MA5106: Fourier Analysis

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January 24, 2024

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Chapter 1

Lecture 1

The origins of Fourier analysis lie in solving the heat equation:

$$\Delta u = \partial_t u$$

where Δ denotes the Laplacian.

In order to solve this, Fourier believed for a long time that one could expand a function as a series

$$f \sim \sum_k a_k \sin kx + \sum b_k \cos kx.$$

This is not true. In 1876, Paul Du Bois-Reymond gave an example of a continuous function whose Fourier series does not converge. But in 1966, Carleson showed that given an L^2 function on $[0,1]$, the points at which the Fourier series does not converge has measure 0.

There are many applications to PDEs, in solving the

Laplace Equation: $\Delta u = 0$,

Heat Equation: $\partial_t u = \Delta u$,

Wave Equation: $\partial_{tt} u = \Delta u$.

Definition 1.1 (Fourier Series). Given $f \in L^1[a,b]$, its k-th *Fourier coefficient* is defined as

$$\widehat{f}(k) := \frac{1}{L} \int_a^b f(x) \exp\left(-\frac{2\pi i k}{L} x\right)\, dx.$$

where $L = b - a$.

The *Fourier series* of f is given formally by

$$f \sim \sum_{k \in \mathbb{Z}} \widehat{f}(k) \exp\left(\frac{2\pi i k}{L} x\right).$$

The question is whether

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \widehat{f}(k) \exp\left(\frac{2\pi i k}{L} x\right) = f(x)$$

in the following cases:

- if $f \in L^1[a,b]$. Here we cannot expect pointwise convergence because one can just change the value of f at a single point without affecting its Fourier series.

- if $f \in C[a,b]$. This is not true because of an example by Paul Du Bois-Reymond.
- if $f \in C^1[a,b]$ then this is true.
- if $f \in L^2[a,b]$, then there may not be pointwise convergence but there is convergence in the L^2-norm.

There are notions of convergence other than pointwise and uniform. For example Cesàro and Abel. Fejér had proved that for continuous functions, the Cesàro sums converge uniformly to the function, whatever that means.

Example 1.2. Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x$. Then,

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \exp(-ikx)\, dx = \begin{cases} 0 & k = 0 \\ \frac{(-1)^k i}{k} & k \neq 0. \end{cases}$$

The Fourier series is then given by

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} \frac{\sin kx}{k}.$$

Chapter 2

Lecture 2

2.1 Functions on the Unit Circle

Denote

$$S^1 := \{z \in \mathbb{C} \colon |z| = 1\},$$

the unit circle. We parametrize points on the circle as $e^{i\theta}$.

Given a function $F : S^1 \to \mathbb{C}$, using the above remark, we can define a function $f : [-\pi \to \pi] \to \mathbb{C}$ by

$$f(\theta) = F(e^{i\theta}).$$

The continuity and differentiability properties of F correspond to those of f.

Conversely, given a function on an interval on the real line that agree on the endpoints, we can simply push it to the unit circle using something similar. Indeed, if $f : \mathbb{R} \to \mathbb{C}$ is a periodic function on $\mathbb{R}$ of period T, we define a function $F : S^1 \to \mathbb{C}$ by

$$F\left(\exp\left(\frac{2\pi i}{T}\theta\right)\right) = f(\theta).$$

Example 2.1 (Dirichlet Kernel). We define

$$D_N(x) := \sum_{n=-N}^{N} e^{ikx}$$

in $[-\pi \to \pi]$. Simple manipulation shows

$$D_N(x) = \begin{cases} 2N+1 & x = 0 \\ \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} & x \neq 0. \end{cases}$$

This is obviously a smooth function on $[-\pi, \pi]$ and represents a smooth function on the circle because it is a trigonometric polynomial.

Example 2.2 (Poisson Kernel). We define

$$P(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

where $0 \le r < 1$ and $-\pi \le \theta \le \pi$. This obviously converges due to the comparison test. A closed form for this is

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

This is handy in solving the Dirichlet Problem on the unit disk as is seen in a complex analysis course.

2.2 Convergence of Fourier Series I

Let $f \in L^1(S^1)$. Define

$$S_N(f)(\theta) = \sum_{k=-N}^{N} \widehat{f}(k) e^{-ik\theta},$$

the partial sums of the Fourier series.

Theorem 2.3 (Uniqueness of Fourier Series). *Suppose $f \in L^1(S^1)$ and $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$. Then, $f(\omega) = 0$ for all points of continuity ω.*

Proof. Without loss of generality, suppose that $\omega = 0$, f is a real valued function on $[-\pi, \pi]$ and $f(0) > 0$. Since $\cos^k(x)$ is a polynomial in e^{imx}'s, the integral $\int_{-\pi}^{\pi} f(x)\cos^m x \, dx = 0$ for all $m \ge 0$.

Since f is continuous at 0, we can pick a $\pi/2 > \delta > 0$ such that $f(x) \ge f(0)/2$ in $[-\delta, \delta]$. Let $\varepsilon > 0$ be such that $\cos\delta = 1 - 2\varepsilon$. Then, $|\varepsilon + \cos\theta| \le 1 - \varepsilon$ on $[-\pi,\pi] \backslash [-\delta, \delta]$. Now, choose $\eta < \delta$ such that $\varepsilon + \cos\theta \ge 1 + \varepsilon/2$ on $[-\eta, \eta]$. Let $p(\theta) = \varepsilon + \cos\theta$ on $[-\pi, \pi]$.

We have

$$0 = \int_{-\pi}^{\pi} f(\theta)p(\theta)^n \, d\theta = \int_{|\theta| \le \delta} f(\theta)p(\theta)^n \, d\theta + \int_{\delta \le |\theta| \le \pi} f(\theta)p(\theta)^n \, d\theta,$$

whence

$$\left| \int_{\delta \le |\theta| \le \pi} f(\theta)p(\theta)^n \, d\theta \right| = \left| \int_{|\theta| \le \delta} f(\theta)p(\theta)^n \, d\theta \right|.$$

The left hand side is bounded above by

$$\int_{\delta \le |\theta| \le \pi} |f(\theta)||p(\theta)|^n \, d\theta \le (1-\varepsilon)^n \|f\|_1.$$

while the right hand side is bounded below by

$$\delta f(0) \left(1 + \frac{\varepsilon}{2}\right)^n.$$

It is not hard to see that this is not possible for sufficiently large n.

Next, we must argue this for complex valued functions f. Indeed, one can capture the real and complex parts as $(f + \overline{f})/2$ and $(f - \overline{f})/2i$, and the conclusion would follow. ■

Proposition 2.4. *Let $f \in C(S^1)$. Suppose $\sum_{k \in \mathbb{Z}} |\widehat{f}(k)|$ converges. Then the Fourier series of f converges uniformly to f. That is, $S_n(f) \rightrightarrows f$.*

Proof. Using the triangle inequality, we have

$$|S_n(f)(\theta) - S_m(f)(\theta)| \le \sum_{m < |k| \le n} |\widehat{f}(k)|,$$

whence, $\{S_n(f)\}$ is a Cauchy sequence in $C(S^1)$ and hence, the partial sums converge to some continuous function F on the circle.

It remains to show that $F = f$. Note that the Fourier coefficients of F and f are the same. Indeed, due to uniform convergence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)e^{-ik\theta}\,d\theta = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} S_n(f)(\theta)e^{-ik\theta}\,d\theta = \widehat{f}(k).$$

The conclusion follows from the fact that if a continuous function has all Fourier coefficients as 0, then it must be the zero function. ■

Chapter 3

Lecture 3

Proposition 3.1. *Let $f \in C^2(S^1)$. Then, there is a constant $C > 0$ such that*

$$|\widehat{f}(n)| \leq \frac{C}{n^2}$$

for all $n \in \mathbb{Z}\backslash\{0\}$.

Proof. This is a standard integration by parts application. Indeed,

$$\begin{aligned} 2\pi\widehat{f}(n) &= \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta \\ &= -(-in)^{-1}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\, d\theta \\ &= (-in)^{-2}\int_{-\pi}^{\pi} f''(\theta)e^{-in\theta}\, d\theta. \end{aligned}$$

We know that f'' is bounded by M on $[-\pi, \pi]$ (owing to it being continuous). Then,

$$2\pi|\widehat{f}(n)| \leq \frac{1}{n^2} \times 2\pi \times M.$$

This completes the proof. ■

Corollary 3.2. If $f \in C^2(S^1)$, the Fourier series of f converges uniformly to f.

Definition 3.3. A function f on the circle is said to be Hölder continuous of class $\alpha > 0$. if there is a $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all $x, y \in S^1$. This is denoted by $f \in C^{0,\alpha}(S^1)$.

Remark 3.0.1. *The uniform convergence of $S_N f$ to f holds for $f \in C^{0,\alpha}(S^1)$ where $\alpha > 1/2$. This is due to Bernstein.*

3.1 Convolutions

We have

$$\begin{aligned}
S_N f(x) &= \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx} \\
&= \sum_{k=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{ik(x-y)}\, dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-N}^{N} e^{ik(x-y)\, dy} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy \\
&= (f * D_N)(x).
\end{aligned}$$

This is called a convolution.

Definition 3.4. Given $f, g \in L^1(S^1)$, define

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y)\, dy.$$

It is not hard to argue that $f * g \in L^1(S^1)$ using Fubini's Theorem.

Proposition 3.5. *Let $u, v, w \in L^1(S^1)$. Then,*

(a) $u * v = v * u$.

(b) $u * (v + w) = u * v + u * w$.

(c) $(\lambda u) * v = \lambda (u * v)$.

(d) $u * (v * w) = (u * v) * w$.

(e) $\widehat{u * v}(k) = \widehat{u}(k)\widehat{v}(k)$.

(f) *$u * v$ is continuous if $u, v \in L^1(S^1)$ and bounded.*

Proof. We have

$$2\pi(v * u)(x) = \int_{-\pi}^{\pi} v(y) u(x-y)\, dy = \int_{-\pi+x}^{\pi+x} v(x-z) u(z)\, dz = \int_{-\pi}^{\pi} v(x-z) u(z)\, dz = 2\pi(u * v)(x).$$

For (e),

$$\begin{aligned}
\widehat{u * v}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (u * v)(x) e^{-ikx}\, dx \\
&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u(y) v(x-y) e^{-ikx}\, dy dx \\
&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u(y) e^{-iky} v(x-y) e^{-ik(x-y)}\, dy dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{u}(k) v(x-y) e^{-ik(x-y)}\, dx \\
&= \widehat{u}(k)\widehat{v}(k).
\end{aligned}$$

$\blacksquare$

Chapter 4

Lecture 4

Definition 4.1. A sequence of continuous functions $K_n : S^1 \to \mathbb{C}$ are said to be *a family of good kernels* if:

(a)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$$
for all $n \in \mathbb{N}$.

(b) There is a positive real number $M > 0$ such that
$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M$$
for all $n \in \mathbb{N}$.

(c) For every $\delta > 0$,
$$\lim_{n\to\infty}\int_{|x|\geq\delta} |K_n(x)|\,dx = 0$$

Theorem 4.2. *Suppose $\{K_m\}_{m\geq 1}$ is a family of good kernels and $f \in L^1(S^1) \cap L^\infty(S^1)$. Then,*
$$(f * K_m)(x) \to f(x) \quad \text{as } m \to \infty$$
at all points of continuity of f. Moreover, if $f \in C(S^1)$, then the convergence is uniform.

Proof. We have
$$\left|(f * K_n)(x) - f(x)\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x-y) - f(x))K_n(y)\,dy\right|.$$
Using continuity of f at x, one can pick a small enough δ such that $f(x-y) - f(x)$ is small for all $y \in [-\delta, \delta]$. Then break the integral into two parts and use part (c) of the definition of a good kernel. ■

Proposition 4.3. *The Dirichlet kernel is not a good kernel.*

Proof. We have
$$\int_{-\pi}^{\pi} |D_N(x)|\,dx = \int_{-\pi}^{\pi}\left|\frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\frac{x}{2}}\right|\,dx.$$

The integral on the right can be broken down into segments $\left[\frac{k\pi}{N+\frac{1}{2}}, \frac{(k+1)\pi}{N+\frac{1}{2}}\right]$. Integrate over each of them and find a $\Omega(\log N)$ bound on the integral. This shows that it diverges. ■

Recall that L^2 is a Hilbert space with the inner product given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

There is the Hölder Inequality

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2.$$

Chapter 5

Lecture 5

Let $f \in L^2(-\pi, \pi)$ and denote $e_k = e^{ik\theta}$ which is a function on the unit circle. The n-th Fourier coefficient of f is given by

$$\widehat{f}(k) = \langle f, e_k \rangle \qquad \forall k \in \mathbb{Z}.$$

We would show that the $\{e_k\}$'s form an orthonormal basis for $L^2(S^1)$.

Theorem 5.1. *Given $f \in L^2(S^1)$, we have*

1. *$S_n(f) \to f$ in the L^2-norm.*
2. *$\|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2$.*
3. *$\widehat{f}(k) \to 0$ as $|k| \to \infty$.*

Proof. ■

Theorem 5.2. *Let f be a Lipschitz function on S^1. Then, $S_n(f)$ converges pointwise to f.*

Proof. We have

$$S_N(f)(\omega) - f(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(\omega - t) - f(\omega) \right] D_N(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\omega - t) - f(\omega)}{t} \frac{t \sin\left(N + \frac{1}{2}\right) t}{\sin \frac{t}{2}}\, dt.$$

Note that

$$\sin\left(N + \frac{1}{2}\right) t = \sin(Nt) \cos\left(\frac{t}{2}\right) + \cos(Nt) \sin\left(\frac{t}{2}\right).$$

Let

$$F(t) = \frac{f(\omega - t) - f(t)}{t},$$

whence $|F(t)| \leq M$ on S^1, as a result, $F \in L^2$.

Then,

$$S_N(f)(\omega) - f(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(t) t \cos(t/2)}{\sin(t/2)} \sin(Nt) + F(t) t \cos(Nt)\, dt.$$

Note that $\frac{t\cos(t/2)}{\sin(t/2)}$ is continuous and hence, $F(t)\frac{t\cos(t/2)}{\sin(t/2)}$ is again in L^2. Also, $tF(t)$ is in L^2. Invoking the Riemann Lebesgue Lemma, we have the desired conclusion. ■

Remark 5.0.1. *If $f \in C^{0,1}(S^1)$, then $\widehat{f}(k) = \mathcal{O}(1/|k|)$.*