

# Local Cohomology

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## §1 The $I$ -torsion functor

**DEFINITION 1.1.** Let  $R$  be a ring,  $I \trianglelefteq R$  an ideal, and  $M$  an  $R$ -module. Define

$$\Gamma_I(M) := \{x \in M : \text{there is a positive integer } n \in \mathbb{N} \text{ such that } I^n x = 0\} = \bigcup_{n \geq 1} (0 :_M I^n).$$

This is known as the  *$I$ -torsion functor*.

It is clear that  $\Gamma_I(M)$  is a submodule of  $M$  and any  $R$ -linear map  $\varphi : M \rightarrow N$  restricts to an  $R$ -linear map  $\Gamma_I(\varphi) : \Gamma_I(M) \rightarrow \Gamma_I(N)$ . Thus,  $\Gamma_I : {}_R\mathcal{M} \rightarrow {}_R\mathcal{M}$  is a functor.

**LEMMA 1.2.** The functor  $\Gamma_I$  is left-exact.

*Proof.* Let  $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$  be a short exact sequence of  $R$ -modules. ■

**DEFINITION 1.3.** The right derived functors of  $\Gamma_I : {}_R\mathcal{M} \rightarrow {}_R\mathcal{M}$  are called the *local cohomology functors with support in  $I$* .

Henceforth  $R$  is a Noetherian ring unless specified otherwise.

There are some properties of  $\Gamma_I$  which are trivial to verify:

- $\Gamma_I(M) = \Gamma_{\sqrt{I}}(M)$  as submodules of  $M$ .
- If  $\Gamma_I(M) = 0$ , then  $I$  cannot be contained in any associated prime of  $M$ . In particular, if  $M$  is a finite  $R$ -module, then  $\text{Ass}_R(M)$  is finite, and hence, using Prime Avoidance, there is an  $M$ -regular element in  $I$ , that is,  $\text{depth}(I, M) \geq 1$ .
- Given a family of  $R$ -modules  $\{M_\alpha\}_{\alpha \in \Lambda}$ ,  $\Gamma_I\left(\bigoplus_{\alpha \in \Lambda} M_\alpha\right) = \bigoplus_{\alpha \in \Lambda} \Gamma_I(M_\alpha)$  as submodules of  $\bigoplus_{\alpha \in \Lambda} M_\alpha$ .
- If  $S \subseteq R$  is a multiplicative subset, then  $\Gamma_{S^{-1}I}(S^{-1}M) = S^{-1}\Gamma_I(M)$  as submodules of  $S^{-1}M$ .
- For  $\mathfrak{p} \in \text{Spec}(R)$ ,

$$\Gamma_I(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & I \subseteq \mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$

In particular if  $E$  is an injective  $R$ -module, then  $\Gamma_I(E)$  is an injective  $R$ -module, and is a direct summand of  $E$ .

Since all the above isomorphisms are natural, these extend to isomorphisms on local cohomology, that is, for  $i \geq 0$ :

- $H_I^i(M) = \Gamma_{\sqrt{I}}(M)$  as submodules of  $M$ .
- Given a family of  $R$ -modules  $\{M_\alpha\}_{\alpha \in \Lambda}$ ,  $H_I^i\left(\bigoplus_{\alpha \in \Lambda} M_\alpha\right) = \bigoplus_{\alpha \in \Lambda} H_I^i(M_\alpha)$  as submodules of  $\bigoplus_{\alpha \in \Lambda} M_\alpha$ .

- If  $S \subseteq R$  is a multiplicative subset, then  $H_{S^{-1}I}^i(S^{-1}M) = S^{-1}\Gamma_I(M)$  as submodules of  $S^{-1}M$ .

**LEMMA 1.4.** Let  $M$  be an  $R$ -module. If  $\Gamma_I(M) = M$ , then  $\Gamma_I(E_R(M)) = E_R(M)$ .

*Proof.* Suppose not and choose some  $x \in E_R(M) \setminus \Gamma_I(E_R(M))$ . Since  $R$  is Noetherian, there is an associated prime  $\mathfrak{p}$  of  $E_R(M)$  containing  $\text{Ann}_R(x)$ . But since there is no power of  $I$  annihilating  $x$ , we must have  $I \not\subseteq \mathfrak{p}$ .

On the other hand, since  $\text{Ass}_R(E_R(M)) = \text{Ass}_R(M)$ , it follows that there is some  $y \in M$  with  $\mathfrak{p} = \text{Ann}_R(y)$ . Further, since  $\Gamma_I(M) = M$ , there is a positive integer  $n > 0$  such that  $I^n y = 0$ , i.e.,  $I^n \subseteq \mathfrak{p}$ , and hence,  $I \subseteq \mathfrak{p}$ , a contradiction. ■

**COROLLARY 1.5.** Let  $M$  be an  $R$ -module. If  $\Gamma_I(M) = M$ , then  $H^i(M) = 0$  for  $i > 0$ .

*Proof.* Let  $0 \rightarrow M \rightarrow E^*$  be a minimal injective resolution of  $M$ . Due to Lemma 1.4, it follows that  $\Gamma_I(E^i) = E^i$  for  $i \geq 0$ , and the conclusion follows, since the resolution remains unchanged after applying  $\Gamma_I$ . ■

**PROPOSITION 1.6.** Let  $M$  be an  $R$ -module, and set  $N := M/\Gamma_I(M)$ . Then  $\Gamma_I(N) = 0$  and  $H_I^i(N) \cong H_I^i(M)$  for  $i > 0$ .

*Proof.* Set  $L := \Gamma_I(M)$ . Then there is a short exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ , and  $L$  is  $\Gamma_I$ -acyclic. It is then clear from the long exact sequence that  $H_I^i(N) \cong H_I^i(M)$  for  $i > 0$ . Finally, since the induced map  $\Gamma_I(L) \rightarrow \Gamma_I(M)$  is an isomorphism and  $H_I^1(L) = 0$ , it follows that  $\Gamma_I(N) = 0$ . ■

**THEOREM 1.7 (GROTHENDIECK VANISHING THEOREM).** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring,  $I \subseteq R$  an ideal, and  $M$  a finite  $R$ -module. Then  $H_I^j(M) = 0$  for  $j > \dim_R M$ .

*Proof.* We argue by induction on  $d := \dim_R(M)$ . If  $d = 0$ , then  $M$  is Artinian, so that  $\text{Ass}_R(M) = \{\mathfrak{m}\}$ . It follows that every element of  $M$  is annihilated by a power of  $\mathfrak{m}$ , and thus by a power of  $I$ . Consequently,  $\Gamma_I(M) = M$ . Due to Corollary 1.5,  $H_I^i(M) = 0$  for  $i > 0$ , and this establishes the base case.

Suppose now that  $d > 0$ . Set  $N := M/\Gamma_I(M)$ . As we have seen in Proposition 1.6,  $\Gamma_I(N) = 0$  and  $H_I^i(N) \cong H_I^i(M)$  for  $i > 0$ . Further,  $\dim_R N \leq d$ . If this inequality is strict, then we are done due to the induction hypothesis. Hence we may assume that  $\dim_R N = d$ . As we remarked earlier, since  $\Gamma_I(N) = 0$ , and  $N$  is a finite  $R$ -module,  $\text{depth}(I, N) \geq 1$ . Choose an  $N$ -regular element  $a \in I$ . Let  $x \mapsto \bar{x}$  denote the natural surjection  $M \rightarrow M/aM =: \bar{M}$  and  $\mu_a : M \rightarrow M$  be multiplication by  $a$ . The short exact sequence  $0 \rightarrow M \xrightarrow{\mu_a} M \rightarrow \bar{M} \rightarrow 0$  induces a long exact sequence:

$$\cdots \rightarrow H_I^{i-1}(\bar{M}) \rightarrow H_I^i(M) \xrightarrow{\mu_a} H_I^i(M) \rightarrow H_I^i(\bar{M}) \rightarrow \cdots.$$

For  $i > d$ , note that  $i-1 > d-1 = \dim_R \bar{M}$ , so that  $H_I^{i-1}(\bar{M}) = H_I^i(\bar{M}) = 0$ . This shows that  $\mu_a : H_I^i(M) \rightarrow H_I^i(M)$  is an isomorphism of  $R$ -modules. Recall that  $H_I^i(M)$  is  $I$ -torsion and  $a \in I$ . If  $H_I^i(M) \neq 0$ , then for  $n \gg 0$ , the composition  $\mu_a^n$  would have non-trivial kernel, which is absurd since it is an isomorphism. This shows that  $H_I^i(M) = 0$  for  $i > d$ , as desired. ■

**PROPOSITION 1.8.** Let  $(R, \mathfrak{m}, k)$  be a Gorenstein local ring with  $d = \dim R$ . Then

$$H_{\mathfrak{m}}^d(R) \cong E_R(k).$$

*Proof.* It is well-known that the minimal injective resolution of a Gorenstein local ring looks like:

$$0 \rightarrow R \rightarrow \bigoplus_{\text{ht } \mathfrak{p}=0} E_R(R/\mathfrak{p}) \rightarrow \bigoplus_{\text{ht } \mathfrak{p}=1} E_R(R/\mathfrak{p}) \rightarrow \cdots \rightarrow E_R(k) \rightarrow 0.$$

Further, it is clear that

$$\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(k) & \mathfrak{p} = \mathfrak{m} \\ 0 & \text{otherwise,} \end{cases}$$

whence the conclusion follows. ■

It is also possible to characterize the depth of an ideal using the local cohomology modules:

**PROPOSITION 1.9.** Let  $R$  be a Noetherian ring and  $I \leq R$  an ideal. If  $M$  is a finite  $R$ -module such that  $IM \neq M$ , then

$$\text{depth}(I, M) = \inf \{i : H_I^i(M) \neq 0\}.$$

*Proof.* We induct on  $d = \text{depth}(I, M)$ . If  $d = 0$ , then  $I \subseteq \mathfrak{p}$  for some associated prime  $\mathfrak{p}$  of  $M$ . ■

## §2 Čech Cohomology

**DEFINITION 2.1.** Let  $R$  be a Noetherian ring, and  $\underline{a} = a_1, \dots, a_n \in R$ . Let  $\check{C}^\bullet(a_i)$  denote the cochain complex:

$$\cdots 0 \rightarrow R \rightarrow R_{a_i} \rightarrow 0 \rightarrow \cdots,$$

and define  $\check{C}^\bullet(\underline{a})$  to be the cochain complex:

$$\check{C}^\bullet(a_1) \otimes \cdots \otimes \check{C}^\bullet(a_n).$$

Further, if  $M$  is an  $R$ -module, then define  $\check{C}^\bullet(\underline{a}, M) := \check{C}^\bullet(\underline{a}) \otimes M$ . This is known as the *Čech complex*. The cohomology modules of this cochain complex are known as the *Čech cohomology modules* and are denoted by  $\check{H}_\underline{a}^i(M)$ .

**REMARK 2.2.** Since the tensor product of chain complexes is associative and commutative, the order of tensoring above doesn't matter.

**THEOREM 2.3.** Let  $R$  be a Noetherian ring,  $I \trianglelefteq R$  an ideal, and  $\underline{a} := a_1, \dots, a_n \in R$  such that  $\sqrt{(a_1, \dots, a_n)} = \sqrt{I}$ . Then

$$\check{H}_\underline{a}^j(M) \cong H_I^j(M)$$

for every  $R$ -module  $M$ .

*Proof.* ■

**PROPOSITION 2.4.** If  $R \rightarrow S$  is a flat morphism of Noetherian rings,  $M$  an  $R$ -module, and  $I$  an ideal in  $R$ , then

$$H_I^j(M) \otimes_R S \cong H_{IS}^j(M \otimes_R S)$$

as  $S$ -modules.

**DEFINITION 2.5.** Let  $R$  be a Noetherian ring and  $I \trianglelefteq R$  an ideal. Define the *cohomological dimension* of  $I$  in  $R$  to be

$$\text{cdim}(I, R) := \inf \left\{ i : H_I^j(M) = 0 \text{ for all } R\text{-modules } M \text{ and all } j > i \right\}.$$

**PROPOSITION 2.6.**

$$\text{cdim}(I, R) = \inf \left\{ i : H_I^j(R) = 0 \text{ for all } j > i \right\}.$$

**COROLLARY 2.7.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring. Then  $\text{cdim}(I, R) = \text{cdim}(\hat{I}, \hat{R})$ .

*Proof.* This is immediate from Proposition 2.6 and the fact that  $R \rightarrow \hat{R}$  is faithfully flat. ■