Subnormality in Group Theory

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§1 Sylow Theory

§§ The Three Theorems

In this section, we shall state and prove the three Sylow theorems.

Theorem 1.1 (Sylow's First Theorem). Let G be a finite group and p be a prime dividing the order of G with $k \in \mathbb{N}$ such that $p^k \| |G|$. Then, there is a subgroup $P \leqslant G$ with $|P| = p^k$.

We denote the set of all p-Sylow subgroups by $\mathrm{Syl}_p(G)$.

Theorem 1.2 (Sylow's Second Theorem). Let G be a finite group and p be a prime dividing the order of G. Then, all subgroups in $\mathrm{Syl}_p(G)$ are conjugate.

In order to prove the above theorem, we require the following lemmas:

Lemma 1.3. Let G be a finite group, p a prime dividing $|G|$ and $P \in \mathrm{Syl}_p(G)$. If H is a p-group contained in $N_G(P)$, then H is contained in P.

Lemma 1.4. Let G be a finite group, p a prime dividing $|G|$, H a p-subgroup and $P \in \mathrm{Syl}_p(G)$. Then, there is $x \in G$ such that $xHx^{-1} \subseteq P$.

Theorem 1.5 (Sylow's Third Theorem). Let G be a finite grouop and p a prime dividing $|G|$. Let n_p be the cardinality of $\mathrm{Syl}_p(G)$. Then,

1. $n_p = |G|/|N_G(P)|$ for any $P \in \mathrm{Syl}_p(G)$
2. $n_p \mid |G|$
3. $n_p \equiv 1 \pmod{p}$

§§ Some Related Results

Henceforth, unless specified otherwise, G is a finite group and p is a prime dividing the order of G.

Lemma 1.6. Let G be a finite group and P be a p-subgroup of G. Then, there is a p-Sylow subgroup of G containing P.

Proof. Choose any $Q \in \mathrm{Syl}_p(G)$. Using Lemma 1.4, there is $x \in G$ such that $xPx^{-1} \subseteq Q$, and equivalently, $P \subseteq x^{-1}Qx$, which is also a p-Sylow subgroup. This completes the proof. ∎

Corollary 1.7. Let G be a finite group and H a subgroup. If $P \in \mathrm{Syl}_p(H)$, then there is $Q \in \mathrm{Syl}_p(G)$ such that $P = H \cap Q$.

Proof. Since P is a p-subgroup of G, due to Lemma 1.6, there is a p-Sylow subgroup Q containing it. We shall show that $P = H \cap Q$. Obviously, $P \subseteq H \cap Q$, therefore, $v_p(|H \cap Q|) \geqslant v_p(|P|) = v_p(H)$. But since $H \cap Q$ is a subgroup of H, we must have $v_p(|H|) \geqslant v_p(|H \cap Q|)$, as a result, $v_p(|H|) = v_p(|H \cap Q|)$ and $P = H \cap Q$, since $H \cap Q$ is a p-group owing the fact that it is a subgroup of Q. ∎

Theorem 1.8. Let $P \in \mathrm{Syl}_p(G)$ and H be a subgroup of G such that $N_G(P) \subseteq H$. Then, $N_G(H) = H$ and $[G : H] \equiv 1 \pmod{p}$.

Proof. Let $x \in N_G(H)$. Then, $P^x \subseteq H$ and is also an element of $\mathrm{Syl}_p(H)$. Using Theorem 1.2, there is $h \in H$ such that $P^x = P^h$, equivalently, $x^{-1}h \in N_G(P) \subseteq H$, implying that $x \in H$.

Now, we have

$$[G : H] = \frac{[G : N_G(P)]}{[H : N_G(P)]} = \frac{n_p(G)}{n_p(H)} \equiv 1 \pmod{p}$$

∎

In particular, we have the following attractive result:

Corollary 1.9. Let $P \in \mathrm{Syl}_p(G)$. Then, $N_G(N_G(P)) = N_G(P)$.

Theorem 1.10 (Frattini Argument). Let N be a normal subgroup of G and $P \in \mathrm{Syl}_p(N)$, then $G = N_G(P)N$.

Proof. Let $g \in G$. Since $N \trianglelefteq G$, $P^g \subseteq N^g \subseteq N$, $P^g \in \mathrm{Syl}_p(N)$, as a result, there is $n \in N$ such that $(P^g) = P^n$, equivalently, $P^{n^{-1}g} = P$. This immediately implies $n^{-1}g \in N_G(P)$, therefore, $g \in NN_G(P) = N_G(P)N$, completing the proof. ∎

§2 NILPOTENT GROUPS

DEFINITION 2.1 (NILPOTENT GROUPS). A group G is said to be *nilpotent* if there is a finite collection of normal subgroups $H_0, \ldots, H_n$ with

$$1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

and such that

$$H_{i+1}/H_i \subseteq Z(G/H_i)$$

for $0 \leqslant i < n$.

The Upper Central Series and the Lower Central Series are often useful in the analysis of nilpotent groups.

DEFINITION 2.2 (UPPER CENTRAL SERIES). For any group G, define the *Upper Central Series* as a sequence of groups,

$$1 = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots$$

such that

1. Each Z_i is characteristic in G
2. $Z_{i+1}/Z_i = Z(G/Z_i)$

DEFINITION 2.3 (LOWER CENTRAL SERIES). For any group G, define the *Lower Central Series* as a sequence of groups,

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots$$

such that $G_{i+1} = [G, G_i]$

§§ Analyzing The Upper And Lower Central Series

LEMMA 2.4. For all $i \geqslant 0$, let $\pi_i : G \twoheadrightarrow G/Z_i$ denote the projection. Then, $Z_{i+1} = \pi_i^{-1}(Z(G/Z_i))$.

Proof. Obvious. ■

LEMMA 2.5. For all $i \geqslant 0$, Z_i is characteristic in G

Proof. We shall show this by induction on i. The statement is obviously true for $Z_0 = \{1\}$. Suppose we have shown that the statement holds up to $i \geqslant 0$. Let $\varphi : G \to G$ be an automorphism of groups. We now have the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, two heads, "\pi_i"'] \arrow[dr, dotted, "f"] & G \arrow[d, two heads, "\pi_i"] \\
G/Z_i \arrow[r, dotted, "\exists!\ \psi"'] & G/Z_i
\end{tikzcd}$$

Since $\ker \pi_i \circ \varphi = \varphi^{-1}(\ker \pi_i) = Z_i$, due to the **universal property** of the quotient, there is a unique homomorphism $\varphi : G/Z_i \to G/Z_i$ such that the above diagram commutes. Define $f = \pi_i \circ \varphi$. Then, $Z_i = \ker f = \pi_i^{-1}(\ker \psi)$, and thus, $\ker \psi = 1$. This implies that ψ is injective. Further, since π_i is surjective, so is $f = \pi_i \circ \varphi$, implying that ψ must be surjective. As a result, ψ is an automorphism of groups.

Let $g \in Z_{i+1}$, then $\pi_i(\varphi(g)) = \psi(\pi_i(g))$. We know, due to Lemma 2.4, that $\pi(g) \in Z(G/Z_i)$ and therefore, $\psi(\pi_i(g)) \in Z(G/Z_i)$, consequently $\pi_i(\varphi(g)) \in Z(G/Z_i)$ and thus, $\varphi(g) \in Z_{i+1}$.

Since we have shown for all automorphisms $\varphi : G \to G$, that $\varphi(Z_{i+1}) \subseteq Z_{i+1}$, then $\varphi^{-1}(Z_{i+1}) \subseteq Z_{i+1}$. This immediately gives us that $\varphi(Z_{i+1}) = Z_{i+1}$ for all automorphisms $\varphi : G \to G$ and Z_{i+1} is characteristic. ∎

LEMMA 2.6. For all $i \geqslant 0$, we have $[G, Z_{i+1}] \subseteq Z_i$.

Proof. Let $g \in G$ and $x \in Z_{i+1}$. Let $\pi_i : G \to G/Z_i$ be the natural projection. Then,

$$\pi_i([g, x]) = [\pi_i(g), \pi_i(x)] = 1$$

where the last equality follows from the fact that $\pi_i(x) \in \pi_i(Z_{i+1}) = Z(G/Z_i)$. This immediately implies that $[g, x] \in Z_i$ and the desired conclusion. ∎

LEMMA 2.7. For all $i \geqslant 0$, G_i is characteristic in G.

Proof. We shall show this by induction on i. The base case with $G_0 = G$ is trivial. Let $\varphi : G \to G$ be an automorphism of groups. Then, for all $g \in G$ and $x \in G_i$, it is not hard to see that $\varphi([g, x]) = [\varphi(g), \varphi(x)] \in [G, G_i] = G_{i+1}$. Therefore, for all automorphisms $\varphi : G \to G$, $\varphi(G_{i+1}) \subseteq G_{i+1}$. This implies that $\varphi(G_{i+1}) = G_{i+1}$, and completes the induction. ∎

LEMMA 2.8. For all $i \geqslant 0$, $G_i/G_{i+1} \subseteq Z(G/G_{i+1})$.

Proof. Let $\pi_{i+1} : G \to G/G_{i+1}$ denote the natural projection. Let $x \in G_i$ and $g \in G$, then

$$1 = \pi_{i+1}([x, g]) = [\pi_{i+1}(x), \pi_{i+1}(g)]$$

since π_{i+1} is surjective, $\pi_{i+1}(x) \in Z(G/G_{i+1})$. This completes the proof. ∎

THEOREM 2.9. For a group G, the following are equivalent,

1. For some $n \geqslant 0$, $Z_n = G$
2. For some $m \geqslant 0$, $G_m = 1$
3. G is nilpotent

Proof. We shall show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1), which would imply the desired conclusion.

- $\underline{(1) \implies (2)}$: We have a finite series

$$1 = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = G$$

  We shall show, through induction on i, that $G_i \subseteq Z_{n-i}$. The base case with $i = 0$ is obviously true. Using Lemma 2.6, we have, for all $i \leqslant n-1$,

$$G_{i+1} = [G, G_i] \subseteq [G, Z_{n-i}] \subseteq [G, Z_{n-i-1}] \subseteq Z_{i+1}$$

  which completes the induction. Finally, we have $G_n \subseteq Z_0 = 1$, implying the desired conclusion.

- $\underline{(2) \implies (3)}$: Simply define $H_i = G_{n-i}$ for all $0 \leqslant i \leqslant n$. Due to Lemma 2.8, we have that $H_{i+1}/H_i \subseteq Z(G/H_i)$.

- $\underline{(3) \implies (1)}$: We shall show that for all $i \geqslant 0$, $H_i \subseteq Z_i$. The base case with $i = 0$ is trivial. Consider the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, two heads, "\pi_i"] \arrow[d, two heads, "\pi_i'"'] & G/Z_i \\ G/H_i \arrow[ur, dotted, two heads, "\exists!\ \phi"'] & \end{tikzcd}$$

  Since $H_i \subseteq Z_i$, using the universal property of the quotient, there is an epimorphism $\phi : G/H_i \to G/Z_i$ such that the above diagram commutes. Let $x \in H_{i+1}$. Then, $\pi_i'(x) \in Z(G/H_i)$, therefore, for all $g \in G$

$$1 = \phi\left(\pi_i'([g,x])\right) = \pi_i([g,x]) = [\pi_i(g), \pi_i(x)]$$

  Now, since π_i is surjective, $\pi_i(x) \in Z(G/Z_i)$, and thus, $x \in Z_{i+1}$. This implies the desired conclusion.

■

§§ Related Results for Nilpotent Groups

Lemma 2.10. Every finite p-group is nilpotent.

Proof. Let G be a finite p-group. We shall show that the upper central series is finite by showing the proper containment $Z_i \subsetneq Z_{i+1}$ whenever $Z_i \subsetneq G$ which would imply the desired conclusion. Let $\pi_i : G \to G/Z_i$ denote then natural projection. We know, due to Lemma 2.4, that $Z_{i+1} = \pi_i^{-1}(Z(G/Z_i))$ and since G/Z_i is a non-trivial p-group, it must have a non-trivial center, therefore, $Z_i \subsetneq Z_{i+1}$. This completes the proof. ■

Lemma 2.11. Let G be a nilpotent group and H, a proper subgroup of G. Then, $H \subsetneq N_G(H)$.
Note that finiteness of G is NOT required.

Proof. Since G is nilpotent, the upper central series $1 = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = G$ is strictly increasing (with respect to containment). Let k be the maximal index such that $Z_k \subseteq H$, that is to say, $Z_{k+1} \subsetneq H$. Now, using Lemma 2.6,

$$[Z_{k+1}, H] \subseteq [Z_{k+1}, G] \subseteq Z_k \subseteq H$$

as a result, $Z_{k+1} \subseteq N_G(H)$ which completes the proof. ∎

LEMMA 2.12. Let G be a finite nilpotent group. For every prime p dividing the order of G, the p-Sylow subgroup P is normal and therefore unique.

Proof. Recall from the study of Sylow subgroups that $N_G(N_G(P)) = N_G(P)$. This combined with Lemma 2.11 implies that $N_G(P) = G$, and P is normal in G which immediately implies uniqueness. ∎

LEMMA 2.13. Let $G_1, \ldots, G_n$ be nilpotent groups. Then, their direct product $G_1 \times \cdots \times G_n$ is also nilpotent.

Proof. The central series of the product is the pointwise product of the individual central series. ∎

THEOREM 2.14. A finite group is nilpotent if and only if it is a direct product of p-groups.

Proof. Suppose G is a finite nilpotent group, then due to Lemma 2.12, the Sylow subgroups of G are normal and it is well known that in this case, G is the direct product of the Sylow subgroups.

Conversely, if G is the direct product of p-groups, then using Lemma 2.13 and Lemma 2.10, we have that G is nilpotent. ∎

PROPOSITION 2.15. Let G be a finite group. If $H \subsetneq N_G(H)$ for every proper subgroup H of G, then G is nilpotent.

Proof. Let P be a Sylow subgroup of G. Since $N_G(P) = N_G(N_G(P))$, we must have that $N_G(P) = G$, consequently, P is normal in G. It follows that G is a (internal) direct product of its Sylow subgroups, i.e., a direct product of p-groups, each of which is nilpotent. Hence, G is nilpotent. ∎

THEOREM 2.16. Every subgroup and quotient of a nilpotent group is nilpotent.

Proof. Let G be a nilpotent group and H a subgroup of G. Let $H_0 \supseteq H_1 \supseteq \cdots$ be the lower central series of H. We shall show by induction on i, that $H_i \subseteq G_i$. The base case with $i = 0$ is trivial. We now have

$$H_{i+1} = [H, H_i] \subseteq [G, H_i] \subseteq [G, G_i] = G_{i+1}$$

this completes the induction. Finally, since the lower central series of G is finite, the lower central series of H must be finite too, implying that H is nilpotent.

On the other hand, let N be a normal subgroup of G and $G' = G/N$. Let $\pi : G \to G'$ denote the natural projection. We shall show by induction on i that $G'_i = \pi(G_i)$. The base case with $i = 0$ is trivial. We have

$$G'_{i+1} = [G', G'_i] = \pi([G, G_i]) = \pi(G_{i+1})$$

This completes the induction and implies that the lower central series of G' is finite. ∎

LEMMA 2.17. A group G is nilpotent if and only if $G/Z(G)$ is nilpotent.

Proof. One direction of the statement is trivial due to Theorem 2.16. Now suppose $\widetilde{G} = G/Z(G)$ is nilpotent and let $\pi : G \to G/Z(G)$ denote the natural projection. Let $\widetilde{G} = \widetilde{G}_0 \supseteq \widetilde{G}_1 \supseteq \cdots \supseteq \widetilde{G}_n = 1$ denote the lower central series of $\widetilde{G}$. We shall show by induction on i that $G_i \subseteq \pi^{-1}(\widetilde{G}_i)$. We have

$$\pi(G_{i+1}) = \pi([G, G_i]) = [\pi(G), \pi(G_i)] \subseteq [\widetilde{G}, \widetilde{G}_i] = \widetilde{G}_{i+1}$$

This completes the induction and implies the desired conclusion. ■

LEMMA 2.18. Let G be a nilpotent group and N a non-trivial normal subgroup of G. Then, $Z(G) \cap N$ is non-trivial.

Proof. Let $1 = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = G$ denote the upper central series of G. Let k be the unique index such that $Z_k \cap N = 1$ while $Z_{k+1} \cap N \neq 1$. We shall show that $G \cap Z_{k+1} \subseteq Z(G)$. Indeed, we have

$$[G, N \cap Z_{k+1}] \subseteq [G, N] \cap [G, Z_{k+1}] \subseteq N \cap Z_k = 1$$

where we used that for all normal subgroups N, $[G, N] \subseteq N$ and Lemma 2.6.

Since $[G, N \cap Z_{k+1}] = 1$, we must have that $1 \neq N \cap Z_{k+1} \subseteq Z(G)$, which completes the proof. ■

§§ The Fitting Subgroup

DEFINITION 2.19. Let G be a finite group. For every prime p, let $\mathrm{Syl}_p(G)$ denote the collection of all Sylow p-subgroups of G. Define

$$\mathbf{O}(G) = \bigcap_{H \in \mathrm{Syl}_p(G)} H.$$

Since all Sylow p-subgroups of G are conjugate, $\mathbf{O}(G)$ is a normal p-subgroup of G. For distinct primes $p \neq q$, $\mathbf{O}_p(G) \cap \mathbf{O}_q(G) = \{1\}$ and hence, $\mathbf{O}_p(G)$ commutes with $\mathbf{O}_q(G)$.

PROPOSITION 2.20. $\mathbf{O}_p(G)$ contains every normal p-subgroup of G.

Proof. Let $P \trianglelefteq G$ be a normal p-subgroup. It is well-known that there is a Sylow p-subgroup of G containing P. But since all the Sylow p-subgroups of G are conjugate, P must be contained in all of them, and hence, in $\mathbf{O}_p(G)$. ■

Consider the product map

$$\mu : \prod_{p|G} \mathbf{O}_p(G) \longrightarrow G,$$

given by $\mu\left((x_p)\right) = \prod x_p$. We contend that this map is injective. Let H be the image of μ. Since each $\mathbf{O}_p(G)$ is contained in H, their orders must divide the order of H. Further, since they are coprime, we have that the order of H is equal to the order of the product $\prod_p \mathbf{O}_p(G)$ and hence, the map must be injective.

DEFINITION 2.21. The image of μ is denoted by $\mathbf{F}(G)$ and is called the *Fitting subgroup*.

PROPOSITION 2.22. $\mathbf{F}(G)$ is a normal nilpotent subgroup of G. Further, it contains every nilpotent normal subgroup of G.

Proof. Being a product of normal subgroups, $\mathbf{F}(G)$ is normal. It is nilpotent as it is isomorphic to a direct product of p-groups, each of which is nilpotent.

Let $N \trianglelefteq G$ be a normal nilpotent subgroup of G and suppose $P \in \mathrm{Syl}_p(N)$. Then, P is normal in G. For any $g \in G$, P^g is also contained in N (owing to N being normal in G) and has the same cardinality as P, i.e. is a Sylow p-subgroup of N. Consequently, $P = P^g$ and P is normal in G, whence P is contained in $\mathbf{O}_p(G) \subseteq \mathbf{F}(G)$. This shows that all Sylow subgroups of N are contained in $\mathbf{F}(G)$. Since N is the product of its Sylow subgroups, we have shown that N is contained in $\mathbf{F}(G)$. ■

PROPOSITION 2.23. $\mathbf{F}(G)$ is characteristic in G.

Proof. Let $\varphi \in \mathrm{Aut}(G)$. Note that $\varphi(\mathbf{F}(G))$ is also nilpotent and normal in G. Consequently, it must be contained in $\mathbf{F}(G)$, whence the conclusion follows. ■

PROPOSITION 2.24. If $N \trianglelefteq G$, then $\mathbf{F}(N) \subseteq \mathbf{F}(G)$.

Proof. We know that $\mathbf{F}(N)$ is nilpotent and hence, it suffices to show that it is normal in G. For any $g \in G$, the map $x \mapsto g^{-1}xg = x^g$ is an automorphism of N. Since $\mathbf{F}(N)$ is characteristic in N, we have that $\mathbf{F}(N)^g \subseteq \mathbf{F}(N)$, whence the conclusion follows. ■

§3 SUBNORMALITY

DEFINITION 3.1. Let G be a grouop. A subgroup $S \subseteq G$ is said to be *subnormal* in G if there exist subgroups H_i of G such that

$$S = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r = G.$$

In this situation, we write $S \lhd\lhd G$. The smallest integer r for which the above holds is called the *subnormal depth* of S in G.

REMARK 3.2. Note that the definition of a subnormal subgroup behaves well with respect to "contraction". That is, if $S \lhd\lhd G$ and H is any subgroup of G, then $S \cap H \lhd\lhd H$. As a result, if $S, T \lhd\lhd G$, then $S \cap T \lhd\lhd G$.

Now, suppose $\varphi : G \to \overline{G}$ is a surjective group homomorphism and $S \lhd\lhd G$. Then, $\varphi(S) \lhd\lhd \overline{G}$, since the image of a subnormal series under φ is still subnormal.

LEMMA 3.3. Let G be a finite group. Then G is nilpotent if and only if every subgroup of G is subnormal.

Proof. Suppose G is nilpotent and H is a proper subgroup of G. Define $H_0 = H$ and $H_{i+1} = N_G(H_i)$. Then, either $H_{i+1} = G$ or $H_i \subsetneq H_{i+1}$. This gives us a subnormal series for H.

Conversely, suppose every subgroup of G is subnormal and let H be a proper subgroup. There is a sequence

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

In particular, we may assume that $H_i \subsetneq H_{i+1}$ for $0 \leqslant i \leqslant n-1$. Hence, $H \subsetneq H_1 \subseteq N_G(H)$. Due to Proposition 2.15, we see that G must be nilpotent. ■

Proposition 3.4. Let G be a finite group and $H \leqslant G$. Then $H \subseteq \mathbf{F}(G)$ if and only if H is nilpotent and subnormal in G.

Proof. Since $\mathbf{F}(G)$ is nilpotent, if H were contained in $\mathbf{F}(G)$, then it would be niloptent too. Further, due to the preceding lemma, $H \lhd\lhd G$ and $\mathbf{F}(G) \lhd G$, whence $H \lhd\lhd G$.

We prove the converse by induction on $|G|$. If $H = G$, then there is nothing to prove, since G would be nilpotent and $\mathbf{F}(G) = G$. Suppose now that $H \subsetneq G$. There is a subnormal series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

where every successive containment is proper. Set $M = H_{n-1} \lhd G$. The inductive hypothesis applies since H is nilpotent and subnormal in M, consequently, $H \subseteq \mathbf{F}(M) \subseteq \mathbf{F}(G)$, due to Proposition 2.24, thereby completing the proof. ■

Definition 3.5. A *minimal normal subgroup* of a group G is a non-identity normal subgroup of G that does not admit any non-trivial normal subgroups. The *socle* of a *finite* group G is defined to be the subgroup generated by all minimal normal subgroups of G, which is precisely their product.

If M and N are two minimal normal subgroups of G, then $M \cap N = \{1\}$ and hence, every element of M commutes with every element of N. Thus, $\mathrm{Soc}(G)$ is precisely the product of all minimal normal subgroups of G and is a normal subgroup of G. Further, if G is a finite group that is not trivial, then it admits a non-trivial minimal finite group, and hence, $\mathrm{Soc}(G)$ is non-trivial.

Proposition 3.6. Let G be a finite group. Then $\mathrm{Soc}(G)$ is characteristic in G.

Proof. Let $\varphi \in \mathrm{Aut}(G)$. For a minimal normal subgroup M of G, $\varphi(M)$ is also a minimal normal subgroup of G. Consequently, φ permutes the minimal normal subgroups of G and thus stabilizes the socle. ■

Theorem 3.7. Let G be a finite group, $S \lhd\lhd G$, and M a minimal normal subgroup of G. Then $M \subseteq N_G(S)$.

Proof. Induction on $|G|$. If $S = G$, then there is nothing to prove, so we can suppose that $S \subsetneq G$. Since $S \lhd\lhd G$, arguing as in the preceding proof, we can choose a normal subgroup $N \subsetneq G$ such that $S \lhd\lhd N \lhd G$.

If $M \cap N = 1$, then every element of M commutes with every element of N, and hence, $M \subseteq C_G(N) \subseteq C_G(S) \subseteq N_G(S)$. Suppose now that $M \cap N$ is non-trivial. But since M is a minimal normal subgroup, $M = M \cap N$, i.e. $M \subseteq N$.

The inductive hypothesis applies to N, whence every minimal normal subgroup of N normalizes S, consequently, $\mathrm{Soc}(N)$ normalizes S. Therefore, it suffices to show that $M \subseteq \mathrm{Soc}(N)$.

Since N is a finite group and M is a non-trivial normal subgroup of N, it contains a minimal normal subgroup. That is, $M \cap \mathrm{Soc}(N) \neq 1$. Since $\mathrm{Soc}(N)$ is characteristic in N, it must be normal in G. Owing to the minimality of M in G, $M \cap \mathrm{Soc}(N) = M$, that is, $M \subseteq \mathrm{Soc}(N)$ as desired. ■

THEOREM 3.8 (WIELANDT). Let G be a finite group and $S, T \lhd\lhd G$. Then $\langle S, T \rangle \lhd\lhd G$.

Proof. Induction on $|G|$. Suppose G is non-trivial, choose a minimal normal subgroup M of G and set $\overline{G} = G/M$. By abuse of notation, we use the "overbar" to denote the homomorphism $G \to \overline{G}$. Note that

$$\langle \overline{S}, \overline{T} \rangle = \overline{\langle S, T \rangle} = \overline{\langle S, T \rangle M},$$

since M is the kernel of $G \to \overline{G}$. The inductive hypothesis applies to $\overline{G}$ and hence, $\langle \overline{S}, \overline{T} \rangle \lhd\lhd \overline{G}$. There is a natural bijection between the subgroups of G containing M and the subgroups of $\overline{G}$, which preserves normality and hence, subnormality. Therefore, $\langle S, T \rangle M \lhd\lhd G$.

Finally, note that $M \subseteq N_G(S), N_G(T)$ and hence, $M \subseteq N_G(\langle S, T \rangle)$, whence $\langle S, T \rangle \lhd \langle S, T \rangle M \lhd\lhd G$, whence the conclusion follows. ■

LEMMA 3.9. Let G be a group and $H \leqslant G$. If $HH^x = G$ for some $x \in G$, then $H = G$.

Proof. Write $x = uv$, where $u \in H$ and $v \in H^x$. Then $xv^{-1} = u$ and we have

$$H^x = (H^x)^{v^{-1}} = H^{uv^{-1}} = H^u = H.$$

Then $G = HH^x = HH = H$, as desired. ■

THEOREM 3.10 (WIELANDT ZIPPER LEMMA). Let G be a finite group and $S \leqslant G$ such that $S \lhd\lhd H$ for every proper subgroup H of G containing S. If S is not subnormal in G, then there is a unique maximal subgroup of G containing S.

Proof. We induct on $|G : S|$. Since S is not normal, $N_G(S) \subsetneq G$, and thus $N_G(S) \subseteq M$ for some maximal subgroup M of G. We must show that this M is unique. Suppose that $S \subseteq K$ is another maximal subgroup of G. We shall show that $K = M$.

By our hypothesis, $S \lhd\lhd K$. Suppose first that $S \trianglelefteq K$. Then $K \subseteq N_G(S) \subseteq M$ and hence due to maximality, $K = M$, as desired. We can suppose, therefore, that S is not normal in K. Choose the shortest subnormal series

$$S = H_0 \lhd H_1 \lhd \cdots \lhd H_r = K,$$

where $r \geqslant 2$, since S is not normal in K. Also, S is not normal in H_2 since otherwise we could delete H_1 to obtain a shorter subnormal series. Let $x \in H_2$ be such that $S^x \neq S$, and write $T = \langle S, S^x \rangle \supsetneq S$. Note that $T \subseteq K$. Also, $S^x \subseteq H_1^x = H_1 \subseteq N_G(S)$, and thus, $T \subseteq N_G(S) \subseteq M$. Furthermore, we have that $S \lhd T \subsetneq G$.

Note that S^x also satisfies the hypothesis of the theorem because conjugation by x is an automorphism of G. We claim that the subgroup $T = \langle S, S^x \rangle$ also satisfies the same hypothesis. In particular, we need to show that if $T \subseteq H \subsetneq G$, then $T \lll H$ and T is not subnormal in G.

First, if $T \subseteq H \subsetneq G$, then $S \subseteq H$, and thus $S \lll H$, and similarly, $S^x \lll H$, consequently, due to Theorem 3.8, $T \lll H$. Also, $S \lhd T$ and so if $T \lll G$, then it would follow that $S \lll G$, a contradiction. Thus T is not subnormal in G.

Our inductivev hypothesis applies to T since it properly contains S, and hence T is contained in a unique maximal subgroup of G. But since $T \subseteq M$ and $T \subseteq K$, we have that $M = K$, as desired. ∎

Definition 3.11. For a subgroup H of a group G, let H^G denote the smallest normal subgroup of G containing H. This is known as the *normal closure* of H in G.

Theorem 3.12 (Baer). Let G be a finite group and $H \leqslant G$. Then $H \subseteq \mathbf{F}(G)$ if and only if $\langle H, H^x \rangle$ is nilpotent for all $x \in G$.

Proof. If $H \subseteq \mathbf{F}(G)$, then $H^x \subseteq \mathbf{F}(G)$ for every $x \in G$, since $\mathbf{F}(G) \trianglelefteqslant G$. Hence, $\langle H, H^x \rangle \subseteq \mathbf{F}(G)$. But since $\mathbf{F}(G)$ is nilpotent, so is $\langle H, H^x \rangle$.

Conversely, suppose $\langle H, H^x \rangle$ is nilpotent for every $x \in G$. We induct on $|G|$. Taking $x = 1$, we see that H is nilpotent, whence it suffices to prove that $H \lll G$.

Suppose H is not subnormal in G. For any proper subgroup K of G containing H, the induction hypothesis applies to K and hence, $H \subseteq \mathbf{F}(K)$, that is, $H \lll K$. Due to Wielandt's Zipper Lemma, there is a unique maximal subgroup M of G containing H.

If $\langle H, H^x \rangle = G$, then G is nilpotent and $\mathbf{F}(G) = G$, and $H \lll G$, a contradiction. Thus, $\langle H, H^x \rangle \subsetneq G$ for all $x \in G$. This subgroup must be contained in a maximal subgroup of G; but since it contains H, and there is a unique maximal subgroup M containing H, we conclude that $H^x \subseteq M$ for all $x \in G$. Therefore, $H^G \subseteq M \subsetneq G$.

Since H^G is normal and properly contained in G, the induction hypothesis applies and $H \lll H^G \lhd G$, that is, $H \lll G$, a contradiction. This completes the proof. ∎

Theorem 3.13 (Zenkov). Let G be a finite group and $A, B \leqslant G$ be abelian subgroups. If M is a minimal element in the set

$$\{A \cap B^g \colon g \in G\},$$

then $M \subseteq \mathbf{F}(G)$.

Proof. The set $\{A \cap B^g \colon g \in G\}$ remains unchanged upon replacing B with B^g. Therefore, we may assume that $M = A \cap B$. We prove the statement by induction on $|G|$. First, suppose that $G = \langle A, B^g \rangle$ for some $g \in G$. Since A and B^g are abelian, we have $A \cap B^g \subseteq Z(G)$, and hence,

$$A \cap B^g = (A \cap B^g)^{g^{-1}} = A^{g^{-1}} \cap B \subseteq B.$$

It follows that $A \cap B^g \subseteq A \cap B \subseteq M$, and by the minimality of M, we have $M = A \cap B^g \subseteq Z(G) \subseteq F(G)$, as desired.

Next, assume that $\langle A, B^g \rangle \subsetneq G$ for all $g \in G$. To show that M is contained in $\mathbf{F}(G)$, it suffices to show that every Sylow p-subgroup P of M is contained in $\mathbf{F}(G)$ (because every group is generated by its Sylow subgroups). Due to Theorem 3.12, it suffices to show that $\langle P, P^g \rangle$ is nilpotent for every $g \in G$.

Fix $g \in G$, and let $H = \langle A, B^g \rangle \subsetneq G$, and $C = B \cap H$. For $h \in H$, we have

$$A \cap C^h = A \cap (B \cap H)^h = A \cap B^h \cap H = A \cap B^h.$$

In particular, $M = A \cap B = A \cap B \cap H = A \cap C$ is minimal in the set $\{A \cap C^h : h \in H\}$ since its minimal in the larger set $\{A \cap B^g : g \in G\}$. By the inductive hypothesis, $P \subseteq M \subseteq \mathbf{F}(H)$, and hence, $P \subseteq \mathbf{O}_p(H)$, since $\mathbf{O}_p(H)$ is the unique Sylow p-subgroup of $\mathbf{F}(H)$. Also, $P^g \subseteq B^g \subseteq H$, and since $\mathbf{O}_p(H)$ is a normal subgroup, we have that $\mathbf{O}_p(H)P^g$ is a p-group containing $\langle P, P^g \rangle$. In particular, $\langle P, P^g \rangle$ is a p-group, whence is nilpotent, as desired. ■

COROLLARY 3.14. Let A be an abelian subgroup of a non-trivial finite group G, and suppose that $|A| \geqslant |G : A|$. Then $A \cap \mathbf{F}(G)$ is non-trivial.

Proof. If $A = G$, then there is nothing to prove. Suppose now that $A \subsetneq G$. If $g \in G$, then $|A||A^g| = |A|^2 \geqslant |A||G : A| = |G|$. Further, due to Lemma 3.9, $AA^g \subsetneq G$. Hence,

$$|G| > |AA^g| = \frac{|A||A|^g}{|A \cap A^g|} \geqslant \frac{|G|}{|A \cap A^g|},$$

and thus $A \cap A^g$ is non-trivial. Since this holds for all $g \in G$, we can apply Theorem 3.13 to deduce that there is a $g \in G$ such that $A \cap A^g \subseteq \mathbf{F}(G)$, whence $A \cap \mathbf{F}(G)$ is non-trivial. ■

THEOREM 3.15 (LUCCINI). Let A be a proper cyclic subgroup of a finite group G, and let $K = \mathrm{core}_G(A)$. Then $|A : K| < |G : A|$, and in particular, if $|A| \geqslant |G : A|$, then K is non-trivial.

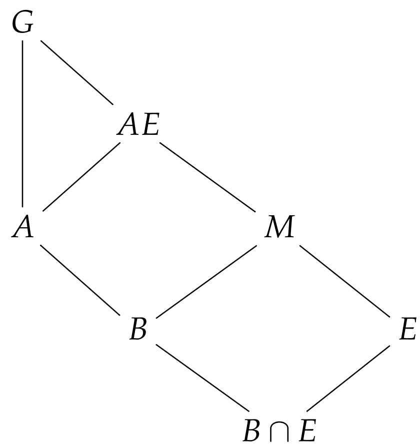
Proof. Induction on $|G|$. Note that A/K is a proper cyclic subgroup of G/K and the core of A/K in G/K is trivial. If K is non-trivial, then the inductive hypothesis applies and we deduce that

$$|A/K| = |A/K : \mathrm{core}_{G/K}(A/K)| < |G/K : A/K| = |G : A|.$$

We may now assume that $K = 1$, and we shall show that $|A| < |G : A|$. Suppose not, that is, $|A| \geqslant |G : A|$. Due to Corollary 3.14, $A \cap \mathbf{F}(G)$ is non-trivial. In particular, $\mathbf{F}(G)$ is non-trivial, so we can choose a minimal normal subgroup E of G with $E \subseteq \mathbf{F}(G)$ (since $\mathbf{F}(G)$ is normal in G). Due to Lemma 2.18, $E \cap Z(\mathbf{F}(G))$ is non-trivial; but since $Z(\mathbf{F}(G))$ is characteristic in $\mathbf{F}(G)$, it is normal in G. Due to the minimality of E, we must have $E \subseteq Z(\mathbf{F}(G))$, in particular, E is abelian. Being abelian, every Sylow subgroup of E is characteristic in G, whence due to minimality, E itself must be a p-group. We contend that E is an elementary abelian p-group. Indeed, consider $\widetilde{E} = \{x^p : x \in E\}$, which is proper and characteristic in E, and hence, is normal in G. Due to minimality of E, $\widetilde{E} = 1$, as desired.

Since $E \subseteq Z(\mathbf{F}(G))$, we see that E normalizes the non-trivial group $A \cap \mathbf{F}(G)$, and of course A normalizes this too. Then $A \cap \mathbf{F}(G) \trianglelefteqslant AE$. Since $\mathrm{core}_G(A) = 1$, we cannot have $AE = G$, else $A \cap \mathbf{F}(G)$ would be contained in the core. It follows that $AE \subsetneq G$.

Set $\overline{G} = G/E$, $\overline{A} = AE/E \subsetneq \overline{G}$, $\overline{M} = \mathrm{core}_{\overline{G}}(\overline{A})$, with $E \subseteq M$ and $M \trianglelefteqslant G$. Note that $M \subseteq AE$, and hence, $AE \subseteq AM \subseteq AE$, whence $AM = AE$. Due to the inductive hypothesis, we must have $|\overline{A} : \overline{M}| < |\overline{G} : \overline{A}|$, that is, $|AE : M| < |G : AE|$.



Let $B = A \cap M$ so that B is cyclic. We have

$$|AE : A| = |AM : A| = |M : A \cap M| = |M : B|,$$

and hence, $|AE : M| = |A : B|$. Therefore,

$$|M : B| = |AE : A| = \frac{|G : A|}{|G : AE|} < \frac{|G : A|}{|AE : M|} = \frac{|G : A|}{|A : B|} \leqslant \frac{|A|}{|A : B|} = |B|.$$

Before we proceed, note that $E \subseteq M \subseteq AE = EA$, and hence, because of what's colloquially known as Dedekind's rule, $M = E(A \cap M) = EB = BE$ (since $E \trianglelefteq G$).

Suppose M is abelian, and let $\varphi : M \to M$ be the endomorphism $\varphi(m) = m^p$. Then $E \subseteq \ker \varphi$ since it is an elementary abelian p-group. It follows that

$$\varphi(M) = \varphi(EB) = \varphi(B) \subseteq B \subseteq A.$$

Now, $M \trianglelefteq G$, and hence, $\varphi(M) \trianglelefteq G$, and we conclude that $\varphi(M) = 1$, since $\mathrm{core}_G(A) = 1$. Then $\varphi(B) = 1$, and since B is cyclic, it follows that $|B| \leqslant p$. Then $|M : B| < |B| \leqslant p$, and since $M/B \cong E/B \cap E$[1], it is a p-group, it follows that $M/B = 1$, that is, $M = B \subseteq A$. But $M \trianglelefteq G$, and since $M \subseteq A$, we have $M = 1$, whence $E = 1$, a contradiction.

It follows that M is non-abelian, and since $M/E \cong B/B \cap E$ is cyclic, we conclude that E is not central in M[2], and so $E \cap Z(M) \subsetneq E$. Again recall that $Z(M)$ is characteristic in M and hence normal in G. Due to the minimality of E, we must have $E \cap Z(M) = 1$, and thus $Z(M)$ is cyclic because the restriction of the surjection $M \twoheadrightarrow M/E$ is injective on $Z(M)$.

Since B is an abelian subgroup of M and $|M : B| < |B|$, due to Corollary 3.14, we have that $B \cap \mathbf{F}(M)$ is non-trivial. Due to Proposition 2.24, $\mathbf{F}(M) \subseteq \mathbf{F}(G)$, and so E centralizes $\mathbf{F}(M)$ because $E \subseteq Z(\mathbf{F}(G))$. Since every element of $B \cap \mathbf{F}(M)$ commutes with every element of B (since B is abelian) and every element of E, we see that $B \cap \mathbf{F}(M)$ is a non-trivial central subgroup of $EB = M$. Since $Z(M)$ is cyclic, we see that $B \cap \mathbf{F}(M) \subseteq Z(M)$ is characteristic in $Z(M) \trianglelefteq G$[3], and hence, $B \cap \mathbf{F}(M)$ is a non-trivial normal subgroup of G contained in A, a contradiction. This completes the proof. ∎

[1] These quotients make sense because M is abelian.

[2] Recall that if $G/Z(G)$ is cyclic, then G is abelian.

[3] Every subgroup of a cyclic group is characteristic.

THEOREM 3.16 (HOROSEVSKII). Let $\sigma \in \mathrm{Aut}(G)$, where G is a non-trivial finite group. Then the order $o(\sigma)$ of σ as an element of $\mathrm{Aut}(G)$ is strictly smaller than $|G|$.

Proof. Let $A = \langle \sigma \rangle \subseteq \mathrm{Aut}(G)$, so that A is a cyclic group of order equal to the order of σ as an element of $\mathrm{Aut}(G)$. Set $\Gamma = G \rtimes_\theta A$, where $\theta : A \to \mathrm{Aut}(G)$ is the obvious inclusion map. We identify G and A with subgroups $G \times \{1\}$ and $\{1\} \times A$ of Γ. Note that the conjugation action of A on G as elements of Γ is given by $g^\tau = \tau(g) \in G$ for $\tau \in A$. By definition of an automorphism, every non-identity element of A acts non-trivially on G, and hence, $A \cap C_\Gamma(G) = 1$.

Since G is non-trivial and A is cyclic, due to Theorem 3.15, $|A : K| < |\Gamma : A|$, where $K = \mathrm{core}_\Gamma(A)$. But then $K \cap G \subseteq A \cap G = 1$, and both K and G are normal in Γ, consequently, their elements commute, that is, $K \subseteq C_\Gamma(G)$. Since $K \subseteq A$, we see that $K \subseteq A \cap C_\Gamma(G) = 1$, that is, K is trivial. Thus,

$$o(\sigma) = |A| = |A : K| < |\Gamma : A| = G,$$

as desired. ∎