Hartshorne Exercises

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Contents

Chapter I

Varieties

§I.1 Affine Varieties

Definition. A topological space X is said to be *irreducible* if whenever $X = X_1 \cup X_2$ where X_1 and X_2 are closed subsets of X, $X = X_1$ or $X = X_2$.

Exercise I.1.1. (a) $A(Y) = k[x,y]/(y^2 - x)$. Consider the surjective map

$$\varphi : k[x,y] \to k[t]$$

sending $x \mapsto t^2$, $y \mapsto t$. Then, $\mathfrak{p} = \ker \varphi$ is a prime ideal containing $(y^2 - x)$. Further, $\operatorname{ht} \mathfrak{p} = \dim k[x,y] - \dim k[t] = 1$. Thus, $\mathfrak{p} = (y^2 - x)$. This establishes the desired isomorphism.

(b) Using analogous reasoning, one can show that $A(Z) \cong k[t, t^{-1}]$. Suppose there is an isomorphism $k[t, t^{-1}] \cong k[x]$. Under this isomorphism, t must map to a unit and hence inside k, a contradiction.

Exercise I.1.2. Consider the map $\varphi : k[x,y,z] \to k[t]$ sending $x \mapsto t$, $y \mapsto t^2$, and $z \mapsto t^3$. Let $\mathfrak{p} = \ker \varphi$, which is a prime ideal with $\operatorname{ht} \mathfrak{p} = \dim k[x,y,z] - \dim k[t] = 2$. Note that $(x^2 - y, x^3 - z) \subseteq \mathfrak{p}$. Now, suppose $f(x,y,z) \in \mathfrak{p}$, then we can view f as an element of $k[x][y,z]$ and write

$$f(x,y,z) = (y - x^2)P + (z - x^3)Q + \underbrace{f(x, x^2, x^3)}_{=0},$$

and hence, $\mathfrak{p} = (y - x^2, z - x^3)$. The conclusion follows.

Exercise I.1.3.

Exercise I.1.4. Since $\mathbb{A}^1$ is not Hausdorff, the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$ is not closed, while the diagonal of $\mathbb{A}^2$ is $Z(x - y)$, which is closed.

Exercise I.1.5. $B \cong k[x_1, \ldots, x_n]/\mathfrak{a}$ for some radical ideal $\mathfrak{a}$. If we set $Y = Z(\mathfrak{a})$, then $B = A(Y)$.

Exercise I.1.6.

- If X is irreducible and $U \subseteq X$ is non-empty open, then $X = (X \setminus U) \cup \overline{U}$ and hence, U is dense. Further, U is irreducible; for if $U = U_1 \cup U_2$ where U_i closed in U, then $U_i = U \cap X_i$ where X_i closed in X. Consequently, $U \subseteq X_1 \cup X_2$. The latter being closed, contains $\overline{U} = X$ and hence, for some i, $X = X_i$, therefore, $U = U_i$.
- If $Y \subseteq X$ (any topological space) is irreducible, then so is $\overline{Y}$; for if $\overline{Y} = Y_1 \cup Y_2$, where Y_i closed in $\overline{Y}$, then Y_i closed in X. Further, $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$, thus, for some i, $Y = Y \cap Y_i$, hence, $Y_i \supseteq Y$ but being closed, $Y_i \supseteq \overline{Y}$.

Exercise I.1.7. (a) This is trivial.

(b) Let $\{U_\alpha\}$ be an open cover of X, a noetherian topological space. If $\mathfrak{M}$ denotes the collection of all finite unions of U_α's, then $\mathfrak{M}$ has a maximal element, which must be all of X.

(c) Let $Y \subseteq X$ and suppose $V_1 \subseteq V_2 \subseteq \cdots$ is an ascending chain of open subsets of Y. There are U_i open in X such that $V_i = U_i \cap Y$. Let $\widetilde{U}_i = \bigcup_{j=1}^{i} U_j$. Note that $\widetilde{U}_i \cap Y = U_i$. Then, $\widetilde{U}_1 \subseteq \widetilde{U}_2 \subseteq \cdots$, and hence, stabilizes at some $\widetilde{U}_N$. It follows that $V_N = V_{N+1} = \cdots$.

(d) Every subspace of a noetherian topological space is noetherian, and hence, quasi-compact, and hence, closed (since the ambient space is Hausdorff). Thus, the topology is discrete. A discrete quasi-compact topology must have a finite underlying set.

EXERCISE I.1.8. There is a prime ideal $\mathfrak{p}$ in $k[x_1, \ldots, x_n]$ such that $Y = Z(\mathfrak{p})$. Similarly, there is an irreducible polynomial $f \in k[x_1, \ldots, x_n]$ such that $H = Z(f)$. Note that $f \notin \mathfrak{p}$, else $Y \subseteq H$.

Let $\mathfrak{q}$ be a minimal prime over $(f) + \mathfrak{p}$. Working in the ring $R/\mathfrak{p}$, $\overline{\mathfrak{q}}$ is minimal over $(\overline{f})$. Due to Krull's Hauptidealsatz, $\operatorname{ht} \overline{\mathfrak{q}} \leqslant 1$. The height must be non-zero since $\overline{\mathfrak{q}} \neq 0$. Thus, $\operatorname{ht} \overline{\mathfrak{q}} = 1$, whence $\dim k[x_1, \ldots, x_n]/\mathfrak{q} = \dim R/\overline{\mathfrak{q}} = \dim R - 1 = r - 1$.

EXERCISE I.1.9. This is again a trivial consequence of the Hauptidealsatz.

EXERCISE I.1.10. (a) Let $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ be a chain of closed irreducible subsets of Y. These are also irreducible as subspaces of X and hence, so are their closures. This gives us a chain

$$\overline{Y}_0 \subseteq \overline{Y}_1 \subseteq \cdots \subseteq \overline{Y}_n.$$

We contend that the inclusions are strict. Suppose $\overline{Y}_i = \overline{Y}_{i+1}$ for some $1 \leqslant i < n$. Thus, the closure of Y_i in Y is equal to that of Y_{i+1} in Y. This is absurd, since the Y_j's are closed in Y. Thus, $\dim X \geqslant n$. Taking sup over all n, we have $\dim X \geqslant \dim Y$.

(b) Due to part (a), we have $\dim X \geqslant \sup \dim U_i$. If $Y_0 \subsetneq \cdots \subsetneq Y_n$ is a chain of closed irreducible subsets of X, choose a $U = U_i$ having non-empty intersection with Y_0. Then, $U \cap Y_j$ is irreducible and dense in Y_j for every j. Note that $U \cap Y_{j-1} \subseteq Y_{j-1} \subsetneq Y_j$. Since Y_{j-1} is closed in Y_j, $U \cap Y_{j-1}$ is not dense in Y_j. Thus, $U \cap Y_{j-1} \subsetneq U \cap Y_j$. Thus, $\dim U \geqslant n$ that is, $\sup \dim U_i \geqslant n$. Taking supremum over n, we obtain the desired conclusion.

(c)

(d) Suppose Y is properly contained in X. Then for any chain of closed irreducibles $Y_0 \subsetneq \cdots \subsetneq Y_n$ in Y, we can append X to get a chain of closed irreducibles in X, in particular, this means $\dim X \geqslant \dim Y + 1$, a contradiction.

(e) Spec (Nagata's monster ring).

§I.2 Projective Varieties

Exercise I.2.1. Let X be the affine algebraic set in $\mathbb{A}^{n+1}$ corresponding to $\mathfrak{a}$. Under the canonical map $\mathbb{A}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$. Since f is homogeneous, f vanishes on X, thus, $f^q \in \mathfrak{a}$ for some $q > 0$ due to the affine nullstellensatz.

Exercise I.2.2. $(i) \implies (ii)$ We look at the affine variety corresponding to $\mathfrak{a}$. There are two possible options for this: either $\varnothing$ or the origin in $\mathbb{A}^{n+1}$. In the former case, due to the weak nullstellensatz, $\mathfrak{a} = S$. In the latter case, $\sqrt{\mathfrak{a}}$ is the ideal corresponding to the origin, that is, $\sqrt{\mathfrak{a}} = (x_0, \dots, x_n) = S_+$.

$(ii) \implies (iii)$ If $\sqrt{\mathfrak{a}} = S$, then $1 \in \mathfrak{a}$, hence, $\mathfrak{a} = S$. If $\sqrt{\mathfrak{a}} = S_+$. There is a sufficiently large positive integer N such that $x_i^N \in \mathfrak{a}$ for $0 \leqslant i \leqslant n$. It is then easy to see that $S_{(n+1)N} \subseteq \mathfrak{a}$.

$(iii) \implies (i)$ If $\mathfrak{a} \supseteq S_d$ for some $d > 0$, then it contains the monomials $x_0^d, \dots, x_n^d$. The projective variety corresponding to this collection of monomials is empty.

Exercise I.2.3. (a) Clear.

(b) Clear.

(c) Clear.

(d) This follows from Exercise *I.2.1*.

(e) Since $Z(I(Y))$ is closed and contains Y, it must contain $\overline{Y}$. Suppose P is a point not contained in $\overline{Y}$. Then, P is not contained in some closed set $Z(\mathfrak{a})$ containing Y, where $\mathfrak{a}$ is a homogeneous ideal. Thus, there is a homogeneous $f \in \mathfrak{a}$ such that $f(P) \neq 0$. But since $f \in I(Y)$, it follows that $P \notin Z(I(Y))$. This completes the proof.

Exercise I.2.4. (a) There are two maps involved here:

$$\begin{aligned} \{\text{Algebraic sets in } \mathbb{P}^n\} &\to \{\text{Homogeneous ideals in } S\} \setminus \{S_+\} \\ Y &\longmapsto I(Y) \end{aligned}$$

and

$$\begin{aligned} \{\text{Homogeneous ideals in } S\} \setminus \{S_+\} &\to \{\text{Algebraic sets in } \mathbb{P}^n\} \\ \mathfrak{a} &\longmapsto Z(\mathfrak{a}). \end{aligned}$$

Due to the preceding exercise, $Z(I(Y)) = \overline{Y} = Y$. On the other hand, if $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. On the other hand, if $Z(\mathfrak{a}) = \varnothing$, then we have shown that $\mathfrak{a} = S$, since it is not equal to S_+. Hence, $I(\varnothing) = S = \mathfrak{a}$, thereby establishing the bijection.

(b) Suppose $I(Y)$ is not a prime ideal. Due to an equivalent characterization of homogeneous prime ideals mentioned in the book, there are homogeneous polynomials $f, g \in S \setminus I(Y)$ such that $fg \in I(Y)$. Then, $Y \subseteq Z(f) \cup Z(g)$ $Y \nsubseteq Z(f), Z(g)$ and hence, Y is not irreducible.

On the other hand, suppose $Y = Y_1 \cup Y_2$, where $Y_1, Y_2 \subsetneq Y$ are closed in Y. Due to the bijection established in (a), $I(Y_i) \supsetneq I(Y)$. Choose $f \in I(Y_1) \setminus I(Y)$ and $g \in I(Y_2) \setminus I(Y)$. Then, $fg \in I(Y)$ and hence, $I(Y)$ is not prime.

(c) $\mathbb{P}^n$ corresponds to (0), which is prime in S.

Exercise I.2.5. (a) Due to (a) and (b) of the preceding exercise, this follows from the fact that S is noetherian.

(b) This is a property of arbitrary noetherian topological spaces and we shall prove it in this generality.

Let X be a noetherian topological space and let Σ be the collection of all closed subspaces of X that cannot be expressed as a finite union of irreducible closed subspaces of X. Suppose Σ is non-empty. Since X is noetherian, choose a minimal element Y of Σ. Y cannot be irreducible, else it would trivially be a finite union of closed irreducibles. Since Y is not irreducible, it can be written as a union of proper closed subsets $Y = Y_1 \cup Y_2$. Due to the minimality of Y, $Y_1, Y_2 \notin \Sigma$, and hence, each can be written as a finite union of closed irreducibles, whence so can Y, a contradiction again. Thus, $\Sigma = \varnothing$.

EXERCISE I.2.6. Let U_i be the open set $\mathbb{P}^n \setminus Z(x_i)$ and set $Y_i = Y \cap U_i \neq \emptyset$, which is closed in U_i and hence, is homeomorphic to an affine variety. We shall treat Y_i as an affine variety.

The "variables" $\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}$ form a set of coordinates on U_i as an affine space. Under this identification, $A(Y_i)$ is the set of all polynomial functions $f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$ which vanish on Y_i. This polynomial can be written in the form

$$\frac{\widetilde{f}(x_0, \dots, x_i, \dots, x_n)}{x_i^N}$$

where $N = \deg f$ and $\widetilde{f}$ is homogeneous. By construction, $\widetilde{f}$ is a homogeneous polynomial vanishing on $Y \cap U_i$ which is dense in Y. But $Z(\widetilde{f})$ must be closed, and thus, $\widetilde{f}$ vanishes on Y.

This gives a canonical ring homomorphism $A(Y_i) \to (S(Y)_{x_i})_0$ given by $f \mapsto \widetilde{f}/x_i^N$. We contend that this homomorphism is bijective. Indeed, if f lies in the kernel of the homomorphism, then $\widetilde{f}/x_i^N = 0$ as an element of $S(Y)_{x_i}$, consequently, $x_i^m \widetilde{f} = 0$ as an element of $S(Y)$. In particular, $\widetilde{f}/x_i^N$ vanishes identically on $Y \cap U_i$, since x_i is nonzero here. To see surjectivity, simply note that every element in the codomain looks like $\widetilde{f}/x_i^N$. This establishes the desired isomorphism.

The dimension of Y_i as a topological space is the dimension of Y_i as an affine variety, which is the dimension of $(S(Y)_{x_i})_0$ as a ring.

Next, we establish the isomorphism $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. There is a map $S(Y) \to A(Y_i)[x_i, x_i^{-1}]$, which sends a polynomial function to $x_i^{\deg} \times \operatorname{poly}(x_0/x_i, \dots, x_n/x_i)$. This is obviously a ring homomorphism. Further, note that x_i is invertible in the image and hence, this factors through $S(Y)_{x_i}$. We shall show that the induced map is an isomorphism of rings. Note that any element in the image looks like a Laurent polynomial of the form

$$\sum_{n \in \mathbb{Z}} f_n\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) x_i^n$$

where f_n vanishes on $Y \cap U_i$. Thus, its homogenization vanishes on Y and hence, is an element of $S(Y)$. It follows that the map defined is surjective. Injectivity is obvious.

Therefore, $\dim Y_i$ is the Krull dimension of $A(Y_i)[x_i, x_i^{-1}]$, which is the transcendence degree of $\operatorname{Frac}(A(Y_i))(x_i)$. This is precisely $1 + \dim Y_i$. But also note that $\dim S(Y)_{x_i}$ is $\dim S(Y)$ by comparing transcendence degrees.

Therefore, we have shown $\dim S(Y) = 1 + \dim Y_i$ whenever $Y_i \neq \emptyset$. Taking supremum over all such Y_i, we obtain the desired conclusion.

EXERCISE I.2.7. (a) $\dim \mathbb{P}^n = \dim k[x_0, \dots, x_n] - 1 = n + 1 - 1 = n$.

(b) We have

$$\dim \overline{Y} = \sup \dim \overline{Y} \cap U_i = \sup \dim Y \cap U_i = \dim Y,$$

where the second equality follows from **Proposition 1.10** in Hartshorne.

EXERCISE I.2.8.

EXERCISE I.2.9. (a) Let $f \in I(Y)$; then $\beta(f)$ is its homogenization. On U_0, $x_0 \neq 0$ and hence, $\beta(f)$ vanishes on $Y \subseteq U_0$. Again, the zero set of $\beta(f)$ is closed in $\mathbb{P}^n$, and hence, vanishes on $\overline{Y}$. Consequently, $\beta(I(Y)) \subseteq I(\overline{Y})$. On the other hand, if $F \in k[x_0, \dots, x_n]$ is a homogeneous polynomial vanishing over $\overline{Y}$, and hence, over Y. Thus, $f = \alpha(F)$ vanishes on Y. Consequently, $F = \beta(f)$, thereby concluding the proof.

(b) I'm not in the mood to write it up.

EXERCISE I.2.10. (a) Trivial.

(b) Since $S(Y) = A(C(Y))$.

(c) We have

$$\dim Y + 1 = \dim S(Y) = \dim A(C(Y)) = \dim C(Y).$$

EXERCISE I.2.12 (THE d-UPLE EMBEDDING). (a) Note that θ is a degree d graded ring homomorphism and hence, the kernel is homogeneous. The kernel is a prime ideal since the image of θ is a subring of an integral domain, whence an integral domain. Note that ρ_d is injective. This will be useful.

(b) Let
$$S = \left\{ (i_0, \ldots, i_n) \colon i_j \geqslant 0,\ i_0 + \cdots + i_n = d \right\},$$
and note that $|S| = N$. We shall henceforth index the y-variables as y_s for $s \in S$. Analogously, elements of $\mathbb{P}^N$ shall be denoted as $[a_s \colon s \in S]$.

Consider the open "affine" $U_{(d,0,\ldots,0)}, U_{(0,d,\ldots,0)}, \ldots$. We contend that these cover $Z(\mathfrak{a})$. Indeed, suppose $[a_s \colon s \in S] \in Z(\mathfrak{a})$. Then, there is some $s = (i_0, \ldots, i_n) \in S$ such that $a_s \neq 0$. Consider the function
$$f(\{y_t \colon t \in S\}) = y^d_{(i_0,\ldots,i_n)} - y^{i_0}_{(d,0,\ldots,0)} \cdots y^{i_n}_{(0,\ldots,0,d)}.$$

Note that $f \in \mathfrak{a}$, and hence, $f(\{a_t \colon t \in S\}) = 0$. That is,
$$0 \neq a^d_s = a^{i_0}_{(d,0,\ldots,0)} \cdots a^{i_n}_{(0,\ldots,0,d)},$$
thus, $[\{a_t \colon t \in S\}]$ lies in one of the aforementioned open sets.

We now construct local inverses for ρ_d. Consider $U_{(d,0,\ldots,0)} \cap Z(\mathfrak{a})$ and take an element $[\{b_s \colon s \in S\}]$ in it. Define $\sigma_0 : U_{(d,0,\ldots,0)} \cap Z(\mathfrak{a}) \to \mathbb{P}^n$ as
$$[\{b_s \colon s \in S\}] \mapsto \left[b_{(d,0,\ldots,0)} : b_{(d-1,1,0,\ldots,0)} : \cdots : b_{(d-1,0,\ldots,0,1)} \right].$$

It is not hard to see that $\rho_d \circ \sigma_0$ is the identity map on its domain. Analogously, construct σ_i for $0 \leqslant i \leqslant n$. Since the U's cover $Z(\mathfrak{a})$, we see that ρ_d must be surjective.

(c) To show that ρ_d is a homeomorphism, it suffices to show that the σ's can be glued together, since each σ_i is a continuous function (owing to it being polynomial in the coordinates).

Indeed, suppose $[\{b_s \colon s \in S\}] \in U_{(d,0,\ldots,0)} \cap U_{(0,d,\ldots,0)}$. Since ρ is injective and $\rho_d(\sigma_0(b)) = \rho_d(\sigma_1(b))$, we have that $\sigma_0(b) = \sigma_1(b)$.

(d)

EXERCISE I.2.14 (THE SEGRE EMBEDDING). $N = (r+1)(s+1) - 1$ and let the homogeneous coordinates of $\mathbb{P}^n$ be $[z_{ij} \colon 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s]$. There is a ring homomorphism
$$\varphi : k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s],$$
sending $z_{ij} \mapsto x_i y_j$. Let $\mathfrak{a} = \ker \varphi$. If $f \in \mathfrak{a}$, then $f(\{x_i y_j \colon 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s\}) = 0$. Since an element in the image ψ looks like $[a_i b_j \colon 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s]$, we have that $\operatorname{im} \psi \subseteq Z(\mathfrak{a})$.

On the other hand, suppose $[c_{ij} \colon 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s] \in Z(\mathfrak{a})$. Without loss of generality, suppose $c_{00} = 1$. Then, set $a_i = c_{i0}$ and $b_j = c_{0j}$ and note that $\psi(\mathbf{a}, \mathbf{b}) = \mathbf{c}$, thereby completing the proof.

§I.3 Morphisms

Definition. A *variety over k* is any affine, quasi-affine, projective, or quasi-projective variety as defined above. If X, Y are two varieties, a *morphism* $\varphi : X \to Y$ is a *continuous* map such that for every open set $V \subseteq Y$, and for every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.

Exercise I.3.17 (Normal Varieties). (a)

(b)

(c) The coordinate ring $k[t^2, t^3]$ is not integrally closed in its fraction field $k(t)$, whence due to (d), the variety is not normal.

(d) This is immediate from the fact that being integrally closed is a local property, see [AM94, Chapter V].

(e) The construction of $\widetilde{Y}$ is obvious. Consider $A(Y) \subseteq K(Y)$ and let $\overline{A(Y)}$ denote the integral closure of the former in the latter. By **Theorem I.3.9A**, this is an affine k-domain, consequently, there is an affine variety $\widetilde{Y}$ such that $\overline{A(Y)} \cong A(\widetilde{Y})$ and the *integral* morphism $A(Y) \to A(\widetilde{Y})$ corresponds to a surjection $\widetilde{Y} \twoheadrightarrow Y$.

Due to **Theorem I.4.3**, we we may first assume that Z is affine and $\varphi : Z \to Y$ is dominant. Since $\varphi(Z)$ is dense in Y, the map $\varphi^* : K(Y) \to K(X)$ is well-defined and injective since it is a morphism of fields. The restriction of this map to $A(\widetilde{Y})$ must have image contained in $A(Z)$, since $A(\widetilde{Y})$ is integral over $A(Y)$ and $A(Z)$ is integrally closed in $K(Z)$. This gives a (unique) map $A(\widetilde{Y}) \to A(Z)$ extending $\varphi^* : A(Y) \to A(Z)$, where uniqueness follows from the fact that $A(\widetilde{Y}) \subseteq K(Y)$, the fraction field of $A(Y)$.

Once we have shown that a uique lift exists for each affine open in Z, it is obvious that these morphisms glue to a global morphism on all of Z, where to glue the morphisms on intersections, we make use of the uniqueness on affine opens; recall again that affine opens constitute a base for the topology on Z.

Exercise I.3.20. (a) Since every variety has a basis of open affine sets (**Theorem I.4.3**), we may assume that Y is affine. Since $A(Y)_{\mathfrak{m}_P}$ is an integrally closed domain in its fraction field $K(Y)$, we have

$$A(Y)_{\mathfrak{m}_P} = \bigcap_{\mathfrak{q} \text{ height 1 in } A(Y)_{\mathfrak{m}_P}} (A(Y)_{\mathfrak{m}_P})_{\mathfrak{q}} = \bigcap_{\substack{\operatorname{ht}\mathfrak{q}=1 \\ \mathfrak{q} \subseteq \mathfrak{m}_P}} A(Y)_{\mathfrak{q}}.$$

Now, f is a rational function and hence, is equal to $\frac{g}{h}$ where $g, h \in A(Y)$ and $h \neq 0$ on $Y \setminus P$. We contend that h is not in any height 1 prime $\mathfrak{q} \subseteq \mathfrak{m}_P$. For if it were, then we could choose a $Q \neq P$ with $\mathfrak{q} \subseteq \mathfrak{m}_Q$, since $\mathfrak{q} \neq \mathfrak{m}_P$, owing to $\operatorname{ht}\mathfrak{m}_P = 2$. It follows that h vanishes at Q, a contradiction. Hence, $h \notin \mathfrak{q}$ for all height 1 primes $\mathfrak{q} \subseteq \mathfrak{m}_P$. Consequently, in the above intersection, $\frac{g}{h} \in A(Y)_{\mathfrak{q}}$ for every such $\mathfrak{q}$, and thus $f \in A(Y)_{\mathfrak{m}_P}$, that is, f is regular at P, as desired.

(b) The rational function $\frac{1}{x}$ on $\mathbb{A}^1$ is regular on $\mathbb{A}^1 \setminus \{0\}$ but does not have a regular extension to $\mathbb{A}^1$. Another way to see this is that the inclusion $\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ correponds to the ring homomorphism $k[x] \to k[x, x^{-1}]$.

Chapter II

Schemes

§II.1 Sheaves

Definition. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of abelian groups on X. The association $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf on X. It is called the *sheaf Hom* and is denoted by $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

Remark. Let $\mathscr{F}$ be a presheaf and $\mathscr{G} \subseteq \mathscr{F}$ a sub-presheaf. For any $P \in X$, there is a natural map $\mathscr{G}_P \to \mathscr{F}_P$ sending an equivalence class $[\langle U, s\rangle] \in \mathscr{F}_P$ to the equivalence class $[\langle U, s\rangle] \in \mathscr{G}_P$, which is a homomorphism of groups. If $[\langle U, s\rangle]$ is in the kernel of this map, then there is a $V \subseteq U$ containing P such that $\operatorname{res}^U_V(s) = 0 \in \mathscr{F}(V)$, consequently, $\operatorname{res}^U_V(s) = 0 \in \mathscr{G}(V)$, since $\mathscr{G}$ is a sub-presheaf. It follows that the induced map is injective and we can identify $\mathscr{G}_P$ with a subgroup of $\mathscr{F}_P$. We shall tacitly make this identification throughout.

Exercise II.1.2. (a) Let $[\langle U, s\rangle] \in \ker \varphi_P$, that is, $[\langle U, \varphi_U(s)\rangle] = 0 \in \mathscr{G}_P$. Hence, there is a neighborhood $V \subseteq U$ of P such that $\operatorname{res}^U_V \varphi_U(s) = 0$, consequently, $\varphi_V(\operatorname{res}^U_V(s)) = 0 \in \mathscr{G}(V)$. It follows that $\operatorname{res}^U_V(s) \in (\ker \varphi)(U)$, and hence, $[\langle U, s\rangle] = [\langle V, \operatorname{res}^U_V(s)\rangle] \in (\ker \varphi)_P$. This shows that $\ker \varphi_P \subseteq (\ker \varphi)_P$.

Conversely, if $[\langle U, s\rangle] \in (\ker \varphi)_P$, then $s \in \ker \varphi_U$, and hence, $\varphi_P([\langle U, s\rangle]) = 0$, as desired. Hence, $\ker \varphi_P = (\ker \varphi)_P$.

Next, we show that $\operatorname{im} \varphi_P = (\operatorname{im} \varphi)_P$. There is a commutative diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \mathscr{G} \\ \operatorname{im}_{\text{pre}} \varphi \arrow[r, "\theta"'] \arrow[ur, hook] & \operatorname{im} \varphi \arrow[u] \end{tikzcd}$$

Note that sheafification θ induces an isomorphism of stalks, and hence, we have a commutative diagram

$$\begin{tikzcd} \mathscr{F}_P \arrow[r, "\varphi_P"] \arrow[d, "\varphi_P"'] & \mathscr{G}_P \\ (\operatorname{im}_{\text{pre}} \varphi)_P \arrow[r, "\theta_P"'] \arrow[ur, hook] & (\operatorname{im} \varphi)_P \arrow[u] \end{tikzcd}$$

Since θ_P is an isomorphism, $(\operatorname{im} \varphi)_P = (\operatorname{im}_{\text{pre}} \varphi)_P \subseteq \mathscr{G}_P$. But the above commutative diagram implies

$$\operatorname{im} \varphi_P \subseteq (\operatorname{im}_{\text{pre}} \varphi)_P \subseteq \operatorname{im} \varphi_P,$$

thereby completing the proof.

(b) We have
$$\ker\varphi = 0 \iff (\ker\varphi)_P = 0\ \forall P \in X \iff \ker\varphi_P = 0\ \forall P \in X.$$
Thus, φ is injective if and only if φ_P is injective for all $P \in X$.

Next, let $\mathscr{H} = \operatorname{im}\varphi \subseteq \mathscr{G}$. If φ is surjective, then $\mathscr{H} = \mathscr{G}$ and since $\operatorname{im}\varphi_P = (\operatorname{im}\varphi)_P = \mathscr{H}_P = \mathscr{G}_P$, we are done. On the other hand, if φ_P is surjective for all P, then $\mathscr{H}_P = \mathscr{G}_P$ for all P, that is, the inclusion map $\iota : \mathscr{H} \hookrightarrow \mathscr{G}$ is a stalk-local isomorphism and hence, an isomorphism of sheaves. It follows that $\mathscr{H} = \mathscr{G}$. To see this, note that there is a map $\sigma : \mathscr{G} \to \mathscr{H}$ that is an inverse to the inclusion. In particular, the composition $\iota \circ \sigma : \mathscr{G} \to \mathscr{G}$ is an isomorphism of sheaves. But the image of this map lies inside $\mathscr{G}$, whence $\mathscr{H} = \mathscr{G}$. This completes the proof.

(c) This is immediate from (a).

REMARK. In Exercise *II.1.2*, we are implicitly using the fact that the sheafification of an injective map of presheaves is injective. This is the content of Exercise *II.1.4*. Once this is established, we may simply treat $\operatorname{im}\varphi$ as a subsheaf of $\mathscr{G}$ and thus, $(\operatorname{im}\varphi)_P$ as a subgroup of $\mathscr{G}_P$ for each $P \in X$.

EXERCISE II.1.3. (a) Due to Exercise *II.1.2*, we know that φ is surjective if and only if φ_P is surjective for all $P \in X$. Suppose now that φ is surjective, $U \subseteq X$ is an open set and $s \in \mathscr{G}(U)$. We can find $[\langle V_P, t_P\rangle] \in \mathscr{F}_P$ such that
$$[\langle U, s\rangle] = \varphi_P\left([\langle V_P, t_P\rangle]\right) = [\langle V_P, \varphi_{V_P}(t_P)\rangle],$$
where $t_P \in \mathscr{F}(V_P)$. Hence, there is a neighborhood W_P of P contained in $U \cap V_P$ such that
$$\operatorname{res}^U_{W_P}(s) = \operatorname{res}^{V_P}_{W_P}\left(\varphi_{V_P}(t_P)\right) = \varphi_{W_P}\left(\operatorname{res}^{V_P}_{W_P}(t_P)\right).$$
Replace V_P by W_P and t_P by $\operatorname{res}^{V_P}_{W_P}(t_P)$ to obtain the desired conclusion.

Conversely, suppose the conclusion holds. We shall show that φ_P is surjective for all P. Let $[\langle U, s\rangle] \in \mathscr{G}_P$ for all $P \in X$. Then, there is an open cover $\{U_i\}$ of U and $t_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(t_i) = \operatorname{res}^U_{U_i}(s)$. Let U_j contain P, then $[\langle U_i, t_i\rangle]$ maps to $[\langle U, s\rangle]$, thereby establishing surjectivity.

(b) Let $X = \mathbb{C}$, $\mathscr{O}$ the sheaf of holomorphic functions, and $\mathscr{O}^*$ the sheaf of nowhere vanishing holomorphic functions. The exponential map $\mathscr{O} \to \mathscr{O}^*$ is surjective because it is surjective on each stalk, indeed, every non-vanishing holomorphic function admits a holomorphic logarithm locally.

The induced map $\mathscr{O}(U) \to \mathscr{O}^*(U)$ is obviously not surjective on $U = \mathbb{C} \setminus \{0\}$, since the function $z \mapsto z$ does not admit a holomorphic logarithm on U.

EXERCISE II.1.4. (a) To avoid circular reasoning, we must prove this without the aid of Exercise *II.1.2*. We describe the unique map $\mathscr{F}^+(U) \to \mathscr{G}^+(U)$. Let $s \in \mathscr{F}^+(U)$. For every $P \in U$, there is a neighborhood V of P contained in U and a $t \in \mathscr{F}(V)$ such that $s(Q) = t_Q \in \mathscr{F}_Q$ for all $Q \in V_P$. Send $s \mapsto \widetilde{s} \in \mathscr{G}^+(U)$, where $\widetilde{s}(P) = \varphi_Q(s(P)) \in \mathscr{G}_P$ for all $P \in X$. To see that this is indeed an element of $\mathscr{G}^+(U)$, consider some $P \in U$ and V_P as before; then, for all $Q \in V_P$, we have
$$\widetilde{s}(Q) = \varphi_Q(t_Q) = \varphi_Q\left([\langle V_P, t\rangle]\right) = [\langle V_P, \varphi_{V_P}(t)\rangle] \in \mathscr{G}_Q.$$
Finally, to show injectivity, we must show injectivity over every open set U. Indeed, suppose $s \in \mathscr{F}^+(U)$ maps to 0, that is, $\widetilde{s} = 0$, that is, $\varphi_P(s(P)) = 0$ for all $P \in X$. But since each φ_P is injective, we see that $s(P) = 0$ for all $P \in X$, that is, $s = 0$, as desired.

(b) That the image presheaf of an injective morphism of sheaves is a subsheaf is trivial.

EXERCISE II.1.5. Hartshorne proves that a morphism of sheaves is an isomorphism if and only if it is an isomorphism stalk locally. But a morphism of stalks is a morphism of abelian groups, and hence, is an isomorphism if and only if it is both injective and surjective. Finally, due to Exercise *II.1.2* (b), the stalk local maps are both injective and surjective if and only if the morphism of sheaves is as such. This completes the proof.

EXERCISE II.1.6.

EXERCISE II.1.16 (FLASQUE SHEAVES). (a) A subspace of an irreducible space is irreducible, and hence, is connected. It follows that $\mathscr{F}(U)$ is the set of constant functions for all open $U \subseteq X$. It follows that $\mathscr{F}$ is flasque.

(b) Let $U \subseteq X$ be an open set. We shall show that $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Since $\mathscr{F} \to \mathscr{F}''$ is surjective, by Exercise II.1.3(a), for each $s \in \mathscr{F}''(U)$, there is an open cover $\{U_i\}_{i\in A}$ of U and $t_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(t_i) = \mathrm{res}^U_{U_i}(s)$. Let $\mathscr{P}$ denote the set

$$\left\{(U_I, t_I)\colon I \subseteq A,\ U_I = \bigcup_{i\in I} U_i,\ t_I \in \mathscr{F}(U_I),\ \mathrm{res}^{U_I}_{U_i} = t_i\ \forall\, i \in I\right\}.$$

Endow this with the structure of a poset $(U_I, t_I) \leqq (U_J, t_J)$ if and only if $I \subseteq J$ and $\mathrm{res}^{U_J}_{U_I} t_J = t_I$. It is obvious that every chain in $\mathscr{P}$ has an upper bound, whence it follows that $\mathscr{P}$ has a maximal element, say (V, t). If $V = U$, then we are done. If not, then there is some index $i \in A$ such that $U_i \not\subset V$. Then,

$$\mathrm{res}^{U_i}_{U_i\cap U_I} t_i = \mathrm{res}^{U_I}_{U_i\cap U_I} t_I \implies \mathrm{res}^{U_i}_{U_i\cap U_I} t_i - \mathrm{res}^{U_I}_{U_i\cap U_I} t_I \in \mathscr{F}'(U_i \cap U_I).$$

Since $\mathscr{F}'$ is flasque, there is an $r \in \mathscr{F}'(U_i \cup U_I)$ such that

$$\mathrm{res}^{U_i\cup U_I}_{U_i\cap U_I} r = \mathrm{res}^{U_i}_{U_i\cap U_I} t_i - \mathrm{res}^{U_I}_{U_i\cap U_I} t_I.$$

Set $t^* = \mathrm{res}^{U_i\cup U_I}_{U_I} r + t_I$ and note that $\mathrm{res}^{U_I}_{U_i\cap U_I} t_I = \mathrm{res}^{U_i}_{U_i\cap U_I} t_i$, and hence, there is some $\widetilde{t} \in \mathscr{F}(U_i \cup U_I)$ that restricts to t_i on U_i and t_I on U_I, whence $(U_i \cup U_I, \widetilde{t}) \in \mathscr{P}$. This contradicts the maximality of (U_I, t_I), and thus $U_I = U$, thereby completing the proof.

(c) Let $V \subseteq U \subseteq X$ be open sets. Then there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0 \end{tikzcd}$$

with the first two vertical arrows as surjections. It follows from the Snake Lemma that the map $\mathscr{F}''(U) \to \mathscr{F}''(V)$ is surjective, that is, $\mathscr{F}''$ is flasque.

(d) If $V \subseteq U \subseteq X$ are open sets, then the restriction map $f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$ is the same as the restriction map $\mathscr{F}(f^{-1}V) \to \mathscr{F}(f^{-1}U)$, which is surjective, since $\mathscr{F}$ is flasque.

(e) This is trivial.

EXERCISE II.1.17 (SUPPORT). If $P \in U$ is such that $s_P = [\langle U, s\rangle] = 0$, then there is a neighborhood V of P contained in U such that $\mathrm{res}^U_V(s) = 0$. Hence, for all $Q \in V$, $s_Q = [\langle V, \mathrm{res}^U_V s\rangle] = 0$. This shows that the complement of $\mathrm{Supp}\, s$ is open, as desired.

§II.2 SCHEMES

EXERCISE II.2.3 (REDUCED SCHEMES).

(a) Suppose X is reduced. Then, every open affine corresponds to a reduced ring. Consequently, the local ring of any point on X is the localisation of a reduced ring and hence, is reduced.

Conversely, suppose $\mathscr{O}_{X,P}$ is reduced for every $P \in X$. Let $U = \operatorname{Spec} A$ be an affine open. The local ring of any point $P \in U$ is a localisation of A at a prime. Since all these rings are reduced, so is A.

Let $U \subseteq X$ be open. Cover U with affine opens $U_i = \operatorname{Spec} A_i$ and let $s \in \mathscr{O}(U)$ be nilpotent. Its image $s_i = \operatorname{res}_{U,U_i}(s)$ is nilpotent in $\mathscr{O}(U_i) = A_i$ and hence, $s_i = 0$. Consequently $s = 0$ due to the identity axiom. This shows that $\mathscr{O}(U)$ is reduced.

(b) The first part follows immediately from the fact that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{red} & \xrightarrow[\phi_{red}]{} & B_{red}. \end{array}$$

Consider the map of locally ringed spaces $(\mathbf{id}, f^\sharp)$, where $f^\sharp : \mathscr{O}_X \to \mathscr{O}_X^{red}$ is the collection of the canonical maps $\mathscr{O}_X(U) \to \mathscr{O}_X^{red}(U)$.

(c) Follows from the fact that any morphism of rings $\phi : A \to B$ with B reduced factors through the natural map $A \to A_{red}$.

EXERCISE II.2.4. Let $\varphi \in \operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X))$. Cover X with affine opens $U_i = \operatorname{Spec} A_i$. The restriction map gives us a homomorphism

$$A \xrightarrow{\varphi} \Gamma(X, \mathscr{O}_X) \xrightarrow{\operatorname{res}^X_{U_i}} \Gamma(U_i, \mathscr{O}_X) = A_i,$$

which induces a map on schemes $\pi_i : U_i \to \operatorname{Spec} A$ where $\pi_i = \operatorname{Spec}(\operatorname{res}^X_{U_i} \circ \varphi)$.

We contend that the maps π_i can be glued. Indeed, for $i \neq j$, cover $U_i \cap U_j$ with affine opens $U_{ijk} = \operatorname{Spec} A_{ijk}$. Now,

$$\pi_i|_{U_{ijk}} = \operatorname{Spec}(\operatorname{res}^{U_i}_{U_{ijk}}) \circ \pi_i = \operatorname{Spec}(\operatorname{res}^{U_i}_{U_{ijk}} \circ \operatorname{res}^X_{U_i} \circ \varphi) = \operatorname{Spec}(\operatorname{res}^X_{U_{ijk}} \circ \varphi).$$

Similarly, $\pi_j|_{U_{ijk}} = \operatorname{Spec}(\operatorname{res}^X_{U_{ijk}} \circ \varphi)$, consequently, the family of morphisms $\{\pi_i\}$ can be glued to a morphism $\pi : X \to \operatorname{Spec} A$. This gives a map

$$\beta : \operatorname{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X)) \to \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A).$$

It is straightforward to verify that α and β are inverses to one another.

EXERCISE II.2.5. Follows from the previous exercise and the fact that $\mathbb{Z}$ is an initial object in the category of rings.

EXERCISE II.2.7. Let $(f, f^\sharp) : \operatorname{Spec} K \to X$ is a morphism of schemes which sends the unique point in $\operatorname{Spec} K$ to $x \in X$. Then, there is an induced map on local rings $f^\sharp_x : \mathscr{O}_x \to K$, which must be local and hence, factor through the maximal ideal of $\mathscr{O}_x$, thereby inducing a map $k(x) \to K$. It is easy to see that this process is reversible.

EXERCISE II.2.9. Let $Z \subseteq X$ be irreducible and closed. Let $U = \operatorname{Spec} A$ be an open affine intersecting Z. Then, $Z \cap U$ is open in Z and hence, is irreducible. Further, it is closed in U and hence, corresponds to a prime ideal $\xi = \mathfrak{p} \in \operatorname{Spec} A$. Note that $\overline{\{\xi\}} \cap U = Z \cap U$ and $\overline{\{\xi\}} \subseteq Z$ since Z is closed.

Let V be any other open set intersecting Z. Then, one can replace V with an open affine Spec B intersecting Z. Suppose $\xi \notin V$. Then,

$$(Z \cap U) \cap (Z \cap V) = Z \cap U \cap V = \overline{\{\xi\}} \cap U \cap V = \varnothing,$$

since the closure of $\{\xi\}$ in U is contained in $U \setminus V$. This is not possible since $Z \cap U$ and $Z \cap V$ are nonempty open sets in an irreducible space. Hence, ξ is a generic point.

Now we argue for uniqueness. Suppose ξ_1 and ξ_2 were two generic points in Z. Consider an affine neighborhood $U = \operatorname{Spec} A$ intersecting Z. Then, $Z \cap U$ must contain ξ_1 and ξ_2. Let ξ_i correspond to a prime $\mathfrak{p}_i$ in A for $i = 1, 2$. Now, $Z \cap U = V(\mathfrak{p}_1) = V(\mathfrak{p}_2)$, consequently, $\mathfrak{p}_1 = \mathfrak{p}_2$, that is, $\xi_1 = \xi_2$. This completes the proof.

Definition. Let $(X, \mathscr{O}_X)$ be a scheme and let $f \in \Gamma(X, \mathscr{O}_X)$. Define X_f to be the set of all $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathscr{O}_{X,x}$. This is known as the *support* of f on X.

Exercise II.2.16.

(a) The set of all $x \in U$ such that $f_x \notin \mathfrak{m}_x$ is the set of all prime ideals $\mathfrak{p}$ in B such that $f/1$ is not in the maximal ideal $\mathfrak{p}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$. Equivalently, $f \notin \mathfrak{p}$. Thus, $X_f \cap U = D(\overline{f})$. Now, since X can be covered with open affines and the intersection of X_f with every open affine is open, X_f must also be open.

(b) Pick a finite open cover $\{U_i = \operatorname{Spec} A_i\}_{i=1}^m$. The restriction of a to $X_f \cap U_i = D(\operatorname{res}^X_{U_i}(f))$ is zero and hence, there is a positive integer n_i such that $\operatorname{res}^X_{U_i}(f^{n_i}a) = 0$. Let $N = \max_{1 \leqslant i \leqslant m} n_i$. Then, $\operatorname{res}^X_{U_i}(f^N a) = 0$. Due to the identity axiom, we must have $f^N a = 0$.

(c) Let $U_i = \operatorname{Spec} A_i$ and let $f_i = \operatorname{res}^X_{U_i}(f)$. Since $X_f \cap U_i = D(f_i)$, there is a $b_i \in A_i = \Gamma(U_i, \mathscr{O}_X)$ such that $\operatorname{res}^X_{U_i \cap X_f}(b) = \frac{b_i}{f_i^{n_i}}$ for some nonnegative integer n_i. Choosing n to be larger than all the n_i's, we get that there is a $b_i \in A_i$ such that $\operatorname{res}^X_{U_i \cap X_f}(f^n b) = \operatorname{res}^{U_i}_{U_i \cap X_f}(b_i)$.

Now consider $b_i - b_j$ on $U_i \cap U_j$, which can be covered by finitely many affine opens $U_{ijk} = \operatorname{Spec} A_{ijk}$. Since $\operatorname{res}^X_{U_i \cap U_j \cap X_f}(b_i - b_j) = 0$, using a similar argument as in (b), there is a positive integer m_{ij} such that $f^{m_{ij}}(b_i - b_j)$ restricts to 0 On $U_i \cap U_j$. Choosing m larger than m_{ij} for all pairs i, j, we have that $f^m(b_i - b_j)$ restricts to 0 on $U_i \cap U_j$. Consequently, $\operatorname{res}^{U_i}_{U_i \cap U_j}(f^m b_i) = \operatorname{res}^{U_j}_{U_i \cap U_j}(f^m b_j)$ and hence, there is a $c \in \Gamma(X, \mathscr{O}_X)$ such that $\operatorname{res}^X_{U_i}(c) = f^m b_i$. Hence, $\operatorname{res}^X_{U_i \cap X_f}(c) = \operatorname{res}^X_{U_i \cap X_f}(f^{n+m} b)$. This completes the proof.

(d) First, we show that $\operatorname{res}^X_{X_f}(f)$ is invertible. Since $f_x \notin \mathfrak{m}_x \subseteq \mathscr{O}_x$ for every $x \in X_f$, we see that the restriction of f to every affine open contained in X_f must be invertible (else it would lie in a prime ideal and hence, in the stalk of some point). Consider an open cover U_i of X_f using affine opens. There is a $g_i \in \Gamma(U_i, \mathscr{O})$ such that $g_i \operatorname{res}^X_{U_i}(f) = 1$. For $i \neq j$, we have

$$\operatorname{res}^{U_i}_{U_i \cap U_j}(g_i) \operatorname{res}^X_{U_i \cap U_j}(f) = 1 = \operatorname{res}^{U_j}_{U_i \cap U_j}(g_j) \operatorname{res}^X_{U_i \cap U_j}(f)$$

and hence, $\operatorname{res}^{U_i}_{U_i \cap U_j}(g_i) = \operatorname{res}^{U_j}_{U_i \cap U_j}(g_j)$ and hence, the g_i's can be lifted to some $g \in \Gamma(X_f, \mathscr{O}_X)$, furthermore $\operatorname{res}^X_{X_f}(f) g = 1$, whence invertibility follows.

Consider the map $\Phi : A_f \to \Gamma(X_f, \mathscr{O}_X)$ given by

$$\frac{a}{f^n} \mapsto \frac{\operatorname{res}^X_{X_f}(a)}{\operatorname{res}^X_{X_f}(f^n)}.$$

If $\Phi(a/f^n) = 0$, then $\mathrm{res}^X_{X_f}(a) = 0$, consequently, due to part (b), there is a positive integer m such that $f^m a = 0$, equivalently, $a/f^n = 0$ in A_f. Hence, Φ is injective.

As for surjectivity, let $b \in \Gamma(X_f, \mathscr{O}_X)$. Due to part (c), there is a positive integer m such that $f^m b = \mathrm{res}^X_{X_f}(a)$ for some $a \in A$ whence $\Phi(a/f^m) = b$. This completes the proof.

EXERCISE II.2.17 (A CRITERION FOR AFFINENESS).

(a) Each $f : f^{-1}U_i \to U_i$ has an inverse $g_i : U_i \to f^{-1}U_i$ that agrees on intersections since inverses are unique. These maps can be glued to give an inverse $g : Y \to X$ of f.

(b) First, note that $X = \bigcup_{i=1}^{n} X_{f_i}$, for if not, then there is an $x \in X$ such that $x \notin X_{f_i}$ for $1 \leqslant i \leqslant n$. Consider an affine open $U = \operatorname{Spec} B$ containing x and let $\mathfrak{p}$ be the prime corresponding to x. According to our hypothesis, $\mathrm{res}^X_U(f_i) \in \mathfrak{p}$ for $1 \leqslant i \leqslant n$. But these restrictions generate the unit ideal, a contradiction.

Being a finite union of affine opens, X is quasi-compact. Further, $X_{f_i} \cap X_{f_j}$ is a distinguished open in X_{f_i} and hence, is quasi-compact. As a result, Exercise *II.2.16* (d) is applicable. Using Exercise *II.2.4* and glueing morphisms just as in part (a), we are done.

DEFINITION. A morphism $f : X \to Y$ of schemes is said to be *dominant* if $f(X)$ is dense in Y.

EXERCISE II.2.18.

(a) Intersection of all prime ideals is the nilradical.

(b) We denote the morphism by $\pi : Y \to X$. If $\pi^\sharp$ is injective, then taking global sections, we obtain that φ is injective. Conversely, suppose φ is injective. It suffices to show that $\varphi^\sharp$ is injective on the $D(f)$'s since these form a base on X. We have

$$\pi^\sharp_{D(f)} : \mathscr{O}_X(D(f)) \to \mathscr{O}(\pi^{-1}(D(f)) \equiv \pi^\sharp_{D(f)} : A_f \to B_f,$$

which is injective. This proves the first part.

Next, we must show that π is dominant if φ is injective. Indeed, suppose $\pi(Y)$ were not dense, then there would be a basic open set $D(f)$ in $\operatorname{Spec} A$ such that $\pi^{-1}D(f) = \varnothing$, equivalently, $f \in \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of B. Hence, f is nilpotent in B, whence nilpotent in A, consequently, $D(f) = \varnothing$. This completes the proof.

(c) We denote the morphism by π. The first part follows from the fact that $\operatorname{Spec} A/\mathfrak{a} \hookrightarrow \operatorname{Spec} A$ is a topological imbedding. The second part is argued in a similar way as (b) by first concluding surjectivity on basic opens $D(f)$. Then, taking stalks, it follows that $\pi^\sharp$ is surjective.

(d)

§II.3 First Properties of Schemes

Lemma II.3.1 (Affine Communication Lemma).

Definition. A morphism $f : X \to Y$ of schemes is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}V_i$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

The morphism f is *of finite type* if in addition each $f^{-1}V_i$ can be covered by a finite number of the U_{ij}.

Definition. A morphism $f : X \to Y$ is a *finite* morphism if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}V_i$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a finite B_i-module.

Exercise II.3.1. Let $\pi : X \to Y$ denote the morphism. We use Lemma *II*.3.1. To this end, we first show that if $\operatorname{Spec} B \subseteq Y$ is an affine open such that $\pi^{-1} \operatorname{Spec} B$ can be covered by affine opens $U_i = \operatorname{Spec} A_i$, each of which is a finitely generated B-algebra, then the same is true for $\operatorname{Spec} B_f$, where $f \in B$. Now, $\pi^{-1} \operatorname{Spec} B_f \subseteq \pi^{-1} \operatorname{Spec} B$ and hence, is contained in $\bigcup U_i$. Consider $\pi^{-1} \operatorname{Spec} B_f \cap U_i$. This can be written as a union of $D(f_{ij})$'s where $f_{ij} \in A_i$. Note that $D(f_{ij}) = \operatorname{Spec}(A_i)_{f_{ij}}$, which is a finitely generated A_i algebra, whence a finitely generated B-algebra, consequently, a finitely generated B_f-algebra. This proves the first condition of Lemma *II*.3.1.

Next, suppose $(1) = (f_1, \ldots, f_n)$ in B and $\operatorname{Spec} B_{f_i}$ has the desired property. Then obviously B has the property, since B_{f_i} is a finitely generated B-algebra, and hence, any finitely generated B_{f_i}-algebra will be a finitely generated B-algebra.

Definition. A morphism $f : X \to Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}V_i$ is quasi-compact for each i.

Exercise II.3.2. Let $\pi : X \to Y$ denote the morphism. We use Lemma *II*.3.1. To this end, it suffices to show that if $\operatorname{Spec} A \subseteq Y$ is an affine open such that $\pi^{-1} \operatorname{Spec} A$ is quasi-compact, then for any $f \in A = \Gamma(\operatorname{Spec} A, \mathscr{O}_A)$, $\pi^{-1} \operatorname{Spec} A_f$ is quasi-compact. We wish to characterize

$$\{P \in \pi^{-1} \operatorname{Spec} A \colon f \notin \pi(p) = \mathfrak{p} \in \operatorname{Spec} A\}.$$

We have the map $\pi_P^\sharp : \mathscr{O}_{Y,\pi(P)} \to \mathscr{O}_{X,P}$. If $f \in \mathfrak{p} = \pi(P)$, then $f \in \mathfrak{m}_{Y,P}$ and hence, $\pi_P^\sharp f \in \mathfrak{m}_{X,P}$ (since $\pi_P^\sharp$ is a local homomorphism). On the other hand, if $f \notin \mathfrak{p}$, then $f/1 = 1/1$ in $\mathscr{O}_{Y,\pi(P)} = A_{\mathfrak{p}}$, consequently, $\pi_P^\sharp f = 1 \notin \mathfrak{m}_{X,P}$.

Thus, the set we are looking for is the *complement* of $(\pi^{-1} \operatorname{Spec} A)_{\pi^\sharp f}$, the latter being closed in the open subscheme $\pi^{-1} \operatorname{Spec} A$, due to Exercise *II*.2.16. Since $\pi^{-1} \operatorname{Spec} A$ is quasi-compact, we can cover it with open affines. Let $U = \operatorname{Spec} B$ be one such affine. Then, $\operatorname{res} \pi^\sharp f \in \mathscr{O}_B$ and the set of desired points $\mathfrak{p}$ are precisely those in $D(\operatorname{res} \pi^\sharp f)$, consequently, is quasi-compact. Being a finite union of quasi-compact sets, the required complement is quasi-compact.

Exercise II.3.3.

(a) $\Longrightarrow$ Obviously a morphism of finite type is locally of finite type. On the other hand, with the notation of the above above definitions, since $f^{-1}V_i$ can be covered by finitely many U_{ij}'s, it is a finite union of quasi-compact spaces, whence is quasi-compact. Thus, f is a quasi-compact morphism.

$\Longleftarrow$ On the other hand, suppose $f : X \to Y$ is locally of finite type and quasi-compact. Then, due to Exercise *II*.3.2, $f^{-1}V_i$ is quasi-compact, whence can be covered by finitely many of the U_{ij}'s. Thus, f is of finite type.

(b)

(c)

EXERCISE II.3.4. Let $\pi : X \to Y$ denote the morphism. We use Lemma II.3.1. Suppose $V = \operatorname{Spec} B$ can be covered by distinguished opens $V_i = \operatorname{Spec} B_{f_i}$ for $1 \leqslant i \leqslant n$ such that each V_i has the desired property. We shall show that V has the desired property. Let $U = \pi^{-1}V_i = \operatorname{Spec} A_i$ where A_i is a finite B_{f_i}-module. Let $A = \Gamma(U, \mathscr{O}_X)$. Then, the morphism π induces a homomorphism $\varphi : B \to A$ of rings making

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d, "\mathrm{res}^U_{U_i}"] \\ B_{g_i} \arrow[r] & A_i \end{tikzcd}$$

commute. Using the above diagram, it is not hard to argue that $U_{g_i} = A_i$, consequently, Exercise II.2.17 shows that U is affine and equal to $\operatorname{Spec} A$.

We have reduced the algebraic geometry problem to the following commutative algebra problem:

> Let $\varphi : B \to A$, let $f_1, \ldots, f_n$ generate the unit ideal in B and let $g_i = \varphi(f_i)$. Suppose A_{g_i} is a finite B_{f_i} module for $1 \leqslant i \leqslant n$. Then A is a finite B-module.

add in

DEFINITION. A morphism $\pi : X \to Y$ is *quasi-finite* if for every $y \in Y$, $\pi^{-1}(y)$ is a finite set.

EXERCISE II.3.5.

(a) This is essentially asking us to show that if B is an A-algebra that is a finite A-module, then for every $\mathfrak{p} \in \operatorname{Spec} A$, the fiber over $\mathfrak{p}$ in B is finite. Recall that the fiber over $\mathfrak{p}$ is precisely $\operatorname{Spec}(\kappa(\mathfrak{p}) \otimes_A B)$, which is the spectrum of a $\kappa(\mathfrak{p})$-algebra that is also a finite $\kappa(\mathfrak{p})$-module, i.e. the spectrum of an artinian ring, whence is finite.

(b) Follows from the commutative algebra fact that integral morphisms induce closed maps on the spectrum.

(c)

add

DEFINITION. A morphism $\pi : X \to Y$, with Y irreducible is *generically finite* if $\pi^{-1}(\eta)$ is a finite set, where η is the generic point of Y.

EXERCISE II.3.7. Let $\pi : X \to Y$ denote the morphism. Let ξ be the generic point of X and η the generic point of Y. First, we show that $\pi(\xi) = \eta$. Indeed,

$$\pi(X) = \pi(\overline{\{\xi\}}) \subseteq \overline{\{\pi(\xi)\}}.$$

But since π is dominant, $\pi(X)$ is dense in Y, consequently, $\pi(\xi)$ must be a generic point, hence, equal to η.

EXERCISE II.3.11 (CLOSED SUBSCHEMES).

(a)

(b) We may suppose, without loss of generality that $Y \subseteq X$. For a point $P \in Y$, choose an open affine neighborhood $U = \operatorname{Spec} C$ of P in Y. Then, there is an $f \in A$ such that $P \in D(f) \cap Y \subseteq U$. We contend that $D(f) \cap Y$ is a distinguished open in U. Indeed, the inclusion $\iota : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ restricted to U induces a map of rings $\varphi : A \to C$. It is easy to see that $\iota^{-1}(D(f)) = D_U(\varphi(f))$, consequently, $D(f) \cap Y$ is a distinguished open in U.

Next, cover X with $D(f_i)$'s such that $D(f_i) \cap Y$ is either affine in Y, or nonempty. Let $\overline{f}_i = \iota^\sharp_X(f_i) \in \Gamma(Y, \mathscr{O}_Y)$. We claim that $Y_{\overline{f}_i} = D(f_i) \cap Y$. Indeed, if $P \in D(f_i) \cap Y$, then there is a surjective map of stalks

$$\mathscr{O}_{X,P} \to \mathscr{O}_{Y,P}$$

sending f_i to $\overline{f}_i$. Since f_i is invertible in the former, it must be invertible in the latter. On the other hand, if $P \in Y_{\overline{f}_i}$, then $\overline{f}_i$ is invertible in the latter whence, cannot lie in the maximal ideal $\mathfrak{m}_{X,P}$, since the above map is a local homomorphism of local rings. This shows that $D(f_i) \cap Y = Y_{f_i}$.

Combining our above discussion with Exercise II.2.17 (b), we have that Y is affine. Next, we must show that Y is obtained as the quotient of an ideal in A. For this, invoke Exercise II.2.18 (d).

Exercise II.3.12.

Exercise II.3.13 (Properties of Morphisms of Finite Type).

Exercise II.3.14. It suffices to assume X is locally of finite type over k. In which case, there is a cover $U_i = \operatorname{Spec} A_i$ of X such that each A_i is a finitely generated k-algebra and hence, a Jacobson ring. Consequently, the closed points of U_i are dense in U_i, whence the closed points of X are dense in X.

As for a counterexample for arbitrary schemes, consider $\operatorname{Spec} A$ where A is a ring such that $\mathfrak{R} \neq \mathfrak{N}$.

§II.4 Separated and Proper Morphisms

Definition. A morphism $\pi : X \to Y$ of schemes is said to be *separated* if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion.

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathbf{id}_X"] \arrow[ddr, bend right, "\mathbf{id}_X"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\
& X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

Definition. A morphism $\pi : X \to Y$ is said to be *universally closed* if it is closed as a continuous map on the underlying topological spaces and for every morphism $Y' \to Y$, the map obtained by *base extension* $X \times_Y Y' \to Y'$ is also closed.

Definition. A morphism $\pi : X \to Y$ is said to be *proper* if it is separated, of finite type and universally closed.

Since a complete proof of the following is not provided in the text, I reproduce it here.

Corollary (Hartshorne, II.4.6). Assume that all schemes are noetherian in the following statements.

(a) Open and closed immersions are separated.

(b) A composition of two separated morphisms is seprated.

(c) Separated morphisms are stable under base extension.

(d) If $\pi : X \to Y$ and $\pi' : X' \to Y'$ are separated morphisms of schemes over a base scheme S, then the *product morphism* $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is also separated.

(e) If $\pi : X \to Y$ and $\varphi : Y \to Z$ are two morphisms and if $\varphi \circ \pi$ is separated, then π is separated.

(f) A morphism $\pi : X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $\pi^{-1}V_i \to V_i$ is separated for each i.

Proof. (a) We show more generally that "a monomorphism of schemes is separated". Let $Y \hookrightarrow X$ be a monomorphism in $\mathfrak{Sch}_{\mathbb{Z}}$. Then, the fiber product $Y \times_X Y$ is precisely Y, given by the following diagram.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y \arrow[r, "\mathbf{id}"] \arrow[d, "\mathbf{id}"'] & Y \arrow[d, hook] \\
& Y \arrow[r, hook] & X
\end{tikzcd}$$

Since $Y \hookrightarrow X$ is a monomorphism, the two maps $Z \to Y$ in the above diagram must be the same and it follows that $Y = Y \times_X Y$. Hence, the diagonal morphism $\Delta : Y \to Y \times_X Y$ is the identity map, whence is a closed immersion.

(b) We use the valuative criterion. Let R be a DVR and K its fraction field. Let $U = \operatorname{Spec} K$ and $T = \operatorname{Spec} R$

and suppose $\pi : X \to Y$ and $\varphi : Y \to Z$ are separated. Let there be a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dd] & X \arrow[d] \\ & Y \arrow[d] \\ T \arrow[r] & Z. \end{tikzcd}$$

Suppose there are two lifts $\psi_1, \psi_2 : T \to X$ making the diagram commute. Then, $\pi \circ \psi_1 = \pi \circ \psi_2$ since $Y \to Z$ is separated. Finally, since $X \to Y$ is separated, we must have $\psi_1 = \psi_2$. This shows that $X \to Z$ is separated.

(c) This is done in the book.

(d) The same idea as in (b) works. Not writing this up because the diagram is too complicated to draw and I'm too lazy.

(e) Again, begin with a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

and suppose there are two lifts $\psi_1, \psi_2 : T \to X$ making the diagram commute. Since $X \to Z$ is separated, we must have that $\psi_1 = \psi_2$. Hence, $X \to Y$ is separated.

(f)

■

§II.5 Sheaves of Modules

Definition. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$. It is said to be *locally free* if X has an open cover by sets U for which $\mathscr{F}|_U$ is a free $\mathscr{O}_X|_U$-module.

Exercise II.5.7.

(a) We reduce this to the affine case since $\mathscr{F}$ is coherent on a noetherian scheme. Thus, we have a finitely generated A-module M and a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module.

Choose a basis $\{\frac{m_1}{1}, \dots, \frac{m_n}{1}\}$ of $M_\mathfrak{p}$ over $A_\mathfrak{p}$ and consider the exact sequence

$$0 \to K \to A^n \to M \to Q \to 0,$$

where the map $A^n \to M$ is the natural map sending $e_i \mapsto m_i$ for $1 \leqslant i \leqslant n$. Localising, we see that $K_\mathfrak{p} = Q_\mathfrak{p} = 0$ and hence, there is an $f \in A \setminus \mathfrak{p}$ such that $K_f = Q_f = 0$ (since both K and Q are finitely generated). Localising the above exact sequence at f, we obtain an isomorphism $A_f^n \xrightarrow{\sim} M_f$. It follows that $\mathscr{F}|_{D(f)}$ is a free sheaf.

(b) Follows immediately from (a).

Exercise II.5.8.

(a)

(b) This is a topological property of connected spaces and has nothing to do with algebraic geometry.

(c) We shall use Exercise II.5.7 (b) to show that $\mathscr{F}$ is locally free. To this end, we need to show that $\mathscr{F}_x$ is a free $\mathscr{O}_{X,x}$-module for each $x \in X$. Let $U = \operatorname{Spec} A$ be an open affine neighborhood of x in X on which φ is constant. Let $\mathfrak{p} \in \operatorname{Spec} A$ be the prime corresponding to the point $x \in U$. Thus, we have a finite A-module M such that $\mathscr{F}|_U = \widetilde{M}$. Using Nakayama's lemma, we can find a minimal generating set $m_1, \dots, m_r \in M_\mathfrak{p}$, where $r = \varphi(x)$, which gives a surjection $A_\mathfrak{p}^r \twoheadrightarrow M_\mathfrak{p}$. This can be localized at each prime $\mathfrak{q} \subseteq \mathfrak{p}$, and hence $m_1, \dots, m_r \in M_\mathfrak{q}$ generate it as an $A_\mathfrak{q}$-module. But since $\varphi(\mathfrak{q}) = r$, it follows that $m_1, \dots, m_r \in M_\mathfrak{q}$ is a minimal generating set for each prime $\mathfrak{q} \subseteq \mathfrak{p}$.

Finally, we claim that $m_1, \dots, m_r$ freely generate $M_\mathfrak{p}$. Indeed, suppose $a_1 m_1 + \cdots + a_r m_r = 0$ for $a_i \in A_\mathfrak{p}$. This equality is true for $M_\mathfrak{q}$ as an $A_\mathfrak{q}$-module and hence, all the coefficients lie in $\mathfrak{q}A_\mathfrak{q}$, therefore, all the coefficients lie in $\mathfrak{q}A_\mathfrak{p}$ for all primes $\mathfrak{q} \subseteq \mathfrak{p}$. But since $A_\mathfrak{p}$ is reduced, it follows that $a_i = 0$ for all $1 \leqslant i \leqslant r$ in $A_\mathfrak{p}$. Hence $M_\mathfrak{p} = \mathscr{F}_x$ is a free $A_\mathfrak{p} = \mathscr{O}_{X,x}$-module, as desired.

Bibliography

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