Completions

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§1 Graded and Filtered Objects

DEFINITION 1.1. Let $(G, +)$ be an Abelian monoid with identity element $0 \in G$. A *G-graded* ring is a ring R together with a direct sum decomposition

$$R = \bigoplus_{i \in G} R_i$$

into additive subgroups, such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in G$.

Similarly, a *graded R-module* is an R-module M together with a direct sum decomposition

$$M = \bigoplus_{i \in G} M_i$$

into additive subgroups, such that $R_i M_j \subseteq M_{i+j}$ for all $i, j \in G$. An element $x \in M$ is said to be *homogeneous* if $x \in M_i$ for some $i \in G$.

Note that any element $x \in M$ can be written uniquely as a sum

$$x = \sum_{i \in G} x_i,$$

where $x_i \in M_i$ for all $i \in G$. The x_i's are called the *homogeneous components* of x. A submodule $N \subseteq M$ is said to be *homogeneous* if it can be generated by homogeneous elements in M.

A homomorphism $f : M \to N$ of graded R-modules is said to be *graded of degree $d \in G$* if $f(M_i) \subseteq N_{i+d}$ for all $i \in G$. A graded homomorphism of degree 0 is said to be just *graded*.

PROPOSITION 1.2. Let $M = \bigoplus_{i \in G} M_i$ be a G-graded R-module and $N \subseteq M$ a submodule. The following are equivalent:

(1) N is a homogeneous submodule of M.

(2) For all $x \in N$, each homogeneous component of x lies in N.

(3) $N = \sum_{i \in G} (N \cap M_i)$.

Proof. (1) $\implies$ (2) Suppose N is generated as an R-module by the homogeneous elements $\{z_j\}$. Then we can write

$$x = \sum_j r_j z_j = \sum_j \left(\sum_{g \in G} r_j^g z_j \right),$$

where we can decompose r_j as

$$r_j = \sum_{g \in G} r_j^g$$

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into its homogeneous components in R. Grouping together components of the same degree shows that every homogeneous component of x lies in N.

(2) $\Longrightarrow$ (3) Clearly, the right hand side is contained in the left hand side. Conversely, if $x \in N$, then we can write $x = \sum_{i \in G} x_i$ where the x_i's are the homogeneous components of x. According to (2), $x_i \in N \cap M_i$ for each $i \in G$, whence x is contained in the right hand side, as desired.

(3) $\Longrightarrow$ (1) Indeed, N is generated as an R-module by the set

$$\bigcup_{i \in G} (N \cap M_i)$$

consisting only of homogeneous elements. ∎

PROPOSITION 1.3. Let $R = \bigoplus_{i \in G} R_i$ be a G-graded ring. Then R_0 is a subring of R and for every graded R-module $M = \bigoplus_{i \in G} M_i$, each M_i is naturally an R_0-module.

Proof. R_0 is an additive subgroup of R and $R_0 R_0 \subseteq R_0$. Thus it suffices to show that $1 \in R_0$. We can decompose x into its homogeneous components as

$$1 = \sum_{i \in G} x_i.$$

For any $j \in G$, we then have

$$x_j = \sum_{i \in G} x_i x_j,$$

where $x_i x_j$ is homogeneous of degree $i + j$. Therefore,

$$x_i x_j = \begin{cases} x_j & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Summing over all $j \in G$, we get

$$x_0 = \sum_{j \in G} x_0 x_j = \sum_{j \in G} x_j = 1.$$

That is, $1 \in R_0$, and hence R_0 is a subring of R. Finally, since $R_0 M_i \subseteq M_i$, it follows that each M_i is naturally an R_0-module. ∎

REMARK 1.4 (QUOTIENT OF GRADED MODULES). Let $M = \bigoplus_{i \in G} M_i$ be a graded module over a graded ring $R = \bigoplus_{i \in G} R_i$, and let $N \subseteq M$ be a graded submodule of M. We can endow the quotient module M/N with a natural grading

$$\bigoplus_{i \in G} M_i / N_i,$$

where the R-module structure is the obvious one. To see that this is indeed isomorphic to M/N as an R-module, consider the graded projection $\pi \colon M \to M/N$ given by

$$\pi \left(\sum_{i \in G} x_i \right) = \sum_{i \in G} x_i \bmod N_i.$$

One can check that this is an R-linear surjective homomorphism with $\ker \pi = N$, which implies the desired conclusion.

Analogously, if $I \lhd R$ is a graded ideal, then R/I is naturally a graded R-module as above, and has a ring structure given by

$$(r_i \bmod I_i)(r_j \bmod I_j) = r_i r_j \bmod I_{i+j}.$$

Henceforth, R/I shall always be thought of a graded ring with the above grading.

Throughout this article, we shall mainly concern ourselves with the case $G = (\mathbb{N}, +, 0)$, and henceforth, a graded ring/module shall refer to an $\mathbb{N}$-graded ring/module. For an $\mathbb{N}$-graded ring $R = \bigoplus_{n \geqslant 0} R_n$, we set

$$R_+ = \bigoplus_{n \geqslant 1} R_n,$$

which is clearly a homogeneous ideal in R and is called the *irrelevant ideal* of R. Often when R_0 is a field, then the irrelevant ideal turns out to be the unique *graded* maximal ideal and is denoted by $\mathfrak{m}_+$ for empasis.

THEOREM 1.5 (GRADED NAKAYAMA). Let $R = \bigoplus_{n \geqslant 0} R_n$ be a graded ring, and $M = \bigoplus_{n \geqslant 0} M_n$ a graded R-module. If $R_+ M = M$, then $M = 0$.

Proof. Let $n \geqslant 0$ be the smallest non-negative integer such that $M_n \neq 0$. Let $0 \neq x_n \in M_n$. Using the fact that $R_+ M = M$, we can write

$$x_n = \sum_\lambda r^\lambda y^\lambda,$$

for some finite set of $r^\lambda \in R_+$ and $y^\lambda \in M$. Writing out each r^λ and y^λ in its homogeneous components and isolating terms of degree n, we get

$$x_n = \sum_\lambda \left(\sum_{i+j=n} r_i^\lambda y_j^\lambda \right).$$

But since $r^\lambda \in R_+$, if $r_i^\lambda \neq 0$ then $i \geqslant 1$, so that $j \leqslant n - 1$, and hence $y_j^\lambda = 0$. Thus $x_n = 0$, a contradiction. This completes the proof. ∎

DEFINITION 1.6. A *filtered ring* is a ring R together with a descending chain of additive subgroups

$$R = R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots,$$

such that $R_n R_m \subseteq R_{n+m}$ for all $n, m \geqslant 0$. In particular, each R_n is an ideal in the ring R.

Let R be a filtered ring as above. A *filtered module* over R is an R-module M together with a descending chain of R-submodules of M

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

such that $R_m M_n \subseteq M_{m+n}$ for all $m, n \geqslant 0$.

A map $f : M \to N$ between filtered R-modules is said to be a filtered homomorphism if $f(M_n) \subseteq N_n$ for all $n \geqslant 0$.

REMARK 1.7. Let M be a filtered module over a filtered ring R as above. For an R-submodule $N \subseteq M$, we define the *induced filtration* on N as $N_n = N \cap M_n$ for all $n \geqslant 0$. Similarly, we define the induced filtration on M/N as

$$\left(\frac{M}{N} \right)_n = \frac{N + M_n}{N}.$$

Equipped with these filtrations, every map in the short exact sequence

$$0 \to N \hookrightarrow M \twoheadrightarrow M/N \to 0$$

is a filtered homomorphism.

DEFINITION 1.8. Let R be a filtered ring and M a filtered R-module as above. Define the *associated graded ring*

$$\mathrm{gr}(R) = \bigoplus_{n \geqslant 0} R_n/R_{n+1}$$

with product structure given by

$$(x + R_{n+1})(y + R_{m+1}) = xy + R_{n+m+1}$$

for all $n, m \geqslant 0$. It is easy to check that $\mathrm{gr}(R)$ is a graded ring.

We further define the *associated graded module*

$$\mathrm{gr}(M) = \bigoplus_{n \geqslant 0} M_n/M_{n+1}$$

which is a graded $\mathrm{gr}(R)$-module with the module structure given by

$$(a + R_{m+1}) \cdot (x + M_{n+1}) = a \cdot x + M_{m+n+1}.$$

If N is another filtered R-module and $f : M \to N$ a filtered homomorphism, then there is an induced graded $\mathrm{gr}(R)$-homomorphism $\mathrm{gr}(f) : \mathrm{gr}(M) \to \mathrm{gr}(N)$ given by

$$\mathrm{gr}(f)(x + M_{n+1}) = f(x) + N_{n+1}.$$

We note that gr, as defined above, is a functor from the category of filtered R-modules to the category of graded $\mathrm{gr}(R)$-modules. Indeed, it is trivial to check that $\mathrm{gr}(\mathbf{id}_M) = \mathbf{id}_{\mathrm{gr}(M)}$ and that $\mathrm{gr}(g \circ f) = \mathrm{gr}(g) \circ \mathrm{gr}(f)$.

PROPOSITION 1.9. Let R be a filtered ring, M and N filtered R-modules, and $f : M \to N$ a filtered homomorphism. If

(i) $\mathrm{gr}(f) : \mathrm{gr}(M) \to \mathrm{gr}(N)$ is injective, and

(ii) $\bigcap_{n \geqslant 0} M_n = 0$,

then f is injective.

Proof. Since $\mathrm{gr}(f)$ is injective, the map $\mathrm{gr}_n(f) : M_n/M_{n+1} \to N_n/N_{n+1}$ is injective for every $n \geqslant 0$. We shall first show by induction on $n \geqslant 0$ that $f^{-1}(N_n) \subseteq M_n$. Clearly $f^{-1}(N_0) \subseteq M_0$. As for the inductive step, note that

$$f^{-1}(N_{n+1}) \subseteq f^{-1}(N_n) \subseteq M_n.$$

Hence,

$$f^{-1}(N_{n+1}) \subseteq f^{-1}(N_{n+1}) \cap M_n \subseteq M_{n+1},$$

where the last containment follows from the fact that $\mathrm{gr}_n(f)$ is injective. As a result,

$$f^{-1}(0) \subseteq f^{-1}\left(\bigcap_{n \geqslant 0} N_n\right) \subseteq \bigcap_{n \geqslant 0} M_n = 0,$$

thereby completing the proof. ∎

THEOREM 1.10. Let $R = \bigoplus_{n \geqslant 0} R_n$ be a graded ring. The following are equivalent:

(1) R is Noetherian.

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(2) R_0 is Noetherian and R is a finitely generated R_0-algebra.

Proof. Clearly (2) $\implies$ (1) due to the Hilbert Basis Theorem. Suppose now that R is Noetherian. Since

$$R_0 \cong R/R_+,$$

so is R_0. Now, R_+ is a finitely generated ideal in R and we may assume that it is generated by homogeneous elements $x_1, \ldots, x_r \in R_+$ of degrees $n_1, \ldots, n_r$ respectively. Let $R' = R_0[x_1, \ldots, x_n] \subseteq R$. We shall show by induction on $n \geq 0$ that $R_n \subseteq R'$. Trivially $R_0 \subseteq R'$. Suppose it is known that $R_k \subseteq R'$ for all $k < n$. An element $x \in R_n$ can be written as an R-linear combination of $x_1, \ldots, x_r$ as

$$x = a_1 x_1 + \cdots + a_r x_r.$$

Since x is a homogeneous element, breaking each a_i into its homogeneous components and grouping terms of the same degree, we may suppose that each a_i is graded of degree $n - n_i$, with the convention that $R_d = 0$ for $d < 0$. Since $n_i \geq 1$ for all $1 \leq i \leq r$, it follows that $a_i \in R'$ for all $1 \leq i \leq r$. Hence $x \in R'$, thereby completing the proof. ∎

DEFINITION 1.11. Let $M = (M_n)_{n \geq 0}$ be a filtered R-module and I an ideal in R. The filtration is said to be an *I-filtration* if $IM_n \subseteq M_{n+1}$ for all $n \geq 0$. Further, an I-filtration is said to be *I-stable* if $IM_n = M_{n+1}$ for all $n \gg 0$.

Let M be a filtered R-module with an I-filtration. We define the *Rees algebra* of R as a subring of the polynomial algebra

$$R^* = \bigoplus_{n \geq 0} I^n T^n \subseteq R[T].$$

That is, R^* consists of all polynomials $a_0 + a_1 T + \cdots + a_n T^n \in R[T]$ such that $a_i \in I^i$ for all $i \geq 0$. Note that if R is Noetherian, then I is a finitely generated ideal, say $I = (a_1, \ldots, a_r)$. Then R^* is precisely the ring

$$R^* = R[a_1 T, \ldots, a_r T] \subseteq R[T],$$

and in particular, is a Noetherian ring.

Similarly, we define an R^*-module

$$M^* = \bigoplus_{n \geq 0} M_n T^n$$

whose elements are formal sums

$$\sum_{\substack{n \geq 0 \\ \text{finite}}} x_n T^n,$$

with the obvious module structure over R^*.

THEOREM 1.12. Let R be a filtered Noetherian ring, I an ideal of R, and M a finitely generated filtered R-module equipped with an I-filtration. The following are equivalent:

(1) The filtration on M is I-stable.

(2) M^* is a finitely generated R^*-module.

Proof. Set
$$M_n^* = M_0 \oplus M_1 T \oplus \cdots \oplus M_n T^n \oplus I M_n T^{n+1} \oplus I^2 M_n T^{n+2} \oplus \cdots,$$
which is clearly an R^*-module. Further, since each M_n is a finite R-module, we can choose a finite R-generating set for the module $M_0 \oplus \cdots \oplus M_n$, which would then be an R^*-generating set for M_n^*. That is, each M_n^* is a finite R^*-module.

Note that the filtration on M being I-stable is equivalent to the ascending chain $(M_n^*)_{n \geqslant 0}$. We also have
$$M^* = \bigcup_{n \geqslant 0} M_n^*.$$
Thus, if the chain stabilizes, then M^* is a finite R^*-module. Conversely, if M^* is a finite R^*-module, then M^* is Noetherian, and hence the chain must stabilize. This completes the proof. ∎

LEMMA 1.13 (ARTIN-REES). Let R be a filtered Noetherian ring, and M a finitely generated filtered R-module equipped with an I-stabie flitration. If N is a submodule of M, then the induced filtration on N is I-stable.

Proof. Let $N_n = N \cap M_n$, which is the induced filtration on N. Clearly this filtration is I-stable. We shall treat N^* as a natural R^*-submodule of M^*. Since the filtration on M is I-stable, due to Theorem 1.12, M^* is a finite R^*-module, but since R^* is Noetherian, N^* is also a finite R^*-module, so that by Theorem 1.12, the filtration on N is I-stable. ∎

THEOREM 1.14 (KRULL INTERSECTION THEOREM). Let R be a Noetherian ring, I an ideal of R, and M a finite R-module. Then the module
$$\bigcap_{n \geqslant 0} I^n M$$
consists of precisely those elements that are annihilated by some element in $1 + I$.

Proof. Let
$$N = \bigcap_{n \geqslant 0} I^n M,$$
which is a submodule of M. In view of Lemma 1.13, this filtration is I-stable. That is, for $n \gg 0$,
$$IN = I\left(N \cap I^n M\right) = N \cap I^{n+1} M = N.$$
By Nakayama's lemma, N is annihilated by an element of the form $1 + a$, where $a \in I$.

Conversely, suppose $x \in M$ is such that $(1+a)x = 0$ for some $a \in I$. Then
$$x = -ax = a^2 x = -a^3 x = \cdots,$$
and hence $x \in \bigcap_{n \geqslant 0} I^n M = N$. ∎

COROLLARY 1.15. If R is a Noetherian ring, I an ideal of R contained in the Jacobson radical, and M a finite R-module, then
$$\bigcap_{n \geqslant 0} I^n M = 0.$$

Proof. This follows from the fact that every element in $1 + I$ is invertible. ∎

COROLLARY 1.16. Let R be a Noetherian domain, and I an ideal in R. Then
$$\bigcap_{n \geqslant 0} I^n = 0.$$

Proof. This follows from the fact that every element in $1 + I$ is a non-zerodivisor. ∎

§2 Completions

We first recall the theory of inverse limits. The category of R-modules is complete, that is, it admits all limits and colimits. Consequently, inverse limits[1] always exist. We shall mainly concern ourselves with $\mathbb{N}$-indexed inverse systems.

Consider an inverse system of the form

$$\cdots \to M_n \xrightarrow{\theta_n} M_{n-1} \to \cdots \to M_1 \xrightarrow{\theta_1} M_0.$$

The *inverse limit* of this system is, by definition, an R-module M together with R-linear maps $f_i \colon M \to M_i$ for all $i \geqslant 0$ such that $\theta_n \circ f_n = f_{n-1}$ for all $n \geqslant 1$ having the following universal property: Given any R-module N with R-linear maps $g_i \colon N \to M_i$ such that $\theta_n \circ g_n = g_{n-1}$ for all $n \geqslant 1$, there exists a unique R-linear map $\varphi \colon N \to M$ such that $f_n \circ \varphi = g_n$ for all $n \geqslant 0$.

The inverse limit of the above inverse system is often denoted by

$$\varprojlim_n M_n.$$

PROPOSITION 2.1. The inverse limit exists.

Proof. Set

$$M = \left\{ (x_n)_{n \geqslant 0} \in \prod_{n \geqslant 0} M_n : \theta_n(x_n) = x_{n-1} \text{ for all } n \geqslant 1 \right\}.$$

It is straightforward to check that M has the desired universal property. ∎

INTERLUDE 2.2 (COMPLETION THROUGH CAUCHY SEQUENCES). Let M be an R-module equipped with a filtration $(M_n)_{n \geqslant 0}$. Define the *completion* of M to be the equivalence classes of Cauchy sequences $(x_n)_{n \geqslant 0}$ in M with

$$(x_n) \sim (y_n) \iff \forall \, m \geqslant 0, \, \exists \, n_0 \geqslant 0 \text{ such that } x_n - y_n \in M_m \quad \forall \, n \geqslant n_0.$$

That this is an equivalence relation is easy to check. Let $\widehat{M}$ denote the set of equivalence classes. There is a natural R-module structure on $\widehat{M}$ given by

$$r \cdot [(x_n)] = [(r \cdot x_n)].$$

This is clearly well-defined since each M_n is an R-module, in particular if $x_n - y_n \in M_m$, then $r x_n - r y_n \in M_m$ for all $r \in R$.

Next, given an R-module M equipped with a filtration $(M_n)_{n \geqslant 0}$, there is a natural inverse system

$$\cdots \to \frac{M}{M_n} \xrightarrow{\theta_n} \frac{M}{M_{n-1}} \to \cdots \to \frac{M}{M_1} \xrightarrow{\theta_1} \frac{M}{M_0} = 0.$$

For now, set $\widetilde{M} = \varprojlim_n M/M_n$. An element of $\widetilde{M}$ looks like $(x_n + M_n)_{n \geqslant 0}$ such that $x_n - x_{n-1} \in M_{n-1}$ for all $n \geqslant 1$. Therefore, $x_n - x_m \in M_m$ whenever $n \geqslant m$. Define a map $\alpha \colon \widetilde{M} \to \widehat{M}$ which sends

$$(x_n + M_n)_{n \geqslant 0} \longmapsto [(x_n)_{n \geqslant 0}].$$

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[1] which are just limits in a category

This map is well-defined, indeed, if $x_n \equiv y_n \bmod M_n$, then $x_n - y_n \in M_n$ for all $n \geqslant 0$, that is, $x_n - y_n \in M_m$ for all $m \geqslant n$, so that $(x_n) \sim (y_n)$. The map α is also clearly R-linear. We claim that α is an isomorphism.

Let $[(y_n)] \in \widehat{M}$. Set $x_0 = y_0$. Since this sequence is Cauchy, there exists an $n_1 \geqslant 0$ such that $y_m - y_n \in M_1$ for all $m, n \geqslant n_1$. Set $x_1 = y_{n_1}$. Again, there exists an $n_2 > n_1$ such that $y_m - y_n \in M_2$ for all $m, n \geqslant n_2$. Set $x_2 = y_{n_2}$ and continue this process. Then the sequence $(x_n)_{n \geqslant 0}$ is such that $x_{n+1} - x_n \in M_n$ for all $n \geqslant 0$. This is a subsequence of (y_n) and is also Cauchy. Further, $(x_n) \sim (y_n)$. Indeed, for any $m \geqslant 0$, there exists an $\ell \geqslant 0$ such that $y_p - y_q \in M_m$ for all $p, q \geqslant \ell$. For some $r \gg 0$, we have $n_s \geqslant \ell$, and hence, for all $s > \max\{r, \ell\}$,

$$x_s - y_s = y_{n_s} - y_s \in M_m.$$

This shows that α is surjective, since α maps $(x_n + M_n)$ to $[(y_n)]$. Next, to see that α is injective, suppose α sends $(x_n + M_n)_{n \geqslant 0}$ to 0 in $\widetilde{M}$, that is, $(x_n) \sim (0)$, i.e., for all $m \geqslant 0$, $x_n \in M_m$ for all $n \gg 0$. But by construction, we have $x_n - x_m \in M_m$ for all $n \geqslant m \geqslant 0$, and hence $x_m \in M_m$ for all $m \geqslant 0$. This shows that α is injective, and hence α is an isomorphism.

To summarize, we have proved:

THEOREM 2.3. Let $M = (M_n)_{n \geqslant 0}$ be a filtered R-module. Then the natural map $\alpha \colon \varprojlim_n M/M_n \to \widehat{M}$ defined above is an isomorphism of R-modules.

INTERLUDE 2.4 (FUNCTORIAL BEHAVIOR OF COMPLETION). Let $f \colon M \to N$ be an R-linear map between filtered R-modules $M = (M_n)_{n \geqslant 0}$ and $N = (N_n)_{n \geqslant 0}$. If $(x_n)_{n \geqslant 0}$ is a Cauchy sequence in M, then clearly $(f(x_n))_{n \geqslant 0}$ is a Cauchy sequence in N. Further, if $(x_n) \sim (y_n)$ in M, then $(f(x_n)) \sim (f(y_n))$ in N. This defines a map $\widehat{f} \colon \widehat{M} \to \widehat{N}$ sending

$$[(x_n)_{n \geqslant 0}] \longmapsto [(f(x_n))_{n \geqslant 0}].$$

This determines a functor from the category of filtered R-modules to the category of R-modules.

Similarly, f induces a morphism of inverse systems

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & M/M_n & \xrightarrow{\theta_n} & M/M_{n-1} & \longrightarrow \cdots \longrightarrow & M/M_1 & \xrightarrow{\theta_1} & M/M_0 \\
& \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\cdots \longrightarrow & N/N_n & \xrightarrow{\theta_n} & N/N_{n-1} & \longrightarrow \cdots \longrightarrow & N/N_1 & \xrightarrow{\theta_1} & N/N_0
\end{array}
$$

where $f_n \colon M/M_n \to N/N_n$ is the natural map taking $x + M_n \mapsto f(x) + N_n$ for all $n \geqslant 0$. The universal property furnishes a map

$$\widetilde{f} \colon \varprojlim_n M/M_n \to \varprojlim_n N/N_n$$

which sends $(x_n + M_n)_{n \geqslant 0} \longmapsto (f(x_n) + N_n)_{n \geqslant 0}$. This construction clearly defines a functor $\varprojlim$ from the category of filtered R-modules to the category of R-modules.

The isomorphism $\alpha \colon \varprojlim M/M_n \to \widehat{M}$ defined above is a natural isomorphism between these functors. Indeed, it remains to verify that for every filtered homomorphism $f \colon M \to N$, the diagram

$$
\begin{array}{ccc}
\widehat{M} & \xrightarrow{\widehat{f}} & \widehat{N} \\
\downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_N} \\
\varprojlim_n M/M_n & \xrightarrow{\widetilde{f}} & \varprojlim_n N/N_n
\end{array}
$$

commutes, but that is clear.

THEOREM 2.5. Let $\{(M_n'),(\theta_n')\}$, $\{(M_n),(\theta_n)\}$, and $\{(M_n''),(\theta_n'')\}$ be inverse systems of R-modules. Suppose there are short exact sequences

$$0 \to M_n' \xrightarrow{f_n} M_n \xrightarrow{g_n} M_n'' \to 0$$

such that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_{n+1}' & \xrightarrow{f_{n+1}} & M_{n+1} & \xrightarrow{g_{n+1}} & M_{n+1}'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \theta_{n+1}'} & & \downarrow{\scriptstyle \theta_{n+1}} & & \downarrow{\scriptstyle \theta_{n+1}''} & & \\
0 & \longrightarrow & M_n' & \xrightarrow{f_n} & M_n & \xrightarrow{g_n} & M_n'' & \longrightarrow & 0
\end{array}
$$

commutes for each $n \geq 0$.

  (1) The sequence

$$0 \to \varprojlim_n M_n' \xrightarrow{\tilde{f}} \varprojlim_n M_n \xrightarrow{\tilde{g}} \varprojlim_n M_n''$$

   is exact.

  (2) If θ_n' is surjective for all $n \geq 1$, then the sequence

$$0 \to \varprojlim_n M_n' \xrightarrow{\tilde{f}} \varprojlim_n M_n \xrightarrow{\tilde{g}} \varprojlim_n M_n'' \to 0$$

   is exact.

Proof. ∎

COROLLARY 2.6. Let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be a short exact sequence of R-modules, and let $\{M_n\}$ be a filtration on M, which induces natural filtrations on M' and M'', which are $\{M' \cap f^{-1}(M_n)\}$ and $\{g(M_n)\}$ respectively. Let $\widehat{M'}$, $\widehat{M}$, and $\widehat{M''}$ denote completions with respect to these filtrations. Then the sequence

$$0 \to \widehat{M'} \xrightarrow{\hat{f}} \widehat{M} \xrightarrow{\hat{g}} \widehat{M''} \to 0$$

is exact.

Proof. For each $n \geq 1$, the sequence of induced maps

$$0 \to \frac{M'}{M' \cap f^{-1}(M_n)} \xrightarrow{f_n} \frac{M}{M_n} \xrightarrow{g_n} \frac{M''}{g(M_n)} \to 0$$

is exact. Further, these induced maps clearly commute with the canonical projections in the inverse systems. Taking inverse limits and using Theorem 2.5 (2), it follows that

$$0 \to \varprojlim_n \frac{M'}{M' \cap f^{-1}(M_n)} \xrightarrow{\tilde{f}} \varprojlim_n \frac{M}{M_n} \xrightarrow{\tilde{g}} \varprojlim_n \frac{M''}{g(M_n)} \to 0$$

is exact. Using the natural isomorphism between the two completion functors as in Interlude 2.4, the sequence

$$0 \to \widehat{M'} \xrightarrow{\hat{f}} \widehat{M} \xrightarrow{\hat{g}} \widehat{M''} \to 0$$

is exact. ∎

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Let $M = (M_n)_{n \geqslant 0}$ be a filtered R-module. Let $n \geqslant 0$ and let $\widehat{M}_n$ denote the completion of M_n with respect to the filtration induced from M. In view of Corollary 2.6, the map $\iota : M_n \to M$ induces an injective map $\widehat{\iota} : : \widehat{M}_n \to \widehat{M}$. The image of $\widehat{\iota}$ is the set of all equivalence of Cauchy sequences containing one representative, all of whose elements lie in M_n. Identify this submodule of $\widehat{M}$ with $\widehat{M}_n$. Clearly $\widehat{M}_0 = \widehat{M}$, and therefore, we have obtained a filtration on $\widehat{M} = (\widehat{M}_n)_{n \geqslant 0}$.

CLAIM. There is a natural isomorphism $M/M_n \cong \widehat{M}/\widehat{M}_n$.

Consider the map $M \to \widehat{M}$ sending $x \in M$ to the (equivalence class of) the constant sequence $[(x)]$. This is clearly an R-module homomorphism and induces an R-linear map $\beta : M/M_n \to \widehat{M}/\widehat{M}_n$. We shall show that β is an isomorphism. Indeed, in view of Corollary 2.6, there is a short exact sequence

$$0 \to \widehat{M}_n \xrightarrow{\widehat{\iota}} \widehat{M} \xrightarrow{\widehat{\pi}} \widehat{\left(\frac{M}{M_n}\right)} \to 0$$

of R-modules where the completions are taken with respect to the induced filtrations. But note that the induced filtration on M/M_n is "trivial", that is, $(M_n + M_m)/M_n = 0$ for all $m \geqslant n$.

Thus, the completion $\widehat{M/M_n}$ is naturally isomorphic to M/M_n where each element of M/M_n is identified with the (equivalence class of) the constant Cauchy sequence corresponding to it. This proves the claim. $\square$

It is important to note that the map $\beta : M/M_n \to \widehat{M}/\widehat{M}_n$ defined above is natural, that is, the diagram

$$
\begin{array}{ccc}
\dfrac{M}{M_{n+1}} & \longrightarrow & \dfrac{M}{M_n} \\[2mm]
\beta_{n+1} \downarrow & & \downarrow \beta_n \\[2mm]
\dfrac{\widehat{M}}{\widehat{M}_{n+1}} & \longrightarrow & \dfrac{\widehat{M}}{\widehat{M}_n}
\end{array}
$$

commutes for all $n \geqslant 0$. This is trivial to verify. As a result, there is an induced isomorphism

$$\widehat{\widehat{M}} \cong \widehat{M},$$

where the former completion is taken with respect to the induced filtration $(\widehat{M}_n)_{n \geqslant 0}$. On must keep in mind that the isomorphism is induced by the β_n's. That is, it sends a Cauchy sequence (x_n) in $\widehat{M}$ to the Cauchy sequence consisting of constant Cauchy sequences corresponding to each x_n.

§3 I-adic filtrations

DEFINITION 3.1. Let R be a ring, I an ideal in R, and M an R-module. The *I-adic filtration* on M is defined as $M_n = I^n M$. Defining an analogous filtration $R_n = I^n$ on R, it follows that M is a filtered R-module. The completion of M with respect to this filtration is called the *I-adic filtration*.

We note that the I-adic completion of R as an R-module has a natural ring structure. Indeed, an element of $\widehat{R} = \varprojlim_n R/I^n$ looks like $(x_n + I^n)_{n \geqslant 0}$. We define

$$\left(x_n + I^n\right)_{n \geqslant 0} \cdot \left(y_n + I^n\right)_{n \geqslant 0} = \left(x_n y_n + I^n\right)_{n \geqslant 0}.$$

That this defines a valid ring structure is straightforward to check. Further, with this structure, the natural map $R \to \widehat{R}$ defined above is a ring homomorphism.

Next, if M is an R-module, then $\widehat{M} = \varprojlim_n M/I^n M$ is naturally a $\widehat{R}$-module with the module structure given by

$$\left(a_n + I^n\right)_{n \geqslant 0} \cdot \left(x_n + I^n M\right) = \left(a_n \cdot x_n + I^n M\right)_{n \geqslant 0}.$$

Note that although we have defined both ring and module structure above using inverse limits, we can make similar definitions using Cauchy sequences by moving back-and-forth through the isomorphism $\alpha \colon \varprojlim_n M/M_n \to \widehat{M}$ defined in the preceding section. In this case, the ring structure is

$$[(x_n)_{n \geqslant 0}] \cdot [(y_n)_{n \geqslant 0}] = [(x_n y_n)_{n \geqslant 0}],$$

and the module structure is

$$[(a_n)_{n \geqslant 0}] \cdot [(y_n)_{n \geqslant 0}] = [(a_n \cdot y_n)_{n \geqslant 0}].$$

Further, if $f \colon M \to N$ is R-linear, then $f(I^n M) \subseteq I^n N$ and hence, there is an induced map $\widehat{f} \colon \widehat{M} \to \widehat{N}$ between I-adic filtrations, and this map is easily checked to be $\widehat{R}$-linear. It follows that I-adic completion defines a functor from the category of R-modules to the category of $\widehat{R}$-modules.

DEFINITION 3.2. Let M be an R-module. Two filtrations $(M_n)_{n \geqslant 0}$ and $(M'_n)_{n \geqslant 0}$ are said to be *equivalent*, if there exists an integer $k \geqslant 0$ such that $M_{n+k} \subseteq M'_n$ and $M'_{n+k} \subseteq M_n$ for all $n \geqslant 0$.

Equipped with these two filtrations, there are two completions $\widehat{M}$ and $\widehat{M}'$ respectively. Let $(x_n)_{n \geqslant 0}$ be a Cauchy sequence in M with respect to the filtration (M_n). We claim that this is a Cauchy sequence in M with respect to the other filtration too. Indeed, let $r \geqslant 0$. Then there exists an $n_0 \geqslant 0$ such that for all $m, n \geqslant n_0$, $x_m - x_n \in M_{r+k} \subseteq M'_r$. Furthermore, if $(x_n) \sim (y_n)$ are two equivalent Cauchy sequences with respect to the former filtration, then similarly it follows that they are equivalent under the latter filtration. Thus, we have defined a map $\widehat{M} \to \widehat{M}'$ which is "identity" on the Cauchy sequences. This is clearly R-linear. Analogously, there is a map $\widehat{M}' \to \widehat{M}$ which is again "identity" on the Cauchy sequences. Since both compositions of these maps are the identity maps, it follows that they are isomorphisms. We have proved:

PROPOSITION 3.3. Completions of a module with respect to equivalent filtrations are naturally isomorphic.

As always, it is important to keep in mind what exactly the natural isomorphism is.

PROPOSITION 3.4. Let R be a Noetherian ring, M a finite R-module, I an ideal of R, and N a submodule of M. Then the two filtrations $(I^n N)_{n \geqslant 0}$ and $(I^n M \cap N)_{n \geqslant 0}$ on N are equivalent.

Proof. Clearly $I^n N \subseteq I^n M \cap N$. In view of Lemma 1.13, the I-filtration $(I^n M \cap N)_{n \geqslant 0}$ is I-stable. Thus, there exists an $n_0 \geqslant 0$ such that for all $n \geqslant n_0$,

$$I^{n+k} M \cap N = I^k \left(I^n M \cap N\right) \subseteq I^k N,$$

for all $k \geqslant 0$. In particular, we have, for all $n \geqslant 0$, the inclusions

$$I^{n+n_0} M \cap N \subseteq I^n N,$$

and trivially,

$$I^{n+n_0} N \subseteq I^n M \cap N.$$

Thus, the two filtrations are equivalent. ∎

THEOREM 3.5. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of R-modules, and let I be an ideal in R. Then the induced sequence

$$0 \to \widehat{M'} \xrightarrow{\widehat{f}} \widehat{M} \xrightarrow{\widehat{g}} \widehat{M''} \to 0$$

between I-adic completions is exact.

Proof. Let M be equipped with the I-adic filtration. In view of Corollary 2.6, there is an exact sequence of completed modules, when M' and M'' are given the induced filtrations. Note that since g is surjective, the induced filtration on M'' is precisely that I-adic filtration.

On the other hand, due to Proposition 3.4, the induced filtration on M' is equivalent to the I-adic filtration through the natural isomorphism defined above. Using the explicit description of this isomorphism, it follows that the sequence $0 \to \widehat{M'} \xrightarrow{\widehat{f}} \widehat{M} \xrightarrow{\widehat{g}} \widehat{M''} \to 0$ is exact, where all completions are taken with respect to the I-adic filtration. ∎

Let M be an R-module and I an ideal in R. Then there is a natural R-linear map

$$\varphi_M : \widehat{R} \otimes_R M \to \widehat{M}$$

sending

$$[(a_n)_{n \geqslant 0}] \cdot x \longmapsto [(a_n \cdot x)_{n \geqslant 0}].$$

Note that this is equivalently defined at the level of inverse systems as

$$\left(a_n + I^n\right)_{n \geqslant 0} \cdot x \longmapsto \left(a_n \cdot x + I^n M\right)_{n \geqslant 0}.$$

Clearly this map is $\widehat{R}$-linear. Further, it is trivial to check that this map is a natural transformation between the functor $\widehat{R} \otimes_R -$ and the completion functor.

THEOREM 3.6. Let R be a ring, M a finite R-module, and I an ideal in R.

(1) The map φ_M defined above is surjective.

(2) Further, if R is Noetherian, then φ_M is an isomorphism.

Proof. First, we show that φ_M is an isomorphism when M is a finite free module. Since the I-adic filtration on a finite free module breaks up into filtrations on each of its components, it suffices to first prove this for $M = R$, where it is trivial that φ_R is an isomorphism.

Next, since M is a finite R-module, there is a short exact sequence

$$0 \to K \xrightarrow{f} F \xrightarrow{g} M \to 0,$$

where F is a finite free R-module. In view of Theorem 3.5 and the fact that φ_M is a natural transformation, there is a commutative diagram of $\widehat{R}$-modules

$$
\begin{array}{ccccccccc}
\widehat{R} \otimes_R N & \longrightarrow & \widehat{R} \otimes_R F & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \\
\downarrow{\varphi_N} & & \wr \downarrow{\varphi_F} & & \downarrow{\varphi_M} & & \\
0 \longrightarrow \widehat{N} & \xrightarrow{\widehat{f}} & \widehat{F} & \xrightarrow{\widehat{g}} & \widehat{M} & \longrightarrow & 0
\end{array}
$$

with both rows exact since the tensor product is an exact functor. The Snake Lemma furnishes an exact sequence

$$\ker \varphi_N \to 0 \to \ker \varphi_M \to \operatorname{coker} \varphi_N \to 0 \to \operatorname{coker} \varphi_M \to 0.$$

In particular $\operatorname{coker} \varphi_M = 0$, that is, φ_M is surjective. Further, if R is Noetherian, then N is a finite R-module, whence φ_N is surjective because of (1), as a result, $\ker \varphi_M = 0$, that is, φ_M is an isomorphism. ∎

COROLLARY 3.7. Let R be a Noetherian ring and I an ideal in R. Then the I-adic completion $\widehat{R}$ is a flat R-algebra.

THEOREM 3.8. Let R be a Noetherian ring, I an ideal in R, and $\widehat{R}$ the I-adic completion. Then

(1) The image of the inclusion $\widehat{J} \hookrightarrow \widehat{R}$ is the extension ideal $J\widehat{R}$.

(2) With the above identification, $\left(\widehat{I}\right)^n = \widehat{I^n}$ for all $n \geq 0$.

(3) $I^n/I^{n+1} \cong \widehat{I}^n/\widehat{I}^{n+1}$ as R-modules. In particular, $R/I \cong \widehat{R}/\widehat{I}$ as rings.

(4) $\widehat{I}$ is contained in the Jacobson radical of $\widehat{R}$.

Proof. There is a commutative diagram

$$
\begin{array}{ccc}
\widehat{R} \otimes_R J & \longrightarrow & \widehat{R} \otimes_R R \\
\wr \downarrow & & \downarrow \wr \\
\widehat{J} & \longrightarrow & \widehat{R}
\end{array}
$$

where the vertical arrows are isomorphisms due to Theorem 3.6. The image of the bottom arrow is then the image of the composition of the top arrow and the right arrow, and this is immediately seen to be the extension of the ideal J to $\widehat{R}$. This proves (1). The assersion (2) follows immediately from (1), since

$$\widehat{I^n} = I^n\widehat{R} = \left(I\widehat{R}\right)^n = \widehat{I}^n$$

as ideals in $\widehat{R}$. Assersion (3) follows by taking I-adic completions in

$$0 \to I^{n+1} \to I^n \to \frac{I^n}{I^{n+1}} \to 0,$$

we have the isomorphism

$$\frac{\widehat{I}^n}{\widehat{I}^{n+1}} \cong \left(\widehat{\frac{I^n}{I^{n+1}}}\right) = \frac{I^n}{I^{n+1}}.$$

As we have seen in the preceding section, the ring $\widehat{R}$ is complete with respect to the $\widehat{I}$-adic filtration. Hence, for any $a \in \widehat{I}$, the sequence of partial sums

$$s_n = 1 + a + \cdots + a^n$$

is Cauchy with respect to the $\widehat{I}$-topology, and hence, is equivlaent to a constant sequence, that is, the sequence converges to some $s \in \widehat{R}$. It is then easy to argue that $(1-a)s = 1$, so that $1 - a$ is a unit in $\widehat{R}$; consequently, $\widehat{I}$ is contained in the Jacobson radical of $\widehat{R}$. ∎

PROPOSITION 3.9. Let $(R, \mathfrak{m}, k)$ be a local ring. Then the $\mathfrak{m}$-adic completion $\widehat{R}$ is also local. Further if R is Noetherian, then the maximal ideal of $\widehat{R}$ is $\widehat{\mathfrak{m}}$.

Proof. Consider the ring homomorphism

$$\widehat{R} = \varprojlim_n R/\mathfrak{m}^n \to R/\mathfrak{m},$$

which is clearly surjective. Set $\mathfrak{M}$ to be the kernel, which is a maximal ideal in $\widehat{R}$. We claim that this is the unique maximal ideal. Indeed, if $(x_n + \mathfrak{m}^n)_{n \geq 0} \notin \mathfrak{M}$, then each $x_n \notin \mathfrak{m}$, thus is a unit in R, and thus, setting $y_n = x_n^{-1}$, we obtain an inverse $(y_n + \mathfrak{m}^n)_{n \geq 0}$ for the above element in $\widehat{R}$.

If R is a Noetherian ring, then in view of Theorem 3.8 (3), $\widehat{R}/\widehat{\mathfrak{m}} \cong R/\mathfrak{m}$ so that $\widehat{\mathfrak{m}}$ is a maximal ideal, and as argued above, it must be the unique maximal ideal in $\widehat{R}$. ∎