Homological methods in Commutative Algebra

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§1 Regular Sequences

§§ Regular sequences and the Koszul complex

Definition 1.1. Let A be a ring and M an A-module. An element $a \in A$ is said to be *M-regular* if a is a non zero-divisor on M. A sequence $a_1, \dots, a_n$ of elements of A is an *M-sequence* if

(1) Each a_i is $M/(a_1, \dots, a_{i-1})M$-regular.

(2) $M \neq (a_1, \dots, a_n)M$.

Definition 1.2. Let A be a ring and $x_1, \dots, x_n \in A$. We define a complex $K_\bullet$ by setting $K_0 = A$, $K_p = 0$ for $p > n$ or $p < 0$, and

$$K_p = \bigoplus_{1 \leqslant i < \cdots < i_p \leqslant n} A e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

For $1 \leqslant p \leqslant n$, define $K_p \to K_{p-1}$ by

$$d\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{i=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_r} \wedge \cdots \wedge e_{i_p},$$

and extend linearly to K_p. This is known as the *Koszul complex*.

Proposition 1.3. The Koszul complex is indeed a complex.

Proof. $d \circ d : K_1 \to K_{-1}$ is obviously the zero map. Now, let $p \geqslant 2$, we shall show that $(d \circ d)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = 0$. Note that the above can be written as a linear combination of the basis elements of K_{p-2}. Consider the basis element $e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_a} \wedge \cdots \wedge \widehat{e}_{i_b} \wedge \cdots \wedge e_{i_p}$. We shall show that its coefficient is 0.

Indeed, its coefficient is contributed by

$$e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_a} \wedge \cdots \wedge e_{i_p} \quad \text{and} \quad e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_b} \wedge \cdots \wedge e_{i_p},$$

each of which has coefficient $(-1)^{a-1}x_{i_a}$ and $(-1)^{b-1}x_{i_b}$ respectively. The coefficient of our desired basis element in the differential of the first is $(-1)^{b-2}x_{i_b}$ and in the second is $(-1)^{a-1}x_{i_a}$. Thus, the coefficient of our desired basis element in the differential of $e_{i_1} \wedge \cdots \wedge e_{i_p}$ is

$$(-1)^{a-1}x_{i_a}(-1)^{b-2}x_{i_b} + (-1)^{b-1}x_{i_b}(-1)^{a-1}x_{i_a} = 0,$$

thereby completing the proof. ■

DEFINITION 1.4. Let $C_\bullet$ and $D_\bullet$ be complexes of A-modules. Define their *tensor product* $(C \otimes D)_\bullet$ by

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes_A D_j.$$

The boundary maps are given by $d : (C \otimes D)_n \to (C \otimes D)_{n-1}$

$$d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy \qquad x \in C_i,\ y \in C_j.$$

PROPOSITION 1.5. There is an isomorphism of complexes $(C \otimes D)_\bullet \cong (D \otimes C)_\bullet$.

Proof. If $x \otimes y \in (C \otimes D)_n$ with $x \in C_i$ and $y \in D_j$, then send this element to $(-1)^{ij} y \otimes x \in (D \otimes C)_n$. It is not hard to check that this is indeed a chain map. That this is an isomorphism of chain complexes follows from the fact that for every n, $(C \otimes D)_n \to (D \otimes C)_n$ is an isomorphism. ■

PROPOSITION 1.6. Let $x_1, \ldots, x_n \in A$. Then $K_\bullet(x_1, \ldots, x_n) \cong K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$ as complexes.

Proof. We prove this by induction on n. The base case with $n = 1$ is tautological. Suppose now that $n \geqslant 1$. We shall show that $K_\bullet(x_1, \ldots, x_n) \otimes K_\bullet(x_{n+1}) \cong K_\bullet(x_1, \ldots, x_{n+1})$. Write the complex $K_\bullet(x_{n+1})$ as

$$0 \longrightarrow Ae_{n+1} \xrightarrow{e_{n+1} \mapsto x_{n+1}} A \longrightarrow 0.$$

Then, $(K(x_1, \ldots, x_n) \otimes K(x_{n+1}))_p = \big(K_p(x_1, \ldots, x_n) \otimes A\big) \oplus \big(K_{p-1}(x_1, \ldots, x_n) \otimes Ae_{n+1}\big)$. There is a natural isomorphism

$$(K(x_1, \ldots, x_n) \otimes K(x_{n+1}))_p \longrightarrow K_p(x_1, \ldots, x_{n+1}),$$

which sends $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes 1$ to $e_{i_1} \wedge \cdots \wedge e_{i_p}$ in $K_p(x_1, \ldots, x_n)$, and sends $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes e_{n+1}$ to $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \wedge e_{n+1}$ in $K_p(x_1, \ldots, x_{n+1})$.

It remains to check that the map defined above is indeed a chain map. Indeed, under the differential in the tensor complex, $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes 1$ maps to $d(e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes 1$, which maps to $e(e_{i_1} \wedge \cdots \wedge e_{i_p})$ under the aforementioned isomorphism. On the other hand, the starting element maps to $e_{i_1} \wedge \cdots \wedge e_{i_p}$ under the isomorphism first and then maps to $d(e_{i_1} \wedge \cdots \wedge e_{i_p})$ under the differential.

Next, if we begin with $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes e_{n+1}$, then under the differential, it maps to

$$d(e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}) \otimes e_{n+1} + (-1)^{p-1} x_{n+1} e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes 1,$$

which maps to

$$d(e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}) \wedge e_{n+1} + (-1)^{p-1} x_{n+1} e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}$$

under the isomorphism. On the other hand, the starting element maps to $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \wedge e_{n+1}$ under the isomorphism, which maps to the above element under the differential. This completes the proof. ■

DEFINITION 1.7. Let $\underline{x} = x_1, \dots, x_n$ be a sequence in A. For an A-module M, set

$$K_\bullet(\underline{x}, M) = K(\underline{x}) \otimes M.$$

The homology groups of this complex are denoted by $H_p(\underline{x}, M)$. Similarly, for a complex $C_\bullet$ of A-modules, set $C_\bullet(\underline{x}) = C_\bullet \otimes K_\bullet(\underline{x})$.

PROPOSITION 1.8. Let $\underline{x} = x_1, \dots, x_n$ be a sequence in A. Then

$$H_0(\underline{x}, M) = M/(\underline{x})M \qquad H_n(\underline{x}, M) \cong \{\xi \in M : x_1\xi = \cdots = x_n\xi = 0\}.$$

Proof. The assertion about $H_0(\underline{x}, M)$ is trivial. $H_n(\underline{x}, M)$ is precisely the kernel of the map $K_n(\underline{x}, M) \to K_{n-1}(\underline{x}, M)$, which is given by

$$\xi e_1 \wedge \cdots \wedge e_n \longmapsto \sum_{i=1}^{n} (-1)^{i-1} x_i \xi e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n,$$

where $\xi e_{i_1} \wedge \cdots \wedge e_{i_p} \in K_p(\underline{x}, M)$ is shorthand for $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \xi \in K_p(\underline{x}, M)$.

The right hand side of the above equation is zero if and only if each $x_i\xi$ is zero, whence the conclusion follows. ■

THEOREM 1.9. Let $C_\bullet$ be a complex of A-modules and $x \in A$. Then, there is a short exact sequence of complexes

$$0 \to C_\bullet \to C_\bullet(x) \to C'_\bullet \to 0,$$

where $C'_{p+1} = C_p$ is the (upward) shift of the complex $C_\bullet$. The homology long exact sequence so obtained looks like

$$\cdots \to H_p(C_\bullet) \to H_p(C_\bullet(x)) \to H_{p-1}(C_\bullet) \xrightarrow{(-1)^{p-1}x} H_{p-1}(C_\bullet) \to \cdots.$$

Further, we have $x \cdot H_p(C_\bullet(x)) = 0$ for all $p \in \mathbb{Z}$.

Proof. Denote the Koszul complex $K_\bullet(x)$ by

$$\cdots \to 0 \to Ae_1 \xrightarrow{e_1 \mapsto x} A \to 0.$$

Thus, we can identify $C_\bullet(x)$ with $C_p \oplus C_{p-1}$ with the boundary map as

$$d(\xi, \eta) = (d\xi + (-1)^{p-1} x\eta, d\eta) \in C_{p-1} \oplus C_{p-2}.$$

Hence, we have a short exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & C_p \arrow[r] \arrow[d] & C_{p-1} \arrow[r] \arrow[d] & \cdots. \\
\cdots \arrow[r] & C_p \oplus C_{p-1} \arrow[r] \arrow[d] & C_{p-1} \oplus C_{p-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{p-1} \arrow[r] & C_{p-2} \arrow[r] & \cdots
\end{tikzcd}$$

That the above commutes is straightforward. It remains to compute the boundary map from $H_{p-1}(C_\bullet) = H_p(C'_\bullet)$ to $H_{p-1}(C_\bullet)$.

Choose a cycle $\eta \in C'_p = C_{p-1}$, that is, $d\eta = 0$. This lifts to $(0,\eta) \in C_p \oplus C_{p-1}$, which maps to $\big((-1)^{p-1}x\eta, 0\big) \in C_{p-1} \oplus C_{p-2}$, which again lifts to $(-1)^{p-1}x\eta$ in C_{p-1}, which is a cycle in C_{p-1}. Hence, the induced map on homologies is multiplication by $(-1)^{p-1}x$.

Finally, we must show that x annihilates $H_p(C_\bullet(x))$ for all p. Choose a cycle $(\xi,\eta) \in C_p \oplus C_{p-1}$, that is, $d\eta = 0$, and $d\xi = (-1)^p x\eta$. Hence,

$$C_{p+1} \ni d(0,(-1)^p\xi) = \big((-1)^p x\xi, (-1)^p d\xi\big) = x \cdot (\xi,\eta).$$

Thus, x annihilates $[(\xi,\eta)] \in H_p(C_\bullet(x))$, whence annihilates all of $H_p(C_\bullet(x))$. ■

COROLLARY 1.10. Let $\underline{x} = x_1,\ldots,x_n$ be a sequence in A. Then $(\underline{x})$ annihilates $H_p(\underline{x},M)$ for every $p \in \mathbb{Z}$.

Proof. It suffices to show that x_n annihilates $H_p(\underline{x},M)$ since the Koszul complex is invariant under permutation of the sequence $\underline{x}$. But this is obvious, since $K_\bullet(\underline{x},M)$ is isomorphic to $K_\bullet(x_1,\ldots,x_{n-1},M) \otimes K_\bullet(x_n)$ due to the commutativity of tensor products of complexes. We are done by invoking the preceding theorem with $C_\bullet = K_\bullet(x_1,\ldots,x_{n-1},M)$ and $x = x_n$. ■

THEOREM 1.11. Let A be a ring, M an A-module, and $x_1,\ldots,x_n$ an M-sequence. Then

$$H_p(\underline{x},M) = 0 \quad \forall\ p > 0, \qquad H_0(\underline{x},M) = M/(\underline{x})M.$$

Proof. Induct on n. The base case with $n = 1$ follows from the fact that $H_1(x_1,M) = (0 :_M x_1) = 0$, since x_1 is M-regular. Now, suppose $n > 1$. If $p > 1$, then there is an exact sequence furnished by Theorem 1.9 by taking $C_\bullet = K_\bullet(x_1,\ldots,x_{n-1},M)$ and $x = x_n$:

$$0 = H_p(x_1,\ldots,x_{n-1},M) \longrightarrow H_p(x_1,\ldots,x_n,M) \longrightarrow H_{p-1}(x_1,\ldots,x_{n-1},M) = 0,$$

whence $H_p(\underline{x},M) = 0$. It remains to establish that $H_1(\underline{x},M) = 0$. Set $M_i = M/(x_1,\ldots,x_i)M$ with the convention that $M_0 = M$. The above long exact sequence again furnishes

$$0 = H_1(x_1,\ldots,x_{n-1},M) \to H_1(\underline{x},M) \to H_0(x_1,\ldots,x_{n-1},M) = M_{n-1} \xrightarrow{x_n} M_{n-1}.$$

But since x_n is a non zero-divisor on M_{n-1}, we see that $H_1(\underline{x},M) = 0$ as desired. ■

THEOREM 1.12. Suppose one of the following two conditions holds:

(α) $(A,\mathfrak{m})$ is a Noetherian local ring, $x_1,\ldots,x_n \in \mathfrak{m}$, and M is a finite A-module.

(β) A is an $\mathbb{N}$-graded ring, M is an $\mathbb{N}$-graded A-module, and $x_1,\ldots,x_n$ are homogeneous elements of positive degree.

Then, if $H_1(\underline{x},M) = 0$ and $M \neq 0$, then $x_1,\ldots,x_n$ is an M-sequence.

Proof. Induction on n. If $n = 1$, then $0 = H_1(x_1,M) = (0 :_M x_1)$, whence x_1 is a non zero-divisor on M. Now suppose $n > 1$. Again, we make use of the exact sequence associated with $K_\bullet(x_1,\ldots,x_{n-1},M) \otimes K_\bullet(x_n)$ to get

$$H_1(x_1,\ldots,x_{n-1},M) \xrightarrow{-x_n} H_1(x_1,\ldots,x_{n-1},M) \to H_1(\underline{x},M) = 0.$$

But since $H_i(x_1,\ldots,x_{n-1},M)$ is a finite A-module in case (α) or a $\mathbb{N}$-graded module in case (β), the above surjection implies, due to Nakayama, that $H_1(x_1,\ldots,x_{n-1},M) = 0$. The induction hypothesis then implies $x_1,\ldots,x_{n-1}$ is an M-sequence.

Now, continuing the above long exact sequence, we get

$$0 = H_1(\underline{x},M) \longrightarrow H_0(x_1,\ldots,x_{n-1},M) = M_{n-1} \xrightarrow{x_n} M_{n-1},$$

where $M_{n-1} = M/(x_1,\ldots,x_{n-1})M$. The above sequence implies x_n is M_{n-1}-regular, whence $x_1,\ldots,x_n$ is an M-sequence, as desired. ■

THEOREM 1.13. Let A be a Noetherian ring, M a finite A-module, and I an ideal of A such that $M \neq IM$. For a given integer $n > 0$, the following conditions are equivalent:

(1) $\mathrm{Ext}^i_A(N,M) = 0$ for all $i < n$ and for any finite A-module N with $\mathrm{Supp}(N) \subseteq V(I)$.

(2) $\mathrm{Ext}^i_A(A/I,M) = 0$ for all $i < n$.

(3) $\mathrm{Ext}^i_A(N,M) = 0$ for all $i < n$ and for some finite A-module N with $\mathrm{Supp}(N) = V(I)$.

(4) There exists an M-sequence of length n contained in I.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (4) First, we show that I contains an M-regular element. Suppose not, then due to prime avoidance, I must be contained in some associated prime $\mathfrak{p} \in \mathrm{Ass}_A(M)$. Thus, there is an injective map $A/\mathfrak{p} \hookrightarrow M$, which upon localizing at $\mathfrak{p}$, we see that $\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \neq 0$. Now, $\mathfrak{p} \in V(I) = \mathrm{Supp}(N)$, whence $N_\mathfrak{p} \neq 0$, and hence, due to Nakayama's lemma, $N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \neq 0$ (since $N_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module). Then, $N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p}$ is a non-zero $\kappa(\mathfrak{p})$-vector space, and consequently, $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p}, \kappa(\mathfrak{p})) \neq 0$ (choose a basis and project onto a coordinate). Now, we can form the composition

$$N_\mathfrak{p} \to N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \to \kappa(\mathfrak{p}) \hookrightarrow M_\mathfrak{p}.$$

The first two maps are surjections and hence, the composition is non-zero. It follows that $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0$. Since N is finite over a Noetherian ring, we have

$$(\mathrm{Hom}_A(N,M))_\mathfrak{p} = \mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0,$$

whence $\mathrm{Ext}^0_A(N,M) = \mathrm{Hom}_A(N,M) \neq 0$, a contradiction to (3). Hence, I contains an M-regular element, say f. If $n = 1$, then we are already done. If $n > 1$, then set $M_1 = M/fM$ and consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M_1 \to 0.$$

The long exact sequence using $\mathrm{Ext}_A(N,-)$ gives

$$\cdots \to \mathrm{Ext}^{i-1}_A(N,M) \xrightarrow{f} \mathrm{Ext}^{i-1}_A(N,M) \to \mathrm{Ext}^{i-1}_A(N,M_1) \to \mathrm{Ext}^i_A(N,M) \to \cdots.$$

For $1 \leqslant i < n$, this implies $\mathrm{Ext}^{i-1}_A(N,M_1) = 0$, and due to the induction hypothesis, there is an M_1-sequence $f_2,\ldots,f_n$ in I. Thus, $f_1,\ldots,f_n$ is an M-sequence in I.

(4) $\Rightarrow$ (1). Induction on n. We shall deal with the base case later. Suppose $n > 1$. Let $\underline{x} = x_1,\ldots,x_n$ be an M-sequence in I. Set $M_1 = M/x_1M$ which fits into a short exact sequence $0 \to M \xrightarrow{x_1} M \to M_1 \to 0$. The sequence $x_2,\ldots,x_n$ is an M_1-sequence in I, whence due to the inductive hypothesis, $\mathrm{Ext}^i_A(N,M_1) = 0$ for all $i < n-1$. The long exact sequence corresponding to $\mathrm{Ext}_A(N,-)$ gives us

$$0 = \mathrm{Ext}^{i-1}_A(N,M_1) \to \mathrm{Ext}^i_A(N,M) \xrightarrow{x_1} \mathrm{Ext}^i(N,M)$$

for all $0 \leqslant i < n$, with the convention that $\mathrm{Ext}^{-1}(N,M_1) = 0$. But note that $\mathrm{Ext}^i_A(N,-)$ is annihilated by $\mathrm{Ann}_A(N)$. But since $\mathrm{Supp}(N) = V(\mathrm{Ann}_A(N)) \subseteq V(I)$, we conclude that $I \subseteq \sqrt{I} \subseteq \sqrt{\mathrm{Ann}_A(N)}$. In particular, a sufficiently large power of x_1 annihilates N, whence, annihilates $\mathrm{Ext}^i_A(N,M)$. But since multiplication by x_1 is injective, we must have that $\mathrm{Ext}^i_A(N,M) = 0$ for $i < n$, thereby completing the proof. ■

THEOREM 1.14. Let A be a Noetherian ring, I an ideal of A, and M a finite A-module such that $M \neq IM$. Then the length of any maximal M-sequence contained in I is the same, say n, and n is determined by

$$\mathrm{Ext}^i_A(A/I,M) = 0 \quad \forall\ i < n \quad \text{and} \quad \mathrm{Ext}^n_A(A/I,M) \neq 0.$$

We write $n = \mathrm{depth}(I,M)$ and call n the *I-depth* of M.

Proof. Let $\underline{a} = a_1,\dots,a_n$ be a maximal M-sequence in I. Suppose $\mathrm{Ext}^n_A(A/I,M) = 0$. Define $M_i = M/(a_1,\dots,a_i)M$. Using the short exact sequence $0 \to M \xrightarrow{a_1} M \to M_1 \to 0$, we have an exact sequence

$$0 = \mathrm{Ext}^{n-1}_A(A/I,M) \to \mathrm{Ext}^{n-1}_A(A/I,M_1) \to \mathrm{Ext}^n_A(A/I,M) = 0,$$

whence $\mathrm{Ext}^{n-1}_A(A/I,M_1) = 0$; and since $a_2,\dots,a_n$ is an M_1-sequence, $\mathrm{Ext}^i_A(A/I,M_1) = 0$ for $i < n-1$. Arguing similarly, we get that $\mathrm{Ext}^0_A(A/I,M_n) = 0$. Due to the preceding theorem, I must contain an M_n-regular element, contradicting the maximality of $\underline{a}$. Thus, $\mathrm{Ext}^n_A(A/I,M) \neq 0$ and $\mathrm{Ext}^i_A(A/I,M) = 0$ for $i < n$.

On the other hand, if $\underline{b} = b_1,\dots,b_m$ is a maximal M-sequence, then due to the above paragraph, $\mathrm{Ext}^m_A(A/I,M) \neq 0$ and $\mathrm{Ext}^i_A(A/I,M) = 0$ for $i < m$. In particular, this means that $m = n$.

Finally, suppose n satisfies the conditions given in the theorem. Then, due to the preceding theorem, there is an M-sequence $\underline{a} = a_1,\dots,a_n$ in I. Further, since $\mathrm{Ext}^n_A(A/I,M) \neq 0$, this sequence must be maximal, else it could be extended and again, due to the preceding theorem $\mathrm{Ext}^n_A(A/I,M) = 0$. This completes the proof. ■

REMARK 1.15. The above theorem can be phrased more succinctly as

$$\mathrm{depth}(I,M) = \inf\left\{i :\ \mathrm{Ext}^i_A(A/I,M) \neq 0\right\}.$$

In particular, if $(A,\mathfrak{m},k)$ is a Noetherian local ring, then we write $\mathrm{depth}(\mathfrak{m},M)$ as $\mathrm{depth}\, M$ and

$$\mathrm{depth}\, M = \inf\left\{i :\ \mathrm{Ext}^i_A(k,M) \neq 0\right\}.$$

THEOREM 1.16 (DEPTH SENSITIVITY OF KOSZUL COMPLEX). Let A be a Noetherian ring, $I = (y_1,\dots,y_n)$ an ideal of A, and M a finite A-module such that $M \neq IM$. If

$$q = \sup\left\{i : H_i(\underline{y},M) \neq 0\right\},$$

then $\mathrm{depth}(I,M) = n - q$.

Proof. We shall argue by induction on $s = \mathrm{depth}(I,M)$. If $s = 0$, then every element of I is a zero-divisor on M, whence by prime avoidance, there is an associated prime $\mathfrak{p} \in \mathrm{Ass}_A(M)$ such that $I \subseteq \mathfrak{p}$. By definition, there is some $0 \neq \xi \in M$ such that $\mathfrak{p} = \mathrm{Ann}_A(\xi)$, and hence, $I\xi = 0$. Recall that $H_n(\underline{y},M) = (0 :_M (\underline{y})) = (0 :_M I) \neq 0$, since it contains ξ. Thus, $q = n$.

Now, suppose $s > 0$, then $H_n(\underline{y},M) = 0$, since some element of I is a non zero-divisor on M. In particular, this means $q < n$. Let $\underline{x} = x_1,\dots,x_s$ be a maximal M-sequence in I. There is a short exact sequence $0 \to M \xrightarrow{x_1} M \to M_1 \to 0$, where $M_1 = M/x_1M$. Since every element in the Koszul comples $K_\bullet(\underline{y})$ is a free module, tensoring with the above short exact sequence will give a short exact sequence of complexes

$$0 \to K_\bullet(\underline{y},M) \xrightarrow{x_1} K_\bullet(\underline{y},M) \to K_\bullet(\underline{y},M_1) \to 0.$$

The associated long exact sequence looks like

$$H_i(\underline{y},M) \xrightarrow{x_1} H_i(\underline{y},M) \to H_i(\underline{y},M_1) \to H_{i-1}(\underline{y},M) \xrightarrow{x_1} H_{i-1}(\underline{y},M)$$

for all i. Recall that $I = (\underline{y})$ annihilates $H_i(\underline{y},M)$ for all i, and hence the image of the first map and the kernel of the last map in the above sequence is 0, therby giving us a short exact sequence

$$0 \to H_i(\underline{y},M) \to H_i(\underline{y},M_1) \to H_{i-1}(\underline{y},M) \to 0, \qquad \forall\ i \in \mathbb{Z}.$$

Now, note that if $H_i(\underline{y},M_1) = 0$, then $H_i(\underline{y},M) = H_{i-1}(\underline{y},M) = 0$. Hence, $H_{q+1}(\underline{y},M_1) \neq 0$, but for $i > q+1$, $H_i(\underline{y},M_1) = 0$. Now, $\mathrm{depth}(I,M_1) = s-1$, since $x_2,\dots,x_n$ is a maximal M_1-sequence in I, for if not, then the original sequence $\underline{x}$ could be extended to a larger M-sequence in I. By the induction hypothesis, we have $q+1 = n-(s-1)$, and thus, $s = n-q$. ■

REMARK 1.17. In other words, $\operatorname{depth}(I,M)$ is the number of successive zero terms from the left in the sequence

$$H_n(\underline{y},M),\ H_{n-1}(\underline{y},M),\ \ldots,\ H_0(\underline{y},M) = M/IM \neq 0.$$

§§ Cohen-Macaulay Rings

THEOREM 1.18 (ISCHEBECK). Let $(A,\mathfrak{m})$ be a Noetherian local ring, M and N be non-zero finite A-modules, and suppose $\operatorname{depth} M = k$ and $\dim N = r$. Then

$$\operatorname{Ext}^i_A(N,M) = 0 \qquad \text{for } i < k-r.$$

Proof. We shall first prove the statement of the theorem when $N = A/\mathfrak{p}$. If $\dim N = r = 0$, then $N = A/\mathfrak{m}$. Using Remark 1.15, we have that

$$k = \operatorname{depth} M = \inf\left\{ i :\ \operatorname{Ext}^i_A(N,M) \neq 0 \right\}.$$

Hence, for all $i < k = k-r$, we have that $\operatorname{Ext}^i_A(N,M) = 0$.

Suppose now that $r > 0$. Then $\mathfrak{p}$ is not maximal, so we can choose some $x \in \mathfrak{m} \setminus \mathfrak{p}$. This gives us a short exact sequence

$$0 \to N \xrightarrow{\cdot x} N \to N' \to 0,$$

where $N' = N/xN = A/(\mathfrak{p},x)$. Since $\dim N' < \dim N$, the induction hypothesis applies to N'. For each $i < k-r$, we obtain a long exact sequence

$$\operatorname{Ext}^i_A(N',M) \to \operatorname{Ext}^i_A(N,M) \xrightarrow{\cdot x} \operatorname{Ext}^i_A(N,M) \to \operatorname{Ext}^{i+1}_A(N',M) = 0.$$

The induction hypothesis implies $\operatorname{Ext}^{i+1}_A(N',M) = 0$, whence due to Nakayama's lemma, $\operatorname{Ext}^i_A(N,M) = 0$, as desired. ■

COROLLARY 1.19. Let $(A,\mathfrak{m})$ be a Noetherian local ring, M a finite A-module, and $\mathfrak{p} \in \operatorname{Ass}_A(M)$. Then $\dim A/\mathfrak{p} \geqslant \operatorname{depth} M$.

Proof. If $\dim A/\mathfrak{p} < \dim M$, then due to Theorem 1.18

$$\operatorname{Hom}_A(A/\mathfrak{p},M) = \operatorname{Ext}^0_A(A/\mathfrak{p},M) = 0,$$

which is absurd, since $\mathfrak{p} \in \operatorname{Ass}_A(M)$. ■

DEFINITION 1.20. Let $(A,\mathfrak{m},k)$ be a Noetherian local ring, and M a finite A-module. We say that M is a *Cohen-Macaulay module* if $M \neq 0$ and $\operatorname{depth} M = \dim M$, or if $M = 0$. If A is a Cohen-Macaulay module over itself, then it is said to be a Cohen-Macaulay (local) ring.

THEOREM 1.21. Let A be a Noetherian local ring, and M a finite A-module.

(1) If M is a CM-module, then for any $\mathfrak{p} \in \operatorname{Ass}_A(M)$ we have

$$\dim A/\mathfrak{p} = \dim M = \operatorname{depth} M.$$

Hence M has no embedded associated primes.

(2) If $a_1,\ldots,a_r \in \mathfrak{m}$ is an M-sequence and we set $M' = M/(a_1,\ldots,a_r)$ then

$$M \text{ is a CM-module over } A \iff M' \text{ is a CM-module over } A.$$

(3) If M is a CM-module over A, then $M_\mathfrak{p}$ is a CM-module over $A_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$, and if $M_\mathfrak{p} \neq 0$ then

$$\operatorname{depth}(\mathfrak{p}, M) = \operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p}.$$

Proof. (1) We have

$$\dim M = \sup\{\dim A/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_A(M)\} \geqslant \inf\{\dim A/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_A(M)\} \geqslant \operatorname{depth} M.$$

Since $\dim M = \operatorname{depth} M$, the conclusion follows.

(2) This follows immediately from the fact that $\operatorname{depth} M' = \operatorname{depth} M - r$ and $\dim M' = \dim M - r$.

(3) It suffices to consider the case $\mathfrak{p} \in \operatorname{Supp}_A(M)$, that is, $\mathfrak{p} \supseteq \operatorname{Ann}_A(M)$. Since every M-regular sequence contained in $\mathfrak{p}$ is an $M_\mathfrak{p}$-regular sequence contained in $\mathfrak{p}A_\mathfrak{p}$, we have the obvious inequalities

$$\dim M_\mathfrak{p} \geqslant \operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geqslant \operatorname{depth}(\mathfrak{p}, M_\mathfrak{p}).$$

We shall show that $\dim M_\mathfrak{p} = \operatorname{depth}(\mathfrak{p}, M)$, whence all the desired conclusions would follow. The proof is by induction on $\operatorname{depth}(\mathfrak{p}, M)$. For the base case, we have $\operatorname{depth}(\mathfrak{p}, M) = 0$, which, due to prime avoidance, means that $\mathfrak{p}$ is contained in an associated prime of M. Since M has no embedded associated primes, we must have that $\mathfrak{p}$ is an associated prime. As a result, $\dim M_\mathfrak{p} = 0 = \operatorname{depth}(\mathfrak{p}, M)$.

Suppose now that $\operatorname{depth}(\mathfrak{p}, M) > 0$; choose an M-regular element $a \in \mathfrak{p}$ and set $M' = M/aM$. Then

$$\operatorname{depth}(\mathfrak{p}, M') = \operatorname{depth}(\mathfrak{p}, M) - 1,$$

and M' is a CM-module over A due to (2). Further, note that $M'_\mathfrak{p} = M_\mathfrak{p}/aM_\mathfrak{p} \neq 0$ due to Nakayama's lemma. Thus, the induction hypothesis applies and using the fact that $a \in A_\mathfrak{p}$ is $M_\mathfrak{p}$-regular, we have

$$\dim M_\mathfrak{p} - 1 = \dim M_\mathfrak{p}/aM_\mathfrak{p} = \dim M'_\mathfrak{p} = \operatorname{depth}(\mathfrak{p}, M') = \operatorname{depth}(\mathfrak{p}, M) - 1,$$

whence the desideratum follows. ■

§§ Base Change Theorems

LEMMA 1.22. Let A be a ring, M an A-module, and $n \geqslant 0$ an integer. Then

$$\operatorname{inj\,dim} M \leqslant n \iff \operatorname{Ext}_A^{n+1}(A/I, M) = 0 \quad \text{for all ideals } I.$$

If A is Noetherian, then we can replace "for all ideals" by "for all prime ideals" in the above equivalence.

Proof. The forward direction is trivial by considering an injective resolution of length $\leqslant n$ and constructing the left derived functors of $\operatorname{Hom}_A(A/I, -)$.

We prove the converse. If $n = 0$, then $\operatorname{Ext}_A^1(A/I, M) = 0$, which is equivalent to Baer's criterion for injectivity. Thus M is injective, that is, $\operatorname{inj\,dim} M = 0 \leqslant n$. Now, suppose $n > 0$. Consider an injective resolution of length $n-1$ and let K be the cokernel of the last map. That is,

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to K_n \to 0,$$

where every E^i is injective. We claim that K is injective. To see this, break down the above exact sequence into short exact sequences of the form

$$0 \to K_0 \to E^0 \to K_1 \to 0 \qquad 0 \to K_1 \to E^1 \to K_2 \to 0,$$

and so on, with the convention that $K_0 = M$. The long exact sequence for $\operatorname{Ext}_A(A/I,-)$ on the first short exact sequence gives

$$0 = \operatorname{Ext}^n_A(A/I,E^0) \to \operatorname{Ext}^n_A(A/I,K_1) \to \operatorname{Ext}^{n+1}_A(A/I,K_0) = 0,$$

whence $\operatorname{Ext}^n(A/I,K_1) = 0$. Proceeding similarly with the other exact sequences, one can show that $\operatorname{Ext}^1_A(A/I,K_n) = 0$, for every ideal I of A. Hence, K_n is injective, i.e., $\operatorname{inj}\dim M \leqslant n$. ■

Lemma 1.23. Let A be a ring, M and N two A-modules, and $x \in A$. Suppose that x is both A-regular and M-regular, and that $xN = 0$. Set $B = A/xA$ and $\overline{M} = M/xM$. Then

(1) $\operatorname{Hom}_A(N,M) = 0$ and $\operatorname{Ext}^{n+1}_A(N,M) \cong \operatorname{Ext}^n_B(N,\overline{M})$ for all $n \geqslant 0$.

(2) $\operatorname{Ext}^n_A(M,N) \cong \operatorname{Ext}^n_B(\overline{M},N)$ for all $n \geqslant 0$.

(3) $\operatorname{Tor}^A_n(M,N) \cong \operatorname{Tor}^B_n(\overline{M},N)$ for all $n \geqslant 0$.

Proof. (1) If $f : N \to M$ is A-linear, then for any $n \in N$, $xf(n) = f(xn) = 0$, and since x is M-regular, $f(n) = 0$. Thus $f = 0$, as desired. Now, set $T^n(N) = \operatorname{Ext}^{n+1}_A(N,M)$. Then, the collection $(T^n)_{n\geqslant 0}$ is a contravariant δ-functor from the category $\mathfrak{Mod}_B$ to the category $\mathfrak{Mod}_A$. Further, the short exact sequence

$$0 \to M \xrightarrow{x} M \to \overline{M} \to 0$$

furnishes a long exact sequence

$$0 = \operatorname{Hom}_A(N,M) \to \operatorname{Hom}_A(N,\overline{M}) \xrightarrow{\delta} \operatorname{Ext}^1_A(N,M) \xrightarrow{x} \operatorname{Ext}^1_A(N,M) \to \cdots.$$

Since x annihilates N, it must annihilate $\operatorname{Ext}^1_A(N,M)$, and so the above exact sequences reduces to

$$0 \to \operatorname{Hom}_A(N,\overline{M}) \xrightarrow{\delta} \operatorname{Ext}^1_A(N,M) \to 0.$$

Thus δ is a natural isomorphism between the functors T^0 and $\operatorname{Ext}^1_A(-,M)$. Now, it suffices to show that the collection $(T^n)_{n\geqslant 0}$ constitutes a universal δ-functor, whence it suffices to show that $T^n(P) = 0$ for every projective B-module P and $n \geqslant 1$; since then it would be coeffaceable by projectives and due to a theorem of Grothendieck, it would be universal.

This is equivalent to showing that $\operatorname{Ext}^n_A(P,M) = 0$ where P is a direct sum of copies of A/xA and $n \geqslant 2$. But note that $\operatorname{proj}\dim_A A/xA \leqslant 1$, and hence $\operatorname{Ext}^n_A(A/xA,M) = 0$ for all A-modules M and $n \geqslant 2$, as desired. This proves (1).

(2) We contend that $\operatorname{Tor}^A_n(M,B) = 0$ for all $n > 0$. Since $\operatorname{proj}\dim_A B \leqslant 1$, it immediately follows that $\operatorname{Tor}^A_n(M,B) = 0$ for $n > 1$. For $n = 1$, the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

furnishes a long exact sequence

$$0 = \operatorname{Tor}^A_1(M,A) \to \operatorname{Tor}^A_1(M,B) \to M \xrightarrow{x} M \to \overline{M} \to 0.$$

Since x is M-regular, we have that $\operatorname{Tor}^A_1(M,A) = 0$.

Now, let $P_\bullet \to M \to 0$ be a free resolution of M. Because of the preceding paragraph, the sequence $P_\bullet \otimes_A B \to M \otimes_A B \to 0$ is exact, so that $P_\bullet \otimes B$ is a free resolution of the B-module $M \otimes B \cong \overline{M}$. From the Hom-Tensor adjunction, note that there are natural isomorphisms

$$\operatorname{Hom}_A(P_\bullet,N) = \operatorname{Hom}_A(P_\bullet,\operatorname{Hom}_B(B,N)) \cong \operatorname{Hom}_B(P_\bullet \otimes_A B,N).$$

Therefore,

$$\operatorname{Ext}^n_A(M,N) = H^n(\operatorname{Hom}_A(P_\bullet,N)) = H^n(\operatorname{Hom}_B(P_\bullet \otimes_A B, N)) = \operatorname{Ext}^n_B(\overline{M},N),$$

as desired.

(3) Continuing with the notation of (2), we have

$$\operatorname{Tor}^A_n(M,N) = H_n(P_\bullet \otimes_A N) = H_n((P_\bullet \otimes_A B)\otimes_B N) = \operatorname{Tor}^B_n(\overline{M},N),$$

thereby completing the proof. ■

§2 REGULAR RINGS

§§ Regular Rings

DEFINITION 2.1. Let $(A,\mathfrak{m},k)$ be a local ring and let M be a finite A-module. An exact sequence

$$\cdots \to L_i \xrightarrow{d_i} L_{i-1} \xrightarrow{d_{i-1}} \cdots \to L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M \to 0$$

is called a *minimal (free) resolution* of M if

- each L_i is a finite free A-module
- $0 = \overline{d}_i : L_i \otimes_A k \to L_{i-1} \otimes_A k$, or equivalently $d_i L_i \subseteq \mathfrak{m} L_{i-1}$ for all $i \geqslant 1$, and
- $\overline{\varepsilon} : L_0 \otimes_A k \to M \otimes_A k$ is an isomorphism.

It is easy to see that a minimal free resolution exists for every finite module over a Noetherian local ring; at each stage simply take a minimal generating set of the kernel and continue.

LEMMA 2.2. Let $(A,\mathfrak{m},k)$ be a local ring, and M a finite A-module. Suppose $L_\bullet$ is a minimal resolution of M; then

(1) $\dim_k \operatorname{Tor}^A_i(M,k) = \operatorname{rank} L_i$ for all i.

(2) $\operatorname{proj\,dim}_A M = \sup\{i : \operatorname{Tor}^A_i(M,k) \neq 0\} \leqslant \operatorname{proj\,dim}_A k$,

(3) if $M \neq 0$ and $\operatorname{proj\,dim}_A M = r < \infty$, then for any finite A-module $N \neq 0$, we have $\operatorname{Ext}^r_A(M,N) \neq 0$.

Proof. (1) This follows immediately from the fact that $\overline{d}_i = 0$ for all $i \geqslant 1$.

(2) The second inequality is straightforward. For if $\operatorname{proj\,dim}_A k = \infty$, then there is nothing to prove. If $\operatorname{proj\,dim}_A k < \infty$, then take a projective resolution of this length and tensor with A to conclude.

From (1) it immediately follows that $\operatorname{proj\,dim}_A M \leqslant \sup\{i : \operatorname{Tor}^A_i(M,k) \neq 0\}$, since this quantity is precisely the length of the minimal free resolution of M. If $\operatorname{proj\,dim}_A M = \infty$, then there is nothing to prove. If $\operatorname{proj\,dim}_A M < \infty$, then take a projective resolution of M achieving this length and tensor with k whence it follows that $\sup\{i : \operatorname{Tor}^A_i(M,k) \neq 0\} \leqslant \operatorname{proj\,dim}_A M$, as desired.

(3) Applying $\mathrm{Hom}_A(-,N)$ to the resolution $L_\bullet \to M$, we obtain a complex which ends with

$$\mathrm{Hom}_A(L_{r-1},N) \xrightarrow{d_r^*} \mathrm{Hom}_A(L_r,N) \to 0,$$

where $\mathrm{Ext}_A^r(M,N)$ is the cokernel of the above map. Since each L_i is free, we can write $\mathrm{Hom}_A(L_i,N)$ as a direct sum of some copies of N and we can express every boundary map $d_i : L_i \to L_{i-1}$ as a matrix with entries in $\mathfrak{m}$. It follows that d_i^* is given by the same matrix (with entries in $\mathfrak{m}$). Hence, the image of d_r^* is contained in $\mathfrak{m}\mathrm{Hom}_A(L_r,N)$, which is properly contained in $\mathrm{Hom}_A(L_r,N)$ by Nakayama's lemma. This completes the proof. ∎

Remark 2.3. The above proof also shows that the minimal resolution is indeed the one that achieves the projective dimension of a module.

Theorem 2.4 (Auslander-Buchsbaum). Let A be a Noetherian local ring and $M \neq 0$ a finite A-module. If $\operatorname{proj\,dim}_A M < \infty$, then

$$\operatorname{proj\,dim}_A M + \operatorname{depth} M = \operatorname{depth} A.$$

Proof. We shall induct on $h = \operatorname{proj\,dim}_A M$. If $h = 0$, then M is a free module, and there is nothing to prove. If $h = 1$, then the minimal resolution looks like

$$0 \to A^m \xrightarrow{\varphi} A^n \to M \to 0,$$

where φ is given by an $n \times m$ matrix with entries in $\mathfrak{m}$. ∎

Lemma 2.5. Let A be a ring and $n \geqslant 0$ an integer. Then the following are equivalent:

(1) $\operatorname{proj\,dim}_A M \leqslant n$ for every A-module M,

(2) $\operatorname{proj\,dim}_A M \leqslant n$ for every finite A-module M,

(3) $\operatorname{inj\,dim}_A N \leqslant n$ for every A-module N, and

(4) $\mathrm{Ext}_A^{n+1}(M,N) = 0$ for all A-modules M and N.

Proof. All implications are straightforward. ∎

Definition 2.6. The *global dimension* of a ring is defined as

$$\operatorname{gl\,dim} A = \sup\{\operatorname{proj\,dim} M : M \text{ is an } A\text{-module}\}.$$

Due to Lemma 2.5, the above supremum can also be taken over all finite A-modules. Further, if $(A,\mathfrak{m},k)$ is a Noetherian local ring, due to Lemma 2.2 (2), we have

$$\operatorname{gl\,dim} A = \operatorname{proj\,dim}_A k.$$

Recall that the *embedding dimension* of a Noetherian local ring $(A,\mathfrak{m},k)$ is defined to be

$$\operatorname{emb\,dim} A = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Theorem 2.7 (Serre). Let $(A,\mathfrak{m},k)$ be a Noetherian local ring. Then the following are equivalent

(1) A is regular;

(2) $\mathrm{gl\ dim}\, A = \dim A$;

(3) $\mathrm{gl\ dim}\, A < \infty$.

Proof. (1) $\implies$ (2) Choose a regular system of parameters $x_1, \dots, x_n \in \mathfrak{m}$, so that $n = \dim A$. Since $\underline{x} = x_1, \dots, x_n$ is an A-sequence, it follows from Theorem 1.11 that $K_\bullet(\underline{x})$ is exact, whence it is a free resolution of k. Note further that the transition matrices in the Koszul complex have entries lying in $\mathfrak{m}$, whence the Koszul complex is a minimal free resolution of $\mathfrak{m}$. Thus,

$$\mathrm{gl\ dim}\, A = \mathrm{proj\ dim}_A\, k = n = \dim A,$$

as desired.

(2) $\implies$ (3) is clear. We shall show that (3) $\implies$ (1). Let $\mathrm{gl\ dim}\, A = r < \infty$, and set $\mathrm{emb\ dim}\, A = s$. We shall show that A is regular by induction on s. If $s = 0$, then $\mathfrak{m} = 0$, and hence, A is a field, so it is regular.

Suppose now that $s > 0$. We claim that $\mathfrak{m} \notin \mathrm{Ass}_A(A)$. If not, then consider a minimal resolution of k,

$$0 \to L_r \to L_{r-1} \to \cdots \to L_0 \to k \to 0,$$

where the maps are given by matrices with entries in $\mathfrak{m}$. Now, there is some $0 \neq a \in A$ such that $\mathfrak{m} = \mathrm{Ann}_A(a)$. It follows that the element $(a, a, \dots, a) \in L_r$ lies in the kernel of the map $L_r \to L_{r-1}$, a contradiction.

Thus $\mathfrak{m} \notin \mathrm{Ass}_A(A)$. Choose

$$x \in \mathfrak{m} \setminus \left(\mathfrak{m}^2 \cup \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(A)} \mathfrak{p} \right).$$

using prime avoidance[1]. Then x is A-regular, hence also $\mathfrak{m}$-regular. Setting $B = A/xA$, and using Lemma 1.23 (2), we have $\mathrm{Ext}^i_A(\mathfrak{m}, N) = \mathrm{Ext}^i_B(\mathfrak{m}/x\mathfrak{m}, N)$ for all B-modules N. Hence, $\mathrm{Ext}^{r+1}_B(\mathfrak{m}/x\mathfrak{m}, N) = 0$ for every B-module N; that is, $\mathrm{proj\ dim}_B\, \mathfrak{m}/x\mathfrak{m} \leqslant r$.

Next, we show that the natural surjection $\mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}/xA$ splits as A-modules (and hence as B-modules). First, choose a minimal generating set $x, x_2, \dots, x_s$ of $\mathfrak{m}$ and set $\mathfrak{b} = (x_2, \dots, x_s)$. Note that $\mathfrak{b} \cap xA \subseteq x\mathfrak{m}$. Indeed, if $y = a_2 x_2 + \cdots + a_n x_n = ax \in \mathfrak{b} \cap xA$, then looking at the equality modulo $\mathfrak{m}$, we see that $a \in \mathfrak{m}$, whence $x \in \mathfrak{b} \cap x\mathfrak{m} \subseteq x\mathfrak{m}$. Now, consider the chain of natural maps

$$\frac{\mathfrak{m}}{xA} = \frac{\mathfrak{b} + xA}{xA} \xrightarrow{\sim} \frac{\mathfrak{b}}{\mathfrak{b} \cap xA} \to \frac{\mathfrak{m}}{x\mathfrak{m}} \to \frac{\mathfrak{m}}{xA}.$$

Their composition is the identity, and hence, the surjection $\mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}/xA$ splits. In particularly, this means that

$$\mathrm{proj\ dim}_B\, \mathfrak{m}/xA \leqslant \mathrm{proj\ dim}_B\, \mathfrak{m}/x\mathfrak{m} \leqslant r.$$

Because of the exact sequence $0 \to \mathfrak{m}/xA \to B \to k \to 0$, we see that $\mathrm{gl\ dim}\, B = \mathrm{proj\ dim}_B\, k \leqslant r + 1$. Since $\mathrm{emb\ dim}\, B = r - 1$, the induction hypothesis gives that B is a regular local ring. Now, since x is A-regular, $\dim B = \dim A - 1$, and therefore,

$$\mathrm{emb\ dim}\, A = \mathrm{emb\ dim}\, B + 1 = \dim B + 1 = \dim A,$$

whence A is a regular local ring, as desired. ∎

THEOREM 2.8 (SERRE). Let A be a regular local ring and $\mathfrak{p}$ a prime ideal of A. Then $A_\mathfrak{p}$ is a regular local ring.

[1] TODO: Add in the statement

Proof. If proj $\dim_A k < \infty$, then localizing a finite projective resolution of k at $\mathfrak{p}$, we obtain the desired conclusion. ∎

Definition 2.9. A *regular ring* is a Noetherian ring such that the localization at every prime is a regular local ring.

§§ Finite Free Resolutions

Lemma 2.10 (Schanuel). Let A be a ring and M an A-module. Suppose that

$$0 \to K \to P \to M \to 0 \quad \text{and} \quad 0 \to K' \to P' \to M \to 0$$

are exact sequences with P and P' projective. Then $K \oplus P' \cong K' \oplus P$.

Proof. Since P and P' are projective, there are maps

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r, "\alpha"] \arrow[d, shift right, "\lambda"'] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r] & P' \arrow[r, "\alpha'"'] \arrow[u, shift right, "\lambda'"'] & M \arrow[r] & 0 \end{tikzcd}$$

$\lambda\colon P \to P'$ and $\lambda'\colon P' \to P$ making the square on the right commute. Adding in the summands P' and P to the respective rows, we obtain another commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \oplus P' \arrow[r] \arrow[d, dotted, "\theta"] & P \oplus P' \arrow[r, "{(\alpha,0)}"] \arrow[d, shift right, "\varphi"'] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \oplus P \arrow[r] & P \oplus P' \arrow[r, "{(0,\alpha')}"'] \arrow[u, shift right, "\psi"'] & M \arrow[r] & 0 \end{tikzcd}$$

where $\varphi\colon P \oplus P' \to P \oplus P'$ is defined by

$$\varphi \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \mathbf{id}_P & -\lambda' \\ \lambda & \mathbf{id}_{P'} - \lambda \circ \lambda' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix},$$

and

$$\psi \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \mathbf{id}_P - \lambda' \circ \lambda & \lambda' \\ -\lambda & \mathbf{id}_{P'} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}.$$

One can check that $\varphi \circ \psi = \psi \circ \varphi = \mathbf{id}_{P \oplus P'}$, so that φ and ψ are isomorphisms. There is a map $\theta\colon K \oplus P' \to K' \oplus P$ making the entire diagram above commute. Using the five-lemma or otherwise on this diagram, one concludes that θ is an isomorphism. ∎

Lemma 2.11 (Generalized Schanuel). Let A be a ring and M an A-module. Suppose

$$0 \to P_n \to \cdots \to P_0 \to M \to 0 \quad \text{and } 0 \to Q_n \to \cdots \to Q_0 \to M \to 0$$

are two exact sequences with P_i and Q_i projective for $0 \leqslant i \leqslant n-1$, then

$$P_0 \oplus Q_1 \oplus \cdots \cong Q_0 \oplus P_1 \oplus \cdots.$$

Proof. Induct on n. The base case with $n=0$ is precisely Lemma 2.10. Let K denote the kernel of $P_0 \to M$ and K' the kernel of $Q_0 \to M$. Due to Lemma 2.10, $K \oplus Q_0 \cong K' \oplus P_0$. Add in the summands Q_0 and P_0 to the respective resolutions as follows:

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r] & P_1 \oplus Q_0 \arrow[r] & K \oplus Q_0 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \oplus P_0 \arrow[r] & K' \oplus P_0 \arrow[r] & 0.
\end{tikzcd}$$

Using the inductive hypothesis, we have the desired isomorphism. ■

DEFINITION 2.12. A *finite free resolution* of an A-module M is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

such that each F_i is a finite free A-module. If M admits a finite free resolution as above, we define its *Euler number* to be

$$\chi_A(M) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank}_A F_i.$$

Clearly, due to Lemma 2.11, $\chi(M)$ is independent of the chosen finite free resolution. Further, if M admits an FFR over A, then for any prime ideal $\mathfrak{p} \subseteq A$, $M_{\mathfrak{p}}$ admits an FFR over $A_{\mathfrak{p}}$ and $\chi_A(M) = \chi_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.

PROPOSITION 2.13. Let $(A,\mathfrak{m})$ be a local ring such that for each finite subset $E \subseteq \mathfrak{m}$ there exists $0 \neq y \in A$ with $yE = 0$. Then the only A-modules having an FFR over A are the (finite rank) free modules.

Proof. Let $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ be an FFR of M, and set $N = \operatorname{coker}(F_n \to F_{n-1})$. Our first goal will be to show that N is a free module of finite rank. Clearly, N is a finite A-module. If it were not free, then it would admit a minimal free resolution of the form $0 \to L_1 \to L_0 \to N \to 0$, since it already admits a free resolution of length 1. Using Lemma 2.10, we have $L_1 \oplus F_{n-1} \cong L_0 \oplus F_n$, so that L_1 is a finite rank free module. Treating L_1 as a submodule of L_0, we can write down a basis for L_1 in terms of a basis for L_0 with coefficients in $\mathfrak{m}$ since the resolution is minimal. Thus, there would exist $0 \neq y \in A$ annihilating all those coefficients, whence $yL_1 = 0$, a contradiction. Thus N must be a finite rank free A-module.

Coming back to the proof at hand, workin backwards from the given free resolution and replacing the map $F_n \to F_{n-1}$ by $\operatorname{coker}(F_n \to F_{n-1})$ at each stage, we reduce to the case $0 \to F_1 \to F_0 \to M \to 0$, which we have handled above. Hence, M is a finite rank free module over A. Conversely, it is clear that every finite rank free A-module has an FFR. ■

THEOREM 2.14. Let A be a ring. If an A-module M admits an FFR, then $\chi_A(M) \geqslant 0$.

Proof. Let $\mathfrak{p}$ be a minimal prime of A. Since $\chi_A(M) = \chi_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, we can replace A by $A_{\mathfrak{p}}$ and M by $M_{\mathfrak{p}}$ and assume that $(A,\mathfrak{m})$ is a local ring whose maximal ideal is equal to the nilradical. We claim that the hypothesis of Proposition 2.13 is satisfied. Indeed, let $x_1,\ldots,x_r \in \mathfrak{m}$. We shall induct on r to show that there exists $0 \neq y \in A$ such that $yx_i = 0$ for all $1 \leqslant i \leqslant r$. If $r = 1$, then the nilpotence of x_1 implies the existence of such a y. Suppose $r > 1$, then using the inductive hypothesis, there exists $0 \neq z \in A$ such that $zx_1 = \cdots = zx_{r-1} = 0$. Let $j \geqslant 1$ be the minimal integer such that $zx_r^j = 0$, which exists since x_r is nilpotent. Choosing $y = zx_r^{j-1} \neq 0$, we have that $yx_i = 0$ for $1 \leqslant i \leqslant r$. As a consequence of Proposition 2.13, we see that M is finite free, so that $\chi_A(M) \geqslant 0$. ■

COROLLARY 2.15. Let A be a ring and suppose there is an injective A-linear map $A^m \hookrightarrow A^n$, then $m \leqslant n$.

Proof. Let $M = \operatorname{coker}(A^m \hookrightarrow A^n)$. Then M has a finite free resolution and $\chi_A(M) = n - m \geqslant 0$ due to Theorem 2.14, thereby completing the proof. ■

THEOREM 2.16 (AUSLANDER-BUCHSBAUM). Let A be a Noetherian ring and M an A-module admitting an FFR. The following are equivalent:

(1) $\operatorname{Ann}_A(M) \neq 0$.

(2) $\chi_A(M) = 0$.

(3) $\operatorname{Ann}_A(M)$ contains an A-regular element.

Proof. (1) $\implies$ (2) Let $I = \operatorname{Ann}_A(M) \neq 0$ and set $J = \operatorname{Ann}_A(I)$. Suppose $\chi_A(M) > 0$. Choose any $\mathfrak{p} \in \operatorname{Ass}_A(A)$. Then $\chi_{A_\mathfrak{p}}(M_\mathfrak{p}) > 0$, and hence $M_\mathfrak{p} \neq 0$. Further, since $\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}_{A_\mathfrak{p}}(A_\mathfrak{p})$, it follows from Proposition 2.13 that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Now note that $IA_\mathfrak{p} = \operatorname{Ann}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$, so that $J \not\subseteq \mathfrak{p}$. Since this holds for every $\mathfrak{p} \in \operatorname{Ass}_A(A)$, it follows from Prime Avoidance that J must contain an A-regular element. But since $J \cdot I = 0$, we would have $I = 0$, a contradiction. Thus $\chi_A(M) = 0$.

(2) $\implies$ (3) If $\chi_A(M) = 0$, then as argued above, for every $\mathfrak{p} \in \operatorname{Ass}_A(A)$, $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module and $\chi_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$, whence $M_\mathfrak{p} = 0$. Since M is a finite A-module, this must imply that $\operatorname{Ann}_A(M) \not\subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Ass}_A(A)$. This is equivalent to stating that $\operatorname{Ann}_A(M)$ contains an A-regular element.

(3) $\implies$ (1) is clear. This completes the proof. ■

DEFINITION 2.17. An A-module M is said to be *stably free* if there exist finite free A-modules F and F' such that $M \oplus F \cong F'$ as A-modules.

Clearly, every stably free module is projective and has an FFR, $0 \to F \to F' \to M \to 0$. Conversely, we also have:

LEMMA 2.18. A finite projective module having an FFR is stably free.

Proof. We shall induct on the length of the FFR. The base cases of length 0 and 1 are trivial. Suppose now that

$$0 \to F_n \to \cdots \to F_0 \to P \to 0$$

is a finite free resolution of a projective A-module M with $n \geqslant 2$. Let $K = \ker(F_0 \to P)$. Then $0 \to K \to F_0 \to P \to 0$ splits, so that K is a finite projective module admitting an FFR of length $n-1$. Using the inductive hypothesis, K is stably free, that is, there are finite free modules F and F' such that $K \oplus F \cong F'$. Hence,

$$P \oplus F' \cong P \oplus K \oplus F \cong F_0 \oplus F,$$

so that P is also stably free. ■

LEMMA 2.19. Let A be a Noetherian ring. If every finite A-module admits an FFR, then A is a regular ring.

Proof. Let $\mathfrak{p}$ be a prime ideal in A. According to the hypothesis, the A-module $A/\mathfrak{p}$ admits an FFR. Localizing this resolution at $\mathfrak{p}$, one obtains an FFR of $\kappa(\mathfrak{p})$ over $A_\mathfrak{p}$, whence $A_\mathfrak{p}$ is a regular local ring. Thus A is a regular ring. ■

§§ Unique Factorization Domains

THEOREM 2.20. Let A be a Noetherian domain. Then A is a UFD if and only if every height 1 prime ideal in A is principal.

Proof. Suppose A is a UFD and $\mathfrak{p}$ a height 1 prime ideal in A. Choose any $0 \neq a \in \mathfrak{p}$ and factorize $a = \pi_1 \cdots \pi_n$ into irreducibles, which are the same things as primes in this case. Since $\mathfrak{p}$ is a prime ideal, there exists a $\pi_i \in \mathfrak{p}$. This gives a chain of prime ideals $(0) \subseteq (\pi_i) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is height 1, it follows that $\mathfrak{p} = (\pi_i)$, i.e., is principal.

Conversely, suppose every height 1 prime ideal in A is principal. Every Noetherian domain is a factorization domain, therefore, it suffices to show that all irreducibles in A are prime. Let $0 \neq a \in A$ be an irreducible element and choose a prime ideal $\mathfrak{p}$ minimal among those containing the ideal (a). Due to the Hauptidealsatz, $\operatorname{height}\mathfrak{p} = 1$, so that $\mathfrak{p} = (b)$ for some $0 \neq b \in A$, whence there exists $0 \neq c \in A$ such that $a = bc$. Since A is irreducible, c must be a unit, and hence $(a) = \mathfrak{p}$, i.e., a is a prime element in A, thereby completing the proof. ■

THEOREM 2.21. Let A be a Noetherian domain, Γ a set of prime elements of A, and S the multiplicative set generated by Γ. If $S^{-1}A$ is a UFD, then so is A.

Proof. ■

LEMMA 2.22. Let A be an integral domain, and $\mathfrak{a}$ an ideal of A such that $\mathfrak{a} \oplus A^n \cong A^{n+1}$ for some $n \geqslant 0$. Then $\mathfrak{a}$ is a principal ideal.

Proof. ■