Riemann-Roch for Riemann Surfaces

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Abstract

This is an attempt to present a relatively self-contained proof of the Riemann-Roch theorem for compact Riemann surfaces. We assume that the reader is familiar with the language of sheaves and the basics of Riemann surfaces. The exposition herein is mainly taken from two books, [For12] and [NN00]. The latter is closely followed while proving the finiteness of cohomology.

Contents

§1 Čech Cohomology

Definition 1.1. Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. Define the *q-th cochain group* of $\mathscr{F}$ with respect to $\mathfrak{U}$ as

$$C^q(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

We denote elements of this cochain group by $\left(f_{i_0, \dots, i_q}\right)_{(i_0, \dots, i_q) \in I^{q+1}}$ and these are called q-cochains.

Next, define the *coboundary operators*

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}) \qquad \delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$$

as follows:

- For $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathscr{F})$, let $\delta\left((f_i)_{i \in I}\right) = (g_{ij})_{i,j \in I} \in C^2(\mathfrak{U}, \mathscr{F})$ where
$$g_{ij} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$
- For $(f_{ij})_{i,j \in I} \in C^1(\mathfrak{U}, \mathscr{F})$, let $\delta\left((f_{ij})_{i,j \in I}\right) = (g_{ijk})_{i,j,k \in I}$ where
$$g_{ijk} = f_{jk}|_{U_i \cap U_j \cap U_k} - f_{ik}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

The coboundary operators are group homomorphisms and let

$$Z^1(\mathfrak{U}, \mathscr{F}) = \ker\left(C^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathscr{F})\right) \qquad B^1(\mathfrak{U}, \mathscr{F}) = \operatorname{im}\left(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F})\right).$$

The elements of $Z^1(\mathfrak{U}, \mathscr{F})$ are called 1-cocycles and the elements of $B^1(\mathfrak{U}, \mathscr{F})$ are called 1-coboundaries. It is easy to see that $(f_{ij})_{i,j\in I}$ is a 1-cocycle if and only if

$$f_{ik}|_{U_i\cap U_j\cap U_k} = f_{jk}|_{U_i\cap U_j\cap U_k} + f_{ij}|_{U_i\cap U_j\cap U_k} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

The above relation is called the *cocycle relation*. Indeed, if $(f_{ij})_{i,j\in I}$ is a 1-cocycle, then taking $i = j$, we see that

$$f_{ii}|_{U_i\cap U_k} = 0 \qquad \forall k \in I.$$

Since the U_k's cover U_i, using the identity axiom, we have that $f_{ii} = 0 \in \mathscr{F}(U_i)$. As a consequence, we also see that

$$f_{ji}|_{U_i\cap U_j\cap U_k} + f_{ij}|_{U_i\cap U_j\cap U_k} = 0.$$

Again, using the same argument, we have that $f_{ij} + f_{ji} = 0 \in \mathscr{F}(U_i \cap U_j)$. It immediately follows from the above discussion that $\delta \circ \delta = 0$ as a map $C^0(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$.

Definition 1.2. The group

$$H^1(\mathfrak{U}, \mathscr{F}) := \frac{Z^1(\mathfrak{U}, \mathscr{F})}{B^1(\mathfrak{U}, \mathscr{F})}$$

is called the *1-st cohomology group* with coefficients in $\mathscr{F}$ *with respect to the covering* $\mathfrak{U}$.

Definition 1.3. Let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_k)_{k\in K}$ be two open covers of X. We say that $\mathfrak{V}$ is *finer* than $\mathfrak{U}$ if every V_k is contained in some U_i.

Thus, there is a map $\tau : K \to I$ such that $V_k \subseteq U_{\tau(k)}$. This defines a mapping

$$t^{\mathfrak{U}}_{\mathfrak{V}} : Z^1(\mathfrak{U}, \mathscr{F}) \to Z^1(\mathfrak{V}, \mathscr{F})$$

as follows: for $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$, let $t^{\mathfrak{U}}_{\mathfrak{V}}\left((f_{ij})\right) = (g_{kl})$, where

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k\cap V_l} \qquad \forall\, k, l \in K.$$

To see that this map is indeed well-defined, we need only check that (g_{kl}) is a 1-cocycle. To this end, we must check that the cocycle condition is satisfied. Indeed, for indices $k, l, m \in K$, we have

$$\begin{aligned} g_{km}|_{V_k\cap V_l\cap V_m} &= f_{\tau(k)\tau(m)}|_{V_k\cap V_l\cap V_m} \\ &= \left(f_{\tau(k)\tau(l)}|_{U_{\tau(k)}\cap U_{\tau(l)}\cap U_{\tau(m)}} + f_{\tau(l)\tau(m)}|_{U_{\tau(k)}\cap U_{\tau(l)}\cap U_{\tau(m)}}\right)|_{V_k\cap V_l\cap V_m} \\ &= f_{\tau(k)\tau(l)}|_{V_k\cap V_l\cap V_m} + f_{\tau(k)\tau(l)}|_{V_k\cap V_l\cap V_m} \\ &= g_{kl}|_{V_k\cap V_l\cap V_m} + g_{lm}|_{V_k\cap V_l\cap V_m}, \end{aligned}$$

as desired. Further, we claim that the above map takes 1-coboundaries to 1-coboundaries. Indeed, suppose $(f_{ij})_{i,j\in I}$ is a 1-coboundary, that is, there is some $(f_i)_{i\in I}$ such that

$$f_{ij} = f_i|_{U_i\cap U_j} - f_j|_{U_i\cap U_j}.$$

Let $(g_k)_{k\in K}$ be such that $g_k = f_{\tau(k)}|_{V_k}$. Then $\delta\left((g_k)\right) = (g_{kl})$ where

$$\begin{aligned} g_{kl} &= g_k|_{V_k\cap V_l} - g_l|_{V_k\cap V_l} \\ &= f_{\tau(k)}|_{V_k\cap V_l} - f_{\tau(l)}|_{V_k\cap V_l} \\ &= f_{\tau(k)\tau(l)}|_{V_k\cap V_l}, \end{aligned}$$

that is, $(g_k l) = t^{\mathfrak{U}}_{\mathfrak{V}}\left((f_{ij})\right)$, that is, $t^{\mathfrak{U}}_{\mathfrak{V}}$ takes 1-coboundaries to 1-coboundaries. This induces a map

$$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F}).$$

LEMMA 1.4. The map $t_{\mathfrak{V}}^{\mathfrak{U}}$ induced on cohomology is independent of the choice of $\tau : K \to I$.

Proof. Suppose $\widetilde{\tau} : K \to I$ is another such mapping. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and let

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k \cap V_l} \qquad \text{and} \qquad \widetilde{g}_{kl} = f_{\widetilde{\tau}(k)\widetilde{\tau}(l)}|_{V_k \cap V_l}.$$

We must show that the cocycles (g_{kl}) and $(\widetilde{g}_{kl})$ are cohomologous, that is, their difference lies in $B^1(\mathfrak{V}, \mathscr{F})$. Define

$$h_k = f_{\tau(k),\widetilde{\tau}(k)}|_{V_k} \in \mathscr{F}(V_k).$$

Then, on $V_k \cap V_l$, we have

$$\begin{aligned} g_{kl} - \widetilde{g}_{kl} &= f_{\tau(k)\tau(l)} - f_{\widetilde{\tau}(k)\widetilde{\tau}(l)} \\ &= f_{\tau(k)\tau(l)} + f_{\tau(l),\widetilde{\tau}(k)} - f_{\tau(l)\widetilde{\tau}(k)} - f_{\widetilde{\tau}(k)\widetilde{\tau}(l)} \\ &= f_{\tau(k)\widetilde{\tau}(k)} - f_{\tau(l)\widetilde{\tau}(l)} \\ &= h_k - h_l. \end{aligned}$$

Whence $(g_{kl} - \widetilde{g}_{kl})$ is a coboundary, as desired. ∎

LEMMA 1.5. The map $t_{\mathfrak{V}}^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is injective.

Proof. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ is a 1-cocycle whose image in $Z^1(\mathfrak{V}, \mathscr{F})$ is a 1-coboundary. Then, there is some $(g_k) \in C^1(\mathfrak{U}, \mathscr{F})$ such that

$$f_{\tau(k)\tau(l)}|_{V_k \cap V_l} = g_k|_{V_k \cap V_l} - g_l|_{V_k \cap V_l}.$$

Then on $U_i \cap V_k \cap V_l$, we have

$$g_k - g_l = f_{\tau(k)\tau(l)} = f_{\tau(k)i} + f_{i\tau(l)} = f_{i\tau(l)} - f_{i\tau(k)}.$$

Hence $f_{i\tau(k)} + g_k = f_{i\tau(l)} + g_l$ on $U_i \cap V_k \cap V_l$. The gluability axiom applied to the open cover $\{U_i \cap V_k\}_{k \in K}$ furnishes a $h_i \in \mathscr{F}(U_i)$ such that

$$h_i = f_{i\tau(k)} + g_k \qquad \text{on } U_i \cap V_k \text{ for all } k \in K.$$

Then, on $U_i \cap U_j \cap V_k$ we have

$$f_{ij} = f_{i\tau(k)} + f_{\tau(k)j} = f_{i\tau(k)} + g_k - f_{j\tau(k)} - g_k = h_i - h_j.$$

Since $\{U_i \cap U_j \cap V_k\}$ forms an open cover of $U_i \cap U_j$, using the identity axiom, we have that $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. Thus, $(f_{ij}) \in B^1(\mathfrak{U}, \mathscr{F})$, thereby completing the proof. ∎

DEFINITION 1.6. If $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$ are open covers of X, then it is easy to see that $t_{\mathfrak{W}}^{\mathfrak{V}} \circ t_{\mathfrak{V}}^{\mathfrak{U}} = t_{\mathfrak{W}}^{\mathfrak{U}}$. This gives us a directed system of cohomology groups, and we define

$$H^1(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}).$$

REMARK 1.7. Note that since the $t_{\mathfrak{V}}^{\mathfrak{U}}$ are all injective, the natural map $H^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ is injective. In particular, this means that $H^1(X, \mathscr{F}) = 0$ if and only if $H^1(\mathfrak{U}, \mathscr{F}) = 0$ for every open cover $\mathfrak{U}$ of X.

THEOREM 1.8 (LERAY). Let $\mathscr{F}$ be a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X such that $H^1(U_i, \mathscr{F}) = 0$ for every $i \in I$. Then for every open covering $\mathfrak{V} = (V_\alpha)_{\alpha \in A} < \mathfrak{U}$, the mapping

$$t_{\mathfrak{V}}^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is an isomorphism. The covering $\mathfrak{U}$ is called a *Leray covering* of X for the sheaf $\mathscr{F}$.

Proof. Let $\tau : A \to I$ be such that $V_\alpha \subseteq U_{\tau(\alpha)}$ for every $\alpha \in A$. Since we know that $t^{\mathfrak{U}}_{\mathfrak{V}}$ is injective, we would like to show that it is surjective. Let $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathscr{F})$. The family $(U_i \cap V_\alpha)_{\alpha\in A}$ is an open covering of U_i, which we denote by $U_i \cap \mathfrak{V}$. By assumption and Remark 1.7, we know that $H^1(U_i \cap \mathfrak{V}, \mathscr{F}) = 0$, that is, there exist $g_{i\alpha} \in \mathscr{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{on } U_i \cap V_\alpha \cap V_\beta.$$

On the intersection $U_i \cap U_j \cap V_\alpha \cap V_\beta$, we have

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}.$$

Using the gluability axiom on the open cover $\{U_i \cap U_j \cap V_\alpha\}_{\alpha\in A}$, there exist elements $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

We claim that f_{ij} satisfies the cocycle condition. Obviously, from the above description, on $U_i \cap U_j \cap U_k \cap V_\alpha$ we have that $F_{ik} = F_{ij} + F_{jk}$. Using the identity axiom, we see that this equality holds on $U_i \cap U_j \cap U_k$. Thus, $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. Let $h_\alpha = g_{\tau(\alpha)\alpha}|_{V_\alpha} \in \mathscr{F}(V_\alpha)$. The on $V_\alpha \cap V_\beta$, we have

$$\begin{aligned} F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} &= \left(g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}\right) - \left(g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta}\right) \\ &= g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} \\ &= h_\beta - h_\alpha, \end{aligned}$$

whence $(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta})$ is a coboundary, thereby completing the proof. ∎

COROLLARY 1.9. If $\mathfrak{U}$ is a Leray covering of X, then $H^1(\mathfrak{U}, \mathscr{F}) \cong H^1(X, \mathscr{F})$.

§2 THE FINITENESS THEOREM

§§ Laurent Schwartz's Theorem

THEOREM 2.1 (CLOSED RANGE THEOREM). Let $u : E \to F$ be a continuous linear map between Banach spaces. Then $u(E)$ is closed in F if and only if $u^*(F^*)$ is closed in E^*.

Proof. See [Rud91, Theorem 4.14]. ∎

THEOREM 2.2 (SCHAUDER). Let $u : E \to F$ be a continuous linear map between Banach spaces. Then u is compact if and only if u^* is.

Proof. See [Rud91, Theorem 4.19]. ∎

LEMMA 2.3. Let E, F be Banach spaces and let $u : E \to F$ be a continuous linear map. Suppose that u is injective and that $u(E)$ is closed. Let $v : E \to F$ be a compact continuous linear map. Then $\ker(u + v)$ is finite-dimensional and $(u + v)(E)$ is closed in F.

Proof. Let $N = \ker(u + v)$. To see that this is finite-dimensional, it suffices to show that the closed unit ball in N is compact. To this end, let (x_n) be a sequence in the closed unit ball of N. Since v is compact, there is a subsequence (x_{n_k}) such that $(v(x_{n_k}))$ converges, as a result, $u(x_{n_k}) = -v(x_{n_k})$ also converges. Since $u(E)$ is closed in F, it is a Banach space and $u : E \to u(E)$ is a bijection, whence, due to the "bounded inverse theorem", there is a constant $c > 0$ such that $\|u(x)\| \geqslant c\|x\|$ for all $x \in E$, consequently, for $k, l \geqslant 1$,

$$\|x_{n_k} - x_{n_l}\| \leqslant \frac{1}{c}\|u(x_{n_k}) - u(x_{n_l})\|,$$

whence (x_{n_k}) is Cauchy, and thus converges. This shows that N is finite-dimensional.

Owing to N being finite-dimensional, there is a closed subspace N' of E such that $E = N \oplus N'$. Since $(u+v)(E) = (u+v)(N')$, it suffices to show that the latter is closed in F. Let (x_n) be a sequence in N' such that $(u+v)(x_n)$ converges in F; we show that the limit lies in $(u+v)(N')$. First, we claim that the sequence (x_n) is bounded. If not, then we can move to a subsequence and assume that $0 \neq x_n$ for all n and $\|x_n\| \to \infty$ as $n \to \infty$. Set $x_n' = x_n/\|x_n\|$. Then $\|x_n'\| = 1$ and $(u+v)(x_n') \to 0$ as $n \to \infty$. Since v is compact, there is a subsequence (x_{n_k}') such that $v(x_{n_k}')$ converges, consequently,

$$u(x_{n_k}') = (u+v)(x_{n_k}') - v(x_{n_k}')$$

also converges. As we argued in the preceding paragraph using the "bounded inverse theorem", this means that (x_{n_k}') converges. It follows that there is some $x_0 \in N'$ with $\|x_0\| = 1$ and $(u+v)(x_0) = 0$, that is, $x_0 \in N \cap N' = \{0\}$, a contradiction. Hence, (x_n) is a bounded sequence in E.

Compactness of v implies that there is a subsequence (x_{n_k}) such that $v(x_{n_k})$ converges in F; and since $(u+v)(x_n)$ was assumed to be convergent, we see that $u(x_{n_k})$ is convergent too. Again, using the "bounded inverse theorem", we have that (x_{n_k}) is convergent to some $x_0 \in N'$. Hence,

$$\lim_{n\to\infty}(u+v)(x_n) = \lim_{k\to\infty}(u+v)(x_{n_k}) = (u+v)(x_0) \in (u+v)(E),$$

as desired. ■

THEOREM 2.4 (L. SCHWARTZ). Let E, F be Banach space and let $u, v : E \to F$ be continuous linear maps. Suppose that u is surjective and that v is compact. Then $F' = (u+v)(E)$ is closed and F/F' is finite-dimensional.

Proof. Due to Theorem 2.1, it suffices to show that $(u^* + v^*)(F^*)$ is closed in E^*. Due to Theorem 2.2, we know that v^* is compact, and due to Theorem 2.1, we know that $u^*(F^*)$ is closed in E^*. Further, since u is surjective, it is easy to see that u^* must be injective. Thus, due to Lemma 2.3, we see that $(u^*+v^*)(F^*)$ is closed in E^*, as desired. We have shown that F' is closed in F.

To show that F/F' is finite-dimensional, we shall show that its closed unit ball is compact. Indeed, let (w_n) be a sequence in the closed unit ball of F/F', and choose preimages (x_n) in F satisfying $\|x_n\| \leqslant 2$. Since $u : E \to F$ is surjective, it is a consequence of the open mapping theorem, that there is a constant $M > 0$, independent of the sequence chosen, and a sequence (y_n) in E such that $\|y_n\| \leqslant M$ and $u(y_n) = x_n$. Since v is compact, there is a subsequence (x_{n_k}) such that $z_k = v(x_{n_k})$ converges in F to some $\tilde{z}$. We can write

$$y_{n_k} = u(x_{n_k}) + v(x_{n_k}) - z_k = (u+v)(x_{n_k}) - z_k,$$

and hence, $-z_k$ maps to w_{n_k} in F/F'. Since the former converges, so does the latter. It follows that (w_n) admits a convergent subsequence, consequently, the closed unit ball in F/F' is compact, whence F/F' is finite-dimensional. This completes the proof. ■

§§ The Finiteness Theorem

The objective of this (sub)section is to prove the following theorem:

THEOREM 2.5. Let X be a compact Riemann surface, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of X. Then $H^1(\mathfrak{U}, \mathscr{O})$ is a finite-dimensional vector space.

Since X is compact, there is a finite covering $\mathfrak{V} = \{V_\alpha\}$ such that each V_α is an *analytic disc*[1] and such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Due to Theorem 1.8, we know that $H^1(\mathfrak{U}, \mathscr{O}) \cong H^1(\mathfrak{V}, \mathscr{O})$ and hence, it suffices to show that the latter is a finite-dimensional vector space.

We may therefore suppose that $\mathfrak{U} = \{U_i\}_{i\in I}$ where I is finite, and that there are charts $\phi_i : U_i \to D \subseteq \mathbb{C}$, there $D = B(0,1)$, the open unit disc in $\mathbb{C}$. For any $r < 1$, set $U_i(r) = \phi_i^{-1}(B(0,r))$. There is a $0 < r_0 < 1$ such that $\bigcup_{i\in I} U_i(r_0) = X$. Thus, for $r_0 < r < 1$, we have that $\bigcup_{i\in I} U_i(r) = X$. Denote by $\mathfrak{U}(r)$ the finite covering $\{U_i(r)\}_{i\in I}$ of X.

[1] An open set $U \subseteq X$ is said to be a analytic disc if there is a chart $\phi : U \to D$ where D is the open unit disc in $\mathbb{C}$.

Let

$$Z_b^1(r) = \left\{ (f_{ij})_{i,j\in I} \in Z^1(\mathfrak{U}(r), \mathscr{O}) \colon \sup_{x\in U_i(r)\cap U_j(r)} |f_{ij}(x)| < \infty \quad \text{for all } i,j \in I \right\},$$

and for $\xi = (f_{ij})_{i,j\in I} \in Z_b^1(r)$, define

$$\|\xi\| := \|\xi\|_r = \max_{i,j\in I} \sup_{x\in U_i(r)\cap U_j(r)} |f_{ij}(x)|.$$

The above gives $Z_b^1(r)$ the structure of a normed linear space. It is easy to see that $Z_b^1(r)$ is in fact a Banach space since a uniformly Cauchy sequence of holomorphic functions converges to a holomorphic function.

Next, define

$$B_b^1(r) = Z_b^1(r) \cap B^1(\mathfrak{U}(r), \mathscr{O}),$$

and

$$C_b^0(r) = \left\{ (f_i)_{i\in I} \in C^0(\mathfrak{U}(r), \mathscr{O}) \colon \sup_{x\in U_i(r)} |f_i(x)| < \infty \quad \text{for all } i \in I \right\}.$$

If $\eta = (f_i)_{i\in I} \in C_b^0(r)$, we also set

$$\|\eta\| = \|\eta\|_r = \max_{i\in I} \sup_{x\in U_i(r)} |f_i(x)|.$$

Then $C_b^0(r)$ is also a Banach space under the above norm for the same reason $Z_b^1(r)$ was.

Lemma 2.6. Let $\delta : C^0(\mathfrak{U}(r), \mathscr{O}) \to B^1(\mathfrak{U}(r), \mathscr{O})$ be the coboundary operator as defined earlier. We have

$$\delta^{-1}\left(B_b^1(r)\right) = C_b^0(r).$$

Proof. Obviously, the image of $C_b^0(r)$ under δ is contained in $B_b^1(r)$ since the indexing set I is finite. Conversely, let $(f_i)_{i\in I} \in C^0(\mathfrak{U}(r), \mathscr{O})$ be such that

$$M_{ij} = \sup_{x\in U_i(r)\cap U_j(r)} |f_i(x) - f_j(x)| < \infty \quad \text{for all } i,j \in I.$$

Since $U_i(r)$ is relatively compact in U_i, it suffices to show that every $a \in \partial U_i(r)$ has a neighborhood V in U_i such that

$$\sup_{x\in V\cap U_i(r)} |f_i(x)| < \infty.$$

Indeed, since $\partial U_i(r) = \overline{U_i(r)} \setminus U_i(r)$ is compact, it can be covered with finitely many such V's and hence, there is a constant $M > 0$ such that for each $a \in \partial U_i(r)$, we have

$$\limsup_{\substack{x\to a\\ x\in U_i(r)}} |f_i(x)| < M,$$

consequently, from the Maximum Modulus Principle (see [Con78, Theorem VI.1.4]), f_i is bounded on $U_i(r)$, that is, $(f_i)_{i\in I} \in C_b^0(r)$.

Now, let $a \in \partial U_i(r)$, choose an index $j \in I$ such that $a \in U_j(r)$, and let V be a neighborhood of a that is relatively compact in $U_j(r)$. We have, for $x \in V \cap U_i(r)$,

$$\sup_{x\in V\cap U_i(r)} |f_i(x)| \leqslant \sup_{x\in V\cap U_i(r)} |f_i(x) - f_j(x)| + \sup_{x\in V} |f_j(x)| \leqslant M_{ij} + \sup_{x\in V} |f_j(x)| < \infty,$$

since V is relatively compact in $U_j(r)$. The conclusion follows. ∎

THEOREM 2.7 (MONTEL). Let X be a Riemann surface and let $\mathcal{F} \subseteq \mathscr{O}(X)$ be a subset such that for each compact set $K \subseteq X$

$$\sup_{f \in \mathcal{F}} \sup_{x \in K} |f(x)| < \infty.$$

Then any sequence $(f_n)_{n \geqslant 1}$ of functions in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of K.

Proof Sketch. Cover X by countably many charts. Using a diagonal argument, choose a subsequence $(f_\nu)_\nu$ that converges uniformly on compact subsets of each chart. Since any compact subset of X can be covered by finitely many compact sets, each contained in a chart, we have the desired conclusion. ■

LEMMA 2.8. For $r_0 < r < 1$, the vector space

$$Z_b^1(r)/B_b^1(r)$$

is finite-dimensional.

Proof. Choose ρ such that $r < \rho < 1$. Since $\mathfrak{U}(r)$ is a refinement of $\mathfrak{U}$ through the identity map $\tau : I \to I$, due to Theorem 1.8, we have that the restriction map

$$Z^1(\mathfrak{U}, \mathscr{O}) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad (f_{ij})_{i,j \in I} \longmapsto \left(f_{ij}|_{U_i(r) \cap U_j(r)}\right)_{i,j \in I}$$

induces an isomorphism $H^1(\mathfrak{U}, \mathscr{O}) \to H^1(\mathfrak{U}(r), \mathscr{O})$. Thus, the map

$$Z^1(\mathfrak{U}, \mathscr{O}) \oplus C^0(\mathfrak{U}(r), \mathscr{O}) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad ((f_{ij})_{i,j \in I}, (g_i)_{i \in I}) \longmapsto \left(f_{ij}|_{U_i(r) \cap U_j(r)}\right)_{i,j \in I} + \delta\left((g_i)_{i \in I}\right).$$

Since every $f \in \mathscr{O}(U_i \cap U_j)$ is bounded on $U_i(\rho) \cap U_j(\rho)$, owing to the latter being relatively compact in the former, it follows that the map

$$Z_b^1(\rho) \oplus C^0(\mathfrak{U}(r), \mathscr{O}) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad ((f_{ij})_{i,j \in I}, (g_i)_{i \in I}) \longmapsto \left(f_{ij}|_{U_i(r) \cap U_j(r)}\right)_{i,j \in I} + \delta\left((g_i)_{i \in I}\right)$$

is surjective too. We claim that the restriction of the above map

$$u : Z_b^1(\rho) \oplus C_b^0(r) \to Z_b^1(r) \qquad ((f_{ij})_{i,j \in I}, (g_i)_{i \in I}) \longmapsto \left(f_{ij}|_{U_i(r) \cap U_j(r)}\right)_{i,j \in I} + \delta\left((g_i)_{i \in I}\right)$$

is surjective. Indeed, for some $(f_{ij})_{i,j \in I} \in Z_b^1(r)$, choose an element $(\widetilde{f}_{ij})_{i,j \in I} \in Z^1(\mathfrak{U}, \mathscr{O})$ which maps to the image of (f_{ij}) in $H^1(\mathfrak{U}(r), \mathscr{O})$. The difference

$$(f_{ij})_{i,j \in I} - \left(\widetilde{f}_{ij}|_{U_i(r) \cap U_j(r)}\right)_{i,j \in I} \in B_b^1(r),$$

since $U_i(r) \cap U_j(r)$ is relatively compact in $U_i \cap U_j$. From Lemma 2.6, we know that the preimage of the above map under δ lies inside $C_b^0(r)$, as desired.

Let $v : Z_b^1(\rho) \oplus C_b^0(r) \to Z_b^1(r)$ be given by the map

$$v\left((f_{ij})_{i,j \in I}, (g_i)_{i \in I}\right) = \left(-f_{ij}|_{U_i(r) \cap U_j(r)}\right)_{i,j \in I}.$$

Note that $U_i(r) \cap U_j(r)$ is relatively compact in $U_i(\rho) \cap U_j(\rho)$, whence by Theorem 2.7 it is easy to argue that v is compact[2].

Finally, by Theorem 2.4, the continuous linear mapping $u + v : Z_b^1(\rho) \oplus C_b^0(r) \to Z_b^1(r)$ given by

$$(u + v)\left((f_{ij})_{i,j \in I}, (g_i)_{i \in I}\right) = \delta\left((g_i)_{i \in I}\right)$$

has closed image with finite codimension. Since the image of $u + v$ is precisely $B_b^1(r)$, we have the desideratum. ■

[2]Probably add more details here?

Lemma 2.9. $\dim H^1(\mathfrak{U}(r), \mathscr{O}) < \infty$.

Proof. Let $\tau : I \to I$ be the identity map. Due to Theorem 1.8, the induced map $\tau^* : H^1(\mathfrak{U}, \mathscr{O}) \to H^1(\mathfrak{U}(r), \mathscr{O})$ is an isomorphism. The map

$$Z^1(\mathfrak{U}, \mathscr{O}) \to Z^1_b(\rho) \qquad (f_{ij})_{i,j\in I} \longmapsto \left(f_{ij}|_{U_i(\rho)\cap U_j(\rho)}\right)$$

induces a map $\alpha : H^1(\mathfrak{U}, \mathscr{O}) \to Z^1_b(\rho)/B^1_b(\rho)$. Similarly, the map

$$Z^1_b(\rho) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad (f_{ij})_{i,j\in I} \longmapsto \left(f_{ij}|_{U_i(r)\cap U_j(r)}\right)$$

induces a map $\beta : Z^1_b(\rho)/B^1_b(\rho) \to H^1(\mathfrak{U}(r), \mathscr{O})$ such that $\tau^* = \beta \circ \alpha$. In particular, this means that β is surjective. Due to Lemma 2.8, $H^1(\mathfrak{U}(r), \mathscr{O})$ is finite-dimensional. ■

Proof of Theorem 2.5. Follows immediately from Theorem 1.8 and Lemma 2.9. ■

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