Coxeter and Tits Systems

Swayam Chube

Last Updated: March 25, 2025

§1 Coxeter Systems

Let W denote a group and $S \subseteq W$ a generating set such that $1 \notin S$ and $S = S^{-1}$. Fix this pair throughout this section, and we refer to such a pair as a *generating pair*.

Definition 1.1. Let $w \in W$. The length of w with respect to S, denoted by $\ell_S(w)$ (often abbreviated to $\ell(w)$) is the smallest integer $q \geqslant 0$ such that w is the product of a sequence of q elements of S. A *reduced representation* of W with respect to S is any sequence $\mathbf{s} = (s_1, \dots, s_q)$ of elements of S such that $w = s_1 \cdots s_q$ and $q = \ell_S(w)$.

Clearly, if $w, w' \in W$, then

$$\begin{aligned} \ell(ww') &\leqslant \ell(w) + \ell(w'), \\ \ell(w^{-1}) &= \ell(w), \\ |\ell(w) - \ell(w')| &\leqslant \ell(ww'^{-1}). \end{aligned}$$

Definition 1.2. (W, S) is said to be a *Coxeter system* if every element in S has order at most 2, and it satisfies the following condition:

(Cox) For $s, s' \in S$, let $1 \leqslant m(s, s') \leqslant \infty$ be the order of $ss' \in W$ and let

$$I = \{(s, s') \colon m(s, s') < \infty\}.$$

Then

$$W = \left\langle s \in S \colon (ss')^{m(s,s')} = 1,\ (s, s') \in I \right\rangle$$

is a presentation for the group W.

Remark 1.3. Consider the function $f : S \to \{-1, 1\}$ given by $f(s) = -1$ for each $s \in S$. If $s, s' \in S$ such that $m = m(s, s') < \infty$, then $(f(s)f(s'))^m = 1$ almost tautologically. Hence, this function induces a map $\operatorname{sgn} : W \to \{-1, 1\}$ known as the *signature* of W. It is clear that $\operatorname{sgn}(w) = (-1)^{\ell(w)}$.

Proposition 1.4. Assume that (W, S) is a Coxeter system. Then, two elements $s, s' \in S$ are conjugate in W if and only if the following condition is satisfied:

(Con) There exists a finite sequence $(s_1, \dots, s_q)$ of elements of S such that $s_1 = s$, $s_q = s'$ and $s_j s_{j+1}$ is of *finite* odd order for $1 \leqslant j < q$.

Proof. First, if $s, s' \in S$ such that $p = ss'$ is of finite order $2n + 1$, then

$$sps^{-1} = p^{-1} \implies sp^n s^{-1} = p^{-n},$$

so that

$$p^n s p^{-n} = p^n p^n s = p^{-1} s = s',$$

and s' is conjugate to s. In particular, this shows that if (Con) is satisfied, then (s, s') is a pair of conjugates in W.

For each $s \in S$, let A_s be the set of $s' \in S$ satisfying (Con); clearly, every $s' \in A_s$ is conjugate of s. Let $f : S \to \{-1, 1\}$ that is equal to 1 on A_s and to -1 in $S \setminus A_s$. We shall show that this map can be extend to a group homomorphism $W \to \{-1, 1\}$. Indeed, let $s', s'' \in S$ with $m = m(s, s') < \infty$. If m is odd, then s' and s'' are conjugate so either both in A_s or both in $S \setminus A_s$, and hence $f(s')f(s'') = 1$, in particular, $(f(s')f(s''))^m = 1$. On the other hand, if m is even, then clearly $(f(s')f(s''))^m = 1$. Consequently, to (Cox), the map f extends to a group homomorphism $W \to \{-1, 1\}$.

Finally, let s' be a conjugate of s in W. Since $s \in \ker f$, so does s', hence $s' \in A_s$. ∎

Definition 1.5. Let (W, S) be a Coxeter system and let T be the set of conjugates in W of elements of S. For any sequence $\mathbf{s} = (s_1, \ldots, s_q)$ of elements of S, denote by $\Phi(\mathbf{s})$ the sequence $(t_1, \ldots, t_q)$ of elements of T defined by

$$t_j = (s_1 \cdots s_{j-1}) s_j (s_1 \cdots s_{j-1})^{-1} = (s_1 \cdots s_{j-1}) s_j (s_{j-1} \cdots s_1).$$

Then $t_1 = s_1$ and $s_1 \cdots s_q = t_q \cdots t_1$. For $t \in T$, denote by $n(\mathbf{s}, t)$ the number of indices $1 \leqslant j \leqslant q$ for which $t_j = t$. Finally, set

$$R = \{-1, 1\} \times T.$$

Lemma 1.6. (1) Let $w \in W$ and $t \in T$. The number $(-1)^{n(\mathbf{s},t)}$ has the same value $\eta(w, t)$ for all sequences $\mathbf{s} = (s_1, \ldots, s_q)$ in S such that $w = s_1 \cdots s_q$.

(2) For $w \in W$, let $U_w : R \to R$ be given by

$$U_w(\varepsilon, t) = \left(\varepsilon \eta(w^{-1}, t), wtw^{-1}\right).$$

The map $w \mapsto U_w$ is a homomorphism from W to the group of permutations of R, $\mathfrak{Sym}(R)$.

Proof. For $s \in S$, define a map $U_s : R \to R$ by

$$U_s(\varepsilon, t) = \left(\varepsilon(-1)^{\delta_{s,t}}, sts^{-1}\right),$$

where $\delta_{s,t}$ is the Kronecker symbol. Clearly, $U_s^2 = \mathbf{id}_R$, and hence U_s is a permutation of R.

For a sequence $\mathbf{s} = (s_1, \ldots, s_q)$ in S, put $w = s_q \cdots s_1$ and $U_{\mathbf{s}} = U_{s_q} \cdots U_1$. We shall show by induction that

$$U_{\mathbf{s}}(\varepsilon, t) = \left(\varepsilon(-1)^{n(\mathbf{s},t)}, wtw^{-1}\right). \tag{1}$$

This is clear if $q = 0, 1$. For $q > 1$, put $\mathbf{s}' = (s_1, \ldots, s_{q-1})$ and

$$w' = s_{q-1} \cdots s_1.$$

Using the induction hypothesis, we can write

$$U_{\mathbf{s}}(\varepsilon, t) = U_{s_q}\left(\varepsilon(-1)^{n(\mathbf{s}',t)}, w'tw'^{-1}\right) = \left(\varepsilon(-1)^{n(\mathbf{s}',t)+\delta_{s_q, w'tw'^{-1}}}, wtw^{-1}\right).$$

But since $\Phi(\mathbf{s}) = \left(\Phi(\mathbf{s}'), w'tw'^{-1}\right)$, the formula (1) follows.

Now let $s, s' \in S$ be such that $p = ss'$ has finite order m. Let $\mathbf{s} = (s_1, \ldots, s_{2m})$ where

$$s_j = \begin{cases} s & j \text{ is odd} \\ s' & j \text{ is even.} \end{cases}$$

Then $s_{2m} \cdots s_1 = p^{-m} = 1$ and

$$t_j = (s_1 \cdots s_{j-1}) s_j (s_{j-1} \cdots s_1) = p^{j-1} s \qquad \text{for } 1 \leqslant j \leqslant 2m.$$

Sinc p is of order m, the elements $t_1, \ldots, t_m$ are distinct and $t_{j+m} = t_j$ for $1 \leqslant j \leqslant m$. The integer $n(\mathbf{s}, t)$ is equal to either 0 or 2 and due to (1), we have that $U_{\mathbf{s}} = \mathbf{id}_R$, i.e., $(U_s U_{s'})^m = \mathbf{id}_R$. Thus, by (Cox), there is a group homomorphism $W \to \mathfrak{Sym}(R)$ given by $w \mapsto U_w$, extending the mapping $s \mapsto U_s$. It follows that $U_w = U_{\mathbf{s}}$ for every sequence $\mathbf{s} = (s_1, \ldots, s_q)$ such that $w = s_q \cdots s_1$. Both conclusions of the lemma follow hence. ∎

Lemma 1.7. Let $\mathbf{s} = (s_1, \ldots, s_q)$, $\Phi(\mathbf{s}) = (t_1, \ldots, t_q)$ and $w = s_1 \cdots s_q$. Let T_w be the set of elements of T such that $\eta(w, t) = -1$. Then $\mathbf{s}$ is a reduced representation of w if and only if the t_i are distinct, and in that case, $T_w = \{t_1, \ldots, t_q\}$ and $\# T_w = \ell(w)$.

Proof. Clearly $T_w \subseteq \{t_1, \ldots, t_q\}$. Taking $\mathbf{s}$ to be a reduced representation, it follows that $\# T_w \leqslant \ell(w)$. Further, if the t_i's are distinct, then $\eta(w, t) = -1$ if and only if $t \in \{t_1, \ldots, t_q\}$, so that $T_w = \{t_1, \ldots, t_q\}$ and $q = \# T_w \leqslant \ell(w)$. Hence, $\mathbf{s}$ is a reduced representation.

On the other hand, suppose $t_i = t_j$ for some $i < j$. Then

$$s_i = (s_i \cdots s_{j-1}) s_j (s_i \cdots s_{j-1})^{-1};$$

consequently,

$$w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} \cdots s_{j+1} \cdots s_q,$$

whence $\mathbf{s}$ is not a reduced representation of w, as desired. ∎

Lemma 1.8. Let $w \in W$ and $s \in S$ be such that $\ell(sw) \leqslant \ell(w)$. For any sequence $\mathbf{s} = (s_1, \dots, s_q)$ of elements of S with $w = s_1 \cdots s_q$, there exists an index $1 \leqslant j \leqslant q$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

Proof. Let p be the length of w and $w' = sw$. Due to Remark 1.3, $\ell(w') \equiv \ell(w) + 1 \pmod 2$. The hypothesis $\ell(w') \leqslant \ell(w)$ and the relation

$$|\ell(w) - \ell(w')| \leqslant \ell(ww'^{-1}) = \ell(s) = 1,$$

and hence, $\ell(w') = p - 1$. Let $w' = s'_1 \cdots s'_{p-1}$ be a reduced representation of w' and put $\mathbf{s} = (s, s'_1, \dots, s'_{p-1})$ and $\Phi(\mathbf{s}') = (t'_1, \dots, t'_p)$. Since $\mathbf{s}'$ is a reduced representation of w, due to Lemma 1.7, the t_j's must be distinct and $n(\mathbf{s}', s) = 1$ since $t_1 = s$. Further, since both $\mathbf{s}$ and $\mathbf{s}'$ represent w, due to Lemma 1.6, we must have $n(\mathbf{s}, s) \equiv n(\mathbf{s}', s) \pmod 2$, whence $n(\mathbf{s}, s) \neq 0$. Consequently, s is equal to one of the t_j's. The lemma then follows immediately. ■

§§ The Exchange Condition

Definition 1.9. Let W be a group and $S \subseteq W$ a generating set such that $S^{-1} = S$ and every element in S has order at most 2. The *exchange condition* is the following assertion about (W, S):

(Exc) Let $w \in W$ and $s \in S$ be such that $\ell(sw) \leqslant \ell(w)$. For any reduced representation $w = s_1 \cdots s_q$, there exists an index $1 \leqslant j \leqslant q$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

Proposition 1.10. Let (W, S) be a pair as in Definition 1.9 and satisfying (Exc). Let $s \in S$, $w \in W$ and $w = s_1 \cdots s_q$ be a reduced representation of w. Then one of the following must hold:

(i) $\ell(sw) = \ell(w) + 1$ and $sw = ss_1 \cdots s_q$ is a reduced representation of sw, or

(ii) $\ell(sw) = \ell(w) - 1$ and there exists an index $1 \leqslant j \leqslant q$ such that $sw = s_1 \cdots s_{j-1}s_{j+1} \cdots s_q$ is a reduced representation of sw and $w = ss_1 \cdots s_{j-1}s_{j+1} \cdots s_q$ is a reduced representation of w.

Proof. Let $w' = sw$. We know that

$$|\ell(w) - \ell(w')| \leqslant \ell(s) = 1.$$

Suppose first that $\ell(w') > \ell(w)$. Then $\ell(w') = q + 1$ and $w' = ss_1 \cdots s_q$ whence this is also a reduced representation.

Next, suppose $\ell(w') \leqslant \ell(w)$. Due to (Exc), there exists an index $1 \leqslant j \leqslant q$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

Then $w = ss_1 \cdots s_{j-1}s_{j+1} \cdots s_q$. Since $\ell(w') \geqslant q - 1$, we must have $\ell(w') = q - 1$ and that the above representation is reduced. ■

Lemma 1.11. Let (W, S) be a pair as in Definition 1.9 and satisfying (Exc). Let $w \in W$ have length $q \geqslant 1$, let D be the set of all reduced representations of w, and let $F : D \to E$.

Assume that $F(\mathbf{s}) = F(\mathbf{s}')$ if the elements $\mathbf{s} = (s_1, \dots, s_q)$ and $\mathbf{s}' = (s'_1, \dots, s'_q)$ of D satisfy one of the following:

(i) $s_1 = s'_1$ or $s_q = s'_q$; or

(ii) there exist s and s' in S such that $s_j = s'_k = s$ and $s_k = s'_j = s'$ for j odd and k even.

Then F is constant.

Proof. The proof proceeds in two steps:
Step 1. Let $\mathbf{s}, \mathbf{s}' \in D$ and put $\mathbf{t} = (s'_1, s_1, \dots, s_{q-1})$. We shall show that if $F(\mathbf{s}) \neq F(\mathbf{s}')$ then $\mathbf{t} \in D$ and $F(\mathbf{t}) \neq F(\mathbf{s})$.

Indeed, $w = s'_1 \cdots s'_q$ and $s'_1 w = s'_2 \cdots s'_q$, so that $\ell(s'_1 w) < q = \ell(w)$. Due to Proposition 1.10 (ii), there is an index $1 \leqslant j \leqslant q$ such that $\mathbf{u} = (s'_1, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_q)$ belongs to D. Due to condition (i), we have $F(\mathbf{u}) = F(\mathbf{s}')$. If $j \neq q$, then we would also have $F(\mathbf{u}) = F(\mathbf{s})$ due to condition (i), contrary to our hypothesis that $F(\mathbf{s}) \neq F(\mathbf{s}')$. Thus $j = q$ and hence $\mathbf{t} = \mathbf{u} \in D$ and $F(\mathbf{t}) = F(\mathbf{s}') \neq F(\mathbf{s})$, as desired.

Step 2. Let $\mathbf{s}, \mathbf{s}' \in D$. For $0 \leqslant j \leqslant q+1$, define a sequence $\mathbf{s}_j$ of q-elements of S as:

$$\begin{aligned}
\mathbf{s}_0 &= (s'_1, \dots, s'_q)\\
\mathbf{s}_1 &= (s_1, \dots, s_q)\\
\mathbf{s}_{q+1-k} &= \begin{cases} (s_1, s'_1, \dots, s_1, s'_1, s_1, s_2, \dots, s_k) & q-k \text{ even and } 0 \leqslant k \leqslant q \\ (s'_1, s_1, \dots, s_1, s'_1, s_1, s_2, \dots, s_k) & q-k \text{ odd and } 0 \leqslant k \leqslant q \end{cases}
\end{aligned}$$

Let (H_j) denote the assertion:

$$\text{“}\mathbf{s}_j \in D,\ \mathbf{s}_{j+1} \in D \text{ and } F(\mathbf{s}_j) \neq F(\mathbf{s}_{j+1})\text{''}.$$

Due to **Step 1**, $(H_j) \implies (H_{j+1})$ for $0 \leqslant j \leqslant q$, and due to condition (ii), (H_q) is false. Hence (H_0) is false, so that $F(\mathbf{s}) = F(\mathbf{s}')$, thereby completing the proof. ∎

Proposition 1.12. Let M be a monoid and $f : S \to M$. Set

$$a(s, s') = \begin{cases} (f(s)f(s'))^l & m(s, s') = 2l \\ (f(s)f(s'))^l f(s) & m(s, s') = 2l+1 \\ 1 & m(s, s') = \infty. \end{cases}$$

If $a(s, s') = a(s', s)$ whenever $s \neq s'$ in S, then there exists a map $g : W \to M$ such that

$$g(w) = f(s_1) \cdots f(s_q)$$

for every reduced representation $w = s_1 \cdots s_q$ of $w \in W$.

Proof. For $w \in W$, let D_w be the set of all reduced representations of w and $F_w : D_w \to M$ given by

$$F_w(s_1, \dots, s_q) = f(s_1) \cdots f(s_q).$$

We shall argue by induction on $\ell(w)$ that F_w is a constant function. The base cases $\ell(w) = 0, 1$ are trivial. Suppose now that $q = \ell(w) \geqslant 2$ and the inductive hypothesis has been proven for all lengths $< q$. In light of Lemma 1.11, it suffices to show that $F_w(\mathbf{s}) = F_w(\mathbf{s}')$ in both conditions of the aforementioned lemma.

(i) This is quite straightforward using the inductive hypothesis and the equality

$$F_w(s_1, \dots, s_q) = f(s_1)F_{w'}(s_2, \dots, s_q) = F_{w''}(s_1, \dots, s_{q-1})f(s_q).$$

(ii) This is a bit cumbersome. See [Bou08, pg. 9] ∎

Theorem 1.13. Let (W, S) be a pair such that S generates W, $1 \notin S$, $S^{-1} = S$ and every element in S has order at most 2. Then (W, S) is a Coxeter system if and only if it satisfies (Exc).

Proof. We have already seen that a Coxeter system satisfies (Exc). Conversely, suppose (W, S) is a pair as in 1.9 and satisfies (Exc). To show that (W, S) is a Coxeter system, it suffices to show that it has the desired *universal property* of its presentation.

Indeed, let G be a group and $f : S \to G$ be a map such that $(f(s)f(s'))^{m(s,s')} = 1$ whenever $m(s, s') < \infty$. Due to Proposition 1.12, there exists a map $g : W \to G$ such that

$$g(w) = f(s_1) \cdots f(s_q)$$

whenever $w = s_1 \cdots s_q$ is a reduced representation of w. It suffices to show that g is a group homomorphism. To this end, since S generates W, it suffices to show that

$$g(sw) = f(s)g(w) \qquad \forall\ s \in S,\ \forall\ w \in W.$$

Due to Proposition 1.10, there are two possible cases:

(i) If $\ell(sw) = \ell(w)+1$ then choosing a reduced representation $w = s_1 \cdots s_q$, it follows that $sw = ss_1 \cdots s_q$ is a reduced representation of sw. Hence

$$g(sw) = f(s)f(s_1) \cdots f(s_q) = f(s)g(w).$$

(ii) If $\ell(sw) = \ell(w) - 1$ put $w' = sw$. Then $w = sw'$ and $\ell(sw') = \ell(w') + 1$. Due to case (i), $g(sw') = f(s)g(w')$, i.e., $f(s)g(w) = g(sw)$ since $f(s)^2 = 1$. ∎

Proposition 1.14. Let (W, S) be a Coxeter system. For $s \in S$, set

$$P_s = \{w \in W\colon \ell(sw) > \ell(w)\}.$$

(I) $\bigcap_{s \in S} P_s = \{1\}$.

(II) For any $s \in S$, the sets P_s and sP_s form a partition of W.

(III) Let $s, s' \in S$ and let $w \in W$. If $w \in P_s$ and $ws' \notin P_s$ then $sw = ws'$.

Proof. (I) Let $1 \neq w \in W$ and let $w = s_1 \cdots s_q$ be a reduced representation of w with $q \geqslant 1$. Clearly $s_1 w = s_2 \cdot s_q$ is a reduced representation of $s_1 w$, so that $w \notin P_{s_1}$.

(II) Let $w \in W$ and $s \in S$. Due to Proposition 1.10, there are two cases to handle:

(i) $\ell(sw) = \ell(w) + 1$: then $w \in P_s$.

(ii) $\ell(sw) = \ell(w) - 1$: then setting $w' = sw$, we see that $\ell(sw') = \ell(w') + 1$, so that $w' \in P_s$ and $w \in sP_s$.

To see that $P_s \cap sP_s = \emptyset$, suppose $w \in P_s \cap sP_s$. Then $w = sw'$ where $w' \in P_s$, so that $\ell(w) = \ell(sw') > \ell(w')$. But since $w' = sw$ and $w \in P_s$, we must have $\ell(w') = \ell(sw) > \ell(w)$, a contradiction.

(III) Let $q = \ell(w)$. Since $w \in P_s$, it follows that $\ell(sw) = q + 1$ and from $ws' \notin P_s$ it follows that $sws' \in P_s$, so that $q+1 \geqslant \ell(ws') = \ell(sws')+1$ and hence $\ell(sws') \leqslant q$. Further, since $\ell(sws') = \ell(sw) \pm 1$, we must have $\ell(sws') = q$ and $\ell(ws') = q + 1$.

Let $w = s_1 \cdots s_q$ be a reduced representation of w and set $s_{q+1} = s'$. Then $ws' = s_1 \cdots s_{q+1}$ is a reduced representation of ws'. Due to (Exc) and the fact that $\ell(sws') \leqslant \ell(ws')$, there is an index $1 \leqslant j \leqslant q+1$ such that

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j.$$

If $1 \leqslant j \leqslant q$, we would have $sw = s_1 \cdots s_{j-1} s_{j+1} \cdot s_q$, contradicting the fact that $\ell(sw) = q + 1$. Thus $j = q + 1$, i.e., $sw = ws'$, as desired. ∎

Proposition 1.15. Let (W, S) be a generating pair such that every element in S has order at most 2. Let $(P_s)_{s \in S}$ be a family of subsets of W satisfying (III) and the following additional conditions:

(I') $1 \in P_s$ for all $s \in S$.

(II') The sets P_s and sP_s are disjoint for all $s \in S$.

Then (W, S) is a Coxeter system and

$$P_s = \{s \in S\colon \ell(sw) > \ell(w)\}.$$

Proof. Let $s \in S$ and $w \in W$. There are two cases:

(i) $w \notin P_s$. Clearly, $w \neq 1$, so $q = \ell(w) \geqslant 1$. Let $w = s_1 \cdots s_q$ be a reduced representation of w. Set

$$w_j = s_1 \cdots s_j \qquad 1 \leqslant j \leqslant q,$$

and $w_0 = 1$. Since $w_0 \in P_s$ and $w_q \notin P_s$, there is an index $1 \leqslant j \leqslant q$ such that $w_{j-1} \in P_s$ but $w_j \notin P_s$. Since $w_j = w_{j-1} s_j$, using (III), $sw_{j-1} = w_{j-1} s_j = w_j$. Therefore,

$$sw = s_1 \cdots s_{j-1} s_{j+1} \cdots s_q$$

so that $\ell(sw) < \ell(w)$.

(ii) $w \in P_s$. Put $w' = sw$, so that $w' \notin P_s$ due to (II'). Then by (i), we have $\ell(w) = \ell(sw') < \ell(w') = \ell(sw)$.

In particular, this shows that $P_s = \{w \in W\colon \ell(sw) > \ell(w)\}$. Finally, to show that (W, S) is a Coxeter system, in light of Theorem 1.13, we shall show that it satisfies (Exc). Indeed, let $w \in W$ and $s \in S$ such that $\ell(sw) \leqslant \ell(w)$. Then $w \notin P_s$ and repeating the same argument as in (i), we see that (Exc) is satisfied. ∎

Henceforth, let (W, S) be a Coxeter system. For any subset $X \subseteq S$, we denote by W_X the subgroup of W generated by X.

Proposition 1.16. Let $w \in W$. There exists a subset S_w of S such that $S_w = \{s_1, \dots, s_q\}$ for *any* reduced representation $w = s_1 \cdots s_q$.

Proof. Let M denote the monoid of subsets of S with the union operation. Set $f(s) = \{s\}$ for $s \in S$. In the notation of Proposition 1.12, if $m(s, s') < \infty$, then $a(s, s') = \{s, s'\} = a(s', s)$. And if $m(s, s') = \infty$, then $a(s, s') = a(s', s) = 1$. Thus, the map f extends to a map $g : W \to M$ with the properties stated in the Proposition. It is clear now that the proof is complete. ∎

Corollary 1.17. For any subset $X \subseteq S$,

$$W = \{w \in W \colon S_w \subseteq X\} .$$

Proof. Clearly $S_{w^{-1}} = S_w$ and due to Proposition 1.10, $S_{sw} \subseteq \{s\} \cup S_w$ for $s \in S$ and $w \in W$; so that $S_{ww'} \subseteq S_w \cup S_{w'}$. Therefore, the set

$$U = \{w \in W \colon S_w \subseteq X\}$$

is a subgroup of W containing X and hence must be equal to W_X. ∎

Corollary 1.18. For any subset $X \subseteq S$, we have $W_X \cap S = X$.

Proof. This follows from the fact that $S_s = \{s\}$ for every $s \in S$. ∎

Corollary 1.19. The set S is a minimal generating set of W.

Proof. Follows from the preceding Corollary. ∎

Corollary 1.20. For any subset $X \subseteq S$ and $w \in W_X$, $\ell_X(w) = \ell_S(w)$.

Proof. Any reduced representation of w must have all elements contained in X. ∎

Theorem 1.21. (1) For any subset $X \subseteq S$, the pair (W_X, X) is a Coxeter system.

(2) Let $(X_i)_{i \in I}$ be a family of subsets of S. If $X = \bigcap_{i \in I} X_i$, then $W_X = \bigcap_{i \in I} W_{X_i}$.

(3) Let X and X' be two subsets of S. Then $W_X \subseteq W_{X'}$ (resp. $W_X = W_{X'}$) if and only if $X \subseteq X'$ (resp. $X = X'$).

Proof. To see (1), it suffices to show that (W_X, X) satisfies (Exc). Indeed, let $x \in X$ and $w \in W_X$ such that $\ell_X(xw) \leqslant \ell_X(w)$ and let $w = x_1 \cdots x_q$ be a reduced representation of w. Due to Corollary 1.20, there is an index $1 \leqslant j \leqslant q$ such that

$$xx_1 \cdots x_{j-1} = x_1 \cdots x_{j-1} x_j .$$

Thus (X, W_X) satisfies (Exc) and thus is a Coxeter system due to Theorem 1.13.

As for (2), any $w \in \bigcap_{i \in I} W_{X_i}$, $S_w \subseteq X_i$ for each $i \in I$ and hence $S_w \subseteq X$, so that $w \in W_X$. The inclusion $W_X \subseteq \bigcap_{i \in I} W_{X_i}$ trivial and hence, we have equality.

Finally, for (3), if $W_X \subseteq W_{X'}$, then

$$X = W_X \cap S \subseteq W_{X'} \cap S = X',$$

and conversely, if $X \subseteq X'$, then the inclusion $W_X \subseteq W_{X'}$ is clear. Once this has been established, the assertion about equality is trivial. ∎

§2 Tits Systems

Definition 2.1. A *Tits system* is a tuple (G, B, N, S), where G is a group, B and N are two subgroups of G and S is a subset of $W := N/(B \cap N)$, satisfying the following axioms:

(Tits 1) The set $B \cup N$ generates G and $T := B \cap N$ is a normal subgroup of N.

(Tits 2) The set S generates the group W and every element of S has order at most 2.

(Tits 3) $sBw \subseteq BwB \cup BswB$ for $s \in S$ and $w \in W$.

(Tits 4) For all $s \in S$, $sBs \not\subseteq B$.

The group W is called the *Weyl group* of the Tits system.

Remark 2.2. Note that every $w \in W$ denotes a coset and as such, is a subset of B. Therefore, all products wB and Bw are defined to be products of sets, that is,

$$wB = \bigcup_{a \in w} aB, \quad Bw = \bigcup_{a \in W} Ba, \quad \text{and} \quad BwB = \bigcup_{a \in w} BaB.$$

Since $T \subseteq B$, we clearly have $wB = aB$ for each $a \in w$, therefore, it suffices to interpret the above formulas by treating $W \subseteq B$ through a (likely non-canonical) lift.

For any $w \in W$, let $C(w)$ denote the double coset BwB. It is clear that

$$C(1) = B, \quad B(ww') \subseteq C(w)C(w'), \quad \text{and} \quad C(w^{-1}) = C(w)^{-1}.$$

Due to (Tits 3), we have

$$C(s)C(w) \subseteq C(w) \cup C(sw).$$

Moreover, since $C(sw) \subseteq C(s)C(w)$, and the latter is a union of double cosets, there are only two possibilities

$$C(s)C(w) = \begin{cases} C(sw) & C(w) \not\subseteq C(s)C(w) \\ C(w) \cup C(sw) & C(w) \subseteq C(s)C(w). \end{cases} \tag{2}$$

Due to (Tits 4), $B \neq C(s)C(s)$, so that

$$C(s)C(s) = B \cup C(s).$$

It follows that $B \cup C(s)$ is closed under inversion and multiplication, and hence is a subgroup of G. Multiplying both sides of the above by $C(w)$, and using (2),

$$C(s)C(s)C(w) = BC(w) \cup C(s)C(w) = C(w) \cup C(s)C(w) = C(w) \cup C(sw). \tag{3}$$

Taking inverses of all the above formulas and replacing w^{-1} by w, we obtain

$$\begin{aligned} C(w)C(s) &\subseteq C(w) \cup C(ws) \\ C(w)C(s) &= \begin{cases} C(ws) & C(w) \not\subseteq C(w)C(s) \\ C(w) \cup C(ws) & C(w) \subseteq C(w)C(s) \end{cases} \\ C(w)C(s)C(s) &= C(w) \cup C(ws). \end{aligned}$$

Lemma 2.3. Let $s_1, \dots, s_q \in S$ and let $w \in W$. We have

$$C(s_1 \cdots s_q)C(w) \subseteq \bigcup_{\substack{1 \leqslant i_1 < \cdots < i_p \leqslant q \\ 0 \leqslant p \leqslant q}} C(s_{i_1} \cdots s_{i_p} w).$$

Proof. Argue by induction on $q \geqslant 0$. The base case $q = 0$ is trivial. For the induction step, use

$$C(s_1 \cdots s_q)C(w) \subseteq C(s_1)C(s_2 \cdots s_q)C(w),$$

the induction hypothesis, and

$$C(s_1)C(s_{j_1} \cdots s_{j_p} w) \subseteq C(s_1 s_{j_1} \cdots s_{j_p} w) \cup C(s_{j_1} \cdots s_{j_p} w)$$

to complete the proof. ∎

Theorem 2.4. ([Mac71, 2.3.1]) $G = BWB$. The map $w \mapsto C(w)$ is a bijection between W and $B \backslash G / B$, the set of double cosets of G with respect to B.

Proof. Clearly BWB is stable under inversion and due to Lemma 2.3, it is stable under products too. It follows that BWB is a subgroup of G containing B and N, therefore, $BWB = G$ due to (Tits 1).

Surjectivity of the map $C : W \to B \backslash G / B$ is clear from the fact that $G = BWB$. It remains to show that C is injective. We shall argue by induction on $q \geqslant 0$ that:

$$\text{“if } w \neq w' \in W \text{ and } \ell(w) \geqslant \ell(w') = q, \text{ then } C(w) \neq C(w')\text{”}.$$

In the base case $q = 0$, $w' = 1$. If $BwB = B$, then $w \in B$, so that $w = 1$. Suppose now that $q \geqslant 1$ and $\ell(w) \geqslant \ell(w') = q$. There exists $s \in S$ such that $\ell(sw') = q - 1$. Thus,

$$\ell(w) > \ell(sw') \qquad \ell(sw) \geqslant \ell(w) - 1 \geqslant q - 1 = \ell(sw').$$

As a result of the inductive hypothesis, $C(w) \neq C(sw')$ and $C(sw) \neq C(sw')$; hence

$$C(sw') \cap (C(s)C(w)) \subseteq C(sw') \cap (C(sw) \cup C(w)) = \emptyset,$$

and $C(sw') \subseteq C(s)C(w')$, in particular, $C(sw') \cap (C(s)C(w)) \neq \emptyset$. It follows that $C(w) \neq C(w')$. ■

Theorem 2.5. ([Mac71, 2.3.7]) The pair (W, S) is a Coxeter system. Moreover, for $s \in S$ and $w \in W$,

$$C(s)C(w) = C(sw) \iff \ell(sw) > \ell(w).$$

Proof. For $s \in S$, set

$$P_s = \{w \in W \colon C(sw) = C(s)C(w)\}.$$

We shall verify that the P_s satisfy the conditions of Proposition 1.15. Condition (I') is clearly satisfied.

To verify (II'), suppose $w \in P_s \cap sP_s$, we would then have $w, sw \in P_s$, so that

$$C(s)C(w) = C(sw) \qquad C(s)C(sw) = C(w),$$

that is, $C(s)C(s)C(w) = C(w)$, which in light of (3) implies $C(sw) = C(w)$, a contradiction to Theorem 2.4.

Finally, we verify (III). Let $s, s' \in S$ and $w, w' \in W$ with $w' = ws'$ and $w \in P_s$ but $w' \notin P_s$. Hence

$$C(sw) = C(s)C(w) \quad \text{ and } \quad C(w') \subseteq C(s)C(w') = C(s)w'B,$$

due to 2. As a result, there exist $b, b', b'' \in B$ such that $bw'B = b'sb''w'B$, whence $w'^{-1}b'sb''w' \in B$, in particular, $w'B = b'sb''w'B$, therefore, $C(w') \cap C(s)w' \neq \emptyset$.

The relation $w = w's'$ implies

$$C(sw) = C(s)w's'B.$$

We have seen that $C(w')C(s') \subseteq C(w') \cup C(w's')$, which implies

$$C(w')s'B \subseteq C(ws') \cup C(w).$$

Since $C(s)w'$ meets $C(w')$, it follows that $C(sw) = C(s)w's'B$ meets $C(w')s'B \subseteq C(ws') \cup C(w)$. Therefore, $C(sw)$ is equal to one of the double cosets $C(ws')$ or $C(w)$. Since $sw \neq w$, in conjunction with Theorem 2.4, we must have $sw = ws'$, as desired. ■

Corollary 2.6. Let $w_1, \dots, w_q \in W$ and let $w = w_1 \cdots w_q$. If

$$\ell(w) = \ell(w_1) + \cdots + \ell(w_q),$$

then

$$C(w) = C(w_1) \cdots C(w_q).$$

Proof. Take reduced representations for each of the w_i's. The concatenation of these representations must form a reduced representation of w. It is clear from the theorem that given a reduced representation $s_1 \cdots s_n$ of w, we must have $C(w) = C(s_1) \cdots C(s_n)$. The corollary follows hence. ■

Corollary 2.7. For each $w \in W$, let T_w be as in Lemma 1.7. If $t \in T_w$, then $C(t) \subseteq C(w)C(w^{-1})$.

Proof. Choose a reduced representation $w = s_1 \cdots s_q$, then due to Lemma 1.7, $T_w = \{t_1, \dots, t_q\}$, where

$$t_j = (s_1 \cdots s_{j-1})s_j(s_1 \cdots s_{j-1})^{-1}$$

and we have $s_1 \cdots s_j = t_j \cdots t_1$.

Let $t \in T_w$ and say $1 \leqslant j \leqslant q$ is such that $t = t_j$. Set $w' = s_1 \dots s_{j-1}$ and $w'' = s_{j+1} \cdots s_q$. Then we have

$$w = w'sw'', \quad \ell(w) = \ell(w') + \ell(w'') + 1, \quad \text{and} \quad t = w'sw'^{-1}.$$

Due to Corollary 2.6,

$$C(w)C(w^{-1}) = C(w')C(s)C(w'')C(w''^{-1})C(s)C(w'^{-1}) \supseteq C(w')C(s)C(s)C(w'^{-1}).$$

But we know that $C(s) \subseteq B \cup C(s) = C(s)C(s)$, and hence

$$C(t) \subseteq C(w')C(s)C(w'^{-1}) = C(w')C(s)C(s)C(w'^{-1}) \subseteq C(w)C(w^{-1}),$$

as desired. ■

Corollary 2.8. Let $w \in W$ and let H_w be the subgroup of G generated by $C(w)C(w^{-1})$. Then

(i) For any reduced representation $w = s_1 \cdots s_q$, $C(s_j) \subseteq H_w$ for $1 \leqslant j \leqslant q$.

(ii) The group H_w contains $C(w)$ and is generated by $C(w)$.

Proof. (i) We induct on $j \geqslant 1$. The base case is clear from Corollary 2.7. Suppose now that $j > 1$. Let $t = (s_1 \cdots s_{j-1})s_j(s_1 \cdots s_{j-1})^{-1}$. Then due to Lemma 1.7 $t \in T_w$ and $C(t) \subseteq H_w$ due to Corollary 2.7. Using the induction hypothesis and

$$C(s_j) \subseteq C((s_1 \cdots s_{j-1})^{-1})C(t)C(s_1 \cdots s_{j-1}) \subseteq H_w,$$

as desired.

(ii) By Corollary 2.6, we have that $C(w) = C(s_1) \cdots C(s_q)$, and hence $C(w) \subseteq H_w$. This completes the proof. ■

Definition 2.9. For any subset $X \subseteq S$, denote by W_X the subgroup of W generated by X and by G_X the set $BW_XB \subseteq G$. Set $G_\emptyset = B$.

Theorem 2.10. (i) ([Mac71, 2.3.2])For $X \subseteq X$, G_X is a subgroup of G generated by $\bigcup_{s\in X} C(s)$.

(ii) ([Mac71, 2.3.3]) The map $X \mapsto G_X$ is a bijection from $\mathscr{P}(S)$ to the set of subgroups of G containing B.

(iii) Let $(X_i)_{i\in I}$ be a family of subsets of X. If $X = \bigcap_{i\in I} X_i$, then $G_X = \bigcap_{i\in I} G_{X_i}$.

(iv) Let X and Y be two subsets of X. Then $G_X \subseteq G_Y$ (resp. $G_X = G_Y$) if and only if $X \subseteq Y$.

Proof. (i) Clearly $G_X = G_X^{-1}$ and Lemma 2.3 shows that $G_XG_X \subseteq G_X$. Hence, G_X is a subgroup of G. Further, due to Corollary 2.6 it is clear that G_X is generated by $\bigcup_{s\in X} C(s)$.

(ii) Since the map $X \mapsto W_X$ is injective and there is a bijection between W and $B\backslash G/B$, it follows that the map $X \mapsto G_X$ is injective.

Conversely, let H be a subgroup of G containing B. Let

$$U = \{w \in W \colon C(w) \subseteq H\},$$

and let $X = U \cap S$. Clearly U is a subgroup of W so that $W_X \subseteq U$ and $G_X \subseteq H$. On the other hand, let $u \in U$ and $u = s_1 \cdots s_q$ bea reduced representation of u. By Corollary 2.8, $C(s_j) \subseteq H$, and hence $s_j \in X$ for $1 \leqslant j \leqslant q$. Thus, $u \in W_X$, and since $H = \bigcup_{u\in U} C(u)$, it follows that $H \subseteq G_X$, thereby proving (ii).

(iii) Clear.

(iv) Clear. ■

Corollary 2.11. $S = \{w \in W \colon w \neq 1,\ B \cup C(w) \text{ is a subgroup of } G\}$.

Proof. Clearly, for any $s \in S$, $B \cup C(s)$ forms a subgroup of G because we have already shown that $C(s)C(s) \subseteq B \cup C(s)$. Conversely, if $w \in W$ is such that $B \cup C(w)$ forms a subgroup of G, then this subgroup is equal to BW_XB, where $W_X = \{1, w\}$ (recall the bijection between W and double cosets). Thus, X generates the group $\{1, w\}$, and hence $\#X = 1$ i.e., $w \in S$. ■

Proposition 2.12. ([Mac71, 2.3.5]) Let $X, Y \subseteq X$ and $w \in W$. Then

$$G_XwG_Y = BW_XwW_YB.$$

Proof. Clearly $BW_XwW_YB \subseteq G_XwG_Y$. We prove the other inclusion. Let $s_1, \dots, s_q \in X$ and $t_1, \dots, t_p \in Y$. Then, due to Lemma 2.3, it follows that

$$C(s_1 \cdots s_q)C(w)C(t_1 \cdots t_p) \subseteq BW_XwW_YB,$$

and therefore

$$G_XwG_Y \subseteq BW_XwW_YB,$$

thereby completing the proof. ■

Proposition 2.13. Let $g \in G$ and $X \subseteq S$. If $gBg^{-1} \subseteq G_X$, then $g \in G_X$.

Proof. Let $w \in W$ be such that $g \in C(w)$. Since B is a subgroup of G, the fact that $gBg^{-1} \subseteq G_X$ implies $C(w)C(w^{-1}) \subseteq G_X$. In the notation of Corollary 2.8, we have $H_w \subseteq G_X$, so that $C(w) \subseteq G_X$, whence $g \in G_X$. ■

Definition 2.14. A subgroup of G is said to be *parabolic* if it contains a conjugate of B.

Proposition 2.15. Let P be a subgroup of G.

(i) P parabolic if and only if there exists a subset $X \subseteq S$ such that P is conjugate to G_X.

(ii) ([Mac71, 2.3.4]) Let $X, X' \subseteq S$ and $g, g' \in G$ be such that $P = gG_Xg^{-1} = g'G_{X'}g'^{-1}$. Then $X = X'$ and $g'g^{-1} \in P$.

Proof. (i) Immediate from Theorem 2.10.

(ii) We have

$$g^{-1}g'Bg'^{-1}g \subseteq g^{-1}g'G_{X'}g'^{-1}g = G_X,$$

and hence, due to Proposition 2.13, it follows that $g^{-1}g' \in G_X$, whence $G'_X = G_X$, so that $X = X'$ due to Theorem 2.10. Finally,

$$g'g^{-1} = gg^{-1}g'g^{-1} \in gG_Xg^{-1} = P,$$

thereby completing the proof. ■

Theorem 2.16. (i) Let P_1 and P_2 be two parabolic subgroups of G whose intersection is parabolic and let $g \in G$ be such that $gP_1g^{-1} \subseteq P_2$. Then $g \in P_2$ and $P_1 \subseteq P_2$.

(ii) Two parabolic subgroups whose intersection is parabolic are not conjugate unless they are equal.

(iii) Let Q_1 and Q_2 be two parabolic subgroups of G contained in a subgroup Q of G. Then any $g \in G$ such that $gQ_1g^{-1} = Q_2$ belongs to Q.

(iv) ([Mac71, 2.3.6]) Every parabolic subgroup is self-normalizing.

Proof. For (i), since the intersection is parabolic, there is an $h \in G$ such that $hBh^{-1} \subseteq P_1 \cap P_2$. As a result, $h^{-1}P_1h = G_{X_1}$ and $h^{-1}P_2h = G_{X_2}$ for some $X_1, X_2 \subseteq S$. Our hypothesis implies

$$ghG_{X_1}(gh)^{-1} \subseteq hG_{X_2}h^{-1} \implies (h^{-1}gh)G_{X_1}(h^{-1}gh)^{-1} \subseteq G_{X_2} \implies (h^{-1}gh)B(h^{-1}gh) \subseteq G_{X_2},$$

so that $h^{-1}gh \in G_{X_2}$ due to Proposition 2.13, i.e., $G_{X_1} \subseteq G_{X_2}$, therefore, $P_1 \subseteq P_2$. Finally, since $h^{-1}gh \in G_{X_2}$, we must have $g \in P_2$, proving (i).

Assersion (ii) is immediate from (i). Assersion (iii) follows from (i) because Q is a parabolic such that $Q_1 \cap Q = Q_1$ is parabolic and $gQ_1g^{-1} \subseteq Q$. Assersion (iv) is an immediate consequence of (iii). ■

References

[Bou08] N. Bourbaki. *Lie Groups and Lie Algebras: Chapters 4-6*. Elements of Mathematics. Springer Berlin Heidelberg, 2008.

[Mac71] I.G. Macdonald. *Spherical Functions on a Group of P-adic Type*. Publications of the Ramanujan Institute. Ramanujan Institute for Advanced Study in Mathematics, University of Madras, 1971.