Completions

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§1 Graded and Filtered Objects

DEFINITION 1.1. Let $(G,+)$ be an Abelian monoid with identity element $0 \in G$. A *G-graded* ring is a ring R together with a direct sum decomposition

$$R = \bigoplus_{i \in G} R_i$$

into additive subgroups, such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in G$.

Similarly, a *graded R-module* is an R-module M together with a direct sum decomposition

$$M = \bigoplus_{i \in G} M_i$$

into additive subgroups, such that $R_i M_j \subseteq M_{i+j}$ for all $i, j \in G$. An element $x \in M$ is said to be *homogeneous* if $x \in M_i$ for some $i \in G$.

Note that any element $x \in M$ can be written uniquely as a sum

$$x = \sum_{i \in G} x_i,$$

where $x_i \in M_i$ for all $i \in G$. The x_i's are called the *homogeneous components* of x. A submodule $N \subseteq M$ is said to be *homogeneous* if it can be generated by homogeneous elements in M.

A homomorphism $f : M \to N$ of graded R-modules is said to be *graded of degree $d \in G$* if $f(M_i) \subseteq N_{i+d}$ for all $i \in G$. A graded homomorphism of degree 0 is said to be just *graded*.

PROPOSITION 1.2. Let $M = \bigoplus_{i \in G} M_i$ be a G-graded R-module and $N \subseteq M$ a submodule. The following are equivalent:

(1) N is a homogeneous submodule of M.

(2) For all $x \in N$, each homogeneous component of x lies in N.

(3) $N = \sum_{i \in G} (N \cap M_i)$.

Proof. (1) $\implies$ (2) Suppose N is generated as an R-module by the homogeneous elements $\{z_j\}$. Then we can write

$$x = \sum_j r_j z_j = \sum_j \left(\sum_{g \in G} r_j^g z_j \right),$$

where we can decompose r_j as

$$r_j = \sum_{g \in G} r_j^g$$

into its homogeneous components in R. Grouping together components of the same degree shows that every homogeneous component of x lies in N.

(2) $\Longrightarrow$ (3) Clearly, the right hand side is contained in the left hand side. Conversely, if $x \in N$, then we can write $x = \sum_{i\in G} x_i$ where the x_i's are the homogeneous components of x. According to (2), $x_i \in N \cap M_i$ for each $i \in G$, whence x is contained in the right hand side, as desired.

(3) $\Longrightarrow$ (1) Indeed, N is generated as an R-module by the set

$$\bigcup_{i\in G} (N \cap M_i)$$

consisting only of homogeneous elements. ∎

PROPOSITION 1.3. Let $R = \bigoplus_{i\in G} R_i$ be a G-graded ring. Then R_0 is a subring of R and for every graded R-module $M = \bigoplus_{i\in G} M_i$, each M_i is naturally an R_0-module.

Proof. R_0 is an additive subgroup of R and $R_0R_0 \subseteq R_0$. Thus it suffices to show that $1 \in R_0$. We can decompose x into its homogeneous components as

$$1 = \sum_{i\in G} x_i.$$

For any $j \in G$, we then have

$$x_j = \sum_{i\in G} x_i x_j,$$

where $x_i x_j$ is homogeneous of degree $i + j$. Therefore,

$$x_i x_j = \begin{cases} x_j & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Summing over all $j \in G$, we get

$$x_0 = \sum_{j\in G} x_0 x_j = \sum_{j\in G} x_j = 1.$$

That is, $1 \in R_0$, and hence R_0 is a subring of R. Finally, since $R_0 M_i \subseteq M_i$, it follows that each M_i is naturally an R_0-module. ∎

REMARK 1.4 (QUOTIENT OF GRADED MODULES). Let $M = \bigoplus_{i\in G} M_i$ be a graded module over a graded ring $R = \bigoplus_{i\in G} R_i$, and let $N \subseteq M$ be a graded submodule of M. We can endow the quotient module M/N with a natural grading

$$\bigoplus_{i\in G} M_i/N_i,$$

where the R-module structure is the obvious one. To see that this is indeed isomorphic to M/N as an R-module, consider the graded projection $\pi\colon M \to M/N$ given by

$$\pi\left(\sum_{i\in G} x_i\right) = \sum_{i\in G} x_i \bmod N_i.$$

One can check that this is an R-linear surjective homomorphism with $\ker \pi = N$, which implies the desired conclusion.

Analogously, if $I \trianglelefteq R$ is a graded ideal, then R/I is naturally a graded R-module as above, and has a ring structure given by

$$(r_i \bmod I_i)(r_j \bmod I_j) = r_i r_j \bmod I_{i+j}.$$

Henceforth, R/I shall always be thought of a graded ring with the above grading.

Throughout this article, we shall mainly concern ourselves with the case $G = (\mathbb{N}, +, 0)$, and henceforth, a graded ring/module shall refer to an $\mathbb{N}$-graded ring/module. For an $\mathbb{N}$-graded ring $R = \bigoplus_{n\geqslant 0} R_n$, we set

$$R_+ = \bigoplus_{n\geqslant 1} R_n,$$

which is clearly a homogeneous ideal in R and is called the *irrelevant ideal* of R. Often when R_0 is a field, then the irrelevant ideal turns out to be the unique *graded* maximal ideal and is denoted by $\mathfrak{m}_+$ for empasis.

THEOREM 1.5 (GRADED NAKAYAMA). Let $R = \bigoplus_{n\geqslant 0} R_n$ be a graded ring, and $M = \bigoplus_{n\geqslant 0} M_n$ a graded R-module. If $R_+M = M$, then $M = 0$.

Proof. Let $n \geqslant 0$ be the smallest non-negative integer such that $M_n \neq 0$. Let $0 \neq x_n \in M_n$. Using the fact that $R_+M = M$, we can write

$$x_n = \sum_\lambda r^\lambda y^\lambda,$$

for some finite set of $r^\lambda \in R_+$ and $y^\lambda \in M$. Writing out each r^λ and y^λ in its homogeneous components and isolating terms of degree n, we get

$$x_n = \sum_\lambda \left(\sum_{i+j=n} r_i^\lambda y_j^\lambda \right).$$

But since $r^\lambda \in R_+$, if $r_i^\lambda \neq 0$ then $i \geqslant 1$, so that $j \leqslant n-1$, and hence $y_j^\lambda = 0$. Thus $x_n = 0$, a contradiction. This completes the proof. ■

DEFINITION 1.6. A *filtered ring* is a ring R together with a descending chain of additive subgroups

$$R = R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots,$$

such that $R_nR_m \subseteq R_{n+m}$ for all $n, m \geqslant 0$. In particular, each R_n is an ideal in the ring R.

Let R be a filtered ring as above. A *filtered module* over R is an R-module M together with a descending chain of R-submodules of M

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

such that $R_mM_n \subseteq M_{m+n}$ for all $m, n \geqslant 0$.

A map $f : M \to N$ between filtered R-modules is said to be a filtered homomorphism if $f(M_n) \subseteq N_n$ for all $n \geqslant 0$.

REMARK 1.7. Let M be a filtered module over a filtered ring R as above. For an R-submodule $N \subseteq M$, we define the *induced filtration* on N as $N_n = N \cap M_n$ for all $n \geqslant 0$. Similarly, we define the induced filtration on M/N as

$$\left(\frac{M}{N}\right)_n = \frac{N + M_n}{N}.$$

Equipped with these filtrations, every map in the short exact sequence

$$0 \to N \hookrightarrow M \twoheadrightarrow M/N \to 0$$

is a filtered homomorphism.

DEFINITION 1.8. Let R be a filtered ring and M a filtered R-module as above. Define the *associated graded ring*

$$\operatorname{gr}(R) = \bigoplus_{n \geqslant 0} R_n / R_{n+1}$$

with product structure given by

$$(x + R_{n+1})(y + R_{m+1}) = xy + R_{n+m+1}$$

for all $n, m \geqslant 0$. It is easy to check that $\operatorname{gr}(R)$ is a graded ring.

We further define the *associated graded module*

$$\operatorname{gr}(M) = \bigoplus_{n \geqslant 0} M_n / M_{n+1}$$

which is a graded $\operatorname{gr}(R)$-module with the module structure given by

$$(a + R_{m+1}) \cdot (x + M_{n+1}) = a \cdot x + M_{m+n+1}.$$

If N is another filtered R-module and $f \colon M \to N$ a filtered homomorphism, then there is an induced graded $\operatorname{gr}(R)$-homomorphism $\operatorname{gr}(f) \colon \operatorname{gr}(M) \to \operatorname{gr}(N)$ given by

$$\operatorname{gr}(f)(x + M_{n+1}) = f(x) + N_{n+1}.$$

We note that gr, as defined above, is a functor from the category of filtered R-modules to the category of graded $\operatorname{gr}(R)$-modules. Indeed, it is trivial to check that $\operatorname{gr}(\mathbf{id}_M) = \mathbf{id}_{\operatorname{gr}(M)}$ and that $\operatorname{gr}(g \circ f) = \operatorname{gr}(g) \circ \operatorname{gr}(f)$.

PROPOSITION 1.9. Let R be a filtered ring, M and N filtered R-modules, and $f \colon M \to N$ a filtered homomorphism. If

(i) $\operatorname{gr}(f) \colon \operatorname{gr}(M) \to \operatorname{gr}(N)$ is injective, and

(ii) $\bigcap_{n \geqslant 0} M_n = 0$,

then f is injective.

Proof. Since $\operatorname{gr}(f)$ is injective, the map $\operatorname{gr}_n(f) \colon M_n/M_{n+1} \to N_n/N_{n+1}$ is injective for every $n \geqslant 0$. We shall first show by induction on $n \geqslant 0$ that $f^{-1}(N_n) \subseteq M_n$. Clearly $f^{-1}(N_0) \subseteq M_0$. As for the inductive step, note that

$$f^{-1}(N_{n+1}) \subseteq f^{-1}(N_n) \subseteq M_n.$$

Hence,

$$f^{-1}(N_{n+1}) \subseteq f^{-1}(N_{n+1}) \cap M_n \subseteq M_{n+1},$$

where the last containment follows from the fact that $\operatorname{gr}_n(f)$ is injective. As a result,

$$f^{-1}(0) \subseteq f^{-1}\left(\bigcap_{n \geqslant 0} N_n\right) \subseteq \bigcap_{n \geqslant 0} M_n = 0,$$

thereby completing the proof. ■

THEOREM 1.10. Let $R = \bigoplus_{n \geqslant 0} R_n$ be a graded ring. The following are equivalent:

(1) R is Noetherian.

(2) R_0 is Noetherian and R is a finitely generated R_0-algebra.

Proof. Clearly (2) $\implies$ (1) due to the Hilbert Basis Theorem. Suppose now that R is Noetherian. Since

$$R_0 \cong R/R_+,$$

so is R_0. Now, R_+ is a finitely generated ideal in R and we may assume that it is generated by homogeneous elements $x_1, \ldots, x_r \in R_+$ of degrees $n_1, \ldots, n_r$ respectively. Let $R' = R_0[x_1, \ldots, x_n] \subseteq R$. We shall show by induction on $n \geqslant 0$ that $R_n \subseteq R'$. Trivially $R_0 \subseteq R'$. Suppose it is known that $R_k \subseteq R'$ for all $k < n$. An element $x \in R_n$ can be written as an R-linear combination of $x_1, \ldots, x_r$ as

$$x = a_1 x_1 + \cdots + a_r x_r.$$

Since x is a homogeneous element, breaking each a_i into its homogeneous components and grouping terms of the same degree, we may suppose that each a_i is graded of degree $n - n_i$, with the convention that $R_d = 0$ for $d < 0$. Since $n_i \geqslant 1$ for all $1 \leqslant i \leqslant r$, it follows that $a_i \in R'$ for all $1 \leqslant i \leqslant r$. Hence $x \in R'$, thereby completing the proof. ■

DEFINITION 1.11. Let $M = (M_n)_{n \geqslant 0}$ be a filtered R-module and I an ideal in R. The filtration is said to be an *I-filtration* if $IM_n \subseteq M_{n+1}$ for all $n \geqslant 0$. Further, an I-filtration is said to be *I-stable* if $IM_n = M_{n+1}$ for all $n \gg 0$.

Let M be a filtered R-module with an I-filtration. We define the *Rees algebra* of R as a subring of the polynomial algebra

$$R^* = \bigoplus_{n \geqslant 0} I^n T^n \subseteq R[T].$$

That is, R^* consists of all polynomials $a_0 + a_1 T + \cdots + a_n T^n \in R[T]$ such that $a_i \in I^i$ for all $i \geqslant 0$. Note that if R is Noetherian, then I is a finitely generated ideal, say $I = (a_1, \ldots, a_r)$. Then R^* is precisely the ring

$$R^* = R[a_1 T, \ldots, a_r T] \subseteq R[T],$$

and in particular, is a Noetherian ring.

Similarly, we define an R^*-module

$$M^* = \bigoplus_{n \geqslant 0} M_n T^n$$

whose elements are formal sums

$$\sum_{\substack{n \geqslant 0 \\ \text{finite}}} x_n T^n,$$

with the obvious module structure over R^*.

THEOREM 1.12. Let R be a filtered Noetherian ring, I an ideal of R, and M a finitely generated filtered R-module equipped with an I-filtration. The following are equivalent:

(1) The filtration on M is I-stable.

(2) M^* is a finitely generated R^*-module.

Proof. Set
$$M_n^* = M_0 \oplus M_1 T \oplus \cdots \oplus M_n T^n \oplus I M_n T^{n+1} \oplus I^2 M_n T^{n+2} \oplus \cdots,$$
which is clearly an R^*-module. Further, since each M_n is a finite R-module, we can choose a finite R-generating set for the module $M_0 \oplus \cdots \oplus M_n$, which would then be an R^*-generating set for M_n^*. That is, each M_n^* is a finite R^*-module.

Note that the filtration on M being I-stable is equivalent to the ascending chain $(M_n^*)_{n\geqslant 0}$. We also have
$$M^* = \bigcup_{n\geqslant 0} M_n^*.$$
Thus, if the chain stabilizes, then M^* is a finite R^*-module. Conversely, if M^* is a finite R^*-module, then M^* is Noetherian, and hence the chain must stabilize. This completes the proof. ■

Lemma 1.13 (Artin-Rees). Let R be a filtered Noetherian ring, and M a finitely generated filtered R-module equipped with an I-stabie flitration. If N is a submodule of M, then the induced filtration on N is I-stable.

Proof. Let $N_n = N \cap M_n$, which is the induced filtration on N. Clearly this filtration is I-stable. We shall treat N^* as a natural R^*-submodule of M^*. Since the filtration on M is I-stable, due to Theorem 1.12, M^* is a finite R^*-module, but since R^* is Noetherian, N^* is also a finite R^*-module, so that by Theorem 1.12, the filtration on N is I-stable. ■

Theorem 1.14 (Krull Intersection Theorem). Let R be a Noetherian ring, I an ideal of R, and M a finite R-module. Then the module
$$\bigcap_{n\geqslant 0} I^n M$$
consists of precisely those elements that are annihilated by some element in $1+I$.

Proof. Let
$$N = \bigcap_{n\geqslant 0} I^n M,$$
which is a submodule of M. In view of Lemma 1.13, this filtration is I-stable. That is, for $n \gg 0$,
$$IN = I\left(N \cap I^n M\right) = N \cap I^{n+1} M = N.$$
By Nakayama's lemma, N is annihilated by an element of the form $1+a$, where $a \in I$.

Conversely, suppose $x \in M$ is such that $(1+a)x = 0$ for some $a \in I$. Then
$$x = -ax = a^2 x = -a^3 x = \cdots,$$
and hence $x \in \bigcap_{n\geqslant 0} I^n M = N$. ■

Corollary 1.15. If R is a Noetherian ring, I an ideal of R contained in the Jacobson radical, and M a finite R-module, then
$$\bigcap_{n\geqslant 0} I^n M = 0.$$
Proof. This follows from the fact that every element in $1+I$ is invertible. ■

Corollary 1.16. Let R be a Noetherian domain, and I an ideal in R. Then
$$\bigcap_{n\geqslant 0} I^n = 0.$$
Proof. This follows from the fact that every element in $1+I$ is a non-zerodivisor. ■