Spectral Sequences

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> *"It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up."*
>
> Ravi Vakil

Throughout this article, we shall work with cohomological spectral sequences. The analogous statements for homological statements can be obtained by either negating the indices or simply reversing the arrows.

Definition 1. A *differential bigraded module* over a ring R is a collection of R-modules, $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$, together with an R-linear map $d\colon E^{\bullet,\bullet} \to E^{\bullet,\bullet}$, called the *differential* of bidegree $(s, -s+1)$ for some $s \in \mathbb{Z}$, and satisfying $d \circ d = 0$.

We can take the *homology* of a differential bigraded module as

$$H^{p,q}\left(E^{\bullet,\bullet}, d\right) = \frac{\ker\left(d\colon E^{pq} \to E^{p+s,q-s+1}\right)}{\operatorname{im}\left(d\colon E^{p-s,q+s-1} \to ^{p,q}\right)}.$$

Definition 2. A (cohomological) *spectral sequence* is a collection of differential bigraded R-modules $\{E_r^{\bullet,\bullet}, d_r\}_{r\geqslant 1}$ where the differential d_r has degree $(r, -r+1)$ for all $r \geqslant 1$ such that $E_{r+1}^{p,q} \cong H^{p,q}\left(E_r^{\bullet,\bullet}, d_r\right)$.

The differential bigraded module $(E_r^{\bullet,\bullet}, d_r)$ is called the *r-th page* of the spectral sequence. Note that the knowledge of the r-th page of a spectral sequene enables one to determine $E_{r+1}^{p,q}$ for all $p, q \in \mathbb{Z}$ but not the differential d_{r+1}.

We shall now describe the construction of what is known as the *limiting page* $E_\infty^{\bullet,\bullet}$ of the spectral sequence. Henceforth, a submodule of a bigraded module $\{E^{\bullet,\bullet}\}$ is a collection $\widetilde{E}^{\bullet,\bullet}$ such that $\widetilde{E}^{p,q}$ is a submodule of $E^{p,q}$ for all $p, q \in \mathbb{Z}$. In order to construct the limiting page, we shall define a sequence of submodules of $E_2^{\bullet,\bullet}$:

$$B_2^{\bullet,\bullet} \subseteq B_3^{\bullet,\bullet} \subseteq \cdots B_n^{\bullet,\bullet} \subseteq \cdots \subseteq Z_n^{\bullet,\bullet} \subseteq \cdots \subseteq Z_3^{\bullet,\bullet} \subseteq Z_2^{\bullet,\bullet}$$

and set

$$B_\infty^{\bullet,\bullet} = \bigcup_{n\geqslant 2} B_n^{\bullet,\bullet} \quad \text{and} \quad Z_\infty^{\bullet,\bullet} = \bigcap_{n\geqslant 2} Z_n^{\bullet,\bullet}.$$

Begin by setting $Z_2 = \ker d_2$ and $B_2 = \operatorname{im} d_2$. Note that we can identify E_3 with Z_2/B_2, and under this identification, $\ker d_3 = Z_3/B_2$ and $\operatorname{im} d_3 = B_3/B_2$ for some submodules Z_3 and B_3 of E_2. Continuing this way, we obtain our desired chain of inclusions. Finally, define

$$E_\infty^{\bullet,\bullet} = \frac{Z_\infty^{\bullet,\bullet}}{B_\infty^{\bullet,\bullet}}. \tag{†}$$

DEFINITION 3. Let $H^\bullet$ be a graded R-module. A (decreasing) filtration $F^\bullet$ on $H^\bullet$ is a sequence of graded submodules $\{F^pH^\bullet\}_{p\in\mathbb{Z}}$ such that $F^{p+1}H^\bullet \subseteq F^pH^\bullet$ for all $p \in \mathbb{Z}$. There is an associated bigraded module $E_0^{\bullet,\bullet}(H^\bullet, F)$ given by

$$E_0^{p,q}(H^\bullet, F) = \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}} \qquad \forall p,q \in \mathbb{Z}.$$

DEFINITION 4. A spectral sequence $\{E_r^{\bullet,\bullet}, d_r\}_{r\geqslant 1}$ is said to *converge* to a graded R-module $H^\bullet$ if there is a (decreasing) filtration $F^\bullet$ of $H^\bullet$ such that

$$E_\infty^{p,q} \cong E_0^{p,q}(H^\bullet, F),$$

where $E_\infty^{\bullet,\bullet}$ is the limiting sheet of the spectral sequence as constructed in (†).

DEFINITION 5. An R-module A is said to be a *filtered differential graded module* if

- A is an (internal) direct sum of submodules $A = \bigoplus_{n\in\mathbb{Z}} A^n$,
- there is an R-linear map $d\colon A \to A$ of degree 1 satisfying $d \circ d = 0$, and
- A has a (decreasing) filtration $F^\bullet$ and the differential d respects the filtration, that is, $d(F^pA) \subseteq F^pA$.

The above datum is equivalent to being given a cochain complex $A^\bullet$ with a (decreasing) filtration in the sense of subcomplexes, that is, $F^pA^\bullet$ is a subcomplex of $A^\bullet$ and $F^{p+1}A^\bullet \subseteq F^pA^\bullet$. We shall identify these two notions of a filtered differential graded module henceforth.

The graded homology object $H(A,d) = \bigoplus_{n\in\mathbb{Z}} H^n(A,d)$ inherits an *induced filtration*, that is,

$$F^pH(A,d) = \operatorname{im}\left(H(F^pA,d) \to H(A,d)\right).$$

THEOREM 6. Each filtered differential graded module $(A, d, F^\bullet)$ determines a spectral sequence $\{E_r^{\bullet,\bullet}, d_r\}_{r\geqslant 1}$ with d_r of bidegree $(r, -r+1)$ and

$$E_1^{p,q} \cong H^{p+q}\left(\frac{F^pA}{F^{p+1}A}\right).$$

Suppse further that the filtration is *bounded*, that is, for each dimension n, there are values $s = s(n)$ and $t = t(n)$ so that

$$0 = F^sA^n \subseteq F^{s-1}A^n \subseteq \cdots \subseteq F^{t+1}A^n \subseteq F^tA^n = A^n,$$

then the spectral sequence converges to $H(A,d)$ with the induced filtration, that is,

$$E_\infty^{p,q} \cong \frac{F^pH^{p+q}(A,d)}{F^{p+1}H^{p+q}(A,d)}.$$

Proof. Define the following objects:

$$\begin{aligned}
Z_r^{p,q} &= F^pA^{p+q} \cap d^{-1}\left(F^{p+r}A^{p+q+1}\right)\\
B_r^{p,q} &= F^pA^{p+q} \cap d\left(F^{p-r}A^{p+q-1}\right)\\
Z_\infty^{p,q} &= \ker d \cap F^pA^{p+q}\\
B_\infty^{p,q} &= \operatorname{im} d \cap F^pA^{p+q}.
\end{aligned}$$

The elements of $Z_r^{p,q}$ are precisely the elements of F^pA^{p+q} that have boundaries in $F^{p+r}A^{p+q+1}$, and similarly, the elements of $B_r^{p,q}$ are precisely the elements of F^pA^{p+q} that are the boundaries of elements in $F^{p-r}A^{p+q-1}$. Clearly, we have

$$B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_1^{p,q} \subset Z_0^{p,q}.$$

Furthermore,

$$d\left(Z_r^{p-r,q+r-1}\right) = d\left(F^{p-r}A^{p+q-1} \cap d^{-1}\left(F^pA^{p+q}\right)\right) \subseteq F^pA^{p+q} \cap d\left(F^{p-r}A^{p+q-1}\right) = B_r^{p,q}.$$

Note that for $r > s(p+q+1)-p$, we have $Z_r^{p,q} = Z_\infty^{p,q}$ and for $r > p-t(p+q-1)$, we have $B_r^{p,q} = B_\infty^{p,q}$. Hence,

$$Z_\infty^{p,q} = \bigcap_{r\geqslant 0} Z_r^{p,q} \quad \text{and} \quad B_\infty^{p,q} = \bigcup_{r\geqslant 0} B_r^{p,q} \qquad \text{for all } p,q\in\mathbb{Z}.$$

For $0 \leqslant r \leqslant \infty$, define

$$E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}},$$

and let $\eta_r^{p,q}: Z_r^{p,q} \to E_r^{p,q}$ be the canonical surjection, where $\ker\eta_r^{p,q} = Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}$. First, note that

$$d\left(Z_r^{p-r,q+r-1}\right) = d\left(F^{p-r}A^{p+q-1} \cap d^{-1}\left(F^pA^{p+q}\right)\right) \subseteq F^pA^{p+q} \cap d\left(F^{p-r}A^{p+q-1}\right) = B_r^{p,q},$$

so that

$$d\left(Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}\right) = d\left(Z_{r-1}^{p+1,q-1}\right) + d\left(B_{r-1}^{p,q}\right) \subseteq B_{r-1}^{p+r,q-r+1} \subseteq \ker\eta_r^{p+r,q-r+1}.$$

This induces a map $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ for all $p,q\in\mathbb{Z}$ and $r\geqslant 0$:

$$\begin{tikzcd}
Z_r^{p,q} \arrow[r, "d"] \arrow[d, "\eta_r^{p,q}"'] & Z_r^{p+r,q-r+1} \arrow[d, "\eta_r^{p+r,q-r+1}"] \\
E_r^{p,q} \arrow[r, "d_r"'] & E_r^{p+r,q-r+1}.
\end{tikzcd}$$

Moreover, since $d\circ d = 0$, it follows that $d_r\circ d_r = 0$. In order to complete the proof, there are three things that we must establish:

(I) $H^{p,q}(E_r^{\bullet,\bullet}) = E_{r+1}^{p,q}$,

(II) $E_1^{p,q} \cong H^{p+q}\left(F^pA/F^{p+1}A\right)$, and

(III) $E_\infty^{p,q} \cong F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d)$.

$$\begin{tikzcd}
Z_r^{p+1,q-1}+B_r^{p,q} \arrow[r, hook] & Z_{r+1}^{p,q} \arrow[r, hook] \arrow[d] & Z_r^{p,q} \arrow[r] \arrow[d, "\eta_r^{p,q}"] & Z_r^{p+r,q-r+1} \arrow[d, "\eta_r^{p+r,q-r+1}"] \\
& \ker d_r \arrow[r] \arrow[d] & E_r^{p,q} \arrow[r, "d_r"'] & E_r^{p+r,q-r+1} \\
& H^{p,q}(E_r^{\bullet,\bullet},d_r) \arrow[d] & & \\
& 0 & &
\end{tikzcd}$$

These computations, although necessary, are not particularly enlightening. We take it on faith that a tenacious (under-)graduate student can verify them as and when required. ∎

Exact Couples

DEFINITION 7. A tuple $\langle D, E, i, j, k\rangle$ is said to be an *exact couple* if the diagram

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is exact at each object, that is, $\operatorname{im} i = \ker j$, $\operatorname{im} j = \ker k$, and $\operatorname{im} k = \ker i$.

Given an exact couple $\langle D, E, i, j, k\rangle$, define $d = j \circ k \colon E \to E$. Note that $d \circ d = 0$, and set

$$D' = \operatorname{im} i \subseteq D \quad \text{and} \quad E' = \frac{\ker d}{\operatorname{im} d}.$$

Note that E' is a subquotient of E. Define $i' \colon D' \to D'$ to be the restriction of i to D'. Next, set $j' \colon D' \to E'$ as

$$j'(i(x)) = j(x) \bmod dE \in E'.$$

We must verify that j' is well-defined. Indeed, if $i(x) = i(y)$, then $x - y \in \ker i = \operatorname{im} k$, i.e., there is a $z \in E$ such that $x - y = k(z)$, consequently, $j(x - y) = d(z) \in dE$, whence $j'(i(x)) = j'(i(y))$, as desired. Finally, define $k' \colon E' \to D'$ by

$$k'(e \bmod dE) = k(e).$$

To see that this is well-defined, suppose $e - f \in dE$, then $e - f = d(z)$ for some $z \in E$. Then $k(d(z)) = k(j(k(z))) = 0$, whence $k(e) = k(f)$, as desired.

THEOREM 8. The tuple $\langle D', E', i', j', k'\rangle$ is an exact couple. This is called the *derived couple* of $\langle D, E, i, j, k\rangle$.

Proof. Omitted. ∎

This process can be iterated, begin by setting

$$\langle D^1, E^1, i^{(1)}, j^{(1)}, k^{(1)}\rangle = \langle D, E, i, j, k\rangle,$$

and set

$$\langle D^{r+1}, E^{r+1}, i^{(r+1)}, j^{(r+1)}, k^{(r+1)}\rangle = \langle D^r, E^r, i^{(r)}, j^{(r)}, k^{(r)}\rangle'$$

for all $r \geqslant 1$.

The next theorem describes a procedure to obtain a spectral sequence from an exact couple of bigraded modules over a ring.

THEOREM 9. Suppose $D^{\bullet,\bullet} = \{D^{p,q}\}$ and $E^{\bullet,\bullet} = \{E^{p,q}\}$ are bigraded modules over R equipped with homomorphisms

$$\begin{tikzcd} D^{\bullet,\bullet} \arrow[rr, "i"] & & D^{\bullet,\bullet} \arrow[dl, "j"] \\ & E^{\bullet,\bullet} \arrow[ul, "k"] & \end{tikzcd}$$

where $\deg i = (-1, 1)$, $\deg j = (0, 0)$, and $\deg k = (1, 0)$. This data determines a cohomological spectral sequence with $E_r = (E^{\bullet,\bullet})^r$ and $d_r = j^{(r)} \circ k^{(r)}$.

Proof. Only the degree of the map d_r must be verified, which is straightforward. ∎

Double Complexes

DEFINITION 10. A cohomological *double complex* is a triple $(\{M^{p,q}\}_{p,q\in\mathbb{Z}}, d^v, d^h)$, where $\{M^{p,q}\}_{p,q\in\mathbb{Z}}$ is a bigraded collection of R-modules, and for all $p,q\in\mathbb{Z}$, and there are maps

$$d^h : M^{p,q} \to M^{p+1,q} \quad \text{and} \quad d^v : M^{p,q} \to M^{p,q+1}$$

$$\begin{tikzcd} M^{p,q+1} \arrow[r, "d^h"] & M^{p+1,q+1} \\ M^{p,q} \arrow[u, "d^v"] \arrow[r, "d^h"'] & M^{p+1,q} \arrow[u, "d^v"'] \end{tikzcd}$$

such that

$$d^h \circ d^h = 0, \quad d^v \circ d^v = 0, \quad \text{and} \quad d^h \circ d^v + d^v \circ d^h = 0.$$

Every cohomological double complex has its associated *totalization*, which is a cochain complex $T^\bullet = \mathrm{Tot}^{\oplus}(M^{\bullet,\bullet}, d^h, d^v)$ with

$$T^n = \bigoplus_{i+j=n} T^{i,j}$$

and maps $d^h + d^v : T^n \to T^{n+1}$.

Associated to the category of double complexes are two natural (co)homology functors $H_I^{\bullet,\bullet}$ and $H_{II}^{\bullet,\bullet}$ given by

$$H_I^{p,q}(M) = \frac{\ker d^h : M^{p,q} \to M^{p+1,q}}{\operatorname{im} d^h : M^{p-1,q} \to M^{p,q}},$$

and

$$H_{II}^{p,q}(M) = \frac{\ker d^v : M^{p,q} \to M^{p,q+1}}{\operatorname{im} d^v : M^{p,q-1} \to M^{p,q}}.$$

Note that $H_I^{\bullet,\bullet}(M)$ is a double complex with the vertical maps induced by d^v (up to a sign), and all horizontal maps are zero. Similarly $H_{II}^{\bullet,\bullet}(M)$ is another double complex with zeros as vertical maps and the horizontal maps being induced by d^h. It is therefore possible to take homology once again, that is,

$$H_{II}^{\bullet,\bullet}\left(H_I^{\bullet,\bullet}(M)\right) \quad \text{and} \quad H_I^{\bullet,\bullet}\left(H_{II}^{\bullet,\bullet}(M)\right).$$

Next, we can associate two natural (decreasing) filtrations on the totalization of a double complex, namely:

$$F_{\mathrm{I}}^p T^n = \bigoplus_{\substack{i+j=n \\ i\geqslant p}} M^{i,j} \quad \text{and } F_{\mathrm{II}}^q T^n = \bigoplus_{\substack{i+j=n \\ j\geqslant q}} M^{i,j}.$$

In light of Theorem 6, each of $F_{\mathrm{I}}^\bullet$ and $F_{\mathrm{II}}^\bullet$ has an associated spectral sequence which we denote by ${}_{\mathrm{I}}E_\bullet^{\bullet,\bullet}$ and ${}_{\mathrm{II}}E_\bullet^{\bullet,\bullet}$ respectively.

THEOREM 11. With the above notation, we have

$${}_{\mathrm{I}}E_2^{\bullet,\bullet} \cong H_I^{\bullet,\bullet}(H_{II}^{\bullet,\bullet}(M)) \quad \text{and} \quad {}_{\mathrm{II}}E_2^{\bullet,\bullet} \cong H_{II}^{\bullet,\bullet}\left(H_I^{\bullet,\bullet}(M)\right).$$

Proof. Omitted. ∎

LEMMA 12. Let M be a finite projective module over a commutative ring A. Then for any module N, the natural map

$$\operatorname{Hom}_A(M,A)\otimes_A N \to \operatorname{Hom}_A(M,N) \qquad \varphi\otimes x \longmapsto \big(z\mapsto \varphi(z)\cdot x\big).$$

is an isomorphism.

Proof. The map is clearly an isomorphism when M is a finite free module. Since projective modules are direct summands of free modules, it is straightforward to conclude. ∎

THEOREM 13 (ISCHEBECK). Let A be a Noetherian ring and M a finite A-module admitting a finite resolution by finite projective A-modules. Then for any A-module N, there is a spectral sequence

$$E_2^{p,q} = \operatorname{Tor}_{-q}\big(\operatorname{Ext}^p(M,A),N\big) \Longrightarrow \operatorname{Ext}^\bullet(M,N).$$

Proof. Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of M and N respectively with $P_\bullet$ consisting of only finitely many non-zero elements and all modules in $P_\bullet$ are finitely generated. Consider the double complex $C^{p,q} = \operatorname{Hom}(P_p,Q_{-q}) \cong \operatorname{Hom}(P_p,A)\otimes_A Q_{-q}$. There are two spectral sequences associated with this double complex in the sense of Theorem 11, denote them by ${}_{\mathrm{I}}E_\bullet^{\bullet,\bullet}$ and ${}_{\mathrm{II}}E_\bullet^{\bullet,\bullet}$ following the statement of the theorem.

Note that $H_{II}^{\bullet,\bullet}(M)$ is the double complex with $H_{II}^{p,q}(M) = \operatorname{Tor}_{-q}\big(\operatorname{Hom}_A(P_p,A),N\big)$. But since $\operatorname{Hom}_A(P_p,A)$ is a finite projective A-module, we note that

$$H_{II}^{p,q}(M) = \begin{cases}\operatorname{Hom}(P_p,A)\otimes_A N & q=0\\ 0 & \text{otherwise.}\end{cases}$$

Taking $H_I^{\bullet,\bullet}$ of this, we obtain

$${}_{\mathrm{I}}E_2^{p,q} = \begin{cases}\operatorname{Ext}^p(M,N) & q=0\\ 0 & \text{otherwise.}\end{cases}$$

So this spectral sequence degenerates at the second page, and hence this page is also the limiting page. Since $P_\bullet$ is a finite resolution, in view of Theorem 6, this spectral sequence converges to the (associated graded of) the cohomology object $H^\bullet$ of $\operatorname{Tot}^\oplus(M)$. By definition, we have

$${}_{\mathrm{I}}E_\infty^{p,q} = \frac{F_{\mathrm{I}}^p H^{p+q}}{F_{\mathrm{I}}^{p+1}H^{p+q}}.$$

Fix $p+q=n$ and note that on the "antidiagonal" $p+q=n$ on the limiting sheet, there is exactly one non-zero term, namely ${}_{\mathrm{I}}E_\infty^{p,0}$, and hence, it follows that $H^n \cong \operatorname{Ext}^n(M,N)$.

A similarly straightforward computation shows that

$${}_{\mathrm{II}}E_2^{p,q} \cong \operatorname{Tor}_{-q}(\operatorname{Ext}^p(M,A),N),$$

thereby completing the proof. ∎

THEOREM 14 (ISCHEBECK). Let A be a Noetherian ring, M a finite A-module, and N any A-module. Assume that M admits a finite projective resolution, or that N admits a finite injective resolution. Then there is a spectral sequence

$$E_{p,q}^2 = \operatorname{Ext}^{-q}\big(\operatorname{Ext}^p(M,A),N\big) \Longrightarrow \operatorname{Tor}_\bullet(M,N).$$

Proof. If you can read German, see [Isc69], and if you understand it, please explain it to me. ∎

Foxby's Theorem

The goal of this section will be to prove the following theorem of Foxby:

Theorem 15 (Foxby). Let $(A,\mathfrak{m},k)$ be a Noetherian local ring. If A admits a non-zero finite module M with finite injective and projective dimension, then A is Gorenstein.

For a finite A-module M and $s\in\mathbb{Z}$, set

$$i_M(s)=\dim_k \operatorname{Ext}^s(k,M) \quad \text{and } p_M(s)=\dim_k \operatorname{Tor}_s(k,M).$$

These have associated power series:

$$I_M(X)=\sum_{s\in\mathbb{Z}} i_M(s)X^s \quad \text{and} \quad P_M(X)=\sum_{s\in\mathbb{Z}} p_M(s)X^s.$$

Theorem 16. Let M be a finite module over a Noetherian local ring $(A,\mathfrak{m},k)$. Then

(1) If $\operatorname{proj\,dim}_A M<\infty$, then $I_M(X)=I_A(X)P_M(X^{-1})$.

(2) If $\operatorname{inj\,dim}_A M<\infty$, then $P_M(X)=I_A(X)I_M(X^{-1})$.

Proof. We shall prove (1). Consider take a minimal free resolutions of k and M, and consider the Ischebeck spectral sequence as in Theorem 13. Note that on the page ${}_{\mathrm{II}}E_2^{\bullet,\bullet}$, the maps d_2 are in some sense restrictions of the maps in the totalization of the double complex considered. Since we began with a minimal free resolutions of k and M, the horizontal and vertical differentials are given by matrices with entries in the maximal ideal $\mathfrak{m}$, and recalling the construction of d_2 from the proof Theorem 6 and using the fact that every element on the considered E_2-page is a k-vector space, it follows that all the maps there are identically zero. This means that the spectral sequence degenerates at the second page, whence this page is isomorphic to the limiting page ${}_{\mathrm{II}}E_\infty^{\bullet,\bullet}$. We can now read off the dimension of $\operatorname{Ext}^n(k,M)$ by summing over the diagonal:

$$\dim_k \operatorname{Ext}^n(k,M)=\sum_{p+q=n}\dim_k \operatorname{Tor}_{-q}\left(\operatorname{Ext}^p(k,A),M\right)=\sum_{p+q=n}\dim_k \operatorname{Ext}^p(k,A)\dim_k \operatorname{Tor}_{-q}(k,M),$$

which is precisely the content of (1). An analogous proof works for (2) using Theorem 14. ∎

Proof of Foxby's Theorem. Let $d=\dim A$. Then due to the Auslander-Buchsbaum formula, we have that $\operatorname{proj\,dim}_A M\le d$ and due to Ischebeck's formula (the one about injective dimension) $\operatorname{inj\,dim}_A M=d$, that is, $\deg I_M=d$. We can expand the equality $I_M(X)=I_A(X)P_M(X^{-1})$ as

$$p_M(n)=i_A(n)i_M(0)+\cdots+i_A(n+d)i_M(d).$$

The left hand side is zero for $n>\operatorname{proj\,dim}_A M$ and hence all the terms on the right hand side are identically zero for $n>\operatorname{proj\,dim}_A M$. In particular, $i_A(n+d)i_M(d)=0$ for all $n>\operatorname{proj\,dim}_A M$. Thus $i_A(n+d)=0$ for all $n>\operatorname{proj\,dim}_A M$ since $i_M(d)\neq 0$ (recall that this was an easy consequence of Bass' Lemma). In particular, this means that I_A is a polynomial, so that $\operatorname{inj\,dim}_A A<\infty$, equivalently, A is Gorenstein. ∎

References

[Isc69] Friedrich Ischebeck. Eine dualität zwischen den funktoren ext und tor. *Journal of Algebra*, 11(4):510–531, 1969.