Commutative Algebra

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May 11, 2023

Abstract

This document mainly contains terse notes of commutative algebra and solutions to exercises from [1]. The three main references were [1], [3] and [4].

Except for in the chapter on modules, all rings are assumed to be commutative unless stated otherwise. We use a uniform convention to represent a commutative ring with A and a general ring with R. Similarly, we represent modules by one of M, N, P. A maximal ideal is generally denoted by $\mathfrak{m}$ while a prime ideal is denoted by $\mathfrak{p}$.

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Part I

Theory Building

Chapter 1

Rings and Ideals

1.1 Nilradical and Jacobson radical

Definition 1.1 (Multiplicatively Closed). A subset $S \subseteq A$ is said to be *multiplicatively closed* if

(a) $1 \in S$

(b) for all $x, y \in S$, $xy \in S$

Proposition 1.2. *Let $S \subsetneq A \backslash \{0\}$ be a multiplicatively closed subset. Then, there is a prime ideal $\mathfrak{p}$ disjoint from S.*

1.2 Local Rings

Definition 1.3. A commutative ring A is said to be local if it has a unique maximal ideal.

Proposition 1.4. *A is local if and only if the subset of non-units form an ideal.*

Obviously, a field k is a local ring. On the other hand, the polynomial ring $k[x]$ is not local, since both x and $1 - x$ are non-units but their sum is a unit.

We contend that the ring $A = k[x_1, x_2, \ldots]/(x_1, x_2, \ldots)^2$ is local. Indeed, let π denote the canonical map $k[x_1, x_2, \ldots] \to A$ and $\mathfrak{m}$ be maximal in A. Then, $\pi^{-1}(\mathfrak{m})$ is maximal in $k[x_1, x_2, \ldots]$ and contains $(x_1, x_2, \ldots)^2$, therefore, contains $(x_1, x_2, \ldots)$. But the latter is maximal and therefore, $\pi^{-1}(\mathfrak{m}) = (x_1, x_2, \ldots)$ whence the maximal ideal is unique. Thus A is local..

1.3 Operations on Ideals

1.4 The Zariski Topology

Chapter 2

Modules

2.1 Introduction

Throughout this section, R denotes a general ring which need not be commutative.

Definition 2.1 (Module). A left R-module is an abelian group $(M,+)$ along with a ring action, that is, a ring homomorphism $\mu : R \to \text{End}(M)$.

Henceforth, unless specified otherwise, an *R-module* refers to a *left R-module*. Trivially note that R is an R-module, so is any ideal in R and so is every quotient ring R/I where I is an ideal in R. When R is a field, an R-module is the same as a vector space.

Every abelian group G trivially forms a $\mathbb{Z}$-module. Using this and the forthcoming *Structure Theorem for Finitely Generated Modules over a PID*, we obtain the *Structure Theorem for Finitely Generated Abelian Groups*.

Definition 2.2 (Submodule). Let M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of R.

Proposition 2.3 (Submodule Criteria). *Let M be an R-module. Then $\varnothing \subsetneq N \subseteq M$ is a submodule if and only if for all $x, y \in N$ and $r \in R$, $x + ry \in N$.*

Proof. Straightforward definition pushing. ∎

Definition 2.4 (Module Homomorphism). Let M, N be R-modules. A *module homomorphism* is a group homomorphism $\phi : M \to N$ such that for all $x \in M$ and $r \in R$, $\phi(rx) = r\phi(x)$.

In other words, a module homomorphism is simply an R-linear map.

Proposition 2.5 (Homomorphism Criteria). *Let M, N be R-modules. Then $\phi : M \to N$ is an R-module homomorphism if and only if for all $x, y \in M$ and $r \in R$, $\phi(x + ry) = \phi(x) + r\phi(y)$.*

Proof. Straightforward definition pushing. ∎

It is not hard to see, using the above proposition and the submodule criteria that the image of an R-module under a homomorphism is a submodule.

Definition 2.6 (Kernel, Cokernel). Let $\phi : M \to N$ be an R-module homomorphism. We define

$$\ker \phi = \{x \in M \mid \phi(x) = 0\} \qquad \operatorname{coker} \phi = N/\phi(M)$$

For an R-module M, define the annihilator of M in R as

$$\mathrm{Ann}_R(M) = \{r \in R \mid rx = 0 \; \forall x \in M\}$$

It is trivial to check that $\mathrm{Ann}(M)$ is a left ideal in R, and if R were commutative, it would be an ideal.

Proposition 2.7. *If I is an ideal contained in $\mathrm{Ann}_A(M)$, then M is naturally an A/I-module.*

Proof. Define the action $(a + I) \cdot m = a \cdot m$. It is easy to check that this action is well defined. Further,

$$(a + I) \cdot ((b + I) \cdot m) = (a + I) \cdot (bm) = (ab) \cdot m = ((a + I)(b + I)) \cdot m$$

This completes the proof. ∎

In particular, if $I = \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then M forms a vector space over $A/\mathfrak{m}$.

2.2 Free Modules

Throughout this section, R denotes a general ring which need not be commutative. The content of this section is taken from [2].

We define the free module using a universal property and then provide a construction for it. This should establish uniqueness.

Definition 2.8 (Universal Property of Free Modules). Let S be a non-empty set. A *free module on S* is an R-module F together with a mapping $f : S \to F$ such that for every R-module M and every set map $g : S \to M$, there is a unique R-module homomorphism $h : F \to M$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f"'] & M \\ F \arrow[ur, dotted, "\exists ! h"'] & \end{tikzcd}$$

Let F be the set of all set functions $\phi : S \to R$ which takes nonzero values at finitely many elements of S. This has the structure of an R-module. Define the set map $f : S \to F$ by

$$f(s)(t) = \begin{cases} 1 & s = t \\ 0 & \text{otherwise} \end{cases}$$

We contend that (F, f) is a free module on S. Indeed, let $g : S \to M$ be a set map where M is an R-module. Define the linear map $h : F \to M$ by

$$h(f(s)) = g(s)$$

Since every element in F can uniquely be written as a linear combination of elements in $\{f(s)\}_{s \in S}$, we have successfully defined a module homomorphism $h : F \to M$ such that $g = h \circ f$. The uniqueness of this map is quite obvious. Hence, (F, f) is a free module on S.

Definition 2.9 (Basis). Let M be an R-module. Then $S \subseteq M$ is said to be a *basis* if it is linearly independent and generates M.

It is important to note that not every minimal generating set is a basis. Take for example the $\mathbb{Z}$-module $\mathbb{Z}$. Notice that $\{2, 3\}$ is a minimal generating set but is not a basis for it is not linearly independent.

2.2.1 Over a PID

Throughout this (sub)section, let R denote a PID.

Theorem 2.10. *Let F be a free R-module. If $H \leq F$ is a submodule, then H is free and* $\dim H \leq \dim F$.

Proof. Let $\{e_i\}_{i \in I}$ be a basis for F. Denote the projection map of the i-th coordinate by $p_i : F \to R$. Due to the Well Ordering Theorem, we can impose a well order $(I, \leqq)$ on I. Let F_i be the submodule generated by $\{e_j \mid j \leqq i\}$ and $H_i = H \cap F_i$. Now, $p_i(H_i)$ is an ideal in R, and therefore, is of the form $a_i R$ for some $a_i \in R$. Of course, it is possible that $a_i = 0$. If $a_i \neq 0$, then pick some $h_i \in H_i$ such that $p_i(h_i) = a_i$, on the other hand, if $a_i = 0$, then set $h_i = 0$. It is not hard to see from this definition that $p_i(h_j) = 0$ whenever $j < i$.

We contend that the set $S = \{h_i \neq 0 \mid i \in I\}$ forms a basis for H, this would immediately imply that $\dim H \leq \dim F$. First, we shall show that S is linearly independent. We shall do this by transfinite induction. The base case is trivial. Suppose the induction hypothsis holds for $S_i = \{h_j \in S \mid j < i\}$. If a linear combination of the elements of S_{i+1} is zero, then the coefficient of h_i must be nonzero. Therefore, we may write

$$bh_i = \sum_{k=1}^{n} a_{j_k} h_{j_k}$$

For some $a_{j_1}, \ldots, a_{j_n}, b \in R$. Upon projecting using p_i, we obtain $ba_i = 0$, consequently, $b = 0$, and S_{i+1} is linearly independent.

It is not hard to argue that the h_i's span H. Pick some $h \in H$. Note that only finitely many of the $p_i(h)$'s will be nonzero. Let them be $i_1 < \cdots < i_n$. Now work backwards from i_n to determine the coefficients of h_{i_k} for each $1 \leq k \leq n$. ■

2.3 Finitely Generated Modules

Definition 2.11 (Finitely Generated Module). An R-module M is said to be finitely generated if there is a finite subset S of M which generates M. That is, there is no proper submodule N of M containing S.

Proposition 2.12. *An R-module M is finitely generated if M is isomorphic to a quotient of $R^{\oplus n}$ for some positive integer n.*

Proof. We shall only prove the forward direction since the converse is trivial to prove. Suppose M is finitely generated. Then, it is generated by a finite subset $S = \{x_1, \ldots, x_m\}$. Define the R-module homomorphism $\phi : R^{\oplus n} \to M$ by $(r_1, \ldots, r_n) \mapsto r_1 x_1 + \cdots + r_n x_n$. From the first isomorphism theorem, we have $M \cong R^{\oplus n} / \ker \phi$. ■

Proposition 2.13. *Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A. Let $\phi \in$* $\mathrm{End}(M)$ *such that*

$\phi(M) \subseteq \mathfrak{a}M$. Then, there are $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ such that

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0$$

as an element of $\operatorname{End}(M)$*, where a_k is treated as the homomorphism $x \mapsto a_k x$ in* $\operatorname{End}(M)$*.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a generating set for M. Then, for all $1 \le i \le n$, there are coefficients $\{a_{i1}, \ldots, a_{in}\}$ in $\mathfrak{a}$ such that

$$\phi(x_i) = \sum_{j=1}^{n} a_{ij} x_j$$

We may rewrite this as

$$\sum_{j=1}^{n} (\phi\delta_{ij} - a_{ij}) x_j = 0$$

Let B denote the matrix $(\phi\delta_{ij} - a_{ij})_{1 \le i,j \le n}$. Then, multiplying by $\operatorname{adj}(B)$, we see that $\det(B)(x_j) = 0$ for all $1 \le j \le n$ where $\det(B)$ is viewed as an element in $\operatorname{End}(M)$ and thus, is the zero map in $\operatorname{End}(M)$. It is not hard to see that $\det(B)$ is in the required form. ■

Lemma 2.14 (Nakayama). *Let M be a finitely generated module and $\mathfrak{a} \subseteq \mathfrak{R}$ be an ideal such that $M = \mathfrak{a}M$. Then, $M = 0$.*

Proof. Let $\phi = \mathbf{id}$ be the identity homomorphism in $\operatorname{End}(M)$. Using Proposition 2.13, there are coefficients $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ satisfying the statement of the proposition. As a result, $x = 1 + a_{n-1} + \ldots + a_0$ is the zero endomorphism. But since $a_{n-1} + \ldots + a_0 \in \mathfrak{a} \subseteq \mathfrak{R}$, x is a unit and hence, $M = 0$. ■

Corollary. Let M be a finitely generated A-module, N a submodule of M and $\mathfrak{a} \subseteq \mathfrak{R}$ an ideal. If $M = \mathfrak{a}M + N$ then $M = N$.

Proof. We have $M/N = \mathfrak{a}M/N$, consequently, $M/N = 0$ and $M = N$ due to Lemma 2.14. ■

Lemma 2.15. *Let $(A, \mathfrak{m})$ be local and $k = A/\mathfrak{m}$. Let M be a finitely generated A-module. Let $\{\overline{x}_1, \ldots, \overline{x}_n\}$ be elements in $M/\mathfrak{m}$ that form a basis for $M/\mathfrak{m}$ as a k-vector space. Then, $\{x_1, \ldots, x_n\}$ generates M.*

Proof. Let N be the submodule generated by $\{x_1, \ldots, x_n\}$. Then, the composition $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is surjective, consequently, $M = N + \mathfrak{m}M$ whence, it follows that $M = N$. ■

2.4 Hom **Modules and Functors**

For R-modules M, N, we denote the set of all R-module homomorphisms from M to N by $\operatorname{Hom}_R(M, N)$. When the choice of the ring R is clear from the context, we shall denote this set by $\operatorname{Hom}(M, N)$.

Proposition 2.16. *Let M, N be A-modules. Then* $\operatorname{Hom}(M, N)$ *has the structure of an A-module.*

Proof. It is obvious that $\operatorname{Hom}(M, N)$ has the structure of an abelian group. Define the natural action by $(af)(x) = af(x)$. It is not hard to see that this action is well defined. ■

Proposition 2.17. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of A-modules. Then, for any A-module N, we have a natural isomorphism*

$$\operatorname{Hom}_A\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\right) = \prod_{\lambda\in\Lambda} \operatorname{Hom}_A(M_\lambda, N)$$

Proof. Since the direct sum is the product in $A - \mathbf{Mod}$, the conclusion follows from the universal property. ■

Theorem 2.18. *Let $\phi : M \to N$ be an A-module homomorphism. Then, for every R-module P, there is an induced A-module homomorphism $\overline{\phi} : \operatorname{Hom}(N, P) \to \operatorname{Hom}(M, P)$ and an induced A-module homomorphism $\widetilde{\phi} : \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$.*

Equivalently phrased, $\operatorname{Hom}(-, P)$ *is a contravariant functor while* $\operatorname{Hom}(P, -)$ *is a covariant functor.*

Proof. We shall prove only the first half of the assertion since the second half follows from a similar proof. Define $\overline{\phi}$ using the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, dotted, "f\circ\phi"'] & N \arrow[d, "f"] \\ & P \end{tikzcd}$$

To see that this is indeed an R-module homomorphism, we need only verify that for all $f, g \in \operatorname{Hom}(N, P)$ and all $r \in R$, $(f + rg) \circ \phi = f \circ \phi + rg \circ \phi$ which is trivial to check. ■

2.5 Exact Sequences

Definition 2.19. A sequence of module homomorphisms

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is said to be exact at N if $\operatorname{im} f = \ker g$. A short exact sequence is a sequence of module homomorphisms:

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

which is exact at M, N and P.

It is not hard to see that the sequence in the definition is short exact if and only if f is injective, g is surjective and $\operatorname{im} f = \ker g$.

2.5.1 Diagram Chasing Poster Children

2.6 Tensor Product

Definition 2.20 (Bilinear Map). Let M, N, P be A-modules. A map $T : M \times N \to P$ is said to be bilinear if for each $x \in M$, the mapping $T_x : N \to P$ given by $y \mapsto T(x, y)$ is A-linear and for each $y \in N$, the mapping $T_y : M \to P$ given by $x \mapsto T(x, y)$ is A-linear.

Fix two A-modules M and N. Let $\mathscr{C}$ denote the category of bilinear maps $T : M \times N \to P$ where P is any A-module. A morphism between two bilinear maps $f : M \times N \to P_1$ and $g : M \times N \to P_2$ in this category is a module homomorphism $\phi : P_1 \to P_2$ such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[d, "g"'] & P_1 \arrow[dl, dotted, "\phi"] \\
P_2 &
\end{tikzcd}$$

A universal object in $\mathscr{C}$ is called the tensor product of M and N and is denoted by $M \otimes N$. In other words, the tensor product is an initial object in the category $\mathscr{C}$.

Definition 2.21 (Universal Property of the Tensor Product). Let M, N, P be A-modules and $T : M \times N \to P$ be a bilinear map. Then, there is a unique A-module homomorphism $\phi : M \otimes N \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[r, "T"] \arrow[d, "\varphi"'] & P \\
M \otimes N \arrow[ur, dotted, "\exists!\phi"'] &
\end{tikzcd}$$

Of course, having the universal property would imply that the tensor product, if it exists, is unique upto a unique isomorphism. We shall now construct a tensor product of M and N.

Constructing the Tensor Product

Let F be the free A-module on $M \times N$. Let us denote the basis elements of F by $e_{(x,y)}$ where $x \in M$ and $y \in N$. Now, for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $a \in A$, let D denote the submodule generated by elements of the form:

$$\begin{aligned}
&e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)} \\
&e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)} \\
&e_{(ax,y)} - ae_{(x,y)} \\
&e_{(x,ay)} - ae_{(x,y)}
\end{aligned}$$

Let $G = F/D$ and let $\varphi : M \times N \to G$ be the composition of the following maps:

$$M \times N \hookrightarrow F \twoheadrightarrow G$$

Let $T : M \times N \to P$ be a bilinear map. Consider the following commutative diagram:

$$\begin{tikzcd}
M \times N \arrow[r, "T"] \arrow[d, hook, "\iota"'] & P \\
F \arrow[r, "\pi"'] \arrow[ur, dotted, "\exists!f"] & G \arrow[u, dotted, "\exists!\phi"']
\end{tikzcd}$$

To show that existence of ϕ, we must show that $D \subseteq \ker f$, since we can then finish using the universal property of the kernel. But this is trivial to check and follows from the fact that T is a bilinear map and completes the construction.

Similarly, we define the tensor product for a finite sequence of A-modules $\{M_i\}_{i=1}^n$. That is, given a multilinear map $T : \prod_{i=1}^{n} M_i \to P$, there is a unique A-module homomorphism ϕ such that the following diagram commutes:

$$\begin{tikzcd} M_1 \times \cdots \times M_n \arrow[r, "T"] \arrow[d, "\varphi"'] & P \\ M_1 \otimes \cdots \otimes M_n \arrow[ur, dotted, "\exists!\phi"'] & \end{tikzcd}$$

Properties of Tensor Product

Given two modules M and N with the canonical map $\varphi : M \times N \to M \otimes N$, we denote by $m \otimes n$, the element $\varphi(m, n)$ in $M \otimes N$.

Proposition 2.22. *Let M, N, P be A-modules. Then,*

(a) $M \otimes N \cong N \otimes M$

(b) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P) \cong M \otimes N \otimes P$

(c) $M \oplus N \otimes P \cong (M \otimes P) \oplus (N \otimes P)$

(d) $A \otimes M \cong M$

Proof. (a) First, we shall show that there are well defined homomorphisms $M \otimes N \to N \otimes M$ and $N \otimes M \to M \otimes N$ mapping $m \otimes n \mapsto n \otimes m$ and $n \otimes m \mapsto m \otimes n$ respectively. This is best done using the universal property. Let $T : M \times N \to N \times M$ be the isomorphism $m \times n \mapsto n \times m$. Consider now the following commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "T"] \arrow[d, "\varphi"'] & N \times M \arrow[d, "\varphi'"] \\ M \otimes N & N \otimes M \end{tikzcd}$$

Since both φ' and T are bilinear, so is $\varphi \circ T$, consequently, there is a unique induced homomorphism $f : M \otimes N \to N \otimes M$ making the diagram commute, consequently, $f(m \otimes n) = \varphi'(T(m \times n)) = n \otimes m$. Similarly, there is a homomorphism $g : N \otimes M \to M \otimes N$ such that $g(n \otimes m) = m \otimes m$. It is not hard to see that $g \circ f = \mathbf{id}_{M \otimes N}$ and $f \circ g = \mathbf{id}_{N \otimes M}$, consequently, they are isomorphisms.

(b)

(c)

(d) Consider the map $T : A \times M \to M$ given by $(a, m) \mapsto am$. It is not hard to see that this map is bilinear, consequently, there is a map $f : A \otimes M \to M$ such that the following diagram commutes:

$$\begin{tikzcd} A \times M \arrow[r, "T"] \arrow[d, "\varphi"'] & M \\ A \otimes M \arrow[ur, dotted, "f"'] & \end{tikzcd}$$

Note that $f(a \otimes m) = am$ by definition. Consider the map $g : M \to A \otimes M$ given by $g(m) = 1 \otimes m$. It is not hard to see that g is a well defined module homomorphism. Further, since $f \circ g$ and $g \circ f$ are the identity homomorphisms, they both must be isomorphisms.

■

Example 1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$ for all $m, n \in \mathbb{N}$. In particular, if m and n are coprime, then $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$.

Proof. Consider the module homomorphism $T : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. ∎

Let $f : M \to M'$ and $g : N \to N'$ be A-module homomorphisms. Then, the map $\Phi : M \times N \to M' \otimes N'$ given by $\Phi(m,n) = f(m) \otimes g(n)$. It is not hard to see that Φ is bilinear. Consequently, it induces a map $f \otimes g : M \otimes N \to M' \otimes N'$ such that

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$$

Further, if $f' : M' \to M''$ and $g' : N' \to N''$ are A-module homomorphisms, then we have another map $f' \otimes g' : M' \otimes N' \to M'' \otimes N''$ such that

$$(f' \otimes g')(x \otimes y) = f'(x) \otimes g'(y)$$

Now, it is not hard to see that $(f' \circ f') \otimes (g' \circ g)$ and $(f' \otimes g') \circ (f \otimes g)$ agree on the elementary tensors, therefore, agree on all of $M \otimes N$.

2.7 Right Exactness

Proposition 2.23. *Let M, N, P be A-modules. Then, there is a natural isomorphism:*

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P)) \cong \mathrm{Hom}_A(M \otimes_A N, P)$$

Proof. Consider the map

$$\theta : \mathrm{Hom}_A(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P))$$

given by $\theta(\alpha)(m)(n) = \alpha(m \otimes n)$. Now, pick some $\eta \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P))$. Define the map $\zeta : M \times N \to P$ given by $\zeta(m,n) = \eta(m)(n)$. Obviously, ζ is bilinear and induces a map $\delta : M \otimes_A N \to P$ such that $\delta(m \otimes n) = \eta(m)(n)$. Call the map sending $\eta \mapsto \delta$ as β where

$$\beta : \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P)) \to \mathrm{Hom}_A(M \otimes_A N, P)$$

and $\beta(\eta)(m \otimes n) = \eta(m)(n)$.

We contend that θ and β are inverses to one another. Indeed,

$$((\beta \circ \theta)(\alpha))(m \otimes n) = \theta(\alpha)(m)(n) = \alpha(m \otimes n)$$

and

$$((\theta \circ \beta)(\eta))(m)(n) = \beta(\eta)(m \otimes n) = \eta(m)(n)$$

whence the conclusion follows. ∎

In particular, we see that the functor $- \otimes_A N$ is the left adjoint of the functor $\mathrm{Hom}_A(N, -)$, consequently, $\mathrm{Hom}_A(N, -)$ is the right adjoint of $- \otimes_A N$.

Theorem 2.24. *The functor $- \otimes_A N$ is right exact. That is, given a exact sequence*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

the sequence

$$M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \longrightarrow 0$$

Proof. Since the given sequence is exact, so is

$$\operatorname{Hom}_A(M'', \operatorname{Hom}_A(N,P)) \xrightarrow{\overline{g}} \operatorname{Hom}_A(M, \operatorname{Hom}_A(N,P)) \xrightarrow{\overline{f}} \operatorname{Hom}_A(M', \operatorname{Hom}_A(N,P)) \longrightarrow 0$$

but from Proposition 2.23, so is

$$\operatorname{Hom}_A(M'' \otimes_A N, P) \longrightarrow \operatorname{Hom}_A(M \otimes_A N, P) \longrightarrow \operatorname{Hom}_A(M' \otimes_A N, P) \longrightarrow 0$$

Since the above sequence is exact for all modules P, we have the desired conclusion. ■

The tensor product is not left exact. Conider the sequence of $\mathbb{Z}$-modules

$$0 \hookrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

where $f(m) = 2m$. Upon tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

Note that

$$(f \otimes 1)(m \otimes \overline{n}) = 2m \otimes \overline{n} = m \otimes (2\overline{n}) = m \otimes 0 = 0$$

Therefore, the sequence cannot be exact.

2.8 Flat Modules

Definition 2.25 (Flat Module).

Theorem 2.26. *Let N be a A-module. Then, the following are equivalent*

(a) *N is flat*

(b) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then the tensored sequence*

$$0 \longrightarrow M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \longrightarrow 0$$

is exact.

(c) *If $f : M' \to M$ is injective, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective*

(d) *If $f : M' \to M$ is injective and M, M' are finitely generated, then $f \otimes_A 1 : M' \otimes_A N \to M \otimes_A N$ is injective.*

Proof.

(a)

■

2.9 Projective Modules

Theorem 2.27. *For an A-module P, the following are equivalent:*

(a) Every map $f : P \to M''$ can be lifted to $\widetilde{f} : P \to M$ in the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, "\widetilde{f}"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & M'' \arrow[r, two heads] & 0 \end{tikzcd}$$

(b) Every short exact sequence $0 \to M' \to M \to P \to 0$ splits

(c) There is a module M such that $P \oplus M$ is free

(d) The functor $\mathrm{Hom}_A(P, -)$ is exact.

Proof.

$(a) \implies (b)$**:** Taking $M'' = P$ and $f = \mathbf{id}_P$, we have the desired conclusion.

$(b) \implies (c)$**:** Let F denote the free module on the set P. Then, the map $\Phi : F \to P$ given by $\Phi(e_x) = x$ for all $x \in P$ is a surjective A-module homomorphism. We have the following short exact sequence:

$$0 \to \ker \Phi \xrightarrow{\iota} F \xrightarrow{\Phi} P \to 0$$

This is known to split and thus, $F = \psi(P) \oplus \ker \Phi$ where $\psi : P \to F$ is the section.

$(c) \implies (d)$**:** Let $M' \to M \to M''$ be an exact sequence of modules and K be an A-module such that $P \oplus K = F \cong A^{\Lambda}$. Then, the induced sequence

$$\prod_{\lambda \in \Lambda} M' \to \prod_{\lambda \in \Lambda} M \to \prod_{\lambda \in \Lambda} M''$$

is exact. We have seen that there is a natural isomorphism $\mathrm{Hom}_A(A, M) \xrightarrow{\sim} M$, consequently, there is a natural isomorphism

$$\mathrm{Hom}_A(A^{\oplus \Lambda}, M) \xrightarrow{\sim} \prod_{\lambda \in \Lambda} M$$

whence it follows that the sequence

$$\mathrm{Hom}_A(A^{\oplus \Lambda} A, M') \to \mathrm{Hom}_A(A^{\oplus \Lambda} A, M) \to \mathrm{Hom}_A(A^{\oplus \Lambda}, M'')$$

But since $\mathrm{Hom}_A(A^{\oplus \Lambda}, M) \cong \mathrm{Hom}_A(P, M) \oplus \mathrm{Hom}_A(K, M)$, we have the desired conclusion.

$(d) \implies (a)$**:** Trivial.

■

Definition 2.28 (Projective Module). An A-module P satisfying any one of the four equivalent conditions of Theorem 2.27 is said to be a *projective A-module*.

In particular, from Theorem 2.27(c), we see that every free module is projective.

Lemma 2.29. *A finitely generated projective module P over a local ring $(A, \mathfrak{m})$ is free.*

Proof. Let $\{\overline{x}_1, \dots, \overline{x}_n\}$ be a basis for $M/\mathfrak{m}M$ as a k-vector space where $k = A/\mathfrak{m}$. As we have seen earlier, $\{x_1, \dots, x_n\}$ generates M. Let F be the free module with basis $\{e_1, \dots, e_n\}$ and $\Phi : F \to M$ be the module homomorphism given by $\Phi(e_i) = x_i$ and $K = \ker \Phi$. Since M is projective, there is a module homomorphism $\psi : M \to F$ satisfying $\Phi \circ \psi = \mathbf{id}_M$ and $F = K \oplus \psi(M)$.

We contend that $K = \mathfrak{m}K$. Indeed, let $x \in K$, then $x = \sum r_i e_i$ for a unique choice $\{r_1, \dots, r_n\}$. Then, $\sum r_i x_i = 0$, consequently, $r_i \in \mathfrak{m}$ for all i. Since $F = K \oplus \psi(M)$, we may write $e_i = u_i + v_i$ for some $u_i \in K$ and $v_i \in \psi(M)$. As a result,

$$x - \sum r_i u_i = \sum r_i v_i \in \ker \Phi \cap \psi(M) = \{0\}$$

and the conclusion follows.

Finally due to Lemma 2.14, we must have that $K = 0$ whence M is free. ■

Chapter 3

Localization

3.1 Rings of Fractions

Define the relation $\sim_S$ on $A \times S$ by $(a, s) \sim_S (a', s')$ if there is $t \in S$ such that $t(s'a - sa') = 0$. That this is an equivalence relation is easy to verify. We shall use a/s to denote the equivalence class $[(a, s)]$ in $A \times S/\sim_S$.

Consider the operations:

$$\frac{a}{s} + \frac{a'}{s'} = \frac{s'a + sa'}{ss'} \qquad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$$

It is not hard to see that these are well defined and endow $A \times S/\sim_S$ with a ring structure. We denote this ring by $S^{-1}A$ and is called the *ring of fractions* of A by S.

There is a natural ring homomorphism $\varphi : A \to S^{-1}A$ given by $\varphi(x) = x/1$. When A is an integral domain and $S = A \backslash \{0\}$, $S^{-1}A$ is precisely the field of fractions. Recall that if $\mathfrak{p}$ is a prime ideal in A, then $S = A \backslash \mathfrak{p}$ is a multiplicatively closed subset of A. We denote the ring $S^{-1}A$ by $A_\mathfrak{p}$.

Theorem 3.1. *The ring $A_\mathfrak{p}$ is local.*

Proof. Let $S = A \backslash \mathfrak{p}$ and define

$$\mathfrak{m} = \left\{ \frac{a}{s} \,\middle|\, a \in \mathfrak{p},\ s \in S \right\}$$

It is not hard to see that $\mathfrak{m}$ is an ideal in $A_\mathfrak{p}$. We contend that $\mathfrak{m}$ is the ideal of non-units in $A_\mathfrak{p}$. Indeed, if $a/s \in \mathfrak{m}$ is a unit, then there is $b/t \in A_\mathfrak{p}$ such that $(ab)/(st) = 1$, consequently, there is $w \in S$ such that $w(ab - st) = 0$, whence $wst \in \mathfrak{p}$, a contradiction.

On the other hand, if $a/s \notin \mathfrak{m}$, then a/s is a unit since $(a/s) \cdot (s/a) = 1$. Now, since the collection of all non-units forms an ideal, the ring must be local due to Proposition 1.4. ∎

Similarly, when we let $S = \{a^n\}_{n \geq 0}$ for some $a \in A$, we denote $S^{-1}A$ by A_a.

There is a degenerate case, when we allow $0 \in S$, notice that the ring $S^{-1}A$ is the zero ring, since for all $a/s \in S^{-1}A$, we have $0(as) = 0$, therefore, $a/s = 0/s$.

3.1.1 Universal Property

Fix a multiplicative subset $S \subseteq A$. Let $\mathscr{C}$ denote the category with objects as pairs (ϕ, B) where $\phi : A \to B$ is a ring homomorphism such that $\phi(s)$ is a unit in B for all $s \in S$. A morphism in this category is a map $f : (\phi, B) \to (\psi, C)$ making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"'] & C \\ B \arrow[ur, "f"'] & \end{tikzcd}$$

The ring of fractions is an initial object in this category. Therefore, we have the following universal property. We shall verify in the "proof" that our construction of the field of fractions does satisfy this property and is therefore an initial object in $\mathscr{C}$.

Proposition 3.2. *Let $f : A \to B$ be a ring homomorphism such that $f(s)$ is a unit in B for all $s \in S$. Then there is a unique ring homomorphism $g : S^{-1}A \to B$ making the following diagram commute*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"'] & B \\ S^{-1}A \arrow[ur, dotted, "\exists! g"'] & \end{tikzcd}$$

Proof. Define the map $g : S^{-1}A \to B$ by $g(a/s) = g(a)g(s)^{-1}$. To see that this map is well defined, note that if $a/s = a'/s'$, then there is $t \in S$ such that $t(s'a - sa') = 0$, consequently, $g(t)(g(s')g(a) - g(s)g(a')) = 0$. As a result, $g(a)g(s)^{-1} = g(a')g(s')^{-1}$. From this, it follows immediately that g is a ring homomorphism making the diagram commute.

As for uniqueness, note that for all $a/s \in S^{-1}A$,

$$g(a/s) = g(a/1)g(1/s) = g(a/1)g(s/1)^{-1} = f(a)f(s)^{-1}$$

which is fixed by the choice of f. This completes the proof. ■

3.2 Modules of Fractions

Let M be an A-module and $S \subseteq A$ be a multiplicatively closed subset. Define the relation $\sim_S$ on $M \times S$ by $(m, s) \sim_S (m', s')$ if and only if there is $t \in S$ such that $t(s'm - sm') = 0$. That this is an equivalence relation is easy to verify. We shall use m/s to denote the equivalence class $[(m, s)]$ in $M \times S/ \sim_S$.

Chapter 4

Noetherian and Artinian Rings and Modules

4.1 Chain Conditions

4.2 Noetherian Rings

Lemma 4.1. *If A is Noetherian and $\phi : A \to B$ is an epimorphism, then B is also Noetherian.*

Theorem 4.2 (Hilbert Basis Theorem). *If A is Noetherian, then so is $A[x]$.*

Note that the converse is also true since $A \cong A[x]/(x)$.

Proof. We shall show that every ideal in $A[x]$ is finitely generated. Suppose not and let $I \subseteq A[x]$ be an ideal that is not finitely generated. Choose $f_1 \in I$ with minimum degree. Now, inductively, choose $f_{k+1} \in I \backslash (f_1, \ldots, f_k)$ with the minimum degree. Obviously, this process goes on indefinitely, since we have assumed I to not be finitely generated. We now have

$$
\begin{aligned}
f_1 &= a_1 x^{d_1} + \text{lower degree terms} \\
f_2 &= a_2 x^{d_2} + \text{lower degree terms} \\
&\vdots \\
f_n &= a_n x^{d_n} + \text{lower degree terms} \\
&\vdots
\end{aligned}
$$

with $d_1 \leq d_2 \leq \cdots$. We also have the following ascending chain of ideals in A,

$$(a_1) \subseteq (a_1, a_2) \subseteq \cdots$$

Therefore, there is $n \in \mathbb{N}$ such that $(a_1, \ldots, a_n) = (a_1, \ldots, a_n, a_{n+1})$. Consequently, we may write a_{n+1} as a linear combination of $a_1, \ldots, a_n$, say

$$a_{n+1} = b_1 a_1 + \cdots + b_n a_n$$

for some $b_1, \ldots, b_n \in A$. Let

$$g = f_{n+1} - (b_1 x^{d_{n+1} - d_1} f_1 + \cdots + b_n x^{d_{n+1} - d_n} f_n)$$

It is not hard to argue that $g \in I \backslash (f_1, \ldots, f_n)$, but $\deg g \leq \deg f_{n+1}$, a contradiction. This completes the proof. ■

An analogous theorem, with an analogous proof is true wherein $A[x]$ is replaced by $A[\![x]\!]$.

4.3 Artinian Rings

Bibliography

[1] Michael Francis Atiyah and I. G. MacDonald. *Introduction to Commutative Algebra.* Addison-Wesley-Longman, 1969.

[2] T. S. Blyth. *Module Theory: An approach to linear algebra*. 1990.

[3] David S. Dummit and Richard M. Foote. *Abstract algebra*. Wiley, New York, 3rd ed edition, 2004.

[4] Serge Lang. *Algebra*. Springer, New York, NY, 2002.