Algebraic Geometry

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December 10, 2023

Contents

Chapter 1

Varieties

Throughout this chapter, k denotes an algebraically closed field, A denotes the polynomial ring $k[x_1, \ldots, x_n]$ in n-variables and R denotes the graded ring $k[x_0, \ldots, x_n]$ with the standard homogeneous polynomial grading.

1.1 Affine Varieties

Definition 1.1. For a subset $T \subseteq A$, define

$$Z(T) := \{p \in \mathbb{A}^n \mid f(p) = 0, \forall p \in T\}.$$

This is called the *zero-set* of T. Conversely, for a subset $S \subseteq \mathbb{A}^n$, define

$$\mathscr{I}(S) := \{f \in A \mid f(p) = 0, \forall p \in S\}.$$

This is called the *ideal generated* by S.

Definition 1.2 (Algebraic Set). A subset Y of $\mathbb{A}^n$ is an *algebraic set* if there is a subset $T \subseteq A$ such that $Y = Z(T)$.

Theorem 1.3. *Let $T_i \subseteq A$, $\mathfrak{a} \trianglelefteq A$ an ideal and $Y_i \subseteq \mathbb{A}^n$.*

(a) $Z(T) = Z((T))$ *where (T) is the ideal generated by T in A.*

(b) $Z(T_1 T_2) = Z(T_1) \cup Z(T_2)$.

(c) $Z(\bigcup T_i) = \bigcap Z(T_i)$. *Hence, the collection of all algebraic sets in $\mathbb{A}^n$ can be identified with the collection of closed sets in some topology on $\mathbb{A}^n$. This is called the* Zarkiski Topology *on $\mathbb{A}^n$.*

(d) *If $T_1 \subseteq T_2$, then $Z(T_1) \supseteq Z(T_2)$.*

(e) *If $Y_1 \subseteq Y_2$, then $\mathscr{I}(Y_1) \supseteq \mathscr{I}(Y_2)$.*

(f) $\mathscr{I}(Y_1 \cup Y_2) = \mathscr{I}(Y_1) \cap \mathscr{I}(Y_2)$.

(g) $\mathscr{I}(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(h) $Z(\mathscr{I}(Y)) = \overline{Y}$, *the closure of Y in the Zariski Topology.*

(i) *There is an inclusion reversing bijection between the radical ideals of A and the algebraic sets in $\mathbb{A}^n$.*

Definition 1.4 (Irreducible). A topological space X is said to be *irreducible* if it cannot be written as the union of two proper closed subspaces.

Proposition 1.5. *Let X be a topological space and $Y \subseteq X$ an irreducible subspace. Then, $\overline{Y}$ is irreducible.*

Proof. Suppose $\overline{Y} = Y_1 \cup Y_2$ where Y_1 and Y_2 are proper closed subspaces of $\overline{Y}$. Then, $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$. Since Y is irreducible, either $Y \subseteq Y_1$ or $Y \subseteq Y_2$, consequently, $\overline{Y} \subseteq Y_1$ or $\overline{Y} \subseteq Y_2$, a contradiction. ∎

Proposition 1.6. *Let X be an irreducible topological space and $W \subseteq X$ a non-empty open subset. Then, W is dense and irreducible.*

Proof. ∎

Definition 1.7 (Affine Variety, Quasi-Affine Variety). An *affine algebraic variety* (or simply *affine variety*) is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is called a *quasi-affine variety*.

Remark 1.1.1. *Let Y be a quasi-affine variety. Then, there is an affine variety X that contains Y and Y is open in X. Then, Y is dense in X and hence, $\overline{Y} = X$, where the closure is taken in $\mathbb{A}^n$.*

Proposition 1.8. *An algebraic set in $\mathbb{A}^n$ is irreducible if and only if its corresponding ideal is prime in A.*

Proof. ∎

Definition 1.9 (Coordinate Ring). For an algebraic set $Y \subseteq \mathbb{A}^n$, we define the *affine coordinate ring* $A[Y] := A/\mathscr{I}(Y)$.

Remark 1.1.2. *Note that $A[Y]$ is always a reduced, finitely generated k-algebra and is an integral domain if and only if Y is irreducible.*

Definition 1.10 (Noetherian Space). A topological space X is said to be *noetherian* if it has the ascending chain condition on open sets.

Proposition 1.11. *A subspace of a noetherian topological space is noetherian.*

Proof. Let X be noetherian and $Y \subseteq X$. Let $U_1 \subseteq U_2 \subseteq \ldots$ be an ascending chain of open subsets of Y, then there are open V_i in X such that $V_i \cap Y = U_i$. Let $W_i = \bigcup_{1 \leq j \leq i} V_j$. Then, $W_1 \subseteq W_2 \subseteq \ldots$ an $W_i \cap Y = U_i$. Since X is noetherian, the chain $\{W_i\}$ stabilizes, consequently, so does the chain $\{U_i\}$. ∎

Proposition 1.12. *A noetherian topological space is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover. Let $\mathscr{A}$ be the collection of all finite unions of U_α's. Then, $\mathscr{A}$ must contain a maximal element, which must be X. Thus, X is compact. ■

Corollary 1.13. A Hausdorff noetherian space X is finite with the discrete topology.

Proof. Due to the preceeding result, every subspace of X is compact (and therefore closed) and hence, every subspace of X is open. Thus, X has the discrete topology. A discrete compact set must be finite. ■

Proposition 1.14. *A noetherian topological space can be expressed as a finite union $X = Y_1 \cup \cdots \cup Y_n$ of irreducible closed subspaces. If we require that $Y_i \not\subseteq Y_j$ for $i \neq j$, then the Y_i's are uniquely determined and are caled the **irreducible components** of X.*

Proof. Let Σ be the poset of closed subspaces of X that cannot be expressed as a finite union of irreducible subspaces. If this poset is non-empty, then it admits a minimal element, say Z. Note that Z cannot be irreducible, hence, $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are proper closed subspaces of Z, whence, are a finite union of irreducible subspaces. Consequently, Z is a finite union of irreducible subspaces.

Suppose we have two minimal representations, $X = Y_1 \cup \cdots \cup Y_n = Y_1' \cup \cdots \cup Y_m'$. Then, $Y_i = (Y_i \cap Y_1') \cup \ldots (Y_i \cap Y_m')$. Since Y_i is irreducible, there is an index j such that $Y_i \subseteq Y_j'$. Similarly, there is an index l such that $Y_j' \subseteq Y_l$. Therefore, $Y_i \subseteq Y_l$, consequently, $i = l$. This completes the proof. ■

Definition 1.15 (Dimension). Let X be a topological space. Then,

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \cdots \subsetneq Y_n \subseteq X, \ \text{each } Y_i \text{ is irreducible}\}.$$

Lemma 1.16. *Let X be a topological space.*

(a) If Y is a subspace of X, then $\dim Y \leq \dim X$.

(b) $\{U_i\}_{i \in I}$ an open cover of X. Then,
$$\dim X = \sup_{i \in I} \dim U_i.$$

(c) If Y is a closed subspace of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, *then $Y = X$.*

Proof. (a) Let $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq X$ be a chain of closed, irreducible subspaces of Y. Then, consider the chain of closures in X,
$$\overline{Z}_0 \subseteq \cdots \subseteq \overline{Z}_n \subseteq X.$$
We contend that the inclusion $\overline{Z}_i \subseteq \overline{Z}_{i+1}$ is strict. Indeed, if $\overline{Z}_i = \overline{Z}_{i+1}$, then $\overline{Z}_i \cap Y = \overline{Z}_{i+1} \cap Y$, which is absurd, since the Z_j's are closed in Y. Hence, $\dim X \geq n$ and it follows that $\dim X \geq \dim Y$.

(b) From part (a), we know that $\dim X \geq \sup_i \dim U_i$. We shall show the inequality in the other direction.

Let $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq X$ be a chain of closed, irreducible subspaces of X. Pick some point $x_0 \in Z_0$ and let U_i be an element of the open cover containing x_0. Consider the sequence of closed subspaces

$Z_0 \cap U_i \subseteq \cdots \subseteq Z_n \cap U_i$. Each $Z_j \cap U_i$ is an open subspace of Z_j and hence, is irreducible and dense in Z_j.

Next, we contend that the inclusions $Z_j \cap U_i \subseteq Z_{j+1} \cap U_i$ are strict. Indeed, if $Z_j \cap U_i = Z_{j+1} \cap U_i = Y$, then, Y is dense in both Z_{j+1} and Z_j but Z_j is a proper closed subspace of Z_{j+1}, a contradiction. Thus, $\dim U_i \geq n$. That is, $\sup_i \dim U_i \geq \dim X$. The conclusion follows.

(c) ______ Add in later

■

Proposition 1.17. *If Y is an algebraic set, then $\dim Y = \dim A[Y]$, where the latter is the Krull dimension.*

Proof. Immediate from definition. ■

Proposition 1.18. *Let Y be a quasi-affine variety, then $\dim Y = \dim \overline{Y}$.*

Proof. Note that Y is open in $\overline{Y}$ as we have argued in Remark 1.1.1. Suppose $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq Y$ is a sequence of closed irreducible subsets of Y, then $\overline{Z}_0 \subsetneq \cdots \subsetneq \overline{Z}_n \subseteq \overline{Y}$is a sequence of closed irreducible subsets of $\overline{Y}$. Thus, $\dim \overline{Y} \geq \dim Y$.

Conversely, suppose $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq \overline{Y}$ is a chain of closed irreducible subsets of $\overline{Y}$. Then, each $Z_i \cap Y$ is an open subset of Z_i whence is irreducible. Further, if we have $Z_i \cap Y = Z_{i+1} \cap Y$ for some Y, then $Z_{i+1} = (Z_{i+1} \backslash Y) \cup Z_i$, both of which are closed, a contradiction. Thus, $Z_{i+1} \cap Y \neq Z_i \cap Y$ and $\dim Y \geq \dim \overline{Y}$. This completes the proof. ■

Proposition 1.19. *The Zariski topology on $\mathbb{A}^2$ is not the same as the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.*

Proof. Note that the diagonal Δ in $\mathbb{A}^2$ is $Z((x-y))$ and hence, is closed. On the other hand, $\mathbb{A}^1$ is not Hausdorff, whence, the diagonal Δ in $\mathbb{A}^1 \times \mathbb{A}^1$ is not closed. The conclusion follows. ■

We shall see how to form the product of two varieties in an upcoming section.

1.2 Projective Varieties

We recall a bit about homogeneous ideals first.

Definition 1.20. Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring. An ideal $\mathfrak{a} \trianglelefteq R$ is said to be *homogeneous* if

$$\mathfrak{a} = \bigoplus_{n \geq 0} (\mathfrak{a} \cap R_n)$$

as an abelian group.

Proposition 1.21. *An ideal $\mathfrak{a} \trianglelefteq R$ is homogeneous if and only if $\mathfrak{a}$ can be generated by homogeneous elements.*

Proof. The forward direction is trivial. Conversely, suppose $\mathfrak{a}$ is generated by $F = \bigcup F_i$ where each $F_i \subseteq R_i$. Obviously, $\bigoplus_{n\geq 0}(\mathfrak{a} \cap R_n) \subseteq \mathfrak{a}$. A generic element of $\mathfrak{a}$ is of the form

$$a = \sum_{f\in F} r_f f = \sum_{f\in F}\sum_{i=0}^{\infty} r_{f,i} f$$

where $r_{f,i} \in R_i$. Consequently, $r_{f,i} f$ is a homogeneous element in some R_j and also lies in $\mathfrak{a}$. Thus, $\mathfrak{a} \subseteq \bigoplus_{n\geq 0}(\mathfrak{a} \cap R_n)$. This completes the proof. ■

Proposition 1.22. *Homogeneous ideals are closed under sum, product, intersection and radicals.*

Proof. The first three are obvious. Let $\mathfrak{a}$ be a homogeneous ideal, $\mathfrak{b} = \sqrt{\mathfrak{a}}$ and let $x^m \in \mathfrak{a}$ for some positive integer m. We can write $x = x_{i_1} + \cdots + x_{i_k}$ where $i_1 < \cdots < i_k$ and $x_{i_j} \in R_{i_j}$. Then, x^m has a non-zero component in R_{mi_k}, which is $x_{i_k}^m$. Thus, $x_{i_k}^m \in \mathfrak{a}$, consequently, $x_{i_k} \in \mathfrak{b}$. Then, we have that $x - x_{i_k}$ also lies in $\mathfrak{b}$. Using this, we can argue that all the x_{i_j}'s lie in $\mathfrak{b}$, whence, $\mathfrak{b} \subseteq \bigoplus_{n\geq 0}(\mathfrak{b} \cap R_n)$ and hence, equality holds. This completes the proof. ■

Proposition 1.23. *A homogeneous ideal $\mathfrak{p} \trianglelefteq R$ is prime if and only if for all homogeneous elements $f, g \in R$, $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.*

Proof. We shall prove only the reverse direction. Suppose $f, g \in R$, $fg \in \mathfrak{p}$ but $f, g \notin \mathfrak{p}$. Let $f = f_1 + \cdots + f_n$ and $g = g_1 + \cdots + g_m$ where each f_i, g_j is homogeneous and are arranged according to increasing homogeneous degree. Let f_{n_0} be the largest such that $f_{n_0} \notin \mathfrak{p}$, similarly, choose g_{m_0}. Then, $fg \in \mathfrak{p}$ implies

$$(f_1 + \cdots + f_{n_0})(g_1 + \cdots + g_{m_0}) \in \mathfrak{p}.$$

If we expand the left hand side, $f_{n_0} g_{m_0}$ has the largest homogeneous degree among all the terms and hence, must lie in $\mathfrak{p}$ (since the latter is a homogeneous ideal). Thus, either $f_{n_0} \in \mathfrak{p}$ or $g_{m_0} \in \mathfrak{p}$ according to our assumptions, a contradiction. This completes the proof. ■

Definition 1.24. The *projective n-space over k*, denote $\mathbb{P}^n$ is defined as the set of equivalence classes of the set

$$\underbrace{k \times \cdots \times k}_{n \text{ times}} \setminus \{(0, \ldots, 0)\},$$

under the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \iff \exists \lambda \in k^\times,\ y_i = \lambda x_i \text{ for every } 0 \leq i \leq n.$$

Let $S = k[x_0, \ldots, x_n]$ with the standard grading $S = \bigoplus_{n\geq 0} S_n$ where S_n is the additive abelian subgroup consisting of homogeneous degree n polynomials in S.

Definition 1.25 (Algebraic Set). A subset Y of $\mathbb{P}^n$ is said to be an *algebraic set* if there is a set T of homogeneous elements of S such that $Y = Z(T)$. Now, let $Y \subseteq \mathbb{P}^n$. Define

$$\mathscr{I}(Y) := \{f \in S^h \mid f(p) = 0, \forall p \in Y\},$$

where $S^h = \bigcup_{n\geq 0} S_n$ is the set of all homogeneous polynomials in S.

Henceforth, we endow $\mathbb{P}^n$ with the Zariski topology.

Proposition 1.26. *Algebraic sets are closed under finite unions and arbitrary intersections. Therefore, the Zariski topology is defined to be the collection of complements of algebraic sets in $\mathbb{P}^n$.*

Proof. Note that $Z(T_1T_2) = Z(T_1) \cup Z(T_2)$ and $Z(\cup T_i) = \bigcap Z(T_i)$. ∎

Example 1.27. Let us consider $\mathbb{P}^1$ with the Zariski topology. Note that every homogeneous ideal can be generated by finitely many homogeneous polynomials. It suffices to find $Z(f)$ for a single homogeneous polynomial $f(x, y)$.

If $[a_0 : a_1] \in Z(f)$, then note that neither of the a_i's can be zero. Thus, $f(1, a_1/a_0) = 0$ and hence, a_1/a_0 can take finitely many values. Hence, the Zariski topology on $\mathbb{A}^1$ is precisely the cofinite topology.

Proposition 1.28. *Let $T_i \subseteq S^h$, $\mathfrak{a} \trianglelefteq A$ and $Y_i \subseteq \mathbb{P}^n$.*

(a) If $T_1 \subseteq T_2 \subseteq S^h$, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2 \subseteq \mathbb{P}^n$, then $\mathscr{I}(Y_1) \supseteq \mathscr{I}(Y_2)$.

(c) $\mathscr{I}(Y_1 \cup Y_2) = \mathscr{I}(Y_1) \cap \mathscr{I}(Y_2)$.

(d) If $\mathfrak{a} \trianglelefteq S$ is a homogeneous ideal, then $\mathscr{I}(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) $Z(\mathscr{I}(Y)) = \overline{Y}$.

Proof. (a), (b) and (c) are trivial and (e) follows easily. Consider (d). Let $f \in \mathscr{I}(Z(\mathfrak{a}))$. Note that f is a homogeneous polynomial in S. Consider the algebraic set $V \subseteq \mathbb{A}^{n+1}$ generated by $\mathfrak{a}$. Since f vanishes on $Z(\mathfrak{a}) \subseteq \mathbb{P}^n$, it must vanish on $V \subseteq \mathbb{A}^{n+1}$. As a result, there is a positive integer q such that $f^q \in \mathfrak{a}$. The conclusion follows. ∎

Proposition 1.29. *The follwoing are equivalent:*

(a) $Z(\mathfrak{a}) = \varnothing$.

(b) $\sqrt{\mathfrak{a}}$ is either S or S_+.

(c) $S_d \subseteq \mathfrak{a}$ for some $d > 0$.

Proof. $(a) \implies (b)$ Let $V \subseteq \mathbb{A}^{n+1}$ be the affine algebraic zero set of $\mathfrak{a}$. We have either $V = \varnothing$ or $V = \{(0, \ldots, 0)\}$. Then (b) follows from Hilbert's Nullstellensatz.

$(b) \implies (c)$. Trivial.

$(c) \implies (a)$. Note that $\mathfrak{a}$ contains x_i^d for every $0 \leq i \leq n$. The conclusion follows. ∎

Corollary 1.30. There is a $1-1$ inclusion-reversing bijection between projective algebraic subsets of $\mathbb{P}^n$ and homogeneous radical ideals of S not equal to S_+.

Proposition 1.31. *A projective algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $\mathscr{I}(Y)$ is a prime ideal.*

Proof. Suppose Y is irreducible. Let f, g be homogeneous elements such that $fg \in \mathscr{I}(Y)$. Then, $Z(f) \cup Z(g) \supseteq Y$, whence, Y is contained in either $Z(f)$ or $Z(g)$, consequently, either $f \in \mathscr{I}(Y)$ or $g \in \mathscr{I}(Y)$ whence $\mathscr{I}(Y)$ is prime.

Conversely, suppose $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are closed subsets of $\mathbb{P}^n$. Then, $\mathscr{I}(Y) = \mathscr{I}(Y_1) \cap \mathscr{I}(Y_2)$, consequently, $\mathscr{I}(Y) = \mathscr{I}(Y_i)$ for some $i \in \{1, 2\}$, which follows from the fact that $\mathscr{I}(Y)$ is prime. ■

Corollary 1.32. $\mathbb{P}^n$ is irreducible.

Definition 1.33 (Projective Variety, Quasi-Projective Variety). A *projective variety* is an irreducible algebraic set in $\mathbb{P}^n$. A *quasi-projective variety* is an open subset of a projective variety.

Theorem 1.34. *Let U_i denote the open set $\mathbb{P}^n \backslash Z(\{x_i\})$. The sets $\{U_i\}_{i=0}^n$ cover $\mathbb{P}^n$. Consider the map $\varphi_i : U_i \to \mathbb{A}^n$ given by*

$$\varphi_i((a_0, \dots, a_n)) = \left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

Then, φ_i is a homeomorphism.

Proof. We shall prove this for $i = 0$ and denote φ_0 by $\varphi : U_0 \to \mathbb{A}^n$. ■

Theorem 1.35. *Let Y be a projective n-variety with homogeneous coordinate ring $S(Y)$. Then,* $\dim S(Y) = \dim Y + 1$.

Proof. Let $U_i = \mathbb{P}^n \backslash Z(x_i)$. We have seen that each U_i is homeomorphic to $\mathbb{A}^n$ under the map $\varphi_i : U_i \to \mathbb{A}^n$ as defined above. Let $Y_i = \varphi_i(U_i \cap Y) \subseteq \mathbb{A}^n$. Note further that Y_i is irreducible owing to $U_i \cap Y$ being irreducible (since it is an open subset of Y which is irreducible). Thus, Y_i is an affine n-variety.

Note that we can identify $A(Y_i)$ with the subring of degree 0 elements in $S(Y)_{x_i}$. Further, note that

$$S(Y)_{x_i} = \left(S(Y)_{x_i}\right)_0 [x_i, x_i^{-1}] \cong A(Y_i)[x_i, x_i^{-1}].$$

Note that $\dim S(Y)_{x_i} = \dim S(Y)$ since they both have the same fraction fields. Then, it follows that

$$\dim S(Y) = \dim S(Y)_{x_i} = \dim A(Y_i)[x_i, x_i^{-1}] = \dim A(Y_i) + 1 = \dim Y_i + 1.$$

The equality $\dim A(Y_i)[x_i, x_i^{-1}] = \dim A(Y_i) + 1$ follows by looking at the transcendence degree of the fraction fields.

Finally, note that $\dim Y_i = \dim(Y \cap U_i)$ and hence, $\dim Y = \sup \dim Y_i$, consequently, $\dim Y + 1 = \dim S(Y)$. This completes the proof. ■

Corollary 1.36. Let Y be a quasi-projective variety. Then, $\dim Y = \dim \overline{Y}$.

Proof. Let $U_i = \mathbb{P}^n \backslash Z(x_i)$ for $0 \leq i \leq n$. Note that $Y \cap U_i$ is a quasi-affine variety whose closure is $\overline{Y} \cap U_i$. Thus,

$$\dim Y = \sup_i \dim(Y \cap U_i) = \sup_i \dim(\overline{Y} \cap U_i) = \dim \overline{Y}.$$ ■

Definition 1.37 (Cone over a Projective Variety). Let $Y \subseteq \mathbb{P}^n$ be a projective algebraic set. Let φ : $\mathbb{A}^{n+1} \backslash \{(0,\ldots,0)\} \to \mathbb{P}^n$ be given by $\theta((a_0,\ldots,a_n)) = [(a_0,\ldots,a_n)]$. The *affine cone over* Y is defined to be

$$C(Y) := \theta^{-1}(Y) \cup \{(0,\ldots,0)\}.$$

Definition 1.38 (Segre Embedding). Define the map $\psi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ where $N = (m+1)(n+1) - 1$ by

$$\psi\left([a_0 : \cdots : a_{n+1}], [b_0 : \cdots : b_{m+1}]\right) = [[a_i b_j]_{i,j}].$$

This is called the *Segre embedding*

Proposition 1.39. *With notation as above,* $\operatorname{im}\psi$ *is a subvariety of* $\mathbb{P}^N$.

Proof. Consider the k-algebra homomorphism $\phi : k[\{z_{ij}\}] \to k[x_0,\ldots,x_n,y_0,\ldots,y_m]$ given by $\phi(z_{ij}) = x_i y_j$. Let $\mathfrak{p} = \ker\phi$. This is obviously a prime ideal in $k[\{z_{ij}\}]$.

Indeed, if $f \in \mathfrak{p}$, then $f(\{x_i y_j\}) = 0$. We can write $f = \sum_{d\geq 0} f_d$. Then, $f_d(\{x_i y_j\}) = 0$ for every $d \geq 0$. Thus, every $f_d \in \mathfrak{p}$ and it follows that $\mathfrak{p}$ is homogeneous. We may now talk about $Z(\mathfrak{p})$ as a projective variety in $\mathbb{P}^N$.

We contend that $Z(\mathfrak{p}) = \operatorname{im}\psi$. The inclusion $\operatorname{im}\psi \subseteq Z(\mathfrak{p})$ is obvious. Now suppose $[\{c_{ij}\}] \in Z(\mathfrak{p})$. Without loss of generality, suppose $c_{00} \neq 0$. By normalizing coordinates, we may suppose that $c_{00} = 1$. Define $[a_0 : \cdots : a_n]$ and $[b_0 : \cdots : b_m]$ as follows.

$$a_i = \begin{cases} 1 & i = 0 \\ c_{i0} & i > 0 \end{cases} \quad \text{and} \quad b_j = \begin{cases} 1 & j = 0 \\ c_{0j} & j > 0 \end{cases}.$$

Now, note that $z_{ij}z_{00} - z_{i0}z_{0j} \in \mathfrak{p}$ and hence, $c_{ij} = c_{i0}c_{0j} = a_i b_j$. Thus,

$$[\{c_{ij}\}] = \psi\left([a_0 : \cdots : a_n], [b_0 : \cdots : b_m]\right),$$

this completes the proof. ∎

We shall use the Segre embedding to define the product of two (quasi-)projective varieties.

1.3 Morphisms

We begin by defining regular maps on varieties. The definitions will be different for afine and projective varieties.

Definition 1.40 (Regular Map). Let Y be a quasi-affine variety in $\mathbb{A}^n$. A function $f : Y \to k$ is *regular at a point* $P \in Y$ if there is an open neighborhood U of P in Y and polynomials $g, h \in A$ such that h does not vanish on U and $f = g/h$ on U. The map f is said to be *regular on* Y if it is regular at every point of Y.

Let Y be a quasi-projective variety in $\mathbb{P}^n$. A function $f : Y \to k$ is *regular at a point* $P \in Y$ if there is an open neighborhood U of P in Y and homogeneous polynomials $g, h \in S^h$ of equal degree such that h does not vanish on U and $f = g/h$ on U.

Theorem 1.41. *A regular function is continuous when k is identified with $\mathbb{A}^1$ or $\mathbb{P}^1$.*

Proof. First, suppose Y is a quasi-affine variety. Identify k with $\mathbb{A}^1$ and let $\varphi : Y \to k$ be a regular map on Y. It suffices to show that the inverse image of a closed set in $\mathbb{A}^1$ is closed in Y. But closed sets in $\mathbb{A}^1$ are precisely finite subsets of k. Hence, it suffices to show that the inverse image of a singleton in $\mathbb{A}^1$ is closed in Y. Let $a \in k$.

There is an open cover of Y such that on every open set of the cover, φ is of the form f/g. Pick such an open set U. Then, $\varphi^{-1}(\{a\}) \cap U = Z(f - ag) \cap U$, which is closed in U. Thus, $\varphi^{-1}(\{a\})$ is closed in Y.

Now, suppose Y is a quasi-projective variety. Note that when k is identified with $\mathbb{P}^1$, it has the cofinite topology and a proof similar to the one in the preceeding paragraphs works. ∎

Henceforth, a *variety* refers to either a quasi-affine or quasi-projective variety. When a result explicitly depends on the type of variety, we shall mention it.

Corollary 1.42. Let Y be a variety. If $f, g : Y \to k$ are regular functions that agree on an open subset of Y, then $f = g$ on Y.

Proof. Let $X = \{y \in Y \mid f(y) = g(y)\}$. We know that X is closed and contains an open subset U of Y. But U is dense in Y (since Y is irreducible) and hence, $X = Y$. ∎

Definition 1.43 (Morphism). Let X and Y be varieties over k (can be quasi-affine or quasi-projective). A *morphism* $\varphi : X \to Y$ is a continuous map such that for every open set $V \subseteq Y$ and for every regular function $f : V \to K$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular on $\varphi^{-1}(V)$.

$$\begin{tikzcd} X \supseteq \varphi^{-1}(V) \arrow[r, "\varphi"] \arrow[dr, dotted, "f\circ\varphi"'] & V \arrow[d, "f"] \\ & k \end{tikzcd}$$

Proposition 1.44. *The composition of two morphisms is a morphism. The identity map on a variety is a morphism.*

Proof. Let X, Y, Z be varieties and $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms. Let $V \subseteq Z$ be an open set and $f : V \to k$ be a regular function on V.

$$\begin{tikzcd} \varphi^{-1}\psi^{-1}V \arrow[r] \arrow[drr, "f\circ\psi\circ\varphi"'] & \psi^{-1}V \arrow[r] \arrow[dr, "f\circ\psi"] & V \arrow[d, "f"] \\ & & k \end{tikzcd}$$

Applying the definition of a morphism twice, we see that $f \circ \psi \circ \varphi$ is a regular function. This completes the proof. ∎

Definition 1.45 (Ring of Regular Functions). Let Y be a variety. Then, the set of regular functions on Y, denoted $\mathscr{O}(Y)$ forms a ring known as the *ring of regular functions on Y*.

On the other hand, given any $P \in Y$, there is the *local ring of P on Y*, which is the ring of germs of regular functions at P. This is denoted by $\mathscr{O}_{P,Y}$ or just $\mathscr{O}_P$ if the variety is clear from the context.

Remark 1.3.1. *Here, we explicitly define the ring of germs at P. Consider the set of all pairs $\langle U, f\rangle$ where U is a neighborhood of P in Y. Next, define the relation $\langle U, f\rangle \sim \langle V, g\rangle$ if $f = g$ on $U \cap V$.*

To see that this is an equivalence relation, we need only verify transitivity. Indeed, suppose $\langle U, f\rangle \sim \langle V, g\rangle$ and $\langle V, g\rangle \sim \langle W, h\rangle$. Then, $f = g = h$ on $U \cap V \cap W$ which is a neighborhood of P. Since they agree on an open set, $f = h$ on $U \cap W$.

Next, we must show that $\mathscr{O}_P$ is local. Let $\mathfrak{m}$ denote the collection of germs that vanish at P. This is obviously a maximal ideal in $\mathscr{O}_P$. If $[\langle U, f\rangle] \in \mathscr{O}_P$ does not vanish at P, then there is a neighborhood V of P contained in U on which f does not vanish and f is a quotient of polynomials. Thus, $1/f$ is also a quotient of polynomials on V and is a well-defined inverse of the germ $[\langle V, f\rangle]$. Hence, $\mathscr{O}_P$ is local.

Lemma 1.46. *Let $X \subseteq \mathbb{A}^n$ be open and hence, a quasi-affine variety. If $f : X \to k$ is a regular function then $f = g/h$ for some $g, h \in A = k[x_1, \ldots, x_n]$.*

Proof. Let $U \subseteq X$ be an open set such that $f = g/h$ (reduced) on U for some polynomials $g, h \in A$. We contend that $f(P) = g(P)/h(P)$ for all $P \in X$. Indeed, there is a neighborhood V of P in X such that $f = g'/h'$ (reduced) on V for some $g', h' \in A$.

The intersection $U \cap V$ is non-empty and $g/h = g'/h'$ on $U \cap V$. Consequently, $U \cap V \subseteq Z(gh' - g'h)$. But $U \cap V$ is open and dense in $\mathbb{A}^n$, whence $gh = g'h$ as polynomials in A. Using the fact that A is a unique factorization domain and $(g, h) = (g', h') = 1$, we have that $g = g'$ and $h = h'$. This completes the proof. ∎

Corollary 1.47. Let $X = \mathbb{A}^n \backslash \{(0, \ldots, 0)\}$. Then, $\mathscr{O}(X) \cong k[x_1, \ldots, x_n]$.

Proof. Let $f : X \to k$ be a regular function. Due to Lemma 1.46, $f = g/h$ (reduced) on X for some $g, h \in k[x_1, \ldots, x_n]$. Note that h does not vanish on X and hence, must be constant. Therefore, $f \in k[x_1, \ldots, x_n]$. The conclusion follows. ∎

Definition 1.48 (Function Field). Let Y be a variety. The *function field of Y*, denoted $K(Y)$ is defined as the set of equivalence classes of the collection of pairs $\langle U, f\rangle$ where $U \subseteq Y$ is open and $f : U \to k$ is a regular function on U. The equivalence relation is defined as:

$$\langle U, f\rangle \sim \langle V, g\rangle \iff f = g \text{ on } U \cap V.$$

The elements of $K(Y)$ are called *rational functions on Y*.

Remark 1.3.2. *That $K(Y)$ is a well defined ring, one can argue as in Remark 1.3.1. We show that this ring is a field. Indeed, suppose $\langle U, f\rangle$ is an element in $K(Y)$ with f not identically 0 on U. Then, there is an open set $V \subseteq U$ such that f is non-zero on V. Then, $\langle V, 1/f\rangle$ is an inverse of $\langle U, f\rangle$ in $K(V)$.*

Remark 1.3.3. *Let Y be a variety. There is a natural map $\mathscr{O}(Y) \to \mathscr{O}_P$ that sends a regular function on Y to the equivalence class of the pair $\langle Y, f\rangle$, which is obviously an injective map.*

Next, there is also a map $\mathscr{O}_P \to K(Y)$ that sends $[\langle U, f\rangle] \mapsto [\langle U, f\rangle]$. Again, this is obviously injective. Hence, we have an inclusion of rings

$$\mathscr{O}(Y) \subseteq \mathscr{O}_{P,Y} \subseteq K(Y)$$

and we shall often treat $\mathscr{O}(Y)$ and $\mathscr{O}_P$ as subrings of $K(Y)$.

Theorem 1.49. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Then,*

(a) $\mathscr{O}(Y) \cong A(Y)$.

(b) for each $P \in Y$, $\mathscr{O}_P \cong A(Y)_{\mathfrak{m}_P}$ where $\mathfrak{m}_P$ is the maximal ideal of functions in $A(Y)$ vanishing at P. Consequently, $\dim \mathscr{O}_P = \dim Y$.

(c) $K(Y)$ is isomorphic to the fraction field of $A(Y)$. In particular, it is a finitely generated field extension of k of transcendence degree $\dim Y$.

Proof. First, note that every maximal ideal in $A(Y)$ is of the form $\mathfrak{m}_P$ for some $P \in Y$. There is a natural map $A(Y)_{\mathfrak{m}_P} \to \mathscr{O}_P$ that sends f/g to $[\langle Y \backslash Z(g), f/g \rangle]$. This is obviously both injective and surjective and therefore, an isomorphism. This establishes (b).

Now, there is a natural map $A \to \mathscr{O}(Y)$ that sends a polynomial to the regular function defined by it on Y. The kernel of this map is precisely $\mathscr{I}(Y)$ and hence, induces an injective ring homomorphism $\alpha : A(Y) \to \mathscr{O}(Y)$. Upon identifying $A(Y)$ with a subring of $\mathscr{O}(Y)$ and using the fact that $\mathscr{O}_P = A(Y)_{\mathfrak{m}_P}$, we have

$$A(Y) \subseteq \mathscr{O}(Y) \subseteq \bigcap_{P \in Y} \mathscr{O}_P \subseteq \bigcap_{\mathfrak{m}_P} A(Y)_{\mathfrak{m}_P} = A(Y),$$

where the last equality follows from the fact that A is an integral domain. This establishes (a).

Finally, from (b) and the fact that $\mathscr{O}_P \subseteq K(Y)$, it would follow that $K(Y)$ is precisely the fraction field of $A(Y)$. ■

Remark 1.3.4 (The Functor $\mathscr{O}$). *We shall quickly see that $\mathscr{O}$ is a functor from the category of k-varieties, $\mathfrak{Var}_k$ to the category of finitely generated k-algebras* **FinAlg**$_k$*. Let $\varphi : X \to Y$ be a morphism of varieties. Then, if $f \in \mathscr{O}(Y)$, then $f \circ \varphi \in \mathscr{O}(X)$. This induces a map $\varphi_* : \mathscr{O}(Y) \to \mathscr{O}(X)$. Thus, $\mathscr{O}$ is a contravariant functor.*

Theorem 1.50. *Let X be any variety and Y an affine variety. Then, there is a natural bijection of sets*

$$\alpha : \mathrm{Hom}_{\mathfrak{Var}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Alg}}(\mathscr{O}(Y), \mathscr{O}(X)).$$

Proof. ■

Corollary 1.51. The functor $X \mapsto \mathscr{O}(X) \equiv A(X)$ is a contravariant equivalence of categories between $\mathfrak{Var}_k$ and **FinAlg**$_k$.

Example 1.52. We contend that the quasi-affine variety $\mathbb{A}^n \backslash \{(0, \ldots, 0)\}$ is not isomorphic to an affine variety. Let $X = \mathbb{A}^n \backslash \{(0, \ldots, 0)\}$, and suppose X is affine. The inclusion morphism $\iota : X \hookrightarrow \mathbb{A}^n$ induces a restriction map $\iota_* : \mathscr{O}(\mathbb{A}^n) \to \mathscr{O}(X) = k[x_1, \ldots, x_n]$, which is an isomorphism. From the equivalence of categories, it follows that ι must be an isomorphism too, which is absurd, since it is not even surjective.

Definition 1.53 (Locally Closed). A subspace of a topological space is said to be *locally closed* if it is open in its closure. Equivalently, if it is the intersection of an open set and a closed set.

Definition 1.54. If X is a quasi-affine (resp. quasi-projective) variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (resp. quasi-projective) variety and is said to be a *subvariety* of X.

1.3.1 Product of Varieties

Definition 1.55 (Product of Affine Varieties). If $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are affine-varieties, their product is defined to be the topological space $X \times Y \subseteq \mathbb{A}^{n+m}$ in the *subspace topology*.

Proposition 1.56. *With notation as above,*

(a) $X \times Y \subseteq \mathbb{A}^{n+m}$ *is a closed irreducible subspace, and hence, an affine variety.*

(b) $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) $\dim X \times Y = \dim X + \dim Y$.

Proof. (a)

■

Definition 1.57 (Product of Quasi-Projective Varieties).

1.4 Rational Maps

Lemma 1.58. *Let X, Y be varieties and $\varphi, \psi : X \to Y$ be morphisms. If there is a non-empty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$, then $\varphi = \psi$.*

Proof. ■

Definition 1.59 (Rational Map).

Chapter 2

Schemes

2.1 Sheaves

Throughout this section, we shall work with sheaves of abelian groups. The definitions carry over to any abelian category with pretty much the same proofs.

Definition 2.1 (Presheaf). Let X be a topological space. A *presheaf of abelian groups on* X is a contravariant functor $\mathscr{F} : \mathfrak{Top}(X) \to \mathbf{AbGrp}$ where $\mathfrak{Top}(X)$ denotes the poset category of open sets in X ordered by inclusion.

For an open set $U \subseteq X$, we refer to $\mathscr{F}(U)$ as the *sections* of the presheaf $\mathscr{F}$ over the open set U. Sometimes, we use the notation $\Gamma(U, \mathscr{F})$ to denote $\mathscr{F}(U)$.

Definition 2.2 (Sheaf). A presheaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfies the following additional conditions,

Identity Axiom: if U is an open set, $\{V_i\}$ an open cover of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$.

Gluability Axiom: if U is an open set, $\{V_i\}$ an open cover of U, and if there are elements $s_i \in \mathscr{F}(V_i)$ for each i, such that for each pair i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i.

Definition 2.3 (Direct Limit). Let $\mathscr{C}$ be a category, $(I, \leq)$ a directed set and a collection $\langle \{A_i\}, \{f_{ij} : A_i \to A_j\}_{i \leq j} \rangle$ with the following properties:

(a) f_{ii} is the identity on A_i, and

(b) $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

The *direct limit* of the above direct system (if it exists), denoted $\varinjlim A_i$ is an object $\langle X, \{\phi_i\} \rangle$ where $\phi_i : A_i \to X$ are morphisms that satisfy, for all $i \leq j$, $\phi_i = \phi_j \circ f_{ij}$. This object is universal in the sense that if $\langle Y, \{\psi_i\} \rangle$ is another object with morphisms such that $\psi_i = \psi_j \circ f_{ij}$ then there is a unique morphism $u : X \to Y$ satisfying $\psi_i = u \circ \phi_i$ for all i.

Remark 2.1.1. *Now suppose $\mathscr{C}$ is the category of abelian groups. We show that this category always admits a direct limit. Indeed, suppose $\langle \{A_i\}, \{f_{ij}\} \rangle$ is a direct system. Let*

$$X = \bigsqcup_{i \in I} A_i \Big/ \sim$$

where $\langle i, a_i\rangle \sim \langle j, a_j\rangle$ if and only if there is some $k \geq i, j$ such that $f_{ik}(a_i) = f_{jk}(a_j)$. The group operation is defined as, $[\langle i, a_i\rangle] \cdot [\langle j, a_j\rangle] = [\langle k, a_k\rangle]$ where $k \geq i, j$ and $a_k = f_{ik}(a_i) f_{jk}(a_j)$.

First, note that the group operation is well defined. Indeed, suppose $[\langle i, a_i\rangle] = [\langle p, a_p\rangle]$, $[\langle j, a_j\rangle] = [\langle q, a_q\rangle]$ and suppose that we chose $[\langle r, a_r\rangle]$ to be $[\langle p, a_p\rangle] \cdot [\langle q, a_q\rangle]$. There is an index $l \geq k, r$ sufficiently "large" such that $f_{il}(a_i) = f_{pl}(a_p)$ and $f_{jl}(a_j) = f_{ql}(a_q)$. The conclusion now follows.

Consider maps $\phi_i : A_i \to X$ given by $\phi_i(a_i) = [\langle i, a_i\rangle]$. This is obviously a morphism and $\phi_j \circ f_{ij} = \phi_i$ for all $i \leq j$. Now, suppose $\langle Y, \{\psi_i\}\rangle$ is another such object. Define $u : X \to Y$ by

$$u([\langle i, a_i\rangle]) = \psi_i(a_i).$$

Again, it is not hard to see that this map is well defined and is the unique map making everything commute.

Definition 2.4 (Stalk of a Presheaf). Let $\mathscr{F}$ be a presheaf on a topological space X and $P \in X$. Consider the directed set of neighborhoods of P in X ordered by *reverse inclusion*. The direct limit $\varinjlim \mathscr{F}(U)$ over this directed system is called the *stalk of $\mathscr{F}$ at P*, denoted $\mathscr{F}_P$.

Definition 2.5 (Morphism of Presheaves/Sheaves). Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X. A *morphism* $\varphi : \mathscr{F} \to \mathscr{G}$ is a *natural transformation* between these two functors. In other words, it is a collection of maps $\{\varphi(U)\}_{U \in \mathfrak{Top}(X)}$ making the following diagram commute

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"'] & \mathscr{G}(V)
\end{tikzcd}$$

for all $U, V \in \mathfrak{Top}(X)$. A morphism of presheaves is an *isomorphism* if and only if it admits an inverse.

Proposition 2.6. *Let $\mathscr{F}, \mathscr{G}$ be presheaves on X and $\varphi : \mathscr{F} \to \mathscr{G}$ a morphism. Then, for each $P \in X$, there is an induced map $\varphi : \mathscr{F}_P \to \mathscr{G}_P$ given by*

$$\varphi_P([\langle U, s\rangle]) = [\langle U, \varphi_U(s)\rangle].$$

Proof. Straightforward. ∎

Proposition 2.7. *Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ be morphisms of presheaves. Then, $(\psi \circ \varphi)_P = \psi_P \circ \varphi_P$.*

Proof. Let $[\langle U, s\rangle] \in \mathscr{F}_P$. Then,

$$\psi_P \circ \varphi_P([\langle U, s\rangle]) = [\langle U, \psi_U(\varphi_U(s))\rangle] = [\langle U, (\psi \circ \varphi)_U(s)\rangle].$$

This completes the proof. ∎

Theorem 2.8. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Then, φ is an isomorphism if and only if the induced map on the stalk $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for every $P \in X$.*

Proof. The forward direction is trivial and we shall show the converse, for which it suffices to show that φ_U is an isomorphism for every $U \in \mathfrak{Top}(X)$.

First, we show injectivity. Let $s \in \mathscr{F}(U)$ that maps to 0 in $\mathscr{G}(U)$. Let $P \in U$. Then, $\varphi_P([\langle U, s\rangle]) = [\langle U, \varphi_U(s)\rangle] = 0$ whence $[\langle U, s\rangle] = 0$ in $\mathscr{F}_P$. Hence, there is a neighborhood V_P of P in U on which s restricts to 0. Note that the V_P's form an open cover of U and due to the identity axiom, $s = 0$ on U, from which injectivity follows.

Next, we show surjectivity. Let $t \in \mathscr{G}(U)$ and $P \in U$. Since φ_P is surjective, there is some $[\langle V_P, s_P\rangle]$ with $V_P \subseteq U$ in $\mathscr{F}_P$ that maps to $[\langle U, t\rangle]$ in $\mathscr{G}_P$. Thus,

$$[\langle V_P, \varphi_{V_P}(s_P)\rangle] = [\langle U, t\rangle].$$

Thus, we may shrink V_P such that $\varphi_{V_P}(s_P) = t|_{V_P}$, where we have redefined s_P to be its restriction to the shrinked V_P.

Now, let P, Q be points in U. We have

$$\varphi_{V_P\cap V_Q}(s_P|_{V_P\cap V_Q}) = \varphi_{V_P}(s_P)|_{V_P\cap V_Q} = t|_{V_P\cap V_Q}.$$

Similarly, we have $\varphi_{V_P\cap V_Q}(s_Q|_{V_P\cap V_Q}) = t|_{V_P\cap V_Q}$. Using the injectivity of $\varphi_{V_P\cap V_Q}$ that we proved in the earlier paragraph, we have $s_P|_{V_P\cap V_Q} = s_Q|_{V_P\cap V_Q}$. Using the gluability axiom, we have an $s \in \mathscr{F}(U)$ that restricts to s_P on V_P for every P.

Finally, we verify that $\varphi_U(s) = t$. Indeed, $\{V_P\}$ is a cover of U and $t|_{V_P} = \varphi_U(s)|_{V_P}$. From the identity axiom, it follows that $t = \varphi_U(s)$. This completes the proof. ■

Definition 2.9 (Sheafification). Let $\mathscr{F}$ be a presheaf on X. Then, there is a sheaf $\mathscr{F}^+$ and a morphism of presheaves $\theta : \mathscr{F} \to \mathscr{F}^+$ with the property that for any other sheaf $\mathscr{G}$ and morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$ making the following diagram commute:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \\ \mathscr{F}^+ \arrow[ur, dotted, "\exists!\psi"'] & \end{tikzcd} .$$

We have not yet constructed the sheafification map. In order to do so, we introduce another construction first.

Definition 2.10 (Espace étalé). Let $\mathscr{F}$ be a presheaf on X. We define a topological space $\text{Spé}(\mathscr{F})$ known as the *espace étalé* of the presheaf. The underlying set is

$$\text{Spé}(\mathscr{F}) = \bigsqcup_{P\in X} \mathscr{F}_P.$$

Let $\pi : \text{Spé}(\mathscr{F}) \to X$ be the map that sends $s \in \mathscr{F}_P$ to P. For every $U \in \mathfrak{Top}(X)$ and $s \in \mathscr{F}(U)$, there is an induced map $\bar{s} : U \to \text{Spé}(\mathscr{F})$ that sends $P \mapsto s_P := [\langle U, s\rangle] \in \mathscr{F}_P$. Note that $\pi \circ \bar{s} = \mathbf{id}_U$.

The topology on $\text{Spé}(\mathscr{F})$ is the finest topology such that for every U and every $s \in \mathscr{F}(U)$, the map $\bar{s}$ is continuous.

Proposition 2.11. *A basis for the topology on* $\text{Spé}(\mathscr{F})$ *is given by the collection of sets of the form*

$$\{(P, s_P) \mid P \in U\},$$

where U ranges over the open sets in X and $s \in \mathscr{F}(U)$.

Proof. Call the above collection of sets $\mathscr{B}$. First, we verify that this is a basis for a topology. Indeed, consider the intersection

$$\{(P, s_P) \mid P \in U\} \cap \{(Q, t_Q) \mid Q \in V\}.$$

If (p, σ) lies in the above intersection, then there is a neighborhood W of p contained in $U \cap V$ such that $s|_W = t|_W$. Then, for any $q \in W$, we must have $s_q = t_q$. As a result,

$$\{(q, s_q) \mid q \in W\} \subseteq \{(P, s_P) \mid P \in U\} \cap \{(Q, t_Q) \mid Q \in V\}.$$

This establishes that the collection is indeed a basis.

Similarly, it is not hard to see that if Spé($\mathscr{F}$) is endowed with the topology generated by the above basis, then every map $\bar{s}$ is continuous for every section $s \in \mathscr{F}(U)$. Hence, the topology on Spé($\mathscr{F}$) is finer than the topology generated by the above basis.

Let $V \subseteq$ Spé($\mathscr{F}$) be an open set and let it contain a germ (P, s_P) where $s \in \mathscr{F}(U)$ and U is a neighborhood of P. Let $W = \bar{s}^{-1}(V)$, which is a neighborhood of P. For every $Q \in U \cap W$, $\bar{s}(Q) \in V$, and hence, $(Q, s_Q) \in V$. In particular,

$$\{(Q, s_Q) \mid Q \in U \cap W\} \subseteq V.$$

The conclusion follows. ■

Corollary 2.12. Let U be an open set in X. Then, $\Gamma(U, \text{Spé}(\mathscr{F}))$, the set of continuous sections, has the structure of an abelian group.

Proof. Let $f, g : U \to$ Spé($\mathscr{F}$) be continuous maps. Define $(f+g)(P) = f(P) + g(P) \in \mathscr{F}_P$. Consider the basic open set $B = \{(P, s_P) \mid P \in W\}$ where $W \subseteq U$ is open and $s \in \mathscr{F}(W)$.

Now, let $P \in (f+g)^{-1}(B)$. Thus, $f(P) + g(P) = s_P \in \mathscr{F}_P$. Suppose $f(P) = (P, t_P)$ and $g(P) = (P, r_P)$, where both t and r lie in $\mathscr{F}(V)$ for a sufficiently small neighborhood V of P in U. Thus, $t + r$ agrees with s on some neighborhood of P contained in V, whence, $(f+g)^{-1}(B)$ contains a neighborhood around P. Thus, is open. This completes the proof. ■

Remark 2.1.2. *From the above proof, we we note that if $f \in \Gamma(U, \text{Spé}(\mathscr{F}))$ and $P \in U$, then there is a neighborhood V of P contained in V and $t \in \mathscr{F}(V)$ such that $f(Q) = t_Q$ for every $Q \in V$. That is, every continuous section locally looks like the germs of a section in the presheaf.*

Proposition 2.13. *The sheafification map $\theta : \mathscr{F} \to \mathscr{F}^+$ exists and is unique up to a unique isomorphism.*

Proof. Let $\mathscr{F}^+(U) = \Gamma(U, \text{Spé}(\mathscr{F}))$, the collection of all continuous sections $U \to$ Spé($\mathscr{F}$). If $V \subseteq U$, then there is an obvious map $\Gamma(V, \text{Spé}(\mathscr{F})) \to \Gamma(U, \text{Spé}(\mathscr{F}))$ which restricts a section on V to that on U. It follows that $\mathscr{F}^+$ is a presheaf. That the sheaf axioms are satisfied is trivial to check.

Define the map $\theta : \mathscr{F} \to \mathscr{F}^+$ by defining $\theta_U : \mathscr{F}(U) \to \mathscr{F}^+(U)$ as $s \mapsto \bar{s}$, with the standard notation used previously. It is straightforward to check that this is indeed a morphism of presheaves.

It remains to verify that θ has the universal property of sheafification. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves where $\mathscr{G}$ is a sheaf. Let U be open in X and $f \in \mathscr{F}^+(U) = \Gamma(U, \text{Spé}(\mathscr{F}))$. We must define $\psi_U(f) \in \mathscr{G}(U)$. We use the gluing axiom to do so.

For every point $P \in U$, there is a neighborhood V_P of P in U and a corresponding $t^P \in \mathscr{F}(V_P)$ such that $(t^P)_x = f(x)$ for all $x \in V_P$. Let $Q \in U$. Then, on $W = V_P \cap V_Q$, we see that $\varphi_W(t^P) = \varphi_W(t^Q)$. Hence, the gluing property applies and there is a $t \in \mathscr{G}(U)$ such that $t|_{V_P} = \varphi_{V_P}(t^P)$. This defines a map of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$. ■

Proposition 2.14. *For all $P \in X$, there is an isomorphism $\mathscr{F}_P \cong \mathscr{F}_P^+$. That is, sheafification preserves stalks.*

Proof. Define the map $\alpha : \mathscr{F}_P \to \mathscr{F}_P^+$ that sends a germ $[\langle U, s\rangle] \in \mathscr{F}_P$ to $[\langle U, \bar{s}\rangle] \in \mathscr{F}_P^+$. We first show that this is well defined. If $[\langle U, s\rangle] = [\langle V, t\rangle]$, then there is a neighborhood W of P contained in $U \cap V$ such that $s|_W = t|_W$. Note that $\bar{s} = \bar{t}$ on W and hence, $[\langle W, \bar{s}\rangle] = [\langle W, \bar{t}\rangle]$.

This map is obviously injective. To see surjectivity, simply recall that any section in $\Gamma(U, \text{Spé}(\mathscr{F}))$ locally looks like $\bar{s}$ on some neighborhood of P contained in U. This completes the proof. ∎

Definition 2.15 (Subsheaf). A sheaf $\mathscr{G}$ is said to be a *subsheaf* of a sheaf $\mathscr{F}$ if for every open $U \subseteq X$, $\mathscr{G}(U) \subseteq \mathscr{F}(U)$ and the restriction maps of $\mathscr{G}$ are induced by the restriction maps of $\mathscr{F}$.

Proposition 2.16 (Kernel Sheaf). *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Define the map $\mathscr{F}' : \mathfrak{Top}(X) \to$* **AbGrp** *by $\mathscr{F}'(U) = \ker \varphi_U \subseteq \mathscr{F}(U)$. Then, $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$ and is called the* ***kernel*** *of φ and is denoted by $\ker \varphi$.*

Proof. Obviously, $\mathscr{F}'$ is a presheaf with the restriction maps as those induced by $\mathscr{F}$. It remains to verify the sheaf axioms. The identity axiom is trivial to verify. Suppose now that $\{V_i\}$ is an open cover of U and $s_i \in \mathscr{F}'(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then, there is an $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$. Note that $\varphi_U(s)|_{V_i} = 0$ for every i and hence, $\varphi_U(s) = 0$ from the identity axiom and it follows that $s \in \mathscr{F}'(U)$. This completes the proof. ∎

Proposition 2.17. $\ker \varphi_P \cong (\ker \varphi)_P$ *for all $P \in X$. Therefore, φ is injective if and only if φ_P is injective for all $P \in X$.*

Proof. ∎

Definition 2.18 (Image Sheaf). Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then, the map $\mathscr{G}' : \mathfrak{Top}(X) \to$ **AbGrp** given by $\mathscr{G}'(U) = \operatorname{im} \varphi_U$ is a presheaf on X. The *image sheaf* of φ is defined to be the sheafification of $\mathscr{G}'$ and is denoted by $\operatorname{im} \varphi$.

Definition 2.19 (Injective/Surjective Morphisms). A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves is said to be *injective* if $\ker \varphi = 0$ and surjective if $\operatorname{im} \varphi = \mathscr{G}$.

Lemma 2.20. *Let $\mathscr{F}, \mathscr{G}$ be presheaves on X and $\varphi : \mathscr{F} \to \mathscr{G}$ a morphism such that φ_U is injective for every open subset U of X. Then, $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is also injective.*

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d] \arrow[dr, dashed] & \mathscr{G} \arrow[d] \\
\mathscr{F}^+ \arrow[r, dotted, "\exists!"'] & \mathscr{G}^+
\end{tikzcd}$$

Proof. It suffices to check this at the level of stalks where it follows from the fact that sheafification preserves stalks. ∎

Remark 2.1.3. *From Lemma 2.20, we know that the map* $\operatorname{im}\varphi \to \mathscr{G}$ *is an injective morphism of sheaves and hence,* $\operatorname{im}\varphi$ *can be identified with a subsheaf of* $\mathscr{G}$*. Henceforth, we shall treat* $\operatorname{im}\varphi$ *as a subsheaf of* $\mathscr{G}$*.*

Definition 2.21 (Exact Sequence). A sequence of morphisms

$$\mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

of sheaves is said to be *exact* at $\mathscr{F}$ if $\operatorname{im}\varphi = \ker\psi$. In general, a sequence of morphisms

$$\cdots \to \mathscr{F}_{i-1} \to \mathscr{F}_i \to \mathscr{F}_{i+1} \to \cdots$$

is said to be *exact* if it is exact at every $\mathscr{F}_i$.

Proposition 2.22. $\operatorname{im}\varphi_P = (\operatorname{im}\varphi)_P$ *for every* $P \in X$*. Therefore,* φ *is surjective if and only if* φ_P *is surjective for all* $P \in X$*.*

Proof. Let $\mathscr{H}$ denote the image presheaf of φ in $\mathscr{G}$. It is known that sheafification preserves stalks and thus, $(\operatorname{im}\varphi)_P = \mathscr{H}_P$. It suffices to show that $\mathscr{H}_P = \operatorname{im}\varphi_P$, which is obvious. Since any germ in $\mathscr{H}_P$ is of the form $[\langle U, t\rangle]$ where t is in the image of φ_U. On the other hand, an element in the image of φ_P is of the form $[\langle U, \varphi_U(s)\rangle]$. This completes the proof. ■

Theorem 2.23. *A sequence of morphisms* $\mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$ *is exact if and only if for every* $P \in X$*, the induced sequence of abelian groups* $\mathscr{F}'_P \xrightarrow{\varphi_P} \mathscr{F}_P \xrightarrow{\psi_P} \mathscr{F}''_P$ *is exact. That is, exactness can be checked at stalks.*

Proof. The forward direction is obvious. Consider the converse. We have $(\psi \circ \varphi)_P = \psi_P \circ \varphi_P = 0$ for every $P \in X$. Consequently, $\operatorname{im}\varphi_P \subseteq \ker\psi_P$ for all $P \in X$ and hence, $\operatorname{im}\varphi \subseteq \ker\psi$. Finally, note that since equality holds at each stalk, we must have $\ker\psi = \operatorname{im}\varphi$ and the sequence is exact. ■

Lemma 2.24. *A morphism* $\varphi : \mathscr{F} \to \mathscr{G}$ *of sheaves is surjective if and only if for all* $U \subseteq X$ *open and* $s \in \mathscr{G}(U)$*, there is a covering* $\{U_i\}$ *of* U*, and there are elements* $t_i \in \mathscr{F}(U_i)$ *such that* $\varphi_{U_i}(t_i) = s|_{U_i}$ *for all* i*.*

Proof. From Theorem 2.23, φ is surjective if and only if φ_P is surjective for all $P \in X$. Suppose φ is surjective. Let $s \in \mathscr{F}(U)$. Then, for every $P \in U$, there is a neighborhood U_P of P contained in U and $t^P \in \mathscr{F}(U_P)$ such that $\varphi_{U_P}(t^P) = s|_{U_P}$ (this follows from surjectivity of the stalk map).

Conversely, the given condition, precisely implies that the map φ_P is surjective at the level of stalks for all $P \in X$. This completes the proof. ■

Example 2.25. The following example shows that a surjective morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves need not give rise to a surjective morphism $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ of abelian groups for all open $U \subseteq X$.

Let $X = \mathbb{C}\backslash\{0\}$, and for an open $U \subseteq X$, let $\mathscr{O}(U)$ denote the additive group of holomorphic functions on X and $\mathscr{O}^*(X)$ the multiplicative group of nowhere vanishing holomorphic functions on X. Consider the map $\exp : \mathscr{O} \to \mathscr{O}^*$ given by $\exp_U(f) = e^{2\pi i f} \in \mathscr{O}^*(U)$.

Note that the map on stalks $\exp_P : \mathscr{O}_P \to \mathscr{O}^*_P$ is surjective since there is always a branch of the logarithm locally. Thus, exp is a surjective morphism of sheaves. But note that setting $U = X$ and taking $\mathbf{id}_X \in \mathscr{O}^*(X)$, there is no function $f \in \mathscr{O}(X)$ that maps to $\mathbf{id}_X$ under $\exp_X$.

Definition 2.26 (Sheaf $\mathscr{H}om$). Let $\mathscr{F}, \mathscr{G}$ be sheaves on X. The map $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf on X known as the *sheaf of local morphisms of $\mathscr{F}$ to $\mathscr{G}$*, or "sheaf Hom" for short and is denoted by $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

Remark 2.1.4. *We haven't explicitly mentioned this but the restriction of a sheaf to an open subset is still a sheaf (the axioms can be verified easily).*

Proposition 2.27. *$\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is indeed a sheaf.*

Proof. Obviously, $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is an abelian group. ■

Complete this

Definition 2.28 (Pushforward). Let $\mathscr{F}$ be a sheaf on X and $f : X \to Y$ be a continuous function. Define the map $f_*\mathscr{F} : \mathfrak{Top}(Y) \to \mathbf{AbGrp}$ by $V \mapsto \mathscr{F}(f^{-1}V)$.

Proposition 2.29. *The pushforward is indeed a sheaf.*

Proof. The pushforward is obviously a presheaf. Let $\{V_i\}$ be an open cover of V and $s \in f_*\mathscr{F}(V)$ such that $s|_{V_i} = 0$ for every i. Note that $s \in \mathscr{F}(f^{-1}V)$ and $s|_{f^{-1}V_i} = 0$, consequently, $s = 0$. Now suppose $s_i \in f_*\mathscr{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Note that $\{f^{-1}V_i\}$ forms an open cover of $f^{-1}V$ and hence, there is $s \in \mathscr{F}(f^{-1}V)$ such that $s|_{f^{-1}V_i} = s_i$. Hence, $s \in f_*\mathscr{F}(V)$ such that $s|_{V_i} = s_i$. This completes the proof. ■

Definition 2.30 (Section Functor). Let $U \subseteq X$ be an open set. Then, the map $\mathscr{F} \mapsto \mathscr{F}(U)$ is functorial and is denoted by $\Gamma(U, -)$. This is the *section functor on U*.

Proposition 2.31. *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be a short exact sequence of sheaves. Then, the induced map*

$$0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$$

is exact. That is, $\Gamma(U, -)$ is a left exact functor.

Proof. The injectivity of $\mathscr{F}'(U) \to \mathscr{F}(U)$ is immediate from the definition. Further, note that $\varphi(\mathscr{F}')$ is already a subsheaf of $\mathscr{F}$ since the map is injective. Therefore, exactness at $\mathscr{F}(U)$ follows. This completes the proof. ■

Definition 2.32 (Flasque Sheaves). A sheaf $\mathscr{F}$ is said to be *flasque* if whenever $V \subseteq U$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

Theorem 2.33. *Let $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ be sheaves on X and $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ an exact sequence of sheaves.*

(a) If $\mathscr{F}'$ is flasque, then for any open $U \subseteq X$, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact.

(b) If $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then so is $\mathscr{F}''$.

(c) If $f : X \to Y$ is a continuous map and $\mathscr{F}$ is flasque, then $f_*\mathscr{F}$ is a flasque sheaf on Y.

Proof. (a) First, note that the section functor $\Gamma(U, -)$ is left exact and hence it suffices to show the surjectivity of the induced map $\mathscr{F}(U) \to \mathscr{F}''(U)$. Let $s \in \mathscr{F}''(U)$. Then, due to Lemma 2.24, there is an open cover $\{U_i\}_{i \in I}$ of U and $t_i \in \mathscr{F}(U_i)$ such that $\psi_{U_i}(t_i) = s|_{U_i}$. Consider the collection of all pairs $\{(J, t_J)\}$ where $J \subseteq I$ and $t_J \in \mathscr{F}\left(\bigcup_{j \in J} U_j\right)$ such that $t_J|_{U_j} = t_j$ for every $j \in J$. This has the structure of a poset with $(J, t_J) \leqq (J', t_{J'})$ if $J \subseteq J'$ and $t_{J'}$ restricted to $\bigcup_{j \in J} U_j$ is equal to t_J.

Using the gluability axiom, it follows that every chain in the aforementioned poset has a maximal element, say (K, t_K). Let $V = \bigcup_{k \in K} U_k$. We contend that $V = U$. Suppose not, then there is a U_i that is not contained in V. Let $W = V \cap U_i$. The images under ψ_W of $t_K|_W$ and $t_i|_W$ are the same. Therefore, $t_K|_W - t_i|_W \in \ker g_W = \mathscr{F}'(W)$. Since $\mathscr{F}'$ is flasque, there is a $z \in \mathscr{F}'(U_i)$ such that $\varphi_{U_i}(z)|_W = t_K|_W - t_i|_W$. Set $y = \varphi_{U_i}(z)$ and $t_i' = t_i + y \in \mathscr{F}(U_i)$, then, $t_i'|_W = t_K|_W$ and hence, there is a corresponding $t^* \in \mathscr{F}(V \cup U_i)$ that restricts to t_K and t_i' on V and U_i respectively. Therefore, $(K \cup \{i\}, t^*) \geq (K, t_K)$ thereby contradicting the maximality of (K, t_K) and hence, $V = U$ and the conclusion follows.

(b) Let $U \subseteq V$ be open subsets of X. Then, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] \arrow[d] & \mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{F}''(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] & \mathscr{F}(U) \arrow[r] & \mathscr{F}''(U) \arrow[r] & 0
\end{tikzcd}$$

where the first two vertical maps are surjective. It follows from the Snake Lemma that so is the third.

(c) Obvious. ■