Homological methods in Commutative Algebra

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January 6, 2025

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§1 Regular Sequences

§§ Regular sequences and the Koszul complex

Definition 1.1. Let A be a ring and M an A-module. An element $a \in A$ is said to be *M-regular* if a is a non zero-divisor on M. A sequence $a_1, \dots, a_n$ of elements of A is an *M-sequence* if

(1) Each a_i is $M/(a_1, \dots, a_{i-1})M$-regular.

(2) $M \neq (a_1, \dots, a_n)M$.

Definition 1.2. Let A be a ring and $x_1, \dots, x_n \in A$. We define a complex $K_\bullet$ by setting $K_0 = A$, $K_p = 0$ for $p > n$ or $p < 0$, and

$$K_p = \bigoplus_{1 \leqslant i < \cdots < i_p \leqslant n} A e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

For $1 \leqslant p \leqslant n$, define $K_p \to K_{p-1}$ by

$$d\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{i=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_r} \wedge \cdots \wedge e_{i_p},$$

and extend linearly to K_p. This is known as the *Koszul complex*.

PROPOSITION 1.3. The Koszul complex is indeed a complex.

Proof. $d \circ d : K_1 \to K_{-1}$ is obviously the zero map. Now, let $p \geqslant 2$, we shall show that $(d \circ d)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = 0$. Note that the above can be written as a linear combination of the basis elements of K_{p-2}. Consider the basis element $e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_a} \wedge \cdots \wedge \widehat{e}_{i_b} \wedge \cdots \wedge e_{i_p}$. We shall show that its coefficient is 0.

Indeed, its coefficient is contributed by

$$e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_a} \wedge \cdots \wedge e_{i_p} \quad \text{and} \quad e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_b} \wedge \cdots \wedge e_{i_p},$$

each of which has coefficient $(-1)^{a-1} x_{i_a}$ and $(-1)^{b-1} x_{i_b}$ respectively. The coefficient of our desired basis element in the differential of the first is $(-1)^{b-2} x_{i_b}$ and in the second is $(-1)^{a-1} x_{i_a}$. Thus, the coefficient of our desired basis element in the differential of $e_{i_1} \wedge \cdots \wedge e_{i_p}$ is

$$(-1)^{a-1} x_{i_a} (-1)^{b-2} x_{i_b} + (-1)^{b-1} x_{i_b} (-1)^{a-1} x_{i_a} = 0,$$

thereby completing the proof. ∎

DEFINITION 1.4. Let $C_\bullet$ and $D_\bullet$ be complexes of A-modules. Define their *tensor product* $(C \otimes D)_\bullet$ by

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes_A D_j.$$

The boundary maps are given by $d : (C \otimes D)_n \to (C \otimes D)_{n-1}$

$$d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy \qquad x \in C_i,\ y \in C_j.$$

PROPOSITION 1.5. There is an isomorphism of complexes $(C \otimes D)_\bullet \cong (D \otimes C)_\bullet$.

Proof. If $x \otimes y \in (C \otimes D)_n$ with $x \in C_i$ and $y \in D_j$, then send this element to $(-1)^{ij} y \otimes x \in (D \otimes C)_n$. It is not hard to check that this is indeed a chain map. That this is an isomorphism of chain complexes follows from the fact that for every n, $(C \otimes D)_n \to (D \otimes C)_n$ is an isomorphism. ∎

PROPOSITION 1.6. Let $x_1, \ldots, x_n \in A$. Then $K_\bullet(x_1, \ldots, x_n) \cong K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$ as complexes.

Proof. We prove this by induction on n. The base case with $n = 1$ is tautological. Suppose now that $n \geqslant 1$. We shall show that $K_\bullet(x_1, \ldots, x_n) \otimes K_\bullet(x_{n+1}) \cong K_\bullet(x_1, \ldots, x_{n+1})$. Write the complex $K_\bullet(x_{n+1})$ as

$$0 \longrightarrow Ae_{n+1} \xrightarrow{e_{n+1} \mapsto x_{n+1}} A \longrightarrow 0.$$

Then, $\left(K(x_1, \ldots, x_n) \otimes K(x_{n+1})\right)_p = \left(K_p(x_1, \ldots, x_n) \otimes A\right) \oplus \left(K_{p-1}(x_1, \ldots, x_n) \otimes Ae_{n+1}\right)$. There is a natural isomorphism

$$\left(K(x_1, \ldots, x_n) \otimes K(x_{n+1})\right)_p \longrightarrow K_p(x_1, \ldots, x_{n+1}),$$

which sends $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes 1$ to $e_{i_1} \wedge \cdots \wedge e_{i_p}$ in $K_p(x_1, \ldots, x_n)$, and sends $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes e_{n+1}$ to $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \wedge e_{n+1}$ in $K_p(x_1, \ldots, x_{n+1})$.

It remains to check that the map defined above is indeed a chain map. Indeed, under the differential in the tensor complex, $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes 1$ maps to $d(e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes 1$, which maps to $e(e_{i_1} \wedge \cdots \wedge e_{i_p})$ under the aforementioned isomorphism. On the other hand, the starting element maps to $e_{i_1} \wedge \cdots \wedge e_{i_p}$ under the isomorphism first and then maps to $d(e_{i_1} \wedge \cdots \wedge e_{i_p})$ under the differential.

Next, if we begin with $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes e_{n+1}$, then under the differential, it maps to

$$d(e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}) \otimes e_{n+1} + (-1)^{p-1} x_{n+1} e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \otimes 1,$$

which maps to

$$d(e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}) \wedge e_{n+1} + (-1)^{p-1} x_{n+1} e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}$$

under the isomorphism. On the other hand, the starting element maps to $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \wedge e_{n+1}$ under the isomorphism, which maps to the above element under the differential. This completes the proof. ■

DEFINITION 1.7. Let $\underline{x} = x_1, \ldots, x_n$ be a sequence in A. For an A-module M, set

$$K_\bullet(\underline{x}, M) = K(\underline{x}) \otimes M.$$

The homology groups of this complex are denoted by $H_p(\underline{x}, M)$. Similarly, for a complex $C_\bullet$ of A-modules, set $C_\bullet(\underline{x}) = C_\bullet \otimes K_\bullet(\underline{x})$.

PROPOSITION 1.8. Let $\underline{x} = x_1, \ldots, x_n$ be a sequence in A. Then

$$H_0(\underline{x}, M) = M/(\underline{x})M \qquad H_n(\underline{x}, M) \cong \{\xi \in M \colon x_1\xi = \cdots = x_n\xi = 0\}.$$

Proof. The assertion about $H_0(\underline{x}, M)$ is trivial. $H_n(\underline{x}, M)$ is precisely the kernel of the map $K_n(\underline{x}, M) \to K_{n-1}(\underline{x}, M)$, which is given by

$$\xi e_1 \wedge \cdots \wedge e_n \longmapsto \sum_{i=1}^{n} (-1)^{i-1} x_i \xi e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n,$$

where $\xi e_{i_1} \wedge \cdots \wedge e_{i_p} \in K_p(\underline{x}, M)$ is shorthand for $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \xi \in K_p(\underline{x}, M)$.

The right hand side of the above equation is zero if and only if each $x_i\xi$ is zero, whence the conclusion follows. ■

THEOREM 1.9. Let $C_\bullet$ be a complex of A-modules and $x \in A$. Then, there is a short exact sequence of complexes

$$0 \to C_\bullet \to C_\bullet(x) \to C'_\bullet \to 0,$$

where $C'_{p+1} = C_p$ is the (upward) shift of the complex $C_\bullet$. The homology long exact sequence so obtained looks like

$$\cdots \to H_p(C_\bullet) \to H_p(C_\bullet(x)) \to H_{p-1}(C_\bullet) \xrightarrow{(-1)^{p-1}x} H_{p-1}(C_\bullet) \to \cdots.$$

Further, we have $x \cdot H_p(C_\bullet(x)) = 0$ for all $p \in \mathbb{Z}$.

Proof. Denote the Koszul complex $K_\bullet(x)$ by

$$\cdots \to 0 \to Ae_1 \xrightarrow{e_1 \mapsto x} A \to 0.$$

Thus, we can identify $C_\bullet(x)$ with $C_p \oplus C_{p-1}$ with the boundary map as

$$d(\xi, \eta) = (d\xi + (-1)^{p-1} x\eta, d\eta) \in C_{p-1} \oplus C_{p-2}.$$

Hence, we have a short exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & C_p \arrow[r] \arrow[d] & C_{p-1} \arrow[r] \arrow[d] & \cdots . \\
\cdots \arrow[r] & C_p \oplus C_{p-1} \arrow[r] \arrow[d] & C_{p-1} \oplus C_{p-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{p-1} \arrow[r] & C_{p-2} \arrow[r] & \cdots
\end{tikzcd}$$

That the above commutes is straightforward. It remains to compute the boundary map from $H_{p-1}(C_\bullet) = H_p(C'_\bullet)$ to $H_{p-1}(C_\bullet)$.

Choose a cycle $\eta \in C'_p = C_{p-1}$, that is, $d\eta = 0$. This lifts to $(0, \eta) \in C_p \oplus C_{p-1}$, which maps to $\big((-1)^{p-1} x\eta, 0\big) \in C_{p-1} \oplus C_{p-2}$, which again lifts to $(-1)^{p-1} x\eta$ in C_{p-1}, which is a cycle in C_{p-1}. Hence, the induced map on homologies is multiplication by $(-1)^{p-1}x$.

Finally, we must show that x annihilates $H_p(C_\bullet(x))$ for all p. Choose a cycle $(\xi, \eta) \in C_p \oplus C_{p-1}$, that is, $d\eta = 0$, and $d\xi = (-1)^p x\eta$. Hence,

$$C_{p+1} \ni d(0, (-1)^p \xi) = ((-1)^p x\xi, (-1)^p d\xi) = x \cdot (\xi, \eta).$$

Thus, x annihilates $[(\xi, \eta)] \in H_p(C_\bullet(x))$, whence annihilates all of $H_p(C_\bullet(x))$. ■

COROLLARY. Let $\underline{x} = x_1, \ldots, x_n$ be a sequence in A. Then $(\underline{x})$ annihilates $H_p(\underline{x}, M)$ for every $p \in \mathbb{Z}$.

Proof. It suffices to show that x_n annihilates $H_p(\underline{x}, M)$ since the Koszul complex is invariant under permutation of the sequence $\underline{x}$. But this is obvious, since $K_\bullet(\underline{x}, M)$ is isomorphic to $K_\bullet(x_1, \ldots, x_{n-1}, M) \otimes K_\bullet(x_n)$ due to the commutativity of tensor products of complexes. We are done by invoking the preceding theorem with $C_\bullet = K_\bullet(x_1, \ldots, x_{n-1}, M)$ and $x = x_n$. ■

THEOREM 1.10. Let A be a ring, M an A-module, and $x_1, \ldots, x_n$ an M-sequence. Then

$$H_P(\underline{x}, M) = 0 \quad \forall\, p > 0, \qquad H_0(\underline{x}, M) = M/(\underline{x})M.$$

Proof. Induct on n. The base case with $n = 1$ follows from the fact that $H_1(x_1, M) = (0 :_M x_1) = 0$, since x_1 is M-regular. Now, suppose $n > 1$. If $p > 1$, then there is an exact sequence furnished by Theorem 1.9 by taking $C_\bullet = K_\bullet(x_1, \ldots, x_{n-1}, M)$ and $x = x_n$:

$$0 = H_p(x_1, \ldots, x_{n-1}, M) \longrightarrow H_p(x_1, \ldots, x_n, M) \longrightarrow H_{p-1}(x_1, \ldots, x_{n-1}, M) = 0,$$

whence $H_p(\underline{x}, M) = 0$. It remains to establish that $H_1(\underline{x}, M) = 0$. Set $M_i = M/(x_1, \ldots, x_i)M$ with the convention that $M_0 = M$. The above long exact sequence again furnishes

$$0 = H_1(x_1, \ldots, x_{n-1}, M) \to H_1(\underline{x}, M) \to H_0(x_1, \ldots, x_{n-1}, M) = M_{n-1} \xrightarrow{x_n} M_{n-1}.$$

But since x_n is a non zero-divisor on M_{n-1}, we see that $H_1(\underline{x}, M) = 0$ as desired. ■

THEOREM 1.11. Suppose one of the following two conditions holds:

(α) $(A, \mathfrak{m})$ is a Noetherian local ring, $x_1, \ldots, x_n \in \mathfrak{m}$, and M is a finite A-module.

(β) A is an $\mathbb{N}$-graded ring, M is an $\mathbb{N}$-graded A-module, and $x_1, \ldots, x_n$ are homogeneous elements of positive degree.

Then, if $H_1(\underline{x}, M) = 0$ and $M \neq 0$, then $x_1, \ldots, x_n$ is an M-sequence.

Proof. Induction on n. If $n = 1$, then $0 = H_1(x_1, M) = (0 :_M x_1)$, whence x_1 is a non zero-divisor on M. Now suppose $n > 1$. Again, we make use of the exact sequence associated with $K_\bullet(x_1, \ldots, x_{n-1}, M) \otimes K_\bullet(x_n)$ to get

$$H_1(x_1, \ldots, x_{n-1}, M) \xrightarrow{-x_n} H_1(x_1, \ldots, x_{n-1}, M) \to H_1(\underline{x}, M) = 0.$$

But since $H_i(x_1, \ldots, x_{n-1}, M)$ is a finite A-module in case (α) or a $\mathbb{N}$-graded module in case (β), the above surjection implies, due to Nakayama, that $H_1(x_1, \ldots, x_{n-1}, M) = 0$. The induction hypothesis then implies $x_1, \ldots, x_{n-1}$ is an M-sequence.

Now, continuing the above long exact sequence, we get

$$0 = H_1(\underline{x}, M) \longrightarrow H_0(x_1, \ldots, x_{n-1}, M) = M_{n-1} \xrightarrow{x_n} M_{n-1},$$

where $M_{n-1} = M/(x_1, \ldots, x_{n-1})M$. The above sequence implies x_n is M_{n-1}-regular, whence $x_1, \ldots, x_n$ is an M-sequence, as desired. ■

THEOREM 1.12. Let A be a Noetherian ring, M a finite A-module, and I an ideal of A such that $M \neq IM$. For a given integer $n > 0$, the following conditions are equivalent:

(1) $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$ and for any finite A-module N with $\mathrm{Supp}(N) \subseteq V(I)$.

(2) $\mathrm{Ext}^i_A(A/I, M) = 0$ for all $i < n$.

(3) $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$ and for some finite A-module N with $\mathrm{Supp}(N) = V(I)$.

(4) There exists an M-sequence of length n contained in I.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (4) First, we show that I contains an M-regular element. Suppose not, then due to prime avoidance, I must be contained in some associated prime $\mathfrak{p} \in \mathrm{Ass}_A(M)$. Thus, there is an injective map $A/\mathfrak{p} \hookrightarrow M$, which upon localizing at $\mathfrak{p}$, we see that $\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \neq 0$. Now, $\mathfrak{p} \in V(I) = \mathrm{Supp}(N)$, whence $N_\mathfrak{p} \neq 0$, and hence, due to Nakayama's lemma, $N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \neq 0$ (since $N_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module). Then,

$N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p}$ is a non-zero $\kappa(\mathfrak{p})$-vector space, and consequently, $\operatorname{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p}, \kappa(\mathfrak{p})) \neq 0$ (choose a basis and project onto a coordinate). Now, we can form the composition

$$N_\mathfrak{p} \to N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \to \kappa(\mathfrak{p}) \hookrightarrow M_\mathfrak{p}.$$

The first two maps are surjections and hence, the composition is non-zero. It follows that $\operatorname{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0$. Since N is finite over a Noetherian ring, we have

$$(\operatorname{Hom}_A(N, M))_\mathfrak{p} = \operatorname{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0,$$

whence $\operatorname{Ext}^0_A(N, M) = \operatorname{Hom}_A(N, M) \neq 0$, a contradiction to (3). Hence, I contains an M-regular element, say f. If $n = 1$, then we are already done. If $n > 1$, then set $M_1 = M/fM$ and consider the short exact sequence

$$0 \to M \xrightarrow{f} M \to M_1 \to 0.$$

The long exact sequence using $\operatorname{Ext}_A(N, -)$ gives

$$\cdots \to \operatorname{Ext}^{i-1}_A(N, M) \xrightarrow{f} \operatorname{Ext}^{i-1}_A(N, M) \to \operatorname{Ext}^{i-1}_A(N, M_1) \to \operatorname{Ext}^i_A(N, M) \to \cdots.$$

For $1 \leqslant i < n$, this implies $\operatorname{Ext}^{i-1}_A(N, M_1) = 0$, and due to the induction hypothesis, there is an M_1-sequence $f_2, \dots, f_n$ in I. Thus, $f_1, \dots, f_n$ is an M-sequence in I.

(4) $\Rightarrow$ (1). Induction on n. We shall deal with the base case later. Suppose $n > 1$. Let $\underline{x} = x_1, \dots, x_n$ be an M-sequence in I. Set $M_1 = M/x_1M$ which fits into a short exact sequence $0 \to M \xrightarrow{x_1} M \to M_1 \to 0$. The sequence $x_2, \dots, x_n$ is an M_1-sequence in I, whence due to the inductive hypothesis, $\operatorname{Ext}^i_A(N, M_1) = 0$ for all $i < n - 1$. The long exact sequence corresponding to $\operatorname{Ext}_A(N, -)$ gives us

$$0 = \operatorname{Ext}^{i-1}_A(N, M_1) \to \operatorname{Ext}^i_A(N, M) \xrightarrow{x_1} \operatorname{Ext}^i(N, M)$$

for all $0 \leqslant i < n$, with the convention that $\operatorname{Ext}^{-1}(N, M_1) = 0$. But note that $\operatorname{Ext}^i_A(N, -)$ is annihilated by $\operatorname{Ann}_A(N)$. But since $\operatorname{Supp}(N) = V(\operatorname{Ann}_A(N)) \subseteq V(I)$, we conclude that $I \subseteq \sqrt{I} \subseteq \sqrt{\operatorname{Ann}_A(N)}$. In particular, a sufficiently large power of x_1 annihilates N, whence, annihilates $\operatorname{Ext}^i_A(N, M)$. But since multiplication by x_1 is injective, we must have that $\operatorname{Ext}^i_A(N, M) = 0$ for $i < n$, thereby completing the proof. ■

THEOREM 1.13. Let A be a Noetherian ring, I an ideal of A, and M a finite A-module such that $M \neq IM$. Then the length of any maximal M-sequence contained in I is the same, say n, and n is determined by

$$\operatorname{Ext}^i_A(A/I, M) = 0 \quad \forall\, i < n \quad \text{and} \quad \operatorname{Ext}^n_A(A/I, M) \neq 0.$$

We write $n = \operatorname{depth}(I, M)$ and call n the *I-depth* of M.

Proof. Let $\underline{a} = a_1, \ldots, a_n$ be a maximal M-sequence in I. Suppose $\mathrm{Ext}^n_A(A/I, M) = 0$. Define $M_i = M/(a_1, \ldots, a_i)M$. Using the short exact sequence $0 \to M \xrightarrow{a_1} M \to M_1 \to 0$, we have an exact sequence

$$0 = \mathrm{Ext}^{n-1}_A(A/I, M) \to \mathrm{Ext}^{n-1}_A(A/I, M_1) \to \mathrm{Ext}^n_A(A/I, M) = 0,$$

whence $\mathrm{Ext}^{n-1}_A(A/I, M_1) = 0$; and since $a_2, \ldots, a_n$ is an M_1-sequence, $\mathrm{Ext}^i_A(A/I, M_1) = 0$ for $i < n-1$. Arguing similarly, we get that $\mathrm{Ext}^0_A(A/I, M_n) = 0$. Due to the preceding theorem, I must contain an M_n-regular element, contradicting the maximality of $\underline{a}$. Thus, $\mathrm{Ext}^n_A(A/I, M) \neq 0$ and $\mathrm{Ext}^i_A(A/I, M) = 0$ for $i < n$.

On the other hand, if $\underline{b} = b_1, \ldots, b_m$ is a maximal M-sequence, then due to the above paragraph, $\mathrm{Ext}^m_A(A/I, M) \neq 0$ and $\mathrm{Ext}^i_A(A/I, M) = 0$ for $i < m$. In particular, this means that $m = n$.

Finally, suppose n satisfies the conditions given in the theorem. Then, due to the preceding theorem, there is an M-sequence $\underline{a} = a_1, \ldots, a_n$ in I. Further, since $\mathrm{Ext}^n_A(A/I, M) \neq 0$, this sequence must be maximal, else it could be extended and again, due to the preceding theorem $\mathrm{Ext}^n_A(A/I, M) = 0$. This completes the proof. ∎

REMARK 1.14. The above theorem can be phrased more succinctly as

$$\operatorname{depth}(I, M) = \inf\left\{i\colon\ \mathrm{Ext}^i_A(A/I, M) \neq 0\right\}.$$

In particular, if $(A, \mathfrak{m}, k)$ is a Noetherian local ring, then we write $\operatorname{depth}(\mathfrak{m}, M)$ as $\operatorname{depth} M$ and

$$\operatorname{depth} M = \inf\left\{i\colon\ \mathrm{Ext}^i_A(k, M) \neq 0\right\}.$$

THEOREM 1.15 (DEPTH SENSITIVITY OF KOSZUL COMPLEX). Let A be a Noetherian ring, $I = (y_1, \ldots, y_n)$ an ideal of A, and M a finite A-module such that $M \neq IM$. If

$$q = \sup\left\{i\colon\ H_i(\underline{y}, M) \neq 0\right\},$$

then $\operatorname{depth}(I, M) = n - q$.

Proof. We shall argue by induction on $s = \operatorname{depth}(I, M)$. If $s = 0$, then every element of I is a zero-divisor on M, whence by prime avoidance, there is an associated prime $\mathfrak{p} \in \mathrm{Ass}_A(M)$ such that $I \subseteq \mathfrak{p}$. By definition, there is some $0 \neq \xi \in M$ such that $\mathfrak{p} = \mathrm{Ann}_A(\xi)$, and hence, $I\xi = 0$. Recall that $H_n(\underline{y}, M) = (0 :_M (\underline{y})) = (0 :_M I) \neq 0$, since it contains ξ. Thus, $q = n$.

Now, suppose $s > 0$, then $H_n(\underline{y}, M) = 0$, since some element of I is a non zero-divisor on M. In particular, this means $q < n$. Let $\underline{x} = x_1, \ldots, x_s$ be a maximal M-sequence in I. There is a short exact sequence $0 \to M \xrightarrow{x_1} M \to M_1 \to 0$, where $M_1 = M/x_1M$. Since every element in the Koszul comples $K_\bullet(\underline{y})$ is a free module, tensoring with the above short exact sequence will give a short exact sequence of complexes

$$0 \to K_\bullet(\underline{y}, M) \xrightarrow{x_1} K_\bullet(\underline{y}, M) \to K_\bullet(\underline{y}, M_1) \to 0.$$

The associated long exact sequence looks like

$$H_i(\underline{y}, M) \xrightarrow{x_1} H_i(\underline{y}, M) \to H_i(\underline{y}, M_1) \to H_{i-1}(\underline{y}, M) \xrightarrow{x_1} H_{i-1}(\underline{y}, M)$$

for all i. Recall that $I = (\underline{y})$ annihilates $H_i(\underline{y}, M)$ for all i, and hence the image of the first map and the kernel of the last map in the above sequence is 0, therby giving us a short exact sequence

$$0 \to H_i(\underline{y}, M) \to H_i(\underline{y}, M_1) \to H_{i-1}(\underline{y}, M) \to 0, \qquad \forall\, i \in \mathbb{Z}.$$

Now, note that if $H_i(\underline{y}, M_1) = 0$, then $H_i(\underline{y}, M) = H_{i-1}(\underline{y}, M) = 0$. Hence, $H_{q+1}(\underline{y}, M_1) \neq 0$, but for $i > q+1$, $H_i(\underline{y}, M_1) = 0$. Now, $\operatorname{depth}(I, M_1) = s-1$, since $x_2, \ldots, x_n$ is a maximal M_1-sequence in I, for if not, then the original sequence $\underline{x}$ could be extended to a larger M-sequence in I. By the induction hypothesis, we have $q+1 = n-(s-1)$, and thus, $s = n-q$. ■

REMARK 1.16. In other words, $\operatorname{depth}(I, M)$ is the number of successive zero terms from the left in the sequence

$$H_n(\underline{y}, M),\ H_{n-1}(\underline{y}, M),\ \ldots,\ H_0(\underline{y}, M) = M/IM \neq 0.$$

§§ Gorenstein Rings

LEMMA 1.17. Let A be a ring, M an A-module, and $n \geqslant 0$ an integer. Then

$$\operatorname{inj\,dim} M \leqslant n \iff \operatorname{Ext}_A^{n+1}(A/I, M) = 0 \quad \text{for all ideals } I.$$

If A is Noetherian, then we can replace "for all ideals" by "for all prime ideals" in the above equivalence.

Proof. The forward direction is trivial by considering an injective resolution of length $\leqslant n$ and constructing the left derived functors of $\operatorname{Hom}_A(A/I, -)$.

We prove the converse. If $n = 0$, then $\operatorname{Ext}_A^1(A/I, M) = 0$, which is equivalent to Baer's criterion for injectivity. Thus M is injective, that is, $\operatorname{inj\,dim} M = 0 \leqslant n$. Now, suppose $n > 0$. Consider an injective resolution of length $n-1$ and let K be the cokernel of the last map. That is,

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to K_n \to 0,$$

where every E^i is injective. We claim that K is injective. To see this, break down the above exact sequence into short exact sequences of the form

$$0 \to K_0 \to E^0 \to K_1 \to 0 \qquad 0 \to K_1 \to E^1 \to K_2 \to 0,$$

and so on, with the convention that $K_0 = M$. The long exact sequence for $\operatorname{Ext}_A(A/I, -)$ on the first short exact sequence gives

$$0 = \operatorname{Ext}_A^n(A/I, E^0) \to \operatorname{Ext}_A^n(A/I, K_1) \to \operatorname{Ext}_A^{n+1}(A/I, K_0) = 0,$$

whence $\operatorname{Ext}^n(A/I, K_1) = 0$. Proceeding similarly with the other exact sequences, one can show that $\operatorname{Ext}_A^1(A/I, K_n) = 0$, for every ideal I of A. Hence, K_n is injective, i.e., $\operatorname{inj\,dim} M \leqslant n$. ■

LEMMA 1.18. Let A be a ring, M and N two A-modules, and $x \in A$. Suppose that x is both A-regular and M-regular, and that $xN = 0$. Set $B = A/xA$ and $\overline{M} = M/xM$. Then

(1) $\operatorname{Hom}_A(N, M) = 0$ and $\operatorname{Ext}_A^{n+1}(N, M) \cong \operatorname{Ext}_B^n(N, \overline{M})$ for all $n \geqslant 0$.

(2) $\operatorname{Ext}_A^n(M, N) \cong \operatorname{Ext}_B^n(\overline{M}, N)$ for all $n \geqslant 0$.

(3) $\operatorname{Tor}_n^A(M, N) \cong \operatorname{Tor}_n^B(\overline{M}, N)$ for all $n \geqslant 0$.

Proof. (1) If $f : N \to M$ is A-linear, then for any $n \in N$, $xf(n) = f(xn) = 0$, and since x is M-regular, $f(n) = 0$. Thus $f = 0$, as desired. Now, set $T^n(N) = \operatorname{Ext}_A^{n+1}(N, M)$. Then, the collection $(T^n)_{n\geqslant 0}$ is a contravariant δ-functor from the category $\mathfrak{Mod}_B$ to the category $\mathfrak{Mod}_A$. Further, the short exact sequence

$$0 \to M \xrightarrow{x} M \to \overline{M} \to 0$$

furnishes a long exact sequence

$$0 = \operatorname{Hom}_A(N, M) \to \operatorname{Hom}_A(N, \overline{M}) \xrightarrow{\delta} \operatorname{Ext}_A^1(N, M) \xrightarrow{x} \operatorname{Ext}_A^1(N, M) \to \cdots .$$

Since x annihilates N, it must annihilate $\operatorname{Ext}_A^1(N, M)$, and so the above exact sequences reduces to

$$0 \to \operatorname{Hom}_A(N, \overline{M}) \xrightarrow{\delta} \operatorname{Ext}_A^1(N, M) \to 0.$$

Thus δ is a natural isomorphism between the functors T^0 and $\operatorname{Ext}_A^1(-, M)$. Now, it suffices to show that the collection $(T^n)_{n\geqslant 0}$ constitutes a universal δ-functor, whence it suffices to show that $T^n(P) = 0$ for every projective B-module P and $n \geqslant 1$; since then it would be coeffaceable by projectives and due to a theorem of Grothendieck, it would be universal.

This is equivalent to showing that $\operatorname{Ext}_A^n(P, M) = 0$ where P is a direct sum of copies of A/xA and $n \geqslant 2$. But note that $\operatorname{proj\,dim}_A A/xA \leqslant 1$, and hence $\operatorname{Ext}_A^n(A/xA, M) = 0$ for all A-modules M and $n \geqslant 2$, as desired. This proves (1).

(2) We contend that $\operatorname{Tor}_n^A(M, B) = 0$ for all $n > 0$. Since $\operatorname{proj\,dim}_A B \leqslant 1$, it immediately follows that $\operatorname{Tor}_n^A(M, B) = 0$ for $n > 1$. For $n = 1$, the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

furnishes a long exact sequence

$$0 = \operatorname{Tor}_1^A(M, A) \to \operatorname{Tor}_1^A(M, B) \to M \xrightarrow{x} M \to \overline{M} \to 0.$$

Since x is M-regular, we have that $\operatorname{Tor}_1^A(M, A) = 0$.

Now, let $P_\bullet \to M \to 0$ be a free resolution of M. Because of the preceding paragraph, the sequence $P_\bullet \otimes_A B \to M \otimes_A B \to 0$ is exact, so that $P_\bullet \otimes B$ is a free resolution of the B-module $M \otimes B \cong \overline{M}$. From the Hom-Tensor adjunction, note that there are natural isomorphisms

$$\mathrm{Hom}_A(P_\bullet, N) = \mathrm{Hom}_A(P_\bullet, \mathrm{Hom}_B(B, N)) \cong \mathrm{Hom}_B(P_\bullet \otimes_A B, N).$$

Therefore,

$$\mathrm{Ext}_A^n(M, N) = H^n\left(\mathrm{Hom}_A(P_\bullet, N)\right) = H^n\left(\mathrm{Hom}_B(P_\bullet \otimes_A B, N)\right) = \mathrm{Ext}_B^n(\overline{M}, N),$$

as desired.

(3) Continuing with the notation of (2), we have

$$\mathrm{Tor}_n^A(M, N) = H_n\left(P_\bullet \otimes_A N\right) = H_n\left((P_\bullet \otimes_A B) \otimes_B N\right) = \mathrm{Tor}_n^B(\overline{M}, N),$$

thereby completing the proof. ■

§2 Regular Rings

§§ Regular Rings

Definition 2.1. Let $(A, \mathfrak{m}, k)$ be a local ring and let M be a finite A-module. An exact sequence

$$\cdots \to L_i \xrightarrow{d_i} L_{i-1} \xrightarrow{d_{i-1}} \cdots \to L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M \to 0$$

is called a *minimal (free) resolution* of M if

- each L_i is a finite free A-module
- $0 = \overline{d}_i : L_i \otimes_A k \to L_{i-1} \otimes_A k$, or equivalently $d_i L_i \subseteq \mathfrak{m} L_{i-1}$ for all $i \geqslant 1$, and
- $\overline{\varepsilon} : L_0 \otimes_A k \to M \otimes_A k$ is an isomorphism.

It is easy to see that a minimal free resolution exists for every finite module over a Noetherian local ring; at each stage simply take a minimal generating set of the kernel and continue.

Lemma 2.2. Let $(A, \mathfrak{m}, k)$ be a local ring, and M a finite A-module. Suppose $L_\bullet$ is a minimal resolution of M; then

(1) $\dim_k \mathrm{Tor}_i^A(M, k) = \operatorname{rank} L_i$ for all i.

(2) $\operatorname{proj\,dim}_A M = \sup\left\{i\colon\ \mathrm{Tor}_i^A(M, k) \neq 0\right\} \leqslant \operatorname{proj\,dim}_A k$,

(3) if $M \neq 0$ and $\operatorname{proj\,dim}_A M = r < \infty$, then for any finite A-module $N \neq 0$, we have $\mathrm{Ext}_A^r(M, N) \neq 0$.

Proof. (1) This follows immediately from the fact that $\overline{d}_i = 0$ for all $i \geqslant 1$.

(2) The second inequality is straightforward. For if $\operatorname{proj\,dim}_A k = \infty$, then there is nothing to prove. If $\operatorname{proj\,dim}_A k < \infty$, then take a projective resolution of this length and tensor with A to conclude.

From (1) it immediately follows that $\operatorname{proj\,dim}_A M \leqslant \sup\left\{i\colon \operatorname{Tor}_i^A(M,k) \neq 0\right\}$, since this quantity is precisely the length of the minimal free resolution of M. If $\operatorname{proj\,dim}_A M = \infty$, then there is nothing to prove. If $\operatorname{proj\,dim}_A M < \infty$, then take a projective resolution of M achieving this length and tensor with k whence it follows that $\sup\left\{i\colon \operatorname{Tor}_i^A(M,k) \neq 0\right\} \leqslant \operatorname{proj\,dim}_A M$, as desired.

(3) Applying $\operatorname{Hom}_A(-,N)$ to the resolution $L_\bullet \to M$, we obtain a complex which ends with

$$\operatorname{Hom}_A(L_{r-1},N) \xrightarrow{d_r^*} \operatorname{Hom}_A(L_r,N) \to 0,$$

where $\operatorname{Ext}_A^r(M,N)$ is the cokernel of the above map. Since each L_i is free, we can write $\operatorname{Hom}_A(L_i,N)$ as a direct sum of some copies of N and we can express every boundary map $d_i : L_i \to L_{i-1}$ as a matrix with entries in $\mathfrak{m}$. It follows that d_i^* is given by the same matrix (with entries in $\mathfrak{m}$). Hence, the image of d_r^* is contained in $\mathfrak{m}\operatorname{Hom}_A(L_r,N)$, which is properly contained in $\operatorname{Hom}_A(L_r,N)$ by Nakayama's lemma. This completes the proof. ■

THEOREM 2.3 (AUSLANDER-BUCHSBAUM). Let A be a Noetherian local ring and $M \neq 0$ a finite A-module. If $\operatorname{proj\,dim}_A M < \infty$, then

$$\operatorname{proj\,dim}_A M + \operatorname{depth} M = \operatorname{depth} A.$$

Proof. ■