Algebraic Topology

Swayam Chube

June 4, 2023

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Chapter 1

The Fundamental Group

1.1 Fundamental Groupoid and Group

Definition 1.1 (Homotopy). Let X and Y be topological spaces. A homotopy is a continuous function $H : X \times I \to Y$. A *homotopy* between two functions $f, g : X \to Y$ is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Definition 1.2 (Homotopy of Paths). Let X be a topological space and $f, g : I \to X$ be paths. Then, f and g are said to be *path homotopic* if there is a continuous function $H : I \times I \to X$ such that $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ for all $s \in I$. We denote this by $f \simeq_p g$.

Proposition 1.3. *The relation $\simeq$ on the set of all paths in X is an equivalence relation.*

Proposition 1.4. *Let $f : I \to X$ be a path and $\varphi : I \to I$ be a continuous function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, $f \simeq_p f \circ \varphi$.*

Proof. Define the function $\Phi : I \times I \to X$ by

$$\Phi(s, t) = f(t\varphi(s) + (1 - t)s)$$

It is not hard to see that Φ is a path homotopy between f and $f \circ \varphi$. ∎

Consider the set of all equivalence classes of paths in X under the equivalence relation $\simeq_p$. Define the operation $*$ on pairs of equivalence classes $[f]$ and $[g]$ where $f(1) = g(0)$ by

$$[f] * [g] = [f * g]$$

where

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 < t \leq 1 \end{cases}$$

Proposition 1.5. *The operation $*$ is associative. That is,*

$$[f] * ([g] * [h]) = ([f] * [g]) * h$$

Proof. Note that $[f] * ([g] * [h])$ is the equivalence class containing the path:

$$\alpha(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(4t-2) & 1/2 < t \leq 3/4 \\ h(4t-3) & 3/4 < t \leq 1 \end{cases}$$

Consider the piecewise linear function $\varphi : [0,1] \to [0,1]$ that maps $[0,1/2]$ to $[0,1/4]$, $[1/2,3/4]$ to $[1/4,1/2]$ and $[1/2,1]$ to $[3/4,1]$, then through $\alpha \circ \varphi$, the conclusion follows. ∎

Definition 1.6 (Fundamental Group). Let $\pi_1(X, x_0)$ be the set of equivalence classes of paths $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$. It is not hard to see from the discussion above that $\pi_1(X, x_0)$ has a group structure. This is known as the *fundamental group*.

Let $\mathbf{Top}_*$ denote the category of pointed topological spaces, that is, the category wherein objects are pairs (X, x_0) where $x_0 \in X$ and a morphism $f : (X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ with $f(x_0) = y_0$.

Proposition 1.7. *Let $f : (X, x_0) \to (Y, y_0)$ be a morphism in $\mathbf{Top}_*$. Then, the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\alpha] \mapsto [f \circ \alpha]$ is a homomorphism of groups. Further, if*

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

then $(g \circ f)_* = g_* \circ f_*$.

Proof. If H is a path homotopy between α_1 and α_2 in X, then $f \circ H$ is a homotopy between $f \circ \alpha_1$ and $f \circ \alpha_2$ in Y. Thus, the map f_* is well defined. Next, suppose $[\alpha], [\beta] \in \pi_1(X, x_0)$, then, it is not hard to see that $(f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta)$, consequently, f_* is a homomorphism of groups. The final assertion is obvious from the definition. ∎

As a result, we see that π_1 is a (covariant) functor from $\mathbf{Top}_*$ to $\mathbf{Grp}$.

Theorem 1.8. *Let X be path connected and $x_0, x_1 \in X$. Let $\alpha : I \to X$ be a path from x_0 to x_1. Then, the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $[f] \mapsto [\overline{\alpha} * f * \alpha]$ is a group isomorphism.*

Proof. It is easy to see that $\hat{\alpha}$ is a homomorphism. The surjectivity and injectivity of this map are obvious. ∎

Proposition 1.9. *Let X be path connected and $h : X \to Y$ be a continuous map. If $x_0, x_1 \in X$ with $\alpha : I \to X$*

a path between them and $\beta = h \circ \alpha$, then we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \hat{\alpha}\downarrow & & \downarrow\hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

Proof. Let $[f] \in \pi_1(X, x_0)$. Then,

$$\hat{\beta} \circ (h_{x_0})_*([f]) = \hat{\beta}([h \circ f]) = [\overline{\beta} * h \circ f * \beta]$$

and

$$(h_{x_1})_* \circ \hat{\alpha}([f]) = (h_{x_1})_*([\overline{\alpha} * f * \alpha]) = [\overline{\beta} * h \circ f * \beta]$$

This completes the proof. ■

1.2 Computing Fundamental Groups

Theorem 1.10. *For $x_0 \in X$ and $y_0 \in Y$, $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the natural projection maps and p_*, q_* the induced homomorphisms. Let $\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ be the homomorphism given by $\Phi([f]) = (p_*([f]), q_*([f]))$. We shall show that Φ is both injective and surjective.

Since p and q are covering maps, both p_* and q_* are injective, consequently, so is Φ. Let $([f], [g]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$. Consider the function $h : I \to X \times Y$, $h(t) = f(t) \times g(t)$. It is not hard to see that $\Phi([h]) = ([f], [g])$. ■

Corollary. $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Thus, the fundamental group of a torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

1.3 Retracts and Deformation Retracts

Definition 1.11 (Basepoint Preserving Homotopy). A homotopy $H : (X, x_0) \times I \to (Y, y_0)$ is said to be basepoint preserving if $H(x_0, t) = y_0$ for all $t \in I$.

Proposition 1.12. *Let $H : (X, x_0) \times I \to (Y, y_0)$ be a basepoint preserving homotopy between $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (X, x_0) \to (Y, y_0)$. Then $\phi_* = \psi_*$.*

Proof. Choose some $[f] \in \pi_1(X, x_0)$. We would like to show that $\phi \circ f$ and $\psi \circ f$ are path homotopic. It is not hard to see that $H \circ f$ is the required homotopy. ■

Definition 1.13 (Retract). If $A \subseteq X$, then a retraction of X onto A is a continuous map $r : X \to A$ such that $r|_A$ is the identity map of A. If such a map r exists then A is a *retract* of X.

Definition 1.14 (Deformation Retract). If $A \subseteq X$, then A is said to be a *deformation retract* of X if there is a map $H : X \times I \to X$ such that $H(\cdot, 0) = \mathbf{id}_X$ and $H(x, 1) \in A$ for all $x \in X$. Moreover, the restriction $H|_{A\times\{1\}} = \mathbf{id}_A$.

A deformation retract is said to be *strong* if $H(a, t) = a$ for all $a \in A$ and $t \in I$.

It is evident, from the definition that if A is a deformation retract of X, then it is a retract of X.

Theorem 1.15. *Let $i : A \to X$ be the inclusion map and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ be the induced homomorphism for some $a_0 \in A \subseteq X$.*

(a) If A is a retract of X, then i_* is a monomorphism

(b) If A is a deformation retract of X, then i_* is an isomorphism

In both the above cases, the basepoint for X is chosen inside A.

Proof.

(a) Let $r : X \to A$ be the retract. Then $r \circ i = \mathbf{id}_A$. Then $r_* \circ i_* = \mathbf{id}_*$, therefore i_* is injective.

(b) Let $H : X \times I \to X$ be the deformation retract and $r : X \to A$ be $H|_{X\times\{1\}}$. Obviously, $r \circ i = \mathbf{id}_A$, consequently, i_* is injective. Let $[f] \in \pi_1(X, a_0)$. Then, $\Phi : I \times I \to X$ given by $\Phi(s, t) = H(f(s), t)$ is a homotopy between f and a loop in A. Hence, i_* is surjective and thus, an isomorphism.

■

Definition 1.16 (Homotopy Equivalence). A continuous map $\varphi : X \to Y$ is said to be a *homotopy equivalence* if there is a map $\psi : Y \to X$ such that $\varphi \circ \psi \simeq \mathbf{id}_Y$ and $\psi \circ \varphi \simeq \mathbf{id}_X$. In this case, the spaces X and Y are said to be *homotopy equivalent* or said to have the same *homotopy type*.

Theorem 1.17. *Let $\varphi : X \to Y$ be a homotopy equivalence. Then, for any $x_0 \in X$, the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to (Y, \varphi(x_0))$ is an isomorphism.*

Proof. ■

1.4 Seifert-van Kampen's Theorem

Theorem 1.18 (Siefert-van Kampen). *Let $X = U \cup V$ where U and V are open in X. Further, suppose U, V and $U \cap V$ are nonempty and path connected. Let H be a group, $x_0 \in U \cap V$ and*

$$\phi_1 : \pi_1(U, x_0) \to H \qquad \phi_2 : \pi_1(V, x_0) \to H$$

be homomorphisms. Finally, let i_1, i_2, j_1, j_2 be the homomorphisms of fundamental groups induced by inclusion *maps. Then, there is a unique map $\Phi : \pi_1(X, x_0) \to H$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dotted, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Notice how the diagram resembles that of a pushout in a general category and hence, has the universal property and hence, the object, if it exists is unique up to a unique isomorphism. In the special case that $U \cap V$ is simply connected, that is, has a trivial fundamental group, the commutative diagram reduces to that of a coproduct. And it is well known that the coproduct in the category of groups is the free product.

Proof. Let $\mathscr{L}(U, x_0), \mathscr{L}(V, x_0), \mathscr{L}(U \cap V, x_0)$ denote the set of loops in U, V and $U \cap V$. The path homotopy class of a path f in X, U, V and $U \cap V$ is denoted by $[f]$, $[f]_U$, $[f]_V$ and $[f]_{U\cap V}$ respectively. The proof proceeds in multiple steps. The main idea is to first define a set map ρ on the set of loops contained completely in either U or V, then extend it to a set map σ on the set of paths contained completely in either U or V and finally extend it to a set map τ on the set of all paths in X.

Once the map τ is defined, we shall show that $\tau(f) = \tau(g)$ whenever $f \simeq_p g$ and therefore, τ would descend to a group homomorphism from $\pi_1(X, x_0)$ to H.

Step 1: Defining the set map $\rho : \mathscr{L}(U, x_0) \cup \mathscr{L}(V, x_0) \to H$.

This has quite a natural definition:

$$\rho(f) = \begin{cases} \phi_1([f]_U) & f \text{ is contained completely in } U \\ \phi_2([f]_V) & f \text{ is contained completely in } V \end{cases}$$

For a loop contained in $U \cap V$, the map ρ is well defined due to the commutativity of the diagram. It is not hard to see that if $f, g \in \mathscr{L}(U, x_0)$, then $\rho(f * g) = \rho(f)\rho(g)$.

Step 2: Extend the map ρ to a map $\sigma : \mathscr{P}(U) \cup \mathscr{P}(V) \to H$.

For each $x \in X$, fix a path α_x from x_0 to x such that whenever x lies in U, V or $U \cap V$, α_x lies completely in U, V or $U \cap V$ respectively.

Let f be a path from x_1 to x_2 that lies completely in U or completely in V. Define

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})$$

Now, let f and g be paths completely contained in U. If $f \simeq_p g$ in U, then $\alpha_{x_1} * f * \alpha_{x_2}^{-1} \simeq_p \alpha_{x_1} * g * \alpha_{x_2}^{-1}$ in U and from the definition of ρ, we see that

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1}) = \rho(\alpha_{x_1} * g * \alpha_{x_2}^{-1}) = \sigma(g)$$

Next, if f is a path from x_1 to x_2 and g is a path from x_2 to x_3 (both contained in U), then

$$\begin{aligned}
\sigma(f * g) &= \rho(\alpha_{x_1} * f * g * \alpha_{x_3}^{-1}) \\
&= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1} * \alpha_{x_2} * g * \alpha_{x_3}^{-1}) \\
&= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})\rho(\alpha_{x_2} * g * \alpha_{x_3}^{-1}) = \sigma(f)\sigma(g)
\end{aligned}$$

Step 3: Extend the map σ to a map $\tau : \mathscr{P}(X) \to H$

Let $f : I \to X$ be a path. It is not hard to argue, using Lebesgue's Number Lemma, that there is a mesh δ such that for every partition $0 = s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ of $[0,1]$ with mesh less than δ, $f([s_i, s_{i+1}])$ is completely contained in either U or V for $0 \le i \le n-1$.

Denote by f_i, the restriction of f to $[s_i, s_{i+1}]$. Define

$$\tau(f, P) = \sigma(f_0) \cdots \sigma(f_{n-1})$$

We contend that the map $\tau(f, P)$ is independent of the partition chosen, so long as its mesh is less than δ. To do so, we first show that refining a partition with mesh less than δ does not change the image under τ, for which, it suffices to show that adding a single point to the partition does not change the image. Indeed, let $c \in (s_i, s_{i+1})$ be added to the partition. But since $f([s_i, c])$ and $f([c, s_{i+1}])$ lie completely either in U or in V, we have that $\sigma(f|_{[s_i,c]})\sigma(f|_{[c,s_{i+1}]}) = \sigma(f|_{[s_i,s_{i+1}]})$ whence the conclusion follows.

Now, let P_1 and P_2 be two partitions of $[0,1]$ with mesh less than δ. Then $P_1 \cup P_2$ is a partition that refines both P_1 and P_2, consequently,

$$\tau(f, P_1) = \tau(f, P_1 \cup P_2) = \tau(f, P_2)$$

which establishes our claim.

Step 4: If $f \simeq_p g$ in X, then $\tau(f) = \tau(g)$.

Let $F : I \times I \to X$ be a path homotopy between f and g. Using the Lebesgue Number Lemma, there are partitions $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $f([s_i, s_{i+1}] \times [t_i, t_{i+1}])$ is completely contained in either U or V.

Step 5: $\tau(f * g) = \tau(f)\tau(g)$

Let P be a partition of $f * g$ such that $(f * g)([s_i, s_{i+1}])$ is completely contained in either U or V. Define $P^* = P \cup \{1/2\}$. It is not hard to see, using P^* that τ is multiplicative.

Step 6: Constructing the homomorphism Φ.

Restrict the map τ to $\tau : \mathscr{L}(X, x_0) \to H$. From **Step 4**, it follows that there is a map $\Phi : \pi_1(X, x_0) \to H$ and from **Step 5**, we get that Φ is a homomorphism.

The above argument establishes the existence of a group homomorphism $\Phi : \pi_1(X, x_0) \to H$ making the diagram commute. We must now show that the map Φ is unique. But this follows from the fact that the generators of Φ are precisely the images of the generators of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ under the homomorphisms j_1 and j_2 respectively. ■

Chapter 2

Covering Spaces

Definition 2.1 (Covering Space). A covering space of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the condition that there is an open cover $\{U_\alpha\}$ of X such that for each $\alpha \in J$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically by p to U_α.

Notice that for each $x \in X$, the subspace $p^{-1}(x)$ of $\widetilde{X}$ has the discrete topology.

Proposition 2.2. *Let $p : \widetilde{X} \to X$ be a covering map where X is connected. If for some $x \in X$, $|p^{-1}(x)| = n \in \mathbb{N}$, then for all $x' \in X$, $|p^{-1}(x')| = n$.*

Proof. Follows from the fact that the map $x \mapsto |p^{-1}(x)|$ is a continuous map from X to $\mathbb{N}$. ■

2.1 Lifting Properties

Definition 2.3 (Lift). Let $f : Y \to X$ be a continuous and $p : \widetilde{X} \to X$ be a covering map. A *lift* of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p \circ \widetilde{f}$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 2.4. *Let Y be connected and $p : \widetilde{X} \to X$ a covering map. If $f : Y \to X$ is a continuous map having two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$, that agree at some point in Y, then they agree on all of Y.*

Proof. Let

$$A = \{y \in Y \mid \widetilde{f}_1(y) = \widetilde{f}_2(y)\}$$

We shall show that A is clopen in Y, whence we would be done owing to A being nonempty. Let $y \in A$ and $x = f(y)$. There is a neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of $\{V_\alpha\}$ which are homeomorphically mapped to U. Let V_β be the one containing $\widetilde{x} = \widetilde{f}_1(y) = \widetilde{f}_2(y)$. Then, due to continuity, there is a neighborhood N of y that is mapped into V_β by both $\widetilde{f}_1$ and $\widetilde{f}_2$. Then, for all $z \in N$, $p \circ \widetilde{f}_1(z) = p \circ \widetilde{f}_2(z)$ but since p is injective on V_β, we must have $\widetilde{f}_1(z) = \widetilde{f}_2(z)$, consequently, $N \subseteq A$ and A is open.

On the other hand, if $y \notin A$, then $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$ lie in distinct open sets V_{β_1} and V_{β_2}, consequently, for all $z \in N = \widetilde{f}_1^{-1}(V_{\beta_1}) \cap \widetilde{f}_2^{-1}(V_{\beta_2})$, $\widetilde{f}_1(z) \neq \widetilde{f}_2(z)$, thereby completing the proof. ■

Theorem 2.5 (Homotopy Lifting Property). *Let $p : \widetilde{X} \to X$ be a covering map and $F : Y \times I \to X$ a continuous map. Let $\widetilde{F}_0 : Y \to \widetilde{X}$ be a lift of $F|_{X \times \{0\}}$. Then, there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}|_{X \times \{0\}} = \widetilde{F}_0$.*

Proof. ■

Proposition 2.6 (Path Lifting). *Let $f : I \to X$ be a path and let $x_0 = f(0)$. For any $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x}_0$.*

Proposition 2.7. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Theorem 2.8 (Lifting Criterion). *Let Y be path connected and locally path connected and $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Then, for any continuous map $f : (Y, y_0) \to (X, x_0)$, a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. ■

2.2 The Universal Cover

Definition 2.9 (Semilocally Simply-Connected). A topological space X is said to be *semilocally simply-connected* if each point $x \in X$ has a neighborhood U such that the inclusion induced homomorphism $i_* : \pi(U, x) \to \pi(X, x)$ is trivial.

Henceforth, a topological space is said to be *unfathomably based* if it is path-connected, locally path-connected and semilocally simply-connected.

Theorem 2.10. *If X is unfathomably based, then there is a simply connected space $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$.*

Proof. Pick a basepoint $x_0 \in X$. Define

$$\widetilde{X} = \{[\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0\}$$

and the function $p : \widetilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.

Let $\mathscr{U}$ denote the set of all path connected open sets $U \subseteq X$ such that the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial. Indeed, if $V \subseteq U \in \mathscr{U}$ is path connected and open, then the homomorphism induced by the inclusion $V \hookrightarrow X$ is the composition of the homomorphisms induced by $V \hookrightarrow U \hookrightarrow X$ and since the latter is trivial, the composition is trivial, consequently, $V \in \mathscr{U}$.

We contend that $\mathscr{U}$ forms a basis for the topology on X. Indeed, let W be a neighborhood of x, then there is a neighhborhood U of x such that the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial. Since X is locally path connected, there is a path connected neighborhood V of x that is contained in $U \cap W$, whence the conclusion follows.

We shall now topologize $\widetilde{X}$. Let γ be a path in X from x_0 and $U \in \mathscr{U}$ contain $\gamma(1)$. Define the set

$$U_{[\gamma]} = \{[\gamma * \eta] \mid \eta : I \to U,\ \eta(0) = \gamma(1)\}$$

where the equivalence classes are in X. Since U is path connected, $p : U_{[\gamma]} \to U$ is surjective. Moreover, since the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial, any two paths from $\gamma(1)$ to any point $x \in U$ are homotopic in X.

We contend that if $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$. Obviously, there is a path $\eta : I \to U$ such that $\gamma' = \gamma * \eta$, whence it follows that $\gamma' * \mu = \gamma * \eta * \mu$ and thus, $U_{[\gamma']} \subseteq U_{[\gamma]}$. On the other hand, $[\gamma * \mu] = [\gamma * \eta * \overline{\eta} * \mu]$ whereby the conclusion follows.

Next, we claim that the collection $\{U_\gamma\}$ forms a basis for a topology on $\widetilde{X}$. Suppose $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ where $U, V \in \mathscr{U}$, then $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Since $\mathscr{U}$ forms a basis, there is $W \in \mathscr{U}$ such that $W \subseteq U \cap V$, consequently, $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$. This proves our claim.

Consider the bijection $p : U_{[\gamma]} \to U$, we contend that this is a homeomorphism. For any basis element $V_{[\gamma']} \subseteq U_{[\gamma]}$, we have $p(V_{[\gamma']}) = V$, consequently, p is an open map. On the other hand, if $V \subseteq U$ is an open set, then $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for some $[\gamma'] \in U_{[\gamma]}$ with $\gamma'(1) \in V$. Since $V_{[\gamma']} \subseteq U_{[\gamma']} = U_{[\gamma]}$, we see that the restriction of p is continuous and therefore a homeomorphism.

Using the local formulation of continuity, we have that $p : \widetilde{X} \to X$ is a continuous map. Any $x \in X$ has a neighborhood $U \in \mathscr{U}$, consequently, $p^{-1}(U) = \bigcup U_{[\gamma]}$ where $[\gamma]$ ranges over all paths from x_0 to some point in U. It is not hard to argue that the sets $U_{[\gamma]}$ must partition $p^{-1}(U)$, whereby p is a covering map.

Finally, we must show that $\widetilde{X}$ is simply connected. Pick the base point $[x_0] \in \widetilde{X}$. First, we show that $\widetilde{X}$ is path connected. Let $[\gamma] \in \widetilde{X}$. Define $\gamma_t : I \to X$ by

$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \le s \le t \\ \gamma(t) & t < s \le 1 \end{cases}$$

It suffices to show that the map $\varphi : I \to \widetilde{X}$ given by $\varphi(t) = [\gamma_t]$ is continuous. Using the Lebesgue Number Lemma, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_{i-1}, t_i]) \subseteq U_i \in \mathscr{U}$. Let $p_i : U_{i[\gamma_{t_i}]} \to U_i$ be the restriction of p, which is a homeomorphism. Then, for all $t \in [t_{i-1}, t_i]$, $\varphi(t) = p_i^{-1}(\gamma(t))$ and continuity follows from the Pasting Lemma.

Next, we show $\pi_1(\widetilde{X}, [x_0]) = 0$. Since p_* is injective, it suffices to show that the image of p_* is trivial. Let γ be a loop in the image of p_*. Then, the map $t \mapsto [\gamma_t]$ is a lift of γ as we have seen earlier and is unique due to Theorem 2.5. Now, since the lift is a loop, we must have

$$[x_0] = [\gamma_1] = [\gamma]$$

consequently, γ is nulhomotopic. This completes the proof. ■

Theorem 2.11. *Suppose X is unfathomably based. Then for every subgroup $H \subseteq \pi_1(X, x_0)$, there is a covering space $p : (X_H, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$.*

If $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are covering spaces, then an *isomorphism between* them is a homeomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ f$.

Theorem 2.12. *Let (X, x_0) be path connected and locally path connected and $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$*

be covering spaces. Then, for $\widetilde{x}_1 \in p_1^{-1}(x_0)$ and $\widetilde{x}_2 \in p_2^{-1}(x_0)$, there is an isomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.

Proof. We prove the converse, since the forward direction is trivial. Using Theorem 2.8, there are lifts $\widetilde{p}_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ and $\widetilde{p}_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ of p_1 and p_2 respectively. This give us $p_1 = p_2 \circ \widetilde{p}_1$ and $p_2 = p_1 \circ \widetilde{p}_2$, whereby $p_1 \circ (\widetilde{p}_2 \circ \widetilde{p}_1) = p_1$. Note that this implies $\widetilde{p}_2 \circ \widetilde{p}_1$ is a lift of the map p_1, but since $\mathbf{id}_{(\widetilde{X}_1, \widetilde{x}_1)}$ is also a lift, and agree on $\widetilde{x}_1$, we must have that $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathbf{id}_{(\widetilde{X}_1, \widetilde{x}_1)}$ and similarly, $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathbf{id}_{(\widetilde{X}_2, \widetilde{x}_2)}$. This implies the desired conclusion. ■

Let X be path connected and $p : \widetilde{X} \to X$ a path connected covering space. Pick some basepoint $x_0 \in X$ and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$. Let $\widetilde{\gamma} : I \to \widetilde{X}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$ and $\gamma = p \circ \widetilde{\gamma}$. Let $H_0 = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ and $H_1 = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$. Let $g = [\gamma] \in \pi_1(X, x_0)$.

If $[f] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$, then $[\overline{\widetilde{\gamma}} * f * \widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x}_1)$. Consequently, $g^{-1} H_0 g \subseteq H_1$. On the other hand, if $[f] \in \pi_1(\widetilde{X}, \widetilde{x}_1)$, then $[\widetilde{\gamma} * f * \overline{\widetilde{\gamma}}] \in \pi_1(\widetilde{X}, \widetilde{x}_1)$. This gives us that $g H_1 g^{-1} \subseteq H_0$, in conclusion, $H_1 = g^{-1} H_0 g$.

In conclusion, we have proved the following classification theorem.

Theorem 2.13.

2.3 Deck Transformations and Covering Space Actions

2.3.1 Deck Transformations

Definition 2.14. For a covering space $p : \widetilde{X} \to X$, the isomorphisms $f : X \to X$ are called *deck transformations*. These form a group $G(\widetilde{X})$ under composition.

A covering space $p : \widetilde{X} \to X$ is said to be *normal* if for all $x \in X$ and each pair $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$, there is a deck transformation that maps $\widetilde{x} \mapsto \widetilde{x}'$.

Remark. *If $\widetilde{X}$ is path connected, then any two deck transformations agreeing on a single point must agree everywhere.*

Theorem 2.15. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X, and let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X, x_0)$. Then,*

(a) the covering space is normal if and only if H is normal in $\pi_1(X, x_0)$

(b) $G(\widetilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

Proof. Suppose the covering is normal, let $g^{-1} H g$ be a conjugate of H in $\pi_1(X, x_0)$. Then, there is correspondingly $\widetilde{x}_1 \in p^{-1}(x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = g^{-1} H g$. Since the covering is normal, there is a deck transformation $f : \widetilde{X} \to \widetilde{X}$ taking $\widetilde{x}_0$ to $\widetilde{x}_1$. From Theorem 2.12, we must have that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$, whereby $g^{-1} H g = H$ and $H \trianglelefteq \pi_1(X, x_0)$.

Conversely, suppose $H \trianglelefteq \pi_1(X, x_0)$ and let $\widetilde{x}_1 \in p^{-1}(x_0)$. From Theorem 2.13, we have that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ is conjugate to H but since H is normal, the former is equal to H. As a result, from Theorem 2.12, there is a deck transformation taking x_0 to x_1, consequently, the covering space is normal.

Note that given $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$, there is a unique deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$. Now, given some $[\gamma] \in N(H)$, there is a lift $\widetilde{\gamma} : I \to \widetilde{X}$ such that $\widetilde{\gamma}(0) = \widetilde{x}_0$. Define now the function $\phi : N(H) \to G(\widetilde{X})$ by $\phi([\gamma]) = \widetilde{\gamma}(1)$. Let $[\gamma], [\delta] \in N(H)$ with $\sigma = \phi([\gamma])$ and $\tau = \phi([\delta])$. Then, it is not hard to see that $\gamma * \delta$ lifts to $\widetilde{\gamma} * \sigma(\widetilde{\delta})$, which corresponds to the deck transformation $\sigma \circ \tau$, implying that ϕ is a homomorphism. Moreover, ϕ is also surjective, for if there is a deck transformation σ taking $\widetilde{x}_0$ to $\widetilde{x}_1$, then $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = H$. Now, let $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ with $\gamma = p \circ \widetilde{\gamma}$. This implies $[\gamma] \in N(H)$, consequently, $\phi([\gamma]]) = \sigma$.

We now contend that $\ker \phi = H$. Obviously $H \subseteq \ker \phi$. On the other hand, if $[\gamma] \in \ker \phi$, then γ lifts to a loop based at $\widetilde{x}_0$, whereby, $[\gamma] \in H$. The proof is finished by invoking the first isomorphism theorem. ■

2.3.2 Covering Space Actions

Definition 2.16 (Covering Space Action). A *group action* of G on a topological space Y is a homomorphism $\varphi : G \to \mathrm{Aut}_{\mathbf{Top}}(Y)$. A *covering space action* is a group action of G on Y such that for each $y \in Y$, there is a neighborhood U of y such that for all $g_1, g_2 \in G$, $g_1U \cap g_2U \neq \varnothing$, if and only if $g_1 = g_2$.

We may rephrase the definition of a covering space action as:

> A *covering space action* of G on Y is a group action such that for each $y \in Y$, there is a neighborhood U of y such that for all $g \in G$, $U \cap gU \neq \varnothing$ if and only if $g = 1_G$.

Proposition 2.17. *The group action of the group of deck transformations, $G(\widetilde{X})$, of a covering space $p : \widetilde{X} \to X$ is a covering space action.*

Proof. ■

Theorem 2.18. *Let G act on Y through a covering space action.*

(a) The quotient map $p : Y \to Y/G$ given by $p(y) = Gy$ is a normal covering space.[a]*.*

(b) If Y is path connected, then G is the group of deck transformations of the covering space $p : Y \to Y/G$

(c) If Y is path connected and locally path connected, then $G \cong \pi_1(Y/G, Gy_0)/p_*(\pi_1(Y, y_0))$.

[a]Hence the nomenclature

Proof. (a) Let $Gy \in Y/G$. Since G acts through a covering space action, there is a neighborhood U of Y such that the collection $\{gU \mid g \in G\}$ is that of disjoint open sets. Obviously, $V = \bigsqcup_{g \in G} gU$ is a saturated open set, whereby, $p(V)$ is open in Y/G and a neighborhood of Gy. We contend that the restriction $p : U \to p(V)$ is a homeomorphism. Indeed, if $W \subseteq U$ is open, then $p(W) \subseteq p(V)$ is open, since $p(W) = p\left(\bigsqcup_{g \in G} gW\right)$ and the term within the brackets is a saturated open set. This immediately implies that p is a covering map.

Furthermore, for any $g_1y, g_2y \in Gy$, there is the action $g_2g_1^{-1}$ taking g_1y to g_2y whereby, the covering space is normal.

(b) Obviously, each element of G is a deck transformation. On the other hand, if $f : Y \to Y$ is a deck transformation, then for any $y \in Y$, $f(y) \in Gy$, whereby, there is $g \in G$ such that $gy = f(y)$. From Remark 2.3.1, we have that $g = f$, implying the desired conclusion.

(c) This follows from Theorem 2.15.

■

Chapter 3

Homology