Covering Spaces

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§1 Covering Spaces

Lemma 1.1 (Uniqueness of Lifts). Let $p : E \to B$ be a covering map and $f : X \to B$ a continuous map from a connected topological space X. If $\widetilde{f} : X \to E$ is a lift of f, then it is unique.

Proof. ■

Lemma 1.2 (Path Lifting). Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\widetilde{f} : [0,1] \to E$ beginning at e_0.

Proof. The uniqueness of the lift follows from Lemma 1.1. Begin with an open cover $\mathscr{U}$ of B such that each $U \in \mathscr{U}$ is evenly covered by p. The collection $\{f^{-1}U : U \in \mathscr{U}\}$ is an open cover of $[0,1]$. Using the Lebesgue Number Lemma, we can subdivide $[0,1]$ as

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

such that $[s_i, s_{i+1}] \in f^{-1}U$ for some $U \in \mathscr{U}$.

We construct the lift $\widetilde{f}$ inductively. Suppose $\widetilde{f}$ has been defined on $[s_0, s_i]$ for $0 \leqslant i \leqslant n-1$. Choose $U \in \mathscr{U}$ such that $[s_i, s_{i+1}] \subseteq f^{-1}U$. There is a unique open set $V \subseteq p^{-1}U$ such that $\widetilde{f}(s_i) \in V$ and $p|_V : V \to U$ is a homeomorphism. Define $\widetilde{f}$ to be $p|_V^{-1} \circ f$ on $[s_i, s_{i+1}]$, on which it is obviously continuous. Due to the pasting lemma, we obtain a continuous function $\widetilde{f}$ on $[s_0, s_{i+1}]$. ■

Lemma 1.3 (Homotopy Lifting). Let $p : E \to B$ be a covering map, $p(e_0) = b_0$, and $F : I \times I \to B$ be a homotopy with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map $\widetilde{F} : I \times I \to E$ such that $\widetilde{F}(0,0) = e_0$. If F is a path homotopy, then $\widetilde{F}$ is a path homotopy.

Proof. Again, the uniqueness of the lift follows from Lemma 1.1. Begin with an oepn cover $\mathscr{U}$ of B such that each $U \in \mathscr{U}$ is evenly covered by p. The collection $\{F^{-1}U : U \in \mathscr{U}\}$ is an open cover of $I \times I$. Using the Lebesgue Number Lemma, we can find subdivisions

$$0 = s_0 < s_1 < \cdots < s_n = 1 \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_n = 1,$$

such that the image of each rectangle $R_{i,j} = [s_i, s_{i+1}] \times [t_i, t_{i+1}]$ under f is contained in some $U \in \mathscr{U}$. As in the preceding proof, we shall construct the lift $\widetilde{F}$ inductively in the following fashion:

$$R_{0,0}, \ldots, R_{n-1,0}, R_{0,1}, \ldots, R_{n-1,1}, \ldots R_{n-1,n-1}.$$

First, using path lifting, we can define $\widetilde{F}$ on $I \times \{0\}$ and $\{0\} \times I$. Due to the pasting lemma, $\widetilde{F}$ is continuous on $I \times \{0\} \cup \{0\} \times I$.

We shall now describe how to define $\widetilde{F}$ on $R_{i,j}$ given that $\widetilde{F}$ has been defined for all preceding rectangles in the sequence. Let A denote the union of these rectangles and the left and bottom edges of $I \times I$. In particular, this means that $\widetilde{F}$ has been defined on the left and bottom edges of $R_{i,j}$; let their union be denoted by L. Let $U \in \mathscr{U}$ be such that $R_{i,j} \subseteq F^{-1}U$. Since U is an evenly covered neighborhood, we can write

$p^{-1}U = \bigsqcup V_\alpha$, where $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for each α. The since L is connected, its image under $\widetilde{F}$ is connected and contained in $\bigsqcup V_\alpha$. Let V_α be the unique slice over U containing $\widetilde{F}(L)$. Finally, define $\widetilde{F} = p|_{V_\alpha}^{-1} \circ F$ on $R_{i,j}$. Due to the pasting lemma, it is clear that this extension of $\widetilde{F}$ is continuous on $A \cup R_{i,j}$. This constructs a lift $\widetilde{F} : I \times I \to E$, as required.

Finally, if F is a path homotopy, then F is constant on $\{0\} \times I$ and $\{1\} \times I$, both of which are connected. Further, since $\widetilde{F}$ lifts F, the image $\widetilde{F}(\{0\} \times I)$ is contained in $p^{-1}(b_0)$, which is a discrete, and in particular, totally disconnected set. It follows that $\widetilde{F}$ is constant on $\{0\} \times I$. Similarly one can argue that $\widetilde{F}$ is constant on $\{1\} \times I$. Hence, $\widetilde{F}$ is a path homotopy too. ■

COROLLARY 1.4. Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. Let $f, g : I \to B$ be two paths in B from b_0 to b_1 and let $\widetilde{f}, \widetilde{g} : I \to B$ be their (unique) lifts to paths in E beginning at e_0. If f and g are path homotopic, then $\widetilde{f}$ and $\widetilde{g}$ end at the same point and are path homotopic.

Proof. ■

§2 Classification of Covering Spaces

§§ Existence of Covers

THEOREM 2.1. Let B be a path connected, locally path connected, semilocally simply connected topological space and $b_0 \in B$ a basepoint. Given a subgroup $H \subseteq \pi_1(B, b_0)$, there is a path connected covering $p : E \to B$ and $e_0 \in p^{-1}(b_0)$ such that

$$p_*(\pi_1(E_0, e_0)) = H.$$

Proof. The construction proceeds in several bite-sized steps.
Step 1. Construction of E. Let $\mathscr{P}$ denote the set of all paths in B that begin at b_0. Define $\alpha \sim \beta$ in $\mathscr{P}$ if and only if

$$\alpha(1) = \beta(1) \quad \text{and} \quad [\alpha * \overline{\beta}] \in H.$$

It is clear that the relation $\sim$ is an equivalence relation. Let $E = \mathscr{P}/\sim$ be the set of all equivalence classes under this relation. For $\alpha \in \mathscr{P}$, we use $\alpha^\sharp$ to denote its equivalence class in E. Define the map $p : E \to B$ by $p(\alpha^\sharp) = \alpha(1)$.

Before we proceed, we note two observations:

(1) If $[\alpha] = [\beta]$ as elements of the fundamental groupoid, then $\alpha^\sharp = \beta^\sharp$. Indeed, since $[\alpha * \overline{\beta}] = e_{b_0} \in H$.

(2) If $\alpha^\sharp = \beta^\sharp$, then $(\alpha * \delta)^\sharp = (\beta * \delta)^\sharp$ for any path δ beginning at $\alpha(1) = \beta(1)$. Again, this is straightforward, since

$$\left[(\alpha * \delta) * \overline{(\beta * \delta)}\right] = [\alpha * \overline{\beta}] \in H.$$

Step 2. Topologizing E. Let $\alpha \in \mathscr{P}$ and let U be a path connected neighborhood of $\alpha(1)$. Define

$$B(U, \alpha) = \left\{(\alpha * \delta)^\sharp : \delta \text{ is a path in } U \text{ beginning at } \alpha(1)\right\}.$$

Obviously, $\alpha^\sharp \in B(U, \alpha)$, which can be seen by taking δ to be the constant path at $\alpha(1)$. We contend that the sets $B(U, \alpha)$ form a basis for a topology on E.

First, if $\beta^\sharp \in B(U, \alpha)$, then $\beta^\sharp = (\alpha * \delta)^\sharp$ for some path δ in U beginning at $\alpha(1)$. Now, using the aforementioned observations,

$$(\beta * \overline{\delta})^\sharp = \left(\alpha * \delta * \overline{\delta}\right)^\sharp = \alpha^\sharp,$$

where the first equality follows from (2) while the second equality follows from (1). Consequently, $\alpha^\sharp \in B(U, \beta)$. Next, we show that $B(U, \beta) \subseteq B(U, \alpha)$. Indeed, any element of $B(U, \beta)$ is of the form $(\beta * \gamma)^\sharp$ for some path γ in U beginning at $\beta(1)$. But since $\beta^\sharp = (\alpha * \delta)^\sharp$, using observation (2) above, we have

$$(\beta * \gamma)^\sharp = (\alpha * \delta * \gamma)^\sharp \in B(U, \alpha).$$

Hence $B(U,\beta) \subseteq B(U,\alpha)$. But we argued that $\alpha^\sharp \in B(U,\beta)$, and hence, $B(U,\alpha) \subseteq B(U,\beta)$, whence $B(U,\beta) = B(U,\alpha)$, whenever $\beta^\sharp \in B(U,\alpha)$.

Finally, we show that the sets $B(U,\alpha)$ form a basis for a topology on E. Indeed, suppose $\beta^\sharp \in B(U_1,\alpha_1) \cap B(U_2,\alpha_2)$. Then, by definition, $\beta(1) \in U_1 \cap U_2$. Since B is locally path connected, there is a path connected neighborhood V of $\beta(1)$ contained in $U_1 \cap U_2$. It is clear that $B(V,\beta) \subseteq B(U_i,\beta) = B(U_i,\alpha_i)$ for $i \in \{1,2\}$ due to the conclusion of the preceding paragraph. It follows hence that the $B(U,\alpha)$'s form a basis for a topology on E.

Step 3. p is a continuous open map. It is easy to see that the image of $B(U,\alpha)$ lies in U, conversely, given any $x \in U$, there is a path δ in U from $\alpha(1)$ to x, whence the image of $(\alpha * \delta)^\sharp$ under p is x. It follows that the image of $B(U,\alpha)$ under p is all of U. Hence p is an open map.

To show continuity of p, we show that it is continuous at each $\alpha^\sharp \in E$. Let $b = p(\alpha^\sharp)$ and let W be a neighborhood of b in B. Since B is locally path connected, there is a path connected neighborhood U of b contained in W. Then as we have seen, $B(U,\alpha)$ is a neighborhood of $\alpha^\sharp$ that maps to U under p, whence p is continuous at $\alpha^\sharp$; consequently, p is a continuous open map, as desired.

Step 4. p is a covering map. We shall show that every $b \in B$ has an evenly covered neighborhood. Since B is semilocally simply connected, there is a neighborhood U of b such that the induced map $\pi_1(U,b) \to \pi_1(B,b)$ is trivial. Let S denote the set of all paths in B from b_0 to b. We shall show that $p^{-1}(U)$ is the union of all $B(U,\alpha)$ where $\alpha \in S$.

Obviously, the inclusion $\bigcup_{\alpha \in S} B(U,\alpha) \subseteq p^{-1}(U)$. Conversely, if $\beta^\sharp \in p^{-1}(U)$, then $\beta(1) \in U$. Choose a path δ in U from b to $\beta(1)$, and let α be the path $\beta * \overline{\delta} \in \mathscr{P}$. Then $[\beta] = [\alpha * \delta]$, so that $\beta^\sharp = (\alpha * \delta)^\sharp$, that is, $\beta \in B(U,\alpha)$. It follows that

$$p^{-1}(U) = \bigcup_{\alpha \in S} B(U,\alpha).$$

Further, note that if $B(U,\alpha_1) \cap B(U,\alpha_2)$ is non-empty, then as we have seen in *Step 2*, $B(U,\alpha_1) = B(U,\alpha_2)$. It follows that $p^{-1}(U)$ is a disjoint union of $B(U,\alpha)$ where α ranges over a suitable subset of S.

Finally to show that p is a covering map, we must show that the restriction $p : B(U,\alpha) \to U$ is a homeomorphism, where $\alpha \in S$. Since we know that it is a continuous open map, it suffices to show that it is a bijection. Surjectivity has already been shown so we must only establish injectivity. Indeed, suppose

$$p\left((\alpha * \delta_1)^\sharp\right) = p\left((\alpha * \delta_2)^\sharp\right)$$

for some paths δ_1, δ_2 in U that begin at $b = \alpha(1)$. Of course, we must have $\delta_1(1) = \delta_2(1)$. Further, since the inclusion induced map $\pi_1(U,b) \to \pi_1(B,b)$ is trivial, the loop $\delta_1 * \overline{\delta}_2$ is path homotopic to the constant loop e_b in B based at b; consequently, $[\delta_1] = [\delta_2]$ in B, and hence $[\alpha * \delta_1] = [\alpha * \delta_2]$ in B. Using observation (1), we get that $(\alpha * \delta_1)^\sharp = (\alpha * \delta_2)^\sharp$, thereby proving injectivity. This gives us our original desideratum.

Step 5. Lifting paths to E. Let $e_0 \in E$ denote $(e_{b_0})^\sharp$, the equivalence class of the constant path at b_0. Let $\alpha : I \to B$ be a path in B beginning at b_0. We shall construct a continuous lift $\widetilde{\alpha} : I \to E$ beginning at e_0.

For $c \in [0,1]$, let $\alpha_c : I \to B$ be given by $\alpha_c(t) = \alpha(ct)$. This is a path in B beginning at b_0 and ending at $\alpha(c)$. Set $\widetilde{\alpha}(c) = (\alpha_c)^\sharp \in E$. Note that

$$p\left(\widetilde{\alpha}(c)\right) = \alpha_c(1) = \alpha(c) \implies \alpha = p \circ \widetilde{\alpha},$$

whence $\widetilde{\alpha}$ is indeed a lifting of α. It remains to show that $\widetilde{\alpha}$ is a continuous map.

Indeed, let $c \in [0,1]$ and choose a basic neighborhood $B(U,\alpha_c)$ of $\widetilde{\alpha}(c) = (\alpha_c)^\sharp$; where U is a path connected neighborhood of $\alpha_c(1) = \alpha(c)$. Since α is continuous, there is an $\varepsilon > 0$ such that $\alpha(t) \in U$ whenever $|c - t| < \varepsilon$. Now, for any $d \in [0,1]$ with $|c - d| < \varepsilon$, let $\delta : I \to B$ be the path given by

$$\delta(t) = \alpha\left((1-t)c + td\right).$$

It is easy to see that $[\alpha_c * \delta] = [\alpha_d]$, since the former is just a reparametrization of the latter. Consequently, using observation (1),

$$\widetilde{\alpha}(d) = (\alpha_d)^\sharp = (\alpha_c * \delta)^\sharp = \alpha_d^\sharp \implies \alpha_d^\sharp \in B(U,\alpha_c).$$

This shows that $\widetilde{\alpha}$ is continuous, as desired.

Step 6. E is path connected. If $\alpha^\sharp \in E$, then the path $\alpha : I \to B$ lifts to a path $\widetilde{\alpha} : I \to E$ from e_0 to $\alpha^\sharp$, thereby establishing path connectedness.

Step 7. $p_*(\pi_1(E, e_0)) = H$. Due to the uniqueness of path liftings (given an initial point), any loop in E based at e_0 is of the form $\widetilde{\alpha}$ for some loop α in B based at b_0. Note that

$$(e_{b_0})^\sharp = e_0 = \widetilde{\alpha}(1) = \alpha^\sharp,$$

and hence, $[\alpha * \bar{e}_{b_0}] \in H$, that is, $[\alpha] \in H$. This shows that $p_*(\pi_1(E, e_0)) \subseteq H$.

Conversely, if $[\alpha] \in H$, then again $\alpha^\sharp = (e_{b_0})^\sharp$, since $[\alpha * \bar{e}_{b_0}] \in H$. Consequently, $\widetilde{\alpha}$ is a loop in E based at e_0. It follows hence that $[\alpha] \in p_*(\pi_1(E, e_0))$, consequently, $H = p_*(\pi_1(E, e_0))$, as desired. ■