MA 534: HOMEWORK 3

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1. Problem 1

For $\varepsilon > 0$, define the functions $F_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$F_\varepsilon(z) := \begin{cases} \sqrt{z^2+\varepsilon^2} - \varepsilon & z > 0 \\ 0 & z \leqslant 0. \end{cases}$$

It is clear that F_ε is a continuously differentiable function on $\mathbb{R}$ with

$$F'_\varepsilon(z) = \begin{cases} \frac{z}{\sqrt{z^2+\varepsilon^2}} & z > 0 \\ 0 & z \leqslant 0. \end{cases}$$

Furthermore,

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(z) = \begin{cases} z & z > 0 \\ 0 & z \leqslant 0, \end{cases}$$

and

$$\lim_{\varepsilon \to 0^+} F'_\varepsilon(z) = \begin{cases} 1 & z > 0 \\ 0 & z \leqslant 0. \end{cases}$$

Now, for any test function $\varphi \in C_c^\infty(\Omega)$, we have, using integration by parts with respect to the variable x_j with $1 \leqslant j \leqslant n$,

$$\int_\Omega F_\varepsilon(u) \frac{\partial \varphi}{\partial x_j}\, dx = -\int_\Omega F'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi\, dx$$

Note that $\frac{\partial \varphi}{\partial x_j}$ is of compact support in Ω, and since $|F_\varepsilon(z)| \leqslant |z|$, it is clear from the dominated convergence theorem that

$$\lim_{\varepsilon \to 0^+} \int_\Omega F_\varepsilon(u) \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega \lim_{\varepsilon \to 0^+} F_\varepsilon(u) \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega u^+ \frac{\partial \varphi}{\partial x_j}\, dx.$$

Next, note that $|F'_\varepsilon(z)| \leqslant 1$ and hence, the dominated convergence theorem applies again to give

$$\lim_{\varepsilon \to 0^+} \int_\Omega F'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi\, dx = \int_\Omega \lim_{\varepsilon \to 0^+} F'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi\, dx = \int_{\{u>0\}} \frac{\partial u}{\partial x_j} \varphi\, dx.$$

In conclusion,

$$\int_\Omega u^+ \frac{\partial \varphi}{\partial x_j}\, dx = -\int_{u>0} \frac{\partial u}{\partial x_j} \varphi\, dx.$$

Hence,

$$\frac{\partial u^+}{\partial x_j} = \begin{cases} \frac{\partial u}{\partial x_j} & u > 0 \\ 0 & u \leqslant 0, \end{cases}$$

almost everywhere. Similarly, using the fact that $u^- = (-u)^+$, we get

$$\frac{\partial u^-}{\partial x_j} = \begin{cases} -\frac{\partial u}{\partial x_j} & -u > 0 \\ 0 & -u \leqslant 0 \end{cases} = \begin{cases} -\frac{\partial u}{\partial x_j} & u < 0 \\ 0 & u \geqslant 0, \end{cases}$$

almost everywhere. Finally, note that

$$\left| \frac{\partial u^+}{\partial x_j} \right| \leqslant \left| \frac{\partial u}{\partial x_j} \right|,$$

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almost everywhere, therefore,

$$\int_\Omega |u^+|^2\,dx + \int_\Omega |Du^+|^2\,dx \leqslant \int_\Omega |u|^2\,dx + \int_\Omega |Du|^2\,dx < \infty,$$

i.e., $u^+ \in H^1(\Omega)$. Similarly, $u^- \in H^1(\Omega)$; and since $|u| = u^+ + u^-$, it follows that $|u| \in H^1(\Omega)$.

2. Problem 2

Suppose first that $p = \infty$, then $u \in L^\infty(\mathbb{R}^n)$, i.e., u is bounded on $\mathbb{R}^n$. It follows from Liouville's theorem that u must be constant. Conversely, note that every constant function on $\mathbb{R}^n$ is trivially harmonic and in L^∞. Thus, a harmonic function on $\mathbb{R}^n$ is in $L^\infty(\mathbb{R}^n)$ if and only if it is constant.

Next, let $1 \leqslant p < \infty$. Let $x \in \mathbb{R}^n$. Then, using the mean value property of harmonic functions, we have

$$\begin{aligned}|u(x)| &= \frac{n}{\omega_n}\left|\int_{B(x,1)} u(y)\,dy\right| \\ &\leqslant \frac{n}{\omega_n}\int_{B(x,1)} |u(y)|\,dy.\end{aligned}$$

If $p = 1$, then the above inequality shows that

$$|u(x)| \leqslant \frac{n}{\omega_n}\int_{\mathbb{R}^n} |u(y)|\,dy = \frac{n}{\omega_n}\|u\|_{L^1(\mathbb{R}^n)}.$$

Thus, u is a bounded harmonic function on $\mathbb{R}^n$, whence, due to Liouville's theorem, u must be a constant function. But a constant function is in $L^1(\mathbb{R}^n)$ if and only if it is identically zero, so $u \equiv 0$. Clearly, if $u \equiv 0$, then u is a harmonic function in $L^1(\mathbb{R}^n)$.

Finally, suppose $1 < p < \infty$ and let q denote its conjugate exponent, that is, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, using Hölder's inequality we have

$$\begin{aligned}|u(x)| &\leqslant \frac{n}{\omega_n}\int_{B(x,1)} |u(y)|\,dy \\ &\leqslant \frac{n}{\omega_n}\left(\int_{B(x,1)} |u(y)|^p\,dy\right)^{\frac{1}{p}}\left(\int_{B(x,1)} 1\,dy\right)^{\frac{1}{q}} \\ &\leqslant \left(\frac{n}{\omega_n}\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n} |u(y)|^p\right)^{\frac{1}{p}} \\ &= \left(\frac{n}{\omega_n}\right)^{\frac{1}{p}}\|u\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Thus, u is a bounded harmonic function on $\mathbb{R}^n$, whence due to Liouville's theorem, u must be a constant function. But a constant function is in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if it is identically zero, so $u \equiv 0$. Clearly if $u \equiv 0$, then it is a harmonic function in $L^p(\mathbb{R}^n)$.

3. Problem 3

I shall assume $\mathscr{L} := -\sum_{1\leqslant i,j\leqslant n} a_{ij}\partial_{ij}$ is uniformly elliptic and that the derivatives of the coefficinet functions a_{ij} are bounded on Ω. Note that $\mathscr{L}u = 0$, and hence

$$0 = D(\mathscr{L}u) = -\sum_{1\leqslant i,j\leqslant n} u_{ij}Da_{ij} - \sum_{1\leqslant i,j\leqslant n} a_{ij}Du_{ij}.$$

Set $v : \Omega \to \mathbb{R}$ to be

$$v(x) = |Du(x)|^2 + \lambda u(x)^2,$$

where $\lambda > 0$ will be fixed later. Then

$$\begin{aligned}
\mathscr{L}v &= -\sum_{1\leqslant i,j\leqslant n} a_{ij}\partial_{ij}\left(Du\cdot Du+\lambda u^2\right)\\
&= -\sum_{1\leqslant i,j\leqslant n} a_{ij}\left(2Du_{ij}\cdot Du+2Du_i\cdot Du_j+2\lambda u u_{ij}+2\lambda u_i u_j\right)\\
&= -2\sum_{1\leqslant i,j\leqslant n} a_{ij}Du_{ij}\cdot Du-2\sum_{1\leqslant i,j\leqslant n} a_{ij}Du_i\cdot Du_j-2\lambda u\underbrace{\sum_{1\leqslant i,j\leqslant n} a_{ij}u_{ij}}_{=0}-2\lambda\sum_{1\leqslant i,j\leqslant n} a_{ij}u_iu_j\\
&= 2\sum_{1\leqslant i,j\leqslant n} u_{ij}Da_{ij}\cdot Du-2\sum_{1\leqslant i,j\leqslant n} a_{ij}Du_i\cdot Du_j-2\lambda\sum_{1\leqslant i,j\leqslant n} a_{ij}u_iu_j.
\end{aligned}$$

Since the derivatives of a_{ij}, the Hessain of u, and the gradient of u are bounded, the sum of the first two terms in the above expression are bounded in absolute value. Finally, since $\mathscr{L}$ is uniformly elliptic, the matrix (a_{ij}) is uniformly positive definite, in the sense that there is a $\theta > 0$ such that

$$\sum_{1\leqslant i,j\leqslant n} a_{ij}u_iu_j \geqslant \theta|Du|^2.$$

In particular, this means that the last term is at most $-2\lambda\theta|Du|^2$. Thus, we can choose $\lambda \gg 0$ such that $\mathscr{L}u \leqslant 0$. Finally, we invoke the weak maximum principle to obtain

$$\begin{aligned}
\||Du|^2\|_{L^\infty(\Omega)} &\leqslant \||Du|^2+\lambda u^2\|_{L^\infty(\Omega)}\\
&\leqslant \|v\|_{L^\infty(\Omega)}^2\\
&= \|v\|_{L^\infty(\partial\Omega)}^2\\
&= \||Du|^2+\lambda u^2\|_{L^\infty(\partial\Omega)}\\
&\leqslant \|Du\|_{L^\infty(\partial\Omega)}^2+\lambda\|u\|_{L^\infty(\partial\Omega)}^2\\
&\leqslant C\left(\|Du\|_{L^\infty(\partial\Omega)}+\|u\|_{L^\infty(\partial\Omega)}\right)^2,
\end{aligned}$$

where $C := \max\{1,\lambda\}$. This implies the desired conclusion.

4. Problem 4

Define the quantities $H, D, N : (0,1) \to (0,\infty)$ as

$$\begin{aligned}
H(r) &:= \int_{\partial B_r} |u(x)|^2\,dx\\
D(r) &:= \int_{\partial B_r} u(x)\frac{\partial u}{\partial \nu}(x)\,ds(x) = \int_{B_r} |\nabla u(x)|^2\,dx\\
N(r) &:= \frac{rD(r)}{H(r)}.
\end{aligned}$$

Note that the equality in the definition of $D(r)$ follows from Green's first identity, since u is harmonic, and thus $\Delta u = 0$.

Claim. $D'(r) = \dfrac{n-2}{r}D(r) + 2\displaystyle\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}(x)\right|^2\,ds(x)$.

Proof. Performing the substitution $x = ry$ and using

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\frac{x}{r},$$

we can write

$$D(r) = \int_{\partial B_1} u(ry)(\nabla u(ry)\cdot y)r^{n-1}\,ds(y).$$

Differentiating using the product rule, we obtain

$$\begin{aligned}
D'(r) &= \int_{\partial B_1} (\nabla u(ry)\cdot y)^2 r^{n-1} + (n-1)\int_{\partial B_1} u(ry)(\nabla u(ry)\cdot y) r^{n-2}\, ds(y) + \int_{\partial B_1} u(ry)\frac{d}{dr}(\nabla u(ry)\cdot y) r^{n-1}\, ds(y) \\
&= \int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}(x)\right|^2 ds(x) + \frac{n-1}{r}\underbrace{\int_{\partial B_r} u(x)\frac{\partial u}{\partial \nu}(x)\, ds(x)}_{D(r)} + \int_{\partial B_1} u(ry)\frac{d}{dr}(\nabla u(ry)\cdot y) r^{n-1}\, ds(y).
\end{aligned}$$

We now simplify the last term on the right hand side. Indeed, we can write

$$\nabla u(ry)\cdot y = \sum_{i=1}^{n} \partial_i u(ry) y_i,$$

and thus,

$$\frac{d}{dr}(\nabla u(ry)\cdot y) = \frac{d}{dr}\sum_{i=1}^{n} \partial_i u(ry) y_i = \sum_{1\leqslant i,j\leqslant n} y_i y_j \partial_i \partial_j u(ry).$$

Hence, we have that the last term is equal to

$$\int_{\partial B_1} u(ry)\left(\sum_{1\leqslant i,j\leqslant n} y_i y_j \partial_i\partial_j u(ry)\right) r^{n-1}\, ds(y) = \frac{1}{r^2}\int_{\partial B_r} u(x)\left(\sum_{1\leqslant i,j\leqslant n} x_i x_j u_{ij}(x)\right) ds(x),$$

where we shall use the shorthand u_{ij} to denote $\partial_i\partial_j u$ henceforth. Note that since u is harmonic, it is C^∞, and hence, the order of differentiation does not really matter.

We can write the above quantity as follows, and then using Green's first identity, we obtain

$$\int_{\partial B_r} \sum_{i=1}^{n} \frac{x_i}{r} u(x)\frac{\partial u_i}{\partial \nu}(x)\, ds(x) = \sum_{i=1}^{n}\left(\int_{B_r} \nabla\left(\frac{x_i}{r}u(x)\right)\cdot \nabla u_i(x)\, dx - \int_{B_r} \frac{x_i}{r} u(x)\Delta u_i(x)\, dx\right).$$

Since each u_i is harmonic, the second term is zero and we are left with only the first term, which is equal to

$$\sum_{i=1}^{n}\int_{B_r}\sum_{j=1}^{n} \partial_j\left(\frac{x_i}{r}u(x)\right)\partial_j u_i(x)\, dx = \sum_{i=1}^{n}\int_{B_r}\sum_{j=1}^{n}\left(\frac{x_i}{r}u_j(x) + \frac{1}{r}\delta_{ij}u(x)\right)u_{ij}(x)\, dx,$$

where δ_{ij} denotes the Kronecker symbol. The above is equal to

$$\sum_{1\leqslant i,j\leqslant n}\int_{B_r}\frac{x_i}{r}u_{ij}(x)u_j(x)\, dx + \sum_{i=1}^{n}\int_{B_r}\frac{1}{r}u_{ii}(x)u(x)\, dx.$$

The second integral is equal to $\frac{1}{r}\int_{B_r}\Delta u(x)u(x)\, dx = 0$, since u is harmonic. Hence, we are left with

$$\sum_{1\leqslant i,j\leqslant n}\int_{B_r}\frac{x_i}{r}u_{ij}(x)u_j(x)\, dx. \tag{♣}$$

Consider the function

$$w(x) = \sum_{i=1}^{n}\frac{x_i}{r}u_i(x)$$

defined on the unit ball B_1. Note that $w(x) = \frac{\partial u}{\partial \nu}(x)$ on ∂B_r. Thus, we can write

$$\begin{aligned}
\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}(x)\right|^2 \, ds(x) &= \int_{\partial B_r} w(x)\frac{\partial u}{\partial \nu}(x) \, ds(x) \\
&= \int_{B_r} \nabla w(x) \cdot \nabla u(x) - w(x)\Delta u(x) \, dx \\
&= \int_{B_r} \nabla w(x) \cdot \nabla u(x) \, dx \\
&= \int_{B_r} \sum_{j=1}^{n} \partial_j \left(\sum_{i=1}^{n} \frac{x_i}{r} u_i(x) \right) \cdot u_j(x) \, dx \\
&= \int_{B_r} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\frac{1}{r}\delta_{ij} u_i(x) + \frac{x_i}{r} u_{ij}(x) \right) u_j(x) \, dx \\
&= \sum_{1 \leqslant i,j \leqslant n} \int_{B_r} \frac{x_i}{r} u_{ij}(x) u_j(x) \, dx + \sum_{i=1}^{n} \frac{1}{r} \int_{B_r} (u_i(x))^2 \, dx \\
&= \sum_{1 \leqslant i,j \leqslant n} \int_{B_r} \frac{x_i}{r} u_{ij}(x) u_j(x) \, dx + \sum_{i=1}^{n} \frac{1}{r} \int_{B_r} (u_i(x))^2 \, dx.
\end{aligned}$$

Thus, the quantity in (♣) is equal to

$$\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}(x)\right|^2 \, ds(x) - \frac{1}{r}\int_{B_r} |\nabla u(x)|^2 \, dx = \int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}(x)\right|^2 \, ds(x) - \frac{1}{r} D(r).$$

Substituting this back into the expression for $D'(r)$, we get

$$D'(r) = \frac{n-2}{r} D(r) + 2\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}(x)\right|^2 \, ds(x),$$

as desired. ■

Claim. $H'(r) = \dfrac{n-1}{r} H(r) + 2D(r)$.

Proof. Performing the substitution $x = ry$, we have

$$H(r) = \int_{\partial B_1} |u(ry)|^2 r^{n-1} \, ds(y).$$

Differentiating, we obtain

$$\begin{aligned}
H'(r) &= (n-1)\int_{\partial B_1} |u(ry)|^2 r^{n-2} \, ds(y) + 2\int_{\partial B_1} r^{n-1} u(ry)(\nabla u(ry) \cdot y) \, ds(y) \\
&= \frac{n-1}{r}\int_{\partial B_1} |u(x)|^2 \, ds(x) + 2\int_{\partial B_r} u(x)\left(\nabla u(x) \cdot \frac{x}{r}\right) \, ds(x) \\
&= \frac{n-1}{r}\int_{\partial B_1} |u(x)|^2 \, ds(x) + 2\int_{\partial B_r} u(x)\frac{\partial u}{\partial \nu}(x) \, ds(x) \\
&= \frac{n-1}{r} H(r) + 2D(r),
\end{aligned}$$

as desired. ■

Coming back to the problem at hand, we would like to show that

$$N(r) = \frac{rD(r)}{H(r)}$$

is an increasing function of r. To this end, we shall show that $N'(r) \geqslant 0$ for $r \in (0,1)$. Indeed,

$$\begin{aligned}
N'(r) &= \frac{\big(D(r)+rD'(r)\big)H(r)-rD(r)H'(r)}{H(r)^2} \\
&= \frac{\Big((n-1)D(r)+2r\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}(x)\right|^2\,ds(x)\Big)H(r)-D(r)\Big((n-1)H(r)+2rD(r)\Big)}{H(r)^2} \\
&= \frac{2rH(r)\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}(x)\right|^2\,ds(x)-2rD(r)^2}{H(r)^2}.
\end{aligned}$$

Using the Cauchy Schwarz inequality, we have

$$H(r)\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}(x)\right|^2\,ds(x) \geqslant \left(\int_{\partial B_r}\left|u(x)\frac{\partial u}{\partial \nu}(x)\right|\,ds(x)\right)^2 \geqslant D(r)^2,$$

whence $N'(r) \geqslant 0$. Thus $N(r)$ is a non-decreasing function of r.

As for the limit computation, note that the quotient can be written as

$$N(r) = \frac{r^2}{n}\frac{\frac{n}{\omega_n r^n}\int_{B_r}|\nabla u|^2}{\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r}u^2}.$$

Since u is C^∞, we have that both u^2 and $|\nabla u|^2$ are continuous functions, and hence, the numerator converges to $|\nabla u(0)|^2$ and the denominator converges to $u(0)^2$. Again, since $r \to 0^+$, it is clear that $N(r) \to 0$.

5. Problem 5

For $R > r$, let $A(r)$ denote the annulus

$$A(r) := \left\{x \in \mathbb{R}^n : r < |x| < R\right\}.$$

Note that

$$\partial A(r) = \left\{x \in \mathbb{R}^n : |x| = r\right\} \cup \left\{x \in \mathbb{R}^n : |x| = R\right\}.$$

The (weak) maximum principle gives us

$$\sup_{r\leqslant|x|\leqslant R}|u(x)| = \sup_{x\in\partial A(r)}|u(x)| = \sup_{|x|=R}|u(x)|,$$

where the last equality follows from the fact that $u \equiv 0$ on ∂B_r.

Let $x_0 \in \mathbb{R}^n \setminus \overline{B}_r$. We shall show that $u(x_0) = 0$. Indeed, for $R > |x_0|$, we have

$$|u(x_0)| \leqslant \sup_{r\leqslant|x|\leqslant R}|u(x)| = \sup_{|x|=R}|u(x)|.$$

But according to the hypothesis on u, we have

$$\lim_{R\to\infty}\sup_{|x|=R}|u(x)| = 0,$$

so that $0 \leqslant |u(x_0)| \leqslant 0$, that is, $u(x_0) = 0$. It follows that u is identically 0 on $\mathbb{R}^n \setminus B_r$, as desired.

6. Problem 6

By translating, along the Y-axis, we may assume that $\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < y < \frac{\pi}{2}\right\}$. Next, set

$$v(x,y) = u\left(\frac{x}{2},\frac{y}{2}\right).$$

Then, v is a function defined on the domain $\widetilde{\Omega} = \left\{(x,y) \in \mathbb{R}^2 : 0 < y < \pi\right\}$. Further,

$$\Delta v = 4\Delta u = 0,$$

and hence, v is a harmonic function. The growth condition on v is clearly $v(x) = o\left(e^{\frac{|x|}{2}}\right)$. Next, extend the function v to the domain $\Gamma = \left\{(x,y) \in \mathbb{R}^2 : -\pi < y < \pi\right\}$ using the Schwarz reflection principle. That is, define

$$v(x,y) = -v(x,-y) \qquad \forall\ x \in \mathbb{R},\ -\pi < y < 0.$$

Note that this does not change the growth condition on v. We contend that this extended v is a harmonic function. It is clear by taking the Laplacian that v is harmonic on $\Gamma \setminus \mathbb{R} \times \{0\}$, and thus has the mean value property here. Let $\mathbf{x} = (x_0, 0) \in \mathbb{R} \times \{0\} \subseteq \Gamma$. Consider a sufficiently small ball $B(\mathbf{x}, r) \subseteq \Gamma$. Then,

$$\int_{B(\mathbf{x},r)} v\ dx = \int_{B(\mathbf{x},r)\cap\{y>0\}} v\ dx + \int_{B(\mathbf{x},r)\cap\{y<0\}} v\ dx = 0,$$

since the second term is the negative of the first term. Since $v(\mathbf{x}) = 0$, it is clear that v satisfies the mean value property on all of Γ, therefore, it is harmonic on Γ.

For a fixed $x \in \mathbb{R}$, the function $v(x, \cdot)$ is a C^∞ function of y, and hence, the Fourier series and all its derivatives converge uniformly. We can then write

$$v(x,y) = \sum_{k=1}^{\infty} C_k(x)\sin(ky) + \sum_{k=0}^{\infty} D_k(x)\cos(ky).$$

But since $v(x, \cdot)$ is an odd function of y, there is no cosine part in the Fourier expansion. We are left with

$$v(x,y) = \sum_{k=1}^{\infty} C_k(x)\sin(ky).$$

Recall that

$$C_k(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} v(x,y)\sin(ky)\ dy = \frac{2}{\pi}\int_0^{\pi} v(x,y)\sin(ky)\ dy.$$

In particular, since it is the integral of a C^∞-function, it is a C^∞-function of x. Therefore, we can differentiate under the integral sign to obtain

$$C_k''(x) = \frac{2}{\pi}\int_0^{\pi} \partial_x^2 v(x,y)\sin(ky)\ dy = -\frac{2}{\pi}\int_0^{\pi} \partial_y^2 v(x,y)\sin(ky)\ dy.$$

Integrating by parts, we have

$$\begin{aligned}\int_0^{\pi} \partial_y^2 v(x,y)\sin(ky)\ dy &= \partial_y v(x,y)\sin(ky)\Big|_0^{\pi} - k\int_0^{\pi} \partial_y v(x,y)\cos(ky)\ dy \\ &= v(x,y)\cos(ky)\Big|_0^{\pi} + k^2\int_0^{\pi} v(x,y)\sin(ky)\ dy.\end{aligned}$$

Hence,

$$C_k''(x) = \frac{2k^2}{\pi}\int_0^{\pi} v(x,y)\sin(ky)\ dy = k^2 C_k(x).$$

Solving this ordinary differential equation, we note that

$$C_k(x) = A_k e^{kx} + B_k e^{-kx}.$$

But since $v(x,y)$ is $o(e^{\frac{|x|}{2}})$, using the integral Representation of $C_k(x)$ as above, we have that $C_k(x)$ is also $o(e^{\frac{|x|}{2}})$ for all $k \geqslant 1$. This is possible if and only if $A_k = B_k = 0$ for all $k \geqslant 1$. That is, C_k is identically 0 for all $k \geqslant 1$, and hence, $v \equiv 0$ on Γ so that $u \equiv 0$ on Ω. This completes the proof.

7. Problem 7

Define the function $\varphi : (0,\infty) \to \mathbb{R}$ by

$$\varphi(r) = \frac{1}{|\partial B(x_0,r)|}\int_{\partial B(x_0,r)} u(y)\ dy.$$

Performing the substitution $y = x_0 + rz$, we obtain

$$\varphi(r) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B(0,1)} u(x_0+rz) r^{n-1}\ dz = \frac{1}{\omega_n}\int_{\partial B(0,1)} u(x_0+rz)\ dz.$$

Differentiating the above,

$$\varphi'(r) = \frac{1}{\omega_n}\int_{\partial B(0,1)} \nabla u(x_0+rz)\cdot z\ dz = \frac{1}{\omega_n r^{n-1}}\int_{\partial B(x_0,r)} \nabla u(y)\cdot \frac{y-x_0}{r}\ dy = \frac{1}{\omega_n r^{n-1}}\int_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu}(y)\ ds(y).$$

Using Green's first identity, we have

$$\varphi'(r) = \frac{1}{\omega_n r^{n-1}}\int_{B(x_0,r)} \Delta u(y)\ dy = \frac{1}{\omega_n r^{n-1}}\int_{B(x_0,r)} 1\ dy = \frac{r}{n}.$$

Solving this ordinary differential equation, we obtain

$$\varphi(r) = \frac{r^2}{2n} + C$$

for all $r \in (0,\infty)$, where $C \in \mathbb{R}$ is a constant. Note that $u \geqslant 0$, and hence $\varphi(r) \geqslant 0$ for all $r > 0$. Hence, $C = \lim_{r\to 0^+} \varphi(r) \geqslant 0$, in particular,

$$\varphi(r) \geqslant \frac{r^2}{2n} \qquad \forall\ r > 0.$$

Let

$$M(r) := \sup_{|x-x_0|=r} u(x).$$

Then

$$M(r) - \varphi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0,r)} M(r) - u(y)\, ds(y) \geqslant 0,$$

so that $M(r) \geqslant \frac{r^2}{2n}$. Finally, using the weak maximum principle for subharmonic functions, and recalling that $u \geqslant 0$, we get

$$\sup_{\overline{B(x_0,r)}} u = \sup_{|x-x_0|=r} u(x) = M(x) \geqslant \frac{r^2}{2n},$$

as desired.

8. Problem 8

Let $x^* \in \overline{\Omega}$ be a point of maxima, i.e.,

$$u(x^*) = \sup_{x\in\overline{\Omega}} u(x) \geqslant 0,$$

where the inequality follows since $u|_{\partial\Omega} \equiv 0$. Suppose $u(x^*) > 1$. Then $x^* \notin \partial\Omega$, since u vanishes identically there. Thus $x^* \in \Omega$, whence $\Delta u(x^*) \leqslant 0$. This forces

$$u(x^*)^3 - u(x^*) \leqslant 0 \implies u(x^*) \in (-\infty, -1] \cup [0,1].$$

But since $u(x^*) \geqslant 0$, we must have that $u(x^*) \in [0,1]$, in particular, $u(x^*) \leqslant 1$, a contradiction. Thus $u(x^*) \leqslant 1$.

Similarly, let $x_* \in \overline{\Omega}$ be a point of minima, i.e.,

$$u(x_*) = \inf_{x\in\overline{\Omega}} u(x) \leqslant 0,$$

where the inequality follows since $u|_{\partial\Omega} \equiv 0$. Suppose $u(x_*) < -1$. Then $x_* \notin \partial\Omega$, since u vanishes identically there. Thus $x_* \in \Omega$, whence $\Delta u(x_*) \geqslant 0$. This forces

$$u(x_*)^3 - u(x_*) \geqslant 0 \implies u(x_*) \in [-1,0] \cup [1,\infty).$$

But since $u(x_*) \leqslant 0$, we must have that $u(x_*) \in [-1,0]$, in particular, $u(x_*) \geqslant -1$, a contradiction. Thus $u(x_*) \geqslant -1$. In conclusion, we have that for any $x \in \Omega$,

$$-1 \leqslant u(x_*) \leqslant u(x) \leqslant u(x^*) \leqslant 1,$$

as desired.

Suppose now that there is some point $x_0 \in \Omega$ with $u(x_0) = 1$. This is clearly a point of maxima because $-1 \leqslant u \leqslant 1$ on Ω. Let

$$\omega := \{x \in \Omega : u(x) = 1\}.$$

Since u is a continuous function, ω is closed in Ω and is non-empty as it contains x_0. We claim that ω is open in Ω, whence it would follow that $u \equiv 1$ on Ω since Ω is connected, a contradiction, since $u \equiv 0$ on $\partial\Omega$.

9. Problem 9

(a) This is an immediate consequence of the mean value property. Indeed, since u is harmonic, for $x \in \mathbb{R}^n$, we have
$$|u(x)| = \left| \frac{n}{\omega_n} \int_{B(x,1)} u(y)\, dy \right| \leqslant \frac{n}{\omega_n} \int_{B(x,1)} |u(y)|\, dy \leqslant \frac{nC}{\omega_n}.$$
That is, u is a bounded harmonic function on $\mathbb{R}^n$, and hence is constant due to Liouville's theorem.

(b) Suppose $u(x) = \phi(|x|)$ is indeed a harmonic function on $\mathbb{R}^n$, where $\phi : [0,\infty) \to \mathbb{R}$. Using the mean value property, for $r > 0$, we can write
$$u(0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,r)} u(y)\, ds(y) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,r)} \phi(r)\, ds(y) = \phi(r).$$
Thus, $\phi(r) = u(0)$ for all $r > 0$. Thus, u is constant on $\mathbb{R}^n \setminus \{0\}$. But since u is continuous, u must be constant on all of $\mathbb{R}^n$, as desired.

10. Problem 10

Let
$$M = \sup_{\partial\Omega} |\varphi|.$$
Then $-M \leqslant \varphi(x) \leqslant M$ for all $x \in \partial\Omega$.

Let $x^* \in \overline{\Omega}$ be a point of maxima, i.e.,
$$u(x^*) = \sup_{x \in \overline{\Omega}} u(x).$$
If $x^* \in \Omega$, then it is an interior point, and hence $u(x^*)^3 = \Delta u(x^*) \leqslant 0$, that is, $u(x^*) \leqslant 0$. Else if $x^* \in \partial\Omega$, then we must have that
$$\frac{\partial u}{\partial \nu}(x^*) \geqslant 0 \implies a(x^*)u(x^*) \leqslant \varphi(x^*) \leqslant M \implies u(x^*) \leqslant \frac{M}{a(x^*)} \leqslant \frac{M}{a_0}.$$
In either case, the inequality
$$u(x^*) \leqslant \frac{M}{a_0}$$
holds true.

On the other hand, let $x_* \in \overline{\Omega}$ be a point of minima, i.e.,
$$u(x_*) = \inf_{x \in \overline{\Omega}} u(x).$$
If $x_* \in \Omega$, then it is an interior point, and hence $u(x_*)^3 = \Delta u(x_*) \geqslant 0$, that is, $u(x_*) \geqslant 0$. Else if $x_* \in \partial\Omega$, then we must have that
$$\frac{\partial u}{\partial \nu}(x_*) \leqslant 0 \implies a(x_*)u(x_*) \geqslant \varphi(x_*) \geqslant -M \implies u(x_*) \geqslant \frac{-M}{a(x_*)} \geqslant -\frac{M}{a_0}.$$
In either case, the inequality
$$u(x_*) \geqslant -\frac{M}{a_0}$$
holds true. Hence,
$$\sup_{\overline{\Omega}} |u| = \max\left\{|u(x^*)|, |u(x_*)|\right\} \leqslant \frac{M}{a_0} = \frac{1}{a_0} \sup_{\partial\Omega} |\varphi|,$$
thereby completing the proof.