Results in Combinatorics

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Chapter 1

Results on Families of Sets

Lemma 1.1 (Lubell-Yamamoto-Meshalkin). Let n be a positive integer and $\mathcal{F}$ be a family of subsets of $\{1, \ldots, n\}$ such that no set in $\mathcal{F}$ is contained in some other set in $\mathcal{F}$. Then,

$$\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1$$

Proof. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ and π_i be the set of all permutations of $\{1, \ldots, n\}$ such that the first $|A_i|$ elements of π_i are the elements of A_i. It is not hard to see that $|\pi_i| = |A_i|!(n - |A_i|)!$. Further, we note that any permutation σ of $\{1, \ldots, n\}$ may be in at most one of the π_i's. Thus, double counting the pairs (π_i, σ), we obtain:

$$\sum_{i=1}^{m} |A_i|!(n - |A_i|)! \leq n!$$

and we have the desired conclusion. ■

Theorem 1.2 (Bollobás, 1965). Let $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ be two families of subsets of $\{1, \ldots, n\}$ such that

- $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$
- $A_i \cap B_j = \emptyset$ for all $1 \leq i, j \leq m$ and $i \neq j$

Then
$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

Proof. Let π_i be the set of all permutations of $\{1, \ldots, n\}$ such that the elements of A_i occur before the elements of B_i (note that this is possible since $A_i \cap B_i = \emptyset$). Then
$$|\pi_i| = \binom{n}{|A_i| + |B_i|} \cdot |A_i|! \cdot |B_i|! \cdot (n - |A_i| - |B_i|)! = \frac{n!}{\binom{|A_i|+|B_i|}{|A_i|}}$$

Finally, we note that for $i \neq j$, $\pi_i \cap \pi_j = \emptyset$, which is not hard to show. This implies that
$$\sum_{i=1}^{m} |\pi_i| \leq n!$$
giving us the desired conclusion. ■

Lemma 1.3. Let $\mathcal{F}$ be an r-uniform family of sets. Such that the intersection of any k sets, with $k \leq r+1$ is non-empty. Then, the intersection of all sets in $\mathcal{F}$ is non-empty.

Proof. Suppose not. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a minimal counter-example to the statement. Obviously, $m \geq r+2$ and for each i, $\bigcup_{j \neq i} A_j$ is non-empty (since $\mathcal{F}$ is a minimal counter-example) and thus, let b_i be one such element in said intersection. Suppose all b_i's are distinct. Then, $\{b_2, \ldots, b_m\} \subseteq A_1$, implying that $|A_1| \geq m - 1 \geq r+1$, which is not possible. Thus, there must exist indices i and j with $i \neq j$ such that $b_i = b_j$. It is not hard to conclude from here that all sets in $\mathcal{F}$ must contain b_i. ■

Theorem 1.4 (Erdös-Ko-Rado, 1961). Let n be a positive integer, X be an n-element set and $k \leq n/2$ be a positive integer. Further, let $\mathcal{F}$ be a k-uniform, intersecting family of subsets of X. Then
$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Proof. Let $\mathcal{F} = \{A_1, \dots, A_m\}$. Without loss of generality, let $X = \{1, \dots, n\}$ and π_i be the set of all cyclic permutations of X in which the elements of A_i occur consecutively. Of course, $|\pi_i| = k!(n-k)!$ for each $1 \le i \le m$. Further, we note that any cyclic permutation of X may occur in at most k of the π_i's. (This is an interesting argument). Then, double counting the pair (C, π_i) where C is a cyclic permutation, we have

$$m \cdot k!(n-k)! \le k \cdot (n-1)!$$

this completes the proof. ■

Theorem 1.5 (Benny Sudakov). Let $\mathscr{A} = \{A_1, \dots, A_m\}$ and $\mathscr{B} = \{B_1, \dots, B_p\}$ be families of distinct subsets of $\{1, \dots, n\}$ such that $|A_i \cap B_j|$ is an odd number for all permissible i and j. Then $mp \le 2^{n-1}$.

Chapter 2

Combinatorial Nullstellensatz

Theorem 2.1 (Alon-Tarsi, 1992). Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[x_1, \ldots, x_n]$. Suppose $\deg(f) = d = \sum_{i=1}^{n} d_i$ and the coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ is non-zero. Let $L_1, \ldots, L_n$ be subsets of $\mathbb{F}$ with $|L_i| > d_i$. Then there exist $a_i \in L_i$ for each i such that $f(a_1, \ldots, a_n) \neq 0$.

Proof. The proof is by induction on n. The base case with $n = 1$ follows from the fact that a polynomial of degree n may have at most n zeros in a field, counting multiplicity. Suppose now that $n > 1$. Let us assume that $|L_n| = d_n + 1$. Define the polynomial $h(x)$ as follows:

$$\prod_{t \in L_n} (x_n - t) = x^{d_n+1} - h(x)$$

One notes that for all $t \in L_n$, $h(t) = t^{d_n+1}$. Let us now define $\tilde{f}$ as the remainder obtained on dividing f by $x_n^{d_n+1} - h(x_n)$. We first note that the coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ does not change. Further, the degree of x_n in any term in $\tilde{f}$ is at most d_n. We may now group the terms of $\tilde{f}$ with respect to powers of x_n, that is

$$\tilde{f}(x_1, \ldots, x_{n-1}) = \sum_{i=0}^{d_n} g_i(x_1, \ldots, x_{n-1}) x_n^i$$

Let us now focus on $g_{d_n}(x_1, \ldots, x_{n-1})$, in which the coefficient of the term $\prod_{i=1}^{n-1} x_i^{d_i}$ is equal to the coefficient of the term $\prod_{i=1}^{n} x_i^{d_i}$ in $\tilde{f}(x_1, \ldots, x_n)$, which we have concluded to be non-zero. Therefore, due to the induction hypothesis, there must exist $a_1 \in L_1, \cdots, a_{n-1} \in L_{n-1}$ such that $g_{d_n}(a_1, \ldots, a_{n-1}) \neq 0$. Finally, we can choose a suitable x_n in L_n such that the polynomial

$$\sum_{i=0}^{d_n} g_i(a_1, \ldots, a_{n-1}) x_n^i$$

is non-zero, since its leading coefficient is non-zero. This finishes the proof. ■

Theorem 2.2 (Cauchy-Davenport). Let p be a prime and $A, B \subseteq \mathbb{Z}_p$. Then

$$|A+B| \geq \min\{p, |A|+|B|-1\}$$

where

$$A+B = \{a+b \mid a \in A,\ b \in B\}$$

Proof. First, suppose $|A|+|B| > p$. Then, for any $g \in \mathbb{Z}_p$, the sets A, $g-B$ must intersect. Suppose now that $|A|+|B| \leq p$. Assume for the sake of contradiction that the cardinality of $C = A+B$ is less than $|A|+|B|-1$. Consider the polynoimal:

$$f(x,y) = \prod_{c \in C}(x+y-c)$$

the degree of this polynoimal is $|C| \leq |A|+|B|-2$, as a result, there exist $t_A, t_B \in \mathbb{Z}$ such that $t_A + t_B = |C|$, $t_A \leq |A|-1$, $t_B \leq |B|-1$ and the coefficient of $x^{t_A}y^{t_B}$ is non-zero, this is because any binomial coefficient of the form $\binom{|C|}{x}$ is not divisible by p. Then, due to the Combinatorial Nullstellensatz, there exists $(a,b) \in A \times B$ such that $f(a,b) \neq 0$, a contradiction. This finishes the proof. ■

Example (IMO 2007/6). Let n be a positive integer. Consider

$$S = \{(x,y,z) \mid x,y,z \in \{0,1,\ldots,n\},\ x+y+z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0,0,0)$.

Proof. The answer is $3n$, with the planes being $x+y+z=i$ for $1 \le i \le 3n$. Suppose $k < 3n$ is achievable by the set of planes $\{a_i x + b_i y + c_i z - d_i = 0\}$. Consider now the polynomial:

$$P(x,y,z) = \alpha \prod_{i=1}^{n} (x-i)(y-i)(z-i) - \prod_{i=1}^{k} (a_i x + b_i y + c_i z - d_i)$$

with α chosen such that $P(0,0,0)=0$. First, note that $\deg(P) = 3n$, since we have assumed $k < 3n$ and the coefficient of $x^n y^n z^n$ is $\alpha \neq 0$ since none of the d_i's can be zero. Then, due to the Combinatorial Nullstellensatz on the three sets $L_x = L_y = L_z = \{0, \dots, n\}$, we may conclude that there is a solution to $P(x,y,z) \neq 0$ in $L_x \times L_y \times L_z$, a contradiction. ■

Theorem 2.3 (Erdös-Heillbronn). Let p be a prime and $A \subseteq \mathbb{Z}_p$. Then

$$|A \widehat{+} A| \ge \min\{p, 2|A| - 3\}$$

Proof. We may suppose that $2a - 3 < p$ where $a = |A|$ and let $C = A \widehat{+} A$ with $|C| = m$. Consider the polynoimal

$$f(x,y) = (x-y) \prod_{c \in C} (x+y-c)$$

of degree $m+1$. The coefficient of $x^{a-1} y^{m-a+2}$ is

$$\binom{m}{a-2} - \binom{m}{a-1} = \left(\frac{m-2a+3}{m-a+2}\right) \binom{m}{a-2}$$

Now, suppose that $m < 2a-3$, then the coefficient is non-zero and $m - a + 2 < a - 1 < a$. As a result, there is a solution (a_1, a_2) such that $f(a_1, a_2) \neq 0$ and thus $a_1 \neq a_2$. A contradiction. This finishes the proof. ■

Miscellaneous

Example. Let $\sigma \in \mathrm{Sym}(\{1, \ldots, n\})$, and let $\varepsilon(\sigma) = 1$ if σ is even and -1 otherwise. Let $f(\sigma)$ be the number of fixed points of σ. Prove that

$$\sum_{\sigma} \frac{\varepsilon(\sigma)}{1 + f(\sigma)} = (-1)^{n+1} \frac{n}{n+1}$$

Proof. Let $A_{n \times n}$ be the matrix such that

$$a_{ij} = \begin{cases} x & i = j \\ 1 & \text{otherwise} \end{cases}$$

We see that

$$\det(A) = \sum_{\sigma} \varepsilon(\sigma) x^{f(\sigma)}$$

but we know that $\det(A) = (x-1)^{n-1}(x+n-1)$. Then, we may write

$$\int_0^1 \sum_{\sigma} \varepsilon(\sigma) x^{f(\sigma)} \, \mathrm{d}x = \int_0^1 (x-1)^{n-1}(x+n-1) \, \mathrm{d}x$$
$$\sum_{\sigma} \frac{\varepsilon(\sigma)}{1 + f(\sigma)} = (-1)^{n+1} \frac{n}{n+1}$$

This finishes the proof. ■

Example (Putnam 2016/B4). Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^T)$ (as a function of n), where A^T is the transpose of A.