Algebraic Topology

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Contents

Chapter 0

Homological Algebra

This chapter is mainly taken from [Wei94].

0.1 Basic Definitions

Definition 0.1. A *chain complex* C of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with R-module homomorphisms $d_n : C_n \to C_{n-1}$ such that the composition $d_n \circ d_{n-1} = 0$ for each $n \in \mathbb{Z}$. Define the *n-th homology module* of C to be

$$H_n(C) := \ker(d_n) / \operatorname{im}(d_{n+1}).$$

A *morphism* of chain complexes $u : C \to D$ is a collection of R-module homomorphisms $u_n : C_n \to D_n$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & \cdots . \\
\cdots \arrow[r, "d_{n+2}"'] & D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & \cdots
\end{tikzcd}$$

We denote the category of chain complexes of R-modules by $\mathbf{Ch}(R - \mathbf{Mod})$.

Proposition 0.2. *A morphism $u : C \to D$ of chain complexes induces a sequence of R-module homomorphisms between the homology modules, denoted by u_*.*

Proof. ∎

Definition 0.3. An **Ab**-category is a locally small category $\mathscr{A}$ in which $\operatorname{Hom}(A, B)$ has the structure of an abelian group for all $A, B \in \mathscr{A}$ and composition of morphisms distributes over addition. That is, given a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "g"'] & C \arrow[r, "h"] & D ,
\end{tikzcd}$$

we have $h \circ (g + g') \circ f = h \circ g \circ f + h \circ g' \circ f$ in $\operatorname{Hom}(A, D)$.

An *additive functor* $F : \mathscr{A} \to \mathscr{B}$ is a functor between **Ab**-categories such that the induced map

$\mathrm{Hom}(A, A') \to \mathrm{Hom}(FA, FA')$ is a group homomorphism.

An *additive category* is an **Ab**-category $\mathscr{A}$ with a null (zero) object and a product $A \times B$ for every pair $A, B \in \mathscr{A}$.

Proposition 0.4. *In an additive category, finite products are the same as finite coproducts.*

Proof. Let $A, B \in \mathscr{A}$ have a product $A \times B \in \mathscr{A}$ with maps $p : A \times B \to A$ and $q : A \times B \to B$. Consider the pair of maps $\mathbf{id}_A : A \to A$ and $0 : A \to B$. This induces a unique map $i : A \to A \times B$ such that $p \circ i = \mathbf{id}_A$ and $q \circ i = 0$. Similarly, there is a map $j : B \to A \times B$ such that $p \circ j = 0$ and $q \circ j = \mathbf{id}_B$. Note that

$$p \circ (i \circ p + j \circ q) = p \qquad q \circ (i \circ p + j \circ q) = q$$

whence $i \circ p + j \circ q = \mathbf{id}_{A \times B}$.

We contend that the pair (i, j) defines a coproduct of A, B. Indeed, if $D \in \mathscr{A}$ with maps $f : A \to D$ and $g : B \to D$, set $d = f \circ p + g \circ q : A \times B \to D$. We have

$$d \circ i = (f \circ p + g \circ q) \circ i = f \circ p \circ i + g \circ q \circ i = f$$

and similarly, $d \circ j = g$. It now remains to show the uniqueness of d. Suppose $d' : A \times B \to D$ is a morphism, then

$$(d - d') \circ \mathbf{id}_{A \times B} = (d - d') \circ (i \circ p + j \circ q) = 0$$

whereby completing the proof. ■

Definition 0.5 (Kernel, Cokernel). Let $\mathscr{A}$ be an additive category. A *kernel* of a morphism $f : B \to C$ is defined to be a map $i : A \to B$ such that $f \circ i = 0$ and for every other morphism $j : D \to B$ with $f \circ j = 0$, there is a unique morphism $u : D \to A$ such that $j = i \circ u$. This is expressed in the following diagram.

$$\begin{tikzcd}
& B \arrow[dr, "f"] & \\
& A \arrow[u, "i"] \arrow[r, "0"'] & C \\
D \arrow[uur, "j"] \arrow[ur, dotted, "\exists ! u"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

Similarly, a *cokernel* of $f : B \to C$ is defined to be a map $p : C \to A$ such that $p \circ f = 0$ and for any morphism $q : C \to D$ with $q \circ f = 0$, there is a unique map $u : A \to D$ such that $q = u \circ p$. This is expressed in the following diagram.

$$\begin{tikzcd}
& B \arrow[d, "0"] \arrow[dr, "f"] \arrow[ddl, "0"'] & \\
& A \arrow[dl, dotted, "\exists ! u"'] & C \arrow[l, "p"] \arrow[dll, "q"] \\
D & &
\end{tikzcd}$$

Proposition 0.6. *A kernel is always monic and a cokernel is always epic.*

Proof. ■

Definition 0.7. An *abelian category* is an additive category $\mathscr{A}$ such that

1. every morphism in $\mathscr{A}$ has a kernel and a cokernel,
2. every monic in $\mathscr{A}$ is the kernel of its cokernel and
3. every epi in $\mathscr{A}$ is the cokernel of its kernel.

Theorem 0.8 (Freyd-Mitchell Embedding Theorem). *Let $\mathscr{A}$ be a small abelian category. Then, there is a ring R and a full, faithful and exact functor $F : \mathscr{A} \to R - \mathbf{Mod}$.*

In particular, what this means is that in all diagram chases involving objects in a general abelian category, we may treat the objects as elements in $R - \mathbf{Mod}$ for some ring R, which makes our life much easier.

Definition 0.9. Let C and D be chain complexes. Two chain maps $f, g : C \to D$ are said to be *chain homotopic* if there are R-module homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$f_n - g_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n.$$

Proposition 0.10. *If $f, g : C \to D$ are chain homotopic, then $f_* = g_*$.*

Proof. ■

0.1.1 Some Diagram Chasing

Theorem 0.11 (Snake Lemma). *Let A, B, C, A', B', C' be R-modules that fit into the following commutative diagram*

$$\begin{tikzcd}
 & \ker\alpha \arrow[r, dotted] \arrow[d] & \ker\beta \arrow[r, dotted] \arrow[d] & \ker\gamma \arrow[d] & \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}\alpha \arrow[r, dotted] & \operatorname{coker}\beta \arrow[r, dotted] & \operatorname{coker}\gamma &
\end{tikzcd}$$

with exact rows. Then, there is a map $\partial : \ker\gamma \to \operatorname{coker}\alpha$ which makes the induced sequence

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\partial} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$$

exact. Further, if f is injective, then so is the induced map $\ker\alpha \to \ker\beta$ and if g' is surjective, then so is the induced map $\operatorname{coker}\beta \to \operatorname{coker}\gamma$.

Proof. ■

Corollary 0.12 (Five Lemma). Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\eta"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

with exact rows.

(a) If β, δ are injective and α is surjective, then γ is injective.

(b) If β, δ are surjective and η is injective, then γ is surjective.

Theorem 0.13. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of chain complexes. Then, there is a long exact sequence of homology groups given by*

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$$

Proof. Note that the kernel of the map $d : A_n \to Z_{n-1}A$ contains $d(A_{n-1})$, therefore, we have an induced map $\widetilde{d} : A_n/d(A_{n+1}) \to Z_{n-1}A$ given by $d(a_n + d(A_{n+1})) = d(a_n)$ for all $a_n \in A_n$. Note that $\ker \widetilde{d} = H_n(A)$ and $\operatorname{coker} \widetilde{d} = H_{n-1}(A)$. Similarly, define $\widetilde{d}$ for the chain complexes B and C.

We now have a commutative diagram

$$\begin{tikzcd} & A_n/d(A_{n+1}) \arrow[r] \arrow[d, "\widetilde{d}"'] & B_n/d(B_{n+1}) \arrow[r] \arrow[d, "\widetilde{d}"'] & C_n/d(C_{n+1}) \arrow[r] \arrow[d, "\widetilde{d}"'] & 0 \\ 0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) & \end{tikzcd}$$

with exact rows. The conclusion now follows from Theorem 0.11 ■

0.2 Derived Functors

Chapter 1

Topological Preliminaries

1.1 Cell Complexes

Definition 1.1 (Cell Complex). Cell complexes are constructed using an inductive procedure.

(a) Begin with a discrete set X^0, whose points are regarded as 0-cells.

(b) Inductively, form the *n-skeleton* X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} = \partial e^n_\alpha \to X^{n-1}$.

(c) This inductive process can either be stopped at a finite stage or continued indefinitely, setting $X = \bigcup_{n=1}^{\infty} X^n$. In the latter case, X is given the *weak topology*.

Example 1.2 (Real Projective Space $\mathbb{RP}^n$). Recall that $\mathbb{RP}^n$ is defined as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the identification $x \sim \lambda x$. This can equivalently be thought of as the hemisphere D^n with the identification $x \sim -x$ for $\partial D^n = S^{n-1}$. Under this identification, S^{n-1} quotients to $\mathbb{RP}^{n-1}$ whereby, $\mathbb{RP}^n$ is obtained by simply attaching an n-cell to $\mathbb{RP}^{n-1}$ through the quotient map $\varphi : S^{n-1} = \partial D^n \to \mathbb{RP}^{n-1}$. Thus, the cell complex structure of $\mathbb{RP}^n$ is $e^0 \cup e^1 \cup \cdots \cup e^n$, i.e. one cell in each dimension $0 \leq i \leq n$.

Example 1.3 (Complex Projective Plane $\mathbf{CP}^n$).

Definition 1.4. A *subcomplex* of a cell complex X is a closed subspace $A \subseteq X$ that is a union of cells of x. A pair (X, A) consisting of a cell complex X and a subcomplex A is called a *CW pair*.

Remark 1.1.1. *The property of A being a subcomplex depends on the CW structure of X. For example, S^{n-1} is not a subcomplex of S^n with the natural CW structure obtained by gluing two D^n's. But, we may choose a different CW structure for S^n wherein we begin with the equitorial S^{n-1} and attach two D^n's to it, via the obvious boundary map. In this case, S^{n-1} is indeed a subcomplex of S^n.*

1.2 Homotopy Extension Property

Definition 1.5 (Homotopy Extension Property). A pair (X, A) with $A \subseteq X$ is said to have the *homotopy extension property* if for any topological space Y, a map $f_0 : X \to Y$ and a homotopy $H : A \times I \to Y$ such that $H|_{A\times\{0\}} = f_0|_A$, there is an extension of H, $\widetilde{H} : X \times I \to Y$ with $H|_{X\times\{0\}} = f_0$.

Proposition 1.6. *A pair (X, A) with A closed*[a] *in X has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a* retract *of $X \times I$.*

[a]This is superfluous

Proof. Suppose (X, A) has the homotopy extension property. Consider the identity map $\mathbf{id} : X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$. This may be extended to a map $f : X \times I \to X \times \{0\} \cup A \times I$ which restricts to the identity map on $X \times \{0\} \cup A \times I$. This shows that the latter is a retract of the former. ■

Proposition 1.7. *If (X, A) is a CW-pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$ whereby (X, A) has the homotopy extension property.*

Proof. ■

(X, A) CW-pair has homotopy ext property

Proposition 1.8. *If the pair (X, A) has the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Proof. Since A is contractible, there is a homotopy between the inclusion $A \hookrightarrow X$ and the constant map on A. Due to the homotopy extension property, this can be extended to a homotopy $F : X \times I \to X$ such that $F|_{X\times\{0\}} = \mathbf{id}_X$. Let $q : X \twoheadrightarrow X/A$ denote the quotient map and $\widetilde{q} : X \times I \to X/A \times I$ denote the quotient map with the obvious identification.

Consider the composition $q \circ F$. Then, for $a, a' \in A$, $q \circ F(a, t) = q \circ F(a', t)$ for all t whereby this induces a continuous map $\widetilde{F} : X/A \times I \to X/A$. Let $f_1 := F|_{X\times\{1\}}$ and $\widetilde{f}_1 = \widetilde{F}|_{X\times\{1\}}$. Then, f_1 maps all of A to a single point whence it induces a map $g : X/A \to X$ such that $f_1 = g \circ q$.

We contend that $\widetilde{f}_1 = q \circ g$. Indeed, for any $\overline{x} \in X/A$, there is $x \in X$ such that

$$q \circ g(\overline{x}) = q \circ g \circ q(x) = q \circ f_1(x) = \widetilde{f}_1 \circ q(x) = \widetilde{f}_1(\overline{x}).$$

This shows that $g \circ q = f_1 \simeq \mathbf{id}_X$ through F while $q \circ g = \widetilde{f}_1 \simeq \mathbf{id}_{X/A}$ through $\widetilde{F}$ whence the conclusion follows. ■

Corollary 1.9. If (X, A) is a CW-pair with A contractible, then the quotient map $q : X \to A/A$ is a homotopy equivalence.

Example 1.10.

Chapter 2

The Fundamental Group

2.1 Fundamental Groupoid and Group

Definition 2.1 (Homotopy). Let X and Y be topological spaces. A homotopy is a continuous function $H : X \times I \to Y$. A *homotopy* between two functions $f, g : X \to Y$ is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Definition 2.2 (Homotopy of Paths). Let X be a topological space and $f, g : I \to X$ be paths. Then, f and g are said to be *path homotopic* if there is a continuous function $H : I \times I \to X$ such that $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ for all $s \in I$. We denote this by $f \simeq_p g$.

Proposition 2.3. *The relation $\simeq$ on the set of all paths in X is an equivalence relation.*

Proposition 2.4. *Let $f : I \to X$ be a path and $\varphi : I \to I$ be a continuous function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, $f \simeq_p f \circ \varphi$.*

Proof. Define the function $\Phi : I \times I \to X$ by

$$\Phi(s, t) = f(t\varphi(s) + (1 - t)s)$$

It is not hard to see that Φ is a path homotopy between f and $f \circ \varphi$. ■

Consider the set of all equivalence classes of paths in X under the equivalence relation $\simeq_p$. Define the operation $*$ on pairs of equivalence classes $[f]$ and $[g]$ where $f(1) = g(0)$ by

$$[f] * [g] = [f * g]$$

where

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 < t \leq 1 \end{cases}$$

Proposition 2.5. *The operation $*$ is associative. That is,*

$$[f] * ([g] * [h]) = ([f] * [g]) * h$$

Proof. Note that $[f] * ([g] * [h])$ is the equivalence class containing the path:

$$\alpha(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(4t-2) & 1/2 < t \leq 3/4 \\ h(4t-3) & 3/4 < t \leq 1 \end{cases}$$

Consider the piecewise linear function $\varphi : [0,1] \to [0,1]$ that maps $[0,1/2]$ to $[0,1/4]$, $[1/2,3/4]$ to $[1/4,1/2]$ and $[1/2,1]$ to $[3/4,1]$, then through $\alpha \circ \varphi$, the conclusion follows. ∎

Proposition 2.6. *Let $f : I \to X$ be a path and $\overline{f} : I \to X$ be given by $\overline{f}(t) = f(1-t)$ for $t \in I$. Then $f * \overline{f}$ is path homotopic to the constant path at x_0.*

Proof. Define the homotopy

$$H(s,t) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2}(1-t) \\ f(1-t) & \frac{1}{2}(1-t) \leq s \leq \frac{1}{2}(1+t) \\ \overline{f}(2s-1) & \frac{1}{2}(1+t) \leq s \leq 1 \end{cases}.$$

Then, $H(\cdot,0) = f * \overline{f}$ and $H(\cdot,1)$ is the constant path at x_0 which completes the proof. ∎

Definition 2.7 (Fundamental Group). Let $\pi_1(X, x_0)$ be the set of equivalence classes of paths $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$. Then, $\pi_1(X, x_0)$ forms a group under the operation $*$. This is known as the *fundamental group* of X based at x_0.

Let $\mathbf{Top}_*$ denote the category of pointed topological spaces, that is, the category wherein objects are pairs (X, x_0) where $x_0 \in X$ and a morphism $f : (X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ with $f(x_0) = y_0$.

Proposition 2.8. *Let $f : (X, x_0) \to (Y, y_0)$ be a morphism in $\mathbf{Top}_*$. Then, the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\alpha] \mapsto [f \circ \alpha]$ is a homomorphism of groups. Further, if*

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

then $(g \circ f)_* = g_* \circ f_*$.

Proof. If H is a path homotopy between α_1 and α_2 in X, then $f \circ H$ is a homotopy between $f \circ \alpha_1$ and $f \circ \alpha_2$ in Y. Thus, the map f_* is well defined. Next, suppose $[\alpha], [\beta] \in \pi_1(X, x_0)$, then, it is not hard to see that $(f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta)$, consequently, f_* is a homomorphism of groups. The final assertion is obvious from the definition. ∎

As a result, we see that π_1 is a (covariant) functor from $\mathbf{Top}_*$ to $\mathbf{Grp}$.

Theorem 2.9. *Let X be path connected and $x_0, x_1 \in X$. Let $\alpha : I \to X$ be a path from x_0 to x_1. Then, the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $[f] \mapsto [\overline{\alpha} * f * \alpha]$ is a group isomorphism.*

Proof. It is easy to see that $\hat{\alpha}$ is a homomorphism. The surjectivity and injectivity of this map are obvious. ■

Proposition 2.10. *Let X be path connected and $h : X \to Y$ be a continuous map. If $x_0, x_1 \in X$ with $\alpha : I \to X$ a path between them and $\beta = h \circ \alpha$, then we have the following commutative diagram:*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \hat{\alpha}\downarrow & & \downarrow\hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

Proof. Let $[f] \in \pi_1(X, x_0)$. Then,

$$\hat{\beta} \circ (h_{x_0})_*([f]) = \hat{\beta}([h \circ f]) = [\overline{\beta} * h \circ f * \beta]$$

and

$$(h_{x_1})_* \circ \hat{\alpha}([f]) = (h_{x_1})_*([\overline{\alpha} * f * \alpha]) = [\overline{\beta} * h \circ f * \beta]$$

This completes the proof. ■

2.2 Retracts and Deformation Retracts

Definition 2.11. A *retraction* of a space X onto a subspace A is a map $r : X \to X$ such that $\mathrm{im}(r) = A$ and $r|_A = \mathbf{id}_A$. A *deformation retraction* is a homotopy $H : X \times I \to X$ between $\mathbf{id}_X$ and a retraction r from X onto A. That is, $H|_{X\times\{0\}} = \mathbf{id}_X$ and $H|_{X\times\{1\}} = r$. A deformation retract is said to be *strong* if $H|_{A\times\{t\}} = \mathbf{id}_A$ for all $t \in I$.

Proposition 2.12. *If a space X retracts onto a subspace A and $x_0 \in A$, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \to X$ is injective. If A isa deformation retract of X, then i_* is an isomorphism.*

Proof. Let $r : X \to A$ denote the (restriction of the) retraction of X onto A. Then, $r \circ i = \mathbf{id}_A$ whence $r_* \circ i_* = \mathbf{id}_{\pi_1(A,x_0)}$ whence i_* must be injective.

Now, let $H : X \times I \to X$ be a deformation retraction of X onto A. It suffices to show that i_* is surjective. Indeed, let $f : I \to X$ be a loopt based at x_0. Then, the map $\Phi : I \times I \to X$ given by $\Phi(s,t) = H(f(s), t)$ is a path homotopy between f and $g = H|_{I\times\{1\}}$. Since $g \in \pi_1(A, x_0)$, we see that i_* must be surjective. ■

Definition 2.13 (Homotopy Equivalence). A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there is a continuous map $g : Y \to X$ such that $g \circ f$ is homotopic to $\mathbf{id}_X$ and $f \circ g$ is homotopic to $\mathbf{id}_Y$.

2.3 van Kampen's Theorem

The following formulation has been taken from [Mun02]

Theorem 2.14 (van Kampen). *Let $X = U \cup V$ where U and V are open in X. Further, suppose U, V and $U \cap V$ are nonempty and path connected. Let H be a group, $x_0 \in U \cap V$ and*

$$\phi_1 : \pi_1(U, x_0) \to H \qquad \phi_2 : \pi_1(V, x_0) \to H$$

be homomorphisms. Finally, let i_1, i_2, j_1, j_2 be the homomorphisms of fundamental groups induced by inclusion maps. Then, there is a unique map $\Phi : \pi_1(X, x_0) \to H$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dotted, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Notice how the diagram resembles that of a pushout in a general category and hence, has the universal property and hence, the object, if it exists is unique up to a unique isomorphism. In the special case that $U \cap V$ is simply connected, that is, has a trivial fundamental group, the commutative diagram reduces to that of a coproduct. And it is well known that the coproduct in the category of groups is the free product.

Proof. Let $\mathscr{L}(U, x_0), \mathscr{L}(V, x_0), \mathscr{L}(U \cap V, x_0)$ denote the set of loops in U, V and $U \cap V$. The path homotopy class of a path f in X, U, V and $U \cap V$ is denoted by $[f]$, $[f]_U$, $[f]_V$ and $[f]_{U \cap V}$ respectively. The proof proceeds in multiple steps. The main idea is to first define a set map ρ on the set of loops contained completely in either U or V, then extend it to a set map σ on the set of paths contained completely in either U or V and finally extend it to a set map τ on the set of all paths in X.

Once the map τ is defined, we shall show that $\tau(f) = \tau(g)$ whenever $f \simeq_p g$ and therefore, τ would descend to a group homomorphism from $\pi_1(X, x_0)$ to H.

Step 1: Defining the set map $\rho : \mathscr{L}(U, x_0) \cup \mathscr{L}(V, x_0) \to H$.

This has quite a natural definition:

$$\rho(f) = \begin{cases} \phi_1([f]_U) & f \text{ is contained completely in } U \\ \phi_2([f]_V) & f \text{ is contained completely in } V \end{cases}$$

For a loop contained in $U \cap V$, the map ρ is well defined due to the commutativity of the diagram. It is not hard to see that if $f, g \in \mathscr{L}(U, x_0)$, then $\rho(f * g) = \rho(f)\rho(g)$.

Step 2: Extend the map ρ to a map $\sigma : \mathscr{P}(U) \cup \mathscr{P}(V) \to H$.

For each $x \in X$, fix a path α_x from x_0 to x such that whenever x lies in U, V or $U \cap V$, α_x lies completely in U, V or $U \cap V$ respectively.

Let f be a path from x_1 to x_2 that lies completely in U or completely in V. Define

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})$$

Now, let f and g be paths completely contained in U. If $f \simeq_p g$ in U, then $\alpha_{x_1} * f * \alpha_{x_2}^{-1} \simeq_p \alpha_{x_1} * g * \alpha_{x_2}^{-1}$ in U and from the definition of ρ, we see that

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1}) = \rho(\alpha_{x_1} * g * \alpha_{x_2}^{-1}) = \sigma(g)$$

Next, if f is a path from x_1 to x_2 and g is a path from x_2 to x_3 (both contained in U), then

$$\begin{aligned}\sigma(f * g) &= \rho(\alpha_{x_1} * f * g * \alpha_{x_3}^{-1}) \\ &= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1} * \alpha_{x_2} * g * \alpha_{x_3}^{-1}) \\ &= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})\rho(\alpha_{x_2} * g * \alpha_{x_3}^{-1}) = \sigma(f)\sigma(g)\end{aligned}$$

Step 3: Extend the map σ to a map $\tau : \mathscr{P}(X) \to H$

Let $f : I \to X$ be a path. It is not hard to argue, using Lebesgue's Number Lemma, that there is a mesh δ such that for every partition $0 = s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ of $[0,1]$ with mesh less than δ, $f([s_i, s_{i+1}])$ is completely contained in either U or V for $0 \le i \le n-1$.

Denote by f_i, the restriction of f to $[s_i, s_{i+1}]$. Define

$$\tau(f, P) = \sigma(f_0)\cdots\sigma(f_{n-1})$$

We contend that the map $\tau(f, P)$ is independent of the partition chosen, so long as its mesh is less than δ. To do so, we first show that refining a partition with mesh less than δ does not change the image under τ, for which, it suffices to show that adding a single point to the partition does not change the image. Indeed, let $c \in (s_i, s_{i+1})$ be added to the partition. But since $f([s_i, c])$ and $f([c, s_{i+1}])$ lie completely either in U or in V, we have that $\sigma(f|_{[s_i,c]})\sigma(f|_{[c,s_{i+1}]}) = \sigma(f|_{[s_i,s_{i+1}]})$ whence the conclusion follows.

Now, let P_1 and P_2 be two partitions of $[0,1]$ with mesh less than δ. Then $P_1 \cup P_2$ is a partition that refines both P_1 and P_2, consequently,

$$\tau(f, P_1) = \tau(f, P_1 \cup P_2) = \tau(f, P_2)$$

which establishes our claim.

Step 4: If $f \simeq_p g$ in X, then $\tau(f) = \tau(g)$.

Let $F : I \times I \to X$ be a path homotopy between f and g. Using the Lebesgue Number Lemma, there are partitions $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $f([s_i, s_{i+1}] \times [t_i, t_{i+1}])$ is completely contained in either U or V.

Step 5: $\tau(f * g) = \tau(f)\tau(g)$

Let P be a partition of $f * g$ such that $(f * g)([s_i, s_{i+1}])$ is completely contained in either U or V. Define $P^* = P \cup \{1/2\}$. It is not hard to see, using P^* that τ is multiplicative.

Step 6: Constructing the homomorphism Φ.

Restrict the map τ to $\tau : \mathscr{L}(X, x_0) \to H$. From **Step 4**, it follows that there is a map $\Phi : \pi_1(X, x_0) \to H$ and from **Step 5**, we get that Φ is a homomorphism.

The above argument establishes the existence of a group homomorphism $\Phi : \pi_1(X, x_0) \to H$ making the diagram commute. We must now show that the map Φ is unique. But this follows from the fact that the generators of Φ are precisely the images of the generators of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ under the homomorphisms j_1 and j_2 respectively. ■

2.3.1 Alternate Formulation of van Kampen's Theorem

The following formulation and proof has been taken from [Hat00]. The upshot of this formulation is that it gives a recipe for computing the presentation of the fundamental group which is hard to see from the previous formulation.

Let X be a topological space and $\{A_\alpha\}_{\alpha \in J}$ be an open cover of path connected subspaces of X. Let $x_0 \in X$ be a basepoint such that $x_0 \in A_\alpha$ for each $\alpha \in J$. The inclusion $A_\alpha \hookrightarrow X$ induces a group homomorphism

$j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ where we have dropped the basepoint to avoid clutter. Similarly, the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ induces a group homomorphism $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$.

Due to the Universal Property of Free Products, the group homomorphisms j_α induce a group homomorphism

$$\Phi : *_{\alpha \in J} \pi_1(A_\alpha) \to \pi_1(X).$$

Proposition 2.15. *If each intersection $A_\alpha \cap A_\beta$ is path connected, then Φ is surjective.*

Sketch of Proof. The proof of surjectivity follows the same proof of **Step2** in the proof of Theorem 2.14. It suffices to show that any element in $\pi_1(X)$ can be represented as the product of finitely many elements of $j_\alpha\left(\pi_1(A_\alpha)\right)$.

Take any loop $f : I \to X$ based at x_0 and then using the Lebesgue Number Lemma, find a partition $0 \le t_0 < \cdots < t_n = 1$ of I such that the image $f([t_i, t_{i+1}])$ is completely contained in some A_α for each $0 \le i \le n-1$. Now, join the endpoints of each such path to x_0, which can be done since each $A_\alpha \cap A_\beta$ is path connected. This immediately gives us a decomposition of $[f]$. ∎

Proposition 2.16. *If in addition to the hypothesis of , each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of the surjection Φ is generated by the set*

$$\left\{ i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \middle| \, \omega \in \pi_1(A_\alpha, A_\beta) \text{ for all } \alpha, \beta \in J \right\}.$$

Proof. ∎

Proof of general van kampen

Chapter 3

Covering Spaces

Definition 3.1 (Covering Space). A covering space of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the condition that there is an open cover $\{U_\alpha\}$ of X such that for each $\alpha \in J$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically by p to U_α.

Notice that for each $x \in X$, the subspace $p^{-1}(x)$ of $\widetilde{X}$ has the discrete topology.

Proposition 3.2. *Let $p : \widetilde{X} \to X$ be a covering map where X is connected. If for some $x \in X$, $|p^{-1}(x)| = n \in \mathbb{N}$, then for all $x' \in X$, $|p^{-1}(x')| = n$.*

Proof. The map $x \mapsto |p^{-1}(x)|$ is locally constant and thus continuous. Owing to X being connected and $\mathbb{N}$ having the discrete topology, the aforementioned map must be constant. ∎

3.1 Lifting Properties

Definition 3.3 (Lift). Let $f : Y \to X$ be a continuous and $p : \widetilde{X} \to X$ be a covering map. A *lift* of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p \circ \widetilde{f}$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 3.4. *Let Y be connected and $p : \widetilde{X} \to X$ a covering map. If $f : Y \to X$ is a continuous map having two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$, that agree at some point in Y, then they agree on all of Y.*

Proof. Let
$$A = \{y \in Y \mid \widetilde{f}_1(y) = \widetilde{f}_2(y)\}$$
We shall show that A is clopen in Y, whence we would be done owing to A being nonempty. Let $y \in A$ and $x = f(y)$. There is a neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of $\{V_\alpha\}$ which are homeomorphically mapped to U. Let V_β be the one containing $\widetilde{x} = \widetilde{f}_1(y) = \widetilde{f}_2(y)$. Then, due to continuity,

there is a neighborhood N of y that is mapped into V_β by both $\widetilde{f}_1$ and $\widetilde{f}_2$. Then, for all $z \in N$, $p \circ \widetilde{f}_1(z) = p \circ \widetilde{f}_2(z)$ but since p is injective on V_β, we must have $\widetilde{f}_1(z) = \widetilde{f}_2(z)$, consequently, $N \subseteq A$ and A is open.

On the other hand, if $y \notin A$, then $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$ lie in distinct open sets V_{β_1} and V_{β_2}, consequently, for all $z \in N = \widetilde{f}_1^{-1}(V_{\beta_1}) \cap \widetilde{f}_2^{-1}(V_{\beta_2})$, $\widetilde{f}_1(z) \neq \widetilde{f}_2(z)$, thereby completing the proof. ■

Theorem 3.5 (Homotopy Lifting Property). *Let $p : \widetilde{X} \to X$ be a covering map and $F : Y \times I \to X$ a continuous map. Let $\widetilde{F}_0 : Y \to \widetilde{X}$ be a lift of $F|_{X \times \{0\}}$. Then, there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}|_{X \times \{0\}} = \widetilde{F}_0$.*

Proof. The first step is to define a lift $\widetilde{F}$ on the strip $N \times I$ where N is a neighborhood of some point $y \in Y$.

Fix some $y_0 \in Y$. Each point $y_0 \times t$ has a neighborhood $N_t \times (a_t, b_t)$ which maps maps into an evenly covered neighborhood of $F(y_0 \times t)$. Note that the strip $\{y_0\} \times I$ is compact and is thus covered by finitely many of the N_t's, whence we may choose a neighborhood N of y_0 in Y and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each $N \times [t_i, t_{i+1}]$ is contained in $N_t \times (a_t, b_t)$ for some $t \in I$.

Now suppose $\widetilde{F}$ has been constructed on $N \times [0, t_i]$. We already have a lift for $N \times \{0\}$ and this is our base case. The space $N \times [t_i, t_{i+1}]$ is mapped into an evenly covered neighborhood U by F. Let $\widetilde{U} \subseteq \widetilde{X}$ be the unique open set in $\widetilde{X}$ containing the point $\widetilde{F}(y_0 \times t_i)$. There is a neighborhood N' of y_0 such that $N' \times \{t_i\}$ is mapped into $\widetilde{U}$ by $\widetilde{F}$. Replace N by N' henceforth. The composition $p^{-1} \circ F$ now lifts F on $N' \times [t_i, t_{i+1}]$ and since it agrees with $\widetilde{F}$ on $N' \times \{t_i\}$, we have an extension to $\widetilde{F}$ on $N \times [0, t_{i+1}]$, which is continuous due to the Pasting Lemma.

Now, for each $y \in Y$, we have constructed a lift $\widetilde{F}_y$ on $N_y \times I$ where N_y is some neighborhood of y. We must now argue that we can indeed paste these lifts together. Let $y \in N_{y'} \cap N_{y''}$. Since $\{y\} \times I$ is connected and $\widetilde{F}_{y'}$ and $\widetilde{F}_{y''}$ are two lifts which agree at $y \times 0 \in \{y\} \times I$, both the lifts must agree throughout due to Theorem 3.4. This also establishes the uniqueness of the lift $\widetilde{F}$ whereby completing the proof. ■

Corollary 3.6 (Path Lifting). Let $f : I \to X$ be a path and let $x_0 = f(0)$. For any $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x}_0$.

Corollary 3.7 (Path Homotopy Lifting). Let $H : I \times I \to X$ be a path homotopy. Then, the unique lift $\widetilde{H} : I \times I \to \widetilde{X}$, is also a path homotopy.

Proof. Since the image $\widetilde{H}(\{0\} \times I)$ is connected and a subset of the discrete fiber of $p^{-1}(\widetilde{H}(0 \times 0))$, it must be a single point. Similarly argue for the image $\widetilde{H}(\{1\} \times I)$. This completes the proof. ■

Proposition 3.8. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Proof. It suffices to show that $\ker p_*$ is trivial. Indeed, let $f : I \to \widetilde{X}$ be such that $p_*([f]) = 1_{\pi_1(X, x_0)}$. Thus, there is a path homotopy $F : I \times I \to X$ such that $F|_{I \times \{1\}}$ is the constant map at x_0 while $F|_{I \times \{0\}}$ is the map $p \circ f$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ I \times I \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

We have a lift $\widetilde{F} : I \times \{0\} \to \widetilde{X}$ of the bottom edge given by $\widetilde{F}(t \times 0) = f(t)$ and due to Theorem 3.5, this can be extended to a lift $\widetilde{F} : I \times I \to \widetilde{X}$. Consider the connected subspace $Y = \{0\} \times I \cup I \times \{1\} \cup \{1\} \times I$ of $I \times I$. The restriction $\widetilde{F}|_Y$ maps into $p^{-1}(x_0)$, which has the discrete topology, whereby the restriction must be a constant map equal to $\widetilde{x}_0$ since $\widetilde{F}(0 \times 0) = \widetilde{x}_0$. Thus, $\widetilde{F}$ must be a path homotopy between f and the constant path $\widetilde{x}_0$, thereby completing the proof. ∎

Proposition 3.9. *Let $\widetilde{X}$ and X be path connected spaces with a covering map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Then, there is a bijection between the right cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ and $p^{-1}(x_0)$.*

Proof. ∎ right coset to p^{-1} bijection

Theorem 3.10 (Lifting Criterion). *Let Y be path connected and locally path connected and $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Then, for any continuous map $f : (Y, y_0) \to (X, x_0)$, a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proof. The forward direction is trivial, we shall prove the converse. Let $y \in Y$ and γ denote a path from y_0 to y in Y. The image $f \circ \gamma$ is a path in X beginning at x_0 and has a lift $\widetilde{f \circ \gamma}$ to a path in $\widetilde{X}$ beginning at $\widetilde{x}_0$. Define the map $\widetilde{f} : Y \to \widetilde{X}$ by $\widetilde{f}(y) = \widetilde{f \circ \gamma}(1)$.

First, we must show that this is a well defined map, independent of the choice of γ. Indeed, let γ' be another path in Y from y_0 to y. Then, $\gamma' * \overline{\gamma}$ is a loop in γ' based at y_0, whence $f \circ (\gamma' * \overline{\gamma})$ is a loop in X based at x_0. We must now show that this can be lifted to a *loop*[1] in $\widetilde{X}$ based at $\widetilde{x}_0$.

According to our hypothesis, $\left[f \circ \gamma' * \overline{f \circ \gamma}\right] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ whence there is a path homotopy $H : I \times I \to X$ between $p \circ \widetilde{h}$ and $f \circ \gamma' * \overline{f \circ \gamma}$ where $\widetilde{h}$ is a loop in $\widetilde{X}$. Due to Corollary 3.7, this lifts to a homotpy of paths in $\widetilde{X}$. Due to the uniqueness of path lifting, $\widetilde{H}$ is a path homotopy between the loop $\widetilde{h}$ and some other loop $\widetilde{\alpha}$ in $\widetilde{X}$ that maps to $f \circ \gamma' * \overline{f \circ \gamma}$ under p.

Considering the first and second halves of $\widetilde{\alpha}$, we see that they are $\widetilde{f \circ \gamma'}$ and $\overline{\widetilde{f \circ \gamma'}}$ respectively. Therefore, $\widetilde{f \circ \gamma'}(1) = \widetilde{f \circ \gamma}(1)$ and the map $\widetilde{f}$ is well defined.

Finally, we must show that $\widetilde{f}$ is continuous. Let $y \in Y$ and $U \subseteq X$ be an evenly covered open neighborhood of $f(y)$. Choose an open path connected neighborhood V of y that is contained in $p^{-1}(U)$. Let $\widetilde{U}$ be the open set in $\widetilde{X}$ that is homeomorphically mapped to U by p and contains $\widetilde{f}(y)$. We shall show that $\widetilde{f}(V) \subseteq \widetilde{U}$, which would imply $\widetilde{f}|_V = p^{-1} \circ f$, thereby implying local continuity and thus the continuity of $\widetilde{f}$.

First, fix some path γ from y_0 to y in Y. Let $y' \in V$ and choose some path η from y to y' contained in V. The path $\gamma * \eta$ is a path from y_0 to y'. The composition $p^{-1} \circ \eta$ is a path from $\widetilde{f}(y)$ in $\widetilde{U}$, moreover, the composition $\widetilde{f \circ \gamma} * p^{-1} \circ \eta$ is a path from $\widetilde{x}_0$ lifting $\gamma * \eta$, whence $p^{-1} \circ \eta(1) = \widetilde{f}(1)$ and $\widetilde{f}(V) \subseteq \widetilde{U}$. This completes the proof. ∎

3.2 The Universal Cover

Definition 3.11 (Semilocally Simply-Connected). A topological space X is said to be *semilocally simply-connected* if each point $x \in X$ has a neighborhood U such that the inclusion induced homomorphism $i_* : \pi(U, x) \to \pi(X, x)$ is trivial.

[1] We can always lift this to a path but that will not suffice in this case

Henceforth, a topological space is said to be *nice* if it is path-connected, locally path-connected and semilocally simply-connected.

Theorem 3.12. *If X is nice, then there is a simply connected space $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$.*

Proof. Pick a basepoint $x_0 \in X$. Define

$$\widetilde{X} = \{[\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0\}$$

and the function $p : \widetilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.

Let $\mathscr{U}$ denote the set of all path connected open sets $U \subseteq X$ such that the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial. Indeed, if $V \subseteq U \in \mathscr{U}$ is path connected and open, then the homomorphism induced by the inclusion $V \hookrightarrow X$ is the composition of the homomorphisms induced by $V \hookrightarrow U \hookrightarrow X$ and since the latter is trivial, the composition is trivial, consequently, $V \in \mathscr{U}$.

We contend that $\mathscr{U}$ forms a basis for the topology on X. Indeed, let W be a neighborhood of x, then there is a neighhborhood U of x such that the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial. Since X is locally path connected, there is a path connected neighborhood V of x that is contained in $U \cap W$, whence the conclusion follows.

We shall now topologize $\widetilde{X}$. Let γ be a path in X from x_0 and $U \in \mathscr{U}$ contain $\gamma(1)$. Define the set

$$U_{[\gamma]} = \{[\gamma * \eta] \mid \eta : I \to U,\ \eta(0) = \gamma(1)\}$$

where the equivalence classes are in X. Since U is path connected, $p : U_{[\gamma]} \to U$ is surjective. Moreover, since the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial, any two paths from $\gamma(1)$ to any point $x \in U$ are homotopic in X.

We contend that if $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$. Obviously, there is a path $\eta : I \to U$ such that $\gamma' = \gamma * \eta$, whence it follows that $\gamma' * \mu = \gamma * \eta * \mu$ and thus, $U_{[\gamma']} \subseteq U_{[\gamma]}$. On the other hand, $[\gamma * \mu] = [\gamma * \eta * \overline{\eta} * \mu]$ whereby the conclusion follows.

Next, we claim that the collection $\{U_\gamma\}$ forms a basis for a topology on $\widetilde{X}$. Suppose $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ where $U, V \in \mathscr{U}$, then $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Since $\mathscr{U}$ forms a basis, there is $W \in \mathscr{U}$ such that $W \subseteq U \cap V$, consequently, $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$. This proves our claim.

Consider the bijection $p : U_{[\gamma]} \to U$, we contend that this is a homeomorphism. For any basis element $V_{[\gamma']} \subseteq U_{[\gamma]}$, we have $p(V_{[\gamma']}) = V$, consequently, p is an open map. On the other hand, if $V \subseteq U$ is an open set, then $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for some $[\gamma'] \in U_{[\gamma]}$ with $\gamma'(1) \in V$. Since $V_{[\gamma']} \subseteq U_{[\gamma']} = U_{[\gamma]}$, we see that the restriction of p is continuous and therefore a homeomorphism.

Using the local formulation of continuity, we have that $p : \widetilde{X} \to X$ is a continuous map. Any $x \in X$ has a neighborhood $U \in \mathscr{U}$, consequently, $p^{-1}(U) = \bigcup U_{[\gamma]}$ where $[\gamma]$ ranges over all paths from x_0 to some point in U. It is not hard to argue that the sets $U_{[\gamma]}$ must partition $p^{-1}(U)$, whereby p is a covering map.

Finally, we must show that $\widetilde{X}$ is simply connected. Pick the base point $[x_0] \in \widetilde{X}$. First, we show that $\widetilde{X}$ is path connected. Let $[\gamma] \in \widetilde{X}$. Define $\gamma_t : I \to X$ by

$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \le s \le t \\ \gamma(t) & t < s \le 1 \end{cases}$$

It suffices to show that the map $\varphi : I \to \widetilde{X}$ given by $\varphi(t) = [\gamma_t]$ is continuous. Using the Lebesgue Number Lemma, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_{i-1}, t_i]) \subseteq U_i \in \mathscr{U}$. Let $p_i : U_{i[\gamma_{t_i}]} \to U_i$ be the restriction of p, which is a homeomorphism. Then, for all $t \in [t_{i-1}, t_i]$, $\varphi(t) = p_i^{-1}(\gamma(t))$ and continuity follows from the Pasting Lemma.

Next, we show $\pi_1(\widetilde{X}, [x_0]) = 0$. Since p_* is injective, it suffices to show that the image of p_* is trivial. Let γ be a loop in the image of p_*. Then, the map $t \mapsto [\gamma_t]$ is a lift of γ as we have seen earlier and is unique due to Theorem 3.5. Now, since the lift is a loop, we must have

$$[x_0] = [\gamma_1] = [\gamma]$$

consequently, γ is nulhomotopic. This completes the proof. ∎

Theorem 3.13. *Suppose X is nice. Then for every subgroup $H \subseteq \pi_1(X, x_0)$, there is a covering space $p : (X_H, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$.*

Proof. For $[\gamma], [\gamma'] \in \widetilde{X}$, define the relation $[\gamma] \sim_H [\gamma']$ to mean $\gamma(1) = \gamma'(1)$ and $[\gamma * \overline{\gamma'}] \in H$. This is obviously an equivalence relation. Let X_H denote the quotient space $X/\sim_H$ with $q : \widetilde{X} \to X_H$ the quotient map. Consider now the map $p : X_H \to X$ which is induced as shown in the following diagram.

$$\begin{tikzcd} \widetilde{X} \arrow[r, two heads, "\pi"] \arrow[d, two heads, "q"'] & X \\ X_H \arrow[ur, dotted, "\exists! p"'] & \end{tikzcd}$$

Let $U \subseteq X$ be an open neighborhood. Then, $p^{-1}(U)$ is a disjoint union $\bigsqcup U_{[\gamma]}$ where $[\gamma]$ is an equivalence class of paths with $\gamma(1) \in U$. Note that $[\gamma] \sim_H [\gamma']$ if and only if $[\gamma * \eta] \sim_H [\gamma' * \eta]$. Hence, if any two points in distinct neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified, then so are the entire neighborhoods. Hence, $p : X_H \to X$ is a covering map.

Choose the basepoint $\widetilde{x}_0 \in X_H$ the equivalence class under $\sim_H$ containing the point $[e_{x_0}]$ where e_{x_0} is the constant path at x_0. Let γ be a loop in X based at x_0. This lifts to a path from $[e_{x_0}]$ to $[\gamma]$ in $\widetilde{X}$. This lift maps to a loop in X_H if and only if $[e_{x_0}] \sim_H [\gamma]$ or equivalently, $[\gamma] \in H$. In particular, this means that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$. This completes the proof. ∎

Definition 3.14. If $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are covering spaces, then an *isomorphism between* them is a homeomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ f$.

Theorem 3.15. *Let (X, x_0) be path connected and locally path connected and $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be covering spaces. Then, for $\widetilde{x}_1 \in p_1^{-1}(x_0)$ and $\widetilde{x}_2 \in p_2^{-1}(x_0)$, there is an isomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

Proof. We prove the converse, since the forward direction is trivial. Using Theorem 3.10, there are lifts $\widetilde{p}_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ and $\widetilde{p}_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ of p_1 and p_2 respectively. This give us $p_1 = p_2 \circ \widetilde{p}_1$ and $p_2 = p_1 \circ \widetilde{p}_2$, whereby $p_1 \circ (\widetilde{p}_2 \circ \widetilde{p}_1) = p_1$. Note that this implies $\widetilde{p}_2 \circ \widetilde{p}_1$ is a lift of the map p_1, but since $\mathbf{id}_{(\widetilde{X}_1, \widetilde{x}_1)}$ is also a lift, and agree on $\widetilde{x}_1$, we must have that $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathbf{id}_{(\widetilde{X}_1, \widetilde{x}_1)}$ and similarly, $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathbf{id}_{(\widetilde{X}_2, \widetilde{x}_2)}$. This implies the desired conclusion. ∎

Theorem 3.16. *Let X be path connected and locally path connected. Then, there is a bijection between the isomorphism classes of path connected covering spaces $p : \widetilde{X} \to X$ (ignoring basepoints) and conjugacy classes of subgroups of $\pi_1(X)$*[a].

[a]The basepoint can be ignored since X is path connected.

Proof. Fix, once and for all, a basepoint $x_0 \in X$ and let $G = \pi_1(X, x_0)$. Suppose we have an isomorphism of covering spaces (ignoring basepoints)

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

given by the above diagram with the basepoints $\widetilde{x}_1 \in p_1^{-1}(x_0)$ and $\widetilde{x}_2 \in p_2^{-1}(x_0)$. Then, due to the commutativity of the diagram, we have

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_* \circ f_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$

We now contend that changing the basepoint in a covering space $p : \widetilde{X} \to X$ conjugates the image under p_*. Indeed, let $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$ where x_0 is the chosen basepoint of X and let $\widetilde{\gamma}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Let $H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$ and let $g = p_*([\widetilde{\gamma}])$.

If $\widetilde{f}$ is a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, then

$$g^{-1} p_*([\widetilde{f}]) g = p_*([\overline{\widetilde{\gamma}} * \widetilde{f} * \widetilde{\gamma}]) \in H_1$$

and thus, $g^{-1} H_0 g \subseteq H_1$. Similarly, one can show that $g H_1 g^{-1} \subseteq H_0$, implying that H_0 and H_1 are conjugate subgroups of G.

Conversely, suppose $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are covering maps with $H_i = (p_1)_*(\pi_1(\widetilde{X}_i, \widetilde{x}_i))$ and there is $g \in G$ such that $H_2 = g^{-1} H_1 g$. Let γ be a loop in the equivalence class corresponding to g, then this has a lift $\widetilde{\gamma}$ in $\widetilde{X}_1$ and let $\widetilde{y}_1 = \widetilde{\gamma}(1) \in p_1^{-1}(x_0)$.

It is not hard to see that $(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{y}_1)) = H_2$ whence there is a basepoint preserving isomorphism (since the covering spaces are path connected) $f : (\widetilde{X}_1, \widetilde{y}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ thereby proving the theorem. ■

In conclusion, we have proved the following classification theorem.

Theorem 3.17. *Let X be nice and $x_0 \in X$ a chosen basepoint. Then there is a bijection between the set of basepoint preserving isomorphism classes of path connected covering spaces and the set of subgroups of $\pi_1(X, x_0)$.*

On the other hand, if basepoints are ignored, then there is a bijecion between the isomorphism classes of covering spaces of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

3.2.1 Action of π_1 on a fiber

Let $p : \widetilde{X} \to X$ be a covering map and $x_0 \in X$. We shall first define an action of the group $\pi_1(X, x_0)$ on $p^{-1}(x_0)$. For a loop γ based at x_0, define the function $L_\gamma : p^{-1}(x_0) \to p^{-1}(x_0)$ as follows: Choose some $\widetilde{x}_0 \in p^{-1}(x_0)$ and let $\widetilde{\gamma}$ denote the *unique* lift of γ to a path in $\widetilde{X}$ that begins at $\widetilde{x}_0$. Define $L_\gamma(\widetilde{x}_0) := \widetilde{\gamma}(1)$.

First, note that L_γ is a bijection since it has an inverse given by $L_{\overline{\gamma}}$. Now, suppose γ and γ' are path homotopic loops, that is, $[\gamma] = [\gamma']$. Choose some $\widetilde{x}_0 \in p^{-1}(x_0)$. Then, due to Corollary 3.7, the lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are path homotopic too, whence $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$ and thus $L_\gamma = L_{\gamma'}$.

Finally, suppose γ and η are two loops based at x_0 and $\widetilde{x}_0 \in p^{-1}(x_0)$. Let $\widetilde{x}_1 = \widetilde{\gamma}(1)$ and $\widetilde{\eta}$ be the unique lift of η to a path in $\widetilde{X}$ beginning at $\widetilde{x}_1$. Then, $\widetilde{\gamma} * \widetilde{\eta}$ is the *unique* lift of $\gamma * \eta$ to $\widetilde{X}$ whence, $L_{\gamma * \eta} = L_\eta \circ L_\gamma$.

Consider now the map $\Phi : \pi_1(X, x_0) \to \mathfrak{S}\left(p^{-1}(x_0)\right)$ given by $[\gamma] \mapsto L_{\overline{\gamma}} = L_\gamma^{-1}$. Then,

$$\Phi([\gamma] * [\eta]) = L_{\overline{\gamma * \eta}} = L_{\overline{\eta} * \overline{\gamma}} = L_{\overline{\gamma}} \circ L_{\overline{\eta}} = \Phi([\gamma]) \circ \Phi([\eta]),$$

whence Φ is a group homomorphism and defines an *action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$*.

3.3 Deck Transformations and Covering Space Actions

3.3.1 Deck Transformations

Definition 3.18 (Deck Transformations, Normal Coverings). For a covering space $p : \widetilde{X} \to X$, the isomorphisms $f : X \to X$ are called *deck transformations*. These form a group $G(\widetilde{X})$ under composition.

A covering space $p : \widetilde{X} \to X$ is said to be *normal* if for all $x \in X$ and each pair $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$, there is a deck transformation that maps $\widetilde{x} \mapsto \widetilde{x}'$.

Proposition 3.19. *Let $\widetilde{X}$ be connected. Then, $G(\widetilde{X})$ acts freely on $\widetilde{X}$. In particular, a deck transformation, in this case, is completely determined by where it sends a single point.*

Proof. Let $f \in G(\widetilde{X})$ have a fixed point $\widetilde{x}_0$. Then, this is a lift for $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ as seen from the following diagram.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (\widetilde{X}, \widetilde{x}_0) \arrow[ur, "{f, \mathbf{id}}"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

But since f and $\mathbf{id}$ agree at $\widetilde{x}_0$, they must agree everywhere due to Theorem 3.4. ■

Theorem 3.20. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X, and let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X, x_0)$. Then,*

(a) the covering space is normal if and only if H is normal in $\pi_1(X, x_0)$

(b) $G(\widetilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

Proof. Suppose the covering is normal, let $g^{-1}Hg$ be a conjugate of H in $\pi_1(X, x_0)$. Then, there is correspondingly $\widetilde{x}_1 \in p^{-1}(x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = g^{-1}Hg$. Since the covering is normal, there is a deck transformation $f : \widetilde{X} \to \widetilde{X}$ taking $\widetilde{x}_0$ to $\widetilde{x}_1$. From Theorem 3.15, we must have that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$, whereby $g^{-1}Hg = H$ and $H \trianglelefteq \pi_1(X, x_0)$.

Conversely, suppose $H \trianglelefteq \pi_1(X, x_0)$ and let $\widetilde{x}_1 \in p^{-1}(x_0)$. From Theorem 3.17, we have that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ is conjugate to H but since H is normal, the former is equal to H. As a result, from Theorem 3.15, there is a deck transformation taking x_0 to x_1, consequently, the covering space is normal.

Note that given $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$, there is a unique deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$. Now, given some $[\gamma] \in N(H)$, there is a lift $\widetilde{\gamma} : I \to \widetilde{X}$ such that $\widetilde{\gamma}(0) = \widetilde{x}_0$. Define now the function $\phi : N(H) \to G(\widetilde{X})$ by $\phi([\gamma]) = \widetilde{\gamma}(1)$. Let $[\gamma], [\delta] \in N(H)$ with $\sigma = \phi([\gamma])$ and $\tau = \phi([\delta])$. Then, it is not hard to see that $\gamma * \delta$ lifts to $\widetilde{\gamma} * \sigma(\widetilde{\delta})$, which corresponds to the deck transformation $\sigma \circ \tau$, implying that ϕ is a homomorphism. Moreover, ϕ is also surjective, for if there is a deck transformation σ taking $\widetilde{x}_0$ to $\widetilde{x}_1$, then $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = H$. Now, let $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ with $\gamma = p \circ \widetilde{\gamma}$. This implies $[\gamma] \in N(H)$, consequently, $\phi([\gamma]]) = \sigma$.

We now contend that $\ker \phi = H$. Obviously $H \subseteq \ker \phi$. On the other hand, if $[\gamma] \in \ker \phi$, then γ lifts to a loop based at $\widetilde{x}_0$, whereby, $[\gamma] \in H$. The proof is finished by invoking the first isomorphism theorem. ■

3.3.2 Covering Space Actions

Definition 3.21 (Covering Space Action). A *group action* of G on a topological space Y is a homomorphism $\varphi : G \to \mathrm{Aut}_{\mathbf{Top}}(Y)$. A *covering space action* is a group action of G on Y such that for each $y \in Y$, there is a neighborhood U of y such that for all $g_1, g_2 \in G$, $g_1 U \cap g_2 U \neq \varnothing$, if and only if $g_1 = g_2$.

We may rephrase the definition of a covering space action as:

A *covering space action* of G on Y is a group action such that for each $y \in Y$, there is a neighborhood U of y such that for all $g \in G$, $U \cap gU \neq \varnothing$ if and only if $g = 1_G$.

Theorem 3.22. *Let G act on Y through a covering space action.*

(a) The quotient map $p : Y \to Y/G$ given by $p(y) = Gy$ is a normal covering space.[a].

(b) If Y is path connected, then G is the group of deck transformations of the covering space $p : Y \to Y/G$

(c) If Y is path connected and locally path connected, then $G \cong \pi_1(Y/G, Gy_0)/p_*(\pi_1(Y, y_0))$.

[a] Hence the nomenclature

Proof. (a) Let $Gy \in Y/G$. Since G acts through a covering space action, there is a neighborhood U of Y such that the collection $\{gU \mid g \in G\}$ is that of disjoint open sets. Obviously, $V = \bigsqcup_{g\in G} gU$ is a saturated open set, whereby, $p(V)$ is open in Y/G and a neighborhood of Gy. We contend that the restriction $p : U \to p(V)$ is a homeomorphism. Indeed, if $W \subseteq U$ is open, then $p(W) \subseteq p(V)$ is open, since $p(W) = p\left(\bigsqcup_{g\in G} gW\right)$ and the term within the brackets is a saturated open set. This immediately implies that p is a covering map.

Furthermore, for any $g_1y, g_2y \in Gy$, there is the action $g_2g_1^{-1}$ taking g_1y to g_2y whereby, the covering space is normal.

(b) Obviously, each element of G is a deck transformation. On the other hand, if $f : Y \to Y$ is a deck transformation, then for any $y \in Y$, $f(y) \in Gy$, whereby, there is $g \in G$ such that $g(y) = f(y)$. From Proposition 3.19, we have that $g = f$, implying the desired conclusion.

(c) This follows from Theorem 3.20. ■

Example 3.23 (Fundamental group of S^1). Let the additive group $\mathbb{Z}$ act on $\mathbb{R}$ by translations. Then, $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle S^1. The action of $\mathbb{Z}$ is properly discontinuous and $\mathbb{R}$ is simply connected and thus

$$\mathbb{Z} \cong \pi_1(S^1, s_0)/p_*(\pi_1(\mathbb{R}, x_0)) \cong \pi_1(S^1, s_0).$$

Remark 3.3.1. *Similarly, one can obtain the fundamental group of the torus $S^1 \times S^1$ by considering the additive action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{C}$ which is also properly discontinuous. Note that this also gives the torus the structure of a Riemann surface.*

Example 3.24 (Fundamental group of $\mathbb{RP}^n$). Let $n \geq 2$. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on S^n wherein the nontrivial action is given by $x \mapsto -x$. Obviously this is a covering space action, whereby

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^n, x_0)/p_*(\pi_1(S^n, s_0)) \cong \pi_1(\mathbb{RP}^n, x_0)$$

where x_0 is the orbit of s_0 and we're done.

Now, consider the case $n = 1$. We know that $\mathbb{RP}^1$ is homeomorphic to the circle and thus has fundamental group isomorphic to $\mathbb{Z}$.

Chapter 4

Homology

4.1 The Setup

Definition 4.1 (Standard and Singular n-simplices). The standard n-simplex, denoted $\Delta^n \subseteq \mathbb{R}^{n+1}$ is given by

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}.$$

Denote the $n+1$ vertices of Δ^n by $v_0^n, \dots, v_n^n$ where $v_i^n = (0, \dots, 1, \dots, 0)$ in which 1 occurs at the i-th position. Further, define the i-th face map $\mathfrak{f}_i^n : \Delta^n \to \Delta^{n+1}$ for $0 \leq i \leq n+1$, first on the vertices of Δ^n by

$$\mathfrak{f}_i^n(v_j^n) = \begin{cases} v_j^{n+1} & j < i \\ v_{j+1}^{n+1} & j \geq i \end{cases}$$

and then extend linearly to all of Δ^n.

Given a topological space X, a *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$. Denote by $S_n(X)$, the set of all singular n-simplices in X and let $C_n(X)$ denote the *free abelian group* on $S_n(X)$.

Definition 4.2 (The Singular Complex). Let X be a topological space. Define the map $\partial_n : C_n(X) \to C_{n-1}(X)$ by first defining it on $S_n(X)$,

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \mathfrak{f}_i^{n-1}$$

and then extending to all of $C_n(X)$ using the universal property of free modules.

Remark 4.1.1. *One can check that if $i \leq j \leq n$, then $\mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} = \mathfrak{f}_{j+1}^n \circ \mathfrak{f}_i^{n-1}$.*

Proposition 4.3. $\partial_n \circ \partial_{n+1} = 0$ *for $n \geq 1$.*

Proof. It suffices to check this on $S_{n+1}(X)$, the generator of $C_{n+1}(X)$. Indeed,

$$\begin{aligned}
\partial_n \circ \partial_{n+1}(\sigma) &= \sum_{i=0}^{n+1} (-1)^i \partial_n(\sigma \circ \mathfrak{f}_i^n) \\
&= \sum_{i=0}^{n+1} (-1)^i \sum_{j=0}^{n} (-1)^j \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} \\
&= \sum_{i=0}^{n+1} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} + \sum_{i=0}^{n+1} \sum_{j=i}^{n} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} \\
&= \sum_{i=0}^{n+1} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} + \sum_{i=0}^{n} \sum_{j=i}^{n} \sigma \circ f_{j+1}^n \circ f_i^{n-1} = 0.
\end{aligned}$$

■

Definition 4.4 (Singular Homology Groups). For a topological space X, the homology groups corresponding to the singular chain complex $C_\bullet(X)$ are called the *singular homology groups*.

Let X be a topological space. The standard 0-simplex is just the point $x = 1$ in $\mathbb{R}^1$. Thus, $S_0(X)$ can be identified with the underlying set of X, consequently, $C_0(X)$ can be identified with the free abelian group on X. Define the map $\varepsilon : S_0(X) \to \mathbb{Z}$ by $\varepsilon(x) = 1$ for each $x \in X$ and extend this to $C_0(X)$ through the universal property. It is evident that the map $\varepsilon : C_0(X) \to \mathbb{Z}$ is a surjection. Furthermore, $\varepsilon \circ \partial_1 = 0$, and thus, we may augment the singular chain complex as follows:

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

which we denote by $\widetilde{C}_\bullet(X)$ and the corresponding homology groups by $\widetilde{H}_n(X)$ which are called the *reduced homology groups*. Note that $H_n(X) = \widetilde{H}_n(X)$ for $n > 0$ therefore, the only difference observed is in $\widetilde{H}_0(X)$.

4.2 Some Functorial Properties

Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. There is an induced map $f_n : S_n(X) \to S_n(Y)$ given by $\sigma \mapsto f \circ \sigma$. This can be extended to a map $f_n : C_n(X) \to C_n(Y)$ through the universal property of free modules as follows.

$$\begin{tikzcd}
S_n(X) \arrow[r, "f_n"] \arrow[d, hook] & S_n(Y) \arrow[d, hook] \\
C_n(X) \arrow[r, dotted, "\exists! f_n"'] & C_n(Y)
\end{tikzcd}$$

We denote the sequence of maps $\{f_n\}_{n=0}^{\infty}$ by $f_\sharp$.

Proposition 4.5. *Given the setup as above,*

(a) $f_\sharp : C_\bullet(X) \to C_\bullet(Y)$ *is a chain map.*

(b) $g_\sharp \circ f_\sharp = (g \circ f)_\sharp$.

(c) If $\mathbf{id} : X \to X$ *is the identity map, then* $\mathbf{id}_\sharp$ *is a collection of identity maps on* $C_n(X)$ *for each nonnegative integer* n*.*

Proof. (a) We need to show that $\partial^Y_{n+1} \circ f_{n+1} = f_n \circ \partial^X_n : C_{n+1}(X) \to C_n(Y)$. It suffices to check the equality on elements of $S_{n+1}(X)$, owing to the universal property. Indeed, for $\sigma \in S_{n+1}(X)$, we have

$$\partial^Y_{n+1}(f \circ \sigma) = \sum_{i=0}^{n+1} (-1)^i f \circ \sigma \circ \mathfrak{f}^n_i$$

$$f_n \circ \partial^X_{n+1}(\sigma) = f_n \left(\sum_{i=0}^{n+1} (-1)^i \sigma \circ \mathfrak{f}^n_i \right) = \sum_{i=0}^{n+1} (-1)^i f \circ \sigma \circ \mathfrak{f}^n_i.$$

(b) Since $g_n \circ f_n = (g \circ f)_n$ on the elements of $S_n(X)$, the equality must hold on all of $C_n(X)$. We are implicitly using the universal property here.

(c) Trivial. ∎

Since $f_\sharp$ is a chain map, it induces a group homomorphism $H_n(X) \to H_n(Y)$ on the homology groups, which we denote by f_* or $(f_*)_n$. We shall try to avoid the latter for the sake of brevity.

f_* is functorial

.

We shall establish some notation to make our life easier. If $p_0, \ldots, p_k \in \mathbb{R}^k$ are points, then we denote by $[p_0, \ldots, p_k]$ the unique linear map $\tau : \Delta^k \to \mathbb{R}^k$ that maps $v^k_i \mapsto p_i$. In particular, this map is given by

$$\alpha_0 v^k_0 + \cdots + \alpha_k v^k_k \mapsto \alpha_0 p_0 + \cdots + \alpha_k p_k.$$

Now, let $A \subseteq \mathbb{R}^n$ be a convex subset. Given a map $\sigma : A \to X$ and $p_0, \ldots, p_k \in A$, we denote by $\sigma|_{[p_0,\ldots,p_k]}$ the composition $\sigma \circ [p_0, \ldots, p_k]$.

Theorem 4.6. *Let* $f, g : X \to Y$ *be homotopic maps. Then,* $f_* = g_*$*.*

Proof. We have a map $F : X \times I \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. We shall construct a chain homotopy $P : C_\bullet(X) \to C_\bullet(Y)$ between the maps f and g. ∎

chain homotopy. only computation remains

Corollary 4.7. Let $f : X \to Y$ be a homotopy equivalence. Then, f_* is an isomorphism of groups.

Proof. There is a continuous map $g : Y \to X$ such that $g \circ f \simeq \mathbf{id}_X$ and $f \circ g \simeq \mathbf{id}_Y$. Thus, $g_* \circ f_* = \mathbf{id}_*$ and $f_* \circ g_* = \mathbf{id}_*$. The conclusion follows. ∎

Definition 4.8 (Relative Homology Groups). Let X be a topological space and $A \subseteq X$ a subspace. There is a canonical inclusion $\iota_n : C_n(A) \hookrightarrow C_n(X)$. Denote by $C_n(X, A)$ the abelian group $\operatorname{coker} \iota_n$. There is an induced map $\partial_n : \operatorname{coker} \iota_n \to \operatorname{coker} \iota_{n-1}$ giving us a chain complex $C_\bullet(X, A)$. The homology groups corresponding to this chain complex are called *relative homology groups* and denoted by $H_n(X, A)$.

We now have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A) \xrightarrow{\iota} C_\bullet(X) \longrightarrow C_\bullet(X, A) \longrightarrow 0$$

which, due to Theorem 0.13 gives us a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(X, A) \longrightarrow 0$$

4.3 Barycentric Subdivision and Excision

Definition 4.9. Let $\mathscr{U}$ be a collection of subspaces of X whose interiors form an open cover of X and let $C_n^{\mathscr{U}}(X)$ be the subgroup of X generated by singular n-simplices σ whose image is contained in some set in $\mathscr{U}$. Denote the homology groups of this chain complex by $H_n^{\mathscr{U}}(X)$.

Theorem 4.10. *The inclusion $\iota : C_n^{\mathscr{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, $\iota_* : H_n^{\mathscr{U}}(X) \to H_n(X)$ is an isomorphism for all n.*

Proof. The proof of this theorem involves the procecure of *Barycentric Subdivision*. There are four steps involved in carrying this out.
Barycentric Subdivision of Simplices

We perform this subdivision inductively. First, begin with a 0-simplex, which is just a point. There is nothing to do in this case.

Suppose now that $[v_0, \dots, v_n]$ is a simplex and that we have subdivided each face $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ into smaller simplices. Let

$$b = \frac{1}{n+1} \sum_{i=0}^{n+1} v_i$$

denote the barycenter of $[v_0, \dots, v_n]$. Then, construct the simplices

$$[b, w_0, \dots, w_{n-1}]$$

where $[w_0, \dots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of some face $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ of $[v_0, \dots, v_n]$.

We now claim that the diameter of any simplex in the subdivision of $[v_0, \dots, v_n]$ is atmost $\frac{n}{n+1}$ times the diameter of $[v_0, \dots, v_n]$. First, we establish an inequality. Let $v \in [v_0, \dots, v_n]$ and $\sum_i t_i v_i$ be another point. Then,

$$\left| v - \sum_i t_i v_i \right| = \sum_i \left| t_i (v - v_i) \right| \le \sum_i t_i |v - v_i| \le \max_i |v - v_i|.$$

Let w_i, w_j be two vertices of a simplex in the barycentric subdivision of $[v_0, \dots, v_n]$. If none of them is the barycenter b of $[v_0, \dots, v_n]$, then,

$$|w_i - w_j| \le \frac{n-1}{n} \operatorname{diam}[w_0, \dots, w_{n-1}] \le \frac{n}{n+1} \operatorname{diam}[b, w_0, \dots, w_{n-1}] \le \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n].$$

On the other hand, suppose $w_j = b$ and let b_i denote the barycenter of $[v_0, \dots, \widehat{v_k}, \dots, v_n]$, consequently,

$$b = \frac{n}{n+1} b_k + \frac{1}{n+1} v_k.$$

In particular, this means that

$$|b - v_k| \le \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n].$$

As a result, $|b - w_i| \le \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n]$.
Barycentric Subdivision of Linear Chains

Let $Y \subseteq \mathbb{R}^N$ be a convex set. Let $LC_n(Y)$ denote the free abelian group on the set of all linear maps $\Delta^n \to Y$. Note that the restriction of the boundary map $\partial : C_n(Y) \to C_{n-1}(Y)$ takes $LC_n(Y)$ to $LC_{n-1}(Y)$. Therefore, we have chain complex of linear chains, which we denote by $LC_\bullet(Y)$. Augment this chain complex by setting $LC_{-1}(Y)$ as $\mathbb{Z}$, generated by the empty simplex $[\varnothing]$. The boundary map is given by $\partial([v_0]) = [\varnothing]$.

For each $b \in Y$, denote again by $b : LC_n(Y) \to LC_{n+1}(Y)$, the group homomorphism $b([w_0, \dots, w_n]) = [b, w_0, \dots, w_n]$ and extend linearly. Then,

$$\partial(b([w_0, \dots, w_n])) = [w_0, \dots, w_n] - \sum_{i=0}^{n} (-1)^i [b, w_0, \dots, \widehat{w}_i, \dots, w_n] = [w_0, \dots, w_n] - b(\partial([w_0, \dots, w_n])).$$

Extending linearly, we have that for all $\alpha \in LC_n(Y)$,

$$\partial(b(\alpha)) + b(\partial(\alpha)) = \alpha \iff \partial \circ b + b \circ \partial = \mathbf{id}.$$

In particular, this means that b is a chain homotopy between the identity mamp and the zero map on the augmented chain complex $LC_\bullet(Y)$.

Next, we define a subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$. For any linear map $\lambda : \Delta^n \to Y$, denote by b_λ, the image of the barycenter of Δ^n under λ. Define the operator S inductively as

$$S(\lambda) = b_\lambda(S(\partial\lambda)).$$

The base case of the induction is given by $S([\varnothing]) = [\varnothing]$. Consider the map $\lambda : \Delta^0 \to Y$ given by $[w_0]$. Then,

$$S([w_0]) = w_0(S([\varnothing])) = [w_0]$$

and thus S is the identity map on both $LC_0(Y)$ and $LC_{-1}(Y)$.

We contend that S is a chain map, that is, $\partial S = S\partial$. This is easily verified in the case of $LC_0(Y)$. We prove it for all $LC_n(Y)$ by induction on n. Indeed, let $\lambda \in LC_n(Y)$ for $n \geq 1$, then

$$\begin{aligned}
\partial(S(\lambda)) &= \partial(b_\lambda(S(\partial\lambda))) \\
&= S(\partial\lambda) - b_\lambda(\partial(S(\partial\lambda))) \\
&= S(\partial\lambda) - b_\lambda(\partial(\partial(S\lambda))) \\
&= S(\partial\lambda),
\end{aligned}$$

which completes the induction.

Now that we have a chain map S, we shall show that it is chain homotopic to the identity map by defining a chain homotopy $T : LC_n(Y) \to LC_{n+1}(Y)$. Begin by defining $T : LC_{-1}(Y) \to LC_0(Y)$ as the zero map. Now, for a linear map $\lambda : \Delta^n \to Y$, define

$$T(\lambda) := b_\lambda(\lambda - T(\partial\lambda)).$$

We need to verify that $\partial T + T\partial = \mathbf{id} - S$, in order to show that T is indeed a chain homotopy. Again, we show this by induction on n. The base case on $LC_{-1}(Y)$ is trivial. Now, suppose $\lambda \in LC_n(Y)$ for $n \geq 0$. Then,

$$\begin{aligned}
\partial(T(\lambda)) &= \partial(b_\lambda(\lambda - T(\partial\lambda))) \\
&= \lambda - T(\partial\lambda) - b_\lambda(\partial\lambda - \partial(T(\partial\lambda))) \\
&= \lambda - T(\partial\lambda) - b_\lambda\left(S(\partial\lambda) + T(\partial(\partial\lambda))\right) \\
&= \lambda - T(\partial\lambda) - b_\lambda(S(\partial\lambda)) \\
&= \lambda - T(\partial\lambda) - S(\lambda),
\end{aligned}$$

which completes the induction.

Barycentric Subdivision of General Chains

Define $S : C_(X) \to C_n(X)$ by

$$S(\sigma) = \sigma_\sharp(S(\Delta^n)),$$

where $\sigma_\sharp : C_n(\Delta^n) \to C_n(X)$. Here, it is important to note that $S(\Delta^n)$ denotes $S(\mathbf{id}_{\Delta^n})$ where $S : LC_n(\Delta^n) \to LC_n(\Delta^n)$ is the barycentric subdivision operator on linear chains as we have seen in the previous part. This is obviously valid usage since $\mathbf{id}$ is a linear map from Δ^n to Δ^n.

First, we must show that S is a chain map. Indeed, for any $\sigma : \Delta^n \to X$,

$$\begin{aligned}\partial(S(\sigma)) &= \partial(\sigma_\sharp(S(\Delta^n))) = \sigma_\sharp(\partial(S(\Delta^n))) = \sigma_\sharp(S(\partial(\Delta^n))) \\ &= \sigma_\sharp\left(S\left(\sum_{i=0}^n (-1)^i \Delta_i^n\right)\right) \qquad \text{where } \Delta_i^n \text{ denotes the } i\text{-th face of } \Delta^n \\ &= \sum_{i=0}^n (-1)^i \sigma_\sharp(S(\Delta_i^n)) \\ &= \sum_{i=0}^n (-1)^i S(\sigma|_{\Delta_i^n}) \\ &= S\left(\sum_{i=0}^n (-1)^i \sigma_{\Delta_i^n}\right) = S(\partial\sigma).\end{aligned}$$

Now that we have shown that S is a chain map, we must construct a chain homotopy between **id** and S, which is done by extending the definition of the operator T as we had seen in the previous part to $T : C_n(X) \to C_{n+1}(X)$, by

$$T(\sigma) = \sigma_\sharp(T(\Delta^n)),$$

where Δ^n again denotes the identity map $\mathbf{id} : \Delta^n \to \Delta^n$. That this is a chain homotopy is seen by computing:

$$\begin{aligned}\partial(T(\sigma)) &= \partial(\sigma_\sharp(T(\Delta^n))) = \sigma_\sharp(\partial(T(\Delta^n))) \\ &= \sigma_\sharp(\Delta^n - S(\Delta^n) - T(\partial(\Delta^n))) \\ &= \sigma - S(\sigma) - \sigma_\sharp(T(\partial\Delta^n)) \\ &= \sigma - S(\sigma) - T(\partial\sigma).\end{aligned}$$

This shows that T is a chain map.

Iterated Barycentric Subdivision

Let $S^m : C_n(X) \to C_n(X)$ denote the iteration $\underbrace{S \circ \cdots \circ S}_{n \text{ times}}$ and let $D_m : C_n(X) \to C_{n+1}(X)$ be given by

$$D_m = \sum_{i=0}^{m-1} T \circ S^i.$$

After a short computation, one immediately sees that D_m is a chain homotopy between S^m and **id**. Now, using the Lebesgue Number Lemma, it is not hard to argue that for each $\sigma : \Delta^n \to X$, there is an m such that $S^m(\sigma)$ lies in $C_n^{\mathscr{U}}(X)$. Denote by $m(\sigma)$ the smallest positive integer m such that this $S^m(\sigma)$ is in $C_n^{\mathscr{U}}(X)$.

Define the map $D : C_n(X) \to C_{n+1}(X)$ by setting $D(\sigma) = D_{m(\sigma)}(\sigma)$ and extend linearly. Consider the map $\rho : C_n(X) \to C_n(X)$ given by

$$\rho = \mathbf{id} - (\partial D + D\partial).$$

We contend that ρ is a chain map. Indeed,

$$\partial\rho(\sigma) = \partial(\sigma) - \partial(D(\partial\sigma)) \quad \text{and} \quad \rho(\partial\sigma) = \partial(\sigma) - \partial(D(\partial\sigma)).$$

Next, we claim that the image of σ under ρ lies in $C_n^{\mathscr{U}}(X)$. Indeed,

$$\begin{aligned}\rho(\sigma) &= \sigma - \partial(D(\sigma)) - D(\partial(\sigma)) \\ &= \sigma - \partial(D_{m(\sigma)}(\sigma)) - D(\partial(\sigma)) \\ &= S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma).\end{aligned}$$

By definition, $S^{m(\sigma)}(\sigma) \in C_n^{\mathscr{U}}(X)$. Note that

$$D_{m(\sigma)}(\partial\sigma) = \sum_{i=0}^n (-1)^i D_{m(\sigma)}(\sigma_i)$$

and

$$D(\partial\sigma) = \sum_{i=0}^{n}(-1)^i D_{m(\sigma_i)}(\sigma_i),$$

where σ_i is the restriction of σ to the face $[v_0, \dots, \widehat{v}_i, \dots, v_n]$. In particular, this means that $m(\sigma_i) \le m(\sigma)$ for each $0 \le i \le n$.

We have

$$D_{m(\sigma)}(\sigma_i) - D_{m(\sigma_i)}(\sigma_i) = \sum_{j > m(\sigma_i)} T \circ S^j(\sigma) \in C_n^{\mathscr{U}}(X),$$

since T takes $C_{n-1}^{\mathscr{U}}(X) \to C_n^{\mathscr{U}}(X)$. In particular, this means that $\rho(\sigma) \in C_n^{\mathscr{U}}(X)$.

We may now view ρ as a chain map $C_\bullet(X) \to C_\bullet^{\mathscr{U}}(X)$. Recall that $\iota : C_n^{\mathscr{U}}(X) \to C_n(X)$ denotes the inclusion. As we have seen earlier, $\partial D + D\partial = \mathbf{id} - \iota\rho$ and obviously $\rho\iota = 1$. Thus, ρ and ι are quasi-isomorphisms, which completes the proof. ∎

Theorem 4.11 (Excision Theorem). *Let $Z \subseteq A \subseteq X$ be such that $\overline{Z} \subseteq \operatorname{Int} A$. Then, the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X\backslash Z, A\backslash Z) \xrightarrow{\sim} H_n(X, A)$.*

Equivalently phrased, if A, B are subspaces of X whose interiors cover X, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $H_n(B, A\cap B) \xrightarrow{\sim} H_n(X, A)$ for all n.

Note that the equivalence can be seen by setting $B = X\backslash Z$. Then, the condition $\overline{Z} \subseteq \operatorname{Int} A$ is equivalent to saying $X = \operatorname{Int} A \cup \operatorname{Int} B$.

Proof. Let $\mathscr{U} = \{A, B\}$, which is a cover whose interior is also a cover. We use $C_n(A+B)$ to denote $C_n^{\mathscr{U}}(X)$, which is a more suggestive notation. Recall from the previous proof, the operators D, ρ and ι. We had $\partial D + D\partial = \mathbf{id} - \iota\rho$. Note that all of ∂, D, ρ take $C_n(A)$ to $C_n(A)$ and thus, they induce quotient maps

$$\begin{aligned} \widetilde{D} &: C_n(X)/C_n(A) \to C_{n+1}(X)/C_{n+1}(A) \\ \widetilde{\rho} &: C_n(X)/C_n(A) \to C_n(A+B)/C_n(A) \\ \widetilde{\iota} &: C_n(A+B)/C_n(A) \to C_n(X)/C_n(A). \end{aligned}$$

Note that the equality $\partial\widetilde{D} + \widetilde{D}\partial = \mathbf{id} - \widetilde{\iota}\widetilde{\rho}$. In particular, $\widetilde{\iota}$ is a quasi-isomorphism. In particular, $H_n(A + B, A) \cong H_n(X, A)$.

Consider now the following diagram.

$$\begin{tikzcd} C_n(B) \arrow[r, hook] \arrow[d, two heads] & C_n(A+B) \arrow[r, two heads] & C_n(A+B)/C_n(A) \\ C_n(B)/C_n(A\cap B) \arrow[urr, dotted] & & \end{tikzcd}$$

From the first isomorphism theorem, it is evident that the induced map is an isomorphism of abelian groups and thus induces an isomorphism on the homology groups. This, put together with the previous conclusion gives us

$$H_n(B, A\cap B) \cong H_n(A+B, A) \cong H_n(X, A).$$ ∎

Definition 4.12. A pair (X, A) is said to be *locally retractive*, if there is a neighborhood V of A such that A is a deformation retract of V.

Lemma 4.13. *For a locally retractive pair* (X, A)*, the quotient map* $q : (X, A) \to (X/A, A/A)$ *induces isomorphisms* $q_* : H_n(X, A) \to H_n(X/A, A/A)$*. Recall further that* $H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$*.*

Proof. ■

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