

Category Theory

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Chapter 1

Introduction and Elementary Definitions

1.1 Preliminary Definitions

1.1.1 Categories

Definition 1.1 (Category). A category \mathcal{A} consists of

1. a collection $\text{ob}(\mathcal{A})$ of objects
2. for each $A, B \in \text{ob}(\mathcal{A})$ a collection $\mathcal{A}(A, B)$ of morphisms from A to B
3. for each $A, B, C \in \text{ob}(\mathcal{A})$, a composition function

$$\circ : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C)$$

mapping $(g, f) \mapsto g \circ f$.

4. for each $A \in \text{ob}(\mathcal{A})$, an element id_A of $\mathcal{A}(A, A)$ called the identity on A .

satisfying the following:

associativity: for each $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$

identity: for each $f \in \mathcal{A}(A, B)$, we have $f \circ \text{id}_A = f = \text{id}_B \circ f$

Every category \mathcal{A} also has the associated *opposite category* \mathcal{A}^{op} where $\text{ob}(\mathcal{A}) = \text{ob}(\mathcal{A}^{\text{op}})$ and for each $A, B \in \text{ob}(\mathcal{A})$, $\mathcal{A}^{\text{op}}(A, B) = \mathcal{A}(B, A)$.

For example, **Set** is the category of sets with morphisms as set maps.

Definition 1.2 (Product Category). For every pair of categories \mathcal{A} and \mathcal{B} , there is the product category $\mathcal{A} \times \mathcal{B}$ where

- (a) $\text{ob}(\mathcal{A} \times \mathcal{B}) = \text{ob}(\mathcal{A}) \times \text{ob}(\mathcal{B})$
- (b) $(\mathcal{A} \times \mathcal{B})((A_1, B_1), (A_2, B_2)) = \mathcal{A}(A_1, A_2) \times \mathcal{B}(B_1, B_2)$ for all $A_1, A_2 \in \text{ob}(\mathcal{A})$ and $B_1, B_2 \in \text{ob}(\mathcal{B})$
- (c) For $(f_1, g_1) \in (\mathcal{A} \times \mathcal{B})((A_1, B_1), (A_2, B_2))$ and $(f_2, g_2) \in (\mathcal{A} \times \mathcal{B})((A_2, B_2), (A_3, B_3))$, $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$
- (d) $\text{id}_{(A, B)} = (\text{id}_A, \text{id}_B)$ for all $(A, B) \in \text{ob}(\mathcal{A} \times \mathcal{B})$

Definition 1.3 (Isomorphism). A morphism $f \in \mathcal{A}(A, B)$ is said to be an *isomorphism* if there is $g \in \mathcal{A}(B, A)$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

An isomorphism in **Set** is simply a bijection while an isomorphism in **Top** is a homeomorphism.

Definition 1.4 (Small, Locally Small). A category \mathcal{A} is said to be *small* if $\text{ob}(\mathcal{A})$ is a set and $\mathcal{A}(A, B)$ is a set for all $A, B \in \mathcal{A}$. Similarly, \mathcal{A} is said to be *locally small* if $\mathcal{A}(A, B)$ is a set for all $A, B \in \mathcal{A}$.

Definition 1.5 (Mono, Epi). In a category \mathcal{A} , an arrow $f : A \rightarrow B$ is called a/an:

mono if for any $C \in \mathcal{A}$ given any $g, h : C \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$

epi if for any $C \in \mathcal{A}$ given any $g, h : B \rightarrow C$, $g \circ f = h \circ f$ implies $g = h$

It is important to note that $\text{mono} + \text{epi} \neq \text{iso}$. For example, let **P** be a poset category. Then, every arrow $p \leq q$ is a mono and an epi but not all arrows are isos. Similarly, in **CRing**, the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a mono and an epi but not an iso.

Proposition 1.6. *Epis in Grp are precisely surjective group homomorphisms.*

Proof. ■

Definition 1.7 (Initial, Terminal). In a category \mathcal{A} , an object $\mathbf{0}$ is said to be *initial* if for each $A \in \mathcal{A}$, $\mathcal{A}(\mathbf{0}, A)$ is a singleton. Similarly, an object $\mathbf{1}$ is said to be *terminal* if for each $A \in \mathcal{A}$, $\mathcal{A}(A, \mathbf{1})$ is a singleton.

In **Set**, the empty set is the initial object while every singleton is a terminal object. In **CRing**, \mathbb{Z} is an initial object while the zero ring is a terminal object.

Proposition 1.8. *Initial (terminal) objects are unique upto a unique isomorphism.*

Proof. Let $\mathbf{0}$ and $\mathbf{0}'$ be initial objects in \mathcal{A} . Then, there are unique morphisms $f : \mathbf{0} \rightarrow \mathbf{0}'$ and $g : \mathbf{0}' \rightarrow \mathbf{0}$. Since $g \circ f \in \mathcal{A}(\mathbf{0}, \mathbf{0}')$, we must have $g \circ f = \text{id}_{\mathbf{0}}$ and similarly, $f \circ g = \text{id}_{\mathbf{0}'}$. Hence f and g are isomorphisms. Uniqueness follows from the definition of initial objects.

An analogous proof works for terminal objects. ■

1.1.2 Functors

Definition 1.9 (Functor). Let \mathcal{A} and \mathcal{B} be categories. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ consists of

- a function $\text{ob}(\mathcal{A}) \rightarrow \text{ob}(\mathcal{B})$ written as $A \mapsto F(A)$
- for each $A, A' \in \mathcal{A}$, a function $\mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A'))$, written as $f \mapsto F(f)$

satisfying the following axioms

covariancy: $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in \mathcal{A}

identity consistency: $F(\text{id}_A) = \text{id}_{F(A)}$ whenever $A \in \mathcal{A}$

Such a functor is sometimes also called a **covariant functor**.

Let \mathbf{Top}_* denote the category of topological spaces equipped with a basepoint. Let π be the map that maps a pointed topological space (X, x_0) to its fundamental group $\pi(X, x_0)$. We claim that this is a covariant functor. Let $\phi : (X, x_0) \rightarrow (Y, y_0)$ be a continuous function. One knows from algebraic topology that the above continuous map induces a homomorphism $\phi_* : \pi(X, x_0) \rightarrow \pi(Y, y_0)$ given by $[f] \mapsto [\phi \circ f]$. It is not hard to see that this is a covariant functor.

Definition 1.10 (Contravariant Functor). Let \mathcal{A} and \mathcal{B} be categories. A contravariant functor from \mathcal{A} to \mathcal{B} is a functor $F : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$.

Let \mathbf{Top} be the category of topological spaces. For a topological space X , let $C(X)$ denote the ring of continuous functions $X \rightarrow \mathbb{R}$. That is, $C(X) \in \mathbf{Ring}$. We claim that $C(X)$ is a contravariant functor from \mathbf{Top} to \mathbf{Ring} . Indeed, let $f : X \rightarrow Y$ be a continuous function. Then, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & \mathbb{R} \end{array}$$

The continuous function f induces a map $f_* : C(Y) \rightarrow C(X)$ given by $g \mapsto g \circ f$. It is not hard to see now that the functor C is a contravariant functor from \mathbf{Top} to \mathbf{Ring} which maps a morphism f to a morphism f_* .

Definition 1.11 (Presheaf). A presheaf is a contravariant functor from \mathcal{A} to \mathbf{Set} . That is, a functor $F : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$.

Let X be a topological space and let $\mathcal{O}(X)$ denote the category of open subsets of X with inclusion morphisms. This gives $\mathcal{O}(X)$ the structure of a poset. Consider now the map $F : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Set}$ given by

$$F(U) = \{\text{continuous functions } U \rightarrow \mathbb{R}\}$$

That this is a functor follows from the fact that if $U \subseteq V$, then the restriction of a continuous function $f : V \rightarrow \mathbb{R}$ to U is continuous.

Definition 1.12. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is *faithful* if for each $A, A' \in \mathcal{A}$, the map $\mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A'))$ given by $f \mapsto F(f)$ is injective.

Similarly, it is said to be *full* if the map is surjective.

1.1.3 Natural Transformations

Definition 1.13 (Natural Transformation). Let \mathcal{A} and \mathcal{B} be categories and let $F, G : \mathcal{A} \rightarrow \mathcal{B}$ be functors. A *natural transformation* $\alpha : F \rightarrow G$ is a family $\left(F(A) \xrightarrow{\eta_A} G(A) \right)_{A \in \mathcal{A}}$ of maps in \mathcal{B} such that for every map $A \xrightarrow{f} A'$ in \mathcal{A} , the following diagram commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \eta_A \downarrow & & \downarrow \eta_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

The maps η_A are called the *components* of η . When η_A is an isomorphism for all $A \in \mathcal{A}$, then η is said to be a natural isomorphism.

Consider **CRing**, the category of commutative rings and **Mon**, the category of monoids. Consider the covariant functor $M_n : \mathbf{CRing} \rightarrow \mathbf{Mon}$ that maps a commutative ring R to the monoid $M_n(R)$ of $n \times n$ matrices with entries from R .

Consider now the forgetful functor $U : \mathbf{CRing} \rightarrow \mathbf{Mon}$ that maps a ring R to its multiplicative monoid. It is not hard to see that \det_n is a natural transformation from $M_n \rightarrow U$.

1.1.4 The Category of Functors

Let \mathcal{A} and \mathcal{B} be categories. Denote by $[\mathcal{A}, \mathcal{B}]$, the category with objects as functors from \mathcal{A} to \mathcal{B} and morphisms as natural transformations between functors. That this is a category follows from the fact that the composition of two natural transformations is a natural transformation.

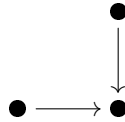
In particular, the category $[\mathcal{A}^{\text{op}}, \mathbf{Set}]$ is called the *presheaf category* on \mathcal{A} .

Chapter 2

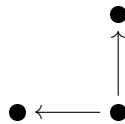
Limits and Colimits

Definition 2.1 (Diagram). A *diagram* is a functor $D : \mathcal{I} \rightarrow \mathcal{A}$ where \mathcal{I} is some indexing category. The category \mathcal{I} is sometimes called the *shape category*.

For example, \mathcal{I} could be given by



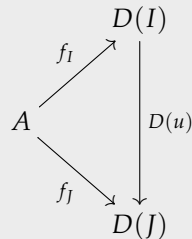
The corresponding diagram is called a *pullback diagram*. The dual to this is the *pushout diagram* given by the indexing category:



Definition 2.2 (Cone). Let $D : \mathcal{I} \rightarrow \mathcal{A}$ be a diagram. A *cone* on D is an object $A \in \mathcal{A}$, the *vertex* of the cone, together with a family

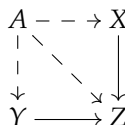
$$\left(A \xrightarrow{f_I} D(I) \right)_{I \in \mathcal{I}}$$

of maps in \mathcal{A} such that for each $I, J \in \text{ob}(\mathcal{I})$, and $u \in \mathcal{I}(I, J)$, the following diagram commutes.



We shall denote such a cone by the shorthand $(A, \{f_I\}_{I \in \mathcal{I}})$.

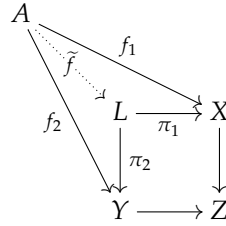
For example, a cone over the a pullback diagram is



2.1 Limits

Definition 2.3 (Limit). Let $D : \mathcal{I} \rightarrow \mathcal{A}$ be a diagram. A *limit* of D is a cone $(L, \{p_I\}_{I \in \mathcal{I}})$ such that for any other cone $(A, \{f_I\}_{I \in \mathcal{I}})$, there is a unique map $\tilde{f} : A \rightarrow L$ such that for all $I \in \mathcal{I}$, $p_I \circ \tilde{f} = f_I$. The maps p_I are called the *projections* of the limit.

For example, a limit over a pullback diagram is



Definition 2.4 (Product). Let \mathcal{I} be a shape category with no morphisms other than the identity morphisms. Then, a *product* in a category \mathcal{A} is a limit over a diagram $D : \mathcal{I} \rightarrow \mathcal{A}$.

In particular if \mathcal{I} is empty, then a limit over a diagram $D : \mathcal{I} \rightarrow \mathcal{A}$ is simply the *final object*.

Chapter 3

Adjoints

Definition 3.1. Let \mathcal{A} and \mathcal{B} be locally small categories and $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$ be functors. We say that F is *left adjoint* to G and G is *right adjoint* to F if there is a natural isomorphism between (bi)functors $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \mathbf{Set}$.

$$\text{Hom}_{\mathcal{B}}(F-, -) \cong \text{Hom}_{\mathcal{A}}(-, G-)$$

We denote this adjunction by $F \dashv G$.

Upon unraveling the definition, we see that for every morphism $(f, g) : (A, B) \rightarrow (A', B')$ in $\mathcal{A} \times \mathcal{B}$, the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}}(FA, B) & \longrightarrow & \text{Hom}_{\mathcal{B}}(FA', B') \\ \eta_{(A, B)} \downarrow & & \downarrow \eta_{(A', B')} \\ \text{Hom}_{\mathcal{A}}(A, GB) & \longrightarrow & \text{Hom}_{\mathcal{A}}(A', GB') \end{array}$$

In particular, we require that for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there is a set bijection

$$\text{Hom}_{\mathcal{B}}(FA, B) \longleftrightarrow \text{Hom}_{\mathcal{A}}(A, GB)$$

Chapter 4

Representables

4.1 Representable Functors

Definition 4.1. Let \mathcal{A} be a locally small category. Then, for each $A \in \mathcal{A}$, define the functor $H_A : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$ given by $H_A(B) = \mathcal{A}(B, A)$. Similarly, define $H^A : \mathcal{A} \rightarrow \mathbf{Set}$ by $H^A(B) = \mathcal{A}(A, B)$.

Notice that $H_\bullet : \mathcal{A}^{\text{op}} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$ is a functor. This is known as the *Yoneda Embedding* of \mathcal{A} .

4.2 Yoneda's Lemma

Theorem 4.2. Let \mathcal{A} be a locally small category and $X \in \text{ob}([\mathcal{A}^{\text{op}}, \mathbf{Set}])$. Then,

$$[\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X) \cong X(A)$$

naturally in $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{\text{op}}, \mathbf{Set}]$.

In particular, we are looking at the functors

$$\begin{aligned} [\mathcal{A}^{\text{op}}, \mathbf{Set}](H_{\bullet_1}, \bullet_2) : \mathcal{A}^{\text{op}} \times [\mathcal{A}^{\text{op}}, \mathbf{Set}] &\rightarrow \mathbf{Set} \\ \bullet_2(\bullet_1) : \mathcal{A}^{\text{op}} \times [\mathcal{A}^{\text{op}}, \mathbf{Set}] &\rightarrow \mathbf{Set} \end{aligned}$$

and would like to show that they are naturally isomorphic.

Proof. In other words, for each pair $(A, X) \in \mathcal{A}^{\text{op}} \times [\mathcal{A}^{\text{op}}, \mathbf{Set}]$, we must define a map

$$\theta_{A,X} : [\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X) \rightarrow X(A)$$

and show that this is a natural bijection. The most natural way to define this map is

$$\theta_{A,X}(\alpha) = \alpha_A(\text{id}_A) \quad \alpha \in [\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X)$$

In order to show that $\theta_{A,X}$ is a bijection, we must construct its inverse. Let

$$\phi_{A,X} : X(A) \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X)$$

be defined by constructing a natural transformation $\tilde{x} \in [\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X)$ for each $x \in X(A)$. In particular, we must define, for each $B \in \mathcal{A}$, a map $\tilde{x}_B : H_A(B) = \mathcal{A}(B, A) \rightarrow X(B)$. We do this by simply choosing $\tilde{x}_B(f) = (X(f))(x) \in X(B)$ for each $f \in \mathcal{A}(B, A) = \mathcal{A}^{\text{op}}(A, B)$. Because $X(f)$ is a map from $X(A)$ to $X(B)$ in \mathbf{Set} .

We shall now show that \tilde{x} is a natural transformation. To do so, we must show that the following square commutes for each $B, B' \in \text{ob}(\mathcal{A})$ and $g \in \mathcal{A}^{\text{op}}(B, B') = \mathcal{A}(B', B)$

$$\begin{array}{ccc} H_A(B) = \mathcal{A}(B, A) & \xrightarrow{H_A(g)} & H_A(B') = \mathcal{A}(B', A) \\ \tilde{x}_B \downarrow & & \downarrow \tilde{x}_{B'} \\ X(B) & \xrightarrow{X(g)} & X(B') \end{array}$$

For any $h \in \mathcal{A}(B, A)$, we have $(H_A(g))(f) = f \circ g \in \mathcal{A}(B', A)$. Then, $\tilde{x}_{B'}(f \circ g) = (X(f \circ g))(x)$. On the other hand, we have

$$(X(g))(\tilde{x}_B(f)) = (X(g))((X(f))(x))$$

But by (contravariant) functoriality of X , we see that $X(f \circ g) = X(g) \circ X(f)$ and therefore, the square commutes.

Next, we must show that $\theta_{A,X}$ and $\phi_{A,X}$ are inverses to one another, which would establish that they are bijections. Pick some $x \in X(A)$. We must show that $\theta_{A,X}(\phi_{A,X}(x)) = x$. Let us, for the ease of notation, denote $\theta_{A,X} = (\cdot)$ and we already have the notation $\phi_{A,X}(x) = \tilde{x}$.

Pick any $\alpha \in [\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X)$. Then, $\tilde{\alpha}$ is an element of $[\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X)$, therefore, it suffices to show, for all $B \in \text{ob}(\mathcal{A})$ that $(\tilde{\alpha}) = \alpha_B$. Indeed, let $f \in H_A(B) = \mathcal{A}^{\text{op}}(A, B) = \mathcal{A}(B, A)$

$$(\tilde{\alpha}_B)(f) = (X(f))(\tilde{\alpha}) = (X(f))(\alpha_A(\mathbf{id}_A))$$

Now, since α is a natural transformation between the functors H_A and X , we have the following commutative diagram:

$$\begin{array}{ccc} H_A(A) & \xrightarrow{H_A(f)} & H_A(B) \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ X(A) & \xrightarrow{X(f)} & X(B) \end{array}$$

Therefore, for $\mathbf{id}_A \in H_A(A)$, we have

$$(X(f))(\alpha_A(\mathbf{id}_A)) = \alpha_B(\mathbf{id}_A \circ f) = \alpha_B(f)$$

and the conclusion follows.

We have now shown that $\theta_{A,X}$ and $\phi_{A,X}$ are bijections. Lastly, we must show that they are natural. TODO: Add in later ■