Differential Topology

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Contents

Part I

Multivariable Calculus

Chapter 1

Differentiation

Definition 1.1. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be *differentiable* at $a \in U$ if there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to \mathbf{0}} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0$$

The linear transformation T is called the *derivative* of f at a and is denoted by $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$.

The following proposition establishes the uniqueness of the derivative at a point, if it exists.

Proposition 1.2. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in U$. Then, there is a unique linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to \mathbf{0}} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0$$

Proof. Let $\mu, \lambda : \mathbb{R}^n \to \mathbb{R}^m$ be two linear transformations satisfying the requirements. Then, we have

$$\|\lambda(h) - \mu(h)\| \leq \|f(a+h) - f(a) - \mu(h)\| + \|f(a+h) - f(a) - \lambda(h)\|$$

Consequently,

$$\lim_{h \to \mathbf{0}} \frac{\|\lambda(h) - \mu(h)\|}{\|h\|} \leq \lim_{h \to \mathbf{0}} \frac{\|\lambda(h) - \mu(h)\| \leq \|f(a+h) - f(a) - \mu(h)\|}{\|h\|} + \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0$$

Now, let $x \in \mathbb{R}^n$. Then,

$$0 = \lim_{t \to 0} \frac{\|\mu(tx) - \lambda(tx)\|}{\|tx\|} = \frac{\|\mu(x) - \lambda(x)\|}{\|x\|}$$

This completes the proof. ∎

Theorem 1.3 (Chain Rule). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be functions differentiable at a and $b = f(a)$ respectively. Then, the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a) = Dg(b) \circ Df(a)$$

Proof. ∎

Proposition 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given by $f = (f_1, \dots, f_m)$. Then f is differentiable if and only if each $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable and*

$$Df(a) = \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_m(a) \end{bmatrix}$$

Proof. Suppose f is differentiable and π_i denote the projection on the i-th coordinate. Since π_i is differentiable, so is $f_i = \pi_i \circ f$. Conversely suppose each f_i is differentiable and let

$$A = \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_m(a) \end{bmatrix}$$

Then, for $h = (h_1, \dots, h_n) \in \mathbb{R}^n$, we have

$$\begin{aligned} \frac{\|f(a+h) - f(a) - Ah\|}{\|h\|} &= \frac{\left\| \begin{bmatrix} f_1(a+h) - f_1(a) - Df_1(a)h \\ \vdots \\ f_m(a+h) - f_m(a) - Df_m(a)h \end{bmatrix} \right\|}{\|h\|} \\ &\leq \sum_{i=1}^{m} \frac{\|f_i(a+h) - f(a) - Df_i(a)(h)\|}{\|h\|} \end{aligned}$$

whence the limit tends to 0 as $h \to \mathbf{0}$ which completes the proof. ■

Definition 1.5 (Partial Derivatives). Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. The limit

$$\lim_{h \to 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

if it exists is called the *i-th partial derivative of f at a* and is denoted by $D_i f(a)$. We also define *mixed partial derivatives* of f at a by

$$D_{i,j} f(a) = D_i(D_j f)(a).$$

Theorem 1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}$. If $D_{i,j} f$ and $D_{j,i} f$ are continuous in an open set containing a, then*

$$D_{i,j} f(a) = D_{j,i} f(a)$$

Proof. The proof uses Fubini's Theorem and is therefore postponed. ■

Lemma 1.7. *Let $A \subseteq \mathbb{R}^n$ be a closed rectangle. If the maximum (resp. minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof. Let $a = (a_1, \dots, a_n)$ and $h_i(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$. Then h_i has a maximum (resp. minimum) at a_i, is defined in an open interval containing a_i and is differentiable at a_i, whence from the calculus of a single variable, we see that $0 = h_i'(a_i) = D_i f(a)$, which completes the proof. ■

Theorem 1.8. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ and is given by $f = (f_1, \ldots, f_m)$, then $D_j f_i(a)$ exists for $1 \le i \le m$ and $1 \le j \le n$ and $Df(a)$ is the $m \times n$ matrix $\left[D_j f_i(a)\right]_{i,j}$.*

Proof. Since f is differentiable, $Df(a)$ is the matrix obtained by stackig $Df_i(a)$ as rows. Therefore, it suffices to prove the statement of the theorem in the case $m = 1$, that is $f : \mathbb{R}^n \to \mathbb{R}$ is given to be differentiable.

Consider the map $h : \mathbb{R} \to \mathbb{R}^n$ given by

$$h(x) = (a_1, \ldots, x, \ldots, a_n).$$

Then, due to Theorem 1.3,

$$D_j f(a) = D(f \circ h)(a_j) = Df(h(a_j))Dh(a_j) = Df(a) \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

This completes the proof. ∎

Theorem 1.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$ with $f = (f_1, \ldots, f_m)$. If there is an open set U containing a on which $D_j f_i$ exists and is continuous at a for $1 \le i \le m$ and $1 \le j \le n$, then f is differentiable at a.*

Proof. Due to Proposition 1.4, we may suppose that $m = 1$. Let $r > 0$ such that $B(a, r) \subseteq U$ and h be sufficiently small such that $a + h \in B(a, r)$. Then,

$$f(a+h) - f(a) = f(a_1 + h_1, \ldots, a_n) - f(a_1, \ldots, a_n) + \cdots + f(a_1 + h_1, \ldots, a_n + h_n) - f(a_1 + h_1, \ldots, a_{n-1} + h_{n-1}, a_n)$$

Using the mean value theorem, we have

$$f(a_1 + h_1, \ldots, a_i + h_i, \ldots, a_n) - f(a_1 + h_1, \ldots, a_{i-1} + h_{i-1}, \ldots, a_n) = h_i D_i f(c_i)$$

where $c_i = (a_1 + h_1, \ldots, b_i, a_{i+1}, \ldots, a_n) \in B(a, r)$ for some $b_i \in (a_i, a_i + h_i)$.

Let $\varepsilon > 0$ be given. Using uniform continuity on some (bounded) closed (and therefore compact) rectangle contained in U, we may choose an $r > 0$ such that whenever $|x - y| \le r$, $|D_i f(x) - D_i f(y)| < \varepsilon/n$ for each $1 \le i \le n$. Note that this can be done because all the D_i's are continuous on U. Then, we have, for any $\|h\| < r$,

$$\begin{aligned}
\frac{\left\|f(a+h) - f(a) - \sum_{i=1}^n h_i D_i f(a)\right\|}{\|h\|} &= \frac{\left\|\sum_{i=1}^n h_i D(c_i) - \sum_{i=1}^n h_i D_i f(a)\right\|}{\|h\|} \\
&\le \sum_{i=1}^n \frac{\|h_i(D_i f(c_i) - D_i f(a))\|}{\|h\|} \\
&\le \sum_{i=1}^n \|D_i f(c_i) - D_i f(a)\| < \varepsilon
\end{aligned}$$

This completes the proof. ∎

1.1 Inverse and Implicit Function Theorem

Lemma 1.10.

Theorem 1.11 (Inverse Function Theorem). *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $a \in U$ such that $\det(Df(a)) \neq 0$. Then, there is an open set V containing a and an open set W containing $f(a)$ such that the restriction $f : V \to W$ is a diffeomorphism.*

Proof. Upon composing f with a suitable linear transformation[1], we may suppose, without loss of generality that $Df(a) = \mathbf{id}_{n\times n}$. Then, we have

$$0 = \lim_{h\to\mathbf{0}} \frac{\|f(a+h) - f(a) - h\|}{\|h\|}$$

and thus, we may shrink U to a small enough open set such that $f(x) \neq f(a)$ for all $x \in U$. Since f is continuously differentiable, the function $\det Df(x)$ is a continuous function, and since $\det Df(a) \neq 0$, we may shrink U further such that $\det Df(x) \neq 0$ for all $x \in U$.

Using the continuity and therefore uniform continuity of $D_j f_i$ for each pair i, j, we may choose a closed rectangle A in U such that for all $x, y \in A$,

$$|D_j f_i(x) - D_j f_i(y)| < \frac{1}{2n^2}.$$

Consider now the function $g(x) = f(x) - x$. This is also continuously differentiable and for $x, y \in A$,

$$|D_j g_i(x) - D_j g_i(y)| = |D_j f_i(x) - D_j f_i(y)| < \frac{1}{2n^2}$$

Thus, using Lemma 1.10, we have for $x_1, x_2 \in A$,

$$\|(f(x_1) - f(x_2)) - (x_1 - x_2)\| = \|(f(x_1) - x_1) - (f(x_2) - x_2)\| = \|g(x_1) - g(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|,$$

consequently,

$$\|f(x_1) - f(x_2)\| \geq \frac{1}{2}\|x_1 - x_2\|$$

Thus f restricted to A is an injective map.

Let γ denote the boundary of A. Since $f(\gamma)$ is a compact set not containing $f(a)$, there is $d > 0$ such that for all $x \in \gamma$, $|f(a) - f(x)| \geq d$. Let $W = B(f(a), d/2)$. We contend that for every $y \in W$, there is a *unique* $x \in A$ such that $f(x) = y$.

Indeed, consider the function

$$h(x) = \|f(x) - y\|^2 = \sum_{i=1}^{n} |f_i(x) - y_i|^2.$$

Since f is a continuous function, so is h and since A is compact, there is a point $x_0 \in A$ at which h attains its minimum. First, notice that x_0 may not lie on γ since for all $x \in \gamma$, by construction, we have $\|f(a) - y\| < d/2 < \|f(x) - y\|$.

Since x_0 lies in the interior of A and the partials $D_j h$ exist for all j, we have

$$0 = D_j h(x_0) = 2\sum_{i=1}^{n} (f_i(x_0) - y_i) D_j f_i(x_0).$$

[1] We may do this as $\det Df(a) \neq 0$.

Equivalently, we may write this in matrix form as

$$0 = \begin{bmatrix} D_1 f_1(x_0) & \cdots & D_1 f_n(x_0) \\ \vdots & \ddots & \vdots \\ D_n f_1(x_0) & \cdots & D_n f_n(x_0) \end{bmatrix} \begin{bmatrix} f_1(x_0) - y_1 \\ \vdots \\ f_n(x_0) - y_n \end{bmatrix} = Df(x_0) \begin{bmatrix} f_1(x_0) - y_1 \\ \vdots \\ f_n(x_0) - y_n \end{bmatrix}.$$

We have $\det Df(x_0) \neq 0$ since $x_0 \in A \subseteq U$, and thus $f_i(x_0) = y_i$, equivalently, $f(x_0) = y$. The uniqueness follows from the injectivity of f on A.

Let $V = f^{-1}(W) \cap \operatorname{int}(A)$. Henceforth, we work with the restriction $f : V \to W$, which we have shown to be a continuously differentiable bijection. It remains to show that the inverse is continuously differentiable. Let $p : W \to V$ denote the inverse of f. Then, we have

$$\|p(y_1) - p(y_2)\| \leq 2\|y_1 - y_2\|$$

for all $y_1, y_2 \in W$ whence continuity of p follows. It remains to show the differentiability of p. ■

p is differentiable

We note that the condition on the continuity of the derivative cannot be dropped from the hypothesis of Theorem 1.11. Indeed, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This is differentiable on $\mathbb{R}$ with $f'(0) \neq 0$, but the derivative,

$$f'(x) = \begin{cases} \frac{1}{2} - \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

is not continuous at $x = 0$. For sufficiently large N, consider the point $x_N = 2/(2N+1)\pi$. It is not hard to argue that $f'(x_N) < 0$ whence f is not injective in any neighborhood containing 0. Thus it may not have an inverse, let alone a differentiable one.

Theorem 1.12 (Implicit Function Theorem).

Add in later

Chapter 2

Integration

Definition 2.1 (Oscillation). Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be bounded and $a \in A$. For every $\delta > 0$, define

$$M(a, f, \delta) = \sup\{f(x) \mid x \in A,\ \|x - a\| < \delta\} \qquad m(a, f, \delta) = \inf\{f(x) \mid x \in A,\ \|x - a\| < \delta\}$$

The *oscillation* of f at a is defined by

$$o(f, a) = \lim_{\delta \to 0}(M(a, f, \delta) - m(a, f, \delta))$$

We impose the boundedness condition on f to make sure that both $M(a, f, \delta)$ and $m(a, f, \delta)$ are well defined real numbers. Note that upon fixing a, the function $M(a, f, \cdot)$ is a decreasing function of $\delta > 0$ and $m(a, f, \cdot)$ is an increasing function of $\delta > 0$ whereby, the limit exists, since $M(a, f, \cdot) - m(a, f, \cdot)$ is a decreasing function of δ and is bounded below by 0.

Proposition 2.2. *A bounded function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous at $a \in A$ if and only if $o(f, a) = 0$.*

Proof. Suppose f is continuous at a. Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $\|x - a\| < \delta$ and $x \in A$. Then, for all such x, $M(a, f, \delta) - m(a, f, \delta) < 2\varepsilon$, consequently, $o(f, a) = 0$.

Conversely, suppose $o(f, a) = 0$. Let $\varepsilon > 0$ be given. Then, there is a $\delta > 0$ such that $M(a, f, \delta) - m(a, f, \delta) < \varepsilon$. Then, for all $x \in A$ with $\|x - a\| < \delta$, we have

$$-\varepsilon < -(M(a, f, \delta) - m(a, f, \delta)) \le f(x) - f(a) \le M(a, f, \delta) - m(a, f, \delta) < \varepsilon.$$

This completes the proof. ■

Theorem 2.3. *Let $A \subseteq \mathbb{R}^n$ be closed. If $f : A \to \mathbb{R}$ is a bounded function and $\varepsilon > 0$, then the set $B = \{x \in A \mid o(f, x) \ge \varepsilon\}$ is closed.*

Proof. We shall show that $\mathbb{R}^n \backslash B$ is open. If $x \in \mathbb{R}^n \backslash B$ and $x \notin A$, then there is trivially an open rectangle containing x disjoint from A and thus from B. On the other hand, if $x \in A$, then there is a $\delta > 0$ such that $M(x, f, \delta) - m(x, f, \delta) < \varepsilon$. Let C be an open rectangle contained in the open ball $B(x, \delta)$ in $\mathbb{R}^n$ (this may contain points not in A). Let $y \in C \cap A$. Choose δ' such that $B(y, \delta') \subseteq C$. Then, $M(y, f, \delta') < M(x, f, \delta)$ and $m(y, f, \delta') \ge m(x, f, \delta)$ whence $M(y, f, \delta') - m(y, f, \delta') < \varepsilon$ and $y \notin B$. This completes the proof. ■

2.1 The Setup

We borrow the idea of partitions from the Riemann Integral of a function of one variable.

Definition 2.4 (Partition). Let $A \subseteq \mathbb{R}^n$ be a closed rectangle, i.e. $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$. A partition of A is a collection $P = (P_1, \ldots, P_n)$ where each P_i given by $a = t_0^{(i)} < t_1^{(i)} < \cdots < t_{m_i}^{(i)} = b$ is a partition of the interval $[a_i, b_i]$.

Rectangles of the form

$$[t_{r_i}^{(1)}, t_{r_i+1}^{(1)}] \times \cdots \times [t_{r_n}^{(n)}, t_{r_n+1}^{(n)}]$$

are called *subrectangles of the partition* P. The collection of subrectangles of P is denoted by $\mathscr{S}(P)$. A partition $P' = (P_1', \ldots, P_n')$ is said to *refine* P if each P_i' refines P_i.

Definition 2.5 (Integral). Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a closed rectangle A and let P be a partition of A. For each $S \in \mathscr{S}(P)$ define

$$m_S(f) := \inf\{f(x) \mid x \in S\} \quad \text{and} \quad M_S(f) := \sup\{f(x) \mid x \in S\}.$$

Using this, we define the *upper and lower sums of f for the partition P* as

$$L(f, P) := \sum_{S \in \mathscr{S}(P)} m_S(f) v(S) \quad \text{and} \quad U(f, P) := \sum_{S \in \mathscr{S}(P)} M_S(f) v(S).$$

The function f is said to be *integrable over A* if

$$\mathbf{L}\int_A f := \sup_{P \in \mathscr{P}(A)} L(f, P) = \inf_{P \in \mathscr{P}(A)} U(f, P) =: \mathbf{U}\int_A f.$$

This common value is called the *integral of f over A* and is denoted by either

$$\int_A f \quad \text{or} \quad \int_A f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n.$$

Lemma 2.6. *Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ is a closed rectangle and $P, P' \in \mathscr{P}(A)$.*

(a) If P' refines P, then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

(b) $L(f, P') \leq U(f, P)$.

Proof. (a) Straightforward computation.

(b) Let $P'' = P \cup P' := (P_1 \cup P_1', \ldots, P_n \cup P_n')$. Then P'' refines both P and P' whence

$$L(f, P') \leq L(f, P'') \leq U(f, P'') \leq U(f, P).$$

■

Proposition 2.7. *Let $A \subseteq \mathbb{R}^n$ be a closed rectangle and $f : A \to \mathbb{R}$ a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$, there is $P \in \mathscr{P}(A)$ such that $U(f, P) - L(f, P) < \varepsilon$.*

Proof. Suppose f is integrable. Then, there are partitions $P, P' \in \mathscr{P}(A)$ such that

$$\int_A f - \frac{\varepsilon}{2} < L(f, P) \leq U(f, P') < \int_A f + \frac{\varepsilon}{2}.$$

Let $P'' \in \mathscr{P}$ refine both P and P'. Then,

$$\int_A f - \frac{\varepsilon}{2} < L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P') < \int_A f + \frac{\varepsilon}{2}$$

whence $U(f,P'') - L(f,P'') < \varepsilon$. The converse is trivial to prove. ■

Lemma 2.8. *Let $A \subseteq \mathbb{R}$ be a closed rectangle, $f : A \to \mathbb{R}$ a bounded function and $\varepsilon > 0$ such that $o(f,x) < \varepsilon$ for all $x \in A$. Then there is a partition $P \in \mathscr{P}(A)$ such that $U(f,P) - L(f,P) < \varepsilon v(A)$.*

Proof. ■

Definition 2.9 (Integration over Jordan measurable sets). Let $C \subseteq \mathbb{R}^n$ be a Jordan measurable set and $f : A \to \mathbb{R}^n$ a bounded function on a closed rectangle A containing C. Then, we *define*

$$\int_C f = \int_A \chi_C \cdot f.$$

2.2 Fubini's Theorem

Theorem 2.10 (Fubini). *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed rectangles and $f : A \times B \to R$ be a bounded integrable function. Denote by g_x the function $f(x,\cdot) : B \to \mathbb{R}$ and let*

$$\mathfrak{L}(x) = \mathbf{L}\int_B g_x \quad \text{and} \quad \mathfrak{U}(x) = \mathbf{U}\int_B g_x.$$

Then $\mathfrak{L}, \mathfrak{U} : A \to \mathbb{R}$ are integrable and

$$\int_A \mathfrak{L} = \int_{A\times B} f = \int_A \mathfrak{U}.$$

Proof. Let P be a partition of $A \times B$. Then, P is of the form (P_A, P_B) where P_A is a partition of A and P_B is a partition of B. Then, every subrectangle in $\mathscr{S}(P)$ is of the form $S_A \times S_B$ where $S_A \in \mathscr{S}(P_A)$ and $S_B \in \mathscr{S}(P_B)$.

$$\begin{aligned}
L(f,P) &= \sum_{S\in\mathscr{S}(P)} m_S(f)v(S) \\
&= \sum_{S_A\in\mathscr{S}(P_A)} \sum_{S_B\in\mathscr{S}(P_B)} m_{S_A\times S_B}(f)v(S_A\times S_B) \\
&= \sum_{S_A\in\mathscr{S}(P_A)} \left(\sum_{S_B\in\mathscr{S}(P_B)} m_{S_A\times S_B}(f)v(S_B) \right) v(S_A) \\
&\leq \sum_{S_A\in\mathscr{S}(P_A)} \left(\sum_{S_B\in\mathscr{S}(P_B)} m_{S_B}(g_x)v(S_B) \right) v(S_A) \\
&\leq \sum_{S_A\in\mathscr{S}(P_A)} \left(\mathbf{L}\int_B g_x \right) v(S_A) = \sum_{S_A\in\mathscr{S}(P_A)} \mathfrak{L}(x)v(S_A)
\end{aligned}$$

for all $x \in S_A$. Therefore,

$$L(f,P) \leq \sum_{S_A\in\mathscr{S}(P_A)} m_{S_A}\left(\mathfrak{L}(x)\right) v(S_A) = L(\mathfrak{L}, P_A).$$

Using a similar argument, we obtain $U(f,P) \geq U(\mathfrak{U}, P_A)$ whence

$$L(f,P) \leq L(\mathfrak{L}, P_A) \leq \underbrace{U(\mathfrak{L}, P_A) \leq U(\mathfrak{U}, P_A)}_{\mathfrak{L} \leq \mathfrak{U} \text{ for all } x \in A} \leq U(f,P).$$

Since f is integrable, for every $\varepsilon > 0$, there is a partition P of $A \times B$ such that $U(f,P) - L(f,P) < \varepsilon$ whence $U(\mathfrak{L}, P_A) - L(\mathfrak{L}, P_A) < \varepsilon$, implying that $\mathfrak{L}$ is integrable over A and

$$\int_{A \times B} f = \int_A \mathfrak{L}.$$

A similar argument can be applied for $\mathfrak{U}$. This completes the proof. ■

2.3 Partitions of Unity

2.3.1 The Bump Function

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

It is not hard to argue that $f \in C^\infty(\mathbb{R})$. Consider now $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = f(1-x)f(1+x).$$

Then $g \in C^\infty(\mathbb{R})$ and g is nonzero only on $(-1,1)$ and is positive there. Define

$$h(x) = \frac{\int_0^x g\left(\frac{x+1}{2}\right)\, dx}{\int_0^1 g\left(\frac{x+1}{2}\right)\, dx}.$$

Then $h \in C^\infty(\mathbb{R})$ such that $h(x) = 0$ for all $x \leq 0$ and $h(x) = 1$ for all $x \geq 1$.

Let now $U \subseteq \mathbb{R}^n$ be open and $C \subseteq U$ a compact subset. For each $a \in C$, there is an $\varepsilon_a > 0$ such that the cube

$$a \in \underbrace{[a_1 - \varepsilon_a, a_1 + \varepsilon_a] \times \cdots \times [a_n - \varepsilon_a, a_n + \varepsilon_a]}_{Q_a} \subseteq U.$$

Consider the function $F_a : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F_a(x) = \prod_{i=1}^n f\left(\frac{x_i - a_i}{\varepsilon_a}\right).$$

Then, $F_a(x) > 0$ for all $x \in \operatorname{Int} Q_a$ and $F_a(x) = 0$ for all $x \notin Q_a$. The collection $\{\operatorname{Int} Q_a\}_{a \in C}$ forms an open cover of C whence has a finite subcover, say $\{Q_{a_1}, \ldots, Q_{a_m}\}$. Let

$$F(x) = \sum_{i=1}^m F_{a_i}(x).$$

Then, $F(x) > 0$ for all $x \in C$ and $F(x) = 0$ for all $x \notin Q := \bigcup_{i=1}^m Q_{a_i}$, which is a closed (in fact, compact) set contained in U.

Let $\delta := \inf_{x \in C} F(x)$. Since C is compact, this minimum is achieved somewhere in C and thus is nonzero. Consider the composition $G : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$G(x) := h\left(F(x)/\delta\right).$$

Then $G(x)$ is a C^∞ function such that

- $G(x) = 1$ for all $x \in C$,
- $G(x) = 0$ for all $x \notin Q$,
- and thus $\operatorname{Supp}(G) \subseteq Q \subseteq U$ is a compact set.

This is called the *bump function*.

2.3.2 Constructing Partitions of Unity

Lemma 2.11. *Let $U \subseteq \mathbb{R}^n$ be an open set. Then, there is an ascending chain of compact sets $K_1 \subseteq K_2 \subseteq \cdots$ such that $K_i \subseteq \operatorname{Int} K_{i+1}$ and $U \subseteq \bigcup_{i=1}^{\infty} K_i$.*

Proof. ■

Theorem 2.12. *Let $A \subseteq \mathbb{R}^n$ and $\mathscr{U}$ be an open cover of A. Then, there is a collection Φ of $C^\infty(\mathbb{R})$ functions with the following properties:*

(a) For each $x \in A$ and $\varphi \in \Phi$, $0 \leq \varphi(x) \leq 1$.

(b) For each $\varphi \in \Phi$, there is an open set $U \in \mathscr{U}$ such that $\operatorname{Supp}(\varphi) \subseteq U$.

(c) The collection $\{\operatorname{Supp}(\varphi) \mid \varphi \in \Phi\}$ is a locally finite collection of compact sets.

(d) For each $x \in A$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$. This makes sense since only finitely many of the φ are nonzero for any $x \in A$.

Such a collection is called a partition of unity for A subordinate to $\mathscr{U}$.

Proof. There are three steps in this proof. First, we construct a partition of unity in the case when A is compact. Then, for an open A, we use the compact exhaustion of A to construct a partition of unity and finally, the case for an arbitrary A follows immediately, as we shall see.

Case 1. A is compact.

Case 2. A is open.

Case 3. A is arbitrary. ■

Remark 2.3.1. *Since $\mathbb{R}^n$ is second countable, every open cover of A can be reduced to a countable open cover of A whence we may choose our partition of unity to contain only countably many terms.*

Definition 2.13 (Extended Integral). An open cover $\mathscr{U}$ of an open set $A \subseteq \mathbb{R}^n$ is said to be *admissible* if each $U \in \mathscr{U}$ is contained in A. Let $f : A \to \mathbb{R}$ be such that for all $x \in A$, f is bounded in some open set containing x and the set of discontinuities of f in A has measure 0, then, f is said to be *integrable in the extended sense* if the sum

$$\int_{\varphi \in \Phi} \varphi \cdot |f|$$

converges for some countable partition of unity subordinate to $\mathscr{U}$. The integral of f is now defined as

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f.$$

Recall that due to Remark 2.3.1, we know that every admissible open cover admits a countable partition of unity subordinate to it.

Theorem 2.14. *Let $A \subseteq \mathbb{R}^n$ be open, $f : A \to \mathbb{R}$ be a function.*

(a) Let Ψ be another partition of unity subordinate to an admissible cover $\mathscr{V}$ of A, then $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$ also converges and

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\psi \in \Psi} \int_A \psi \cdot F.$$

(b) If A and f are bounded, then f is integrable in the extended sense.

(c) If A is Jordan-measurable and f is bounded, then this definition of $\int_A f$ agrees with the old one.

Proof. ■

2.4 Change of Variables

Theorem 2.15. *Let $A \subseteq \mathbb{R}^n$ be an open set and $g : A \to \mathbb{R}^n$ an injective, continuously differentiable function such that $\det(Dg(x)) \neq 0$ for all $x \in A$. If $f : g(A) \to \mathbb{R}$ is integrable, then*

$$\int_{g(A)} f = \int_A (f \circ g) |\det Dg|$$

Proof. ■ Add in later

Part II

Manifolds

Chapter 3

Smooth Manifolds

3.1 Topological manifolds

Definition 3.1 (Locally Euclidean). A topological space X is said to be *locally Euclidean of dimension n* if every $x \in X$ has a neighborhood $U \subseteq X$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 3.2 (Manifold). A *topological manifold of dimension n* is a topological space which is Hausdorff, second countable and locally Euclidean of dimension n.

Remark 3.1.1. *Recall from Algebraic Topology, that an open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$ only if $m = n$. This is proved using the* Excision Theorem. *Therefore, the dimension of a manifold is unambiguously defined.*

Definition 3.3 (Chart). Let M be a topological n-manifold. A *coordinate chart* on M is a pair (U, φ) where U is an open set of M and $\varphi : U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The chart (U, φ) is said to be *centered* at $p \in M$ if $\varphi(p) = 0$.

If $\widehat{U}$ is an open ball in $\mathbb{R}^n$ then U is said to be a *coordinate ball*, and similarly, if $\widehat{U}$ is an open cube in $\mathbb{R}^n$, then U is said to be a *coordinate cube*.

The map φ is called a *local coordinate map* and the component functions $(x^1, \ldots, x^n)$ of φ defined by $\varphi(p) = (x^1(p), \ldots, x^n(p))$ are called *local coordinates* on U.

An *atlas* $\mathscr{A}$ is a collection $\{(U_i, \varphi_i)\}_{i \in I}$ such that $\{U_i\}_{i \in I}$ forms an open cover of M.

Example 3.4 (The Product Manifold). Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$ respectively. Then, the topological space $M_1 \times \cdots \times M_k$ is Hausdorff and second countable. Further, let $\mathbf{p} = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$. Then, for each index j, there is a coordinate chart (U_j, φ_j) containing p_j. It is not hard to argue that

$$\varphi_1 \times \cdots \times \varphi_k : M_1 \times \cdots \times M_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is an embedding and thus $M_1 \times \cdots \times M_k$ is a manifold of dimension $n_1 + \cdots + n_k$.

From the above example, we see that the n-dimensional torus $\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n\text{-times}}$ is an n-dimensional topological manifold.

3.1.1 Some Topological Properties

Lemma 3.5. *Manifolds are locally compact Hausdorff.*

Proof. Straightforward. ∎

Lemma 3.6. *Manifolds are paracompact.*

Proof. Every regular Lindelöf space is paracompact. ∎

3.2 Smooth Structure

Definition 3.7 (Smooth Atlas). Let M be a topological n-manifold. The charts (U, φ) and (V, ψ) on M are said to be *smoothly compatible* if

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^n$$

is a diffeomorphism. An atlas $\mathscr{A}$ is said to be a *smooth atlas* if any two charts in $\mathscr{A}$ are smoothly compatible. An atlas $\mathscr{A}$ is said to be *maximal* if it is a maximal element in the poset of all atlases on M.

Definition 3.8 (Smooth Manifold). Let M be a topological n-manifold. A *smooth structure* on M is a maximal smooth atlas. A *smooth manifold* is a pair $(M, \mathscr{A})$ where M is a topological manifold and $\mathscr{A}$ is a smooth structure on M.

Remark 3.2.1. *There exist topological manifolds that admit no smooth structures at all. There is one such compact* 10*-dimensional manifold due to Kervaire.*

Proposition 3.9. *Let M be a topological manifold.*

(a) Every smooth atlas $\mathscr{A}$ for M is contained in a unique maximal smooth atlas.

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas[a].

[a] This is equivalent to requiring the charts in both the atlases to be compatible with one another.

Proof. (a) Let $\overline{\mathscr{A}}$ denote the set of all charts on M which are smoothly compatible with every chart in $\mathscr{A}$. This obviously contains $\mathscr{A}$. We contend that this is a smooth structure on M.

Let (U, φ) and (V, ψ) be two elements of $\overline{\mathscr{A}}$ we shall show that they are smoothly compatible. We need only check this when both are not in $\mathscr{A}$. We shall show that $\psi \circ \varphi^{-1}$ is smooth. The same proof would show that $\varphi \circ \psi^{-1}$ is smooth whereby both are diffeomorphisms.

Let $x \in \varphi(U \cap V)$. Then there is a unique $p \in U \cap V$ with $\varphi(p) = x$. Let (W, θ) be a chart in $\mathscr{A}$ with $x \in W$. Since this chart is smoothly compatible with (U, φ) and (V, ψ), the maps

$$\psi \circ \theta^{-1} : \theta(W \cap V) \to \psi(W \cap V) \quad \text{and} \quad \theta \circ \varphi^{-1} : \varphi(W \cap U) \to \theta(W \cap U)$$

are smooth, whence the composition

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$$

is smooth on a neighborhood of x. Since this is true for all $x \in \varphi(U \cap V)$, we have that $\overline{\mathscr{A}}$ is a smooth atlas.

Now, if $\mathscr{B}$ is any other smooth atlas containing $\mathscr{A}$, then every chart in $\mathscr{B}$ is smoothly compatible with every chart in $\mathscr{A}$ whence $\mathscr{B} \subseteq \overline{\mathscr{A}}$. This proves both uniqueness and maximality.

(b) Let $\mathscr{A}$ and $\mathscr{B}$ be the two atlases on M. Due to (a), $\overline{\mathscr{A} \cup \mathscr{B}}$ is a smooth structure containing $\mathscr{A}$ and $\mathscr{B}$. Due to uniqueness of the smooth structure, we are done.

■

Remark 3.2.2. *It is* not necessary *that a topological manifold admits exactly one smooth structure. Take for example the topological manifold $\mathbb{R}$ and two homeomorphisms $\mathbf{id}_{\mathbb{R}}$ and $\psi : \mathbb{R} \to \mathbb{R}$ given by $\psi(x) = x^3$. We have two atlases $\{\mathbf{id}_{\mathbb{R}}\}$ and $\{\psi\}$ on $\mathbb{R}$, and thus they give rise to two smooth structures on $\mathbb{R}$. We note that these structures are not the same since $\mathbf{id}_{\mathbb{R}}$ and ψ are not smoothly compatible. Indeed, $\mathbf{id}_{\mathbb{R}} \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}$ is the map $x \mapsto x^{1/3}$ which is not smooth.*

Definition 3.10. If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a *smooth chart* and the corresponding coordinate map φ is called a *smooth coordinate map*.

A *smooth coordinate domain* is the domain of some smooth coordinate chart. A *smooth coordinate ball* is a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space.

A set $B \subseteq M$ is called a *regular coordinate ball* if there is a smooth coordinate ball $B' \supseteq \overline{B}$ and a smooth coordinate map $\varphi : B' \to \mathbb{R}^n$ such that or some positive reals $r < r'$,

$$\varphi(B) = B(0, r), \quad \varphi(\overline{B}) = \overline{B(0, r)}, \quad \varphi(B') = B(0, r').$$

In particular, every regular coordinate ball is *precompact* in M.

Proposition 3.11. *Every smooth manifold has a countable basis of regular coordinate balls.*

Proof. It suffices to ind a basis of regular coordinate balls since a countable basis can then be extracted from it, as is well known. For any $p \in M$, let $\varphi_p : U_p \to \widehat{U}_p$ be a smooth coordinate map with $p \in U_p$. Let $r_p > 0$ be such that $B(\varphi_p(p), r_p) \subseteq \widehat{U}_p$. It is not hard to see that the collection

$$\left\{\varphi_p^{-1}(B(\varphi_p(p), r)) \mid 0 < r < r_p,\ p \in M\right\}$$

forms a basis and each element is a regular coordinate ball. This completes the proof. ■

Lemma 3.12 (Smooth Manifold Chart Lemma). *Let M be a set, and suppose a collection $\{(U_\alpha, \varphi_\alpha)\}$ is given such that*

(a) Each $\varphi_\alpha : U_\alpha \to \widehat{U}_\alpha \subseteq \mathbb{R}^n$ is a bijection where $\widehat{U}_\alpha$ is an open subset of $\mathbb{R}^n$.

(b) For each α, β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.

(c) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.

(d) A countable subcollection of $\{U_\alpha\}$ covers M.

(e) Whenever $p \neq q$ are distinct points in M, either there is some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. We begin by first topologizing M. Let

$$\mathcal{B} := \{\varphi_\alpha^{-1}(V) \mid V \subseteq \widehat{U}_\alpha \text{ is open}\}.$$

We contend that $\mathcal{B}$ forms a basis for some topology on M. Indeed, let $V \subseteq \widehat{U}_\alpha$ and $W \subseteq \widehat{U}_\beta$ be open sets and $p \in \varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$. We have

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W))$$

which is an element of $\mathcal{B}$ since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth and thus continuous map.

Along with this topology, each φ_α is an open continuous map which is also a bijection on $\widehat{U}_\alpha$ whence it is a homeomorphism. That M is Hausdorff follows almost immediately from (e). Indeed, let $p \neq q \in M$. If there are disjoint U_α, U_β containing p and q respectively, then we have a separation. If not, then $p, q \in U_\alpha$ for some α. Let V, W be a separation of $\varphi_\alpha(p), \varphi_\alpha(q)$ in $\widehat{U}_\alpha$, then $\varphi_\alpha^{-1}(V)$ and $\varphi_\alpha^{-1}(W)$ forms a separation of p, q in U_α.

Next, we must show that M is second countable. Note that each U_α is second countable, owing to it being homeomorphic to an open subset of $\mathbb{R}^n$, and since a countable number of U_α's cover M, we have that M is second countable.

Finally, (c) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas and is therefore contained in a unique smooth structure. This completes the proof. ■

The above lemma will be useful in defining the tangent bundle on a smooth manifold.

3.3 Manifolds with Boundary

Definition 3.13 (Manifold with Boundary). An *n-dimensional manifold with boundary* is a second countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or an open subset of $\mathbb{H}^n$ in the subspace topology.

A *chart on M* is a pair (U, φ) where $U \subseteq M$ is an open subset and $\varphi : U \to \widehat{U}$ is a homeomorphism onto either an open subset of $\mathbb{R}^n$ or an open subset of $\mathbb{H}^n$. In the former case, the chart is called an *interior chart* and in the latter case, it is called a *boundary chart*.

A point $p \in M$ is called an *interior point of M* if it is in the domain of some interior chart and similarly, it is called a *boundary point of M* if it is in the domain of a boundary chart (U, φ) such that $\varphi(p) \in \partial\mathbb{H}^n$.

The set of all interior points in M is denoted by $\operatorname{Int} M$ and the set of all boundary pionts in M is denoted by ∂M.

Remark 3.3.1. *From the above definitions, it is obvious that every manifold is a manifold with boundary but the converse is not true. This is illustrated in the following theorem, whose proof we postpone. In particular, if the boundary of a manifold with boundary is nontrivial, then it is not a manifold.*

Add link to proof

Theorem 3.14 (Topological Invariance of Boundary). *Let M be a topological manifold with boundary. Then, $M = \partial M \sqcup \operatorname{Int} M$. That is, the boundary and interior of M are disjoint sets whose union is all of M.*

Proposition 3.15. *Let M be a topological n-manifold with boundary. Then*

(a) $\operatorname{Int} M$ *is an open subset of M and a topological n-manifold.*

(b) ∂M *is a closed subset of M and a topological $(n-1)$-manifold.*

(c) *M is a topological manifold if and only if $\partial M = \varnothing$.*

(d) *If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.*

3.4 Smooth Maps

Definition 3.16. Let M and N be smooth manifolds with or without boundary and $A \subseteq M$. A map $F : A \to N$ is said to be *smooth on* A if for every $p \in A$ there is an open neighborhood $W \subseteq M$ and a smooth map $\widetilde{F} : W \to N$ whose restriction to $W \cap A$ agrees with F.

3.5 Partition of Unity

Definition 3.17 (Partition of Unity). Let M be a topological space and $\mathscr{U}$ an open cover of M indexed by a set J. A *partition of unity subordinate to* $\mathscr{U}$ is an indexed family (ψ_α) of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

1. $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in J$ and $x \in M$.
2. $\operatorname{Supp}(\psi_\alpha) \subseteq U_\alpha$ or each $\alpha \in J$
3. The set $\{\operatorname{Supp}(\psi_\alpha)\}$ is locally finite.
4. $\sum_{\alpha \in J} \psi_\alpha(x) = 1$ for all $x \in M$.

A partition of unity is said to be *smooth* if each ψ_α is a smooth function.

Theorem 3.18. *Let M be a smooth manifold with or without boundary and $\mathscr{U} = (U_\alpha)_{\alpha \in J}$ be an indexed open cover of M. Then there is a smooth partition of unity subordinate to $\mathscr{U}$.*

Definition 3.19 (Bump function). Let M be a topological space, $A \subseteq M$ a closed subset and $U \subseteq M$ an open subset containing A. A continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for A supported in U* if $0 \leq \psi \leq 1$ on M, $\psi|_A = 1$ and $\operatorname{Supp} \psi \subseteq U$.

Proposition 3.20. *Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset $U \subseteq M$ containing A, there is a smooth bump function or A supported in U.*

Proof. The collection $\{U, M \backslash A\}$ is an open cover of M and thus has a smooth partition of unity $\{\psi_1, \psi_2\}$ subordinate to it with $\operatorname{Supp}(\psi_1) \subseteq U$ and $\psi_1 + \psi_2 = 1$ on M. Since $\operatorname{Supp}(\psi_2) \subseteq M \backslash A$, we have $\psi_2|_A = 0$ whence $\psi_1|_A = 1$ and thus ψ_1 is the desired smooth bump function. ■

Lemma 3.21 (Extension Lemma for Smooth Maps into $\mathbb{R}^k$). *Let M be a smooth manifold with or without boundary, $A \subseteq M$ a closed subset and $f : A \to \mathbb{R}^k$ a smooth function. For any open subset U containing A, there is a smooth function $\widetilde{f} : M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and $\operatorname{Supp} f \subseteq U$.*

Proof. For each $a \in A$, by definition, there is an open neighborhood W_a of a and a function $\widetilde{f}_p$: ■

Complete proof. Simple application of POU

Chapter 4

Tangent Spaces

4.1 Tangent Vectors

4.1.1 On $\mathbb{R}^n$

Definition 4.1. Let $a \in \mathbb{R}^n$. A map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is said to be a *derivation at a* if it is linear over $\mathbb{R}$ and satisfies the product rule:

$$w(fg) = f(a)w(g) + g(a)w(f).$$

Denote by $T_a\mathbb{R}^n$ the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. This is obviously a vector space under the operations:

$$(w_1 + w_2)(f) = w_1(f) + w_2(f) \quad \text{and} \quad (cw)(f) = cw(f)$$

for all $w_1, w_2 \in T_a\mathbb{R}^n$ and $c \in \mathbb{R}$.

Lemma 4.2. *Suppose $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$.*

(a) If f is a constant function, then $w(f) = 0$.

(b) If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Proof. (a) Let $f \equiv c \in \mathbb{R}$. First, consider the constant function $g \equiv 1 \in \mathbb{R}$. Note that $g = g^2$ and thus

$$w(g) = w(g^2) = g(a)w(g) + g(a)w(g) = 2w(g)$$

whence $w(g) = 0$ and $w(f) = cw(g) = 0$.

(b) Trivial. ■

For a vector $v \in \mathbb{R}^n$ and a point $a \in \mathbb{R}^n$, let $D_v|_a$ denote the *directional derivative* at a in the direction of v, which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(a + vt)\Big|_{t=0}$$

It is not hard to see that $D_v|_a \in T_a\mathbb{R}^n$. Indeed, if $f, g \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
D_v|_a(fg) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f(a+vt)g(a+vt)\right)\Big|_{t=0} \\
&= f(a)\frac{\mathrm{d}}{\mathrm{d}t} g(a+vt)\Big|_{t=0} + g(a)\frac{\mathrm{d}}{\mathrm{d}t} f(a+vt)\Big|_{t=0} \\
&= f(a)D_v|_a(g) + g(a)D_v|_a(f).
\end{aligned}$$

Proposition 4.3. *The map $v_a \mapsto D_v|_a$ is an isomorphism of vector spaces from $\mathbb{R}^n_a \to T_a\mathbb{R}^n$.*

Proof. Call this map Φ. The fact that Φ is a linear transformation follows from

$$D_v|_a(f) = v \cdot \nabla(f)(a).$$

Next, we shall show that it is injective. Let $v_a \in \mathbb{R}^n_a$ such that $D_v|_a \equiv 0$. Consider the function $\pi_j : \mathbb{R}^n \to \mathbb{R}$ which is the projection on the j-th coordinate. This is obviously in $C^\infty(\mathbb{R}^n)$. Then, we have

$$0 = D_v|_a(\pi_j) = \frac{\mathrm{d}}{\mathrm{d}t}(a^j + v^j t) = v^j$$

whence $v = 0$ and the kernel of Φ is trivial.

Lastly, we must show that Φ is a surjection. Let $w \in T_a\mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$. Due to , we may write Reference Taylor's Theorem

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{j=1}^{n}\sum_{i=1}^{n}(x^i - a^i)(x^j - a^j)\int_0^1 (1-t)\frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\ \mathrm{d}t.$$

Evaluating this at $x = a$, we have that

$$w(f)(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) w(x^i) = D_v|_a(f)$$

where $v = (w(x^1), \ldots, w(x^n))$. This completes the proof. ∎

Corollary 4.4. For $a \in \mathbb{R}^n$, the n derivations

$$\left.\frac{\partial}{\partial x^1}\right|_a, \ldots, \left.\frac{\partial}{\partial x^n}\right|_a$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

4.1.2 On a Manifold

Definition 4.5. Let M be a smooth manifold with or without boundary and let $p \in M$. A linear map $w : C^\infty(M) \to \mathbb{R}$ is said to be a *derivation at p* if it obeys the product rule:

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M).$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM is a vector space called the *tangent space to M at p*. An element of T_pM is called a *tangent vector to M at p*.

Lemma 4.6. *Let M be a smooth manifold with or without boundary, $p \in M$, $w \in T_pM$and $f, g \in C^\infty(M)$.*

(a) If f is a constant function then $w(f) = 0$.

(b) If $f(p) = g(p) = 0$, then $w(fg) = 0$.

Proof. Same as the proof for $\mathbb{R}^n$. ∎

Lemma 4.7. *Let $p \in M$ and $f, g \in C^\infty(M)$ such that $f = g$ in some open neighborhood of p. Then, for any $v \in T_pM$, $v(f) = v(g)$.*

Proof. Let $h = f - g \in C^\infty(M)$ and $p \subseteq U$ be a neighborhood on which h vanishes. The collection $\{M \backslash \{p\}, U\}$ is an open cover of M whence there is a smooth partition of unity $\{\psi, \psi'\}$ subordinate to it.

Note that for all $x \in M \backslash U$, $\psi(x) = 1$ and $\psi'(x) = 0$ whence $\psi \cdot h = h$ on all of M and $\psi(p) = 0 = h(p)$ and thus

$$0 = v(\psi \cdot h) = v(h) = v(f) - v(g).$$

■

4.2 Differential of a Smooth Map

Definition 4.8 (Differential). Let M and N be smooth manifolds with or without boundary and $F : M \to N$ a smooth map. For each $p \in M$, the *differential of F at p* is the map

$$dF_p : T_pM \to T_{F(p)}N$$

given by

$$dF_p(v)(f) = v(f \circ F)$$

for all $f \in C^\infty(N)$.

Proposition 4.9. *The map $dF_p : T_pM \to T_{F(p)}N$ is a linear transformation.*

Proof. Let $v \in T_pM$. Then, for $f, g \in C^\infty(N)$ and $c \in \mathbb{R}$, we have

$$dF_p(v)(f + cg) = v((f + cg) \circ F) = v(f \circ F + cg \circ F) = v(f \circ F) + v(cg \circ F) = v(f \circ F) + cv(g \circ F),$$

and

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F). \end{aligned}$$

Thus $dF_p(v)$ is indeed a derivation on N at p.

Next, we must show that dF_p is a linear transformation. Indeed, if $v, w \in T_pM$ and $c \in \mathbb{R}$, we have for all $f \in C^\infty(N)$,

$$dF_p(v + cw)(f) = (v + cw)(f \circ F) = v(f \circ F) + cw(f \circ F) = dF_p(v) + cdF_p(w).$$

This completes the proof. ■

Proposition 4.10. *Let M, N and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$.*

(a) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P.$

(b) $d(\mathbf{id}_M)_p = \mathbf{id}_{T_pM} : T_pM \to T_pM.$

(c) *If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

Proof. (a) This is almost *by definition*. Let $f \in C^\infty(P)$ and $v \in T_pM$. Then,

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = v((f \circ G) \circ F)$$

and

$$dG_{F(p)}(dF_p(v))(f) = dF_p(v)(f \circ G) = v(f \circ G \circ F).$$

(b) For any $v \in T_pM$ and $f \in C^\infty(M)$,

$$d(\mathbf{id}_M)_p(v)(f) = v(f \circ \mathbf{id}_M) = v(f).$$

(c) Let $G = F^{-1}$. From (a) and (b),

$$\mathbf{id}_{T_pM} = dG_{F(p)} \circ dF_p,$$

whence the conclusion follows. ■

In particular, Proposition 4.10 shows that the map $T : \mathbf{Diff}_* \to \mathbf{Vec}$ which maps

$$(M, p) \mapsto T_pM \quad \text{and} \quad \left[F : (M, p) \to (N, F(p))\right] \mapsto \left[dF_p : T_pM \to T_{F(p)}N\right]$$

is a *covariant functor*. We shall see a similar functor from **Diff** to **Diff** in an upcoming section.

Lemma 4.11. *Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset (and thus a manifold in its own right) and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism of vector spaces.*

Proof. ■

Proposition 4.12. *Let M be a smooth n-manifold (without boundary). Then for any $p \in M$, T_pM is an n-dimensional vector space.*

Proof. Let (U, φ) be a smooth chart containing p. Due to the preceeding lemma, T_pU is isomorphic to T_pM as vector spaces. Thus, it suffices to show that T_pU is an n-dimensional vector space. We have a diffeomorphism $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ whence $d\varphi_p : T_pU \to T_{\varphi(p)}\widehat{U}$ is an isomorphism of vector spaces but since the latter is isomorphic to $\mathbb{R}^n$ (as a vector space) as we have seen earlier, we are done. ■

4.3 The Tangent Bundle

Definition 4.13 (Tangent Bundle). Let M be a smooth manifold with or without boundary. The *tangent bundle of M*, denoted by TM is defined as

$$TM = \coprod_{p \in M} T_pM.$$

This is equipped with the natural projection $\pi : TM \to M$ which maps every vector in T_pM to $p \in M$.

Topology on TM for a manifold.

Let M be a smooth manifold (without boundary). We shall use Lemma 3.12 to construct a smooth structure on TM. Let (U, φ) be a smooth chart for M

Definition 4.14 (Global Differential). Let M and N be smooth manifold with or without boundary and Let $F : M \to N$ be a smooth map. The *global differential* or *global tangent map* is a map $dF : TM \to TN$ which maps $v \in T_pM$ to $dF_p(v) \in T_{F(p)}N$.

In other words, the global differential obtained by simply stitching together the dF_p's for all $p \in M$.

Chapter 5

Submersions and Immersions

5.1 Maps of Constant Rank

Definition 5.1. Let $F : M \to N$ be a map between smooth manifolds with or without boundary. For a point $p \in M$, define the *rank of F at p* to be the rank of the linear transformation $dF_p : T_pM \to T_{F(p)}N$. If F has the same rank r at all points in M, then it is said to have *constant rank* and we write $\operatorname{rank} F = r$.

The map F is called a *smooth submersion* if $\operatorname{rank} F = \dim N$ and a *smooth immersion* if $\operatorname{rank} F = \dim M$.

Proposition 5.2. *Let $F : M \to N$ be a smooth map between smooth manifolds with or without boundary and $p \in M$. If dF_p is a surjection, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is an injection, then p has a neighborhood U such that $F|_U$ is an immersion.*

Definition 5.3 (Local Diffeomorphism). A map $F : M \to N$ between smooth manifolds with or without boundary is called a *local diffeomorphism* if every $p \in M$ has a neighborhood U such that $F(U)$ is open in N and the restriction $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 5.4 (Inverse Function Theorem for Manifolds). *Let $F : M \to N$ be a smooth map between smooth manifolds (without boundary). If $p \in M$ is a point such that dF_p is invertible, then there are neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof. Let (U, φ) and (V, ψ) be smooth charts for M and N centered at p and $F(p)$ respectively. Then,

$$\widehat{F} := \psi \circ F \circ \varphi^{-1} : \widehat{U} = \varphi(U) \subseteq \mathbb{R}^n \to \widehat{V} = \psi(V) \subseteq \mathbb{R}^n$$

is a smooth map with $\widehat{F}(0) = 0$. Since φ and ψ are diffeomorphisms, the linear transformations $d(\varphi^{-1})_0$ and $d\psi_{F(p)}$ are invertible and thus the composition

$$d\widehat{F}_p = d\psi_{F(p)} \circ dF_p \circ d(\varphi^{-1})_0$$

is invertible. Thus, due to Theorem 1.11, there are open subsets $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ such that the restriction $\widehat{F}|_{\widehat{V}_0} : \widehat{U}_0 \to \widehat{V}_0$ is a diffeomorphism. Let $U_0 := \varphi^{-1}(\widehat{U}_0)$ and $V_0 := \psi^{-1}(\widehat{V}_0)$. Then, F restricts to a diffeomorphism of U_0 to V_0. This completes the proof. ∎

5.1.1 The Rank Theorems

Theorem 5.5 (Local Rank Theorem). *Let $F : M \to N$ be a smooth map between smooth manifolds (without boundary) of dimensions m and n respectively with constant rank r. For each $p \in M$, there exist smooth charts (U, φ) for M centered at p and (V, ψ) or N centered at $F(p)$ such that $F(U) \subseteq V$ in which F has a coordinate representation of the form*

$$\widehat{\ }(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

Chapter 6

Vector Fields

Definition 6.1 (Vector Field). Let M be a smooth manifold with or without boundary. A *vector field on* M is a continuous section $X : M \to TM$ of the map $\pi : TM \twoheadrightarrow M$.

A vector field is said to be *smooth* if the section $X : M \to TM$ is a smooth map of manifolds. A *rough* vector field is simply a section $X : M \to TM$ which need not be continuous. The value of a vector field at a point is denoted by either $X(p)$ or X_p.

The support of X is defined to be

$$\operatorname{Supp} X = \overline{\{p \in M \mid X_p \neq 0\}}.$$

A vector field is said to be *compactly supported* if $\operatorname{Supp} X \subseteq M$ is compact.

In other words, a section $X : M \to TM$ is said to be a

vector field if it is a morphism in **Top**.

smooth vector field if it is a morphism in **Diff**.

rough vector field if it is a morphism in **Set**.

Chapter 7

Vector Bundles

7.1 Vector Bundles

Definition 7.1. Let M be a topological space. A *real vector bundle of rank k over M* is a topological space E together with a continuous surjection $\pi : E \twoheadrightarrow M$ satisfying the following conditions:

(a) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

(b) For every $p \in M$, there is a neighborhood U of p in M and a homeomorophism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a *local trivialization of E over U* satisfying the following additional conditions:

- If $\pi_U : U \times \mathbb{R}^k \twoheadrightarrow U$ is the natural projection, then $\pi_U \circ \Phi = \pi$.
- For each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

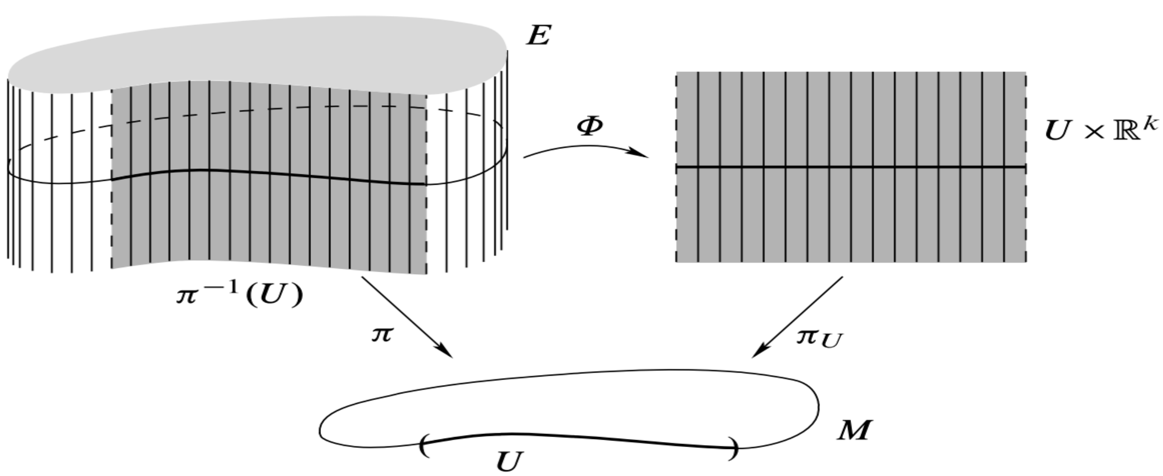

Figure 7.1: A local trivialization of a vector Bundle

Chapter 8

Tensors and Differential Forms

Throughout this chapter, all vector spaces are assumed to be over $\mathbb{R}$. They will usually be finite dimensional but we shall explicitly mention this in order to avoid confusion.

8.1 Tensors

Definition 8.1. Let $V_1, \ldots, V_k$ and W be vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is said to be alternating if for each $1 \le i \le k$,

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = a_i F(v_1, \ldots, v_i, \ldots, v_k) + a_i' F(v_1, \ldots, v_i', \ldots, v_k).$$

We denote by $\mathscr{L}(V_1, \ldots, V_k; W)$ the set of all multilinear maps from $V_1 \times \cdots \times V_k \to W$.

For linear map $f_i : V_i \to \mathbb{R}$ for $1 \le i \le k$, define the multilinear map

$$f_1 \otimes \cdots \otimes f_k : V_1 \times \cdots \times V_k \to \mathbb{R}$$

by $(f_1 \otimes \cdots \otimes f_k)(v_1, \ldots, v_k) = f_1(v_1) \cdots f_k(v_k)$. This notation is a consequence of the forthcoming Theorem 8.2.

Remark 8.1.1 (Constructing the Tensor Product). *In this remark we recall a construction from module theory. Let $V_1, \ldots, V_k$ be vector spaces. Let $\mathfrak{F}(V_1 \times \cdots \times V_k)$ denote the free vector space on $V_1 \times \cdots \times V_k$. Let W denote the subspace spanned by elements*

$$\mathbf{e}_{(v_1,\ldots,v_i+v_i',\ldots,v_k)} - \mathbf{e}_{(v_1,\ldots,v_k)} - \mathbf{e}_{(v_1,\ldots,v_i',\ldots,v_k)}$$
$$\mathbf{e}_{(v_1,\ldots,av_i,\ldots,v_k)} - a\mathbf{e}_{(v_1,\ldots,v_k)}$$

for $1 \le i \le k$ and $a \in \mathbb{R}$. Then, the vector space $\mathfrak{F}(V_1 \times \cdots \times V_k)/W$ is called the tensor product of $V_1, \ldots, V_k$ *and is denoted by $V_1 \otimes \cdots \otimes V_k$. The tensor product has the property that every multilinear map from $V_1 \times \cdots \times V_k$ factors through it.*

Theorem 8.2. *Let $V_1, \ldots, V_k$ be finite dimensional vector spaces. Then, there is a natural isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \cong \mathscr{L}(V_1, \ldots, V_k; \mathbb{R}).$$

Proof. Consider the map $\Phi : V_1^* \times \cdots \times V_k^* \to \mathscr{L}(V_1, \ldots, V_k; \mathbb{R})$ given by

$$\Phi(f_1, \ldots, f_k)(v_1, \ldots, v_k) = f_1(v_1) \cdots f_k(v_k).$$

It is not hard to see that Φ is a multilinear map. Thus, this induces a map

$$\varphi : V_1^* \otimes \cdots \otimes V_k^* \to \mathscr{L}(V_1, \ldots, V_k; \mathbb{R}).$$

The fact that φ is an isomorphism follows from the fact that it maps the basis of $V_1^* \otimes \cdots \otimes V_k^*$ to the basis of $\mathscr{L}(V_1 \times \cdots \times V_k, \mathbb{R})$. ■

8.1.1 Covariant and Contravariant Tensors

Definition 8.3 (Covariant, Contravariant Tensor). Let V be a finite dimensional vector space. If k is a positive integer, a *covariant k-tensor on V* is an element of the k-fold tensor product $T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k-\text{times}}$. The number k is called the *rank* of the aforementioned covariant tensor.

Similarly, a *contravariant k-tensor on V* is an element of the k-fold tensor product $T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k-\text{times}}$. Again, the number k is called the *rank* of the contravariant tensor.

Due to the natural isomorphism of Theorem 8.2, we may identify a covariant k-tensor as a multilinear map $V_1 \times \cdots \times V_k \to \mathbb{R}$ and similarly, we may identify a contravariant k-tensor as a multilinear map $V_1^* \times \cdots \times V_k^* \to \mathbb{R}$. We shall switch between these identifications to suit our needs.

8.1.2 Symmetric and Alternating Tensors

Definition 8.4 (Symmetric, Alternating Tensor). Let V be a finite dimensional vector space. A covariant k-tensor α on V is said to be *symmetric* if for every $\sigma \in \mathscr{S}_k$,

$$\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Similarly, it is said to be *alternating* if

$$\alpha(v_1, \ldots, v_k) = \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

8.1.3 Tensor Fields on a Manifold