Riemann-Roch for Riemann Surfaces

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§1 Čech Cohomology

Definition 1.1. Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. Define the *q-th cochain group* of $\mathscr{F}$ with respect to $\mathfrak{U}$ as

$$C^q(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

We denote elements of this cochain group by $\left(f_{i_0, \dots, i_q}\right)_{(i_0, \dots, i_q) \in I^{q+1}}$ and these are called q-cochains.

Next, define the *coboundary operators*

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}) \qquad \delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$$

as follows:

- For $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathscr{F})$, let $\delta\left((f_i)_{i \in I}\right) = (g_{ij})_{i,j \in I} \in C^2(\mathfrak{U}, \mathscr{F})$ where
$$g_{ij} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$
- For $(f_{ij})_{i,j \in I} \in C^1(\mathfrak{U}, \mathscr{F})$, let $\delta\left((f_{ij})_{i,j \in I}\right) = (g_{ijk})_{i,j,k \in I}$ where
$$g_{ijk} = f_{jk}|_{U_i \cap U_j \cap U_k} - f_{ik}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

The coboundary operators are group homomorphisms and let

$$Z^1(\mathfrak{U}, \mathscr{F}) = \ker\left(C^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathscr{F})\right) \qquad B^1(\mathfrak{U}, \mathscr{F}) = \operatorname{im}\left(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F})\right).$$

The elements of $Z^1(\mathfrak{U}, \mathscr{F})$ are called 1-cocycles and the elements of $B^1(\mathfrak{U}, \mathscr{F})$ are called 1-coboundaries. It is easy to see that $(f_{ij})_{i,j \in I}$ is a 1-cocycle if and only if

$$f_{ik}|_{U_i \cap U_j \cap U_k} = f_{jk}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

The above relation is called the *cocycle relation*. Indeed, if $(f_{ij})_{i,j \in I}$ is a 1-cocycle, then taking $i = j$, we see that

$$f_{ii}|_{U_i \cap U_k} = 0 \qquad \forall k \in I.$$

Since the U_k's cover U_i, using the identity axiom, we have that $f_{ii} = 0 \in \mathscr{F}(U_i)$. As a consequence, we also see that

$$f_{ji}|_{U_i \cap U_j \cap U_k} + f_{ij}|_{U_i \cap U_j \cap U_k} = 0.$$

Again, using the same argument, we have that $f_{ij} + f_{ji} = 0 \in \mathscr{F}(U_i \cap U_j)$. It immediately follows from the above discussion that $\delta \circ \delta = 0$ as a map $C^0(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$.

Definition 1.2. The group

$$H^1(\mathfrak{U}, \mathscr{F}) := \frac{Z^1(\mathfrak{U}, \mathscr{F})}{B^1(\mathfrak{U}, \mathscr{F})}$$

is called the *1-st cohomology group* with coefficients in $\mathscr{F}$ *with respect to the covering* $\mathfrak{U}$.

Definition 1.3. Let $\mathfrak{U} = (U_i)_{i \in I}$ and $\mathfrak{V} = (V_k)_{k \in K}$ be two open covers of X. We say that $\mathfrak{V}$ is *finer* than $\mathfrak{U}$ if every V_k is contained in some U_i.

Thus, there is a map $\tau : K \to I$ such that $V_k \subseteq U_{\tau(k)}$. This defines a mapping

$$t_{\mathfrak{V}}^{\mathfrak{U}} : Z^1(\mathfrak{U}, \mathscr{F}) \to Z^1(\mathfrak{V}, \mathscr{F})$$

as follows: for $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$, let $t_{\mathfrak{V}}^{\mathfrak{U}}\left((f_{ij})\right) = (g_{kl})$, where

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k \cap V_l} \qquad \forall\, k, l \in K.$$

To see that this map is indeed well-defined, we need only check that (g_{kl}) is a 1-cocycle. To this end, we must check that the cocycle condition is satisfied. Indeed, for indices $k, l, m \in K$, we have

$$\begin{aligned} g_{km}|_{V_k \cap V_l \cap V_m} &= f_{\tau(k)\tau(m)}|_{V_k \cap V_l \cap V_m} \\ &= \left(f_{\tau(k)\tau(l)}|_{U_{\tau(k)} \cap U_{\tau(l)} \cap U_{\tau(m)}} + f_{\tau(l)\tau(m)}|_{U_{\tau(k)} \cap U_{\tau(l)} \cap U_{\tau(m)}}\right)|_{V_k \cap V_l \cap V_m} \\ &= f_{\tau(k)\tau(l)}|_{V_k \cap V_l \cap V_m} + f_{\tau(k)\tau(l)}|_{V_k \cap V_l \cap V_m} \\ &= g_{kl}|_{V_k \cap V_l \cap V_m} + g_{lm}|_{V_k \cap V_l \cap V_m}, \end{aligned}$$

as desired. Further, we claim that the above map takes 1-coboundaries to 1-coboundaries. Indeed, suppose $(f_{ij})_{i,j \in I}$ is a 1-coboundary, that is, there is some $(f_i)_{i \in I}$ such that

$$f_{ij} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}.$$

Let $(g_k)_{k \in K}$ be such that $g_k = f_{\tau(k)}|_{V_k}$. Then $\delta\left((g_k)\right) = (g_{kl})$ where

$$\begin{aligned} g_{kl} &= g_k|_{V_k \cap V_l} - g_l|_{V_k \cap V_l} \\ &= f_{\tau(k)}|_{V_k \cap V_l} - f_{\tau(l)}|_{V_k \cap V_l} \\ &= f_{\tau(k)\tau(l)}|_{V_k \cap V_l}, \end{aligned}$$

that is, $(g_k l) = t_{\mathfrak{V}}^{\mathfrak{U}}\left((f_{ij})\right)$, that is, $t_{\mathfrak{V}}^{\mathfrak{U}}$ takes 1-coboundaries to 1-coboundaries. This induces a map

$$t_{\mathfrak{V}}^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F}).$$

Lemma 1.4. The map $t_{\mathfrak{V}}^{\mathfrak{U}}$ induced on cohomology is independent of the choice of $\tau : K \to I$.

Proof. Suppose $\widetilde{\tau} : K \to I$ is another such mapping. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and let

$$g_{kl} = f_{\tau(k)\tau(l)}|_{V_k \cap V_l} \qquad \text{and} \qquad \widetilde{g}_{kl} = f_{\widetilde{\tau}(k)\widetilde{\tau}(l)}|_{V_k \cap V_l}.$$

We must show that the cocycles (g_{kl}) and $(\widetilde{g}_{kl})$ are cohomologous, that is, their difference lies in $B^1(\mathfrak{V}, \mathscr{F})$. Define

$$h_k = f_{\tau(k), \widetilde{\tau}(k)}|_{V_k} \in \mathscr{F}(V_k).$$

Then, on $V_k \cap V_l$, we have

$$\begin{aligned} g_{kl} - \widetilde{g}_{kl} &= f_{\tau(k)\tau(l)} - f_{\widetilde{\tau}(k)\widetilde{\tau}(l)} \\ &= f_{\tau(k)\tau(l)} + f_{\tau(l),\widetilde{\tau}(k)} - f_{\tau(l)\widetilde{\tau}(k)} - f_{\widetilde{\tau}(k)\widetilde{\tau}(l)} \\ &= f_{\tau(k)\widetilde{\tau}(k)} - f_{\tau(l)\widetilde{\tau}(l)} \\ &= h_k - h_l. \end{aligned}$$

Whence $(g_{kl} - \widetilde{g}_{kl})$ is a coboundary, as desired. ■

LEMMA 1.5. The map $t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is injective.

Proof. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ is a 1-cocycle whose image in $Z^1(\mathfrak{V}, \mathscr{F})$ is a 1-coboundary. Then, there is some $(g_k) \in C^1(\mathfrak{U}, \mathscr{F})$ such that

$$f_{\tau(k)\tau(l)}|_{V_k \cap V_l} = g_k|_{V_k \cap V_l} - g_l|_{V_k \cap V_l}.$$

Then on $U_i \cap V_k \cap V_l$, we have

$$g_k - g_l = f_{\tau(k)\tau(l)} = f_{\tau(k)i} + f_{i\tau(l)} = f_{i\tau(l)} - f_{i\tau(k)}.$$

Hence $f_{i\tau(k)} + g_k = f_{i\tau(l)} + g_l$ on $U_i \cap V_k \cap V_l$. The gluability axiom applied to the open cover $\{U_i \cap V_k\}_{k \in K}$ furnishes a $h_i \in \mathscr{F}(U_i)$ such that

$$h_i = f_{i\tau(k)} + g_k \qquad \text{on } U_i \cap V_k \text{ for all } k \in K.$$

Then, on $U_i \cap U_j \cap V_k$ we have

$$f_{ij} = f_{i\tau(k)} + f_{\tau(k)j} = f_{i\tau(k)} + g_k - f_{j\tau(k)} - g_k = h_i - h_j.$$

Since $\{U_i \cap U_j \cap V_k\}$ forms an open cover of $U_i \cap U_j$, using the identity axiom, we have that $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. Thus, $(f_{ij}) \in B^1(\mathfrak{U}, \mathscr{F})$, thereby completing the proof. ■

DEFINITION 1.6. If $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$ are open covers of X, then it is easy to see that $t^{\mathfrak{V}}_{\mathfrak{W}} \circ t^{\mathfrak{U}}_{\mathfrak{V}} = t^{\mathfrak{U}}_{\mathfrak{W}}$. This gives us a directed system of cohomology groups, and we define

$$H^1(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}).$$

REMARK 1.7. Note that since the $t^{\mathfrak{U}}_{\mathfrak{V}}$ are all injective, the natural map $H^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ is injective. In particular, this means that $H^1(X, \mathscr{F}) = 0$ if and only if $H^1(\mathfrak{U}, \mathscr{F}) = 0$ for every open cover $\mathfrak{U}$ of X.

THEOREM 1.8 (LERAY). Let $\mathscr{F}$ be a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X such that $H^1(U_i, \mathscr{F}) = 0$ for every $i \in I$. Then for every open covering $\mathfrak{V} = (V_\alpha)_{\alpha \in A} < \mathfrak{U}$, the mapping

$$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is an isomorphism. The covering $\mathfrak{U}$ is called a *Leray covering* of X for the sheaf $\mathscr{F}$.

Proof. Let $\tau : A \to I$ be such that $V_\alpha \subseteq U_{\tau(\alpha)}$ for every $\alpha \in A$. Since we know that $t^{\mathfrak{U}}_{\mathfrak{V}}$ is injective, we would like to show that it is surjective. Let $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathscr{F})$. The family $(U_i \cap V_\alpha)_{\alpha \in A}$ is an open covering of U_i, which we denote by $U_i \cap \mathfrak{V}$. By assumption and Remark 1.7, we know that $H^1(U_i \cap \mathfrak{V}, \mathscr{F}) = 0$, that is, there exist $g_{i\alpha} \in \mathscr{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{ on } U_i \cap V_\alpha \cap V_\beta.$$

On the intersection $U_i \cap U_j \cap V_\alpha \cap V_\beta$, we have

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}.$$

Using the gluability axiom on the open cover $\{U_i \cap U_j \cap V_\alpha\}_{\alpha \in A}$, there exist elements $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

We claim that f_{ij} satisfies the cocycle condition. Obviously, from the above description, on $U_i \cap U_j \cap U_k \cap V_\alpha$ we have that $F_{ik} = F_{ij} + F_{jk}$. Using the identity axiom, we see that this equality holds on $U_i \cap U_j \cap U_k$. Thus, $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. Let $h_\alpha = g_{\tau(\alpha)\alpha}|_{V_\alpha} \in \mathscr{F}(V_\alpha)$. The on $V_\alpha \cap V_\beta$, we have

$$\begin{aligned} F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} &= \left(g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}\right) - \left(g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta}\right) \\ &= g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} \\ &= h_\beta - h_\alpha, \end{aligned}$$

whence $(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta})$ is a coboundary, thereby completing the proof. ■

COROLLARY 1.9. If $\mathfrak{U}$ is a Leray covering of X, then $H^1(\mathfrak{U}, \mathscr{F}) \cong H^1(X, \mathscr{F})$.

§2 THE FINITENESS THEOREM

§§ Laurent Schwartz's Theorem

THEOREM 2.1 (CLOSED RANGE THEOREM). Let $u : E \to F$ be a continuous linear map between Banach spaces. Then $u(E)$ is closed in F if and only if $u^*(F^*)$ is closed in E^*.

Proof. See [Rud91, Theorem 4.14]. ■

THEOREM 2.2 (SCHAUDER). Let $u : E \to F$ be a continuous linear map between Banach spaces. Then u is compact if and only if u^* is.

Proof. See [Rud91, Theorem 4.19]. ■

LEMMA 2.3. Let E, F be Banach spaces and let $u : E \to F$ be a continuous linear map. Suppose that u is injective and that $u(E)$ is closed. Let $v : E \to F$ be a compact continuous linear map. Then $\ker(u + v)$ is finite-dimensional and $(u + v)(E)$ is closed in F.

Proof. Let $N = \ker(u + v)$. To see that this is finite-dimensional, it suffices to show that the closed unit ball in N is compact. To this end, let (x_n) be a sequence in the closed unit ball of N. Since v is compact, there is a subsequence (x_{n_k}) such that $\left(v(x_{n_k})\right)$ converges, as a result, $u(x_{n_k}) = -v(x_{n_k})$ also converges. Since $u(E)$ is closed in F, it is a Banach space and $u : E \to u(E)$ is a bijection, whence, due to the "bounded inverse theorem", there is a constant $c > 0$ such that $\|u(x)\| \geqslant c\|x\|$ for all $x \in E$, consequently, for $k, l \geqslant 1$,

$$\|x_{n_k} - x_{n_l}\| \leqslant \frac{1}{c}\|u(x_{n_k}) - u(x_{n_l})\|,$$

whence (x_{n_k}) is Cauchy, and thus converges. This shows that N is finite-dimensional.

Owing to N being finite-dimensional, there is a closed subspace N' of E such that $E = N \oplus N'$. Since $(u + v)(E) = (u + v)(N')$, it suffices to show that the latter is closed in F. Let (x_n) be a sequence in N' such that $(u + v)(x_n)$ converges in F; we show that the limit lies in $(u + v)(N')$. First, we claim that the sequence (x_n) is bounded. If not, then we can move to a subsequence and assume that $0 \neq x_n$ for all n and $\|x_n\| \to \infty$ as $n \to \infty$. Set $x'_n = x_n / \|x_n\|$. Then $\|x'_n\| = 1$ and $(u + v)(x'_n) \to 0$ as $n \to \infty$. Since v is compact, there is a subsequence (x'_{n_k}) such that $v(x'_{n_k})$ converges, consequently,

$$u(x'_{n_k}) = (u + v)(x'_{n_k}) - v(x'_{n_k})$$

also converges. As we argued in the preceding paragraph using the "bounded inverse theorem", this means that (x'_{n_k}) converges. It follows that there is some $x_0 \in N'$ with $\|x_0\| = 1$ and $(u + v)(x_0) = 0$, that is, $x_0 \in N \cap N' = \{0\}$, a contradiction. Hence, (x_n) is a bounded sequence in E.

Compactness of v implies that there is a subsequence (x_{n_k}) such that $v(x_{n_k})$ converges in F; and since $(u+v)(x_n)$ was assumed to be convergent, we see that $u(x_{n_k})$ is convergent too. Again, using the "bounded inverse theorem", we have that (x_{n_k}) is convergent to some $x_0 \in N'$. Hence,

$$\lim_{n\to\infty}(u+v)(x_n) = \lim_{k\to\infty}(u+v)(x_{n_k}) = (u+v)(x_0) \in (u+v)(E),$$

as desired. ■

THEOREM 2.4 (L. SCHWARTZ). Let E, F be Banach space and let $u, v : E \to F$ be continuous linear maps. Suppose that u is surjective and that v is compact. Then $F' = (u+v)(E)$ is closed and F/F' is finite-dimensional.

Proof. Due to Theorem 2.1, it suffices to show that $(u^* + v^*)(F^*)$ is closed in E^*. Due to Theorem 2.2, we know that v^* is compact, and due to Theorem 2.1, we know that $u^*(F^*)$ is closed in E^*. Further, since u is surjective, it is easy to see that u^* must be injective. Thus, due to Lemma 2.3, we see that $(u^*+v^*)(F^*)$ is closed in E^*, as desired. We have shown that F' is closed in F.

To show that F/F' is finite-dimensional, we shall show that its closed unit ball is compact. Indeed, let (w_n) be a sequence in the closed unit ball of F/F', and choose preimages (x_n) in F satisfying $\|x_n\| \leqslant 2$. Since $u : E \to F$ is surjective, it is a consequence of the open mapping theorem, that there is a constant $M > 0$, independent of the sequence chosen, and a sequence (y_n) in E such that $\|y_n\| \leqslant M$ and $u(y_n) = x_n$. Since v is compact, there is a subsequence (x_{n_k}) such that $z_k = v(x_{n_k})$ converges in F to some $\widetilde{z}$. We can write

$$y_{n_k} = u(x_{n_k}) + v(x_{n_k}) - z_k = (u+v)(x_{n_k}) - z_k,$$

and hence, $-z_k$ maps to w_{n_k} in F/F'. Since the former converges, so does the latter. It follows that (w_n) admits a convergent subsequence, consequently, the closed unit ball in F/F' is compact, whence F/F' is finite-dimensional. This completes the proof. ■

§§ The Finiteness Theorem

The objective of this (sub)section is to prove the following theorem:

THEOREM 2.5. Let X be a compact Riemann surface, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of X. Then $H^1(\mathfrak{U}, \mathscr{O})$ is a finite-dimensional vector space.

Since X is compact, there is a finite covering $\mathfrak{V} = \{V_\alpha\}$ such that each V_α is an *analytic disc*[1] and such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Due to Theorem 1.8, we know that $H^1(\mathfrak{U}, \mathscr{O}) \cong H^1(\mathfrak{V}, \mathscr{O})$ and hence, it suffices to show that the latter is a finite-dimensional vector space.

We may therefore suppose that $\mathfrak{U} = \{U_i\}_{i\in I}$ where I is finite, and that there are charts $\phi_i : U_i \to D \subseteq \mathbb{C}$, there $D = B(0,1)$, the open unit disc in $\mathbb{C}$. For any $r < 1$, set $U_i(r) = \phi_i^{-1}(B(0,r))$. There is a $0 < r_0 < 1$ such that $\bigcup_{i\in I} U_i(r_0) = X$. Thus, for $r_0 < r < 1$, we have that $\bigcup_{i\in I} U_i(r) = X$. Denote by $\mathfrak{U}(r)$ the finite covering $\{U_i(r)\}_{i\in I}$ of X.

Let

$$Z_b^1(r) = \left\{ (f_{ij})_{i,j\in I} \in Z^1(\mathfrak{U}(r), \mathscr{O}) \colon \sup_{x\in U_i(r)\cap U_j(r)} |f_{ij}(x)| < \infty \quad \text{for all } i,j \in I \right\},$$

and for $\xi = (f_{ij})_{i,j\in I} \in Z_b^1(r)$, define

$$\|\xi\| := \|\xi\|_r = \max_{i,j\in I} \sup_{x\in U_i(r)\cap U_j(r)} |f_{ij}(x)|.$$

The above gives $Z_b^1(r)$ the structure of a normed linear space. It is easy to see that $Z_b^1(r)$ is in fact a Banach space since a uniformly Cauchy sequence of holomorphic functions converges to a holomorphic function.

Next, define

$$B_b^1(r) = Z_b^1(r) \cap B^1(\mathfrak{U}(r), \mathscr{O}),$$

[1] An open set $U \subseteq X$ is said to be a analytic disc if there is a chart $\phi : U \to D$ where D is the open unit disc in $\mathbb{C}$.

and

$$C_b^0(r) = \left\{ (f_i)_{i\in I} \in C^0(\mathfrak{U}(r), \mathscr{O}) \colon \sup_{x\in U_i(r)} |f_i(x)| < \infty \quad \text{for all } i \in I \right\}.$$

If $\eta = (f_i)_{i\in I} \in C_b^0(r)$, we also set

$$\|\eta\| = \|\eta\|_r = \max_{i\in I} \sup_{x\in U_i(r)} |f_i(x)|.$$

Then $C_b^0(r)$ is also a Banach space under the above norm for the same reason $Z_b^1(r)$ was.

LEMMA 2.6. Let $\delta : C^0(\mathfrak{U}(r), \mathscr{O}) \to B^1(\mathfrak{U}(r), \mathscr{O})$ be the coboundary operator as defined earlier. We have

$$\delta^{-1}\left(B_b^1(r)\right) = C_b^0(r).$$

Proof. Obviously, the image of $C_b^0(r)$ under δ is contained in $B_b^1(r)$ since the indexing set I is finite. Conversely, let $(f_i)_{i\in I} \in C^0(\mathfrak{U}(r), \mathscr{O})$ be such that

$$M_{ij} = \sup_{x\in U_i(r)\cap U_j(r)} |f_i(x) - f_j(x)| < \infty \quad \text{for all } i, j \in I.$$

Since $U_i(r)$ is relatively compact in U_i, it suffices to show that every $a \in \partial U_i(r)$ has a neighborhood V in U_i such that

$$\sup_{x\in V\cap U_i(r)} |f_i(x)| < \infty.$$

Indeed, since $\partial U_i(r) = \overline{U_i(r)} \setminus U_i(r)$ is compact, it can be covered with finitely many such V's and hence, there is a constant $M > 0$ such that for each $a \in \partial U_i(r)$, we have

$$\limsup_{\substack{x\to a\\ x\in U_i(r)}} |f_i(x)| < M,$$

consequently, from the Maximum Modulus Principle (see [Con78, Theorem VI.1.4]), f_i is bounded on $U_i(r)$, that is, $(f_i)_{i\in I} \in C_b^0(r)$.

Now, let $a \in \partial U_i(r)$, choose an index $j \in I$ such that $a \in U_j(r)$, and let V be a neighborhood of a that is relatively compact in $U_j(r)$. We have, for $x \in V \cap U_i(r)$,

$$\sup_{x\in V\cap U_i(r)} |f_i(x)| \leqslant \sup_{x\in V\cap U_i(r)} |f_i(x) - f_j(x)| + \sup_{x\in V} |f_j(x)| \leqslant M_{ij} + \sup_{x\in V} |f_j(x)| < \infty,$$

since V is relatively compact in $U_j(r)$. The conclusion follows. ∎

THEOREM 2.7 (MONTEL). Let X be a Riemann surface and let $\mathcal{F} \subseteq \mathscr{O}(X)$ be a subset such that for each compact set $K \subseteq X$

$$\sup_{f\in\mathcal{F}} \sup_{x\in K} |f(x)| < \infty.$$

Then any sequence $(f_n)_{n\geqslant 1}$ of functions in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of K.

Proof Sketch. Cover X by countably many charts. Using a diagonal argument, choose a subsequence $(f_\nu)_\nu$ that converges uniformly on compact subsets of each chart. Since any compact subset of X can be covered by finitely many compact sets, each contained in a chart, we have the desired conclusion. ∎

LEMMA 2.8. For $r_0 < r < 1$, the vector space

$$Z_b^1(r)/B_b^1(r)$$

is finite-dimensional.

Proof. Choose ρ such that $r < \rho < 1$. Since $\mathfrak{U}(r)$ is a refinement of $\mathfrak{U}$ through the identity map $\tau : I \to I$, due to Theorem 1.8, we have that the restriction map

$$Z^1(\mathfrak{U}, \mathscr{O}) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad (f_{ij})_{i,j\in I} \longmapsto \left(f_{ij}|_{U_i(r)\cap U_j(r)}\right)_{i,j\in I}$$

induces an isomorphism $H^1(\mathfrak{U}, \mathscr{O}) \to H^1(\mathfrak{U}(r), \mathscr{O})$. Thus, the map

$$Z^1(\mathfrak{U}, \mathscr{O}) \oplus C^0(\mathfrak{U}(r), \mathscr{O}) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad ((f_{ij})_{i,j\in I}, (g_i)_{i\in I}) \longmapsto \left(f_{ij}|_{U_i(r)\cap U_j(r)}\right)_{i,j\in I} + \delta\left((g_i)_{i\in I}\right).$$

Since every $f \in \mathscr{O}(U_i \cap U_j)$ is bounded on $U_i(\rho) \cap U_j(\rho)$, owing to the latter being relatively compact in the former, it follows that the map

$$Z_b^1(\rho) \oplus C^0(\mathfrak{U}(r), \mathscr{O}) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad ((f_{ij})_{i,j\in I}, (g_i)_{i\in I}) \longmapsto \left(f_{ij}|_{U_i(r)\cap U_j(r)}\right)_{i,j\in I} + \delta\left((g_i)_{i\in I}\right)$$

is surjective too. We claim that the restriction of the above map

$$u : Z_b^1(\rho) \oplus C_b^0(r) \to Z_b^1(r) \qquad ((f_{ij})_{i,j\in I}, (g_i)_{i\in I}) \longmapsto \left(f_{ij}|_{U_i(r)\cap U_j(r)}\right)_{i,j\in I} + \delta\left((g_i)_{i\in I}\right)$$

is surjective. Indeed, for some $(f_{ij})_{i,j\in I} \in Z_b^1(r)$, choose an element $(\widetilde{f}_{ij})_{i,j\in I} \in Z^1(\mathfrak{U}, \mathscr{O})$ which maps to the image of (f_{ij}) in $H^1(\mathfrak{U}(r), \mathscr{O})$. The difference

$$(f_{ij})_{i,j\in I} - \left(\widetilde{f}_{ij}|_{U_i(r)\cap U_j(r)}\right)_{i,j\in I} \in B_b^1(r),$$

since $U_i(r) \cap U_j(r)$ is relatively compact in $U_i \cap U_j$. From Lemma 2.6, we know that the preimage of the above map under δ lies inside $C_b^0(r)$, as desired.

Let $v : Z_b^1(\rho) \oplus C_b^0(r) \to Z_b^1(r)$ be given by the map

$$v\left((f_{ij})_{i,j\in I}, (g_i)_{i\in I}\right) = \left(-f_{ij}|_{U_i(r)\cap U_j(r)}\right)_{i,j\in I}.$$

Note that $U_i(r) \cap U_j(r)$ is relatively compact in $U_i(\rho) \cap U_j(\rho)$, whence by Theorem 2.7 it is easy to argue that v is compact[2].

Finally, by Theorem 2.4, the continuous linear mapping $u + v : Z_b^1(\rho) \oplus C_b^0(r) \to Z_b^1(r)$ given by

$$(u+v)\left((f_{ij})_{i,j\in I}, (g_i)_{i\in I}\right) = \delta\left((g_i)_{i\in I}\right)$$

has closed image with finite codimension. Since the image of $u + v$ is precisely $B_b^1(r)$, we have the desideratum. ∎

Lemma 2.9. $\dim H^1\left(\mathfrak{U}(r), \mathscr{O}\right) < \infty$.

Proof. Let $\tau : I \to I$ be the identity map. Due to Theorem 1.8, the induced map $\tau^* : H^1(\mathfrak{U}, \mathscr{O}) \to H^1(\mathfrak{U}(r), \mathscr{O})$ is an isomorphism. The map

$$Z^1(\mathfrak{U}, \mathscr{O}) \to Z_b^1(\rho) \qquad (f_{ij})_{i,j\in I} \longmapsto \left(f_{ij}|_{U_i(\rho)\cap U_j(\rho)}\right)$$

induces a map $\alpha : H^1(\mathfrak{U}, \mathscr{O}) \to Z_b^1(\rho)/B_b^1(\rho)$. Similarly, the map

$$Z_b^1(\rho) \to Z^1(\mathfrak{U}(r), \mathscr{O}) \qquad (f_{ij})_{i,j\in I} \longmapsto \left(f_{ij}|_{U_i(r)\cap U_j(r)}\right)$$

induces a map $\beta : Z_b^1(\rho)/B_b^1(\rho) \to H^1(\mathfrak{U}(r), \mathscr{O})$ such that $\tau^* = \beta \circ \alpha$. In particular, this means that β is surjective. Due to Lemma 2.8, $H^1(\mathfrak{U}(r), \mathscr{O})$ is finite-dimensional. ∎

Proof of Theorem 2.5. Follows immediately from Theorem 1.8 and Lemma 2.9. ∎

[2]Probably add more details here?

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