Category Theory

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December 13, 2024

Contents

Part I
General Categories

§1 Categories, Functors, and Natural Transformations

Definition 1.1. A *category* $\mathscr{A}$ consists of:

- a collection $\mathrm{ob}(\mathscr{A})$ of objects,
- for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of *maps*, *arrows*, or *morphisms* from A to B,
- for each $A, B, C \in \mathrm{ob}(\mathscr{A})$, a function

$$\mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C) \qquad (g, f) \mapsto g \circ f,$$

  called *composition*, and
- for each $A \in \mathrm{ob}(\mathscr{A})$, an element $1_A \in \mathscr{A}(A, A)$, called the *identity* on A,

satisfying the following axioms:

associativity: for each $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$,

identity: for each $f \in \mathscr{A}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Abusing notation, we often write $A \in \mathscr{A}$ instead of $A \in \mathrm{ob}(\mathscr{A})$.

Definition 1.2. A map $f : A \to B$ in a category $\mathscr{A}$ is said to be an *isomorphism* if there exists a map $g : B \to A$ in $\mathscr{A}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Definition 1.3. Every category $\mathscr{A}$ has an *opposite* category $\mathscr{A}^{op}$ given by simply reversing the arrows of $\mathscr{A}$. Note that $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{A}^{op})$.

Definition 1.4. Let $\mathscr{A}, \mathscr{B}$ be categories. A (covariant) *functor* $F : \mathscr{A} \to \mathscr{B}$ consists of:

- a function $\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B})$ written as $A \mapsto F(A)$, and
- for each $A, A' \in \mathscr{A}$, a function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$ written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$, and
- $F(1_A) = 1_{F(A)}$ for every $A \in \mathscr{A}$.

A *contravariant functor* from $\mathscr{A}$ to $\mathscr{B}$ is simply a functor $F : \mathscr{A}^{op} \to \mathscr{B}$.

Definition 1.5. Let $\mathscr{A}$ be a category. A *presheaf* on $\mathscr{A}$ is a functor $\mathscr{A}^{op} \to \mathbf{Set}$.

Definition 1.6. A functor $F : \mathscr{A} \to \mathscr{B}$ is *faithful* (resp. *full*) if for each $A, A' \in \mathscr{A}$, the function

$$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A')) \qquad f \mapsto F(f)$$

is injective (resp. surjective).

Definition 1.7. Let $\mathscr{A}$ be a category. A *subcategory* $\mathscr{S}$ of $\mathscr{A}$ consists of a subclass $\mathrm{ob}(\mathscr{A})$ of $\mathrm{ob}(\mathscr{A})$ together with, for each $S, S' \in \mathrm{ob}(\mathscr{S})$, a subclass $\mathscr{S}(S, S')$ of $\mathscr{A}(S, S')$ such that $\mathscr{S}$ is closed under composition and identities. It is a *full* subcategory if $\mathscr{S}(S, S') = \mathscr{A}(S, S')$ for all $S, S' \in \mathrm{ob}(\mathscr{S})$.

Remark 1.8. For a subcategory $\mathscr{S}$ of $\mathscr{A}$, there is a natural "inclusion" functor $\mathscr{S} \to \mathscr{A}$. The subcategory is said to be full if this functor is full.

Definition 1.9. Let $\mathscr{A}, \mathscr{B}$ be categories and $F, G : \mathscr{A} \to \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \implies G$ is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes. The maps α_A are called the *components* of α.

If $\alpha : F \implies G$ and $\beta : G \implies H$ are natural transformations, then we can define a natural transformation $\beta \circ \alpha : F \implies H$ by setting $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for every $A \in \mathscr{A}$. Further, there is also the identity natural transformation $1_F : F \implies F$ where every component is the identity map on $F(A)$ for every $A \in \mathscr{A}$.

Definition 1.10. For categories $\mathscr{A}, \mathscr{B}$, we construct the *functor category* $[\mathscr{A}, \mathscr{B}]$ or $\mathscr{B}^{\mathscr{A}}$ whose objects are functors $\mathscr{A} \to \mathscr{B}$ and morphisms are natural transformations between functors.

An isomorhpism in $[\mathscr{A}, \mathscr{B}]$ is called a *natural isomorphism* between two functors. We say that two functors are *naturally isomorphic* if there exists a natural isomorphism between them.

Lemma 1.11. Let $\alpha : F \implies G$, where $F, G : \mathscr{A} \to \mathscr{B}$ are functors. Then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathscr{A}$.

Proof. Suppose α is a natural isomorphism. Then, there is a $\beta : G \implies F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$ in $[\mathscr{A}, \mathscr{B}]$. Thus, $\beta_A \circ \alpha_A = 1_{F(A)}$ and $\alpha_A \circ \beta_A = 1_{G(A)}$ for every $A \in \mathscr{A}$. It follows that both β_A and α_A are isomorphisms.

Conversely, suppose every α_A is an isomorhpism. Let β_A denote its inverse. This gives a collection of maps $\beta = (\beta_A)_{A \in \mathscr{A}}$. It suffices to show that $\beta : G \implies F$ is a natural transformation, which reduces to proving the desired square commutes and is trivial. ■

Definition 1.12. Given functors $F, G : \mathscr{A} \to \mathscr{B}$, we say that

$$F(A) \cong G(A) \textit{ naturally in } A$$

if F and G are naturally isomorphic.

Definition 1.13. An *equivalence* between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, together with natural isomorphisms

$$\eta : 1_{\mathscr{A}} \implies G \circ F \quad \text{and} \quad \varepsilon : F \circ G \implies 1_{\mathscr{B}}.$$

If this happens, we say that $\mathscr{A}$ and $\mathscr{B}$ are *equivalent*, and write $\mathscr{A} \simeq \mathscr{B}$. We also say that the functors F and G are *equivalences*.

Definition 1.14. A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$.

Theorem 1.15. A functor is an equivalence if and only if it is full, faithful, and essentially surjective on objects.

Proof. Suppose $F : \mathscr{A} \to \mathscr{B}$ is an equivalence of categories. Then, there is a functor $G : \mathscr{B} \to \mathscr{A}$ and natural isomorphisms $\eta : GF \implies 1_{\mathscr{A}}$ and $\varepsilon : 1_{\mathscr{B}} \implies FG$. Hence, there are commuting squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\ G(F(A)) \arrow[r, "G(F(f))"'] & G(F(A')) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(G(B)) \arrow[r, "F(G(g))"] \arrow[d, "\varepsilon_B"'] & F(G(B')) \arrow[d, "\varepsilon_{B'}"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

for each $f \in \mathscr{A}(A, A')$ and $g \in \mathscr{B}(B, B')$. Faithfulness follows from the fact that the η_*'s and ε_*'s are isomorphisms. To see fullness, replace f by $\widetilde{f} = \eta_{A'}^{-1} \circ f \circ \eta_A$ and note that $G(F(\widetilde{f})) = f$ and similarly in the other commuting square. Essential surjectivity on objects follows immediately by looking at the second square, taking $B = B'$ and $g = 1_B$.

Conversely, suppose F is full, faithful, and essentially surjective on objects. For each $B \in \mathscr{B}$, choose an object $G(B) \in \mathscr{A}$ and an isomorphism $\varepsilon_B : F(G(B)) \to B$. For $B \xrightarrow{g} B'$ in $\mathscr{B}$, using the full, faithfulness of F, choose $G(g) \in \mathscr{A}(G(B), G(B'))$ such that $F(G(g)) = \varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B$.

To see that G is functorial, consider $B \xrightarrow{g} B' \xrightarrow{g''} B''$ in $\mathscr{B}$. Note that

$$F(G(g'g)) = \varepsilon_{B''}^{-1} g' g \varepsilon_B = \varepsilon_{B''}^{-1} g' \varepsilon_{B'} \varepsilon_{B'}^{-1} g \varepsilon_B = F(G(g'))F(G(g)) = F(G(g')G(g)),$$

whence, due to the faithfulness of F, $G(g' \circ g) = G(g') \circ G(g)$. Similarly, one can show that $G(1_B) = 1_{G(B)}$.

We now construct a natural isomorphism $\eta : 1_{\mathscr{A}} \implies GF$. For each $A \in \mathscr{A}$, $\eta_{F(A)} : F(G(F(A))) \to F(A)$, whence due to the full, faithfulness of F, there is a unique ε_A :

$A \to G(F(A))$ such that $F(\eta_A) = \varepsilon_{F(A)}^{-1}$ and a unique $\gamma_A : G(F(A)) \to A$ such that $F(\gamma_A) = \varepsilon_{F(A)}$.

Consider the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(\eta_A)"] \arrow[d, "F(f)"'] & F(G(F(A))) \arrow[r, "\varepsilon_{F(A)}"] \arrow[d, "F(G(F(f)))"'] & F(A) \arrow[d, "F(f)"] \\
F(A') \arrow[r, "F(\eta_{A'})"'] & F(G(F(A'))) \arrow[r, "\varepsilon_{F(A')}"'] & F(A')
\end{tikzcd}$$

where the outer and left square commute. Since ε's are isomorphisms, we see that the left square commutes too, that is,

$$F\left(G(F(f)) \circ \eta_A\right) = F(G(F(f))) \circ F(\eta_A) = F(\eta_{A'}) \circ F(f) = F(\eta_{A'} \circ f),$$

but since F is full and faithful, $\eta_{A'} \circ f = G(F(f)) \circ \eta_A$.

It remains to show that the η's are all isomorhpisms. But this is trivial, since

$$F(\gamma_A \circ \eta_A) = F(\gamma_A) \circ F(\eta_A) = \varepsilon_{F(A)} \circ F(\eta_A) = 1_{F(A)}$$

and

$$F(\eta_A \circ \gamma_A) = F(\eta_A) \circ F(\gamma_A) = F(\eta_A) \circ \varepsilon_{F(A)} = 1_{F(G(F(A)))}.$$

Again, due to the full and faithfulness of F, we have $\gamma_A \circ \eta_A = 1_A$ and $\eta_A \circ \gamma_A = 1_{G(F(A))}$, thereby completing the proof. ■

§2 Adjoints

Definition 2.1. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. We say that F is *left adjoint* to G and G is *right adjoint* to F, and write $F \dashv G$, if

$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B))$$

naturally in A and B.

We denote the above bijection by using an "overbar", that is,

$$\left(F(A) \xrightarrow{g} B\right) \mapsto \left(A \xrightarrow{\overline{g}} G(B)\right) \quad \text{and} \quad \left(A \xrightarrow{f} G(B)\right) \mapsto \left(F(A) \xrightarrow{\overline{f}} B\right),$$

where $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. We call $\overline{f}$ the *transpose* of f.

Naturality in A and B means that given $B \xrightarrow{q} B'$ in $\mathscr{B}$, we have

$$\overline{\left(F(A) \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A \xrightarrow{\overline{g}} G(B) \xrightarrow{G(q)} G(B')\right)$$

and given $A' \xrightarrow{p} A$ in $\mathscr{A}$, we have

$$\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} = \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B\right).$$

These conditions can be interpreted informally in a nicer way. If $B \xrightarrow{q} B'$ is a map in $\mathscr{B}$, then this induces a natural map

$$\mathscr{B}(F(A), B) \longrightarrow \mathscr{B}(F(A), B') \qquad g \longmapsto q \circ g$$

and

$$\mathscr{A}(A, G(B)) \longrightarrow \mathscr{A}(A, G(B')) \qquad f \mapsto G(q) \circ f.$$

We would like like

$$\begin{tikzcd}
\mathscr{B}(F(A), B) \arrow[r] \arrow[d] & \mathscr{A}(A, G(B)) \arrow[d] \\
\mathscr{B}(F(A), B') \arrow[r] & \mathscr{A}(A, G(B'))
\end{tikzcd}$$

to commute, where the horizontal maps are the "overbar" bijections and the vertical ones are the naturally induced maps as discussed above. Note that the commutativity of the above square is equivalent to

$$\overline{q \circ g} = G(q) \circ \overline{g},$$

for every $g \in \mathscr{B}(F(A), B)$. This is precisely the condition imposed on "overbar" above.

Similarly, if $A' \xrightarrow{p} A$ is a map in $\mathscr{A}$, then this induces a natural map

$$\mathscr{B}(F(A), B) \longrightarrow \mathscr{B}(F(A'), B) \qquad g \mapsto g \circ F(p),$$

and

$$\mathscr{A}(A, G(B)) \longrightarrow \mathscr{A}(A', G(B)) \qquad f \mapsto f \circ p.$$

We would like

$$\begin{tikzcd}
\mathscr{B}(F(A), B) \arrow[d] & \mathscr{A}(A, G(B)) \arrow[l] \arrow[d] \\
\mathscr{B}(F(A'), B) & \mathscr{A}(A', G(B)) \arrow[l]
\end{tikzcd}$$

to commute, where the horizontal maps are the "overbar" bijections and the vertical ones are the naturally induced maps discussed above. Note that the commutativity of the above square is equivalent to

$$\overline{f} \circ F(p) = \overline{f \circ p}.$$

Again, this is precisely the condition imposed on "overbar" above.

DEFINITION 2.2. Let $\mathscr{A}$ be a category. An object $I \in \mathscr{A}$ is *initial* for every $A \in \mathscr{A}$, there is a unique map $I \to A$. An object $T \in \mathscr{A}$ is *terminal* or *final* if for every $A \in \mathscr{A}$, there is a unique map $A \to T$.

Proposition 2.3. Initial and terminal objects are unique up to a unique isomorphism.

Proof. Suppose I and I' are initial objects. There are unique maps $I \xrightarrow{f} I'$ and $I' \xrightarrow{f'} I$. Note that $I \xrightarrow{f' \circ f} I$; but since this map is unique, it must be equal to 1_I. Similarly, $f \circ f' = 1_{I'}$, whence both f and f' are isomorphisms. The uniqueness follows trivially. An analogous proof works for terminal objects. ■

§3 Representables

Definition 3.1. Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. Define a functor $H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ as follows:

- for objects $B \in \mathscr{A}$, put $H^A(B) = \mathscr{A}(A, B)$, and
- for maps $B \xrightarrow{g} B'$ in $\mathscr{A}$, define

$$H^A(g) = \mathscr{A}(A, g) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$$

  by $p \mapsto g \circ p$ for all $A \xrightarrow{p} B$.

Consider the map from $\mathscr{A}$ to $[\mathscr{A}, \mathbf{Set}]$ given by $A \mapsto H^A$, which we denote by $H^\bullet$. If $A' \xrightarrow{f} A$ is a morphism in $\mathscr{A}$, then there is a natural transformation $H^f : H^A \implies H^{A'}$ given by $\left(H^f(B)\right)_{B \in \mathscr{A}}$, where

$$H^f(B) : \mathscr{A}(A, B) \longrightarrow \mathscr{A}(A', B) \qquad p \longmapsto p \circ f.$$

To see that this is indeed a natural transformation, let $B \xrightarrow{g} B'$ in $\mathscr{A}$. We would like

$$\begin{tikzcd} \mathscr{A}(A, B) \arrow[r, "H^A(g)"] \arrow[d, "H^f(B)"'] & \mathscr{A}(A, B') \arrow[d, "H^f(B')"] \\ \mathscr{A}(A', B) \arrow[r, "H^{A'}(g)"'] & \mathscr{A}(A', B') \end{tikzcd}$$

to commute. Indeed, some $p \in \mathscr{A}(A, B)$ mapsto $g \circ p$ in $\mathscr{A}(A, B')$ and to $g \circ p \circ f$ in $\mathscr{A}(A', B')$. On the other hand, p mapsto $p \circ f$ in $\mathscr{A}(A', B)$ and then to $g \circ p \circ f$ in $\mathscr{A}(A', B')$, as desired.

Definition 3.2. Let $\mathscr{A}$ be a locally small category. The functor $H^\bullet : \mathscr{A}^{op} \to [\mathscr{A}, \mathbf{Set}]$ is defined

- on objects A by $H^\bullet(A) = H^A$, and
- on morphisms $A' \xrightarrow{f} A$ in $\mathscr{A}$ by $H^\bullet(f) = H^f : H^A \implies H^{A'}$.

Similarly, we have the dual definitions.

Definition 3.3. Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. Define a functor $H_A = \mathscr{A}(-, A) : \mathscr{A}^{op} \to \mathbf{Set}$ as follows:

- for objects $B \in \mathscr{A}$, put $H_A(B) = \mathscr{A}(B, A)$, and
- for maps $B' \xrightarrow{g} B$ in $\mathscr{A}$, define

$$H_A(g) = \mathscr{A}(g, A) : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$$

  by $p \mapsto p \circ g$ for all $B \xrightarrow{p} A$.

Definition 3.4. Let $\mathscr{A}$ be a locally small category. The functor $H_\bullet : \mathscr{A} \to [\mathscr{A}^{op}, \mathbf{Set}]$ is defined

- on objects A by $H_\bullet(A) = H_A$, and
- on morphisms $A \xrightarrow{f} A'$ by $H_\bullet(f) = H_f : H_A \implies H_{A'}$ given by

$$H_f(B) : H_A(B) = \mathscr{A}(B, A) \longrightarrow \mathscr{A}(B, A') = H_{A'}(B) \qquad p \longmapsto f \circ p.$$

This is known as the *Yoneda embedding* of $\mathscr{A}$.

Theorem 3.5 (Yoneda's Lemma). Let $\mathscr{A}$ be a locally small category and $X : \mathscr{A}^{op} \to \mathbf{Set}$ be a functor. Then, there is a bijection

$$[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \xrightarrow{\sim} X(A) \qquad \alpha \mapsto \alpha_A(1_A),$$

which is natural in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \mathbf{Set}]$.

Proof. Let the map $[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \to X(A)$ defined above be denoted by $\alpha \mapsto \widehat{\alpha}$. We define a map in the opposite direction: for each $x \in X(A)$, let $\widetilde{x} : H_A \implies X$ be given by $(\widetilde{x}_B)_{B \in \mathscr{A}}$, where

$$\widetilde{x}_B : H_A(B) = \mathscr{A}(B, A) \to X(B) \qquad \widetilde{x}_B(f) = (X(f))\,(x) \in X(B).$$

We first show that this is indeed a natural transformation from H_A to X. Let $B' \xrightarrow{g} B$ be an arrow in A. We would like to conclude that

$$\begin{tikzcd} H_A(B) \arrow[r, "- \circ g"] \arrow[d, "\widetilde{x}_B"'] & H_A(B') \arrow[d, "\widetilde{x}_{B'}"] \\ X(B) \arrow[r, "X(g)"'] & X(B') \end{tikzcd}$$

commutes. Indeed, let $f \in H_A(B) = \mathscr{A}(B, A)$. It maps to $f \circ g \in H_A(B')$, which maps to $(X(f \circ g))(x) \in X(B')$. On the other hand, f maps to $(X(f))(x) \in X(B)$, which maps to $(X(g))((X(f))(x)) = X(f \circ g)(x)$, since X is contravariant on $\mathscr{A}$.

Next, we show that the previously defined maps $\alpha \mapsto \widehat{\alpha}$ and $x \mapsto \widetilde{x}$ are inverses to one another. Let $\alpha : H_A \implies X$ be a natural transformation and fix some $B \in \mathscr{A}$. Then, for any $f \in H_A(B) = \mathscr{A}(B, A)$, we have

$$\widetilde{\widehat{\alpha}}_B(f) = (Xf)(\widehat{\alpha}) = (Xf)(\alpha_A(1_A)) \in X(B).$$

It remains to show that $(Xf)(\alpha_A(1_A)) = \alpha_B(f)$, where $f \in H_A(B) = \mathscr{A}(B, A)$. Note that α is a natural transformation and hence, there is a commutative square corresponding to the map $B \xrightarrow{f} A$ as follows:

$$\begin{tikzcd} H_A(A) \arrow[r, "- \circ f"] \arrow[d, "\alpha_A"'] & H_A(B) \arrow[d, "\alpha_B"] \\ X(A) \arrow[r, "Xf"'] & X(B). \end{tikzcd}$$

Under the above square, 1_A first maps to $f \in \mathscr{A}(B, A)$ under the horizontal map and then maps to $\alpha_B(f)$. On the other hand, 1_A maps to $\alpha_A(1_A)$ under the vertical map and then to $(Xf)(\alpha_A(1_A))$ under the horizontal map, which gives us what we wanted due to commutativity.

On the other hand, if $x \in X(A)$, then

$$\widehat{\widetilde{x}} = \widetilde{x}_A(1_A) = (X(1_A))(x) = 1_{X(A)}(x) = x.$$

This shows that the two maps are inverses to one another.

Finally, we must show naturality of $\widetilde{\cdot}$ and $\widehat{\cdot}$. If we show naturality of even one of them, we have shown the other, since the components are all isomorphisms (invoke Lemma 1.11). We prove naturality of $\widehat{\cdot}$.

First, consider naturality in A. Let $B \xrightarrow{f} A$ be an arrow in $\mathscr{A}$, which induces a map $H_f : H_B \implies H_A$ in $[\mathscr{A}^{op}, \mathbf{Set}]$, which in turn induces a map

$$- \circ H_f : [\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \longrightarrow [\mathscr{A}^{op}, \mathbf{Set}](H_B, X).$$

We would like to show that

$$\begin{tikzcd} {[\mathscr{A}^{op}, \mathbf{Set}](H_A, X)} \arrow[r, "- \circ H_f"] \arrow[d, "\widehat{\cdot}"'] & {[\mathscr{A}^{op}, \mathbf{Set}](H_B, X)} \arrow[d, "\widehat{\cdot}"] \\ X(A) \arrow[r, "Xf"'] & X(B) \end{tikzcd}$$

commutes. Indeed, consider $\alpha : H_A \implies X$. First, under the horizontal map, it goes to $\alpha \circ H_f$, which under the vertical map goes to $(\alpha \circ H_f)_B(1_B) \in X(B)$. On the other hand, under the vertical map, α first goes to $\alpha_A(1_A)$, which under the horizontal map goes to $(Xf)(\alpha_A(1_A))$.

Now, note that

$$(\alpha \circ H_f)_B(1_B) = \alpha_B((H_f)_B(1_B)) = \alpha_B(f) = (Xf)(\alpha_A(1_A)),$$

the last of which is an equality that we argued earlier while showing that the maps were inverses.

Next, we must argue for naturality in X. Suppose $\theta : X \implies X'$ is a natural transformation, where $X' : \mathscr{A}^{op} \to \mathbf{Set}$ is a functor. We would like to show that the square

$$\begin{tikzcd}
{[\mathscr{A}^{op}, \mathbf{Set}](H_A, X)} \arrow[r, "\theta \circ -"] \arrow[d, "\hat{\cdot}"'] & {[\mathscr{A}^{op}, \mathbf{Set}](H_A, X')} \arrow[d, "\hat{\cdot}"] \\
X(A) \arrow[r, "\theta_A"'] & X'(A)
\end{tikzcd}$$

Let $\alpha : H_A \implies X$. This maps to $\theta \circ \alpha$ under the horizontal map and goes to $(\theta \circ \alpha)_A(1_A)$ under the vertical map. On the other hand, it first goes to $\alpha_A(1_A)$ under the vertical map which maps to $\theta_A(\alpha_A(1_A))$ under the horizontal map. These two are obviously equal, since $(\theta \circ \alpha)_A = \theta_A \circ \alpha_A$. This completes the proof. ■

Part II
Abelian Categories