Coxeter and Tits Systems

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§1 Coxeter Systems

Let W denote a group and $S \subseteq W$ a generating set such that $1 \in S$ and $S = S^{-1}$. Fix this pair throughout this section.

Definition 1.1. Let $w \in W$. The length of w with respect to S, denoted by $\ell_S(w)$ (often abbreviated to $\ell(w)$) is the smallest integer $q \geqslant 0$ such that w is the product of a sequence of q elements of S. A *reduced representation* of W with respect to S is any sequence $\mathbf{s} = (s_1, \ldots, s_q)$ of elements of S such that $w = s_1 \cdots s_q$ and $q = \ell_S(w)$.

Clearly, if $w, w' \in W$, then

$$\begin{aligned} \ell(ww') &\leqslant \ell(w) + \ell(w'), \\ \ell(w^{-1}) &= \ell(w), \\ |\ell(w) - \ell(w')| &\leqslant \ell(ww'^{-1}). \end{aligned}$$

Definition 1.2. (W, S) is said to be a *Coxeter system* if every element in S has order at most 2, and it satisfies the following condition:

(Cox) For $s, s' \in S$, let $1 \leqslant m(s, s') \leqslant \infty$ be the order of $ss' \in W$ and let

$$I = \{(s, s') \colon m(s, s') < \infty\}.$$

Then

$$W = \left\langle s \in S \colon (ss')^{m(s,s')} = 1,\ (s, s') \in I \right\rangle$$

is a presentation for the group W.

Remark 1.3. Consider the function $f : S \to \{-1, 1\}$ given by $f(s) = -1$ for each $s \in S$. If $s, s' \in S$ such that $m = m(s, s') < \infty$, then $(f(s)f(s'))^m = 1$ almost tautologically. Hence, this function induces a map $\operatorname{sgn} : W \to \{-1, 1\}$ known as the *signature* of W. It is clear that $\operatorname{sgn}(w) = (-1)^{\ell(w)}$.

Proposition 1.4. Assume that (W, S) is a Coxeter system. Then, two elements $s, s' \in S$ are conjugate in W if and only if the following condition is satisfied:

(Con) There exists a finite sequence $(s_1, \ldots, s_q)$ of elements of S such that $s_1 = s$, $s_q = s'$ and $s_j s_{j+1}$ is of *finite* odd order for $1 \leqslant j < q$.

Proof. First, if $s, s' \in S$ such that $p = ss'$ is of finite order $2n + 1$, then

$$sps^{-1} = p^{-1} \implies sp^n s^{-1} = p^{-n},$$

so that

$$p^n s p^{-n} = p^n p^n s = p^{-1} s = s',$$

and s' is conjugate to s. In particular, this shows that if (Con) is satisfied, then (s, s') is a pair of conjugates in W.

For each $s \in S$, let A_s be the set of $s' \in S$ satisfying (Con); clearly, every $s' \in A_s$ is conjugate of s. Let $f : S \to \{-1, 1\}$ that is equal to 1 on A_s and to -1 in $S \setminus A_s$. We shall show that this map can be extend to a group homomorphism $W \to \{-1, 1\}$. Indeed, let $s', s'' \in S$ with $m = m(s, s') < \infty$. If m is odd, then s' and s'' are conjugate so either both in A_s or both in $S \setminus A_s$, and hence $f(s')f(s'') = 1$, in particular, $(f(s')f(s''))^m = 1$. On the other hand, if m is even, then clearly $(f(s')f(s''))^m = 1$. Consequently, to (Cox), the map f extends to a group homomorphism $W \to \{-1, 1\}$.

Finally, let s' be a conjugate of s in W. Since $s \in \ker f$, so does s', hence $s' \in A_s$. ■

Definition 1.5. Let (W, S) be a Coxeter system and let T be the set of conjugates in W of elements of S. For any sequence $\mathbf{s} = (s_1, \dots, s_q)$ of elements of S, denote by $\Phi(\mathbf{s})$ the sequence $(t_1, \dots, t_q)$ of elements of T defined by

$$t_j = (s_1 \cdots s_{j-1})s_j(s_1 \cdots s_{j-1})^{-1} = (s_1 \cdots s_{j-1})s_j(s_{j-1} \cdots s_1).$$

Then $t_1 = s_1$ and $s_1 \cdots s_q = t_q \cdots t_1$. For $t \in T$, denote by $n(\mathbf{s}, t)$ the number of indices $1 \leqslant j \leqslant q$ for which $t_j = t$. Finally, set

$$R = \{-1, 1\} \times T.$$

Lemma 1.6. (1) Let $w \in W$ and $t \in T$. The number $(-1)^{n(\mathbf{s},t)}$ has the same value $\eta(w, t)$ for all sequences $\mathbf{s} = (s_1, \dots, s_q)$ in S such that $w = s_1 \cdots s_q$.

(2) For $w \in W$, let $U_w : R \to R$ be given by

$$U_w(\varepsilon, t) = \left(\varepsilon\eta(w^{-1}, t), wtw^{-1}\right).$$

The map $w \mapsto U_w$ is a homomorphism from W to the group of permutations of R, $\mathfrak{Sym}(R)$.

Proof. For $s \in S$, define a map $U_s : R \to R$ by

$$U_s(\varepsilon, t) = \left(\varepsilon(-1)^{\delta_{s,t}}, sts^{-1}\right),$$

where $\delta_{s,t}$ is the Kronecker symbol. Clearly, $U_s^2 = \mathbf{id}_R$, and hence U_s is a permutation of R.

For a sequence $\mathbf{s} = (s_1, \dots, s_q)$ in S, put $w = s_q \cdots s_1$ and $U_{\mathbf{s}} = U_{s_q} \cdots U_1$. We shall show by induction that

$$U_{\mathbf{s}}(\varepsilon, t) = \left(\varepsilon(-1)^{n(\mathbf{s},t)}, wtw^{-1}\right). \tag{1}$$

This is clear if $q = 0, 1$. For $q > 1$, put $\mathbf{s}' = (s_1, \dots, s_{q-1})$ and

$$w' = s_{q-1} \cdots s_1.$$

Using the induction hypothesis, we can write

$$U_{\mathbf{s}}(\varepsilon, t) = U_{s_q}\left(\varepsilon(-1)^{n(\mathbf{s}',t)}, w'tw'^{-1}\right) = \left(\varepsilon(-1)^{n(\mathbf{s}',t)+\delta_{s_q,w'tw'^{-1}}}, wtw^{-1}\right).$$

But since $\Phi(\mathbf{s}) = \left(\Phi(\mathbf{s}'), w'tw'^{-1}\right)$, the formula (1) follows.

Now let $s, s' \in S$ be such that $p = ss'$ has finite order m. Let $\mathbf{s} = (s_1, \dots, s_{2m})$ where

$$s_j = \begin{cases} s & j \text{ is odd} \\ s' & j \text{ is even.} \end{cases}$$

Then $s_{2m} \cdots s_1 = p^{-m} = 1$ and

$$t_j = (s_1 \cdots s_{j-1})s_j(s_{j-1} \cdots s_1) = p^{j-1}s \qquad \text{for } 1 \leqslant j \leqslant 2m.$$

Sinc p is of order m, the elements $t_1, \dots, t_m$ are distinct and $t_{j+m} = t_j$ for $1 \leqslant j \leqslant m$. The integer $n(\mathbf{s}, t)$ is equal to either 0 or 2 and due to (1), we have that $U_{\mathbf{s}} = \mathbf{id}_R$, i.e., $(U_s U_{s'})^m = \mathbf{id}_R$. Thus, by (Cox), there is a group homomorphism $W \to \mathfrak{Sym}(R)$ given by $w \mapsto U_w$, extending the mapping $s \mapsto U_s$. It follows that $U_w = U_{\mathbf{s}}$ for every sequence $\mathbf{s} = (s_1, \dots, s_q)$ such that $w = s_q \cdots s_1$. Both conclusions of the lemma follow hence. ∎

Lemma 1.7. Let $\mathbf{s} = (s_1, \dots, s_q)$, $\Phi(\mathbf{s}) = (t_1, \dots, t_q)$ and $w = s_1 \cdots s_q$. Let T_w be the set of elements of T such that $\eta(w, t) = -1$. Then $\mathbf{s}$ is a reduced representation of w if and only if the t_i are distinct, and in that case, $T_w = \{t_1, \dots, t_q\}$ and $\#T_w = \ell(w)$.

Proof. Clearly $T_w \subseteq \{t_1, \dots, t_q\}$. Taking $\mathbf{s}$ to be a reduced representation, it follows that $\#T_w \leqslant \ell(w)$. Further, if the t_i's are distinct, then $\eta(w, t) = -1$ if and only if $t \in \{t_1, \dots, t_q\}$, so that $T_w = \{t_1, \dots, t_q\}$ and $q = \#T_w \leqslant \ell(w)$. Hence, $\mathbf{s}$ is a reduced representation.

On the other hand, suppose $t_i = t_j$ for some $i < j$. Then

$$s_i = (s_i \cdots s_{j-1})s_j(s_i \cdots s_{j-1})^{-1};$$

consequently,

$$w = s_1 \cdots s_{i-1}s_{i+1} \cdots s_{j-1} \cdots s_{j+1} \cdots s_q,$$

whence $\mathbf{s}$ is not a reduced representation of w, as desired. ∎

Lemma 1.8. Let $w \in W$ and $s \in S$ be such that $\ell(sw) \leqslant \ell(w)$. For any sequence $\mathbf{s} = (s_1, \dots, s_q)$ of elements of S with $w = s_1 \cdots s_q$, there exists an index $1 \leqslant j \leqslant q$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

Proof. Let p be the length of w and $w' = sw$. Due to Remark 1.3, $\ell(w') \equiv \ell(w) + 1 \pmod 2$. The hypothesis $\ell(w') \leqslant \ell(w)$ and the relation

$$|\ell(w) - \ell(w')| \leqslant \ell(ww'^{-1}) = \ell(s) = 1,$$

and hence, $\ell(w') = p - 1$. Let $w' = s'_1 \cdots s'_{p-1}$ be a reduced representation of w' and put $\mathbf{s} = (s, s'_1, \dots, s'_{p-1})$ and $\Phi(\mathbf{s}') = (t'_1, \dots, t'_p)$. Since $\mathbf{s}'$ is a reduced representation of w, due to Lemma 1.7, the t_j's must be distinct and $n(\mathbf{s}', s) = 1$ since $t_1 = s$. Further, since both $\mathbf{s}$ and $\mathbf{s}'$ represent w, due to Lemma 1.6, we must have $n(\mathbf{s}, s) \equiv n(\mathbf{s}', s) \pmod 2$, whence $n(\mathbf{s}, s) \neq 0$. Consequently, s is equal to one of the t_j's. The lemma then follows immediately. ∎

§§ The Exchange Condition

Definition 1.9. Let W be a group and $S \subseteq W$ a generating set such that $S^{-1} = S$ and every element in S has order at most 2. The *exchange condition* is the following assertion about (W, S):

(Exc) Let $w \in W$ and $s \in S$ be such that $\ell(sw) \leqslant \ell(w)$. For any reduced representation $w = s_1 \cdots s_q$, there exists an index $1 \leqslant j \leqslant q$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

Proposition 1.10. Let (W, S) be a pair as in Definition 1.9 and satisfying (Exc). Let $s \in S$, $w \in W$ and $w = s_1 \cdots s_q$ be a reduced representation of w. Then one of the following must hold:

(i) $\ell(sw) = \ell(w) + 1$ and $sw = ss_1 \cdots s_q$ is a reduced representation of sw, or

(ii) $\ell(sw) = \ell(w) - 1$ and there exists an index $1 \leqslant j \leqslant q$ such that $sw = s_1 \cdots s_{j-1} s_{j+1} \cdots s_q$ is a reduced representation of sw and $w = ss_1 \cdots s_{j-1} s_{j+1} \cdots s_q$ is a reduced representation of w.

Proof. Let $w' = sw$. We know that

$$|\ell(w) - \ell(w')| \leqslant \ell(s) = 1.$$

Suppose first that $\ell(w') > \ell(w)$. Then $\ell(w') = q + 1$ and $w' = ss_1 \cdots s_q$ whence this is also a reduced representation.

Next, suppose $\ell(w') \leqslant \ell(w)$. Due to (Exc), there exists an index $1 \leqslant j \leqslant q$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

Then $w = ss_1 \cdots s_{j-1} s_{j+1} \cdots s_q$. Since $\ell(w') \geqslant q - 1$, we must have $\ell(w') = q - 1$ and that the above representation is reduced. ∎

Lemma 1.11. Let (W, S) be a pair as in Definition 1.9 and satisfying (Exc). Let $w \in W$ have length $q \geqslant 1$, let D be the set of all reduced representations of w, and let $F : D \to E$.

Assume that $F(\mathbf{s}) = F(\mathbf{s}')$ if the elements $\mathbf{s} = (s_1, \dots, s_q)$ and $\mathbf{s}' = (s'_1, \dots, s'_q)$ of D satisfy one of the following:

(i) $s_1 = s'_1$ or $s_q = s'_q$; or

(ii) there exist s and s' in S such that $s_j = s'_k = s$ and $s_k = s'_j = s'$ for j odd and k even.

Then F is constant.

Proof. The proof proceeds in two steps:
Step 1. Let $\mathbf{s}, \mathbf{s}' \in D$ and put $\mathbf{t} = (s'_1, s_1, \dots, s_{q-1})$. We shall show that if $F(\mathbf{s}) \neq F(\mathbf{s}')$ then $\mathbf{t} \in D$ and $F(\mathbf{t}) \neq F(\mathbf{s})$.

Indeed, $w = s'_1 \cdots s'_q$ and $s'_1 w = s'_2 \cdots s'_q$, so that $\ell(s'_1 w) < q = \ell(w)$. Due to Proposition 1.10 (ii), there is an index $1 \leqslant j \leqslant q$ such that $\mathbf{u} = (s'_1, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_q)$ belongs to D. Due to condition (i), we have $F(\mathbf{u}) = F(\mathbf{s}')$. If $j \neq q$, then we would also have $F(\mathbf{u}) = F(\mathbf{s})$ due to condition (i), contrary to our hypothesis that $F(\mathbf{s}) \neq F(\mathbf{s}')$. Thus $j = q$ and hence $\mathbf{t} = \mathbf{u} \in D$ and $F(\mathbf{t}) = F(\mathbf{s}') \neq F(\mathbf{s})$, as desired.

Step 2. Let $\mathbf{s}, \mathbf{s}' \in D$. For $0 \leqslant j \leqslant q+1$, define a sequence $\mathbf{s}_j$ of q-elements of S as:

$$\mathbf{s}_0 = (s_1', \dots, s_q')$$
$$\mathbf{s}_1 = (s_1, \dots, s_q)$$
$$\mathbf{s}_{q+1-k} = \begin{cases} (s_1, s_1', \dots, s_1, s_1', s_1, s_2, \dots, s_k) & q-k \text{ even and } 0 \leqslant k \leqslant q \\ (s_1', s_1, \dots, s_1, s_1', s_1, s_2, \dots, s_k) & q-k \text{ odd and } 0 \leqslant k \leqslant q \end{cases}$$

Let (H_j) denote the assertion:

$$\text{“}\mathbf{s}_j \in D,\ \mathbf{s}_{j+1} \in D \text{ and } F(\mathbf{s}_j) \neq F(\mathbf{s}_{j+1})\text{”}.$$

Due to **Step 1**, $(H_j) \implies (H_{j+1})$ for $0 \leqslant j \leqslant q$, and due to condition (ii), (H_q) is false. Hence (H_0) is false, so that $F(\mathbf{s}) = F(\mathbf{s}')$, thereby completing the proof. ■

Proposition 1.12. Let M be a monoid and $f : S \to M$. Ste

$$a(s, s') = \begin{cases} (f(s)f(s'))^l & m(s,s') = 2l \\ (f(s)f(s'))^l\, f(s) & m(s,s') = 2l+1 \\ 1 & m(s,s') = \infty. \end{cases}$$

If $a(s,s') = a(s',s)$ whenever $s \neq s'$ in S, then there exists a map $g : W \to M$ such that

$$g(w) = f(s_1) \cdots f(s_q)$$

for every reduced decomposition $w = s_1 \cdots s_q$ of $w \in W$.

Proof. ■

Theorem 1.13. Let (W, S) be a pair such that S generates W, $1 \in S$, $S^{-1} = S$ and every element in S has order at most 2. Then (W, S) is a Coxeter system if and only if it satisfies (Exc).

Proof. ■