MA5106: HOMEWORK 1

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1. Problem 1

(a) Let $n \in \mathbb{Z}$ be non-zero. The n-th Fourier coefficient is given by

$$\begin{aligned}
a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|e^{-inx}\ dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{0} (-x)e^{-inx}\ dx + \int_0^{\pi} xe^{-inx}\ dx\right) \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} xe^{inx}\ dx + \int_0^{\pi} xe^{-inx}\ dx\right) \\
&= \frac{1}{2\pi}\int_0^{\pi} x\left(e^{inx} + e^{-inx}\right)\ dx \\
&= \frac{1}{2\pi}\int_0^{\pi} 2x\cos(nx)\ dx = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\ dx \\
&= \frac{1}{\pi}\left[\frac{nx\sin(nx) + \cos(nx)}{n^2}\right]_0^{\pi} \\
&= \frac{(-1)^n - 1}{n^2\pi} = \begin{cases} 0 & n \text{ is even} \\ \frac{-2}{n^2\pi} & n \text{ is odd.} \end{cases}
\end{aligned}$$

On the other hand, if $n = 0$, then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\ dx = \frac{\pi}{2}.$$

(b) Let $n \in \mathbb{Z}$ be non-zero. The n-th Fourier coefficient is given by

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-inx}\ dx + \frac{1}{2\pi}\int_0^{\pi} f(x)e^{-inx}\ dx.
\end{aligned}$$

Note that f is an odd function on $[-\pi, \pi]$, that is, $f(-x) = -f(x)$. Making the substitution $x = -y$ in the first integral, we have

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_0^\pi f(-y) e^{iny} \, dy + \frac{1}{2\pi} \int_0^\pi f(x) e^{-inx} \, dx \\ &= \frac{1}{2\pi} \int_0^\pi -f(y) e^{iny} \, dy + \frac{1}{2\pi} \int_0^\pi f(x) e^{-inx} \, dx \\ &= \frac{1}{2\pi} \int_0^\pi \frac{\pi - x}{2} \left(e^{-inx} - e^{inx} \right) \, dx. \end{aligned}$$

Perform the substitution $y = \pi - x$ to get

$$\begin{aligned} a_n &= \frac{1}{4\pi} \int_0^\pi y \left(e^{-in(\pi - y)} - e^{in(\pi - y)} \right) \, dy \\ &= \frac{(-1)^n}{4\pi} \int_0^\pi y (e^{iny} - e^{-iny}) \, dy \\ &= \frac{(-1)^n 2i}{4\pi} \int_0^\pi y \sin(ny) \, dy \\ &= \frac{(-1)^n i}{2\pi} \left[\frac{\sin(ny) - ny\cos(ny)}{n^2} \right]_0^\pi \\ &= \frac{(-1)^n i}{2\pi} \frac{(-n\pi) \cdot (-1)^n}{n^2} = \frac{-i}{2n}. \end{aligned}$$

As for $n = 0$, we have

$$f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0,$$

since f is an odd function.

Problem 2

Note that

$$f(x) = e^{i(\pi - x)/2} = e^{i\pi/2} e^{-ix/2} = i e^{-ix/2}.$$

First, we obtain the Fourier coefficients, that is,

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\int_0^{2\pi} ie^{-ix/2}e^{-inx}\ dx \\
&= \frac{i}{2\pi}\int_0^{2\pi} e^{-i\left(n+\frac{1}{2}\right)x}\ dx \\
&= \frac{i}{2\pi}\frac{1}{(-i)\left(n+\frac{1}{2}\right)}\left[e^{-i\left(n+\frac{1}{2}\right)x}\right]_0^{2\pi} \\
&= \frac{i}{2\pi}\frac{-2}{(-i)\left(n+\frac{1}{2}\right)} = \frac{2}{(2n+1)\pi}
\end{aligned}$$

Using Parseval's Formula, we have

$$\sum_{n\in\mathbb{Z}} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|e^{-i\frac{\pi-x}{2}}\right|^2\ dx = 1.$$

Therefore,

$$1 = \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{1}{(2n+1)^2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

This gives,

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Problem 3

(a) This is an exercise in change of variables. Set $y = x+\xi$ to obtain

$$\int_{-\pi}^{\pi} f(x+\xi)\ dx = \int_{-\pi+\xi}^{\pi+\xi} f(y)\ dy.$$

Note that

$$\int_{-\pi+\xi}^{\pi+\xi} f(y)\ dy + \int_{-\pi}^{-\pi+\xi} f(y)\ dy = \int_{-\pi}^{\pi+\xi} f(y)\ dy = \int_{-\pi}^{\pi} f(y)\ dy + \int_{\pi}^{\pi+\xi} f(y)\ dy.$$

But, using the periodicity of f, we have, using the substitution $y = z+2\pi$,

$$\int_{\pi}^{\pi+\xi} f(y)\ dy = \int_{-\pi}^{-\pi+\xi} f(z+2\pi)\ dz = \int_{-\pi}^{-\pi+\xi} f(z)\ dz.$$

Hence,

$$\int_{-\pi+\xi}^{\pi+\xi} f(y)\ dy = \int_{-\pi}^{\pi} f(y)\ dy.$$

(b) We have

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

Invoke the substitution $y = x - \frac{\pi}{n}$. Then,

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(y + \frac{\pi}{n}\right) e^{-in\left(y+\frac{\pi}{n}\right)}\, dy = \frac{-1}{2\pi} \int_{-\pi}^{\pi} f\left(y + \frac{\pi}{n}\right) e^{-iny}\, dy,$$

where the last equality follows from part (a). Now, simply add the two equivalent formulations of $\widehat{f}(n)$ and divide by 2. This would give

$$\widehat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[f(y) - f\left(y + \frac{\pi}{n}\right)\right] e^{-iny}\, dy.$$

Problem 4

I shall prove (b), from which (a) would follow. Using integration by parts, we have

$$\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \left[f(x) \frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x) \frac{e^{-inx}}{in}\, dx = \frac{1}{in} \widehat{f}'(n).$$

Iterating this k-times, we obtain

$$\widehat{f}(n) = \frac{1}{(in)^k} \widehat{f^{(k)}}(n).$$

Therefore,

$$\lim_{|n|\to\infty} |n^k \widehat{f}(n)| = \lim_{|n|\to\infty} \widehat{f^{(k)}}(n) = 0$$

from the Riemann-Lebesgue lemma. The conclusion follows.

Problem 5

(a) Let ω denote e^{ix}. We are computing

$$\begin{aligned}\sum_{n=-N}^{N}\omega^n &= \omega^{-N}\sum_{n=0}^{2N}\omega^n \\ &= \omega^{-N}\frac{\omega^{2N+1}-1}{\omega-1} \\ &= \frac{\omega^{N+1}-\omega^{-N}}{\omega-1} \\ &= \frac{\omega^{N+1/2}-\omega^{-(N+1/2)}}{\omega^{1/2}-\omega^{-1/2}} \\ &= \frac{2i\sin\left(N+\frac{1}{2}\right)x}{2i\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(N+1/2\right)x}{\sin(x/2)}.\end{aligned}$$

(b) We have

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}D_n(x) &= \frac{1}{N}\sum_{n=0}^{N-1}\frac{2\sin(x/2)\sin(n+1/2)x}{2\sin^2(x/2)} \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\frac{\cos(nx)-\cos(n+1)x}{2\sin^2(x/2)} \\ &= \frac{1}{N}\frac{1-\cos(Nx)}{2\sin^2(x/2)} = \frac{\sin^2(Nx/2)}{N\sin^2(x/2)}.\end{aligned}$$

(c) We may suppose that $N > 1$ since $\log 1 = 0$. Note that D_N is an even function and hence,

$$\begin{aligned}\int_{-\pi}^{\pi}|D_N(x)|\,dx &= 2\int_0^{\pi}|D_N(x)|\,dx \\ &\geq \sum_{n=0}^{N-1}\int_{\frac{n\pi}{N+\frac{1}{2}}}^{\frac{(n+1)\pi}{N+\frac{1}{2}}}\left|\frac{\sin(N+1/2)x}{\sin(x/2)}\right|\,dx\end{aligned}$$

Therefore, it suffices to show that $\int_0^{\pi}|D_N(x)| \geq c\log N$ for some constant $c > 0$. On the interval $\left[\frac{n\pi}{N+\frac{1}{2}}, \frac{(n+1)\pi}{N+\frac{1}{2}}\right]$,

$$\sin(x/2) \leq \sin\left(\frac{(n+1)\pi}{2N+1}\right) \leq \frac{(n+1)\pi}{2N+1}$$

since $\sin(x/2)$ is an increasing function on $[0,\pi]$ and $\sin x \le x$ on $[0,\infty)$. Hence,

$$\int_{\frac{n\pi}{N+\frac{1}{2}}}^{\frac{(n+1)\pi}{N+\frac{1}{2}}} \left|\frac{\sin(N+1/2)x}{\sin(x/2)}\right| \; dx \ge \frac{2N+1}{(n+1)\pi}\int_{\frac{n\pi}{N+\frac{1}{2}}}^{\frac{(n+1)\pi}{N+\frac{1}{2}}} |\sin(N+1/2)x| \; dx$$
$$= \frac{2N+1}{(n+1)\pi}\frac{1}{N+\frac{1}{2}}\int_{n\pi}^{(n+1)\pi} |\sin(y)| \; dy$$
$$= \frac{2}{(n+1)\pi} \times 2 = \frac{4}{(n+1)\pi}.$$

Consequently,

$$\int_0^\pi |D_N(x)| \; dx \ge \frac{4}{\pi}\sum_{n=0}^{N-1}\frac{1}{n+1} = \frac{4}{\pi}H_N,$$

where H_N is the N-th Harmonic number. For $N \ge 2$, it is well known that

$$H_N \ge \int_1^{N+1} \frac{1}{x} \; dx = \log(N+1) \ge \log N.$$

This completes the proof.

(d) This is immediate, since

$$\int_{-\pi}^{\pi} D_N(x) \; dx = \sum_{n=-N}^{N}\int_{-\pi}^{\pi} e^{inx} \; dx.$$

But for non-zero n, we have

$$\int_{-\pi}^{\pi} e^{inx} \; dx = \frac{1}{in}\left[e^{inx}\right]_{-\pi}^{\pi} = 0$$

and for $n = 0$, we have $\int_{-\pi}^{\pi} 1 \; dx = 2\pi$. Therefore,

$$\int_{-\pi}^{\pi} D_N(x) \; dx = 2\pi.$$