

Derived and Triangulated Categories

Swayam Chube

Last Updated: July 10, 2025

Contents

1 Localization of Categories	1
1.1 Localizing Classes	1
1.2 Localization and Subcategories	7
1.3 Localizing Additive Categories	8
1.4 Localization of Abelian Categories	11

We fix some notation before proceeding. Categories will usually denoted by calligraphic symbols such as $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. The opposite category of a category \mathcal{A} is denoted by \mathcal{A}^{op} . Corresponding to each object $A \in \mathcal{A}$, there is an object $A^{\text{op}} \in \mathcal{A}^{\text{op}}$ and corresponding to each morphism $f: A \rightarrow B$ in \mathcal{A} , there is a morphism $f^{\text{op}} \in \mathcal{A}^{\text{op}}$. If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in \mathcal{A} , then $C^{\text{op}} \xrightarrow{g^{\text{op}}} B^{\text{op}} \xrightarrow{f^{\text{op}}} A^{\text{op}}$ with $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$.

§1 Localization of Categories

THEOREM 1.1. Let \mathcal{A} be a category, and S be a class of morphisms in \mathcal{A} . Then there is a category $\mathcal{A}[S^{-1}]$ and a functor $Q: \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$ such that for every functor $F: \mathcal{A} \rightarrow \mathcal{B}$ such that $F(s)$ is an isomorphism in \mathcal{B} for each $s \in S$, there is a unique functor $G: \mathcal{A}[S^{-1}] \rightarrow \mathcal{B}$ making

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ Q \downarrow & \nearrow \exists! G & \\ \mathcal{A}[S^{-1}] & & \end{array}$$

commute. Further, the pair $(\mathcal{A}[S^{-1}], Q)$ is unique up to a unique isomorphism of categories and is called the *localization* of \mathcal{A} by the class of morphisms S .

§§ Localizing Classes

Quite generally, the category $\mathcal{A}[S^{-1}]$ is quite ugly and difficult to work with. Therefore, we restrict ourselves to a more manageable class S of localizing morphisms.

DEFINITION 1.2. Let \mathcal{A} be a category. A class of morphisms S in \mathcal{A} is said to be a *localizing class* if

(LC1) For any object $M \in \mathcal{A}$, $\text{id}_M \in S$.

(LC2) If s, t are composable morphisms in S , then so is their composition.

(LC3) (a) Every diagram of the form

$$\begin{array}{ccc} & & L \\ & & \downarrow s \\ M & \xrightarrow{f} & N \end{array}$$

with $f \in \text{Mor}(\mathcal{A})$ and $s \in S$ can be enlarged to a commutative square

$$\begin{array}{ccc} K & \xrightarrow{g} & L \\ \downarrow t & & \downarrow s \\ M & \xrightarrow{f} & N \end{array}$$

for some $K \in \mathcal{A}$, $g \in \text{Mor}(\mathcal{A})$, and $s \in S$.

(b) Every diagram of the form

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \downarrow s & & \\ & & L \end{array}$$

with $f \in \text{Mor}(\mathcal{A})$ and $s \in S$ can be enlarged to a commutative square

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \downarrow s & & \downarrow t \\ L & \xrightarrow{g} & K \end{array}$$

with $K \in \mathcal{A}$, $g \in \text{Mor}(\mathcal{A})$, and $t \in S$.

(LC4) Let $f, g: M \rightarrow N$ be two morphisms in \mathcal{A} . Then

$$\exists s \in S \text{ such that } s \circ f = s \circ g \iff \exists t \in S \text{ such that } f \circ t = g \circ t.$$

Clearly, if S is a localizing class in \mathcal{A} , then $S^{\text{op}} = \{s^{\text{op}}: s \in S\}$ is a localizing class in \mathcal{A}^{op} .

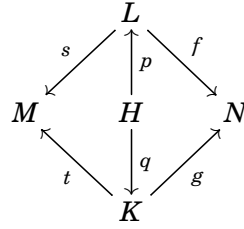
Our goal will now be to describe $\mathcal{A}[S^{-1}]$ given that S is a localizing class of morphisms in \mathcal{A} . Define a *left roof* between two objects M and N in \mathcal{A} to be a diagram of the form

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array}$$

where $s \in S$ and $f \in \text{Mor}(\mathcal{A})$. Two left roofs

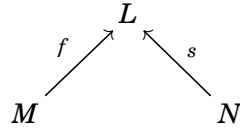
$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \quad \text{and} \quad \begin{array}{ccc} & K & \\ t \swarrow & & \searrow g \\ M & & N \end{array}$$

are said to be *equivalent* if there exists an object $H \in \mathcal{A}$ and morphisms $p: H \rightarrow L$ and $q: H \rightarrow K$ making the diagram

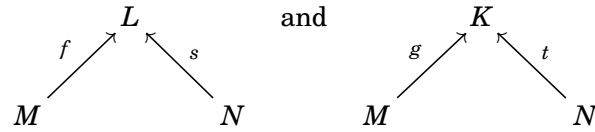


commute and $s \circ p = q \circ t \in S$. It can be checked that this is indeed an equivalence relation.

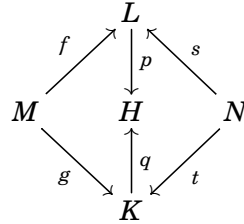
Analogously a *right roof* between two objects M and N in \mathcal{A} is defined to be a diagram of the form



where $s \in S$ and $f \in \text{Mor}(\mathcal{A})$. Clearly there is a natural bijection between the left roofs in \mathcal{A} and the right roofs in \mathcal{A}^{op} with respect to S and S^{op} respectively. Two right roofs

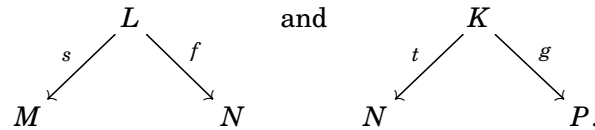


are said to be equivalent if there exists an object $H \in \mathcal{A}$ and morphisms $p: L \rightarrow H$ and $q: K \rightarrow H$ such that

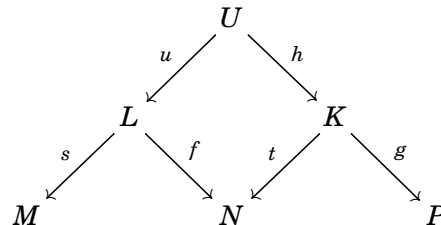


commutes. Again, it can be checked that this is indeed an equivalence relation. In fact, a shorter way to conclude this is to move from \mathcal{A} to \mathcal{A}^{op} since left roofs are mapped to right roofs. It is clear that two left roofs in \mathcal{A} are equivalent if and only if the corresponding right roofs in \mathcal{A}^{op} are equivalent.

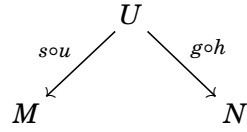
Next, we show how to “compose” two equivalence classes of left roofs. Begin by considering two representatives



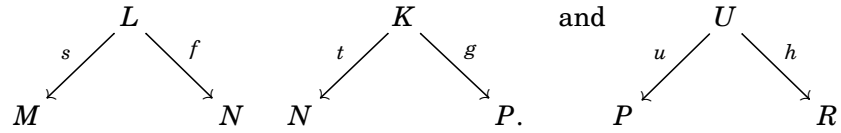
According to **(LC3)a**, we obtain a commutative diagram



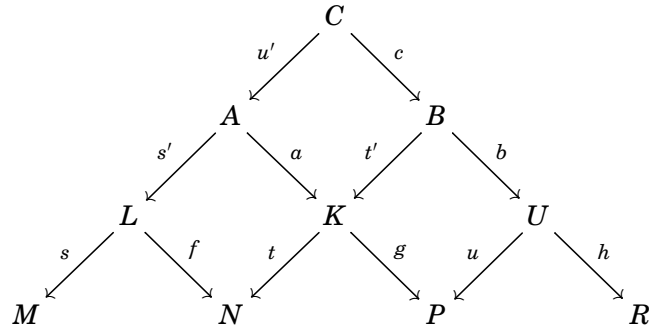
with $u \in S$. Define the composition of the aforementioned equivalence classes to be the left roof



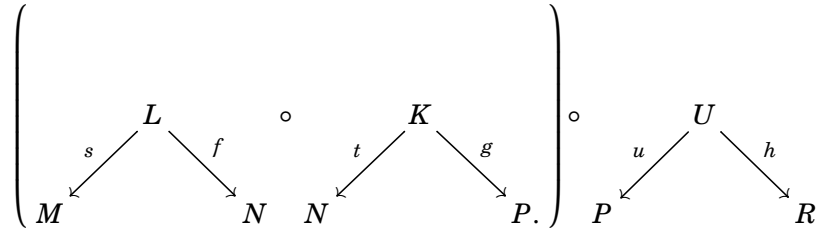
since $s \circ u \in S$. It is a bit tedious, but it can be checked that this “composition” is well-defined. Once this is done, it is clear that the “composition” must be associative. That is, given three representatives



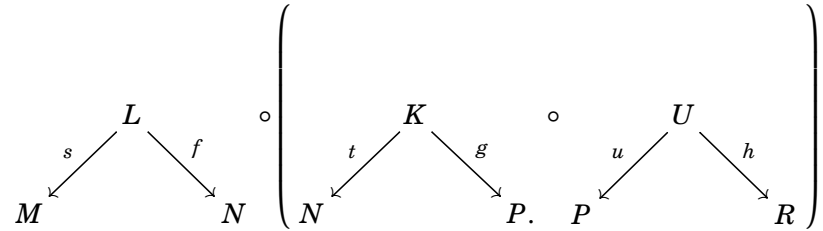
using 2 **(LC3)** α repeatedly, we can complete this to a commutative diagram



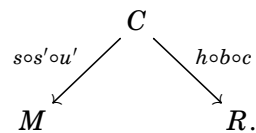
and so it is clear that either composition



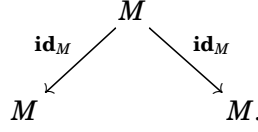
or



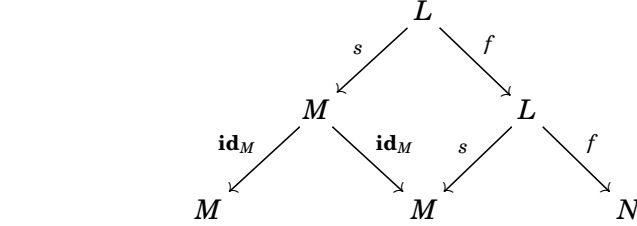
is equal to the equivalence class of the left roof



Finally, for each $M \in \mathcal{A}$, consider the left roof



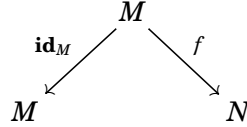
For any left roof $\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array}$, one can compute their composition using the diagram



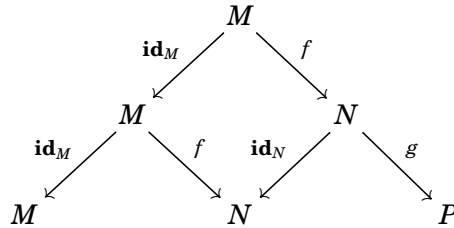
which yields the latter left roof.

Thus, we can define a category \mathcal{A}_S where $\text{ob}(\mathcal{A}_S) = \text{ob}(\mathcal{A})$, and $\text{Mor}_{\mathcal{A}_S}(M, N)$ is the set of equivalence classes of left roofs equipped with composition and identity maps as defined above.

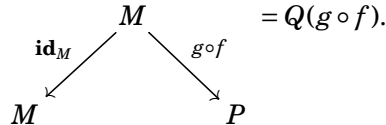
There is a natural functor $Q: \mathcal{A} \rightarrow \mathcal{A}_S$ which is the identity on objects and sends a morphism $f: M \rightarrow N$ in \mathcal{A} to the equivalence class of the left roof



in \mathcal{A}_S . Indeed, it clearly takes id_M to the roof representing the identity at M in \mathcal{A}_S ; further, if $M \xrightarrow{f} N \xrightarrow{g} P$ are two composable morphisms, then we have a commutative diagram



so that the composition of the bottom two left roofs is



Thus Q is indeed a functor. Finally, we claim that the pair (Q, \mathcal{A}_S) has the universal property of localization. Indeed, let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor sending every $s \in S$ to an isomorphism $F(s)$ in \mathcal{B} . Define a functor $G: \mathcal{A}_S \rightarrow \mathcal{B}$ such that

$$G(A) = F(A) \quad \text{for every object } A \in \mathcal{A},$$

and

$$G \left(\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \right) = F(f) \circ F(s)^{-1}.$$

It is easily checked that this is well-defined on the equivalence class of left roofs. That G is a functor is also a trivial verification, and by construction, $F = G \circ Q$.

Finally, we must show that such a G is unique. Indeed, if $F = G \circ Q$, for each object $A \in \mathcal{A}$, we must have $F(A) = G(Q(A)) = G(A)$. Now, a left roof

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array}$$

can be decomposed as the composition

$$\begin{array}{ccccc} & L & & L & \\ s \swarrow & & \text{id}_L \searrow & \text{id}_L \swarrow & \searrow f \\ M & & L & & N \end{array}$$

which is easy to see by completing the diagram above by putting an L at the peak and identity morphisms from it to both the L 's below it. But note that

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow \text{id}_L \\ M & & L \end{array} \quad \text{is the inverse of} \quad \begin{array}{ccc} & L & \\ \text{id}_L \swarrow & & \searrow s \\ L & & M \end{array}$$

so that

$$G \left(\begin{array}{ccc} & L & \\ s \swarrow & & \searrow \text{id}_L \\ M & & L \end{array} \right) = G \left(\begin{array}{ccc} & L & \\ \text{id}_L \swarrow & & \searrow s \\ L & & M \end{array} \right)^{-1} = F(s)^{-1},$$

and hence

$$G \left(\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \right) = F(f) \circ F(s)^{-1},$$

which completes the proof of uniqueness. We have therefore shown:

THEOREM 1.3. Let \mathcal{A} be a category and S be a localizing class of morphisms in \mathcal{A} . Then the functor $Q: \mathcal{A} \rightarrow \mathcal{A}_S$ as described above is the localization of the category \mathcal{A} at S .

§§ Localization and Subcategories

THEOREM 1.4. Let \mathcal{A} be a category, $\mathcal{B} \subseteq \mathcal{A}$ a full subcategory, and S a localizing class of morphisms in \mathcal{A} . Suppose

(LS1) $S_{\mathcal{B}} = S \cap \text{Mor}(\mathcal{B})$ is a localizing class in \mathcal{B} , and

(LS2) for each morphism $s: N \rightarrow M$ in S with $M \in \mathcal{B}$, there exists a morphism $u: P \rightarrow N$ with $P \in \mathcal{B}$ such that $s \circ u \in S$.

Then the induced functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \rightarrow \mathcal{A}[S^{-1}]$ is fully faithful.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad} & \mathcal{A} \\ Q_B \downarrow & & \downarrow Q_A \\ \mathcal{B}[S_{\mathcal{B}}^{-1}] & \longrightarrow & \mathcal{A}[S^{-1}] \end{array}$$

Proof. Since $S_{\mathcal{B}}$ is a localizing class in \mathcal{B} , by tracing the arrows in the commutative diagram of functors above, the map $\mathcal{B}[S_{\mathcal{B}}^{-1}] \rightarrow \mathcal{A}[S^{-1}]$ explicitly sends a roof in \mathcal{B} to the equivalence class of the same roof in $\mathcal{A}[S^{-1}]$.

First, we show that the map is full. Let

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array}$$

be a left roof in $\mathcal{A}[S^{-1}]$ with $M, N \in \mathcal{B}$. Then due to (LS2), there exists $U \in \mathcal{B}$ and a morphism $u: U \rightarrow L$ such that $s \circ u \in S$, and hence in $S_{\mathcal{B}}$.

To see that the map is faithful, suppose two left roofs

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \quad \text{and} \quad \begin{array}{ccc} & K & \\ t \swarrow & & \searrow g \\ M & & N \end{array}$$

in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ are equivalent in $\mathcal{A}[S^{-1}]$, that is, there exists an object $H \in \mathcal{A}$, and morphisms $p: H \rightarrow L$ and $q: H \rightarrow K$ in \mathcal{A} such that

$$\begin{array}{ccccc} & & L & & \\ & s \swarrow & \uparrow p & \searrow f & \\ M & & H & & N \\ & t \swarrow & \downarrow q & \searrow g & \\ & & K & & \end{array}$$

commutes and $s \circ p = t \circ q \in S$. Hence, there exists an object $U \in \mathcal{B}$ and a morphism $u: U \rightarrow H$ such that $s \circ p \circ u = t \circ q \circ u \in S$, and hence in $S_{\mathcal{B}}$. Thus, the diagram

$$\begin{array}{ccccc} & & L & & \\ & s \swarrow & \uparrow p \circ u & \searrow f & \\ M & & U & & N \\ & t \swarrow & \downarrow q \circ u & \searrow g & \\ & & K & & \end{array}$$

commutes and consists of morphisms in \mathcal{B} . Thus, the two roofs are equivalent in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$. ■

§§ Localizing Additive Categories

We begin by showing that one can “take common denominators” for morphisms in $\mathcal{A}[S^{-1}]$.

LEMMA 1.5. Let \mathcal{A} be a category (not necessarily additive) and S a localizing class of morphisms in \mathcal{A} . Let

$$\begin{array}{ccc} & L_i & \\ s_i \swarrow & & \searrow f_i \\ M & & N \end{array}$$

be left roofs in \mathcal{A} representing morphisms $\varphi_i: M \rightarrow N$ in $\mathcal{A}[S^{-1}]$ for $1 \leq i \leq n$ respectively. Then there exists an object $L \in \mathcal{A}$ and morphisms $L \xrightarrow{s} M \in S$, and $g_i: L \rightarrow N$ for $1 \leq i \leq n$ such that

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow g_i \\ M & & N \end{array}$$

represents φ_i for $1 \leq i \leq n$.

Proof. We prove this by induction on n . The base case $n = 1$ is trivial. Suppose now that $n > 1$ and that the statement has been proven for $n - 1$. Hence, there exists an object K and a morphism $K \xrightarrow{t} M \in S$ such that

$$\begin{array}{ccc} & K & \\ t \swarrow & & \searrow h_i \\ M & & N \end{array}$$

represents φ_i for $1 \leq i \leq n - 1$. Using 2 (LC3)a, there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{v} & L_n \\ u \downarrow & & \downarrow s_n \\ K & \xrightarrow{t} & M \end{array}$$

with $u \in S$. Set $s = s_n \circ v = t \circ u \in S$. Then the diagram

$$\begin{array}{ccccc} & & K & & \\ & t \swarrow & & \searrow h_i & \\ M & & U & & N \\ & \swarrow s & \parallel & \nearrow h_i \circ u & \\ & & U & & \end{array}$$

commutes for $1 \leq i \leq n - 1$, and

$$\begin{array}{ccccc} & & L_n & & \\ & s_n \swarrow & & \searrow f_n & \\ M & & U & & N \\ & \swarrow s_n \circ v & \parallel & \nearrow f_n \circ v & \\ & & U & & \end{array}$$

commutes with $s_n \circ v = s \in S$. Set $g_i = h_i \circ u$ for $1 \leq i \leq n-1$ and $g_n = f_n \circ v$; then

$$\begin{array}{ccc} & U & \\ s \swarrow & & \searrow g_i \\ M & & N \end{array}$$

represents φ_i for $1 \leq i \leq n$, thereby completing the proof. \blacksquare

Now let \mathcal{A} be an *additive category* and S a localizing class of morphisms in \mathcal{A} . We shall show that $\mathcal{A}[S^{-1}]$ is naturally an additive category. For objects $M, N \in \mathcal{A}[S^{-1}]$ and morphisms

$$\varphi = \begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \quad \text{and} \quad \psi = \begin{array}{ccc} & K & \\ t \swarrow & & \searrow g \\ M & & N \end{array}$$

in $\mathcal{A}[S^{-1}]$; using Lemma 1.5, we can find an object U and morphisms $U \xrightarrow{u} M \in S$ and $f', g': U \rightarrow N$ such that

$$\varphi = \begin{array}{ccc} & U & \\ u \swarrow & & \searrow f' \\ M & & N \end{array} \quad \text{and} \quad \psi = \begin{array}{ccc} & U & \\ u \swarrow & & \searrow g' \\ M & & N. \end{array}$$

Define

$$\varphi + \psi = \begin{array}{ccc} & U & \\ u \swarrow & & \searrow f' + g' \\ M & & N. \end{array}$$

Note that there are three choices being made here: the choice of the representatives for φ and ψ , and choice of “common denominator” for both morphisms. It follows that $\text{Mor}_{\mathcal{A}[S^{-1}]}(M, N)$ has the structure of an abelian group. Further, it must be checked that

$$\chi \circ (\varphi + \psi) = \chi \circ \varphi + \chi \circ \psi \quad \text{and} \quad (\varphi + \psi) \circ \chi = \varphi \circ \chi + \psi \circ \chi$$

for suitably composable morphisms χ, φ, ψ in $\mathcal{A}[S^{-1}]$. The zero object in $\text{Mor}_{\mathcal{A}[S^{-1}]}(M, N)$ is given by the morphism

$$\begin{array}{ccc} & M & \\ \text{id}_M \swarrow & & \searrow 0 \\ M & & N \end{array}.$$

Finally, given objects $M, N \in \mathcal{A}[S^{-1}]$, define their direct sum/direct product to be the object $M \oplus N$ where the direct sum is taken in \mathcal{A} , and the canonical projections and injections are the images of those in \mathcal{A} . Again, it is straightforward, but must be checked, that these have the desired universal properties. In this way, $\mathcal{A}[S^{-1}]$ has been given a natural additive structure.

Finally, note that the localization functor $Q: \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$ is an additive functor. Indeed, if $f, g: M \rightarrow N$ are morphisms, then

$$Q(f) = \begin{array}{ccc} & M & \\ \text{id}_M \swarrow & & \searrow f \\ M & & N \end{array} \quad \text{and} \quad Q(g) = \begin{array}{ccc} & M & \\ \text{id}_M \swarrow & & \searrow g \\ M & & N \end{array}$$

so that by definition,

$$Q(f) + Q(g) = \begin{array}{ccc} & M & \\ \text{id}_M \swarrow & & \searrow f+g \\ M & & N \end{array} = Q(f + g).$$

Finally, we have

THEOREM 1.6. Let \mathcal{A} be an additive category and S a localizing class of morphisms in \mathcal{A} . Then the category $\mathcal{A}[S^{-1}]$ is naturally an additive category and the localizing functor $Q: \mathcal{A} \rightarrow \mathcal{A}[S^{-1}]$ is additive.

Further, given any additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ such that $F(s)$ is an isomorphism in \mathcal{B} for each $s \in S$, there exists a unique additive functor $G: \mathcal{A}[S^{-1}] \rightarrow \mathcal{B}$ making

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ Q \downarrow & \nearrow G & \\ \mathcal{A}[S^{-1}] & & \end{array}$$

commute.

Proof. We have already proved the first part of the theorem. As for the second part, suppose $\varphi, \psi: M \rightarrow N$ are two morphisms in $\mathcal{A}[S^{-1}]$. Using Lemma 1.5, we may suppose that they are represented by

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array} \quad \text{and} \quad \begin{array}{ccc} & L & \\ s \swarrow & & \searrow g \\ M & & N \end{array}$$

respectively. As a result,

$$G(\varphi + \psi) = F(f + g)F(s)^{-1} = F(f)F(s)^{-1} + F(g)F(s)^{-1} = G(\varphi) + G(\psi),$$

so that G is an additive functor. That G is unique has already been argued. ■

LEMMA 1.7. Let $\varphi: M \rightarrow N$ be a morphism in $\mathcal{A}[S^{-1}]$ represented by a left roof

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ M & & N \end{array}.$$

Then the following are equivalent:

- (1) $\varphi = 0$.
- (2) There exists $t \in S$ such that $t \circ f = 0$.
- (3) There exists $t \in S$ such that $f \circ t = 0$.

Proof. Clearly (2) and (3) are equivalent due to (LC4). Now if $\varphi = 0$, then $Q(f) \circ Q(s)^{-1} = 0$, so that $Q(f) = 0$, i.e.,

$$\begin{array}{ccc} & L & \\ \text{id}_L \swarrow & & \searrow f \\ L & & N \end{array}$$

represents 0. Hence, there exists an object H and morphisms $p, q: H \rightarrow L$ such that

$$\begin{array}{ccccc}
 & & L & & \\
 & \swarrow \text{id}_L & \uparrow p & \searrow f & \\
 L & & H & & N \\
 & \swarrow \text{id}_L & \downarrow q & \searrow 0 & \\
 & & L & &
 \end{array}$$

commutes and $p = q \in S$. The commutativity implies $f \circ p = 0$, so that (1) \implies (2).

Conversely, suppose $f \circ t = 0$ for some $t: H \rightarrow L \in S$. Then the diagram

$$\begin{array}{ccccc}
 & & L & & \\
 & \swarrow s & \uparrow t & \searrow f & \\
 M & & H & & N \\
 & \swarrow \text{id}_M & \downarrow s \circ t & \searrow 0 & \\
 & & M & &
 \end{array}$$

commutes with $s \circ t \in S$. This shows that $\varphi = 0$, thereby completing the proof. ■

COROLLARY 1.8. Let M be an object in \mathcal{A} . Then the following are equivalent:

- (1) $Q(M) = 0$.
- (2) There exists an object $N \in \mathcal{A}$ such that the zero morphism $N \xrightarrow{0} M$ is in S .
- (3) There exists an object $N \in \mathcal{A}$ such that the zero morphism $M \xrightarrow{0} N$ is in S .

Proof. The equivalence of (2) and (3) follows from an immediate application of 2 (LC3)a and 2 (LC3)b. Now if $Q(M) = 0$, then $Q(\text{id}_M) = 0$, so that by Lemma 1.7 there exists $s \in S$ with $\text{id}_M \circ s = 0$, and hence $s = 0$. This proves (2).

Conversely, if there is an object $N \in \mathcal{A}$ with $N \xrightarrow{0} M \in S$, then the image of this map, which is the zero map $Q(N) \xrightarrow{0} Q(M)$ must be an isomorphism. Thus $Q(N) = Q(M) = 0$. ■

LEMMA 1.9. Let $f: M \rightarrow N$ be a morphism in \mathcal{A} . Then

- (1) If f is monic, then so is $Q(f)$.
- (2) If f is epic, then so is $Q(f)$.

Proof. ■

§§ Localization of Abelian Categories