Homology Theory

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Abstract

This is meant to be a quick review of Homology Theory. We closely follow [Rot13].

CONTENTS

Part I
Homology

§1 The Setup of Singular Homology

Definition 1.1. The *standard n-simplex* is the "convex hull" of the standard basis vectors $e_0, \ldots, e_n$ in $\mathbb{R}^{n+1}$ and is denoted by Δ^n. That is,

$$\Delta^n = \{t_0 e_0 + \cdots + t_n e_n \mid 0 \leqslant t_i \leqslant 1,\ t_0 + \cdots + t_n = 1\}.$$

An *orientation* of Δ^n is a linear ordering of its vertices. Two orientations are said to be the same if, as permutations of $e_0, \ldots, e_n$, they have the same parity.

Given an orientation of Δ^n, there is an *induced orientation* of its faces, defined by orienting the i-th face in the sense $(-1)^i[e_0, \ldots, \widehat{e}_i, \ldots, e_n]$.

For each $n \geqslant 1$ and $0 \leqslant i \leqslant n$, define the *$i$-th face map*

$$\varepsilon_i = \varepsilon_i^n : \Delta^{n-1} \to \Delta^n$$

to be the affine map taking the vertices $\{e_0, \ldots, e_{n-1}\}$ to the vertices $\{e_0, \ldots, \widehat{e}_i, \ldots, e_n\}$ preserving the displayed orderings.

Let X be a topological space. For each $n \geqslant 0$, let $S_n(X)$ denote the *free abelian group* generated by

$$\{\sigma : \Delta^n \to X \mid \sigma \text{ is continuous}\}.$$

Definition 1.2. If $\sigma : \Delta^n \to X$ is continuous, and $n > 0$, then its *boundary* is given by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon_i^n \in S_{n-1}(X).$$

If $n = 0$, define $\partial_0 \sigma = 0$. The universal property of free abelian groups allows us to define group homomorphisms $\partial_n : S_n(X) \to S_{n-1}(X)$ with the convention that $S_{-1} = 0$.

Proposition 1.3. If $0 \leqslant k < j \leqslant n+1$, the face maps satisfy

$$\varepsilon_j^{n+1} \circ \varepsilon_k^n = \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n$$

Proof. Both maps agree on the e_i's for $0 \leqslant i \leqslant n-1$. ∎

Theorem 1.4. For all $n \geqslant 0$, we have $\partial_n \circ \partial_{n+1} = 0$.

Proof. Let $\sigma : \Delta^{n+1} \to X$ be continuous and $n \geqslant 1$.

$$\begin{aligned}
\partial_n \partial_{n+1} \sigma &= \partial_n \left(\sum_{j=0}^{n+1} (-1)^j \sigma \circ \varepsilon_j^{n+1} \right) \\
&= \sum_{j=0}^{n+1} \sum_{k=0}^{n} (-1)^{j+k} \sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n \\
&= \sum_{0 \leqslant j \leqslant k \leqslant n} (-1)^{j+k} \sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n + \sum_{n+1 \geqslant j > k \geqslant 0} (-1)^{j+k} \sigma \circ \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n
\end{aligned}$$

We can change the indexing in the second sum by setting $j = p+1$ and $k = q$ to get

$$\partial_n \partial_{n+1} \sigma = \sum_{0 \leqslant j \leqslant k \leqslant n} (-1)^{j+k} \sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n + \sum_{0 \leqslant q \leqslant p \leqslant n} (-1)^{p+q+1} \sigma \circ \varepsilon_q^{n+1} \circ \varepsilon_p^n.$$

It is easy to see that the above sum is 0. This completes the proof. ∎

This gives us the *singular chain complex*,

$$\cdots \longrightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \longrightarrow \cdots S_0(X) \longrightarrow 0.$$

The homology groups of the above complex are called the *singular homology groups*, and are denoted by

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

It is customary to denote $\ker \partial_n$ by $Z_n(X)$ and $\operatorname{im} \partial_{n+1}$ by $B_n(X)$.

NOTATION. Let $f : X \to Y$ be a continuous. For every n-simplex $\sigma : \Delta^n \to X$ in X, the composition $f \circ \sigma : \Delta^n \to Y$ is an n-simplex in Y. There is a unique group homomorphism extending $\sigma \mapsto f \circ \sigma$. We denote this map by $f_\sharp : S_n(X) \to S_n(Y)$.

PROPOSITION 1.5. $f_\sharp$ is a chain map.

Proof. We must show that the following diagram commutes.

$$\begin{tikzcd}
S_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\sharp"'] & S_{n-1}(X) \arrow[d, "f_\sharp"] \\
S_n(Y) \arrow[r, "\partial_n"'] & S_{n-1}(Y)
\end{tikzcd}$$

Let $\sigma \in S_n(X)$. Then,

$$f_\sharp \partial_n \sigma = f_\sharp \left(\sum_{j=0}^{n} (-1)^j \sigma \circ \varepsilon_j^n \right) = \sum_{j=0}^{n} (-1)^j f \circ \sigma \circ \varepsilon_j^n.$$

On the other hand,

$$\partial_n f_\sharp \sigma = \partial_n (f \circ \sigma) = \sum_{j=0}^{n} (-1)^j f \circ \sigma \circ \varepsilon_j^n. \qquad \blacksquare$$

NOTATION. Therefore, $f_\sharp : S_\bullet(X) \to S_\bullet(Y)$ is a chain map and hence, induces a map on the homology groups, $H_n(f) : H_n(X) \to H_n(Y)$ given by

$$H_n(f)(\zeta + B_n(X)) = f_\sharp(\zeta) + B_n(Y),$$

for $\zeta \in Z_n(X)$.

It is not hard to see that $g_\sharp \circ f_\sharp = (g \circ f)_\sharp$, and hence, $H_n(g \circ f) = H_n(g) \circ H_n(f)$, that is, H_n is a *functor* from the category of topological spaces to the category of (abelian) groups.

§§ Homotopy Invariance

THEOREM 1.6. If X is a bounded convex subspace of Euclidean space, then $H_n(X) = 0$ for all $n \geqslant 1$.

Proof. Fix a point $b \in X$. For every n-simplex $\sigma : \Delta^n \to X$, define the "cone over σ with vertex b" to be the $n+1$-simplex $b \cdot \sigma : \Delta^{n+1} \to X$ as follows

$$(b \cdot \sigma)(t_0, \ldots, t_{n+1}) = \begin{cases} b & t_0 = 1 \\ t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right) & t_0 \neq 1. \end{cases}$$

A routine argument shows that $b \cdot \sigma$ is continuous.

Define $c_n : S_n(X) \to S_{n+1}(X)$ to be the unique group homomorphism extending $\sigma \mapsto b \cdot \sigma$. We claim that for all $n \geqslant 1$ and every n-simplex σ in X,

$$\partial_{n+1} c_n(\sigma) + c_{n-1} \partial_n(\sigma) = \sigma.$$

We must first compute the faces of $c_n(\sigma)$ for $n \geqslant 1$ and $0 \leqslant i \leqslant n+1$. If $i = 0$, then

$$((b \cdot \sigma)\varepsilon_0^{n+1})(t_0, \ldots, t_n) = (b \cdot \sigma)(0, t_0, \ldots, t_n) = \sigma(t_0, \ldots, t_n).$$

On the other hand, if $1 \leqslant i \leqslant n+1$, then

$$((b \cdot \sigma)\varepsilon_i^{n+1})(t_0, \ldots, t_n) = (b \cdot \sigma)(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n).$$

If $t_0 = 1$, then the right hand side is equal to b. Otherwise,

$$\begin{aligned}
((b \cdot \sigma)\varepsilon_i^{n+1})(t_0, \ldots, t_n) &= (b \cdot \sigma)(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \\
&= t_0 b + (1-t_0)\sigma\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{i-1}}{1-t_0}, 0, \frac{t_i}{1-t_0}, \ldots, \frac{t_n}{1-t_0}\right) \\
&= t_0 b + (1-t_0)\sigma\varepsilon_{i-1}^n\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_n}{1-t_0}\right) \\
&= c_{n-1}(\sigma\varepsilon_{i-1}^n)(t_0, \ldots, t_n).
\end{aligned}$$

Thus,

$$(c_n\sigma)\varepsilon_0^{n+1} = \sigma \quad \text{and} \quad (c_n\sigma)\varepsilon_i^{n+1} = c_{n-1}(\sigma\varepsilon_{i-1}^n) \quad i > 0.$$

This gives us

$$\begin{aligned}
\partial_{n+1}c_n(\sigma) &= \sum_{i=0}^{n+1}(-1)^i(c_n\sigma)\varepsilon_i^{n+1} \\
&= \sigma + \sum_{i=1}^{n+1}(-1)^i c_{n-1}(\sigma\varepsilon_{i-1}^n) \\
&= \sigma - \sum_{j=0}^{n}(-1)^j c_{n-1}(\sigma\varepsilon_j^n) \\
&= \sigma - c_{n-1}\partial_n\sigma,
\end{aligned}$$

thereby completing the proof. ∎

Porism 1.7. Let X be convex and let $\gamma = \sum m_i\sigma_i \in S_n(X)$. If $b \in X$, then

$$\partial(b \cdot \gamma) = \begin{cases} \gamma - b \cdot \partial\gamma & n > 0 \\ \left(\sum m_i\right) b - \gamma & n = 0. \end{cases}$$

Lemma 1.8. Let X be a space and for $i = 0, 1$, let $\lambda_i^X : X \to X \times I$ be defined by $x \mapsto (x, i)$. If $H_n(\lambda_0^X) = H_n(\lambda_1^X)$, then $H_n(f) = H_n(g)$ whenever $f, g : X \to Y$ are homotopic.

Proof. Let $F : X \times I \to Y$ be a homotopy between f and g. Then, $F \circ \lambda_0^X = f$ and $F \circ \lambda_1^X = g$. This gives us

$$H_n(f) = H_n(F\lambda_0^X) = H_n(F)H_n(\lambda_0^X) = H_n(F)H_n(\lambda_1^X) = H_n(F\lambda_1^X) = H_n(g),$$

for all $n \geqslant 0$. ∎

Theorem 1.9 (Homotopy Invariance of H_n). If $f, g : X \to Y$ are homotopic, then $H_n(f) = H_n(g)$ for all $n \geqslant 0$.

Proof. Due to Lemma 1.8, it suffices to show that $H_n(\lambda_0^X) = H_n(\lambda_1^X)$ for all $n \geqslant 0$. To this end, we construct a chain homotopy $P_n^X : S_n(X) \to S_{n+1}(X \times I)$ satisfying

$$\lambda_{1\sharp}^X - \lambda_{0\sharp}^X = \partial_{n+1}P_n^X + P_{n-1}^X\partial_n$$

for all spaces X. Further, we require it to satisfy a "naturality" condition, that is, for every $\sigma : \Delta^n \to X$

$$\begin{tikzcd} S_n(\Delta^n) \arrow[r, "P_n^{\Delta^n}"] \arrow[d, "\sigma_\sharp"'] & S_{n+1}(\Delta^n \times I) \arrow[d, "(\sigma\times 1)_\sharp"] \\ S_n(X) \arrow[r, "P_n^X"'] & S_{n+1}(X \times I) \end{tikzcd}$$

commutes.

Obviously $P_{-1}^X = 0$, since $S_{-1}(X) = 0$. For $\sigma : \Delta^0 \to X$, define $P_0^X(\sigma) : \Delta^1 \to X \times I$ by $t \mapsto (\sigma(e_0), t)$ where we use t to parametrize Δ^1 through $t \mapsto (1-t)e_0 + te_1$, which is obviously a homeomorphism. It is a routine exercise to verify that this defnition satisfies both conditions we needed.

Suppose now that $n \geqslant 1$. Henceforth, Δ denotes Δ^n. First, we show that for every $\gamma \in S_n(X)$, $(\lambda_{1\sharp}^\Delta - \lambda_{0\sharp}^\Delta - P_{n-1}^\Delta \partial_n)(\gamma) \in Z_n(\Delta^n \times I)$. Indeed,

$$\begin{aligned} \partial_n(\lambda_{1\sharp}^\Delta - \lambda_{0\sharp}^\Delta - P_{n-1}^\Delta \partial_n) &= \lambda_{1\sharp}^\Delta \partial_n - \lambda_{0\sharp}^\Delta \partial_n - \partial_n P_{n-1}^\Delta \partial_n \\ &= \lambda_{1\sharp}^\Delta \partial_n - \lambda_{0\sharp}^\Delta \partial_n - \left(\lambda_{1\sharp}^\Delta - \lambda_{0\sharp}^\Delta - P_{n-2}^\Delta \partial_{n-1}\right)\partial_n \\ &= 0, \end{aligned}$$

where we have used the induction hypothesis to obtain the second equality.

Let $\delta : \Delta^n \to \Delta^n$ denote the identity map. Then, $\delta \in S_n(\Delta^n)$ whence $(\lambda_{1\sharp}^\Delta - \lambda_{0\sharp}^\Delta - P_{n-1}^\Delta \partial_n)(\delta) \in Z_n(\Delta^n \times I)$. We have seen in Theorem 1.6 that $Z_n(\Delta^n \times I) = B_n(\Delta^n \times I)$, consequently, there is $\beta_{n+1} \in S_{n+1}(\Delta^n \times I)$ such that

$$\partial_{n+1}\beta_{n+1} = (\lambda_{1\sharp}^\Delta - \lambda_{0\sharp}^\Delta - P_{n-1}^\Delta \partial_n)(\delta).$$

Define $P_n^X : S_n(X) \to S_{n+1}(X \times I)$ to be the unique group homomorophism extending

$$P_n^X(\sigma) = (\sigma \times 1)_\sharp(\beta_{n+1}),$$

where $\sigma : \Delta^n \to X$ is an n-simplex in X. It remains to verify the two conditions for P_n. Before we proceed, we note that

$$(\sigma \times 1)\lambda_i^\Delta = \lambda_i^X \sigma : \Delta^n \to X \times I.$$

Now, let σ be an n-simplex in X. Then,

$$\begin{aligned} \partial_{n+1}P_n^X(\sigma) &= \partial_{n+1}(\sigma \times 1)_\sharp(\beta_{n+1}) \\ &= (\sigma \times 1)_\sharp \partial_{n+1}(\beta_{n+1}) \\ &= (\sigma \times 1)_\sharp \left(\lambda_{1\sharp}^\Delta - \lambda_{0\sharp}^\Delta - P_{n-1}^\Delta \partial_n\right)(\delta) \\ &= (\sigma \times 1)_\sharp \left(\lambda_1^\Delta - \lambda_0^\Delta - P_{n-1}^\Delta \partial_n(\delta)\right) \\ &= (\sigma \times 1)\lambda_1^\Delta - (\sigma \times 1)\lambda_0^\Delta - (\sigma \times 1)_\sharp P_{n-1}^\Delta \partial_n(\delta) \\ &= \lambda_1^\Delta \sigma - \lambda_0^\Delta \sigma - P_{n-1}^X \partial_n \sigma(\delta) \\ &= \left(\lambda_1^\Delta - \lambda_0^X - P_{n-1}^X \partial_n\right)(\sigma). \end{aligned}$$

This verifies the first equation.

Next, we verify "naturality". Let $\tau : \Delta^n \to \Delta^n$ be an n-simplex. Then, for every $\sigma : \Delta^n \to X$,

$$(\sigma \times 1)_\sharp P_n^\Delta(\tau) = (\sigma \times 1)_\sharp(\tau \times 1)_\sharp(\beta_{n+1}) = (\sigma\tau \times 1)_\sharp(\beta_{n+1}).$$

On the other hand,

$$P_n^X \sigma_\sharp(\tau) = P_n^X(\sigma\tau) = (\sigma\tau \times 1)_\sharp(\beta_{n+1}).$$

This completes the proof. ∎

PORISM 1.10. If $f : X \to Y$ is continuous, then the following diagram commutes.

$$\begin{tikzcd} S_n(X) \arrow[r, "p_n^X"] \arrow[d, "f_\sharp"'] & S_{n+1}(X \times I) \arrow[d, "(f\times 1)_\sharp"] \\ S_n(Y) \arrow[r, "p_n^Y"'] & S_{n+1}(Y \times I) \end{tikzcd}$$

Proof. Let σ be an n-simplex in X. We know that the outer rectangle and the upper square in the following diagram commute.

$$\begin{tikzcd} S_n(\Delta^n) \arrow[r, "p_n^{\Delta^n}"] \arrow[d, "\sigma_\sharp"'] & S_{n+1}(\Delta^n \times I) \arrow[d, "(\sigma\times 1)_\sharp"] \\ S_n(X) \arrow[r, "p_n^X"'] \arrow[d, "f_\sharp"'] & S_{n+1}(X \times I) \arrow[d, "(f\times 1)_\sharp"] \\ S_n(Y) \arrow[r, "p_n^Y"'] & S_{n+1}(Y \times I) \end{tikzcd}$$

This would, after a straightforward diagram chase, imply that the lower square also commutes. ∎

§§ Relative Homology Groups

DEFINITION 1.11. Let X be a topological space and $A \subseteq X$ a subspace. The inclusion $j : A \hookrightarrow X$ defines an inclusion $j_\sharp : S_\bullet(A) \hookrightarrow S_\bullet(X)$. This gives us an induced complex, $S_\bullet(X)/S_\bullet(A)$,

$$\begin{tikzcd} \cdots \arrow[r] & S_n(A) \arrow[r] \arrow[d, hook] & S_{n-1}(A) \arrow[r] \arrow[d, hook] & \cdots \\ \cdots \arrow[r] & S_n(X) \arrow[r] \arrow[d] & S_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & S_n(X)/S_n(A) \arrow[r] & S_{n-1}(X)/S_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

The homology groups of the induced complex are denoted by $H_n(X, A)$. These are known as the *relative homology groups*. We often denote the boundary maps of the induced complex by $\overline{\partial}_n$.

LEMMA 1.12 (EXACT TRIANGLE LEMMA). If $0 \to (S'_\bullet, \partial') \xrightarrow{i} (S_\bullet, \partial) \xrightarrow{p} (S''_\bullet, \partial'') \to 0$ is exact, then there is a long exact sequence

$$\cdots \to H_n(S'_\bullet) \xrightarrow{i_*} H_n(S_\bullet) \xrightarrow{p_*} H_n(S''_\bullet) \xrightarrow{d} H_{n-1}(S'_\bullet) \to \cdots.$$

where the map $d_n : H_n(S''_\bullet) \to H_{n-1}(S''_\bullet)$ is given by

$$[z''_n] \mapsto \left[i_{n-1}^{-1}\partial_n p_n^{-1} z''_n\right],$$

where $[\cdot]$ denotes the equivalence class of a cycle in the homology group. Diagramatically, we pull back z''_n as follows:

$$\begin{tikzcd} & & S_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & S''_n \arrow[r] & 0 \\ 0 \arrow[r] & S'_{n-1} \arrow[r, "i_n"'] & S_{n-1} & & \end{tikzcd}$$

Further, the maps d are natural.

Proof. This is a relatively straightforward diagram chase. I'll probably add the details in someday. ■

DEFINITION 1.13. A *map of pairs* $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

From the defintion, f induces maps $f_\sharp : S_\bullet(A) \to S_\bullet(B)$ and $f_\sharp : S_\bullet(X) \to S_\bullet(Y)$ making the following diagram commute.

$$\begin{tikzcd} S_n(A) \arrow[r] \arrow[d, "f_\sharp"'] & S_n(X) \arrow[d, "f_\sharp"] \\ S_n(B) \arrow[r] & S_n(Y) \end{tikzcd}$$

Consequently, there is an induced map, $\overline{f}_\sharp : S_\bullet(X, A) \to S_\bullet(Y, B)$, which follows from the universal property of the cokernel. This, in turn makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & S_n(A) \arrow[r] \arrow[d, "f_\sharp"'] & S_n(X) \arrow[r] \arrow[d, "f_\sharp"] & S_n(X, A) \arrow[r] \arrow[d, "f_\sharp"'] & 0 \\ 0 \arrow[r] & S_n(B) \arrow[r] & S_n(Y) \arrow[r] & S_n(Y, B) \arrow[r] & 0 \end{tikzcd}$$

THEOREM 1.14 (EXACT SEQUENCE FOR PAIRS). If A is a subspace of X, then there is a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X, A) \xrightarrow{d} H_{n-1}(A) \to \cdots .$$

Moreover, if $f : (X, A) \to (Y, B)$ then there is the following commutative diagram (naturality).

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"'] & H_n(X) \arrow[r] \arrow[d, "f_*"'] & H_n(X, A) \arrow[r] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"'] & \cdots \\ \cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots \end{tikzcd}$$

Proof. Follows from the discussion above and Lemma 1.12. ■

THEOREM 1.15 (EXACT SEQUENCE FOR TRIPLES). If $A' \subseteq A \subseteq X$ are subspaces, then there is a long exact sequence

$$\cdots \to H_n(A, A') \to H_n(X, A') \to H_n(X, A) \to H_{n-1}(X, A) \to \cdots ,$$

where the maps (other than the connecting map) are induced by $(A, A') \to (X, A')$ and $(X, A') \to (X, A)$.

Moreover, if $f : (X, A, A') \to (Y, B, B')$ is a map of pairs, then there is a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A, A') \arrow[r] \arrow[d, "f_*"'] & H_n(X, A') \arrow[r] \arrow[d, "f_*"'] & H_n(X, A) \arrow[r] \arrow[d, "f_*"'] & H_{n-1}(A, A') \arrow[r] \arrow[d, "f_*"'] & \cdots \\ \cdots \arrow[r] & H_n(B, B') \arrow[r] & H_n(Y, B') \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B, B') \arrow[r] & \cdots \end{tikzcd}$$

Proof. Using the third isomorphism theorem, there is a short exact sequence

$$0 \to S_\bullet(A)/S_\bullet(A') \to S_\bullet(X)/S_\bullet(A') \to S_\bullet(X)/S_\bullet(A) \to 0.$$

The first map is induced by the inclusion $(A, A') \hookrightarrow (X, A')$ and it is easy to see that the second map is induced by the inclusion $(X, A') \hookrightarrow (X, A)$. The remainder follows from Lemma 1.12. ■

§2 EXCISION AND MAYER-VIETORIS

THEOREM 2.1 (EXCISION I). Let $U \subseteq \overline{U} \subseteq A^\circ \subseteq\subseteq A \subseteq X$. Then, the inclusion $i : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism of relative homology groups for all $n \geqslant 0$.

THEOREM 2.2 (EXCISION II). Let X_1 and X_2 be subspaces of X with $X = X_1^\circ \cup X_2^\circ$. Then, the inclusion $j : (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2) = (X_1 \cup X_2, X_2)$ induces isomorphisms $j_* : H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2)$.

§§ Barycentric Subdivision and the Proof of Excision

DEFINITION 2.3. Let $n \geqslant 1$. Points $p_0, \dots, p_n \in \mathbb{R}^n$ are said to be *affine independent* if $\{p_1 - p_0, \dots, p_n - p_0\}$ is a linearly independent subset of $\mathbb{R}^n$.

An *affine n-simplex* Σ^n in $\mathbb{R}^n$ is the convex hull of an affine independent set $\{p_0, \dots, p_n\} \subseteq \mathbb{R}^n$. The *barycenter* of Σ^n is defined to be

$$b = \frac{p_0 + \cdots + p_n}{n+1}.$$

An *i-face* of Σ^n is a simplex spanned by some $i+1$ elements of $\{p_0, \dots, p_n\}$.

DEFINITION 2.4. The *barycentric subdivision* of an affine n-simplex Σ^n, denoted by $\operatorname{Sd}\Sigma^n$ is a family of affine n-simplexes defined inductively for $n \geqslant 0$ as follows:

(a) $\operatorname{Sd}\Sigma^0 = \Sigma^0$.

(b) If $\varphi_0, \dots, \varphi_{n+1}$ are the n-faces of Σ^{n+1} and if b is the barycenter of Σ^{n+1}, then $\operatorname{Sd}\Sigma^{n+1}$ consists of all the $(n+1)$-simplexes spanned by b and n-simplexes in $\operatorname{Sd}\varphi_i$, $0 \leqslant i \leqslant n+1$.

DEFINITION 2.5. Let E be a convex subset of Euclidean space. Then, *barycentric subdivision* is a homomorphism $\operatorname{Sd}_n : S_n(E) \to S_n(E)$ defined inductively on n-simplexes $\tau : \Delta^n \to E$ as follows:

(a) If $n = 0$, then $\operatorname{Sd}_0(\tau) = \tau$.

(b) If $n > 0$, then $\operatorname{Sd}_n(\tau) = \tau(b_n) \cdot \operatorname{Sd}_{n-1}(\partial\tau)$, where b_n is the barycenter of Δ^n.

where $b \cdot \sigma$ refers to the "cone construction" from the proof of Theorem 1.6.

DEFINITION 2.6. If X is any topological space, then the n-th *barycentric subdivision*, for $n \geqslant 0$, is the homomorphism $\operatorname{Sd}_n : S_n(X) \to S_n(X)$ extending the map on n-simplexes $\sigma : \Delta^n \to X$ given by

$$\operatorname{Sd}_n(\sigma) = \sigma_\sharp \operatorname{Sd}_n(\delta^n),$$

where $\delta^n : \Delta^n \to \Delta^n$ is the identity map and $\sigma_\sharp : S_n(\Delta^n) \to S_n(X)$ is the induced map.

REMARK 2.7. It is easy to see that both definitions agree when X is a convex subset of Euclidean space.

LEMMA 2.8. If $f : X \to Y$ is continuous, then

$$\begin{tikzcd} S_n(X) \arrow[r, "\operatorname{Sd}"] \arrow[d, "f_\sharp"'] & S_n(X) \arrow[d, "f_\sharp"] \\ S_n(Y) \arrow[r, "\operatorname{Sd}"'] & S_n(Y) \end{tikzcd}$$

Proof. Immediate from the above definition. ∎

PROPOSITION 2.9. $\operatorname{Sd} : S_\bullet(X) \to S_\bullet(X)$ is a chain map.

Proof. First, suppose X is a convex subset of Euclidean space and let $\tau : \Delta^n \to X$ be an n-simplex. We shall prove, by induction on $n \geqslant 0$, that $\operatorname{Sd}_{n-1}\partial_n\tau = \partial_n \operatorname{Sd}_n \tau$. If $n = 0$, then there is nothing to prove. If $n > 0$, then

$$\begin{aligned} \partial_n \operatorname{Sd}_n \tau &= \partial_n \left(\tau(b_n) \cdot \operatorname{Sd}_{n-1}\partial_n\tau\right) \\ &= \operatorname{Sd}_{n-1}\partial_n\tau - \tau(b_n) \cdot (\partial_{n-1}\operatorname{Sd}_{n-1}\partial_n\tau) \\ &= \operatorname{Sd}_{n-1}\partial_n\tau - \tau(b_n) \cdot (\operatorname{Sd}_{n-2}\partial_{n-1}\partial_n\tau) \\ &= \operatorname{Sd}_{n-1}\partial_n\tau, \end{aligned}$$

where the second equality follows from Porism 1.7

Now, let X be any topological space. Let $\sigma : \Delta^n \to X$ be an n-simplex.

$$\begin{aligned}\partial_n \operatorname{Sd}_n(\sigma) &= \partial_n \sigma_\sharp \operatorname{Sd}_n(\delta^n)\\ &= \sigma_\sharp \partial_n \operatorname{Sd}_n(\delta^n)\\ &= \sigma_\sharp \operatorname{Sd}_{n-1} \partial_n(\delta^n)\\ &= \operatorname{Sd}_{n-1} \sigma_\sharp \partial_n(\delta^n)\\ &= \operatorname{Sd}_{n-1} \partial_n \sigma_\sharp(\delta^n)\\ &= \operatorname{Sd}_{n-1} \partial_n \sigma.\end{aligned}$$

This completes the proof. ∎

PROPOSITION 2.10. For each $n \geqslant 0$, $H_n(\operatorname{Sd}) : H_n(X) \to H_n(X)$ is the identity.

Proof. We shall construct a chain homotopy between Sd and 1. First, suppose X is a convex subset of Euclidean space. We chall construct a chain homotopy $T_n : S_n(X) \to S_{n+1}(X)$ by induction on n. If $n = 0$, define T_0 to be the zero map. It is obvious that

$$\partial_1 T_0 = 1 - \operatorname{Sd}_0,$$

since Sd_0 is the identity map.

Let $n \geqslant 1$ and $\gamma \in S_n(X)$. Note that $\gamma - \operatorname{Sd}_n \gamma - T_{n-1}\partial_n \gamma$ is a cycle. Indeed,

$$\begin{aligned}\partial_n\left(\gamma - \operatorname{Sd}_n \gamma - T_{n-1}\partial_n\gamma\right) &= \partial_n\gamma - \operatorname{Sd}_{n-1}\partial_n\gamma - \partial_n T_{n-1}\partial_n\gamma\\ &= \partial_n\gamma - \operatorname{Sd}_{n-1}\partial_n\gamma - \left(1 - \operatorname{Sd}_{n-1} - T_{n-2}\partial_{n-1}\right)\partial_n\gamma\\ &= 0,\end{aligned}$$

where the second equality uses the induction hypothesis.

Define $T_n\gamma = b \cdot (\gamma - \operatorname{Sd}_n \gamma - T_{n-1}\partial_n\gamma)$ where b is a a fixed point in X. Uing Porism 1.7,

$$\partial_{n+1} T_n \gamma = \gamma - \operatorname{Sd}_n \gamma - T_{n-1}\partial_n \gamma.$$

This proves the statement for the case when X is convex.

Suppose now that X is any topological space. If $\sigma : \Delta^n \to X$ is an n-simplex, define T_n to be the unique group homomorphism $T_n : S_n(X) \to S_{n+1}(X)$ extending

$$T_n(\sigma) = \sigma_\sharp T_n(\delta^n) \in S_{n+1}(X),$$

where $\delta^n : \Delta^n \to \Delta^n$ is the identity map.

First, we show a "naturality" of T_n. Let $f : X \to Y$ be continuous. We contend that

$$\begin{tikzcd} S_n(X) \arrow[r, "f_\sharp"] \arrow[d, "T_n"'] & S_n(Y) \arrow[d, "T_n"] \\ S_{n+1}(X) \arrow[r, "f_\sharp"'] & S_{n+1}(Y) \end{tikzcd}$$

commutes. Let $\sigma : \Delta^n \to X$ be an n-simplex in X. Then,

$$T_n f_\sharp \sigma = T_n(f \circ \sigma) = (f \circ \sigma)_\sharp T_n(\delta^n) = f_\sharp \sigma_\sharp T_n(\delta^n) = f_\sharp T_n(\sigma).$$

Finally, we show that T_n is the desired chain homotopy. Let $\sigma : \Delta^n \to X$ be an n-simplex. Then,

$$\begin{aligned}\partial_{n+1} T_n \sigma &= \partial_{n+1}\sigma_\sharp T_n(\delta^n)\\ &= \sigma_\sharp \partial_{n+1} T_n(\delta^n)\\ &= \sigma_\sharp\left(\delta^n - \operatorname{Sd}_n \delta^n - T_{n-1}\partial_n\delta^n\right)\\ &= \sigma - \sigma_\sharp \operatorname{Sd}_n \delta^n - \sigma_\sharp T_{n-1}\partial_n \delta^n\\ &= \sigma - \operatorname{Sd}_n \sigma - T_{n-1}\partial_n \sigma.\end{aligned}$$

This completes the proof. ∎

DEFINITION 2.11. If E is a subspace of Euclidean space, and if $\gamma = \sum_j m_j\sigma_j \in S_n(E)$, where all $m_j \neq 0$, then define the *mesh* of γ to be

$$\operatorname{mesh}\gamma = \sup_j \left(\operatorname{diam}\sigma_j(\Delta^n)\right).$$

The chain γ is said to be *affine* if each $\sigma_j : \Delta^n \to E$ is affine.

THEOREM 2.12. If E is a subspace of some Euclidean space and γ is an affine n-chain in E, then for all integers $q \geqslant 1$,

$$\operatorname{mesh}\operatorname{Sd}^q\gamma = \left(\frac{n}{n+1}\right)^q \operatorname{mesh}\gamma.$$

Proof. Straightforward induction. The base case is an exercise in triangle inequalities. ■

LEMMA 2.13. If $X_1, X_2 \subseteq X$ with $X = X_1^\circ \cup X_2^\circ$, and if σ is an n-simplex in X, then there is an integer $q \geqslant 1$ with

$$\operatorname{Sd}^q\sigma \in S_n(X_1) + S_n(X_2),$$

where we treat $S_n(X_1)$ and $S_n(X_2)$ as submodules of $S_n(X)$.

Proof. Let $\sigma : \Delta^n \to X$ be an n-simplex in X. Then, $\{\sigma^{-1}X_1^\circ, \sigma^{-1}X_2^\circ\}$ forms an open cover for Δ^n and hence, admits a Lebesgue number $\lambda > 0$. Thus, there is a $q \geqslant 1$ with $\operatorname{mesh}\operatorname{Sd}_n^q(\delta^n) < \lambda$.

Note that $\operatorname{Sd}_n^q(\sigma) = \sigma_\sharp\operatorname{Sd}^q(\delta^n)$. Let $\operatorname{Sd}^q(\delta^n) = \sum_j m_j\tau_j$. Then, for each j, $\operatorname{diam}\tau_j(\Delta^n) < \lambda$, whence, $\sigma\tau_j(\Delta^n) \subseteq X_i$ for some $i \in \{1,2\}$. Consequently, $\sigma_\sharp\operatorname{Sd}^q(\delta^n) \in S_n(X_1) + S_n(X_2)$. ■

LEMMA 2.14. Let $X_1, X_2 \subseteq X$. If the inclusion $S_\bullet(X_1) + S_\bullet(X_2) \hookrightarrow S_\bullet(X)$ induces isomorphisms in homology, then excision holds for the subspaces X_1 and X_2 of X.

Proof. We have a commutative diagram:

$$\begin{tikzcd}
\frac{S_\bullet(X_1)}{S_\bullet(X_1\cap X_2)} \arrow[rr, "k"] \arrow[dr, "\ell"'] & & \frac{S_\bullet(X)}{S_\bullet(X_2)} \\
& \frac{S_\bullet(X_1)+S_\bullet(X_2)}{S_\bullet(X_2)} \arrow[ur, "j"'] &
\end{tikzcd}$$

where k is induced by $(X_1, X_1 \cap X_2) \hookrightarrow (X_1, X_2)$ and ℓ and j are the obvious natural maps. Since $S_\bullet(X_1 \cap X_2) = S_\bullet(X_1) \cap S_\bullet(X_2)$, the map ℓ is an isomorphism and hence, so is $H_n(\ell)$.

It remains to show that $H_n(j)$ is an isomorphism. Indeed, we have a short exact sequence of complexes:

$$0 \longrightarrow \frac{S_\bullet(X_1) + S_\bullet(X_2)}{S_\bullet(X_2)} \xrightarrow{j} \frac{S_\bullet(X)}{S_\bullet(X_2)} \longrightarrow \frac{S_\bullet(X)}{S_\bullet(X_1) + S_\bullet(X_2)} \longrightarrow 0.$$

Since $H_n(S_\bullet(X)/S_\bullet(X_1) + S_\bullet(X_2)) = 0$ for all $n \geqslant 0$, from the long exact sequence for homology, we deduce that $H_n(j)$ is an isomorphism. This completes the proof. ■

Proof of Theorem 2.2. ■

§3 THE UNIVERSAL COEFFICIENTS

THEOREM 3.1 (UNIVERSAL COEFFICIENT THEOREM FOR HOMOLOGY). Let X be a topological space and G an abelian group. Then, there are *split exact sequences* for all $n \geqslant 0$:

$$0 \to H_n(X) \otimes_{\mathbb{Z}} G \xrightarrow{\alpha} H_n(X; G) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X); G) \to 0,$$

where α is defined on the pure tensors by $[z] \otimes g \mapsto [z \otimes g]$.

In particular,

$$H_n(X; G) \cong (H_n(X) \otimes_{\mathbb{Z}} G) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. We prove this in a more general setting, for any complex $(C_\bullet, \partial)$ of free abelian groups. Let B_n, Z_n have their usual meanings. Then, we have a short exact sequence

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{d_n} B_{n-1} \to 0,$$

where d_n is just ∂_n with the codomain restricted to B_{n-1}. Since $B_\bullet$ is a complex of flat (in fact, free) $\mathbb{Z}$-modules, we have a short exact sequence of complexes:

$$0 \longrightarrow \mathscr{Z}_\bullet \otimes_{\mathbb{Z}} G \xrightarrow{i_\bullet \otimes 1} C_\bullet \otimes_{\mathbb{Z}} G \xrightarrow{d_\bullet \otimes 1} \mathscr{B}_\bullet \times_{\mathbb{Z}} G \longrightarrow 0$$

where $\mathscr{Z}_n = Z_n$ and $\mathscr{B}_n = B_{n-1}$. The boundary maps in both complexes $\mathscr{Z}_\bullet$ and $\mathscr{B}_\bullet$ are the zero maps and hence, the homology groups are readily computed. This gives us a long exact sequence

$$\cdots \to B_n \otimes G \xrightarrow{\Delta_{n+1}} Z_n \otimes G \xrightarrow{(i_n \otimes 1)_*} H_n(C_\bullet \otimes G) \xrightarrow{(d_n \otimes 1)_*} B_{n-1} \otimes G \to \cdots$$

We shall now explicitly compute the connecting homomorphism. It is given by the following diagram:

$$\begin{tikzcd}
 & & C_n \otimes G \arrow[r, "d_n \otimes 1"] \arrow[d, "\partial_n \otimes 1"'] & B_{n-1} \otimes G \arrow[r] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r, "i_{n-1} \otimes 1"'] & C_{n-1} \otimes G & &
\end{tikzcd}$$

The image of $b_{n-1} \otimes g \in B_{n-1} \otimes G$ under the connecting homomorphism is given by

$$\Delta_n(b_{n-1} \otimes g) = (i_{n-1} \otimes 1)^{-1}(\partial_n \otimes 1)(d_n \otimes 1)^{-1}(b_{n-1} \otimes g) = i_{n-1}^{-1} \partial_n d_n^{-1} b_{n-1} \otimes g.$$

But d_n is the codomain restriction of ∂_n and hence, the above simplifies to $i_{n-1} b_{n-1}$ where b_{n-1} is treated as an element of C_{n-1}, and hence, the entire calculation above gives $(j_{n-1} \otimes 1)(b_{n-1} \otimes g)$, where $j_{n-1} : B_{n-1} \hookrightarrow Z_{n-1}$ is the inclusion.

The long exact sequence now looks like

$$\cdots \to B_n \otimes G \xrightarrow{j_n \otimes 1} Z_n \otimes G \xrightarrow{(i_n \otimes 1)_*} H_n(C_\bullet \otimes G) \xrightarrow{(d_n \otimes 1)_*} B_{n-1} \otimes G \to \cdots$$

This gives a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & Z_n \otimes G / \operatorname{im}(j_n \otimes 1) \arrow[r, "\alpha"] & H_n(C_\bullet \otimes G) \arrow[r] \arrow[drr, "(d_n \otimes 1)_*"'] & \ker(j_n \otimes 1) \arrow[r] \arrow[dr, hook] & 0 \\
 & Z_n \otimes G \arrow[u, two heads] \arrow[ur, "(i_n \otimes 1)_*"'] & & & B_{n-1} \otimes G
\end{tikzcd}$$

where α is given by $z_n \otimes g + \operatorname{im}(j_n \otimes 1) \mapsto [z_n \otimes g]$ where $[\cdot]$ denotes the equivalence class in $H_n(C_\bullet \otimes G)$.

We have the canonical short exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C_\bullet) \to 0.$$

Tensoring this with G, and using the Tor long exact sequence, we have an exact sequence

$$0 \to Tor_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), G) \to B_{n-1} \otimes G \xrightarrow{j_{n-1} \otimes 1} Z_{n-1} \otimes G \xrightarrow{p_{n-1} \otimes 1} H_{n-1}(C_\bullet) \otimes G \to 0,$$

since $\operatorname{Tor}(Z_{n-1}, G) = 0$, owing to Z_{n-1} being flat. Note that $p_{n-1}(z_{n-1}) = [z_{n-1}]$ where $[\cdot]$ denotes the equivalence class of z_{n-1} in $H_{n-1}(C_\bullet)$.

This, in particular, gives us isomorphisms

$$Z_{n-1} \otimes G / \operatorname{im}(j_n \otimes 1) \xrightarrow{\sim} H_{n-1}(C_\bullet) \otimes G,$$
$$\ker(j_n \otimes 1) \xrightarrow{\sim} \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), G).$$

where the first isomorphism is induced by $p_{n-1} \otimes 1$. Substituting this into the short exact sequence we had obtained earlier, we get

$$0 \to H_{n-1}(C_\bullet) \otimes G \xrightarrow{\beta} H_n(C_\bullet \otimes G) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}, G) \to 0.$$

where the map β is given by

$$\beta([z_{n-1}] \otimes g) = \alpha\,(z_{n-1} \otimes g) = [z_{n-1} \otimes g] \in H_{n-1}(C_\bullet \otimes G).$$

This proves the first assertion. ■ prove the second assertion

§4 Cellular Homology

Lemma 4.1. Let X be a CW-complex.

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.

(b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > \dim X$.

(c) The map $H_k(X^n) \xrightarrow{i_*} H_k(X)$ induced by the inclusion $i : X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.

Proof. (a) This follows from the fact that (X^n, X^{n-1}) is a *good pair* and hence, $H_k(X^n, X^{n-1}) = H_k(X^n/X^{n-1})$, where X^n/X^{n-1} is a wedge of n-spheres.

(b) Consider the long exact sequence for the pair (X^n, X^{n-1}),

$$\cdots \to H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}) \to \cdots.$$

Hence, for $k > n$, we have an isomorphism $H_k(X^{n-1}) \xrightarrow{\sim} H_k(X^n)$, since both the relative homology groups on the ends vanish.

Consider the following sequence of inclusion-induced homomorphisms

$$H_k(X^0) \xrightarrow{\sim} H_k(X^1) \xrightarrow{\sim} \ldots \xrightarrow{\sim} H_k(X^{k-1}) \to H_k(X^k) \to \ldots$$

where all maps other than those into and out of $H_k(X^k)$ are isomorphisms. The map into $H_k(X^k)$ is injective while the map out of $H_k(X^k)$ is surjective. Since $H_k(X^0) = 0$ for $k \geqslant 1$, the conclusion follows.

(c) Suppose first that X is finite-dimensional, that is,

$$\varnothing = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^N = X.$$

Then, for $k < n$, we have seen in the proof of (b) that $H_k(X^n)$ is isomorphic to $H_k(X^N) = H_k(X)$. On the other hand, if $k = n$, the map $H_k(X^n) \to H_k(X^N)$ is a composition of a surjection and a sequence of isomorphisms and hence, is a surjection.

Now, suppose X is not finite-dimensional. ■ infinite proof

We now construct the *cellular chain complex* using portions of the long exact sequence for pairs as follows:

$$\begin{tikzcd}
 & & & H_n(X^{n+1}) \cong H_n(X) & & & \\
 & & H_n(X^n) \arrow[ur, two heads] \arrow[dr, hook, "j_n"] & & & & \\
 {} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\Delta_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\Delta_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & {} \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, hook, "j_{n-1}"'] \arrow[dr, two heads] & & \\
 & & & & & H_{n-1}(X^n) \cong H_{n-1}(X) &
\end{tikzcd}$$

Note that the groups in the cellular chain complex are all free abelian due to Lemma 4.1 (a).

Definition 4.2. The homology groups of the cellular chain complex are called the *cellular homology groups*, denoted by $H_n^{CW}(X)$.

Theorem 4.3. For a CW-complex X, $H_n^{CW}(X) \cong H_n(X)$ for all $n \geqslant 0$.

Proof. From the chain complex diagram drawn above, $H_n(X)$ is isomorphic to $H_n(X^n)/\operatorname{im}\Delta_{n+1}$. Further, since j_n is injective, $H_n(X^n)$ is mapped isomorphically onto $\operatorname{im} j_n$ and $\operatorname{im}\Delta_{n+1}$ is mapped isomorphically onto $\operatorname{im} d_{n+1}$. Therefore,

$$H_n(X) \cong \frac{\operatorname{im} j_n}{\operatorname{im} d_{n+1}}.$$

Note that $\ker d_n = \ker \Delta_n$ and from the long exact sequence for the pair (X^n, X^{n-1}), we deduce that $\ker \Delta^n = \operatorname{im} j_n$. This completes the proof. ■

Part II
Cohomology

§5 Singular Cohomology

Definition 5.1. Fix an abelian group G. Let X be a topological space and $(S_\bullet(X), \partial)$ the singular chain complex. We define the *singular cochain complex* to be

$$\cdots \to \operatorname{Hom}_{\mathbb{Z}}(S_n(X), G)) \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(\partial_{n+1}, G)} \operatorname{Hom}_{\mathbb{Z}}(S_{n+1}(X), G) \to \cdots.$$

We denote the "differentials" of the above complex by $\delta.$ and the groups by $S^n(X)$.

The "cohomology groups" corresponding to the above chain complex are known as the *singular cohomology groups*, denoted by $H^n(X; G)$

Let $f : X \to Y$ be a continuous map. Then, $f_\sharp : S_\bullet(X) \to S_\bullet(Y)$ is a chain map, whence, there is an induced map $f^\sharp = \operatorname{Hom}(f_\sharp, G) : S^\bullet(Y) \to S^\bullet(X)$ and since the former was a chain map, so is the latter.

Now, if we have a sequence of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, then it is easy to see that $f^\sharp \circ g^\sharp = (g \circ f)^\sharp$. We have proved:

PROPOSITION 5.2. $H^n(-;G)$ is a contravariant functor from the category of topological spaces to the category of abelian groups. ■

DEFINITION 5.3. Let $A \subseteq X$. We define the *relative cohomology groups* to be those associated to the chain complex

$$\cdots \to \mathrm{Hom}_{\mathbb{Z}}\left(S_n(X)/S_n(A), G\right) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\overline{\partial}_{n+1}, G)} \mathrm{Hom}_{\mathbb{Z}}\left(S_{n+1}(X)/S_{n+1}(A), G\right) \to \cdots.$$

These are denoted by $H^n(X, A; G)$, for $n \geqslant 0$. The above chain complex is denoted by $(S^\bullet(X, A; G), \overline{\delta}^n)$

THEOREM 5.4. If A is a subspace of X, then there is a long exact sequence

$$\cdots \to H^n(X, A; G) \xrightarrow{p_n^*} H^n(X; G) \xrightarrow{i_n^*} H^n(A; G) \xrightarrow{d_n} H^{n+1}(X, A; G) \to \cdots,$$

where the connecting homomorphisms, namely the d_n's are natural.

Proof. We have a short exact sequence of complexes

$$0 \to S_\bullet(A) \xrightarrow{i_\sharp} S_\bullet(X) \xrightarrow{p_\sharp} S_\bullet(X)/S_\bullet(A) \to 0.$$

We know that all groups in the above exact sequence are free and hence, $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ gives us another short exact sequence of complexes

$$0 \to S^\bullet(X, A; G) \xrightarrow{p^\sharp} S^\bullet(X; G) \xrightarrow{i^\sharp} S^\bullet(X; G) \to 0.$$

The conclusion follows from Lemma 1.12. ■

THEOREM 5.5 (EXCISION). Let X_1 and X_2 be subspaces of X with $X = X_1^\circ \cup X_2^\circ$. Then, the inclusion $j : (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces isomorphisms for all $n \geqslant 0$,

$$j^* : H^n(X, X_2; G) \xrightarrow{\sim} H^n(X_1, X_1 \cap X_2; G).$$

Proof. *TODO: First write up for homology and then the same idea works for cohomology.* ■

§6 The Cup Product

DEFINITION 6.1. For $0 \leqslant i \leqslant d$, define maps $\lambda_i^d, \mu_i^d : \Delta^i \to \Delta^d$ by

$$\lambda_i^d(t_0, \ldots, t_i) = (t_0, \ldots, t_i, 0, \ldots, 0) \quad \text{and} \quad \mu_i^d(t_0, \ldots, t_i) = (0, \ldots, 0, t_0, \ldots, t_i).$$

These maps are called the *front face* and *back face* maps respectively.

PROPOSITION 6.2. (a) $\lambda_d^{d+1} = \varepsilon_{d+1}^{d+1}$ and $\mu_d^{d+1} = \varepsilon_0^{d+1}$.

(b) $\lambda_{n+m}^d \lambda_n^{n+m} = \lambda_n^d$ and $\mu_{n+m}^d \mu_n^{n+m} = \mu_n^d$.

(c) $\mu_{m+k}^{n+m+k} \lambda_m^{m+k} = \lambda_{n+m}^{n+m+k} \mu_m^{n+m}$.

(d)

$$\varepsilon_i^{d+1} \lambda_p^d = \begin{cases} \lambda_{p+1}^{d+1} \varepsilon_i^{p+1} & i \leqslant p \\ \lambda_p^{d+1} & i \geqslant p+1 \end{cases}$$

$$\varepsilon_i^{d+1} \mu_q^d = \begin{cases} \mu_q^{d+1} & i \leqslant d-q \\ \mu_{q+1}^{d+1} \varepsilon_{i+q-d}^{q+1} & i \geqslant d-q+1. \end{cases}$$

Proof. Omitted owing to its obviousness. ■

NOTATION. Henceforth, for $\varphi \in S^n(X, G)$ and $c \in S_n(X)$, we write (c, φ) for $\varphi(c) \in G$. In this notation, we have

$$(c, f^\sharp\varphi) = (f_\sharp c, \varphi) \quad \text{in particular,} \quad (\sigma, f^\sharp\varphi) = (f\sigma, \varphi).$$

Further, if $c \in S_{n+1}(X)$, then

$$(c, \delta_n\varphi) = (\partial_{n+1}c, \varphi).$$

DEFINITION 6.3. Let X be a topological space and R a ring, which is naturally a $\mathbb{Z}$-module. If $\varphi \in S^n(X; R)$ and $\theta \in S^m(X; R)$, define their *cup product* $\varphi \smile \theta \in S^{n+m}(X; R)$ by

$$(\sigma, \varphi \smile \theta) = (\sigma\lambda_n^{n+m}, \varphi)(\sigma\mu_m^{n+m}, \theta).$$

Extend this to a map $\smile: S(X; R) \times S(X; R) \to S(X; R)$ by defining

$$\left(\sum_i \varphi_i\right) \smile \left(\sum_j \theta_j\right) = \sum_{i,j} \varphi_i \smile \theta_j$$

where $\varphi_i \in S^i(X; R)$ and $\theta_j \in S^j(X; R)$.

PROPOSITION 6.4. $S(X; R)$ is a graded ring with multiplication given by the cup product.

Proof. Verifying distributivity is straightforward. We show associativity next. Let $\varphi \in S^n(X; R)$, $\theta \in S^m(X; R)$ and $\psi \in (S^k(X; R)$. Then, for any $(n + m + k)$-simplex σ,

$$\begin{aligned}(\sigma, \varphi \smile (\theta \smile \psi)) &= (\sigma\lambda_n^{n+m+k}, \varphi)(\sigma\mu_{m+k}^{n+m+k}, \theta \smile \psi) \\ &= (\sigma\lambda_n^{n+m+k}, \varphi)(\sigma\mu_{m+k}^{n+m+k}\lambda_m^{m+k})(\sigma\mu_{m+k}^{n+m+k}\mu_k^{m+k}, \psi).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\sigma, (\varphi \smile \theta) \smile \psi) &= (\sigma\lambda_{n+m}^{n+m+k}, \varphi \smile \theta)(\sigma\mu_k^{n+m+k}, \psi) \\ &= (\sigma\lambda_{n+m}^{n+m+k}\lambda_n^{n+m}, \varphi)(\sigma\lambda_{n+m}^{n+m+k}\mu_m^{n+m}, \theta)(\sigma\mu_k^{n+m+k}, \psi)\end{aligned}$$

Using Proposition 6.2, we see that the the above quantity is the same as the one derived earlier.

Let $e \in S^0(X; R)$ be such that $(\sigma, e) = 1$ for all $\sigma \in S_0(X)$. It is easy to see that e is a multiplicative identity for $\cup$, thereby completing the proof. ■

PROPOSITION 6.5. If $f : X \to Y$ is a continuous map, then $f^\sharp : S(Y; R) \to S(X; R)$ is a graded ring homomorphism.

Proof. Let $\varphi \in S^n(Y; R)$ and $\theta \in S^m(Y; R)$. Then, for any $(m + n)$-simplex σ in X,

$$\begin{aligned}(\sigma, f^\sharp(\varphi \smile \theta)) &= (f\sigma, \varphi \smile \theta) \\ &= (f\sigma\lambda_n^{n+m}, \varphi)(f\sigma\mu_m^{n+m}, \theta) \\ &= (\sigma\lambda_n^{n+m}, f^\sharp\varphi)(\sigma\mu_m^{n+m}, f^\sharp\theta) \\ &= (\sigma, f^\sharp\varphi \smile f^\sharp\theta).\end{aligned}$$

Finally, we must show that the identity maps to the identity. Indeed, let e' denote the identity of $S(Y; R)$. ■

LEMMA 6.6. If $\varphi \in S^n(X; R)$ and $\theta \in S^m(X; R)$, then

$$\delta(\varphi \smile \theta) = \delta\varphi \smile \theta + (-1)^n\varphi \smile \delta\theta.$$

Proof. For any $(n+m+1)$-simplex σ in X,

$$\begin{aligned}(\sigma, \delta(\varphi \smile \theta)) &= (\partial\sigma, \varphi \smile \theta)\\ &= \sum_{i=0}^{n+m+1} (-1)^i (\sigma\varepsilon_i^{n+m+1}, \varphi \smile \theta)\\ &= \sum_{i=0}^{n+m+1} (-1)^i (\sigma\varepsilon_i^{n+m+1}\lambda_n^{n+m}, \varphi)(\sigma\varepsilon_i^{n+m+1}\mu_n^{n+m}\theta).\end{aligned}$$

We invoke Proposition 6.2 (d) with $d = n+m$, $p = n$ and $q = m$ to get

$$\begin{aligned}&= \sum_{i=0}^{n}(-1)^i(\sigma\lambda_{n+1}^{n+m+1}\varepsilon_i^{n+1})(\sigma\mu_m^{n+m+1}) + \sum_{i=n+1}^{n+m+1}(-1)^i(\sigma\lambda_n^{n+m+1}, \varphi)(\sigma\mu_{m+1}^{n+m+1}\varepsilon_{i-n}^{m+1}, \theta)\\ &= \sum_{i=0}^{n}(-1)^i(\sigma\lambda_{n+1}^{n+m+1}\varepsilon_i^{n+1})(\sigma\mu_m^{n+m+1}) + (-1)^n\sum_{j=1}^{m+1}(-1)^j(\sigma\lambda_n^{n+m+1}, \varphi)(\sigma\mu_{m+1}^{n+m+1}\varepsilon_j^{m+1}, \theta).\end{aligned}$$

On the other hand, the right hand side of the theorem gives us

$$\begin{aligned}&= (\sigma\lambda_{n+1}^{n+m+1}, \delta\varphi)(\sigma\mu_m^{n+m+1}, \theta) + (-1)^n(\sigma\lambda_n^{n+m+1}, \varphi)(\sigma\mu_{m+1}^{n+m+1}, \delta\theta)\\ &= \sum_{i=0}^{n+1}(-1)^i(\sigma, \lambda_{n+1}^{n+m+1}\varepsilon_i^{n+1}, \varphi)(\sigma\mu_m^{n+m+1}, \theta) + (-1)^n(\sigma\lambda_n^{n+m+1}, \varphi)\sum_{j=0}^{m+1}(-1)^j(\sigma\mu_{m+1}^{n+m+1}\varepsilon_j^{m+1}, \theta).\end{aligned}$$

Note that

$$(\sigma, \lambda_{n+1}^{n+m+1}\varepsilon_{n+1}^{n+1}, \varphi)(\sigma\mu_m^{n+m+1}, \theta) - (\sigma\lambda_n^{n+m+1}, \varphi)\sum_{j=0}^{m+1}(-1)^j(\sigma\mu_{m+1}^{n+m+1}\varepsilon_0^{m+1}, \theta) = 0.$$

This completes the proof. ■

PROPOSITION 6.7. The cup product descends to a map $\smile: H(X;R) \times H(X;R) \to H(X;R)$, thereby giving $H(X;R)$ the structure of a ring.

Proof. Let $Z(X;R)$ denote $\bigoplus_{n \geqslant 0} Z^n(X;R)$ and $B(X;R)$ denote $\bigoplus_{n \geqslant 0} B^n(X;R)$. Let $n, m \geqslant 0$, and consider $\smile$: $H^n(X;R) \times H^m(X;R) \to H^{n+m}(X;R)$ given by

$$[\varphi] \smile [\theta] = [\varphi \smile \theta].$$

First, we show that this is well defined. Indeed, let $\varphi' = \varphi + \alpha$ and $\theta' = \theta + \beta$ where $\alpha \in B^n(X;R)$ and $\beta \in B^m(X;R)$. Then,

$$\varphi' \smile \theta' = \varphi \smile \theta + \alpha \smile \theta + \varphi \smile \beta + \alpha \smile \beta.$$

Let $\alpha = \delta\omega$ and $\beta = \delta\eta$ for some $\omega \in S^{n+1}(X;R)$ and $\eta \in S^{m+1}(X;R)$. The above is

$$\varphi \smile \theta + \delta\omega \smile \theta + \varphi \smile \delta\eta + \delta\omega \smile \delta\eta.$$

It is easy to see that

$$\begin{aligned}\delta\omega \smile \theta &= \delta(\omega \smile \theta)\\ \varphi \smile \delta\eta &= (-1)^n\delta(\varphi \smile \eta)\\ \delta\omega \smile \delta\eta &= \delta(\omega \smile \delta\eta).\end{aligned}$$

This shows that $\smile$ is well-defined and bilinear. The remaining proof proceeds just as before. ■

References

[Rot13] J.J. Rotman. *An Introduction to Algebraic Topology*. Graduate Texts in Mathematics. Springer New York, 2013.