1. The prime subfield of F is isomorphic to $\mathbb{F}_p$. Suppose the dimension of F over $\mathbb{F}_p$ is $n > 0$. Then, there is a basis $\{\alpha_1, \ldots, \alpha_n\}$ where each $\alpha_i \in F$. Any element in F may be written as $\sum_{i=1}^{n} c_i\alpha_i$ with $c_i \in \mathbb{F}_p$. This immediately implies the conclusion.

2. The main idea is as follows. Let $f(x) \in \mathbb{F}_p[x]$ be irreducible and α be a root of $f(x)$ in some extension E of $\mathbb{F}_p$. Consider the field $\mathbb{F}_p(\alpha)$, which has degree $\deg f$ over $\mathbb{F}_p$ and therefore is a field of size $p^{\deg f}$.

3. (a) $\underline{1+i}$: $f(x) = x^2 + 2x + 2$

   (b) $\underline{2+\sqrt{3}}$: $g(x) = x^2 - 4x + 1$

   (c) $\underline{1+\sqrt[3]{2}+\sqrt[3]{4}}$: $h(x) = x^3 - 3x^2 - 3x - 1$

5. Trivial

6. 2 since $\sqrt{3+2\sqrt{2}} = 1 + \sqrt{2}$.

7. Obviously, $F(\alpha^2) \subseteq F(\alpha)$. It suffices to show that $[F(\alpha) : F(\alpha^2)] = 1$. Indeed, we have the equality $n = [F(\alpha) : F] = [F(\alpha) : F(\alpha^2)][F(\alpha^2) : F]$. But since α is a root of $m_\alpha(x) = x^2 - \alpha^2 \in F(\alpha)[x]$, we must have $[F(\alpha) : F(\alpha^2)] \leq 2$. It cannot be equal to 2 since n is odd and therefore must be equal to 1, giving us the desired conclusion.

8. Obviously R is an integral domain. It suffices to show that every non-zero element in R has an inverse. Indeed, let $a \in R\backslash\{0\}$. Since K/F is algebraic, a must be algebraic over F. Then, there is a monic polynomial $m_a(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_0 \neq 0$ such that $a_0 \neq 0$ and $m_a(a) = 0$. It is now obvious that the inverse of a must lie in R.

11. Note that since both β and $\sqrt[3]{2}$ have the same minimal polynomial $m(x) = x^3 - 2$ and therefore $\mathbb{Q}(\beta) \cong \mathbb{Q}(\sqrt[3]{2})$. The isomorphism maps -1 to -1. And thus if -1 is the sum of squares in $\mathbb{Q}(\beta)$ then it is also a sum of squares in $\mathbb{Q}(\sqrt[3]{2})$ a contradiction. This finishes the proof.

12. Trivial. Just note the parities of $f(2k)$ and $f(2k+1)$ for $k \in \mathbb{Z}$.

13. Suppose $\mathbb{C} \subsetneq R$. Let $a \in R\backslash\mathbb{C}$ and let R have dimension n. Then, the elements $1, a, \ldots, a^n$ are not linearly independent and therefore there is a polynomial $f \in \mathbb{C}[x]$ such that $f(a) = 0$ but this is contradictory to the fact that $\mathbb{C}$ is algebraically closed.