

Hartshorne Exercises

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December 4, 2024

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Chapter I

Varieties

§I.1 AFFINE VARIETIES

DEFINITION. A topological space X is said to be *irreducible* if whenever $X = X_1 \cup X_2$ where X_1 and X_2 are closed subsets of X , $X = X_1$ or $X = X_2$.

EXERCISE I.1.1. (a) $A(Y) = k[x, y]/(y^2 - x)$. Consider the surjective map

$$\varphi : k[x, y] \rightarrow k[t]$$

sending $x \mapsto t^2$, $y \mapsto t$. Then, $\mathfrak{p} = \ker \varphi$ is a prime ideal containing $(y^2 - x)$. Further, $\text{ht } \mathfrak{p} = \dim k[x, y] - \dim k[t] = 1$. Thus, $\mathfrak{p} = (y^2 - x)$. This establishes the desired isomorphism.

(b) Using analogous reasoning, one can show that $A(Z) \cong k[t, t^{-1}]$. Suppose there is an isomorphism $k[t, t^{-1}] \cong k[x]$. Under this isomorphism, t must map to a unit and hence inside k , a contradiction.

EXERCISE I.1.2. Consider the map $\varphi : k[x, y, z] \rightarrow k[t]$ sending $x \mapsto t$, $y \mapsto t^2$, and $z \mapsto t^3$. Let $\mathfrak{p} = \ker \varphi$, which is a prime ideal with $\text{ht } \mathfrak{p} = \dim k[x, y, z] - \dim k[t] = 2$. Note that $(x^2 - y, x^3 - z) \subseteq \mathfrak{p}$. Now, suppose $f(x, y, z) \in \mathfrak{p}$, then we can view f as an element of $k[x][y, z]$ and write

$$f(x, y, z) = (y - x^2)P + (z - x^3)Q + \underbrace{f(x, x^2, x^3)}_{=0},$$

and hence, $\mathfrak{p} = (y - x^2, z - x^3)$. The conclusion follows.

EXERCISE I.1.3.

EXERCISE I.1.4. Since \mathbb{A}^1 is not Hausdorff, the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$ is not closed, while the diagonal of \mathbb{A}^2 is $Z(x - y)$, which is closed.

EXERCISE I.1.5. $B \cong k[x_1, \dots, x_n]/\mathfrak{a}$ for some radical ideal \mathfrak{a} . If we set $Y = Z(\mathfrak{a})$, then $B = A(Y)$.

EXERCISE I.1.6. • If X is irreducible and $U \subseteq X$ is non-empty open, then $X = (X \setminus U) \cup \overline{U}$ and hence, U is dense. Further, U is irreducible; for if $U = U_1 \cup U_2$ where U_i closed in U , then $U_i = U \cap X_i$ where X_i closed in X . Consequently, $U \subseteq X_1 \cup X_2$. The latter being closed, contains $\overline{U} = X$ and hence, for some i , $X = X_i$, therefore, $U = U_i$.

- If $Y \subseteq X$ (any topological space) is irreducible, then so is \overline{Y} ; for if $\overline{Y} = Y_1 \cup Y_2$, where Y_i closed in \overline{Y} , then Y_i closed in X . Further, $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$, thus, for some i , $Y = Y \cap Y_i$, hence, $Y_i \supseteq Y$ but being closed, $Y_i \supseteq \overline{Y}$.

EXERCISE I.1.7. (a) This is trivial.

- (b) Let $\{U_\alpha\}$ be an open cover of X , a noetherian topological space. If \mathfrak{M} denotes the collection of all finite unions of U_α 's, then \mathfrak{M} has a maximal element, which must be all of X .
- (c) Let $Y \subseteq X$ and suppose $V_1 \subseteq V_2 \subseteq \cdots$ is an ascending chain of open subsets of Y . There are U_i open in X such that $V_i = U_i \cap Y$. Let $\tilde{U}_i = \bigcup_{j=1}^i U_j$. Note that $\tilde{U}_i \cap Y = V_i$. Then, $\tilde{U}_1 \subseteq \tilde{U}_2 \subseteq \cdots$, and hence, stabilizes at some \tilde{U}_N . It follows that $V_N = V_{N+1} = \cdots$.
- (d) Every subspace of a noetherian topological space is noetherian, and hence, quasi-compact, and hence, closed (since the ambient space is Hausdorff). Thus, the topology is discrete. A discrete quasi-compact topology must have a finite underlying set.

EXERCISE I.1.8. There is a prime ideal \mathfrak{p} in $k[x_1, \dots, x_n]$ such that $Y = Z(\mathfrak{p})$. Similarly, there is an irreducible polynomial $f \in k[x_1, \dots, x_n]$ such that $H = Z(f)$. Note that $f \notin \mathfrak{p}$, else $Y \subseteq H$.

Let \mathfrak{q} be a minimal prime over $(f) + \mathfrak{p}$. Working in the ring R/\mathfrak{p} , $\overline{\mathfrak{q}}$ is minimal over (\overline{f}) . Due to Krull's Hauptidealsatz, $\text{ht } \overline{\mathfrak{q}} \leq 1$. The height must be non-zero since $\overline{\mathfrak{q}} \neq 0$. Thus, $\text{ht } \overline{\mathfrak{q}} = 1$, whence $\dim k[x_1, \dots, x_n]/\mathfrak{q} = \dim R/\overline{\mathfrak{q}} = \dim R - 1 = r - 1$.

EXERCISE I.1.9. This is again a trivial consequence of the Hauptidealsatz.

EXERCISE I.1.10. (a) Let $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ be a chain of closed irreducible subsets of Y . These are also irreducible as subspaces of X and hence, so are their closures. This gives us a chain

$$\overline{Y}_0 \subseteq \overline{Y}_1 \subseteq \cdots \subseteq \overline{Y}_n.$$

We contend that the inclusions are strict. Suppose $\overline{Y}_i = \overline{Y}_{i+1}$ for some $1 \leq i < n$. Thus, the closure of Y_i in Y is equal to that of Y_{i+1} in Y . This is absurd, since the Y_j 's are closed in Y . Thus, $\dim X \geq n$. Taking sup over all n , we have $\dim X \geq \dim Y$.

- (b) Due to part (a), we have $\dim X \geq \sup \dim U_i$. If $Y_0 \subsetneq \cdots \subsetneq Y_n$ is a chain of closed irreducible subsets of X , choose a $U = U_i$ having non-empty intersection with Y_0 . Then, $U \cap Y_j$ is irreducible and dense in Y_j for every j . Note that $U \cap Y_{j-1} \subseteq Y_{j-1} \subsetneq Y_j$. Since Y_{j-1} is closed in Y_j , $U \cap Y_{j-1}$ is not dense in Y_j . Thus, $U \cap Y_{j-1} \subsetneq U \cap Y_j$. Thus, $\dim U \geq n$ that is, $\sup \dim U_i \geq n$. Taking supremum over n , we obtain the desired conclusion.

- (c)
- (d) Suppose Y is properly contained in X . Then for any chain of closed irreducibles $Y_0 \subsetneq \cdots \subsetneq Y_n$ in Y , we can append X to get a chain of closed irreducibles in X , in particular, this means $\dim X \geq \dim Y + 1$, a contradiction.
- (e) Spec (Nagata's monster ring).

§I.2 PROJECTIVE VARIETIES

EXERCISE I.2.1. Let X be the affine algebraic set in \mathbb{A}^{n+1} corresponding to \mathfrak{a} . Under the canonical map $\mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$. Since f is homogeneous, f vanishes on X , thus, $f^q \in \mathfrak{a}$ for some $q > 0$ due to the affine nullstellensatz.

EXERCISE I.2.2. (i) \implies (ii) We look at the affine variety corresponding to \mathfrak{a} . There are two possible options for this: either \emptyset or the origin in \mathbb{A}^{n+1} . In the former case, due to the weak nullstellensatz, $\mathfrak{a} = S$. In the latter case, $\sqrt{\mathfrak{a}}$ is the ideal corresponding to the origin, that is, $\sqrt{\mathfrak{a}} = (x_0, \dots, x_n) = S_+$.

(ii) \implies (iii) If $\sqrt{\mathfrak{a}} = S$, then $1 \in \mathfrak{a}$, hence, $\mathfrak{a} = S$. If $\sqrt{\mathfrak{a}} = S_+$. There is a sufficiently large positive integer N such that $x_i^N \in \mathfrak{a}$ for $0 \leq i \leq n$. It is then easy to see that $S_{(n+1)N} \subseteq \mathfrak{a}$.

(iii) \implies (i) If $\mathfrak{a} \supseteq S_d$ for some $d > 0$, then it contains the monomials x_0^d, \dots, x_n^d . The projective variety corresponding to this collection of monomials is empty.

EXERCISE I.2.3. (a) Clear.

(b) Clear.

(c) Clear.

(d) This follows from Exercise [I.2.1](#).

(e) Since $Z(I(Y))$ is closed and contains Y , it must contain \overline{Y} . Suppose P is a point not contained in \overline{Y} . Then, P is not contained in some closed set $Z(\mathfrak{a})$ containing Y , where \mathfrak{a} is a homogeneous ideal. Thus, there is a homogeneous $f \in \mathfrak{a}$ such that $f(P) \neq 0$. But since $f \in I(Y)$, it follows that $P \notin Z(I(Y))$. This completes the proof.

EXERCISE I.2.4. (a) There are two maps involved here:

$$\begin{aligned} \{\text{Algebraic sets in } \mathbb{P}^n\} &\rightarrow \{\text{Homogeneous ideals in } S\} \setminus \{S_+\} \\ Y &\longmapsto I(Y) \end{aligned}$$

and

$$\begin{aligned} \{\text{Homogeneous ideals in } S\} \setminus \{S_+\} &\rightarrow \{\text{Algebraic sets in } \mathbb{P}^n\} \\ \mathfrak{a} &\longmapsto Z(\mathfrak{a}). \end{aligned}$$

Due to the preceding exercise, $Z(I(Y)) = \bar{Y} = Y$. On the other hand, if $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. On the other hand, if $Z(\mathfrak{a}) = \emptyset$, then we have shown that $\mathfrak{a} = S$, since it is not equal to S_+ . Hence, $I(\emptyset) = S = \mathfrak{a}$, thereby establishing the bijection.

- (b) Suppose $I(Y)$ is not a prime ideal. Due to an equivalent characterization of homogeneous prime ideals mentioned in the book, there are homogeneous polynomials $f, g \in S \setminus I(Y)$ such that $fg \in I(Y)$. Then, $Y \subseteq Z(f) \cup Z(g)$ $Y \not\subseteq Z(f), Z(g)$ and hence, Y is not irreducible.

On the other hand, suppose $Y = Y_1 \cup Y_2$, where $Y_1, Y_2 \subsetneq Y$ are closed in Y . Due to the bijection established in (a), $I(Y_i) \supsetneq I(Y)$. Choose $f \in I(Y_1) \setminus I(Y)$ and $g \in I(Y_2) \setminus I(Y)$. Then, $fg \in I(Y)$ and hence, $I(Y)$ is not prime.

- (c) \mathbb{P}^n corresponds to (0) , which is prime in S .

EXERCISE I.2.5. (a) Due to (a) and (b) of the preceding exercise, this follows from the fact that S is noetherian.

- (b) This is a property of arbitrary noetherian topological spaces and we shall prove it in this generality.

Let X be a noetherian topological space and let Σ be the collection of all closed subspaces of X that cannot be expressed as a finite union of irreducible closed subspaces of X . Suppose Σ is non-empty. Since X is noetherian, choose a minimal element Y of Σ . Y cannot be irreducible, else it would trivially be a finite union of closed irreducibles. Since Y is not irreducible, it can be written as a union of proper closed subsets $Y = Y_1 \cup Y_2$. Due to the minimality of Y , $Y_1, Y_2 \notin \Sigma$, and hence, each can be written as a finite union of closed irreducibles, whence so can Y , a contradiction again. Thus, $\Sigma = \emptyset$.

EXERCISE I.2.6. Let U_i be the open set $\mathbb{P}^n \setminus Z(x_i)$ and set $Y_i = Y \cap U_i \neq \emptyset$, which is closed in U_i and hence, is homeomorphic to an affine variety. We shall treat Y_i as an affine variety.

The “variables” $\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}$ form a set of coordinates on U_i as an affine space. Under this identification, $A(Y_i)$ is the set of all polynomial functions $f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$ which vanish on Y_i . This polynomial can be written in the form

$$\frac{\tilde{f}(x_0, \dots, x_i, \dots, x_n)}{x_i^N}$$

where $N = \deg f$ and \tilde{f} is homogeneous. By construction, \tilde{f} is a homogeneous polynomial vanishing on $Y \cap U_i$ which is dense in Y . But $Z(\tilde{f})$ must be closed, and thus, \tilde{f} vanishes on Y .

This gives a canonical ring homomorphism $A(Y_i) \rightarrow (S(Y)_{x_i})_0$ given by $f \mapsto \tilde{f}/x_i^N$. We contend that this homomorphism is bijective. Indeed, if f lies in the kernel of the homomorphism, then $\tilde{f}/x_i^N = 0$ as an element of $S(Y)_{x_i}$, consequently, $x_i^m \tilde{f} = 0$ as an element of $S(Y)$. In particular, \tilde{f}/x_i^N vanishes identically on $Y \cap U_i$, since x_i is nonzero

here. To see surjectivity, simply note that every element in the codomain looks like \tilde{f}/x_i^N . This establishes the desired isomorphism.

The dimension of Y_i as a topological space is the dimension of Y_i as an affine variety, which is the dimension of $(S(Y)_{x_i})_0$ as a ring.

Next, we establish the isomorphism $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. There is a map $S(Y) \rightarrow A(Y_i)[x_i, x_i^{-1}]$, which sends a polynomial function to $x_i^{\deg} \times \text{poly}(x_0/x_i, \dots, x_n/x_i)$. This is obviously a ring homomorphism. Further, note that x_i is invertible in the image and hence, this factors through $S(Y)_{x_i}$. We shall show that the induced map is an isomorphism of rings. Note that any element in the image looks like a Laurent polynomial of the form

$$\sum_{n \in \mathbb{Z}} f_n \left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right) x_i^n$$

where f_n vanishes on $Y \cap U_i$. Thus, its homogenization vanishes on Y and hence, is an element of $S(Y)$. It follows that the map defined is surjective. Injectivity is obvious.

Therefore, $\dim Y_i$ is the Krull dimension of $A(Y_i)[x_i, x_i^{-1}]$, which is the transcendence degree of $\text{Frac}(A(Y_i))(x_i)$. This is precisely $1 + \dim Y_i$. But also note that $\dim S(Y)_{x_i}$ is $\dim S(Y)$ by comparing transcendence degrees.

Therefore, we have shown $\dim S(Y) = 1 + \dim Y_i$ whenever $Y_i \neq \emptyset$. Taking supremum over all such Y_i , we obtain the desired conclusion.

EXERCISE I.2.7. (a) $\dim \mathbb{P}^n = \dim k[x_0, \dots, x_n] - 1 = n + 1 - 1 = n$.

(b) We have

$$\dim \bar{Y} = \sup \dim \bar{Y} \cap U_i = \sup \dim Y \cap U_i = \dim Y,$$

where the second equality follows from **Proposition 1.10** in Hartshorne.

EXERCISE I.2.8.

EXERCISE I.2.9. (a) Let $f \in I(Y)$; then $\beta(f)$ is its homogenization. On U_0 , $x_0 \neq 0$ and hence, $\beta(f)$ vanishes on $Y \subseteq U_0$. Again, the zero set of $\beta(f)$ is closed in \mathbb{P}^n , and hence, vanishes on \bar{Y} . Consequently, $\beta(I(Y)) \subseteq I(\bar{Y})$. On the other hand, if $F \in k[x_0, \dots, x_n]$ is a homogeneous polynomial vanishing over \bar{Y} , and hence, over Y . Thus, $f = \alpha(F)$ vanishes on Y . Consequently, $F = \beta(f)$, thereby concluding the proof.

(b) I'm not in the mood to write it up.

EXERCISE I.2.10. (a) Trivial.

(b) Since $S(Y) = A(C(Y))$.

(c) We have

$$\dim Y + 1 = \dim S(Y) = \dim A(C(Y)) = \dim C(Y).$$

EXERCISE I.2.14. $N = (r + 1)(s + 1) - 1$ and let the homogeneous coordinates of \mathbb{P}^n be $[z_{ij} : 0 \leq i \leq r, 0 \leq j \leq s]$. There is a ring homomorphism

$$\varphi : k[\{z_{ij}\}] \rightarrow k[x_0, \dots, x_r, y_0, \dots, y_s],$$

sending $z_{ij} \mapsto x_i y_j$. Let $\mathfrak{a} = \ker \varphi$. If $f \in \mathfrak{a}$, then $f(\{x_i y_j : 0 \leq i \leq r, 0 \leq j \leq s\}) = 0$. Since an element in the image ψ looks like $[a_i b_j : 0 \leq i \leq r, 0 \leq j \leq s]$, we have that $\text{im } \psi \subseteq Z(\mathfrak{a})$.

On the other hand, suppose $[c_{ij} : 0 \leq i \leq r, 0 \leq j \leq s] \in Z(\mathfrak{a})$. Without loss of generality, suppose $c_{00} = 1$. Then, set $a_i = c_{i0}$ and $b_j = c_{0j}$ and note that $\psi(\mathbf{a}, \mathbf{b}) = \mathbf{c}$, thereby completing the proof.

Chapter II

Schemes

§II.1 SHEAVES

DEFINITION. Let \mathcal{F} and \mathcal{G} be sheaves of abelian groups on X . The association $U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf on X . It is called the *sheaf Hom* and is denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

EXERCISE II.1.15.

§II.2 SCHEMES

EXERCISE II.2.3 (REDUCED SCHEMES).

- (a) Suppose X is reduced. Then, every open affine corresponds to a reduced ring. Consequently, the local ring of any point on X is the localisation of a reduced ring and hence, is reduced.

Conversely, suppose $\mathcal{O}_{X,P}$ is reduced for every $P \in X$. Let $U = \text{Spec } A$ be an affine open. The local ring of any point $P \in U$ is a localisation of A at a prime. Since all these rings are reduced, so is A .

Let $U \subseteq X$ be open. Cover U with affine opens $U_i = \text{Spec } A_i$ and let $s \in \mathcal{O}(U)$ be nilpotent. Its image $s_i = \text{res}_{U,U_i}(s)$ is nilpotent in $\mathcal{O}(U_i) = A_i$ and hence, $s_i = 0$. Consequently $s = 0$ due to the identity axiom. This shows that $\mathcal{O}(U)$ is reduced.

- (b) The first part follows immediately from the fact that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{\text{red}} & \xrightarrow{\phi_{\text{red}}} & B_{\text{red}}. \end{array}$$

Consider the map of locally ringed spaces (id, f^\sharp) , where $f^\sharp : \mathcal{O}_X \rightarrow \mathcal{O}_X^{\text{red}}$ is the collection of the canonical maps $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X^{\text{red}}(U)$.

- (c) Follows from the fact that any morphism of rings $\phi : A \rightarrow B$ with B reduced factors through the natural map $A \rightarrow A_{\text{red}}$.

EXERCISE II.2.4. Let $\varphi \in \text{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X))$. Cover X with affine opens $U_i = \text{Spec } A_i$. The restriction map gives us a homomorphism

$$A \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \xrightarrow{\text{res}_{U_i}^X} \Gamma(U_i, \mathcal{O}_X) = A_i,$$

which induces a map on schemes $\pi_i : U_i \rightarrow \text{Spec } A$ where $\pi_i = \text{Spec}(\text{res}_{U_i}^X \circ \varphi)$.

We contend that the maps π_i can be glued. Indeed, for $i \neq j$, cover $U_i \cap U_j$ with affine opens $U_{ijk} = \text{Spec } A_{ijk}$. Now,

$$\pi_i|_{U_{ijk}} = \text{Spec}(\text{res}_{U_{ijk}}^{U_i}) \circ \pi_i = \text{Spec}(\text{res}_{U_{ijk}}^{U_i} \circ \text{res}_{U_i}^X \circ \varphi) = \text{Spec}(\text{res}_{U_{ijk}}^X \circ \varphi).$$

Similarly, $\pi_j|_{U_{ijk}} = \text{Spec}(\text{res}_{U_{ijk}}^X \circ \varphi)$, consequently, the family of morphisms $\{\pi_i\}$ can be glued to a morphism $\pi : X \rightarrow \text{Spec } A$. This gives a map

$$\beta : \text{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X)) \rightarrow \text{Hom}_{\text{Sch}}(X, \text{Spec } A).$$

It is straightforward to verify that α and β are inverses to one another.

EXERCISE II.2.5. Follows from the previous exercise and the fact that \mathbb{Z} is an initial object in the category of rings.

EXERCISE II.2.7. Let $(f, f^\#) : \text{Spec } K \rightarrow X$ is a morphism of schemes which sends the unique point in $\text{Spec } K$ to $x \in X$. Then, there is an induced map on local rings $f_x^\# : \mathcal{O}_x \rightarrow K$, which must be local and hence, factor through the maximal ideal of \mathcal{O}_x , thereby inducing a map $k(x) \rightarrow K$. It is easy to see that this process is reversible.

EXERCISE II.2.9. Let $Z \subseteq X$ be irreducible and closed. Let $U = \text{Spec } A$ be an open affine intersecting Z . Then, $Z \cap U$ is open in Z and hence, is irreducible. Further, it is closed in U and hence, corresponds to a prime ideal $\xi = \mathfrak{p} \in \text{Spec } A$. Note that $\overline{\{\xi\}} \cap U = Z \cap U$ and $\overline{\{\xi\}} \subseteq Z$ since Z is closed.

Let V be any other open set intersecting Z . Then, one can replace V with an open affine $\text{Spec } B$ intersecting Z . Suppose $\xi \notin V$. Then,

$$(Z \cap U) \cap (Z \cap V) = Z \cap U \cap V = \overline{\{\xi\}} \cap U \cap V = \emptyset,$$

since the closure of $\{\xi\}$ in U is contained in $U \setminus V$. This is not possible since $Z \cap U$ and $Z \cap V$ are nonempty open sets in an irreducible space. Hence, ξ is a generic point.

Now we argue for uniqueness. Suppose ξ_1 and ξ_2 were two generic points in Z . Consider an affine neighborhood $U = \text{Spec } A$ intersecting Z . Then, $Z \cap U$ must contain ξ_1 and ξ_2 . Let ξ_i correspond to a prime \mathfrak{p}_i in A for $i = 1, 2$. Now, $Z \cap U = V(\mathfrak{p}_1) = V(\mathfrak{p}_2)$, consequently, $\mathfrak{p}_1 = \mathfrak{p}_2$, that is, $\xi_1 = \xi_2$. This completes the proof.

DEFINITION. Let (X, \mathcal{O}_X) be a scheme and let $f \in \Gamma(X, \mathcal{O}_X)$. Define X_f to be the set of all $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal \mathfrak{m}_x of the local ring $\mathcal{O}_{X,x}$. This is known as the *support* of f on X .

EXERCISE II.2.16.

- The set of all $x \in U$ such that $f_x \notin \mathfrak{m}_x$ is the set of all prime ideals \mathfrak{p} in B such that $f/1$ is not in the maximal ideal $\mathfrak{p}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$. Equivalently, $f \notin \mathfrak{p}$. Thus, $X_f \cap U = D(\bar{f})$. Now, since X can be covered with open affines and the intersection of X_f with every open affine is open, X_f must also be open.
- Pick a finite open cover $\{U_i = \text{Spec } A_i\}_{i=1}^m$. The restriction of a to $X_f \cap U_i = D(\text{res}_{U_i}^X(f))$ is zero and hence, there is a positive integer n_i such that $\text{res}_{U_i}^X(f^{n_i}a) = 0$. Let $N = \max_{1 \leq i \leq m} n_i$. Then, $\text{res}_{U_i}^X(f^N a) = 0$. Due to the identity axiom, we must have $f^N a = 0$.
- Let $U_i = \text{Spec } A_i$ and let $f_i = \text{res}_{U_i}^X(f)$. Since $X_f \cap U_i = D(f_i)$, there is a $b_i \in A_i = \Gamma(U_i, \mathcal{O}_X)$ such that $\text{res}_{U_i \cap X_f}^X(b) = \frac{b_i}{f_i^{n_i}}$ for some nonnegative integer n_i . Choosing n to be larger than all the n_i 's, we get that there is a $b_i \in A_i$ such that $\text{res}_{U_i \cap X_f}^X(f^n b) = \text{res}_{U_i \cap X_f}^{U_i}(b_i)$.

Now consider $b_i - b_j$ on $U_i \cap U_j$, which can be covered by finitely many affine opens $U_{ijk} = \text{Spec } A_{ijk}$. Since $\text{res}_{U_i \cap U_j \cap X_f}^X(b_i - b_j) = 0$, using a similar argument as in (b), there is a positive integer m_{ij} such that $f^{m_{ij}}(b_i - b_j)$ restricts to 0 on $U_i \cap U_j$. Choosing m larger than m_{ij} for all pairs i, j , we have that $f^m(b_i - b_j)$ restricts to 0 on $U_i \cap U_j$. Consequently, $\text{res}_{U_i \cap U_j}^{U_i}(f^m b_i) = \text{res}_{U_i \cap U_j}^{U_j}(f^m b_j)$ and hence, there is a $c \in \Gamma(X, \mathcal{O}_X)$ such that $\text{res}_{U_i}^X(c) = f^m b_i$. Hence, $\text{res}_{U_i \cap X_f}^X(c) = \text{res}_{U_i \cap X_f}^X(f^{n+m}b)$. This completes the proof.

- (d) First, we show that $\text{res}_{X_f}^X(f)$ is invertible. Since $f_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_x$ for every $x \in X_f$, we see that the restriction of f to every affine open contained in X_f must be invertible (else it would lie in a prime ideal and hence, in the stalk of some point). Consider an open cover U_i of X_f using affine opens. There is a $g_i \in \Gamma(U_i, \mathcal{O})$ such that $g_i \text{res}_{U_i}^X(f) = 1$. For $i \neq j$, we have

$$\text{res}_{U_i \cap U_j}^{U_i}(g_i) \text{res}_{U_i \cap U_j}^X(f) = 1 = \text{res}_{U_i \cap U_j}^{U_j}(g_j) \text{res}_{U_i \cap U_j}^X(f)$$

and hence, $\text{res}_{U_i \cap U_j}^{U_i}(g_i) = \text{res}_{U_i \cap U_j}^{U_j}(g_j)$ and hence, the g_i 's can be lifted to some $g \in \Gamma(X_f, \mathcal{O}_X)$, furthermore $\text{res}_{X_f}^X(f)g = 1$, whence invertibility follows.

Consider the map $\Phi : A_f \rightarrow \Gamma(X_f, \mathcal{O}_X)$ given by

$$\frac{a}{f^n} \mapsto \frac{\text{res}_{X_f}^X(a)}{\text{res}_{X_f}^X(f^n)}.$$

If $\Phi(a/f^n) = 0$, then $\text{res}_{X_f}^X(a) = 0$, consequently, due to part (b), there is a positive integer m such that $f^m a = 0$, equivalently, $a/f^n = 0$ in A_f . Hence, Φ is injective.

As for surjectivity, let $b \in \Gamma(X_f, \mathcal{O}_X)$. Due to part (c), there is a positive integer m such that $f^m b = \text{res}_{X_f}^X(a)$ for some $a \in A$ whence $\Phi(a/f^m) = b$. This completes the proof.

EXERCISE II.2.17 (A CRITERION FOR AFFINENESS).

- (a) Each $f : f^{-1}U_i \rightarrow U_i$ has an inverse $g_i : U_i \rightarrow f^{-1}U_i$ that agrees on intersections since inverses are unique. These maps can be glued to give an inverse $g : Y \rightarrow X$ of f .

- (b) First, note that $X = \bigcup_{i=1}^n X_{f_i}$, for if not, then there is an $x \in X$ such that $x \notin X_{f_i}$ for $1 \leq i \leq n$. Consider an affine open $U = \text{Spec } B$ containing x and let \mathfrak{p} be the prime corresponding to x . According to our hypothesis, $\text{res}_U^X(f_i) \in \mathfrak{p}$ for $1 \leq i \leq n$. But these restrictions generate the unit ideal, a contradiction.

Being a finite union of affine opens, X is quasi-compact. Further, $X_{f_i} \cap X_{f_j}$ is a distinguished open in X_{f_i} and hence, is quasi-compact. As a result, Exercise [II.2.16](#)

(d) is applicable. Using Exercise II.2.4 and glueing morphisms just as in part (a), we are done.

DEFINITION. A morphism $f : X \rightarrow Y$ of schemes is said to be *dominant* if $f(X)$ is dense in Y .

EXERCISE II.2.18.

- (a) Intersection of all prime ideals is the nilradical.
- (b) We denote the morphism by $\pi : Y \rightarrow X$. If $\pi^\#$ is injective, then taking global sections, we obtain that φ is injective. Conversely, suppose φ is injective. It suffices to show that $\varphi^\#$ is injective on the $D(f)$'s since these form a base on X . We have

$$\pi_{D(f)}^\# : \mathcal{O}_X(D(f)) \rightarrow \mathcal{O}(\pi^{-1}(D(f))) \equiv \pi_{D(f)}^\# : A_f \rightarrow B_f,$$

which is injective. This proves the first part.

Next, we must show that π is dominant if φ is injective. Indeed, suppose $\pi(Y)$ were not dense, then there would be a basic open set $D(f)$ in $\text{Spec } A$ such that $\pi^{-1}D(f) = \emptyset$, equivalently, $f \in \mathfrak{q}$ for every prime ideal \mathfrak{q} of B . Hence, f is nilpotent in B , whence nilpotent in A , consequently, $D(f) = \emptyset$. This completes the proof.

- (c) We denote the morphism by π . The first part follows from the fact that $\text{Spec } A/\mathfrak{a} \hookrightarrow \text{Spec } A$ is a topological imbedding. The second part is argued in a similar way as (b) by first concluding surjectivity on basic opens $D(f)$. Then, taking stalks, it follows that $\pi^\#$ is surjective.
- (d)

§II.3 FIRST PROPERTIES OF SCHEMES

LEMMA II.3.1 (AFFINE COMMUNICATION LEMMA).

DEFINITION. A morphism $f : X \rightarrow Y$ of schemes is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \text{Spec } B_i$ such that for each i , $f^{-1}V_i$ can be covered by open affine subsets $U_{ij} = \text{Spec } A_{ij}$, where each A_{ij} is a finitely generated B_i -algebra.

The morphism f is *of finite type* if in addition each $f^{-1}V_i$ can be covered by a finite number of the U_{ij} .

DEFINITION. A morphism $f : X \rightarrow Y$ is a *finite* morphism if there exists a covering of Y by open affine subsets $V_i = \text{Spec } B_i$ such that for each i , $f^{-1}V_i$ is affine, equal to $\text{Spec } A_i$, where A_i is a finite B_i -module.

EXERCISE II.3.1. Let $\pi : X \rightarrow Y$ denote the morphism. We use Lemma II.3.1. To this end, we first show that if $\text{Spec } B \subseteq Y$ is an affine open such that $\pi^{-1}\text{Spec } B$ can be covered by affine opens $U_i = \text{Spec } A_i$, each of which is a finitely generated B -algebra, then the same is true for $\text{Spec } B_f$, where $f \in B$. Now, $\pi^{-1}\text{Spec } B_f \subseteq \pi^{-1}\text{Spec } B$ and hence, is contained in $\bigcup U_i$. Consider $\pi^{-1}\text{Spec } B_f \cap U_i$. This can be written as a union of $D(f_{ij})$'s where $f_{ij} \in A_i$. Note that $D(f_{ij}) = \text{Spec}(A_i)_{f_{ij}}$, which is a finitely generated A_i algebra, whence a finitely generated B -algebra, consequently, a finitely generated B_f -algebra. This proves the first condition of Lemma II.3.1.

Next, suppose $(1) = (f_1, \dots, f_n)$ in B and $\text{Spec } B_{f_i}$ has the desired property. Then obviously B has the property, since B_{f_i} is a finitely generated B -algebra, and hence, any finitely generated B_{f_i} -algebra will be a finitely generated B -algebra.

DEFINITION. A morphism $f : X \rightarrow Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}V_i$ is quasi-compact for each i .

EXERCISE II.3.2. Let $\pi : X \rightarrow Y$ denote the morphism. We use Lemma II.3.1. To this end, it suffices to show that if $\text{Spec } A \subseteq Y$ is an affine open such that $\pi^{-1}\text{Spec } A$ is quasi-compact, then for any $f \in A = \Gamma(\text{Spec } A, \mathcal{O}_A)$, $\pi^{-1}\text{Spec } A_f$ is quasi-compact. We wish to characterize

$$\{P \in \pi^{-1}\text{Spec } A : f \notin \pi(p) = \mathfrak{p} \in \text{Spec } A\}.$$

We have the map $\pi_p^\sharp : \mathcal{O}_{Y, \pi(p)} \rightarrow \mathcal{O}_{X, p}$. If $f \in \mathfrak{p} = \pi(p)$, then $f \in \mathfrak{m}_{Y, p}$ and hence, $\pi_p^\sharp f \in \mathfrak{m}_{X, p}$ (since π_p^\sharp is a local homomorphism). On the other hand, if $f \notin \mathfrak{p}$, then $f/1 = 1/1$ in $\mathcal{O}_{Y, \pi(p)} = A_{\mathfrak{p}}$, consequently, $\pi_p^\sharp f = 1 \notin \mathfrak{m}_{X, p}$.

Thus, the set we are looking for is the *complement* of $(\pi^{-1}\text{Spec } A)_{\pi^\sharp f}$, the latter being closed in the open subscheme $\pi^{-1}\text{Spec } A$, due to Exercise II.2.16. Since $\pi^{-1}\text{Spec } A$ is quasi-compact, we can cover it with open affines. Let $U = \text{Spec } B$ be one such affine. Then, $\text{res } \pi^\sharp f \in \mathcal{O}_B$ and the set of desired points \mathfrak{p} are precisely those in $D(\text{res } \pi^\sharp f)$, consequently, is quasi-compact. Being a finite union of quasi-compact sets, the required complement is quasi-compact.

EXERCISE II.3.3.

(a) \implies Obviously a morphism of finite type is locally of finite type. On the other hand, with the notation of the above definitions, since $f^{-1}V_i$ can be covered by finitely many U_{ij} 's, it is a finite union of quasi-compact spaces, whence is quasi-compact. Thus, f is a quasi-compact morphism.

\Leftarrow On the other hand, suppose $f : X \rightarrow Y$ is locally of finite type and quasi-compact. Then, due to Exercise II.3.2, $f^{-1}V_i$ is quasi-compact, whence can be covered by finitely many of the U_{ij} 's. Thus, f is of finite type.

(b)

(c)

EXERCISE II.3.4. Let $\pi : X \rightarrow Y$ denote the morphism. We use Lemma II.3.1. Suppose $V = \text{Spec } B$ can be covered by distinguished opens $V_i = \text{Spec } B_{f_i}$ for $1 \leq i \leq n$ such that each V_i has the desired property. We shall show that V has the desired property. Let $U = \pi^{-1}V_i = \text{Spec } A_i$ where A_i is a finite B_{f_i} -module. Let $A = \Gamma(U, \mathcal{O}_X)$. Then, the morphism π induces a homomorphism $\varphi : B \rightarrow A$ of rings making

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \text{res}_{U_i}^U \\ B_{f_i} & \longrightarrow & A_i \end{array}$$

commute. Using the above diagram, it is not hard to argue that $U_{g_i} = A_i$, consequently, Exercise II.2.17 shows that U is affine and equal to $\text{Spec } A$.

We have reduced the algebraic geometry problem to the following commutative algebra problem:

Let $\varphi : B \rightarrow A$, let f_1, \dots, f_n generate the unit ideal in B and let $g_i = \varphi(f_i)$. Suppose A_{g_i} is a finite B_{f_i} module for $1 \leq i \leq n$. Then A is a finite B -module.

add
in

DEFINITION. A morphism $\pi : X \rightarrow Y$ is *quasi-finite* if for every $y \in Y$, $\pi^{-1}(y)$ is a finite set.

EXERCISE II.3.5.

(a) This is essentially asking us to show that if B is an A -algebra that is a finite A -module, then for every $\mathfrak{p} \in \text{Spec } A$, the fiber over \mathfrak{p} in B is finite. Recall that the fiber over \mathfrak{p} is precisely $\text{Spec } (\kappa(\mathfrak{p}) \otimes_A B)$, which is the spectrum of a $\kappa(\mathfrak{p})$ -algebra that is also a finite $\kappa(\mathfrak{p})$ -module, i.e. the spectrum of an artinian ring, whence is finite.

(b) Follows from the commutative algebra fact that integral morphisms induce closed maps on the spectrum.

(c)

add

DEFINITION. A morphism $\pi : X \rightarrow Y$, with Y irreducible is *generically finite* if $\pi^{-1}(\eta)$ is a finite set, where η is the generic point of Y .

EXERCISE II.3.7. Let $\pi : X \rightarrow Y$ denote the morphism. Let ξ be the generic point of X and η the generic point of Y . First, we show that $\pi(\xi) = \eta$. Indeed,

$$\pi(X) = \pi(\overline{\{\xi\}}) \subseteq \overline{\{\pi(\xi)\}}.$$

But since π is dominant, $\pi(X)$ is dense in Y , consequently, $\pi(\xi)$ must be a generic point, hence, equal to η .

EXERCISE II.3.11 (CLOSED SUBSCHEMES).

(a)

(b) We may suppose, without loss of generality that $Y \subseteq X$. For a point $P \in Y$, choose an open affine neighborhood $U = \text{Spec } C$ of P in Y . Then, there is an $f \in A$ such that $P \in D(f) \cap Y \subseteq U$. We contend that $D(f) \cap Y$ is a distinguished open in U . Indeed, the inclusion $\iota : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ restricted to U induces a map of rings $\varphi : A \rightarrow C$. It is easy to see that $\iota^{-1}(D(f)) = D_U(\varphi(f))$, consequently, $D(f) \cap Y$ is a distinguished open in U .

Next, cover X with $D(f_i)$'s such that $D(f_i) \cap Y$ is either affine in Y , or nonempty. Let $\bar{f}_i = \iota_X^\#(f_i) \in \Gamma(Y, \mathcal{O}_Y)$. We claim that $Y_{\bar{f}_i} = D(f_i) \cap Y$. Indeed, if $P \in D(f_i) \cap Y$, then there is a surjective map of stalks

$$\mathcal{O}_{X,P} \rightarrow \mathcal{O}_{Y,P}$$

sending f_i to \bar{f}_i . Since f_i is invertible in the former, it must be invertible in the latter. On the other hand, if $P \in Y_{\bar{f}_i}$, then \bar{f}_i is invertible in the latter whence, cannot lie in the maximal ideal $\mathfrak{m}_{X,P}$, since the above map is a local homomorphism of local rings. This shows that $D(f_i) \cap Y = Y_{\bar{f}_i}$.

Combining our above discussion with Exercise II.2.17 (b), we have that Y is affine. Next, we must show that Y is obtained as the quotient of an ideal in A . For this, invoke Exercise II.2.18 (d).

EXERCISE II.3.12.

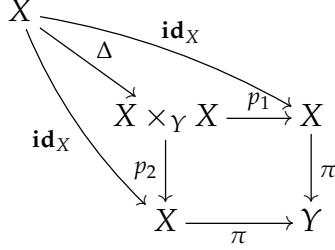
EXERCISE II.3.13 (PROPERTIES OF MORPHISMS OF FINITE TYPE).

EXERCISE II.3.14. It suffices to assume X is locally of finite type over k . In which case, there is a cover $U_i = \text{Spec } A_i$ of X such that each A_i is a finitely generated k -algebra and hence, a Jacobson ring. Consequently, the closed points of U_i are dense in U_i , whence the closed points of X are dense in X .

As for a counterexample for arbitrary schemes, consider $\text{Spec } A$ where A is a ring such that $\mathfrak{R} \neq \mathfrak{N}$.

§II.4 SEPARATED AND PROPER MORPHISMS

DEFINITION. A morphism $\pi : X \rightarrow Y$ of schemes is said to be *separated* if the diagonal morphism $\Delta : X \rightarrow X \times_Y X$ is a closed immersion.



DEFINITION. A morphism $\pi : X \rightarrow Y$ is said to be *universally closed* if it is closed as a continuous map on the underlying topological spaces and for every morphism $Y' \rightarrow Y$, the map obtained by *base extension* $X \times_Y Y' \rightarrow Y'$ is also closed.

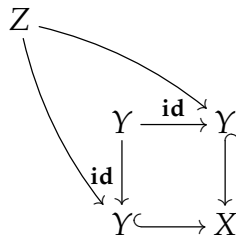
DEFINITION. A morphism $\pi : X \rightarrow Y$ is said to be *proper* if it is separated, of finite type and universally closed.

Since a complete proof of the following is not provided in the text, I reproduce it here.

COROLLARY (HARTSHORNE, II.4.6). Assume that all schemes are noetherian in the following statements.

- (a) Open and closed immersions are separated.
- (b) A composition of two separated morphisms is separated.
- (c) Separated morphisms are stable under base extension.
- (d) If $\pi : X \rightarrow Y$ and $\pi' : X' \rightarrow Y'$ are separated morphisms of schemes over a base scheme S , then the *product morphism* $\pi \times \pi' : X \times_S X' \rightarrow Y \times_S Y'$ is also separated.
- (e) If $\pi : X \rightarrow Y$ and $\varphi : Y \rightarrow Z$ are two morphisms and if $\varphi \circ \pi$ is separated, then π is separated.
- (f) A morphism $\pi : X \rightarrow Y$ is separated if and only if Y can be covered by open subsets V_i such that $\pi^{-1}V_i \rightarrow V_i$ is separated for each i .

Proof. (a) We show more generally that “a monomorphism of schemes is separated”. Let $Y \hookrightarrow X$ be a monomorphism in $\mathcal{S}ch_{\mathbb{Z}}$. Then, the fiber product $Y \times_X Y$ is precisely Y , given by the following diagram.



Since $Y \hookrightarrow X$ is a monomorphism, the two maps $Z \rightarrow Y$ in the above diagram must be the same and it follows that $Y = Y \times_X Y$. Hence, the diagonal morphism $\Delta : Y \rightarrow Y \times_X Y$ is the identity map, whence is a closed immersion.

- (b) We use the valuative criterion. Let R be a DVR and K its fraction field. Let $U = \operatorname{Spec} K$ and $T = \operatorname{Spec} R$ and suppose $\pi : X \rightarrow Y$ and $\varphi : Y \rightarrow Z$ are separated. Let there be a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Z. \end{array}$$

Suppose there are two lifts $\psi_1, \psi_2 : T \rightarrow X$ making the diagram commute. Then, $\pi \circ \psi_1 = \pi \circ \psi_2$ since $Y \rightarrow Z$ is separated. Finally, since $X \rightarrow Y$ is separated, we must have $\psi_1 = \psi_2$. This shows that $X \rightarrow Z$ is separated.

- (c) This is done in the book.
- (d) The same idea as in (b) works. Not writing this up because the diagram is too complicated to draw and I'm too lazy.
- (e) Again, begin with a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & Z \end{array}$$

and suppose there are two lifts $\psi_1, \psi_2 : T \rightarrow X$ making the diagram commute. Since $X \rightarrow Z$ is separated, we must have that $\psi_1 = \psi_2$. Hence, $X \rightarrow Y$ is separated.

- (f)



§II.5 SHEAVES OF MODULES

DEFINITION. An \mathcal{O}_X -module \mathcal{F} is said to be *free* if it is isomorphic to a direct sum of copies of \mathcal{O}_X . It is said to be *locally free* if X has an open cover by sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$ -module.

EXERCISE II.5.7.

- (a) We reduce this to the affine case since \mathcal{F} is coherent on a noetherian scheme. Thus, we have a finitely generated A -module M and a prime ideal $\mathfrak{p} \in \text{Spec } A$ such that $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module.

Choose a basis $\{\frac{m_1}{1}, \dots, \frac{m_n}{1}\}$ of $M_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$ and consider the exact sequence

$$0 \rightarrow K \rightarrow A^n \rightarrow M \rightarrow Q \rightarrow 0,$$

where the map $A^n \rightarrow M$ is the natural map sending $e_i \mapsto m_i$ for $1 \leq i \leq n$. Localising, we see that $K_{\mathfrak{p}} = Q_{\mathfrak{p}} = 0$ and hence, there is an $f \in A \setminus \mathfrak{p}$ such that $K_f = Q_f = 0$ (since both K and Q are finitely generated). Localising the above exact sequence at f , we obtain an isomorphism $A_f^n \xrightarrow{\sim} M_f$. It follows that $M_{\mathfrak{q}}$ is a free $A_{\mathfrak{q}}$ module for all $\mathfrak{q} \in D(f)$.

- (b) Follows immediately from (a).
(c) Let \mathcal{F}^{\vee} denote the dual sheaf. Recall that

$$\mathcal{F}^{\vee}(U) = \text{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U).$$

This gives a natural map $\mathcal{F}(U) \otimes \mathcal{F}^{\vee}(U) \rightarrow \mathcal{O}_X(U)$ given by

$$s \otimes \varphi \mapsto \varphi_U(s).$$

It is easy to check that this is a morphism of presheaves $\mathcal{F} \otimes \mathcal{F}^{\vee} \rightarrow \mathcal{O}_X$ and since the latter is a sheaf, it factors through the sheafification inducing a map on the tensor sheaf.

We contend that this induced map is an isomorphism. To this end, it suffices to show that the induced morphism on stalks is an isomorphism.