

Complex Analysis

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Abstract

The main reference for these notes is [2], which I find much more readable than [1].

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Chapter 1

Introduction

1.1 Preliminaries

Definition 1.1. Let $\{a_n\}$ be a real sequence. Define the limit superior and the limit inferior of a sequence to be

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_n, a_{n+1}, \dots\}$$
$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup\{a_n, a_{n+1}, \dots\}$$

Proposition 1.2. \mathbb{C} is complete.

Proof. Let $\{z_n = x_n + iy_n\}$ be a Cauchy sequence in \mathbb{C} . For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|z_n - z_m| < \varepsilon$, and thus, $|x_n - x_m| < \varepsilon$ and $|y_n - y_m| < \varepsilon$. Consequently, both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy and converge and therefore, so does $\{z_n\}$. ■

1.2 Power Series

Definition 1.3 (Power Series). Let $a \in \mathbb{C}$. A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n(z - a)^n$ where $\{a_n\}$ is an infinite sequence of complex numbers.

Example 1. The power series $\sum_{n=0}^{\infty} z^n$ converges if $|z| < 1$ and diverges if $|z| > 1$.

Proof. Suppose $|z| < 1$. We shall show that the sequence of partial sums is Cauchy. Indeed, for $m \geq n$, we have

$$|z^n + \dots + z^m| < |z|^n \frac{1}{1 - |z|}$$

On the other hand, if $|z| > 1$, we shall show that the sequence is not Cauchy. If s_n denotes the n -th partial sum of the series, we note that

$$|s_{n+1} - s_n| = |z|^{n+1}$$

This completes the proof. ■

Theorem 1.4. For a given power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, define the number $R \in [0, \infty]$ by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

then

- (a) if $|z-a| < R$, the series converges absolutely
- (b) if $|z-a| > R$, the series diverges
- (c) if $0 < r < R$, then the series converges uniformly on $\bar{B}(a, r)$

This R is known as the radius of convergence of the power series.

Proof. For simplicity, let $a = 0$ (this does not affect the correctness of the proof).

- (a) Since $|z| < R$, there is a real number r such that $|z| < r < R$. Consequently, by definition, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n|^{1/n} < \frac{1}{r}$. In other words, for all $n \geq N$, $|z|^n |a_n| < 1$. It is evident from here that the partial sums form a Cauchy sequence.
- (b) If $|z| > R$, there is a positive real number r such that $|z| > r > R$, consequently, there is a subsequence $\{n_k\}$ such that $|a_{n_k}|^{1/n_k} r > 1$. If A_n denotes the partial sums of the sequence, then $|A_{n_k} - A_{n_k-1}| > 1$ and thus, the sequence is not Cauchy, and therefore, divergent.
- (c) There is a positive real number ρ such that $r < \rho < R$ and a natural number N such that for all $n \geq N$, $|a_n| < \frac{1}{\rho^n}$. Consequently, for all $z \in \bar{B}(0, r)$, $|a_n z^n| < \left(\frac{r}{\rho}\right)^n$ and we are done due to the Weierstrass M-test.

■

Theorem 1.5 (Mertens). Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be complex sequences such that

- (a) $\sum a_n$ converges absolutely and $\sum b_n$ converges
- (b) $\sum a_n = A$ and $\sum b_n = B$
- (c) $\{c_n\}$ is the Cauchy product of $\{a_n\}$ and $\{b_n\}$

Then, $\sum c_n$ converges to AB .

Proof. Define A_n , B_n and C_n in the obvious way. Further, let $\beta_n = B_n - B$. Then, we have

$$\begin{aligned} C_n &= \sum_{k=0}^n a_k B_{n-k} \\ &= \sum_{k=0}^n a_k (B + \beta_{n-k}) \\ &= BA_n + \sum_{k=0}^n a_k \beta_{n-k} \end{aligned}$$

Let $\gamma_n = \sum_{k=0}^n a_k \beta_{n-k}$. We shall show $\lim_{n \rightarrow \infty} \gamma_n = 0$. Let $\varepsilon > 0$ be given. Let $\alpha = \sum_{n=0}^{\infty} |a_n|$ (since it is known that it converges absolutely). From (b), we know that $\beta_n \rightarrow 0$, therefore, there is N such that $|\beta_n| < \varepsilon/\alpha$ for

all $n \geq N$. Consequently, we have

$$\begin{aligned} |\gamma_n| &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon \alpha \end{aligned}$$

Which immediately gives us

$$\limsup_{n \rightarrow \infty} |\gamma_n| \leq \varepsilon \alpha$$

and since ε was arbitrary, we have the desired conclusion. ■

1.3 Analytic Functions

Definition 1.6. If $G \subset \mathbb{C}$ is open, and $f : G \rightarrow \mathbb{C}$ then f is *differentiable* at a point $a \in G$ if

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists. The value of this limit is denoted by $f'(a)$ and is called the *derivative* of f at a . If f is differentiable at each point of G we say that f is differentiable on G . If f' is continuous then we say that f is *continuously differentiable*.

Proposition 1.7. If $f : G \rightarrow \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a .

Proof. One line:

$$\lim_{z \rightarrow a} |f(z) - f(a)| = \lim_{z \rightarrow a} \frac{|f(z) - f(a)|}{|z - a|} |z - a| = \lim_{z \rightarrow a} \left| \frac{f(z) - f(a)}{z - a} \right| \lim_{z \rightarrow a} |z - a| = 0$$
■

Definition 1.8 (Analytic Function). A function $f : G \rightarrow \mathbb{C}$ is *analytic* if f is continuously differentiable on G .

Theorem 1.9 (Chain Rule). Let f and g be analytic on G and Ω respectively and suppose $f(G) \subseteq \Omega$. Then $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

for all $z \in G$.

Proof. Define the function $h \equiv g \circ f : G \rightarrow \mathbb{C}$. We shall show that the function h is differentiable at every point $a \in G$ and that the derivative at a equals $g'(f(a))f'(a)$. Notice that the latter implies analyticity.

Let $z = f(a)$. Then, by definition, we have functions $u : G \rightarrow \mathbb{C}$ and $v : \Omega \rightarrow \mathbb{C}$ with $\lim_{x \rightarrow a} u(x) = 0$ and $\lim_{x \rightarrow z} v(x) = 0$ satisfying

$$\begin{aligned} f(x) - f(a) &= (x - a)(f'(a) + u(x)) \\ g(x) - g(z) &= (x - z)(g'(z) + v(x)) \end{aligned}$$

Note that

$$\begin{aligned}
h(x) - h(a) &= g(f(x)) - g(f(a)) \\
&= (f(x) - f(a))(g'(z) + v(f(x))) \\
&= (x - a)(f'(a) + u(x))(g'(z) + v(f(x)))
\end{aligned}$$

Taking the limit gives the desired result. ■

Theorem 1.10. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then

(a) For each $k \geq 1$, the series

$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z-a)^{n-k} \quad (\star)$$

has radius of convergence R

(b) The function f is infinitely differentiable on $B(a, R)$ and furthermore, $f^{(k)}(z)$ is given by the series (\star) for all $k \geq 1$ and $|z-a| < R$

(c) For $n \geq 0$,

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

Proof. It suffices to prove the theorem for $a = 0$.

(a) We shall prove it for $k = 1$ since the general case would follow inductively. Since $\lim_{n \rightarrow \infty} n^{1/(n-1)} = 1$, it suffices to show that

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/(n-1)}$$

Note that we may write

$$f(z) = a_0 + z \underbrace{\sum_{n=1}^{\infty} a_n z^{n-1}}_{g(z)}$$

It is not hard to argue that both $f(z)$ and $g(z)$ have the same radius of convergence, and thus $\limsup |a_n|^{1/n} = \limsup |a_n|^{1/(n-1)}$.

(b) Again, we shall only show this for $k = 1$ since the general case would follow inductively. Define

$$s_n = \sum_{k=0}^n a_k z^k \quad \text{and} \quad e_n = \sum_{k=n+1}^{\infty} a_k z^k$$

Obviously, $f = s_n + e_n$ for all $n \in \mathbb{N}$. Let $g(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Let $w \in B(0, R)$ and choose a positive real number r such that $0 < |w| < r < R$. Let $\delta > 0$ be chosen such that $B(w, \delta) \subseteq B(0, r)$. Choose any $\varepsilon > 0$.

Then, we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \left(\frac{s_n(z) - s_n(w)}{z - w} - g(w) \right) + \frac{e_n(z) - e_n(w)}{z - w}$$

Note that

$$\left| \frac{e_n(z) - e_n(w)}{z - w} \right| \leq \sum_{k=n+1}^{\infty} |z^{k-1} + \dots + w^{k-1}| \leq \sum_{k=n+1}^{\infty} kr^{k-1}$$

Since the series on the right is the trailing sum of a convergent series, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\sum_{k=n+1}^{\infty} kr^{k-1} < \varepsilon/3$.

Similarly, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|s'_n(w) - g(w)| < \varepsilon/3$. Finally, there is $\delta' > 0$ such that for all $z \in B(w, \delta')$,

$$\left| \frac{s_n(z) - s_n(w)}{z - w} - s'_n(w) \right| < \frac{\varepsilon}{3}$$

Putting these together, we see that for all $z \in B(w, \min\{\delta, \delta'\})$, and $n \geq \max\{N_1, N_2\}$

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq \left| \frac{s_n(z) - s_n(w)}{z - w} - s'_n(w) \right| + |s'_n(w) - g(w)| + \left| \frac{e_n(z) - e_n(w)}{z - w} \right| \leq \varepsilon$$

And we are done.

(c) Straightforward. ■

Corollary. If the series $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ has radius of convergence $R > 0$ then $f(z)$ is analytic in $B(a, R)$.

1.4 Cauchy Riemann Equations

Let $f : G \rightarrow \mathbb{C}$ be analytic and let $u(x, y) = \Re f(x + iy)$ and $v(x, y) = \Im f(x + iy)$. Then, we must have, for all $z \in G$,

$$\lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(z + ih) - f(z)}{ih}$$

The analyticity of f implies the differentiability of u and v and thus, the above equality is equivalent to

$$u_x + iv_x = f'(z) = \frac{1}{i} (u_y + iv_y)$$

or,

$$u_x = v_y \quad \text{and} \quad u_y + v_x = 0 \tag{CR}$$

Suppose u and v have continuous partial derivatives, in which case, recall that second order mixed derivatives exist and do not depend on the order of derivatives taken, that is, $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

Straightforward algebraic manipulation would yield

$$u_{xx} + u_{yy} = 0$$

In other words, u and v are harmonic conjugates.

Theorem 1.11. Let $G \subseteq \mathbb{C}$ and $u, v : G \rightarrow \mathbb{R}$ have continuous partial derivatives. Then $f : G \rightarrow \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if u and v satisfy (CR).

Proof. Suppose the functions u and v satisfy the hypothesis of the theorem. Let $z = x + iy$. We shall show that

$$\lim_{s+it \rightarrow 0} \frac{f(z + (s + it)) - f(z)}{s + it}$$

exists.

Define

$$\begin{aligned}\varphi(s, t) &= (u(x + s, y + t) - u(x, y)) - (u_x(x, y)s + u_y(x, y)t) \\ \psi(s, t) &= (v(x + s, y + t) - v(x, y)) - (v_x(x, y)s + v_y(x, y)t)\end{aligned}$$

It is not hard to see, using **CR**, that

$$\varphi(s, t) + i\psi(s, t) = f(z + (s + it)) - f(z) - (s + it)(u_x(x, y) + iv_x(x, y))$$

and hence, it would suffice to show that

$$\lim_{s+it \rightarrow 0} \frac{\varphi(s, t) + i\psi(s, t)}{s + it} = 0$$

We have

$$u(x + s, y + t) - u(x, y) = u(x + s, y + t) - u(x, y + t) + u(x, y + t) - u(x, y)$$

Due to the Mean Value Theorem, there are real numbers s_1 and t_1 with $|s_1| < s$ and $|t_1| < t$ such that

$$u(x + s, y + t) - u(x, y) = u_x(x + s_1, y + t)s + u_y(x, y + t_1)t$$

Thus,

$$\varphi(s, t) = (u_x(x + s_1, y + t) - u_x(x, y))s + (u_y(x, y + t_1) - u_y(x, y))t$$

Using continuity, it is not hard to see that

$$\lim_{s+it \rightarrow 0} \frac{\varphi(s, t)}{s + it} = 0$$

and a similar result can be obtained for $\psi(s, t)$.

This completes the proof. ■

Theorem 1.12. Let G be either the whole complex plane \mathbb{C} or some open disk. If $u : G \rightarrow \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.

Proof. ■

1.5 Analytic Functions as Mappings

We shall suppose in this section that all paths are continuously differentiable.

Theorem 1.13. If $f : G \rightarrow \mathbb{C}$ is analytic, then f preserves angles at each point $z_0 \in G$ where $f'(z_0) \neq 0$.

Proof. Straightforward. ■

Maps which preserve angles are known as **conformal maps**. Thus, if f is analytic on $G \subseteq \mathbb{C}$ and $f'(z) \neq 0$ for all $z \in G$, it is conformal.

Definition 1.14. A mapping of the form $S(z) = \frac{az + b}{cz + d}$ where $S : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is called a *linear fractional transformation*. If a, b, c, d are such that $ad - bc \neq 0$, then $S(z)$ is called a Möbius Transformation.

A Möbius Transformation is invertible, where

$$S^{-1}(z) = \frac{dz - b}{-cz + a}$$

Chapter 2

Complex Integration

2.1 Riemann Stieltjes Integral

The following definition is taken from [3]

Definition 2.1. Let $[a, b]$ be a given interval. By a partition P of $[a, b]$ we mean a finite set of points x_0, x_1, \dots, x_n where

$$a = x_0 \leq x_1 \leq \dots \leq x_n = b$$

Let $\alpha : [a, b] \rightarrow \mathbb{R}$ be monotonically increasing. Corresponding to each partition P of $[a, b]$, write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \quad \text{for } 1 \leq i \leq n$$

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. For each partition $[x_{i-1}, x_i]$, let

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$$

Define

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i \quad L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$$

and

$$\int_a^b f d\alpha = \inf_{P \in \mathcal{P}} U(P, f, \alpha) \quad \int_a^b f d\alpha = \sup_{P \in \mathcal{P}} L(P, f, \alpha)$$

If the above two values are equal, we say that f is *Riemann-Stieltjes integrable* with respect to α on $[a, b]$ and denote the common value as $\int_a^b f d\alpha$.

Definition 2.2. A function $\gamma : [a, b] \rightarrow \mathbb{C}$ for $[a, b] \subseteq \mathbb{R}$ is of *bounded variation* if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < \dots < t_m = b\}$ of $[a, b]$

$$v(\gamma, P) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \leq M$$

The total variation of γ , $V(\gamma)$ is defined by

$$V(\gamma) = \sup_{P \in \mathcal{P}([a,b])} v(\gamma, P)$$

Proposition 2.3. $\gamma : [a, b] \rightarrow \mathbb{C}$ is of bounded variation if and only if $\Re \gamma$ and $\Im \gamma$ are of bounded variation.

Proof. Follows from the following inequality:

$$\max\{|u(t_k) - u(t_{k-1})|, |v(t_k) - v(t_{k-1})|\} \leq |\gamma(t_k) - \gamma(t_{k-1})| \leq |u(t_k) - u(t_{k-1})| + |v(t_k) - v(t_{k-1})|$$

■

Proposition 2.4. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation. Then

- (a) If P and Q are partitions of $[a, b]$ with Q a refinement of P , then $v(\gamma, P) \leq v(\gamma, Q)$
- (b) If $\sigma : [a, b] \rightarrow \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$ then $\alpha\gamma + \beta\sigma$ is of bounded variation and $V(\alpha\gamma + \beta\sigma) \leq |\alpha|V(\gamma) + |\beta|V(\sigma)$

Proof.

1. Let $[t_{i-1}, t_i]$ be an interval in the partition of P . Let $y \in Q \setminus P$ such that $y \in [t_{i-1}, t_i]$. Then, note that

$$|\gamma(t_i) - \gamma(t_{i-1})| \leq |\gamma(t_i) - \gamma(y)| + |\gamma(y) - \gamma(t_i)|$$

giving us the desired conclusion.

2. Similar to above, we have

$$|(\alpha\gamma + \beta\sigma)(t_i) - (\alpha\gamma + \beta\sigma)(t_{i-1})| \leq |\alpha||\gamma(t_i) - \gamma(t_{i-1})| + |\beta||\sigma(t_i) - \sigma(t_{i-1})|$$

Consequently,

$$v(\alpha\gamma + \beta\sigma, P) \leq |\alpha|v(\gamma, P) + |\beta|v(\sigma, P)$$

The conclusion is obvious.

■

Definition 2.5 (Smooth, Piecewise Smooth). A path in a region $G \subseteq \mathbb{C}$ is a continuous function $\gamma : [a, b] \rightarrow G$ for some $[a, b] \subseteq \mathbb{R}$. If $\gamma'(t)$ exists for each $t \in [a, b]$ and $\gamma' : [a, b] \rightarrow \mathbb{C}$ is continuous, then γ is said to be *smooth*. γ is said to be *piecewise smooth* if there is a partition $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$ such that γ is smooth on each subinterval $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.

Proposition 2.6. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise smooth then γ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)| dt$$

Proof. We shall prove the statement in the case when γ is smooth on $[a, b]$. The general case follows from applying our proof to each piecewise smooth subinterval of $[a, b]$.

Let $a = t_0 < t_1 < \dots < t_m = b$ be a partition, denoted by P . Then,

$$\begin{aligned} v(\gamma, P) &= \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) dt \right| \\ &\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt \\ &= \int_a^b |\gamma'(t)| dt \end{aligned}$$

First, this shows that γ is of bounded variation and further, $V(\gamma) \leq \int_a^b |\gamma'(t)| dt$. We shall show the reverse inequality, which would prove the theorem.

Let $\varepsilon > 0$. Since γ' is continuous on $[a, b]$, it must be uniformly continuous, therefore, there is $\delta > 0$ such that whenever $|s - t| < \delta$, we have $|\gamma'(s) - \gamma'(t)| < \varepsilon$.

Let $a = t_0 < t_1 < \dots < t_m = b$ be a partition with mesh smaller than δ . Consequently, for all $1 \leq i \leq m$, we have for all $t \in [t_{i-1}, t_i]$,

$$|\gamma'(t) - \gamma'(t_i)| < \varepsilon \implies |\gamma'(t)| < |\gamma'(t_i)| + \varepsilon$$

Hence,

$$\begin{aligned} \int_{t_{i-1}}^{t_i} |\gamma'(t)| dt &= |\gamma'(t_i)| \Delta t_i + \varepsilon \Delta t_i \\ &= \left| \int_{t_{i-1}}^{t_i} \gamma'(t_i) - \gamma'(t) + \gamma'(t) dt \right| + \varepsilon \Delta t_i \\ &\leq \left| \int_{t_{i-1}}^{t_i} \gamma'(t_i) - \gamma'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} \gamma'(t) dt \right| + \varepsilon \Delta t_i \\ &\leq \varepsilon \Delta t_i + |\gamma(t_i) - \gamma(t_{i-1})| + \varepsilon \Delta t_i \\ &= |\gamma(t_i) - \gamma(t_{i-1})| + 2\varepsilon \Delta t_i \end{aligned}$$

Adding together all these inequalities, we have

$$\int_a^b |\gamma'(t)| dt \leq v(\gamma, P) + 2\varepsilon(b - a) \leq V(\gamma) + 2\varepsilon(b - a)$$

Since ε was arbitrary, we have the desired conclusion. ■

Theorem 2.7. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \rightarrow \mathbb{C}$ is continuous. Then there is a complex number I such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that when P is a partition of $[a, b]$ with $\|P\| < \delta$, then

$$\left| I - \sum_{k=1}^m f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \varepsilon$$

for whatever choice of points $\tau_k \in [t_{k-1}, t_k]$.

This number I is called the *integral of f with respect to γ over $[a, b]$* and is designated by

$$I = \int f d\gamma$$

We first need the following lemma due to Cantor:

Lemma 2.8 (Cantor). Let A_1, A_2, \dots be a sequence of non-empty compact, closed subsets of a topological space X such that $A_1 \supseteq A_2 \supseteq \dots$. Then,

$$\bigcap_{k=0}^{\infty} A_k \neq \emptyset$$

Proof. Suppose $\bigcap_{k=0}^{\infty} A_k = \emptyset$. Define $B_i = X \setminus A_i$, then, $\{B_i\}$ forms an open cover for A_1 , consequently, has a finite subcover, say $\{B_{n_1}, \dots, B_{n_k}\}$. Now, since

$$A_1 \subseteq \bigcup_{i=1}^k B_{n_i} \subseteq \bigcup_{j=1}^{n_k} B_j$$

This immediately implies that

$$A_{n_k} = A \cap \bigcap_{i=1}^{n_k} B_i = \emptyset$$

a contradiction. ■

Proof of Theorem 2.7. Since f is continuous, it must be uniformly continuous. Thus, we can find positive numbers $\delta_1 > \delta_2 > \dots$ such that if $|s - t| < \delta_m$, then $|f(s) - f(t)| < \frac{1}{m}$. Let \mathcal{P}_m denote the collection of all partitions P of $[a, b]$ with $\|P\| < \delta_m$. Note that we have $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \dots$. Finally define F_m to be the closure of

$$\left\{ S(P) := \sum_{k=1}^n f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \mid P \in \mathcal{P}_m, t_{k-1} \leq \tau_k \leq t_k \right\} \quad (\diamond)$$

We shall show that the following hold:

$$\begin{cases} F_1 \supseteq F_2 \supseteq \dots \\ \text{diam } F_m \leq \frac{2}{m} V(\gamma) \end{cases}$$

The first sequence of containments follows trivially from the definition of \mathcal{P}_m . Recall that in a metric space, $\text{diam } \bar{E} = \text{diam } E$ for all $E \subseteq X$. With this in mind, it suffices to show that the diameter of the set (\diamond) is at most $\frac{2}{m} V(\gamma)$.

We shall show that if $P \in \mathcal{P}_m$ and $P \subseteq Q$ are partitions of $[a, b]$, then $|S(P) - S(Q)| < \frac{1}{m} V(\gamma)$.

Choose any interval $[t_{k-1}, t_k]$ in the partition P and let Q refine it as

$$t_{k-1} = s_0 < s_1 < \dots < s_n = t_k$$

Let χ_1, \dots, χ_n be a tagging of the refinement. Then,

$$\begin{aligned} & \left| f(\tau_k) \sum_{i=1}^n \gamma(s_i) - \gamma(s_{i-1}) - \sum_{i=1}^n f(\chi_i)(\gamma(s_i) - \gamma(s_{i-1})) \right| \\ &= \left| \sum_{i=1}^n (f(\tau_k) - f(\chi_i))(\gamma(s_i) - \gamma(s_{i-1})) \right| \\ &\leq \frac{1}{m} \sum_{i=1}^n |\gamma(s_i) - \gamma(s_{i-1})| \end{aligned}$$

Adding together these inequalities for each subinterval $[t_{k-1}, t_k]$, we have that $|S(P) - S(Q)| \leq \frac{1}{m} V(\gamma)$. Let $P, R \in \mathcal{P}_m$ and Q be their common refinement. Then, we have

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma)$$

From this it follows that $\text{diam } F_m \leq \frac{2}{m} V(\gamma)$. Now, since $\text{diam } F_m \rightarrow 0$ as $m \rightarrow \infty$, it must be the case that $\bigcap_{m=1}^{\infty} F_m$ is a singleton set, containing a single complex number, say I .

Let $\varepsilon > 0$, choose $m > \frac{2}{\varepsilon} V(\gamma)$. Choose $\delta = \delta_m$. Since $I \in F_m$, it must be the case that $F_m \subseteq B(I, \varepsilon)$, giving us the desired conclusion. ■

Proposition 2.9. Let $f, g : [a, b] \rightarrow \mathbb{C}$ be continuous functions and let $\gamma, \sigma : [a, b] \rightarrow \mathbb{C}$ be functions of bounded variation. Then for any scalars α and β ,

1. $\int_a^b \alpha f + \beta g \, d\gamma = \alpha \int_a^b f \, d\gamma + \beta \int_a^b g \, d\gamma$
2. $\int_a^b f \, d(\alpha\gamma + \beta\sigma) = \alpha \int_a^b f \, d\gamma + \beta \int_a^b f \, d\sigma$

Proof. ■

Lemma 2.10. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation and let $f : [a, b] \rightarrow \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \dots < t_n = b$ then

$$\int_a^b f \, d\gamma = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f \, d\gamma$$

Theorem 2.11. If γ is piecewise smooth and $f : [a, b] \rightarrow \mathbb{C}$ is continuous, then

$$\int_a^b f \, d\gamma = \int_a^b f(t) \gamma'(t) \, dt$$

Proof. It suffices to consider the case where γ is smooth, since the general statement follows by applying our result to each piecewise smooth component and adding them up using Lemma 2.10.

We have that $\gamma = u + iv$ is smooth where $u, v : [a, b] \rightarrow \mathbb{R}$; thus, both u and v must be smooth, furthermore, $\gamma' = u' + iv'$. As a result, it suffices to prove the theorem for γ being real valued and smooth. We shall require the fact that it is real valued to apply the Mean Value Theorem.

Let $\varepsilon > 0$ and $\delta > 0$ be such that for any partition $P = \{a = t_0 < t_1 < \dots < t_n = b\}$,

$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k) (\gamma(t_k) - \gamma(t_{k-1})) \right| < \frac{\varepsilon}{2}$$

$$\left| \int_a^b f(t) \gamma'(t) \, dt - \sum_{k=1}^n f(\tau_k) \gamma'(\tau_k) (t_k - t_{k-1}) \right| < \frac{\varepsilon}{2}$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$. Using the mean value theorem, choose τ_k such that

$$\gamma'(\tau_k) = \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}$$

Consequently,

$$\left| \int_a^b f \, d\gamma - \int_a^b f(t) \gamma'(t) \, dt \right| < \varepsilon$$

and we have the desired conclusion. ■

Definition 2.12 (Bounded Variation). Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a path. The set $\{\gamma(t) \mid a \leq t \leq b\}$ is called the *trace* of γ and is denoted by $\{\gamma\}$. The path γ is said to be *rectifiable* if it is of bounded variation.

Definition 2.13 (Line Integral). If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and f is a function defined and continuous on the trace of γ . Then, the line integral of f along γ is

$$\int_a^b f(\gamma(t)) d\gamma(t)$$

Theorem 2.14. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and $\varphi : [c, d] \rightarrow [a, b]$ is a continuous non-decreasing function with $\varphi(c) = a$ and $\varphi(d) = b$. Then, for any function f continuous on $\{\gamma\}$,

$$\int_{\gamma} f = \int_{\gamma \circ \varphi} f$$

Proof. Let $\varepsilon > 0$. Then, there is a δ_1 such that for all partitions $P = \{c = s_0 < s_1 < \dots < s_n = d\}$ with $\|P\| < \delta$, and a tagging, $\sigma_k \in [s_{k-1}, s_k]$,

$$\left| \int_{\gamma \circ \varphi} f - \sum_{k=1}^n f(\gamma \circ \varphi(\sigma_k))(\gamma \circ \varphi(s_k) - \gamma \circ \varphi(s_{k-1})) \right| < \frac{\varepsilon}{2}$$

furthermore, whenever $s, t \in [c, d]$ with $|s - t| < \delta_1$, $|\varphi(s) - \varphi(t)| < \delta_2$ (note that we can do this since the function φ is uniformly continuous).

Choose $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ with $\|P\| < \delta_2$ and a tagging $\tau_k \in [t_{k-1}, t_k]$, then

$$\left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| < \frac{\varepsilon}{2}$$

Finally, let $\sigma_k = \varphi(\tau_k)$, then we have through a trivial manipulation that

$$\left| \int_{\gamma} f - \int_{\gamma \circ \varphi} f \right| < \varepsilon$$

■

Definition 2.15. Let $\sigma : [c, d] \rightarrow \mathbb{C}$ and $\gamma : [a, b] \rightarrow \mathbb{C}$ be rectifiable paths. The path σ is *equivalent* to γ if there is a function $\varphi : [c, d] \rightarrow [a, b]$ which is continuous, strictly increasing, and with $\varphi(c) = a$ and $\varphi(d) = b$ such that $\sigma = \gamma \circ \varphi$.

A *curve* is an equivalence class of paths. A trace of a curve is the trace of any one of its members. A curve is smooth (piecewise smooth) if and only if some one of its representatives is smooth (piecewise smooth).

Definition 2.16. If γ is a rectifiable curve then denote by $-\gamma : [-b, -a] \rightarrow \mathbb{C}$ the curve defined by $(-\gamma)(t) = \gamma(-t)$ for $-b \leq t \leq -a$. This may also be denoted by γ^{-1} (although the former is more customary). For some $c \in \mathbb{C}$, let $\gamma + c : [a, b] \rightarrow \mathbb{C}$ denote the curve defined by $(\gamma + c)(t) = \gamma(t) + c$.

Definition 2.17. Let $\gamma[a, b] \rightarrow \mathbb{C}$ be a rectifiable path and for $a \leq t \leq b$, let $|\gamma|(t)$ be $V(\gamma, [a, t])$. That is,

$$|\gamma|(t) = \sup \left\{ \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| : \{a = t_0 < t_1 < \cdots < t_n = t\} \text{ is a partition of } [a, t] \right\}$$

Define

$$\int_{\gamma} f |dz| = \int_a^b f(\gamma(t)) d|\gamma|(t)$$

Proposition 2.18. Let γ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then

- (a) $\int_{\gamma} f = -\int_{-\gamma} f$
- (b) $\left| \int_{\gamma} f \right| \leq \int_{\gamma} |f| |dz| \leq V(\gamma) \sup\{|f(z)| : z \in \{\gamma\}\}$
- (c) If $c \in \mathbb{C}$, then $\int_{\gamma} f(z) dz = \int_{\gamma+c} f(z-c) dz$

Proof. All follow from definitions. ■

Theorem 2.19 (Fundamental Theorem of Calculus for Line Integrals). Let G be open in \mathbb{C} and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \rightarrow \mathbb{C}$ is a continuous function with a primitive $F : G \rightarrow \mathbb{C}$, then

$$\int_{\gamma} f = F(\beta) - F(\alpha)$$

We would require the following lemma in order to prove the above theorem

Lemma 2.20. If G is an open set in \mathbb{C} , $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$ and $|\int_{\gamma} f - \int_{\Gamma} f| < \varepsilon$.

Proof. We shall divide the proof into two cases:

- **Case I:** G is an open disk, say $B(c, r)$

Since $\{\gamma\}$ is compact, there is $\rho > 0$ such that $\{\gamma\} \subseteq \overline{B}(c, \rho) \subseteq G$. Consequently, we shall proceed with the assumption that $G = \overline{B}(c, \rho)$. Therefore, G is compact and f is uniformly continuous on G .

Let $\varepsilon > 0$. Then, there is a δ_1 such that whenever $|s - t| < \delta_1$, $|f(s) - f(t)| < \varepsilon$. Similarly, there is $\delta_2 > 0$ such that whenever $|s - t| < \delta_2$, $|\gamma(s) - \gamma(t)| < \delta_1$.

Furthermore, due to Theorem 2.7, there is a mesh size, δ_3 such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with $\|P\| < \delta_3$,

$$\left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(t_k))(\gamma(t_k) - \gamma(t_{k-1})) \right|$$

Let $\delta = \min\{\delta_2, \delta_3\}$ and $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$ with $\|P\| < \delta$. Define the polygonal path Γ by

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}} ((t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k))$$

which is essentially the straight line joining the points $\gamma(t_{k-1})$ and $\gamma(t_k)$.

First, note that

$$\int_{\Gamma} f = \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t)) dt$$

Then, we have

$$\begin{aligned} \left| \int_{\gamma} f - \int_{\Gamma} f \right| &\leq \varepsilon + \left| \sum_{k=1}^n f(\gamma(t_k))(\gamma(t_k) - \gamma(t_{k-1})) - \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t)) dt \right| \\ &\leq \varepsilon + \left| \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\gamma(t_k)) - f(\Gamma(t)) dt \right| \\ &\leq \varepsilon + \sum_{k=1}^n \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \left| \int_{t_{k-1}}^{t_k} f(\gamma(t_k)) - f(\Gamma(t)) dt \right| \\ &\leq \varepsilon + \varepsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \leq \varepsilon(1 + V(\gamma)) \end{aligned}$$

This completes the proof for the first case.

- Case II: G is arbitrary

Since $\{\gamma\}$ is compact, there is $r > 0$ such that for all $z \in \gamma$, $B(z, r) \subseteq G$. Using uniform continuity, there is $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$. Let $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ be a partition with $\|P\| < \delta$. Define $\gamma_k : [t_{k-1}, t_k] \rightarrow \mathbb{C}$. Note that $\{\gamma_k\} \subseteq B(\gamma(t_{k-1}), r)$ and thus, we can apply Case I to obtain a polygonal path Γ_k such that $|\int_{\gamma_k} f - \int_{\Gamma_k} f| < \varepsilon/n$. The conclusion is now obvious by pasting together all the Γ_k 's. ■

Proof of Theorem 2.19. Again, we divide the proof into two cases:

- Case I: $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise smooth.

Then, we trivially have

$$\int_a^b f(\gamma(t))\gamma'(t) dt = \int_a^b F'(\gamma(t))\gamma'(t) dt = \int_a^b (f \circ \gamma)'(t) dt = F(\gamma(b)) - F(\gamma(a))$$

- Case II: General case

Recall that a polygonal path is piecewise smooth. That is, for any polygonal path Γ that begins at $\gamma(a)$ and ends at $\gamma(b)$, $\int_{\Gamma} f = F(\gamma(b)) - F(\gamma(a))$. Since any rectifiable curve can be approximated by a polygonal path, we have a suitable Γ for every $\varepsilon > 0$ such that

$$\left| \int_{\gamma} f - (F(\beta) - F(\alpha)) \right| = \left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon$$

giving us the desired conclusion. ■

Corollary. Let G , γ and f satisfy the same hypothesis as in Theorem 2.19. If γ is a closed curve, then

$$\int_{\gamma} f = 0$$

Recall that the fundamental theorem of calculus in real analysis claimed that each continuous function had a primitive. This is untrue in complex analysis. Consider the function $f(z) = |z|^2$. That is, $f(x + iy) = x^2 + y^2$. Suppose this has a primitive, say $F = U + iV$. Then, using **CR**, we must have

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = x^2 + y^2 \quad \text{and} \quad \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = 0$$

This implies that $U(x, y) = u(x)$ for some function u , but this gives

$$u'(x) = x^2 + y^2$$

which is obviously not possible.

2.2 Power Series for Analytic Functions

Theorem 2.21 (Leibniz's Rule). Let $\varphi : [a, b] \times [c, d] \rightarrow \mathbb{C}$ be a continuous function and define $g : [c, d] \rightarrow \mathbb{C}$ by

$$g(t) = \int_a^b \varphi(s, t) ds$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a, b] \times [c, d]$ then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) ds$$

Proof. We shall first show that g is continuous. Since φ is continuous, it is uniformly continuous on $[a, b] \times [c, d]$. Choose some $t_0 \in [c, d]$. Then, there is a δ such that whenever $|(s, t) - (s', t')| < \delta$, $|\varphi(s, t) - \varphi(s', t')| < \varepsilon$. Consequently, whenever $|t - t_0| < \delta$, $|g(t) - g(t_0)| < (b - a)\varepsilon$. This implies continuity.

Fix a point $t_0 \in [c, d]$ and choose any $\varepsilon > 0$. Further, denote $\frac{\partial \varphi}{\partial t}$ by φ_2 , which is given to be continuous, and thus, is uniformly continuous on $[a, b] \times [c, d]$. Let $\delta > 0$ be such that whenever $|(s, t) - (s', t')| < \delta$, $|\varphi_2(s', t') - \varphi_2(s, t)| < \varepsilon$. That is,

$$|\varphi_2(s, t) - \varphi_2(s, t_0)| < \varepsilon$$

whenever $|t - t_0| < \delta$ and $a \leq s \leq b$. Therefore, we have

$$\left| \int_{t_0}^t \varphi_2(s, \tau) - \varphi_2(s, t_0) d\tau \right| < \varepsilon |t - t_0|$$

Note that $\Phi(t) = \varphi(s, t) - t\varphi_2(s, t_0)$ is a primitive of $\varphi_2(s, t) - \varphi_2(s, t_0)$. Due to the fundamental theorem of calculus, we must have

$$|\varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0)| \leq \varepsilon |t - t_0|$$

for all $s \in [a, b]$ whenever $|t - t_0| < \delta$. This is equivalent to writing

$$-\varepsilon \geq \frac{\varphi(s, t) - \varphi(s, t_0)}{t - t_0} - \varphi_2(s, t_0) \leq \varepsilon$$

Integrating both sides with respect to s , we have

$$\left| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \varphi_2(s, t_0) ds \right| \leq \varepsilon(b - a)$$

This shows that g is differentiable and

$$g'(t) = \int_a^b \varphi_2(s, t) ds$$

Obviously the right hand side of the above equality is continuous and thus g is continuously differentiable. ■

Example 2. Let z be a complex number with $|z| < 1$. Then,

$$\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} ds$$

and equivalently stated, if $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ is a closed path given by $\gamma(t) = e^{it}$, then

$$\int_{\gamma} \frac{1}{x - z} dx = 2\pi$$

Proof. Define the function

$$g(t) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - tz} ds$$

for $0 \leq t \leq 1$. Note that in this region, the function

$$\varphi(s, t) = \frac{e^{is}}{e^{is} - tz}$$

is well defined, since $|e^{is}| = 1 > |tz|$.

Using Theorem 2.21, we have

$$g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} ds$$

Consider the function

$$\Phi(s) = \frac{iz}{e^{is} - tz}$$

Notice that

$$\Phi'(s) = \frac{ze^{is}}{e^{is} - tz}$$

Then, using Theorem 2.19, $g'(t) = \Phi(2\pi) - \Phi(0) = 0$. Therefore, g is constant. The conclusion follows from calculating $t = 0$. ■

Proposition 2.22. Let $f : G \rightarrow \mathbb{C}$ be analytic and suppose $\overline{B}(a, r) \subseteq G$ where $r > 0$. If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$$

for $|z - a| < r$.

Proof. It is not hard to see that without loss of generality we may suppose that $a = 0$ and $r = 1$. Then, we would like to show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z} ds$$

for $|z| < 1$. This is equivalent to showing

$$\int_0^{2\pi} \left(\frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) ds = 0$$

Define the function

$$\varphi(s, t) = \frac{f(z + t(e^{is} - z))e^{is}}{e^{is} - z} - f(z)$$

and $g(t) = \int_0^{2\pi} \varphi(s, t) ds$. We would like to show that $g(1) = 0$.

Note that the function $\varphi(s, t)$ is well defined and continuously differentiable on the interval $[0, 2\pi] \times [0, 1]$ (it is here that we use the fact that $|z| < 1$). Then,

$$g'(t) = \int_0^{2\pi} f(z + t(e^{is} - z))e^{is} ds$$

Consider the function $\Phi(s) = \frac{1}{it} f(z + t(e^{is} - z))$. Trivially note that $\Phi'(s) = f(z + t(e^{is} - z))e^{is}$. Using the fundamental theorem of calculus, we have

$$g'(t) = \Phi(2\pi) - \Phi(0) = 0$$

Implying that g is constant on $[0, 1]$. Recall that we have already calculated

$$g(0) = \int_0^{2\pi} \frac{f(z)}{e^{is} - z} - f(z) ds = 0$$

This completes the proof. ■

Lemma 2.23. Let γ be a rectifiable curve in \mathbb{C} and suppose that F_n and F are continuous functions on $\{\gamma\}$ such that the sequence $\{F_n\}$ converges uniformly to F . Then

$$\int_{\gamma} F = \lim_{n \rightarrow \infty} \int_{\gamma} F_n$$

Proof. Let $\varepsilon > 0$ be given. Then, there is a positive integer N such that for all $n \geq N$, $|F_n - F| \leq \varepsilon/V(\gamma)$. Then, we have (for all $n \geq N$)

$$\left| \int_{\gamma} F - F_n \right| \leq \int_{\gamma} |F - F_n| |dz| \leq \varepsilon$$

This completes the proof. ■

Theorem 2.24. Let f be analytic in $B(a, R)$; then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a| < R$, where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq R$.

Proof. Let $z \in B(a, R)$. Choose $|z-a| < r < R$ and define γ to be the circle $\partial B(a, r)$. Then, using Proposition 2.22,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw$$

Now, note that

$$\frac{1}{w-z} = \frac{1}{w-a} \cdot \frac{1}{1 - \frac{z-a}{w-a}} = \frac{1}{w-a} \sum_{k=0}^{\infty} \left(\frac{z-a}{w-a} \right)^k$$

Since $w \in \{\gamma\}$, there must exist $M > 0$ such that $|f(w)| < M$ for all $w \in \{\gamma\}$ and thus

$$\frac{|f(w)||z-a|^n}{|w-a|^{n+1}} \leq \frac{M}{r} \left(\frac{|z-a|}{r} \right)^n$$

Due to the Weierstrass M -test, the power series converges uniformly for $w \in \{\gamma\}$. And due to the Weierstrass M -test, the power series converges uniformly for $w \in \{\gamma\}$. Therefore, we may write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} \sum_{k=0}^{\infty} \left(\frac{z-a}{w-a} \right)^k \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{k+1}} dw \right] (z-a)^k \end{aligned}$$

Define

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

Then, the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges to $f(z)$ on $B(a, r)$. Consequently, f is infinitely differentiable at z and thus,

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

Now, the characterization of a_n is independent of γ and therefore r . Consequently, this power series converges to $f(z)$ whenever $|z-a| < R$. Therefore, the radius of convergence must be at least R . ■

Corollary. If $f : G \rightarrow \mathbb{C}$ is analytic and $a \in G$. Then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a| < R$ where $R = d(a, \partial G)$.

Corollary. If $f : G \rightarrow \mathbb{C}$ is analytic, then it is infinitely differentiable.

Corollary. If $f : G \rightarrow \mathbb{C}$ is analytic and $\overline{B}(a, r) \subseteq G$, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

where $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$.

Proposition 2.25 (Cauchy's Estimate). Let f be analytic in $B(a, R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a, R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}$$

Proof. Let $r < R$ and $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$.

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \left| \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} ds \right| \leq \int_{\gamma} \left| \frac{f(w)}{(w-a)^{n+1}} \right| |dw| \leq \frac{n!M}{r^n}$$

The result follows by letting $r \rightarrow R^-$. ■

Proposition 2.26. Let f be analytic in the disk $B(a, R)$ and suppose that γ is a closed rectifiable curve in $B(a, R)$. Then

$$\int_{\gamma} f = 0$$

Proof. It suffices to show that f has a primitive on $B(a, R)$ whence, we would be done by Theorem 2.19. Due to Theorem 2.24, there is a power series representation for f ,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

for $z \in B(a, R)$.

Define the function

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - a)^{n+1}$$

Notice that the radius of convergence of F is equal to that of f and $F' = f$. As a result, F is a primitive for f on $B(a, R)$. ■

2.3 Zeros of Analytic Functions

Definition 2.27 (Entire Function). An *entire function* is a function which is defined and analytic in the whole complex plane \mathbb{C} .

We immediately obtain the following result:

Proposition 2.28. If f is an entire function, then f has a power series expansion with infinite radius of convergence.

Lemma 2.29. No non-constant polynomial is bounded. That is, if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathbb{C}[z]$. Then, $\lim_{z \rightarrow \infty} p(z) = \infty$.

Proof. Trivial. ■

Theorem 2.30 (Liouville). If f is a bounded entire function, then f is constant.

In the proof of Liouville's Theorem, we shall require the following lemma:

Lemma 2.31. If G is open and connected and $f : G \rightarrow \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is constant on G .

Proof. Choose any $z_0 \in G$ and let $\omega_0 = f(z_0)$. Define $A = f^{-1}(\{\omega_0\})$. Obviously, A is closed in G . Choose $a \in A$ and $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq G$. Pick any $z \in B(a, \varepsilon)$ with $a \neq z$. Define $g(t) = f((1-t)a + tz)$. Note that $g'(s) = f'((1-t)a + tz)(z-a) = 0$, consequently, g is constant and therefore, $f(z) = g(1) = g(0) = \omega_0$. Therefore, $B(a, \varepsilon) \subseteq A$ and thus A is open. This shows that A must be equal to G , completing the proof. ■

Proof of Theorem 2.30. Let $M > 0$ be such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Choose any $a \in A$. Then, for any $R > 0$, applying Proposition 2.25, we have

$$|f'(a)| \leq \frac{M}{R}$$

Letting $R \rightarrow \infty$, we have $f'(a) = 0$ for all $a \in \mathbb{C}$. We are now done due to the preceding lemma. ■

We may now prove the fundamental theorem of algebra:

Theorem 2.32 (Fundamental Theorem of Algebra). *If $p(z)$ is a non-constant polynomial then there is a complex number a with $p(a) = 0$.*

Proof. Suppose not. Then, $f(z) = \frac{1}{p(z)}$ is entire. Since $\lim_{z \rightarrow \infty} p(z) = \infty$, $\lim_{z \rightarrow \infty} f(z) = 0$. Therefore, there is ε such that whenever $|z| > \varepsilon$, $|f(z)| < 1$. This immediately implies that f is bounded on \mathbb{C} , consequently is constant. A contradiction. ■

Theorem 2.33. *Let $G \subseteq \mathbb{C}$ be a region, and $f : G \rightarrow \mathbb{C}$ be an analytic function. Then the following are equivalent*

- (a) $f \equiv 0$
- (b) *there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for each $n \geq 0$*
- (c) *the set $f^{-1}(\{0\})$ has a limit point in G*

Proof. It is clear that $(a) \implies (b) \wedge (c)$. We shall show that $(c) \implies (b)$ and $(b) \implies (a)$.

- $(c) \implies (b)$: Let a be a limit point of the set $f^{-1}(\{0\})$. We shall show that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}_0$. Let n be the smallest integer ≥ 1 such that $f^{(r)}(a) = 0$ for all $r < n$. Now, there is $R > 0$ such that $B(a, R) \subseteq G$, and thus there is a power series expansion around a for all $z \in B(a, R)$, given by

$$f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k$$

Define the function

$$g(z) = \sum_{k=0}^{\infty} a_{n+k}(z-a)^k$$

Then $g(a) = a_n \neq 0$. It is not hard to see that $g(z)$ is analytic in $B(a, R)$, as a result, there is some $0 < r < R$ such that $g(z) \neq 0$ for each $z \in B(a, r)$. But since a is a limit point of the set $f^{-1}(\{0\})$, there is some $b \neq a$ in $f^{-1}(\{0\}) \cap B(a, r)$, and we have $0 = f(b) = (b-a)^n g(b)$, a contradiction. This shows that no such $n \in \mathbb{N}$ can exist.

- $(c) \implies (b)$: Let $A = \{z \in G \mid f^{(n)}(z) = 0, \forall n \in \mathbb{N}\}$. We shall show that A is clopen in G . Indeed, let $a \in A$. Since G is open, there is $R > 0$ such that $B(a, R) \subseteq G$. Let $b \in B(a, R)$. Note that f has a power series expansion around a that is valid for all $z \in B(a, R)$. Since $a \in A$, this power series expansion is identically zero, as a result, $f(b) = 0$ and $B(a, R) \subseteq A$ and A is open.

Next, let $\{z_k\}$ be a sequence of points in A converging to $a \in G$. Then, using continuity of $f^{(n)}$, we conclude that $f^{(n)}(a) = \lim f^{(n)}(z_k) = 0$ and A is closed. This completes the proof. ■

Lemma 2.34. Let $G \subseteq \mathbb{C}$ be a region and $f : G \rightarrow \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Then f is constant.

Proof. ■

Theorem 2.35 (Maximum Modulus Theorem). Let $G \subseteq \mathbb{C}$ be a region and $f : G \rightarrow \mathbb{C}$ be an analytic function such that there is $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$. Then f is constant on G .

Proof. Let $r > 0$ be such that $B(a, r) \subseteq G$ and let γ be the curve given by $\gamma(t) = a + re^{it}$. Then, we have

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt \end{aligned}$$

and equivalently,

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})| dt \leq |f(a)|$$

As a result,

$$\int_0^{2\pi} |f(a)| - |f(a + re^{it})| dt = 0$$

since the integrand is a continuous nonnegative function of t , it must be identically zero. As a result, f maps the ball $B(a, r)$ to the circle $|z| = |f(a)|$. Due to Lemma 2.34, f is constant on $B(a, r)$. Since $B(a, r)$ has at least one limit point in G (say a for example), it must be constant on G . ■

2.4 Cauchy's Theorem

Definition 2.36 (Homotopy for Closed Curves). Let $G \subseteq \mathbb{C}$ and $\gamma_0, \gamma_1 : [0, 1] \rightarrow G$ be two closed rectifiable curves. Then γ_0 is *homotopic* to γ_1 in G if there is a continuous function $\Gamma : [0, 1] \times [0, 1] \rightarrow G$ such that

$$\begin{cases} \Gamma(s, 0) = \gamma_0(s) \text{ and } \Gamma(s, 1) = \gamma_1(s) & 0 \leq s \leq 1 \\ \Gamma(0, t) = \Gamma(1, t) & 0 \leq t \leq 1 \end{cases}$$

We denote this by $\gamma_0 \simeq \gamma_1 \pmod{G}$.

Lemma 2.37. The relation \simeq is an equivalence relation over the set of all closed curves in G .

Proof. Standard proof from Algebraic Topology. ■

Theorem 2.38 (Cauchy). Let $G \subseteq \mathbb{C}$ be a region and $f : G \rightarrow \mathbb{C}$ be analytic. Let γ_0 and γ_1 be homotopic closed curves. Then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Proof. Let $\Gamma : I^2 \rightarrow G$ be the homotopy taking γ_0 to γ_1 . Since I^2 is compact, so is $\Gamma(I^2)$. Consequently, due to the Lebesgue Number Lemma, there is $r > 0$ such that for all $a \in \Gamma(I^2)$, $B(a, r) \subseteq G$. Using the uniform continuity of Γ , there is $\delta > 0$ such that whenever $|(s', t') - (s, t)| < \delta$, $|\Gamma(s', t') - \Gamma(s, t)| < r$. Choose $n \in \mathbb{N}$ such that $\sqrt{2}/n < \delta$. Finally, let γ_t denote the curve $\Gamma(s, t)$ where t is fixed and $0 \leq s \leq 1$.

Let $Z_{i,j}$ denote the point $\Gamma\left(\frac{i}{n}, \frac{j}{n}\right)$ and $Q_{i,j}$ denote the square $\left(\frac{i}{n}, \frac{j}{n}\right) \rightarrow \left(\frac{i+1}{n}, \frac{j}{n}\right) \rightarrow \left(\frac{i+1}{n}, \frac{j+1}{n}\right) \rightarrow \left(\frac{i}{n}, \frac{j+1}{n}\right) \rightarrow \left(\frac{i}{n}, \frac{j}{n}\right)$. We shall show that

$$\int_{\Gamma(Q_{i,j})} f = 0$$

which would imply the desired conclusion through a straightforward inductive process.

But since $|z_1 - z_2| < \sqrt{2}/n < \delta$ for all $z_1, z_2 \in Q_{i,j}$, we can conclude that $\Gamma(Q_{i,j}) \subseteq B(Z_{i,j}, r)$, whence we are done due to Proposition 2.26. ■

Corollary. Let $G \subseteq \mathbb{C}$ be a region and γ a closed rectifiable curve in G which is nulhomotopic. Then,

$$\int_{\gamma} f = 0$$

for every analytic function f defined on G .

Corollary. Let $G \subseteq \mathbb{C}$ be a region and γ_0, γ_1 be path homotopic curves. Then,

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for every analytic function f defined on G .

Corollary. If $G \subseteq \mathbb{C}$ is simply connected then $\int_{\gamma} f = 0$ for every closed rectifiable curve $\gamma \subseteq G$ and every analytic function $f : G \rightarrow \mathbb{C}$.

Theorem 2.39. If G is simply connected and $f : G \rightarrow \mathbb{C}$ is analytic in G , then f has a primitive in G .

Proof. Fix some basepoint $a \in G$ and for each $z \in G$, define $F : G \rightarrow \mathbb{C}$ as $F(z) = \int_{\gamma} f$. Due to the previous result, this function is well defined. We shall show that F is a primitive for f on G . Let $z_0 \in G$. Since G is open, there is $r > 0$ such that $\bar{B}(z_0, r) \subseteq G$. Note that this is a convex set centered at z_0 , as a result, all line segments between two points are contained in it. Choose some $z \in B(z_0, r)$. Then,

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0)) dw \\ \implies \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &\leq \left| \frac{1}{z - z_0} \right| \int_{[z_0, z]} |f(w) - f(z_0)| |dw| \end{aligned}$$

Let $\varepsilon > 0$ be given. Note that $\bar{B}(z_0, r)$ is compact in G and thus, f is uniformly continuous. As a result, there is a small enough $r > 0$ such that for all $z \in B(z_0, r)$, $|f(z) - f(z_0)| < \varepsilon$. And thus,

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \varepsilon$$

which implies the desired conclusion. ■

Theorem 2.40 (Morera). Let $G \subseteq \mathbb{C}$ be an open set and $f : G \rightarrow \mathbb{C}$ be a continuous function. If for every triangular path Δ in G , the value of $\int_{\Delta} f = 0$, then f is analytic over G .

Proof. Note that it suffices to show this in the case $G = B(a, R)$ for some $a \in \mathbb{C}$ and $R > 0$, since for every $a \in G$, there is an open ball containing it and showing the analyticity of f every such ball would imply the analyticity of f on G .

Let $[x, y]$ denote the straight line segment that begins at x and ends at y . Define the function $F : G \rightarrow \mathbb{C}$ by

$$F(z) = \int_{[a, z]} f$$

We shall show that $F' = f$, which would imply the analyticity of F and therefore that of f . Choose some $z_0 \in G$. For any $z \in G$, we have

$$F(z) - F(z_0) = \int_{[a, z]} f - \int_{[a, z_0]} f = \int_{[z_0, z]} f$$

Then,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} (f - f(z_0))$$

Choose $r > 0$ such that $\overline{B}(z_0, r) \subseteq G$. Since f is continuous on G , it is uniformly continuous on $\overline{B}(z_0, r)$. Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that whenever $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \varepsilon$. Consequently, for all such z , we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{1}{|z - z_0|} \int_{[z_0, z]} |f(t) - f(z_0)| |dt| \leq \varepsilon$$

This completes the proof. ■

Theorem 2.41 (Goursat). Let $G \subseteq \mathbb{C}$ be an open set and $f : G \rightarrow \mathbb{C}$ be differentiable. Then, f is analytic over G .

Proof. Due to Morera's Theorem, it suffices to show that for every triangular path $\Delta = [a, b, c, a] \subseteq G$, the value $\int_{\Delta} f = 0$.

We shall define a sequence of closed triangular regions $\Delta = \Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \dots$. Obviously, since each triangular region is closed and bounded, it must be compact.

Divide the triangle $\Delta^{(i)}$ into four congruent triangles using the midpoint of each side. Let the smaller triangles be denoted by $\Delta_1, \dots, \Delta_4$. Define

$$j = \operatorname{argmax}_{j \in \{1, \dots, 4\}} \left| \int_{\Delta_j} f \right| \quad \text{and} \quad \Delta^{(i+1)} = \Delta_j$$

We have

$$\begin{cases} \left| \int_{\Delta^{(i)}} f \right| \leq 4 \left| \int_{\Delta^{(i+1)}} f \right| \\ 2 \operatorname{diam} \Delta^{(i+1)} = \operatorname{diam} \Delta^{(i)} \\ 2V(\Delta^{(i+1)}) = V(\Delta^{(i)}) \end{cases}$$

Then, using Lemma 2.8, $\bigcap_{i=0}^{\infty} \Delta^{(i)}$ is singleton, say $\{z_0\}$. Choose some $\varepsilon > 0$. Since f is differentiable at z_0 , there is $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $|z - z_0| < \delta$. Choose $n \in \mathbb{N}$ such that $\text{diam } \Delta^{(n)} = \frac{1}{2^n} \text{diam } \Delta < \delta$. Therefore, $\Delta^{(n)} \subseteq B(z_0, \delta)$. Then, we have

$$\int_{\Delta^{(n)}} f = \int_{\Delta^{(n)}} f(z) - f(z_0) - (z - z_0)f'(z_0) dz$$

whence

$$\begin{aligned} \left| \int_{\Delta^{(n)}} f \right| &= \left| \int_{\Delta^{(n)}} f(z) - f(z_0) - (z - z_0)f'(z_0) dz \right| \\ &\leq \int_{\Delta^{(n)}} |f(z) - f(z_0) - (z - z_0)f'(z_0)| |dz| \\ &\leq \int_{\Delta^{(n)}} \varepsilon |z - z_0| |dz| \\ &\leq \varepsilon \text{diam } \Delta^{(n)} V(\Delta^{(n)}) \\ &= \varepsilon (\text{diam } \Delta) V(\Delta) \frac{1}{4^n} \end{aligned}$$

from which it follows that

$$\left| \int_{\Delta} f \right| \leq 4^n \left| \int_{\Delta^{(n)}} f \right| \leq \varepsilon (\text{diam } \Delta) V(\Delta)$$

Since ε was arbitrary, we have the desired conclusion. ■

Due to Theorem 2.41, we may redefine an analytic function in its more accepted definition.

Definition 2.42 (Analytic). Let $G \subseteq \mathbb{C}$ be open. Then $f : G \rightarrow \mathbb{C}$ is said to be analytic if it is differentiable over G .

2.5 Winding Numbers

Proposition 2.43. If $\gamma : [0, 1] \rightarrow \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$$

is an integer.

Proof. The proof is divided into two parts. First, we prove the statement of the proposition for all piecewise smooth curves.

- Case I: γ is piecewise smooth
 - Case II: γ is an arbitrary rectifiable curve
-

Definition 2.44 (Winding Number). If γ is a closed rectifiable curve in \mathbb{C} then for $a \notin \{\gamma\}$,

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - a} dz$$

is called the *winding number* of γ around a .

Theorem 2.45 (Cauchy's Integral Formula). Let $f : G \rightarrow \mathbb{C}$ be analytic and $\gamma \subseteq G$ be a nulhomotopic rectifiable closed contour. Then, for $a \notin \{\gamma\}$,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} = n(\gamma; a)f(a)$$

Proof. Note that the function $f(z) - f(a)$ is analytic and has a zero at $z = a$, therefore, there is an analytic function $g : G \rightarrow \mathbb{C}$ such that $f(z) - f(a) = g(z)(z - a)$. From here, we have that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(a)}{z-a} = \frac{1}{2\pi i} \int_{\gamma} g(z) = 0$$

and therefore,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(a)}{z-a} = n(\gamma; a)f(a)$$

where the last equality follows from the definition of the winding number. ■

Lemma 2.46. Let $G \subseteq \mathbb{C}$ be a region and $\gamma \subseteq G$ be a closed rectifiable contour and $\varphi : \{\gamma\} \rightarrow \mathbb{C}$ be continuous. For each positive integer m , let

$$F_m(z) = \int_{\gamma} \frac{\varphi(w)}{(w-z)^m} dw$$

Then F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$. Furthermore, $F'_m(z) = mF'_{m+1}(z)$.

Proof. Fix some $a \in \mathbb{C} \setminus \{\gamma\}$. Now, there is $R > 0$ such that $B(a, R) \subseteq \mathbb{C} \setminus \{\gamma\}$. Consider some $z \in B(a, R)$. Then,

$$\begin{aligned} F_m(z) - F_m(a) &= \frac{1}{2\pi i} \int_{\gamma} \varphi(w) \left[\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} \right] dw \\ &= \frac{1}{2\pi i} \int_{\gamma} \varphi(w) \left(\frac{1}{w-z} - \frac{1}{w-a} \right) \left(\sum_{k=0}^{m-1} \frac{1}{(w-z)^k (w-a)^{m-k-1}} \right) dw \\ &= \frac{z-a}{2\pi i} \int_{\gamma} \varphi(w) \left(\sum_{k=1}^m \frac{1}{(w-z)^k (w-a)^{m-k}} \right) dw \end{aligned}$$

■

Theorem 2.47 (Extended Cauchy's Integral Formula). Let $f : G \rightarrow \mathbb{C}$ be an analytic function and $\gamma \subseteq G$ be a closed contour of bounded variation. Then, for every $a \in G \setminus \{\gamma\}$, and every nonnegative integer n ,

$$n(\gamma; a)f^{(n)}(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

Proof. Follows from the above lemma. ■

2.6 The Open Mapping Theorem

Theorem 2.48. Let $G \subseteq \mathbb{C}$ be a region and $f : G \rightarrow \mathbb{C}$ be analytic having zeros a_1, \dots, a_n counting multiplicity in G . Then, for any closed curve $\gamma \subseteq G$, we have

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} = \sum_{k=1}^n n(\gamma; a_k)$$

Proof. Recall that if f has a zero at $z = a$, then there is an analytic function $g : G \rightarrow \mathbb{C}$ such that $f(z) = (z - a)g(z)$. Continuing this way, we have an analytic function $h : G \rightarrow \mathbb{C}$ such that $f(z) = \prod_{k=1}^n (z - a_k)h(z)$. Then,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} = \frac{1}{2\pi i} \int_{\gamma} \sum_{k=1}^n \frac{1}{z - a_k} + \frac{h'(z)}{h(z)}$$

Since the function h has no zeros in G , the function h'/h is analytic on G and therefore, the integral is 0. The conclusion now follows. ■

Lemma 2.49. Let f be analytic on $B(a, R)$ for some $R > 0$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $0 < |\zeta - \alpha| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a, \varepsilon)$.

Proof. ■

In particular, if $m \geq 1$, then for each $\zeta \in B(\alpha, \delta)$, there is a corresponding $\xi \in B(a, \varepsilon)$ such that $f(\xi) = \zeta$. Therefore, $B(\alpha, \delta) \subseteq f(B(a, \varepsilon))$.

Theorem 2.50 (Open Mapping Theorem). Let $G \subseteq \mathbb{C}$ be a region and $f : G \rightarrow \mathbb{C}$ be analytic. Let U be open in G . Then $f(U)$ is open in \mathbb{C} .

Proof. Choose some $a \in U$. Then, there is some $R > 0$ such that $B(a, R) \subseteq U$. Due to Theorem 2.48 and the remark following it, there is $\varepsilon > 0$ and $\delta > 0$ such that $B(f(a), \delta) \subseteq f(B(a, \varepsilon))$. The conclusion is immediate now. ■

Chapter 3

Singularities

Definition 3.1. A function f has an *isolated singularity* at a point $z = a$ if there is $R > 0$ such that f is analytic on $0 < |z - a| < R$. The point a is called a *removable singularity* if there is an analytic function $g : B(a, R) \rightarrow \mathbb{C}$ such that $f(z) = g(z)$ for $0 < |z - a| < R$.

Theorem 3.2. If f has an isolated singularity at a , then the point $z = a$ is a removable singularity if and only if

$$\lim_{z \rightarrow a} (z - a)f(z) = 0$$

Proof. The forward direction is obvious. We shall show the reverse direction, that is, suppose $\lim_{z \rightarrow a} (z - a)f(z) = 0$. There is $R > 0$ such that f is analytic in $0 < |z - a| < R$. Now, define the function $g : B(a, R) \rightarrow \mathbb{C}$ such that $g(z) = (z - a)f(z)$. It is obvious that g is continuous. It suffices to show that g is analytic, since then, there would exist an analytic function h such that $g(z) = (z - a)h(z)$, implying the desired conclusion.

To show that g is analytic, we shall use Morera's Theorem. Let T be a triangle in $B(a, R)$. Note that since this region is convex, it suffices to choose any three points a, b, c in the interior and they would form a valid triangle. Let Δ denote the interior of T . If $a \notin \Delta$, then T is nulhomotopic and due to Theorem 2.38, the integral $\int_T g$ must be zero.

Next, if a is a vertex of the triangle, say $[a, b, c, a]$, then for any points x and y on the line segments $[a, b]$ and $[a, c]$,

$$\int_{[a,b,c,a]} g = \int_{[a,x,y]} g + \int_{[x,b,c,y]} g = \int_{[a,x,y]} g$$

where the last equality follows from Theorem 2.38. Since g is continuous, there is $r > 0$ such that for all $t \in B(a, r)$, $|g(t)| < \varepsilon$. And thus, $|\int_{[a,x,y]} g| < \varepsilon \ell$ where ℓ is the perimeter of T . It is now obvious that the integral must be zero.

Finally, suppose $a \in \Delta$ where $T = [b, c, d, b]$. The integral is now given by

$$\int_{[b,c,d,a]} g = \int_{[a,b,c,a]} g + \int_{[a,c,d,a]} g + \int_{[a,d,b,a]} g = 0$$

This completes the proof. ■

Definition 3.3 (Pole, Essential Singularity). If $z = a$ is an isolated singularity of f , then a is a *pole* of f if $\lim_{z \rightarrow a} |f(z)| = \infty$. If an isolated singularity is neither a pole nor a removable singularity, it is then called an *essential singularity*.

Theorem 3.4. Let $f : G \setminus \{a\} \rightarrow \mathbb{C}$ be analytic with a pole at $z = a$. Then there is an analytic function $g : G \rightarrow \mathbb{C}$ and a positive integer m such that

$$f(z) = \frac{g(z)}{(z-a)^m} \quad \text{on } G \setminus \{a\}$$

and $g(a) \neq 0$.

Proof. Consider the analytic function $h : G \setminus \{a\} \rightarrow \mathbb{C}$ given by $h = \frac{1}{f}$. Then it is obvious that $\lim_{z \rightarrow a} f(z) = 0$, as a result, f has a removable singularity at $z = a$, and thus, there is an analytic function $\tilde{h} : G \rightarrow \mathbb{C}$ such that $h = \tilde{h}$ on G . Now, since $\tilde{h}(a) = 0$, there is a positive integer m and an analytic function $g : G \rightarrow \mathbb{C}$ such that $\tilde{h}(z) = (z-a)^m g(z)$. As a result, we see that

$$f(z) = \frac{1}{(z-a)^m} \frac{1}{g(z)}$$

and the conclusion follows. ■

Definition 3.5. If f has a pole at $z = a$, and m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$, then f is said to have a *pole of order m* at $z = a$.

Bibliography

- [1] L. V. Ahlfors. *Complex Analysis*. McGraw-Hill Book Company, 2 edition, 1966.
- [2] John B. Conway. *Functions of one complex variable*, volume 11 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1978.
- [3] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.