Algebraic Topology

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May 4, 2023

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Chapter 1

The Fundamental Group

1.1 Fundamental Groupoid and Group

Definition 1.1 (Homotopy). Let X and Y be topological spaces. A homotopy is a continuous function $H : X \times I \to Y$. A *homotopy* between two functions $f, g : X \to Y$ is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Definition 1.2 (Homotopy of Paths). Let X be a topological space and $f, g : I \to X$ be paths. Then, f and g are said to be *path homotopic* if there is a continuous function $H : I \times I \to X$ such that $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ for all $s \in I$. We denote this by $f \simeq_p g$.

Proposition 1.3. *The relation $\simeq$ on the set of all paths in X is an equivalence relation.*

Proposition 1.4. *Let $f : I \to X$ be a path and $\varphi : I \to I$ be a continuous function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, $f \simeq_p f \circ \varphi$.*

Proof. Define the function $\Phi : I \times I \to X$ by

$$\Phi(s, t) = f(t\varphi(s) + (1 - t)s)$$

It is not hard to see that Φ is a path homotopy between f and $f \circ \varphi$. ■

Consider the set of all equivalence classes of paths in X under the equivalence relation $\simeq_p$. Define the operation $*$ on pairs of equivalence classes $[f]$ and $[g]$ where $f(1) = g(0)$ by

$$[f] * [g] = [f * g]$$

where

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 < t \leq 1 \end{cases}$$

Proposition 1.5. *The operation $*$ is associative. That is,*

$$[f] * ([g] * [h]) = ([f] * [g]) * h$$

Proof. Note that $[f] * ([g] * [h])$ is the equivalence class containing the path:

$$\alpha(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(4t-2) & 1/2 < t \le 3/4 \\ h(4t-3) & 3/4 < t \le 1 \end{cases}$$

Consider the piecewise linear function $\varphi : [0,1] \to [0,1]$ that maps $[0,1/2]$ to $[0,1/4]$, $[1/2,3/4]$ to $[1/4,1/2]$ and $[1/2,1]$ to $[3/4,1]$, then through $\alpha \circ \varphi$, the conclusion follows. ∎

Definition 1.6 (Fundamental Group). Let $\pi_1(X, x_0)$ be the set of equivalence classes of paths $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$. It is not hard to see from the discussion above that $\pi_1(X, x_0)$ has a group structure. This is known as the *fundamental group*.

Let $\mathbf{Top}_*$ denote the category of pointed topological spaces, that is, the category wherein objects are pairs (X, x_0) where $x_0 \in X$ and a morphism $f : (X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ with $f(x_0) = y_0$.

Proposition 1.7. *Let $f : (X, x_0) \to (Y, y_0)$ be a morphism in $\mathbf{Top}_*$. Then, the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\alpha] \mapsto [f \circ \alpha]$ is a homomorphism of groups. Further, if*

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

then $(g \circ f)_* = g_* \circ f_*$.

Proof. If H is a path homotopy between α_1 and α_2 in X, then $f \circ H$ is a homotopy between $f \circ \alpha_1$ and $f \circ \alpha_2$ in Y. Thus, the map f_* is well defined. Next, suppose $[\alpha], [\beta] \in \pi_1(X, x_0)$, then, it is not hard to see that $(f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta)$, consequently, f_* is a homomorphism of groups. The final assertion is obvious from the definition. ∎

As a result, we see that π_1 is a (covariant) functor from $\mathbf{Top}_*$ to $\mathbf{Grp}$.

Theorem 1.8. *Let X be path connected and $x_0, x_1 \in X$. Let $\alpha : I \to X$ be a path from x_0 to x_1. Then, the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $[f] \mapsto [\overline{\alpha} * f * \alpha]$ is a group isomorphism.*

Proof. It is easy to see that $\hat{\alpha}$ is a homomorphism. The surjectivity and injectivity of this map are obvious. ∎

Proposition 1.9. *Let X be path connected and $h : X \to Y$ be a continuous map. If $x_0, x_1 \in X$ with $\alpha : I \to X$*

a path between them and $\beta = h \circ \alpha$, then we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(h_{x_0})_*"] \arrow[d, "\hat{\alpha}"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \\
\pi_1(X, x_1) \arrow[r, "(h_{x_1})_*"] & \pi_1(Y, y_1)
\end{tikzcd}$$

Proof. Let $[f] \in \pi_1(X, x_0)$. Then,

$$\hat{\beta} \circ (h_{x_0})_*([f]) = \hat{\beta}([h \circ f]) = [\overline{\beta} * h \circ f * \beta]$$

and

$$(h_{x_1})_* \circ \hat{\alpha}([f]) = (h_{x_1})_*([\overline{\alpha} * f * \alpha]) = [\overline{\beta} * h \circ f * \beta]$$

This completes the proof. ∎

1.2 Computing Fundamental Groups

Theorem 1.10. *For $x_0 \in X$ and $y_0 \in Y$, $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the natural projection maps and p_*, q_* the induced homomorphisms. Let $\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ be the homomorphism given by $\Phi([f]) = (p_*([f]), q_*([f]))$. We shall show that Φ is both injective and surjective.

Since p and q are covering maps, both p_* and q_* are injective, consequently, so is Φ. Let $([f], [g]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$. Consider the function $h : I \to X \times Y$, $h(t) = f(t) \times g(t)$. It is not hard to see that $\Phi([h]) = ([f], [g])$. ∎

Corollary. $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Thus, the fundamental group of a torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

1.3 Retracts and Deformation Retracts

Definition 1.11 (Basepoint Preserving Homotopy). A homotopy $H : (X, x_0) \times I \to (Y, y_0)$ is said to be basepoint preserving if $H(x_0, t) = y_0$ for all $t \in I$.

Proposition 1.12. *Let $H : (X, x_0) \times I \to (Y, y_0)$ be a basepoint preserving homotopy between $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (X, x_0) \to (Y, y_0)$. Then $\phi_* = \psi_*$.*

Proof. Choose some $[f] \in \pi_1(X, x_0)$. We would like to show that $\phi \circ f$ and $\psi \circ f$ are path homotopic. It is not hard to see that $H \circ f$ is the required homotopy. ∎

Definition 1.13 (Retract). If $A \subseteq X$, then a retraction of X onto A is a continuous map $r : X \to A$ such that $r\mid_A$ is the identity map of A. If such a map r exists then A is a *retract* of X.

Definition 1.14 (Deformation Retract). If $A \subseteq X$, then A is said to be a *deformation retract* of X if there is a map $H : X \times I \to X$ such that $H(\cdot, 0) = \mathbf{id}_X$ and $H(x, 1) \in A$ for all $x \in X$. Moreover, the restriction $H\,|_{A\times\{1\}} = \mathbf{id}_A$.

A deformation retract is said to be *strong* if $H(a, t) = a$ for all $a \in A$ and $t \in I$.

It is evident, from the definition that if A is a deformation retract of X, then it is a retract of X.

Theorem 1.15. *Let $i : A \to X$ be the inclusion map and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ be the induced homomorphism for some $a_0 \in A \subseteq X$.*

(a) If A is a retract of X, then i_* is a monomorphism

(b) If A is a deformation retract of X, then i_* is an isomorphism

In both the above cases, the basepoint for X is chosen inside A.

Proof.

(a) Let $r : X \to A$ be the retract. Then $r \circ i = \mathbf{id}_A$. Then $r_* \circ i_* = \mathbf{id}_*$, therefore i_* is injective.

(b) Let $H : X \times I \to X$ be the deformation retract and $r : X \to A$ be $H|_{X\times\{1\}}$. Obviously, $r \circ i = \mathbf{id}_A$, consequently, i_* is injective. Let $[f] \in \pi_1(X, a_0)$. Then, $\Phi : I \times I \to X$ given by $\Phi(s, t) = H(f(s), t)$ is a homotopy between f and a loop in A. Hence, i_* is surjective and thus, an isomorphism.

■

Definition 1.16 (Homotopy Equivalence). A continuous map $\varphi : X \to Y$ is said to be a *homotopy equivalence* if there is a map $\psi : Y \to X$ such that $\varphi \circ \psi \simeq \mathbf{id}_Y$ and $\psi \circ \varphi \simeq \mathbf{id}_X$. In this case, the spaces X and Y are said to be *homotopy equivalent* or said to have the same *homotopy type*.

Theorem 1.17. *Let $\varphi : X \to Y$ be a homotopy equivalence. Then, for any $x_0 \in X$, the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to (Y, \varphi(x_0))$ is an isomorphism.*

Proof. ■

1.4 Seifert-van Kampen's Theorem

Theorem 1.18 (Siefert-van Kampen). *Let $X = U \cup V$ where U and V are open in X. Further, suppose U, V and $U \cap V$ are nonempty and path connected. Let H be a group, $x_0 \in U \cap V$ and*

$$\phi_1 : \pi_1(U, x_0) \to H \qquad \phi_2 : \pi_1(V, x_0) \to H$$

be homomorphisms. Finally, let i_1, i_2, j_1, j_2 be the homomorphisms of fundamental groups induced by inclusion *maps. Then, there is a unique map $\Phi : \pi_1(X, x_0) \to H$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dotted, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Notice how the diagram resembles that of a pushout in a general category and hence, has the universal property and hence, the object, if it exists is unique up to a unique isomorphism. In the special case that $U \cap V$ is simply connected, that is, has a trivial fundamental group, the commutative diagram reduces to that of a coproduct. And it is well known that the coproduct in the category of groups is the free product.

Proof. Let $\mathscr{L}(U, x_0), \mathscr{L}(V, x_0), \mathscr{L}(U \cap V, x_0)$ denote the set of loops in U, V and $U \cap V$. The path homotopy class of a path f in X, U, V and $U \cap V$ is denoted by $[f]$, $[f]_U$, $[f]_V$ and $[f]_{U \cap V}$ respectively. The proof proceeds in multiple steps. The main idea is to first define a set map ρ on the set of loops contained completely in either U or V, then extend it to a set map σ on the set of paths contained completely in either U or V and finally extend it to a set map τ on the set of all paths in X.

Once the map τ is defined, we shall show that $\tau(f) = \tau(g)$ whenever $f \simeq_p g$ and therefore, τ would descend to a group homomorphism from $\pi_1(X, x_0)$ to H.

Step 1: Defining the set map $\rho : \mathscr{L}(U, x_0) \cup \mathscr{L}(V, x_0) \to H$.

This has quite a natural definition:

$$\rho(f) = \begin{cases} \phi_1([f]_U) & f \text{ is contained completely in } U \\ \phi_2([f]_V) & f \text{ is contained completely in } V \end{cases}$$

For a loop contained in $U \cap V$, the map ρ is well defined due to the commutativity of the diagram. It is not hard to see that if $f, g \in \mathscr{L}(U, x_0)$, then $\rho(f * g) = \rho(f)\rho(g)$.

Step 2: Extend the map ρ to a map $\sigma : \mathscr{P}(U) \cup \mathscr{P}(V) \to H$.

For each $x \in X$, fix a path α_x from x_0 to x such that whenever x lies in U, V or $U \cap V$, α_x lies completely in U, V or $U \cap V$ respectively.

Let f be a path from x_1 to x_2 that lies completely in U or completely in V. Define

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})$$

Now, let f and g be paths completely contained in U. If $f \simeq_p g$ in U, then $\alpha_{x_1} * f * \alpha_{x_2}^{-1} \simeq_p \alpha_{x_1} * g * \alpha_{x_2}^{-1}$ in U and from the definition of ρ, we see that

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1}) = \rho(\alpha_{x_1} * g * \alpha_{x_2}^{-1}) = \sigma(g)$$

Next, if f is a path from x_1 to x_2 and g is a path from x_2 to x_3 (both contained in U), then

$$\begin{aligned}
\sigma(f * g) &= \rho(\alpha_{x_1} * f * g * \alpha_{x_3}^{-1}) \\
&= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1} * \alpha_{x_2} * g * \alpha_{x_3}^{-1}) \\
&= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})\rho(\alpha_{x_2} * g * \alpha_{x_3}^{-1}) = \sigma(f)\sigma(g)
\end{aligned}$$

Step 3: Extend the map σ to a map $\tau : \mathscr{P}(X) \to H$

Let $f : I \to X$ be a path. It is not hard to argue, using Lebesgue's Number Lemma, that there is a mesh δ such that for every partition $0 = s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ of $[0,1]$ with mesh less than δ, $f([s_i, s_{i+1}])$ is completely contained in either U or V for $0 \le i \le n-1$.

Denote by f_i, the restriction of f to $[s_i, s_{i+1}]$. Define

$$\tau(f, P) = \sigma(f_0) \cdots \sigma(f_{n-1})$$

We contend that the map $\tau(f, P)$ is independent of the partition chosen, so long as its mesh is less than δ. To do so, we first show that refining a partition with mesh less than δ does not change the image under τ, for which, it suffices to show that adding a single point to the partition does not change the image. Indeed, let $c \in (s_i, s_{i+1})$ be added to the partition. But since $f([s_i, c])$ and $f([c, s_{i+1}])$ lie completely either in U or in V, we have that $\sigma(f|_{[s_i,c]})\sigma(f|_{[c,s_{i+1}]}) = \sigma(f|_{[s_i,s_{i+1}]})$ whence the conclusion follows.

Now, let P_1 and P_2 be two partitions of $[0,1]$ with mesh less than δ. Then $P_1 \cup P_2$ is a partition that refines both P_1 and P_2, consequently,

$$\tau(f, P_1) = \tau(f, P_1 \cup P_2) = \tau(f, P_2)$$

which establishes our claim.

Step 4: If $f \simeq_p g$ in X, then $\tau(f) = \tau(g)$.

Let $F : I \times I \to X$ be a path homotopy between f and g. Using the Lebesgue Number Lemma, there are partitions $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $f([s_i, s_{i+1}] \times [t_i, t_{i+1}])$ is completely contained in either U or V.

Step 5: $\tau(f * g) = \tau(f)\tau(g)$

Let P be a partition of $f * g$ such that $(f * g)([s_i, s_{i+1}])$ is completely contained in either U or V. Define $P^* = P \cup \{1/2\}$. It is not hard to see, using P^* that τ is multiplicative.

Step 6: Constructing the homomorphism Φ.

Restrict the map τ to $\tau : \mathscr{L}(X, x_0) \to H$. From **Step 4**, it follows that there is a map $\Phi : \pi_1(X, x_0) \to H$ and from **Step 5**, we get that Φ is a homomorphism.

The above argument establishes the existence of a group homomorphism $\Phi : \pi_1(X, x_0) \to H$ making the diagram commute. We must now show that the map Φ is unique. But this follows from the fact that the generators of Φ are precisely the images of the generators of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ under the homomorphisms j_1 and j_2 respectively. ■

Chapter 2

Covering Spaces

Definition 2.1 (Covering Space). A covering space of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the condition that there is an open cover $\{U_\alpha\}$ of X such that for each $\alpha \in J$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically by p to U_α.

2.1 Lifting Properties

2.2 The Universal Cover

Definition 2.2 (Semilocally Simply-Connected). A topological space X is said to be *semilocally simply-connected* if each point $x \in X$ has a neighborhood U such that the inclusion induced homomorphism $i_* : \pi(U, x) \to \pi(X, x)$ is trivial.

Henceforth, a topological space is said to be *unfathomably based* if it is path-connected, locally path-connected and semilocally simply-connected.

Theorem 2.3. *If X is unfathomably based, then there is a simply connected space $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$.*