

Gorenstein Rings

Notes for the course MA 842: Topics in Algebra II

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§1 Injective Modules

§§ Basic Properties of Injective Modules

Definition 1.1. Let R be a ring. An R -module E is said to be *injective* if for every inclusion of R -modules $N \hookrightarrow M$ and an R -linear map $N \rightarrow E$, there is an R -linear map $M \rightarrow E$ making

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & M \\ & & \downarrow & \nearrow \exists & \\ & & E & & \end{array}$$

commute.

An R -module M is said to be *divisible* if

$$\mu_a : M \longrightarrow M \quad m \longmapsto am$$

is surjective for each non-zero-divisor $a \in R$.

Remark 1.2. It is easy to see that E is injective if and only if given any inclusion of R -modules $N \hookrightarrow M$, the induced map $\text{Hom}_R(M, E) \rightarrow \text{Hom}_R(N, E)$ is surjective. Further, since $\text{Hom}_R(-, E)$ is always left-exact, we have:

An R -module E is injective if and only if $\text{Hom}_R(-, E)$ is an exact functor.

Proposition 1.3. A direct product of injective modules is injective.

Proof. This follows from the natural isomorphism of functors

$$\mathrm{Hom}_R\left(-, \prod_{\lambda \in \Lambda} E_\lambda\right) \cong \prod_{\lambda \in \Lambda} \mathrm{Hom}_R(-, E_\lambda). \quad \blacksquare$$

Proposition 1.4. Every injective R -module is divisible.

Proof. Let E be R -injective, $x \in E$, and $a \in R$ a non-zero-divisor. Let $\varphi : R \rightarrow E$ be the unique R -linear map sending $1 \mapsto x$. Since $R \xrightarrow{\mu_a^R} R$ is injective, there is a map $\tilde{\varphi} : R \rightarrow E$ such that $\tilde{\varphi} \circ \mu_a^R = \varphi$. In particular, $a\tilde{\varphi}(1) = x$, whence $\mu_a^E : E \rightarrow E$ is surjective, as desired. \blacksquare

Theorem 1.5 (Baer's Criterion). Let R be a ring and E an R -module. Then E is injective if and only if for every ideal $I \trianglelefteq R$ and an R -linear map $f : I \rightarrow E$, there is an R -linear map $F : R \rightarrow E$ such that $F|_I = f$.

Proof. The forward implication is clear. We shall prove the converse. Let $0 \rightarrow N \rightarrow M$ be exact and $f : N \rightarrow E$ be an R -linear map. Consider the poset

$$\Omega = \{(P, g) : N \leq P \leq M \text{ and } g : P \rightarrow E \text{ is } R\text{-linear extending } f\},$$

where $(P, g) \leq (P', g')$ if $P \leq P'$ and $g'|_P = g$. Using Zorn's lemma, choose a maximal element $(P, g) \in \Omega$. We claim that $P = M$. Suppose now and choose some $x \in M \setminus P$. Set $I = (P :_R x) \trianglelefteq R$ and consider the map

$$I \longrightarrow E \quad a \mapsto g(ax).$$

This is well-defined and R -linear, whence it extends to an R -linear map $\varphi : R \rightarrow E$. Let $\alpha = \varphi(1)$ and define $F : P + Rx \rightarrow E$ by $F(p + ax) = g(p) + a\alpha$ for all $p \in P$ and $a \in R$. To see that this is well-defined, note that if $p_1 + a_1x = p_2 + a_2x$, then $a_1 - a_2 \in I$, so that

$$g(p_2) - g(p_1) = g((a_1 - a_2)x) = (a_1 - a_2)\alpha \implies g(p_1) + a_1\alpha = g(p_2) + a_2\alpha.$$

The map F is obviously R -linear and extends g , thereby contradicting the maximality of (P, g) . Hence, $P = M$ and E is injective. \blacksquare

Corollary 1.6. An R -module E is injective if and only if $\mathrm{Ext}_R^1(R/I, E) = 0$ for all ideals $I \trianglelefteq R$. \blacksquare

Remark 1.7. We note that it is not sufficient to check the equivalent condition of Theorem 1.5 for finitely generated ideals. Indeed, let $R = \mathcal{O}(\mathbb{C})$ the ring of entire functions, or $R = \mathcal{O}_{\mathbb{Q}}$ the ring of algebraic integers in \mathbb{C} . It is known that R is a non-Noetherian Bézout domain. As such, due to Interlude 1.14, there is a family of R -injectives $\{E_i\}_{i=1}^\infty$ such that $E = \bigoplus_i E_i$ is not injective.

Since each E_i is injective, it is divisible, consequently, E is a divisible R -module. Moreover, since R is a Bézout domain, every finitely generated ideal I in R is principal. It follows now that the equivalent condition of Theorem 1.5 holds for E but E is not injective.

Proposition 1.8. Let R be a PID. An R -module E is injective if and only if it is divisible.

Proof. The forward direction is clear from Proposition 1.4. Conversely, let E be a divisible R -module and let $f : I \rightarrow E$ be R -linear where $I \trianglelefteq R$ is an ideal. If $I = 0$, then $f = 0$ and the zero map $R \rightarrow E$ extends f . If $I \neq 0$, then there is some $0 \neq a \in R$ such that $I = (a)$. If $x = f(a)$, then choose $y \in E$ with $ay = x$ and let $F : R \rightarrow E$ be the unique R -linear map sending 1 to y . It is clear that R extends f and hence E is an injective R -module. \blacksquare

Proposition 1.9. Let R be an integral domain. A torsion-free and divisible R -module is injective.

Proof. Let E be a torsion-free and divisible R -module. We shall use Theorem 1.5 to show that E is injective. Let $0 \neq I \triangleleft R$ be a proper ideal and $f : I \rightarrow E$ be R -linear. Choose some $0 \neq a \in I$ and let $x \in E$ be the unique (since E is torsion-free) element such that $ax = f(a)$. Let $F : R \rightarrow E$ be the unique R -linear map sending $1 \mapsto x$. We contend that F extends f . Indeed, for $0 \neq b \in I$,

$$af(b) = bf(a) = abx \implies f(b) = bx = F(b),$$

as desired. \blacksquare

Lemma 1.10. Let S be an R -algebra and E an injective R -module. Then $\text{Hom}_R(S, E)$ is an injective S -module.

Note. $\text{Hom}_R(S, E)$ is naturally an S -module under the action

$$(s \cdot f)(s') = f(ss') \quad \forall s, s' \in S, f \in \text{Hom}_R(S, E).$$

Proof. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of S -modules. Using the Hom-Tensor adjunction, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_S(M'', \text{Hom}_R(S, E)) & \longrightarrow & \text{Hom}_S(M, \text{Hom}_R(S, E)) & \longrightarrow & \text{Hom}_S(M', \text{Hom}_R(S, E)) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \text{Hom}_R(M' \otimes_S S, E) & \longrightarrow & \text{Hom}_R(M \otimes_S S, E) & \longrightarrow & \text{Hom}_R(M'' \otimes_S S, E) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \text{Hom}_R(M', E) & \longrightarrow & \text{Hom}_R(M, E) & \longrightarrow & \text{Hom}_R(M'', E) \longrightarrow 0 \end{array}$$

The exactness of the bottom row is a consequence of the R -injectivity of E . Thus the top row is exact and we have our desideratum. ■

Theorem 1.11. Every R -module can be embedded inside an R -injective.

Proof. First, we show this for $R = \mathbb{Z}$. Let M be a \mathbb{Z} -module, then $M \cong \bigoplus_I \mathbb{Z}/N$ for some submodule N of $\bigoplus_I \mathbb{Z}$. There is a natural inclusion of \mathbb{Z} -modules $\bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_I \mathbb{Q}$ which induces an inclusion

$$M \cong \frac{\bigoplus_I \mathbb{Z}}{N} \hookrightarrow \frac{\bigoplus_I \mathbb{Q}}{N} =: E$$

Being a quotient of a divisible module, E is divisible and hence \mathbb{Z} -injective.

Now, let R be any ring and M an R -module. Then M is naturally a \mathbb{Z} -module and admits a \mathbb{Z} -linear inclusion $\iota : M \hookrightarrow E$, where E is a \mathbb{Z} -injective. Consider the map

$$\varphi : M \longrightarrow \text{Hom}_{\mathbb{Z}}(R, E) \quad m \longmapsto \varphi_m,$$

where $\varphi_m : R \rightarrow E$ is given by $\varphi_m(r) = f(rm)$. The map φ is obviously R -linear and if $\varphi_m = 0$, then $f(m) = \varphi_m(1) = 0$, i.e., $m = 0$. As a result, φ is injective and we have embedded M inside an injective R -module. ■

Corollary 1.12. Let E be an R -module. Then E is injective if and only if every R -linear inclusion $E \hookrightarrow M$ splits.

Proof. Suppose E is injective.

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \longrightarrow & M \\ & & \parallel & & \nearrow \exists \\ & & E & & \end{array}$$

The above diagram constructs a splitting of $E \hookrightarrow M$.

Conversely, suppose every R -linear inclusion $E \hookrightarrow M$ splits. Due to Theorem 1.11, we may choose M to be injective, so that E is a direct summand of M , whence E is injective. ■

Proposition 1.13. Let R be a Noetherian ring. A direct sum of injective R -modules is injective.

Proof. Let $\{E_\lambda\}_{\lambda \in \Lambda}$ be a collection of R -injectives and $E = \bigoplus_{\lambda \in \Lambda} E_\lambda$. Let $I \triangleleft R$ be a non-zero proper ideal and $f : I \rightarrow E$ an R -linear map. Since I is finitely generated, its image under f is finitely generated in E . Consequently, there is a finite subset $\Lambda_0 \subseteq \Lambda$ such that $f(I) \subseteq \bigoplus_{\lambda \in \Lambda_0} E_\lambda = E_0$. Being a finite direct sum of injectives, E_0 is injective and hence there is a map $F : R \rightarrow E_0$ extending $f : I \rightarrow E_0$. Composing F with the natural inclusion $E_0 \hookrightarrow E$, we obtain our desired extension of f . It now follows from Theorem 1.5 that E is an injective R -module. ■

Interlude 1.14 (Bass-Papp Construction). Let R be a non-Noetherian ring. Choose a strictly increasing chain of proper non-zero ideals

$$0 \neq I_1 \subsetneq I_2 \subsetneq \cdots.$$

For each $n \geq 1$, choose an injective module E_n containing R/I_n , and set $E = \bigoplus_n E_n$. We contend that E is not R -injective.

Let $I = \bigcup_n I_n$. Since each I_n is proper, so is I . Let $f : I \rightarrow E$ be the map given by

$$f(x) = (x \bmod I_1, x \bmod I_2, \dots).$$

If E were injective, then there must exist a map $F : R \rightarrow E$ extending f . Suppose $F(1) = (x_1, x_2, \dots)$. There is a positive integer N such that $x_n = 0$ for all $n \geq N$. Choose $x \in I_{N+1} \setminus I_N$. Since $x \in I$, we have

$$(xx_1, xx_2, \dots) = F(x) = f(x) = (x \bmod I_1, x \bmod I_2, \dots).$$

In particular, $xx_N = 0$, a contradiction. Thus E is not R -injective.

Proposition 1.15. Let (R, \mathfrak{m}, k) be a Noetherian local ring. If $E \neq 0$ is a finitely generated injective R -module, then R is Artinian.

Proof. We shall show that $\dim R = 0$. Suppose not; we contend that there is a prime $\mathfrak{p} \subsetneq \mathfrak{m}$ such that $\text{Hom}_R(R/\mathfrak{p}, E) \neq 0$. Indeed, if there is a non-maximal prime $\mathfrak{p} \in \text{Ass}_R(E)$, then $R/\mathfrak{p} \hookrightarrow E$, giving us the desideratum. On the other hand, if $\text{Ass}_R(E) = \{\mathfrak{m}\}$, then the composition

$$R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{m} \hookrightarrow E$$

gives a non-zero map $R/\mathfrak{p} \rightarrow E$.

Choose $a \in \mathfrak{m} \setminus \mathfrak{p}$; this is a non-zerodivisor on R/\mathfrak{p} and furnishes an exact sequence

$$0 \rightarrow R/\mathfrak{p} \xrightarrow{a} R/\mathfrak{p}.$$

Applying $\text{Hom}_R(-, E)$, we get a surjection

$$\text{Hom}_R(R/\mathfrak{p}, E) \xrightarrow{a} \text{Hom}_R(R/\mathfrak{p}, E) \rightarrow 0.$$

Note that $\text{Hom}_R(R/\mathfrak{p}, E) \cong (0 :_E \mathfrak{p}) \subseteq E$, is a finite R -module. Due to Nakayama's lemma, we must have that $\text{Hom}_R(R/\mathfrak{p}, E) = 0$, a contradiction. Thus $\dim R = 0$, i.e. R is Artinian. ■

Remark 1.16. One cannot drop the local condition in Proposition 1.15. This construction makes use of injective hulls. Let k be an algebraically closed field and

$$R = \frac{k[X, Y]}{(X - X^2, Y - XY)}.$$

Note that R is the coordinate ring of the disjoint union of the origin and the line $x = 1$ in \mathbb{A}_k^2 . In particular, $\dim R = 1$, and R is not Artinian.

Let $\mathfrak{m} = (x, y)$ be the maximal ideal corresponding to the origin. Then $R_{\mathfrak{m}} \cong k$, since it is the local ring of an isolated point. Now,

$$E_R(k) \cong E_{R_{\mathfrak{m}}}(k) \cong E_k(k) = k,$$

so that k is a finitely generated injective R -module.

§§ Essential Extensions and Injective Hulls

Definition 1.17. A containment of R -modules $N \subseteq M$ is said to be *essential* if every non-zero submodule of M intersects N non-trivially.

An injective map $\iota : N \hookrightarrow M$ is said to be essential if $\iota(N) \subseteq M$ is essential.

Remark 1.18. Let $M \subseteq N$ be an essential extension of R -modules and $\varphi : M \hookrightarrow P$ be an R -linear injective map. If φ extends to an R -linear map $\tilde{\varphi} : N \rightarrow P$, then $\tilde{\varphi}$ is injective too. Indeed, if $K = \ker \tilde{\varphi} \neq 0$, then $K \cap M \neq 0$, a contradiction.

Proposition 1.19. Let (R, \mathfrak{m}, k) be a Noetherian local ring. Let M be an Artinian R -module. Then $\text{Soc}_R(M) \subseteq M$ is an essential extension.

Proof. Let $0 \neq K \subseteq M$ be a submodule. Choose $0 \neq x \in K$. Since M is Artinian, the descending chain $Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2x \supseteq \dots$ stabilizes. Let $n \geq 0$ be the least positive integer such that $\mathfrak{m}^n x = \mathfrak{m}^{n+1}x$. Due to Nakayama's lemma, $\mathfrak{m}^n x = 0$, whence $n \geq 1$. It follows that $0 \neq \mathfrak{m}^{n-1}x \subseteq \text{Soc}_R(M) \cap K$, as desired. ■

Corollary 1.20. Let (R, \mathfrak{m}, k, E) be a Noetherian local ring and M an Artinian R -module. If $\dim_k \text{Soc}_R(M) = d$, then $E_R(M) \cong E^{\oplus d}$.

Proof. Since $\text{Soc}_R(M) \cong k^{\oplus d}$, it is clear that $E_R(\text{Soc}_R(M)) \cong E^{\oplus d}$. The inclusion $\text{Soc}_R(M) \hookrightarrow E^{\oplus d}$ can be extended to M to obtain a commutative diagram:

$$\begin{array}{ccc} & M & \\ \uparrow & \searrow & \\ \text{Soc}_R(M) & \hookrightarrow & E_R(\text{Soc}_R(M)) \cong E^{\oplus d} \end{array}$$

where all maps are inclusion. It follows that $M \hookrightarrow E^{\oplus d}$ is an essential extension. Since $E^{\oplus d}$ is an injective module, we have that $E_R(M) \cong E^{\oplus d}$. ■

§2 Matlis Duality

Definition 2.1. Let (R, \mathfrak{m}, k, E) be a Noetherian local ring. For an R -module M , set $M^\vee = \text{Hom}_R(M, E)$. This is known as the *Matlis dual* of a module.

Clearly $(-)^\vee$ is a contravariant exact functor on the category of R -modules. Note that if $I \subseteq \mathfrak{m}$ is an ideal, then as we have seen earlier,

$$E_{R/I}(k) = \text{Hom}_R(R/I, E) = (R/I)^\vee.$$

In particular, taking $I = \mathfrak{m}$, we see that $k^\vee \cong k$ as R -modules.

Lemma 2.2. Let (R, \mathfrak{m}, k, E) be a Noetherian local ring. Then

- (1) If $M \neq 0$, then $M^\vee \neq 0$.
- (2) If $\lambda_R(M) < \infty$, then $\lambda_R(M^\vee) \neq 0$. Moreover, $\lambda_R(M) = \lambda_R(M^\vee)$.

Proof. (1) Let $0 \neq x \in M$. If $I = \text{Ann}_R(x)$, then there is a natural inclusion $R/I \hookrightarrow M$ sending $\bar{1} \mapsto x$. Taking the Matlis dual, we have a surjection

$$M^\vee \twoheadrightarrow (R/I)^\vee = E_{R/I}(k) \neq 0,$$

consequently $M^\vee \neq 0$.

- (2) We shall prove both statements by induction on $\lambda_R(M)$. If $\lambda_R(M) = 0$, then $M = 0$, so that $M^\vee = 0$ and we get $\lambda_R(M) = 0 = \lambda_R(M^\vee)$. Suppose now that $0 < \lambda_R(M) < \infty$. Then $\mathfrak{m} \in \text{Ass}_R(M)$, and we have a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Since length is additive, $\lambda_R(N) = \lambda_R(M) - 1$; hence the induction hypothesis applies and $\lambda_R(N^\vee) = \lambda_R(N)$. Taking the Matlis dual of the above short exact sequence, we have

$$0 \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow k^\vee \longrightarrow 0.$$

Since $k^\vee = 0$, we see that

$$\lambda_R(M^\vee) = \lambda_R(N^\vee) + 1 = \lambda_R(N) + 1 = \lambda_R(M),$$

as desired. ■

Theorem 2.3. Let (R, \mathfrak{m}, k, E) be an Artinian local ring.

(1) E is a faithful finite R -module.

(2) The map

$$\mu : R \longrightarrow \text{Hom}_R(E, E) \quad a \longmapsto \mu_a$$

is an isomorphism of R -modules and rings.

(3) Given a finite R -module M , the natural map

$$\varphi_M : M \longrightarrow M^{\vee\vee} \quad m \longmapsto \text{ev}_m$$

is an isomorphism.

Proof. (1) Suppose $a \in R$ is such that $aE = 0$. Then

$$R^\vee = \text{Hom}_R(R, E) = E = (E :_E a) \cong \text{Hom}_R(R/aR, E) = (R/aR)^\vee.$$

Since R is Artinian, we then have

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R((R/aR)^\vee) = \lambda_R(R/aR) \implies \lambda_R(aR) = 0,$$

consequently, $a = 0$, i.e., E is a faithful R -module.

Next, since R is Artinian, $\mathfrak{m} \in \text{Ass}_R(R)$, consequently, there is an injection $k = R/\mathfrak{m} \hookrightarrow R$. Due to Remark 1.18 extends to an inclusion $E \hookrightarrow R$, consequently, E is a finite R -module.

(2) First note that μ is injective due to (1). But note that

$$\infty > \lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = \lambda_R(E^\vee) = \lambda_R(\text{Hom}_R(E, E)),$$

consequently μ is an isomorphism.

(3) It suffices to show that φ_M is injective since $\lambda_R(M) = \lambda_R(M^{\vee\vee})$. Suppose $0 \neq x \in M$ is such that $\varphi_M(x) = 0$, that is, for all $f \in \text{Hom}_R(M, E)$, $f(x) = 0$. Let $I = \text{Ann}_R(x)$. Now, there is a non-zero map

$$\psi : R/I \twoheadrightarrow R/\mathfrak{m} = k \hookrightarrow E,$$

which extends to a non-zero map $f : M \rightarrow E$ since $R/I \hookrightarrow M$ through $\bar{1} \mapsto x$. Thus, $f(x) = \psi(\bar{1}) \neq 0$, a contradiction. ■

Porism 2.4. Let (R, \mathfrak{m}, k) be a Noetherian local ring and M a finite-length R -module. Then the “evaluation map” $\text{ev} : M \rightarrow M^{\vee\vee}$ is an isomorphism of R -modules.

Proof. As in the preceding proof, ev is injective and due to Lemma 2.2, $\lambda_R(M) = \lambda_R(M^{\vee\vee})$, whence ev is an isomorphism. ■

Interlude 2.5 (On \widehat{R} -modules). Let (R, \mathfrak{m}, k) be a local ring and M an R -module such that $\Gamma_{\mathfrak{m}}(M) = M$. We contend that M is an \widehat{R} -module in a natural way. To this end, we need only define $\widehat{a} \cdot m$ for $\widehat{a} \in \widehat{R}$ and $m \in M$.

Let $\widehat{a} = (a_1, a_2, \dots)$, where we are using the isomorphism

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

Since $\Gamma_{\mathfrak{m}}(M) = M$, there is a positive integer $n \geq 1$ such that $\mathfrak{m}^n m = 0$. Hence, for $k \geq n$, we have $a_k \cdot m = a_n \cdot m$, as $a_k - a_n \in \mathfrak{m}^n$. In light of this, we define $\widehat{a} \cdot m = a_n \cdot m$. We must show that this makes M into an \widehat{R} -module.

Let $m_1, m_2 \in M$ and $\widehat{a} = (a_1, a_2, \dots) \in \widehat{R}$. There are positive integers $n_1, n_2 \geq 1$ such that $\mathfrak{m}^{n_1} m_1 = 0 = \mathfrak{m}^{n_2} m_2$; then $\mathfrak{m}^n m_1 = 0 = \mathfrak{m}^n m_2$ for all $n \geq \max\{n_1, n_2\}$. Hence, for all such $n \geq 1$,

$$\widehat{a} \cdot (m_1 + m_2) = a_n \cdot (m_1 + m_2) = a_n \cdot m_1 + a_n \cdot m_2 = \widehat{a} \cdot m_1 + \widehat{a} \cdot m_2.$$

Next, let $\hat{a}, \hat{b} \in \hat{R}$ and $m \in M$ with

$$\hat{a} = (a_1, a_2, \dots) \quad \text{and} \quad \hat{b} = (b_1, b_2, \dots).$$

There is a positive integer n such that $\mathfrak{m}^n m = 0$. Then

$$(\hat{a} + \hat{b}) \cdot m = (a_n + b_n) \cdot m = a_n \cdot m + b_n \cdot m = \hat{a} \cdot m + \hat{b} \cdot m.$$

Finally, note that $\hat{b} \cdot m = b_n m$ and $\mathfrak{m}^n (\hat{b} \cdot m) = 0$, so that

$$\hat{a} \cdot (\hat{b} \cdot m) = \hat{a} \cdot (b_n \cdot m) = a_n \cdot (b_n \cdot m) = (a_n b_n) \cdot m = (\hat{a} \hat{b}) \cdot m.$$

This shows that M is indeed an \hat{R} -module as described above. Further, since $R \rightarrow \hat{R}$ is the diagonal map, it follows that the \hat{R} -module structure on M agrees with the R -module structure through the diagonal map. In particular, this means that:

A subset of M is an R -submodule if and only if it is an \hat{R} -submodule.

As a result, M is Noetherian (resp. Artinian) as an R -module if and only if it is so as an \hat{R} -module.

Interlude 2.6 (On maps between \mathfrak{m} -power torsion modules). Again, let (R, \mathfrak{m}, k) be a local ring and suppose M and N are R -modules such that $\Gamma_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(N)$. By Interlude 2.5, we know that they are \hat{R} -modules in a natural way. Let $\varphi : M \rightarrow N$ be an R -linear map. We contend that φ is also \hat{R} -linear. Indeed, let $m \in M$ and $\hat{a} = (a_1, a_2, \dots) \in \hat{R}$. There is a positive integer $n \geq 1$ such that $\mathfrak{m}^n m = 0$, and hence, $\mathfrak{m}^n \varphi(m) = 0$. It follows that

$$\varphi(\hat{a} \cdot m) = \varphi(a_n \cdot m) = a_n \cdot \varphi(m) = \hat{a} \cdot \varphi(m),$$

as desired.

Theorem 2.7. Let (R, \mathfrak{m}, k, E) be a Noetherian local ring.

- (1) $\Gamma_{\mathfrak{m}}(E) = E$, and hence E is an \hat{R} module and for every R -module M .
- (2) $E \cong E_{\hat{R}}(k)$ as \hat{R} -modules.
- (3) $R^{\vee\vee} = \text{Hom}_R(E, E) \cong \hat{R}$ as R -algebras.
- (4) E is an Artinian R -module.

Proof. (1) That E is an \hat{R} -module follows immediately from Interlude 2.5.

- (2) The containment $k \subseteq E$ is an essential extension of R -modules, both of which are \mathfrak{m} -power torsion. Due to Interlude 2.5, it follows that it is an essential extension of \hat{R} -modules too. Now, due to Remark 1.18, there is a commutative diagram of inclusions

$$\begin{array}{ccc} & E & \\ \uparrow & \searrow & \\ k & \longrightarrow & E_{\hat{R}}(k), \end{array}$$

where all maps are \hat{R} -linear. It follows that $E \hookrightarrow E_{\hat{R}}(k)$ is an essential extension of \hat{R} -modules, and consequently, an essential extension of R -modules. Since E is R -injective, we must have that the inclusion is an isomorphism of R -modules. Finally, due to Interlude 2.6, this is an isomorphism of \hat{R} -modules.

- (3) For every positive integer $n \geq 1$, set $E_n = (0 :_E \mathfrak{m}^n)$. Note that $E_1 \subseteq E_2 \subseteq \dots$, and $E = \bigcup_n E_n$. Define $\Phi : \hat{R} \rightarrow \text{End}_R(E)$ as follows: for $\hat{a} = (a_1, a_2, \dots) \in \hat{R}$, let $\Phi(\hat{a}) = f \in \text{End}_R(E)$ where f is given by

$$f(x) = a_n x \quad \text{if } x \in E_n.$$

First we must show that the above map is well-defined. Indeed, if $m < n$ and $x \in E_m \subseteq E_n$, then $a_m - a_n \in \mathfrak{m}^m$, whence $(a_m - a_n)x = 0$, i.e., $a_mx = a_nx$. That the map f is R -linear is clear from its definition.

That the map Φ is R -linear is also clear. We claim that Φ is a ring homomorphism. Let $\hat{a} = (a_1, a_2, \dots), \hat{b} = (b_1, b_2, \dots) \in \hat{R}$ and set $f = \Phi(\hat{a}), g = \Phi(\hat{b})$, and $h = \Phi(\hat{a}\hat{b})$. If $x \in E_n$, then

$$h(x) = (a_nb_n)x = f(g(x)) \implies h = f \circ g,$$

thus Φ is a ring homomorphism.

Finally, we show that Φ is bijective, so that it is an isomorphism of R -algebras. If $\hat{a} \in \hat{R}$ is such that $\Phi(\hat{a}) = 0$, then $a_n \in \text{Ann}_R(E_n)$ for every positive integer n . But recall that

$$E_n \cong \text{Hom}_R(R/\mathfrak{m}^n, E) \cong E_{R/\mathfrak{m}^n}(k),$$

which is a faithful R/\mathfrak{m}^n -module due to Theorem 2.3. As a result, $\text{Ann}_R(E_n) = \mathfrak{m}^n$, i.e., $a_n \in \mathfrak{m}^n$ for all $n \geq 1$; in other words, $\hat{a} = 0$. This proves the injectivity of Φ .

Next, we must show surjectivity of Φ . Let $f \in \text{End}_R(E)$, then f restricts to an R -linear endomorphism of $E_n \cong E_{R/\mathfrak{m}^n}(k)$. Due to Theorem 2.3, the restriction of f to each E_n is multiplication by some element $a_n \in R/\mathfrak{m}^n$. Further, it is clear that under the canonical surjection $R/\mathfrak{m}^n \twoheadrightarrow R/\mathfrak{m}^{n-1}$, a_n maps to a_{n-1} , so that $\hat{a} = (a_1, a_2, \dots) \in \hat{R}$ and $\Phi(\hat{a}) = f$. Thus Φ is surjective, as desired.

As a final subtle point, we must check that the R -algebra structure on \hat{R} is the canonical one. The natural map $R \rightarrow \text{End}_R(E)$ is $a \mapsto \mu_a$, the “multiplication by a ” map. From our definition of Φ , it is clear that $\Phi^{-1}(\mu_a) = (a, a, \dots)$, which is precisely the image of a under the canonical map $R \rightarrow \hat{R}$.

(4) Let $M_1 \supseteq M_2 \supseteq \dots$ be a chain of R -submodules in E . There are commutative diagrams

$$\begin{array}{ccc} M_{j+1} & \xhookrightarrow{\iota_{j+1}} & E \\ \downarrow & \searrow \iota_j & \\ M_j & & \end{array}$$

whose Matlis dual furnishes commutative diagrams

$$\begin{array}{ccc} \hat{R} = E^\vee & \xrightarrow{\varphi_j} & M_j^\vee \\ & \searrow \varphi_{j+1} & \downarrow \\ & & M_{j+1}^\vee \end{array}$$

Note that all Matlis duals are \mathfrak{m} -power torsions and hence due to Interlude 2.6, the φ_j 's are \hat{R} -linear. Let $I_j = \ker \varphi_j \subseteq \hat{R}$, which is an ideal. Due to the commutative diagram, it is clear that there is an ascending chain $I_j \subseteq I_{j+1}$. Since \hat{R} is Noetherian, this chain stabilizes, say $I_n = I_{n+1} = \dots$.

Then due to the first isomorphism theorem, $M_j^\vee \twoheadrightarrow M_{j+1}^\vee$ is an isomorphism for all $j \geq n$. Let $C_j = \text{coker}(M_{j+1} \hookrightarrow M_j)$. The exactness of the Matlis dual gives $C_j^\vee = 0$, which, due to Lemma 2.2, implies that $C_j = 0$, that is, $M_{j+1} \hookrightarrow M_j$ is an isomorphism for all $j \geq n$, i.e., the descending chain stabilizes, as desired. ■

Interlude 2.8 (The Matlis Dual is a module over \hat{R}). Let (R, \mathfrak{m}, k, E) be a Noetherian local ring and M an R -module. The Matlis dual $M^\vee = \text{Hom}_R(M, E)$ is naturally a $\hat{R} = \text{End}_R(E)$ -module: for $f \in M^\vee$ and $\varphi \in \text{End}_R(E)$, define $\varphi \cdot f = \varphi \circ f$. It is easy to check that this \hat{R} -module structure on M^\vee extends the R -module structure through the canonical map $R \rightarrow \text{End}_R(E)$, $a \mapsto \mu_a$.

Now, if $f : M \rightarrow N$ is an R -linear map of R -modules, then $f^\vee : N^\vee \rightarrow M^\vee$ is \widehat{R} -linear. Indeed, for $\varphi \in N^\vee$, and $\psi \in \widehat{R} = \text{End}_R(E)$, we have

$$f^\vee(\psi \cdot \varphi) = f^\vee(\psi \circ \varphi) = \psi \circ \varphi \circ f = \psi \cdot f^\vee(\varphi),$$

as desired.

Theorem 2.9 (Matlis Duality, version 1). Let (R, \mathfrak{m}, k, E) be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{finitely generated} \\ \widehat{R}\text{-modules} \end{array} \right\} \begin{array}{c} \xrightarrow{(-)^\vee} \\ \xleftarrow{(-)^\vee} \end{array} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{Artinian } R\text{-modules} \end{array} \right\}.$$

Proof. Let M be an Artinian R -module and let $d = \dim_k \text{Soc}_R(M)$. Due to Corollary 1.20, $E_R(M) \cong E^{\oplus d}$, so that there is an inclusion $M \hookrightarrow E^{\oplus d}$, which upon taking the Matlis dual furnishes an \widehat{R} -linear surjection $\widehat{R}^{\oplus d} \twoheadrightarrow M^\vee$. Thus M^\vee is a finite \widehat{R} -module.

Conversely, suppose M is a finite \widehat{R} -module. Thus, there is a surjection $\widehat{R}^{\oplus n} \twoheadrightarrow M$. Taking the Matlis dual, we obtain an injection $M^\vee \hookrightarrow (\widehat{R}^\vee)^{\oplus n}$.

There is a natural “evaluation map” $\text{ev} : M \rightarrow M^{\vee\vee}$, which we shall show is an isomorphism. That ev is injective follows in the same way as Theorem 2.3 (3). Next, since $\lambda_R(M) < \infty$, we have that $\lambda_R(M) = \lambda_R(M^\vee) = \lambda_R(M^{\vee\vee})$, whence ev is an isomorphism. ■

Theorem 2.10. Let (R, \mathfrak{m}, k, E) be a Noetherian local ring. Then the following are equivalent:

- (1) R is self-injective
- (2) $R \cong E$ as R -modules.
- (3) R is Artinian and $\dim_k \text{Soc}_R(R) = 1$.

Proof. (1) \implies (2) Due to Proposition 1.15, R must be an Artinian local ring, and hence, from Proposition 1.19, $\text{Soc}_R(R) \subseteq R$ is an essential extension. It follows that R is the injective hull of $\text{Soc}_R(R) \cong k^{\oplus d}$ for some positive integer d . Hence, $R \cong E^{\oplus d}$ as R -modules, and comparing lengths, we have

$$\lambda_R(R) = d\lambda_R(E) = d\lambda_R(R^\vee) = d\lambda_R(R),$$

whence $d = 1$ and $R \cong E$.

(2) \implies (3) Due to Theorem 2.7 (4), R is Artinian. Using a length argument as above, we can show that $\dim_k \text{Soc}_R(R) = 1$.

(3) \implies (1) Again, since $k = \text{Soc}_R(R) \subseteq R$ is essential, we have that $R \hookrightarrow E = E_R(k)$. Using a length argument, it follows that this inclusion must be an isomorphism, whence R is self-injective. ■

Theorem 2.11 (Matlis Duality, version 2). Let (R, \mathfrak{m}, k) be a Noetherian local ring. Then there is a bijective correspondence

$$\left\{ \begin{array}{c} \mathfrak{m}\text{-primary ideals} \\ \text{in } R \end{array} \right\} \begin{array}{c} \xrightarrow{(0 :_E -)} \\ \xleftarrow{(0 :_R -)} \end{array} \left\{ \begin{array}{c} \text{finitely generated} \\ R\text{-submodules of } E \end{array} \right\}.$$

Proof. We must first show that the above maps are indeed defined between those sets. Let I be \mathfrak{m} -primary in R . Then

$$(0 :_E I) \cong \text{Hom}_R(R/I, E) = (R/I)^\vee.$$

As a result, $\lambda_R((0 :_E I)) = \lambda_R(R/I) < \infty$, so that $(0 :_E I)$ is a finite R -module.

On the other hand, let $W \subseteq E$ be a finite R -submodule. Taking the Matlis dual of the exact sequence $0 \rightarrow W \rightarrow E$, one obtains an \widehat{R} -linear (due to Interlude 2.8) surjection $\varphi : \widehat{R} \twoheadrightarrow W^\vee$. Further, since $\lambda_R(W) < \infty$, we have $\lambda_R(W^\vee) = \lambda_R(W) < \infty$. Set $I = (0 :_R W)$ and $J = (0 :_R W^\vee)$; note that both I and J are \mathfrak{m} -primary. This shows that both the maps in the theorem are well-defined.

Claim. $I = J$

Since I annihilates W , it must also annihilate W^\vee , so that $I \subseteq J$. Now, since J annihilates W^\vee , it annihilates $W^{\vee\vee} \cong W$ (due to Porism 2.4), so that $J \subseteq I$; as a result, $I = J$. ♠

Finally, we show that the given maps are inverses to one another. Let $I \trianglelefteq R$ be \mathfrak{m} -primary. Then $(0 :_E I) \cong \text{Hom}_R(R/I, E) \cong E_{R/I}(k)$, whence due to Theorem 2.3 (1), $(0 :_R (0 :_E I)) = I$. Next, let $W \subseteq E$ be a finite R -submodule. Clearly $W \subseteq (0 :_E (0 :_R W))$. Further, recall that $\widehat{R} \twoheadrightarrow W^\vee$ and $\lambda_{\widehat{R}}(W^\vee) = \lambda_R(W) < \infty^1$, the kernel of the surjection is $\widehat{\mathfrak{m}}$ -primary, and hence, factors through $\widehat{R}/\widehat{\mathfrak{m}}^n$ for some positive integer n . But since $\widehat{R}/\widehat{\mathfrak{m}}^n \cong R/\mathfrak{m}^n$ as R -modules, it follows that W^\vee is a cyclic R -module. In particular, $W^\vee \cong R/J = R/I$. In particular,

$$\lambda_R((0 :_E (0 :_R W))) = \lambda_R((R/I)^\vee) = \lambda_R(R/I) = \lambda_R(R/J) = \lambda_R(W^\vee) = \lambda_R(W),$$

whence $W = (0 :_E (0 :_R W))$, thereby completing the proof. ■

§3 Injective Resolutions

§§ Bass's Lemma and ramifications

Definition 3.1. Let M be an R -module. An *injective resolution* for M is an exact complex

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots,$$

where each E^n is an injective R -module. The resolution is often denoted succinctly as $0 \rightarrow M \rightarrow E^\bullet$.

We say that M has finite injective dimension if M has an injective resolution $0 \rightarrow M \rightarrow E^\bullet$ and an integer $N \geq 0$ such that $E^n = 0$ for $n \geq N$. We define

$$\text{inj dim}_R M = \inf \{n : 0 \rightarrow M \rightarrow E^0 \rightarrow \dots \rightarrow E^n \rightarrow 0 \text{ is an injective resolution of } M\}.$$

If M does not have finite injective dimension, then set $\text{inj dim}_R M = \infty$.

Remark 3.2. It is possible to create a “canonical” injective resolution by successively taking injective hulls. Set $E^0 = E_R(M)$ and for $i \geq 0$, define

$$E^{i+1} = E_R(\text{coker}(E^{i-1} \rightarrow E^i)),$$

with the convention that $E^{-1} = M$. We call this the *minimal injective resolution* of M .

Lemma 3.3. Let R be a Noetherian ring and $0 \rightarrow M \xrightarrow{\theta} E$ be an inclusion of R -modules with E injective. Then the inclusion is an injective hull of M if and only if

$$\text{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \xrightarrow{\theta_\mathfrak{p}} \text{Hom}_R(R/\mathfrak{p}, E)_\mathfrak{p}$$

is an isomorphism for all $\mathfrak{p} \in \text{Spec}(R)$.

Proof. Owing to the left exactness of $\text{Hom}_R(R/\mathfrak{p}, -)$ and the exactness of localization, the map $\theta_\mathfrak{p}$ is injective for each $\mathfrak{p} \in \text{Spec}(R)$. Hence, it suffices to show that E is injective if and only if $\theta_\mathfrak{p}$ is surjective for each $\mathfrak{p} \in \text{Spec}(R)$.

Recall that there are canonical isomorphisms

$$\text{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \xrightarrow{\sim} \text{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \quad \frac{\psi}{s} \mapsto \left(\frac{a}{t} \mapsto \frac{\psi(a)}{st} \right),$$

where we are identifying $\kappa(\mathfrak{p})$ with the quotient field of R/\mathfrak{p} . Hence, surjectivity of $\theta_\mathfrak{p}$ is equivalent to the surjectivity of

$$\text{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \rightarrow \text{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_\mathfrak{p}).$$

Henceforth, we shall identify M with a submodule of E , so that θ is simply the inclusion map.

¹Since every \widehat{R} -submodule of W^\vee is also an R -submodule, it follows that W^\vee is both Noetherian and Artinian as an \widehat{R} -module.

Suppose first that $M \xrightarrow{\theta} E$ is an injective hull and let $0 \neq \varphi \in \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p)$. Using the above isomorphism, we can write $\varphi = \psi/s$ for some $\psi \in \text{Hom}_R(R/\mathfrak{p}, E)$ and $s \in R \setminus \mathfrak{p}$. Let $\psi(\bar{1}) = z \in E$ and $a \in R$ such that $0 \neq az \in M$. Note that $a \in R \setminus \mathfrak{p}$, since $\mathfrak{p} \subseteq \text{Ann}_R(z)$ ². Define

$$\bar{\varphi} : R/\mathfrak{p} \longrightarrow M \quad \bar{1} \longmapsto az.$$

This is well-defined, since \mathfrak{p} annihilates $az \in M$. We claim that

$$\varphi = \frac{\bar{\varphi}}{as} \in \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p).$$

Indeed, for $x/t \in \kappa(\mathfrak{p})$ we have

$$\left(\frac{\bar{\varphi}}{as}\right)\left(\frac{x}{t}\right) = \frac{\bar{\varphi}(x)}{ast} = \frac{xaz}{ast} = \frac{xz}{st} = \left(\frac{\psi}{s}\right)\left(\frac{x}{t}\right) = \varphi\left(\frac{x}{t}\right),$$

as desired. This shows that $\text{Hom}_{R_p}(\kappa(\mathfrak{p}), M_p) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p)$ is surjective.

Conversely, suppose the aforementioned map is surjective. We shall show that E is the injective hull of M . To this end, it suffices to show that the inclusion $M \subseteq E$ is essential. Let $0 \neq N \subseteq E$ be a submodule and $\mathfrak{p} \in \text{Ass}_R(N)$. There is an injective map

$$0 \rightarrow R/\mathfrak{p} \longrightarrow N \quad \bar{1} \longmapsto z.$$

Since $\mathfrak{p} = \text{Ann}_R(z)$, it suffices to find $a \in R \setminus \mathfrak{p}$ such that $az \in M$. Consider the map

$$\varphi : \kappa(\mathfrak{p}) \longrightarrow E_p \quad \bar{1} \longmapsto z/1.$$

The surjectivity of θ_p furnishes a $\psi : \kappa(\mathfrak{p}) \rightarrow M_p$ such that $\theta_p(\psi) = \varphi$. In particular, this means that

$$\frac{z}{1} = \varphi(\bar{1}) = \psi(\bar{1}) \in M_p,$$

whence there is some $a \in R \setminus \mathfrak{p}$ such that $az \in M$, as desired. ■

Corollary 3.4. Let R be a Noetherian ring and $0 \rightarrow M \rightarrow E^\bullet$ be an injective resolution of an R -module M . Then E^\bullet is minimal if and only if the natural maps

$$\text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \longrightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^{n+1})$$

are identically zero for all $n \geq 0$ and for all $\mathfrak{p} \in \text{Spec}(R)$.

Proof. Let $K^n = \ker(E^n \rightarrow E^{n+1})$. Then there is an exact sequence $0 \rightarrow K^n \rightarrow E^n \rightarrow E^{n+1}$. Using Lemma 3.3, E^n is the injective hull of C^n if and only if

$$\Phi : \text{Hom}_{R_p}(\kappa(\mathfrak{p}), C_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \text{ is an isomorphism.}$$

But the left-exactness of Hom and exactness of localization implies that the sequence

$$0 \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), C_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^{n+1})$$

is exact. Thus Φ is an isomorphism if and only if the map $\text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^n) \rightarrow \text{Hom}_{R_p}(\kappa(\mathfrak{p}), E_p^{n+1})$ is the zero map, as desired. ■

Corollary 3.5. Let R be a Noetherian ring and M an R -module. Let $0 \rightarrow M \rightarrow E^\bullet$ be the minimal injective resolution of M . Then

$$E^j = \bigoplus_{\mathfrak{p}} E_R(R/\mathfrak{p})^{a_j(\mathfrak{p})} \quad \text{and} \quad a_j(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_p}^j(\kappa(\mathfrak{p}), M_p).$$

In particular, if M is a finite R -module, $a_j(\mathfrak{p}) < \infty$ for all $j \geq 0$ and $\mathfrak{p} \in \text{Spec}(R)$.

²Note that $\mathfrak{p} = \text{Ann}_R(z)$, for if not, then $\varphi = 0$.

Proof. ■

Definition 3.6. Let R be a Noetherian ring and M a finite R -module. For $j \geq 0$ and $\mathfrak{p} \in \text{Spec}(R)$, define the *j -th Bass number* as

$$\mu_j(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}).$$

Remark 3.7. We can now justify the name “minimal injective resolution”. In particular, we shall show that the length of the minimal injective resolution is precisely the injective dimension of a module.

Let R be a Noetherian ring and M a finite R -module. Let $0 \rightarrow M \rightarrow E^\bullet$ be the minimal injective resolution in the sense of Remark 3.2. Let $0 \leq \ell \leq \infty$ denote the length of the resolution. Clearly $\text{inj dim}_R M \leq \ell$. If $\text{inj dim}_R M = \infty$, then $\ell \leq \text{inj dim}_R M$ so that $\ell = \text{inj dim}_R M$.

On the other hand, if $\text{inj dim}_R M = n < \infty$, then using this injective resolution to compute the Ext's, we see that for $j > n$, and $\mathfrak{p} \in \text{Spec}(R)$,

$$\text{Ext}_R^j(R/\mathfrak{p}, M) = 0 \implies \mu_j(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = 0.$$

That is, $E^j = 0$ for all $j > n$ and hence, $\ell \leq n$. It follows that $\ell = \text{inj dim}_R M$.

Lemma 3.8 (Bass). Let R be a Noetherian ring and M a finite R -module. Let $\mathfrak{p} \subsetneq \mathfrak{q}$ be primes in R such that $\text{ht}(\mathfrak{q}/\mathfrak{p}) = 1$. If for some $j \geq 0$, $\mu_j(\mathfrak{p}, M) \neq 0$, then $\mu_{j+1}(\mathfrak{q}, M) \neq 0$.

Proof. Localizing at \mathfrak{q} , we may assume that (R, \mathfrak{m}, k) is a Noetherian local ring and $\text{ht}(\mathfrak{m}/\mathfrak{p}) = 1$. If $a \in \mathfrak{m} \setminus \mathfrak{p}$, then $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$, and we have a short exact sequence

$$0 \rightarrow R/\mathfrak{p} \xrightarrow{a} R/\mathfrak{p} \rightarrow R/(\mathfrak{p} + (a)) \rightarrow 0.$$

This gives rise to a long exact sequence

$$\cdots \rightarrow \text{Ext}_R^j(R/\mathfrak{p}, M) \xrightarrow{a} \text{Ext}_R^j(R/\mathfrak{p}, M) \rightarrow \text{Ext}_R^{j+1}(R/(\mathfrak{p} + (a)), M) \rightarrow \cdots,$$

for all $j \geq 0$.

$$\mu_j(\mathfrak{p}, M) \neq 0 \implies \text{Ext}_{R_{\mathfrak{p}}}^j(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0 \implies \text{Ext}_R^j(R/\mathfrak{p}, M) \neq 0.$$

Since the Ext's are finite R -modules, Nakayama's lemma implies that $\text{Ext}_R^{j+1}(R/(\mathfrak{p} + (a)), M) \neq 0$.

Since $\sqrt{\mathfrak{p} + (a)} = \mathfrak{m}$, the R -module $R/(\mathfrak{p} + (a))$ is finite Artinian, so that it has a composition series with successive quotients isomorphic to $R/\mathfrak{m} = k$. Now, if $\text{Ext}_R^{j+1}(k, M) \neq 0$, then through the short exact sequences induced by the composition series, it would follow that $\text{Ext}_R^{j+1}(R/(\mathfrak{p} + (a)), M) = 0$, a contradiction. But since $R \setminus \mathfrak{m}$ consists of only units, we have that

$$0 \neq \text{Ext}_R^{j+1}(k, M) = \text{Ext}_{R_{\mathfrak{m}}}^{j+1}(\kappa(\mathfrak{m}), M_{\mathfrak{m}}),$$

and hence $\mu_{j+1}(\mathfrak{m}, M) \neq 0$. ■

Remark 3.9. Let R be a Noetherian ring and M a finite R -module.

(i) If $\mu_i(\mathfrak{p}, M) \neq 0$, then for all primes $\mathfrak{q} \supseteq \mathfrak{p}$ with $\text{ht}(\mathfrak{q}/\mathfrak{p}) = h < \infty$, $\mu_{i+h}(\mathfrak{q}, M) \neq 0$.

(ii) Since $\mu_0(\mathfrak{p}, M) \neq 0$ if and only if $\mathfrak{p} \in \text{Ass}_R(M)$, using (i) and Remark 3.7, we conclude that

$$\text{inj dim}_R M \geq \sup \{\dim R/\mathfrak{p} : \mathfrak{p} \in \text{Ass}_R(M)\} = \dim M.$$

(iii) If (R, \mathfrak{m}, k, E) is a Noetherian local ring with $0 \rightarrow M \rightarrow E^\bullet$ as the minimal injective resolution. If $E^n \neq 0$ and $E^j = 0$ for all $j > n$, then we must have that

$$\mu_n(\mathfrak{p}, M) \neq 0 \iff \mathfrak{p} = \mathfrak{m}.$$

In particular, $E^n = E^{\mu_j(\mathfrak{m}, M)}$.

Corollary 3.10. Let (R, \mathfrak{m}, k) be a Noetherian local ring and M a finite R -module. Then

$$\text{inj dim}_R M = \infty \iff \mu_j(\mathfrak{m}, M) \neq 0 \text{ for infinitely many } j \geq 0.$$

Proof. Let $0 \rightarrow M \rightarrow E^\bullet$ denote the minimal injective resolution. Since $\mu_j(\mathfrak{m}, M) = \dim_k \text{Ext}_R^j(k, M)$, it is clear that if the supremum on the right hand side is infinite, then so is the length of the minimal injective resolution, which is the injective dimension of M .

Conversely, if $\text{inj dim}_R M = \infty$, then $E^j \neq 0$ for infinitely many $j \geq 0$. We claim that for every integer $N \geq 0$, there is a $j \geq N$ with $\mu_j(\mathfrak{m}, M) \neq 0$. Indeed, there is an index $i \geq N$ with $E^i \neq 0$. Choose $\mathfrak{p} \in \text{Spec}(R)$ with $\mu_i(\mathfrak{p}, M) \neq 0$. Using Lemma 3.8, setting $j = i + \text{ht}(\mathfrak{m}/\mathfrak{p})$, we must have that $\mu_j(\mathfrak{m}, M) \neq 0$, as desired. ■

Theorem 3.11. Let (R, \mathfrak{m}, k) be a Noetherian local ring and M a finite R -module. Then

$$\text{inj dim}_R M = \sup \left\{ j : \text{Ext}_R^j(k, M) \neq 0 \right\}.$$

Proof. If $\text{inj dim}_R M = \infty$, then due to Corollary 3.10, $\text{Ext}_R^j(k, M) \neq 0$ for infinitely many $j \geq 0$, so that the supremum on the right hand side is infinite.

Suppose now that $\text{inj dim}_R M = n < \infty$. Clearly, $\text{Ext}_R^j(k, M) = 0$ for $j > n$ and hence,

$$\sup \left\{ j : \text{Ext}_R^j(k, M) \neq 0 \right\} \leq n = \text{inj dim}_R M.$$

Let $0 \rightarrow M \rightarrow E^\bullet$ denote the minimal injective resolution. Due to Remark 3.9 (ii), we know that $\text{Ext}_R^n(k, M) \neq 0$, and hence,

$$\sup \left\{ j : \text{Ext}_R^j(k, M) \neq 0 \right\} = n = \text{inj dim}_R M,$$

as desired. ■

Corollary 3.12. Let (R, \mathfrak{m}, k) be a regular local ring. If M is a finite R -module, then $\text{inj dim}_R M < \infty$.

Proof. Since R is regular local, $\text{proj dim}_R k < \infty$ and hence for any finite R -module M , $\text{Ext}_R^j(k, M) = 0$ for $j \gg 0$. It follows from Theorem 3.11 that $\text{inj dim}_R M < \infty$. ■

Corollary 3.13. Let (R, \mathfrak{m}, k) be a Noetherian local ring. Then $\text{inj dim}_R k < \infty$ if and only if R is a regular local ring.

Proof. If $\text{inj dim}_R k < \infty$, then $\text{Ext}_R^j(k, k) = 0$ for $j \gg 0$. Hence, the Betti numbers $\beta_j(k) = \dim_k \text{Ext}_R^j(k, k) = 0$ for $j \gg 0$, whence $\text{proj dim}_R k < \infty$, that is, R is a regular local ring.

Conversely, if R is a regular local ring, then $\text{proj dim}_R k < \infty$, so that $\text{Ext}_R^j(k, k) = 0$ for $j \gg 0$, consequently, $\text{inj dim}_R k < \infty$. ■

§4 Gorenstein Rings

§§ Modules of finite injective dimension

Definition 4.1. A Noetherian local ring (R, \mathfrak{m}, k) is said to be a *Gorenstein local ring* if $\text{inj dim}_R R < \infty$.

Proposition 4.2. If (R, \mathfrak{m}, k) is a Gorenstein local ring and $\mathfrak{p} \in \text{Spec}(R)$, then $R_{\mathfrak{p}}$ is a Gorenstein local ring.

Proof. Since $\text{inj dim}_R R < \infty$, the minimal injective resolution of R is finite, say of length n :

$$0 \rightarrow R \rightarrow E^0 \rightarrow \cdots \rightarrow E^n \rightarrow 0.$$

Localizing at \mathfrak{p} , one obtains a finite injective resolution of $R_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$ -module. Thus $R_{\mathfrak{p}}$ is a Gorenstein local ring. ■

This allows us to make the following

Definition 4.3. A Noetherian ring R is said to be *Gorenstein* if $R_{\mathfrak{p}}$ is a Gorenstein local ring for all $\mathfrak{p} \in \text{Spec}(R)$.

Due to Proposition 4.2, every Gorenstein local ring is a Gorenstein ring.

Proposition 4.4. A regular ring is Gorenstein.

Proof. It suffices to show this in the local case. Let (R, \mathfrak{m}, k) be a regular local ring. Then $\text{gl dim } R = \text{proj dim}_R k < \infty$. This means that $\text{Ext}_R^j(k, M) = 0$ for $j \gg 0$; which due to Theorem 3.11 implies $\text{inj dim}_R M < \infty$ for each finite R -module M . In particular, $\text{inj dim}_R R < \infty$, whence R is a Gorenstein local ring, as desired. ■

Remark 4.5. Note that if R is a Noetherian ring such that $\text{inj dim}_R R < \infty$, then

$$\text{inj dim}_{R_{\mathfrak{p}}} R_{\mathfrak{p}} \leq \text{inj dim}_R R < \infty,$$

so that R is a Gorenstein ring. **What about the converse?**

Theorem 4.6 (Ischebeck's Formula). Let (R, \mathfrak{m}, k) be a Noetherian local ring and M, N be finite R -modules. If $\text{inj dim}_R N < \infty$, then

$$\text{inj dim}_R N = \text{depth } M + \sup \{i: \text{Ext}_R^i(M, N) \neq 0\}.$$

Proof. Due to Theorem 3.11, we know that Ischebeck's formula is true for $M = k$. Next, we prove this by induction on $\text{depth } M$.

Suppose first that $\text{depth } M = 0$. Then $\mathfrak{m} \in \text{Ass}_R(M)$, and hence there is a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow C \longrightarrow 0.$$

Let $t = \text{inj dim}_R N$ and consider the long exact sequence induced:

$$\cdots \rightarrow \text{Ext}_R^t(C, N) \rightarrow \text{Ext}_R^t(M, N) \rightarrow \text{Ext}_R^t(k, N) \rightarrow \text{Ext}_R^{t+1}(C, N) = 0.$$

Due to Theorem 3.11, $\text{Ext}_R^t(k, N) \neq 0$, and hence $\text{Ext}_R^t(M, N) \neq 0$ since it surjects onto the former. It follows that $\sup \{i: \text{Ext}_R^i(M, N) \neq 0\} = t = \text{inj dim}_R N$. This shows that Ischebeck's formula holds when $\text{depth } M = 0$.

Suppose now that $\text{depth } M > 0$. Let $a \in \mathfrak{m}$ be a non-zero-divisor on M ; this gives a short exact sequence

$$0 \rightarrow M \xrightarrow{a} M \rightarrow \overline{M} \rightarrow 0,$$

where $\overline{M} = M/aM$. Set $t = \text{inj dim}_R N$ and $d = \text{depth } M > 0$. Then $\text{depth } \overline{M} = d - 1$. The induction hypothesis gives

$$\sup \{i: \text{Ext}_R^i(\overline{M}, N) \neq 0\} = t - d + 1.$$

The short exact sequence above gives a long exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(M, N) \xrightarrow{a} \text{Ext}_R^i(M, N) \rightarrow \text{Ext}_R^{i+1}(\overline{M}, N) \rightarrow \text{Ext}_R^{i+1}(M, N) \rightarrow \cdots.$$

If $i > t - d$, then $\text{Ext}_R^{i+1}(\overline{M}, N) = 0$, and due to Nakayama's lemma, $\text{Ext}_R^i(M, N) = 0$. On the other hand, for $i = t - d$, $\text{Ext}_R^{i+1}(\overline{M}, N) \neq 0$ but $\text{Ext}_R^{i+1}(M, N) = 0$. Thus $\text{Ext}_R^i(M, N)$ surjects onto a non-zero module, whence it must be non-zero too. We have shown

$$\sup \{i: \text{Ext}_R^i(M, N) \neq 0\} = t - d = \text{inj dim}_R N - \text{depth } M,$$

as desired. ■

Corollary 4.7. Let (R, \mathfrak{m}, k) be a Noetherian local ring and M a finite R -module. If $\text{inj dim}_R M < \infty$, then $\text{inj dim}_R M = \text{depth } R$.

Proof. Using Ischebeck's formula,

$$\text{inj dim}_R M = \text{depth } R + \sup \{i: \text{Ext}_R^i(R, M) \neq 0\} = \text{depth } R,$$

as desired. ■

Corollary 4.8. A Gorenstein ring is Cohen-Macaulay.

Proof. It suffices to prove this in the local case (R, \mathfrak{m}, k) . Due to Corollary 4.7, $\text{inj dim}_R R = \text{depth } R$. But due to Remark 3.9 (ii), $\text{inj dim}_R R \geq \dim R$. It follows that $\text{depth } R = \dim R$ and hence R is Cohen-Macaulay. ■

Corollary 4.9. A Gorenstein Artinian local ring is self-injective.

Proof. Due to Corollary 4.7, $\text{inj dim}_R R = \text{depth } R = 0$. ■

Definition 4.10. Let (R, \mathfrak{m}, k) be a Noetherian local ring, M a Cohen-Macaulay R -module, and $\underline{a} = a_1, \dots, a_s \in \mathfrak{m}$ a maximal M -sequence. Then $M/\underline{a}M$ is Artinian and we can define

$$\text{type}(M) = \dim_k (\text{Soc}_R(M/\underline{a}M)).$$

We must argue that this definition is independent of the chosen maximal M -sequence. We begin with a

Lemma 4.11. Let R be a ring, M and N be R -modules, and $a \in \text{Ann}_R(M)$ be a non-zero-divisor on N . Then

$$\text{Ext}_R^{j+1}(M, N) \cong \text{Ext}_R^j(M, N/aN) \quad \forall j \geq 0.$$

Proof. Consider the short exact sequence

$$0 \rightarrow N \xrightarrow{\cdot a} N \rightarrow N/aN \rightarrow 0.$$

This gives rise to a long exact sequence

$$\cdots \rightarrow \text{Ext}_R^j(M, N) \xrightarrow{\cdot a} \text{Ext}_R^j(M, N) \rightarrow \text{Ext}_R^j(M, N/aN) \rightarrow \text{Ext}_R^{j+1}(M, N) \xrightarrow{\cdot a} \text{Ext}_R^{j+1}(M, N) \rightarrow \cdots.$$

Since a annihilates M , both the above “multiplication by a ” maps have zero image. In particular, this gives an exact sequence

$$0 \rightarrow \text{Ext}_R^j(M, N/aN) \rightarrow \text{Ext}_R^{j+1}(M, N) \rightarrow 0,$$

as desired. ■

We return to the setup of Definition 4.10. Using the above Lemma, we have

$$\text{Soc}_R(M/\underline{a}M) \cong \text{Hom}_R(R/\mathfrak{m}, M/\underline{a}M) \cong \text{Ext}_R^0(k, M) \cong \text{Ext}_R^s(k, M).$$

This characterization is independent of the maximal regular sequence, as desired.

Interlude 4.12 (Constructing the minimal injective resolution of M/aM over R/aR).

Let R be a Noetherian ring, M a finite R -module, and $a \in R$ a non-zero-divisor on both M and R . Let $0 \rightarrow M \rightarrow E^\bullet$ be the minimal injective resolution of M over R . Set $\bar{R} = R/aR$ and $\bar{M} = M/aM$. Consider the short exact sequence

$$0 \rightarrow R \xrightarrow{\cdot a} R \rightarrow \bar{R} \rightarrow 0$$

of R -modules. Then $\text{proj dim}_R \bar{R} \leq 1$ so that $\text{Ext}_R^j(\bar{R}, M) = 0$ for all $j > 1$. The above sequence also gives

$$0 \rightarrow \text{Hom}_R(\bar{R}, M) \rightarrow \text{Hom}_R(R, M) \xrightarrow{\cdot a} \text{Hom}_R(R, M) \rightarrow \text{Ext}_R^1(\bar{R}, M) \rightarrow 0.$$

It follows that $\text{Ext}_R^1(\bar{R}, M) \cong \bar{M}$.

Now, consider the complex

$$0 \rightarrow \underbrace{\text{Hom}_R(\bar{R}, M)}_{=0} \rightarrow \text{Hom}_R(\bar{R}, E^0) \rightarrow \text{Hom}_R(\bar{R}, E^1) \rightarrow \text{Hom}_R(\bar{R}, E^2) \rightarrow \cdots.$$

Since $\text{Ext}_R^j(\bar{R}, M) = 0$ for $j \geq 2$, the above complex is exact at $\text{Hom}_R(\bar{R}, E^j)$ for $j \geq 2$. Further, since $\text{Ass}_R(M) = \text{Ass}_R(E^0)$, it follows that a is a non-zero-divisor on E^0 , so that $\text{Hom}_R(\bar{R}, E^0) = 0$. Therefore,

$$\ker(\text{Hom}_R(\bar{R}, E^1) \rightarrow \text{Hom}_R(\bar{R}, E^2)) \cong \text{Ext}_R^1(\bar{R}, M) \cong \bar{M}.$$

Set $\bar{\mu}^j = \text{Hom}_R(\bar{R}, E^j)$. Then $\bar{\mu}^j$ is an injective \bar{R} -module and

$$0 \rightarrow \bar{M} \rightarrow \bar{\mu}^1 \rightarrow \bar{\mu}^2 \rightarrow \dots$$

is an injective resolution of \bar{M} over \bar{R} .

Finally, we claim that the above resolution is the minimal resolution of \bar{M} over \bar{R} . Let $\bar{\mathfrak{p}}$ be a prime in \bar{R} . We must show that the map

$$\text{Hom}_{\bar{R}_{\bar{\mathfrak{p}}}}(\kappa(\bar{\mathfrak{p}}), \bar{\mu}_{\bar{\mathfrak{p}}}^j) \rightarrow \text{Hom}_{\bar{R}_{\bar{\mathfrak{p}}}}(\kappa(\bar{\mathfrak{p}}), \bar{\mu}_{\bar{\mathfrak{p}}}^{j+1})$$

is the zero map. But note that the above is the localization of the map

$$\text{Hom}_{\bar{R}}(\bar{R}/\bar{\mathfrak{p}}, \bar{\mu}^j) \rightarrow \text{Hom}_{\bar{R}}(\bar{R}/\bar{\mathfrak{p}}, \bar{\mu}^{j+1}),$$

which, due to the Hom-Tensor adjunction is canonically isomorphic to

$$\text{Hom}_R(\bar{R}/\bar{\mathfrak{p}} \otimes_{\bar{R}} \bar{R}, \bar{\mu}^j) \rightarrow \text{Hom}_R(\bar{R}/\bar{\mathfrak{p}} \otimes_{\bar{R}} \bar{R}, \bar{\mu}^{j+1}).$$

Finally, since $\bar{R}/\bar{\mathfrak{p}}$ is the same as R/\mathfrak{p} as R -modules, the above map is the same as

$$\text{Hom}_R(R/\mathfrak{p}, E^j) \rightarrow \text{Hom}_R(R/\mathfrak{p}, E^{j+1}).$$

But it is known that this map is identically zero when localized at \mathfrak{p} , as desired.

Theorem 4.13. Let (R, \mathfrak{m}, k, E) be a Noetherian local ring, M a finite R -module, and $a \in R$ a non-zero-divisor on both M and R . Set $\bar{M} = M/aM$ and $\bar{R} = R/aR$. Then

$$\text{inj dim}_R M < \infty \iff \text{inj dim}_{\bar{R}} \bar{M} < \infty.$$

In this case, $\text{inj dim}_R M = \text{inj dim}_{\bar{R}} \bar{M} + 1$.

Proof. Suppose first that $\text{inj dim}_R M < \infty$. It is clear from Interlude 4.12 that $\text{inj dim}_{\bar{R}} \bar{M} < \infty$ and $\text{inj dim}_{\bar{R}} \bar{M} = \text{inj dim}_R M - 1$.

On the other hand, if $\text{inj dim}_R M = \infty$, then $\mu_j(\mathfrak{m}, M) \neq 0$ for infinitely many $j \geq 0$. But recall that $\text{Hom}_R(\bar{R}, E) = E_{\bar{R}}(k)$. Hence, if $E \mid E^j$ for some $j \geq 0$, then $E_{\bar{R}}(k) \mid \bar{\mu}^j$. That is, for $j \geq 1$,

$$\mu_j(\mathfrak{m}, M) \neq 0 \implies \mu_{j-1}(\bar{\mathfrak{m}}, \bar{M}) \neq 0.$$

Hence, $\text{inj dim}_{\bar{R}} \bar{M} = \infty$. This completes the proof. ■

Corollary 4.14. Let (R, \mathfrak{m}, k) be a Noetherian local ring and $a \in R$ a non-zero-divisor. Then R is Gorenstein if and only if R/aR is Gorenstein. ■

Proposition 4.15. Let (R, \mathfrak{m}, k) be a Noetherian local ring. Then the following are equivalent:

- (1) R is Gorenstein
- (2) R is Cohen-Macaulay and $\text{type}(R) = 1$.

Proof. Let $\underline{a} = a_1, \dots, a_s \in \mathfrak{m}$ be a maximal R -sequence. Due to Corollary 4.14, it suffices to show the equivalence for $R/\underline{a}R$. So R is a depth zero Noetherian local ring. Clearly if R is Gorenstein, then it is self-injective and hence $\text{type}(R) = 1$. Conversely, if R is Cohen-Macaulay, then $\dim R = \text{depth } R = 0$, so that R is an Artinian local ring with $\text{type}(R) = 1$, whence R is self-injective, in particular, Gorenstein. ■

§§ A closer look at the Artinian case

Theorem 4.16. Let (R, \mathfrak{m}, k, E) be an Artinian local ring. Then the following are equivalent:

- (1) $\text{idim}_R(R) < \infty$,
- (2) R is self-injective,
- (3) $R \cong E$ as R -modules,
- (4) The ideal $(0) \triangleleft R$ is irreducible,
- (5) $\dim_k(\text{Soc}_R(R)) = 1$,
- (6) for all ideals $I \subseteq R$, $(0 :_R (0 :_R I)) = I$.

Proof. (3) \implies (2) \implies (1) is clear. The implication (1) \implies (3) follows from Corollary 4.7 so that R is self-injective, and hence, due to Theorem 2.10, $R \cong E$ as R -modules.

(3) \implies (6) is a consequence of Theorem 2.11.

(6) \implies (5) If $0 \neq a \in \text{Soc}_R(R)$, then $\text{Ann}_R(a) = \mathfrak{m}$. As a result,

$$\text{Soc}_R(R) = (0 :_R \mathfrak{m}) = (0 :_R (0 :_R a)) = (a),$$

whence $\dim_k(\text{Soc}_R(R)) = 1$.

(5) \implies (3) is again a consequence of Theorem 2.10.

(5) \implies (4) If $0 \neq I$ is any ideal of R , then $I \cap \text{Soc}_R(R) \neq 0$, and hence, $\text{Soc}_R(R) \subseteq I$, since the former is a simple R -module. In particular, this means that the intersection of two non-trivial ideals of R must contain the socle, and hence, must be non-zero; i.e., (0) is an irreducible ideal.

(4) \implies (5) If $\dim_k(\text{Soc}_R(R)) \neq 1$, then $\dim_k(\text{Soc}_R(R)) \geq 2^3$. Let $a, b \in \text{Soc}_R(R)$ be linearly independent over k . Then $(a) = ka$ and $(b) = kb$. Thus $(a) \cap (b) = (0)$, i.e., (0) is not an irreducible ideal, a contradiction. ■

Proposition 4.17. The following are equivalent to the (equivalent) conditions of Theorem 4.16:

- (7) R has a unique minimal non-zero ideal,
- (8) $\text{proj dim}_R E < \infty$,
- (9) E is free,
- (10) E is cyclic,
- (11) Given any submodule $W \subseteq E$, $(0 :_R (0 :_R W)) \cong W$,
- (12) E has a unique maximal proper submodule.

Proof. (7) \implies (4) Let $\mathfrak{a} \triangleleft R$ be the unique non-zero minimal ideal. Let $I \triangleleft R$ be a non-zero ideal. Since R is Artinian, I contains a minimal non-zero ideal, say \mathfrak{b} , which, due to uniqueness, must be equal to \mathfrak{a} . Hence, every non-zero ideal of R contains \mathfrak{a} . It follows that (0) is an irreducible ideal.

(5) \implies (7) It is clear that $\text{Soc}_R(R)$ is a minimal ideal. Further, since $\text{Soc}_R(R) \subseteq R$ is essential, the socle must be contained in every non-zero ideal as was argued in the preceding proof.

(8) \implies (1) follows by taking a finite projective (and hence free) resolution of E and then taking its Matlis dual, which gives a finite injective resolution of R .

(1) \implies (8) follows similarly by taking a finite injective resolution of R and then taking its Matlis dual.

(9) \implies (3) Suppose $E \cong R^{\oplus d}$. Then

$$\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(E) = d\lambda_R(R) \implies d = 1.$$

Thus $R \cong E$. The implication (3) \implies (9) is clear.

(10) \implies (3) If $E \cong R/I$, then

$$\lambda_R(R) - \lambda_R(I) = \lambda_R(E) = \lambda_R(R^\vee) = \lambda(R),$$

whence $\lambda_R(I) = 0$, i.e., $I = 0$. Thus $R \cong E$. The converse (3) \implies (10) is once again clear.

Through Theorem 2.11, the equivalence (7) \iff (12) is clear, thereby completing the proof. ■

³Since $\mathfrak{m} \in \text{Ass}_R(R)$.

§§ Fibres of a flat map

Theorem 4.18. Let $\varphi : (R, \mathfrak{m}_R, k) \rightarrow (S, \mathfrak{m}_S, \ell)$ be a flat map with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. Then

- (1) $\dim R + \dim S/\mathfrak{m}_R S = \dim S$.
- (2) if $\mathfrak{m}_R S = \mathfrak{m}_S$, then for any R -module M of finite length, $\lambda_R(M) = \lambda_S(M \otimes_R S)$.
- (3) if $\underline{a} = a_1, \dots, a_n \in \mathfrak{m}_S$ is $S/\mathfrak{m}_R S$ -regular, then a_1, \dots, a_n is S -regular and $R \rightarrow S/(\underline{a})$ is flat.
- (4) $\text{depth } R + \text{depth } S/\mathfrak{m}_R S = \text{depth } S$.
- (5) S is Cohen-Macaulay if and only if R and $S/\mathfrak{m}_R S$ are so.
- (6) S is Gorenstein if and only if R and $S/\mathfrak{m}_R S$ are so.

Proof. (1) Induct on $\dim R$. If R is Artinian, then $\mathfrak{m}_R S$ is nilpotent and hence $\mathfrak{m}_R S \subseteq \mathfrak{N}(S)$, so that $\dim S/\mathfrak{m}_R S = \dim S$. This prove the assertion when $\dim R = 0$.

Suppose now that $\dim R > 0$ and let $\mathfrak{N} = \mathfrak{N}(R)$. The map $R/\mathfrak{N} \rightarrow S/\mathfrak{N}S$ is flat and $\mathfrak{N}S \subseteq \mathfrak{N}(S)$, so that

$$\dim R = \dim R/\mathfrak{N} \quad \text{and} \quad \dim S = \dim S/\mathfrak{N}S.$$

Replacing R and S by R/\mathfrak{N} and $S/\mathfrak{N}S$ respectively, we may assume that R is reduced. In particular, this means that the zero ideal in R is the intersection of all its (finitely many) minimal primes. Since $\dim R > 0$, the maximal ideal is not minimal and using prime avoidance, choose a non-zero-divisor $a \in \mathfrak{m}_R$. The map $R/aR \rightarrow S/aS$ is flat, $\dim R/aR = \dim R - 1$ and the fibre of this map is still $S/\mathfrak{m}_R S$. The induction hypothesis implies

$$\dim S = \dim S/aS + 1 = \dim R/aR + \dim S/\mathfrak{m}_R S + 1 = \dim R + \dim S/\mathfrak{m}_R S.$$

- (2) Let $n = \lambda_R(M)$. If $n = 0$, then $M = 0$, and there is nothing to prove. Suppose $n > 0$. Then, there is a composition series

$$M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_n = 0,$$

giving rise to short exact sequences

$$0 \rightarrow M_{i+1} \rightarrow M_i \rightarrow k \rightarrow 0 \quad 0 \leq i \leq n-1.$$

Applying the functor $- \otimes_R S$ and using the fact that $\mathfrak{m}_R S = \mathfrak{m}_S$ so that $k \otimes_R S = \ell$ (as S -modules), we obtain

$$0 \rightarrow M_{i+1} \otimes_R S \rightarrow M_i \otimes_R S \rightarrow \ell \rightarrow 0,$$

therefore, $\lambda_S(M \otimes_R S) = \lambda_R(M)$.

- (3) It is clear that it suffices to prove the assertion for $n = 1$. Let $a \in \mathfrak{m}_S$ be $S/\mathfrak{m}_R S$ -regular. We must show that a is S -regular and $R \rightarrow S/aS$ is flat.

Set $d_n = \dim_k \mathfrak{m}_R^n / \mathfrak{m}_R^{n+1}$. There are short exact sequences

$$0 \rightarrow \mathfrak{m}_R^n / \mathfrak{m}_R^{n+1} \rightarrow R/\mathfrak{m}_R^{n+1} \rightarrow R/\mathfrak{m}_R^n \rightarrow 0.$$

Applying $- \otimes_R S$, we obtain short exact sequences

$$0 \rightarrow (S/\mathfrak{m}_R S)^{\oplus d_n} \rightarrow S/\mathfrak{m}_R^{n+1} S \rightarrow S/\mathfrak{m}_R^n S \rightarrow 0$$

of S -modules. Inducting on n , it is easy to show that a is a non-zero-divisor on $S/\mathfrak{m}_R^n S$ for all $n \geq 1$ ⁴. Suppose a is a zero-divisor on S , then there exists $0 \neq s \in S$ such that $as = 0$. By Krull's Intersection Theorem,

$$\bigcap_{n \geq 1} \mathfrak{m}_R^n S \subseteq \bigcap_{n \geq 1} \mathfrak{m}_S^n = 0,$$

⁴If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of modules and a is a non-zero-divisor on M' and M'' , then it is a non-zero-divisor on M . This follows from the fact that $\text{Ass}_R(M) \subseteq \text{Ass}_R(M') \cup \text{Ass}_R(M'')$.

consequently, there is some $n \geq 1$ such that $s \notin \mathfrak{m}_R^n S$. In particular, $0 \neq \bar{s} \in S/\mathfrak{m}_R^n S$. But since a is a non-zero-divisor on S , we cannot have $a\bar{s} = 0$, a contradiction. Thus a is a non-zero-divisor on S .

It remains to show that $R \rightarrow S/aS$ is flat, i.e., we must show that

$$\mathrm{Tor}_1^R(M, S/aS) = 0 \quad \text{for all } R\text{-modules } M,$$

equivalently (using standard reduction techniques), since R is Noetherian, it suffices to show that

$$\mathrm{Tor}_1^R(R/\mathfrak{p}, S/aS) = 0 \quad \text{for all } \mathfrak{p} \in \mathrm{Spec}(R).$$

Set $\bar{R} = R/\mathfrak{p}$ and $\bar{S} = S/\mathfrak{p}S$ and note that $\bar{R} \rightarrow \bar{S}$ is flat and a morphism of local rings, further $\bar{S}/\mathfrak{m}_{\bar{R}}\bar{S} \cong S/\mathfrak{m}_R S$ as rings and S -modules. Now, a is a non-zero-divisor on $S/\mathfrak{m}_R S$, and hence is a non-zero-divisor on $\bar{S}/\mathfrak{m}_{\bar{R}}\bar{S}$. Thus, $\bar{s} \in \bar{S}$ is a non-zero-divisor on $\bar{S}/\mathfrak{m}_{\bar{R}}\bar{S}$ and because of what we have argued in the preceding paragraph, $a\bar{s} \in \bar{S}$ is a non-zero-divisor.

Consider the short exact sequence

$$0 \rightarrow S \xrightarrow{a} S \rightarrow S/aS \rightarrow 0,$$

and applying $- \otimes_R R/\mathfrak{p}$, we obtain

$$0 = \mathrm{Tor}_1^R(S, R/\mathfrak{p}) \rightarrow \mathrm{Tor}_1^R(S/aS, R/\mathfrak{p}) \rightarrow \bar{S} \xrightarrow{a} \bar{S} \rightarrow S/aS \otimes_R R/\mathfrak{p} \rightarrow 0.$$

Thus $\mathrm{Tor}_1^R(S/aS, R/\mathfrak{p}) = 0$, as desired. This completes the proof of (3).

- (4) Let $\underline{a} = a_1, \dots, a_s \in \mathfrak{m}_R$ be a maximal R -sequence. Since $R \rightarrow S$ is flat, \underline{a} is an S -sequence, therefore, replacing R and S by $R/\underline{a}R$ and $S/\underline{a}S$, we may assume $\mathrm{depth}_R R = 0$. Note that the fibre of the map does not change during this reduction. Now, let $\underline{b} = b_1, \dots, b_r \in S$ be $S/\mathfrak{m}_R S$ -regular. Using (3), we know that \underline{b} is S -regular and the map $R \rightarrow S/\underline{b}S$ is flat. Let $\bar{S} = S/\underline{b}S$, then

$$\mathrm{depth}_S S - \mathrm{depth}_S S/\mathfrak{m}_R S = \mathrm{depth} \bar{S} - \mathrm{depth} \bar{S}/\mathfrak{m}_{\bar{R}}\bar{S},$$

and hence, we may replace S by $S/\underline{b}S$ and assume that $\mathrm{depth}_R R = \mathrm{depth}_S S/\mathfrak{m}_R S = 0$. It remains to show that $\mathrm{depth}_S S = 0$ in this situation.

Since $\mathrm{depth}_R R = 0$, there is an injection $S/\mathfrak{m}_R \hookrightarrow R$, which upon tensoring with S and using flatness, gives an injection $S/\mathfrak{m}_R S \hookrightarrow S$ as S -modules. But $\mathrm{depth}_S S/\mathfrak{m}_R S = 0$ implies there is an injection $S/\mathfrak{m}_S \hookrightarrow S/\mathfrak{m}_R S$, and hence there is an injection $S/\mathfrak{m}_S \hookrightarrow S$. Thus $\mathfrak{m}_S \in \mathrm{Ass}_S(S)$, i.e., $\mathrm{depth}_S S = 0$, as desired.

- (5) Immediate from (1) and (4).

- (6) In light of (5), we can assume that S , R , and $S/\mathfrak{m}_R S$ are all Cohen-Macaulay. Let $\underline{a} \in R$ be a maximal R -sequence. Replacing R and S by $R/\underline{a}R$ and $S/\underline{a}S$, we may assume that R is Artinian (recall that $\mathrm{depth} = \dim$ for Cohen-Macaulay rings). Now let $\underline{b} = b_1, \dots, b_s \in \mathfrak{m}_S$ be $S/\mathfrak{m}_R S$ -regular. Then due to (3), \underline{b} is S -regular and $R \rightarrow S/\underline{b}S$. Again replacing S by $S/\underline{b}S$, we may assume that $S/\mathfrak{m}_R S$ is Artinian too. Due to (1), we conclude that S is Artinian too.

We shall show that

$$\mathrm{type}(S) = \mathrm{type}(R) \mathrm{type}(S/\mathfrak{m}_R S),$$

which implies the desideratum. Let $r = \mathrm{type}(R)$, then

$$k^{\oplus r} = \mathrm{Soc}_R(R) \cong \mathrm{Hom}_R(R/\mathfrak{m}_R, R).$$

Applying $- \otimes_R S$, we obtain⁵

$$(S/\mathfrak{m}_R S)^{\oplus r} \cong \mathrm{Hom}_S(S/\mathfrak{m}_R S, S) \cong (0 :_S \mathfrak{m}_R S).$$

⁵If M is a finitely presented R -module and N is any R -module, then for every flat algebra $R \rightarrow S$, there is a canonical isomorphism

$$\mathrm{Hom}_R(M, N) \otimes_R S \cong \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S).$$

Observe that $\text{Soc}_S(S) \subseteq (0 :_S \mathfrak{m}_R S)$, and hence

$$\text{Soc}_S(S) \subseteq \text{Soc}_S((0 :_S \mathfrak{m}_R S)) \subseteq \text{Soc}_S(S),$$

consequently,

$$\text{Soc}_S(S/\mathfrak{m}_R S)^{\oplus r} \cong \text{Soc}_S((S/\mathfrak{m}_R S)^{\oplus r}) \cong \text{Soc}_S(S)$$

therefore,

$$r \text{ type}(S/\mathfrak{m}_R S) = \text{type}(S),$$

as desired. ■

Corollary 4.19. Let (R, \mathfrak{m}, k) be a Noetherian local ring. Then R is Gorenstein if and only if \widehat{R} is so.

Proof. This follows immediately from Theorem 4.18 (6), since $\widehat{R}/\widehat{\mathfrak{m}}\widehat{R} \cong k$, which is always Gorenstein. ■

Corollary 4.20. If R is Gorenstein, then so is $R[X]$.

Proof. Let $\mathfrak{P} \in \text{Spec}(R[X])$ and set $\mathfrak{p} = R \cap \mathfrak{P}$. It is clear that $(R[X])_{\mathfrak{P}} \cong (R_{\mathfrak{p}}[X])_{\mathfrak{P}}$, and hence we can assume that (R, \mathfrak{m}, k) is a Noetherian local ring and $\mathfrak{P} \cap R = \mathfrak{m}$. The map $R \rightarrow (R[X])_{\mathfrak{P}}$ is a flat local homomorphism, since a composition of flat maps is flat. Thus, it suffices to show that

$$\frac{(R[X])_{\mathfrak{P}}}{\mathfrak{m}(R[X])_{\mathfrak{P}}}$$

is a Gorenstein ring. But the above is isomorphic to

$$(R[X]/\mathfrak{m}[X])_{\mathfrak{P}} \cong (k[X])_{\mathfrak{P}},$$

which is obviously Gorenstein. This completes the proof. ■

§§ Canonical Module

Lemma 4.21 (Depth Lemma). Let R be a Noetherian ring, $I \triangleleft R$ a proper ideal and $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ a short exact sequence of finite R -modules such that $IM \neq M' \neq M''$, $IM \neq M$, and $IM'' \neq M''$. Let $m = \text{depth}(I, M)$, $m' = \text{depth}(I, M')$, and $m'' = \text{depth}(I, M'')$.

$$(1) \quad m < m'' \implies m' = m.$$

$$(2) \quad m > m'' \implies m' = m'' + 1.$$

$$(3) \quad m = m'' \implies m' \geq m.$$

Proof. Let T^i denote the functor $\text{Ext}_R^i(R/I, -)$. We shall use Rees's characterization of depth as

$$\text{depth}(I, N) = \sup \{i : \text{Ext}_R^i(R/I, N) \neq 0\},$$

where N is a finite R -module. The given short exact sequence gives rise to a long exact sequence

$$\cdots \rightarrow T^{i-1}(M'') \rightarrow T^i(M') \rightarrow T^i(M) \rightarrow T^i(M'') \rightarrow T^{i+1}(M') \rightarrow \cdots,$$

with the convention that $T^i = 0$ for $i < 0$.

- (1) Let $i < m$, then a part of the long exact sequence looks like $T^{i-1}(M'') \rightarrow T^i(M') \rightarrow T^i(M)$ where both $T^{i-1}(M'')$ and $T^i(M)$ are zero, so that $T^i(M') = 0$. If $i = m$, then we have an exact sequence

$$0 = T^{m-1}(M'') \rightarrow T^m(M') \rightarrow T^m(M) \rightarrow T^m(M'') = 0,$$

whence $T^m(M') \cong T^m(M) \neq 0$, i.e., $m' = m$.

- (2) Let $i \leq m'' < m$. Then the exact sequence $0 = T^{i-1}(M'') \rightarrow T^i(M') \rightarrow T^i(M) = 0$ gives $T^i(M') = 0$. On the other hand, there is an exact sequence $0 = T^{m''}(M) \rightarrow T^{m''}(M'') \rightarrow T^{m''+1}(M')$ and since $T^{m''}(M'') \neq 0$ we must have $T^{m''+1}(M') \neq 0$, i.e., $m' = m'' + 1$.
- (3) Let $i < m = m''$, then the exact sequence $0 T^{i-1}(M'') \rightarrow T^i(M') \rightarrow T^i(M) = 0$ gives $T^i(M') = 0$, i.e., $m' \geq m$, thereby completing the proof \blacksquare

Theorem 4.22. Let (S, \mathfrak{m}, k) be a Gorenstein local ring with $d = \dim S$. Let $\text{CM}_S(i)$ denote the class of Cohen-Macaulay S -modules of dimension i . Then for $M \in \text{CM}_S(i)$,

- (1) $\text{Ext}_S^j(M, S) = 0$ for $j \neq d - i$.
- (2) $\text{Ext}_S^{d-i}(M, S) \in \text{CM}_S(i)$.
- (3) $\text{Ext}_S^{d-i}(\text{Ext}_S^{d-i}(M, S), S) \cong M$ as S -modules.

Proof. Since $M \in \text{CM}_S(i)$,

$$i = \dim(S / \text{Ann}_S(M)) = \dim S - \text{ht Ann}_S(M) \implies \text{ht}(\text{Ann}_S(M)) = d - i,$$

where we used the fact that Gorenstein rings are Cohen-Macaulay. Again, since S is Cohen-Macaulay, there is an S -regular sequence $a_1, \dots, a_{d-i} \in \text{Ann}_S(M)$. A standard “Ext-shifting” argument then gives that for $j < d - i$,

$$\text{Ext}_S^j(M, S) \cong \text{Hom}_S(M, S/(a_1, \dots, a_j)S) = 0,$$

since a_{j+1} is a non-zerodivisor on $S/(a_1, \dots, a_j)S$ but annihilates M . Further, we also have that

$$\text{Ext}_S^{d-i}(M, S) \cong \text{Hom}_{\bar{S}}(M, \bar{S}),$$

where $\bar{S} = S/(a_1, \dots, a_{d-i})S$.

Now, since $\text{inj dim}_S S = d < \infty$, using Theorem 4.6, we have

$$\text{depth}_S M + \sup \{j : \text{Ext}_S^j(M, S) \neq 0\} = d,$$

so that $\text{Ext}_S^j(M, S) = 0$ for $j > d - i$. This completes the proof of (1).

Note that \bar{S} is Gorenstein and M is a maximal Cohen-Macaulay \bar{S} -module. Replacing S by \bar{S} , we can assume that M is a maximal Cohen-Macaulay S -module, and we would like to show that

- (2) $M^* := \text{Hom}_S(M, S) \in \text{CM}_S(d)$, and
- (3) $M^{**} \cong M$ as S -modules.

If M is free, then both the above conclusions are trivial. Suppose now that M is not free; then we must have $\text{proj dim}_S M = \infty$, else, due to the Auslander-Buchsbaum Formula, $\text{proj dim}_S M = \text{depth } S - \text{depth } M = 0$, a contradiction to our assumption that M is not free.

Let $\dots \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \rightarrow M \rightarrow 0$ be a free resolution of M and set $N = \ker \varphi_1$. This gives a short exact sequence

$$0 \rightarrow N \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Due to Lemma 4.21, since $\text{depth}_S M = \text{depth}_S F_0$,

$$\text{depth}_S N \geq \text{depth}_S M = \dim S,$$

so that N is a maximal Cohen-Macaulay S -module. Due to (1), $\text{Ext}_S^j(M, S) = 0$ for $j > 0$ and we get an exact complex

$$0 \rightarrow M^* \rightarrow F_0^* \xrightarrow{\varphi_1^*} F_1^* \rightarrow \dots$$

\blacksquare