The Universal Enveloping Algebra and The Poincaré-Birkhoff-Witt Theorem

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Abstract

In this article, we define and construct the universal enveloping algebra of a Lie algebra, then, state and prove the Poincaré-Birkhoff-Witt Theorem.

§1 The Universal Enveloping Algebra

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over k. A *universal enveloping algebra* is a pair $(\mathfrak{U}, i)$ where $\mathfrak{U}$ is an associative algebra (over k, with identity) and $i : \mathfrak{g} \to \mathfrak{U}$ is a homomorphism of Lie algebras such that for any associative algebra $\mathfrak{A}$ (over k, with identity) and any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{A}$, there is a unique k-algebra homomorphism $\widetilde{\varphi} : \mathfrak{U} \to \mathfrak{A}$ making the following diagram commute.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "i"'] & \mathfrak{A} \\ \mathfrak{U} \arrow[ur, dashed, "\exists!\ \widetilde{\varphi}"'] & \end{tikzcd}$$

§§ Construction

Let $\mathfrak{T}$ denote the *tensor algebra* over $\mathfrak{g}$, that is,

$$\mathfrak{T} = \bigoplus_{n \geqslant 0} \mathfrak{g}^{\otimes n}.$$

There is a map $\mu : \mathfrak{g}^{\otimes n} \times \mathfrak{g}^{\otimes m} \to \mathfrak{g}^{\otimes m+n}$ given by

$$\mu(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \ldots x_n \otimes y_1 \otimes \cdots \otimes y_m$$

and extending linearly. This gives $\mathfrak{T}$ the structure of a k-algebra.

Let $\mathfrak{K}$ denote the ideal in $\mathfrak{T}$ generated by all elements of the form

$$[x, y] - x \otimes y + y \otimes x$$

for $x, y \in \mathfrak{g}$. Set $\mathfrak{U} = \mathfrak{T}/\mathfrak{K}$ and let $\iota : \mathfrak{g} \to \mathfrak{U}$ be the composition

$$\mathfrak{g} \longrightarrow \mathfrak{T} \xrightarrow{\pi} \mathfrak{U}.$$

THEOREM 1.2. $(\mathfrak{U}, \iota)$ is a universal enveloping algebra for $\mathfrak{g}$.

Proof. Let $\varphi : \mathfrak{g} \to \mathfrak{A}$ be a homomorphism of Lie algebras where $\mathfrak{A}$ is an associative k-algebra. The universal property of the tensor algebra extends this to a k-algebra homomorphism $\widetilde{\varphi} : \mathfrak{T} \to \mathfrak{A}$.

Note that

$$\widetilde{\varphi}(x \otimes y - y \otimes x) = \widetilde{\varphi}(x \otimes y) - \widetilde{\varphi}(y \otimes x) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) = \varphi([x, y]),$$

whence $\widetilde{\varphi}$ vanishes on $\mathfrak{K}$, thereby inducing a unique (due to the universal property of the kernel) map $\widetilde{\widetilde{\varphi}} : \mathfrak{U} \to \mathfrak{A}$, thereby completing the proof. ■

§2 THE POINCARÉ-BIRKHOFF-WITT THEOREM

Let $u_1, \ldots, u_n$ be a k-basis of $\mathfrak{g}$. A *monomial* in $\mathfrak{T}$ is an element of the form

$$u_{i_1} \otimes \cdots \otimes u_{i_n}$$

for $n \geqslant 1$. The number n is said to be the *degree* of the monomial. The *index* of the monomial is given by

$$\operatorname{ind}(u_{i_1} \otimes \cdots \otimes u_{i_n}) = \sum_{j<k} \eta_{jk}$$

where

$$\eta_{jk} = \begin{cases} 0 & i_j \leqslant i_k \\ 1 & i_j > i_k \end{cases}$$

A monomial is said to be *standard* if its index is 0. Let $\mathfrak{g}_n$ denote the vector space spanned by monomials of degree n and let $\mathfrak{g}_{n,i}$ denote the subspace of $\mathfrak{g}_n$ spanned by monomials of degree n and index $\leqslant i$.

LEMMA 2.1. Every element of $\mathfrak{T}$ is congruent modulo $\mathfrak{K}$ to a k-linear combination of 1 and standard monomials.

Proof. Straightforward induction on the index and degree of standard monomials. ■

Let $\mathfrak{P}$ denote the vector space spanned by $u_{i_1} \ldots u_{i_n}$ where $i_1 \leqslant \ldots \leqslant i_n$. These are to be interpreted as formal symbols without meaning.

LEMMA 2.2. There is a k-linear map $\sigma : \mathfrak{T} \to \mathfrak{P}$ such that

$$\sigma(1) = 1 \quad \text{and} \quad \sigma(u_{i_1} \otimes \cdots \otimes u_{i_n}) = u_{i_1} \ldots u_{i_n}.$$

if $i_1 \leqslant \ldots \leqslant i_n$. Further,

$$\sigma(u_{j_1} \otimes \cdots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \cdots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes \cdots \otimes u_{j_n} - u_{j_1} \otimes \ldots u_{j_{k+1}} \otimes u_{j_k} \otimes \ldots u_{j_n}).$$

Proof. We induct on degree and index, in that order. Suppose a linear map σ has been defined on $k \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$. It is easy to extend this to $k \oplus \cdots \oplus \mathfrak{g}_{n,0}$ by setting

$$\sigma(u_{i_1} \otimes \ldots u_{i_n}) = u_{i_1} \ldots u_{i_n}.$$

Now, suppose σ has already been defined for $k \oplus \ldots \mathfrak{g}_{n,i-1}$. Suppose $j_k > j_{k+1}$. Then, define

$$\sigma(u_{j_1} \otimes \cdots \otimes u_{j_n}) = \sigma(u_{j_1} \otimes \cdots \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes \cdots \otimes u_{j_n}) + \sigma(u_{j_1} \otimes \cdots \otimes [u_{j_k}, u_{j_{k+1}}] \otimes \cdots \otimes u_{j_n}).$$

The right hand side is well-defined because the first term on the right has index at most $i-1$ and the second term on the right is a linear combination of monomials of smaller degree.

We must show that this is a well-defined assignment of σ, that is the right hand choice is independent of the pair of inversion chosen. To this end, let $j_l > j_{l+1}$. We must consider two cases.

Case 1: $l > k+1$. Set $u_{j_k} = u$, $u_{j_{k+1}} = v$, $u_{j_l} = w$, $u_{j_{l+1}} = x$.

We would like to show

$$\begin{aligned}&\sigma(\ldots v \otimes u \otimes \cdots \otimes w \otimes x \ldots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes w \otimes x \ldots)\\ &= \\ &\sigma(\ldots u \otimes v \otimes \ldots x \otimes w \ldots) + \sigma(\ldots u \otimes v \otimes \cdots \otimes [x,w] \otimes \ldots).\end{aligned}$$

We can expand the left hand side of the above equality using the induction hypothesis as

$$\begin{aligned}&\sigma(\ldots v \otimes u \otimes \cdots \otimes x \otimes w \ldots) + \sigma(v \otimes u \otimes \cdots \otimes [x,w] \otimes \ldots)\\ &+ \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes x \otimes w \ldots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes [w,x] \otimes \ldots).\end{aligned}$$

The right hand side can be written as

$$\begin{aligned}&\sigma(\ldots v \otimes u \otimes \cdots \otimes x \otimes w \ldots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes x \otimes w \ldots)\\ &+ \sigma(\ldots v \otimes u \otimes \cdots \otimes [x,w] \otimes \ldots) + \sigma(\cdots \otimes [u,v] \otimes \cdots \otimes [x,w] \otimes \ldots).\end{aligned}$$

This completes the proof in this case.

Case 2: $l = k+1$. We write $u_{j_k} = u$, $u_{j_{k+1}} = v = u_{j_l}$ and $u_{j_{l+1}} = w$. We want to show the equality

$$\sigma(\ldots v \otimes u \otimes w \ldots) + \sigma(\ldots [u,v] \otimes w \ldots) = \sigma(\ldots u \otimes w \otimes v \ldots) + \sigma(\ldots u \otimes [v,w] \ldots).$$

The left hand side can be expanded further as

$$\begin{aligned}&\sigma(\ldots v \otimes w \otimes u \ldots) + \sigma(\ldots v \otimes [u,w] \ldots) + \sigma(\ldots [u,v] \otimes w \ldots)\\ &= \sigma(\ldots w \otimes v \otimes u \ldots) + \sigma(\ldots [v,w] \otimes u \ldots) + \sigma(\ldots v \otimes [u,w] \ldots) + \sigma(\ldots [u,v] \otimes w \ldots).\end{aligned}$$

Similarly, the right hand side can be expanded as

$$\begin{aligned}
&\sigma(\dots w \otimes u \otimes v \dots) + \sigma(\dots [u,w] \otimes v \dots) + \sigma(\dots u \otimes [v,w] \dots) \\
&= \sigma(\dots w \otimes v \otimes u \dots) + \sigma(\dots w \otimes [u,w] \dots) + \sigma(\dots [u,w] \otimes v \dots) + \sigma(u \otimes [v,w] \dots).
\end{aligned}$$

It remains to show the equality:

$$\begin{aligned}
&\sigma(\dots [v,w] \otimes u \dots) + \sigma(\dots v \otimes [u,w] \dots) + \sigma(\dots [u,v] \otimes w \dots) \\
&= \sigma(\dots w \otimes [u,w] \dots) + \sigma(\dots [u,w] \otimes v \dots) + \sigma(u \otimes [v,w] \dots),
\end{aligned}$$

which reduces to

$$\sigma(\dots [[v,w],u] \dots) + \sigma(\dots [v,[u,w]] \dots) + \sigma(\dots [[u,v],w] \dots) = 0,$$

which follows from Jacobi's Identity. This completes the proof in this case.

Now that σ is well-defined for monomials, we can extend it linearly to $\mathfrak{g}_{n,i}$, thereby completing the induction. ∎

THEOREM 2.3 (POINCARÉ-BIRKHOFF-WITT THEOREM). The cosets of 1 and the standard monomials form a basis for $\mathfrak{U} = \mathfrak{T}/\mathfrak{K}$.

Proof. We have shown that the standard monomials and 1 span $\mathfrak{U}$. It remains to show linear independence. This follows from the preceding lemma, since the $u_{i_1} \dots u_{i_n}$'s are linearly independent in $\mathfrak{P}$. ∎

§3 PROPERTIES OF THE UNIVERSAL ENVELOPING ALGEBRA

DEFINITION 3.1. A ring R is said to be *filtered* if it is equipped with an increasing sequence $\mathscr{R} = \{R_i\}_{i \geqslant 0}$ of abelian subgroups such that

(a) $\bigcup_{i \geqslant 0} R_i = R$.

(b) For all $i, j \geqslant 0$, $R_i R_j \subseteq R_{i+j}$.

Each filtration of a ring gives rise to an *associated graded ring*,

$$\mathrm{Gr}_{\mathscr{R}}(R) = \bigoplus_{i \geqslant 0} R_i / R_{i-1},$$

with the convention that $R_{i-1} = 0$.

For any $a \in R$, there is a non-negative integer n such that $a \in R_n$ but $a \notin R_{n-1}$. The homogeneous element $\overline{b} = b + R_{n-1} \in \mathrm{Gr}_{\mathscr{R}}(R)$ is called the *leading term* of b. If $b = 0$, we take its leading term to be 0.

LEMMA 3.2. Let R be a filtered ring with an increasing filtration $\{R_i\}_{i \geqslant 0}$ and let G denote the corresponding associated graded.

(a) If G is a domain, then so is R.

(b) If G is left (resp. right) noetherian, then so is R.

Proof. (a) Suppose $a, b \in R \setminus \{0\}$ such that $ab = 0$ in R. Let $\overline{a}, \overline{b}$ denote the leading terms of a and b respectively. Then, $\overline{a}\overline{b} = 0$, a contradiction.

(b) Let I be a left-ideal in R. We shall show that I is finitely generated. Let $\overline{I}$ denote the abelian group generated by the leading terms of elements of I. It is easy to see that $\overline{I}$ is a left ideal in G (whence, is a homogeneous left ideal). Since G is left noetherian, there are $b_1, \ldots, b_n \in I$ such that $\overline{b}_1, \ldots, \overline{b}_n$ generate $\overline{I}$ as a left ideal in G where $\overline{b}_i$ is the leading term of b_i.

We contend that the b_i's generate I. Let $b \in I$. Then, $\overline{b}$, the leading term of b is a linear combination of the form

$$\overline{b} = \sum_i \overline{a}_i \overline{b}_i$$

where $\overline{a}_i \in G$. Since the left hand side is homogeneous, we may choose the a_i's to be homogeneous in G, consequently, the $\overline{a}_i$'s are leading terms of some $a_i \in R$.

From the above equality, we deduce that $\overline{b}$ is the leading term of $\sum_i a_i b_i$ whence, $b - \sum_i a_i b_i$ has leading term of homogeneous degree smaller than that of $\overline{b}$. An induction argument finishes the proof. ■

If $\mathfrak{g}$ is a finite-dimensional Lie algebra over k (no restriction), then its universal enveloping algebra $\mathfrak{U}$ is equipped with a canonical filtration:

$$\mathfrak{U}^{(n)} = k \oplus \mathfrak{g} \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^n.$$

Using the Poincaré-Birkhoff-Witt theorem, it is not hard to see that the associated graded corresponding to the above filtration is isomorphic to $k[X_1, \ldots, X_n]$ where $n = \dim_k \mathfrak{g}$.

THEOREM 3.3. The universal enveloping algebra of a finite-dimensional Lie algebra over k is a left (and right) Noetherian domain.

Proof. Follows from Lemma 3.2 and the discussion above. ■

§4 FREE LIE ALGEBRAS

DEFINITION 4.1. Let X be a set. A *free algebra over k* on X is a pair $(\mathfrak{L}(X), \iota)$ where $\iota : X \to \mathfrak{L}(X)$ is such that for any map of sets $\varphi : X \to \mathfrak{g}$ where $\mathfrak{g}$ is a Lie algebra over k, there is a unique Lie algebra homomorphism $\widetilde{\varphi} : \mathfrak{L}(X) \to \mathfrak{g}$ satisfying

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & \mathfrak{g} \\ \mathfrak{L}(X) \arrow[ur, dashed, "\exists!\widetilde{\varphi}"'] & \end{tikzcd}$$

§§ Construction

Let $\mathfrak{F}(X)$ denote the free k-algebra generated by X. Then, $\mathfrak{F}(X)$ has the structure of a Lie algebra. Let $\mathfrak{L}(X)$ denote the Lie subalgebra of $\mathfrak{F}(X)$ generated by X. We contend that $X \hookrightarrow \mathfrak{L}(X)$ is the free algebra on X.

Let $\varphi : X \to \mathfrak{g}$ be a map of sets where $\mathfrak{g}$ is a Lie algebra over k. Then, we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathfrak{F}(X) \arrow[r, dashed, "\exists ! \widetilde{\varphi}"'] & \mathfrak{U}(\mathfrak{g}) \end{tikzcd}$$

Where $\widetilde{\varphi}$ restricts to φ on $X \subseteq \mathfrak{F}(X)$. Note that φ is also a Lie algebra homomorphism. Therefore, $\widetilde{\varphi}^{-1}\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{F}(X)$ containing X and hence, $\mathfrak{L}(X) \subseteq \widetilde{\varphi}^{-1}\mathfrak{g}$. It follows that $\widetilde{\varphi}$ restricts to a Lie algebra homomorphism $\widetilde{\varphi} : \mathfrak{L}(X) \to \mathfrak{g}$. The uniqueness follows since X generates $\mathfrak{L}(X)$. This completes the proof of existence.

The above discussion also shows:

PROPOSITION 4.2. $\mathfrak{F}(X)$ is the universal enveloping algebra of $\mathfrak{L}(X)$.