MATH 591 Homework 4

Swayam Chube

swayamc@umich.edu

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§1 Problem 1

Using an easy first step from the proof of the Whitney embedding theorem, we may suppose that $M \subseteq \mathbb{R}^N$ is a submanifold. Let $F \colon M \to \mathbb{R}^N$ denote the inclusion, which has components $F = (F^1, \dots, F^N)$. If $N \leq 2m$, then we are already done. Suppose henceforth that $N > 2m$.

Since $T_p \mathbb{R}^N \cong \mathbb{R}^N$, and the latter has a standard Hermitian inner product on it, so does $T_p M \subseteq T_p \mathbb{R}^N$. We shall equip $T_p M$ with this inner product henceforth for all $p \in M$. In other words, we are now working with a Riemannian structure on M.

Let

$$UM = \left\{ (p,v) \in TM \subseteq M \times \mathbb{R}^N : \|v\| = 1 \right\}$$

denote the unit bundle. We shall first show that this is a submanifold of the tangent bundle. This is an application of the regular value theorem. Consider the map $\Phi \colon TM \to \mathbb{R}$ given by the composite map

$$TM \xrightarrow{\widetilde{F}} T\mathbb{R}^N \xrightarrow{(x,v) \mapsto \|v\|^2} \mathbb{R}.$$

We shall show that $1 \in \mathbb{R}$ is a regular value of Φ. Let $(p,v) \in \Phi^{-1}(1)$. Choose local coordinates $(x^1, \dots, x^m, r^1, \dots, r^m)$ on TM around (p,v). First note that in this local coordinate system, the representation of $\widetilde{F}$ is given by

$$\widetilde{F}(x^1, \dots, x^m, r^1, \dots, r^m) = \left(F^1(x), \dots, F^m(x), \sum_{i=1}^{m} \frac{\partial F^1}{\partial x^i} r^i, \dots, \sum_{i=1}^{m} \frac{\partial F^m}{\partial x^i} r^i \right).$$

Therefore, in these local coordinate system, Φ is represented by

$$\Phi(x^1, \dots, x^m, r^1, \dots, r^m) = \sum_{k=1}^{m} \left(\sum_{i=1}^{m} \frac{\partial F^k}{\partial x^i} r^i \right)^2.$$

Taking partials with respect to $r^1, \dots, r^m$, we have

$$\frac{\partial \Phi}{\partial r^j} = 2 \sum_{k=1}^{m} \frac{\partial F^k}{\partial x^j} \left(\sum_{i=1}^{m} \frac{\partial F^k}{\partial x^i} r^i \right).$$

Then

$$\sum_{j=1}^{m} r^j \frac{\partial \Phi}{\partial r^j} = 2 \sum_{j=1}^{m} \sum_{k=1}^{m} r^j \frac{\partial F^k}{\partial r^j} \left(\sum_{i=1}^{m} \frac{\partial F^k}{\partial x^i} r^i \right) = 2 \sum_{k=1}^{m} \left(\sum_{i=1}^{m} \frac{\partial F^k}{\partial r^i} r^i \right)^2.$$

Since $\Phi(p,v) = 1$, the above sum evaluated at (p,v) is precisely 2, which is easily seen from the local coordinate representation of $\widetilde{F}$ or Φ as computed above. In particular, this means that at last one of the partial derivatives $\frac{\partial \Phi}{\partial r^j}$ is non-zero. Thus the differential of Φ at (p,v) is surjective, and 1 is a regular value of Φ. Hence, due to the Regular Value Theorem, $UM = \Phi^{-1}(1)$ is a submanifold of M.

Now

§2 Problem 2

We may suppose that $M \subseteq \mathbb{R}^N$ is a submanifold for some $N > 0$. Since $T_p \mathbb{R}^N \cong \mathbb{R}^N$ naturally, there is a natural Hermitian inner product structure on $T_p \mathbb{R}^N$ which we denote by $\langle \cdot, \cdot \rangle$ – this inner product descends to $T_p M$.

For any $p \in M$, there is a natural isomorphism $T_p \mathbb{R}^N \to T_p^* \mathbb{R}^N$ given by $v \mapsto \langle v, \cdot \rangle$. Let $\iota \colon M \to \mathbb{R}^N$ denote the smooth inclusion. Then there is a commutative diagram

$$
\begin{array}{ccc}
T_p M & \xrightarrow{\ \iota_{*,p}\ } & T_p \mathbb{R}^N \\
\downarrow & & \downarrow \\
T_p^* M & \xleftarrow[\ \iota_{*,p}^*\ ]{} & T_p^* \mathbb{R}^N
\end{array}
$$

where the vertical arrows are the natural isomorphisms described above. The map $\iota_{*,p}$ splits canonically through the orthogonal projection $\pi_p \colon T_p \mathbb{R}^n \to T_p M$. We contend that the diagram

$$
\begin{array}{ccc}
T_p M & \xleftarrow{\ \pi_p\ } & T_p \mathbb{R}^N \\
\downarrow & & \downarrow \\
T_p^* M & \xleftarrow[\ \iota_{*,p}^*\ ]{} & T_p^* \mathbb{R}^N
\end{array}
$$

commutes. Indeed, if $v \in T_p \mathbb{R}^N$, we can write $v = \pi_p(v) + v^\perp$, where $v^\perp$ is in the orthogonal complement of $T_p M$ in $T_p \mathbb{R}^N$. Under the vertical isomorphism $T_p M \to T_p^* M$, $\pi_p(v)$ maps to the linear functional $\langle \pi_p(v), \cdot \rangle$. On the other hand, $v \in T_p \mathbb{R}^n$ maps to the linear functional $\langle v, \cdot \rangle$ in $T_p^* \mathbb{R}^N$, which maps to $\langle \pi_p(v), \iota_{*,p}(\cdot) \rangle$ in $T_p^* M$. Now note that for any $u \in T_p^* M$,

$$\langle v, u \rangle = \langle \pi_p(v) + v^\perp, u \rangle = \langle \pi_p(v), u \rangle = \langle \pi_p(v), \iota_{*,p}(u) \rangle,$$

whence the diagram commutes. Now, using the above diagram, note that $df_p \in T_p^* \mathbb{R}^N$ maps to $df_p \in T_p^* M$ under $\iota_{*,p}^*$. If $r^1, \dots, r^N$ denotes the standard coordinate system on $\mathbb{R}^N$, then the basis $\left\{ dr_p^1, \dots, dr_p^N \right\}$ of $T_p^* \mathbb{R}^N$ is dual to the basis $\left\{ \frac{\partial}{\partial r^1}\big|_p, \dots, \frac{\partial}{\partial r^N}\big|_p \right\}$ with respect to the inner product $\langle \cdot, \cdot \rangle$, where

$$dr_p^i \left(\frac{\partial}{\partial r^j}\bigg|_p \right) = \delta_{ij}.$$

Therefore, $(df_{\mathbf{a}})_p = a^1 dr_p^1 + \cdots + a^N dr_p^N \in T_p^* \mathbb{R}^N$ corresponds to

$$a^1 \frac{\partial}{\partial r^1}\bigg|_p + \cdots + a^N \frac{\partial}{\partial r^N}\bigg|_p$$

in $T_p \mathbb{R}^N$ under the vertical isomorphism. When we identify $T_p M$ with the subspace $\iota_{*,p}(T_p M)$ of $T_p \mathbb{R}^N$, which in turn is identified with $\mathbb{R}^N$, we see that π_p sends the above to $\pi_p(\mathbf{a}) \in T_p M$. By the commutativity of the diagram, $\pi_p(\mathbf{a})$ corresponds to $\iota_{*,p}^* \left((df_{\mathbf{a}})_p \right) = (df_{\mathbf{a}})_p \in T_p^* M$ under the vertical isomorphism.

Thus, by identifying $T^* M$ with TM through the Riemannian metric, the given map $F \colon \mathbb{R}^N \times M \to T^* M$ becomes the map $F \colon \mathbb{R}^N \times M \to TM$ given by

$$F(\mathbf{a}, p) = \left(p, \pi_p(\mathbf{a}) \right) \in TM = \left\{ (p, v) \colon p \in M,\ v \in T_p M \subseteq T_p \mathbb{R}^N \cong \mathbb{R}^N \right\}.$$

We claim that F is transversal to the zero section $Z \colon M \to TM$. Indeed, suppose $F(\mathbf{a}, p) \in Z$, so that $\pi_p(\mathbf{a}) = 0$, i.e., $\mathbf{a}$ is perpendicular to $T_p M \subseteq \mathbb{R}^N$.

Choose local coordinates $(r^1, \dots, r^N)$ on $\mathbb{R}^N$ around $\mathbf{a}$ and $(x^1, \dots, x^m)$ on M around p. These also give rise to local coordinates on TM as a subspace of $M \times \mathbb{R}^N$, where we choose the "same" local coordinates. Then the

differential of F is given by

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
| & | & \cdots & | & & & & \\
\pi_p(e_1) & \pi_p(e_2) & \cdots & \pi_p(e_N) & & & & \\
| & | & \cdots & | & & & &
\end{pmatrix}_{2m \times (N+m)}
$$

where $e_1,\ldots,e_N$ are the standard basis vectors of $\mathbb{R}^N$, and the blank space in the matrix is irrelevant. The top right $m \times m$ block is an identity matrix, while the bottom left $N \times N$ block consists of the column vectors $\pi_p(e_1),\ldots,\pi_p(e_N)$. Since π_p is a surjective linear map and $e_1,\ldots,e_N$ constitute a basis of $\mathbb{R}^N$, their images span T_pM, in particular, span an m-dimensional vector space. On the other hand, the right most m column vectors constituting the identity matrix are linearly independent, from the first m columns and linearly independent of each other. Therefore, the column rank of this matrix is $2m$. It follows that $F_{*,(\mathbf{a},p)}$ is transversal to Z.

Hence, by Thom's parametric transversality theorem, for almost all $\mathbf{a} \in T_pM$, the specialization $F_{\mathbf{a}}$ is transverse to the zero section. Identifying back T^*M with TM, we see that for almost all $\mathbf{a} \in \mathbb{R}^N$, the map $F_{\mathbf{a}}$, which is the section $s_{f_{\mathbf{a}}} : M \to T^*M$ is transverse to the zero section, so that $f_{\mathbf{a}}$ is a Morse function.

§3 Problem 3

§§ Part (a)

For any $p = (p^1,\ldots,p^n) \in \mathbb{R}^n$,

$$
X_p^A = \begin{pmatrix} p^1 & \cdots & p^n \end{pmatrix}
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
\left.\frac{\partial}{\partial x^1}\right|_p \\
\vdots \\
\left.\frac{\partial}{\partial x^n}\right|_p
\end{pmatrix},
$$

therefore, for any $f \in C^\infty(\mathbb{R}^n)$,

$$
\left(X^A f\right)(p) = \begin{pmatrix} p^1 & \cdots & p^n \end{pmatrix}
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f}{\partial x^1}(p) \\
\vdots \\
\frac{\partial}{\partial x^n}(p)
\end{pmatrix}
= \sum_{i,j=1}^{n} p^i a_{ij} \frac{\partial f}{\partial x^j}(p).
$$

Now suppose $b = (b_{ij})$ is another $n \times n$ matrix, then

$$
\begin{aligned}
\left(X^A X^B f\right)(p) &= \sum_{i,j=1}^{n} p^i a_{ij} \frac{\partial}{\partial x^j} \left(\sum_{k,l=1}^{n} x^k b_{kl} \frac{\partial f}{\partial x^l} \right)(p) \\
&= \sum_{i,j,k,l=1}^{n} p^i a_{ij} b_{kl} \left(\delta_{jk} \frac{\partial f}{\partial x^l}(p) + p^k \frac{\partial^2 f}{\partial x^j \partial x^l}(p) \right) \\
&= \sum_{i,j,k,l=1}^{n} \delta_{jk} p^i a_{ij} b_{kl} \frac{\partial f}{\partial x^l}(p) + \sum_{i,j,k,l=1}^{n} p^i p^k a_{ij} b_{kl} \frac{\partial^2 f}{\partial x^i \partial x^l}(p).
\end{aligned}
$$

Similarly,

$$
\left(X^B X^A f\right)(p) = \sum_{i,j,k,l=1}^{n} \delta_{jk} p^i b_{ij} a_{kl} \frac{\partial f}{\partial x^l}(p) + \sum_{i,j,k,l=1}^{n} p^i p^k b_{ij} a_{kl} \frac{\partial^2 f}{\partial x^i \partial x^l}(p).
$$

Note that

$$
\sum_{i,j,k,l=1}^{n} p^i p^k a_{ij} b_{kl} \frac{\partial^2 f}{\partial x^i \partial x^l}(p) = \sum_{i,j,k,l=1}^{n} p^i p^k b_{ij} a_{kl} \frac{\partial^2 f}{\partial x^i \partial x^l}(p),
$$

3

which can be seen by interchanging $i \leftrightarrow k$ and $j \leftrightarrow l$ in the indexing. Therefore,

$$\left([X^A,X^B]f\right)(p) = \left(\sum_{i,j,l=1}^n p^i a_{ij} b_{jl} - p^i b_{ij} a_{jl}\right) \frac{\partial f}{\partial x^l}(p) = \sum_{i,l=1}^n p^i \left(\sum_{j=1}^n a_{ij} b_{jl} - b_{ij} a_{jl}\right) \frac{\partial f}{\partial x^l}(p) = X^{[A,B]}f(p),$$

since the (i,l)-th entry of $[A,B]$ is

$$\sum_{j=1}^n a_{ij} b_{jl} - b_{ij} a_{jl}.$$

§§ Part (b)

Let $\gamma \colon \mathbb{R} \to \mathbb{R}^n$ be given by $\gamma(t) = e^{tA}$. Recall the power-series representation

$$\gamma(t) = \sum_{k=0}^\infty \frac{1}{k!} t^k A^k,$$

and since this converges uniformly on compacta, its derivative is given by

$$\gamma'(t) = \sum_{k=1}^\infty \frac{1}{(k-1)!} t^{k-1} A^k = A e^{tA} = e^{tA} A.$$

Suppose now that v is a vector in $\mathbb{R}^n$, and consider the curve $\gamma_v(t) = v e^{tA} = v\gamma(t)$. Then

$$\gamma_v'(t) = v\gamma'(t) = v e^{tA} A = \gamma_v(t) A = X^A_{\gamma_v(t)},$$

which shows that γ_v is an integral curve of X^A for all vectors v.

§4 Problem 4

We quote the following result from [Lee12, Lemma 5.34], which is a simple application of partitions of unity:

LEMMA 4.1. Suppose M is a smooth manifold, $S \subseteq M$ a smooth submanifold, and $f \in C^\infty(S)$.

(a) If S is embedded, then there exists a neighborhood U of S in M and a smooth function $\widetilde{f} \in C^\infty(U)$ such that $\widetilde{f}|_S = f$.

(b) If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M.

LEMMA 4.2. Let $\iota \colon S \to M$ denote the smooth embedding. For any $X \in \mathfrak{X}_S(M)$, there is a unique $Y \in \mathfrak{X}(M)$ such that X is ι-related to Y.

Proof. Let Y denote the *apriori* rough vector field on S such that $\iota_{*,p}(X_p) = Y_p$ for all $p \in S$. Since each $\iota_{*,p}$ is injective, and X is tangent to S, such a rough vector field Y exists. It remains to show that Y is smooth. To this end, it suffices to show that for every $f \in C^\infty(S)$, $Yf \in C^\infty(S)$. In view of Lemma 4.1, there is a neighborhood U of S in M, and a smooth function $F \in C^\infty(U)$ such that $F|_S = f$. Since U is an open subset of M, the restriction of the smooth vector field X to U is still a smooth vector field. By abuse of notation, we shall continue to denote this vector field by X.

Let $p \in S$, then there is a curve $\gamma \colon (-\varepsilon,\varepsilon) \to S \subseteq M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Then, by definition,

$$Y_p([f]) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t)) = X_p([F]),$$

where the first equality follows since γ is completely contained in S. Therefore, $Yf = XF|_S$. Now since $\iota \colon S \to U$ is smooth, so is Yf. This shows that Y is a smooth vector field on S.

Finally, the uniqueness of such a Y follows from the fact that $\iota_{*,p}$ is injective, and $X_p \in \iota_{*,p}(T_pS)$ for all $p \in S$. $\blacksquare$

§§ Part (a)

Let $X_1, X_2 \in \mathfrak{X}_S(M)$, then by Lemma 4.2, there are unique $Y_1, Y_2 \in \mathfrak{X}(S)$ such that Y_i is ι-related to X_i for $i = 1, 2$. As we have seen in class, $[Y_1, Y_2]$ is ι-related to $[X_1, X_2]$, and therefore, by definition, $[X_1, X_2] \in \mathfrak{X}_S(M)$.

§§ Part (b)

We note that for each $X \in \mathfrak{X}_S(M)$, $\rho(X)$ is the *unique* $Y \in \mathfrak{X}(S)$ such that Y is ι-related to X. In our proof of part (a), we argued that if $X_1, X_2 \in \mathfrak{X}_S(M)$, and $Y_i = \rho(X_i)$ for $i = 1, 2$, then $[Y_1, Y_2]$ is ι-related to $[X_1, X_2]$, so that $\rho([X_1, X_2]) = [Y_1, Y_2] = [\rho(X_1), \rho(X_2)]$, i.e., ρ is a Lie algebra morphism.

§§ Part (c)

Suppose there was a surface $S \subseteq \mathbb{R}^3$ such that $X, Y \in \mathfrak{X}_S(\mathbb{R}^3)$. Then due to part (b), $[X, Y] \in \mathfrak{X}_S(\mathbb{R}^3)$, now note that

$$
\begin{aligned}
Z := [X, Y] &= \left[\frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \right] \\
&= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] + \left[y \frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right] - \left[\frac{\partial}{\partial x}, x \frac{\partial}{\partial z} \right] - \left[y \frac{\partial}{\partial z}, x \frac{\partial}{\partial z} \right] \\
&= 0 - \frac{\partial}{\partial z} - \frac{\partial}{\partial z} - 0 \\
&= -2 \frac{\partial}{\partial z}.
\end{aligned}
$$

For any $p \in S$, we have that

$$
\left. \frac{\partial}{\partial x} \right|_p + y(p) \left. \frac{\partial}{\partial z} \right|_p, \left. \frac{\partial}{\partial y} \right|_p - x(p) \left. \frac{\partial}{\partial z} \right|_p, -2 \left. \frac{\partial}{\partial z} \right|_p \in T_p S \subseteq T_p \mathbb{R}^3.
$$

Since $x(p), y(p) \in \mathbb{R}$, taking $\mathbb{R}$-linear combinations, of X_p and Z_p, and Y_p and Z_p, we have that

$$
\left. \frac{\partial}{\partial x} \right|_p, \left. \frac{\partial}{\partial y} \right|_p, \left. \frac{\partial}{\partial z} \right|_p \in T_p S \subseteq T_p \mathbb{R}^3.
$$

Since these three tangent vectors constitute a basis of $T_p \mathbb{R}^3$, we have that $T_p S = T_p \mathbb{R}^3$, which is absurd, since $\dim T_p S = \dim S = 2$ and $\dim T_p \mathbb{R}^3 = 3$. Thus X and Y are not tangent to any surface in $\mathbb{R}^3$.

References

[Lee12] J. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, 2012.