||Af|| \geqslant \sup_{n \in \mathbb{N}} n = \infty,$$

i.e., A is not bounded and hence not continuous.

9. Problem 9

Define the map $T:V\to C(E)$ given by $Tf=f|_E$. Obviously, T is a linear map, for if $f,g\in V$ and $c\in \mathbb{K}$, then

$$T(f + cg) = (f + cg)|_E = f|_E + cg|_E = Tf + cTg.$$

Further, for any $g \in C(E)$, according to the hypothesis of the question, there is an $f \in V$ such that $g = f|_E$, whence $T: V \to C(E)$ is a surjective linear map. Finally, for any $f \in V$,

$$||Tf|| = ||f|_E|| \le ||f||,$$

consequently, *T* is a bounded linear functional, that is, *T* is continuous.

LEMMA 9.1. Let $T: X \to Y$ be a surjective linear map between Banach spaces. Then there is a constant c > 0 such that for every $y \in Y$, there is an $x \in X$ with $||x|| \le c||y||$ such that Tx = y.

Proof. Due to the open mapping theorem, T is an open map, consequently, $T(B_X(0,1))$ is an open set in Y containing 0. Thus, there is an r > 0 such that $B_Y(0,r) \subseteq T(B_X(0,1))$. Let $y \in Y$. If y = 0, then set x = 0. If $y \neq 0$, then consider $z = \frac{ry}{2\|y\|}$, where $\|z\| < r$. Hence, there is a $w \in B_X(0,1)$ such that Tw = z. Set $x = \frac{2\|y\|}{r}w \in X$ and note that

$$Tx = \frac{2\|y\|}{r}Tw = y,$$

and

$$||x|| = \frac{2||y||}{r}||w|| \leqslant \frac{2}{r}||y||.$$

Pick $c = \frac{2}{r}$. We have shown that for every $y \in Y$, there is an $x \in X$ with $||x|| \le c||y||$.

Since both V and C(E) are Banach spaces, the conclusion follows immediately from Lemma 9.1.

10. Problem 10

Let $J: X \to X^{**}$ denote the canonical isometry given by $x \mapsto \operatorname{ev}_x$, the evaluation map at x. That this is indeed an isometry has been argued in class. Let $x^{**} \in X^{**}$ denote the image of $x \in X$ under the map J. Let $f \in X^*$, then for any $x \in S$, we have

$$|x^{**}(f)| = |f(x)| \leqslant M_f$$

for some constant $M_f > 0$, since f(S) is a bounded subset of $\mathbb C$ according to the hypothesis. Due to the *Uniform Boundedness Principle* (or Banach-Steinhaus Theorem), there is a constant M > 0 such that $\|x^{**}\| \leq M$ for all $x \in S$. Note that the theorem applies since X^* is a Banach space as we have argued in class. Finally, since J is an isometry, for every $x \in S$, we have

$$||x|| = ||J(x)|| = ||x^{**}|| \le M,$$

that is,

$$\sup \{||x|| \colon x \in S\} \leqslant M < \infty,$$

as desired.

11. Problem 10

For each $x \in X$, since the sequence $(T_n(x))$ converges, it is bounded in Y, that is, there is an $M_x > 0$ such that $||T_nx|| \le M_x$ for all $n \ge 1$. Since X is Banach space, the *Uniform Boundedness Principle* applies and there is an M > 0 such that $||T_n|| \le M$ for all $n \ge 1$. In particular, this means that $||T_n(x)|| \le M||x||$ for all $x \in X$. As a result, for all $x \in X$,

$$||T(x)|| = \left\|\lim_{n\to\infty} T_n(x)\right\| = \lim_{n\to\infty} ||T_n(x)|| \le M||x||.$$

Finally, we show that *T* is a linear functional. Indeed, if $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$, then

$$T(\alpha x + \beta y) = \lim_{n \to \infty} T_n(\alpha x + \beta y) = \lim_{n \to \infty} \alpha T_n(x) + \beta T_n(y) = \alpha T(x) + \beta T(y).$$

This shows that $T: X \to Y$ is a bounded linear functional, as desired.

REFERENCES

- [Rud53] W. Rudin. *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1953.
- [Rud87] W. Rudin. *Real and Complex Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1987.