Determinants

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§1 Triangulation, Diagonalization, and Primary Decomposition

§§ Eigenvalues and Eigenvectors

DEFINITION 1.1. Let V be a vector space over the field F and let $T: V \to V$ be a linear map. A *eigenvalue* of T is a scalar $\lambda \in F$ such that there is a non-zero vector $\alpha \in V$ with $T\alpha = \lambda\alpha$.

If λ is an eigenvalue of T, then

(i) any $\alpha \in V$ such that $T\alpha = \lambda\alpha$ is called an eigenvector of T associated to the eigenvalue λ.

(ii) the collection of all $\alpha \in V$ such that $T\alpha = \lambda\alpha$ is called the *eigenspace* of T associated to the eigenvalue λ.

THEOREM 1.2. Let $T: V \to V$ be a linear map on a finite-dimensional space V and let $\lambda \in F$. The following are equivalent:

(1) λ is an eigenvalue of T.

(2) The operator $T - \lambda I$ is not invertible.

(3) $\det(T - \lambda I) = 0$.

Proof. Trivial. ■

DEFINITION 1.3. Let n be a positive integer and A an $n \times n$ matrix with entries in F. The *characteristic polynomial* of A is defined to be $\chi_A(X) = \det(X \cdot I - A) \in F[X]$.

Given a linear map $T: V \to V$ where V is a finite-dimensional vector space over F, define the characteristic polynomial of T to be the characteristic polynomial of its matrix representation with respect to any basis of V.

REMARK 1.4. The definition and Theorem 1.2 immediately imply that $\lambda \in F$ is an eigenvalue if and only if $\chi_T(\lambda) = 0$.

DEFINITION 1.5. Let $T: V \to V$ be a linear map on a finite-dimensional vector space V. We say that T is *diagonalizable* if there is a basis for V, each vector of which is an eigenvector of T.

Remark 1.6. It is clear from the definition that T is diagonalizable if and only if there is a basis of V with respect to which T is given by a diagonal matrix.

Lemma 1.7. Let $T\colon V \to V$ be a linear map on a finite-dimensional vector space V over F. Let $\lambda_1, \dots, \lambda_k \in F$ be the distinct eigenvalues of T and let W_i denote the eigenspace of T associated with λ_i for $1 \leqslant i \leqslant k$. If $W = W_1 + \cdots + W_k$, then

$$\dim W = \dim W_1 + \cdots + \dim W_k.$$

Proof. It suffices to show that the given sum is direct. Indeed, suppose $\beta_i \in W_i$ for $1 \leqslant i \leqslant k$ are such that $\beta_1 + \cdots + \beta_k = 0$. Using Lagrange's method of interpolation, choose a polynomial $h_i(X) \in F[X]$ such that

$$h_i(\lambda_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$0 = h_i(T)\big(\beta_1 + \cdots + \beta_k\big) = \beta_i$$

for $1 \leqslant i \leqslant n$. ∎

As a result, we obtain:

Theorem 1.8. Let $T\colon V \to V$ be a linear operator on a finite-dimensional vector space V over a field F. Let $\lambda_1, \dots, \lambda_k \in F$ be the distinct eigenvalues of T and let W_i be the eigenspace of T associated with λ_i for $1 \leqslant i \leqslant k$. Then the following are equivalent:

(1) T is diagonalizable.

(2) The characteristic polynomial for T is

$$\chi(X) = (X - \lambda_1)^{d_1} \cdots (X - \lambda_k)^{d_k},$$

where $\dim W_i = d_i$ for $1 \leqslant i \leqslant k$.

(3) $\dim W_1 + \cdots + \dim W_k = \dim V$.

Proof. The implication (1) $\Longrightarrow$ (2) is clear by considering the matrix representation of T with respect to a suitable basis. Further, the implication (2) $\Longrightarrow$ (3) is clear from the the fact that the degree of the characteristic polynomial is equal to the dimension of V. Finally, the implication (3) $\Longrightarrow$ (1) follows from Lemma 1.7, since that would imply $V = W_1 + \cdots + W_k$, that is, V has a basis consisting of eigenvectors of T. ∎

§§ The Minimal Polynomial

Definition 1.9. Let $T\colon V \to V$ be a linear operator on a finite-dimensional vector space V over a field F. Let $\mathfrak{A}$ denote the set of all polynomials $f(X) \in F[X]$ such that $f(T) = 0$ as a linear operator. Clearly $\mathfrak{A}$ is an ideal in $F[X]$. The unique[1] monic generator of $\mathfrak{A}$ is called the *minimal polynomial* for T.

Remark 1.10. Since $F[X]$ is a Euclidean domain with the Euclidean function given by the degree map, the minimal polynomial is the unique monic polynomial in $\mathfrak{A}$ having the smallest degree.

[1] Because $(F[X])^\times = F^\times$.

PROPOSITION 1.11. Let $T: V \to V$ be a linear operator on a finite-dimensional vector space V over a field F. Then $\lambda \in F$ is a root of the characteristic polynomial of T if and only if it is a root of the minimal polynomial of T.

Proof. Let $p(X) \in F[X]$ be the minimal polynomial for T and let $\chi(X) \in F[X]$ denote the characteristic polynomial. Suppose first that $p(\lambda) = 0$. Then $p(X) = (X - \lambda)q(X)$ for some polynomial $q(X) \in F[X]$. Since $\deg q < \deg p$, we must have $q(T) \neq 0$. Choose a vector $\beta \in V$ such that $\alpha := q(T)\beta \neq 0$. Then

$$0 = p(T)\beta = (T - \lambda I)q(T)\beta = (T - \lambda I)\alpha,$$

so that λ is an eigenvalue of T, whence $\chi(\lambda) = 0$.

Conversely, suppose $\chi(\lambda) = 0$, that is, λ is an eigenvalue of T, so there exists a non-zero vector $\alpha \in V$ with $T\alpha = \lambda\alpha$. Then

$$0 = p(T)\alpha = p(\lambda)\alpha \implies p(\lambda) = 0,$$

thereby completing the proof. ■

THEOREM 1.12 (CAYLEY-HAMILTON). Let $T: V \to V$ be a linear operator on a finite-dimensional vector space V over a field F. If $\chi(X) \in F[X]$ denotes the characteristic polynomial of T, then $\chi(T) = 0$.

Proof. ■