Hartshorne Exercises

Swayam Chube

December 2, 2024

Contents

I Schemes 2

1 Sheaves 2

2 Schemes 3

3 First Properties of Schemes 7

4 Separated and Proper Morphisms 10

5 Sheaves of Modules 12

Part I
Schemes

1 Sheaves

DEFINITION. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of abelian groups on X. The association $U \mapsto \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf on X. It is called the *sheaf Hom* and is denoted by $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

EXERCISE 1.15.

2 Schemes

EXERCISE 2.3 (REDUCED SCHEMES).

(a) Suppose X is reduced. Then, every open affine corresponds to a reduced ring. Consequently, the local ring of any point on X is the localisation of a reduced ring and hence, is reduced.

Conversely, suppose $\mathscr{O}_{X,P}$ is reduced for every $P \in X$. Let $U = \operatorname{Spec} A$ be an affine open. The local ring of any point $P \in U$ is a localisation of A at a prime. Since all these rings are reduced, so is A.

Let $U \subseteq X$ be open. Cover U with affine opens $U_i = \operatorname{Spec} A_i$ and let $s \in \mathscr{O}(U)$ be nilpotent. Its image $s_i = \mathrm{res}_{U,U_i}(s)$ is nilpotent in $\mathscr{O}(U_i) = A_i$ and hence, $s_i = 0$. Consequently $s = 0$ due to the identity axiom. This shows that $\mathscr{O}(U)$ is reduced.

(b) The first part follows immediately from the fact that there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{red} \arrow[r, "\phi_{red}"'] & B_{red}. \end{tikzcd}$$

Consider the map of locally ringed spaces $(\mathbf{id}, f^\sharp)$, where $f^\sharp : \mathscr{O}_X \to \mathscr{O}_X^{red}$ is the collection of the canonical maps $\mathscr{O}_X(U) \to \mathscr{O}_X^{red}(U)$.

(c) Follows from the fact that any morphism of rings $\phi : A \to B$ with B reduced factors through the natural map $A \to A_{red}$.

EXERCISE 2.4. Let $\varphi \in \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X))$. Cover X with affine opens $U_i = \operatorname{Spec} A_i$. The restriction map gives us a homomorphism

$$A \xrightarrow{\varphi} \Gamma(X, \mathscr{O}_X) \xrightarrow{\mathrm{res}^X_{U_i}} \Gamma(U_i, \mathscr{O}_X) = A_i,$$

which induces a map on schemes $\pi_i : U_i \to \operatorname{Spec} A$ where $\pi_i = \operatorname{Spec}(\mathrm{res}^X_{U_i} \circ \varphi)$.

We contend that the maps π_i can be glued. Indeed, for $i \neq j$, cover $U_i \cap U_j$ with affine opens $U_{ijk} = \operatorname{Spec} A_{ijk}$. Now,

$$\pi_i|_{U_{ijk}} = \operatorname{Spec}(\mathrm{res}^{U_i}_{U_{ijk}}) \circ \pi_i = \operatorname{Spec}(\mathrm{res}^{U_i}_{U_{ijk}} \circ \mathrm{res}^X_{U_i} \circ \varphi) = \operatorname{Spec}(\mathrm{res}^X_{U_{ijk}} \circ \varphi).$$

Similarly, $\pi_j|_{U_{ijk}} = \operatorname{Spec}(\mathrm{res}^X_{U_{ijk}} \circ \varphi)$, consequently, the family of morphisms $\{\pi_i\}$ can be glued to a morphism $\pi : X \to \operatorname{Spec} A$. This gives a map

$$\beta : \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X)) \to \mathrm{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A).$$

It is straightforward to verify that α and β are inverses to one another.

EXERCISE 2.5. Follows from the previous exercise and the fact that $\mathbb{Z}$ is an initial object in the category of rings.

EXERCISE 2.7. Let $(f, f^\sharp) : \operatorname{Spec} K \to X$ is a morphism of schemes which sends the unique point in $\operatorname{Spec} K$ to $x \in X$. Then, there is an induced map on local rings $f_x^\sharp : \mathscr{O}_x \to K$, which must be local and hence, factor through the maximal ideal of $\mathscr{O}_x$, thereby inducing a map $k(x) \to K$. It is easy to see that this process is reversible.

EXERCISE 2.9. Let $Z \subseteq X$ be irreducible and closed. Let $U = \operatorname{Spec} A$ be an open affine intersecting Z. Then, $Z \cap U$ is open in Z and hence, is irreducible. Further, it is closed in U and hence, corresponds to a prime ideal $\xi = \mathfrak{p} \in \operatorname{Spec} A$. Note that $\overline{\{\xi\}} \cap U = Z \cap U$ and $\overline{\{\xi\}} \subseteq Z$ since Z is closed.

Let V be any other open set intersecting Z. Then, one can replace V with an open affine $\operatorname{Spec} B$ intersecting Z. Suppose $\xi \notin V$. Then,

$$(Z \cap U) \cap (Z \cap V) = Z \cap U \cap V = \overline{\{\xi\}} \cap U \cap V = \varnothing,$$

since the closure of $\{\xi\}$ in U is contained in $U \setminus V$. This is not possible since $Z \cap U$ and $Z \cap V$ are nonempty open sets in an irreducible space. Hence, ξ is a generic point.

Now we argue for uniqueness. Suppose ξ_1 and ξ_2 were two generic points in Z. Consider an affine neighborhood $U = \operatorname{Spec} A$ intersecting Z. Then, $Z \cap U$ must contain ξ_1 and ξ_2. Let ξ_i correspond to a prime $\mathfrak{p}_i$ in A for $i = 1, 2$. Now, $Z \cap U = V(\mathfrak{p}_1) = V(\mathfrak{p}_2)$, consequently, $\mathfrak{p}_1 = \mathfrak{p}_2$, that is, $\xi_1 = \xi_2$. This completes the proof.

DEFINITION. Let $(X, \mathscr{O}_X)$ be a scheme and let $f \in \Gamma(X, \mathscr{O}_X)$. Define X_f to be the set of all $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathscr{O}_{X,x}$. This is known as the *support* of f on X.

EXERCISE 2.16.

(a) The set of all $x \in U$ such that $f_x \notin \mathfrak{m}_x$ is the set of all prime ideals $\mathfrak{p}$ in B such that $f/1$ is not in the maximal ideal $\mathfrak{p}B_\mathfrak{p}$ in $B_\mathfrak{p}$. Equivalently, $f \notin \mathfrak{p}$. Thus, $X_f \cap U = D(\bar{f})$. Now, since X can be covered with open affines and the intersection of X_f with every open affine is open, X_f must also be open.

(b) Pick a finite open cover $\{U_i = \operatorname{Spec} A_i\}_{i=1}^m$. The restriction of a to $X_f \cap U_i = D(\operatorname{res}^X_{U_i}(f))$ is zero and hence, there is a positive integer n_i such that $\operatorname{res}^X_{U_i}(f^{n_i} a) = 0$. Let $N = \max\limits_{1 \leqslant i \leqslant m} n_i$. Then, $\operatorname{res}^X_{U_i}(f^N a) = 0$. Due to the identity axiom, we must have $f^N a = 0$.

(c) Let $U_i = \operatorname{Spec} A_i$ and let $f_i = \operatorname{res}^X_{U_i}(f)$. Since $X_f \cap U_i = D(f_i)$, there is a $b_i \in A_i = \Gamma(U_i, \mathscr{O}_X)$ such that $\operatorname{res}^X_{U_i \cap X_f}(b) = \frac{b_i}{f_i^{n_i}}$ for some nonnegative integer n_i. Choosing n to be larger than all the n_i's, we get that there is a $b_i \in A_i$ such that $\operatorname{res}^X_{U_i \cap X_f}(f^n b) = \operatorname{res}^{U_i}_{U_i \cap X_f}(b_i)$.

Now consider $b_i - b_j$ on $U_i \cap U_j$, which can be covered by finitely many affine opens $U_{ijk} = \operatorname{Spec} A_{ijk}$. Since $\operatorname{res}^X_{U_i \cap U_j \cap X_f}(b_i - b_j) = 0$, using a similar argument as in (b), there is a positive integer m_{ij} such that $f^{m_{ij}}(b_i - b_j)$ restricts to 0 On $U_i \cap U_j$. Choosing m larger than m_{ij} for all pairs i, j, we have that $f^m(b_i - b_j)$ restricts to 0 on $U_i \cap U_j$. Consequently, $\mathrm{res}^{U_i}_{U_i \cap U_j}(f^m b_i) = \mathrm{res}^{U_j}_{U_i \cap U_j}(f^m b_j)$ and hence, there is a $c \in \Gamma(X, \mathscr{O}_X)$ such that $\mathrm{res}^X_{U_i}(c) = f^m b_i$. Hence, $\mathrm{res}^X_{U_i \cap X_f}(c) = \mathrm{res}^X_{U_i \cap X_f}(f^{n+m} b)$. This completes the proof.

(d) First, we show that $\mathrm{res}^X_{X_f}(f)$ is invertible. Since $f_x \notin \mathfrak{m}_x \subseteq \mathscr{O}_x$ for every $x \in X_f$, we see that the restriction of f to every affine open contained in X_f must be invertible (else it would lie in a prime ideal and hence, in the stalk of some point). Consider an open cover U_i of X_f using affine opens. There is a $g_i \in \Gamma(U_i, \mathscr{O})$ such that $g_i \,\mathrm{res}^X_{U_i}(f) = 1$. For $i \neq j$, we have

$$\mathrm{res}^{U_i}_{U_i \cap U_j}(g_i)\,\mathrm{res}^X_{U_i \cap U_j}(f) = 1 = \mathrm{res}^{U_j}_{U_i \cap U_j}(g_j)\,\mathrm{res}^X_{U_i \cap U_j}(f)$$

and hence, $\mathrm{res}^{U_i}_{U_i \cap U_j}(g_i) = \mathrm{res}^{U_j}_{U_i \cap U_j}(g_j)$ and hence, the g_i's can be lifted to some $g \in \Gamma(X_f, \mathscr{O}_X)$, furthermore $\mathrm{res}^X_{X_f}(f) g = 1$, whence invertibility follows.

Consider the map $\Phi : A_f \to \Gamma(X_f, \mathscr{O}_X)$ given by

$$\frac{a}{f^n} \mapsto \frac{\mathrm{res}^X_{X_f}(a)}{\mathrm{res}^X_{X_f}(f^n)}.$$

If $\Phi(a/f^n) = 0$, then $\mathrm{res}^X_{X_f}(a) = 0$, consequently, due to part (b), there is a positive integer m such that $f^m a = 0$, equivalently, $a/f^n = 0$ in A_f. Hence, Φ is injective.

As for surjectivity, let $b \in \Gamma(X_f, \mathscr{O}_X)$. Due to part (c), there is a positive integer m such that $f^m b = \mathrm{res}^X_{X_f}(a)$ for some $a \in A$ whence $\Phi(a/f^m) = b$. This completes the proof.

EXERCISE 2.17 (A CRITERION FOR AFFINENESS).

(a) Each $f : f^{-1}U_i \to U_i$ has an inverse $g_i : U_i \to f^{-1}U_i$ that agrees on intersections since inverses are unique. These maps can be glued to give an inverse $g : Y \to X$ of f.

(b) First, note that $X = \bigcup_{i=1}^{n} X_{f_i}$, for if not, then there is an $x \in X$ such that $x \notin X_{f_i}$ for $1 \leqslant i \leqslant n$. Consider an affine open $U = \operatorname{Spec} B$ containing x and let $\mathfrak{p}$ be the prime corresponding to x. According to our hypothesis, $\mathrm{res}^X_U(f_i) \in \mathfrak{p}$ for $1 \leqslant i \leqslant n$. But these restrictions generate the unit ideal, a contradiction.

Being a finite union of affine opens, X is quasi-compact. Further, $X_{f_i} \cap X_{f_j}$ is a distinguished open in X_{f_i} and hence, is quasi-compact. As a result, Exercise 2.16 (d) is applicable. Using Exercise 2.4 and glueing morphisms just as in part (a), we are done.

DEFINITION. A morphism $f : X \to Y$ of schemes is said to be *dominant* if $f(X)$ is dense in Y.

EXERCISE 2.18.

(a) Intersection of all prime ideals is the nilradical.

(b) We denote the morphism by $\pi : Y \to X$. If $\pi^\sharp$ is injective, then taking global sections, we obtain that φ is injective. Conversely, suppose φ is injective. It suffices to show that $\varphi^\sharp$ is injective on the $D(f)$'s since these form a base on X. We have

$$\pi^\sharp_{D(f)} : \mathscr{O}_X(D(f)) \to \mathscr{O}(\pi^{-1}(D(f)) \equiv \pi^\sharp_{D(f)} : A_f \to B_f,$$

which is injective. This proves the first part.

Next, we must show that π is dominant if φ is injective. Indeed, suppose $\pi(Y)$ were not dense, then there would be a basic open set $D(f)$ in Spec A such that $\pi^{-1}D(f) = \varnothing$, equivalently, $f \in \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of B. Hence, f is nilpotent in B, whence nilpotent in A, consequently, $D(f) = \varnothing$. This completes the proof.

(c) We denote the morphism by π. The first part follows from the fact that Spec $A/\mathfrak{a} \hookrightarrow$ Spec A is a topological imbedding. The second part is argued in a similar way as (b) by first concluding surjectivity on basic opens $D(f)$. Then, taking stalks, it follows that $\pi^\sharp$ is surjective.

(d)

3 First Properties of Schemes

LEMMA 3.1 (AFFINE COMMUNICATION LEMMA).

DEFINITION. A morphism $f : X \to Y$ of schemes is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}V_i$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

The morphism f is *of finite type* if in addition each $f^{-1}V_i$ can be covered by a finite number of the U_{ij}.

DEFINITION. A morphism $f : X \to Y$ is a *finite* morphism if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}V_i$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a finite B_i-module.

EXERCISE 3.1. Let $\pi : X \to Y$ denote the morphism. We use Lemma 3.1. To this end, we first show that if $\operatorname{Spec} B \subseteq Y$ is an affine open such that $\pi^{-1} \operatorname{Spec} B$ can be covered by affine opens $U_i = \operatorname{Spec} A_i$, each of which is a finitely generated B-algebra, then the same is true for $\operatorname{Spec} B_f$, where $f \in B$. Now, $\pi^{-1} \operatorname{Spec} B_f \subseteq \pi^{-1} \operatorname{Spec} B$ and hence, is contained in $\bigcup U_i$. Consider $\pi^{-1} \operatorname{Spec} B_f \cap U_i$. This can be written as a union of $D(f_{ij})$'s where $f_{ij} \in A_i$. Note that $D(f_{ij}) = \operatorname{Spec}(A_i)_{f_{ij}}$, which is a finitely generated A_i algebra, whence a finitely generated B-algebra, consequently, a finitely generated B_f-algebra. This proves the first condition of Lemma 3.1.

Next, suppose $(1) = (f_1, \dots, f_n)$ in B and $\operatorname{Spec} B_{f_i}$ has the desired property. Then obviously B has the property, since B_{f_i} is a finitely generated B-algebra, and hence, any finitely generated B_{f_i}-algebra will be a finitely generated B-algebra.

DEFINITION. A morphism $f : X \to Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}V_i$ is quasi-compact for each i.

EXERCISE 3.2. Let $\pi : X \to Y$ denote the morphism. We use Lemma 3.1. To this end, it suffices to show that if $\operatorname{Spec} A \subseteq Y$ is an affine open such that $\pi^{-1} \operatorname{Spec} A$ is quasi-compact, then for any $f \in A = \Gamma(\operatorname{Spec} A, \mathscr{O}_A)$, $\pi^{-1} \operatorname{Spec} A_f$ is quasi-compact. We wish to characterize

$$\{P \in \pi^{-1} \operatorname{Spec} A \colon f \notin \pi(\mathfrak{p}) = \mathfrak{p} \in \operatorname{Spec} A\}.$$

We have the map $\pi_P^\sharp : \mathscr{O}_{Y,\pi(P)} \to \mathscr{O}_{X,P}$. If $f \in \mathfrak{p} = \pi(P)$, then $f \in \mathfrak{m}_{Y,P}$ and hence, $\pi_P^\sharp f \in \mathfrak{m}_{X,P}$ (since $\pi_P^\sharp$ is a local homomorphism). On the other hand, if $f \notin \mathfrak{p}$, then $f/1 = 1/1$ in $\mathscr{O}_{Y,\pi(P)} = A_\mathfrak{p}$, consequently, $\pi_P^\sharp f = 1 \notin \mathfrak{m}_{X,P}$.

Thus, the set we are looking for is the *complement* of $(\pi^{-1} \operatorname{Spec} A)_{\pi^\sharp f}$, the latter being closed in the open subscheme $\pi^{-1} \operatorname{Spec} A$, due to Exercise 2.16. Since $\pi^{-1} \operatorname{Spec} A$ is quasi-compact, we can cover it with open affines. Let $U = \operatorname{Spec} B$ be one such affine. Then, $\operatorname{res} \pi^\sharp f \in \mathscr{O}_B$ and the set of desired points $\mathfrak{p}$ are precisely those in $D(\operatorname{res} \pi^\sharp f)$, consequently, is quasi-compact. Being a finite union of quasi-compact sets, the required complement is quasi-compact.

EXERCISE 3.3.

(a) $\Longrightarrow$ Obviously a morphism of finite type is locally of finite type. On the other hand, with the notation of the above above definitions, since $f^{-1}V_i$ can be covered by finitely many U_{ij}'s, it is a finite union of quasi-compact spaces, whence is quasi-compact. Thus, f is a quasi-compact morphism.

$\Longleftarrow$ On the other hand, suppose $f : X \to Y$ is locally of finite type and quasi-compact. Then, due to Exercise 3.2, $f^{-1}V_i$ is quasi-compact, whence can be covered by finitely many of the U_{ij}'s. Thus, f is of finite type.

(b)

(c)

EXERCISE 3.4. Let $\pi : X \to Y$ denote the morphism. We use Lemma 3.1. Suppose $V = \operatorname{Spec} B$ can be covered by distinguished opens $V_i = \operatorname{Spec} B_{f_i}$ for $1 \leqslant i \leqslant n$ such that each V_i has the desired property. We shall show that V has the desired property. Let $U = \pi^{-1}V_i = \operatorname{Spec} A_i$ where A_i is a finite B_{f_i}-module. Let $A = \Gamma(U, \mathscr{O}_X)$. Then, the morphism π induces a homomorphism $\varphi : B \to A$ of rings making

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d, "\mathrm{res}^{U}_{U_i}"] \\ B_{g_i} \arrow[r] & A_i \end{tikzcd}$$

commute. Using the above diagram, it is not hard to argue that $U_{g_i} = A_i$, consequently, Exercise 2.17 shows that U is affine and equal to $\operatorname{Spec} A$.

We have reduced the algebraic geometry problem to the following commutative algebra problem:

> Let $\varphi : B \to A$, let $f_1, \dots, f_n$ generate the unit ideal in B and let $g_i = \varphi(f_i)$. Suppose A_{g_i} is a finite B_{f_i} module for $1 \leqslant i \leqslant n$. Then A is a finite B-module.

add in

DEFINITION. A morphism $\pi : X \to Y$ is *quasi-finite* if for every $y \in Y$, $\pi^{-1}(y)$ is a finite set.

EXERCISE 3.5.

(a) This is essentially asking us to show that if B is an A-algebra that is a finite A-module, then for every $\mathfrak{p} \in \operatorname{Spec} A$, the fiber over $\mathfrak{p}$ in B is finite. Recall that the fiber over $\mathfrak{p}$ is precisely $\operatorname{Spec}(\kappa(\mathfrak{p}) \otimes_A B)$, which is the spectrum of a $\kappa(\mathfrak{p})$-algebra that is also a finite $\kappa(\mathfrak{p})$-module, i.e. the spectrum of an artinian ring, whence is finite.

(b) Follows from the commutative algebra fact that integral morphisms induce closed maps on the spectrum.

(c) add

DEFINITION. A morphism $\pi : X \to Y$, with Y irreducible is *generically finite* if $\pi^{-1}(\eta)$ is a finite set, where η is the generic point of Y.

EXERCISE 3.7. Let $\pi : X \to Y$ denote the morphism. Let ξ be the generic point of X and η the generic point of Y. First, we show that $\pi(\xi) = \eta$. Indeed,

$$\pi(X) = \pi(\overline{\{\xi\}}) \subseteq \overline{\{\pi(\xi)\}}.$$

But since π is dominant, $\pi(X)$ is dense in Y, consequently, $\pi(\xi)$ must be a generic point, hence, equal to η.

EXERCISE 3.11 (CLOSED SUBSCHEMES).

(a)

(b) We may suppose, without loss of generality that $Y \subseteq X$. For a point $P \in Y$, choose an open affine neighborhood $U = \operatorname{Spec} C$ of P in Y. Then, there is an $f \in A$ such that $P \in D(f) \cap Y \subseteq U$. We contend that $D(f) \cap Y$ is a distinguished open in U. Indeed, the inclusion $\iota : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ restricted to U induces a map of rings $\varphi : A \to C$. It is easy to see that $\iota^{-1}(D(f)) = D_U(\varphi(f))$, consequently, $D(f) \cap Y$ is a distinguished open in U.

Next, cover X with $D(f_i)$'s such that $D(f_i) \cap Y$ is either affine in Y, or nonempty. Let $\overline{f}_i = \iota_X^\sharp(f_i) \in \Gamma(Y, \mathscr{O}_Y)$. We claim that $Y_{\overline{f}_i} = D(f_i) \cap Y$. Indeed, if $P \in D(f_i) \cap Y$, then there is a surjective map of stalks

$$\mathscr{O}_{X,P} \to \mathscr{O}_{Y,P}$$

sending f_i to $\overline{f}_i$. Since f_i is invertible in the former, it must be invertible in the latter. On the other hand, if $P \in Y_{\overline{f}_i}$, then $\overline{f}_i$ is invertible in the latter whence, cannot lie in the maximal ideal $\mathfrak{m}_{X,P}$, since the above map is a local homomorphism of local rings. This shows that $D(f_i) \cap Y = Y_{f_i}$.

Combining our above discussion with Exercise 2.17 (b), we have that Y is affine. Next, we must show that Y is obtained as the quotient of an ideal in A. For this, invoke Exercise 2.18 (d).

EXERCISE 3.12.

EXERCISE 3.13 (PROPERTIES OF MORPHISMS OF FINITE TYPE).

EXERCISE 3.14. It suffices to assume X is locally of finite type over k. In which case, there is a cover $U_i = \operatorname{Spec} A_i$ of X such that each A_i is a finitely generated k-algebra and hence, a Jacobson ring. Consequently, the closed points of U_i are dense in U_i, whence the closed points of X are dense in X.

As for a counterexample for arbitrary schemes, consider $\operatorname{Spec} A$ where A is a ring such that $\mathfrak{R} \neq \mathfrak{N}$.

4 Separated and Proper Morphisms

Definition. A morphism $\pi : X \to Y$ of schemes is said to be *separated* if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion.

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathbf{id}_X"] \arrow[ddr, bend right, "\mathbf{id}_X"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "\pi"] \\
& X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

Definition. A morphism $\pi : X \to Y$ is said to be *universally closed* if it is closed as a continuous map on the underlying topological spaces and for every morphism $Y' \to Y$, the map obtained by *base extension* $X \times_Y Y' \to Y'$ is also closed.

Definition. A morphism $\pi : X \to Y$ is said to be *proper* if it is separated, of finite type and universally closed.

Since a complete proof of the following is not provided in the text, I reproduce it here.

Corollary (Hartshorne, II.4.6). Assume that all schemes are noetherian in the following statements.

(a) Open and closed immersions are separated.

(b) A composition of two separated morphisms is seprated.

(c) Separated morphisms are stable under base extension.

(d) If $\pi : X \to Y$ and $\pi' : X' \to Y'$ are separated morphisms of schemes over a base scheme S, then the *product morphism* $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is also separated.

(e) If $\pi : X \to Y$ and $\varphi : Y \to Z$ are two morphisms and if $\varphi \circ \pi$ is separated, then π is separated.

(f) A morphism $\pi : X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $\pi^{-1}V_i \to V_i$ is separated for each i.

Proof. (a) We show more generally that "a monomorphism of schemes is separated". Let $Y \hookrightarrow X$ be a monomorphism in $\mathfrak{Sch}_{\mathbb{Z}}$. Then, the fiber product $Y \times_X Y$ is precisely Y, given by the following diagram.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y \arrow[r, "\mathbf{id}"] \arrow[d, "\mathbf{id}"] & Y \arrow[d, hook] \\
& Y \arrow[r, hook] & X
\end{tikzcd}$$

Since $Y \hookrightarrow X$ is a monomorphism, the two maps $Z \to Y$ in the above diagram must be the same and it follows that $Y = Y \times_X Y$. Hence, the diagonal morphism $\Delta : Y \to Y \times_X Y$ is the identity map, whence is a closed immersion.

(b) We use the valuative criterion. Let R be a DVR and K its fraction field. Let $U = \operatorname{Spec} K$ and $T = \operatorname{Spec} R$ and suppose $\pi : X \to Y$ and $\varphi : Y \to Z$ are separated. Let there be a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dd] & X \arrow[d] \\ & Y \arrow[d] \\ T \arrow[r] & Z. \end{tikzcd}$$

Suppose there are two lifts $\psi_1, \psi_2 : T \to X$ making the diagram commute. Then, $\pi \circ \psi_1 = \pi \circ \psi_2$ since $Y \to Z$ is separated. Finally, since $X \to Y$ is separated, we must have $\psi_1 = \psi_2$. This shows that $X \to Z$ is separated.

(c) This is done in the book.

(d) The same idea as in (b) works. Not writing this up because the diagram is too complicated to draw and I'm too lazy.

(e) Again, begin with a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

and suppose there are two lifts $\psi_1, \psi_2 : T \to X$ making the diagram commute. Since $X \to Z$ is separated, we must have that $\psi_1 = \psi_2$. Hence, $X \to Y$ is separated.

(f)

$\blacksquare$

5 Sheaves of Modules

DEFINITION. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$. It is said to be *locally free* if X has an open cover by sets U for which $\mathscr{F}|_U$ is a free $\mathscr{O}_X|_U$-module.

EXERCISE 5.7.

(a) We reduce this to the affine case since $\mathscr{F}$ is coherent on a noetherian scheme. Thus, we have a finitely generated A-module M and a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module.

Choose a basis $\{\frac{m_1}{1}, \dots, \frac{m_n}{1}\}$ of $M_\mathfrak{p}$ over $A_\mathfrak{p}$ and consider the exact sequence

$$0 \to K \to A^n \to M \to Q \to 0,$$

where the map $A^n \to M$ is the natural map sending $e_i \mapsto m_i$ for $1 \leqslant i \leqslant n$. Localising, we see that $K_\mathfrak{p} = Q_\mathfrak{p} = 0$ and hence, there is an $f \in A \setminus \mathfrak{p}$ such that $K_f = Q_f = 0$ (since both K and Q are finitely generated). Localising the above exact sequence at f, we obtain an isomorphism $A_f^n \xrightarrow{\sim} M_f$. It follows that $M_\mathfrak{q}$ is a free $A_\mathfrak{q}$ module for all $\mathfrak{q} \in D(f)$.

(b) Follows immediately from (a).

(c) Let $\mathscr{F}^\vee$ denote the dual sheaf. Recall that

$$\mathscr{F}^\vee(U) = \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{O}_X|_U).$$

This gives a natural map $\mathscr{F}(U) \otimes \mathscr{F}^\vee(U) \to \mathscr{O}_X(U)$ given by

$$s \otimes \varphi \mapsto \varphi_U(s).$$

It is easy to check that this is a morphism of presheaves $\mathscr{F} \otimes \mathscr{F}^\vee \to \mathscr{O}_X$ and since the latter is a sheaf, it factors through the sheafification inducing a map on the tensor sheaf.

We contend that this induced map is an isomorphism. To this end, it suffices to show that the induced morphism on stalks is an isomorphism.