MA 534: HOMEWORK 2

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1. Problem 1

Let $\rho \in C_c^\infty(\mathbb{R})$ be identically 1 on a neighborhood of 0. Let Q be a compact subset of $\mathbb{R}$ containing the support of ρ. Identify $\mathbb{R}^{n-1}$ with the subspace $\{x \in \mathbb{R}^n \colon x_n = 0\} \subseteq \mathbb{R}^n$. First note that the support of u is contained in the hyperplane $\mathbb{R}^{n-1}$. Indeed, if $x \notin \mathbb{R}^{n-1}$, then $x_n > 0$. Choose an open ball U containing x and disjoint from $\mathbb{R}^{n-1}$. Then, $x_n \neq 0$ on all of U and hence, for every $\varphi \in C_c^\infty(U)$, we have

$$(u, \varphi) = \left(x_n u, \frac{\varphi(x)}{x_n} \right) = 0,$$

which makes sense because $\varphi(x)/x_n$ is well-defined, smooth and compactly supported on U. It follows that the support of u is contained in the hyperplane $\mathbb{R}^{n-1}$.

Claim. If $f \in C_c^\infty(\mathbb{R}^n)$ is such that $f|_{\mathbb{R}^{n-1}}$, then $(u, f) = 0$.

Proof. Define the function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{f(x)}{x_n} & x_n \neq 0 \\ \partial_n f(x_1, \ldots, x_{n-1}, 0) & x_n = 0. \end{cases}$$

Obviously g is smooth on $\mathbb{R}^n \setminus \mathbb{R}^{n-1}$. Now, fix some $\xi = (\xi_1, \ldots, \xi_{n-1}, 0) \in \mathbb{R}^{n-1}$ and consider a sequence $x^n \in \mathbb{R}^n \setminus \mathbb{R}^{n-1}$ converging to ξ. Then,

$$\lim_{n \to \infty} g(x^n) = \lim_{n \to \infty} \frac{f(x^n)}{x_n^n} = \partial_n f(\xi),$$

since f is smooth. Thus g is well-defined and smooth on $\mathbb{R}^n$. Further, if $R > 0$ is such that f is supported inside the open set $B(0, R)$, then for all $x \notin B(0, R)$, we obviously have that both $f(x) = 0$ and $\partial_n f(x) = 0$. Hence, g is also supported inside $B(0, R)$. This shows that g is compact. Finally, note that $x_n g(x) = f(x)$ for all $x \in \mathbb{R}^n$; indeed, this equality is obvious for $x \notin \mathbb{R}^{n-1}$ and for $x \in \mathbb{R}^{n-1}$, since $x_n = 0$, we have $x_n g(x) = 0 = f(x)$. Thus, we have

$$(u, f) = (u, x_n g) = (x_n u, g) = 0,$$

as desired. ■

Next, define $v \in \mathscr{D}'(\mathbb{R}^{n-1})$ by

$$(v, \varphi) = (u, \rho(x_n)\varphi(x_1, \ldots, x_{n-1})) \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^{n-1}).$$

To see that v is indeed a distribution, let $K \subseteq \mathbb{R}^{n-1}$ and suppose $\varphi \in C_c^\infty(K)$. Then, $\rho(x_n)\varphi(x_1, \ldots, x_{n-1})$ is supported inside the compact set $K \times Q$. Since u is a distribution, there is a positive integer N and a constant $C > 0$ such that

$$|(u, \psi)| \leqslant C \sup_{\substack{|\alpha| \leqslant N \\ x \in K \times Q}} |\partial^\alpha \psi(x)|$$

Thus,

$$|(v, \varphi)| \leqslant C \sup_{\substack{|\alpha| \leqslant N \\ x \in K \times Q}} |\partial^\alpha \rho(x_n)\varphi(x_1, \ldots, x_{n-1})|.$$

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Let $M > 0$ be such that $|\partial^\alpha \rho| \leqslant M$ on $\mathbb{R}$ for all $\alpha \leqslant N$, and set

$$\widetilde{M} = \sup_{\substack{|\alpha| \leqslant N \\ x \in K}} |\partial^\alpha \varphi(x)|.$$

Now, for $x \in K \times Q$, we have

$$\begin{aligned} |\partial^\alpha \rho(x_n) \varphi(x_1, \dots, x_{n-1})| &= \left| \sum_{|\beta+\gamma| \leqslant N} \frac{(\beta+\gamma)!}{\beta!\gamma!} \partial^\beta \rho(x_n) \partial^\gamma \varphi(x_1, \dots, x_{n-1}) \right| \\ &\leqslant \sum_{|\beta+\gamma| \leqslant N} \frac{(\beta+\gamma)!}{\beta!\gamma!} \left| \partial^\beta \rho(x_n) \right| |\partial^\gamma \varphi(x_1, \dots, x_{n-1})| \\ &\leqslant M\widetilde{M} \underbrace{\sum_{|\beta+\gamma| \leqslant N} \frac{(\beta+\gamma)!}{\beta!\gamma!}}_{\widetilde{C}} = M\widetilde{M}\widetilde{C}. \end{aligned}$$

Hence,

$$|(v, \varphi)| \leqslant C\widetilde{C}M \sup_{\substack{|\alpha| \leqslant N \\ x \in K}} |\partial^\alpha \varphi(x)|,$$

whence v is a distribution. Finally, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$(v \otimes \delta, \varphi) = \big(v(x'), (\delta(x_n), \varphi)\big) = (v(x'), \varphi(x', 0)) = (u, \rho(x_n)\varphi(x_1, \dots, x_{n-1}, 0)).$$

Note that $\psi(x) = \varphi(x) - \rho(x_n)\varphi(x_1, \dots, x_{n-1}, 0)$ vanishes on the hyperplane $\{x \in \mathbb{R}^n \colon x_n = 0\}$ and hence, due to the claim above, $(u, \psi) = 0$. This gives

$$(u, \rho(x_n)\varphi(x_1, \dots, x_{n-1}, 0)) = (u, \varphi).$$

It follows that $v(x') \otimes \delta(x_n) = u$, as desired.

2. Problem 2

First, we claim that $\operatorname{Supp} u \subseteq \{0\}$. Indeed, if $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$, then

$$(u, \varphi) = \left((x_1 + ix_2)u, \frac{\varphi}{x_1 + ix_2} \right) = 0,$$

since $\varphi/(x_1 + ix_2) \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ as $x_1 + ix_2 \neq 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$. Thus, $\operatorname{Supp} u \subseteq \{0\}$. It follows that u has an expression of the form

$$u = \sum_{\alpha, \beta \geqslant 0} c_{\alpha\beta} \partial_1^\alpha \partial_2^\beta \delta,$$

where the above sum is finite. We shall now identify $\mathbb{R}^2$ with $\mathbb{C}$ and define the differential operators

$$\partial = \partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right) \quad \text{and} \quad \overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}\left(\partial_1 + i\partial_2\right).$$

Using a simple change of variables formula, we can write our expression for u as

$$u = \sum_{\alpha, \beta \geqslant 0} a_{\alpha\beta} \partial^\alpha \overline{\partial}^\beta \delta,$$

where the above sum is finite. Our initial condition on u translates to $zu = 0$. Recall that we have

$$\partial z = 1 \quad \overline{\partial} z = 0 \quad \partial \overline{z} = 0 \quad \overline{\partial}\overline{z} = 1.$$

This shows that

$$\partial^\alpha \overline{\partial}^\beta (z^m \overline{z}^n) = \begin{cases} \alpha!\beta! & \alpha = m,\ \beta = n \\ 0 & \text{otherwise.} \end{cases}$$

Let ρ be a cutoff function that is identically 1 in a neighborhood of 0. For $k \geqslant 1$ and $l \geqslant 0$, we have

$$(u, z^k \overline{z}^l \rho) = \sum_{\alpha, \beta \geqslant 0} a_{\alpha\beta} (\partial^\alpha \overline{\partial}^\beta \delta, z^k \overline{z}^l \rho) = (-1)^{k+l} k! l! a_{kl}$$

due to what we noted above. But since $k \geqslant 1$ we have

$$(u, z^k \bar{z}^l \rho) = (zu, z^{k-1}\bar{z}^l \rho) = 0,$$

whence $a_{kl} = 0$. This leaves

$$u = \sum_{\beta \geqslant 0} a_\beta \bar{\partial}^\beta \delta,$$

where the above sum is finite and a_β are constants. Conversely, if u is of the above form, then for any $\varphi \in C_c^\infty(\mathbb{C})$, we have

$$(zu, \varphi) = (u, z\varphi) = \sum_{\beta \geqslant 0} (-1)^\beta a_\beta \left(u, \bar{\partial}^\beta (z\varphi)\right).$$

If $\beta = 0$, then $(\delta, z\varphi) = 0$ since the function vanishes at 0. On the other hand, if $\beta \geqslant 1$, then using the fact that $\bar{\partial} z = 0$, we get $\bar{\partial}^\beta (z\varphi) = z\bar{\partial}^\beta \varphi$, which vanishes at 0 again. Consequently, we see that $zu = 0$.

Hence, $zu = 0$ if and only if $u = \sum_{\beta \geqslant 0} a_\beta \bar{\partial}^\beta \delta$ for some constants a_β and the sum being finite. Substituting the expression for $\bar{\partial}$ in the above equation, we have our desired expression for u:

$$u = \sum_{0 \leqslant \beta \leqslant N} a_\beta \left(\frac{\partial_1 + i\partial_2}{2}\right)^\beta \delta,$$

for some $N \geqslant 0$ and $a_\beta \in \mathbb{C}$.

3. Problem 3

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then we have

$$(f_j, \varphi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(x) \int_{[-j,j]^n} e^{ix\cdot\xi} \, d\xi \, dx.$$

Since φ is compactly supported, its support is contained in some compact cube Q. So the above integral is essentially equal to

$$(f_j, \varphi) = \frac{1}{(2\pi)^n} \int_Q \int_{[-j,j]^n} \varphi(x) e^{ix\cdot\xi} \, d\xi \, dx = \frac{1}{(2\pi)^n} \int_{[-j,j]^n} \int_Q \varphi(x) e^{ix\cdot\xi} \, dx \, d\xi = \frac{1}{(2\pi)^n} \int_{[-j,j]} \widehat{\varphi}(-\xi) \, d\xi.$$

Note that the second equality follows from Fubini's theorem which applies since we are integrating an L^1 function on a finite measure space. Making the change of variables $\xi = -\eta$, we have

$$(f_j, \varphi) = \frac{1}{(2\pi)^n} \int_{[-j,j]^n} \widehat{\varphi}(\eta) \, d\eta.$$

Using the dominated convergence theorem (since $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^n)$) on the functions $\chi_{[-j,j]^n}(x)\widehat{\varphi}(x)$, we have

$$\lim_{j\to\infty} (f_j, \varphi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\eta) \, d\eta = \varphi(0),$$

where the last equality follows from the Fourier inversion formula. This shows that $f_j \to \delta$ as $j \to \infty$, as desired.

4. Problem 4

Let $\varphi \in \mathscr{S}(\mathbb{R})$. Then there is a constant $M > 0$ such that

$$(1 + x^2)|\varphi(x)| \leqslant M \qquad \forall\, x \in \mathbb{R}.$$

As a result, for $j > 1$,

$$|(f_j, \varphi)| = \left| \int_{j-1}^j \varphi(x) \, dx \right| \leqslant \int_{j-1}^j |\varphi(x)| \, dx \leqslant M \int_{j-1}^j \frac{1}{1+x^2} \, dx = M \arctan\left(\frac{1}{j^2 - j + 1}\right),$$

obviously the quantity on the right goes to 0 as $j \to \infty$. Thus, $(f_j, \varphi) \to 0$ as $j \to \infty$, that is, $f_j \to 0$ in $\mathscr{S}'(\mathbb{R})$.

On the other hand, for $m < n$, we have

$$|f_m - f_n| = \chi_{[m-1,m]} + \chi_{[n-1,n]},$$

so that

$$\|f_m - f_n\|_p = \begin{cases} 2^{1/p} & 1 \leqslant p < \infty \\ 1 & p = \infty. \end{cases}$$

Thus, (f_j) does not converge in L^p for $1 \leqslant p \leqslant \infty$.

5. Problem 5

Let $\varphi \in \mathscr{S}(\mathbb{R})$ and $u = |x|^{-a}$ where $0 < a < n$. Then

$$(\widehat{u}, \varphi) = (u, \widehat{\varphi}) = \int_{\mathbb{R}^n} \frac{1}{|x|^a} \widehat{\varphi}(x)\, dx.$$

Recall the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt.$$

Performing the substitution $t = |x|^2 y$, we get

$$\Gamma(s) = \int_0^\infty |x|^{2s} y^{s-1} e^{-|x|^2 y}\, dy.$$

Taking $s = \frac{a}{2}$, we get

$$\frac{1}{|x|^a} = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty y^{\frac{a}{2}-1} e^{-|x|^2 y}\, dy.$$

Thus,

$$(u, \widehat{\varphi}) = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \int_{\mathbb{R}^n} \widehat{\varphi}(x) \int_0^\infty y^{\frac{a}{2}-1} e^{-|x|^2 y}\, dy\, dx = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty y^{\frac{a}{2}-1} \int_{\mathbb{R}^n} \widehat{\varphi}(x) e^{-|x|^2 y}\, dx\, dy.$$

Recall that for $\alpha > 0$, we have

$$\widehat{x \mapsto e^{-\alpha|x|^2}} = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\alpha}}.$$

Taking $\alpha = \frac{1}{4y}$, we get that

$$\widehat{x \mapsto e^{-\frac{|x|^2}{4y}}} = (4\pi y)^{\frac{n}{2}} e^{-|x|^2 y},$$

that is,

$$\widehat{x \mapsto \frac{1}{(4\pi y)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4y}}} = e^{-|x|^2 y}.$$

Now, using Parseval's theorem and the above expression, we can write

$$\int_{\mathbb{R}^n} \widehat{\varphi}(x) e^{-|x|^2 y}\, dx = (2\pi)^n \int_{\mathbb{R}^n} \frac{1}{(4\pi y)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4y}} \varphi(x)\, dx = \int_{\mathbb{R}^n} \left(\frac{\pi}{y}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4y}} \varphi(x)\, dx$$

Substituting this in our original equation, we have

$$(u, \widehat{\varphi}) = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \int_0^\infty y^{\frac{a}{2}-1} \left(\frac{\pi}{y}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{y}} \varphi(x)\, dx\, dy = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \int_{\mathbb{R}^n} \pi^{\frac{n}{2}} \varphi(x) \int_0^\infty y^{\frac{a-n}{2}-1} e^{-\frac{|x|^2}{4y}}\, dy\, dx.$$

Perform the substitution $s = \frac{|x|^2}{4y}$, so that

$$\begin{aligned} (u, \widehat{\varphi}) &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{a}{2}\right)} \int_{\mathbb{R}^n} \varphi(x) \int_0^\infty \left(\frac{|x|^2}{4s}\right)^{\frac{a-n}{2}-1} e^{-s} \frac{|x|^2}{4s^2}\, ds \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{a}{2}\right)} \int_{\mathbb{R}^n} \varphi(x) |x|^{a-n} 2^{n-a} \int_0^\infty s^{\frac{n-a}{2}-1} e^{-s}\, ds\, dx \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{a}{2}\right)} 2^{n-a} \Gamma\left(\frac{n-a}{2}\right) \int_{\mathbb{R}^n} \frac{1}{|x|^{n-a}} \varphi(x)\, dx. \end{aligned}$$

Thus,

$$\widehat{u} = \frac{\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} \pi^{\frac{n}{2}} 2^{n-a} \frac{1}{|x|^{n-a}}.$$

6. Problem 6

First, we compute the Fourier transform of $u = \mathrm{p.\,v.}\,\frac{1}{x}$. Note that $xu = 1$, which is a fact we have seen in the last assignment. If $1 \in \mathscr{S}'(\mathbb{R})$ denotes the constant function 1, then

$$(\widehat{1}, \varphi) = (1, \widehat{\varphi}) = 2\pi\varphi(0)\ \forall \varphi \in \mathscr{S}(\mathbb{R}),$$

where the last equality follows from the Fourier inversion formula. Thus $\widehat{1} = 2\pi\delta$. This gives

$$(2\pi\delta, \varphi) = (\widehat{1}, \varphi) = (\widehat{xu}, \varphi) = (xu, \widehat{\varphi}) = (u, x\widehat{\varphi}) = (u, -i\widehat{\varphi'}) = (\widehat{u}, -i\varphi') = (\widehat{u}', i\varphi).$$

Thus, it follows that $\widehat{u}' = -2\pi i\delta$. Consider the distribution $\operatorname{sgn} \in \mathscr{S}'(\mathbb{R})$, given by

$$\operatorname{sgn}(\xi) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

Note that the derivative of this distribution is given by

$$(\operatorname{sgn}', \varphi) = -(\operatorname{sgn}, \varphi') = -\left(\int_0^\infty \varphi' - \int_{-\infty}^0 \varphi'\right) = -\left(-\varphi(0) - \varphi(0)\right) = 2\varphi(0),$$

wehnce $\operatorname{sgn}' = 2\delta$. Consequently, $(\widehat{u} + i\pi \operatorname{sgn})' = 0$. As we have seen in the last assignment, this means that $\widehat{u} + i\pi \operatorname{sgn}$ is a constant, say $c \in \mathbb{C}$. Now, if $\varphi \in \mathscr{S}'(\mathbb{R})$ is an even function, then

$$(\widehat{u}, \varphi) = (u, \widehat{\varphi}) = 0,$$

since $\widehat{\varphi}$ is an even function too; recall that

$$(u, \psi) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{\psi(x) - \psi(-x)}{x}\, dx = 0.$$

Further, it is not hard to see that $(i\pi \operatorname{sgn}, \varphi) = 0$. Hence, we must have that $(c, \varphi) = 0$ for every even function in the Schwartz class, whence $c = 0$. It follows that $\widehat{u} = -i\pi \operatorname{sgn}$. Now,

$$\widehat{u * u} = \widehat{u} \cdot \widehat{u} = -\pi^2 \operatorname{sgn}^2 = -\pi^2 \cdot 1,$$

since $\operatorname{sgn}^2 = 1$ a.e. on $\mathbb{R}$. Now, taking the inverse Fourier transform, we have

$$(u * u, \varphi) = (\widehat{u * u}^\vee, \varphi) = (\widehat{u * u}, \varphi^\vee) = (-\pi^2 \cdot 1, \varphi^\vee) = -\pi^2 \int_{\mathbb{R}} \varphi^\vee = -\pi^2 \varphi(0),$$

where the last equality follows from the fact that $\widehat{\varphi^\vee} = \varphi$ and evaluation of the Fourier transform at $\xi = 0$. This shows that $u * u = -\pi^2\delta$, as desired.

7. Problem 7

Taking the Fourier transform, we have that $P(\xi)\widehat{u}(\xi) = 0$ where $\widehat{u}(\xi) \in \mathscr{S}'(\mathbb{R}^n)$. I assume now that P is a homogeneous polynomial, so that $P(\xi) \neq 0$ whenever $\xi \neq 0$. Thus, for $\xi_0 \neq 0$, take a neighborhood U of ξ_0 which does not contain 0, so that $\frac{1}{P(\xi)}$ is a smooth function on that neighborhood. Hence, for all $\varphi \in \mathscr{D}'(U)$, we have

$$(\widehat{u}, \varphi) = \left(P(\xi)\widehat{u}, \frac{\varphi(\xi)}{P(\xi)}\right) = 0.$$

Thus, $\operatorname{Supp} \widehat{u} \subseteq \{0\}$. As we have seen in class, this implies that u is a polynomial.

Note that if we do not assume that P is homogeneous, then we can only conclude that the variety of P is compact in $\mathbb{R}^n$, since the top homogeneous component of P is non-vanishing for non-zero inputs. Consequently, for all points outside this compact variety, $\widehat{u}$ is zero in a neighborhood of those points. It follows that $\widehat{u}$ is compactly supported. Usin the Fourier inversion formula, since $\widehat{u}$ is compactly supported, we can write $u = (\widehat{u}, e^{ix\cdot\xi})$, whence u is given by a smooth function.

8. Problem 8

Since A is a symmetric positive definite matrix, there is an orthogonal matrix U such that $A = U^\top DU$ where D is a diagonal matrix consisting of the eigenvalues of A, repeated according to their multiplicity. We can then compute the Fourier transform of this function as

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-(x,Ax)}e^{-ix\cdot\xi}\,dx = \int_{\mathbb{R}^n} e^{-(Ux,DUx)}e^{-i(x,\xi)}\,dx.$$

Performing the substitution $x = U^\top y$, we have

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-(y,Dy)}e^{-i(y,U\xi)}\,dy.$$

Let $\psi(x) = e^{-(x,Dx)}$. Then $\widehat{\varphi}(\xi) = \widehat{\psi}(U\xi)$. Thus, it suffices to compute $\widehat{\psi}$. Let $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_j > 0$ for $1 \leqslant j \leqslant n$. Set $y_i = \sqrt{\lambda_i}x_i$ to get

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^n} e^{-(x,Dx)}e^{-(x,\xi)}\,dx = \frac{1}{\sqrt{\lambda_1\cdots\lambda_n}}\int_{\mathbb{R}^n} e^{-\|y\|^2}e^{y\cdot\left(\frac{\xi_1}{\sqrt{\lambda_1}},\dots,\frac{\xi_n}{\sqrt{\lambda_n}}\right)}\,dy = \frac{\pi^{\frac{n}{2}}}{\sqrt{\lambda_1\cdots\lambda_n}}\exp\left(-\frac{1}{4}\sum_{j=1}^n\frac{\xi_j^2}{\lambda_j}\right),$$

where we have used the fact that the Fourier transform of the Gaussian $e^{-\|x\|^2}$ is

$$\pi^{\frac{n}{2}}\exp\left(-\frac{1}{4}\|\xi\|^2\right).$$

9. Problem 9

Suppose there is such a $\Lambda \in \mathscr{D}'(\mathbb{R})$. Let u denote the localization of Λ to $(0,\infty)$. Since $u \in \mathscr{D}'(0,\infty)$, we can wwrite

$$u' + \frac{1}{2x^2}u = 0 \implies \left(\exp\left(-\frac{1}{4x^2}\right)u\right)' = 0.$$

As we have seen in the first assignment, this means that $\exp\left(-\frac{1}{4x^2}\right)u$ is a constant; consequently, $u = c\exp\left(\frac{1}{4x^2}\right)$. We shall show that there is no distribution $\Lambda \in \mathscr{D}'(\mathbb{R})$ that localizes to $u = \exp\left(\frac{1}{4x^2}\right)$ on $(0,\infty)$.

Suppose Λ is such a distribution, then the seminorm estimate on the compact set $K = [0,1]$ furnishes a constant $C > 0$ and a non-negative integer m such that

$$|(\Lambda,\varphi)| \leqslant C\sup_{\substack{\alpha\leqslant m\\ x\in K}}|\partial^\alpha\varphi(x)| \qquad \forall\,\varphi \in C_c^\infty(K).$$

Let ρ be a non-negative compactly supported function on the real line taking values in $[0,1]$ that is identically 1 on $[-1,1]$ and has support contained inside $(-2,2)$. Set $\rho_N \in C_c^\infty(0,\infty)$ as

$$\rho_N(x) = \rho\left(4N\left(x - \frac{1}{N}\right)\right).$$

Henceforth, suppose N is a very large positive integer, say $N > m + 100$. Then ρ_N is supported inside the open interval $\left(\frac{1}{2N}, \frac{3}{2N}\right) \subseteq [0,1]$ and ρ_N is identically 1 on the interval $\left[\frac{3}{4N}, \frac{5}{4N}\right]$. Therefore,

$$(u,\rho_N) \geqslant \int_{\frac{3}{4N}}^{\frac{5}{4N}}\exp\left(\frac{1}{4x^2}\right)\,dx \geqslant \frac{1}{2N}\times\exp\left(\frac{4N^2}{25}\right).$$

On the other hand, for $\alpha \leqslant m$, we have

$$\partial^\alpha\rho_N(x) = (4N)^\alpha\partial^\alpha\rho\left(4N\left(x - \frac{1}{N}\right)\right).$$

Since ρ is compactly supported, there is an $M > 0$ such that

$$|\partial^\alpha\rho(x)| \leqslant M \qquad \forall\,x \in \mathbb{R},\ \forall\,0 \leqslant \alpha \leqslant m.$$

Thus, for all $x \in \mathbb{R}$ and $0 \leqslant \alpha \leqslant m$, we get

$$|\partial^\alpha\rho_N(x)| \leqslant (4N)^\alpha M \leqslant (4N)^m M.$$

Finally, using the seminorm estimate, we get that

$$\frac{1}{2N}\exp\left(\frac{4N^2}{25}\right) \leqslant (4N)^m CM \implies \exp\left(\frac{4N^2}{25}\right) \leqslant 2^m(2N)^{m+1}CM,$$

for all positive integers $N > m+100$. This is absurd, since the left hand side grows exponentially, while the right hand side is a polynomial of degree at most $m+1$. It follows that there is no such distribution $\Lambda \in \mathscr{D}'(\mathbb{R})$ which restricts to u on $(0,\infty)$.

Hence, there is no such distribution $\Lambda \in \mathscr{D}'(\mathbb{R})$ restricting to $c\exp\left(\frac{1}{4x^2}\right)$ on $(0,\infty)$ for some constant $c \neq 0$. As a result, the only solution to the differential equation in the problem is the identically 0 distribution.

10. Problem 10

The Fourier transform of u is a continuous function on $\mathbb{C}^n$ and the Fourier transform of v is an analytic function on $\mathbb{C}^n$ since v is compactly supported. Further, we have

$$0 = \widehat{u*v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi) \qquad \forall\, \xi \in \mathbb{C}^n.$$

If $\widehat{u}$ is not identically 0, then there is a $\xi_0 \in \mathbb{C}^n$ such that $\widehat{u}(\xi_0) \neq 0$. Consequently, there is a neighborhood U of ξ_0 in $\mathbb{C}^n$ on which $\widehat{u}$ is nonzero. But since $\widehat{u}\widehat{v} = 0$, we must have that $\widehat{v} = 0$ on U. The identity theorem for complex analytic functions then yields that $\widehat{v} = 0$ on all of $\mathbb{C}^n$, whence by Fourier inversion, $v = 0$ as a distribution.

On the other hand if $\widehat{u} = 0$, then again by Fourier inversion, $u = 0$ as a distribution and hence as an element of L^1. This completes the proof.

11. Problem 11

Note that $u = e^x\cos(e^x)$ is the derivative of $\cos(e^x)$. Thus, for any $\varphi \in \mathscr{S}(\mathbb{R})$, using integration by parts, we have

$$(u,\varphi) = \int_{\mathbb{R}} e^x\cos(e^x)\varphi(x)\,dx = \int_{\mathbb{R}} \varphi(x)\frac{d}{dx}\sin(e^x)\,dx = -\int_{\mathbb{R}} \varphi'(x)\sin(e^x)\,dx.$$

Let

$$M = \sup_{x\in\mathbb{R}}(1+x^2)|\varphi'(x)|.$$

Note that

$$M \leqslant \sup_{x\in\mathbb{R}}|\varphi'(x)| + \sup_{x\in\mathbb{R}} x^2|\varphi'(x)| \leqslant 2\sup_{\substack{|\alpha|\leqslant 2\\ |\beta|\leqslant 1}}|x^\alpha\partial^\beta\varphi(x)|.$$

Further,

$$|(u,\varphi)| \leqslant \int_{\mathbb{R}}|\varphi'(x)\sin(e^x)|\,dx \leqslant \int_R |\varphi'(x)|\,dx \leqslant M\int_{\mathbb{R}}\frac{1}{1+x^2}\,dx = \pi M \leqslant 2\pi\sup_{\substack{|\alpha|\leqslant 2\\ |\beta|\leqslant 1}}|x^\alpha\partial^\beta\varphi(x)|.$$

This shows that u is a tempered distribution.

12. Problem 12

Let $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ with $x_i \neq 0$. Then due to the mean value property, there is a constant c between 0 and x_i such that

$$\frac{f(x_1,\dots,x_i,\dots,x_n) - f(x_1,\dots,0,\dots,x_n)}{x_i} = \partial_i f(x_1,\dots,c,\dots,x_n) = 0.$$

Thus, $f(x_1,\dots,x_i,\dots,x_n) = f(x_1,\dots,0,\dots,x_n)$ for all $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. But since f is in Schwartz class, we must have

$$0 = \lim_{x_i\to\infty} f(x_1,\dots,x_n) = \lim_{x_i\to\infty} f(x_1,\dots,0,\dots,x_n).$$

This shows that f vanishes on the hyperplane $\{x \in \mathbb{R}^n : x_i = 0\}$. But because of our first observation, we see that for any $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, we have

$$f(x) = f(x_1,\dots,0,\dots,x_n) = 0,$$

that is, $f = 0$.

13. Problem 13

Note that $C_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$. We shall show that $C_c^\infty(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$, whence it would immediately follow that $\mathscr{S}(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$.

Let $\varphi \in C^\infty(\mathbb{R}^n)$. For every positive integer n, let $\rho_n \in C_c^\infty(\mathbb{R}^n)$ be identically 1 on the open ball $B(0,n)$ with support contained in the open ball $B(0,2n)$. Define $\varphi_n = \rho_n \varphi$. We claim that $\varphi_n \to \varphi$ in the topology of $C^\infty(\mathbb{R}^n)$.

Indeed, if $K \subseteq \mathbb{R}^n$ is a compact set, then there is a positive integer N such that $K \subseteq B(0,N)$. Then for all $n \geqslant N$, $\varphi - \varphi_n$ is identically 0 in a neighborhood of K. Thus, $\partial^\alpha \varphi - \partial^\alpha \varphi_n$ is identically 0 on a neighborhood of K for all $n \geqslant N$. It follows that $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on K. Thus $\varphi_n \to \varphi$ in the topology of $C^\infty(\mathbb{R}^n)$. This shows that $C_c^\infty(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$.