

Riemann-Roch for Riemann Surfaces

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§1 ČECH COHOMOLOGY

DEFINITION 1.1. Let X be a topological space and \mathcal{F} a sheaf of abelian groups on X . Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X . Define the *q -th cochain group* of \mathcal{F} with respect to \mathfrak{U} as

$$C^q(\mathfrak{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q}).$$

We denote elements of this cochain group by $(f_{i_0, \dots, i_q})_{(i_0, \dots, i_q) \in I^{q+1}}$ and these are called q -cochains.

Next, define the *coboundary operators*

$$\delta : C^0(\mathfrak{U}, \mathcal{F}) \rightarrow C^1(\mathfrak{U}, \mathcal{F}) \quad \delta : C^1(\mathfrak{U}, \mathcal{F}) \rightarrow C^2(\mathfrak{U}, \mathcal{F})$$

as follows:

- For $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathcal{F})$, let $\delta((f_i)_{i \in I}) = (g_{ij})_{i, j \in I} \in C^2(\mathfrak{U}, \mathcal{F})$ where

$$g_{ij} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_j).$$

- For $(f_{ij})_{i, j \in I} \in C^1(\mathfrak{U}, \mathcal{F})$, let $\delta((f_{ij})_{i, j \in I}) = (g_{ijk})_{i, j, k \in I}$ where

$$g_{ijk} = f_{jk}|_{U_i \cap U_j \cap U_k} - f_{ik}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} \in \mathcal{F}(U_i \cap U_j \cap U_k).$$

The coboundary operators are group homomorphisms and let

$$Z^1(\mathfrak{U}, \mathcal{F}) = \ker \left(C^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathcal{F}) \right) \quad B^1(\mathfrak{U}, \mathcal{F}) = \text{im} \left(C^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathcal{F}) \right).$$

The elements of $Z^1(\mathfrak{U}, \mathcal{F})$ are called 1-cocycles and the elements of $B^1(\mathfrak{U}, \mathcal{F})$ are called 1-coboundaries. It is easy to see that $(f_{ij})_{i, j \in I}$ is a 1-cocycle if and only if

$$f_{ik}|_{U_i \cap U_j \cap U_k} = f_{jk}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} \in \mathcal{F}(U_i \cap U_j \cap U_k).$$

The above relation is called the *cocycle relation*. Indeed, if $(f_{ij})_{i,j \in I}$ is a 1-cocycle, then taking $i = j$, we see that

$$f_{ii}|_{U_i \cap U_k} = 0 \quad \forall k \in I.$$

Since the U_k 's cover U_i , using the identity axiom, we have that $f_{ii} = 0 \in \mathcal{F}(U_i)$. As a consequence, we also see that

$$f_{ji}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} = 0.$$

Again, using the same argument, we have that $f_{ij} + f_{ji} = 0 \in \mathcal{F}(U_i \cap U_j)$. It immediately follows from the above discussion that $\delta \circ \delta = 0$ as a map $C^0(\mathfrak{U}, \mathcal{F}) \rightarrow C^2(\mathfrak{U}, \mathcal{F})$.

DEFINITION 1.2. The group

$$H^1(\mathfrak{U}, \mathcal{F}) := \frac{Z^1(\mathfrak{U}, \mathcal{F})}{B^1(\mathfrak{U}, \mathcal{F})}$$

is called the *1-st cohomology group* with coefficients in \mathcal{F} *with respect to the covering* \mathfrak{U} .

DEFINITION 1.3. Let $\mathfrak{U} = (U_i)_{i \in I}$ and $\mathfrak{V} = (V_k)_{k \in K}$ be two open covers of X . We say that \mathfrak{V} is *finer* than \mathfrak{U} if every V_k is contained in some U_i .

Thus, there is a map $\tau : K \rightarrow I$ such that $V_k \subseteq U_{\tau(k)}$. This defines a mapping

$$t_{\mathfrak{V}}^{\mathfrak{U}} : Z^1(\mathfrak{U}, \mathcal{F}) \rightarrow Z^1(\mathfrak{V}, \mathcal{F})$$

as follows: for $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$, let $t_{\mathfrak{V}}^{\mathfrak{U}}((f_{ij})) = (g_{kl})$, where

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k \cap V_l} \quad \forall k, l \in K.$$

To see that this map is indeed well-defined, we need only check that (g_{kl}) is a 1-cocycle. To this end, we must check that the cocycle condition is satisfied. Indeed, for indices $k, l, m \in K$, we have

$$\begin{aligned} g_{km}|_{V_k \cap V_l \cap V_m} &= f_{\tau(k)\tau(m)}|_{V_k \cap V_l \cap V_m} \\ &= \left(f_{\tau(k)\tau(l)}|_{U_{\tau(k)} \cap U_{\tau(l)} \cap U_{\tau(m)}} + f_{\tau(l)\tau(m)}|_{U_{\tau(k)} \cap U_{\tau(l)} \cap U_{\tau(m)}} \right) |_{V_k \cap V_l \cap V_m} \\ &= f_{\tau(k)\tau(l)}|_{V_k \cap V_l \cap V_m} + f_{\tau(k)\tau(l)}|_{V_k \cap V_l \cap V_m} \\ &= g_{kl}|_{V_k \cap V_l \cap V_m} + g_{lm}|_{V_k \cap V_l \cap V_m}, \end{aligned}$$

as desired. Further, we claim that the above map takes 1-coboundaries to 1-coboundaries. Indeed, suppose $(f_{ij})_{i,j \in I}$ is a 1-coboundary, that is, there is some $(f_i)_{i \in I}$ such that

$$f_{ij} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}.$$

Let $(g_k)_{k \in K}$ be such that $g_k = f_{\tau(k)}|_{V_k}$. Then $\delta((g_k)) = (g_{kl})$ where

$$\begin{aligned} g_{kl} &= g_k|_{V_k \cap V_l} - g_l|_{V_k \cap V_l} \\ &= f_{\tau(k)}|_{V_k \cap V_l} - f_{\tau(l)}|_{V_k \cap V_l} \\ &= f_{\tau(k)\tau(l)}|_{V_k \cap V_l}, \end{aligned}$$

that is, $(g_{kl}) = t_{\mathfrak{V}}^{\mathfrak{U}}((f_{ij}))$, that is, $t_{\mathfrak{V}}^{\mathfrak{U}}$ takes 1-coboundaries to 1-coboundaries. This induces a map

$$t_{\mathfrak{V}}^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathcal{F}) \rightarrow H^1(\mathfrak{V}, \mathcal{F}).$$

LEMMA 1.4. The map $t_{\mathfrak{V}}^{\mathfrak{U}}$ induced on cohomology is independent of the choice of $\tau : K \rightarrow I$.

Proof. Suppose $\tilde{\tau} : K \rightarrow I$ is another such mapping. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$ and let

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k \cap V_l} \quad \text{and} \quad \tilde{g}_{kl} = f_{\tilde{\tau}(k)\tilde{\tau}(l)}|_{V_k \cap V_l}.$$

We must show that the cocycles (g_{kl}) and (\tilde{g}_{kl}) are cohomologous, that is, their difference lies in $B^1(\mathfrak{V}, \mathcal{F})$. Define

$$h_k = f_{\tau(k), \tilde{\tau}(k)}|_{V_k} \in \mathcal{F}(V_k).$$

Then, on $V_k \cap V_l$, we have

$$\begin{aligned} g_{kl} - \tilde{g}_{kl} &= f_{\tau(k)\tau(l)} - f_{\tilde{\tau}(k)\tilde{\tau}(l)} \\ &= f_{\tau(k)\tau(l)} + f_{\tau(l), \tilde{\tau}(k)} - f_{\tau(l)\tilde{\tau}(k)} - f_{\tilde{\tau}(k)\tilde{\tau}(l)} \\ &= f_{\tau(k)\tilde{\tau}(k)} - f_{\tau(l)\tilde{\tau}(l)} \\ &= h_k - h_l. \end{aligned}$$

Whence $(g_{kl} - \tilde{g}_{kl})$ is a coboundary, as desired. ■

LEMMA 1.5. The map $t_{\mathfrak{V}}^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathcal{F}) \rightarrow H^1(\mathfrak{V}, \mathcal{F})$ is injective.

Proof. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$ is a 1-cocycle whose image in $Z^1(\mathfrak{V}, \mathcal{F})$ is a 1-coboundary. Then, there is some $(g_k) \in C^1(\mathfrak{U}, \mathcal{F})$ such that

$$f_{\tau(k)\tau(l)}|_{V_k \cap V_l} = g_k|_{V_k \cap V_l} - g_l|_{V_k \cap V_l}.$$

Then on $U_i \cap V_k \cap V_l$, we have

$$g_k - g_l = f_{\tau(k)\tau(l)} = f_{\tau(k)i} + f_{i\tau(l)} = f_{i\tau(l)} - f_{i\tau(k)}.$$

Hence $f_{i\tau(k)} + g_k = f_{i\tau(l)} + g_l$ on $U_i \cap V_k \cap V_l$. The gluability axiom applied to the open cover $\{U_i \cap V_k\}_{k \in K}$ furnishes a $h_i \in \mathcal{F}(U_i)$ such that

$$h_i = f_{i\tau(k)} + g_k \quad \text{on } U_i \cap V_k \text{ for all } k \in K.$$

Then, on $U_i \cap U_j \cap V_k$ we have

$$f_{ij} = f_{i\tau(k)} + f_{\tau(k)j} = f_{i\tau(k)} + g_k - f_{j\tau(k)} - g_k = h_i - h_j.$$

Since $\{U_i \cap U_j \cap V_k\}$ forms an open cover of $U_i \cap U_j$, using the identity axiom, we have that $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. Thus, $(f_{ij}) \in B^1(\mathfrak{U}, \mathcal{F})$, thereby completing the proof. ■

DEFINITION 1.6. If $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$ are open covers of X , then it is easy to see that $t_{\mathfrak{W}}^{\mathfrak{V}} \circ t_{\mathfrak{V}}^{\mathfrak{U}} = t_{\mathfrak{W}}^{\mathfrak{U}}$. This gives us a directed system of cohomology groups, and we define

$$H^1(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathcal{F}).$$

REMARK 1.7. Note that since the $t_{\mathfrak{U}}^{\mathfrak{U}}$ are all injective, the natural map $H^1(\mathfrak{U}, \mathcal{F}) \rightarrow H^1(X, \mathcal{F})$ is injective. In particular, this means that $H^1(X, \mathcal{F}) = 0$ if and only if $H^1(\mathfrak{U}, \mathcal{F}) = 0$ for every open cover \mathfrak{U} of X .

THEOREM 1.8 (LERAY). Let \mathcal{F} be a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X such that $H^1(U_i, \mathcal{F}) = 0$ for every $i \in I$. Then for every open covering $\mathfrak{V} = (V_\alpha)_{\alpha \in A} < \mathfrak{U}$, the mapping

$$t_{\mathfrak{V}}^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathcal{F}) \rightarrow H^1(\mathfrak{V}, \mathcal{F})$$

is an isomorphism. The covering \mathfrak{U} is called a *Leray covering* of X for the sheaf \mathcal{F} .

Proof. Let $\tau : A \rightarrow I$ be such that $V_\alpha \subseteq U_{\tau(\alpha)}$ for every $\alpha \in A$. Since we know that $t_{\mathfrak{V}}^{\mathfrak{U}}$ is injective, we would like to show that it is surjective. Let $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathcal{F})$. The family $(U_i \cap V_\alpha)_{\alpha \in A}$ is an open covering of U_i , which we denote by $U_i \cap \mathfrak{V}$. By assumption and Remark 1.7, we know that $H^1(U_i \cap \mathfrak{V}, \mathcal{F}) = 0$, that is, there exist $g_{i\alpha} \in \mathcal{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{on } U_i \cap V_\alpha \cap V_\beta.$$

On the intersection $U_i \cap U_j \cap V_\alpha \cap V_\beta$, we have

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}.$$

Using the gluability axiom on the open cover $\{U_i \cap U_j \cap V_\alpha\}_{\alpha \in A}$, there exist elements $F_{ij} \in \mathcal{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

We claim that f_{ij} satisfies the cocycle condition. Obviously, from the above description, on $U_i \cap U_j \cap U_k \cap V_\alpha$ we have that $F_{ik} = F_{ij} + F_{jk}$. Using the identity axiom, we see that this equality holds on $U_i \cap U_j \cap U_k$. Thus, $(F_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$. Let $h_\alpha = g_{\tau(\alpha)\alpha}|_{V_\alpha} \in \mathcal{F}(V_\alpha)$. The on $V_\alpha \cap V_\beta$, we have

$$\begin{aligned} F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} &= \left(g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha} \right) - \left(g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta} \right) \\ &= g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} \\ &= h_\beta - h_\alpha, \end{aligned}$$

whence $(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta})$ is a coboundary, thereby completing the proof. ■

COROLLARY 1.9. If \mathfrak{U} is a Leray covering of X , then $H^1(\mathfrak{U}, \mathcal{F}) \cong H^1(X, \mathcal{F})$.