

# Gorenstein Rings

Notes for the course MA 842: Topics in Algebra II

H. Ananthnarayan

SCRIBE: Swayam Chube

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## §1 Injective Modules

### §§ Basic Properties of Injective Modules

**DEFINITION 1.1.** Let  $R$  be a ring. An  $R$ -module  $E$  is said to be *injective* if for every inclusion of  $R$ -modules  $N \hookrightarrow M$  and an  $R$ -linear map  $N \rightarrow E$ , there is an  $R$ -linear map  $M \rightarrow E$  making

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & M \\ & & \downarrow & \nearrow \exists & \\ & & E & & \end{array}$$

commute.

An  $R$ -module  $M$  is said to be *divisible* if

$$\mu_a : M \longrightarrow M \quad m \longmapsto am$$

is surjective for each non-zero-divisor  $a \in R$ .

**REMARK 1.2.** It is easy to see that  $E$  is injective if and only if given any inclusion of  $R$ -modules  $N \hookrightarrow M$ , the induced map  $\text{Hom}_R(M, E) \rightarrow \text{Hom}_R(N, E)$  is surjective. Further, since  $\text{Hom}_R(-, E)$  is always left-exact, we have:

An  $R$ -module  $E$  is injective if and only if  $\text{Hom}_R(-, E)$  is an exact functor.

**PROPOSITION 1.3.** Every injective  $R$ -module is divisible.

*Proof.* Let  $E$  be  $R$ -injective,  $x \in E$ , and  $a \in R$  a non-zero-divisor. Let  $\varphi : R \rightarrow E$  be the unique  $R$ -linear map sending  $1 \mapsto x$ . Since  $R \xrightarrow{\mu_a^R} R$  is injective, there is a map  $\tilde{\varphi} : R \rightarrow E$  such that  $\tilde{\varphi} \circ \mu_a^R = \varphi$ . In particular,  $a\tilde{\varphi}(1) = x$ , whence  $\mu_a^E : E \rightarrow E$  is surjective, as desired. ■

**THEOREM 1.4 (BAER'S CRITERION).** Let  $R$  be a ring and  $E$  an  $R$ -module. Then  $E$  is injective if and only if for every ideal  $I \trianglelefteq R$  and an  $R$ -linear map  $f : I \rightarrow E$ , there is an  $R$ -linear map  $F : R \rightarrow E$  such that  $F|_I = f$ .

*Proof.* The forward implication is clear. We shall prove the converse. Let  $0 \rightarrow N \rightarrow M$  be exact and  $f : N \rightarrow E$  be an  $R$ -linear map. Consider the poset

$$\Omega = \{(P, g) : N \leq P \leq M \text{ and } g : P \rightarrow E \text{ is } R\text{-linear extending } f\},$$

where  $(P, g) \leq (P', g')$  if  $P \leq P'$  and  $g'|_P = g$ . Using Zorn's lemma, choose a maximal element  $(P, g) \in \Omega$ . We claim that  $P = M$ . Suppose now and choose some  $x \in M \setminus P$ . Set  $I = (P :_R x) \trianglelefteq R$  and consider the map

$$I \longrightarrow E \quad a \mapsto g(ax).$$

This is well-defined and  $R$ -linear, whence it extends to an  $R$ -linear map  $\varphi : R \rightarrow E$ . Let  $\alpha = \varphi(1)$  and define  $F : P + Rx \rightarrow E$  by  $F(p + ax) = g(p) + a\alpha$  for all  $p \in P$  and  $a \in R$ . To see that this is well-defined, note that if  $p_1 + a_1x = p_2 + a_2x$ , then  $a_1 - a_2 \in I$ , so that

$$g(p_2) - g(p_1) = g((a_1 - a_2)x) = (a_1 - a_2)\alpha \implies g(p_1) + a_1\alpha = g(p_2) + a_2\alpha.$$

The map  $F$  is obviously  $R$ -linear and extends  $g$ , thereby contradicting the maximality of  $(P, g)$ . Hence,  $P = M$  and  $E$  is injective. ■

**COROLLARY 1.5.** An  $R$ -module  $E$  is injective if and only if  $\text{Ext}_R^1(R/I, E) = 0$  for all ideals  $I \trianglelefteq R$ . ■

**REMARK 1.6.** We note that it is not sufficient to check the equivalent condition of Theorem 1.4 for finitely generated ideals. Indeed, let  $R = \mathcal{O}(\mathbb{C})$  the ring of entire functions, or  $R = \mathcal{O}_{\overline{\mathbb{Q}}}$  the ring of algebraic integers in  $\mathbb{C}$ . It is known that  $R$  is a non-Noetherian Bézout domain. As such, due to Interlude 1.12, there is a family of  $R$ -injectives  $\{E_i\}_{i=1}^{\infty}$  such that  $E = \bigoplus_i E_i$  is not injective.

Since each  $E_i$  is injective, it is divisible, consequently,  $E$  is a divisible  $R$ -module. Moreover, since  $R$  is a Bézout domain, every finitely generated ideal  $I$  in  $R$  is principal. It follows now that the equivalent condition of Theorem 1.4 holds for  $E$  but  $E$  is not injective.

**PROPOSITION 1.7.** Let  $R$  be a PID. An  $R$ -module  $E$  is injective if and only if it is divisible.

*Proof.* ■

**LEMMA 1.8.** Let  $S$  be an  $R$ -algebra and  $E$  an injective  $R$ -module. Then  $\text{Hom}_R(S, E)$  is an injective  $S$ -module.

*Note.*  $\text{Hom}_R(S, E)$  is naturally an  $S$ -module under the action

$$(s \cdot f)(s') = f(ss') \quad \forall s, s' \in S, f \in \text{Hom}_R(S, E).$$

*Proof.* Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be a short exact sequence of  $S$ -modules. Using the Hom-Tensor adjunction, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_S(M'', \text{Hom}_R(S, E)) & \longrightarrow & \text{Hom}_S(M, \text{Hom}_R(S, E)) & \longrightarrow & \text{Hom}_S(M', \text{Hom}_R(S, E)) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \text{Hom}_R(M' \otimes_S S, E) & \longrightarrow & \text{Hom}_R(M \otimes_S S, E) & \longrightarrow & \text{Hom}_R(M'' \otimes_S S, E) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \text{Hom}_R(M', E) & \longrightarrow & \text{Hom}_R(M, E) & \longrightarrow & \text{Hom}_R(M'', E) \longrightarrow 0 \end{array}$$

The exactness of the bottom row is a consequence of the  $R$ -injectivity of  $E$ . Thus the top row is exact and we have our desideratum. ■

**THEOREM 1.9.** Every  $R$ -module can be embedded inside an  $R$ -injective.

*Proof.* First, we show this for  $R = \mathbb{Z}$ . Let  $M$  be a  $\mathbb{Z}$ -module, then  $M \cong \bigoplus_I \mathbb{Z}/N$  for some submodule  $N$  of  $\bigoplus_I \mathbb{Z}$ . There is a natural inclusion of  $\mathbb{Z}$ -modules  $\bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_I \mathbb{Q}$  which induces an inclusion

$$M \cong \frac{\bigoplus_I \mathbb{Z}}{N} \hookrightarrow \frac{\bigoplus_I \mathbb{Q}}{N} =: E$$

Being a quotient of a divisible module,  $E$  is divisible and hence  $\mathbb{Z}$ -injective.

Now, let  $R$  be any ring and  $M$  an  $R$ -module. Then  $M$  is naturally a  $\mathbb{Z}$ -module and admits a  $\mathbb{Z}$ -linear inclusion  $\iota : M \hookrightarrow E$ , where  $E$  is a  $\mathbb{Z}$ -injective. Consider the map

$$\varphi : M \longrightarrow \text{Hom}_{\mathbb{Z}}(R, E) \quad m \longmapsto \varphi_m,$$

where  $\varphi_m : R \rightarrow E$  is given by  $\varphi_m(r) = f(rm)$ . The map  $\varphi$  is obviously  $R$ -linear and if  $\varphi_m = 0$ , then  $f(m) = \varphi_m(1) = 0$ , i.e.,  $m = 0$ . As a result,  $\varphi$  is injective and we have embedded  $M$  inside an injective  $R$ -module. ■

**COROLLARY 1.10.** Let  $E$  be an  $R$ -module. Then  $E$  is injective if and only if every  $R$ -linear inclusion  $E \hookrightarrow M$  splits.

*Proof.* Suppose  $E$  is injective.

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \longrightarrow & M \\ & & \parallel & \nearrow \exists & \\ & & E & & \end{array}$$

The above diagram constructs a splitting of  $E \hookrightarrow M$ .

Conversely, suppose every  $R$ -linear inclusion  $E \hookrightarrow M$  splits. Due to Theorem 1.9, we may choose  $M$  to be injective, so that  $E$  is a direct summand of  $M$ , whence  $E$  is injective. ■

**PROPOSITION 1.11.** Let  $R$  be a Noetherian ring. A direct sum of injective  $R$ -modules is injective.

*Proof.* Let  $\{E_\lambda\}_{\lambda \in \Lambda}$  be a collection of  $R$ -injectives and  $E = \bigoplus_{\lambda \in \Lambda} E_\lambda$ . Let  $I \triangleleft R$  be a non-zero proper ideal and  $f : I \rightarrow E$  an  $R$ -linear map. Since  $I$  is finitely generated, its image under  $f$  is finitely generated in  $E$ . Consequently, there is a finite subset  $\Lambda_0 \subseteq \Lambda$  such that  $f(I) \subseteq \bigoplus_{\lambda \in \Lambda_0} E_\lambda = E_0$ . Being a finite direct sum of injectives,  $E_0$  is injective and hence there is a map  $F : R \rightarrow E_0$  extending  $f : I \rightarrow E_0$ . Composing  $F$  with the natural inclusion  $E_0 \hookrightarrow E$ , we obtain our desired extension of  $f$ . It now follows from Theorem 1.4 that  $E$  is an injective  $R$ -module. ■

**INTERLUDE 1.12 (BASS-PAPP CONSTRUCTION).** Let  $R$  be a non-Noetherian ring. Choose a strictly increasing chain of proper non-zero ideals

$$0 \neq I_1 \subsetneq I_2 \subsetneq \cdots$$

For each  $n \geq 1$ , choose an injective module  $E_n$  containing  $R/I_n$ , and set  $E = \bigoplus_n E_n$ . We contend that  $E$  is not  $R$ -injective.

Let  $I = \bigcup_n I_n$ . Since each  $I_n$  is proper, so is  $I$ . Let  $f : I \rightarrow E$  be the map given by

$$f(x) = (x \bmod I_1, x \bmod I_2, \dots).$$

If  $E$  were injective, then there must exist a map  $F : R \rightarrow E$  extending  $f$ . Suppose  $F(1) = (x_1, x_2, \dots)$ . There is a positive integer  $N$  such that  $x_n = 0$  for all  $n \geq N$ . Choose  $x \in I_{N+1} \setminus I_N$ . Since  $x \in I$ , we have

$$(xx_1, xx_2, \dots) = F(x) = f(x) = (x \bmod I_1, x \bmod I_2, \dots).$$

In particular,  $x \bmod I_N = xx_N = 0$ , a contradiction. Thus  $E$  is not  $R$ -injective.

**PROPOSITION 1.13.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring. If  $E \neq 0$  is an finitely generated injective  $R$ -module, then  $R$  is Artinian.

*Proof.* We shall show that  $\dim R = 0$ . Suppose not; we contend that there is a prime  $\mathfrak{p} \subsetneq \mathfrak{m}$  such that  $\text{Hom}_R(R/\mathfrak{p}, E) \neq 0$ . Indeed, if there is a non-maximal prime  $\mathfrak{p} \in \text{Ass}_R(E)$ , then  $R/\mathfrak{p} \hookrightarrow E$ , giving us the desideratum. On the other hand, if  $\text{Ass}_R(E) = \{\mathfrak{m}\}$ , then the composition

$$R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{m} \hookrightarrow E$$

gives a non-zero map  $R/\mathfrak{p} \rightarrow E$ .

Choose  $a \in \mathfrak{m} \setminus \mathfrak{p}$ ; this is a non-zero-divisor on  $R/\mathfrak{p}$  and furnishes an exact sequence

$$0 \rightarrow R/\mathfrak{p} \xrightarrow{a} R/\mathfrak{p}.$$

Applying  $\text{Hom}_R(-, E)$ , we get a surjection

$$\text{Hom}_R(R/\mathfrak{p}, E) \xrightarrow{a} \text{Hom}_R(R/\mathfrak{p}, E) \rightarrow 0.$$

Note that  $\text{Hom}_R(R/\mathfrak{p}, E) \cong (0 :_E \mathfrak{p}) \subseteq E$ , is a finite  $R$ -module. Due to Nakayama's lemma, we must have that  $\text{Hom}_R(R/\mathfrak{p}, E) = 0$ , a contradiction. Thus  $\dim R = 0$ , i.e.  $R$  is Artinian. ■

**REMARK 1.14.** One cannot drop the local condition in Proposition 1.13. This construction makes use of injective hulls. Let  $k$  be an algebraically closed field and

$$R = \frac{k[X, Y]}{(X - X^2, Y - XY)}.$$

Note that  $R$  is the coordinate ring of the disjoint union of the origin and the line  $x = 1$  in  $\mathbb{A}_k^2$ . In particular,  $\dim R = 1$ , and  $R$  is not Artinian.

Let  $\mathfrak{m} = (x, y)$  be the maximal ideal corresponding to the origin. Then  $R_{\mathfrak{m}} \cong k$ , since it is the local ring of an isolated point. Now,

$$E_R(k) \cong E_{R_{\mathfrak{m}}}(k) \cong E_k(k) = k,$$

so that  $k$  is a finitely generated injective  $R$ -module.

## §§ Essential Extensions and Injective Hulls

**DEFINITION 1.15.** A containment of  $R$ -modules  $N \subseteq M$  is said to be *essential* if every non-zero submodule of  $M$  intersects  $N$  non-trivially.

An injective map  $\iota : N \hookrightarrow M$  is said to be essential if  $\iota(N) \subseteq M$  is essential.

**REMARK 1.16.** Let  $M \subseteq N$  be an essential extension of  $R$ -modules and  $\varphi : M \hookrightarrow P$  be an  $R$ -linear injective map. If  $\varphi$  extends to an  $R$ -linear map  $\tilde{\varphi} : N \rightarrow P$ , then  $\tilde{\varphi}$  is injective too. Indeed, if  $K = \ker \tilde{\varphi} \neq 0$ , then  $K \cap M \neq 0$ , a contradiction.

**PROPOSITION 1.17.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring. Let  $M$  be an Artinian  $R$ -module. Then  $\text{Soc}_R(M) \subseteq M$  is an essential extension.

*Proof.* Let  $0 \neq K \subseteq M$  be a submodule. Choose  $0 \neq x \in K$ . Since  $M$  is Artinian, the descending chain  $Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2x \supseteq \cdots$  stabilizes. Let  $n \geq 0$  be the least positive integer such that  $\mathfrak{m}^n x = \mathfrak{m}^{n+1}x$ . Due to Nakayama's lemma,  $\mathfrak{m}^n x = 0$ , whence  $n \geq 1$ . It follows that  $0 \neq \mathfrak{m}^{n-1}x \subseteq \text{Soc}_R(M) \cap K$ , as desired. ■

**COROLLARY 1.18.** Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring and  $M$  an Artinian  $R$ -module. If  $\dim_k \text{Soc}_R(M) = d$ , then  $E_R(M) \cong E^{\oplus d}$ .

*Proof.* Since  $\text{Soc}_R(M) \cong k^{\oplus d}$ , it is clear that  $E_R(\text{Soc}_R(M)) \cong E^{\oplus d}$ . The inclusion  $\text{Soc}_R(M) \hookrightarrow E^{\oplus d}$  can be extended to  $M$  to obtain a commutative diagram:

$$\begin{array}{ccc} & M & \\ \uparrow & \searrow & \\ \text{Soc}_R(M) & \hookrightarrow & E_R(\text{Soc}_R(M)) \cong E^{\oplus d} \end{array}$$

where all maps are inclusion. It follows that  $M \hookrightarrow E^{\oplus d}$  is an essential extension. Since  $E^{\oplus d}$  is an injective module, we have that  $E_R(M) \cong E^{\oplus d}$ . ■

## §2 Matlis Duality

**DEFINITION 2.1.** Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring. For an  $R$ -module  $M$ , set  $M^\vee = \text{Hom}_R(M, E)$ . This is known as the *Matlis dual* of a module.

Clearly  $(-)^\vee$  is a contravariant exact functor on the category of  $R$ -modules. Note that if  $I \subseteq \mathfrak{m}$  is an ideal, then as we have seen earlier,

$$E_{R/I}(k) = \text{Hom}_R(R/I, E) = (R/I)^\vee.$$

In particular, taking  $I = \mathfrak{m}$ , we see that  $k^\vee \cong k$  as  $R$ -modules.

**LEMMA 2.2.** Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring. Then

- (1) If  $M \neq 0$ , then  $M^\vee \neq 0$ .
- (2) If  $\lambda_R(M) < \infty$ , then  $\lambda_R(M^\vee) \neq 0$ . Moreover,  $\lambda_R(M) = \lambda_R(M^\vee)$ .

*Proof.* (1) Let  $0 \neq x \in M$ . If  $I = \text{Ann}_R(x)$ , then there is a natural inclusion  $R/I \hookrightarrow M$  sending  $\bar{1} \mapsto x$ . Taking the Matlis dual, we have a surjection

$$M^\vee \twoheadrightarrow (R/I)^\vee = E_{R/I}(k) \neq 0,$$

consequently  $M^\vee \neq 0$ .

- (2) We shall prove both statements by induction on  $\lambda_R(M)$ . If  $\lambda_R(M) = 0$ , then  $M = 0$ , so that  $M^\vee = 0$  and we get  $\lambda_R(M) = 0 = \lambda_R(M^\vee)$ . Suppose now that  $0 < \lambda_R(M) < \infty$ . Then  $\mathfrak{m} \in \text{Ass}_R(M)$ , and we have a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Since length is additive,  $\lambda_R(N) = \lambda_R(M) - 1$ ; hence the induction hypothesis applies and  $\lambda_R(N^\vee) = \lambda_R(N)$ . Taking the Matlis dual of the above short exact sequence, we have

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow k^\vee \longrightarrow 0.$$

Since  $k^\vee = 0$ , we see that

$$\lambda_R(M^\vee) = \lambda_R(N^\vee) + 1 = \lambda_R(N) + 1 = \lambda_R(M),$$

as desired. ■

**THEOREM 2.3.** Let  $(R, \mathfrak{m}, k, E)$  be an Artinian local ring.

- (1)  $E$  is a faithful finite  $R$ -module.
- (2) The map

$$\mu : R \longrightarrow \text{Hom}_R(E, E) \quad a \longmapsto \mu_a$$

is an isomorphism of  $R$ -modules and rings.

- (3) Given a finite  $R$ -module  $M$ , the natural map

$$\varphi_M : M \longrightarrow M^{\vee\vee} \quad m \longmapsto \text{ev}_m$$

is an isomorphism.

*Proof.* (1) Suppose  $a \in R$  is such that  $aE = 0$ . Then

$$R^\vee = \text{Hom}_R(R, E) = E = (E :_E a) \cong \text{Hom}_R(R/aR, E) = (R/aR)^\vee.$$

Since  $R$  is Artinian, we then have

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R((R/aR)^\vee) = \lambda_R(R/aR) \implies \lambda_R(aR) = 0,$$

consequently,  $a = 0$ , i.e.,  $E$  is a faithful  $R$ -module.

Next, since  $R$  is Artinian,  $\mathfrak{m} \in \text{Ass}_R(R)$ , consequently, there is an injection  $k = R/\mathfrak{m} \hookrightarrow R$ . Due to Remark 1.16 extends to an inclusion  $E \hookrightarrow R$ , consequently,  $E$  is a finite  $R$ -module.

(2) First note that  $\mu$  is injective due to (1). But note that

$$\infty > \lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = \lambda_R(E^\vee) = \lambda_R(\operatorname{Hom}_R(E, E)),$$

consequently  $\mu$  is an isomorphism.

(3) It suffices to show that  $\varphi_M$  is injective since  $\lambda_R(M) = \lambda_R(M^{\vee\vee})$ . Suppose  $0 \neq x \in M$  is such that  $\varphi_M(x) = 0$ , that is, for all  $f \in \operatorname{Hom}_R(M, E)$ ,  $f(x) = 0$ . Let  $I = \operatorname{Ann}_R(x)$ . Now, there is a non-zero map

$$\psi : R/I \twoheadrightarrow R/\mathfrak{m} = k \hookrightarrow E,$$

which extends to a non-zero map  $f : M \rightarrow E$  since  $R/I \hookrightarrow M$  through  $\bar{1} \mapsto x$ . Thus,  $f(x) = \psi(\bar{1}) \neq 0$ , a contradiction. ■

**INTERLUDE 2.4 (ON  $\widehat{R}$ -MODULES).** Let  $(R, \mathfrak{m}, k)$  be a local ring and  $M$  an  $R$ -module such that  $\Gamma_{\mathfrak{m}}(M) = M$ . We contend that  $M$  is an  $\widehat{R}$ -module in a natural way. To this end, we need only define  $\widehat{a} \cdot m$  for  $\widehat{a} \in \widehat{R}$  and  $m \in M$ .

Let  $\widehat{a} = (a_1, a_2, \dots)$ , where we are using the isomorphism

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

Since  $\Gamma_{\mathfrak{m}}(M) = M$ , there is a positive integer  $n \geq 1$  such that  $\mathfrak{m}^n m = 0$ . Hence, for  $k \geq n$ , we have  $a_k \cdot m = a_n \cdot m$ , as  $a_k - a_n \in \mathfrak{m}^n$ . In light of this, we define  $\widehat{a} \cdot m = a_n \cdot m$ . We must show that this makes  $M$  into an  $\widehat{R}$ -module.

Let  $m_1, m_2 \in M$  and  $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$ . There are positive integers  $n_1, n_2 \geq 1$  such that  $\mathfrak{m}^{n_1} m_1 = 0 = \mathfrak{m}^{n_2} m_2$ ; then  $\mathfrak{m}^n m_1 = 0 = \mathfrak{m}^n m_2$  for all  $n \geq \max\{n_1, n_2\}$ . Hence, for all such  $n \geq 1$ ,

$$\widehat{a} \cdot (m_1 + m_2) = a_n \cdot (m_1 + m_2) = a_n \cdot m_1 + a_n \cdot m_2 = \widehat{a} \cdot m_1 + \widehat{a} \cdot m_2.$$

Next, let  $\widehat{a}, \widehat{b} \in \widehat{R}$  and  $m \in M$  with

$$\widehat{a} = (a_1, a_2, \dots) \quad \text{and} \quad \widehat{b} = (b_1, b_2, \dots).$$

There is a positive integer  $n$  such that  $\mathfrak{m}^n m = 0$ . Then

$$(\widehat{a} + \widehat{b}) \cdot m = (a_n + b_n) \cdot m = a_n \cdot m + b_n \cdot m = \widehat{a} \cdot m + \widehat{b} \cdot m.$$

Finally, note that  $\widehat{b} \cdot m = b_n m$  and  $\mathfrak{m}^n (\widehat{b} \cdot m) = 0$ , so that

$$\widehat{a} \cdot (\widehat{b} \cdot m) = \widehat{a} \cdot (b_n \cdot m) = a_n \cdot (b_n \cdot m) = (a_n b_n) \cdot m = (\widehat{a}\widehat{b}) \cdot m.$$

This shows that  $M$  is indeed an  $\widehat{R}$ -module as described above. Further, since  $R \rightarrow \widehat{R}$  is the diagonal map, it follows that the  $\widehat{R}$ -module structure on  $M$  agrees with the  $R$ -module structure through the diagonal map. In particular, this means that:

A subset of  $M$  is an  $R$ -submodule if and only if it is an  $\widehat{R}$ -submodule.

As a result,  $M$  is Noetherian (resp. Artinian) as an  $R$ -module if and only if it is so as an  $\widehat{R}$ -module.

**INTERLUDE 2.5 (ON MAPS BETWEEN  $\mathfrak{m}$ -POWER TORSION MODULES).** Again, let  $(R, \mathfrak{m}, k)$  be a local ring and suppose  $M$  and  $N$  are  $R$ -modules such that  $\Gamma_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(N)$ . By Interlude 2.4, we know that they are  $\widehat{R}$ -modules in a natural way. Let  $\varphi : M \rightarrow N$  be an  $R$ -linear map. We contend that  $\varphi$  is also  $\widehat{R}$ -linear. Indeed, let  $m \in M$  and  $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$ . There is a positive integer  $n \geq 1$  such that  $\mathfrak{m}^n m = 0$ , and hence,  $\mathfrak{m}^n \varphi(m) = 0$ . It follows that

$$\varphi(\widehat{a} \cdot m) = \varphi(a_n \cdot m) = a_n \cdot \varphi(m) = \widehat{a} \cdot \varphi(m),$$

as desired.

**THEOREM 2.6.** Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring.

- (1)  $\Gamma_{\mathfrak{m}}(E) = E$ , and hence  $E$  is an  $\widehat{R}$  module and for every  $R$ -module  $M$ ,  $M^\vee$  is  $\mathfrak{m}$ -power torsion.
- (2)  $E \cong E_{\widehat{R}}(k)$  as  $\widehat{R}$ -modules.
- (3)  $R^{\vee\vee} = \text{Hom}_R(E, E) \cong \widehat{R}$  as  $R$ -modules.
- (4)  $E$  is an Artinian  $R$ -module.

*Proof.* (1) That  $E$  is an  $\widehat{R}$ -module follows immediately from Interlude 2.4. Finally,  $M^\vee = \text{Hom}_R(M, E)$  is  $\mathfrak{m}$ -power torsion because  $E$  is so.

- (2) The containment  $k \subseteq E$  is an essential extension of  $R$ -modules, both of which are  $\mathfrak{m}$ -power torsion. Due to Interlude 2.4, it follows that it is an essential extension of  $\widehat{R}$ -modules too. Now, due to Remark 1.16, there is a commutative diagram of inclusions

$$\begin{array}{ccc} & E & \\ \uparrow & \searrow & \\ k & \longrightarrow & E_{\widehat{R}}(k), \end{array}$$

where all maps are  $\widehat{R}$ -linear. It follows that  $E \hookrightarrow E_{\widehat{R}}(k)$  is an essential extension of  $\widehat{R}$ -modules, and consequently, an essential extension of  $R$ -modules. Since  $E$  is  $R$ -injective, we must have that the inclusion is an isomorphism of  $R$ -modules. Finally, due to Interlude 2.5, this is an isomorphism of  $\widehat{R}$ -modules.

- (3)
- (4) Let  $M_1 \supseteq M_2 \supseteq \cdots$  be a chain of  $R$ -submodules in  $E$ . There are commutative diagrams

$$\begin{array}{ccc} M_{j+1} & \xhookrightarrow{\iota_{j+1}} & E \\ \downarrow & \searrow \iota_j & \\ M_j & & \end{array}$$

whose Matlis dual furnishes commutative diagrams

$$\begin{array}{ccc} \widehat{R} = E^\vee & \xrightarrow{\varphi_j} & M_j^\vee \\ & \searrow \varphi_{j+1} & \downarrow \\ & & M_{j+1}^\vee \end{array}$$

Note that all Matlis duals are  $\mathfrak{m}$ -power torsions and hence due to Interlude 2.5, the  $\varphi_j$ 's are  $\widehat{R}$ -linear. Let  $I_j = \ker \varphi_j \subseteq \widehat{R}$ , which is an ideal. Due to the commutative diagram, it is clear that there is an ascending chain  $I_j \subseteq I_{j+1}$ . Since  $\widehat{R}$  is Noetherian, this chain stabilizes, say  $I_n = I_{n+1} = \cdots$ .

Then due to the first isomorphism theorem,  $M_j^\vee \twoheadrightarrow M_{j+1}^\vee$  is an isomorphism for all  $j \geq n$ . Let  $C_j = \text{coker}(M_{j+1} \hookrightarrow M_j)$ . The exactness of the Matlis dual gives  $C_j^\vee = 0$ , which, due to Lemma 2.2, implies that  $C_j = 0$ , that is,  $M_{j+1} \hookrightarrow M_j$  is an isomorphism for all  $j \geq n$ , i.e., the descending chain stabilizes, as desired.  $\blacksquare$

**THEOREM 2.7 (MATLIS DUALITY, VERSION 1).** Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{finitely generated} \\ \widehat{R}\text{-modules} \end{array} \right\} \begin{array}{c} \xrightarrow{(-)^\vee} \\ \xleftarrow{(-)^\vee} \end{array} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{Artinian } R\text{-modules} \end{array} \right\}.$$

*Proof.* Let  $M$  be an Artinian  $R$ -module and let  $d = \dim_k \text{Soc}_R(M)$ . Due to Corollary 1.18,  $E_R(M) \cong E^{\oplus d}$ , so that there is an inclusion  $M \hookrightarrow E^{\oplus d}$ , which upon taking the Matlis dual furnishes an  $\widehat{R}$ -linear surjection  $\widehat{R}^{\oplus d} \twoheadrightarrow M^\vee$ . Thus  $M^\vee$  is a finite  $\widehat{R}$ -module.

Conversely, suppose  $M$  is a finite  $\widehat{R}$ -module. Thus, there is a surjection  $\widehat{R}^{\oplus n} \twoheadrightarrow M$ . Taking the Matlis dual, we obtain an injection  $M^\vee \hookrightarrow (\widehat{R}^\vee)^{\oplus n}$ .

There is a natural “evaluation map”  $\text{ev} : M \rightarrow M^{\vee\vee}$ , which we shall show is an isomorphism. That  $\text{ev}$  is injective follows in the same way as Theorem 2.3 (3). Next, since  $\lambda_R(M) < \infty$ , we have that  $\lambda_R(M) = \lambda_R(M^\vee) = \lambda_R(M^{\vee\vee})$ , whence  $\text{ev}$  is an isomorphism. ■

**THEOREM 2.8.** Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring. Then the following are equivalent:

- (1)  $R$  is self-injective
- (2)  $R \cong E$  as  $R$ -modules.
- (3)  $R$  is Artinian and  $\dim_k \text{Soc}_R(R) = 1$ .

*Proof.* (1)  $\implies$  (2) Due to Proposition 1.13,  $R$  must be an Artinian local ring, and hence, from Proposition 1.17,  $\text{Soc}_R(R) \subseteq R$  is an essential extension. It follows that  $R$  is the injective hull of  $\text{Soc}_R(R) \cong k^{\oplus d}$  for some positive integer  $d$ . Hence,  $R \cong E^{\oplus d}$  as  $R$ -modules, and comparing lengths, we have

$$\lambda_R(R) = d\lambda_R(E) = d\lambda_R(R^\vee) = d\lambda_R(R),$$

whence  $d = 1$  and  $R \cong E$ .

(2)  $\implies$  (3) Due to Theorem 2.6 (4),  $R$  is Artinian. Using a length argument as above, we can show that  $\dim_k \text{Soc}_R(R) = 1$ .

(3)  $\implies$  (1) Again, since  $k = \text{Soc}_R(R) \subseteq R$  is essential, we have that  $R \hookrightarrow E = E_R(k)$ . Using a length argument, it follows that this inclusion must be an isomorphism, whence  $R$  is self-injective. ■

## §3 Injective Resolutions

### §§ Bass’s Lemma and ramifications

**DEFINITION 3.1.** Let  $M$  be an  $R$ -module. An *injective resolution* for  $M$  is an exact complex

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots,$$

where each  $E^n$  is an injective  $R$ -module. The resolution is often denoted succinctly as  $0 \rightarrow M \rightarrow E^\bullet$ .

We say that  $M$  has finite injective dimension if  $M$  has an injective resolution  $0 \rightarrow M \rightarrow E^\bullet$  and an integer  $N \geq 0$  such that  $E^n = 0$  for  $n \geq N$ . We define

$$\text{inj dim}_R M = \inf \left\{ n : 0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^n \rightarrow 0 \text{ is an injective resolution of } M \right\}.$$

If  $M$  does not have finite injective dimension, then set  $\text{inj dim}_R M = \infty$ .

**REMARK 3.2.** It is possible to create a “canonical” injective resolution by successively taking injective hulls. Set  $E^0 = E_R(M)$  and for  $i \geq 0$ , define

$$E^{i+1} = E_R \left( \text{coker} \left( E^{i-1} \rightarrow E^i \right) \right),$$

with the convention that  $E^{-1} = M$ . We call this the *minimal injective resolution* of  $M$ .



**LEMMA 3.3.** Let  $R$  be a Noetherian ring and  $0 \rightarrow M \xrightarrow{\theta} E$  be an inclusion of  $R$ -modules with  $E$  injective. Then the inclusion is an injective hull of  $M$  if and only if

$$\mathrm{Hom}_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \xrightarrow{\theta_{\mathfrak{p}}} \mathrm{Hom}_R(R/\mathfrak{p}, E)_{\mathfrak{p}}$$

is an isomorphism for all  $\mathfrak{p} \in \mathrm{Spec}(R)$ .

*Proof.* Owing to the left exactness of  $\mathrm{Hom}_R(R/\mathfrak{p}, -)$  and the exactness of localization, the map  $\theta_{\mathfrak{p}}$  is injective for each  $\mathfrak{p} \in \mathrm{Spec}(R)$ . Hence, it suffices to show that  $E$  is injective if and only if  $\theta_{\mathfrak{p}}$  is surjective for each  $\mathfrak{p} \in \mathrm{Spec}(R)$ .

Recall that there are canonical isomorphisms

$$\mathrm{Hom}_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \quad \frac{\psi}{s} \longmapsto \left( \frac{a}{t} \mapsto \frac{\psi(a)}{st} \right),$$

where we are identifying  $\kappa(\mathfrak{p})$  with the quotient field of  $R/\mathfrak{p}$ . Hence, surjectivity of  $\theta_{\mathfrak{p}}$  is equivalent to the surjectivity of

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \rightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}}).$$

Henceforth, we shall identify  $M$  with a submodule of  $E$ , so that  $\theta$  is simply the inclusion map.

Suppose first that  $M \xrightarrow{\theta} E$  is an injective hull and let  $0 \neq \varphi \in \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}})$ . Using the above isomorphism, we can write  $\varphi = \psi/s$  for some  $\psi \in \mathrm{Hom}_R(R/\mathfrak{p}, E)$  and  $s \in R \setminus \mathfrak{p}$ . Let  $\psi(\bar{1}) = z \in E$  and  $a \in R$  such that  $0 \neq az \in M$ . Note that  $a \in R \setminus \mathfrak{p}$ , since  $\mathfrak{p} \subseteq \mathrm{Ann}_R(z)$ <sup>1</sup>. Define

$$\bar{\varphi} : R/\mathfrak{p} \longrightarrow M \quad \bar{1} \longmapsto az.$$

This is well-defined, since  $\mathfrak{p}$  annihilates  $az \in M$ . We claim that

$$\varphi = \frac{\bar{\varphi}}{as} \in \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}}).$$

Indeed, for  $x/t \in \kappa(\mathfrak{p})$  we have

$$\left( \frac{\bar{\varphi}}{as} \right) \left( \frac{x}{t} \right) = \frac{\bar{\varphi}(x)}{ast} = \frac{xaz}{ast} = \frac{xz}{st} = \left( \frac{\psi}{s} \right) \left( \frac{x}{t} \right) = \varphi \left( \frac{x}{t} \right),$$

as desired. This shows that  $\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \rightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}})$  is surjective.

Conversely, suppose the aforementioned map is surjective. We shall show that  $E$  is the injective hull of  $M$ . To this end, it suffices to show that the inclusion  $M \subseteq E$  is essential. Let  $0 \neq N \subseteq E$  be a submodule and  $\mathfrak{p} \in \mathrm{Ass}_R(N)$ . There is an injective map

$$0 \rightarrow R/\mathfrak{p} \longrightarrow N \quad \bar{1} \longmapsto z.$$

Since  $\mathfrak{p} = \mathrm{Ann}_R(z)$ , it suffices to find  $a \in R \setminus \mathfrak{p}$  such that  $az \in M$ . Consider the map

$$\varphi : \kappa(\mathfrak{p}) \longrightarrow E_{\mathfrak{p}} \quad \bar{1} \longmapsto z/1.$$

The surjectivity of  $\theta_{\mathfrak{p}}$  furnishes a  $\psi : \kappa(\mathfrak{p}) \rightarrow M_{\mathfrak{p}}$  such that  $\theta_{\mathfrak{p}}(\psi) = \varphi$ . In particular, this means that

$$\frac{z}{1} = \varphi(\bar{1}) = \psi(\bar{1}) \in M_{\mathfrak{p}},$$

whence there is some  $a \in R \setminus \mathfrak{p}$  such that  $az \in M$ , as desired. ■

**COROLLARY 3.4.** Let  $R$  be a Noetherian ring and  $0 \rightarrow M \rightarrow E^{\bullet}$  be an injective resolution of an  $R$ -module  $M$ . Then  $E^{\bullet}$  is minimal if and only if the natural maps

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}}^n) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}}^{n+1})$$

are identically zero for all  $n \geq 0$  and for all  $\mathfrak{p} \in \mathrm{Spec}(R)$ .

<sup>1</sup>Note that  $\mathfrak{p} = \mathrm{Ann}_R(z)$ , for if not, then  $\varphi = 0$ .

*Proof.* Let  $K^n = \ker(E^n \rightarrow E^{n+1})$ . Then there is an exact sequence  $0 \rightarrow K^n \rightarrow E^n \rightarrow E^{n+1}$ . Using Lemma 3.3,  $E^n$  is the injective hull of  $C^n$  if and only if

$$\Phi : \text{Hom}_{R_p}(\kappa(\mathfrak{p}), C_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \text{ is an isomorphism.}$$

But the left-exactness of  $\text{Hom}$  and exactness of localization implies that the sequence

$$0 \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), C_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^{n+1})$$

is exact. Thus  $\Phi$  is an isomorphism if and only if the map  $\text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^{n+1})$  is the zero map, as desired. ■

**COROLLARY 3.5.** Let  $R$  be a Noetherian ring and  $M$  an  $R$ -module. Let  $0 \rightarrow M \rightarrow E^\bullet$  be the minimal injective resolution of  $M$ . Then

$$E^j = \bigoplus_{\mathfrak{p}} E_R(R/\mathfrak{p})^{a_j(\mathfrak{p})} \quad \text{and} \quad a_j(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_p}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}).$$

In particular, if  $M$  is a finite  $R$ -module,  $a_j(\mathfrak{p}) < \infty$  for all  $j \geq 0$  and  $\mathfrak{p} \in \text{Spec}(R)$ .

*Proof.* ■

**REMARK 3.6.** We can now justify the name “minimal injective resolution”. In particular, we shall show that the length of the minimal injective resolution is precisely the injective dimension of a module.

Let  $R$  be a Noetherian ring and  $M$  a finite  $R$ -module. Let  $0 \rightarrow M \rightarrow E^\bullet$  be the minimal injective resolution in the sense of Remark 3.2. Let  $0 \leq \ell \leq \infty$  denote the length of the resolution. Clearly  $\text{inj dim}_R M \leq \ell$ . If  $\text{inj dim}_R M = \infty$ , then  $\ell \leq \text{inj dim}_R M$  so that  $\ell = \text{inj dim}_R M$ .

On the other hand, if  $\text{inj dim}_R M = n < \infty$ , then using this injective resolution to compute the  $\text{Ext}$ ’s, we see that for  $j > n$ , and  $\mathfrak{p} \in \text{Spec}(R)$ ,

$$\text{Ext}_R^j(R/\mathfrak{p}, M) = 0 \implies \mu_j(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_p}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = 0.$$

That is,  $E^j = 0$  for all  $j > n$  and hence,  $\ell \leq n$ . It follows that  $\ell = \text{inj dim}_R M$ .

**DEFINITION 3.7.** Let  $R$  be a Noetherian ring and  $M$  a finite  $R$ -module. For  $j \geq 0$  and  $\mathfrak{p} \in \text{Spec}(R)$ , define the  *$j$ -th Bass number* as

$$\mu_j(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_p}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}).$$

**LEMMA 3.8 (BASS).** Let  $R$  be a Noetherian ring and  $M$  a finite  $R$ -module. Let  $\mathfrak{p} \subsetneq \mathfrak{q}$  be primes in  $R$  such that  $\text{ht}(\mathfrak{q}/\mathfrak{p}) = 1$ . If for some  $j \geq 0$ ,  $\mu_j(\mathfrak{p}, M) \neq 0$ , then  $\mu_{j+1}(\mathfrak{q}, M) \neq 0$ .

*Proof.* Localizing at  $\mathfrak{q}$ , we may assume that  $(R, \mathfrak{m}, k)$  is a Noetherian local ring and  $\text{ht}(\mathfrak{m}/\mathfrak{p}) = 1$ . If  $a \in \mathfrak{m} \setminus \mathfrak{p}$ , then  $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$ , and we have a short exact sequence

$$0 \rightarrow R/\mathfrak{p} \xrightarrow{a} R/\mathfrak{p} \rightarrow R/(\mathfrak{p} + (a)) \rightarrow 0.$$

This gives rise to a long exact sequence

$$\cdots \rightarrow \text{Ext}_R^j(R/\mathfrak{p}, M) \xrightarrow{a} \text{Ext}_R^j(R/\mathfrak{p}, M) \rightarrow \text{Ext}_R^{j+1}(R/(\mathfrak{p} + (a)), M) \rightarrow \cdots,$$

for all  $j \geq 0$ .

$$\mu_j(\mathfrak{p}, M) \neq 0 \implies \text{Ext}_{R_p}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0 \implies \text{Ext}_R^j(R/\mathfrak{p}, M) \neq 0.$$

Since the  $\text{Ext}$ ’s are finite  $R$ -modules, Nakayama’s lemma implies that  $\text{Ext}_R^{j+1}(R/(\mathfrak{p} + (a)), M) \neq 0$ .

Since  $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$ , the  $R$ -module  $R/(\mathfrak{p} + (a))$  is finite Artinian, so that it has a composition series with successive quotients isomorphic to  $R/\mathfrak{m} = k$ . Now, if  $\text{Ext}_R^{j+1}(k, M) \neq 0$ , then through the short

exact sequences induced by the composition series, it would follow that  $\text{Ext}_R^{j+1}(R/(\mathfrak{p} + (a)), M) = 0$ , a contradiction. But since  $R \setminus \mathfrak{m}$  consists of only units, we have that

$$0 \neq \text{Ext}_R^{j+1}(k, M) = \text{Ext}_{R_{\mathfrak{m}}}^{j+1}(\kappa(\mathfrak{m}), M_{\mathfrak{m}}),$$

and hence  $\mu_{j+1}(\mathfrak{m}, M) \neq 0$ . ■

**REMARK 3.9.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and  $M$  a finite  $R$ -module.

(i) If  $\mu_i(\mathfrak{p}, M) \neq 0$  and  $h = \text{ht}(\mathfrak{q}/\mathfrak{p})$ , then  $\mu_{i+h}(\mathfrak{q}, M) \neq 0$ .

**COROLLARY 3.10.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and  $M$  a finite  $R$ -module. Then

$$\text{inj dim}_R M = \infty \iff \mu_j(\mathfrak{m}, M) \neq 0 \text{ for infinitely many } j \geq 0.$$

**THEOREM 3.11.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and  $M$  a finite  $R$ -module. Then

$$\text{inj dim}_R M = \sup \left\{ j : \text{Ext}_R^j(k, M) \neq 0 \right\}.$$

*Proof.* ■

### §§ Modules of finite injective dimension

**DEFINITION 3.12.** A Noetherian local ring  $(R, \mathfrak{m}, k)$  is said to be a *Gorenstein local ring* if  $\text{inj dim}_R R < \infty$ .