Commutative Algebra

Swayam Chube

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Abstract

This document mainly contains terse notes of commutative algebra and solutions to exercises from [1]. The three main references were [1], [3] and [4].

Except for in the chapter on modules, all rings are assumed to be commutative unless stated otherwise. We use a uniform convention to represent a commutative ring with A and a general ring with R. Similarly, we represent modules by one of M, N, P. A maximal ideal is generally denoted by $\mathfrak{m}$ while a prime ideal is denoted by $\mathfrak{p}$.

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2.1 Introduction

Throughout this section, R denotes a general ring which need not be commutative.

Definition 2.1 (Module). A left R-module is an abelian group $(M, +)$ along with a ring action, that is, a ring homomorphism $\mu : R \to \text{End}(M)$.

Henceforth, unless specified otherwise, an *R-module* refers to a *left R-module*. Trivially note that R is an R-module, so is any ideal in R and so is every quotient ring R/I where I is an ideal in R. When R is a field, an R-module is the same as a vector space.

Every abelian group G trivially forms a $\mathbb{Z}$-module. Using this and the forthcoming *Structure Theorem for Finitely Generated Modules over a PID*, we obtain the *Structure Theorem for Finitely Generated Abelian Groups*.

Definition 2.2 (Submodule). Let M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of R.

Proposition 2.3 (Submodule Criteria). *Let M be an R-module. Then $\varnothing \subsetneq N \subseteq M$ is a submodule if and only if for all $x, y \in N$ and $r \in R$, $x + ry \in N$.*

Proof. Straightforward definition pushing. ■

Definition 2.4 (Module Homomorphism). Let M, N be R-modules. A *module homomorphism* is a group homomorphism $\phi : M \to N$ such that for all $x \in M$ and $r \in R$, $\phi(rx) = r\phi(x)$.

In other words, a module homomorphism is simply an R-linear map.

Proposition 2.5 (Homomorphism Criteria). *Let M, N be R-modules. Then $\phi : M \to N$ is an R-module homomorphism if and only if for all $x, y \in M$ and $r \in R$, $\phi(x + ry) = \phi(x) + r\phi(y)$.*

Proof. Straightforward definition pushing. ■

It is not hard to see, using the above proposition and the submodule criteria that the image of an R-module under a homomorphism is a submodule.

For R-modules M, N, we denote the set of all R-module homomorphisms from M to N by $\mathrm{Hom}_R(M, N)$. When the choice of the ring R is clear from the context, we shall denote this set by $\mathrm{Hom}(M, N)$.

Proposition 2.6. *Let M, N be R-modules. Then $\mathrm{Hom}(M, N)$ forms an R-module.*

Proof. It is obvious that $\mathrm{Hom}(M, N)$ has the structure of an abelian group. Define the natural action by $(rf)(x) = rf(x)$. It is not hard to see that this action is well defined. ■

Proposition 2.7. *Let $\phi : M \to N$ be an R-module homomorphism. Then, for every R-module P, there is an induced R-module homomorphism $\overline{\phi} : \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P)$ and an induced R-module homomorphism $\widetilde{\phi} : \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$.*

Equivalently phrased, $\mathrm{Hom}(-, P)$ is a contravariant functor while $\mathrm{Hom}(P, -)$ is a covariant functor.

Proof. We shall prove only the first half of the assertion since the second half follows from a similar proof. Define $\overline{\phi}$ using the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, dotted, "f \circ \phi"'] & N \arrow[d, "f"] \\ & P \end{tikzcd}$$

To see that this is indeed an R-module homomorphism, we need only verify that for all $f, g \in \mathrm{Hom}(N, P)$ and all $r \in R$, $(f + rg) \circ \phi = f \circ \phi + rg \circ \phi$ which is trivial to check. ■

Definition 2.8 (Kernel, Cokernel). Let $\phi : M \to N$ be an R-module homomorphism. We define

$$\ker \phi = \{x \in M \mid \phi(x) = 0\} \qquad \operatorname{coker} \phi = N/\phi(M)$$

For an R-module M, define the annihilator of M in R as

$$\mathrm{Ann}(M) = \{r \in R \mid rx = 0 \; \forall x \in M\}$$

It is trivial to check that $\mathrm{Ann}(M)$ is a left ideal in R, and if R were commutative, it would be an ideal.

2.2 Free Modules

Throughout this section, R denotes a general ring which need not be commutative. The content of this section is taken from [2].

We define the free module using a universal property and then provide a construction for it. This should establish uniqueness.

Definition 2.9. Let S be a non-empty set. A *free module on S* is an R-module F together with a mapping $f : S \to F$ such that for every R-module M and every set map $g : S \to M$, there is a unique R-module homomorphism $h : F \to M$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f"'] & M \\ F \arrow[ur, dotted, "\exists! h"'] & \end{tikzcd}$$

2.3 Finitely Generated Modules

Definition 2.10 (Finitely Generated Module). An R-module M is said to be finitely generated if there is a finite subset S of M which generates M. That is, there is no proper submodule N of M containing S.

Proposition 2.11. *An R-module M is finitely generated if M is isomorphic to a quotient of $R^{\oplus n}$ for some positive integer n.*

Proof. We shall only prove the forward direction since the converse is trivial to prove. Suppose M is finitely generated. Then, it is generated by a finite subset $S = \{x_1, \dots, x_m\}$. Define the R-module homomorphism $\phi : R^{\oplus n} \to M$ by $(r_1, \dots, r_n) \mapsto r_1x_1 + \cdots + r_nx_n$. From the first isomorphism theorem, we have $M \cong R^{\oplus n} / \ker \phi$. ∎

Proposition 2.12. *Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A. Let $\phi \in \operatorname{End}(M)$ such that $\phi(M) \subseteq \mathfrak{a}M$. Then, there are $a_0, \dots, a_{n-1} \in \mathfrak{a}$ such that*

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0$$

as an element of $\operatorname{End}(M)$, where a_k is treated as the homomorphism $x \mapsto a_kx$ in $\operatorname{End}(M)$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a generating set for M. Then, for all $1 \leq i \leq n$, there are coefficients $\{a_{i1}, \ldots, a_{in}\}$ in $\mathfrak{a}$ such that

$$\phi(x_i) = \sum_{j=1}^{n} a_{ij} x_j$$

We may rewrite this as

$$\sum_{j=1}^{n} (\phi\delta_{ij} - a_{ij}) x_j = 0$$

Let B denote the matrix $(\phi\delta_{ij} - a_{ij})_{1 \leq i,j \leq n}$. Then, multiplying by $\mathrm{adj}(B)$, we see that $\det(B)(x_j) = 0$ for all $1 \leq j \leq n$ where $\det(B)$ is viewed as an element in $\mathrm{End}(M)$ and thus, is the zero map in $\mathrm{End}(M)$. It is not hard to see that $\det(B)$ is in the required form. ■

Lemma 2.13 (Nakayama). *Let M be a finitely generated module and $\mathfrak{a} \subseteq \mathfrak{R}$ be an ideal such that $M = \mathfrak{a}M$. Then, $M = 0$.*

Proof. Let $\phi = \mathbf{id}$ be the identity homomorphism in $\mathrm{End}(M)$. Using Proposition 2.12, there are coefficients $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ satisfying the statement of the proposition. As a result, $x = 1 + a_{n-1} + \ldots + a_0$ is the zero endomorphism. But since $a_{n-1} + \ldots + a_0 \in \mathfrak{a} \subseteq \mathfrak{R}$, x is a unit and hence, $M = 0$. ■

Over a PID

Throughout this section, let R denote a principal ideal domain.

2.4 Exact Sequences

Definition 2.14. A sequence of module homomorphisms

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is said to be exact at N if $\mathrm{im}\, f = \ker g$. A short exact sequence is a sequence of module homomorphisms:

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

which is exact at M, N and P.

It is not hard to see that the sequence in the definition is short exact if and only if f is injective, g is surjective and $\mathrm{im}\, f = \ker g$.

Theorem 2.15. *For all R-modules X,* $\mathrm{Hom}(X, -)$ *is a left exact functor. That is,* $0 \longrightarrow M \longrightarrow N \longrightarrow P$ *is exact if and only if* $0 \longrightarrow \mathrm{Hom}(X, M) \longrightarrow \mathrm{Hom}(X, N) \longrightarrow \mathrm{Hom}(X, P)$ *is exact.*

Proof. Consider the following commutative diagram where the row is exact.

$$\begin{tikzcd} & & X \arrow[dl, "u"'] \arrow[d, "v"] & \\ 0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r, "g"] & P \end{tikzcd}$$

Let $u \in \ker \overline{f}$. Since f is injective, it is obvious that u must be the trivail homomorphism. Next, we must show that $\mathrm{im}\, \overline{f} = \ker \overline{g}$. First, note that $\overline{f} \circ \overline{g} = \overline{f \circ g} = 0$ since $\mathrm{Hom}(X, -)$ is a covariant functor. Finally, suppose $v \in \ker \overline{g}$. Then, $g \circ v = 0$, consequently, $\mathrm{im}\, v \subseteq \mathrm{im}\, f$. Now, since f is injective, $f^{-1}(\mathrm{im}\, v)$ is a submodule of M and hence, the map $w : X \to M$ given by $x \mapsto f^{-1}(v(x))$ is well defined and $f \circ w = v$.

For the converse, simply note that $\mathrm{Hom}(R, M)$ is isomorphic to M. ∎

Diagram Chasing

2.5 Tensor Product

Throughout this section, R denotes a general ring which need not be commutative.

Definition 2.16 (Bilinear Map). Let M, N, P be R-modules. A map $T : M \times N \to P$ is said to be bilinear if for each $x \in M$, the mapping $T_x : N \to P$ given by $y \mapsto T(x, y)$ is R-linear and for each $y \in N$, the mapping $T_y : M \to P$ given by $x \mapsto T(x, y)$ is R-linear.

Fix two R-modules M and N. Let $\mathscr{C}$ denote the category of bilinear maps $T : M \times N \to P$ where P is any R-module. A morphism between two bilinear maps $f : M \times N \to P_1$ and $g : M \times N \to P_2$ in this category is a module homomorphism $\phi : P_1 \to P_2$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "g"'] & P_1 \arrow[dl, dotted, "\phi"] \\ P_2 & \end{tikzcd}$$

A universal object in $\mathscr{C}$ is called the tensor product of M and N and is denoted by $M \otimes N$. In other words, the tensor product is an initial object in the category $\mathscr{C}$.

Definition 2.17 (Universal Property of the Tensor Product). Let M, N, P be R-modules and $T : M \times N \to P$ be a bilinear map. Then, there is a unique R-module homomorphism $\phi : M \otimes N \to P$ such that the following diagram commutes:

$$
\begin{tikzcd}
M \times N \arrow[r, "T"] \arrow[d, "\varphi"'] & P \\
M \otimes N \arrow[ur, dotted, "\exists!\phi"'] &
\end{tikzcd}
$$

Of course, having the universal property would imply that the tensor product, if it exists, is unique upto a unique isomorphism. We shall now construct a tensor product of M and N.

Constructing the Tensor Product

Let F be the free R-module on $M \times N$. Let us denote the basis elements of F by $e_{(x,y)}$ where $x \in M$ and $y \in N$. Now, for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r \in R$, let D denote the submodule generated by elements of the form:

$$
\begin{gathered}
e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)} \\
e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)} \\
e_{(rx,y)} - re_{(x,y)} \\
e_{(x,ry)} - re_{(x,y)}
\end{gathered}
$$

Let $G = F/D$ and let $\varphi : M \times N \to G$ be the composition of the following maps:

$$M \times N \hookrightarrow F \twoheadrightarrow G$$

Let $T : M \times N \to P$ be a bilinear map. Consider the following commutative diagram:

$$
\begin{tikzcd}
M \times N \arrow[r, "T"] \arrow[d, hook, "\iota"'] & P \\
F \arrow[r, "\pi"'] \arrow[ur, dotted, "\exists! f"] & G \arrow[u, dotted, "\exists!\phi"']
\end{tikzcd}
$$

To show that existence of ϕ, we must show that $D \subseteq \ker f$, since we can then finish using the universal property of the kernel. But this is trivial to check and follows from the fact that T is a bilinear map and completes the construction.

Similarly, we define the tensor product for a finite sequence of R-modules $\{M_i\}_{i=1}^n$. That is, given a multilinear map $T : \prod_{i=1}^{n} M_i \to P$, there is a unique R-module homomorphism ϕ such that the following diagram commutes:

$$\begin{tikzcd}
M_1 \times \cdots \times M_n \arrow[r, "T"] \arrow[d, "\varphi"'] & P \\
M_1 \otimes \cdots \otimes M_n \arrow[ur, dotted, "\exists!\phi"'] &
\end{tikzcd}$$

Properties of Tensor Product

Given two modules M and N with the canonical map $\varphi : M \times N \to M \otimes N$, we denote by $m \otimes n$, the element $\varphi(m, n)$ in $M \otimes N$.

Proposition 2.18. *Let M, N, P be A-modules. Then,*

(a) $M \otimes N \cong N \otimes M$

(b) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P) \cong M \otimes N \otimes P$

(c) $M \oplus N \otimes P \cong (M \otimes P) \oplus (N \otimes P)$

(d) $A \otimes M \cong M$

Further, in each case, the isomorphism is unique.

Proof. In each case, it suffices to show that both modules have the same universal property, which would imply a unique isomorphism between the two modules.

(a) Consider the map $T : M \times N \to N \times M$ given by $(m, n) \mapsto (n, m)$. Let $\varphi : M \times N \to M \otimes N$ and $\varphi' : N \times M \to N \otimes M$ be the canonical morphisms. Consider now the following commutative diagram:

$$\begin{tikzcd}
M \times N \arrow[r, "T", "\sim"'] \arrow[d, "\varphi"'] & N \times M \arrow[d, "\varphi'"] \\
M \otimes N \arrow[r, dotted, shift right, "\phi"'] & N \otimes M \arrow[l, dotted, shift right, "\phi'"']
\end{tikzcd}$$

Define the map $\phi(m \otimes n) = n \otimes m$ and $\phi'(n \otimes m) = m \otimes n$. It is not hard to see that ϕ and ϕ' make the diagram commute. Further, since $\varphi' \circ T$ is bilinear, ϕ is the unique morphism making the diagram commute and similarly for ϕ'. Finally, since ϕ and ϕ' are

■

Bibliography

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