Tutorials

Swayam Chube
Problems By: Prof. Ronnie Sebastian

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1 Tutorial 1

1. Since $\emptyset, X \in \mathcal{T}_i$ for all i, they are also elements of $\mathcal{T}$. Let $\{A_\alpha\}$ be a collection of open subsets in $\mathcal{T}$. Then, for all $i \in I$, $\{A_\alpha\} \subseteq \mathcal{T}$, therefore $\bigcup_\alpha A_\alpha \in \mathcal{T}_i$ for all $i \in I$ and therefore is also an element of $\mathcal{T}$. Finally, let $\{A_1, \ldots, A_n\}$ be a finite collection of elements in $\mathcal{T}$, then by a similar argument, $\bigcup_{i=1}^n A_i \in \mathcal{T}$ and therefore, $\mathcal{T}$ is a topology.

   Let $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\emptyset, \{a, b\}, \{c, d\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{a, c\}, \{b, d\}, X\}$. But $\{a, b, c\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$, therefore, it is not a topology on X.

2. In the first case, existence is guaranteed by Zorn's lemma. As for uniqueness, suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two such distinct topologies, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is properly contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ and also contains $\mathcal{T}_i$ for all $i \in I$, contradicting the minimality of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

   The second case has been answered in the previous problem.

3. Trivial.

4.

5.

6.

7. Just consider $\bigcup_{x \in A} U_x$.