Set Theory

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Chapter 1

The Zermelo-Fraenkel Axioms

1.1 Axioms of Set Theory

We shall discuss Zermelo-Fraenkel Set Theory, which is a first order theory, with signature $\mathsf{ZF} = (\varnothing, \{\in\})$. That is, there are no function symbols and the only predicate is the "belongs to" relation.

ZF0 (Nonempty Domain) There is at least one set.

$$\exists x (x = x)$$

This axiom is redundunt since **ZF7** guarantees the existence of an infinite set and thus the domain of discourse must be nonempty.

ZF1 (Extensionality) Informally speaking, a set is determined uniquely by its elements.

$$\forall x \forall y (\forall z (z \in x \iff z \in y) \implies x = y)$$

ZF2 (Foundation/Regularity) This states that any nonempty set contains an element that is disjoint from it.

$$\forall x \left[\exists y (y \in x) \implies \exists y (y \in x \land \neg \exists z (z \in x \land z \in y))\right]$$

ZF3 (Comprehension) Informally speaking, this axiom allows us to define sets in the set-builder notation. Let ϕ be a valid first order formula with free variables $w_1, \ldots, w_n, x, z$. Then

$$\forall z \forall w_1, \ldots, w_n \exists y \forall x \left(x \in y \iff x \in z \land \phi\right)$$

Notice how this is the same as writing

$$y = \{x \in z \mid \phi\}$$

ZF4 (Pairing) Informally, this states that given two sets x and y, there is a set $z = \{x, y\}$.

$$\forall x \forall y \exists z \forall w (w \in z \iff (w = x \lor w = y))$$

ZF5 (Union) This axiom allows us to take a union of a collection of sets.

$$\forall \mathscr{F} \exists A \forall y (x \in y \land y \in \mathscr{F} \implies x \in A)$$

ZF6 (Replacement Scheme) Let ϕ be a valid formula without Y as a free variable. Then,

$$\forall A (\forall x \in A \ \exists! y \ \phi(x, y) \implies \exists Y \ \forall x \in A \ \exists y \in Y \phi(x, y))$$

Informally speaking, this allows us to replace the elements of a set to obtain a new set.

ZF7 (Infinity) There is an infinite inductive set.

$$\exists x (\varnothing \in x \land \forall y \in x (S(y) \in x))$$

ZF8 (Power Set) Every set has a set containing all its subsets. It is important to note that this need not be **the** power set.

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

We have been a bit sloppy in stating the axioms. Notice that our signature does not contain a predicate $\subseteq$ or the successor function S, neither do we know, a priori, of the existence of **the** empty set.

To define the formula $\subseteq (x, y)$, use

$$\subseteq (x, y) := \forall z (z \in x \implies z \in y)$$

As for the successor function, given any set x, using **ZF4**, there is a set $y = \{x\}$. Using **ZF5**, we may define $S(y) := x \cup y$. Finally, using **ZF0** and **ZF3**, we know of the existence of the empty set as

$$\exists x (x = x \wedge \exists y \forall z (z \in x \iff z \in y \wedge z \neq z))$$

Further, due to **ZF1**, the empty set is unique.

1.2 Consequences of the Axioms

Theorem 1.1. *There is no universal set. That is,*

$$\neg \exists z \forall x (x \in z)$$

Proof. If there were a universal set, then using **ZF3**, we may construct the set $y = \{x \in z \mid x \notin x\}$. Then, it is not hard to argue that

$$y \in y \iff y \notin y,$$

a contradiction. ■

Definition 1.2 (Power Set). Let x be a set. Due to **ZF8**, there is a set z containing all the subsets of x. Using Comprehension, we may construct

$$\mathscr{P}(x) := \{y \in z \mid y \subseteq x\}.$$

This is known as the **power set** of x.

Definition 1.3. Let $\mathscr{F}$ denote a set. Let A be a set satisfying **ZF5**. Define

$$\bigcup \mathscr{F} := \{x \in A \mid \exists y \in \mathscr{F} (x \in y)\}$$

and

$$\bigcap \mathscr{F} := \{x \in A \mid \forall y \in \mathscr{F} (x \in y)\}.$$

1.3 Relations, Functions and Well Ordering

Definition 1.4 (Ordered Pair). For sets x, y, define the ordered pair $\langle x, y \rangle$ by

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

The set on the right is constructed by using the pairing axiom twice.

Definition 1.5 (Cartesian Product). Let A and B be sets. Using Replacement, we may define, for each $y \in B$,

$$A \times \{y\} := \{z \mid \exists x \in A(z = \langle x, y\rangle)\}.$$

Again, by Replacement, define the set

$$\mathscr{F} := \{z \mid \exists y \in B(z = A \times \{y\})\}.$$

Finally, define

$$A \times B := \bigcup \mathscr{F}.$$

Definition 1.6 (Relation, Function). Let A be a set. A relation R on A is a subset of $A \times A$. Define the domain and range of a relation as

$$\mathrm{dom}(R) := \{x \in A \mid \exists y(\langle x, y\rangle \in R)\} \qquad \mathrm{ran}(R) := \{y \mid \exists x(\langle x, y\rangle \in R)\}.$$

We write xRy to denote $\langle x, y\rangle \in R$.

A relation f is said to be a function if

$$\forall x \in \mathrm{dom}(f) \exists! y \in \mathrm{ran}(f)(\langle x, y\rangle \in f).$$

We use $f : A \to B$ to denote a function f with $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.

Definition 1.7 (Total Ordering, Well Ordering). A *total ordering* is a pair $\langle A, R\rangle$ where A is a set and R is a relation that is irreflexive, transitive and satisfies trichotomy.

We say R *well-orders* A if $\langle A, R\rangle$ is a total ordering and every non empty subset of A has an R-least element.

We use $\mathrm{pred}(A, x, R)$ to denote the set $\{y \in A \mid yRx\}$.

Lemma 1.8. *Let $\langle A, R\rangle$ be a well-ordering. Then for all $x \in A$, $\langle A, R\rangle \not\cong \langle \mathrm{pred}(A, x, R), R\rangle$.*

Proof. Suppose $\langle A, R\rangle \cong \langle \mathrm{pred}(A, x, R), R\rangle$ and let $f : A \to \mathrm{pred}(A, x, R)$ be the order isomorphism. Let x be the R-least element of the set

$$\{y \in A \mid f(y) \neq y\},$$

which obviously exists since the aforementioned set is nonempty. If $xRf(x)$, there is some $y \in A$ with yRx and $f(y) = x \neq y$ a contradiction to the choice of x. On the other hand, if $f(x)Rx$, then $f(f(x)) \neq f(x)$ since f is injective, a contradiction to the choice of x. This completes the proof. ∎

Theorem 1.9. *Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be two well-orderings. Then exactly one of the following holds:*

(a) $\langle A, R\rangle \cong \langle B, S\rangle$.

(b) $\exists y \in B\, (\langle A, R\rangle \cong \langle \mathrm{pred}(B, y, S), S\rangle)$.

(c) $\exists x \in A\, (\langle \mathrm{pred}(A, x, R), R\rangle \cong \langle B, S\rangle)$.

Proof. Let

$$f := \{\langle v, w\rangle \mid v \in A,\ w \in B,\ \langle \mathrm{pred}(A, v, R), R\rangle \cong \langle \mathrm{pred}(B, w, S), S\rangle\}.$$

Due to the preceeding lemma, if $\langle v_1, w\rangle, \langle v_2, w\rangle \in f$, then $v_1 = v_2$. Similarly, if $\langle v, w_1\rangle, \langle v, w_2\rangle \in f$, then $w_1 = w_2$. Hence, f is an injective function.

It is not hard to argue that f is an order isomorphism from an initial segment of A to an initial segment of B. Both these segments may not be proper else we could find another isomorphism from an initial segment of A to an initial segment of B by extending one of the isomorphisms in f. This completes the proof. ■

Chapter 2

Ordinal Numbers

2.1 Transitive Sets

Definition 2.1. A set x is said to be *transitive* if

$$\forall y \forall z (z \in y \land y \in x \implies z \in x).$$

Proposition 2.2. *A set x is transitive if and only if*

$$\forall y (y \in x \implies y \subseteq x).$$

Proof. Suppose x is transitive and $y \in x$. Since for all $z \in y$, $z \in x$, we must have $y \subseteq x$. The converse is trivial. ■

Proposition 2.3. *If x is a transitive set, then so is $x \cup \{x\}$.*

Proof. ■

Proposition 2.4. *If x is a transitive set, then so is $\mathscr{P}(x)$.*

Proof. ■

Proposition 2.5. *If $\mathscr{F}$ is a family of transitive sets, then so is $\bigcup \mathscr{F}$.*

Proof. ■

Proposition 2.6. *If x is a transitive set, then so is every $z \in x$.*

Proof. ■

2.2 Ordinals

Definition 2.7 (Ordinal). A set x is said to be an *ordinal* if it is transitive and well ordered by $\in$. That is, the pair $\langle x, \in_x \rangle$ is a well ordering, where

$$\in_x := \{\langle v, w \rangle \in x \times x \mid v \in w\}.$$

Theorem 2.8 (Properties of Ordinals).

(a) If x is an ordinal and $y \in x$, then y is an ordinal and $y = \operatorname{pred}(x, y)$.

(b) If $x \cong y$ are ordinals, then $x = y$.

(c) If x, y are ordinals, then exactly one of the following is true: $x = y$, $x \in y$ or $y \in x$.

(d) If C is a nonempty set of ordinals, then $\exists x \in C\ \forall y \in C (x \in y \vee x = y)$. That is, every nonempty set of ordinals has a minimum element.

Proof. (a) Due to Proposition 2.6, y is a transitive and owing to it being the subset of a well ordered set, it is well ordered too, hence an ordinal.

(b) Let $f : x \to y$ be an isomorphism. Let

$$A := \{z \in x \mid f(z) \neq z\}.$$

Suppose A is nonempty, then it has a least element, say $w \in x$. If $v \in w$, then $v = f(v) \in f(w)$ whence $w \subseteq f(w)$. On the other hand, if $v \in f(w)$, then there is some $u \in w$ such that $v = f(u) = u \in w$ and thus $f(w) = w$, a contradiction.

(c) Follows from Theorem 1.9.

(d) First note that it suffices to find $x \in C$ with $x \cap C = \varnothing$ for if $y \in C$ is another ordinal with $x \neq y$, then $y \notin x$ lest $x \cap C \neq \varnothing$.

Pick any $x \in C$. If $x \cap C = \varnothing$, then we are done. Else, let $x' \in x \cap C$ be the $\in$-least element. It is not hard to argue that $x' \cap C = \varnothing$ and we are done.

■

Lemma 2.9. *If A is a transitive set of ordinals, then A is an ordinal.*

Proof. We must first show that the membership relation $\in_A$ is a linear order. This follows from Theorem 2.8 (c) and the fact that A is a transitive set. Lastly, to see that A is well ordered, simply invoke Theorem 2.8 (d). ■

Theorem 2.10. *If $\langle A, R \rangle$ is a well ordering, then there is a unique ordinal C such that $\langle A, R \rangle \cong C$.*

Proof. Let

$$B := \{a \in A \mid \exists x_a (x_a \text{ is an ordinal } \wedge \langle \operatorname{pred}(A, a, R), R \rangle \cong x_a)\},$$
$$f := \{\langle b, x_b \rangle \mid b \in B\}.$$

First, note that for all $b \in B$, x_b, since it exists must be unique and thus f is a well defined function with $\operatorname{dom}(f) = B$.

Let $C = \operatorname{ran}(f)$. We contend that C is an ordinal. Let $y \in x \in C$ and $a \in B$ be such that $g : \operatorname{pred}(A, a, R) \to x$ is an isomorphism. Then, there is some $b \in \operatorname{pred}(A, a, R)$ with $g(b) = y$. It is not hard to see that the restriction $g : \operatorname{pred}(A, b, R) \to y$ is an isomorphism whence $y \in C$ and thus C is an ordinal due to the preceeding lemma.

The function $f : B \to C$ is obviously a surjection. We contend that it is an isomorphism. Indeed, let $a, b \in B$ with aRb and $g : \operatorname{pred}(A, b, R) \to x_b$ be *the* isomorphism. If $y = g(a)$, then the restriction $g : \operatorname{pred}(A, a, R) \to y$ is an isomorphism whence $f(a) = y \in x = f(b)$ and f is an order isomorphism.

Suppose $B \neq A$. Let $b \in A \backslash B$ be the R-least element. Then, $\operatorname{pred}(A, b, R) \subseteq B$. Now suppose $B \neq \operatorname{pred}(A, b, R)$, consequently, there is some $b' \in B \backslash \operatorname{pred}(A, b, R)$, then bRb' and if there is an order isomorphism from $\operatorname{pred}(A, b', R)$ to some ordinal x, then there must be one from $\operatorname{pred}(A, b, R)$ as we have argued earlier, a contradiction.

Thus, either $B = A$ or $B = \operatorname{pred}(A, b, R)$ for some $b \in A$. In the latter case, the function f is an order isomorphism between $\operatorname{pred}(A, b, R)$ and an ordinal C whence $b \in B$, a contradiction. Thus $B = A$ and the proof is complete. ■

Definition 2.11. If $\langle A, R \rangle$ is a well ordering, then $\operatorname{type}(A, R)$ is the unique ordinal C such that $\langle A, R \rangle \cong C$.