

Functional Analysis

Swayam Chube

December 1, 2024

§1 PRELIMINARIES

LEMMA 1.1 (RIESZ LEMMA). Let X be a normed linear space and $Y \subsetneq X$ a proper closed subspace. Then, for every $0 < \alpha < 1$, there is an $x \in X \setminus Y$ such that $\|x\| = 1$ and $\text{dist}(x, Y) > \alpha$.

§2 COMPLETENESS ARGUMENTS

THEOREM 2.1 (RUDIN, EXERCISE 4.26). Let X and Y be Banach spaces. The set of all surjective bounded linear operators in $\mathcal{B}(X, Y)$ forms an open subset.

Proof. Let $T : X \rightarrow Y$ be a surjective linear operator. By the open mapping theorem, there is an $r > 0$ such that $B_Y(0, 2r) \subseteq T(B_X(0, 1))$. If $0 \neq y \in Y$, then $\frac{ry}{\|y\|} \in B_Y(0, 2r)$, consequently, there is an $x' \in X$ with $\|x'\| < 1$ and $Tx' = \frac{ry}{\|y\|}$, thus, $x = \frac{\|y\|}{r}x'$ maps to y under T . Note that $\|x\| < \frac{\|y\|}{r}$. For the sake of brevity, let $t = 1/r$.

Let $\delta = \frac{1}{2t} > 0$ and $S \in \mathcal{B}(X, Y)$ such that $\|T - S\| < \delta$. We shall show that S is surjective, for which, it would suffice to show that the image of S contains the unit ball of Y . Indeed, let $y_0 \in Y$ with $\|y_0\| \leq 1$. Choose an $x_0 \in X$ such that $\|x_0\| < t$ and $Tx_0 = y_0$. Setting $y_1 = y_0 - Sx_0$, we have

$$\|y_1\| = \|(T - S)x_0\| \leq \delta t.$$

Again, choose $x_1 \in X$ such that $Tx_1 = y_1$ and $\|x_1\| < t\|y_1\| = \delta t^2$. Setting $y_2 = y_1 - Sx_1$, we have

$$\|y_2\| = \|(T - S)x_1\| \leq \delta^2 t^2$$

and so on. We have thus constructed two sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ such that

- $Tx_n = y_n$,
- $y_{n+1} = y_n - Sx_n$ for $n \geq 0$, and
- $\|x_n\| < \delta^n t^{n+1}$ and $\|y_n\| \leq \delta^n t^n$.

Let $x = \sum_{n=0}^{\infty} x_n$, which converges since $\sum_{n=0}^{\infty} \|x_n\|$ does. Hence,

$$Sx = \lim_{n \rightarrow \infty} \sum_{i=0}^n Sx_i = \sum_{i=0}^{\infty} y_i - y_{i+1} = y_0,$$

thereby completing the proof. ■

§3 THE HAHN-BANACH THEOREMS

LEMMA 3.1 (DOMINATED EXTENSION THEOREM). Let X be a real vector space with a subspace M . Suppose $p : X \rightarrow \mathbb{R}$ satisfies

$$p(x + y) \leq p(x) + p(y) \quad \text{and} \quad p(tx) = tp(x) \quad \forall x, y \in M, \forall t \geq 0.$$

Let $f : X \rightarrow \mathbb{R}$ be a linear functional such that $f(x) \leq p(x)$ for all $x \in M$. Then, there is a linear functional $\Lambda : X \rightarrow \mathbb{R}$ such that $\Lambda x = f(x)$ for all $x \in M$ and

$$-p(-x) \leq \Lambda x \leq p(x) \quad \forall x \in X.$$

Proof. If $M = X$, then there is nothing to prove. Suppose now that M is a proper subspace of X and choose $x_1 \in X \setminus M$. For $x, y \in M$, we have

$$f(x) + f(y) = f(x + y) \leq p(x + y) \leq p(x - x_1) + p(y + x_1),$$

and hence,

$$f(x) - p(x - x_1) \leq -f(y) + p(y + x_1) \quad \forall x, y \in M.$$

Let α denote the supremum of the left hand side in the above inequality as x ranges over M . Note that α is finite as the left hand side is always bounded above by $p(x_1)$. Let $M_1 = M + \mathbb{R}x_1$ and define $f_1 : M_1 \rightarrow \mathbb{R}$ by

$$f_1(m + \lambda x_1) = f(m) + \lambda \alpha;$$

in particular, $f_1(x_1) = \alpha$. Note that for $\lambda \neq 0$,

$$\begin{aligned} f_1(m + \lambda x_1) &= |\lambda| f_1(|\lambda|^{-1}m + \text{sgn}(\lambda)x_1) \\ &= |\lambda| f(|\lambda|^{-1}m) + \lambda \alpha \\ &\leq |\lambda| \left(p(|\lambda|^{-1}m + \text{sgn}(\lambda)x_1) - \text{sgn}(\lambda)\alpha \right) \\ &= p(m + \lambda x_1). \end{aligned}$$

This furnishes an extension $f_1 : M_1 \rightarrow \mathbb{R}$ such that $f_1(y) \leq p(y)$ for all $y \in M_1$. One can then extend this, using Zorn's Lemma, to $\Lambda : X \rightarrow \mathbb{R}$ such that $\Lambda x \leq p(x)$ for all $x \in X$. We then have

$$-p(-x) \leq -\Lambda(-x) = \Lambda x \leq p(x),$$

thereby completing the proof. ■

THEOREM 3.2 (HAHN-BANACH EXTENSION THEOREM). Let M be a subspace of a vector space (real or complex) X , p a semi-norm on X , and f a linear functional on M such that $|f(x)| \leq p(x)$ for all $x \in M$. Then f extends to a linear functional Λ on X satisfying $|\Lambda x| \leq p(x)$ for all $x \in X$.

Proof. Suppose first that the field of scalars is \mathbb{R} . Due to the preceding lemma, f can be extended to $\Lambda : X \rightarrow \mathbb{R}$ satisfying

$$-p(x) = -p(-x) \leq \Lambda x \leq p(x) \quad \forall x \in X,$$

that is, $|\Lambda x| \leq p(x)$.

Next, suppose the field of scalars is \mathbb{C} . Let $u = \Re f$. Due to the first part of the proof, u can be extended to a linear functional $U : X \rightarrow \mathbb{R}$ satisfying $|Ux| \leq p(x)$ for all $x \in X$. Define $\Lambda : X \rightarrow \mathbb{C}$ by

$$\Lambda x = u(x) - iu(ix) \quad \forall x \in X.$$

We contend that Λ is the desired functional. Let $x \in X$ and choose an $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\alpha \Lambda x = |\Lambda x|$. Hence,

$$|\Lambda x| = \alpha \Lambda x = \underbrace{\Lambda(\alpha x) = U(\alpha x)}_{\text{because LHS} \in \mathbb{R}_{\geq 0}} \leq p(\alpha x) = p(x).$$

This completes the proof. ■

COROLLARY. Let X be a normed linear space and M a subspace of X . Suppose $f : M \rightarrow \mathbb{K}$ is a bounded linear functional, then there exists a bounded linear functional $\Lambda : X \rightarrow \mathbb{K}$ extending f . Further, $\|f\| = \|\Lambda\|$

Proof. Invoke the preceding result with $p(x) = \|f\| \|x\|$. ■

THEOREM 3.3 (HAHN-BANACH SEPARATION THEOREM). Suppose A and B are disjoint convex subsets of a topological vector space X .

(a) If A is open, there exist $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that

$$\Re \Lambda x < \gamma \leq \Re \Lambda y \quad \forall x \in A, y \in B.$$

(b) If A is compact, B is closed, and X is locally convex, there exist $\Lambda \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\Re \Lambda x < \gamma_1 < \gamma_2 < \Re \Lambda y \quad \forall x \in A, y \in B.$$

Proof. We first prove this theorem when the scalar field is assumed to be \mathbb{R} .

(a) Fix points $a_0 \in A, b_0 \in B$. Set $x_0 = b_0 - a_0$ and $C = A - B + x_0$. Then, C is a convex neighborhood of 0 in X , and thus, admits a Minkowski functional, $p : X \rightarrow \mathbb{R}$ which is subadditive and $p(tx) = tp(x)$ for all $t \geq 0$. Further, since $A \cap B = \emptyset$, $x_0 \notin C$, whence $p(x_0) \geq 1$.

Define a linear functional $f : \mathbb{R}x_0 \rightarrow \mathbb{R}$ by $f(\lambda x_0) = \lambda$ and using the Dominated Extension Theorem, extend this to a functional $\Lambda : X \rightarrow \mathbb{R}$ such that

$$-p(-x) \leq \Lambda x \leq p(x) \quad \forall x \in X.$$

Let $D = C \cap (-C)$, which is a symmetric convex neighborhood of the origin. For any $x \in D$, it is easy to see that $p(x) \leq 1$, whence

$$-1 \leq -p(-x) \leq \Lambda x \leq p(x) \leq 1,$$

and hence, Λ is a continuous linear functional.

Now, for $a \in A$ and $b \in B$,

$$\Lambda a - \Lambda b = \Lambda(a - b) = \Lambda(a - b + x_0) - 1 \leq p(a - b + x_0) - 1 < 0,$$

since $a - b + x_0 \in C$. Hence, $\Lambda a < \Lambda b$ for every $a \in A$ and $b \in B$. Finally, since $\Lambda(A)$ and $\Lambda(B)$ are disjoint convex subsets of \mathbb{R} , both must be intervals with the former to the left of the latter. Further, since the former is an open subset of \mathbb{R} , we immediately obtain the desired conclusion.

- (b) There is a convex, balanced neighborhood V of the origin in X such that $(A + V) \cap (B + V) = \emptyset$. Set $C = A + V$, which is a convex open subset of X , disjoint from B . Due to part (a), there is a linear functional Λ such that $\Lambda(C)$ is to the left of $\Lambda(B)$ and $\Lambda(A)$ sits as a compact interval inside $\Lambda(C)$. The conclusion now is immediate.

We now suppose that the field of scalars is \mathbb{C} ; whence X is also a topological \mathbb{R} -vector space. In both parts (a) and (b), we were able to obtain an \mathbb{R} -linear functional, continuous on X when viewed as a \mathbb{R} -TVS and separating the two sets as desired. Define the \mathbb{C} -linear functional $\Lambda x = u(x) - iu(ix)$ and note that this has the desired separation properties too. ■

COROLLARY. If X is an LCTVS, then X^* separates points on X .

Proof. Let $p, q \in X$. Use Theorem 3.3 (b) with $A = \{p\}$ and $B = \{q\}$. ■

THEOREM 3.4. Let M be a proper closed subspace of a locally convex topological vector space, and $x_0 \in X \setminus M$. There exists a linear functional $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ and $\Lambda x = 0$ for all $x \in M$.

Proof. Using Theorem 3.3(b) with $A = \{x_0\}$ and $B = M$, there is a $\Lambda \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\Re \Lambda x_0 < \gamma_1 < \gamma_2 < \Re \Lambda y \quad \forall y \in M.$$

Since $\Lambda(0) = 0$ and $0 \in M$, we must have that $\Lambda x_0 \neq 0$. Further, since $\lambda y \in M$ for every $\lambda \in \mathbb{K}$, the only way $\Re(\lambda \Lambda y) > \gamma_2$ for every $\lambda \in \mathbb{K}$ is if Λ vanishes on M . Dividing Λ by Λx_0 , we have our desired conclusion. ■

COROLLARY. Let X be an LCTVS and $M \subseteq X$ a subspace. Suppose $f : M \rightarrow \mathbb{K}$ is a continuous linear functional, then there is a $\Lambda \in X^*$ such that $\Lambda|_M = f$.

Proof. ■

§4 WEAK AND WEAK* TOPOLOGIES

LEMMA 4.1. Let X be a \mathbb{K} -vector space and $\Lambda_1, \dots, \Lambda_n, \Lambda$ be linear functionals on X and set

$$N = \{x \in X : \Lambda_i x = 0, \forall 1 \leq i \leq n\}.$$

The following are equivalent:

(a) There are scalars $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ such that

$$\Lambda = \alpha_1 \Lambda_1 + \dots + \alpha_n \Lambda_n.$$

(b) There exists $0 < \gamma < \infty$ such that

$$|\Lambda x| \leq \gamma \max_{1 \leq i \leq n} |\Lambda_i x| \quad \forall x \in X.$$

(c) $\Lambda x = 0$ for every $x \in N$.

Proof. (a) \implies (b) \implies (c) is trivial. It remains to show that (c) \implies (a). Consider the map $\Phi : X \rightarrow \mathbb{K}^n$ given by

$$\Phi(x) = (\Lambda_1 x, \dots, \Lambda_n x)$$

and let $Y \subseteq \mathbb{K}^n$ be its image. Define $\Psi : Y \rightarrow \mathbb{K}$ by

$$\Psi(\Phi(x)) = \Lambda x.$$

That this is well-defined follows from the fact that $N \subseteq \ker \Lambda$. Since we are in a finite-dimensional space, the map Ψ can be extended to a linear map $\Psi : \mathbb{K}^n \rightarrow \mathbb{K}$, which must be of the form

$$(y_1, \dots, y_n) \mapsto \alpha_1 y_1 + \dots + \alpha_n y_n.$$

It then follows that $\Lambda = \alpha_1 \Lambda_1 + \dots + \alpha_n \Lambda_n$. ■

DEFINITION 4.2. Let X be a set and

$$\mathcal{F} = \{f : X \rightarrow Y_f\}$$

a collection of functions. The \mathcal{F} -topology on X is defined to be the coarsest topology such that every $f \in \mathcal{F}$ is continuous.

The set \mathcal{F} is said to *separate points* if for each pair $p \neq q$ in X , there is an $f \in \mathcal{F}$ such that $f(p) \neq f(q)$.

REMARK 4.3. The \mathcal{F} -topology is more explicitly the topology generated by

$$\{f^{-1}(U) : U \subseteq Y_f \text{ is open, } f \in \mathcal{F}\}.$$

PROPOSITION 4.4. If \mathcal{F} is a separating family of functions on a space X , and each Y_f is Hausdorff, then the \mathcal{F} -topology on X is Hausdorff.

Proof. Let $p \neq q$ be points in X and choose $f \in \mathcal{F}$ such that $f(p) \neq f(q)$. Then, there are disjoint neighborhoods U and V of $f(p)$ and $f(q)$ respectively in Y_f . Since each f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighborhoods of p and q in the \mathcal{F} -topology. ■

PROPOSITION 4.5. If X is a compact topological space and \mathcal{F} is a countable family of continuous separating real-valued functions on X , then X is metrizable.

Proof. Let $\mathcal{F} = \{f_n : n \geq 1\}$. We may suppose without loss of generality that $\|f\|_\infty \leq 1$ for each $f \in \mathcal{F}$. It is not hard to check that the function $d : X \times X \rightarrow \mathbb{R}$ given by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |f_n(x) - f_n(y)|$$

is a metric inducing the topology on X . ■

THEOREM 4.6. Let X be a \mathbb{K} -vector space and X' a vector space of linear functionals on X that separates points. The X' -topology τ' on X makes it a locally convex topological vector space whose dual is X' .

Proof. Due to Proposition 4.4, τ' is Hausdorff. Note that the topology is generated by the set

$$\{\Lambda^{-1}(U) : \Lambda \in X', U \subseteq \mathbb{K} \text{ is open}\}.$$

Hence, a base for the topology is given by finite intersections of elements of the above form. Thus, is generated by intersections of the form

$$\Lambda_1^{-1}(U_1) \cap \cdots \cap \Lambda_n^{-1}(U_n),$$

where $U_1, \dots, U_n \subseteq \mathbb{K}$ are open sets. It immediately follows that this base is translation invariant whence, the entire topology is translation invariant. A local base at 0 is given by open sets of the above form, such that $0 \in U_i$ for $1 \leq i \leq n$. We can further refine this local base by choosing open sets of the form

$$V(\Lambda_1, \dots, \Lambda_n; \varepsilon_1, \dots, \varepsilon_n) = \{x \in X : |\Lambda_i x| \leq \varepsilon_i, 1 \leq i \leq n\}.$$

Further, from this description, it is not hard to see that αV is a basic open set whenever $\alpha > 0$ and V a basic open set.

Now that we have established a local base for τ' , we show that (X, τ') is indeed a topological vector space. That τ' is locally convex immediately follows from the above description of a local base. Next, we show that addition is continuous, for which it suffices to show continuity at $(0, 0) \in X \times X$. Let U be a neighborhood of 0 in X , then U contains a basic open set V of the above form. Since $\frac{1}{2}V + \frac{1}{2}V \subseteq V$, we see that addition is continuous.

To see that scalar multiplication is continuous, let $x \in X$, $\alpha \in \mathbb{K}$ and $x + V$ a neighborhood of x . We may suppose that V is a basic open set of the above form. Since V is absorbing, there is an $s > 0$ such that $x \in sV$. Choose r sufficiently small so that $r(r+s) + r|\alpha| < 1$. Then, if $y \in x + rV$, and $|\beta - \alpha| < r$,

$$\beta y - \alpha x = (\beta - \alpha)y + \alpha(y - x) \in r(r+s)V + |\alpha|rV \subseteq V,$$

since $y \in (r + s)V$. Hence, scalar multiplication is continuous and (X, τ') is a locally convex topological vector space.

Finally, let Λ be a continuous linear functional on X and consider a basic open set $V(\Lambda_1, \dots, \Lambda_n, \varepsilon_1, \dots, \varepsilon_n)$ such that $|\Lambda x| < 1$ on V . Thus, there is a $\gamma > 0$ such that

$$|\Lambda x| \leq \gamma \max_{1 \leq i \leq n} |\Lambda_i x|$$

whence, Λ is a linear combination of the Λ_i . ■

DEFINITION 4.7. Let X be a topological vector space whose dual X^* separates points on X (this is true in particular for locally convex TVSs). Then the X^* -topology on X is called the *weak topology* and is denoted by (X, τ_w) or X_w .

Obviously the weak topology is coarser than the original topology. A set $E \subseteq X$ is said to be *weakly bounded* if it is bounded in the weak topology. Similarly, a sequence (x_n) is said to be *weakly convergent* to x if it converges in the weak topology. Since the weak topology is Hausdorff, the limit of any weakly convergent sequence is unique.

PROPOSITION 4.8. Let X be a topological vector space on which X^* separates points. Then

- (a) X_w is a locally convex topological vector space.
- (b) A set $E \subseteq X$ is weakly bounded if and only if every $\Lambda \in X^*$ is bounded on E .
- (c) A sequence (x_n) is weakly convergent to x if and only if $\Lambda x_n \rightarrow \Lambda x$ for every $\Lambda \in X^*$.

Proof. All three assertions are trivial. ■

PROPOSITION 4.9. Let X be a locally convex topological vector space and $E \subseteq X$ a convex subset. Then the weak closure \bar{E}_w is the same as the original closure \bar{E} .

Proof. Since the weak topology is coarser than the original topology, $\bar{E} \subseteq \bar{E}_w$. Now, let $x_0 \in X \setminus \bar{E}$. Due to the Hahn-Banach Separation Theorem, there is an $\Lambda \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\Re \Lambda x_0 < \gamma_1 < \gamma_2 < \Re \Lambda y \quad \forall y \in \bar{E} \supseteq E.$$

Thus, there is a weak neighborhood of x_0 not intersecting E , consequently, $x_0 \notin \bar{E}_w$. This completes the proof. ■

THEOREM 4.10. Suppose X is an infinite-dimensional normed linear space. Then the weak topology on X is not metrizable.

Proof. We shall show that the weak topology (X, w) is not first-countable whence the conclusion would follow. Suppose not, then there is a local base $\{U_n\}$ at 0. For each $n \geq 1$, there is a finite subset $F_n \subseteq X^*$ and $\varepsilon_n > 0$ such that

$$V_n = \{x \in X: |f(x)| < \varepsilon_n, \forall f \in F_n\}.$$

We contend that

$$X^* = \bigcup_{n \geq 1} \text{span}(F_n).$$

Indeed, let $g \in X^*$ and

$$U = \{x \in X: |g(x)| < 1\}.$$

There is an index $n \geq 1$ such that $V_n \subseteq U$. Now, if x is in $\bigcap_{f \in F_n} \ker f$, then so is λx for every $\lambda \in \mathbb{K}$, consequently, $\lambda x \in V_n$ and hence, $|\lambda||g(x)| < 1$ for every $\lambda \in \mathbb{K}$. This forces $g(x) = 0$, that is,

$$\bigcap_{f \in F_n} \ker f \subseteq \ker g,$$

which, in light of Lemma 4.1 gives $g \in \text{span}(F_n)$, proving our claim.

It follows that X^* has at most countable dimension and since X is infinite-dimensional, so is X^* , but this is absurd, since X^* is a Banach space. ■

DEFINITION 4.11. Let X be a topological vector space and X^* . The evaluation functionals induced by X form a separating vector space of functionals. The X -topology induced on X^* by these functionals is called the *weak* topology*.

THEOREM 4.12 (BANACH-ALAOGLU). Let X be a topological vector space and V a neighborhood of 0. The *polar* of V :

$$K = \{\Lambda \in X^*: |\Lambda x| \leq 1, \forall x \in V\} \subseteq X^*$$

is weak*-compact.

Proof. Since V is a neighborhood of the origin, it is absorbing and hence, for each $x \in X$, there is $\gamma(x) > 0$ such that $x \in \gamma(x)V$. For $x \in V$, choose $\gamma(x) \leq 1$. Let D_x denote the compact set

$$D_x = \{z \in \mathbb{K}: |z| \leq \gamma(x)\}, \quad (1)$$

and

$$P = \prod_{x \in X} D_x,$$

which is compact due to Tychonoff's Theorem. Further, for each $\Lambda \in K$ and $x \in X$, since $x/\gamma(x) \in V$, we have $|\Lambda x| \leq |\gamma(x)|$, consequently, the element $(\Lambda x)_{x \in X}$ is an element of P . Thus, we can identify K with a subset of P . Henceforth, we shall denote elements of P as functions $f: X \rightarrow \mathbb{K}$. We shall show that:

- (i) the subspace topology K inherits from P and the weak*-topology on K are the same,
- (ii) with respect to the subspace topology, K is closed in P ;

whence it follows that K is compact.

Let $\Lambda_0 \in K$ and consider a basic open set in the weak*-topology centered at Λ_0 of the form

$$W = \{\Lambda \in X^*: |\Lambda x_i - \Lambda_0 x_i| < \varepsilon, 1 \leq i \leq n\}.$$

In the product topology on P , the following set is open

$$V = \{f \in P: |f(x_i) - \Lambda_0 x_i| < \varepsilon, 1 \leq i \leq n\}.$$

It is not hard to see that $W \cap K = V \cap K$. This shows that the subspace topology induced on K by the product topology is finer than that induced by the weak*-topology.

On the other hand, choose any open set in the product topology in P intersecting K and choose an element Λ_0 in the intersection. The aforementioned open set contains one of the form V as above and since $W \cap K = V \cap K$, we see that the weak*-topology is finer than the subspace topology. This shows that the two topologies are the same.

Finally, we must show that K is closed in P . Let $f_0 \in \bar{K}$, $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$. We contend that $f_0(\alpha x + \beta y) = \alpha f_0(x) + \beta f_0(y)$. Let $\varepsilon > 0$ and

$$V = \{f \in P: |f(z) - f_0(z)| < \varepsilon, z \in \{x, y, \alpha x + \beta y\}\}.$$

There is some $f \in K \cap V$. Then,

$$\begin{aligned} & |f_0(\alpha x + \beta y) - \alpha f(x) - \beta f(y)| \leq \\ & |f_0(\alpha x + \beta y) - f(\alpha x + \beta y)| + |\alpha f(x) - \alpha f_0(x)| + |\beta f(y) - \beta f_0(y)| \\ & \leq (|\alpha| + |\beta| + 1)\varepsilon. \end{aligned}$$

Since the above inequality holds for all $\varepsilon > 0$, we have that f_0 is linear. Further, by construction, f_0 is bounded by 1 on V , since $\gamma(x) \leq 1$ for all $x \in V$ and hence, $f_0 \in X^*$. It follows that $f_0 \in K$ and hence, K is closed in P , thereby completing the proof. ■

PROPOSITION 4.13 (RUDIN, EXERCISE 3.11). Let X be an infinite dimensional Fréchet space. Then X^* with the weak*-topology is of the first category in itself.

Proof. Let $V_n = B(0, 1/n) \subseteq X$ and let K_n denote their respective polars, that is

$$K_n = \{\Lambda \in X^*: |\Lambda x| \leq 1, \forall x \in V_n\}.$$

First, we claim that $X^* = \bigcup_{n=1}^{\infty} K_n$. Indeed, for any $\Lambda \in X^*$, note that the open set $\Lambda^{-1}(B_{\mathbb{K}}(0, 1))$ contains some V_n and hence, $\Lambda \in K_n$.

It remains to now show that these have empty interior. Indeed, suppose K_N has nonempty interior for some $N \in \mathbb{N}$. Since K_N is convex, symmetric, so is its interior. Thus, we have that 0 lies in the interior of K_N . As a result, there is an $\varepsilon > 0$ and $x_1, \dots, x_n \in X$ such that

$$W = \{\Lambda \in X^*: |\Lambda x_i| < \varepsilon, 1 \leq i \leq n\} \subseteq K_N.$$

Since K_N is compact, it is bounded and hence, so is W . But since X^* is infinite-dimensional too, so is $\bigcap_{i=1}^n \ker \text{ev}_{x_i} \subseteq W$ which is contained in a bounded set, whence, must be the trivial subspace.

Next, for any $x \in X$, note that

$$\bigcap_{i=1}^n \ker \text{ev}_{x_i} = \{0\} \subseteq \ker \text{ev}_x,$$

thus x is a linear combination of the x_i 's, that is, X is finite-dimensional, a contradiction. This completes the proof. ■

§§ The Krein-Milman Theorem

DEFINITION 4.14. A subset E of a topological vector space X is said to be *totally bounded* if to every neighborhood V of 0 in X corresponds a finite set F such that $E \subseteq F + V$.

REMARK 4.15. Note that we can require that $F \subseteq E$. Indeed, let V be a neighborhood of 0 and choose a neighborhood W of 0 such that $W + W \subseteq V$. There is a finite set $F \subseteq X$ such that $E \subseteq F + W$. For each $f \in F$ such that $(f + W) \cap E \neq \emptyset$, choose some e in the intersection. For any $w \in W$, we have $f + w - e = (f - e) + w \in W + W \subseteq V$. Hence, $f + W \subseteq e + V$. The collection of all such e 's, say \tilde{F} is such that $E \subseteq \tilde{F} + W$.

THEOREM 4.16. (a) If A_1, \dots, A_n are compact convex sets in a topological vector space X , then $\text{co}(A_1 \cup \dots \cup A_n)$ is compact.

(b) If X is an LCTVS and $E \subseteq X$ is totally bounded, then $\text{co}(E)$ is totally bounded.

(c) If X is a Fréchet space and $K \subseteq X$ is compact, then $\overline{\text{co}}(K)$ is compact.

Proof. (a) Let

$$\Delta = \{(s_1, \dots, s_n) \in \mathbb{R}^n : s_1 + \dots + s_n = 1, s_i \geq 0 \forall 1 \leq i \leq n\}.$$

Let $A = A_1 \times \dots \times A_n$ and define the map $f : \Delta \times A \rightarrow X$ by

$$f(s, a) = s_1 a_1 + \dots + s_n a_n.$$

This is a continuous map since addition and scalar multiplication are continuous on X . Put $K = f(\Delta \times A)$. Then, K is compact and is contained in $\text{co}(A_1 \cup \dots \cup A_n)$.

We shall show that $K = \text{co}(A_1 \cup \dots \cup A_n)$, for which it suffices to show that K is convex (since each A_i is contained in K). Indeed, let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then, for $(s, a), (t, b) \in \Delta \times A$, we have

$$\alpha \sum_{i=1}^n s_i a_i + \beta \sum_{i=1}^n t_i b_i = \sum_{i=1}^n (\alpha s_i + \beta t_i) \cdot \frac{\alpha s_i a_i + \beta t_i b_i}{\alpha s_i + \beta t_i} = f(u, c),$$

where $u = \alpha s + \beta t$ and

$$c_i = \frac{\alpha s_i a_i + \beta t_i b_i}{\alpha s_i + \beta t_i} \in A_i,$$

and we are done.

(b) Let U be a neighborhood of 0 in X and choose a convex, balanced neighborhood V of 0 in X such that $V + V \subseteq U$. There is a finite set $F \subseteq X$ such that $E \subseteq F + V$, whence $E \subseteq \text{co}(F) + V$. Since the latter is convex, we have $\text{co}(E) \subseteq \text{co}(F) + V$.

Due to part (a), $\text{co}(F)$ is compact. The collection $\{f + V : f \in \text{co}(F)\}$ is an open cover of $\text{co}(F)$ and hence, admits a finite subcover, $\text{co}(F) \subseteq F_1 + V$ for some $F_1 \subseteq X$. Therefore,

$$\text{co}(E) \subseteq F_1 + V + V \subseteq F_1 + U,$$

that is, $\text{co}(E)$ is totally bounded.

(c) Due to part (b), $\text{co}(K)$ is totally bounded. Thus, its closure is totally bounded and complete, whence compact. ■

LEMMA 4.17 (CARATHÉODORY). If $E \subseteq \mathbb{R}^n$ and $x \in \text{co}(E)$, then x lies in the convex hull of some subset of E which contains at most $n + 1$ points.

Proof. We shall show that if $k > n$ and $x = \sum_{i=1}^{k+1} t_i x_i$ is a convex combination for some $x_i \in \mathbb{R}^n$, then x is a convex combination of some k of these vectors. This is enough to prove the statement of the theorem.

We may suppose without loss of generality that $t_i > 0$ for $1 \leq i \leq k + 1$. Consider the linear map $\mathbb{R}^{k+1} \rightarrow \mathbb{R}^{n+1}$ given by

$$(a_1, \dots, a_{k+1}) \mapsto \left(\sum_{i=1}^{k+1} a_i x_i, \sum_{i=1}^{k+1} a_i \right).$$

The kernel of this map must be nontrivial and hence, there exists $(a_1, \dots, a_{k+1}) \in \mathbb{R}^{k+1}$ with some $a_i \neq 0$, so that $\sum_{i=1}^{k+1} a_i x_i = 0$ and $\sum_{i=1}^{k+1} a_i = 0$. Set

$$|\lambda| = \min_{1 \leq i \leq k+1} \frac{t_i}{|a_i|},$$

which is finite, since $a_i \neq 0$ for some $1 \leq i \leq k + 1$. Choose the sign of λ so that $\lambda a_j = \lambda_j$ for some $1 \leq j \leq k + 1$. Set $c_i = t_i - \lambda a_i \geq 0$. Then,

$$\sum_{i=1}^{k+1} c_i x_i = \sum_{i=1}^{k+1} t_i x_i - \lambda \sum_{i=1}^{k+1} a_i x_i = x,$$

and

$$\sum_{i=1}^{k+1} c_i = \sum_{i=1}^{k+1} t_i - \lambda \sum_{i=1}^{k+1} a_i = 1.$$

Note that $c_j = 0$ and hence, we have written x as a convex combination of some k of the x_i 's. ■

PROPOSITION 4.18. If $K \subseteq \mathbb{R}^n$ is compact, then so is $\text{co}(K)$.

Proof. Let

$$\Delta = \{(s_1, \dots, s_{n+1}) \in \mathbb{R}^{n+1} : s_1 + \dots + s_{n+1} = 1, s_i \geq 0 \forall 1 \leq i \leq n + 1\}.$$

Due to Carathéodory's lemma, it follows that $x \in \text{co}(K)$ if and only if x is a linear combination of some $n + 1$ elements of K . Thus, the map $\Delta \times K^{n+1} \rightarrow \mathbb{R}^n$ given by

$$(t, x_1, \dots, x_{n+1}) \mapsto t_1 x_1 + \dots + t_{n+1} x_{n+1}$$

is continuous and its image is $\text{co}(K)$. This completes the proof. ■

DEFINITION 4.19. Let X be a \mathbb{K} -vector space and $K \subseteq X$. A non-empty set $S \subseteq K$ is called an *extreme set* of K if whenever $x, y \in K$, $0 < t < 1$ such that $(1 - t)x + ty \in S$, then $x, y \in S$.

The *extreme points* of K are the extreme sets that are singletons. The set of all extreme points of K is denoted by $E(K)$.

LEMMA 4.20. Let X be a topological vector space on which X^* separates points. Suppose A, B are disjoint, nonempty, compact, convex sets in X . Then there exists $\Lambda \in X^*$ such that

$$\sup_{x \in A} \Re \Lambda x < \inf_{y \in B} \Re \Lambda y.$$

Proof. Topologize X with the weak topology, which is coarser than the original topology, and hence, A, B are compact. Now, use the Hahn-Banach separation theorem and the fact that $(X_w)^* = X^*$. ■

THEOREM 4.21 (KREIN-MILMAN). Let X be a topological vector space on which X^* separates points. If $K \subseteq X$ is a nonempty compact convex set in X , then $K = \overline{\text{co}}(E(K))$.

Proof. Let \mathcal{P} denote the poset of all nonempty compact extreme sets of K ordered by inclusion. Note that \mathcal{P} is nonempty, since $K \in \mathcal{P}$. We make the following two observations about \mathcal{P} :

- (a) If $S \neq \emptyset$, is an intersection of elements of \mathcal{P} , then $S \in \mathcal{P}$.
- (b) If $S \in \mathcal{P}$, $\Lambda \in X^*$ and $\mu = \max_{x \in S} \Re \Lambda x$, then

$$S_\Lambda = \{x \in S : \Re \Lambda x = \mu\} \in \mathcal{P}.$$

Observation (a) is obvious. As for (b), first note that S_Λ is closed in S , and hence, in K , thus, is compact. Now, suppose $x, y \in K$ and $t > 0$ such that $tx + (1 - t)y \in S_\Lambda \subseteq S$. Since S is an extreme set of K , $x, y \in S$, consequently, $\Re \Lambda x, \Re \Lambda y \leq \mu$ and

$$\mu = \Re \Lambda(tx + (1 - t)y) \leq t\mu + (1 - t)\mu = \mu,$$

whence $x, y \in S_\Lambda$, thereby proving (b).

Choose some $S \in \mathcal{P}$ and let \mathcal{P}' be the sub-poset of all members of \mathcal{P} that are contained in S . Let Ω be a maximal chain in \mathcal{P}' and let M denote the intersection of all elements of Ω . Since Ω has the finite intersection property and all sets in Ω are compact, $M \neq \emptyset$ and is compact.

We contend that M is a singleton. Indeed, since $M_\Lambda \subseteq M$, due to the minimality of M , we must have that $M_\Lambda = M$ for every $\Lambda \in X^*$. That is, $\Re \Lambda(x - y) = 0$ for all $x, y \in M$ and $\Lambda \in X^*$. Since X^* separates points on X , we must have that $x - y = 0$, that is, M is a singleton.

We have therefore proved that $E(K) \cap S \neq \emptyset$ for every $S \in \mathcal{P}$. Now, since K is convex, $\overline{\text{co}}(E(K)) \subseteq K$, consequently, the former is compact. Suppose now that there is some

$x_0 \in K \setminus \overline{\text{co}}(E(K))$. Applying the preceding lemma with $B = \{x_0\}$ and $A = \overline{\text{co}}(E(K))$, there is a $\Lambda \in X^*$ such that

$$\Re \Lambda x_0 > \sup_{y \in \overline{\text{co}}(E(K))} \Re \Lambda y.$$

Then, $K_\Lambda \in \mathcal{P}$ and is disjoint from $\overline{\text{co}}(E(K))$, a contradiction. Thus, $\overline{\text{co}}(E(K)) = K$, thereby completing the proof. ■

§5 COMPACT OPERATORS

DEFINITION 5.1. A linear map $T : X \rightarrow Y$ between Banach spaces is said to be *compact* if $T(U)$ is relatively compact in Y where U is the unit ball in X .

The following proposition is immediate from the equivalence of compactness and sequential compactness in metric spaces.

PROPOSITION 5.2. T is compact if and only if every bounded sequence (x_n) in X contains a subsequence (x_{n_k}) such that (Tx_{n_k}) converges in Y .

DEFINITION 5.3. The *spectrum* $\sigma(T)$ of an operator $T \in \mathcal{B}(X)$ is the set of all scalars λ such that $T - \lambda I$ is not invertible.

THEOREM 5.4. Let X and Y be Banach spaces.

- (a) If $T \in \mathcal{B}(X, Y)$ and $\dim \mathcal{R}(T) < \infty$, then T is compact.
- (b) If $T \in \mathcal{B}(X, Y)$, T is compact, and $\mathcal{R}(T)$ is closed, then $\dim \mathcal{R}(T) < \infty$.
- (c) The compact operators form a closed subspace of $\mathcal{B}(X, Y)$ in its norm-topology.
- (d) If $T \in \mathcal{B}(X)$, T is compact, and $\lambda \neq 0$ is a scalar, then $\dim \mathcal{N}(T - \lambda I) < \infty$.
- (e) If $\dim X = \infty$, $T \in \mathcal{B}(X)$, and T is compact, then $0 \in \sigma(T)$.
- (f) If $S, T \in \mathcal{B}(X)$, and T is compact, then so are ST and TS .

Proof. (a) Let U denote the unit ball of X . Then $T(U)$ is a bounded subset of $\mathcal{R}(T)$ and since the latter is closed in Y , $\overline{T(U)}$ is a closed and bounded subset of $\mathcal{R}(T)$, consequently, is compact.

- (b) Since $\mathcal{R}(T)$ is closed in Y , it is complete, i.e., a Banach space. Due to the open mapping theorem, $T(U)$ is open in $\mathcal{R}(T)$ with compact closure, whence $\mathcal{R}(T)$ is locally compact, and hence, finite dimensional.
- (c) Let $T_n \rightarrow T$ in $\mathcal{B}(X, Y)$ where each T_n is a compact operator. We shall show that $T(U)$ is totally bounded in Y . Let $\varepsilon > 0$ and choose an N such that $\|T - T_N\| < \varepsilon/3$. Note that $T_N(U)$ is totally bounded in Y , and hence, there are $x_1, \dots, x_n \in U$ such that

$$T_N(U) \subseteq \bigcup_{i=1}^n B_Y(T_N x_i, \varepsilon/3).$$

Now, for any $y \in U$, there is an index $1 \leq i \leq n$ such that $T_N y \in B(T_N x_i, \varepsilon/3)$. As a result,

$$\|Ty - Tx_i\| \leq \|Ty - T_N y\| + \|T_N y - T_N x_i\| + \|T_N x_i - Tx_i\| < \varepsilon.$$

Hence,

$$T(U) \subseteq \bigcup_{i=1}^n B_Y(Tx_i, \varepsilon),$$

and the conclusion follows.

- (d) Let $Y = \mathcal{N}(T - \lambda I)$. Then note that T acts on Y by $y \mapsto \lambda y$. Further, since T is compact and Y is closed in X , the restriction of T to Y is compact and hence, Y must be finite-dimensional.
- (e) If $0 \notin \sigma(T)$, then T is invertible, whence $\mathcal{R}(T)$ is closed but $\dim \mathcal{R}(T) = \infty$, a contradiction.
- (f) This follows from Proposition 5.2. ■

THEOREM 5.5. Suppose X and Y are Banach spaces and $T \in \mathcal{B}(X, Y)$. Then T is compact if and only if $T^* \in \mathcal{B}(Y^*, X^*)$ is compact.

Proof. Suppose first that T is compact and let $\{y_n^*\}$ be a sequence in the unit ball of Y^* . We shall show that $T^* y^* = y^* \circ T$ admits a convergent subsequence in X^* . Let $K = \overline{T(U)} \subseteq Y$, which, according to our assumption is compact in Y . Note that the collection $\{y_n^*\}$ is equicontinuous and pointwise bounded on K . Due to the Arzelá-Ascoli Theorem, there is a subsequence $\{y_{n_k}^*\}$ that converges uniformly on K .

We contend that $\{T^* y_{n_k}^*\}$ converges in the operator norm. Indeed, for any $x \in U$,

$$|(T^* y_{n_k}^*(x) - T^* y_{n_l}^*(x))| = |y_{n_k}^*(Tx) - y_{n_l}^*(Tx)|,$$

and since $Tx \in K$, the conclusion follows.

Conversely, suppose T^* is compact. Consider the natural isometric embeddings $\Phi_X : X \rightarrow X^{**}$ and $\Phi_Y : Y \rightarrow Y^{**}$, which fit into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \Phi_X \downarrow & & \downarrow \Phi_Y \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array} \quad (2)$$

Due to the first part of the proof, T^{**} is compact. Thus, $T^{**}(U^{**})$ is totally bounded in Y^{**} . Next, $\Phi_X(U)$ is contained in U^{**} and hence, $T^{**}\Phi_X(U) = \Phi_Y T(U)$ is totally bounded in Y^{**} . Since Φ_Y is an isometry, it follows that $T(U)$ is totally bounded in Y , thereby completing the proof. ■

DEFINITION 5.6. A closed subspace M of a topological vector space X is said to be *complemented* if there exists a closed subspace N of X such that

$$X = M + N \quad \text{and} \quad M \cap N = \{0\}.$$

In this case, X is said to be the *direct sum* of M and N , denoted by $X = M \oplus N$.

LEMMA 5.7. Let M be a closed subspace of a topological vector space X .

- (a) If X is locally convex and $\dim M < \infty$, then M is complemented in X .
- (b) If $\dim(X/M) < \infty$, then M is complemented in X .

Proof. (a) Let $\{e_1, \dots, e_n\}$ be a basis for M . Every $x \in M$ has a unique representation

$$x = \alpha_1(x)e_1 + \dots + \alpha_n(x)e_n.$$

Note that $\alpha_i(e_j) = 0$ whenever $i \neq j$. Due to the Hahn-Banach Theorem, each α_i can be extended to a continuous linear functional on X . Let $N = \bigcap_{i=1}^n \mathcal{N}(\alpha_i)$. It is not hard to argue that $X = M \oplus N$.

- (b) Let $\pi : X \rightarrow X/M$ be the quotient map, and let $\{e_1, \dots, e_n\}$ be a basis for X/M . Lift this to $\{x_1, \dots, x_n\}$ in X and let N be the vector subspace they span. Again, it is not hard to argue that $X = M \oplus N$. ■

THEOREM 5.8. Let X be a Banach space, $T \in \mathcal{B}(X)$ a compact operator, and $\lambda \neq 0$. Then $T - \lambda I$ has closed range.

Proof. Let $N = \mathcal{N}(T - \lambda I)$, which is a closed subspace of X . Due to Lemma 5.7, admits a complement, say M . Let $S : M \rightarrow X$ be given by $x \mapsto Tx - \lambda x$, which is a bounded linear operator. Since $\mathcal{R}(S) = \mathcal{R}(T - \lambda I)$, it suffices to show that the former is closed.

To this end, we first show that there is a constant $\beta > 0$ such that $\|Sx\| \geq \beta\|x\|$ for all $x \in M$, which is equivalent to

$$\beta = \inf_{\substack{\|x\|=1 \\ x \in M}} \|Sx\| > 0.$$

Suppose not. Then, there is a sequence $x_n \in M$ with $\|x_n\| = 1$, such that $Sx_n \rightarrow 0$ as $n \rightarrow \infty$. Since $T : X \rightarrow X$ is compact, its restriction to M is also compact, whence, there is a subsequence (x_{n_k}) such that $Tx_{n_k} \rightarrow x_0$ for some $x_0 \in X$. Replace x_n with this subsequence. Then, $Tx_n - \lambda x_n \rightarrow 0$ and hence, $\lambda x_n \rightarrow x_0$. As a result,

$$Sx_0 = \lim_{n \rightarrow \infty} S(\lambda x_n) = \lambda \lim_{n \rightarrow \infty} Sx_n = 0.$$

But since S is injective, $x_0 = 0$. This is absurd, since $\|x_0\| = \lim_{n \rightarrow \infty} \|\lambda x_n\| = |\lambda| > 0$. It follows that $\beta > 0$.

Finally, we show that $\mathcal{R}(S)$ is closed in X . Indeed, suppose $y \in \overline{\mathcal{R}(S)}$; then there is a sequence (x_n) in M such that $Sx_n \rightarrow y$, that is (Sx_n) is Cauchy. But since

$$\beta\|x_n - x_m\| \leq \|Sx_n - Sx_m\|,$$

so is (x_n) . Hence, $x_n \rightarrow x_0$ for some $x_0 \in M$; and $Sx_0 = y$. This completes the proof. ■

THEOREM 5.9 (SPECTRUM OF A COMPACT OPERATOR). Let X be a Banach space and $T \in \mathcal{B}(X)$ a compact operator.

(a) Every $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of T .

(b) For every $\lambda \neq 0$, the increasing chain of subspaces

$$\mathcal{N}(T - \lambda I) \subseteq \mathcal{N}((T - \lambda I)^2) \subseteq \dots$$

eventually stabilizes. Further, all these subspaces are finite dimensional.

(c) For every $r > 0$, the set

$$\{\lambda \in \sigma(T) : |\lambda| > r\}$$

is finite.

(d) As a consequence, $\sigma(T)$ is countable and the only possible limit point of $\sigma(T)$ is 0.

Proof. Suppose $\dim X = \infty$, for if $\dim X < \infty$, then all the above statements are trivial as there are only finitely many eigenvalues.

(a) Suppose $0 \neq \lambda \in \sigma(T)$ is not an eigenvalue of T , then $T - \lambda I$ is injective, but not surjective, else, due to the open mapping theorem, it would be invertible. Define

$$Y_n = (T - \lambda I)^n(X).$$

Obviously, $Y_{n+1} \subseteq Y_n$ for all $n \geq 1$. Further, since the restriction of T to each of these subspaces is compact, due to Theorem 5.4 (d), each Y_n is infinite-dimensional and all inclusions are strict.

For each $n \geq 1$, using the Riesz Lemma, choose $y_n \in Y_n \setminus Y_{n+1}$ such that $\|y_n\| = 1$ and

$$\text{dist}(y_n, Y_{n+1}) > \frac{1}{2}.$$

Since T is compact and (x_n) is bounded, the sequence (Tx_n) must admit a convergent subsequence. But for $n < m$, we have

$$\|Tx_n - Tx_m\| = \|(T - \lambda I)x_n + \lambda x_n - (T - \lambda I)x_m - \lambda x_m\|,$$

and since $(T - \lambda I)x_n - (T - \lambda I)x_m - \lambda x_m \in Y_{n+1}$, we conclude that $\|Tx_n - Tx_m\| > \lambda/2$, a contradiction.

(b) If λ is not an eigenvalue, then each $\mathcal{N}((T - \lambda I)^n)$ is the trivial subspace and there is nothing to prove. Suppose now that λ is an eigenvalue of T and set $Y_n = \mathcal{N}((T - \lambda I)^n)$. Obviously $Y_1 \subseteq Y_2 \subseteq \dots$. Further, $(T - \lambda I)^n = S + (-\lambda)^n I$ where S is some compact operator and hence, $\dim Y_n < \infty$. Next, note that if $Y_n = Y_{n+1}$ for some $n \geq 1$, then $Y_n = Y_{n+1} = Y_{n+2} = \dots$.

Suppose now that $Y_n \subsetneq Y_{n+1}$ for every $n \geq 1$. Again, using the Riesz Lemma, choose $y_{n+1} \in Y_{n+1} \setminus Y_n$ such that $\|y_{n+1}\| = 1$ and

$$\text{dist}(y_{n+1}, Y_n) > \frac{1}{2}.$$

Again, since (y_n) is bounded and T is compact, the sequence (Ty_n) must admit a convergent subsequence. But for $2 \leq n < m$, we have

$$\|Ty_n - Ty_m\| = \|(T - \lambda I)y_n + \lambda y_n - (T - \lambda I)y_m - \lambda y_m\|,$$

and since $(T - \lambda I)y_n - (T - \lambda I)y_m + \lambda y_n \in Y_{m-1}$, it follows that $\|Ty_n - Ty_m\| > \lambda/2$, a contradiction.

- (c) Suppose there is an $r > 0$ such that the set $\{\lambda \in \sigma(T) : |\lambda| > r\}$ is infinite. Choose a countable subset $\{\lambda_1, \lambda_2, \dots\}$ with corresponding eigenvectors $\{x_1, x_2, \dots\}$. Let $Y_n = \text{span}\{x_1, \dots, x_n\}$; when then form a strictly increasing chain of closed subspaces.

First, we contend that for $n \geq 2$, $(T - \lambda_n I)(Y_n) \subseteq Y_{n-1}$. Indeed, any element of Y_n can be written uniquely as

$$Y_n \ni y = \alpha_1 x_1 + \dots + \alpha_n x_n.$$

Then, $(T - \lambda_n I)y = \alpha_1 (T - \lambda_n I)x_1 + \dots + \alpha_{n-1} (T - \lambda_n I)x_{n-1}$. And for $1 \leq i \leq n-1$, we have

$$(T - \lambda_i I)(T - \lambda_n I)x_i = (T - \lambda_n I)(T - \lambda_i I)x_i = 0,$$

whence $(T - \lambda_n I)x_i \in Y_i$.

Next, using the Riesz Lemma, for $n \geq 2$, choose $y_n \in Y_n \setminus Y_{n-1}$ such that $\|y_n\| = 1$ and

$$\text{dist}(y_n, Y_{n-1}) > \frac{1}{2}.$$

Since (y_n) is bounded and T is compact, the sequence (Ty_n) admits a convergent subsequence. But for $2 \leq n < m$, we have

$$\|Ty_n - Ty_m\| = \|(T - \lambda_n I)y_n + \lambda_n y_n - (T - \lambda_m I)y_m - \lambda_m y_m\|,$$

and since

$$(T - \lambda_n I)y_n + \lambda_n y_n - (T - \lambda_m I)y_m \in Y_{m-1},$$

we get that $\|Ty_n - Ty_m\| > |\lambda_m|/2 > r/2$, a contradiction.

- (d) Note that

$$\sigma(T) = \{0\} \cup \bigcup_{n \geq 1} \left\{ \lambda \in \sigma(T) : |\lambda| > \frac{1}{n} \right\},$$

and being a countable union of finite sets, $\sigma(T)$ is countable. Next, suppose $0 \neq \mu \in \mathbb{K}$ is a limit point of $\sigma(T)$. There is an $\varepsilon > 0$ such that $|\mu| > \varepsilon$. But since the set of eigenvalues in $\mathbb{K} \setminus \overline{B}(0, \varepsilon)$ is finite, μ cannot be their limit point. This completes the proof. ■

§§ Examples

THEOREM 5.10 (MINKOWSKI'S INTEGRAL INEQUALITY). Let (X, \mathfrak{M}, μ) and $(Y, \mathfrak{N}, \lambda)$ be positive measure spaces. If $f : X \times Y \rightarrow \mathbb{R}$ is non-negative and measurable with respect to the product measure, then for $1 \leq p < \infty$,

$$\left\{ \int_X \left(\int_Y f(x, y) d\lambda(y) \right)^p d\mu(x) \right\}^{\frac{1}{p}} \leq \int_Y \left(\int_X f(x, y)^p d\mu(x) \right)^{\frac{1}{p}} d\lambda(y)$$

Proof. Since $p = 1$ is just Fubini, we assume $p > 1$ and let q be the conjugate exponent to p . Let $H : X \rightarrow \mathbb{R}$ be defined as

$$H(x) = \int_Y f(x, y) d\lambda(y),$$

which is a measurable function on X due to Fubini. We now have the series of inequalities

$$\begin{aligned} \|H\|_p^p &= \int_X \int_Y f(x, y) H(x)^{p-1} d\lambda(y) d\mu(x) \\ &= \int_Y \int_X f(x, y) H(x)^{p-1} d\mu(x) d\lambda(y) \\ &\leq \int_Y \left(\int_X f(x, y)^p d\mu(x) \right)^{\frac{1}{p}} \left(\int_X H(x)^{pq-q} d\lambda(y) \right)^{\frac{1}{q}} d\lambda(y) \\ &= \int_Y \left(\int_X f(x, y)^p d\mu(x) \right)^{\frac{1}{p}} \|H\|_p^{\frac{p}{q}} d\lambda(y) \end{aligned}$$

and hence

$$\|H\|_p \leq \int_Y \left(\int_X f(x, y)^p d\mu(x) \right)^{\frac{1}{p}} d\lambda(y),$$

thereby completing the proof. ■

THEOREM 5.11. Let $1 < p < \infty$ and define the *Hardy operator* $H : L^p(0, \infty) \rightarrow L^p(0, \infty)$ as

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt.$$

Then, H is a non-compact operator with operator norm

$$\|H\| = \frac{p}{p-1}.$$

Proof. For operator norm, take $x^{-1/p} \chi_{[0, N]}$ and let $N \rightarrow \infty$. ■

§6 REFLEXIVE SPACES

DEFINITION 6.1. A normed linear space X is said to be *reflexive* if the natural embedding $\Phi : X \rightarrow X^{**}$ is surjective.

PROPOSITION 6.2. Let X be a normed linear space. The natural embedding $\Phi : X \rightarrow X^{**}$ is a topological imbedding when X is given the weak topology and X^* is given the weak*-topology.

Proof. ■

THEOREM 6.3 (KAKUTANI). A Banach space X is reflexive if and only if its norm-closed unit ball is weakly compact.

Proof. Let B, B^{**} denote the norm-closed unit balls of X and X^{**} respectively. If X were reflexive, then the natural embedding $\Phi : X \rightarrow X^{**}$ is surjective. Due to the preceding result, Φ is a homeomorphism when X is given the weak topology and X^{**} is given the weak*-topology. Since B^{**} is compact in the weak*-topology, and Φ is an isometry, we see that B must be compact in the weak topology.

Conversely, suppose B is compact in the weak topology. Again, due to the preceding proposition, $\Phi(B)$ is compact and convex in the weak*-topology and $\Phi(B) \subseteq B^{**}$. If X were not reflexive, then $\Phi(B) \subsetneq B^{**}$. Choose $x^{**} \in B^{**} \setminus \Phi(B)$. Due to the Hahn-Banach Separation Theorem, there is a linear functional $\Lambda : X^{**} \rightarrow \mathbb{K}$ that is continuous with respect to the weak*-topology on X^* and there are $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\Re \Lambda(x^{**}) < \gamma_1 < \gamma_2 < \Re \Lambda(y) \quad \forall y \in \Phi(B).$$

Note that there is some $0 \neq x^* \in X^*$ such that $\Lambda = \text{ev}_{x^*}$, and hence,

$$\Re x^{**}(x^*) < \gamma_1 < \gamma_2 \leq \inf_{y \in \Phi(B)} \Re y(x^*) = \inf_{x \in B} \Re x^*(x).$$

The rightmost quantity is precisely $-\|x^*\|$. Thus $\Re x^{**}(x^*) < -\|x^*\|$, in particular, $|x^{**}(x^*)| > \|x^*\|$, whence $\|x^{**}\| > 1$, a contradiction, since we chose it inside B^{**} . This completes the proof. ■

COROLLARY. Every closed, bounded convex subset of a reflexive Banach space is weakly compact.

Proof. This follows from the fact that a convex closed subset of an LCTVS is also weakly closed. ■

§7 HILBERT SPACES

DEFINITION 7.1. An *inner product space* is a \mathbb{K} -vector space H together with a function $(\cdot, \cdot) : H \times H \rightarrow \mathbb{K}$ such that

- (i) $(x, y) = \overline{(y, x)}$,
- (ii) $(x + y, z) = (x, z) + (y, z)$,
- (iii) $(\alpha x, y) = \alpha(x, y)$,
- (iv) $(x, x) \geq 0$, and $(x, x) = 0$ if and only if $x = 0$,

for all $x, y, z \in H$ and $\alpha \in \mathbb{K}$.

Obviously, $\|x\| := \sqrt{(x, x)}$ defines a norm on H . If H is complete with respect to this norm, then H is said to be a *Hilbert space*.

PROPOSITION 7.2. Let H be an inner product space and $x, y \in H$. Then,

$$|(x, y)| \leq \|x\| \|y\| \quad \text{and} \quad \|x + y\| \leq \|x\| + \|y\|.$$

Proof. For every $\lambda \in \mathbb{K}$, we have

$$0 \leq (x + \lambda y, x + \lambda y) = |\lambda|^2 \|y\|^2 + \|x\|^2 + 2\Re(x, \lambda y).$$

For every $\alpha \in \mathbb{R}$, we can choose $\lambda \in \mathbb{K}$ such that $|\lambda| = |\alpha|$ and $\Re(x, \lambda y) = \alpha |(x, y)|$. Thus,

$$\alpha^2 \|y\|^2 + 2\alpha(x, y) + \|x\|^2 \geq 0$$

for every $\alpha \in \mathbb{R}$. Thus,

$$4|(x, y)|^2 \leq 4\|x\|^2 \|y\|^2 \implies |(x, y)| \leq \|x\| \|y\|. \quad (3)$$

Finally, note that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re(x, y) \leq \|x\|^2 + \|y\|^2 + 2|(x, y)| \leq (\|x\| + \|y\|)^2,$$

thereby completing the proof. ■

THEOREM 7.3. Let H be a Hilbert space. Every nonempty closed convex $E \subseteq H$ contains a unique x of minimal norm.

Proof. Let

$$d = \inf\{\|x\| : x \in E\}.$$

Choose a sequence (x_n) in E such that $\|x_n\| \rightarrow d$ as $n \rightarrow \infty$. Since E is convex, $\frac{1}{2}(x_n + x_m) \in E$, whence $\|x_n + x_m\| \geq 2d$, for all $m, n \geq 1$.

Next, using the “parallelogram identity”,

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2.$$

Let $\varepsilon > 0$ and choose $N \geq 1$ such that whenever $n \geq N$,

$$d \leq \|x_n\| \leq \sqrt{d^2 + \varepsilon^2}.$$

Thus, for $m, n \geq N$,

$$\|x_n - x_m\|^2 \leq 4d^2 + 4\varepsilon^2 - \|x_n + x_m\|^2 \leq 4\varepsilon^2,$$

thus $\|x_n - x_m\| \leq 2\varepsilon$ whenever $m, n \geq N$. This shows that (x_n) is Cauchy and hence, converges to some $x \in E$. Obviously, $\|x\| = d$.

As for uniqueness, suppose $x, y \in E$ with $\|x\| = \|y\| = d$. Then,

$$0 \leq \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0.$$

Thus, $x = y$, thereby completing the proof. ■

The above theorem fails quite spectacularly for Banach spaces.

EXAMPLE 7.4. Let $X = C[0, 1]$ the \mathbb{R} -vector space of real-valued continuous functions on $[0, 1]$ with the supremum norm. Let

$$M = \left\{ f \in X : \int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt = 1 \right\}.$$

Then, M is a closed convex subset of X but no element of M has minimal norm.

Proof. Obviously, M is convex. To see that it is closed, note that the linear functional

$$T : X \rightarrow \mathbb{R} \quad f \mapsto \int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt$$

is a bounded linear functional, and hence, is continuous. Thus, M is closed too.

Next, for any $f \in M$,

$$1 = \left| \int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt \leq \|f\|_\infty.$$

We contend that

$$\inf \{ \|f\|_\infty : f \in M \} = 1.$$

To see this, fix some $0 < \delta < 1/2$. Define the function

$$f(x) = \begin{cases} 1 + \varepsilon & 0 \leq x \leq \frac{1}{2} - \delta \\ \frac{1+\varepsilon}{\delta} \left(\frac{1}{2} - x \right) & \frac{1}{2} - \delta \leq x \leq \frac{1}{2} + \delta \\ -(1 + \varepsilon) & \frac{1}{2} + \delta \leq x \leq 1. \end{cases}$$

Then,

$$\int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt = (1 + \varepsilon)(1 - 2\delta) + \delta(1 + \varepsilon) = (1 - \delta)(1 + \varepsilon).$$

Choosing

$$\varepsilon = \frac{\delta}{1 - \delta},$$

we get $Tf = 1$. Note that $\|f\|_\infty = 1 + \varepsilon$ and as $\delta \rightarrow 0^+$, we get $\|f\|_\infty \rightarrow 1^+$. This proves our claim.

Finally, suppose $f \in M$ such that $\|f\|_\infty = 1$. Then,

$$0 = \int_0^{1/2} 1 - f(t) dt + \int_{1/2}^1 1 + f(t) dt.$$

Since both integrals are non-negative and the functions are continuous, we must have $f(t) = 1$ whenever $0 \leq t \leq 1/2$ and $f(t) = -1$ whenever $1/2 \leq t \leq 1$, a contradiction. This completes the proof. ■

THEOREM 7.5. Let M be a closed subspace of a Hilbert space H , then $H = M \oplus M^\perp$.

Proof. Since

$$M^\perp = \bigcap_{x \in M} \ker(\cdot, x),$$

it is a closed subspace of H . Obviously, $M \cap M^\perp = \{0\}$. It remains to show that $H = M + M^\perp$. Indeed, let $x \in H$ and let $x_1 \in M$ be the unique element minimizing the distance to x . We contend that $x_2 = x - x_1$ is perpendicular to x_1 .

Indeed, note that for every $y \in M$, we have

$$\|x_2\|^2 \leq \|x_2 + y\|^2 \implies \|y\|^2 + 2\Re(x_2, y) \geq 0,$$

for all $y \in M$. Suppose $(x_2, y) \neq 0$ for some $y \in M$. We can choose y such that $\Re(x_2, y) = -|(x_2, y)|$. Then, replacing y by αy for some $\alpha > 0$, we have $\alpha^2\|y\|^2 - 2\alpha|(x_2, y)| \geq 0$ for all $\alpha > 0$. This is obviously false, and hence, $(x_2, y) = 0$ for all $y \in M$, thereby completing the proof. ■

The above theorem fails for closed subspaces of Banach spaces.

EXAMPLE 7.6. $c_0 \subseteq \ell^\infty$ is not complemented.

Proof. We begin with a claim.

Claim. Let $T : \ell^\infty \rightarrow \ell^\infty$ be a bounded linear operator with $c_0 \subseteq \ker T$. Then there is an infinite subset $A \subseteq \mathbb{N}$ such that $Tx = 0$ whenever x is supported in A .

Proof of Claim: Suppose not. Then, for every infinite subset $A \subseteq \mathbb{N}$, there is an $x \in \ell^\infty$, supported in A such that $Tx \neq 0$. Choose an uncountable collection $\{A_i : i \in I\}$ of infinite subsets of \mathbb{N} with pairwise finite intersections. According to our assumption, there are $x_i \in \ell^\infty$ supported in A_i with $Tx_i \neq 0$ and $\|x_i\| = 1$.

Since I is uncountable, there is an $n \in \mathbb{N}$ such that

$$I_n = \{i \in I : (Tx_i)(n) \neq 0\}$$

is uncountable (because the union of all the I_n 's is I). Further, there is a positive integer k such that

$$I_{n,k} = \left\{ i \in I : |(Tx_i)(n)| \geq \frac{1}{k} \right\}$$

is uncountable (because the union of all the $I_{n,k}$'s is I_n).

Let $J \subseteq I_{n,k}$ be finite and set

$$y = \sum_{j \in J} \operatorname{sgn}((Tx_j)(n)) \cdot x_j.$$

Then,

$$(Ty)(n) = \sum_{j \in J} \operatorname{sgn}((Tx_j)(n)) \cdot (Tx_j)(n) \geq \sum_{j \in J} \frac{1}{k} = \frac{|J|}{k}.$$

Note that for $i \neq j$, $A_i \cap A_j$ is finite and hence, we can write $y = x + z$, where x has finite support and $\|z\| \leq 1$. Thus, $x \in c_0 \subseteq \ker T$ and hence,

$$\frac{|J|}{k} \leq \|Ty\| = \|Tx + Tz\| = \|Tz\| \leq \|T\| \implies |J| \leq k\|T\|,$$

which is absurd, since $I_{n,k}$ is infinite. This proves the claim. \square

Coming back, suppose c_0 were complemented in ℓ^∞ . Then, there would be a projection operator $P : \ell^\infty \rightarrow c_0 \subseteq \ell^\infty$. Set $T = \mathbf{id} - P$. Since $c_0 \subseteq \ker T$, due to the claim above, there is an infinite subset $A \subseteq \mathbb{N}$, such that $Tx = 0$ whenever x is supported in A . Consider $\chi_A \in \ell^\infty$, the characteristic function of the set A . But note that

$$P(\chi_A) = (\mathbf{id} - T)(\chi_A) = \chi_A \notin c_0,$$

a contradiction. This completes the proof. \blacksquare

THEOREM 7.7 (RIESZ REPRESENTATION LEMMA). Let H be a Hilbert space. The natural map $H \rightarrow H^*$ given by $y \mapsto (\cdot, y)$ is an isometric and surjective.

Proof. Obviously, the map is injective and linear. To see isometry, note that $(y, y) = \|y\|^2$, whence $\|(\cdot, y)\| \geq \|y\|$ and due to Cauchy-Schwarz,

$$|(x, y)| \leq \|x\|\|y\| \implies \|(\cdot, y)\| \leq \|y\| \implies \|(\cdot, y)\| = \|y\|.$$

It remains to show surjectivity. Let $\Lambda \neq 0$ be a continuous linear functional on H and $N = \ker \Lambda$. Since N is closed, we can write $H = N \oplus N^\perp$. Choose a nonzero vector $z \in N^\perp$. For any $x \in H$,

$$x - \frac{\Lambda x}{\Lambda z} z \in \ker \Lambda,$$

whence

$$0 = \left(x - \frac{\Lambda x}{\Lambda z} z, z \right) = (x, z) - \frac{\Lambda x}{\Lambda z} \|z\|^2.$$

Thus,

$$\Lambda x = \left(x, \frac{\overline{\Lambda z}}{\|z\|^2} z \right),$$

thereby completing the proof. \blacksquare

THEOREM 7.8. Let H be a Hilbert space and suppose $f : H \times H \rightarrow \mathbb{K}$ is sesquilinear and bounded, that is,

$$M := \sup \{|f(x, y)| : \|x\| = \|y\| = 1\} < \infty,$$

then there exists a unique $S \in \mathcal{B}(H)$ such that

$$f(x, y) = (x, Sy) \quad \forall x, y \in H.$$

Further, $\|S\| = M$.

Proof. Fix $y \in H$ and consider the mapping $x \mapsto f(x, y)$. This is a continuous linear functional on H and hence, is given by $x \mapsto (x, Sy)$ for a unique $Sy \in H$. We claim that the association $y \mapsto Sy$ is linear.

Indeed, if $y_1, y_2 \in H$, then

$$f(\cdot, y_1 + y_2) = f(\cdot, y_1) + f(\cdot, y_2) = f(\cdot, Sy_1) + f(\cdot, Sy_2) = f(\cdot, Sy_1 + Sy_2).$$

Due to uniqueness of $S(y_1 + y_2)$, we see that $S(y_1 + y_2) = Sy_1 + Sy_2$. Next, let $\alpha \in \mathbb{K}$ and $y \in H$. Then,

$$(\cdot, S(\alpha y)) = f(\cdot, \alpha y) = \bar{\alpha} f(\cdot, y) = \bar{\alpha}(\cdot, Sy) = (\cdot, \alpha Sy),$$

whence $S(\alpha y) = \alpha Sy$, i.e., S is linear.

Finally, we must show that $\|S\| = M$. Indeed, for $\|x\| = \|y\| = 1$:

$$|f(x, y)| \leq |(x, Sy)| \leq \|x\| \|Sy\| \leq \|S\|,$$

whence $M \leq \|S\|$. On the other hand, if $Sy \neq 0$, then

$$\|Sy\| = \left(\frac{Sy}{\|Sy\|}, Sy \right) = f\left(\frac{Sy}{\|Sy\|}, y \right) \leq M$$

Taking supremum over $\|y\| = 1$, we have that $\|S\| \leq M \leq \|S\|$, thereby completing the proof. ■

§§ Adjoints

DEFINITION 7.9. Let $T \in \mathcal{B}(H)$. The map $f : H \times H \rightarrow \mathbb{K}$ given by $f(x, y) = (Tx, y)$, is a bounded sesquilinear form on H , whence, there is a $T^* \in \mathcal{B}(H)$ such that

$$(Tx, y) = f(x, y) = (x, T^*y) \quad \forall x, y \in H.$$

Next, note that

$$(x, Ty) = \overline{(y, T^*x)} = (T^*x, y) = (x, T^{**}y) \quad \forall x, y \in H.$$

Hence, $T^{**} = T$. On the other hand,

$$\|T^*\| = \sup\{|(Tx, y)| : \|x\| = \|y\| = 1\} \leq \|T\|.$$

Consequently, $\|T\| = \|T^{**}\| \leq \|T^*\| \leq \|T\|$, whence, $\|T^*\| = \|T\|$.

Similarly, the following identities are easy to show for $S, T \in \mathcal{B}(H)$:

$$(S + T)^* = S^* + T^*, \quad (\alpha S)^* = \bar{\alpha} S^*, \quad \text{and} \quad (ST)^* = T^* S^*.$$

Therefore,

$$\|Tx\|^2 = (Tx, Tx) = (x, T^*Tx) \leq \|T^*T\| \|x\|^2 \quad \forall x \in H.$$

Hence, $\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$, whence $\|T\|^2 = \|T^*T\|$. This makes $\mathcal{B}(H)$ a C^* -algebra.

§§ Compact Self-Adjoint Operators

LEMMA 7.10. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a compact self-adjoint operator. Then

$$\|T\| = \sup\{|\langle Tx, x \rangle| : \|x\| = 1\}.$$

Proof. Let B denote the quantity on the right hand side. Due to the Cauchy-Schwarz Inequality, $B \leq \|T\|$. Let $x \neq 0$ and set $\lambda = \sqrt{\frac{\|Tx\|}{\|x\|}}$.

We have

$$\begin{aligned} \langle Tx, Tx \rangle &= \frac{1}{4} \left| \langle T(\lambda x + \lambda^{-1}Tx), \lambda x + \lambda^{-1}Tx \rangle - \langle T(\lambda x - \lambda^{-1}Tx), \lambda x - \lambda^{-1}Tx \rangle \right| \\ &\leq \frac{1}{4} \left| \langle T(\lambda x + \lambda^{-1}Tx), \lambda x + \lambda^{-1}Tx \rangle \right| + \frac{1}{4} \left| \langle T(\lambda x - \lambda^{-1}Tx), \lambda x - \lambda^{-1}Tx \rangle \right| \\ &\leq \frac{B}{4} \left(\|\lambda x + \lambda^{-1}Tx\|^2 + \|\lambda x - \lambda^{-1}Tx\|^2 \right) \\ &= \frac{B}{2} \left(\|\lambda x\|^2 + \|\lambda^{-1}Tx\|^2 \right) \\ &= B\|x\|\|Tx\|. \end{aligned}$$

Thus, $\|Tx\| \leq B\|x\|$, whence $\|T\| \leq B$, thereby completing the proof. ■

LEMMA 7.11. With the notation of the preceding lemma, either $\|T\|$ or $-\|T\|$ is an eigenvalue of T .

Proof. Due to the preceding lemma, there is a sequence of unit vectors (x_n) in H such that $|\langle Tx_n, x_n \rangle| \rightarrow \|T\|$. Since T is self-adjoint,

$$\overline{\langle Tx, x \rangle} = \langle x, Tx \rangle = \langle Tx, x \rangle,$$

and hence, $\langle Tx, x \rangle \in \mathbb{R}$. Therefore, moving to a subsequence, we may suppose that $\langle Tx_n, x_n \rangle \rightarrow \lambda \in \{\pm\|T\|\}$. Further, since T is compact, we may replace (x_n) with a subsequence such that $Tx_n \rightarrow \lambda y$ for some $y \in H$.

We contend that $x_n \rightarrow y$. First, note that

$$|\langle Tx_n, x_n \rangle| \leq \|Tx_n\|\|x_n\| = \|Tx_n\| \leq \|T\| = |\lambda|.$$

By our choice of the sequence (x_n) , $|\langle Tx_n, x_n \rangle| \rightarrow |\lambda|$ and hence, $\|Tx_n\| \rightarrow |\lambda|$. Next,

$$\begin{aligned} \|\lambda x_n - Tx_n\|^2 &= \langle \lambda x_n - Tx_n, \lambda x_n - Tx_n \rangle \\ &= \lambda^2 + \|Tx_n\|^2 - \langle \lambda x_n, Tx_n \rangle - \langle Tx_n, \lambda x_n \rangle \\ &= \lambda^2 + \|Tx_n\|^2 - 2\lambda \langle Tx_n, x_n \rangle \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$. Hence, $\|\lambda x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, consequently, $x_n \rightarrow y$, thereby completing the proof. ■

§8 BANACH ALGEBRAS

DEFINITION 8.1. A *Banach algebra* is a \mathbb{C} -algebra \mathcal{A} equipped with a norm $\|\cdot\| : \mathcal{A} \rightarrow [0, \infty)$ with respect to which it is a Banach space and

$$\|xy\| \leq \|x\|\|y\| \quad \forall x, y \in \mathcal{A}.$$

The Banach algebra is said to be *unital* if it possesses a multiplicative identity.

An *involution* on an algebra \mathcal{A} is a map

$$\mathcal{A} \rightarrow \mathcal{A} \quad x \mapsto x^*$$

of order 2 that satisfies

$$(x + y)^* = x^* + y^* \quad (\lambda x)^* = \bar{\lambda}x^* \quad (xy)^* = y^*x^*.$$

An algebra equipped with such an involution is called a **-algebra*. A Banach *-algebra that satisfies

$$\|x^*x\| = \|x\|^2 \quad \forall x \in \mathcal{A}$$

is called a *C*-algebra*.

REMARK 8.2. If \mathcal{A} is a C*-algebra, for $x \neq 0$, we have

$$\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\| \implies \|x\| \leq \|x^*\| \leq \|x^{**}\| = \|x\|,$$

whence $\|x\| = \|x^*\|$. That is, the involution is an isometry.

DEFINITION 8.3. If \mathcal{A} and \mathcal{B} are Banach algebras, a *homomorphism* is a bounded linear map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$.

Further, if \mathcal{A} and \mathcal{B} are Banach *-algebras, a **-homomorphism* is a homomorphism of Banach algebras $\phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{A}$.

THEOREM 8.4. Let \mathcal{A} be a unital Banach algebra.

(a) If $|\lambda| > \|x\|$, then $\lambda - x$ is invertible in \mathcal{A} .

(b) If x is invertible, and $\|y\| < \|x^{-1}\|^{-1}$, then $x - y$ is invertible with inverse

$$(x - y)^{-1} = \sum_{n \geq 0} (x^{-1}y)^n x^{-1}.$$

(c) If x is invertible and $\|y\| < \frac{1}{2}\|x^{-1}\|^{-1}$, then

$$\|(x - y)^{-1} - x^{-1}\| < 2\|x^{-1}\|^2\|y\|.$$

(d) $\mathcal{A}^\times \subseteq \mathcal{A}$ is open and $x \mapsto x^{-1}$ on \mathcal{A}^\times is continuous.

Proof. (a) We have

$$(\lambda - x)^{-1} = \lambda^{-1} \left(e - \lambda^{-1} x \right)^{-1} = \sum_{n \geq 0} \lambda^{-(n+1)} x^{-n},$$

which converges because things are Cauchy and all the good stuff.

(b) Again, we can write

$$(x - y)^{-1} = \left(x(e - x^{-1}y) \right)^{-1} = (e - x^{-1}y)^{-1} x^{-1} = \sum_{n \geq 0} (x^{-1}y)^n x^{-1}.$$

(c) Using the above expansion, we can write

$$\|(x - y)^{-1} - x^{-1}\| \leq \sum_{n \geq 0} \|x^{-1}\|^{n+2} \|y\|^{n+1} < 2\|x^{-1}\|^2 \|y\|.$$

(d) Due to part (b), \mathcal{A}^\times is open in \mathcal{A} and due to part (c), $x \mapsto x^{-1}$ is continuous. ■

DEFINITION 8.5. Let \mathcal{A} be a unital Banach algebra and $x \in \mathcal{A}$. The *spectrum* of x is

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}.$$

For $\lambda \notin \sigma(x)$, define the *resolvent* of x as

$$R_x(\lambda) = (\lambda e - x)^{-1} : \mathbb{C} \setminus \sigma(x) \rightarrow \mathcal{A}.$$

PROPOSITION 8.6. For any $x \in \mathcal{A}$, $\sigma(x)$ is a compact subset of \mathbb{C} that is contained in the disk $\{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.

Proof. Obviously, if $|\lambda| > \|x\|$, then $\lambda e - x$ is invertible. Thus, $\sigma(x)$ is contained in the above disk. Consider the map $\lambda \mapsto \lambda e - x$, which is continuous and hence, the preimage of \mathcal{A}^\times is open in \mathbb{C} . As a result, $\sigma(x)$ is closed, thereby completing the proof. ■

PROPOSITION 8.7. R_x is an analytic function. And, $R_x(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. We have

$$\begin{aligned} R_x(\mu) - R_x(\lambda) &= (\mu e - x)^{-1} - (\lambda e - x)^{-1} \\ &= R_x(\mu) ((\lambda e - x) - (\mu e - x)) R_x(\lambda). \end{aligned}$$

Hence,

$$\frac{R_x(\mu) - R_x(\lambda)}{\mu - \lambda} = -R_x(\mu) R_x(\lambda).$$

In the limit $\mu \rightarrow \lambda$, we get

$$R'_x(\lambda) = -R_x(\lambda)^2.$$

As for the second part, simply note that for $|\lambda| > \|x\|$,

$$\|R_x(\lambda)\| = \left\| \sum_{n \geq 0} \lambda^{-(n+1)} x^{-n} \right\| \leq |\lambda|^{-1} \sum_{n \geq 0} |\lambda|^{-n} \|x\|^n = \frac{1}{|\lambda| - \|x\|},$$

which goes to 0 as $\lambda \rightarrow \infty$, thereby completing the proof. ■

THEOREM 8.8 (GELFAND-MAZUR). Let \mathcal{A} be a unital Banach algebra $\sigma(x) \neq \emptyset$ for every $x \in \mathcal{A}$.

Proof. Suppose $\sigma(x) = \emptyset$ for some $x \in \mathcal{A}$. Then, $R_x : \mathbb{C} \rightarrow \mathcal{A}$ is an analytic function. For any $\Lambda \in \mathcal{A}^*$, $\Lambda \circ R_x$ is an entire function and is bounded, since

$$\lim_{\lambda \rightarrow \infty} \Lambda(R_x(\lambda)) = \Lambda \left(\lim_{\lambda \rightarrow \infty} R_x(\lambda) \right) = 0.$$

Due to Liouville's Theorem, $\Lambda \circ R_x$ must be constant on \mathbb{C} and equal to 0. Since this is true for every $\Lambda \in \mathcal{A}^*$, we see that $R(\lambda) = 0$ for every $\lambda \in \mathbb{C}$, which is absurd. This completes the proof. ■

COROLLARY. If \mathcal{A} is a unital Banach algebra in which every nonzero element is invertible, then $\mathcal{A} = \mathbb{C}e$.

Proof. Suppose $x \in \mathcal{A} \setminus \mathbb{C}e$, then $\lambda e - x \neq 0$ for every $\lambda \in \mathbb{C}$, whence, $\lambda e - x$ is invertible for every $\lambda \in \mathbb{C}$, a contradiction. ■

DEFINITION 8.9. Let \mathcal{A} be a unital Banach algebra. For $x \in \mathcal{A}$, the *spectral radius* of x is defined to be

$$\rho(x) := \sup \{ |\lambda| : \lambda \in \sigma(x) \}.$$

We have the obvious inequality $\rho(x) \leq \|x\|$.

THEOREM 8.10 (SPECTRAL RADIUS FORMULA). Let \mathcal{A} be a unital Banach algebra and $x \in \mathcal{A}$. Then,

$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}.$$

Proof. If $\lambda \in \sigma(x)$, then

$$\lambda^n e - x^n = (\lambda e - x) (\lambda^{n-1} e + \dots + x^{n-1}).$$

Consequently, $\lambda^n e - x^n$ cannot be invertible. Hence, $|\lambda|^n \leq \|x^n\|$. In particular, this gives

$$\rho(x) \leq \liminf_{n \rightarrow \infty} \|x^n\|^{1/n}.$$

Next, for $|\lambda| > \|x\|$, we have a Laurent series about infinity:

$$\Lambda \circ R_x(\lambda) = \sum_{n \geq 0} \lambda^{-(n+1)} \Lambda(x^n).$$

Note that $\Lambda \circ R_x$ is analytic on $|\lambda| > \rho(x)$ and hence, the above Laurent series must be valid there too.

Hence, for any $|\lambda| > \rho(x)$, there is a constant $C_\Lambda > 0$ such that

$$|\Lambda(\lambda^{-n} x^n)| = |\lambda^{-n} \Lambda(x^n)| \leq C_\Lambda \quad \forall n \in \mathbb{N}.$$

This holds for all $\lambda \in \mathcal{A}^*$. Thus, the sequence $(\lambda^{-n}x^n)$ is bounded, that is, there is a $C > 0$ such that $\|x^n\| \leq C|\lambda|^n$. Hence,

$$\limsup_{n \rightarrow \infty} \|x^n\|^{1/n} \leq \limsup_{n \rightarrow \infty} C^{1/n}|\lambda| = |\lambda|.$$

Taking infimum over λ , we get

$$\limsup_{n \rightarrow \infty} \|x^n\|^{1/n} \leq \rho(x) \leq \liminf_{n \rightarrow \infty} \|x^n\|^{1/n},$$

thereby completing the proof. ■

DEFINITION 8.11. Let \mathcal{A} be a unital Banach algebra. A *multiplicative functional* on \mathcal{A} is a nonzero homomorphism $h : \mathcal{A} \rightarrow \mathbb{C}$. The set of all multiplicative functionals on \mathcal{A} is called the *spectrum* of \mathcal{A} and is denoted by $\sigma(\mathcal{A})$.

PROPOSITION 8.12. Let \mathcal{A} be a unital Banach algebra and suppose $h \in \sigma(\mathcal{A})$.

- (a) $h(e) = 1$.
- (b) If $x \in \mathcal{A}^\times$, then $h(x) \neq 0$.
- (c) $|h(x)| \leq \rho(x) \leq \|x\|$ for all $x \in \mathcal{A}$. That is, $\|h\| \leq 1$.

Proof. (a) Since $h \neq 0$, there is an $x \in \mathcal{A}$ such that $h(x) \neq 0$. Then,

$$h(x) = h(xe) = h(x)h(e) \implies h(e) = 1.$$

(b) Obviously,

$$1 = h(e) = h(x^{-1}x) = h(x^{-1})h(x) \implies h(x) \neq 0.$$

(c) Suppose $|\lambda| > \rho(x)$. Then, $\lambda e - x \in \mathcal{A}^\times$, consequently,

$$0 \neq h(\lambda e - x) = \lambda - h(x) \implies h(x) \neq \lambda.$$

Since this holds for all $|\lambda| > \rho(x)$, we have $|h(x)| \leq \rho(x) \leq \|x\|$. ■

As a consequence, $\sigma(\mathcal{A})$ is contained in the closed unit ball of \mathcal{A}^* . Equip the latter with the weak*-topology. Using nets, it is easy to see that $\sigma(\mathcal{A})$ is closed in \mathcal{A}^* . Due to Banach-Alaoglu, the closed unit ball of \mathcal{A}^* is weak*-compact and hence, so is $\sigma(\mathcal{A})$ with the subspace topology from the weak*-topology on \mathcal{A}^* . Thus, $\sigma(\mathcal{A})$ is a *compact Hausdorff space*.

PROPOSITION 8.13. Let \mathcal{A} be a commutative unital Banach algebra and $\mathcal{J} \subsetneq \mathcal{A}$ be a proper ideal.

- (a) $\mathcal{J} \subseteq \mathcal{A} \setminus \mathcal{A}^\times$
- (b) $\overline{\mathcal{J}}$ is a proper ideal.

(c) \mathcal{J} is contained in a maximal ideal.

(d) Every maximal ideal is closed.

Proof. The first assertion is obvious. As for the second, note that $\mathcal{A} \setminus \mathcal{A}^\times$ is closed and hence, $\overline{\mathcal{J}} \subseteq \mathcal{A} \setminus \mathcal{A}^\times$. Consequently, $\overline{\mathcal{J}} \neq \mathcal{A}$. To see that it is an ideal, suppose $x \in \overline{\mathcal{J}}$ and $a \in \mathcal{A}$. Then, there is a sequence (x_n) converging to x . Consequently, (ax_n) converges to ax . But each $ax_n \in \mathcal{J}$ and hence, $ax \in \overline{\mathcal{J}}$. This proves (b).

The third assertion is a standard application of Zorn's lemma. As for (d), if \mathcal{M} is a maximal ideal, then $\mathcal{M} \subseteq \overline{\mathcal{M}} \subsetneq \mathcal{A}$ due to (b). The maximality of \mathcal{M} forces $\mathcal{M} = \overline{\mathcal{M}}$, thereby completing the proof. ■

THEOREM 8.14. Let \mathcal{A} be a commutative unital Banach algebra. Then, the map $h \mapsto \ker h$ is a bijective correspondence between $\sigma(\mathcal{A})$ and the set of all maximal ideals in \mathcal{A} .

Proof. The map is obviously an injection. We establish surjectivity. Let \mathcal{M} be a maximal ideal in \mathcal{A} and consider the quotient algebra \mathcal{A}/\mathcal{M} equipped with the norm:

$$\|x + \mathcal{M}\| = \inf \{\|x + y\| : y \in \mathcal{M}\}.$$

This is again a commutative unital Banach algebra in which every non-zero element is invertible (standard fact from ring theory). Due to Gelfand-Mazur, $\mathcal{A}/\mathcal{M} \cong \mathbb{C}(e + \mathcal{M})$. The composition

$$\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{M} \cong \mathbb{C}(e + \mathcal{M}) \cong \mathbb{C}$$

is the required linear functional, thereby proving surjectivity. ■

DEFINITION 8.15. Let \mathcal{A} be a commutative unital Banach algebra. For each $x \in \mathcal{A}$, there is a continuous function $\hat{x} : \sigma(\mathcal{A}) \rightarrow \mathbb{C}$ given by $h \mapsto h(x)$. This gives a map

$$\Gamma_{\mathcal{A}} : \mathcal{A} \rightarrow C(\sigma(\mathcal{A})) \quad x \mapsto \hat{x},$$

known as the *Gelfand transform* on \mathcal{A} .

PROPOSITION 8.16. Let \mathcal{A} be a commutative unital Banach algebra and $x \in \mathcal{A}$.

- (a) The $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ is a homomorphism, and \hat{e} is the constant function 1.
- (b) x is invertible if and only if \hat{x} never vanishes.
- (c) The range of $\hat{x} : \sigma(\mathcal{A}) \rightarrow \mathbb{C}$ is precisely $\sigma(x)$.
- (d) $\|\hat{x}\|_{\sup} = \rho(x) \leq \|x\|$.

Proof. (a) Obvious.

- (b) If x is invertible, then due to (a), so is \hat{x} , whence it never vanishes. On the other hand, if x is not invertible, then it is contained in some maximal ideal \mathfrak{M} , whence, there is an $h \in \sigma(\mathcal{A})$ that vanishes on x . Thus, $\hat{x}(h) = 0$, that is, \hat{x} vanishes somewhere.

(c) Next, suppose $\lambda = \widehat{x}(h) = h(x)$. Then, $h(\lambda e - x) = 0$, hence, $\lambda e - x$ is not invertible, i.e. $\lambda \in \sigma(x)$. Similarly, if $\lambda \in \sigma(x)$, then $\lambda e - x$ is not invertible and hence, \widehat{x} vanishes somewhere, consequently, $h(\lambda e - x) = 0$ for some $h \in \sigma(\mathcal{A})$. This shows that λ is in the range of \widehat{x} .

(d) Follows from (c). ■

DEFINITION 8.17. Let \mathcal{A} be a commutative unital Banach $*$ -algebra. If $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ is a $*$ -homomorphism, then \mathcal{A} is said to be *symmetric*.

REMARK 8.18. Note that \mathcal{A} being symmetric is the same as saying

$$\widehat{x^*} = \overline{\widehat{x}} \quad \forall x \in \mathcal{A}.$$

PROPOSITION 8.19. Let \mathcal{A} be a commutative Banach $*$ -algebra.

- (a) \mathcal{A} is symmetric if and only if \widehat{x} is real-valued whenever $x = x^*$.
- (b) Every C^* -algebra is symmetric.
- (c) If \mathcal{A} is symmetric, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$.

Proof. (a) If \mathcal{A} is symmetric and $x^* = x$, then $\widehat{x} = \widehat{x^*} = \overline{\widehat{x}}$, whence \widehat{x} is real-valued. Next, we prove the converse. For any $x \in \mathcal{A}$, write

$$x = \underbrace{\frac{x + x^*}{2}}_y + \underbrace{\frac{x - x^*}{2}}_z.$$

Note that $y^* = y$ and $z + z^* = 0$. Our hypothesis implies \widehat{y} is real-valued and $\widehat{z} + \overline{\widehat{z}} = 0$. Thus,

$$\widehat{x^*} = \widehat{y^*} + \widehat{z^*} = \widehat{y} - \widehat{z} = \widehat{y} + \overline{\widehat{z}} = \overline{\widehat{x}}.$$

- (b) Let $x \in \mathcal{A}$ be such that $x^* = x$. Suppose $h(x) = \alpha + i\beta$. We shall show that $\beta = 0$. Indeed, for $t \in \mathbb{R}$, let $z = x + ite$. Then,

$$z^*z = (x - ite)(x + ite) = x^2 + t^2e.$$

And hence,

$$|\alpha + (\beta + t)i|^2 = |h(z)|^2 \leq \|z\|^2 = \|z^*z\| = \|x^2 + t^2e\| \leq \|x\|^2 + t^2.$$

That is,

$$\alpha^2 + 2\beta t + \beta^2 \leq \|x\|^2 \quad \forall t \in \mathbb{R}.$$

Thus, $\beta = 0$ and due to (a), \mathcal{A} is symmetric.

- (c) Note that $\Gamma(\mathcal{A})$ contains all the constant functions and thus, the family $\Gamma(\mathcal{A})$ does not vanish at any point. Next, by definition, $\Gamma(\mathcal{A})$ separates points. Further, since Γ is a $*$ -homomorphism, $\Gamma(\mathcal{A})$ is closed under taking conjugates. Thus, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$ due to the Stone-Weierstrass Theorem. ■

PROPOSITION 8.20. Let \mathcal{A} be a commutative unital Banach algebra.

(a) If $x \in \mathcal{A}$, then $\|\hat{x}\|_{\sup} = \|x\|$ if and only if $\|x^{2^k}\| = \|x\|^{2^k}$ for all $k \geq 1$.

(b) $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ is an isometry if and only if $\|x^2\| = \|x\|^2$ for all $x \in \mathcal{A}$.

Proof. (a) This follows immediately from the spectral radius formula.

$$\|\hat{x}\|_{\sup} = \rho(x) = \lim_{k \rightarrow \infty} \|x^{2^k}\|^{1/2^k} = \lim_{k \rightarrow \infty} \|x\|^{2^k \cdot 2^{-k}} = \|x\|.$$

(b) We have

$$\|x^{2^k}\| = \|x^{2^{k-1}}\|^2 = \dots = \|x\|^{2^k}. \quad \blacksquare$$

THEOREM 8.21 (GELFAND-NAIMARK). If \mathcal{A} is a commutative unital C^* -algebra, then $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ is an isometric $*$ -isomorphism.

Proof. That Γ is a $*$ -homomorphism has already been established. We first show that Γ is an isometry. Let $x \in \mathcal{A}$ and set $y = x^*x$. Then, $y^* = y$, so

$$\|y^{2^k}\| = \left\| \left(y^{2^{k-1}} \right)^* y^{2^{k-1}} \right\| = \|y^{2^{k-1}}\|^2 = \dots = \|y\|^{2^k}.$$

Due to part (a) of the preceding result, $\|\hat{y}\|_{\sup} = \|y\|$. But $\hat{y} = \widehat{x^*x} = |\hat{x}|^2$. Hence,

$$\|\hat{x}\|_{\sup}^2 = \|\hat{y}\|_{\sup} = \|y\| = \|x\|^2 \implies \|\hat{x}\|_{\sup} = \|x\|,$$

whence, due to part (b) of the preceding result, Γ is an isometry. Thus, its image is closed in $C(\sigma(\mathcal{A}))$. But we already argued that $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$ and hence, Γ must be surjective. This completes the proof. \blacksquare