MA 824: ASSIGNMENT 1

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1. Problem 1

Some Preliminary Estimates. First, suppose $1 \leqslant p < q < \infty$. Then

$$\|x\|_q^q = \sum_{i=1}^d |x_i|^q = \sum_{i=1}^d |x_i|^p \cdot |x_i|^{q-p} \leqslant \|x\|_q^{q-p} \sum_{i=1}^d |x_i|^p = \|x\|_q^{q-p} \|x\|_p^p,$$

where the first inequality follows from the fact that $|x_i| \leqslant \|x\|_q$ for $1 \leqslant i \leqslant d$. The above shows that $\|x\|_q^p \leqslant \|x\|_p^p$, that is, $\|x\|_q \leqslant \|x\|_p$.

On the other hand, using Hölder's inequality on the measure space $\{1, \ldots, d\}$ equipped with the counting measure, we have

$$\|x^p\|_1 \leqslant \|x^p\|_{\frac{q}{p}} \|\mathbb{1}\|_{\frac{q}{q-p}},$$

where $x^p = (|x_1|^p, \ldots, |x_d|^p) \in \mathbb{C}^d$. Now note that $\|\mathbb{1}\|_{\frac{q}{q-p}} = d^{\frac{q-p}{q}}$, and

$$\|x^p\|_{\frac{q}{p}} = \left(\sum_{i=1}^d |x_i|^q\right)^{\frac{p}{q}} = \|x\|_q^p.$$

This gives us

$$\|x\|_p^p \leqslant \|x\|_q^p d^{\frac{q-p}{q}} \implies \|x\|_p \leqslant d^{\frac{1}{p}-\frac{1}{q}} \|x\|_q,$$

that is,

$$\|x\|_q \leqslant \|x\|_p \leqslant d^{\frac{1}{p}-\frac{1}{q}} \|x\|_q.$$

Note that both inequalities are tight. Indeed, take $x = (1, 0, \ldots, 0) \in \mathbb{C}^d$, then $\|x\|_q = \|x\|_p = 1$. On the other hand, taking $x = (1, 1, \ldots, 1) \in \mathbb{C}^d$, we see that $\|x\|_p = d^{\frac{1}{p}}$ and $\|x\|_q = d^{\frac{1}{q}}$, whence $\|x\|_p = d^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$.

Next, suppose $q = \infty$ and $p < q$. For $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$, there is an index $1 \leqslant i_0 \leqslant d$ such that $|x_{i_0}| = \|x\|_\infty$. Hence, we have

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}} \geqslant |x_{i_0}| = \|x\|_\infty,$$

and

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^d \|x\|_\infty^p\right)^{\frac{1}{p}} = d^{\frac{1}{p}} \|x\|_\infty.$$

The inequalities

$$\|x\|_\infty \leqslant \|x\|_p \leqslant d^{\frac{1}{p}} \|x\|_\infty$$

are tight. Taking $x = (1, 0, \ldots, 0) \in \mathbb{C}^d$, we have $\|x\|_p = \|x\|_\infty = 1$. And taking $x = (1, 1, \ldots, 1) \in \mathbb{C}^d$, we have $\|x\|_p = d^{\frac{1}{p}} = d^{\frac{1}{p}} \|x\|_\infty$, as desired.

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Conclusion. First, if $p = q$, then take $c = C = 1$. Henceforth, we assume that $p \neq q$. Next, suppose $1 \leqslant p, q < \infty$. Then using our estimates from the previous (sub)section,

$$\begin{cases} d^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\|x\|_p & p < q \\ \|x\|_p & p > q \end{cases} \leqslant \|x\|_q \leqslant \begin{cases} \|x\|_p & p < q \\ d^{\left(\frac{1}{q}-\frac{1}{p}\right)}\|x\|_p & p > q. \end{cases}$$

Finally, if $q = \infty$, then $p < \infty$ and we have

$$d^{-\frac{1}{p}} \leqslant \|x\|_q \leqslant \|x\|_p,$$

and if $p = \infty$ so that $q < \infty$, then

$$\|x\|_p \leqslant \|x\|_q \leqslant d^{\frac{1}{q}}\|x\|_p.$$

As we have seen in the preceding (sub)section, all the above estimates are tight, that is, the constants are the best possible.

2. Problem 2

(a) There is a $0 < \lambda < 1$ such that $p = \lambda r + (1-\lambda)s$. Hölder's inequality gives

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\lambda r + (1-\lambda)s}\|_1 \leqslant \|f^{\lambda r}\|_{\frac{1}{\lambda}} \|f^{(1-\lambda)s}\|_{\frac{1}{1-\lambda}}.$$

Note that

$$\|f^{\lambda r}\|_{\frac{1}{\lambda}} = \left(\int_X \left(|f|^{\lambda r}\right)^{\frac{1}{\lambda}}\, d\mu\right)^{\lambda} = \left(\int_X |f|^r\, d\mu\right)^{\lambda} = \|f\|_r^{\lambda r} \leqslant \max\{\|f\|_r, \|f\|_s\}^{\lambda r}$$

and

$$\|f^{(1-\lambda)s}\|_{\frac{1}{1-\lambda}} = \left(\int_X \left(|f|^{(1-\lambda)s}\right)^{\frac{1}{1-\lambda}}\, d\lambda\right)^{1-\lambda} = \|f\|_s^{(1-\lambda)s} \leqslant \max\{\|f\|_r, \|f\|_s\}^{(1-\lambda)s}.$$

Thus

$$\|f\|_p^p \leqslant \max\{\|f\|_r, \|f\|_s\}^{\lambda r + (1-\lambda)s} \implies \|f\|_p \leqslant \max\{\|f\|_r, \|f\|_s\},$$

as desired.

(b) Suppose first that $\|f\|_\infty = \infty$. As a result, for every $C > 0$,

$$\mu\{x \in X \colon |f(x)| \geqslant C\} > 0.$$

Hence,

$$\|f\|_p^p = \int_X |f|^p\, d\mu \geqslant \int_{\{x\colon |f(x)| \geqslant C\}} |f(x)|^p\, d\mu \geqslant \mu\{x \in X \colon |f(x)| \geqslant C\}\, C^p,$$

and hence,

$$\|f\|_p \geqslant C\mu\{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{p}}.$$

Thus, as $p \to \infty$, using the fact that $\mu\{x \in X \colon |f(x)| \geqslant C\} > 0$, we have

$$\liminf_{p\to\infty} \|f\|_p \geqslant C \liminf_{p\to\infty} \mu\{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{p}} = \infty.$$

It follows immediately that $\lim\limits_{n\to\infty} \|f\|_p = \infty$.

Next, suppose $\|f\|_\infty < \infty$. For $p > r$, we have

$$\|f\|_p^p = \int_X |f|^p\, d\mu = \int_X |f|^{p-r}|f|^r\, d\mu \leqslant \|f\|_\infty^{p-r} \int_X |f|^r\, d\mu.$$

Thus,

$$\|f\|_p \leqslant \|f\|_\infty^{1-\frac{r}{p}} \|f\|_r^{\frac{r}{p}}.$$

Since $\|f\|_\infty > 0$, we have $\|f\|_r > 0$, consequently,

$$\limsup_{p\to\infty} \|f\|_p \leqslant \|f\|^\infty \lim_{p\to\infty} \|f\|_\infty^{-\frac{r}{p}} \|f\|_r^{\frac{r}{p}} = \|f\|_\infty.$$

On the other hand, let $0 < C < \|f\|_\infty$, so that
$$\mu\{x \in X \colon |f(x)| \geqslant C\} > 0.$$
An obvious estimate gives us
$$\|f\|_p^p = \int_X |f|^p \, d\mu \geqslant \int_{\{x \in X \colon |f(x)| \geqslant C\}} |f|^p \, d\mu \geqslant C^p \mu\{x \in X \colon |f(x)| \geqslant C\},$$
consequently,
$$\|f\|_p \geqslant C\mu\{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{p}}.$$
Finally, we claim that $0 < \mu\{x \in X \colon |f(x)| \geqslant C\} < \infty$. Indeed, since $\|f\|_\infty > 0$, it is obvious that the above measure is positive. Further, since $\|f\|_r < \infty$, we have
$$C\mu\{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{r}} \leqslant \left(\int_X |f|^r \, d\mu\right)^{\frac{1}{r}} = \|f\|_r < \infty,$$
whence the measure is finite. It follows that
$$\liminf_{p \to \infty} \|f\|_p \geqslant C \liminf_{p \to \infty} \mu\{x \in X \colon |f(x)| \geqslant C\}^{\frac{1}{p}} = C.$$
Since the above inequality holds for all $0 < C < \|f\|_\infty$; taking a supremum over all such C, we get that
$$\|f\|_\infty \geqslant \limsup_{p \to \infty} \|f\|_p \geqslant \liminf_{p \to \infty} \|f\|_p \geqslant \|f\|_\infty,$$
therefore,
$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty,$$
as desired.

(c) Since $r < s$, we have $\frac{1}{r} > \frac{1}{s}$. Choose a $p \geqslant 1$ such that
$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s} \implies 1 = \frac{r}{p} + \frac{r}{s}.$$
Hölder's inequality then gives us
$$\|f^r\|_1 \leqslant \|f^r\|_{\frac{s}{r}} \|\mathbb{1}\|_{\frac{p}{r}},$$
where $\mathbb{1}$ denotes the constant function 1. Now note that
$$\|f^r\|_{\frac{s}{r}} = \left(\int_X |f|^s\right)^{\frac{r}{s}} = \|f\|_s^r,$$
$\|\mathbb{1}\|_{\frac{p}{r}} = 1$, since $\mu(X) = 1$; and
$$\|f^r\|_1 = \int_X |f|^r \, d\mu = \|f\|_r^r.$$
Hence, we have shown that $\|f\|_r^r \leqslant \|f\|_s^r$. If $\|f\|_s = \infty$, then the inequality $\|f\|_r \leqslant \|f\|_s$ is trivial. If $\|f\|_s < \infty$, then taking r-th roots we get $\|f\|_r \leqslant \|f\|_s$, as desired.

3. Problem 3

First, suppose $1 \leqslant p < \infty$. For $i \geqslant 1$, define the "standard basis vectors" $e_i \in \ell^p$ by
$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$
Let $\mathbb{K}$ denote the base field over which ℓ^p is defined. If $\mathbb{K} = \mathbb{R}$, set $Q = \mathbb{Q}$ and if $\mathbb{K} = \mathbb{C}$, then set $Q = \mathbb{Q} + \mathbb{Q}i$. Note that in either case, Q is dense in $\mathbb{K}$.

Set
$$D = \bigcup_{n=1}^{\infty} \{q_1 e_1 + \cdots + q_n e_n \colon q_1, \ldots, q_n \in Q\}.$$

Being a countable union of countable sets, D is itself countable. We contend that D is dense in ℓ^p. Let $x = (x_n) \in \ell^p$ and $\varepsilon > 0$. Since the sum $\sum_{n=1}^{\infty} |x_n|^p$ converges, there is a positive integer N such that the tail sum

$$\sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Let $y = (y_n) \in \ell^p$ be given by

$$y_n = \begin{cases} x_n & n \leqslant N \\ 0 & n > N. \end{cases}$$

Then,

$$\|x - y\|^p = \sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Therefore, $\|x - y\| < \frac{\varepsilon}{2}$. Next, using the density of Q in $\mathbb{K}$, for each $1 \leqslant n \leqslant N$, we can find a $z_n \in Q$ such that

$$|y_n - z_n|^p < \frac{1}{N}\left(\frac{\varepsilon}{2}\right)^p.$$

Setting $z = z_1 e_1 + \cdots + z_N e_N \in D \subseteq \ell^p$,

$$\|y - z\|^p = \sum_{n=1}^{N} |y_n - z_n|^p < \left(\frac{\varepsilon}{2}\right)^p.$$

It follows from the triangle inequality that

$$\|x - z\| \leqslant \|x - y\| + \|y - z\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have shown that for all $x \in \ell^p$ and $\varepsilon > 0$, there is a $z \in D$ wiht $\|x - z\| < \varepsilon$, and hence, D is dense in ℓ^p.

Finally, we show that ℓ^∞ is not separable. For this part of the proof, we write an element $x \in \ell^\infty$ as $(x(n))_{n \geqslant 1}$. Suppose not and there were a countable dense subset $D \subseteq \ell^\infty$. For each subset $S \subseteq \mathbb{N} = \{1, 2, \dots\}$, define $x_S \in \ell^\infty$ as

$$x_S(n) = \begin{cases} 1 & n \in S \\ 0 & \text{otherwise,} \end{cases}$$

and set $U_S = B\left(x_S, \frac{1}{2}\right)$. We claim that the U_S's are pairwise disjoint. Indeed, suppose $S, T \subseteq \mathbb{N}$ are distinct subsets and $y \in U_S \cap U_T$. It follows that

$$\|x_S - x_T\| = \|(x_S - y) - (x_T - y)\| \leqslant \|x_S - y\| + \|x_T - y\| < 1.$$

But for $n \in S \Delta T$, we have that $|x_S(n) - x_T(n)| = 1$, whence $\|x_S - x_T\| \geqslant 1$, a contradiction. Thus, the U_S's are pairwise disjoint. For each $S \subseteq \mathbb{N}$, there is a $z_S \in D$ such that $z_S \in U_S \cap D$, owing to the density of D. Since the U_S's are pairwise disjoint, the z_S's are distinct. But $\mathbb{N}$ has uncountably many subsets (due to Cantor), and D is countable, a contradiction. Hence ℓ^∞ is not separable.

4. Problem 4

Let X be a locally compact normed linear space; then there is a neighborhood V of the origin in X such that $\overline{V}$ is compact. Since V is open, there is an $r > 0$ such that $B(0, r) \subseteq V$, consequently, $\overline{B}(0, r) \subseteq \overline{V}$. Since the latter is compact, so is the former. Further, the map $x \mapsto rx$ is a homeomorphism $X \to X$ with inverse given by $x \mapsto r^{-1}x$. Under the former map, the closed unit ball $\overline{B}(0, 1)$ mapsto $\overline{B}(0, r)$. Since the latter is compact, the former must be too, that is, $\overline{B}(0, 1)$ is compact. Hence, we may suppose without loss of generality that $V = B(0, 1)$.

The following method is due to André Weil and generalizes well to topological vector spaces over complete valued fields. Note that $\overline{V}$ is compact and

$$\overline{V} \subseteq \bigcup_{x \in \overline{V}} \left(x + \frac{1}{2}V\right).$$

Since the latter is an open cover, it contains a finite subcover of $\overline{V}$. That is, there are $x_1, \ldots, x_n \in \overline{V}$ such that

$$\overline{V} \subseteq \bigcup_{i=1}^{n} \left(x_i + \frac{1}{2} V \right).$$

Let Y denote the span of $\{x_1, \ldots, x_n\}$. Being a finite-dimensional subspace of X, Y is closed in X as we have seen in class. The above containment implies

$$V \subseteq \overline{V} \subseteq \bigcup_{i=1}^{n} \left(x_i + \frac{1}{2} V \right) \subseteq Y + \frac{1}{2} V.$$

But then

$$Y + \frac{1}{2} V \subseteq Y + \frac{1}{2} \left(Y + \frac{1}{2} V \right) = Y + \frac{1}{2} Y + \frac{1}{4} V = Y + \frac{1}{4} V,$$

where the last equality follows from the fact that Y is a vector space and hence, $\frac{1}{2} Y = Y$ and $Y + Y = Y$. Inductively, suppose we have shown that

$$V \subseteq Y + \frac{1}{2^m} V$$

for some positive integer m. Then,

$$V \subseteq Y + \frac{1}{2^m} \left(Y + \frac{1}{2} V \right) = Y + \frac{1}{2^m} Y + \frac{1}{2^{m+1}} V = Y + \frac{1}{2^{m+1}} V,$$

where the last equality follows from the fact that Y is a vector space. Consequently, we have

$$V \subseteq \bigcap_{m=1}^{\infty} \left(Y + \frac{1}{2^m} V \right).$$

CLAIM.

$$\overline{Y} = \bigcap_{m=1}^{\infty} \left(Y + \frac{1}{2^m} V \right)$$

Proof. Suppose $y \in \overline{Y}$ and m a positive integer. Then, by definition, there is some $x \in Y$ such that $\|x - y\| < 2^{-m}$, that is, $y - x \in 2^{-m} V$, consequently, $y \in Y + 2^{-m} V$. It follows that $y \in \bigcap_{m=1}^{\infty} (Y + 2^{-m} V)$.

Conversely, if $y \in \bigcap_{m=1}^{\infty} (Y + 2^{-m} V)$ and $r > 0$. Choose a positive integer m such that $2^{-m} < r$. Since $y \in Y + 2^{-m} V$, there is some $x \in Y$ such that $y \in x + 2^{-m} V$, equivalently, $\|y - x\| < 2^{-m} < r$. Hence, $B(y, r) \cap Y \neq \emptyset$. It follows that $y \in \overline{Y}$. ∎

Using the above claim, we have

$$V \subseteq \overline{Y} = Y,$$

since Y is closed in X, owing to it being finite-dimensional. Since Y is a vector space, we see that $\operatorname{Span}(V) \subseteq Y$. Now, for any $0 \neq x \in X$, we have $\frac{x}{2\|x\|} \in V$, since it has norm $\frac{1}{2}$. Hence, $x \in \operatorname{Span}(V)$, in particular, $\operatorname{Span}(V) = X$. This shows that $X \subseteq Y$, that is, $X = Y$, hence X is finite-dimensional, as desired.

5. PROBLEM 5

Before we begin, let $x = (x(i)) \in X$ and $j \in I$. Then, for $1 \leqslant p < \infty$,

$$\|x(i)\|^p \leqslant \sum_{i \in I} \|x(i)\|^p = \|x\|^p \implies \|x(i)\| \leqslant \|x\|.$$

And if $p = \infty$, then obviously $\|x(i)\| \leqslant \|x\|$. All integrals over I henceforth are with respect to the counting measure $(I, \mathscr{P}(I), \mu)$ on I.

Throughout this solution, let $\|\cdot\|_p$ denote the $L^p\,(I, \mathscr{P}(I), \mu)$ norm with $1 \leqslant p \leqslant \infty$. We also identify sequences indexed by I with measurable functions $I \to \mathbb{K}$, so that there is no difference between a sum indexed by I and an integral over I.

(a) Let $1 \leqslant p \leqslant \infty$, $x, y \in \bigoplus_p X_i$, and $\alpha, \beta \in \mathbb{K}$. We shall show that $\alpha x + \beta y \in \bigoplus_p X_i$ and $\|\alpha x + \beta y\| \leqslant |\alpha|\|x\| + |\beta|\|y\|$. Indeed, note that $\|\alpha x + \beta y\|$ is the L^p-norm of the function $f : I \to \mathbb{K}$ given by $f(i) = \|\alpha x(i) + \beta y(i)\|$. The triangle inequality gives us $f(i) \leqslant |\alpha|\|x(i)\| + |\beta|\|y(i)\|$ for all $i \in I$. Let $g, h : I \to \mathbb{K}$ be given by $g(i) = \|x(i)\|$ and $h(i) = \|y(i)\|$. Then $f = |\alpha|g + |\beta|h$. Since the L^p-spaces form a normed linear space, we have

$$\|\alpha x + \beta y\| \leqslant \|f\|_p \leqslant |\alpha|\|g\|_p + |\beta|\|h\|_p = |\alpha|\|x\| + |\beta|\|y\| < \infty.$$

Hence, $\alpha x + \beta y \in \bigoplus_p X_i$ for all $\alpha, \beta \in \mathbb{K}$. This shows that $\bigoplus_p X_i$ is a vector space and the inequality proved above shows that $\|\cdot\|$ is indeed a norm on $\bigoplus_p X_i$.

Finally, we must show that $P_i : X \to X_i$ has norm $\leqslant 1$. That P_i is a linear map is obvious. For any $x \in X$, as we had observed in the paragraph preceding the solution of part (a), $\|x(i)\| \leqslant \|x\|$, and hence, $\|P_i\| \leqslant 1$.

(b) Suppose first that each X_i is a Banach space and let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in X. That is, given any $\varepsilon > 0$, there is a positive integer $N > 0$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geqslant N$. Hence, for $i \in I$, we have $\|x_m(i) - x_n(i)\| < \varepsilon$ for all $m, n \geqslant N$. This shows that the sequence $(x_n(i))_{n\geqslant 1}$ is Cauchy in X_i, therefore converges to some $x(i) \in X_i$ since X_i is Banach. Since the norm is a continuous function on each X_i, it follows that $\|x_n(i)\| \to \|x\|$. Set $x = (x(i))$. We now treat the cases $p < \infty$ and $p = \infty$ separately.

Let $p < \infty$. First, we show that $x \in X$. Indeed, by Fatou's lemma, we have

$$\|x\|^p = \int_I \|x(i)\|^p \, d\mu(i) = \int_I \liminf_{n\to\infty} \|x_n(i)\|^p \, d\mu(i) \leqslant \liminf_{n\to\infty} \int_I \|x_n(i)\|^p \, d\mu = \liminf_{n\to\infty} \|x_n\|^p < \infty,$$

since the sequence (x_n) is bounded in X, owing to it being Cauchy (this is a standard fact from metric spaces). Next, we must show that $x_n \to x$ in X. For $\varepsilon > 0$, there is a positive integer N such that $\|x_m - x_n\| < \varepsilon$ whenever $m, n \geqslant N$. Then, by Fatou's Lemma, for $n \geqslant N$, we have

$$\|x - x_n\|^p = \int_I \liminf_{m\to\infty} \|x_m(i) - x_n(i)\|^p \, d\mu(i) \leqslant \liminf_{m\to\infty} \int_I \|x_m(i) - x_n(i)\| = \liminf_{m\to\infty} \|x_m - x_n\|^p \leqslant \varepsilon^p,$$

that is, $\|x - x_n\| \leqslant \varepsilon$. This shows that $x_n \to x$ in X, and hence X is a Banach space for $1 \leqslant p < \infty$.

Next, let $p = \infty$. First, we show that $x \in X$. Indeed, there is a positive integer N such that $\|x_m - x_n\| < 1$ for all $m, n \geqslant N$. In particular, $\|x_n - x_N\| < 1$ for all $n \geqslant N$, whence $\|x_n(i) - x_N(i)\| < 1$ for all $i \in I$. Consequently, $\|x_n(i)\| < \|x_N(i)\| + 1 \leqslant \|x_N\| + 1$ for all $i \in I$. Since $x_n(i) \to x(i)$ in X_i, we have that

$$\|x(i)\| = \lim_{n\to\infty} \|x_n(i)\| \leqslant \|x_N\| + 1.$$

Taking a supremum over $i \in I$, we get that $\|x\| \leqslant \|x_N\| + 1$, that is, $x \in X$. Finally, we show that $x_n \to x$ in X. Let $\varepsilon > 0$. Then there is a positive integer N such that $\|x_m - x_n\| < \varepsilon$ whenever $m, n \geqslant N$. Since $x_m(i) \to x(i)$ in X_i, we see that for $n \geqslant N$,

$$\|x(i) - x_n(i)\| = \lim_{m\to\infty} \|x_m(i) - x_n(i)\| \leqslant \varepsilon.$$

Taking a supremum over $i \in I$, we have $\|x - x_n\| \leqslant \varepsilon$ for all $n \geqslant N$, whence $x_n \to x$ in X. This shows that X is a Banach space when $p = \infty$.

Conversely, suppose X is Banach; we shall show that each X_i is Banach. Let (x_n) be a Cauchy sequence in X_i. Define a sequence (y_n) in X by

$$y_n(j) = \begin{cases} x_n & j = i \\ 0 & \text{otherwise.} \end{cases}$$

We claim that (y_n) is a Cauchy sequence in X. Indeed, for $p < \infty$, and positive integers m, n, we have

$$\|y_m - y_n\| = \left(\int_I \|y_m(j) - y_n(j)\|^p \, d\mu(j) \right)^{1/p} = \|x_m - x_n\|,$$

and for $p = \infty$,

$$\|y_m - y_n\| = \sup_{j\in I} \|y_m(j) - y_n(j)\| = \|x_m - x_n\|.$$

Thus, (y_n) is a Cauchy sequence in X, since (x_n) is a Cauchy sequence in X_i. Thus, there is some $y = (y(j))$ such that $y_n \to y$ in X. Then, as we have observed at the beginning, $\|y_n(i) - y(i)\| \leqslant \|y_n - y\|$, whence $y_n(i) \to y(i)$ in X_i, that is, $x_n \to y(i)$ in X_i. This shows that X_i is a Banach space, thereby completing the proof.

(c) We first show that the image of a ball $B_X(0,r)$ centered at 0 in X is open under $P_i : X \to X_i$. In fact, we claim that the image of this ball is $B_{X_i}(0,r)$. Indeed, if $x = (x(i)) \in B_X(0,r)$, then $\|x(i)\| \leqslant \|x\| < r$, whence the image of $B_X(0,r)$ under P_i is contained in $B_{X_i}(0,r)$. Conversely, if $x_i \in X_i$ with $\|x_i\| < r$, then setting $y = y(i) \in X$ where

$$y(j) = \begin{cases} x_i & j = i \\ 0 & \text{otherwise,} \end{cases}$$

we note that $\|y\| = \|x_i\|$ in both cases $p < \infty$ and $p = \infty$. Therefore, $y \in B_X(0,r)$. It follows that $B_{X_i}(0,r)$ is contained in the image of $B_X(0,r)$ under P_i, whence $P\left(B_X(0,r)\right) = B_{X_i}(0,r)$.

Now, obviously $P_i : X \to X_i$ is a linear map, for if $c \in \mathbb{C}$ and $x = (x(j)) \in X$, then $P_i\left(cx\right) = P_i\left((cx(j))\right) = cx(j)$ and if $y = (y(j)) \in X$, then

$$P_i(x+y) = P_i\left((x(j)) + (y(j))\right) = P_i\left((x(j) + y(j))\right) = P_i(x) + P_i(y).$$

Let $U \subseteq X$ be an open set. Then, for each $x = (x(i)) \in U$, there is an $r_x > 0$ such that $B_X(x, r_x) \subseteq U$, whence $U = \bigcup_{x \in U} B_X(x, r_x)$. Note that for any $y \in X$ and $r > 0$,

$$P_i\left(B_X(y,r)\right) = P_i\left(y + B_X(0,r)\right) = P_i(y) + P_i\left(B_X(0,r)\right) = y(i) + B_{X_i}(0,r) = B_{X_i}(y(i), r).$$

Consequently,

$$P_i(U) = P_i\left(\bigcup_{x \in U} B_X(x, r_x)\right) = \bigcup_{x \in U} P_i\left(B_X(x, r_x)\right) = \bigcup_{x \in U} B_{X_i}(x(i), r_x),$$

which is an open subset of X_i, as desired.

6. Problem 6

That the dual space of c_0 is isometrically isomorphic to ℓ^1 has been argued in class and I shall not reproduce that argument. We show that the dual space of c isometrically isomorphic to ℓ^1. In this case, we denote an element $x \in \ell^1$ by a sequence indexed by $n \geqslant 0$. This is in contrast to the standard indexing of $n \geqslant 1$. It will be clear why this is done.

Define a map $T : \ell^1 \to c^*$ given by $a = (a(n))_{n \geqslant 0} \longmapsto T_a$ where

$$T_a(x) = a(0)x_\infty + \sum_{n=1}^{\infty} a(n)x(n),$$

where $x_\infty = \lim_{n \to \infty} x(n)$. Obviously, we must have $|x_\infty| \leqslant \|x\|$. We must show that this sum converges, for which it suffices to show absolute convergence. Indeed, every partial sum (of the absolute value sum) is bounded as

$$|a(0)x_\infty| + \sum_{n=1}^{N} |a(n)x(n)| \leqslant \|x\| \left(\sum_{n=0}^{N} |a(n)|\right) \leqslant \|a\|\|x\|,$$

and hence, must converge. It follows that T_a is a well-defined function. That T_a is linear is clear from the definition. To see that it is bounded, we again have that

$$\begin{aligned} |T_a(x)| &= \left| a(0)x_\infty + \sum_{n=1}^{\infty} a(n)x(n) \right| \\ &\leqslant |a(0)||x_\infty| + \sum_{n=1}^{\infty} |a(n)||x(n)| \\ &\leqslant \|x\| \left(\sum_{n=0}^{\infty} |a(n)|\right) = \|a\|\|x\|. \end{aligned}$$

Thus, $T_a \in c^*$ and $\|T_a\| \leqslant \|a\|$. We claim that the map T is linear. Indeed, if $a, b \in \ell^1$ and $\alpha \in \mathbb{K}$, then

$$\begin{aligned} T_{a+\alpha b}(x) &= (a(0) + \alpha b(0))\, x_\infty + \sum_{n=1}^{\infty} (a(n) + \alpha b(n))\, x(n) \\ &= a_0 x_\infty + \sum_{n=1}^{\infty} a(n)x(n) + \alpha \left(b_0 x_\infty + \sum_{n=1}^{\infty} b(n)x(n) \right) \\ &= T_a(x) + \alpha T_b(x) \end{aligned}$$

for all $x \in c$. This shows that $T(a + \alpha b) = T(a) + \alpha T(b)$, whence T is linear. Further, since $\|T_a\| \leqslant \|a\|$, the map $T : \ell^1 \to c^*$ is a bounded linear functional.

Next, we show that T is an isometry. Let $a \in \ell^1$. If $a = 0$, then it is clear that $T_a = 0$, whence $\|T_a\| = 0$. Suppose now that $a \neq 0$. For every $j \geqslant 0$, let

$$z_j = \begin{cases} \frac{\overline{a(j)}}{a(j)} & a(j) \neq 0 \\ 0 & a(j) = 0, \end{cases}$$

whereby $|z_j| \leqslant 1$. Note that the z_j's are chosen so that $z_j a(j) = |a(j)|$. For every positive integer N, let $x_N \in c$ be given by

$$x_N(n) = \begin{cases} z_n & n \leqslant N \\ z_0 & n > N. \end{cases}$$

Since the sequence $x_N(n)$ eventually stabilizes, it lies in c and $\|x_N\| \leqslant 1$. According to our definition,

$$\begin{aligned} |T_a(x_N)| &= \left| |a(0)| + \sum_{j=1}^{N} |a(j)| + \sum_{j=N+1}^{\infty} a(j) z_0 \right| \\ &\geqslant |a(0)| + \sum_{j=1}^{N} |a(j)| - |z_0| \left| \sum_{j=N+1}^{\infty} a(j) \right| \\ &\geqslant |a(j)| + \sum_{j=1}^{N} |a(j)| - \left| \sum_{j=N+1}^{\infty} a(j) \right|, \end{aligned}$$

since $|z_0| \leqslant 1$. But since $\|x_N\| \leqslant 1$, we have

$$\|T_a\| \geqslant |T_a(x_N)| \geqslant |a(0)| + \sum_{j=1}^{N} |a(j)| - \left| \sum_{j=N+1}^{\infty} a(j) \right| \geqslant |a(0)| + \sum_{j=1}^{N} |a(j)| - \sum_{j=N+1}^{\infty} |a(j)| = \|a\| - 2 \sum_{j=N+1}^{\infty} |a(j)|.$$

Since the sum $\sum_{j=0}^{\infty} |a(j)|$ converges, the tail sum goes to 0. In particular, taking $N \to \infty$ in the above inequality, we get

$$\|T_a\| \geqslant \|a\| \implies \|T_a\| = \|a\|,$$

that is, T is an isometry, whence T is injective, as the kernel is trivial; for if $T(x) = 0$, then $\|x\| = \|T(x)\| = 0$, i.e., $x = 0$.

Finally, to show that T is an isometric isomorphism, we must show that T is surjective. Indeed, let $\Lambda \in c^*$ and let the e_i's denote the "standard basis vectors" for c, that is,

$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Set $a(n) = \Lambda(e_n)$ and $a(0) = T(\xi)$, where $\xi(j) = 1$ for all $j \geqslant 1$. We contend that $a \in \ell^1$ and $\Lambda = T_a$. Indeed, for $x = (x(n))_{n \geqslant 1} \in c$, set $x_\infty = \lim_{n \to \infty} x(n)$ and $y = x - x_\infty \xi \in c$. Obviously, we have that

$$\lim_{n \to \infty} y(n) = 0.$$

Let $y_N \in c$ be given by

$$y_N = y(1)e_1 + \cdots + y(N)e_N \in c.$$

Then

$$y - y_N = \left(\underbrace{0, \ldots, 0}_{N \text{ times}}, y(N+1), y(N+2), \ldots \right).$$

Note that

$$\|y - y_N\| = \sup_{n \geqslant N+1} |y(n)| \to 0$$

as $N \to \infty$, since $\lim_{n\to\infty} y(n) = 0$. Since Λ is continuous, we have

$$\Lambda y = \lim_{N\to\infty} \Lambda y_N = \lim_{N\to\infty} \sum_{i=1}^{N} y(i)\Lambda(e_i) = \lim_{N\to\infty} \sum_{i=1}^{N} a(i)y(i) = \sum_{n=1}^{\infty} a(n)y(n).$$

Hence,

$$\Lambda x = \Lambda y + x_\infty \Lambda \xi = a(0)x_\infty + \sum_{n=1}^{\infty} a(n)x(n).$$

Finally, we must show that $a \in \ell^1$. Again, for $n \geqslant 1$, set

$$z_n = \begin{cases} \frac{\overline{a}(n)}{|a(n)|} & a(n) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$w_N = z_1 e_1 + \cdots + z_N e_N \in c.$$

Since $\lim_{n\to\infty} w_N(n) = 0$, we see that

$$|\Lambda w_N| = \left| \sum_{n=1}^{N} z_n a(n) \right| = \sum_{n=1}^{N} |a(n)|.$$

Since each $|z_n| \leqslant 1$, we see that $\|w_N\| \leqslant 1$, consequently,

$$\sum_{n=1}^{N} |a(n)| = |\Lambda w_N| \leqslant \|\Lambda\| \|w_N\| \leqslant \|\Lambda\|.$$

Since the sum is bounded independent of N and the left hand side is a monotonically increasing sequence, it must converge, that is,

$$\sum_{n=1}^{\infty} |a(n)| < \infty \implies \sum_{n=0}^{\infty} |a(n)| < \infty,$$

equivalently, $a \in \ell^1$, as desired. Thus, we have shown that $\Lambda = T_a$ for some $a \in \ell^1$. This proves surjectivity of T and establishes the isometric isomorphism.

Now, we show that c_0 and c are not isometrically isomorphic. Suppose there was an isometric isomorphism $T : c \to c_0$. Set $\xi = (1, 1, \ldots) \in c$. Since T is an isometry, $x = T(\xi)$ must have norm 1. But since $\lim_{n\to\infty} x(n) = 0$, there is a positive integer N such that for all $n \geqslant N$, $|x(n)| < \frac{1}{2}$. But since

$$\sup_{n\in\mathbb{N}} |x(n)| = 1,$$

there is an $n_0 < N$ with $|x(n_0)| = 1$. Define y, z as

$$y(n) = \begin{cases} x(n) & n < N \\ x(n) + \frac{1}{4n} & n \geqslant N \end{cases} \quad \text{and} \quad z(n) = \begin{cases} x(n) & n < N \\ x(n) - \frac{1}{4n} & n \geqslant N. \end{cases}$$

Note that $y(n_0) = z(n_0) = 1$ since $n_0 < N$. Further, for $n \geqslant N$, we have

$$|y(n)| \leqslant |x(n)| + \frac{1}{4n} < \frac{1}{2} + \frac{1}{4} < 1 \qquad |z(n)| \leqslant |x(n)| + \frac{1}{4n} < \frac{1}{2} + \frac{1}{4} < 1,$$

and

$$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} x(n) = 0 = \lim_{n\to\infty} z(n).$$

It follows that $y, z \in c_0$, $\|y\| = \|z\| = 1$, and $x = \frac{1}{2}(y+z)$. Since T is an isometric isomorphism, there exist $\zeta, \eta \in c$ such that $T(\zeta) = y$ and $T(\eta) = z$ and $\|\zeta\| = \|\eta\| = 1$. We also have that

$$T(\xi) = x = \frac{1}{2}(y+z) = \frac{1}{2}(T(\zeta) + T(\eta)) = T\left(\frac{\zeta+\eta}{2}\right),$$

consequently, $\xi = \frac{1}{2}(\zeta + \eta)$, in other words,

$$\zeta(n) + \eta(n) = 2 \quad \forall\, n \in \mathbb{N}.$$

But since $|\zeta(n)|, |\eta(n)| \leqslant 1$, we have that $\zeta(n) = \eta(n) = 1$ for all $n \in \mathbb{N}$, i.e., $\zeta = \eta = \xi$, a contradiction, since $x \neq y$ and $x \neq z$. It follows that c and c_0 are not isometric, thereby completing the proof.

7. Problem 7

First, consider the case when μ is a positive measure. Then,

$$\mu([0,1]) = \int_0^1 d\mu = 0,$$

and hence, $\mu(E) = 0$ for all Borel sets $E \subseteq [0,1]$, that is, $\mu = 0$.

Next, suppose μ is a complex measure. We claim that μ is a regular Borel measure. To this end, we use the following theorem:

Theorem 7.1. *Let X be a locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then λ is regular.*

Proof. See [Rud87, Theorem 2.17] ■

Obviously every open set in $[0,1]$ is σ-compact[1] and for every compact set $K \subseteq [0,1]$, $|\mu|(K) < \infty$, since $|\mu|([0,1]) < \infty$, where $|\mu|$ is the total variation measure. It follows that $|\mu|$ is a positive regular Borel measure on $[0,1]$, and hence, μ is a regular complex Borel measure on $[0,1]$. Thus, the map $T_\mu : C[0,1] \to \mathbb{C}$ given by

$$T_\mu f = \int_0^1 f\, d\mu$$

is a bounded linear functional on $[0,1]$, since the dual space of $C[0,1]$ is identified with the Banach space of all complex regular Borel measures on $[0,1]$. Further, we know that $T_\mu(x^n) = 0$ for all $n \geqslant 0$, and hence by taking finite linear combinations, $T_\mu(p(x)) = 0$ for all polynomials $p(x) \in \mathbb{C}[x]$. Due to Weierstrass' Theorem, the space $\mathbb{C}[x]$ is dense in $C[0,1]$ with respect to the sup-norm. Thus, T_μ is identically zero on a dense subspace of $C[0,1]$, consequently, T_μ must be identically 0. That is, $T_\mu f = 0$ for all $f \in C[0,1]$.

Recall that there is an isometric isomorphism (in particular, a bijection) $\mathscr{M}([0,1]) \to (C[0,1])^*$ given by $\lambda \mapsto T_\lambda$, where T_λ is as defined above and $\mathscr{M}([0,1])$ is the space of all regular complex Borel measures on $[0,1]$ equipped with the total variation norm. Since $\mu \in \mathscr{M}([0,1])$ maps to $0 \in (C[0,1])^*$, we see that $\mu = 0$, as desired.

8. Problem 8

We have seen in class and it is a standard fact from real analysis that the space $Y = C[0,1]$ is complete with respect to the sup-norm. We claim that X is not complete. Note that $X = C^1[0,1]$ is a subspace of Y with the same norm. Thus to show that X is not complete, it suffices to exhibit a sequence in X converging to an element of $Y \setminus X$. Take $f \in Y \setminus X$ given by

$$f(x) = \left|x - \frac{1}{2}\right| \qquad 0 \leqslant x \leqslant 1.$$

This is obviously not an element of X since f is not differentiable at $\frac{1}{2}$. Due to a theorem of Weierstraß, we know that there is a sequence of polynomials $p_n \in Y$, which converge uniformly to f on $[0,1]$, that is, $p_n \to f$ in Y. Since polynomials are infinitely differentiable, they are elements of X. Thus, we have found a sequence of elements of X which converges to an element of $Y \setminus X$. Since every convergent sequence is Cauchy, the

[1] This follows from the fact that every locally compact Hausdorff space admits an exhaustion; since the open subsets of a locally compact Hausdorff (in this case, compact Hausdorff) space are locally compact Hausdorff, we are done.

sequence $\{p_n\}$ is Cauchy in Y, and hence in X (since the norm on X is the restriction of the norm on Y). But p_n cannot converge to some $g \in X$ since that would imply $f = g$ due to the uniqueness of limits in Y; indeed, since convergence in X is the same as convergence in Y. This argument shows that X is not complete.

Now, we show that the map $A : X \to Y$ given by $Af = f'$ has a closed graph but is not continuous. We shall need the following result which is usually covered in a first course on real analysis:

LEMMA 8.1. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad a \leqslant x \leqslant b.$$

Proof. See [Rud53, Theorem 7.17]. ■

First, we show that the graph of A is closed in $X \times Y$. Since $X \times Y$ is a metric space, it suffices to show that the graph of A is sequentially closed. To this end, let (f_n, Af_n) be a sequence in $\mathrm{Graph}(A)$ converging to some $(f, g) \in X \times Y$, that is, $f \in C^1[0, 1]$ and $g \in C[0, 1]$. Let $Af_n = g_n \in Y$. Since $(f_n, g_n) \to (f, g)$, we have that $f_n \to f$ in X and $g_n \to g$ in Y (this is a standard fact about the product topology on metric spaces). Since both X and Y are equipped with the sup-norm on $[0, 1]$, we have that $f_n \to f$ and $g_n \to g$ uniformly on $[0, 1]$. Hence, Lemma 8.1 applies and we get that

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} g_n(x) = g(x).$$

That is, $g = Af$, equivalently, $(f, g) \in \mathrm{Graph}(A)$. This shows that $\mathrm{Graph}(A)$ is closed.

We show that A is not continuous. Clearly A is linear (since taking the derivative is a linear operation). Consider, for positive integers $n \geqslant 1$, the functions $f_n(x) = x^n$. Then $f_n \in X = C^1[0, 1]$ and $g_n = Af_n \in Y$ are given by $g_n(x) = nx^{n-1}$. Obviously, $\|f_n\| = 1$ and $\|Af_n\| = n$, whence

$$\|A\| = \sup_{\|f\|\leqslant 1} \|Af\| \geqslant \sup_{n\in\mathbb{N}} n = \infty,$$

i.e., A is not bounded and hence not continuous.

9. PROBLEM 9

Define the map $T : V \to C(E)$ given by $Tf = f|_E$. Obviously, T is a linear map, for if $f, g \in V$ and $c \in \mathbb{K}$, then

$$T(f + cg) = (f + cg)|_E = f|_E + cg|_E = Tf + cTg.$$

Further, for any $g \in C(E)$, according to the hypothesis of the question, there is an $f \in V$ such that $g = f|_E$, whence $T : V \to C(E)$ is a surjective linear map. Finally, for any $f \in V$,

$$\|Tf\| = \|f|_E\| \leqslant \|f\|,$$

consequently, T is a bounded linear functional, that is, T is continuous.

LEMMA 9.1. Let $T : X \to Y$ be a surjective linear map between Banach spaces. Then there is a constant $c > 0$ such that for every $y \in Y$, there is an $x \in X$ with $\|x\| \leqslant c\|y\|$ such that $Tx = y$.

Proof. Due to the open mapping theorem, T is an open map, consequently, $T(B_X(0, 1))$ is an open set in Y containing 0. Thus, there is an $r > 0$ such that $B_Y(0, r) \subseteq T(B_X(0, 1))$. Let $y \in Y$. If $y = 0$, then set $x = 0$. If $y \neq 0$, then consider $z = \frac{ry}{2\|y\|}$, where $\|z\| < r$. Hence, there is a $w \in B_X(0, 1)$ such that $Tw = z$. Set $x = \frac{2\|y\|}{r} w \in X$ and note that

$$Tx = \frac{2\|y\|}{r} Tw = y,$$

and

$$\|x\| = \frac{2\|y\|}{r} \|w\| \leqslant \frac{2}{r} \|y\|.$$

Pick $c = \frac{2}{r}$. We have shown that for every $y \in Y$, there is an $x \in X$ with $\|x\| \leqslant c\|y\|$. ■

Since both V and $C(E)$ are Banach spaces, the conclusion follows immediately from Lemma 9.1.

10. Problem 10

Let $J : X \to X^{**}$ denote the canonical isometry given by $x \mapsto \mathrm{ev}_x$, the evaluation map at x. That this is indeed an isometry has been argued in class. Let $x^{**} \in X^{**}$ denote the image of $x \in X$ under the map J.

Let $f \in X^*$, then for any $x \in S$, we have

$$|x^{**}(f)| = |f(x)| \leqslant M_f$$

for some constant $M_f > 0$, since $f(S)$ is a bounded subset of $\mathbb{C}$ according to the hypothesis. Due to the *Uniform Boundedness Principle* (or Banach-Steinhaus Theorem), there is a constant $M > 0$ such that $\|x^{**}\| \leqslant M$ for all $x \in S$. Note that the theorem applies since X^* is a Banach space as we have argued in class. Finally, since J is an isometry, for every $x \in S$, we have

$$\|x\| = \|J(x)\| = \|x^{**}\| \leqslant M,$$

that is,

$$\sup \{\|x\| \colon x \in S\} \leqslant M < \infty,$$

as desired.

11. Problem 10

For each $x \in X$, since the sequence $(T_n(x))$ converges, it is bounded in Y, that is, there is an $M_x > 0$ such that $\|T_n x\| \leqslant M_x$ for all $n \geqslant 1$. Since X is Banach space, the *Uniform Boundedness Principle* applies and there is an $M > 0$ such that $\|T_n\| \leqslant M$ for all $n \geqslant 1$. In particular, this means that $\|T_n(x)\| \leqslant M\|x\|$ for all $x \in X$. As a result, for all $x \in X$,

$$\|T(x)\| = \left\|\lim_{n\to\infty} T_n(x)\right\| = \lim_{n\to\infty} \|T_n(x)\| \leqslant M\|x\|.$$

Finally, we show that T is a linear functional. Indeed, if $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$, then

$$T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty} \alpha T_n(x) + \beta T_n(y) = \alpha T(x) + \beta T(y).$$

This shows that $T : X \to Y$ is a bounded linear functional, as desired.

References

[Rud53] W. Rudin. *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1953.

[Rud87] W. Rudin. *Real and Complex Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, 1987.