MA 408: Measure Theory

Swayam Chube

April 11, 2023

Contents

Lebesgue Theory

Chapter 1

Lebesgue Measure

1.1 Outer Measure

The Lebesgue Outer Measure, unlike the Lebesgue Measure is defined for every subset of $\mathbb{R}$. We construct the Lebesgue Measure from the outer measure by restricting it to a class of special subsets.

Definition 1.1 (Interval). An unbounded interval is of one of the following forms:

$$(-\infty, a],\ (-\infty, a),\ [a, \infty),\ (a, \infty)$$

while a bounded interval is of one of the following forms:

$$[a, b],\ (a, b],\ [a, b),\ (a, b)$$

For an unbounded interval, we define $\ell(I) = \infty$ while for a bounded interval, we define $\ell(I) = b - a$.

Definition 1.2 (Outer Measure). Let $A \subseteq \mathbb{R}$. Consider the countable collections $\{I_k\}_{k\in\mathbb{N}}$ of nonempty open bounded intervals that cover A, that is, $A \subseteq \bigcup\limits_{k\in\mathbb{N}} I_k$ and define

$$m^*(A) = \inf\left\{\sum_{k\in\mathbb{N}} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k\in\mathbb{N}} I_k\right\}$$

It is obvious that $m^*(\varnothing) = 0$. Moreover, for sets $A \subseteq B$, the set of covers of B is a subset of covers of A, consequently, $m^*(A) \le m^*(B)$. Further, let $A = \{a_1, a_2, \ldots\}$ be a countable set. Consider the cover $\{I_k\}_{k\in\mathbb{N}}$ where $I_k = \left(a_k - \frac{\varepsilon}{2^{k+1}}, a_k + \frac{\varepsilon}{2^{k+1}}\right)$. Then,

$$0 \le m^*(A) \le \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^k} = \varepsilon$$

Since the above holds for all $\varepsilon > 0$, we conclude that $m^*(A) = 0$.

Proposition 1.3. *Let I be an interval. Then $m^*(I) = \ell(I)$.*

Proof. Let $I = [a, b]$. We shall first show that $m^*(I) = b - a$. It is easy to see that $m^*(I) \le b - a$. Let $\{I_k\}_{k\in\mathbb{N}}$ be a collection of open intervals covering $[a, b]$. Since I is compact, there is a finite subcover, say $\{I_k\}_{k=1}^{n}$.

We shall show that

$$\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) > b - a$$

which would imply the desired conclusion.

Since $a \in I$, there is an open interval, say, $I_1 = (a_1, b_1)$ containing a. If $b_1 > b$, stop here and set $N = 1$. Else, choose an interval, say $I_2 = (a_2, b_2)$ containing b_1. If $b_2 > b$, stop here and set $N = 2$ and so on. This process must terminate since n is finite, consequently, $N \leq n$.

Notice that due to the choice of the a_i and b_i's, $a_{k+1} < b_k$ for all $k \leq N-1$ and $b_N > b$. Then,

$$\sum_{k=1}^{n} \ell(I_k) \geq \sum_{k=1}^{N} \ell(I_k) = (b_N - a_1) + \sum_{k=1}^{N-1} (a_{k+1} - b_k) > b - a$$

Now, let I be any bounded interval. Obviously there exist closed bounded intervals J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2 \quad \text{and} \quad \ell(J_1) + \varepsilon = \ell(I) = \ell(J_2) - \varepsilon$$

Then, we have

$$\ell(I) - \varepsilon = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = \ell(I) + \varepsilon$$

Since the above inequality holds for all $\varepsilon > 0$, we have the desired conclusion.

The case when I is unbounded is easy enough. ■

The following follows readily from the definition.

Proposition 1.4. *The outer measure m^* is translation invariant. That is, for any set $A \subseteq \mathbb{R}$ and $y \in \mathbb{R}$,*

$$m^*(A + y) = m^*(A)$$

Proposition 1.5. *The outer measure m^* is countably subadditive, that is, if $\{E_k\}_{k \in \mathbb{N}}$ is a countable collection of sets, disjoint or not, then*

$$m^*\left(\bigcup_{k \in \mathbb{N}} E_k\right) \leq \sum_{k \in \mathbb{N}} m^*(E_k)$$

Proof. If any of the E_k's has infinite outer measure, then the inequality follows trivially. Henceforth suppose that all E_k have finite outer measure. Let $\varepsilon > 0$ be given. By definition, for every $k \in \mathbb{N}$ there is a collection of open intervals $\{I_{k,i}\}_{i \in \mathbb{N}}$ such that

$$E_k \subseteq \bigcup_{i \in \mathbb{N}} I_{k,i} \quad \text{and} \quad \sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Now, note that $\{I_{k,i}\}_{k,i \in \mathbb{N}}$ is an open cover for $\bigcup_{k \in \mathbb{N}} E_k$, consequently, we have

$$m^*\left(\bigcup_{k \in \mathbb{N}} E_k\right) \leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{k,i}) < \varepsilon + \sum_{k=1}^{\infty} m^*(E_k)$$

Since the above inequality holds for all $\varepsilon > 0$, we have the desired conclusion. ■

As a corollary, we have *finite subadditivity*.

Corollary. Let $\{E_k\}_{k=1}^n$ be a finite collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^n E_k\right) \leq \sum_{k=1}^n m^*(E_k)$$

Proof. Let $E_k = \varnothing$ for all $k > n$ in Proposition 1.5. ■

Theorem 1.6. *The Lebesgue Outer Measure m^* is a metric outer measure. That is, if A and B are bounded subsets of $\mathbb{R}$ that are positively separated, then $m^*(A \cup B) = m^*(A) + m^*(B)$.*

In order to prove the above theorem, we require the following lemmas:

Lemma 1.7. *Let I be an open bounded interval and $\varepsilon, \delta > 0$. Then, there is a finite collection of open bounded intervals $\{I_k\}_{k=1}^\infty$ such that $\operatorname{diam} I_k < \delta$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^\infty \ell(I_k) < \ell(I) + \varepsilon$.*

Lemma 1.8. *Let $E \subseteq \mathbb{R}$ be bounded. Given $\varepsilon, \delta > 0$, there is an open cover of E by bounded intervals $\{I_k\}_{k=1}^\infty$ with $\operatorname{diam} I_k < \delta$ for all $k \in \mathbb{N}$ and*

$$\sum_{k=1}^\infty \ell(I_k) < m^*(E) + \varepsilon$$

The following proof is a sketch.

Proof. Let $\{I_k\}_{k=1}^\infty$ be an open cover by bounded intervals of E such that

$$\sum_{k=1}^\infty \ell(I_k) < m^*(E) + \varepsilon/2$$

For each I_k, construct a finite open cover by bounded intervals $\{J_l\}_{l=1}^N$ for it with intervals of diameter less than δ such that

$$\sum_{l=1}^N \ell(J_l) < \ell(I_k) + \varepsilon/2^{k+1}$$

Then the union of all such filterations is countable and has total length not exceeding $m^*(E) + \varepsilon$. ■

Proof of Theorem 1.6. Let $\alpha = d(A, B)$, $\varepsilon > 0$ and $\delta = \alpha/3$ and let $\{I_k\}_{k=1}^\infty$ be an open cover by bounded intervals such that $\operatorname{diam} I_k < \delta$ and $\sum_{k=1}^\infty \ell(I_k) < m^*(A \cup B) + \varepsilon$. Let $\mathscr{A}$ be the subcollection of intervals that intersect A and similarly define $\mathscr{B}$. It is not hard to show that $\mathscr{A} \cap \mathscr{B} = \varnothing$, whence $\mathscr{A} \cup \mathscr{B}$ is an open cover by bounded intervals of $A \cup B$. Then,

$$m^*(A) + m^*(B) \leq \sum_{I \in \mathscr{A}} \ell(I) + \sum_{I \in \mathscr{B}} \ell(I) < \sum_{k=1}^\infty \ell(I_k) < m^*(A \cup B) + \varepsilon$$

since the above inequality holds for all $\varepsilon > 0$, we have $m^*(A) + m^*(B) \leq m^*(A \cup B)$ from which the conclusion follows. ■

1.2 Constructing the σ-Algebra

Definition 1.9 (Measurable Sets). A set E is said to be *measurable* if for every $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$$

Remark. Since Proposition 1.5 guarantees that $m^*(A) \leq m^*(A \cap E) + m^*(A \backslash E)$, it suffices to show that $m^*(A) \geq m^*(A \cap E) + m^*(A \backslash E)$.

Proposition 1.10. *Any set of outer measure* 0 *is measurable. In particular, any countable set is measurable.*

Proof. Let $E \subseteq \mathbb{R}$ have outer measure 0 and $A \subseteq \mathbb{R}$. Since $A \cap E \subseteq E$, using the monotonicity of m^*, we have that $m^*(A \cap E) = 0$. Now, using Proposition 1.5,

$$m^*(A \backslash E) \leq m^*(A) \leq m^*(A \cap E) + m^*(A \backslash E) = m^*(A \backslash E)$$

This shows that $m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$ and that E is measurable. ■

Proposition 1.11. *The union of a finite collection of measurable sets is measurable.*

Proof. It suffices to show that the union of two measurable sets is measurable since the general result follows from induction. Let E_1 and E_2 be measurable sets. Then, we have

$$\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \\
&= m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2^c) + m^*(A \cap E_1^c \cap E_2^c) \\
&\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)
\end{aligned}$$

This coupled with $m^*(A) \leq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$ implies the desired conclusion. ■

Proposition 1.12. *Let A be any set and $\{E_k\}_{k=1}^n$ a finite* ***disjoint*** *collection of measurable sets. Then*

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

Proof. The equality obviously holds for $n = 1$. Let E_1 and E_2 be disjoint measurable sets. Then,

$$\begin{aligned}
m^*(A \cap (E_1 \cup E_2)) &= m^*(A) - m^*(A \cap E_1^c \cap E_2^c) \\
&= m^*(A) - (m^*(A \cap E_1^c) - m^*(A \cap E_1^c \cap E_2)) \\
&= m^*(A) - m^*(A \cap E_1^c) + m^*(A \cap E_2) \\
&= m^*(A \cap E_1) + m^*(A \cap E_2)
\end{aligned}$$

We now proceed by induction on n. Note that $\bigcup\limits_{k=1}^{n-1} E_k$ and E_n are disjoint. Consequently

$$\begin{aligned}
m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) &= m^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_k\right]\right) + m^*(A \cap E_n) \\
&= \sum_{k=1}^{n-1} m^*(A \cap E_k) + m^*(A \cap E_n)
\end{aligned}$$

This completes the proof. ■

We readily obtain the following by setting $A = \mathbb{R}$ in the above proposition.

Corollary. Let $\{E_k\}_{k=1}^n$ be a finite dsjoint collection of measurable sets. Then

$$m^*\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n m^*(E_k)$$

Definition 1.13 (Algebra). Let X be a set. A collection $\mathcal{F}$ of subsets of X is called an *algebra* if

(a) $X \in \mathcal{F}$

(b) If $A \in \mathcal{F}$, then $X \backslash A \in \mathcal{F}$

(c) Let $n \in \mathbb{N}$ and $\{A_i\}_{i=1}^n$ be such that $A_i \in \mathcal{F}$. Then $\bigcup\limits_{i=1}^n A_i \in \mathcal{F}$

From the previous proposition, we can infer that the measurable sets form an *algebra*.

Proposition 1.14. *A countable union of mesurable sets is measurable.*

Proof. Let E be a countable union of measurable sets, $\{E_k\}_{k\in\mathbb{N}}$. Define

$$E_k' = E_k \backslash \bigcup_{i=1}^k E_i$$

It is not hard to see that $\{E_k'\}$ is a collection of disjoint measurable sets whose union is E. Consequently, without loss of generality, we may suppose that E_k are disjoint.

Define $F_n = \bigcup\limits_{i=1}^n E_i$. Since each F_n is measurable and $F_1 \subseteq F_2 \subseteq \cdots$, we have, for any $A \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap F_n^c) \\ &\geq m^*(A \cap F_n) + m^*(A \cap E^c) \\ &= \sum_{k=1}^n m^*(A \cap E_k) + m^*(A \cap E^c) \end{aligned}$$

Since the above inequality holds for all $n \in \mathbb{N}$, we must have that

$$m^*(A) \geq \sum_{k=1}^\infty m^*(A \cap E_k) + m^*(A \cap E^c) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

and hence E is measurable. ∎

Definition 1.15 (σ-Algebra). Let X be a set. A collection $\mathcal{F}$ of subsets of X is called a *σ-algebra* if

(a) $X \in \mathcal{F}$

(b) If $A \in \mathcal{F}$, then $X \backslash A \in \mathcal{F}$

(c) If $\{A_i\}_{i=1}^\infty$ is a sequence of sets in $\mathcal{F}$, then $\bigcup\limits_{i=1}^\infty A_i \in \mathcal{F}$

From the previous proposition, we can infer that the collection of measurable sets forms a *σ-algebra*.

Proposition 1.16. *Every interval is measurable.*

Proof. We claim that it suffices to show that every interval of the form (a, ∞) is measurable. Indeed, from here we have that every interval of the form $(-\infty, a]$ is measurable. Then,

$$(-\infty, a) = \bigcup_{n=1}^{\infty} \left(-\infty, a - \frac{1}{n}\right]$$

is also measurable whence $[a, \infty)$ is also measurable. Now, note that

$$\begin{aligned} [a, b] &= [a, \infty) \cap (-\infty, b] \qquad & [a, b) &= [a, \infty) \cap (-\infty, b) \\ (a, b] &= (a, \infty) \cap (-\infty, b] \qquad & (a, b) &= (a, \infty) \cap (-\infty, b) \end{aligned}$$

and hence every interval is measurable.

We shall now show that every interval of the form (a, ∞) is measurable, for which it suffices to show that $m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) = m^*(A)$. Let $A' = A \backslash \{a\}$, then

$$m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) = m^*(A) \iff m^*(A' \cap (a, \infty)) + m^*(A' \cap (-\infty, a]) = m^*(A')$$

Therefore, without loss of generality, we may suppose that $a \notin A$. Define $A_1 = A \cap (-\infty, a] = A \cap (-\infty, a)$ and $A_2 = A \cap (a, \infty)$. Let $\{I_k\}_{k \in \mathbb{N}}$ be a collection of open bounded intervals that covers A. Define $I_k' = I_k \cap (-\infty, a)$ and $I_k'' = I_k \cap (a, \infty)$. Note that I_k' and I_k'' are both open bounded intervals and the collections $\{I_k'\}_{k \in \mathbb{N}}$ and $\{I_k''\}_{k \in \mathbb{N}}$ cover A_1 and A_2 respectively. Further, from the definition of I_k' and I_k'', we have that $\ell(I_k) = \ell(I_k') + \ell(I_k'')$. Consequently,

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'') \\ &= \sum_{k=1}^{\infty} \left(\ell(I_k') + \ell(I_k'')\right) \\ &= \sum_{k=1}^{\infty} \ell(I_k) \end{aligned}$$

Since this inequality holds for all covers $\{I_k\}$ of A by open bounded intervals, we must have that $m^*(A_1) + m^*(A_2) \leq m^*(A)$ and (a, ∞) is measurable. This completes the proof. ■

Lemma 1.17. *Every open set is the disjoint union of a countable collection of open intervals.*

Corollary. Open sets, closed sets, G_δ-sets and F_σ-sets are measurable.

Proposition 1.18 (Excision Property). *Let E be a measurable set and $E \subseteq A \subseteq \mathbb{R}$. Then,*

$$m^*(A \backslash E) = m^*(A) - m^*(E)$$

The proof of the above proposition follows from the definition of a measurable set and is omitted.

Theorem 1.19. *Let $E \subseteq \mathbb{R}$. Then the following are equivalent:*

(a) *E is measurable*

(b) *For each $\varepsilon > 0$ there is an open set U containing E for which $m^*(U \backslash E) < \varepsilon$*

(c) There is a G_δ set G containing E for which $m^*(G\backslash E) = 0$

(d) For each $\varepsilon > 0$, there is a closed set F contained in E for which $m^*(E\backslash F) < \varepsilon$

(e) There is an F_σ set F contained in E for which $m^*(E\backslash F) = 0$

Proof. We shall show that $(a) \implies (b) \implies (c) \implies (a)$ along with $(b) \iff (d)$ and $(c) \iff (c)$, which would imply the desired conclusion.

- $\underline{(a) \implies (b)}$: First, suppose E has finite measure. Then, by definition, there is a covering of E with open bounded intervals $\{I_k\}_{k\in\mathbb{N}}$ such that

  $$\sum_{k\in\mathbb{N}} \ell(I_k) < m^*(E) + \varepsilon$$

  But using the countable subadditivity of the outer measure, we have

  $$m^*(E) \leq m^*\left(\bigcup_{k\in\mathbb{N}} I_k\right) \leq \sum_{k\in\mathbb{N}} \ell(I_k) < m^*(E) + \varepsilon$$

  Letting $U = \bigcup_{k\in\mathbb{N}} I_k$, we have

  $$m^*(U\backslash E) = m^*(U) - m^*(E) < \varepsilon$$

  Now suppose E has infinite measure. Let $F_k = E \cap [k, k+1)$. Obviously each F_k is measurable and has finite measure. Reindex the collection $\{F_k\}_{k\in\mathbb{Z}}$ as $\{E_k\}_{k\in\mathbb{N}}$, where $E = \bigsqcup_{k\in\mathbb{N}} E_k$ and each E_k is measurable and has finite measure. Choose an open set U_k containing E_k and $m^*(U_k\backslash E_k) < \varepsilon/2^k$. Let $U = \bigcup_{k\in\mathbb{N}} U_k$. Then,

  $$m^*(U\backslash E) \leq m^*\left(\bigcup_{k\in\mathbb{N}} [U_k\backslash E_k]\right) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

- $\underline{(b) \implies (c)}$: Let U_k be an open set containing E such that $m^*(U_k\backslash E) < 1/k$. Define $G = \bigcap_{k\in\mathbb{N}} U_k$, which is a G_δ-set. Then, $E \subseteq G$ and

  $$m^*(G\backslash E) \leq m^*(U_k\backslash E) < \frac{1}{k} \quad \forall\, k \in \mathbb{N}$$

  whence $m^*(G\backslash E) = 0$.

- $\underline{(c) \implies (a)}$: Since G is measurable, and $G\backslash E$ being a set of outer measure 0 is measurable, we must have that $E = G\backslash(G\backslash E)$ is measurable.

- $\underline{(b) \implies (d)}$: Since E is measurable, so is E^c. Due to (b), there is a G_δ-set G containing E^c with $m^*(G\backslash E^c) < \varepsilon$. Consequently, $F = G^c$ is an F_σ-set contained in E with $E\backslash F = G\backslash E^c$ giving us the desired conclusion.

- $\underline{(d) \implies (b)}$: Similar to above.

- Similarly, one can show $\underline{(c) \iff (e)}$

This completes the proof. ∎

Theorem 1.20. *Let E be a measurable set of finite outer measure. Then for every $\varepsilon > 0$, there is a fiinte disjoint* *collection of open bounded intervals $\{I_k\}_{k=1}^n$ for which if $\mathcal{O} = \bigcup_{k=1}^n I_k$ then*

$$m^*(E\Delta\mathcal{O}) < \varepsilon$$

Proof. Due to Theorem 1.19 (b), there is an open set U containing E such that $m^*(U\backslash E) < \varepsilon/2$. Further, due to Lemma 1.17, there is a countable collection of disjoint (possibly empty) open intervals $\{I_k\}_{k\in\mathbb{N}}$ such that $U = \bigsqcup_{k\in\mathbb{N}} I_k$. Since $m^*(U)$ is finite, so is $\sum_{k\in\mathbb{N}} \ell(I_k)$, consequently, there is $n \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} \ell(I_k) < \varepsilon/2$$

Define $\mathcal{O} = \bigsqcup_{k=1}^{n} I_k$. Then $\mathcal{O} \subseteq U$ and is measurable. Therefore, $m^*(\mathcal{O}\backslash E) \le m^*(U\backslash E) < \varepsilon/2$ and

$$m^*(E\backslash\mathcal{O}) \le m^*(U\backslash\mathcal{O}) = \sum_{k=n+1}^{\infty} \ell(I_k) < \varepsilon/2$$

whence the conclusion follows. ■

Theorem 1.21 (Lebesgue). *A subset E of $\mathbb{R}$ is measurable if and only if for every open bounded interval (a, b),*

$$b - a = m^*((a,b) \cap E) + m^*((a,b)\backslash E)$$

Proof. Let $\varepsilon > 0$. Then by definition, there is a collection of open bounded intervals $\{I_k\}_{k\in\mathbb{N}}$ such that $E \subseteq \mathcal{O} = \bigcup_{k\in\mathbb{N}} I_k$ and

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \varepsilon$$

Then, we have

$$m^*(\mathcal{O}\backslash E) = m^*\left(\bigcup_{k=1}^{\infty} [I_k\backslash E]\right) \le \sum_{k=1}^{\infty} m^*(I_k\backslash E) = \sum_{k=1}^{\infty} [\ell(I_k) - m^*(I_k \cap E)] < m^*(E) + \varepsilon - m^*(\mathcal{O} \cap E) = \varepsilon$$

And we are done due to Theorem 1.19 (b). ■

1.3 The Lebesgue Measure and Properties

Definition 1.22 (Lebesgue Measure). The restriction of the set function outer measure to the collection of measurable sets is called *Lebesgue Measure*. It is denoted by m. Therefore, if E is a measurable set, its Lebesgue Measure is defined as

$$m(E) = m^*(E)$$

Proposition 1.23. *If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets and $A \subseteq \mathbb{R}$, then its union $\bigcup_{k=1}^{\infty} E_k$ is also measurable and*

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

Proof. Obviously, $\bigcup\limits_{k=1}^{\infty} E_k$ is measurable and due to Proposition 1.5,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

Further, due to Proposition 1.12,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \geq m^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

Since the above inequality holds for all $n \in \mathbb{N}$, we must have that

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

implying the desired conclusion. ∎

Corollary. Lebesgue Measure is countably additive. if $\{E_k\}$ is a countable disjoint collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

Proof. Follows from Proposition 1.23 by taking $A = \mathbb{R}$. ∎

Consolidating what we have till now:

The set function Lebesgue Measure (m) defined on the σ-algebra of Lebesgue measurable sets, assigns length to any interval, is translation invariant and countable additive.

Definition 1.24. A countabe collection of sets $\{E_k\}_{k=1}^{\infty}$ is said to be *ascending* if $E_1 \subseteq E_2 \subseteq \cdots$ and is said to be *descending* if $E_1 \supseteq E_2 \supseteq \cdots$.

Theorem 1.25 (Continuity of Measure). *The Lebesgue Measure possesses the folowing continuity properties:*

(a) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$$

(b) If $\{B_k\}_{k=1}^{\infty}$ is an descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$$

Proof. We shall first show (a) and infer (b) from it.

(a) Define $\{C_n = A_n \backslash A_{n-1}\}$ with $A_0 = \varnothing$. It then follows that $\{C_n\}$ is a collection of dijoint measurable sets such that $\bigcup C_n = \bigcup A_n$. Now, due to the countable additivity of measure,

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} m(C_n) = \sum_{n=1}^{\infty} m(A_n) - m(A_{n-1}) = \lim_{n\to\infty} m(A_n)$$

(b) Define $D_n = B_1 \backslash B_n$. Then, $\{D_n\}$ is an ascending chain with $\bigcup D_n = B_1 \backslash \bigcap B_n$. Consequently,

$$\lim_{n\to\infty} m(D_n) = m\left(\bigcup_{n=1}^{\infty} D_n\right) = m\left(B_1 \backslash \bigcap_{n=1}^{\infty} B_n\right) = m(B_1) - m\left(\bigcap_{n=1}^{\infty} B_n\right)$$

Where we require $m(B_1) < \infty$ to use the excision property. Then, we have

$$m(B_1) - \lim_{n\to\infty} m(B_n) = m(B_1) - m\left(\bigcap_{n=1}^{\infty} B_n\right)$$

whence the conclusion follows.

■

The assertion about descending chains may not hold if $m(B_1) = \infty$. Take for example the descending chain $\{(n,\infty)\}_{n=1}^{\infty}$. We have $\bigcap_{n=1}^{\infty}(n,\infty) = \varnothing$ but $m((n,\infty)) = \infty$, and thus the limit $\lim_{n\to\infty} m((n,\infty)) = \infty$.

Lemma 1.26 (Borel-Cantelli). *Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.*

Proof. Let S be the set of all $x \in \mathbb{R}$ belonging to infinitely many of the E_k's. Then,

$$S = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

It is not hard to see that S is measurable.

But then we have

$$m(S) \leq \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$$

where the last equality follows from the fact that $\sum_{k=1}^{\infty} m(E_k)$ is finite. This completes the proof. ■

1.4 Nonmeasurable Sets

Lemma 1.27. *Let $E \subseteq \mathbb{R}$ be a bounded and measurable. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates $\{\lambda + E\}_{\lambda\in\Lambda}$ is disjoint. Then $m(E) = 0$.*

Proof. Using the countable additivity of Lebesgue Measure,

$$\infty > m\left(\bigcup_{\lambda\in\Lambda} [\lambda + E]\right) = \sum_{\lambda\in\Lambda} m(E)$$

therefoere, $m(E) = 0$. ■

Definition 1.28 (Rational Equivalence). Let $E \subseteq \mathbb{R}$. Define an equivalence relation on E by $x \sim_{\mathbb{Q}} y$ if $x - y \in \mathbb{Q}$.

Theorem 1.29 (Vitali). *Any set E of real numbers with positive outer measure contains a subset that is not measurable.*

Proof. Since every set of positive outer measure has a bounded subset with positive outer measure, we may suppose without loss of generality that E is bouned. Let $\mathcal{C}_E$ be a choice set over the equivalence classes over E defined by $\sim_{\mathbb{Q}}$. We shall show that $\mathcal{C}_E$ is not measurable. Suppose it is. Let Λ be a bounded collection of rational numbers. Then, the collection $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ is a collection of disjoint measurable sets. Then, we can say due to the preceeding lemma that $m(\mathcal{C}_E) = 0$.

Let $b \in \mathbb{R}$ such that $E \subseteq [-b, b]$ and choose $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$. We claim that $E \subseteq \bigcup_{\lambda \in \Lambda_0} [\lambda + \mathcal{C}_E]$. Indeed, if $x \in E$, there is $y \in [x]_{\sim_{\mathbb{Q}}} \cap \mathcal{C}_E$ but since $x, y \in [-b, b]$, we must have that $\lambda = x - y \in [-2b, 2b]$ and hence, $x \in \lambda + \mathcal{C}_E$

$$m^*(E) \leq \sum_{\lambda \in \Lambda_0} m(\lambda + \mathcal{C}_E) = 0$$

a contradiction and we have the desired conclusion. ■

Corollary. There are disjoint sets of real numbers A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B)$$

Proof. Suppose not, then for every pair of disjoint sets of real numbers A and B, $m^*(A \cup B) = m^*(A) + m^*(B)$ and therefore, every subset of $\mathbb{R}$ is measurable. A contradiction to Theorem 1.29. ■

1.5 Cantor Set and Cantor Lebesgue Function

We shall first construct a descending chain of closed sets, $\{C_n\}_{n=1}^{\infty}$. First, consider the closed, bounded interval $I = [0, 1]$ and divide it into three intervals of equal length $1/3$ and remove the interior of the middle interval. Call the remaining set C_1. Explicitly, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Repeat this procedure for each of the two intervals in C_1 to obtain C_2 and do this ad infinitum.

Finally, define

$$\mathbf{C} = \bigcap_{n=1}^{\infty} C_n$$

Since each C_k is closed and bounded, it is compact and also measurable. Further, since $\mathbf{C}$ is the intersection of closed, bounded sets, it is closed and bounded and therefore compact and measurable. Finally, due to Cantor's Intersection Theorem, $\mathbf{C}$ is non-empty and measurable.

It is not hard to show using induction that $m(C_n) = \left(\frac{2}{3}\right)^n$ and thus,

$$m(\mathbf{C}) \leq \left(\frac{2}{3}\right)^n$$

Since the above inequality holds for all $n \in \mathbb{N}$, $m(\mathbf{C}) = 0$.

Proposition 1.30. *The Cantor set $\mathbf{C}$ is uncountable.*

Proof. Suppose $\mathbf{C}$ is countable and let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. We shall construct a descending sequence of compact (and therefore closed) sets as follows.

Note that C_1 is the disjoint union of two closed intervals. Let F_1 be the interval not containing c_1. Now, inductively, if we have F_k, note that $C_{k+1} \cap F_k$ is the disjoint union of two closed intervals, then let F_{k+1} be the interval not containing c_{k+1}.

It is not hard to see that F_{k+1} is a descending sequence of closed and bounded, and therefore compact sets. Then, due to Cantor's Intersection Theorem, there is $x \in \bigcap_{k=1}^{\infty} F_k \subseteq \mathbf{C}$. Therefore, $x = c_n$ for some n, but this is a contradiction to the fact that $x = c_n \notin F_n$.

Hence, $\mathbf{C}$ is uncountable. ■

Definition 1.31 (Perfect). A set S is said to be *perfect* if the limit points of S are precisely the points of S.

Proposition 1.32. *The Cantor set* $\mathbf{C}$ *is perfect.*

Proof. Let $x \in \mathbf{C}$ and $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $3^{-N} < \varepsilon$. Since $x \in C_N$, there is a closed interval F of length 3^{-N} that contains x. But due to our constraints, $F \subseteq B(x, \varepsilon)$. Note that $F \cap C_{N+1}$ is a disjoint union of two closed intervals G_1, G_2 of length $3^{-(N+1)}$. Without loss of generality, say $x \in G_2$. Then, consider the descending chain $\{G_1 \cap C_k\}_{k=1}^{\infty}$. Using Cantor's Intersection Theorem, there is some element of $\mathbf{C}$ in G_1, consequently in F and thus $B(x, \varepsilon)$. This completes the proof. ■

Proposition 1.33. *The Cantor set* $\mathbf{C}$ *is nowhere dense.*

Proof. Let $\mathcal{O}$ be an open set in $\mathbb{R}$. If $\mathcal{O} \cap \mathbf{C} = \varnothing$, then we are done. If not, then it contains an interval, say (a, b). If $(a, b) \cap \mathbf{C} = \varnothing$, then we are done. If not, then let $x \in (a, b) \cap \mathbf{C}$. Then, there is some $N \in \mathbb{N}$ such that the interval in C_N containing x is completely contained in (a, b), call this interval I. Then, there is an open set $U \subseteq I$ such that $C_{N+1} \cap I = I \backslash U$. Then, U is an open set in $\mathcal{O}$ that does not intersect $\mathbf{C}$, thereby completing the proof. ■

Lemma 1.34. *Every nonempty perfect set in* $\mathbb{R}$ *is uncountable.*

Proof. We shall prove this by contradiction. Suppose $A \subseteq \mathbb{R}$ is perfect and countable. Therefore, it has an enumeration $A = \{a_1, a_2, \ldots\}$. Let $U_0 = (a_1 - 1, a_1 + 1)$. Let $n \in \mathbb{N}$ be the smallest index greater than 1 such that $a_n \in U_0$. Let U_1 be an open interval containing a_n such that $\overline{U_1} \subseteq U_0$ and $a_1 \notin \overline{U_1}$. Let $F_1 = \overline{U_1}$. Now, define $U_2 = \ldots = U_{n-1} = U_1$ and $F_2 = \ldots = F_{n-1} = F_1$. Repeat this procedure ad infinitum. Now consider the descending chain $\{F_n \cap A\}$, which is such that $a_n \notin F_n \cap A$. But due to Cantor's Intersection Theorem, $A \cap \bigcap\limits_{n=1}^{\infty} F_n$ is nonempty, a contradiction. ■

Now, define $\mathcal{O}_k = [0,1] \backslash C_k$ and $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$. Then, obviously, $\mathcal{O} = [0,1] \backslash \mathbf{C}$. We shall now define a sequence of functions $\{\varphi_k : \mathcal{O}_k \to [0,1]\}$ such that φ_{k+1} is an extension of φ_k. To do this, note that $\mathcal{O}_k$ is a disjoint union of $2^k - 1$ open intervals. On which we give φ_k the successive values

$$\left\{ \frac{1}{2^k}, \ldots, \frac{2^k - 1}{2^k} \right\}$$

It is not hard to see that φ_k and φ_{k+1} agree on $\mathcal{O}_k$. Using this, we may define φ over all of $\mathcal{O}$. Finally, we shall extend this to define φ over points of $\mathbf{C}$ as

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(x) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap (0, x)\}$$

This function φ is known as the Cantor-Lebesgue function.

Proposition 1.35. *The Cantor-Lebesgue function is an increasing continuous function that maps* $[0,1]$ *onto* $[0,1]$. *It is differentiable at every point in* $\mathcal{O}$ *and the value of its derivative is equal to* 0.

Proof. Note that each φ_k is increasing, consequently, for any $x, y \in \mathcal{O}$ with $x < y$, there is some index N such that $x, y \in \mathcal{O}_N$ and thus, $\varphi(x) < \varphi(y)$, whence φ is increasing on $\mathcal{O}$. Now let $x \in \mathbf{C}$ and $y \in \mathcal{O}$ with $x < y$. If $x = 0$, then we trivially have that $\varphi(x) < \varphi(y)$. Now suppose $x > 0$. Then $\varphi(x) = \sup\{\varphi(t) \mid t \in (0,x) \cap \mathcal{O}\}$. But since $\varphi(y) \geq \varphi(t)$ for all $t \in \mathcal{O} \cap (0,x)$, we have that $\varphi(x) \leq \varphi(y)$. When $x, y \in \mathbf{C}$, the conclusion is obvious. Thus φ is increasing.

Next, we shall show that φ is continuous. That φ is continuous on $\mathcal{O}$ is easy to see since it is continuous on each $\mathcal{O}_k$. Let $x_0 \in \mathbf{C}$. Then, for each $\mathcal{O}_k$, x lies between two open intervals of $\mathcal{O}_k$. Let $\varepsilon > 0$ and choose k such that $2^{-k} < \varepsilon$. Choose a_k in the interval just before x and b_k in the interval just after x. Let $\delta = \min\{|a_k - x|, |b_k - x|\}$. Whenever $|x - y| < \delta$, $|\varphi(x) - \varphi(y)| \leq |\varphi(a_k) - \varphi(b_k)| = 2^{-k} < \varepsilon$. This implies continuity of φ at x.

Finally, we may conclude that φ maps $[0,1]$ to $[0,1]$ using the Intermadiate Value Theorem. ■

Theorem 1.36. *Let* $\varphi : [0,1] \to [0,1]$ *be the Cantor-Lebesgue function and define the function* $\psi : [0,1] \to [0,1]$ *by* $\psi(x) = \varphi(x) + x$ *for all* $x \in [0,1]$. *Then* ψ *is a strictly increasing continuous function that maps* $[0,1]$ *onto* $[0,2]$ *and*

(a) *maps the Cantor set* $\mathbf{C}$ *onto a measurable set of positive measure and*

(b) *maps a measurable set, a subsest of* $\mathbf{C}$ *onto a nonmeasurable set.*

Proof. Note that since ψ is a strictly increasing continuous function, it has a continuous inverse, as a result, it is a homeomorphism from $[0,1]$ to $[0,2]$. Then $\psi(\mathbf{C})$ is closed and $\psi(\mathcal{O})$ is open. These two are disjoint sets whose union is $[0,2]$. Let $\{I_k\}$ be the set of intervals that are removed while constructing the Cantor set. Then they are all disjoint and $\psi(I_k)$ is simply a translation of I_k and is therefore homemomorphic to I_k. Consequently,

$$m(\psi(\mathcal{O})) = m\left(\bigcup_{k=1}^{\infty} \psi(I_k)\right) = \sum_{k=1}^{\infty} m(\psi(I_k)) = \sum_{k=1}^{\infty} m(I_k) = m(\mathcal{O}) = 1$$

Thus, $m(\psi(\mathbf{C})) = 1$, which proves (a).

Due to Theorem 1.29, there is a non-measurable set $V \subseteq \psi(\mathbf{C})$. Define $U = \psi^{-1}(V) \subseteq \mathbf{C}$. Since U has measure 0, it is measurable and is mapped to a nonmeasurable set V, which proves (b). ■

Theorem 1.37. *Let* $f : [a,b] \to \mathbb{R}$ *be Lipschitz. Then* f *maps measurable sets to measurable sets.*

Proof.

- f maps sets of measure 0 to sets of measure 0: Let $E \subseteq [a,b]$ have measure 0 and $\varepsilon > 0$. Note that it would suffice to show that $m^*(f(E)) = 0$, since that would immediately imply the measurability of $f(E)$. Let $\{I_k\}$ be a collection of open bounded intervals that cover E such that $\sum_{k=1}^{\infty} \ell(I_k) < \varepsilon$. Since f is a continuous function, it maps intervals to intervals. Now, since $f(E) \subseteq \bigcup f(I_k)$. It follows that

$$m^*(f(E)) \leq \sum_{k=1}^{\infty} \ell(f(E_k)) < c\varepsilon$$

It follows that $m^*(f(E)) = 0$.

- f maps F_σ sets to F_σ sets: Let F be an F_σ set. Then, there is a countable collection $\{A_n\}$ of closed sets such that $F = \bigcup_{n\in\mathbb{N}} A_n$. Note that each A_n is closed and bounded, therefore, its image is compact (since f is continuous) and hence closed and bounded. As a result,

$$f(F) = \bigcup_{n\in\mathbb{N}} f(A_n)$$

and is an F_σ set.

- Putting it together: Let $E \subseteq [a, b]$ be measurable. Then, there is an F_σ set F that is contained in E such that $m(E \backslash F) = 0$. Now,

$$f(E)\backslash f(F) \subseteq \underbrace{f(E\backslash F)}_{\text{has measure 0}}$$

and hence, $m(f(E)\backslash f(F)) = 0$ and thus $f(E)$ is measurable.

■

Lemma 1.38. *Let $f : X \to Y$ be a continuous function between two topological spaces X and Y. Then, for every Borel set B in Y, $f^{-1}(B)$ is a Borel set in X.*

Proof. Define

$$\mathfrak{M} = \{E \subseteq Y \mid f^{-1}(E) \text{ is Borel}\}$$

We claim that $\mathfrak{M}$ is a σ-algebra. If $E \in \mathfrak{M}$, then $f^{-1}(Y\backslash E) = X\backslash f^{-1}(E)$, which is Borel and hence, $Y\backslash E \in \mathfrak{M}$. Similarly, let $\{E_1, E_2, \ldots\} \subseteq \mathfrak{M}$. We have

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n)$$

where the quantity on the right is a Borel set, whence $\bigcup\limits_{n=1}^{\infty} E_n \in \mathfrak{M}$ from which it follows that $\mathfrak{M}$ is a σ-algebra.

Finally, since $\mathcal{T}_Y \subseteq \mathfrak{M}$, it must contain all Borel sets and we have the desired conclusion. ■

Lemma 1.39. *Let $f : I \to \mathbb{R}$ be a strictly increasing function where I is an interval. Then f maps Borel sets to Borel sets.*

Proof. It is not hard to show that f has a continuous inverse from $g : f(I) \to I$. Since I is an interval, so is $f(I)$. Let B be a Borel set contained in I. Then $f(B) = g^{-1}(B)$ is Borel due to the previous lemma. ■

Theorem 1.40. *The Cantor set* **C** *contains a measurable set that is not Borel.*

Proof. Recall that we have shown that the function $\psi = \varphi + \mathbf{id}$ maps a measurable set A to a non measurable set. But since ψ is a strictly increasing continuous function defined on an interval, it must map Borel sets to Borel sets. Consequently, A is not Borel lest $f(A)$ be measurable. ■

Theorem 1.41. *The set of all Lebesgue Measurable subsets of $\mathbb{R}$ has cardinality greater than $\mathbb{R}$.*

Proof. Assuming CH, we have shown that **C** has cardinality equal to $\mathbb{R}$. Since $m(\mathbf{C}) = 0$, every subset of the Cantor set is measurable, consequently, the set of Lebesgue measurable sets contains a set that is in bijection with the power set of $\mathbb{R}$, consequently, it must have cardinality greater than that of $\mathbb{R}$. ■

Theorem 1.42. $\mathbb{Q}$ *is not a* G_δ *set.*

Proof. Suppose $\mathbb{Q}$ were a G_δ set. Then, there would exist a countable collection of open sets $\{U_k\}_{k\in\mathbb{N}}$ such that $\mathbb{Q} = \bigcap\limits_{k\in\mathbb{N}} U_k$. Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ be an enumeration for $\mathbb{Q}$ and define $V_k = U_k \backslash \{q_k\}$. Then $\{V_k\}$ is a collection of nonempty open sets such that $\bigcap\limits_{n=k}^{\infty} V_k = \varnothing$.

Choose some open, bounded interval J_1 with nonzero measure in V_1, which is known to exist since it is a nonempty open set. This interval obviously contains a nonempty closed interval I_1 with nonzero measure. Since I_1 and J_1 contain infinitely many rationals, their intersection with V_2 is nonempty and contains infinitely many rationals. Choose some open interval with nonzero measure in the set $J_1 \cap V_2$ and call this J_2 and similar to above construct I_2 and continue in this fashion.

We would obtain a descending sequence of closed and bounded intervals $I_1 \supseteq I_2 \supseteq \cdots$ such that $I_k \subseteq V_k$. Now, due to Cantor's Intersection Theorem, $\bigcap\limits_{k\in\mathbb{N}} I_k \neq \varnothing$ which is a contradiction. ■

Chapter 2

Lebesgue Measurable Functions

Theorem 2.1. *Let the function f have a measurable domain E. Then the following statements are equivalent:*

(i) For each real number c, the set $E_1(c) = \{x \in E \mid f(x) > c\}$ is measurable

(ii) For each real number c, the set $E_2(c) = \{x \in E \mid f(x) \geq c\}$ is measurable

(iii) For each real number c, the set $E_3(c) = \{x \in E \mid f(x) < c\}$ is measurable

(iv) For each real number c, the set $E_4(c) = \{x \in E \mid f(x) \leq c\}$ is measurable

*Each of the above implies that for each **extended real number** c, the set*

$$\{x \in E \mid f(x) = c\}$$

is measurable.

Proof. We shall show that $(i) \implies (iv) \implies (iii) \implies (ii) \implies (i)$. This follows from the following equalities:

$$\begin{aligned}
E_4(c) &= E \backslash E_1(c) \\
E_3(c) &= \bigcup_{n=1}^{\infty} E_4\left(c - \frac{1}{n}\right) \\
E_2(c) &= E \backslash E_3(c) \\
E_1(c) &= \bigcup_{n=1}^{\infty} E_2\left(c + \frac{1}{n}\right)
\end{aligned}$$

Now, if c is finite, then

$$\{x \in E \mid f(x) = c\} = E_2(c) \cap E_4(c)$$

and is measurable. Next, if $c = \infty$, then

$$\{x \in E \mid f(x) = c\} = \bigcap_{n \in \mathbb{N}} E_1(n)$$

■

We denote the extended real line by $[-\infty, \infty]$. The arithmetic on this real line is well known and we omit its discussion.

Definition 2.2 (Measurable Function). An extended real-valued function $f : E \to [-\infty, \infty]$ defined on E is said to be *Lebesgue measurable*, or just *measurable* provided E is measurable and it satisfies one of the four statements of Theorem 2.1.

Proposition 2.3. *Let E be a measurable set and $f : E \to [-\infty, \infty]$. Then f is measurable if and only if for each open set $\mathcal{O}$, $f^{-1}(\mathcal{O})$ is measurable.*

Proof. Suppose the inverse image of each open set is measurable, consequently, the inverse image of (c, ∞) is open for all $c \in \mathbb{R}$, and due to Theorem 2.1, f is measurable.

Now suppose f is measurable. Then,

$$f^{-1}((a,b)) = f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty))$$

is measurable. Since $\mathcal{O}$ is open, it can be written as the union of countably many disjoint open intervals, and thus, the inverse image of $\mathcal{O}$ can be written as the union of countably many disjoint measurable sets and is measurable. This completes the proof. ■

Corollary. If E is measurable and $f : E \to \mathbb{R}$[a] is continuous, then it is measurable.

[a]Note that we are not talking about an extended real valued function here

Proposition 2.4. *A monotone function that is defined on an interval is measurable.*

Proposition 2.5. *Let $f, g : E \to [-\infty, \infty]$*

(i) If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E

(ii) For a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and $E \backslash D$ are measurable.

Proof. (i) Let $A = \{x \in E \mid f(x) \neq g(x)\}$. It is known that $m(A) = 0$. Then, for any $c \in \mathbb{R}$,

$$\{x \in E \mid g(x) > c\} = \{x \in A \mid g(x) > c\} \cup (\{x \in E \mid f(x) > c\} \cap (E \backslash A))$$

Since $\{x \in A \mid g(x) > c\} \subseteq A$, it has outer measure 0 and is measurable. Further, since measurable sets are closed under intersection, $\{x \in E \mid f(x) > c\} \cap (E \backslash A)$ is measurable. As a result, their union is measurable.

(ii) Simply note that

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f(x) > c\} \cup \{x \in E \backslash D \mid f(x) > c\}$$

■

Assertion (i) of the previous theorem allows us to extend a measurable function to another measurable function. Take for example two extended real valued functions f and g finite on $E \backslash E_0$ where E_0 has measurae 0. Then, $f + g$ is defined on $E \backslash E_0$. If we were to show that $f + g$ is measurable on $E \backslash E_0$, any extension of it to E would also be measurable.

Theorem 2.6. *Let f and g be measurable (extended real valued) functions on E that are finite a.e. on E. Then*

(a) For any α and β, $\alpha f + \beta g$ is measurable on E

(b) fg is measurable on E

Proof. Note that since f, g are finite a.e. on E, we may suppose, due to the discussion preceeding the statement of the theorem that f, g are finite everywhere on E. Then, it shall suffice to show that both αf and $f + g$ are measurable. Without loss of generality, let $\alpha > 0$. Then

$$\{x \in E \mid \alpha f(x) < c\} = \{x \in E \mid f(x) < c/\alpha\}$$

Thus, $\alpha f(x)$ is measurable.

Next, we shall show that $f + g$ is measurable for which it suffices to show that $\{x \in E \mid f(x) + g(x) < c\}$ is measurable. We claim that

$$\{x \in E \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} \{x \in E \mid f(x) < q\} \cap \{x \in E \mid g(x) < c - q\}$$

Indeed, if $f(x) + g(x) < c$, then there is a rational q such that $f(x) < q < c - g(x)$, consequently, $f(x) < q$ and $g(x) < c - q$. Since the rationals are countable, we have the desired conclusion.

Finally, note that

$$\{x \in E \mid f(x)^2 > c\} = \begin{cases} \{x \in E \mid f(x) > \sqrt{c}\} \cup \{x \in E \mid f(x) < -\sqrt{c}\} & c \geq 0 \\ E & c < 0 \end{cases}$$

Thus, f^2 is also measurable. Now, simply note that

$$fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$

whence fg is measurable. ■

Proposition 2.7. *The composition of two measurable functions need not be measurable.*

Proof. Define the function

$$\psi(x) = \begin{cases} \varphi(x) + x & x \in [0,1] \\ 2x & x \notin [0,1] \end{cases}$$

where φ is the Cantor-Lebesgue function. We have already shown that ψ is continuous and strictly increasing, and is therefore a bijection from $\mathbb{R}$ to $\mathbb{R}$. We have also shown that there is $A \subseteq \mathbf{C}$ such that $\psi(A)$ is not measurable. Let χ_A be the characteristic function from $\mathbb{R}$ to $\mathbb{R}$ for the set A. We claim that $f = \chi_A \circ \psi^{-1}$ is not measurable, which would prove the statement of the proposition, since ψ^{-1} is continuous and thus measurable.

Indeed,

$$\{x \in \mathbb{R} \mid f(x) > 0.5\} = \{x \in \mathbb{R} \mid \psi^{-1}(x) \in A\} = \psi(A)$$

which is not measurable. This shows that f is not measurable. ■

Proposition 2.8. *Let E be measurable, $g : E \to \mathbb{R}$ be measurable and $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then the composition $f \circ g : E \to \mathbb{R}$ is measurable.*

Proof. Let $\mathcal{O}$ be an open set in $\mathbb{R}$. Then $(f \circ g)^{-1}(\mathcal{O}) = g^{-1}(f^{-1}(\mathcal{O}))$. Since f is contiuous, $f^{-1}(\mathcal{O})$ is open and thus $g^{-1}(f^{-1}(\mathcal{O}))$ is measurable, implying that $f \circ g$ is measurable. ■

Proposition 2.9. *Let $f : E \to [-\infty, \infty]$ be an extended real valued measurable function on a measurable domain E. Then, for every Borel set B, $f^{-1}(B)$ is measurable.*

Proof. Let $\mathfrak{M} = \{A \subseteq \mathbb{R} \mid f^{-1}(A) \text{ is measurable}\}$. It is not hard to show that $\mathfrak{M}$ is a σ-algebra. Further, $\mathfrak{M}$ contains all the open sets in $\mathbb{R}$, consequently, contains all the Borel sets in $\mathbb{R}$. This completes the proof. ■

Theorem 2.10. *Let E be measurable and $\{f_n : E \to [-\infty, \infty]\}$ be a sequence of measurable functions. Define*

$$g = \sup f_n \quad \text{and} \quad h = \limsup_{n\to\infty} f_n$$

Then g and h are measurable.

Proof. Let $c \in \mathbb{R}$. Then we have

$$\{x \in E \mid g(x) > c\} = \bigcup_{n\in\mathbb{N}} \{x \in E \mid f_n(x) > c\}$$

and

$$\{x \in E \mid h(x) > c\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m=n}^{\infty} \{x \in E \mid f_m(x) > c\}$$

both of which are measurable. This completes the proof. ■

Similarly, one can show that $\inf f_n$ and $\liminf f_n$ are measurable.

Corollary. Let E be measurable and $f, g : E \to [-\infty, \infty]$ be measurable functions. Then $\min\{f, g\}$ and $\max\{f, g\}$ are measurable.

Proof. Note that

$$\max\{f, g\} = \limsup\{f, g, f, g, f, g, \ldots\}$$
$$\min\{f, g\} = \liminf\{f, g, f, g, f, g, \ldots\}$$

■

For a function $f : E \to [-\infty, \infty]$, define $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. It is obious that f^+ and f^- are nonnegative functions and $f = f^+ - f^-$ on E.

2.1 Pointwise Limits and Simple Approximation

We begin by reacalling some definitions from Real Analysis.

Definition 2.11. For a sequence $\{f_n\}$ of functions with common domain E and a function f on E and a subset A of E, we say that

(i) The sequence $\{f_n\}$ converges to f pointwise on A provided $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$

(ii) The sequence $\{f_n\}$ covnerges to f pointwise a.e. on A provided it converges to f pointwise on $A \backslash B$ where $m(B) = 0$

(iii) The sequence $\{f_n\}$ converges to f uniformly on A provided for each $\varepsilon > 0$ there is an index N

for which

$$|f - f_n| < \varepsilon \text{ on } A \text{ for all } n \geq N$$

Recall that the pointwise limit of a Riemann integrable function need not be Riemann integrable. Measurable functions on the other hand are better behaved under pointwise limits.

Proposition 2.12. *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function f. Then f is measurable.*

Proof. Let E_0 be such that $\{f_n\}$ converges pointwise on $E \setminus E_0$. If we show that f is measurable on $E \setminus E_0$, then every extension of f to E is measurable. Therefore, without loss of generality, we may suppose that $\{f_n\}$ converges pointwise on E.

If $\{f_n\}$ converges pointwise, then $f = \limsup f_n$ and is measurable. ■

The **characteristic function** of a set A, denoted χ_A is defined as

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Definition 2.13 (Simple Function). A real-valued function φ defined on a measurable set E is called *simple* if it is measurable and takes only a finite number of distinct values.

It is not hard to infer from the definition that a simple function can be represented as

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k}$$

where E_k's are disjoint and measurable. This is known as the **canonical representation of the simple function**.

Note that the simple functions that we consider are real valued and not extended real valued.

Lemma 2.14 (Simple Approximation Lemma). *Let f be measurable real-valued bounded function on E. Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε defined on E such that*

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$$

Proof. Since f is bounded, there is an open interval (c, d) containing $f(E)$. Consider a partition

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$

such that $y_k - y_{k-1} < \varepsilon$. Define $E_k = f^{-1}([y_{k-1}, y_k))$ and the functions

$$\psi = \sum_{k=1}^{n} y_k \chi_{E_k} \quad \text{and} \quad \varphi = \sum_{k=1}^{n} y_{k-1} \chi_{E_k}$$

Obviously, for all x, $\psi - \varphi < \varepsilon$, further, there is a unique k such that $x \in E_k$. Then $\varphi(x) = y_{k-1} < f(x) < y_k = \psi(x)$. This completes the proof. ■

Theorem 2.15 (Simple Approximation Theorem). *Let E be measurable and $f : E \to [-\infty, \infty]$ is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converge pointwise on E to f and has the property that $|\varphi_n| \leq |f|$ on E for all n. If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.*

Proof. If there is a sequence of simple functions that converge to f, then f is measurable since it is the pointwise limit of measurable functions.

Now suppose f is measurable. We shall first prove the statement in the case when f is nonnegative. Let $E_n = \{x \in E \mid f(x) \leq n\}$, which is measurable. Due to Lemma 2.14, there are simple functions φ_n and ψ_n on E_n such that

$$0 \leq \varphi_n \leq f \leq \psi_n \quad \text{and} \quad 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E_n$$

Extend both φ_n and ψ_n to E by defining $\varphi_n(x) = \psi_n(x) = n$ when $x \in E \backslash E_n$. It is not hard to show that the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ converge pointwise to f on E. Now, let $\varphi_n' = \max\{\varphi_1, \ldots, \varphi_n\}$. Then, $\{\varphi_n'\}$ is an increasing sequence of functions that converge to f.

Now, we shall prove the problem statement for a general function $f = f^+ - f^-$. Since f^+ and f^- are nonnegative functions, there are sequences φ_n^+ and φ_n^- of simple functions satisfying the assertion of the theorem. Note that the functions φ_n^+ and φ_n^- are defined in such a way that they do not both take nonzero values at the same point. Then,

$$|\varphi_n^+ - \varphi_n^-| = \varphi_n^+ + \varphi_n^- \leq f^+ + f^- = |f|$$

which completes the proof. ■

2.1.1 Step Functions

A step function is a special kind of simple function. It is of the form $\sum_{k=1}^{n} \alpha_k \chi_{E_k}$ where each E_k is an interval. One must note that step functions are not as strong as simple functions, that is to say that they do not approximate measurable functions as well as simple functions do.

Theorem 2.16 (Step Approximation Theorem). *Let I be a closed, bounded interval and $f : I \to \mathbb{R}$ a bounded measurable function. Let $\varepsilon, \delta > 0$. Then there is a step function h on I and a measurable subset F of I for which*

$$|h - f| < \varepsilon \text{ and } m(I \backslash F) < \delta$$

To prove the above theorem, we require a series of lemmas.

Lemma 2.17. *Let $E \subseteq I$ be measurable. Let $\varepsilon > 0$. Show that there is a step function $h : I \to \mathbb{R}$ and a measurable subset F of I for which*

$$h = \chi_E \text{ on } F, \text{ and } m(I \backslash F) < \varepsilon$$

Proof. Let U be an open set containing U such that $m(U \backslash E) < \varepsilon/2$. Now, U may be written as as disjiont union of open intervals $\{I_k\}_{k=1}^{\infty}$ where some intervals may be empty. Using the continuity of measure, there is a positive integer N such that

$$\sum_{k=N+1}^{\infty} \ell(I_k) < \varepsilon/2$$

Define $\mathcal{O} = \bigcup_{k=1}^{N} I_k$ and $F = (\mathcal{O} \cap E) \cup (I \backslash U)$. Then,

$$I \backslash F = (U \backslash E) \cup (\mathcal{U} \backslash O)$$

and thus, $m(I \backslash F) \leq m(U \backslash E) + m(\mathcal{U} \backslash O) < \varepsilon$. Finally, define $h = \chi_{\mathcal{O}}$. Since $F \cap \mathcal{O} = \mathcal{O} \cap E \subseteq E$, the restrictions on h are satisfied. ■

Lemma 2.18. *Let $\psi : I \to \mathbb{R}$ be a simple function, $E \subseteq I$ be measurable and $\varepsilon > 0$. Then there is a step function $h : I \to \mathbb{R}$ and a measurable subset F of I for which*

$$h = \psi \text{ on } F, \text{ and } m(I \backslash F) < \varepsilon$$

Proof. Follows from the previous lemma by taking linear combinations. ∎

Proof of Theorem 2.16. Due to Lemma 2.14, there is a simple function $\psi : I \to \mathbb{R}$ such that $0 \leq f - \psi < \varepsilon$. Due to the preceeding lemma, there is a step function $h : I \to \mathbb{R}$ and a measurable subset F of I with $m(I \backslash F) < \delta$ such that $h = \psi$ on F. The conclusion now follows. ∎

2.2 Egoroff and Lusin's Theorems

Theorem 2.19 (Egoroff). *Let E be a measurable set with finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converge pointwise a.e. on E to the extended real-valued function f which is finite a.e. Then for each $\varepsilon > 0$, there is a closed set $F \subseteq E$ for which*

$$\{f_n\} \rightrightarrows f \text{ on } F \text{ and } m(E \backslash F) < \varepsilon$$

Since $\{f_n\} \to f$ pointwise a.e. on E, there is a subset E_0 such that $m(E_0) = 0$ and $\{f_n\} \to f$ pointwise on $E \backslash E_0$. Similarly, there is E_0' such that f is finite on $E \backslash E_0'$ and $m(E_0') = 0$. Let $E' = E \backslash (E_0 \cup E_0')$. Then $m(E') = m(E)$ and $\{f_n\} \to f$ pointwise on E' and f is finite on all of E'. Then, if we prove Egoroff's Theorem on E', we woud have a closed subset F of E' such that $\{f_n\} \rightrightarrows f$ on F and $m(E \backslash F) = m(E' \backslash F) < \varepsilon$, and hence would prove it for the general case. Therefore, without loss of generality, we may suppose that $\{f_n\} \to f$ pointwise on E and is finite on all of E.

In order to prove the reduced statement of Egoroff's Theorem, we require the following lemma,

Lemma 2.20. *Let E have finite measure and $\{f_n\} \to f$ pointwise on E where f is finite on all of E. Then, for every $\eta > 0$ and $\delta > 0$, there is a measurable subset $A \subseteq E$ and index $N \in \mathbb{N}$ such that*

$$|f_n - f| \leq \eta \; \forall \; n \geq N \text{ on } A \text{ and } m(E \backslash A) < \delta$$

Proof. Define the collection of sets $\{F_n\}$ as

$$F_n := \{x \in E \mid |f(x) - f_n(x)| < \eta\}$$

Since the function $|f - f_n|$ is measurable, so is the set F_n. Now, define

$$E_n := \bigcap_{k=n}^{\infty} F_k = \{x \in E \mid |f(x) - f_n(x)| < \eta, \; \forall n \geq N\}$$

Since E_n is the countable intersection of measurable sets, it is measurable. Furthermore, by definition, the collection $\{E_n\}$ forms an ascending chain satisfying

$$\bigcup_{n=1}^{\infty} E_n = E$$

Consequently, $\lim_{n \to \infty} m(E_n) = m(E) < \infty$. Hence, there is $N \in \mathbb{N}$ such that $m(E \backslash E_N) = m(E) - m(E_N) < \delta$. Let $A = E_N$. Then, for all $x \in A$, $|f(x) - f_n(x)| < \eta$ for all $n \geq N$. This completes the proof. ∎

We can now prove Theorem 2.19.

Proof of Theorem 2.19. Due to our discussion above, we may suppose that $f_n \to f$ pointwise on E and f is finite on all of E. Let A_n be such that there is an index N with $|f - f_k| \leq 1/n$ for all $k \geq N$ and $m(E \backslash A_n) < \varepsilon/2^{n+1}$. Define $A = \bigcap_{n=1}^{\infty} A_n$. Then

$$m(E \backslash A) \leq \sum_{n=1}^{\infty} m(E \backslash A_n) < \varepsilon/2$$

We claim that $f_n \rightrightarrows f$ uniformly on A. Choose some $\eta > 0$. Then, there is $N \in \mathbb{N}$ such that $1/N < \eta$. Consequently, for all $x \in A_N$, there is an index M such that for all $k \geq M$, $|f - f_k| < \eta$. But since $A \subseteq A_N$, we have that for all $x \in A$, there is an index M such that for all $k \geq M$, $|f(x) - f_k(x)| < \eta$, which implies uniform convergence.

Now, since A is the countable intersection of measurable sets, it is measurable. As a result, due to Theorem 1.19, there is a closed set $F \subseteq A$ with $m(A \backslash F) < \varepsilon/2$. Thus, $F \subseteq E$ and $m(E \backslash F) < \varepsilon$ and $f_n \rightrightarrows f$ on F. This completes the proof. ∎

Let us consider the case when $m(E) = \infty$. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = x\left(1 - \frac{1}{n}\right)$$

Then, it is not hard to see that $f_n \to f$ on $E = \mathbb{R}$ and f is finite on E. Choose any $\varepsilon > 0$ and let $F \subseteq \mathbb{R}$ be a closed subset of $\mathbb{R}$ such that $m(\mathbb{R} \backslash F) < \varepsilon$. Since $\mathbb{R} \backslash F$ is open in $\mathbb{R}$, it is the disjiont union of open intervals. Further, since it has finite measure, all the disjiont intervals must be bounded. As a result, F is not bounded.

We now claim that f_n may not converge uniformly to f on F. Suppose it did, then, pick some $\delta > 0$. Then, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|f - f_n| < \delta$ on F. But this implies $|x/n| < \delta$ on F, which is absurd, since F is unbounded.

The above discussion shows that Egoroff's theorem may not hold on dropping the finite measure hypothesis.

Next, we come to Lusin's Theorem, illustrating the third of Littlewood's three principles. We shall state the theorem first, then prove a useful lemma and finally prove the theorem.

Lemma 2.21. *Let f be a simple function defined on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E \backslash F) < \varepsilon$.*

Proof. Let $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ where the sets E_i are disjoint. Due to Theorem 1.19, there are closed sets F_i with $m(E_i \backslash F_i) < \varepsilon/n$ and $F_i \subseteq E_i$. Let $F = \bigsqcup_{k=1}^{n} F_i$. Then F is closed. Define the function g on F by $g(x) = a_i$ if $x \in F_i$. Note that this function is well defined because the F_i are disjoint. Then, using Tietze's Extension Theorem, we may extend g to a continuous function on all of $\mathbb{R}$.

Finally, note that

$$m(E \backslash F) = m\left(\bigsqcup_{k=1}^{n} (E_k \backslash F_k)\right) = \sum_{k=1}^{n} m(E_k \backslash F_k) < \varepsilon$$

This completes the proof. ∎

Lusin's Theorem essentially extends the above proposition to general measurable functions.

Theorem 2.22 (Lusin). *Let $f : E \to \mathbb{R}$ be real valued measurable. Then for each $\varepsilon > 0$, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed set $F \subseteq E$ such that $f = g$ on F and $m(E \backslash F) < \varepsilon$.*

Proof. We divide the proof into two cases, one for when $m(E) < \infty$ and the other for when $m(E) = \infty$.

$\underline{m(E) < \infty}$: Due to the Simple Approximation Theorem, there is a sequence $\varphi_n : E \to \mathbb{R}$ of simple functions that converge to f on E. Using the preceeding lemma, there is a sequence of continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ and closed sets F_n such that the restriction of g_n to F_n is φ_n and $m(E \backslash F_n) < \varepsilon/2^{n+2}$. Further, due to Egoroff's Theorem, there is a closed set F_0 that is contained in E such that $\{f_n\}$ converges to f uniformly on F_0 and $m(E\backslash F_0) < \varepsilon/2^2$. Define $F = \bigcap_{n=0}^{\infty}$. Then $m(E\backslash F) \leq \varepsilon/2 < \varepsilon$ and since φ_n converge uniformly on F_0 and thus on F, so do g_n and their pointwise limit g is continuous on F. Finally, due to the Tietze Extension Theorem, this function may be extended to a continuous function $g : \mathbb{R} \to \mathbb{R}$.

$\underline{m(E) = \infty}$: Consider the collection $\{E \cap [k, k+1)\}_{k \in \mathbb{Z}}$, which is a countable collection of disjoint sets whose union is E. Reindex this set as $\{E_n\}$. For each E_n, there is a closed subset F_n such that $m(E_n \backslash F_n) < \varepsilon/2^{n+1}$ and there is a continuous function g_n on E_n which agrees with f on F_n. Since the collection F_n is locally finite, the Pasting Lemma holds and there is a continuous function g on $\bigcup F_n$ which agrees with f. Again, since F_n is locally finite, the union $\bigcup F_n$ is closed and thus, we may use the Tietze Extension Theorem.

■

Chapter 3

Lebesgue Integration

3.1 Lebesgue Integral of Bounded Function on Finite Measure Sets

In this section we shall study only bounded functions on sets of finite measure and show that all Riemann Integrable functions are Lebesgue Integrable. Of course, to prove this result, it suffices to consider a domain of finite measure since the Riemann Integral is defined over a bounded interval.

We shall also show in this section that all bounded measurable functions on a set of finite measure are integrable, due to Lemma 2.14. In the next section, we shall extend our theory of integration to general measurable functions, which need not be finite and which may be defined over sets of infinite measure.

Definition 3.1 (Integral of Simple Functions). Let $\psi : E \to \mathbb{R}$ be a simple function with canonical representation

$$\psi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

Then, we define the integral of ψ over E by

$$\int_E \psi = \sum_{i=1}^{n} \alpha_i m(E_i)$$

Proposition 3.2. *Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set of finite measure E. If $\varphi = \sum_{i=1}^n \alpha_i \chi_{E_i}$ on E, then*

$$\int_E \varphi = \sum_{i=1}^{n} \alpha_i m(E_i)$$

Proof. Trivial. ■

Proposition 3.3 (Simple Linearity and Monotonicity). *Let φ and ψ be simple functions on a set of finite measure E. Then for any $\alpha, \beta \in \mathbb{R}$,*

$$\int_E \alpha\varphi + \beta\psi = \alpha \int_E \varphi + \beta \int_E \psi$$

Further, if $\varphi \leq \psi$ on E, then $\int_E \varphi \leq \int_E \psi$.

Proof. Since φ and ψ take finitely many distinct values, we may represent E as the disjoint union of measurable sets E_i, such that φ and ψ are constant on each E_i. Then, we may write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i} \qquad \psi = \sum_{i=1}^{n} b_i \chi_{E_i}$$

As a result, we have

$$\begin{aligned} \alpha\varphi + \beta\psi &= \sum_{i=1}^{n} (\alpha a_i + \beta b_i)\chi_{E_i} \\ \implies \int_E \alpha\varphi + \beta\psi &= \sum_{i=1}^{n} (\alpha a_i + \beta b_i) m(E_i) = \alpha \sum_{i=1}^{n} a_i m(E_i) + \beta \sum_{i=1}^{n} b_i m(E_i) = \alpha \int_E \varphi + \beta \int_E \psi \end{aligned}$$

Next,

$$\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) \geq 0$$

since $\psi - \varphi \geq 0$. ∎

Definition 3.4 (Upper and Lower Integrals). Let $E \subseteq \mathbb{R}$ have finite measure and $f : E \to \mathbb{R}$ be a bounded function. We define the lower and upper Lebesgue integral of f over E to be

$$\overline{\int_E} f = \sup \left\{ \int_E \varphi \,\middle|\, \varphi \text{ simple and } \varphi \leq f \text{ on } E \right\}$$
$$\underline{\int_E} f = \inf \left\{ \int_E \varphi \,\middle|\, \varphi \text{ simple and } f \leq \varphi \text{ on } E \right\}$$

Then f is said to be *Lebesgue Integrable* over E provided its upper and lower Lebesgue intnegrals over E are equal. The common value is termed the *Lebesgue Integral* of f over E and is denoted by $\int_E f$.

Since every step function is simple we immediately have that **every Riemann Integrable function is Lebesgue Integrable.**

Theorem 3.5. *Let $E \subseteq \mathbb{R}$ have finite measure and $f : E \to \mathbb{R}$ be a bounded measurable function. Then f is integrable over E.*

Proof. Due to Lemma 2.14, there are simple functions φ, ψ such that $\varphi \leq f \leq \psi$ and $\psi - \varphi < 1/n$. Consequently,

$$\int_E \psi - \varphi \leq \frac{1}{n} m(E)$$

This immediately implies the desired conclusion. ∎

Theorem 3.6 (Linearity and Monotonicity). *Let $E \subseteq \mathbb{R}$ have finite measure and $f, g : E \to \mathbb{R}$ be bounded measurable functions. Then for any $\alpha, \beta \in \mathbb{R}$*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

and if $f \leq g$ on E, then

$$\int_E f \leq \int_E g$$

Proof. We shall show linearity in two steps. First, we shall show that $\int_E \alpha f = \alpha \int_E f$. The case for $\alpha = 0$ is trivial. Let us consider the case $\alpha > 0$, since the other case follows analogously.

$$\int_E \alpha f = \sup_{\psi \le \alpha f} \int_E \psi = \sup_{\varphi \le f} \int_E \alpha\varphi = \alpha \int_E f$$

Next, we shall show that $\int_E (f+g) = \int_E f + \int_E g$. First note that since f and g are bounded, so is $f+g$. Now, for every pair of simple functions (ϕ, φ) with $\phi \le f$ and $\varphi \le g$, we have $\phi + \varphi \le f + g$ and thus,

$$\int_E (f+g) = \sup_{\psi \le f+g} \int_E \psi \ge \int_E (\phi + \varphi) = \int_E \phi + \int_E \varphi$$

Taking the supremum on both sides, we have $\int_E (f+g) \ge \int_E f + \int_E g$. A similar inequality can be obtained in the reverse direction and the conclusion follows.

Finally, for monotonicity, note that $g - f \ge 0$ on E and therefore, $\int_E (g-f) \ge 0$ and using linearity, we have $\int_E g \ge \int_E f$. This completes the proof. ■

Proposition 3.7. *Let E have finite measure and $f : E \to \mathbb{R}$ be bounded. Then for any measurable $E_1 \subseteq E$,*

$$\int_{E_1} f = \int_E f\chi_{E_1}$$

Proof. Let φ be a simple function such that $\varphi \le f$ on E_1. Extend φ to the function $\phi : E \to \mathbb{R}$ on E that is defined as

$$\phi(x) = \begin{cases} \varphi(x) & x \in E_1 \\ 0 & x \notin E_1 \end{cases}$$

It is not hard to see that $\phi \le f\chi_{E_1}$ on E and thus

$$\int_E f\chi_{E_1} \ge \sup_{\phi} \int_E \phi = \sup_{\varphi} \int_{E_1} \varphi = \int_{E_1} f$$

Conversely, let φ be a simple function such that $\varphi \ge f$ on E_1, then the extension ϕ is such that $\phi \ge f\chi_{E_1}$ on E and therefore,

$$\int_E f\chi_{E_1} \le \inf_{\phi} \int_E \phi = \inf_{\varphi} \int_{E_1} \varphi = \int_{E_1} f$$

This completes the proof. ■

Corollary. Let E have finite measure and $f : E \to \mathbb{R}$ be bounded. Suppose A and B are disjoint measurable subsets of E. Then,

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof. Note that $\chi_{A\cup B} = \chi_A + \chi_B$. The conclusion is obvious now. ■

Lemma 3.8. *Let E have finite measure and $f : E \to \mathbb{R}$ be bounded. Then*

$$\left| \int_E f \right| \le \int_E |f|$$

Proof. Obviously, we have $-|f| \leq f \leq |f|$, then, using monotonicity and linearity of integration, we have

$$-\int_E |f| \leq \int_E f \leq \int_E |f|$$

The conclusion is now obvious. ■

Proposition 3.9. *Let E have finite measure and $f_n : E \to \mathbb{R}$ be a sequence of bounded measurable functions that converges uniformly to $f : E \to \mathbb{R}$ which is a bounded function. Then,*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Let $\varepsilon > 0$ be given. Since the convergence is uniform, there is an index $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n - f| < \varepsilon/m(E)$. Using the above lemma, for all $n \geq N$, we have

$$\left|\int_E (f_n - f)\right| \leq \int_E |f_n - f| \leq \varepsilon$$

The conclusion is obvious. ■

Theorem 3.10 (Bounded Convergence Theorem). *Let E have finite measure and $f_n : E \to \mathbb{R}$ be a sequence of measurable functions. Suppose $\{f_n\}$ is uniformly pointwise bounded on E. If $\{f_n\} \to f$ pointwise on E, then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. First, since f_n converge pointwise to f, the latter is measurable. There is some $M > 0$ such that for all $n \in \mathbb{N}$, $|f_n| \leq M$. As a result, f is also bounded and therefore, the integral is well defined. Let $\varepsilon > 0$ be given. Due to Theorem 2.19, there is a closed subset F of E with $m(E\backslash F) < \varepsilon/4M$ and $f_n \rightrightarrows f$ on F. Furthermore, due to uniform convergence, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n - f| < \varepsilon/2m(E)$. As a result, for all $n \geq N$, we have

$$\left|\int_E (f_n - f)\right| \leq \left|\int_F (f_n - f) + \int_{E\backslash F} f_n - f\right| \leq \int_F |f_n - f| + \int_{E\backslash F} |f_n - f| \leq \int_F \frac{\varepsilon}{2m(E)} + \int_{E\backslash F} 2M \leq \varepsilon$$

This completes the proof. ■

Note that dropping the uniform boundedness hypothesis will not work. Take for example the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x < 1/n \\ 2n - n^2 x & 1/n \leq x < 2/n \\ 0 & 2/n \leq x \leq 1 \end{cases}$$

It is not hard to see that f_n converges to the zero function pointwise, but $\int_{[0,1]} f_n = 1$ for each $n \in \mathbb{N}$, and hence the sequence of integrals do not converge.

Next, the Bounded Convergence Theorem does not hold for the Riemann Integral. To see this, let $\{q_1, q_2, \ldots\}$ be the enumertion of rationals in $[0,1]$. Define the sequence of Riemann Integrable functions $f_n : [0,1] \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} 1 & x = r_k,\ 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

We see that $|f_n| \leq 1$ for all $n \in \mathbb{N}$ and f_n converges pointwise to $\chi_{\mathbb{Q}\cap[0,1]}$ which is not Riemann integrable.

3.2 Lebesgue Integral of Nonnegative Measurable Functions

In this section we study the integrals of measurable functions that are not necessarily bounded over domains that are not necessarily having finite measure. This is of course an extension of the theory we have built in the previous section.

Definition 3.11 (Support). Let $f : E \to [-\infty, \infty]$ be a measurable function. The *support* of f, denoted by $\text{supp}(f) = \{x \in E \mid f(x) \neq 0\}$. The function h is said to have *finite suppor* if $\text{supp}(f)$ has finite measure.

If $f : E \to \mathbb{R}$ is bounded, measurable and has finite support, we define

$$\int_E f = \int_{\text{supp}(f)} f$$

which is well defined, since $\text{supp}(h)$ is a measurable subset of E and thus Lebesgue measurable, further, since it has finite measure and f is bounded, the integral is as defined in the previous section.

Definition 3.12. Let E be a measurable subset of $\mathbb{R}$ and $f : E \to [0, \infty]$, a nonnegative measurable function on E. We define the integral of f over E by

$$\int_E f = \sup \left\{ \int_E h \,\middle|\, h \text{ bounded, measurable and of finite support } 0 \leq h \leq f \text{ on } E \right\}$$

Theorem 3.13 (Chebychev's Inequality). *Let $E \subseteq \mathbb{R}$ be measurable and $f : E \to [0, \infty]$ be a measurable function. Then for any $\lambda > 0$,*

$$m\left(\{x \in E \mid f(x) \geq \lambda\}\right) \leq \frac{1}{\lambda} \int_E f$$

Proof. Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$. If $m(E_\lambda) = \infty$. Further, define $E_\lambda^{(n)} = [-n, n] \cap E_\lambda$. Due to Theorem 1.25, $\lim_{n\to\infty} E_\lambda^{(n)} = \infty$. Note that the function $\lambda\chi_{E_\lambda^{(n)}} \leq f$ on E and is bounded, measurable with finite support, and therefore,

$$\lambda m(E_\lambda^{(n)}) \leq \int_E f$$

Taking supremum on both sides, we havethe desired conclusion. Next, suppose $m(E_\lambda) < \infty$. Then, the function $\lambda\chi_{E_\lambda}$ is bounded measurable with finite support and is $\leq f$ on E, whence the conclusion follows. ■

Proposition 3.14. *Let $E \subseteq \mathbb{R}$ be measurable and $f : E \to [0, \infty]$ be a measurable function. Then*

$$\int_E f = 0 \text{ if and only if } f = 0 \text{ a.e. on } E$$

Proof. Define the set

$$E_n := \{x \in E \mid f(x) \geq 1/n\}$$

Then, due to Theorem 3.13, $m(E_n) \leq 0$ and thus $m(E_n) = 0$. Finally using the continuity of measure,

$$m(\{x \in E \mid f(x) > 0\}) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) = 0$$

The conclusion follows. ■

Theorem 3.15. *Let $E \subseteq \mathbb{R}$ be measurable and $f, g : E \to [0, \infty]$ be measurable. Then for any $\alpha > 0$ and $\beta > 0$,*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

Moreover, if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

Proof. First, for any h, note that $h \leq f$ if and only if $\alpha h \leq \alpha f$ and it follows that $\int_E \alpha f = \alpha \int_E f$. We would now show that $\int_E f + g = \int_E f + \int_E g$. First, one direction of the inequality is trivial, since for every pair (h, k) of nonnegative bounded measurable functions of finite support with $h \leq f$ and $k \leq g$, we have $h + k \leq f + g$ and hence,

$$\int_E h + \int_E k = \int_E (h + k) \leq \int_E (f + g)$$

and taking the supremum on both sides, we have $\int_E f + \int_E g \leq \int_E (f + g)$. It suffices to show that other direction of the inequality.

Let ℓ be a nonnegative bounded measurable function of finite support satisfying $\ell \leq f + g$. Let us define $h = \min\{\ell, f\}$ and $k = \ell - h$. It is not hard to see that h, k are bounded measurable functions of finite support on E satisfying $h \leq f$ and $k \leq g$. As a result,

$$\int_E \ell = \int_E h + \int_E k \leq \int_E f + \int_E g$$

and taking the supremum, the conclusion follows. The assertion about monotonicity follows from noting that $g - f$ is a nonnegative function on E and therefore has nonnegative integral and finally using linearity. ∎

Lemma 3.16. *Let $E \subseteq \mathbb{R}$ be measurable and $f : E \to [0, \infty]$ be measurable. Then, for a measurable subset E_1 of E,*

$$\int_{E_1} f = \int_E f \chi_{E_1}$$

Proof. ∎

Theorem 3.17 (Additivity over Domains). *Let $E \subseteq \mathbb{R}$ be measurable and $A, B \subseteq E$ be disjoint measurable. Then,*

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof. Follows from the previous lemma. ∎

Corollary. Let $E_0 \subseteq E$ have measure 0. Then

$$\int_E f = \int_{E \setminus E_0} f$$

Lemma 3.18 (Fatou's Lemma). *Let $E \subseteq \mathbb{R}$ be measurable and $\{f_n\}$ be sequence of nonnegative measurable* *functions on E that converge pointwise a.e. on E to f. Then*

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

Proof. First, we may suppose without loss of generality that the convergence is everywhere on E (this is not hard to reason). Next, let h be a nonnegative bounded measurable function of finite support such that $0 \leq h \leq f$. Define $h_n = \min\{f_n, h\}$. Then, $h_n \to h$ on E. Further, each $h_n \leq h$ and is therefore pointwise uniformly bounded (since h is bounded).

Due to Theorem 3.10, $\int_E h_n \to \int_E h$. Furthermore, since $h_n \leq f_n$, we have $\int_E h_n \leq \int_E f_n$ for each $n \in \mathbb{N}$. Taking lim inf, we have

$$\int_E h = \lim_{n\to\infty} \int_E h_n = \liminf_{n\to\infty} \int_E h_n \leq \liminf_{n\to\infty} \int_E f_n$$

The conclusion follows. ∎

Example 1. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \leq f$ on E for each n. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Due to Fatou's Lemma and the fact that $f_n \leq f$, we have

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n \leq \limsup_{n\to\infty} \int_E f_n \leq \int_E f$$

The conclusion now follows. ∎

Theorem 3.19 (Monotone Convergence Theorem). *Let $E \subseteq \mathbb{R}$ be measurable and $\{f_n\}$ an increasing sequence of nonnegative measurable functions on E. Let $f : E \to [0, \infty]$ be such that $f_n \to f$ a.e. on E. Then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Since the sequence $\{f_n\}$ is increasing, so is the sequence $\{\int_E f_n\}$ and therefore, converges to some extended real number. Now, Lemma 3.18 gives us

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n \leq \int_E f$$

The conclusion follows. ∎

Corollary. Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E, then

$$\int_E f = \sum_{n=1}^{\infty} \int_E u_n$$

Proof. Trivial and omitted. ∎

Note that the Monotone Convergence Theorem may not hold for decreasing sequences of positive measurable functions, take for example the sequence $\{f_n = \chi_{(n,\infty)}\}_{n\in\mathbb{N}}$. The pointwise limit of this sequence is the zero function but $\int_E f_n = \infty$ for each $n \in \mathbb{N}$.

The following is a generalization of Lemma 3.18 but we shall use Theorem 3.19 to prove it which in turn depends on Lemma 3.18.

Lemma 3.20 (Generalized Fatou's Lemma). *Let $E \subseteq \mathbb{R}$ be measurable and $\{f_n\}$ be a sequence of nonnegative measurable functions. Then*

$$\int_E \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n$$

Proof. Define the function

$$g_k = \inf_{i \ge k} f_i$$

Then, $g_1 \le g_2 \le \cdots$ and $g_n \to \liminf_{n\to\infty} f_n$. By definition, we also have

$$\int_E g_k \le \int_E f_k$$

Taking $\liminf_{n\to\infty}$ and using Theorem 3.19, we have

$$\int_E \liminf_{n\to\infty} f_n = \lim_{n\to\infty} \int_E g_n = \liminf_{n\to\infty} \int_E g_n \le \liminf_{n\to\infty} \int_E f_n$$

This completes the proof. ■

Definition 3.21 (Integrable). A nonnegative measurable function f on a measurable set E is said to be *integrable* on E provided

$$\int_E f < \infty$$

Proposition 3.22. *Let $E \subseteq \mathbb{R}$ be measurable and f be integrable over E. Then f is finite a.e. on E.*

Proof. Due to Theorem 3.13, for each $n \in \mathbb{N}$,

$$m(\{x \in E \mid f(x) \ge n\}) \le \frac{1}{n} \int_E f$$

since $\int_E f$ is finite, the right hand side tends to 0 as $n \to \infty$. The conclusion follows. ■

Proposition 3.23. *Let $E \subseteq \mathbb{R}$ be measurable and $f : E \to [0, \infty]$ be measurable. Then,*

$$\int_E f = \sup \left\{ \int_E \varphi \;\middle|\; \varphi \text{ simple of finite support and } 0 \le \varphi \le f \right\}$$

Proof. We first show that there is an increasing sequence of nonnegative simple functions with finite support that converges to f. Define $E_n = E \cap [-n, n]$. Consider the simple function φ_n on $\{x \in E_n \mid f(x) \le n\}$, which is measurable and has finite measure since E_n has finite measure. On the remaining E_n, define $\varphi = n$. We now have a sequence of measurable functions, each with finite support that converges to f. To make sure this is increasing, write

$$\psi_n = \max_{1 \le i \le n} \varphi_n$$

Finally, due to the Monotone Convergence Theorem, we have

$$\lim_{n\to\infty} \int_E \varphi_n = \int_E f$$

and the conclusion follows. ■

This establishes that Rudin's definition of the integral is equivalent to Royden's definition of the integeral, albeit the former does it in a more abstract sense.

3.3 General Lebesgue Integral

In this section we shall study the Lebesgue Integral of not necessarily nonnegative measurable functions on a measurable set. This is the integral in its maximum generality. The highlight of this section is the Dominated Convergence Theorem (Theorem 3.29).

Proposition 3.24. *Let $E \subseteq \mathbb{R}$ be measurable and f be a measurable function on E. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.*

Proof. We have $|f| = f^+ + f^-$ and $f^+ \leq |f|$ and $f^- \leq |f|$. The conclusion follows from linearity and monotonicity. ■

Definition 3.25 (Integrable). Let $E \subseteq \mathbb{R}$ be measurable. A measurable function f on E is said to be integrable over E provided $|f|$ is integrable over E. In this case, we define

$$\int_E f = \int_E f^+ - \int_E f^-$$

Proposition 3.26. *Let $E \subseteq \mathbb{R}$ be measurable and let f be integrable over E. Then f is finite a.e. on E and*

$$\int_E f = \int_{E \setminus E_0} f$$

if $E_0 \subseteq E$ and $m(E_0) = 0$.

Proof. Since $|f|$ is integrable, it is finite a.e. on E, thus f is finite a.e. on E. We have

$$\int_{E \setminus E_0} f = \int_{E \setminus E_0} f^+ - \int_{E \setminus E_0} f^- = \int_E f^+ - \int_E f^- = \int_E f$$

■

Proposition 3.27 (Integral Comparison Test). *Let $E \subseteq \mathbb{R}$ be measurable and f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and $|f| \leq g$ on E. Then f is integrable over E and*

$$\left| \int_E f \right| \leq \int_E |f|$$

Proof. Trivial. ■

Theorem 3.28 (Linearity and Monotonicity). *Let $E \subseteq \mathbb{R}$ be measurable and f, g be measurable functions on E. Then, for any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is integrable over E and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

Further, if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

Proof. First, suppose $\alpha > 0$. Then $[\alpha f]^+ = \alpha f^+$ and $[\alpha f]^- = \alpha f^-$. Similarly, when $\alpha < 0$, $[\alpha f]^+ = -\alpha f^-$ and $[\alpha f]^- = -\alpha f^+$. It is not hard to see from here that $\int_E \alpha f = \alpha \int_E f$.

Next, we establish that $\int_E (f+g) = \int_E f + \int_E g$. Since $|f+g| \leq |f| + |g|$, it is integrable due to the Integral Comparison Test. Furthermore, since f and g are integrable, we may suppose without loss of generality that they are finite on E (since they are required to be finite a.e. on E). We have

$$(f+g)^+ - (f+g)^- = (f^+ - f^-) + (g^+ - g^-)$$

rearranging, we obtain

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

Since both sides are nonnegative integrable functions, we have, using the linearity of integration of nonnegative measurable functions,

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+$$

Rearranging the terms, we have the desired conclusion.

Finally, we may suppose without loss of generality that f and g are finite on E, now if $f \leq g$, then $g - f \geq 0$ on E, then

$$0 \leq \int_E (g-f) = \int_E g - \int_E f$$

The conclusion follows. ■

Theorem 3.29 (Dominated Convergence Theorem). *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and $|f_n| \leq g$ on E for all $n \in \mathbb{N}$. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. First, note that $|f| \leq g$ and therefore is integrable due to the Integral Comparison Test. Since each f_n, f and g are finite a.e. and the convergence is pointwise a.e., we may suppose without loss of generality that all the f_n, f and g are finite on E and the convergence is pointwise on all of E. Consider the sequence of nonnegative integrable functions $\{g - f_n\}$ on E. It is not hard to see that this sequence converges pointwise to the nonnegative function $g - f$ on E. As a result, using Lemma 3.18,

$$\int_E (g-f) \leq \liminf_{n\to\infty} \int_E (g - f_n) = \int_E g - \limsup_{n\to\infty} \int_E f_n$$

and thus $\limsup\limits_{n\to\infty} \int_E f_n \leq \int_E f$.

Similarly, consider the sequence of nonnegative integrable functions $\{g + f_n\}$ that converges pointwise to the nonnegative integrable function $g + f$ on E. Using Lemma 3.18, we have

$$\int_E (g+f) \leq \liminf_{n\to\infty} \int_E (g + f_n) = \int_E g + \liminf_{n\to\infty} \int_E f_n$$

and thus $\int_E f \leq \liminf\limits_{n\to\infty} \int_E f_n$. The conclusion follows. ■

3.4 Countable Additivity and Continuity of Integration

Theorem 3.30 (Countable Additivity). *Let $E \subseteq \mathbb{R}$ and $f : E \to [-\infty, \infty]$ be integrable. Let $\{E_n\}$ be a* *disjoint countable collection of measurable subsets of E whose union is E. Then*

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

Proof. Define $A_n = \bigcup_{k=1}^{n} E_k$ and $f_n := f\chi_{A_n}$. Then f_n is measurable on E and $|f_n| \leq |f|$ on E for all $n \in \mathbb{N}$. Further, note that $f_n \to f$ pointwise on E. Since f is measurable, we may invoke the Dominated Convergence Theorem to obtain

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_{A_n} f = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f$$

Since the right hand side is the definition of $\sum\limits_{n=1}^{\infty} \int_{E_n} f$, the proof is complete. ■

Theorem 3.31 (Continuity of Integration).

3.5 The Vitali Convergence Theorem

Theorem 3.32. *Let $f : E \to [-\infty, \infty]$ be measurable. If f is integrable over E, then for each $\varepsilon > 0$, there is $\delta > 0$ such that whenever $A \subseteq E$ is measurable with $m(A) < \delta$, then $\int_E |f| < \varepsilon$.*

Conversely, if $m(E) < \delta$ and for each $\varepsilon > 0$, there is $\delta > 0$ such that whenever $A \subseteq E$ with $m(A) < \delta$, $\int_A |f| < \varepsilon$, then f is integrable on E.

Proof. We shall show this for f^+ and f^- separately, which would immediately imply the result for f. Therefore, we may suppose that $f \geq 0$ on E. Then, by definition, there is a measurable bounded function of finite support f_ε satisfying $0 \leq f_\varepsilon \leq f$ on E and $\int_E f - f_\varepsilon < \varepsilon/2$.

Now, for any $A \subseteq E$, we have

$$\int_A (f - f_\varepsilon) \leq \int_E (f - f_\varepsilon) < \varepsilon/2$$

And hence, $\int_A f < \int_A f_\varepsilon + \varepsilon/2$. Since f_ε is bounded, there is some $M > 0$ such that $0 \leq f_\varepsilon \leq M$ on E and hence, when $m(A) < \varepsilon/2M$, we have

$$\int_A f < \int_A f_\varepsilon + \varepsilon/2 \leq \varepsilon$$

This proves the first assertion.

Conversely, suppose $m(E) < \infty$. Then, there is $\delta > 0$ corresponding to $\varepsilon = 1$. Since we may construct disjoint sets $\{E_i\}_{i=1}^{N}$ such that $E = \bigsqcup\limits_{i=1}^{N} E_i$ and $m(E_i) < \delta$ for each $1 \leq i \leq N$, we have

$$\int_E f = \sum_{i=1}^{N} \int_{E_i} f < N$$

which completes the proof. ■

Definition 3.33 (Uniformly Integrable). A family $\mathcal{F}$ of measurable functions on E is said to be *uniformly integrable over E* provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$, whenever $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \varepsilon$.

Proposition 3.34. *Let $E \subseteq \mathbb{R}$ be measurable and $\mathcal{F} = \{f_i\}_{i=1}^{n}$ be a finite collection of integrable functions on E. Then $\mathcal{F}$ is uniformly integrable over E.*

Proof. Let $\varepsilon > 0$ be given. Since every singleton is uniformly integrable, there is $\delta_i > 0$ corresponding to ε for $\{f_i\}$. Now, just set $\delta = \min_{1 \le i \le n} \delta_i$. ∎

Theorem 3.35 (Vitali Convergence Theorem). *Let $E \subseteq \mathbb{R}$ have finite measure. Suppose the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable over E and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. We shall first show that f is integrable over E. Since the collection $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ corresponding to $\varepsilon = 1$ in the definition of uniform integrability. We may write $E = \bigsqcup_{k=1}^{N} E_k$ where $m(E_k) < \delta$. Therefore,

$$\int_E |f_n| = \sum_{k=1}^{N} \int_{E_n} |f_n| < N$$

Note that since $f_n \to f$ pointwise a.e. on E, we must have that $|f_n| \to |f|$ pointwise a.e. on E. Therefore, due to Fatou's Lemma,

$$\int_E |f| \le \liminf_{n\to\infty} \int_E |f_n| \le N$$

This shows that $|f|$ and hence f is integrable. Therefore, f is finite a.e. on E, hence, we may suppose without loss of generality that f is real valued on E and the convergence is pointwise on E.

Now, we have

$$\left| \int_E f - \int_E f_n \right| = \left| \int_E (f - f_n) \right| \le \int_E |f - f_n|$$

Let $\varepsilon > 0$ be given. Correspondingly, there is $\delta > 0$ corresponding to $\varepsilon/3$ in the definition of uniform integrability. Due to Egoroff's Theorem (Theorem 2.19), there is a subset $A \subseteq E$ with $m(A) < \delta$ such that $E \backslash A$ is closed and the convergence $f_n \to f$ is uniform on E.

Further, since $m(A) < \delta$, we have $\int_A |f_n| < \varepsilon/3$ for each $n \in \mathbb{N}$ and due to Fatou's Lemma, we have $\int_A |f| \le \varepsilon/3$. Finally, there is some $N \in \mathbb{N}$ such that for all $n \ge N$, $|f_n - f| \le \varepsilon/3m(E\backslash A)$. Putting all this together, we have, for all $n \ge N$ that

$$\left| \int_E f - \int_E f_n \right| \le \int_E |f - f_n| \le \int_{E\backslash A} |f_n - f| + \int_A |f_n| + \int_A |f| < \varepsilon$$

This completes the proof. ∎

Chapter 4

L^p Spaces

4.1 Introduction

Let $E \subseteq \mathbb{R}$ be measurable and $\mathcal{F}$ be the set of all extended real valued measurable functions defined on E. We define the equivalence relation $\sim$ on $\mathcal{F}$ by $f \sim g$ if and only if $f = g$ a.e. on E. That this is an equivalence relation is trivial.

For $1 \leq p < \infty$, define $L^p(E)$ to be the collection of equivalence classes $[f]$ for which

$$\int_E |f|^p < \infty$$

It is not hard to see that this property is well defined by taking any representative of an equivalence class. Note that $L^1(E)$ is the collection of equivalence classes of integrable functions on E.

We contend that $L^p(E)$ is an $\mathbb{R}$-vector space. Indeed, note that for real numbers a, b we have

$$|a+b| \leq |a| + |b| \leq 2\max\{|a|, |b|\}$$

and consequently,

$$|a+b|^p \leq 2^p \max\{|a|^p, |b|^p\}$$

This shows that if $[f], [g] \in L^p(E)$, then so does $[f+g]$.

A function $f \in \mathcal{F}$ is said to be *essentially bounded* if there is $M \geq 0$, called an *essential upper bound* for f, for which $|f(x)| \leq M$ for almost all $x \in E$. It is not hard to show that $L^\infty(E)$ is an $\mathbb{R}$-vector space. In conclusion, all the L^p spaces are $\mathbb{R}$-vector spaces for $1 \leq p < \infty$.

4.2 Some Inequalities

Theorem 4.1 (Young). *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For nonnegative reale numbers a, b,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Consider the function $f(x) = \frac{1}{p}x^p + \frac{1}{q} - x$. Note that $f(1) = 0$. Moreover, $f'(x) < 0$ on $(0,1)$ and $f'(x) > 0$ on $(1,\infty)$. As a result, $f(x) \geq 0$ for each $x \in [0,\infty)$. Substituting $x = a/b^{q-1}$ we obtain the desired inequality. ■

Note, if $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then p and q are said to be *conjugates* of one another.

Theorem 4.2 (Hölder). *Let E be a measurable set, $1 \leq p < \infty$ and q the conjugate of p. If $f \in L^p(E)$ and $g \in L^q(E)$, then fg is integrable over E and*

$$\int_E |fg| \leq \|f\|_p \|g\|_q$$

Moreover, if $f \neq 0$, the function $f^* = \|f\|_p^{1-p} \operatorname{sgn}(f)|f|^{p-1}$ belongs to $L^q(E)$ and

$$\int_E ff^* = \|f\|_p$$

and $\|f\|_q = 1$.

Proof. First, we analyze the case $p > 1$. Upon replacing f by $f/\|f\|_p$ and g by $g/\|g\|_q$, we need only show, for $\|f\|_p = 1$ and $\|g\|_q = 1$ that $\int_E |fg| \leq 1$. First, since $|f|^p$ and $|g|^q$ are integrable over E, they are finite a.e. on E. Hence, due to Young's Inequality,

$$|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q} \text{ a.e. on } E$$

Then, due to the Integral Comparison Test, $|fg|$ is integrable on E, further,

$$\int_E |fg| \leq \int_E \left(\frac{|f|^p}{p} + \frac{|g|^q}{q} \right) = 1$$

Next, note that

$$\int_E ff^* = \int_E \|f\|_p^{1-p} |f|^p = \|f\|_p$$

and

$$\|f^*\|_q = \|f\|_p^{1-p} \left(\int_E |f|^p \right)^{\frac{1}{q}} = \|f\|_p^{1-p} \|f\|_p^{p/q} = 1$$

We now prove the statement for $p = 1$. On E, we have $|fg| \leq |f| \|g\|_\infty$, thus $|fg|$ is integrable and

$$\int_E |fg| \leq \int_E |f| \|g\|_\infty \leq \|f\|_1 \|g\|_\infty$$

The second part of the assertion is trivial for $p = 1$. ∎

Theorem 4.3 (Minkowski). *Let $E \subseteq \mathbb{R}$ be measurable and $1 \leq p \leq \infty$. If $f, g \in L^p(E)$, then so does $f + g$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. We have

$$\begin{aligned}
\|f + g\|_p &= \int_E (f + g)(f + g)^* \\
&= \int_E f(f + g)^* + \int_E g(f + g)^* \\
&\leq \left(\|f\|_p + \|g\|_p \right) \|(f + g)^*\|_p \\
&= \|f\|_p + \|g\|_p
\end{aligned}$$

This completes the proof. ∎

Theorem 4.4. *For $1 \le p \le \infty$, $\left(L^p(E), \|\cdot\|_p\right)$ forms a normed $\mathbb{R}$-vector space.*

Proof. That it forms an $\mathbb{R}$-vector space has been established. Further, due to Minkowski's Inequality, we conclude that $\|\cdot\|_p$ is indeed a norm on $L^p(E)$. ■

4.3 Riesz-Fischer Theorem

Definition 4.5 (Rapidly Cauchy Sequences). Let $(V, \|\cdot\|)$ be a normed vector space. A sequence $\{v_n\}$ in V is said to be rapidly Cauchy if there is a sequence $\{\varepsilon_n\}$ of positive reals such that $\sum\limits_{n=1}^{\infty} \varepsilon_n$ converges and $\|v_{n+1} - v_n\| \le \varepsilon_n^2$ for each $n \in \mathbb{N}$.

Obviously, every rapidly Cauchy sequence is Cauchy. The following proposition gives a partial converse.

Proposition 4.6. *Let $(V, \|\cdot\|)$ be a normed vector space. Then every Cauchy sequence has a rapidly Cauchy subsequence.*

Proof. For each $k \in \mathbb{N}$, there is $N_k \in \mathbb{N}$ such that for all $m, n \ge N_k$, $\|v_m - v_n\| < 1/2^k$. We may choose N_k as an increasing sequence, whence it follows that $\{v_{N_k}\}$ is a rapidly Cauchy sequence. ■

Theorem 4.7. *Let E be a measurable set and $1 \le p \le \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.*

Proof. First, by excising a suitable subset of measure 0 from E, we may suppose that all functions in the rapidly Cauchy sequence are real valued. We divide the proof into two cases.

Case 1: $1 \le p < \infty$**.** Let $\{f_n\}$ be a rapidly Cauchy sequence in $L^p(E)$ and $\{\varepsilon_n\}$ be the corresponding sequence such that $\|f_{n+1} - f_n\|_p < \varepsilon_n^2$.

We now have

$$\begin{aligned} m\left(\{x \in E : |f_{k+1}(x) - f_k(x)| \ge \varepsilon_k\}\right) &= m\left(\{x \in E : |f_{k+1}(x) - f_k(x)|^p \ge \varepsilon_k^p\}\right) \\ &\le \frac{1}{\varepsilon_k^p} \int_E |f_{k+1} - f_k|^p \\ &< \varepsilon_k^p \end{aligned}$$

Since $\sum_{n=1}^{\infty} \varepsilon_n$ converges, so does $\sum_{n=1}^{\infty} \varepsilon_n^p$. Then, due to Lemma 1.26, every $x \in E$ belongs to at most finitely many of the above sets. Therefore, there is $E_0 \subseteq E$ of measure 0 such that for each $x \in E \backslash E_0$, there is $K(x) \in \mathbb{N}$ such that for all $k \ge K(x)$, $|f_{k+1}(x) - f_k(x)| < \varepsilon_k$.

From here, it is not hard to see that for each $x \in E \backslash E_0$, the sequence $\{f_n(x)\}$ is Cauchy and therefore, converges in $\mathbb{R}$. Let $f(x)$ be the limit of $\{f_n(x)\}$. Using the triangle inequality, we have

$$\int_E |f_{n+k} - f_n|^p < \left(\sum_{j=n}^{n+k-1} \varepsilon_j^2\right)^p < \left(\sum_{j=n}^{\infty} \varepsilon_j^2\right)^p$$

In the limit $k \to \infty$, due to Fatou's Lemma, we have

$$\int_E |f - f_n|^p \leq \left(\sum_{j=n}^{\infty} \varepsilon_j^2 \right)^p$$

Now, due to the above inequality, we see that $f - f_n \in L^p(E)$, therefore, $f \in L^p(E)$. Further, from the inequality, we infer that $f_n \to f$ under the L^p norm, completing the proof in this case.

■

Theorem 4.8 (Riesz-Fischer). *Let $E \subseteq \mathbb{R}$ be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space.*

Proof. We have shown that every Cauchy sequence in a normed vector space has a rapidly Cauchy subsequence, and from the above result, that subsequence must converge. Hence, every Cauchy sequence in $L^p(E)$ has a convergent subsequence, consequently the Cauchy sequence must converge[1]. ■

[1]This is a well known result

Abstract Theory

Chapter 5

Abstract Measure Spaces

5.1 Measures and Measurable Sets

Definition 5.1 (Measurable Space). A *measurable space* is a pair $(X, \mathfrak{M})$ consisting of a set X and a σ-algebra $\mathfrak{M}$ of subsets of X. A subset E of X is said to be *measurable* if $E \in \mathfrak{M}$.

Definition 5.2 (Measure). A *measure* on a measurable space $(X, \mathfrak{M})$ is an extended real-valued non-negative function $\mu : \mathfrak{M} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and μ is *countably additive*. A *measure space* is a triple $(X, \mathfrak{M}, \mu)$ where μ is a measure on the σ-algebra $\mathfrak{M}$ on X.

The triple $(\mathbb{R}, \mathcal{L}, m)$ is a measure space where $\mathcal{L}$ is the Lebesgue σ-algebra. Similarly, $(\mathbb{R}, \mathcal{B}, m)$ is also a measure space where $\mathcal{B}$ is the Borel σ-algebra.

A rather artificial measure space is constructed by defining the *counting measure* η on X which maps a finite set to its cardinality and an infinite set to ∞. This makes the triple $(X, \mathcal{P}(X), \eta)$ a measure space.

Let X be a set and fix some $x_0 \in X$. Define the measure δ_{x_0} on the power set $\mathcal{P}(X)$ by

$$\delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

That this is a valid measure is evident and is called the *Dirac measure* concentrated at x_0.

Next, we define the *co-countable measure*. Let X be an uncountable set and

$$\Sigma = \{E \subseteq X \mid E \text{ or } E^c \text{ is countable}\}$$

We shall first establish that Σ is indeed a σ-algebra. To do this, we need only show that the set is closed under countable union. Let $\{E_n\}_{n=1}^{\infty}$ be a collection of sets in Σ. If any E_k is such that E_k^c is countable, then $\left(\bigcup_{n=1}^{\infty} E_n\right)^c \subseteq E_k^c$ and is therefore countable. On the other hand, if none of the E_k have a countable complement, then each E_k must be countable. Then, $\bigcup_{n=1}^{\infty} E_n$ is a countable union of countable sets and is therefore countable. Thus, Σ forms a σ-algebra. Define now the function $\mu : \Sigma \to [0, \infty]$ by

$$\mu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E^c \text{ is countable} \end{cases}$$

From our definition, we have $\mu(\varnothing) = 0$. To establish that μ is a valid measure, we need only verify that it is countably additive. Let $\{E_n\}_{n=1}^{\infty}$ be a collection of disjoint measurable sets. If any one of the E_k's have a countable complement, then all the other E_j's, being a subset of E_k^c must be countable and hence have measure 0. Therefore,

$$1 = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = 1$$

On the other hand, if all the E_k's are countable, there is nothing to prove.

Proposition 5.3. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then, the following hold:*

Finite Additivity: *For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,*

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$$

Monotonicity: *If A and B are measurable sets and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*

Excision: *If, moreover $A \subseteq B$ and $m(A) < \infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$*

Countable Monotonicity: *For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E,*

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

Proof. Finite additivity follows from countable additivity by taking $E_{n+1} = \cdots = \varnothing$ whereas monotonicity follows from the equality $\mu(B) = \mu(B \backslash A) + \mu(A)$ and that $\mu(B \backslash A) \geq 0$. Note that excision also follows from the same equality.

Finally, for countable monotonicity, define the following sets:

$$F_n = E_n \backslash \bigcup_{k=1}^{n-1} E_k$$

It is obvious that the collection $\{F_k\}$ is a collection of disjoint measurable sets. Furthermore, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$ and hence,

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

where the last inequality follows from the fact that $F_n \subseteq E_n$ and hence, $\mu(F_n) \leq \mu(E_n)$. This completes the proof. ■

Proposition 5.4 (Continuity of Measure). *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

(a) If $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of measurable sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k)$$

(b) If $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of measurable sets with $\mu(B_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k)$$

Proof.

(a) If there is an index k such that $\mu(A_k) = \infty$, then $\mu(A_n) = \infty$ for all $n \geq k$ and equality holds. Now suppose $\mu(A_n)$ is finite for all $n \in \mathbb{N}$.Let $A_0 = \varnothing$. Next, define $C_n = A_n \backslash A_{n-1}$ for all $n \in \mathbb{N}$. Then, the C_n's are disjoint and $\bigsqcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$. Using countable additivity, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigsqcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} \mu(A_n) - \mu(A_{n-1}) = \lim_{n\to\infty} \mu(A_n)$$

(b)

■

A property $\mathcal{P}$ is said to hold **almost everywhere** on E if there is a measurable subset E_0 of E with $\mu(E_0) = 0$ such that $\mathcal{P}$ holds on $E\backslash E_0$.

Lemma 5.5 (Borel-Cantelli). *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{E_k\}_{k=1}^{\infty}$ a countable collection of measurable sets for which $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all $x \in X$ belong to at most a finite number of the E_k's.*

Proof. It is not hard to see that the set of all $x \in X$ that belong to infinitely many of the E_k's is given by

$$S = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

Since S is a countable intersection of a collection of measurable sets (each being a countable union of measurable sets), is measurable. Note that the sequence of measurable sets $\{A_n\}$, given by

$$A_n = \bigcup_{k=n}^{\infty} E_k$$

is a descending chain such that

$$\mu(A_1) \leq \sum_{k=1}^{\infty} \mu(E_k) < \infty$$

Thus, due to the continuity of measure,

$$\mu(S) = \lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k)$$

It is not obvious that $\mu(S) = 0$. This completes the proof. ■

Definition 5.6 (Finite, σ-finite). Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then μ is said to be *finite* if $\mu(X) < \infty$. Similarly, it is said to be *σ-finite* if X is the union of a countable collection of measurable sets, each having finite measure.

From the definition, it is clear that every finite measure is σ-finite.

The restriction of the Lebesgue measure on $[0, 1]$ is finite and thus, trivially σ-finite while the Lebesgue measure on $\mathbb{R}$ is σ-finite.

On the other hand, the counting measure on $\mathbb{R}$ is not σ-finite and hence, not finite.

Definition 5.7 (Complete Measure Space). A measure space $(X, \mathfrak{M}, \mu)$ is said to be *complete* if for every $E \subseteq X$ with $\mu(E) = 0$, every $F \subseteq E$ is measurable.

The Lebesgue measure on $\mathbb{R}$ is complete while the restriction of the Lebesgue measure to the Borel σ-algebra, $\mathcal{B}$, while a valid measure space, is not complete, since the Cantor set, which is Borel, contains a subset which is not Borel.

Theorem 5.8 (Completion). *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Define the collection $\mathfrak{M}_0$ of subsets of X which may be written in the form $E = A \cup B$ where B is a subset of some $C \subseteq X$ with measure 0. Finally, define $\mu_0(E) = \mu(A)$. Then $(X, \mathfrak{M}_0, \mu_0)$ is a measure space and extends $(X, \mathfrak{M}, \mu)$.*

Proof. There are two parts to this proof. First, we show that $\mathfrak{M}_0$ is a σ-algebra. Next, we show that μ_0 is a valid measure on $\mathfrak{M}_0$ that extends μ.

Let $E \in \mathfrak{M}_0$. Then there is $A \in \mathfrak{M}$ and $B \subseteq C$ with $\mu(C) = 0$ such that $E = A \cup B$. Now, let $D = C \backslash B$. We have

$$E^c = A^c \cap B^c = A^c \cap (C^c \cup B) = (A^c \cap C^c) \cup (A^c \cap B) \in \mathfrak{M}_0$$

Next, let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of sets in $\mathfrak{M}_0$. Then there is a corresponding collection $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{M}$ and $\{B_n\}$ and $\{C_n\}$ where the latter is a collection of sets with μ-measure 0. Then,

$$\bigcup_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \left(\bigcup_{n=1}^{\infty} B_n \right)$$

where $\bigcup\limits_{n=1}^{\infty} B_n \subseteq \bigcup\limits_{n=1}^{\infty} C_n$ and the set on the right hand side has measure 0. Hence, $\mathfrak{M}_0$ is a σ-algebra.

Now, we must show that the function μ_0 is well defined on $\mathfrak{M}_0$. To do this, let $A_1 \cup B_1 = A_2 \cup B_2 = E \in \mathfrak{M}_0$ where $B_1 \subseteq C_1$ and $B_2 \subseteq C_2$, both of which have μ-measure 0. Then, $A_1 \subseteq A_1 \cup B_1 = A_2 \cup B_2 \subseteq A_2 \cup C_2$, and hence, $\mu(A_1) \leq \mu(A_2)$. Similarly, the reverse direction is also seen to hold. Hence, $\mu(A_1) = \mu(A_2)$ and the function μ_0 is well-defined.

Finally, we must show countable additivity. For this, let $\{E_n\}_{n=1}^{\infty}$ be a countable disjoint collection in $\mathfrak{M}_0$ and correspondingly, we have collections $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$. We have

$$\mu_0\left(\bigcup E_n\right) = \mu_0\left(\bigcup A_n \cup \bigcup B_n\right) = \mu\left(\bigcup A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu_0(E_n)$$

This completes the proof. ■

5.2 Carathéodory Measure Induced By Outer Measure

Definition 5.9 (Countably Monotone, Outer Measure). Let X be a set and $\mathcal{S} \subseteq 2^X$. A set function $\mu : S \to [0, \infty]$ is said to be *countably monotone* if whenever a set $E \in \mathcal{S}$ is covered by a countable collection $\{E_k\}_{k=1}^{\infty}$ then

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

A set function $\mu^* : 2^X \to [0, \infty]$ is said to be an *outer measure* if $\mu(\varnothing) = 0$ and μ^* is countably monotone.

It is not hard to show that an outer measure is finitely monotone and therefore, monotone.

Definition 5.10 (Measurable). For an outer measure $\mu^* : 2^X \to [0, \infty]$, a subset E of X is said to be *measurable* with respect to μ^* if for every $A \subseteq X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$$

From the definition of measurability, we see that E is measurable if and only if E^c is measurable.

From the finite monotonicity of μ^*, we obtain $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \backslash E)$. Hence, to show E is measurable, it suffices to sho $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \backslash E)$. Since this inequality trivially holds when $\mu^*(A) = \infty$, we need only verify it in the case $\mu^*(A) < \infty$.

Proposition 5.11. *The union of a finite collection of measurable sets is measurable.*

Proof. We shall show that the union of two measurable sets is measurable and finite union would follow from induction. Let E_1, E_2 be measurable. Then, for any $A \subseteq X$,

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \\ &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c) \\ &\geq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap E_1^c \cap E_2^c) \end{aligned}$$

where the last inequality follows from $E_1 \cup (E_1^c \cap E_2) = E_1 \cup E_2$. ■

Proposition 5.12. *Let $A \subseteq X$ and $\{E_k\}_{k=1}^\infty$ be a finite disjoint collection of measurable sets. Then*

$$\mu^*\left(A \cap \left[\bigcup_{k=1}^n E_k\right]\right) = \sum_{k=1}^n \mu^*(A \cap E_k)$$

Proof. For $n = 1$, there is nothing to prove. We shall prove the statement for $n = 2$ and the general case would then follow from induction.

$$\begin{aligned} \mu^*(A \cap (E_1 \cup E_2)) &= \mu^*(A) - \mu^*(A \cap E_1^c \cap E_2^c) \\ &= \mu^*(A) - [\mu^*(A \cap E_1^c) - \mu^*(A \cap E_1^c \cap E_2)] \\ &= \mu^*(A) - \mu^*(A \cap E_1^c) + \mu^*(A \cap E_2) \\ &= \mu^*(A \cap E_1) + \mu^*(A \cap E_2) \end{aligned}$$

This completes the proof. ■

Proposition 5.13. *The union of a countable collection of measurable sets is measurable.*

Proof. Let $\{E_n\}_{n=1}^\infty$ be a countable collection of measurable sets. Define $E_0 = \varnothing$ and

$$F_n = E_n \backslash \left(\bigcup_{k=1}^n E_k\right)$$

Then $\{F_n\}_{n=1}^\infty$ is a disjoint collection of measurable sets such that $E = \bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$. Let $A \subseteq X$. Define $G_n = \bigcup_{k=1}^n F_k$. We have

$$\mu^*(A) = \mu^*(A \cap G_n) + \mu^*(A \backslash G_n) \geq \mu^*(A \cap G_n) + \mu^*(A \backslash E) = \sum_{k=1}^n \mu^*(A \cap F_k) + \mu^*(A \backslash E)$$

Taking $n \to \infty$, we have the desired conclusion. ■

Proposition 5.14. *Let $\{E_n\}_{n=1}^{\infty}$ be a disjoint collection of measurable sets. Then, for any $A \subseteq X$,*

$$\mu^*\left(A \cap \left[\bigcup_{n=1}^{\infty} E_n\right]\right) = \sum_{n=1}^{\infty} \mu^*(A \cap E_n)$$

Proof. We have

$$\mu^*\left(A \cap \left[\bigcup_{k=1}^{\infty} E_k\right]\right) \geq \mu^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

Then, taking $n \to \infty$, we have the desired conclusion. ∎

Corollary. Let $\mathfrak{M}$ be the collection of measurable sets. Then $\mathfrak{M}$ is a σ-algebra. Further, the restriction of μ^* to $\mathfrak{M}$ makes $(X, \mathfrak{M}, \mu)$ into a complete measure space.

Proof. We need only show that μ is a complete measure. Let $E \subseteq X$ have measure 0. Then, for any $F \subseteq E$, we have $0 \leq \mu^*(F) \leq \mu^*(E) = 0$, as a result, $\mu^*(F) = 0$. Finally, for any $A \subseteq X$,

$$\mu^*(A \cap F) + \mu^*(A \cap F^c) = \mu^*(A \cap F^c) \leq \mu^*(A)$$

and hence F is measurable. ∎

5.3 Constructing Outer Measures

Theorem 5.15. *Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : S \to [0, \infty]$ a set function. Define $\mu^*(\varnothing) = 0$. For $\varnothing \subsetneq E \subseteq X$, define*

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k)$$

where the infimum is taken over countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E with the convention that $\mu^*(E) = \infty$ if there is no cover of E by a countable collection in $\mathcal{S}$.

Proof. It is not hard to see, from the definition that μ^* is monotone. It now suffices to show countable monotonicity. Let $\varepsilon > 0$ and $\{E_n\}_{n=1}^{\infty}$ be a countable collection of measurable sets. Let $E = \bigcup_{n=1}^{\infty} E_n$. If $\mu^*(E_k) = \infty$ for some $k \in \mathbb{N}$, then $\mu^*(E) = \infty$ due to monotonicity. Now suppose $\mu^*(E_n) < \infty$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, there is a countable cover $\{E_{nk}\}_{k=1}^{\infty}$ of E_n such that

$$\mu^*(E_n) < \sum_{k=1}^{\infty} \mu^*(E_{nk}) + \frac{\varepsilon}{2^n}$$

As a result,

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n) < \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu^*(E_{nk}) + \varepsilon$$

In the limit $\varepsilon \to 0$, the conclusion follows. ∎

Definition 5.16. Let $\mathcal{S}$ be a collection of subsets of X and $\mu : \mathcal{S} \to [0, \infty]$ be a set function. The measure

5.4 Carathéodory Extension Theorem

Definition 5.17 (Semi-Algebra). Let X be a nonempty set. A collection $\mathscr{C}$ of subsets of X is said to be a *semi-algebra* if

(a) $\varnothing, X \in \mathscr{C}$

(b) If $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$

(c) For every $A \in \mathscr{C}$, there is $n \in \mathbb{N}$ and disjoint $C_1, \ldots, C_n \in \mathscr{C}$ such that $A^c = \bigcup\limits_{k=1}^{n} C_k$

Definition 5.18 (Algebra). Let X be a nonempty set. A collection $\mathcal{A}$ of subsets of X is said to be an algebra if

(a) $X \in \mathcal{A}$

(b) $\mathcal{A}$ is closed under finite intersections

(c) $\mathcal{A}$ is closed under complements

Proposition 5.19. *Let X be a nonempty set and $\mathscr{C}$ a semialgebra on X. Denote by $\mathcal{A}(\mathscr{C})$, the minimal algebra containing $\mathscr{C}$. Then,*

$$\mathcal{A}(\mathscr{C}) = \{E \subseteq X \mid E = \bigsqcup_{k=1}^{n} C_k,\ C_j \in \mathscr{C}\}$$

Proof. Let $\mathcal{S}$ denote the set of all finite disjoint unions of elements of $\mathscr{C}$. Obviously, any algebra containing $\mathscr{C}$ must contain $\mathcal{S}$. It now suffices to show that $\mathscr{S}$ is an algebra. We shall first show that $\mathcal{S}$ is closed under finite intersection. Indeed, let $\{E_k\}_{k=1}^{n}$ be a collection of sets in $\mathcal{S}$. Then, for each $1 \leq k \leq n$, there is a disjoint collection $\{C_{k,i}\}_{i=1}^{N(k)}$ such that $E_k = \bigcup_{i=1}^{N(k)} C_{k,i}$. Then,

$$\bigcap_{k=1}^{n} E_k = \bigcup_{i_1=1}^{N(1)} \cdots \bigcup_{i_n=1}^{N(n)} \left(\bigcap_{k=1}^{n} C_{k,i_k} \right)$$

It is not hard to see that this is a disjoint union of sets in $\mathscr{C}$ and thus, belongs to $\mathcal{S}$.

We shall now show closure under complements. From the definition of a semi-algebra, we note that for each $A \in \mathscr{C}$, $A^c \in \mathcal{S}$. Finally, for any $E \in \mathcal{S}$, we may write it as a disjoint union $\bigsqcup\limits_{k=1}^{n} C_k$ of sets in $\mathscr{C}$. As a result, $E^c = \bigcap_{k=1}^{n} C_k^c$. Since $C_k^c \in \mathcal{S}$ for each $1 \leq k \leq n$, and $\mathcal{S}$ is closed under intersections, we conclude that $E^c \in \mathcal{S}$. This completes the proof of the theorem. ∎

Definition 5.20 (Measure on a Semi-Algebra). Let X be a nonempty set and $\mathscr{C}$ a semi-algebra on X. A function $\mu : \mathscr{C} \to [0, \infty]$ is said to be a measure if $\mu(\varnothing) = 0$ and μ is countably additive.

Theorem 5.21. *Let X be a nonempty set and μ a measure on a semi-algebra $\mathscr{C}$ on X, there is a unique measure $\widetilde{\mu}$ on $\mathcal{A}(\mathscr{C})$ which extends μ.*

Proof. TODO: Add in later ∎

Definition 5.22 (Monotone Class). Let X be a nonempty set and $\mathcal{M}$ a collection of subsets of X. We say $\mathcal{M}$ is a monotone class if

(a) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ if $A_n \in \mathcal{M}$ for each $n \in \mathbb{N}$ and $\{A_n\}_{n=1}^{\infty}$ is an ascending chain.

(b) $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$ if $A_n \in \mathcal{M}$ for each $n \in \mathbb{N}$ and $\{A_n\}_{n=1}^{\infty}$ is a descending chain.

For a collection of subsets $\mathscr{C}$ of X, define $\mathcal{M}(\mathscr{C})$ to be the smallest monotone class containing $\mathscr{C}$, which is just the intersection of all monotone classes containing $\mathscr{C}$. Note that the intersection is over a nonempty set, since the powerser of X is a monotone class containing $\mathscr{C}$ (trivially).

Proposition 5.23. *Let μ_1 and μ_2 be measures on a measurable space $(X, \mathfrak{M})$. Define the collection*

$$\mathcal{M} = \{E \in \mathfrak{M} \mid \mu_1(E) = \mu_2(E)\}$$

(a) $\varnothing \in \mathcal{M}$

(b) If $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, and $\{A_n\}$ is an ascending chain, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

(c) If μ_1 and μ_2 are finite. Then for $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $\{A_n\}$ being a descending chain, we have $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

Proof. Follows from the continuity of measure. ■

Definition 5.24. For a collection $\mathscr{C}$ of subsets of X, $\mathfrak{M}(\mathscr{C})$ is defined as the smallest σ-algebra containing $\mathscr{C}$, which is equal to the intersection of all the σ-algebras containing $\mathscr{C}$.

Theorem 5.25 (Monotone Class Theorem). *Let $\mathcal{A}$ be an algebra of sets. Then $\mathcal{M}(\mathcal{A}) = \mathfrak{M}(\mathcal{A})$.*

Proof. We shall denote $\mathcal{M}(\mathcal{A})$ by $\mathcal{M}$ and $\mathfrak{M}(\mathcal{A})$ by $\mathfrak{M}$. Since all σ-algebras are monotone classes, we have $\mathcal{M} \subseteq \mathfrak{M}$. We shall show the reverse inclusion. First, we shall show that $\mathcal{M}$ is an algebra.

For each $M \in \mathcal{M}$, define

$$\mathcal{M}(M) = \{E \in \mathcal{M} \mid E \backslash M, E \cap M \in \mathcal{M}\}$$

We contend that $\mathcal{M}(M)$ forms a monotone class. Indeed, suppose $\{E_n\}$ is an ascending chain in $\mathcal{M}(M)$. Then,

$$\left(\bigcup_{n=1}^{\infty} E_n\right) \backslash M = \bigcup_{n=1}^{\infty} (E_n \backslash M)$$
$$\left(\bigcup_{n=1}^{\infty} E_n\right) \cap M = \bigcup_{n=1}^{\infty} (E_n \cap M)$$

And thus, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}(M)$. Similarly, one can show this for descending chains.

Further, by symmetry, also note that if $M \in \mathcal{M}(N)$, then $N \in \mathcal{M}(M)$. Pick any $A \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, we must have $\mathcal{A} \subseteq \mathcal{M}(A)$. Now, since $\mathcal{M}(A)$ is a monotone class and $\mathcal{M}$ is the minimal monotone class containing $\mathcal{A}$, we must have $\mathcal{M} \subseteq \mathcal{M}(A) \subseteq \mathcal{M}$, as a result, $\mathcal{M}(A) = \mathcal{M}$. This means, $M \in \mathcal{M}(A)$ and due to symmetry, $A \in \mathcal{M}(M)$.

Since the choice of $A \in \mathcal{A}$ was arbitrary, we have $\mathcal{A} \subseteq \mathcal{M}(M)$. Again, using the minimality of $\mathcal{M}$, we have $\mathcal{M} = \mathcal{M}(M)$ for each $M \in \mathcal{M}$. From this, we infer that $\mathcal{M}$ is closed under finite intersection and relative complements, and since $X \in \mathcal{M}$ (trivially), we see that $\mathcal{M}$ is an algebra.

Finally, we shall show that $\mathcal{M}$ is a σ-algebra. Indeed, let $\{E_n\}_{n=1}^{\infty}$ be a collection of sets in $\mathcal{M}$. Then, $\{\bigcup_{k=1}^{n} E_k\}$ forms an ascending chain and since $\mathcal{M}$ is a monotone class, $\bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{M}$. This completes the proof. ■

Theorem 5.26. *Let μ be a measure on an algebra $\mathcal{A}$ of subsets of a nonempty set X. If μ is σ-finite, then there is a unique extension $\widetilde{\mu}$ of μ on $\mathfrak{M}(\mathcal{A})$.*

Proof. Since μ is a nonnegative set function, we may take the Carathéodory extension of μ to some σ-algebra. Since that σ-algebra must contain $\mathfrak{M}(\mathcal{A})$, we simply restrict it to $\mathfrak{M}(\mathcal{A})$ to obtain an extension of μ to $\mathfrak{M}(\mathcal{A})$.

We shall now show uniqueness. Let μ_1 and μ_2 be two measures which extend μ. Since μ is σ-finite, so are μ_1 and μ_2. Further, there is a disjoint collection of sets $\{X_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that $X = \bigsqcup_{i=1}^{\infty} X_i$ and $\mu(X_i) < \infty$. Then for any $E \in \mathfrak{M}(\mathcal{A})$, and $i \in \{1, 2\}$,

$$\mu_i(E) = \mu_i\left(E \cap \left(\bigcup_{j=1}^{\infty} X_j\right)\right) = \sum_{j=1}^{\infty} \mu_i(E \cap X_j)$$

Moreover, $\mathfrak{M}(\mathcal{A}) \cap X_i = \mathfrak{M}(\mathcal{A} \cap X_i)$, since $\mathcal{A} \cap X_i$ is an algebra. Hence, if we show that μ_1 and μ_2 agree on each X_i, then they would agree on X. As a result, we may suppose μ is finite and hence, so are μ_1 and μ_2.

Define now

$$\mathcal{M} = \{E \in \mathfrak{M}(\mathcal{A}) \mid \mu_1(E) = \mu_2(E)\}$$

We have shown already that $\mathcal{M}$ must form a monotone class and must contain $\mathcal{A}$. Finally, using Theorem 5.25, have that $\mathfrak{M}(\mathcal{A}) = \mathcal{M}$, which completes the proof. ■

Chapter 6

Abstract Integration

6.1 Measurable Functions

Definition 6.1 (Measurable Function). Let $(X, \mathfrak{M})$ be a measurable space and $f : X \to [-\infty, \infty]$ be an extended real valued function. Then, f is said to be measurable if it satisfies one of the following equivalent conditions for all $c \in \mathbb{R}$

(a) $\{x \in X \mid f(x) < c\}$ is measurable

(b) $\{x \in X \mid f(x) \leq c\}$ is measurable

(c) $\{x \in X \mid f(x) > c\}$ is measurable

(d) $\{x \in X \mid f(x) \geq c\}$ is measurable

Proposition 6.2. *Let $(X, \mathfrak{M}, \mu)$ be a complete measure space and X_0 a measurable subset of X for which $\mu(X \backslash X_0) = 0$. Then a function $f : X \to [-\infty, \infty]$ is measurable if and only if its restriction to X_0 is measurable.*

Proof. We have

$$\{x \in X_0 \mid f(x) > c\} \subseteq \{x \in X \mid f(x) > c\} \subseteq \{x \in X_0 \mid f(x) > c\} \cup (X \backslash X_0)$$

and the conclusion follows. ■

Corollary. Let $(X, \mathfrak{M}, \mu)$ be a complete measure space. If $g, h : X \to [-\infty, \infty]$ are functions such that $g = h$ a.e. on X, then g is measurable if and only if h is measurable.

Proposition 6.3. *Let $(X, \mathfrak{M}, \mu)$ be a complete measure space. Let $f, g : X \to [-\infty, \infty]$ be measurable functions that are finite a.e. on X. Let $\alpha, \beta \in \mathbb{R}$. Then,*

(a) $\alpha f + \beta g$ is measurable

(b) $f \cdot g$ is measurable

(c) $\min\{f, g\}$ and $\max\{f, g\}$ are measurable

Proof. Same as that for Lebesgue measurable functions. ■

Proposition 6.4. *Let $(X, \mathfrak{M})$ be a measurable space and f a real valued measurable function on X. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\varphi \circ f$ is measurable.*

Proof. Same as that for Lebesgue measurable functions. ∎

Theorem 6.5. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. If either the measure space $(X, \mathfrak{M}, \mu)$ is complete or the convergence is pointwise on all of X, then f is measurable.*

Proof. Again, by excising a suitable subset of measure 0, we may suppose that the convergence is everywhere. The remainder of the proof is the same as that for Lebesgue measurable functions. ∎

Lemma 6.6 (Simple Approximation Lemma). *Let $(X, \mathfrak{M})$ be a measurable space and $f : X \to \mathbb{R}$ be bounded and measurable. Then, for each $\varepsilon > 0$, there are simple functions φ_ε, ψ_ε such that $\varphi_\varepsilon \leq f \leq \psi_\varepsilon$ and $\psi_\varepsilon - \varphi_\varepsilon < \varepsilon$ on X.*

Proof. Same as that for Lebesgue measurable functions. ∎

Theorem 6.7 (Simple Approximation Theorem). *Let $(X, \mathfrak{M}, \mu)$ be a measure space and f a measurable function on X. Then, there is a sequence of simple functions $\{\psi_n\}$ on X that converges pointwise on X to f such that $|\psi_n| \leq |f|$ on X for all n. Further,*

(a) If X is σ-finite, then we may choose the sequence $\{\psi_n\}$ so that each ψ_n vanishes outside a set of finite measure

(b) If f is nonnegative, we may choose the sequence $\{\psi_n\}$ to be increasing and each $\psi_n \geq 0$ on X

Proof. We may write $f = f^+ - f^-$ where f^+ and f^- are nonnegative measurable functions on X. First, we shall suppose f is nonnegative. Define

$$E_n = \{x \in X \mid f(x) \leq n\}$$

Then E_n is measurable and f is bounded over E_n. Therefore, due to Lemma 6.6, there is a simple function ψ_n defined on E_n such that $f - \psi_n < 1/n$ on E_n. Extend ψ_n to all of X by giving it the value n on $X \backslash E_n$. It is obvious that $\psi_n \to f$ pointwise on X.

Let us now return to the general case. From the previous paragraph, we infer that there are sequences $\{\psi_n^+\}$ of nonnegative simple functions converging to f^+ and similarly, $\{\psi_n^-\}$ converging to f^-. Now, consider the sequence $\{\psi_n^+ - \psi_n^-\}$. It is not hard to show that it converges to f and satisfies the required properties.

1. Now, if X is σ-finite, there is a countable collection of subsets $\{E_n\}$ of X with finite measure such that $X = \bigcup_{n=1}^{\infty} E_n$. Define $X_n = \bigcup_{k=1}^{n} E_k$. Due to the above discussion, there is a sequence of simple functions $\{\psi_n\}$ that converge pointwise to f. Define now $\varphi_n = \psi_n \chi_{X_n}$. Then φ_n vanishes outside a set of finite measure and converges pointwise to f on X.

2. If f were nonnegative, then we have a sequence of nonnegative measurable functions $\{\psi_n\}$ converging pointwise to f. Define $\varphi_n = \max_{1 \leq k \leq n} \psi_n$. Then $\{\varphi_n\}$ is an increasing sequence of simple functions that converges pointwise to f on X.

∎

Theorem 6.8 (Egoroff). *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions on X that converges pointwise a.e. on X to a function f that is finite a.e. on X. Then for each $\varepsilon > 0$, there is a measurable subset X_ε of X for which $\mu(X \backslash X_\varepsilon) < \varepsilon$ and $\{f_n\} \to f$ uniformly on X_ε.*

Proof. First, we shall excise suitable measure 0 sets and suppose the function f is real valued and convergence is pointwise on X. Notice that this should not change our conclusion. Fix some $N \in \mathbb{N}$. Define

$$A_n = \{x \in X : |f(x) - f_m(x)| < 1/N \; \forall m \geq n\}$$

It is not hard to see that $A_1 \subseteq A_2 \subseteq \cdots$. Moreover, for all $x \in X$, there is $n \in \mathbb{N}$ such that $x \in A_n$. As a result, $\bigcup_{n=1}^{\infty} A_n = X$. Using the continuity of measure, there is an index M_N such that $\mu(X \backslash A_{M_N}) < \varepsilon/2^N$.

Finally, define $X_\varepsilon = \bigcap_{n=1}^{\infty} A_{M_n}$. Then, $\mu(X \backslash X_\varepsilon) < \varepsilon$. We contend that the convergence is uniform on X_ε. Let $\delta > 0$ be given, then there is $N \in \mathbb{N}$ such that $1/N < \delta$. For all $m \geq M_N$ and $x \in X_\varepsilon$, we have $x \in A_{M_N}$, thus, $|f(x) - f_m(x)| < 1/N < \delta$, which completes the proof. ■

6.2 Integration of Nonnegative Measurable Functions

For a nonnegative simple function ψ on X with canonical representation $\sum_{k=1}^{n} c_k \chi_{E_k}$, we define

$$\int_X \psi = \sum_{k=1}^{n} c_k \mu(E_k)$$

With the normal convention of arithmetic in $[0, \infty]$. For a measurable subset $E \subseteq X$, we define $\int_E \psi = \int_X \psi \chi_E$.

For a nonnegative extended real valued measurable function $f : X \to [0, \infty]$, we define

$$\int_X f = \sup \left\{ \int_X \psi \;\middle|\; 0 \leq \psi \leq f, \; \psi \text{ is simple} \right\}$$

For a measurable subset $E \subseteq X$, define $\int_E f = \int_X f \chi_E$.

Some Elementary Properties

TODO: Add when you feel the need to do so. This will probably be when a lab submission is an hour away and your procrastination kicks in.

Theorem 6.9 (Chebyshev's Inequality). *Let $(X, \mathfrak{M}, \mu)$ be a measure space, f a nonnegative measurable function on X, and λ a positive real number. Then*

$$\mu(\{x \in X \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_X f$$

Proof. Let $E_\lambda = \{x \in X \mid f(x) \geq \lambda\}$. Then $\lambda \chi_{E_\lambda} \leq f$ is a simple function and from the definition of the integral,

$$\lambda \mu(E_\lambda) \leq \int_X f$$

and the conclusion follows. ■

Proposition 6.10. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and f a nonnegative measurable function on X for which $\int_X f < \infty$. Then f is finite a.e. on X and $\{x \in X \mid f(x) > 0\}$ is σ-finite.*

Proof. Define $E_n = \{x \in X \mid f(x) \geq n\}$. Then $\mu(E_n) \leq 1/n \int_X f$ and

$$\{x \in X \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} E_n$$

and using the continuity of measure, we see that the measure of the above set is 0.

Next, define $A_n = \{x \in X \mid f(x) \geq 1/n\}$. $\mu(A_n) \leq n \int_X f < \infty$ and

$$\{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n$$

and is therefore σ-finite. ∎

Lemma 6.11 (Fatou). *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of nonnegative measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. Assume f is measurable. Then*

$$\int_X f \leq \liminf_{n\to\infty} \int_X f_n$$

Note that since the measure space may not be complete, it is not implicit that f is measurable from pointwise a.e. convergence.

Proof. We may excise a suitable subset of X such that the convergence is pointwise on the rest of X. Let $\varphi \leq f$ be a simple function. If $\int_X \varphi = 0$, then obviously, $\int_X \varphi \leq \liminf_{n\to\infty} \int_X f_n$. We now consider two cases.

Case 1: $\int_X \varphi = \infty$. Then there is some positive real number a and a measurable $E \subseteq X$ such that $\mu(E) = \infty$ and $\varphi(x) = a$ for all $x \in E$. Define the (measurable) subsets

$$X_n := \{x \in X \mid f_k(x) \geq a/2, \ \forall\, k \geq n\}$$

Since f_k converges pointwise to f and $\varphi \leq f$, $\bigcup_{n=1}^{\infty} X_n \supseteq E$, consequently, using the continuity of measure, $\lim_{n\to\infty} \mu(X_n) = \infty$. Next, using Chebyshev's Inequality,

$$\mu(X_n) \leq \frac{2}{a} \int_{X_n} f_n \leq \frac{2}{a} \int_X f_n$$

and we conclude that $\liminf_{n\to\infty} \int_X f_n = \infty$.

Case 2: $0 < \int_X \varphi < \infty$. Then, φ is nonzero on a set of finite measure, say X_0.

∎

Theorem 6.12 (Monotone Convergence Theorem). *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence of measurable functions converging pointwise a.e. to f, which is measurable on X. Then,*

$$\lim_{n\to\infty} \int_X f_n = \int_X f$$

Proof. Since the convergence is pointwise a.e., we may excise a suitable subset of measure 0 from X such that the convergence is pointwise on the remaining set. Notice that this doesn't change the value of any integral. Therefore, we may suppose that the convergence is pointwise on X. Then, we woud have $f_n \leq f$ for all $n \in \mathbb{N}$. As a result,

$$\int_X f_n \leq \int_X f$$

and consequently,

$$\int_X f \geq \limsup_{n\to\infty} \int_X f_n \geq \liminf_{n\to\infty} \int_X f_n \geq \int_X f$$

where the last inequality is due to Lemma 6.11. This implies the desired conclusion. ∎

Corollary. Let $(X, \mathfrak{M}, \mu)$ be a measurable space and f a nonnegative measurable function on X. Then there is an increasing sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and

$$\lim_{n\to\infty} \int_X \psi_n = \int_X f$$

The proof is omitted due to obviousness.

Proposition 6.13 (Additivity of Integration). *Let $(X, \mathfrak{M}, \mu)$ be a measure space and f, g nonnegative measurable functions. Then for $\alpha, \beta \in \mathbb{R}_{\geq 0}$,*

$$\int_X (\alpha f + \beta g) = \alpha \int_X f + \beta \int_X g$$

Proof. From the simple approximation theorem, there are increasing sequences of simple functions $\{\psi_n\}$ and $\{\varphi_n\}$ converging to f and g respectively. As a result, the increasing sequences of simple functions $\{\alpha\psi_n\}$ and $\{\beta\varphi_n\}$ converge to αf and βg respectively. Then, the sequence $\{\alpha\psi_n + \beta\varphi_n\}$ is an increasing sequence of measurable functions converging to $\alpha f + \beta g$, and thus,

$$\int_X (\alpha f + \beta g) = \lim_{n\to\infty} \int_X (\alpha\psi_n + \beta\varphi_n) = \lim_{n\to\infty} \left[\alpha \int_X \psi_n + \beta \int_X \varphi_n\right] = \alpha \int_X f + \beta \int_X g$$

This completes the proof. ∎

Definition 6.14 (Integrable). Let $(X, \mathfrak{M}, \mu)$ be a measure space. A nonnegative measurable function f is said to be integrable if $\int_X f < \infty$.

Chapter 7

New Measures from Old

7.1 Product Measures

Definition 7.1 (Measurable Rectangles, Product σ-algebra). Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two measure spaces. Subsets of $X \times Y$ of the form $A \times B$ where $A \in \mathscr{A}$ and $B \in \mathscr{B}$ are called *measurable rectangles*. Let $\mathcal{R}$ denote the collection of all measurable rectangles. The σ-algebra $\mathscr{A} \otimes \mathscr{B} := \mathfrak{M}(\mathcal{R})$ is c alled the *product σ-algebra*.

Theorem 7.2. *Let $\eta : \mathcal{R} \to [0, \infty]$ be defined by $\eta(A \times B) = \mu(A) \times \nu(B)$ where $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Then, η is a well defined measure on $\mathcal{R}$. Further, if μ and ν are σ-finite, there is a unique measure $\widetilde{\eta}$ on $A \otimes B$ that extends η.*

Proof. Obviously, $\eta(\varnothing) = 0$. We shall now show that η is countably additive. Indeed, let $\{A_n\}$ be a sequence of sets in $\mathscr{A}$ and $\{B_n\}$ a sequence of sets in $\mathscr{B}$ such that the sequence $\{A_n \times B_n\}$ is disjoint and there is $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $A \times B = \bigcup\limits_{k=1}^{\infty} A_k \times B_k$.

Fix some $x \in A$ and define $S_x := \{n \in \mathbb{N} \mid x \in A_n\}$. Then, for every $y \in B$, $x \times y \in A \times B$ and therefore, there is an index n $x \times y \in A_n \times B_n$. Note that this index must be unique lest the collection $\{A_n \times B_n\}$ not be disjoint. As a result, $B = \bigsqcup_{n \in S_x} B_n$. Now, using countable additivity, we have

$$\nu(B) = \sum_{n \in S_x} \nu(B_n) = \sum_{n=1}^{\infty} \chi_{A_n}(x) \nu(B_n)$$

for each $x \in A$. Then, we may write

$$\chi_A \nu(B) = \sum_{n=1}^{\infty} \chi_{A_n} \nu(B_n)$$

We have

$$\mu(A)\nu(B) = \int_X \chi_A \nu(B)\, d\mu = \int_X \left(\sum_{n=1}^{\infty} \chi_{A_n} \nu(B_n) \right) d\mu = \sum_{n=1}^{\infty} \int_X \chi_{A_n} \nu(B_n)\, d\mu = \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n)$$

Finally, we must show that $\mathcal{R}$ is a semi-algebra. Obviously, $\varnothing, X \times Y \in \mathcal{R}$ and $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$. Finally, for $A \times B \in \mathcal{R}$,

$$(X \times Y) \backslash (A \times B) = A \times (Y \backslash B) \sqcup (X \backslash A) \times B \sqcup (X \backslash A) \times (Y \backslash B)$$

The uniqueness now followes from Carathéodory Extension Theorem. This completes the proof. ∎