Derived and Triangulated Categories

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We fix some notation before proceeding. Categories will usually denoted by calligraphic symbols such as $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$. The opposite category of a category $\mathscr{A}$ is denoted by $\mathscr{A}^{\mathrm{op}}$. Corresponding to each object $A \in \mathscr{A}$, there is an object $A^{\mathrm{op}} \in \mathscr{A}^{\mathrm{op}}$ and corresponding to each morphism $f : A \to B$ in $\mathscr{A}$, there is a morphism $f^{\mathrm{op}} \in \mathscr{A}^{\mathrm{op}}$. If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathscr{A}$, then $C^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} B^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} A^{\mathrm{op}}$ with $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.

§1 Localization of Categories

Theorem 1.1. Let $\mathscr{A}$ be a category, and S be a class of morphisms in $\mathscr{A}$. Then there is a category $\mathscr{A}[S^{-1}]$ and a functor $Q : \mathscr{A} \to \mathscr{A}[S^{-1}]$ such that for every functor $F : \mathscr{A} \to \mathscr{B}$ such that $F(s)$ is an isomorphism in $\mathscr{B}$ for each $s \in S$, there is a unique functor $G : \mathscr{A}[S^{-1}] \to \mathscr{B}$ making

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "Q"'] & \mathscr{B} \\ \mathscr{A}[S^{-1}] \arrow[ur, dotted, "\exists!\, G"'] & \end{tikzcd}$$

commute. Further, the pair $(\mathscr{A}[S^{-1}], Q)$ is unique up to a unique isomorphism of categories and is called the *localization* of $\mathscr{A}$ by the class of morphisms S.

§§ Localizing Classes

Quite generally, the category $\mathscr{A}[S^{-1}]$ is quite ugly and difficult to work with. Therefore, we restrict ourselves to a more managable class S of localizing morphisms.

Definition 1.2. Let $\mathscr{A}$ be a category. A class of morphisms S in $\mathscr{A}$ is said to be a *localizing class* if

(LC1) For any object $M \in \mathscr{A}$, $\mathbf{id}_A \in S$.

(LC2) If s, t are composable morphisms in S, then so is their composition.

(LC3) (a) Every diagram of the form

$$\begin{tikzcd} & L \arrow[d, "s"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

with $f \in \mathrm{Mor}(\mathscr{A})$ and $s \in S$ can be enlarged to a commutative square

$$\begin{tikzcd} K \arrow[r, dotted, "g"] \arrow[d, dotted, "t"'] & L \arrow[d, "s"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

for some $K \in \mathscr{A}$, $g \in \mathrm{Mor}(\mathscr{A})$, and $s \in S$.

(b) Every diagram of the form

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "s"'] & M \\ L & \end{tikzcd}$$

with $f \in \mathrm{Mor}(\mathscr{A})$ and $s \in S$ can be enlarged to a commutative square

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "s"'] & M \arrow[d, dotted, "t"] \\ L \arrow[r, dotted, "g"'] & K \end{tikzcd}$$

with $K \in \mathscr{A}$, $g \in \mathrm{Mor}(\mathscr{A})$, and $t \in S$.

(LC4) Let $f, g \colon M \to N$ be two morphisms in $\mathscr{A}$. Then

$$\exists\, s \in S \text{ such that } s \circ f = s \circ g \iff \exists\, t \in S \text{ such that } f \circ t = g \circ t.$$

Clearly, if S is a localizing class in $\mathscr{A}$, then $S^{\mathrm{op}} = \{s^{\mathrm{op}} : s \in S\}$ is a localizing class in $\mathscr{A}^{\mathrm{op}}$.

Our goal will now be to describe $\mathscr{A}[S^{-1}]$ given that S is a localizing class of morphisms in $\mathscr{A}$. Define a *left roof* between two objects M and N in $\mathscr{A}$ to be a diagram of the form

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where $s \in S$ and $f \in \mathrm{Mor}(\mathscr{A})$. Two left roofs

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

are said to be *equivalent* if there exists an object $H \in \mathscr{A}$ and morphisms $p: H \to L$ and $q: H \to K$ making the diagram

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & H \arrow[u, "p"] \arrow[d, "q"] & N \\ & K \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute and $s \circ p = q \circ t \in S$. It can be checked that this is indeed an equivalence relation.

Analogously a *right roof* between two objects M and N in $\mathscr{A}$ is defined to ba diagram of the form

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N \arrow[ul, "s"'] \end{tikzcd}$$

where $s \in S$ and $f \in \mathrm{Mor}(\mathscr{A})$. Clearly there is a natural bijection between the left roofs in $\mathscr{A}$ and the right roofs in $\mathscr{A}^{\mathrm{op}}$ with respect to S and S^{op} respectively. Two right roofs

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N \arrow[ul, "s"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & K & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"'] \end{tikzcd}$$

are said to be equivalent if there exists an object $H \in \mathscr{A}$ and morphisms $p: L \to H$ and $q: K \to H$ such that

$$\begin{tikzcd} & L \arrow[d, "p"] & \\ M \arrow[ur, "f"] \arrow[dr, "g"'] & H & N \arrow[ul, "s"'] \arrow[dl, "t"] \\ & K \arrow[u, "q"'] & \end{tikzcd}$$

commutes. Again, it can be checked that this is indeed an equivalence relation. In fact, a shorter way to conclude this is to move from $\mathscr{A}$ to $\mathscr{A}^{\mathrm{op}}$ since left roofs are mapped to right roofs. It is clear that two left roofs in $\mathscr{A}$ are equivalent if and only if the corresponding right roofs in $\mathscr{A}^{\mathrm{op}}$ are equivalent.

Next, we show how to "compose" two equivalence classes of left roofs. Begin by considering two representatives

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ N & & P. \end{tikzcd}$$

According to **(LC3)a**, we obtain a commutative diagram

$$\begin{tikzcd} & & U \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & L \arrow[dl, "s"'] \arrow[dr, "f"] & & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ M & & N & & P \end{tikzcd}$$

with $u \in S$. Define the composition of the aforementioned equivalence classes to be the left roof

$$\begin{tikzcd} & U \arrow[dl, "s\circ u"'] \arrow[dr, "g\circ h"] & \\ M & & N \end{tikzcd}$$

since $s \circ u \in S$. It is a bit tedious, but it can be checked that this "composition" is well-defined. Once this is done, it is clear that the "composition" must be associative. That is, given three representatives

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ N & & P. \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "h"] & \\ P & & R \end{tikzcd}$$

using 2 **(LC3)**a repeatedly, we can complete this to a commutative diagram

$$\begin{tikzcd} & & & C \arrow[dl, "u'"'] \arrow[dr, "c"] & & & \\ & & A \arrow[dl, "s'"'] \arrow[dr, "a"] & & B \arrow[dl, "t'"'] \arrow[dr, "b"] & & \\ & L \arrow[dl, "s"'] \arrow[dr, "f"] & & K \arrow[dl, "t"'] \arrow[dr, "g"] & & U \arrow[dl, "u"'] \arrow[dr, "h"] & \\ M & & N & & P & & R \end{tikzcd}$$

and so it is clear that either composition

$$\left(\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \circ \begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ N & & P. \end{tikzcd} \right) \circ \begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "h"] & \\ P & & R \end{tikzcd}$$

or

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \circ \left(\begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ N & & P. \end{tikzcd} \circ \begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "h"] & \\ P & & R \end{tikzcd} \right)$$

is equal to the equivalence class of the left roof

$$\begin{tikzcd} & C \arrow[dl, "s\circ s'\circ u'"'] \arrow[dr, "h\circ b\circ c"] & \\ M & & R. \end{tikzcd}$$

Finally, for each $M \in \mathscr{A}$, consider the left roof

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "\mathbf{id}_M"] & \\ M & & M. \end{tikzcd}$$

For any left roof $\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$, one can compute their composition using the diagram

$$\begin{tikzcd}[column sep=small] & & L \arrow[dl, "s"'] \arrow[dr, "f"] & & \\ & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "\mathbf{id}_M"] & & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & M & & N \end{tikzcd}$$

which yields the latter left roof.

Thus, we can define a category $\mathscr{A}_S$ where $\mathrm{ob}(\mathscr{A}_S) = \mathrm{ob}(\mathscr{A})$, and $\mathrm{Mor}_{\mathscr{A}_S}(M,N)$ is the set of equivalence classes of left roofs equipped with composition and identity maps as defined above.

There is a natural functor $Q\colon \mathscr{A} \to \mathscr{A}_S$ which is the identity on objects and sends a morphism $f\colon M \to N$ in $\mathscr{A}$ to the equivalence class of the left roof

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

in $\mathscr{A}_S$. Indeed, it clearly takes $\mathbf{id}_M$ to the roof representing the identity at M in $\mathscr{A}_S$; further, if $M \xrightarrow{f} N \xrightarrow{g} P$ are two composable morphisms, then we have a commutative diagram

$$\begin{tikzcd}[column sep=small] & & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "f"] & & \\ & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "f"] & & N \arrow[dl, "\mathbf{id}_N"'] \arrow[dr, "g"] & \\ M & & N & & P \end{tikzcd}$$

so that the composition of the bottom two left roofs is

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "g\circ f"] & \\ M & & P \end{tikzcd} = Q(g \circ f).$$

Thus Q is indeed a functor. Finally, we claim that the pair $(Q, \mathscr{A}_S)$ has the universal property of localization. Indeed, let $F\colon \mathscr{A} \to \mathscr{B}$ be a functor sending every $s \in S$ to an isomorphism $F(s)$ in $\mathscr{B}$. Define a functor $G\colon \mathscr{A}_S \to \mathscr{B}$ such that

$$G(A) = F(A) \quad \text{for every object } A \in \mathscr{A},$$

and

$$G\left(\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N\end{tikzcd}\right) = F(f) \circ F(s)^{-1}.$$

It is easily checked that this is well-defined on the equivalence class of left roofs. That G is a functor is also a trivial verification, and by construction, $F = G \circ Q$.

Finally, we must show that such a G is unique. Indeed, if $F = G \circ Q$, for each object $A \in \mathscr{A}$, we must have $F(A) = G(Q(A)) = G(A)$. Now, a left roof

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N\end{tikzcd}$$

can be decomposed as the composition

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "\mathbf{id}_L"] & & L \arrow[dl, "\mathbf{id}_L"'] \arrow[dr, "f"] & \\ M & & L & & N\end{tikzcd}$$

which is easy to see by completing the diagram above by putting an L at the peak and identity morphisms from it to both the L's below it. But note that

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "\mathbf{id}_L"] & \\ M & & L\end{tikzcd} \quad \text{is the inverse of} \quad \begin{tikzcd}[column sep=small] & L \arrow[dl, "\mathbf{id}_L"'] \arrow[dr, "s"] & \\ L & & M\end{tikzcd}$$

so that

$$G\left(\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "\mathbf{id}_L"] & \\ M & & L\end{tikzcd}\right) = G\left(\begin{tikzcd}[column sep=small] & L \arrow[dl, "\mathbf{id}_L"'] \arrow[dr, "s"] & \\ L & & M\end{tikzcd}\right)^{-1} = F(s)^{-1},$$

and hence

$$G\left(\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N\end{tikzcd}\right) = F(f) \circ F(s)^{-1},$$

which completes the proof of uniqueness. We have therefore shown:

THEOREM 1.3. Let $\mathscr{A}$ be a category and S be a localizing class of morphisms in $\mathscr{A}$. Then the functor $Q : \mathscr{A} \to \mathscr{A}_S$ as described above is the localization of the category $\mathscr{A}$ at S.

THEOREM 1.4. Let $\mathscr{A}$ be a category, $\mathscr{B} \subseteq \mathscr{A}$ a full subcategory, and S a localizing class of morphisms in $\mathscr{A}$. Suppose

(**LS**1) $S_{\mathscr{B}} = S \cap \mathrm{Mor}(\mathscr{B})$ is a localizing class in $\mathscr{B}$, and

(**LS**2) for each morphism $s\colon N \to M$ in S with $M \in \mathscr{B}$, there exists a morphism $u\colon P \to N$ with $P \in \mathscr{B}$ such that $s \circ u \in S$.

Then the induced functor $\mathscr{B}[S_{\mathscr{B}}^{-1}] \to \mathscr{A}[S^{-1}]$ is fully faithful.

$$\begin{tikzcd}
\mathscr{B} \arrow[r, hook] \arrow[d, "Q_B"'] & \mathscr{A} \arrow[d, "Q_A"] \\
\mathscr{B}[S_{\mathscr{B}}^{-1}] \arrow[r] & \mathscr{A}[S^{-1}]
\end{tikzcd}$$

Proof. Since $S_{\mathscr{B}}$ is a localizing class in $\mathscr{B}$, by tracing the arrows in the commutative diagram of functors above, the map $\mathscr{B}[S_{\mathscr{B}}^{-1}] \to \mathscr{A}[S^{-1}]$ explicitly sends a roof in $\mathscr{B}$ to the equivalence class of the same roof in $\mathscr{A}[S^{-1}]$.

First, we show that the map is full. Let

$$\begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}$$

be a left roof in $\mathscr{A}[S^{-1}]$ with $M, N \in \mathscr{B}$. Then due to (**LS2**), there exists $U \in \mathscr{B}$ and a morphism $u\colon U \to L$ such that $s \circ u \in S$, and hence in $S_{\mathscr{B}}$.

To see that the map is faithful, suppose two left roofs

$$\begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\
M & & N
\end{tikzcd}$$

in $\mathscr{B}[S_{\mathscr{B}}^{-1}]$ are equivalent in $\mathscr{A}[S^{-1}]$, that is, there exists an object $H \in \mathscr{A}$, and morphisms $p\colon H \to L$ and $q\colon H \to K$ in $\mathscr{A}$ such that

$$\begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
M & H \arrow[u, "p"'] \arrow[d, "q"] & N \\
 & K \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes and $s \circ p = t \circ q \in S$. Hence, there exists an object $U \in \mathscr{B}$ and a morphism $u\colon U \to H$ such that $s \circ p \circ u = t \circ q \circ u \in S$, and hence in $S_{\mathscr{B}}$. Thus, the diagram

$$\begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
M & U \arrow[u, "p \circ u"'] \arrow[d, "q \circ u"] & N \\
 & K \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes and consists of morphisms in $\mathscr{B}$. Thus, the two roofs are equivalent in $\mathscr{B}[S_{\mathscr{B}}^{-1}]$. ■

§§ Localizing Additive Categories

We begin by showing that one can “take common demonimators” for morphisms in $\mathscr{A}[S^{-1}]$.

Lemma 1.5. Let $\mathscr{A}$ be a category (not necessarily additive) and S a localizing class of morphisms in $\mathscr{A}$. Let

$$\begin{tikzcd} & L_i \arrow[dl, "s_i"'] \arrow[dr, "f_i"] & \\ M & & N \end{tikzcd}$$

be left roofs in $\mathscr{A}$ representing morphisms $\varphi_i : M \to N$ in $\mathscr{A}[S^{-1}]$ for $1 \leqslant i \leqslant n$ respectively. Then there exists an object $L \in \mathscr{A}$ and morphisms $L \xrightarrow{s} M \in S$, and $g_i : L \to N$ for $1 \leqslant i \leqslant n$ such that

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "g_i"] & \\ M & & N \end{tikzcd}$$

represents φ_i for $1 \leqslant i \leqslant n$.

Proof. We prove this by induction on n. The base case $n = 1$ is trivial. Suppose now that $n > 1$ and that the statement has been proven for $n-1$. Hence, there exists an object K and a morphism $K \xrightarrow{t} M \in S$ such that

$$\begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "h_i"] & \\ M & & N \end{tikzcd}$$

represents φ_i for $1 \leqslant i \leqslant n-1$. Using 2 (LC3)$a$, there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "v"] \arrow[d, "u"'] & L_n \arrow[d, "s_n"] \\ K \arrow[r, "t"'] & M \end{tikzcd}$$

with $u \in S$. Set $s = s_n \circ v = t \circ u \in S$. Then the diagram

$$\begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "h_i"] & \\ M & U \arrow[u, "u"'] \arrow[d, equal] & N \\ & U \arrow[ul, "s"] \arrow[ur, "h_i \circ u"'] & \end{tikzcd}$$

commutes for $1 \leqslant i \leqslant n-1$, and

$$\begin{tikzcd} & L_n \arrow[dl, "s_n"'] \arrow[dr, "f_n"] & \\ M & U \arrow[u, "v"'] \arrow[d, equal] & N \\ & U \arrow[ul, "s_n \circ v"] \arrow[ur, "f_n \circ v"'] & \end{tikzcd}$$

commutes with $s_n \circ v = s \in S$. Set $g_i = h_i \circ u$ for $1 \leqslant i \leqslant n-1$ and $g_n = f_n \circ v$; then

$$\begin{tikzcd} & U \arrow[dl, "s"'] \arrow[dr, "g_i"] & \\ M & & N \end{tikzcd}$$

represents φ_i for $1 \leqslant i \leqslant n$, thereby completing the proof. ■

Now let $\mathscr{A}$ be an *additive category* and S a localizing class of morphisms in $\mathscr{A}$. We shall show that $\mathscr{A}[S^{-1}]$ is naturally an additive category. For objects $M,N \in \mathscr{A}[S^{-1}]$ and morphisms

$$\varphi = \begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \psi = \begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

in $\mathscr{A}[S^{-1}]$; using Lemma 1.5, we can find an object U and morphisms $U \xrightarrow{u} M \in S$ and $f',g' : U \to N$ such that

$$\varphi = \begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "f'"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \psi = \begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "g'"] & \\ M & & N. \end{tikzcd}$$

Define

$$\varphi + \psi = \begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "f'+g'"] & \\ M & & N. \end{tikzcd}$$

Note that there are three choices being made here: the choice of the representatives for φ and ψ, and choice of "common denominator" for both morphisms. It follows that $\mathrm{Mor}_{\mathscr{A}[S^{-1}]}(M,N)$ has the structure of an abelian group. Further, it must be checked that

$$\chi \circ (\varphi + \psi) = \chi \circ \varphi + \chi \circ \psi \quad \text{and} \quad (\varphi + \psi) \circ \chi = \varphi \circ \chi + \psi \circ \chi$$

for suitably composable morphisms χ, φ, ψ in $\mathscr{A}[S^{-1}]$. The zero object in $\mathrm{Mor}_{\mathscr{A}[S^{-1}]}(M,N)$ is given by the morphism

$$\begin{tikzcd} & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "0"] & \\ M & & N \end{tikzcd} .$$

Finally, given objects $M,N \in \mathscr{A}[S^{-1}]$, define their direct sum/direct product to be the object $M \oplus N$ where the direct sum is taken in $\mathscr{A}$, and the canonical projections and injections are the images of those in $\mathscr{A}$. Again, it is straightforward, but must be checked, that these have the desired universal properties. In this way, $\mathscr{A}[S^{-1}]$ has been given a natural additive structure.

Finally, note that the localization functor $Q : \mathscr{A} \to \mathscr{A}[S^{-1}]$ is an additive functor. Indeed, if $f,g : M \to N$ are morphisms, then

$$Q(f) = \begin{tikzcd} & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad Q(g) = \begin{tikzcd} & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

so that by definition,

$$Q(f)+Q(g) = \begin{tikzcd} & M \arrow[dl, "\mathbf{id}_M"'] \arrow[dr, "f+g"] & \\ M & & N \end{tikzcd} = Q(f+g).$$

Finally, we have

THEOREM 1.6. Let $\mathscr{A}$ be an additive category and S a localizing class of morphisms in $\mathscr{A}$. Then the category $\mathscr{A}[S^{-1}]$ is naturally an additive category and the localizing functor $Q: \mathscr{A} \to \mathscr{A}[S^{-1}]$ is additive.

Further, given any additive functor $F: \mathscr{A} \to \mathscr{B}$ such that $F(s)$ is an isomorphism in $\mathscr{B}$ for each $s \in S$, there exists a unique additive functor $G: \mathscr{A}[S^{-1}] \to B$ making

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "Q"'] & \mathscr{B} \\ \mathscr{A}[S^{-1}] \arrow[ur, "G"'] & \end{tikzcd}$$

commute.

Proof. We have already proved the first part of the theorem. As for the second part, suppose $\varphi, \psi: M \to N$ are two morphisms in $\mathscr{A}[S^{-1}]$. Using Lemma 1.5, we may suppose that they are represented by

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

respectively. As a result,

$$G(\varphi+\psi) = F(f+g)F(s)^{-1} = F(f)F(s)^{-1} + F(g)F(s)^{-1} = G(\varphi)+G(\psi),$$

so that G is an additive functor. That G is unique has already been argued. ∎

LEMMA 1.7. Let $\varphi: M \to N$ be a morphism in $\mathscr{A}[S^{-1}]$ represented by a left roof

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}.$$

Then the following are equivalent:

(1) $\varphi = 0$.

(2) There exists $t \in S$ such that $t \circ f = 0$.

(3) There exists $t \in S$ such that $f \circ t = 0$.

Proof. Clearly (2) and (3) are equivalent due to **(LC4)**. Now if $\varphi = 0$, then $Q(f) \circ Q(s)^{-1} = 0$, so that $Q(f) = 0$, i.e.,

$$\begin{tikzcd} & L \arrow[dl, "\mathbf{id}_L"'] \arrow[dr, "f"] & \\ L & & N \end{tikzcd}$$

represents 0. Hence, there exists an object H and and morphisms $p, q: H \to L$ such that

$$\begin{tikzcd}
& L \arrow[dl, "\mathbf{id}_L"'] \arrow[dr, "f"] & \\
L & H \arrow[u, "p"'] \arrow[d, "q"] & N \\
& L \arrow[ul, "\mathbf{id}_L"] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutes and $p = q \in S$. The commutativity implies $f \circ p = 0$, so that (1) $\Longrightarrow$ (2).

Conversely, suppose $f \circ t = 0$ for some $t: H \to L \in S$. Then the diagram

$$\begin{tikzcd}
& L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
M & H \arrow[u, "t"'] \arrow[d, "s \circ t"] & N \\
& M \arrow[ul, "\mathbf{id}_M"] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutes with $s \circ t \in S$. This shows that $\varphi = 0$, thereby completing the proof. ■

Corollary 1.8. Let M be an object in $\mathscr{A}$. Then the following are equivalent:

(1) $Q(M) = 0$.

(2) There exists an object $N \in \mathscr{A}$ such that the zero morphism $N \xrightarrow{0} M$ is in S.

(3) There exists an object $N \in \mathscr{A}$ such that the zero morphism $M \xrightarrow{0} N$ is in S.

Proof. The equivalence of (2) and (3) follows from an immediate application of 2 (**LC3**)a and 2 (**LC3**)b. Now if $Q(M) = 0$, then $Q(\mathbf{id}_M) = 0$, so that by Lemma 1.7 there exists $s \in S$ with $\mathbf{id}_M \circ s = 0$, and hence $s = 0$. This proves (2).

Conversely, if there is an object $N \in \mathscr{A}$ with $N \xrightarrow{0} M \in S$, then the image of this map, which is the zero map $Q(N) \xrightarrow{0} Q(M)$ must be an isomorphism. Thus $Q(N) = Q(M) = 0$. ■

Lemma 1.9. Let $f: M \to N$ be a morphism in $\mathscr{A}$. Then

(1) If f is monic, then so is $Q(f)$.

(2) If f is epic, then so is $Q(f)$.

Proof. ■

§§ Localization of Abelian Categories