Category Theory

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December 14, 2024

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Part I
General Categories

§1 Categories, Functors, and Natural Transformations

Definition 1.1. A *category* $\mathscr{A}$ consists of:

- a collection $\mathrm{ob}(\mathscr{A})$ of objects,
- for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of *maps*, *arrows*, or *morphisms* from A to B,
- for each $A, B, C \in \mathrm{ob}(\mathscr{A})$, a function

$$\mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C) \qquad (g, f) \mapsto g \circ f,$$

  called *composition*, and
- for each $A \in \mathrm{ob}(\mathscr{A})$, an element $1_A \in \mathscr{A}(A, A)$, called the *identity* on A,

satisfying the following axioms:

associativity: for each $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$,

identity: for each $f \in \mathscr{A}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Abusing notation, we often write $A \in \mathscr{A}$ instead of $A \in \mathrm{ob}(\mathscr{A})$.

Definition 1.2. A map $f : A \to B$ in a category $\mathscr{A}$ is said to be an *isomorphism* if there exists a map $g : B \to A$ in $\mathscr{A}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Definition 1.3. Every category $\mathscr{A}$ has an *opposite* category $\mathscr{A}^{op}$ given by simply reversing the arrows of $\mathscr{A}$. Note that $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{A}^{op})$.

Definition 1.4. Let $\mathscr{A}, \mathscr{B}$ be categories. A (covariant) *functor* $F : \mathscr{A} \to \mathscr{B}$ consists of:

- a function $\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B})$ written as $A \mapsto F(A)$, and
- for each $A, A' \in \mathscr{A}$, a function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$ written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$, and
- $F(1_A) = 1_{F(A)}$ for every $A \in \mathscr{A}$.

A *contravariant functor* from $\mathscr{A}$ to $\mathscr{B}$ is simply a functor $F : \mathscr{A}^{op} \to \mathscr{B}$.

Definition 1.5. Let $\mathscr{A}$ be a category. A *presheaf* on $\mathscr{A}$ is a functor $\mathscr{A}^{op} \to \mathbf{Set}$.

Definition 1.6. A functor $F : \mathscr{A} \to \mathscr{B}$ is *faithful* (resp. *full*) if for each $A, A' \in \mathscr{A}$, the function

$$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A')) \qquad f \mapsto F(f)$$

is injective (resp. surjective).

Definition 1.7. Let $\mathscr{A}$ be a category. A *subcategory* $\mathscr{S}$ of $\mathscr{A}$ consists of a subclass $\mathrm{ob}(\mathscr{A})$ of $\mathrm{ob}(\mathscr{A})$ together with, for each $S, S' \in \mathrm{ob}(\mathscr{S})$, a subclass $\mathscr{S}(S, S')$ of $\mathscr{A}(S, S')$ such that $\mathscr{S}$ is closed under composition and identities. It is a *full* subcategory if $\mathscr{S}(S, S') = \mathscr{A}(S, S')$ for all $S, S' \in \mathrm{ob}(\mathscr{S})$.

Remark 1.8. For a subcategory $\mathscr{S}$ of $\mathscr{A}$, there is a natural "inclusion" functor $\mathscr{S} \to \mathscr{A}$. The subcategory is said to be full if this functor is full.

Definition 1.9. Let $\mathscr{A}, \mathscr{B}$ be categories and $F, G : \mathscr{A} \to \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \implies G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\
G(A) \arrow[r, "G(f)"'] & G(A')
\end{tikzcd}$$

commutes. The maps α_A are called the *components* of α.

If $\alpha : F \implies G$ and $\beta : G \implies H$ are natural transformations, then we can define a natural transformation $\beta \circ \alpha : F \implies H$ by setting $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for every $A \in \mathscr{A}$. Further, there is also the identity natural transformation $1_F : F \implies F$ where every component is the identity map on $F(A)$ for every $A \in \mathscr{A}$.

Definition 1.10. For categories $\mathscr{A}, \mathscr{B}$, we construct the *functor category* $[\mathscr{A}, \mathscr{B}]$ or $\mathscr{B}^{\mathscr{A}}$ whose objects are functors $\mathscr{A} \to \mathscr{B}$ and morphisms are natural transformations between functors.

An isomorhpism in $[\mathscr{A}, \mathscr{B}]$ is called a *natural isomorphism* between two functors. We say that two functors are *naturally isomorphic* if there exists a natural isomorphism between them.

Lemma 1.11. Let $\alpha : F \implies G$, where $F, G : \mathscr{A} \to \mathscr{B}$ are functors. Then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathscr{A}$.

Proof. Suppose α is a natural isomorphism. Then, there is a $\beta : G \implies F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$ in $[\mathscr{A}, \mathscr{B}]$. Thus, $\beta_A \circ \alpha_A = 1_{F(A)}$ and $\alpha_A \circ \beta_A = 1_{G(A)}$ for every $A \in \mathscr{A}$. It follows that both β_A and α_A are isomorphisms.

Conversely, suppose every α_A is an isomorhpism. Let β_A denote its inverse. This gives a collection of maps $\beta = (\beta_A)_{A \in \mathscr{A}}$. It suffices to show that $\beta : G \implies F$ is a natural transformation, which reduces to proving the desired square commutes and is trivial. ∎

Definition 1.12. Given functors $F, G : \mathscr{A} \to \mathscr{B}$, we say that

$$F(A) \cong G(A) \textit{ naturally in } A$$

if F and G are naturally isomorphic.

Definition 1.13. An *equivalence* between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, together with natural isomorphisms

$$\eta : 1_{\mathscr{A}} \implies G \circ F \quad \text{and} \quad \varepsilon : F \circ G \implies 1_{\mathscr{B}}.$$

If this happens, we say that $\mathscr{A}$ and $\mathscr{B}$ are *equivalent*, and write $\mathscr{A} \simeq \mathscr{B}$. We also say that the functors F and G are *equivalences*.

Definition 1.14. A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$.

Theorem 1.15. A functor is an equivalence if and only if it is full, faithful, and essentially surjective on objects.

Proof. Suppose $F : \mathscr{A} \to \mathscr{B}$ is an equivalence of categories. Then, there is a functor $G : \mathscr{B} \to \mathscr{A}$ and natural isomorphisms $\eta : GF \implies 1_{\mathscr{A}}$ and $\varepsilon : 1_{\mathscr{B}} \implies FG$. Hence, there are commuting squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\ G(F(A)) \arrow[r, "G(F(f))"'] & G(F(A')) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(G(B)) \arrow[r, "F(G(g))"] \arrow[d, "\varepsilon_B"'] & F(G(B')) \arrow[d, "\varepsilon_{B'}"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

for each $f \in \mathscr{A}(A, A')$ and $g \in \mathscr{B}(B, B')$. Faithfulness follows from the fact that the η_*'s and ε_*'s are isomorphisms. To see fullness, replace f by $\widetilde{f} = \eta_{A'}^{-1} \circ f \circ \eta_A$ and note that $G(F(\widetilde{f})) = f$ and similarly in the other commuting square. Essential surjectivity on objects follows immediately by looking at the second square, taking $B = B'$ and $g = 1_B$.

Conversely, suppose F is full, faithful, and essentially surjective on objects. For each $B \in \mathscr{B}$, choose an object $G(B) \in \mathscr{A}$ and an isomorphism $\varepsilon_B : F(G(B)) \to B$. For $B \xrightarrow{g} B'$ in $\mathscr{B}$, using the full, faithfulness of F, choose $G(g) \in \mathscr{A}(G(B), G(B'))$ such that $F(G(g)) = \varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B$.

To see that G is functorial, consider $B \xrightarrow{g} B' \xrightarrow{g''} B''$ in $\mathscr{B}$. Note that

$$F(G(g'g)) = \varepsilon_{B''}^{-1} g' g \varepsilon_B = \varepsilon_{B''}^{-1} g' \varepsilon_{B'} \varepsilon_{B'}^{-1} g \varepsilon_B = F(G(g'))F(G(g)) = F(G(g')G(g)),$$

whence, due to the faithfulness of F, $G(g' \circ g) = G(g') \circ G(g)$. Similarly, one can show that $G(1_B) = 1_{G(B)}$.

We now construct a natural isomorphism $\eta : 1_{\mathscr{A}} \implies GF$. For each $A \in \mathscr{A}$, $\eta_{F(A)} : F(G(F(A))) \to F(A)$, whence due to the full, faithfulness of F, there is a unique ε_A :

$A \to G(F(A))$ such that $F(\eta_A) = \varepsilon^{-1}_{F(A)}$ and a unique $\gamma_A : G(F(A)) \to A$ such that $F(\gamma_A) = \varepsilon_{F(A)}$.

Consider the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[d, "F(f)"'] & F(G(F(A))) \arrow[r, "\varepsilon_{F(A)}"] \arrow[d, "F(G(F(f)))"'] & F(A) \arrow[d, "F(f)"] \\ F(A') \arrow[r, "F(\eta_{A'})"'] & F(G(F(A'))) \arrow[r, "\varepsilon_{F(A')}"'] & F(A') \end{tikzcd}$$

where the outer and left square commute. Since ε's are isomorphisms, we see that the left square commutes too, that is,

$$F\left(G(F(f)) \circ \eta_A\right) = F(G(F(f))) \circ F(\eta_A) = F(\eta_{A'}) \circ F(f) = F(\eta_{A'} \circ f),$$

but since F is full and faithful, $\eta_{A'} \circ f = G(F(f)) \circ \eta_A$.

It remains to show that the η's are all isomorhpisms. But this is trivial, since

$$F(\gamma_A \circ \eta_A) = F(\gamma_A) \circ F(\eta_A) = \varepsilon_{F(A)} \circ F(\eta_A) = 1_{F(A)}$$

and

$$F(\eta_A \circ \gamma_A) = F(\eta_A) \circ F(\gamma_A) = F(\eta_A) \circ \varepsilon_{F(A)} = 1_{F(G(F(A)))}.$$

Again, due to the full and faithfulness of F, we have $\gamma_A \circ \eta_A = 1_A$ and $\eta_A \circ \gamma_A = 1_{G(F(A))}$, thereby completing the proof. ■

§2 Adjoints

Definition 2.1. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. We say that F is *left adjoint* to G and G is *right adjoint* to F, and write $F \dashv G$, if

$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B))$$

naturally in A and B.

We denote the above bijection by using an "overbar", that is,

$$\left(F(A) \xrightarrow{g} B\right) \mapsto \left(A \xrightarrow{\overline{g}} G(B)\right) \quad \text{and} \quad \left(A \xrightarrow{f} G(B)\right) \mapsto \left(F(A) \xrightarrow{\overline{f}} B\right),$$

where $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. We call $\overline{f}$ the *transpose* of f.

Naturality in A and B means that given $B \xrightarrow{q} B'$ in $\mathscr{B}$, we have

$$\overline{\left(F(A) \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A \xrightarrow{\overline{g}} G(B) \xrightarrow{G(q)} G(B')\right)$$

and given $A' \xrightarrow{p} A$ in $\mathscr{A}$, we have

$$\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} = \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B\right).$$

These conditions can be interpreted informally in a nicer way. If $B \xrightarrow{q} B'$ is a map in $\mathscr{B}$, then this induces a natural map

$$\mathscr{B}(F(A), B) \longrightarrow \mathscr{B}(F(A), B') \qquad g \longmapsto q \circ g$$

and

$$\mathscr{A}(A, G(B)) \longrightarrow \mathscr{A}(A, G(B')) \qquad f \mapsto G(q) \circ f.$$

We would like like

$$\begin{tikzcd} \mathscr{B}(F(A), B) \arrow[r] \arrow[d] & \mathscr{A}(A, G(B)) \arrow[d] \\ \mathscr{B}(F(A), B') \arrow[r] & \mathscr{A}(A, G(B')) \end{tikzcd}$$

to commute, where the horizontal maps are the "overbar" bijections and the vertical ones are the naturally induced maps as discussed above. Note that the commutativity of the above square is equivalent to

$$\overline{q \circ g} = G(q) \circ \overline{g},$$

for every $g \in \mathscr{B}(F(A), B)$. This is precisely the condition imposed on "overbar" above.

Similarly, if $A' \xrightarrow{p} A$ is a map in $\mathscr{A}$, then this induces a natural map

$$\mathscr{B}(F(A), B) \longrightarrow \mathscr{B}(F(A'), B) \qquad g \mapsto g \circ F(p),$$

and

$$\mathscr{A}(A, G(B)) \longrightarrow \mathscr{A}(A', G(B)) \qquad f \mapsto f \circ p.$$

We would like

$$\begin{tikzcd} \mathscr{B}(F(A), B) \arrow[d] & \mathscr{A}(A, G(B)) \arrow[l] \arrow[d] \\ \mathscr{B}(F(A'), B) & \mathscr{A}(A', G(B)) \arrow[l] \end{tikzcd}$$

to commute, where the horizontal maps are the "overbar" bijections and the vertical ones are the naturally induced maps discussed above. Note that the commutativity of the above square is equivalent to

$$\overline{f} \circ F(p) = \overline{f \circ p}.$$

Again, this is precisely the condition imposed on "overbar" above.

Definition 2.2. Let $\mathscr{A}$ be a category. An object $I \in \mathscr{A}$ is *initial* for every $A \in \mathscr{A}$, there is a unique map $I \to A$. An object $T \in \mathscr{A}$ is *terminal* or *final* if for every $A \in \mathscr{A}$, there is a unique map $A \to T$.

PROPOSITION 2.3. Initial and terminal objects are unique up to a unique isomorphism.

Proof. Suppose I and I' are initial objects. There are unique maps $I \xrightarrow{f} I'$ and $I' \xrightarrow{f'} I$. Note that $I \xrightarrow{f' \circ f} I$; but since this map is unique, it must be equal to 1_I. Similarly, $f \circ f' = 1_{I'}$, whence both f and f' are isomorphisms. The uniqueness follows trivially. An analogous proof works for terminal objects. ■

§3 REPRESENTABLES

DEFINITION 3.1. Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. Define a functor $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ as follows:

- for objects $B \in \mathscr{A}$, put $H^A(B) = \mathscr{A}(A, B)$, and
- for maps $B \xrightarrow{g} B'$ in $\mathscr{A}$, define

  $$H^A(g) = \mathscr{A}(A, g) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$$

  by $p \mapsto g \circ p$ for all $A \xrightarrow{p} B$.

Consider the map from $\mathscr{A}$ to $[\mathscr{A}, \mathbf{Set}]$ given by $A \mapsto H^A$, which we denote by $H^\bullet$. If $A' \xrightarrow{f} A$ is a morphism in $\mathscr{A}$, then there is a natural transformation $H^f : H^A \implies H^{A'}$ given by $\left(H^f(B)\right)_{B \in \mathscr{A}}$, where

$$H^f(B) : \mathscr{A}(A, B) \longrightarrow \mathscr{A}(A', B) \qquad p \longmapsto p \circ f.$$

To see that this is indeed a natural transformation, let $B \xrightarrow{g} B'$ in $\mathscr{A}$. We would like

$$\begin{tikzcd}
\mathscr{A}(A, B) \arrow[r, "H^A(g)"] \arrow[d, "H^f(B)"'] & \mathscr{A}(A, B') \arrow[d, "H^f(B')"] \\
\mathscr{A}(A', B) \arrow[r, "H^{A'}(g)"'] & \mathscr{A}(A', B')
\end{tikzcd}$$

to commute. Indeed, some $p \in \mathscr{A}(A, B)$ mapsto $g \circ p$ in $\mathscr{A}(A, B')$ and to $g \circ p \circ f$ in $\mathscr{A}(A', B')$. On the other hand, p mapsto $p \circ f$ in $\mathscr{A}(A', B)$ and then to $g \circ p \circ f$ in $\mathscr{A}(A', B')$, as desired.

DEFINITION 3.2. Let $\mathscr{A}$ be a locally small category. The functor $H^\bullet : \mathscr{A}^{op} \to [\mathscr{A}, \mathbf{Set}]$ is defined

- on objects A by $H^\bullet(A) = H^A$, and
- on morphisms $A' \xrightarrow{f} A$ in $\mathscr{A}$ by $H^\bullet(f) = H^f : H^A \implies H^{A'}$.

Similarly, we have the dual definitions.

Definition 3.3. Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. Define a functor $H_A = \mathscr{A}(-, A) : \mathscr{A}^{op} \to \mathbf{Set}$ as follows:

- for objects $B \in \mathscr{A}$, put $H_A(B) = \mathscr{A}(B, A)$, and
- for maps $B' \xrightarrow{g} B$ in $\mathscr{A}$, define

$$H_A(g) = \mathscr{A}(g, A) : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$$

  by $p \mapsto p \circ g$ for all $B \xrightarrow{p} A$.

Definition 3.4. Let $\mathscr{A}$ be a locally small category. The functor $H_\bullet : \mathscr{A} \to [\mathscr{A}^{op}, \mathbf{Set}]$ is defined

- on objects A by $H_\bullet(A) = H_A$, and
- on morphisms $A \xrightarrow{f} A'$ by $H_\bullet(f) = H_f : H_A \implies H_{A'}$ given by

$$H_f(B) : H_A(B) = \mathscr{A}(B, A) \longrightarrow \mathscr{A}(B, A') = H_{A'}(B) \qquad p \longmapsto f \circ p.$$

This is known as the *Yoneda embedding* of $\mathscr{A}$.

Theorem 3.5 (Yoneda's Lemma). Let $\mathscr{A}$ be a locally small category and $X : \mathscr{A}^{op} \to \mathbf{Set}$ be a functor. Then, there is a bijection

$$[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \xrightarrow{\sim} X(A) \qquad \alpha \mapsto \alpha_A(1_A),$$

which is natural in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \mathbf{Set}]$.

Proof. Let the map $[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \to X(A)$ defined above be denoted by $\alpha \mapsto \widehat{\alpha}$. We define a map in the opposite direction: for each $x \in X(A)$, let $\widetilde{x} : H_A \implies X$ be given by $(\widetilde{x}_B)_{B \in \mathscr{A}}$, where

$$\widetilde{x}_B : H_A(B) = \mathscr{A}(B, A) \to X(B) \qquad \widetilde{x}_B(f) = (X(f))\,(x) \in X(B).$$

We first show that this is indeed a natural transformation from H_A to X. Let $B' \xrightarrow{g} B$ be an arrow in A. We would like to conclude that

$$\begin{tikzcd} H_A(B) \arrow[r, "- \circ g"] \arrow[d, "\widetilde{x}_B"'] & H_A(B') \arrow[d, "\widetilde{x}_{B'}"] \\ X(B) \arrow[r, "X(g)"'] & X(B') \end{tikzcd}$$

commutes. Indeed, let $f \in H_A(B) = \mathscr{A}(B, A)$. It maps to $f \circ g \in H_A(B')$, which maps to $(X(f \circ g))(x) \in X(B')$. On the other hand, f maps to $(X(f))(x) \in X(B)$, which maps to $(X(g))((X(f))(x)) = X(f \circ g)(x)$, since X is contravariant on $\mathscr{A}$.

Next, we show that the previously defined maps $\alpha \mapsto \widehat{\alpha}$ and $x \mapsto \widetilde{x}$ are inverses to one another. Let $\alpha : H_A \implies X$ be a natural transformation and fix some $B \in \mathscr{A}$. Then, for any $f \in H_A(B) = \mathscr{A}(B, A)$, we have

$$\widetilde{\widehat{\alpha}}_B(f) = (Xf)(\widehat{\alpha}) = (Xf)(\alpha_A(1_A)) \in X(B).$$

It remains to show that $(Xf)(\alpha_A(1_A)) = \alpha_B(f)$, where $f \in H_A(B) = \mathscr{A}(B, A)$. Note that α is a natural transformation and hence, there is a commutative square corresponding to the map $B \xrightarrow{f} A$ as follows:

$$\begin{tikzcd}
H_A(A) \arrow[r, "- \circ f"] \arrow[d, "\alpha_A"'] & H_A(B) \arrow[d, "\alpha_B"] \\
X(A) \arrow[r, "Xf"'] & X(B).
\end{tikzcd}$$

Under the above square, 1_A first maps to $f \in \mathscr{A}(B, A)$ under the horizontal map and then maps to $\alpha_B(f)$. On the other hand, 1_A maps to $\alpha_A(1_A)$ under the vertical map and then to $(Xf)(\alpha_A(1_A))$ under the horizontal map, which gives us what we wanted due to commutativity.

On the other hand, if $x \in X(A)$, then

$$\widehat{\widetilde{x}} = \widetilde{x}_A(1_A) = (X(1_A))(x) = 1_{X(A)}(x) = x.$$

This shows that the two maps are inverses to one another.

Finally, we must show naturality of $\widetilde{\cdot}$ and $\widehat{\cdot}$. If we show naturality of even one of them, we have shown the other, since the components are all isomorphisms (invoke Lemma 1.11). We prove naturality of $\widehat{\cdot}$.

First, consider naturality in A. Let $B \xrightarrow{f} A$ be an arrow in $\mathscr{A}$, which induces a map $H_f : H_B \implies H_A$ in $[\mathscr{A}^{op}, \mathbf{Set}]$, which in turn induces a map

$$- \circ H_f : [\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \longrightarrow [\mathscr{A}^{op}, \mathbf{Set}](H_B, X).$$

We would like to show that

$$\begin{tikzcd}
{[\mathscr{A}^{op}, \mathbf{Set}](H_A, X)} \arrow[r, "- \circ H_f"] \arrow[d, "\widehat{\cdot}"'] & {[\mathscr{A}^{op}, \mathbf{Set}](H_B, X)} \arrow[d, "\widehat{\cdot}"] \\
X(A) \arrow[r, "Xf"'] & X(B)
\end{tikzcd}$$

commutes. Indeed, consider $\alpha : H_A \implies X$. First, under the horizontal map, it goes to $\alpha \circ H_f$, which under the vertical map goes to $(\alpha \circ H_f)_B(1_B) \in X(B)$. On the other hand, under the vertical map, α first goes to $\alpha_A(1_A)$, which under the horizontal map goes to $(Xf)(\alpha_A(1_A))$.

Now, note that

$$(\alpha \circ H_f)_B(1_B) = \alpha_B((H_f)_B(1_B)) = \alpha_B(f) = (Xf)(\alpha_A(1_A)),$$

the last of which is an equality that we argued earlier while showing that the maps were inverses.

Next, we must argue for naturality in X. Suppose $\theta : X \implies X'$ is a natural transformation, where $X' : \mathscr{A}^{op} \to \mathbf{Set}$ is a functor. We would like to show that the square

$$\begin{tikzcd}
{[\mathscr{A}^{op}, \mathbf{Set}](H_A, X)} \arrow[r, "\theta\circ -"] \arrow[d, "\hat{\cdot}"'] & {[\mathscr{A}^{op}, \mathbf{Set}](H_A, X')} \arrow[d, "\hat{\cdot}"] \\
X(A) \arrow[r, "\theta_A"'] & X'(A)
\end{tikzcd}$$

Let $\alpha : H_A \implies X$. This maps to $\theta \circ \alpha$ under the horizontal map and goes to $(\theta \circ \alpha)_A(1_A)$ under the vertical map. On the other hand, it first goes to $\alpha_A(1_A)$ under the vertical map which maps to $\theta_A(\alpha_A(1_A))$ under the horizontal map. These two are obviously equal, since $(\theta \circ \alpha)_A = \theta_A \circ \alpha_A$. This completes the proof. ∎

COROLLARY. For a locally small category $\mathscr{A}$, the Yoneda embedding $H_\bullet : \mathscr{A} \to [\mathscr{A}^{op}, \mathbf{Set}]$ is full and faithful.

Proof. Let $A, A' \in \mathscr{A}$. Due to (the proof of) Theorem 3.5, there is a bijection

$$\hat{\cdot} : [\mathscr{A}^{op}, \mathbf{Set}](H_A, H_{A'}) \to H_{A'}(A) \qquad \alpha \mapsto \alpha_A(1_A).$$

Under this map, H_f maps to

$$\widehat{H_f} = (H_f)_A(1_A) = f \circ 1_A = f.$$

This shows that $f \mapsto H_f$ must be a bijection, thereby completing the proof. ∎

LEMMA 3.6. Let $F : \mathscr{A} \to \mathscr{B}$ be a full, faithful functor, and $A, A' \in \mathscr{A}$. Then

(a) a map $f \in \mathscr{A}$ is an isomorphism if and only if the map $F(f)$ in $\mathscr{B}$ is an isomorphism.

(b) for any isomorphism $g : F(A) \to F(A')$ in $\mathscr{B}$, there is a unique isomorphism $f : A \to A'$ in $\mathscr{A}$ such that $F(f) = g$.

(c) the objects A and A' of $\mathscr{A}$ are isomorphic if and only if the objects $F(A)$ and $F(A')$ of $\mathscr{B}$ are isomorphic.

§4 LIMITS

DEFINITION 4.1. Let $\mathscr{A}$ be a category and $\mathbf{I}$ a small category. A functor $\mathbf{I} \to \mathscr{A}$ is called a *diagram* in $\mathscr{A}$ of *shape* $\mathbf{I}$.

DEFINITION 4.2. Let $\mathscr{A}$ be a category, $\mathbf{I}$ a small category, and $D : \mathbf{I} \to \mathscr{A}$ a diagram in $\mathscr{A}$.

(a) A *cone* on D is an object $A \in \mathscr{A}$ (the *vertex* of the cone) together with a family $\left(A \xrightarrow{f_I} D(I)\right)_{I \in \mathbf{I}}$ of maps in $\mathscr{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the triangle

$$\begin{tikzcd} & D(I) \arrow[dd, "Du"] \\ A \arrow[ur, "f_I"] \arrow[dr, "f_J"'] & \\ & D(J) \end{tikzcd}$$

commutes.

(b) A *limit* of D is a cone $\left(L \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}}$ with the property that for any cone on D, there exists a unique map $\overline{f} : A \to L$ such that $p_I \circ \overline{f} = f_I$ for all $I \in \mathbf{I}$. The maps p_I are called the *projections* of the limit.

EXAMPLE 4.3. The fibered product in a category is an example of a limit. Suppose we wish to define $X \times_Z Y$, where $X, Y, Z \in \mathscr{A}$, a category. Consider the small category $\mathbf{I}$:

$$\begin{tikzcd} & \bullet_1 \arrow[d] \\ \bullet_2 \arrow[r] & \bullet_3 \end{tikzcd}$$

and the functor $D : \mathbf{I} \to \mathscr{A}$ sending $\bullet_1 \mapsto X$, $\bullet_2 \mapsto Y$, and $\bullet_3 \mapsto Z$. The limit (L, p_X, p_Y) over D is called the *fibered product*.

Given any other pair (T, f_X, f_Y) making the following diagram commute:

$$\begin{tikzcd} T \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\ & L \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

there is a unique map $T \to L$ making all triangles in the above diagram commute. The square in the above diagram is often called a *pullback square*.

Similarly, one can construct the product of X and Y. In this case, take $\mathbf{I}$ to be

$$\bullet_1 \qquad \bullet_2$$

and D to be the functor sending $\bullet_1 \mapsto X$ and $\bullet_2 \mapsto Y$. The limit over D is called the *product* of X and Y and is denoted by $X \times Y$.

Note that a product can be defined for an arbitrary collection of objects. Just take a suitable category $\mathbf{I}$ with no arrows (other than the identities) and the desired diagram functor D. A limit over D is the required product.

Example 4.4. We now define equalizers. Consider the category $\mathbf{I}$:

$$\begin{tikzcd} \bullet_1 \arrow[r, bend left] \arrow[r, bend right] & \bullet_2. \end{tikzcd}$$

Let $s, t : X \to Y$ be maps in $\mathscr{A}$ and let $D : \mathbf{I} \to \mathscr{A}$ be the functor sending $\bullet_1 \mapsto X$, $\bullet_2 \mapsto Y$ and the two arrows to s and t. An *equalizer* of s and t is a limit over D.

Definition 4.5. (a) Let $\mathbf{I}$ be a small category. A category $\mathscr{A}$ *has limits of shape* $\mathbf{I}$ if for every diagram $D : \mathbf{I} \to \mathscr{A}$, a limit of D exists.

(b) A category *has all limits* if it has limits of shape $\mathbf{I}$ for all small categories $\mathbf{I}$.

(c) A category is said to be *finite* if it contains only finitely many morphisms.

(d) A *finite limit* is a limit of shape $\mathbf{I}$ for some finite category $\mathbf{I}$.

Proposition 4.6. Let $\mathscr{A}$ be a category.

(a) If $\mathscr{A}$ has all products and equalizers, then $\mathscr{A}$ has all limits.

(b) If $\mathscr{A}$ has binary products, a terminal object and equalizers, then $\mathscr{A}$ has all finite limits.

Proof. We prove (a), since the proof of (b) is analogous. Let $D : \mathbf{I} \to \mathscr{A}$ be a diagram in $\mathscr{A}$. We shall work with the two products

$$\prod_{I \in \mathbf{I}} D(I) \qquad \text{and} \qquad \prod_{\substack{J \xrightarrow{u} K \\ \text{in } \mathbf{I}}} D(K).$$

For each $u : J \to K$ in $\mathbf{I}$, there is a composition

$$\prod_{I \in \mathbf{I}} D(I) \xrightarrow{\mathrm{pr}_J} D(J) \xrightarrow{Du} D(K).$$

The universal property of products furnishes a unique

$$s : \prod_{I \in \mathbf{I}} D(I) \longrightarrow \prod_{\substack{J \xrightarrow{u} K \\ \text{in } \mathbf{I}}} D(K).$$

Similarly, for each $u : J \to K$ in $\mathbf{I}$, there is a composition

$$\prod_{I \in \mathbf{I}} D(I) \xrightarrow{\mathrm{pr}_K} D(K).$$

The universal property of products furnishes a unique

$$t : \prod_{I \in \mathbf{I}} D(I) \longrightarrow \prod_{\substack{J \xrightarrow{u} K \\ \text{in } \mathbf{I}}} D(K).$$

Let $p : L \longrightarrow \prod_{I \in \mathbf{I}} D(I)$ denote their equalizer and denote $p_I = \mathrm{pr}_I \circ p$. We contend that this is the desired limit.

Let $\left(A \xrightarrow{f_I} D(I)\right)_{I \in \mathbf{I}}$ be another cone on D. That is, for $J \xrightarrow{u} K$, we have a commutative diagram:

$$\begin{tikzcd}
 & D(J) \arrow[dd, "Du"] \\
A \arrow[ur, "f_J"] \arrow[dr, "f_K"'] & \\
 & D(K).
\end{tikzcd}$$

The universal property of products first furnishes a unique map

$$f : A \longrightarrow \prod_{I \in \mathbf{I}} D(I)$$

and a unique

$$g : A \longrightarrow \prod_{\substack{J \xrightarrow{u} K \\ \text{in } \mathbf{I}}} D(K),$$

whose K-th component is given by f_K.

We would like to show that

$$\begin{tikzcd}
 & & A \arrow[dll, dotted, bend right, "\exists!"'] \arrow[dl, "f"'] \arrow[dr, "g"] & \\
L \arrow[r] & \prod_{I \in \mathbf{I}} D(I) \arrow[rr, shift left, bend left, "s"] \arrow[rr, shift right, bend right, "t"'] & & \prod_{\substack{J \xrightarrow{u} K \\ \text{in } \mathbf{I}}} D(K)
\end{tikzcd}$$

commutes. We have, for each $J \xrightarrow{u} K$,

$$\mathrm{pr}_K \circ s \circ f = Du \circ \mathrm{pr}_I \circ f = Du \circ f_I = f_K = \mathrm{pr}_K \circ g.$$

Due to the universal property, $s \circ f = g$. Now, the universal property of equalizers furnishes a unique map $A \to L$ making the diagram commute. The composition

$$A \longrightarrow L \longrightarrow \prod_{I \in \mathbf{I}} D(I) \xrightarrow{\mathrm{pr}_I} D(I)$$

is precisely $pr_I \circ f = f_I$. This shows that L is indeed the desired limit. ■

Part II
Abelian Categories