Set Theory

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Abstract

The main reference for this was [Kun80].

Contents

Chapter 1

The Zermelo-Fraenkel Axioms

1.1 Axioms of Set Theory

We shall discuss Zermelo-Fraenkel Set Theory, which is a first order theory, with signature $\mathsf{ZF} = (\varnothing, \{\in\})$. That is, there are no function symbols and the only predicate is the "belongs to" relation.

ZF0 (Nonempty Domain) There is at least one set.

$$\exists x(x = x)$$

This axiom is redundunt since **ZF7** guarantees the existence of an infinite set and thus the domain of discourse must be nonempty.

ZF1 (Extensionality) Informally speaking, a set is determined uniquely by its elements.

$$\forall x \forall y(\forall z(z \in x \iff z \in y) \implies x = y)$$

ZF2 (Foundation/Regularity) This states that any nonempty set contains an element that is disjoint from it.

$$\forall x \left[\exists y(y \in x) \implies \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))\right]$$

ZF3 (Comprehension) Informally speaking, this axiom allows us to define sets in the set-builder notation. Let ϕ be a valid first order formula with free variables $w_1, \ldots, w_n, x, z$. Then

$$\forall z \forall w_1, \ldots, w_n \exists y \forall x \left(x \in y \iff x \in z \wedge \phi\right)$$

Notice how this is the same as writing

$$y = \{x \in z \mid \phi\}$$

ZF4 (Pairing) Informally, this states that given two sets x and y, there is a set $z = \{x, y\}$.

$$\forall x \forall y \exists z \forall w(w \in z \iff (w = x \vee w = y))$$

ZF5 (Union) This axiom allows us to take a union of a collection of sets.

$$\forall \mathscr{F} \exists A \forall y(x \in y \wedge y \in \mathscr{F} \implies x \in A)$$

ZF6 (Replacement Scheme) Let ϕ be a valid formula without Y as a free variable. Then,

$$\forall A(\forall x \in A\ \exists! y\ \phi(x,y) \implies \exists Y\ \forall x \in A\ \exists y \in Y \phi(x,y))$$

Informally speaking, this allows us to replace the elements of a set to obtain a new set.

ZF7 (Infinity) There is an infinite inductive set.

$$\exists x(\varnothing \in x \wedge \forall y \in x(S(y) \in x))$$

ZF8 (Power Set) Every set has a set containing all its subsets. It is important to note that this need not be **the** power set.

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

ZF9 (Choice) Informally, given a collection of nonempty sets X, there is a choice function that chooses one element from each set in X.

$$\forall X \left(\varnothing \notin X \implies \exists f : X \to \bigcup X,\ \forall x \in X (f(x) \in x) \right).$$

We have been a bit sloppy in stating the axioms. Notice that our signature does not contain a predicate $\subseteq$ or the successor function S, neither do we know, a priori, of the existence of **the** empty set.

To define the formula $\subseteq (x, y)$, use

$$\subseteq (x, y) := \forall z (z \in x \implies z \in y)$$

As for the successor function, given any set x, using **ZF4**, there is a set $y = \{x\}$. Using **ZF5**, we may define $S(y) := x \cup y$. Finally, using **ZF0** and **ZF3**, we know of the existence of the empty set as

$$\exists x (x = x \wedge \exists y \forall z (z \in x \iff z \in y \wedge z \neq z))$$

Further, due to **ZF1**, the empty set is unique.

1.2 Consequences of the Axioms

Theorem 1.1. *There is no universal set. That is,*

$$\neg \exists z \forall x (x \in z)$$

Proof. If there were a universal set, then using **ZF3**, we may construct the set $y = \{x \in z \mid x \notin x\}$. Then, it is not hard to argue that

$$y \in y \iff y \notin y,$$

a contradiction. ■

Definition 1.2 (Power Set). Let x be a set. Due to **ZF8**, there is a set z containing all the subsets of x. Using Comprehension, we may construct

$$\mathscr{P}(x) := \{y \in z \mid y \subseteq x\}.$$

This is known as the **power set** of x.

Definition 1.3. Let $\mathscr{F}$ denote a set. Let A be a set satisfying **ZF5**. Define

$$\bigcup \mathscr{F} := \{x \in A \mid \exists y \in \mathscr{F} (x \in y)\}$$

and

$$\bigcap \mathscr{F} := \{x \in A \mid \forall y \in \mathscr{F} (x \in y)\}.$$

1.3 Relations, Functions and Well Ordering

Definition 1.4 (Ordered Pair). For sets x, y, define the ordered pair $\langle x, y \rangle$ by

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

The set on the right is constructed by using the pairing axiom twice.

Definition 1.5 (Cartesian Product). Let A and B be sets. Using Replacement, we may define, for each $y \in B$,

$$A \times \{y\} := \{z \mid \exists x \in A(z = \langle x, y\rangle)\}.$$

Again, by Replacement, define the set

$$\mathscr{F} := \{z \mid \exists y \in B(z = A \times \{y\})\}.$$

Finally, define

$$A \times B := \bigcup \mathscr{F}.$$

Definition 1.6 (Relation, Function). Let A be a set. A relation R on A is a subset of $A \times A$. Define the domain and range of a relation as

$$\mathrm{dom}(R) := \{x \in A \mid \exists y(\langle x, y\rangle \in R)\} \qquad \mathrm{ran}(R) := \{y \mid \exists x(\langle x, y\rangle \in R)\}.$$

We write xRy to denote $\langle x, y\rangle \in R$.

A relation f is said to be a function if

$$\forall x \in \mathrm{dom}(f) \exists! y \in \mathrm{ran}(f)(\langle x, y\rangle \in f).$$

We use $f : A \to B$ to denote a function f with $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.

Definition 1.7 (Total Ordering, Well Ordering). A *total ordering* is a pair $\langle A, R\rangle$ where A is a set and R is a relation that is irreflexive, transitive and satisfies trichotomy.

We say R *well-orders* A if $\langle A, R\rangle$ is a total ordering and every non empty subset of A has an R-least element.

We use $\mathrm{pred}(A, x, R)$ to denote the set $\{y \in A \mid yRx\}$.

Lemma 1.8. *Let $\langle A, R\rangle$ be a well-ordering. Then for all $x \in A$, $\langle A, R\rangle \ncong \langle \mathrm{pred}(A, x, R), R\rangle$.*

Proof. Suppose $\langle A, R\rangle \cong \langle \mathrm{pred}(A, x, R), R\rangle$ and let $f : A \to \mathrm{pred}(A, x, R)$ be the order isomorphism. Let x be the R-least element of the set

$$\{y \in A \mid f(y) \neq y\},$$

which obviously exists since the aforementioned set is nonempty. If $xRf(x)$, there is some $y \in A$ with yRx and $f(y) = x \neq y$ a contradiction to the choice of x. On the other hand, if $f(x)Rx$, then $f(f(x)) \neq f(x)$ since f is injective, a contradiction to the choice of x. This completes the proof. ∎

Theorem 1.9. *Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be two well-orderings. Then exactly one of the following holds:*

(a) $\langle A, R\rangle \cong \langle B, S\rangle$.

(b) $\exists y \in B\left(\langle A, R\rangle \cong \langle \mathrm{pred}(B, y, S), S\rangle\right)$.

(c) $\exists x \in A\left(\langle \mathrm{pred}(A, x, R), R\rangle \cong \langle B, S\rangle\right)$.

Proof. Let

$$f := \{\langle v, w\rangle \mid v \in A,\ w \in B,\ \langle \mathrm{pred}(A, v, R), R\rangle \cong \langle \mathrm{pred}(B, w, S), S\rangle\}.$$

Due to the preceeding lemma, if $\langle v_1, w \rangle, \langle v_2, w \rangle \in f$, then $v_1 = v_2$. Similarly, if $\langle v, w_1 \rangle, \langle v, w_2 \rangle \in f$, then $w_1 = w_2$. Hence, f is an injective function.

It is not hard to argue that f is an order isomorphism from an initial segment of A to an initial segment of B. Both these segments may not be proper else we could find another isomorphism from an initial segment of A to an initial segment of B by extending one of the isomorphisms in f. This completes the proof. ■

Chapter 2

Ordinal Numbers

2.1 Transitive Sets

Definition 2.1. A set x is said to be *transitive* if

$$\forall y \forall z (z \in y \land y \in x \implies z \in x).$$

Proposition 2.2. *A set x is transitive if and only if*

$$\forall y (y \in x \implies y \subseteq x).$$

Proof. Suppose x is transitive and $y \in x$. Since for all $z \in y$, $z \in x$, we must have $y \subseteq x$. The converse is trivial. ■

Proposition 2.3. *If x is a transitive set, then so is $x \cup \{x\}$.*

Proof. ■

Proposition 2.4. *If x is a transitive set, then so is $\mathscr{P}(x)$.*

Proof. ■

Proposition 2.5. *If $\mathscr{F}$ is a family of transitive sets, then so is $\bigcup \mathscr{F}$.*

Proof. ■

Proposition 2.6. *If x is a transitive set, then so is every $z \in x$.*

Proof. ■

2.2 Ordinals

Definition 2.7 (Ordinal). A set x is said to be an *ordinal* if it is transitive and well ordered by $\in$. That is, the pair $\langle x, \in_x \rangle$ is a well ordering, where

$$\in_x := \{\langle v, w \rangle \in x \times x \mid v \in w\}.$$

Theorem 2.8 (Properties of Ordinals).

(a) If x is an ordinal and $y \in x$, then y is an ordinal and $y = \operatorname{pred}(x, y)$.

(b) If $x \cong y$ are ordinals, then $x = y$.

(c) If x, y are ordinals, then exactly one of the following is true: $x = y$, $x \in y$ or $y \in x$.

(d) If C is a nonempty set of ordinals, then $\exists x \in C\ \forall y \in C (x \in y \vee x = y)$. That is, every nonempty set of ordinals has a minimum element.

Proof. (a) Due to Proposition 2.6, y is a transitive and owing to it being the subset of a well ordered set, it is well ordered too, hence an ordinal.

(b) Let $f : x \to y$ be an isomorphism. Let

$$A := \{z \in x \mid f(z) \neq z\}.$$

Suppose A is nonempty, then it has a least element, say $w \in x$. If $v \in w$, then $v = f(v) \in f(w)$ whence $w \subseteq f(w)$. On the other hand, if $v \in f(w)$, then there is some $u \in w$ such that $v = f(u) = u \in w$ and thus $f(w) = w$, a contradiction.

(c) Follows from Theorem 1.9.

(d) First note that it suffices to find $x \in C$ with $x \cap C = \varnothing$ for if $y \in C$ is another ordinal with $x \neq y$, then $y \notin x$ lest $x \cap C \neq \varnothing$.

Pick any $x \in C$. If $x \cap C = \varnothing$, then we are done. Else, let $x' \in x \cap C$ be the $\in$-least element. It is not hard to argue that $x' \cap C = \varnothing$ and we are done. ∎

Lemma 2.9. *If A is a transitive set of ordinals, then A is an ordinal.*

Proof. We must first show that the membership relation $\in_A$ is a linear order. This follows from Theorem 2.8 (c) and the fact that A is a transitive set. Lastly, to see that A is well ordered, simply invoke Theorem 2.8 (d). ∎

Theorem 2.10. *If $\langle A, R \rangle$ is a well ordering, then there is a unique ordinal C such that $\langle A, R \rangle \cong C$.*

Proof. Let

$$\begin{aligned} B &:= \{a \in A \mid \exists x_a (x_a \text{ is an ordinal } \wedge \langle \operatorname{pred}(A, a, R), R \rangle \cong x_a)\}, \\ f &:= \{\langle b, x_b \rangle \mid b \in B\}. \end{aligned}$$

First, note that for all $b \in B$, x_b, since it exists must be unique and thus f is a well defined function with $\operatorname{dom}(f) = B$.

Let $C = \operatorname{ran}(f)$. We contend that C is an ordinal. Let $y \in x \in C$ and $a \in B$ be such that $g : \operatorname{pred}(A, a, R) \to x$ is an isomorphism. Then, there is some $b \in \operatorname{pred}(A, a, R)$ with $g(b) = y$. It is not hard to see that the restriction $g : \operatorname{pred}(A, b, R) \to y$ is an isomorphism whence $y \in C$ and thus C is an ordinal due to the preceeding lemma.

The function $f : B \to C$ is obviously a surjection. We contend that it is an isomorphism. Indeed, let $a, b \in B$ with aRb and $g : \operatorname{pred}(A, b, R) \to x_b$ be *the* isomorphism. If $y = g(a)$, then the restriction $g : \operatorname{pred}(A, a, R) \to y$ is an isomorphism whence $f(a) = y \in x = f(b)$ and f is an order isomorphism.

Suppose $B \neq A$. Let $b \in A \backslash B$ be the R-least element. Then, $\operatorname{pred}(A, b, R) \subseteq B$. Now suppose $B \neq \operatorname{pred}(A, b, R)$, consequently, there is some $b' \in B \backslash \operatorname{pred}(A, b, R)$, then bRb' and if there is an order isomorphism from $\operatorname{pred}(A, b', R)$ to some ordinal x, then there must be one from $\operatorname{pred}(A, b, R)$ as we have argued earlier, a contradiction.

Thus, either $B = A$ or $B = \operatorname{pred}(A, b, R)$ for some $b \in A$. In the latter case, the function f is an order isomorphism between $\operatorname{pred}(A, b, R)$ and an ordinal C whence $b \in B$, a contradiction. Thus $B = A$ and the proof is complete. ∎

Definition 2.11 (Type of a Well Ordering). If $\langle A, R\rangle$ is a well ordering, then $\operatorname{type}(A, R)$ is the unique ordinal C such that $\langle A, R\rangle \cong C$.

Henceforth, we use Greek letters $\alpha, \beta, \gamma, \ldots$ to vary over ordinals. That is, saying $\forall\alpha(\ldots)$ is equivalent to saying $\forall x(x$ is an ordinal $\ldots)$. Further, since the ordinals are well ordered, we write $\alpha < \beta$ to denote $\alpha \in \beta$ and similarly, $\alpha \leq \beta$ means $\alpha \in \beta \vee \alpha = \beta$.

Definition 2.12. Let X be a set of ordinals. Define

$$\sup(X) := \bigcup X \quad \text{and} \quad \min(X) := \bigcap X.$$

Further, for an ordinal α, let $S(\alpha)$ denote the set $\alpha \cup \{\alpha\}$.

Lemma 2.13. *(a)* $\forall\alpha, \beta(\alpha \leq \beta \iff \alpha \subseteq \beta)$.

(b) If X is a set of ordinals, $\sup(X)$ *is the least ordinal $\geq$ all elements of X and if $X \neq \varnothing$,* $\min(X)$ *is the least ordinal in X.*

Proof. (a) The forward direction is obvious. Suppose $\alpha \subseteq \beta$. If $\alpha = \beta$, then we are done. If not, let γ be the $<$-least element of $\beta\backslash\alpha$. We contend that $\gamma = \alpha$. Indeed, if $x \in \gamma$, then $x \notin \beta\backslash\alpha$ lest we contradict the minimality of γ consequently, $x \in \alpha$ whence $\gamma \subseteq \alpha$. On the other hand, since $\alpha = \operatorname{pred}(\beta, \alpha)$, we have $\alpha \leq \gamma$ and thus $\alpha \subseteq \gamma$. This shows that $\alpha = \gamma \in \beta$ and the conclusion follows.

(b) ∎

Lemma 2.14. *For an ordinal α, $S(\alpha)$ is an ordinal, $\alpha < S(\alpha)$ and*

$$\forall\beta(\beta < S(\alpha) \iff \beta \leq \alpha).$$

Definition 2.15 (Successor, Limit Ordinal). An ordinal α is said to be a *successor ordinal* if there is an ordinal β such that $\alpha = S(\beta)$. On the other hand, α is said to be a *limit ordinal* if $\alpha \neq \varnothing$ and α is not a successor ordinal.

2.3 Transfinite Induction and Recursion

2.3.1 Classes but informally

Informally speaking, a class is any collection of the form

$$\{x \mid \phi(x)\}$$

where $\phi(x)$ is a well defined first order formula. As we have seen earlier, the class

$$\{x \mid x = x\}$$

is not a set. A *proper class* is a class which is not a set. One uses boldface letters to denote classes.

Definition 2.16. Denote

$$\mathbf{V} := \{x \mid x = x\} \qquad \mathbf{ON} := \{x \mid x \text{ is an ordinal}\}.$$

To be completely formal, a class is simply a first order formula with one or more free variables. For example, the class of all ordinals can be thought of as the formula

$$\mathbf{ON}(x) = x \text{ is an ordinal.}$$

We can extend this to define functions between classes $\mathbf{A}$ and $\mathbf{B}$. A function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is given by a first order logic formula in two variables $\mathbf{F}(x, y)$ such that

$$\forall x\ \mathbf{A}(x) \implies \exists ! y\ \left(\mathbf{B}(y) \wedge \mathbf{F}(x,y)\right).$$

Theorem 2.17 (Transfinite Induction on ON). *If $\mathbf{C} \subseteq \mathbf{ON}$ and $\mathbf{C} \neq \varnothing$, then $\mathbf{C}$ has a least element.*

Proof. The proof is exactly like Theorem 2.8 (d). ∎

One must note that there is a significant difference between Theorem 2.8 (d) and Theorem 2.17. The former is a single provable statement in ZFC while the latter is a theorem schema which represents an infinite collection of theorems. In particular, suppose the class $\mathbf{C}$ corresponded to a formula $\mathbf{C}(x, z_1, \ldots, z_n)$, then Theorem 2.17 in this case says the following:

$$\begin{aligned} \forall z_1, \ldots, z_n \Big\{ & \left[\forall x (\mathbf{C}(x, z_1, \ldots, z_n) \implies x \text{ is an ordinal}) \wedge \exists x \mathbf{C}(x, z_1, \ldots, z_n)\right] \\ & \implies \Big[\exists x \left(\mathbf{C}(x, z_1, \ldots, z_n) \wedge \forall y (\mathbf{C}(y, z_1, \ldots, z_n) \implies y \geq x)\right)\Big] \Big\}. \end{aligned}$$

And Theorem 2.17 specifies one such formula for each well-formed sentence $\mathbf{C}$.

Theorem 2.18 (Transfinite Recursion on ON). *If $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, then there is a unique $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ such that*

$$\forall \alpha \left(\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)\right).$$

The formal restatement of the above in terms of first order logic is the following:

$$\forall x \exists ! y\ \mathbf{F}(x,y) \implies \Big[\forall \alpha \exists ! y\ \mathbf{G}(\alpha, y) \wedge \forall \alpha \exists x \exists y \left(\mathbf{G}(\alpha, y) \wedge \mathbf{F}(x,y) \wedge x = \mathbf{G} \restriction \alpha\right)\Big]$$

where

$$(x = \mathbf{G} \restriction \alpha) := \text{function}(x) \wedge \text{dom}(x) = \alpha \wedge \left(\forall \beta \in \text{dom}(x)\ \mathbf{G}(\beta, x(\beta))\right).$$

Similarly, one can encode the uniqueness condition.

Proof. ∎

2.4 Ordinal Arithmetic

Addition

Definition 2.19 (Ordinal Addition). If α, β are ordinals, then define $\alpha + \beta = \text{type}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$ where

$$R = \left\{\langle\langle\xi, 0\rangle, \langle\eta, 0\rangle\rangle \mid \xi < \eta < \alpha\right\} \cup \left\{\langle\langle\xi, 0\rangle, \langle\eta, 1\rangle\rangle \mid \xi < \eta < \beta\right\} \cup \left[(\alpha \times \{0\}) \times (\beta \times \{1\})\right].$$

Informally speaking, we construct a new ordinal $\alpha + \beta$ by first "placing" α is a line and then placing β after it linearly. This is best visualized when α and β are finite ordinals.

To see that R indeed gives $\alpha \times \{0\} \cup \beta \times \{1\}$ the structure of a well order, let S be a nonempty subset. If $S \cap \alpha \times \{0\}$ is nonempty, then the minimal element of S exists and is the minimal element of $S \cap \alpha \times \{0\}$. On the other hand, if $S \cap \alpha \times \{0\} = \varnothing$, the minimal element of S exists and is the minimal element of $S \cap \beta \times \{1\}$.

Lemma 2.20. *For ordinals α, β, γ,*

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(b) $\alpha + 0 = \alpha$.

(c) $\alpha + 1 = S(\alpha)$.

(d) $\alpha + S(\beta) = S(\alpha + \beta)$.

(e) If β is a limit ordinal, then $\alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}$.

Proof. We shall only prove (e) since the others are straightforward. First, note that $\alpha + \beta \geq \alpha + \xi$ for every $\xi < \beta$, which is easy to see by setting up an obvious order preserving injection. ∎

complete this argument

Remark 2.4.1. *One must note that ordinal addition is **not commutative**. Indeed,*

$$1 + \omega = \sup\{1 + n \mid n < \omega\} = \omega$$

while

$$\omega + 1 = S(\omega) \neq \omega$$

where the last "non-equality" follows from the axiom of foundation. Thus, $1 + \omega \ncong \omega + 1$.

Multiplication

Definition 2.21. If α, β are ordinals, define $\alpha \cdot \beta = \mathrm{type}(\beta \times \alpha, R)$ where R is the dictionary order, given by

$$R = \left\{ \langle\langle \xi, \eta \rangle, \langle \xi', \eta' \rangle\rangle \;\middle|\; \xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta') \right\}.$$

We must first check that R is indeed a well ordering. That it is a strict linear order is clear. Let $S \subseteq \beta \times \alpha$ be a nonempty subset. Let S_1 be the projection of S onto β. This has a minimum element, say ξ. Consider now the set of all $\eta \in \alpha$ such that $\langle \xi, \eta \rangle \in S$. This is a nonempty subset of α and thus has a minimum element, say δ. Then, $\langle \xi, \delta \rangle$ is a minimum element of S.

Lemma 2.22. *For ordinals α, β, γ,*

(a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(b) $\alpha \cdot 0 = 0$.

(c) $\alpha \cdot 1 = \alpha$.

(d) $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$.

(e) If β is a limit ordinal, then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$.

(f) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Proof. ∎

Proof of ordinal multiplication

Exponentiation

Definition 2.23. For ordinals α, β, we define α^β by recursion on β as

- $\alpha^0 = 1$.
- $\alpha^{\beta+1} = \alpha^\beta \cdot \beta$.
- If β is a limit ordinal, $\alpha^\beta = \sup\{\alpha^\xi \mid \xi < \beta\}$.

Remark 2.4.2. *Interestingly,*

$$2^\omega = \sup\{2^n \mid n < \omega\} = \omega.$$

2.5 Equivalent forms of the Axiom of Choice

Theorem 2.24 (Well Ordering Theorem). *For every nonempty set A, there is a relation $R \subseteq A \times A$ such that R well orders A.*

AC $\implies$ WO

Let A be a set. We shall explicitly construct a well ordering on X using the Axiom of Choice. First, let $f : \mathcal{P}(A)\backslash\{\varnothing\} \to A$ be a choice function and extend it to $f : \mathcal{P}(A) \to A \coprod \{\varnothing\}$ by defining $f(\varnothing) = \varnothing$. We shall now use transfinite recursion to define a function F on the ordinals as follows:

$$\begin{aligned} F(0) &:= f(A) \\ F(\alpha) &:= f\left(\{x \in A \mid \forall \beta \in \alpha (F(\beta) \neq x)\}\right). \end{aligned}$$

First, note that if $F(\alpha) = F(\beta) \neq \varnothing$, then $\alpha = \beta$. Next, we contend that there must be an ordinal α with $F(\alpha) = \varnothing$. For if not, then we may apply the axiom of replacement and that of comprehension to obtain a set of all ordinals, a contradiction to the Burali-Forti paradox.

Let $\mathbf{C}$ denote the class of all ordinals α with $F(\alpha) = \varnothing$. Due to Theorem 2.17, there is a minimal such ordinal, say α_0, then

$$f\left(\{x \in A \mid \forall \beta \in \alpha_0 (F(\beta) \neq x)\}\right) = \varnothing \implies \{x \in A \mid \forall \beta \in \alpha_0 (F(\beta) \neq x)\} = \varnothing.$$

Let $G : A \to \alpha_0$ denote the inverse function of F. Define the relation $R \subseteq A \times A$ by

$$R := \{\langle x, y \rangle \mid G(x) \in G(y)\}.$$

That this is a well ordering is easy to see.

WO $\implies$ AC

This direction, on the other hand, is much easier. Let X denote a collection of sets and let $Y = \bigcup X$. Let R be a well ordering on Y. Define the function $f : X \to Y$ by $f(x) = \min(x)$, the R-least element, which can be chosen since Y has been well ordered and $x \subseteq Y$.

AC $\implies$ Zorn

Let X be a set and $P = (X, \leqq)$ be a poset on it such that every chain in P has an upper bound. Let $f : \mathscr{P}(X)\backslash\{\varnothing\} \to X$ be a choice function.

Suppose P has no maximal element. Then, every chain in P must have a strict upper bound. Let $\mathscr{C}$ be the set of all chains in P. Let $g : \mathscr{C} \to \mathscr{P}(X)$ map a chain in P to the set of all *strict* upper bounds. Consequently, $g(C) \neq \varnothing$ for every chain C in P.

We shall define a class function $F : \mathbf{ON} \to X$ using transfinite recursion. Begin with $F(0) = F(X)$. Now, for any ordinal $\alpha \in \mathbf{ON}$, let C_α denote the chain $\{F(\beta) \mid \beta < \alpha\}$ and define

$$F(\alpha) := f(g(C_\alpha)).$$

It is not hard to see that $F(\alpha) = F(\beta)$ if and only if $\alpha = \beta$ whence we may use Replacement to obtain a *set* of all ordinals, which is absurd.

Zorn $\implies$ AC

Let X be a collection of sets and $Y = \bigcup X$. Let P be the poset of pairs (S, f) where $S \subseteq X$ and $f : S \to Y$ is a function with $f(s) \in s$ for each $s \in S$. We say $(S, f) \leqq (S', f')$ if $S \subseteq S'$ and $f' \restriction_S = f$.

Let $C = \{(S_\alpha, f_\alpha)\}$ be a chain in P. Define the function $f : S := \bigcup_\alpha S_\alpha \to Y$ by $f(x) := f_\alpha(x)$ if $x \in S_\alpha$. Then, (S, f) is an upper bound for the chain C. Thus, due to Zorn's Lemma, P contains a maximal element, say $(\widetilde{S}, F)$. We contend that $\widetilde{S} = X$. For if not, then there is $x \in X \backslash \widetilde{S}$ and the function F can be extended to $\widetilde{S} \cup \{x\}$ by simply choosing an element of x and assigning it to x under F. This contradicts the maximality of $(\widetilde{S}, F)$ and hence, F is the desired choice function.

Chapter 3

Cardinal Numbers

Definition 3.1. Sets A and B are said to be *equinumerous* if there is a bijection $f : A \to B$. This is denoted by $A \approx B$. On the other hand, if there is an injection $f : A \to B$, it is denoted by $A \preceq B$. We write $A \prec B$ if $A \preceq B$ and $B \not\preceq A$.

Theorem 3.2 (Cantor-Schröder-Bernstein). $A \preceq B \wedge B \preceq A \implies A \approx B$.

Definition 3.3. For a set A, $|A|$ is the least α such that $\alpha \approx A$. α is a *cardinal* if and only if $\alpha = |\alpha|$.

From Theorem 2.24, there is a well ordering R on A and thus an ordinal α with an order preserving bijection between $\langle A, R\rangle$ and α, in particular, $A \approx \alpha$. Thus, $|A|$ is defined for every set. Further, note that α is a cardinal if and only if $\forall \beta < \alpha (\beta \not\approx \alpha)$ and for any ordinal α, $|\alpha| \leq \alpha$.

Lemma 3.4. *If* $|\alpha| \leq \beta \leq \alpha$, *then* $|\beta| = |\alpha|$.

Proof. Since $\beta \leq \alpha$, we have $\beta \subseteq \alpha$ and thus $\beta \preceq \alpha$. On the other hand, $|\alpha| \subseteq \beta$. Composing this inclusion with the bijection $\alpha \approx |\alpha|$, we have $\alpha \preceq \beta$. We are done due to Theorem 3.2. ■

Lemma 3.5. *If* $n \in \omega$, *then*

(a) $n \not\approx n + 1$.

(b) $\forall \alpha (\alpha \approx n \implies \alpha = n)$.

Proof. (a) Suppose not. Pick the smallest $n \in \omega$ such that $n \approx n + 1$. Note that $n \neq 0$. We have an injective function $f : n + 1 \to n$. Composing appropriately, we may suppose that $f(n) = n - 1$ where $n \in n + 1$ and $n - 1 \in n$. The restriction $f \restriction_n$ is an injective function from n to $n - 1$ whence by Theorem 3.2, $n - 1 \approx n$, a contradiction.

(b) If $n < \alpha$, then $n + 1 \leq \alpha$ whence $n + 1 \preceq \alpha$. On the other hand, $\alpha \approx n < n + 1$, consequently $\alpha \approx n + 1$, a contradiction to (a).

Now suppose $\alpha < n$. Then, $|n| = |\alpha| \leq \alpha \leq \alpha + 1 \leq n$, consequently, $|\alpha + 1| = |n|$. But since $\alpha + 1 \approx n + 1$, we have $n + 1 \approx n$, a contradiction to (a). Thus $\alpha = n$. ■

Corollary 3.6. ω is a cardinal and so is every ordinal $n < \omega$.

Definition 3.7. A is *finite* if and only if $|A| < \omega$. A is *countable* if and only if $|A| \leq \omega$. We use the shorthand *infinite* to mean "not finite" and *uncountable* to mean "not countable".

Definition 3.8 (Cardinal Arithmetic). For cardinals κ and λ, define

$$\kappa \oplus \lambda := |\kappa \times \{0\} \cup \lambda \times \{1\}|, \quad \kappa \otimes \lambda := |\kappa \times \lambda|.$$

Unlike ordinal arithmetic, the operations $\oplus$ and $\otimes$ are commutative, which is obvious from the definition above. Furthermore, note that

$$|\kappa + \lambda| = |\lambda + \kappa| = \kappa \oplus \lambda \quad \text{and} \quad |\kappa \cdot \lambda| = |\lambda \cdot \kappa| = \kappa \otimes \lambda.$$

Lemma 3.9. *For $m, n \in \omega$, $n \oplus m = n + m < \omega$ and $n \otimes m = n \cdot m < \omega$.*

Proof. ■

Proposition 3.10. *Every infinite cardinal is a limit ordinal.*

Proof. Suppose $\kappa = \alpha + 1$ is a cardinal. Then, α is not a finite ordinal, that is, $\omega < \alpha$ and thus there is an ordinal β such that $\alpha = \omega + \beta$. Consequently, $1 + \alpha = 1 + \omega + \beta = \omega + \beta$ as we have seen previously that $1 + \omega = \omega$. Consequently,

$$|\kappa| = |\alpha + 1| = |1 + \alpha| = |\alpha|,$$

a contradiction to the fact that κ is a cardinal. ■

if $\alpha \leq \beta$ there is an ordinal δ such that $\beta = \alpha + \delta$.

Theorem 3.11 (Tarski). *If κ is an infinite cardinal, then $\kappa \otimes \kappa = \kappa$.*

Proof. We shall prove this statement by transfinite induction on κ. That this statement holds for $\kappa = \omega$ is well known. Suppose now that $\kappa > \omega$ and the statement holds for each cardinal $\lambda < \kappa$.

Note that for an infinite ordinal $\alpha < \kappa$, we have $|\alpha| < \kappa$ and thus

$$|\alpha \times \alpha| = |\alpha| \otimes |\alpha| = |\alpha| < \kappa.$$

Let $\prec$ denote the strict lexicographic ordering on $\kappa \times \kappa$. Define the relation $\trianglelefteq$ on $\kappa \times \kappa$ by $\langle \alpha, \beta \rangle \trianglelefteq \langle \gamma, \delta \rangle$ if and only if

$$\max\{\alpha, \beta\} < \max\{\gamma, \delta\} \text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \langle \alpha, \beta \rangle \prec \langle \gamma, \delta \rangle.$$

That this relation is an ordering is immediate from the definition. We shall now show that this is a well ordering. Let $S \subseteq \kappa \times \kappa$ be nonempty. Using Replacement, construct the set S' which consists of $\max\{\alpha, \beta\}$ for all $\langle \alpha, \beta \rangle \in S$. Since $S' \subseteq \kappa$, it contains a minimum element, say α_0. Using Comprehension, construct the set S'' consisting of all pairs $\langle \alpha, \beta \rangle$ such that $\max\{\alpha, \beta\} = \alpha_0$. Now, $S'' \subseteq \kappa \times \kappa$, and under the lexicographic order, it has a minimum element, which is also the minimum element of S under the ordering $\trianglelefteq$.

Given any $\langle \alpha, \beta \rangle \in \kappa \times \kappa$, the set of all pairs preceeding it in $\langle \kappa \times \kappa, \trianglelefteq \rangle$ is a subset of

$$(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)$$

Since κ is a limit ordinal, we have $\max\{\alpha, \beta\} + 1 < \kappa$ and due to the induction hypothesis, the cardinality of the above set is strictly smaller than κ whence $|\kappa \times \kappa| \leq \kappa$. There is an obvious injection from κ into $\kappa \times \kappa$, forcing $|\kappa \times \kappa| = \kappa$ due to Theorem 3.2. ■

Corollary 3.12. Let κ, λ be infinite cardinals. Then,

(a) $\kappa \oplus \lambda = \kappa \otimes \lambda = \max\{\kappa, \lambda\}$,

(b) $|\kappa^{<\omega}| = \kappa$.

Proof. ■

Theorem 3.13 (Cantor). $\forall X \left(X \prec \mathscr{P}(X)\right)$.

Proof. Suppose not, then $X \approx \mathscr{P}(X)$ for some X, which follows from Theorem 3.2 and the fact that there is a canonical injection from X to $\mathscr{P}(X)$. Let $f : X \to \mathscr{P}(X) \to X$ be a bijection. Using Comprehension, construct the set

$$S := \{x \in X \mid x \notin f(x)\} \subseteq X.$$

Let $s \in X$ be the unique element such that $f(s) = S$. Then,

$$s \in S \iff s \notin S,$$

a contradiction. ■

Theorem 3.14. $\forall \alpha \exists \kappa \left(\kappa > \alpha \text{ is a cardinal}\right)$ *is true in ZF.*

If we were to work in ZFC then we could just well order $\mathscr{P}(\alpha)$ and consider its cardinality.

Proof. The statement is obvious for finite cardinals. Suppose now that $\alpha \geq \omega$. Let

$$W := \{R \in \mathscr{P}(\alpha \times \alpha) \mid R \text{ well orders } \alpha\} S := \{\text{type}(\langle \alpha, R \rangle) \mid R \in W\}.$$

Let $\beta = \sup(S)$. We contend that β is a cardinal and $\beta > \alpha$. First, note that if $\delta \in W$, then $S(\delta) \in W$, consequently, $\beta \notin W$. Further, $\beta \not\approx \alpha$ lest one could find a well ordering on α which is in order preserving bijection with β. Suppose β were not a cardinal. Then, there is some $\gamma < \beta$ with $\gamma \approx \beta$. By definition, there is η such that $\gamma \leq \eta < \beta$ with $\eta \in W$, consequently, $\eta \approx \beta$ but $\alpha \approx \eta$, a contradiction. This completes the proof. ■

Definition 3.15 (Successor, Limit Cardinals). Let α be an ordinal. Denote by α^+ the smallest *cardinal* strictly greater than α. A cardinal κ is said to be a *successor cardinal* if $\kappa = \alpha^+$ for some α. On the other hand, if $\kappa > \omega$ and is not a successor cardinal, then κ is said to be a *limit cardinal*.

Definition 3.16 (Aleph Numbers). Define the numbers $\aleph_\alpha$ by transfinite recursion on α.

(a) $\aleph_0 := \omega$.

(b) $\aleph_{\alpha+1} = (\aleph_\alpha)^+$.

(c) For a limit ordinal λ, define $\aleph_\lambda := \sup\{\aleph_\alpha \mid \alpha < \lambda\}$.

Theorem 3.17. *(a) Each $\aleph_\alpha$ is a cardinal.*

(b) Every infinite cardinal is equal to $\aleph_\alpha$ for some α.

(c) If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.

(d) $\aleph_\alpha$ is a limit cardinal ifa nd only if α is a limit ordinal.

(e) $\aleph_\alpha$ *is a successor cardinal if and only if* α *is a successor ordinal.*

Proof. All of these follow immediately from the definition above. ∎

Remark 3.0.1. *One often writes* ω_α *in place of* $\aleph_\alpha$*. We adopt both conventions and use them interchangeably.*

Lemma 3.18. *If there is a surjective function* $f : X \to Y$*, then* $|Y| \leq |X|$*.*

Proof. Consider the set
$$S = \{f^{-1}(y) \mid y \in Y\},$$
which can be constructed using Replacement. Let $g : Y \to S$ be given by $g(y) = f^{-1}(y)$ and F be a choice function on S. Then, the composition $F \circ g$ is an injective function from Y to X, implying the desired conclusion. ∎

Definition 3.19 (Cardinal Exponentiation). For sets A and B, define
$$A^B := {}^B A := \{f \subseteq \mathscr{P}(B \times A) \mid f \text{ is a function}\}.$$
For cardinals κ and λ, define $\kappa^\lambda := |{}^\lambda\kappa|$.

Theorem 3.20. *Let* $2 \leq \kappa \leq \lambda$ *and* λ *an infinite cardinal. Then,* $\kappa^\lambda = 2^\lambda$*.*

Proof. Obviously, ${}^\lambda 2 \approx \mathscr{P}(\lambda)$ which can be seen by looking at the characteristic function of each subset of λ. Then, we have
$${}^\lambda k \preceq {}^\lambda\lambda \preceq \mathscr{P}(\lambda \times \lambda) \preceq \mathscr{P}(\lambda) \preceq {}^\lambda 2.$$
The conclusion follows from Theorem 3.2. ∎

Theorem 3.21. *Let* $\mathscr{B}(\mathbb{R})$ *denote the Borel* σ*-algebra on* $\mathbb{R}$ *with the standard topology. Then,* $|\mathscr{B}(\mathbb{R})| = 2^{\aleph_0}$*, the cardinality of the continuum.*

Proof. That $\mathscr{B}(\mathbb{R})$ has cardinality at least that of the continuum is straightforward since it contains all singletons. Showing the reverse direction is a bit involved and requires transfinite recursion.

First, note that $\mathbb{R}$ is second countable and thus has a countable abse for its topology, denote this by S_0. For an ordinal $\alpha < \omega_1$, let $S_{\alpha+1}$ denote the collection of all unions of the form
$$\bigcup_i A_i \cup \bigcup_j (\mathbb{R} \backslash B_j)$$
where A_i and B_j are chosen from S_α. Note that if $|S_\alpha| \leq 2^{\aleph_0}$, then the number of these unions that can be formed is at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ since there is a surjection from the set of all functions $\aleph_0 \to S_\alpha$ onto $S_{\alpha+1}$.

On the other hand, if α is a limit ordinal, define
$$S_\alpha = \bigcup_{\lambda < \alpha} S_\lambda.$$

We contend that $S = \bigcup_{\alpha < \omega_1} S_\alpha$ is a σ-algebra. Obviously, S contains $\varnothing$ and $\mathbb{R}$, and is closed under complementation. Let $\{A_n\}_{n=1}^\infty$ be a sequence in S. For each positive integer n, let $\alpha(n)$ denote the minimal ordinal λ such that $A_n \in S_\lambda$. Note that for each n, the cardinality $|\alpha(n)| \leq \omega$. Hence, if $\beta = \sup_{n<\omega} \alpha(n)$, then $|\beta| \leq \omega$, consequently, $\beta < \omega_1$ and $\{A_n \mid n < \omega\} \subseteq S_\beta$ implying that $\bigcup_{n=1}^\infty A_n \in S_{\beta+1} \subseteq S$.

As a result, S contains $\mathscr{B}(\mathbb{R})$ but the cardinality of S is at most
$$|\omega_1| \otimes 2^{\aleph_0} = \aleph_1 \otimes 2^{\aleph_0} \leq 2^{\aleph_0} \otimes 2^{\aleph_0} = 2^{\aleph_0}.$$
This completes the proof. ∎

Theorem 3.22 (Mazurkiewicz, 1914). *There is a subset $A \subseteq \mathbb{R}^2$ which meets* every *line in the plane at* exactly 2 *points.*

Proof. Let $\mathscr{L}$ denote the set of all possible lines in the plane. The cardinality of $\mathscr{L}$ is at least $2^{\aleph_0}$ and atmost $2^{\aleph_0} \otimes 2^{\aleph_0} = 2^{\aleph_0}$. Since this is in bijection with $2^{\aleph_0}$, it has an induced well ordering, which we denote by $\mathscr{L} = \{L_\alpha \mid \alpha < 2^{\aleph_0}\}$.

We shall, using transfinite recursion construct a chain X_α of subsets of $\mathbb{R}^2$ for $\alpha < 2^{\aleph_0}$ such that $|X_\alpha| < 2^{\aleph_0}$ and $|X_\alpha \cap L_\beta| \leq 2$ for each $\beta < 2^{\aleph_0}$.

Begin with $X_0 = \{x_0\}$ for any $x_0 \in \mathbb{R}^2$. Suppose now that the sequence has been constructed for each $\beta < \alpha$ where $\alpha > 0$. Let $Y_\alpha := \bigcup_{\beta<\alpha} X_\beta$. Let S_α denote the set of all lines between two points in Y_α. Note that the cardinality of S_α is strictly smaller than $2^{\aleph_0}$.

Let γ be the smallest ordinal such that $|L_\gamma \cap Y_\alpha| \leq 1$. If no such ordinal exists, then Y_α is the desired set. Suppose such a γ does exist. Then, the set

$$L_\gamma \setminus \underbrace{\left(\bigcup_{L \in S_\alpha} L \cup \bigcup_{\beta<\gamma} L_\beta \cup Y_\alpha \right)}_{T}$$

which is non empty, since the intersection of L_γ with T has cardinality strictly smaller than $2^{\aleph_0}$. Let x_α be one such element in the above set and define $X_\alpha = Y_\alpha \cup \{x_\alpha\}$.

It is not hard to see that X_α satisfies the desired properties and thus we may continue this procedure and obtain $\{X_\alpha \mid \alpha < 2^{\aleph_0}\}$. Let $X = \bigcup_{\alpha<2^{\aleph}_0} X_\alpha$. This is the required set. ∎

Chapter 4

Well Founded Sets

Throughout this chapter, we shall work in ZF$^-$, which is ZF without the axiom of foundation.

Definition 4.1. By transfinite recursion, deifne $R(\alpha)$ for each $\alpha \in \mathbf{ON}$ by

(a) $R(0) = \varnothing$,

(b) $R(\alpha + 1) = \mathscr{P}(R(\alpha))$,

(c) $R(\alpha) = \bigcup_{\lambda < \alpha} R(\lambda)$ when λ is a limit ordinal.

Finally, define the first order formula

$$\mathbf{WF}(x) := \exists \alpha (x \in R(\alpha)).$$

We denote by **WF** the class corresponding to the above formula.

Lemma 4.2. *For each α,*

1. *$R(\alpha)$ is transitive.*
2. *$\forall \xi \leq \alpha (R(\xi) \subseteq R(\alpha))$.*

Proof. We prove both statements by transfinite induction on α. The base case with $\alpha = 0$ is trivial. Suppose $\alpha = \beta + 1$. Since $R(\beta)$ is transitive, so is its power set as we have seen earlier and obviously $R(\beta) \subseteq R(\alpha)$ since $R(\beta) \in R(\alpha)$. Finally, suppose α is a limit ordinal. Then, (b) is immediate and (a) follows from the fact taht the union of transitive sets is transitive. ∎

Remark 4.0.1. *As a consequence of the definition of* **WF**, *for any $x \in$* **WF**, *the least α for which $x \in R(\alpha)$ must be a successor ordinal.*

Definition 4.3. If $x \in$ **WF**, then $\operatorname{rank}(\alpha)$ is the *least* β such tdhat $x \in R(\beta + 1)$.

Lemma 4.4. *For any α,*

$$R(\alpha) = \{x \in \mathbf{WF} \mid \operatorname{rank}(x) < \alpha\}.$$

Proof. Trivial. ∎

Lemma 4.5. *If $y \in$* **WF**, *then*

(a) $\forall x \in y (x \in \mathbf{WF} \wedge \operatorname{rank}(x) < \operatorname{rank}(y))$, *and*

(b) $\operatorname{rank}(y) = \sup\{\operatorname{rank}(x) + 1 \mid x \in y\}$.

Proof. Let $\alpha = \operatorname{rank}(y)$. Then, $y \in R(\alpha + 1) = \mathscr{P}(R(\alpha))$ and thus $y \subseteq \mathscr{R}(\alpha)$, consequently, $x \in R(\alpha)$ and $\operatorname{rank}(x) < \alpha$.

As for the second part, let $\alpha = \sup\{\operatorname{rank}(x) + 1 \mid x \in y\}$. From (a), we know that $\alpha \leq \operatorname{rank}(y)$. Further, each $x \in y$ has rank $< \alpha$ and thus $y \subseteq R(\alpha)$ whence $y \in R(\alpha + 1)$, consequently, $\operatorname{rank}(y) \leq \alpha$. ■

Corollary 4.6. There is no $x \in \mathbf{WF}$ such that $x \in x$.

Proof. If this were true, then $\operatorname{rank}(x) < \operatorname{rank}(x)$, a contradiction. ■

Lemma 4.7. *(a)* $\forall \alpha \in \mathbf{ON}(\alpha \in \mathbf{WF} \wedge \operatorname{rank}(\alpha) = \alpha)$.

(b) $\forall \alpha \in \mathbf{ON}(R(\alpha) \cap \mathbf{ON} = \alpha)$.

Proof. We shall prove (a) using transfinite induction on α. That (a) holds for $\alpha = 0$ is trivial. Now suppose (a) holds for each $\beta < \alpha$. Then, we have

$$\operatorname{rank}(\alpha) = \sup\{\operatorname{rank}(\beta) + 1 \mid \beta < \alpha\} = \sup\{\beta \mid \beta < \alpha\} = \alpha$$

which proves (a). It is easy to see that (b) is immediate from (a). ■

Lemma 4.8. $\forall x(x \in \mathbf{WF} \iff x \subseteq \mathbf{WF})$.

Proof. The forward direction follows from the transitivity of **WF**. As for the reverse direction, let $x \subseteq \mathbf{WF}$ and let

$$\alpha = \sup\{\operatorname{rank}(y) + 1 \mid y \in x\}.$$

Then, $x \subseteq R(\alpha)$, consequently, $x \in R(\alpha + 1)$. ■

Lemma 4.9. *(a)* $\forall n \in \omega(|R(n)| < \omega)$.

(b) $|R(\omega)| = \omega$.

Proof. (a) is immediate from induction on n. Obviously, $\omega \subseteq R(\omega)$. On the other hand, note that $R(\omega)$ is a countable union of countable sets and is thus countable. ■

4.1 Well Founded Relations

Definition 4.10. A relation R is *well-founded* on a set A if

$$\forall X \subseteq A \left[X \neq \varnothing \implies \exists y \in X \left(\neg \exists z \in X \left(zRy \right) \right) \right].$$

For example, if $\langle A, R \rangle$ is a well-ordering, then R is well-founded on A.

Lemma 4.11. *If $A \in \mathbf{WF}$, then $\in$ is well-founded on A.*

Proof. Let $X \subseteq A$ be nonempty and $\alpha = \min\{\operatorname{rank}(y) \mid y \in X\}$. Choose some $y \in X$ with $\operatorname{rank}(y) = \alpha$. Then y is $\in$-minimal in X. ■

Lemma 4.12. *If A is transitive and $\in$ is well-founded on A, then $A \in \mathbf{WF}$.*

Proof. Suppose not. Then equivalently, $A \not\subseteq \mathbf{WF}$, whence $A \setminus \mathbf{WF}$ is nonempty. Let $y \in A \setminus \mathbf{WF}$ be the $\in$-least element of $A \setminus \mathbf{WF}$. If $z \in y$, then $z \in A$ due to the transitivity of A but on the other hand, $z \notin A \setminus \mathbf{WF}$ lest one contradicts the minimality of y. Therefore, $z \in \mathbf{WF}$. Consequently, $y \subseteq \mathbf{WF}$ whence $y \in \mathbf{WF}$, a contradiction. ■

Definition 4.13. Let

$$\bigcup^0 A = A,$$

and for each $0 < n < \omega$, define, recursively,

$$\bigcup^{n+1} A = \bigcup \left(\bigcup^n A \right).$$

Finally, set

$$\operatorname{trcl}(A) := \bigcup \left\{ \bigcup^n A \mid n \in \omega \right\}.$$

Lemma 4.14. *Let A be a set. Then,*

(a) $A \subseteq \operatorname{trcl}(A)$.

(b) $\operatorname{trcl}(A)$ *is transitive.*

(c) If $A \subseteq T$, and T is transitive, then $\operatorname{trcl}(A) \subseteq T$.

(d) If A is transitive, then $\operatorname{trcl}(A) = A$.

(e) $\operatorname{trcl}(A) = A \cup \bigcup \{\operatorname{trcl}(x) \mid x \in A\}$.

Proof. (a) Trivial.

(b) If $x \in \operatorname{trcl}(A)$, then there is some n such that $x \in \bigcup^n A$, therefore $x \subseteq \bigcup^{n+1} A$, whence $x \subseteq \operatorname{trcl}(A)$. Thus $\operatorname{trcl}(A)$ is transitive.

(c) We shall show by induction on n that $\bigcup^n A \subseteq T$. The base case is given to begin with. Suppose $\bigcup^n A \subseteq T$. Then, due to transitivity, $\bigcup^{n+1} A = \bigcup \left(\bigcup^n A \right) \subseteq T$. The conclusion follows.

(d) Follows from (a) and (c) by taking $T = A$.

(e) First, note that if $x \in A$, then $x \in \operatorname{trcl}(A)$ and due to transitivity $x \subseteq \operatorname{trcl}(A)$. From (c), we have $\operatorname{trcl}(x) \subseteq \operatorname{trcl}(A)$. Let

$$T = A \cup \bigcup \{\operatorname{trcl}(x) \mid x \in A\}.$$

Then, it is easy to see that T must be transitive and from what we concluded earlier, $T \subseteq \operatorname{trcl}(A)$ but since $A \subseteq T$, from (c), we must have $\operatorname{trcl}(A) \subseteq T$ whence $\operatorname{trcl}(A) = T$. ■

Theorem 4.15. *For any set A, the following are equivalent:*

(a) $A \in \mathbf{WF}$.

(b) $\operatorname{trcl}(A) \in \mathbf{WF}$.

(c) $\in$ *is well-founded on* $\operatorname{trcl}(A)$.

Proof. ■

4.2 The Axiom of foundation

Recall the axiom of foundation

$$\forall x \left(x \neq \varnothing \implies \exists y \left(y \in x \wedge \neg \exists z \left(z \in x \wedge z \in y \right) \right) \right).$$

or equivalently,

$$\forall x \left(x \neq \varnothing \implies \exists y \left(y \in x \wedge y \cap x = \varnothing \right) \right).$$

Theorem 4.16. *The following are equivalent:*

(a) the Axiom of Foundation.

(b) $\forall A (\in$ is well-founded on $A)$

(c) $\mathbf{V} = \mathbf{WF}$.

Proof. That (a) and (b) are equivalent is immediate from the definition of well-foundedness. Let $A \in \mathbf{V}$. Then, $\in$ is well founded on A and thus on $\mathrm{trcl}(A)$, consequently, $A \in \mathbf{WF}$ and thus $\mathbf{V} = \mathbf{WF}$. The converse is trivial. ■

Add picture of the universe

4.3 Induction and Recursion on Well-Founded Relations

We extend the notion of well-foundedness to classes as follows.

Definition 4.17. $\mathbf{R}$ is well founded on $\mathbf{A}$ if and only if

$$\forall X \subseteq \mathbf{A} \left[X \neq \varnothing \implies \exists y \in X \left(\neg \exists z \in X (z \mathbf{R} y) \right) \right].$$

Definition 4.18. $\mathbf{R}$ is *set-like* on $\mathbf{A}$ if for all $x \in \mathbf{A}$, $\{y \in \mathbf{A} \mid y\mathbf{R}x\}$ is a set. If $\mathbf{R}$ is set-like on $\mathbf{A}$, then

(a) $\mathrm{pred}(\mathbf{A}, x, \mathbf{R}) = \{y \in \mathbf{A} \mid y\mathbf{R}x\}$.

(b) $\mathrm{pred}^0(\mathbf{A}, x, \mathbf{R}) = \mathrm{pred}(\mathbf{A}, x, \mathbf{R})$.

(c) $\mathrm{pred}^{n+1}(\mathbf{A}, x, \mathbf{R}) = \bigcup \{\mathrm{pred}(\mathbf{A}, y, \mathbf{R}) \mid y \in \mathrm{pred}^n(\mathbf{A}, x, \mathbf{R})\}$.

(d) $\mathrm{cl}(\mathbf{A}, x, \mathbf{R}) = \bigcup \{\mathrm{pred}^n(\mathbf{A}, x, \mathbf{R}) \mid n \in \omega\}$.

Lemma 4.19. *If $\mathbf{R}$ is set-like on A and $x \in \mathbf{A}$, then for each $y \in \mathrm{cl}(\mathbf{A}, x, \mathbf{R})$, $\mathrm{pred}(\mathbf{A}, y, \mathbf{R}) \subseteq \mathrm{cl}(\mathbf{A}, x, \mathbf{R})$.*

Proof. There is some nonnegative integer n such that $y \in \mathrm{pred}^n(\mathbf{A}, y, \mathbf{R})$. Then,

$$\mathrm{pred}(\mathbf{A}, y, \mathbf{R}) \subseteq \mathrm{pred}^{n+1}(\mathbf{A}, x, \mathbf{R}).$$

The conclusion follows. ■

Theorem 4.20. *If $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$, then every non-empty subclass $\mathbf{X}$ of $\mathbf{A}$ has an $\mathbf{R}$-minimal element.*

Proof. Pick some $x \in \mathbf{X}$. If this $\mathbf{R}$-minimal, then we are done. If not, then consider $\mathbf{X} \cap \mathrm{cl}(\mathbf{A}, x, \mathbf{R})$ is a nonempty *subset* of A, since $\mathrm{cl}(\mathbf{A}, x, \mathbf{R})$ is a set. This means that it has an $\mathbf{R}$-minimal element, say y. From the previous lemma, y must be $\mathbf{R}$-minimal in $\mathbf{X}$. ■

Remark 4.3.1. *Notice the similarity of the above with Theorem 2.17. This in particular means that we can apply transfinite induction on well-founded set-like relations.*

Theorem 4.21 (Well-Founded Transfinite Recursion). *Assume* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$. *If* $\mathbf{F} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$, *then there is a unique* $\mathbf{G} : \mathbf{A} \to \mathbf{V}$ *such that*

$$\forall x \in \mathbf{A} \left[\mathbf{G}(x) = \mathbf{F}\left(x, \mathbf{G} \restriction \mathrm{pred}(\mathbf{A}, x, \mathbf{R})\right) \right].$$

Definition 4.22. If $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$, define

$$\mathrm{rank}(x, \mathbf{A}, \mathbf{R}) = \sup\{\mathrm{rank}(y, \mathbf{A}, \mathbf{R}) + 1 \mid y\mathbf{R}x \wedge y \in \mathbf{A}\}.$$

Definition 4.23. Let $\mathbf{R}$ be well-founded and set-like on $\mathbf{A}$. Define the *Mostowski collapsing function*, $\mathbf{G}$ of $\mathbf{A}, \mathbf{R}$ by

$$\mathbf{G}(x) = \{\mathbf{G}(y) \mid y \in \mathbf{A} \wedge y\mathbf{R}x\}.$$

The *Mostowski collapse*, $\mathbf{M}$ of $\mathbf{A}, \mathbf{R}$ is defined to be the range of $\mathbf{G}$.

Definition 4.24. $\mathbf{R}$ is said to be *extensional* on $\mathbf{A}$ if

$$\forall x, y \in \mathbf{A} \left(\forall z \in \mathbf{A} \left(z\mathbf{R}x \iff z\mathbf{R}y \right) \implies x = y \right).$$

Informally, this is equivalent to saying that the Axiom of Extensionality is true in $\mathbf{A}$ if $\in$ is interpreted as $\mathbf{R}$.

Theorem 4.25 (Mostowski Collapsing Theorem). *Suppose* $\mathbf{R}$ *is well-founded, set-like, and extensional on* $\mathbf{A}$, *then there is a transitive class* $\mathbf{M}$ *and a bijective map* $\mathbf{G} : \mathbf{A} \to \mathbf{R}$ *such that* $\mathbf{G}$ *is an isomorphism between* $(\mathbf{A}, \mathbf{R})$ *and* $(\mathbf{M}, \in)$. *Furthermore,* $\mathbf{M}$ *and* $\mathbf{G}$ *are unique.*

Bibliography

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