MA5106: Fourier Analysis

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January 10, 2024

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Chapter 1

Lecture 1

The origins of Fourier analysis lie in solving the heat equation:

$$\Delta u = \partial_t u$$

where Δ denotes the Laplacian.

In order to solve this, Fourier believed for a long time that one could expand a function as a series

$$f \sim \sum_k a_k \sin kx + \sum b_k \cos kx.$$

This is not true. In 1876, Paul Du Bois-Reymond gave an example of a continuous function whose Fourier series does not converge. But in 1966, Carleson showed that given an L^2 function on $[0,1]$, the points at which the Fourier series does not converge has measure 0.

There are many applications to PDEs, in solving the

Laplace Equation: $\Delta u = 0$,

Heat Equation: $\partial_t u = \Delta u$,

Wave Equation: $\partial_{tt} u = \Delta u$.

Definition 1.1 (Fourier Series). Given $f \in L^1[a,b]$, its k-th *Fourier coefficient* is defined as

$$\widehat{f}(k) := \frac{1}{L}\int_a^b f(x) \exp\left(-\frac{2\pi i k}{L}x\right)\, dx.$$

where $L = b - a$.

The *Fourier series* of f is given formally by

$$f \sim \sum_{k\in\mathbb{Z}} \widehat{f}(k) \exp\left(\frac{2\pi i k}{L}x\right).$$

The question is whether

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \widehat{f}(k) \exp\left(\frac{2\pi i k}{L}x\right) = f(x)$$

in the following cases:

- if $f \in L^1[a,b]$. Here we cannot expect pointwise convergence because one can just change the value of f at a single point without affecting its Fourier series.

- if $f \in C[a,b]$. This is not true because of an example by Paul Du Bois-Reymond.
- if $f \in C^1[a,b]$ then this is true.
- if $f \in L^2[a,b]$, then there may not be pointwise convergence but there is convergence in the L^2-norm.

There are notions of convergence other than pointwise and uniform. For example Cesàro and Abel. Fejér had proved that for continuous functions, the Cesàro sums converge uniformly to the function, whatever that means.

Example 1.2. Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x$. Then,

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \exp(-ikx)\, dx = \begin{cases} 0 & k = 0 \\ \frac{(-1)^k i}{k} & k \neq 0. \end{cases}$$

The Fourier series is then given by

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} \frac{\sin kx}{k}.$$