Functional Analysis

Swayam Chube

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Contents

Chapter 1

Topological Vector Spaces

1.1 Normed Vector Spaces

Definition 1.1 (Vector Space). A vector space V over a field k is an Abelian group $(V, +)$ along with an action of the field k satisfying

(a) $\alpha(u+v) = \alpha u + \alpha v$ for all $\alpha \in k$ and $u, v \in V$

(b) $1v = v$ for all $v \in V$ where 1 is the multiplicative identity in k

(c) $(\alpha\beta)v = \alpha(\beta v)$ for all $\alpha, \beta \in k$ and $v \in V$

Definition 1.2 (Linear Independence). A finite subset S of a k-vector space V is said to be linearly independent if

$$\sum_{s \in S} \alpha_s s = 0 \Longleftrightarrow \alpha_s = 0 \; \forall s \in S$$

An infinite subset T of V is said to be linearly independent if every finite subset is linearly independent.

Definition 1.3 (Norm, Normed Space). For a $\mathbb{K}$-vector space X, a norm is a continuous function $\|\cdot\| : \mathbb{K} \to \mathbb{R}$ satisfying the following

(a) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = \mathbf{0}$.

(b) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{K}$

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

A vector space equipped with a norm is called a *normed space*.

Proposition 1.4. *Let V be a normed $\mathbb{K}$-vector space. Then the function $d : V \times V \to [0, \infty)$ given by $d(x, y) = \|x - y\|$ is a metric.*

Proof. It suffices to verify the triangle inequality,

$$d(x,y) + d(y,z) = \|x-y\| + \|y-z\| \geq \|(x-y)+(y-z)\| = \|x-z\|$$

and the conclusion follows. ∎

It is important to keep in mind that every norm induces a metric but the converse is not true. Take for example the discrete metric on $\mathbb{R}$. Obviously, $\mathbb{R}$ is an $\mathbb{R}$-vector space but is not normed, for

$$\|2-0\| = 1 \neq 2 = 2\|1-0\|$$

Definition 1.5 (Equivalence of Norms). Two norms $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ on a $\mathbb{K}$-vector space V are said to be *equivalent* if there are positive constants C_1 and C_2 such that

$$\|v\|_{(1)} \leq C_1\|v\|_{(2)} \text{ and } \|v\|_{(2)} \leq C_2\|v\|_{(1)}\ \forall v \in V$$

Proposition 1.6. *The equivalence of norms relation is indeed an equivalence relation.*

Proof. Reflexivity follows from taking $C_1 = C_2 = 1$ and symmetry is implicit in the definition. As for transitivity, suppose

$$\|v\|_{(1)} \leq C_1\|v\|_{(2)} \text{ and } \|v\|_{(2)} \leq C_2\|v\|_{(1)}\ \forall v \in V$$
$$\|v\|_{(2)} \leq C_1\|v\|_{(3)} \text{ and } \|v\|_{(3)} \leq D_2\|v\|_{(2)}\ \forall v \in V$$

Then, $\|v\|_{(1)} \leq C_1D_1\|v\|_{(3)}$ and $\|v\|_{(3)} \leq C_2D_2\|v\|_{(1)}$ for all $v \in V$. This completes the proof. ∎

Proposition 1.7. *Equivalent norms induce the same topology.*

Proof. Trivial. ∎

Theorem 1.8. *Let V be a finite dimensional K-vector space. Then all norms on V are equivalent.*

Proof. We shall show all norms are equivalent to the ℓ_1-norm. Let $\|\cdot\|$ be an arbitrary norm and $\{e_1, \ldots, e_n\}$ be a (finite) basis for V. First, we shall show that the norm function $\|\cdot\|$ is continuous under the ℓ_1-norm $\|\cdot\|_1$. Let $\varepsilon > 0$ be given. Let $v, v' \in V$ and have representations $v = \alpha_1e_1 + \cdots + \alpha_ne_n$ and $v' = \alpha_1'e_1 + \cdots + \alpha_n'v_n$. Then, due to the triange inequality, we have

$$|\|v\| - \|v'\|| \leq \|v - v'\| = \left\|\sum_{i=1}^{n}(\alpha_i - \alpha_i')e_i\right\| \leq \sum_{i=1}^{n}|\alpha_i - \alpha_i'|\|e_i\| \leq \|v-v'\|_1 \max_{1\leq i\leq n}\|e_i\|$$

Let $\delta = \varepsilon / \max_{1\leq i\leq n}\|e_i\|$. As a result, whenever $\|v - v'\|_1 < \delta$, we have $|\|v\| - \|v'\|| < \varepsilon$, implying continuity.

Since the unit sphere under the ℓ_1-norm is compact and $\|\cdot\|$ is continuous, due to the extreme value theorem, there are positive reals C_1 and C_2 such that

$$C_1 = \min_{\|v\|_1=1}\|v\| \qquad C_2 = \max_{\|v\|_1=1}\|v\|$$

Finally, for any $v \in V\backslash\{0\}$, let $u = v/\|v\|_1$. Then $\|v\| = \|v\|_1\|u\|$ and thus,

$$C_1\|v\|_1 \leq \|v\| \leq C_2\|v\|_1$$

This completes the proof. ∎

1.2 Banach Spaces

Definition 1.9 (Banach Space). A *Banach Space* is a normed space which is complete with respect to the induced metric.

For a metric space X, we denote $\mathcal{C}_\infty(X)$ by the set of all bounded functions $f : X \to \mathbb{C}$. That this is a $\mathbb{K}$-vector space is trivial. We define the norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

This norm is well defined since we are considering the set of all bounded functions. That this is a norm is now trivial to check.

Theorem 1.10. *Let X be a metric space. Then $\mathcal{C}_\infty(X)$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}_\infty(X)$ under the sup-norm. Then, it follows that for any $x \in X$, the sequence $\{f_n(x)\}$ is Cauchy and hence has a limit, say $f(x)$. This defines a function $f : X \to \mathbb{C}$. We shall show that $f \in \mathcal{C}_\infty(X)$.

First, to see the boundedness of f, note that there is $N \in \mathbb{N}$ such that for all $m \geq N$, $\|f_m - f_N\| < 1$. As a result, $\|f_m\| < \|f_N\| + 1$ for all $m \geq N$. As a result, $\|f\| \leq \|f_N\| + 1$ and f is bounded.

To see that f is continuous, we shall show that $f_n \to f$ uniformly, and we would be done due to the uniform limit theorem. Let $\varepsilon > 0$ be given. Then, there is some $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|f_m - f_n\| < \varepsilon/2$. Taking the limit $m \to \infty$, we have that for all $n \geq N$, $\|f - f_n\| \leq \varepsilon/2 < \varepsilon$ and thus the convergence is uniform and f is continuous. This completes the proof. ■

We now show an example of a normed space which is not Banach. Take for example the $\mathbb{R}$-vector space $\mathbb{R}^\infty$, the set of all sequences which are eventually 0. Consider the sequence $\{\mathbf{x}_n\}$ given by

$$\mathbf{x}_n(m) = \begin{cases} \frac{1}{m} & m \leq n \\ 0 & \text{otherwise} \end{cases}$$

That this is a Cauchy sequence under the sup-norm is trivial. But the limit of such a sequence is not in $\mathbb{R}^\infty$ which is not hard to argue.

Definition 1.11. For a normed $\mathbb{K}$-vector space V and a sequence $\{v_n\}$ in V, the series $\sum_{n=1}^\infty v_n$ is said to be summable if the partial sums converge. Similarly, it is said to be absolutely summable if the sequence of partial sums $\{\sum_{k=1}^n \|v_k\|\}$ converges.

Theorem 1.12. *A normed $\mathbb{K}$-vector space V is Banach if and only if every absolutely summable series is summable.*

Proof. The forward direction of the assertion is trivial, we shall show the converse. Let V be such that every absolutely summable series is summable and let $\{v_n\}$ be a Cauchy sequence in V. Then, by definition, for all $k \in \mathbb{N}$, there is $N_k \in \mathbb{N}$ such that whenever $m, n \geq N_k$, $\|v_m - v_n\| < 2^{-k}$. Further, one may choose the N_k's in strictly increasing order. Define the sequence $\{u_n\}$ by $u_n = v_{N_{n+1}} - v_{N_{n-1}}$ for all $n \in \mathbb{N}$. Then, the series $\{u_n\}$ is absolutely summable and therefore summable. Let $w = \sum_{n=1}^\infty u_n$ and define $v = w + v_{N_1}$. We shall show that $v_n \to v$. Indeed, for any $n, k \in \mathbb{N}$, we have

$$\|v - v_n\| = \|w - (v_{N_k} - v_{N_1}) + (v_{N_k} - v_n)\| \leq \|w - (v_{N_k} - v_{N_1})\| + \|v_{N_k} - v_n\|$$

First, note that $v_{N_k} - v_{N_1}$ is the $(k-1)$-th partial sum. Let $\varepsilon > 0$ be given. Then, there is large enough k such that $\|w - (v_{N_k} - v_{N_1})\| < \varepsilon/2$ and $\|v_{N_k} - v_n\| < \varepsilon/2$ for all $n \geq N_k$. The conclusion follows. ■

1.3 Operators and Functionals

Definition 1.13 (Linear Operator). Let V, W be $\mathbb{K}$-vector spaces. A linear operator is a map $T : V \to W$ such that

$$T(a_1v_1 + a_2v_2) = a_1T(v_1) + a_2T(v_2) \quad \forall v_1, v_2 \in V,\ a_1, a_2 \in \mathbb{K}$$

We shall mainly concern ourselves with continuous linear operators, that is, linear operators such that $T^{-1}(U)$ is open in V for all open sets U in W[1].

Proposition 1.14. *Let V and W be normed $\mathbb{K}$-vector spaces. The following are equivalent to the continuity of a linear operator $T : V \to W$.*

(a) For every convergent sequence $v_n \to v$ in V, the sequence $T(v_n)$ converges to $T(v)$ in W

(b) For each open set U in W, $T^{-1}(U)$ is open in V

(c) For each closed set A in W, $T^{-1}(A)$ is closed in V

Proof. Straightforward definition pushing. ∎

Definition 1.15 (Bounded Linear Operator). A linear operator $T : V \to W$ between two normed $\mathbb{K}$-vector spaces is said to be *bounded* if there is a constant $C \geq 0$ such that

$$\|T(v)\|_W \leq C\|v\|_V \quad \forall v \in V$$

Proposition 1.16. *A linear map $T : V \to W$ between two normed $\mathbb{K}$-vector spaces is continuous if and only if it is bounded in the sense that there exists a constant $C \geq 0$ such that*

Proof. ∎

As a result, we see that the set of all continuous operators is the same as the set of all bounded operators. We denote this set by $\mathcal{B}(V, W)$.

Proposition 1.17. *Let V and W be normed $\mathbb{K}$-vector spaces. Then, the function $\|\cdot\| : \mathcal{B}(V, W) \to [0, \infty)$ given by*

$$\|T\| = \sup_{\|v\|=1} \|T(v)\|$$

is a norm. As a result, $\mathcal{B}(V, W)$ is a normed vector space.

Proof. Trivial. ∎

Theorem 1.18. *Let W be a $\mathbb{K}$-Banach space and V a normed $\mathbb{K}$-vector space. Then, $\mathcal{B}(V, W)$ with the aforementioned norm is a Banach space.*

[1] This is just the topological definition of a continuous function

Proof. Let $\{T_n\}$ be a Cauchy sequence of linear operators in $\mathcal{B}(V,W)$. Then, for each $v \in V$, $\{T_n(v)\}$ is a Cauchy sequence and therefore converges in W (since it is Banach). Define the map $T : V \to W$ by $T(v) = \lim_{n\to\infty} T_n(v)$. We shall first show that T is a linear operator and then show that it is bounded, which would imply the completeness of $\mathcal{B}(V,W)$.

Let $a_1, a_2 \in \mathbb{K}$ and $v_1, v_2 \in V$. Now, since each sequence $\{T_n(v_1)\}$ and $\{T_n(v_2)\}$ is Cauchy, so is $\{T_n(a_1v_1 + a_2v_2) = a_1T(v_1) + a_2T(v_2)\}$, and converges to $a_1T(v_1) + a_2T(v_2)$, which shows that T is a linear operator.

To see boundedness, note that every Cauchy sequence is bounded, therefore, there is some $C > 0$ such that $\|T_n\| < C$ for all $n \in \mathbb{N}$. As a result, for all $v \in V$,

$$\|T_n(v)\| \leq \|T_n\|\|v\| \leq C\|v\|$$

In the limit $n \to \infty$, we note that $\|T(v)\| \leq C\|v\|$ and the conclusion follows. ∎

Corollary 1.19. The dual space of a normed space is a Banach space.

1.4 Subspaces and Quotients

Definition 1.20 (Subspaces). A subspace W of a $\mathbb{K}$-vector space V is a $\mathbb{K}$-vector space such that $W \subseteq V$. If V is a normed space, then the restriction of the same norm to W is a norm on W and therefore W obtains a natural structure of a normed vector space.

The quotient follows naturally if one is familiar with modules. Since both V and W are $\mathbb{K}$-modules, so is the quotient module V/W and is consequently a vector space.

Definition 1.21 (Seminorm).

1.5 Uniform Boundedness and Open Mapping Theorems

Theorem 1.22 (Uniform Boundedness Principle/Banach-Steinhaus Theorem). *Let B be a Banach space and suppose $\{T_n\}$ is a sequence of bounded linear operators $T_n : B \to V$ where V is a normed space. Suppose that for each $b \in B$, the set $\{T_n(b) \mid n \in \mathbb{N}\}$ is bounded, then $\sup\limits_{n\in\mathbb{N}} \|T_n\|$ is finite.*

Proof. Define for every $N \in \mathbb{N}$,

$$S_N := \left\{b \in B \colon \|b\| \leq 1 \text{ and } \|T_n(b)\|_V \leq N,\ \forall n \in \mathbb{N}\right\}.$$

Then,

$$S_N = \bigcap_{n=1}^{\infty} T_n^{-1}(\overline{B_V(0,N)}) \cap \overline{B_B(0,1)}$$

and is closed. Since $\{T_n(b)\}$ is bounded for every $b \in B$, we have

$$\overline{B_B(0,1)} = \bigcup_{n=1}^{\infty} S_n.$$

Due to the Baire Category Theorem, there is some $N \in \mathbb{N}$ such that S_N has a nonempty interior whence contains a closed ball of the form $\overline{B(v,\delta)}$ for some $\delta > 0$.

Let $w \in B$ with $\|w\| = \delta$. Then, $v + w \in \overline{B(v,\delta)}$ and thus $\|T_n(v+w)\| \leq N$ for every positive integer n. Consequently, for each $n \in \mathbb{N}$,

$$\|T_n(w)\| \leq \|T(v+w)\| + \|T(v)\| \leq 2N.$$

Therefore, $\|T_n\| \leq 2N/\|w\| = 2N/\delta$ for every positive integer n, thereby completing the proof. ∎

Theorem 1.23 (Open Mapping Theorem/Banach-Schauder Theorem). *Let B_1, B_2 be Banach spaces. If $T : B_1 \to B_2$ is a continuous linear operator, then T is an open map.*

Proof. We shall denote open balls in B_i by $B_i(v,r)$ for $i \in \{1,2\}$. The main idea of the proof is to show that 0 lies in the interior of $T(B_1(0,1))$. It is not hard to argue that this would finish the proof. We shall proceed in two steps.

Step I: 0 lies in the interior of $\overline{T(B_1(0,1))}$.

Step II: 0 lies in the interior of $T(B_1(0,1))$.

From the conclusion of **Step I**, we see that for every $v \in B_2$ with $\|v\| = \delta$, there is a sequence $\{u_n\}$ in $B_1(0,1)$ such that $\{T(u_n)\}$ converges to v. In particular, for every $v \in B_2$ with $\|v\| = \delta$, there is $u \in B_1$ with $\|v - T(u)\| < \frac{1}{2}\|v\|$.

Now, pick any $v \in B_2$. Then, $v' = \delta v/\|v\| \in B_2$ with $\|v'\| = \delta$ whence there is $u' \in B_1$ with $\|u'\| < 1$ such that $\|v' - T(u')\| < \|v'\|/2$. Multiplying with $\|v\|/\delta$ we see that there is $u \in B_1$ with $\|u\| < C\|v\|$ such that $\|v - T(u)\| < \|v\|/2$ where $C = 1/\delta$.

Let $w_1 := w \in B_2(0,1)$. Pick some $u_1 := u \in B_1$ such that $\|w_1 - T(u_1)\| < \|w\|/2$. Define $w_2 := w_1 - T(u_1)$ and proceed similarly to obtain a sequence $\{u_n\}_{n=1}^{\infty}$. Note that

$$\|u_j\| \leq C\|w_{j-1}\| = C2^{-(j-1)}\|w\|.$$

Thus, the sequence $\{u_n\}$ is absolutely summable. Furthermore,

$$w - T\left(\sum_{j=1}^{n} u_j\right) = w_1 - \sum_{j=1}^{n}(w_j - w_{j+1}) = w_{n+1}.$$

Define $u := \sum_{j=1}^{\infty} u_n$. Then $T(u) = w$ and

$$\|u\| \leq \sum_{n=1}^{\infty} C2^{-(n-1)}\|w\| = 2C\|w\|.$$

In conclusion, every $w \in B_2(0,1)$ is the image of some $u \in \overline{B_1(0,2C)} \subseteq B_1(0,3C)$. Upon scaling, every $w \in B_2(0,1/3C)$ is the image of some $u \in B_1(0,1)$ and thus 0 lies in the interior of $T(B_1(0,1))$ whereby completing the proof. ∎

Example 1.24. Let B be a $\mathbb{K}$-vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on B that give it the structure of a Banach space. Suppose there is $K > 0$ such that $\|\cdot\|_1 \leq K\|\cdot\|_2$. Then, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

Proof. Consider $\mathbf{id} : (B, \|\cdot\|_2) \to (B, \|\cdot\|_1)$. This is a bounded linear operator and thus continuous. Due to Theorem 1.23, $\mathbf{id}$ is a homeomorphism and the conclusion follows. ∎

Theorem 1.25 (Closed Graph Theorem). *Let B_1 and B_2 be Banach spaces and $T : B_1 \to B_2$ a linear operator. Then T is continuous if and only if*

$$\mathrm{Gr}(T) = \{x \times T(x) \mid x \in B_1\} \subseteq B_1 \times B_2$$

is closed.

Proof. The forward direction is trivial, since B_2 is Hausdorff. Conversely, suppose $\mathrm{Gr}(T) \subseteq B_1 \times B_2$ is closed, then it is a Banach space. Consider the following commutative diagram:

$$\begin{tikzcd} B_1 \times B_2 \arrow[r, two heads, "\pi_2"] & B_2 \\ B_1 \arrow[u, "S"] \arrow[ur, "T"'] & \end{tikzcd}$$

where $S : B_1 \to B_2 \times B_2$ is given by $x \mapsto x \times T(x)$. Let $\pi_1 : \mathrm{Gr}(T) \twoheadrightarrow B_1$ denote the natural projection, which is continuous and $\pi_1 \circ S = \mathbf{id}_{B_1}$. Further, since π_1 is a bijection, due to Theorem 1.23, both π_1 and S are homeomorphisms. In particular, S is continuous. If $\iota : \mathrm{Gr}(T) \hookrightarrow B_1 \times B_2$ is the inclusion map, then $T = \pi_2 \circ \iota \circ S$ is continuous, being the composition of continuous functions. This completes the proof. ∎

1.6 Hahn-Banach Theorem

Lemma 1.26. *Let V be a normed vector space, $M \subseteq V$ a vector subspace, $u : M \to \mathbb{C}$ be a linear map such that $|u(t)| \leq C\|t\|_V$ for all $t \in M$ and finally, let $x \notin M$. Denote by M' the vector subspace $M + x\mathbb{C}$. Then there exists $u' : M' \to \mathbb{C}$ such that $u'|_M = u$ and $|u'(t + ax)| \leq C\|t + ax\|_V$ for all $t \in M$ and $a \in \mathbb{C}$.*

Proof. ∎

Long ass proof

From the above lemma, we have the "Hahn-Banach Theorem".

Theorem 1.27 (Hahn-Banach). *Let V be a normed vector space, $M \subseteq V$ a vector subspace and $u : M \to \mathbb{C}$ be a linear map such that $|u(t)| \leq C\|t\|_V$ for all $t \in M$. Then there is a continuous linear functional $U : V \to \mathbb{C}$ such that $U|_M = u$ and $\|U\| \leq C$.*

Proof. Standard application of Zorn's Lemma. ∎

Chapter 2

Hilbert Spaces

2.1 Inner Product or pre-Hilbert Spaces

Definition 2.1 (Inner Product Space). An *inner product space* H is a $\mathbb{C}$-vector space along with a *Hermitian inner product* $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that

(a) $\lambda_1 v_1 + \lambda_2 v_2, w\rangle = \lambda_1 \langle v_1, w\rangle + \lambda_2 \langle v_2, w\rangle$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v_1, v_2, w \in H$,

(b) $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in H$,

(c) for each $v \in H$, $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

An inner product space is also called a *pre-Hilbert space*.

Theorem 2.2 (Cauchy-Schwarz Inequality). *Let H be an inner product space. Then, for all $u, v \in H$,*

$$|\langle u, v\rangle| \leq \|u\| \|v\|$$

where $\|u\| = \sqrt{\langle u, u\rangle}$.

Proof. Using positive definiteness, for all $t \in \mathbb{R}$, we have

$$0 \leq \langle u + tv, u + tv\rangle = \|u\|^2 + 2\Re\langle u, v\rangle t + t^2 \|v\|^2.$$

This is a quadratic polynomial in t whence its determinant is non-positive. Thus,

$$|\Re\langle u, v\rangle| \leq \|u\| \|v\|$$

for all $u, v \in H$. Let $z = \langle u, v\rangle / |\langle u, v\rangle|$. Then,

$$\|u\| \|v\| = \|u\| \|zv\| \geq \Re\langle u, zv\rangle = \Re\left(\bar{z}\langle u, v\rangle\right) = |\langle u, v\rangle|. \qquad \blacksquare$$

Proposition 2.3. *The function $\|\cdot\| : H \to \mathbb{R}$ given by $\|u\| = \sqrt{\langle u, u\rangle}$ is a norm.*

Proof. It suffices to verify the triangle inequality. Indeed, for $u, v \in H$, due to the Cauchy-Schwarz inequality, we have

$$(\|u\| + \|v\|)^2 = \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \geq \|u\|^2 + 2\Re\langle u, v\rangle + \|v\|^2 = \langle u + v, u + v\rangle = \|u + v\|. \qquad \blacksquare$$

Definition 2.4. Elements $u, v \in H$ are said to be *orthogonal* if $\langle u, v \rangle = 0$. This is denoted by $u \perp v$, which is obviously a reflexive relation. A sequence $\{e_n\}_{n=1}^{\infty}$ is said to be *orthonormal* if

$$\langle e_m, e_n \rangle = \delta_{mn}.$$

Theorem 2.5 (Bessel's Inequality). *If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in an inner product space H, then for any $u \in H$,*

$$\sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2 \leq \|u\|^2.$$

Proof. Define $u_n := \sum_{i=1}^{n} \langle u, e_i \rangle e_i$. Then,

$$\langle u, u_n \rangle = \sum_{i=1}^{n} \langle u, e_i \rangle^2 = \langle u_n, u_n \rangle \implies \langle u - u_n, u_n \rangle = 0.$$

We have

$$\begin{aligned}
\|u\|^2 &= \langle (u - u_n) + u_n, (u - u_n) + u_n \rangle \\
&= \|u - u_n\|^2 + \langle u - u_n, u_n \rangle + \langle u_n, u - u_n \rangle + \|u_n\|^2 \\
&= \|u - u_n\|^2 + \|u_n\|^2 \\
&\geq \|u_n\|^2 = \sum_{i=1}^{n} |\langle u, e_i \rangle|^2.
\end{aligned}$$

Since the last inequality holds for all $n \in \mathbb{N}$, it holds in the limit $n \to \infty$ and the conclusion follows. ∎

2.2 Hilbert Spaces

Definition 2.6 (Hilbert Space). A *Hilbert space* is an inner product space that is complete with respect to the norm induced by the inner product.

Definition 2.7 (Maximal Orthonormal Sequence). An orthonormal sequence $\{e_i\}$ (finite or infinite) in an inner product space is said to be *maximal* if it is maximal with respect to subsequence inclusion.

It is not hard to see that an orthonormal sequence $\{e_i\}$ is maximal if and only if

$$\langle u, e_i \rangle = 0, \ \forall i \implies u = 0.$$

for if not, then the sequence $(u/\|u\|, e_1, e_2, \ldots)$ is an orthonormal sequence containing $\{e_i\}_{i=1}^{\infty}$ as a subsequence.

Lemma 2.8. *If a Hilbert space H is separable, then it contains a maximal orthonormal subset.*

Proof. Let $\{v_i\}_{i=1}^{\infty}$ be a countable dense subset of H. We shall use the Gram-Schmidt Orthonormalization process to construct a maximal orthonormal sequence. Let $e_1 = v_1/\|v_1\|$ and

$$e_{n+1} = \frac{v_{n+1} - \sum_{j=1}^{n} \langle v_{n+1}, e_j \rangle e_j}{\left\| v_{n+1} - \sum_{j=1}^{n} \langle v_{n+1}, e_j \rangle e_j \right\|}.$$

It is not hard to argue that for all positive integers n,

$$\operatorname{Span}(e_1, \ldots, e_n) = \operatorname{Span}(v_1, \ldots, v_n)$$

and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence by construction. We contend that this is a maximal orthonormal sequence. Indeed, suppose $u \in H$ with $u \perp e_i$ for each $i \in \mathbb{N}$, we shall show that $u = 0$.

Since $\{v_i\}_{i=1}^{\infty}$ is dense in H, there is a sequence $\{w_i\}_{j=1}^{\infty}$ converging to u such that each w_k is some v_j and thus a finite linear combination of the e_i's. Due to Theorem 2.5,

$$\|w_k\|^2 = \sum_{i=1}^{\infty} |\langle w_k, e_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle u - w_k, e_i\rangle|^2 \leq \|u - w_k\|^2.$$

Consequently, $\|w_k\|^2 \to 0$ as $k \to \infty$ whence $u = 0$ and thus the sequence $\{e_i\}$ is a maximal orthonormal sequence in H. ■

Definition 2.9 (Orthonormal Basis). A maximal orthonormal sequence in a separable Hilbert space is also called an *orthonormal basis*.

Theorem 2.10. *If $\{e_i\}$ (finite or infinite) is an orthonormal basis in a Hilbert space, then for any $u \in H$, the 'Fourier-Bessel series'*

$$\sum_{i=1}^{\infty} \langle u, e_i\rangle e_i$$

converges to u.

Proof. If $\{e_i\}$ is a finite sequence then the conclusion is obvious. Suppose now that the sequence is infinite. Let

$$u_n = \sum_{i=1}^{n} \langle u, e_i\rangle e_i.$$

We contend that $\{u_n\}$ forms a Cauchy sequence. Let $\varepsilon > 0$ be given. Due to Theorem 2.5, there is a positive integer N such that

$$\sum_{k=N+1}^{\infty} |\langle u, e_k\rangle|^2 < \varepsilon^2.$$

If $m, n \geq N$, with $m < n$, then

$$\|u_n - u_m\|^2 = \sum_{i=m+1}^{n} |\langle u, e_i\rangle|^2 \leq \sum_{i=m+1}^{\infty} |\langle u, e_i\rangle|^2 \leq \sum_{i=N+1}^{\infty} |\langle u, e_i\rangle|^2 < \varepsilon^2$$

whence the conclusion follows. Since H is complte, there is some $w \in H$ to which the partial sums $\{u_n\}$ converge. We shall show that $w - u$ is orthogonal to each e_i. Indeed, for $n \geq i$, due to Theorem 2.5,

$$|\langle w - u_n, e_i\rangle| \leq \|w - u_n\|$$

whence $\lim_{n\to\infty} \langle w - u_n, e_i\rangle = 0$. Consequently,

$$\langle w, e_i\rangle = \lim_{n\to\infty} \langle u_n, e_i\rangle = \langle u, e_i\rangle \implies \langle w - u, e_i\rangle = 0.$$

This completes the proof. ■

Corollary 2.11. If a Hilbert space H contains an orthonormal basis $\{e_i\}$, then H is separable.

Definition 2.12 (Convex). A subset $C \subseteq V$ of a normed vector space is said to be *convex* if whenever $v_1, v_2 \in C$, then $\frac{1}{2}(v_1 + v_2) \in C$.

Proposition 2.13. *Let $C \subseteq H$ be a subset of a Hilbert space which is nonempty, closed and convex. Then there is a unique $v \in C$ such that $\|v\| = \inf_{u \in C} \|u\|$.*

Proof. Let $d = \inf_{u \in C} \|u\|$. Then, there is a sequence $\{u_n\}_{n=1}^{\infty}$ in C such that $\|u_n\| \to d$ as $n \to \infty$. We contend that the sequence $\{u_n\}$ is Cauchy. Indeed, for $m, n \in \mathbb{N}$, we have, due to the Paralellogram Law:

$$\|v_m - v_n\|^2 = 2\|v_m\|^2 + 2\|v_n\|^2 - \|v_m + v_n\|^2 = 2\left(\|v_m\|^2 + \|v_n\|^2 - 2\left\|\frac{v_m + v_n}{2}\right\|^2\right).$$

We may now pick $N \in \mathbb{N}$ such that for all $m, n \geq N$, such that for all $n \geq N$, $\|v_n\|^2 < d^2 + \varepsilon^2/4$. Further, note that $(v_m + v_n)/2 \in C$ due to convexity. Then for $m, n \geq N$,

$$\|v_m - v_n\|^2 < \left(d^2 + \frac{\varepsilon^2}{4} + d^2 + \frac{\varepsilon^2}{4} - 2d^2\right) = \varepsilon^2$$

which implies the desired conclusion. Since H is complete, this sequence converges to some $v \in H$ and since $\|\cdot\| : V \to \mathbb{R}$ is a continuous function, we have

$$\|v\| = \lim_{n \to \infty} \|v_n\| = d.$$

We shall now show uniqueness. Suppose $v, v' \in C$ with $d = \|v\| = \|v'\|$. Then,

$$\|v - v'\|^2 = 2\|v\|^2 + 2\|v'\|^2 - 4\left\|\frac{v + v'}{2}\right\|^2 \leq 0$$

whence $v = v'$. This completes the proof. ■

2.2.1 Orthocomplements and Projections

Proposition 2.14. *If $W \subseteq H$ is a vector subspace of a Hilbert space, then*

$$W^{\perp} = \{u \in H \mid u \perp w, \, \forall w \in W\}$$

is a closed vector subspace of H with $W \cap W^{\perp} = \{0\}$. Moreover, if W is also a closed subspace, then $H = W \oplus W^{\perp}$.

Proof. We have

$$W^{\perp} = \bigcap_{w \in W} \{v \in H \mid \langle v, w \rangle = 0\} = \bigcap_{w \in W} T_w^{-1}(\{0\})$$

which is obviously closed. That it is a subspace is trivial to check. If $u \in W \cap W^{\perp}$, then $\langle u, u \rangle = 0$ whence $u = 0$.

Finally, suppose W is a closed subspace of H. If $W = H$, then $W^\perp = 0$ and $H = W \oplus W^\perp$ and there is nothing more to prove. Let now $u \in H \backslash W$. Consider the closed convex subset $u + W$ of H. There is a unique $v \in C$ such that $\|v\| = \inf_{u' \in C} \|u'\|$. We contend that $v \in W^\perp$.

Indeed, let $\lambda \in \mathbb{C}$ and $w \in W \backslash \{0\}$. Then,

$$\|v\|^2 \le \|v + \lambda w\|^2 = \|v\|^2 + 2\Re(\lambda \langle v, w\rangle) + |\lambda|^2 \|w\|^2 \implies 2\Re(\lambda\langle v, w\rangle) + |\lambda|^2\|w\|^2 \ge 0.$$

Suppose $\langle v, w\rangle \ne 0$. Then, choose $\lambda = t\overline{\langle v, w\rangle}/|\langle v, w\rangle|$. Then, we have, for all $t \in \mathbb{R}_{\ge 0}$,

$$2t|\langle v, w\rangle| + t\|w\|^2 \ge 0.$$

This is possible if and only if $\langle v, w\rangle = 0$. Therefore, for any $u \in H\backslash W$, there is $v \in W^\perp$ such that $u + w = v$ for some $w \in W$ whence $u = v + (-w)$, consequently, $H = W \oplus W^\perp$. ■

2.2.2 Riesz Representation Theorem

Theorem 2.15 (Riesz). *Let H be a Hilbert space and $T : H \to \mathbb{C}$ a continuous functional. Then there is a unique $\phi \in H$ such that for each $v \in H$,*

$$T(v) = \langle v, \phi\rangle.$$

Further, $\|T\| = \|\phi\|$.

Proof. If $T = 0$, then $\phi = 0$ works and is the only choice since $0 = T(\phi) = \langle \phi, \phi\rangle$. Now, suppose $T \ne 0$. Then, $\ker T$ is a closed subspace of H and thus has an orthogonal complement. Choose some $v \in (\ker T)^\perp$ with $\|v\| = 1$ and let $\phi = \overline{T(v)}v$. We shall show that this choice of ϕ works.

Let $u \in H$. Then, $T\left(u - \frac{T(u)}{T(v)}v\right) = 0$ whence $u - \frac{T(u)}{T(v)}v \in \ker T$ and thus

$$\begin{aligned}\langle u, \phi\rangle &= \left\langle u - \frac{T(u)}{T(v)}v, \phi\right\rangle + \left\langle \frac{T(u)}{T(v)}v, \phi\right\rangle \\ &= \frac{T(u)}{T(v)}\langle v, \phi\rangle = T(u).\end{aligned}$$

This proves the existence part. Now, suppose ϕ, ϕ' represent T. Then,

$$\begin{aligned}\langle \phi - \phi', \phi - \phi'\rangle &= \langle \phi, \phi\rangle - \langle \phi', \phi\rangle - \langle \phi, \phi'\rangle + \langle \phi', \phi'\rangle \\ &= T(\phi) - T(\phi') - T(\phi) + T(\phi') = 0\end{aligned}$$

whence uniqueness follows.

Finally, let $u \in H$ with $\|u\| = 1$. Then,

$$\|T(u)\| = |\langle u, \phi\rangle| \le \|u\|\|\phi\| = \|\phi\|$$

and since $T(\phi/\|\phi\|) = \|\phi\|$, we have the desired conclusion. ■