Category Theory

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Contents

Chapter 1

Introduction and Elementary Definitions

1.1 Preliminary Definitions

1.1.1 Categories

Definition 1.1 (Category). A category $\mathscr{A}$ consists of

1. a collection $\mathrm{ob}(\mathscr{A})$ of objects
2. for each $A, B \in \mathrm{ob}(\mathscr{A})$ a collection $\mathscr{A}(A, B)$ of morphisms from A to B
3. for each $A, B, C \in \mathrm{ob}(\mathscr{A})$, a composition function

$$\circ : \mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C)$$

   mapping $(g, f) \mapsto g \circ f$.

4. for each $A \in \mathrm{ob}(\mathscr{A})$, an element $\mathbf{id}_A$ of $\mathscr{A}(A, A)$ called the identity on A.

satisfying the following:

associativity: for each $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$

identity: for each $f \in \mathscr{A}(A, B)$, we have $f \circ \mathbf{id}_A = f = \mathbf{id}_B \circ f$

Every category $\mathscr{A}$ also has the associated *opposite category* $\mathscr{A}^{\mathrm{op}}$ where $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{A}^{\mathrm{op}})$ and for each $A, B \in \mathrm{ob}(\mathscr{A})$, $\mathscr{A}^{\mathrm{op}}(A, B) = \mathscr{A}(B, A)$.

For example, **Set** is the category of sets with morphisms as set maps.

Definition 1.2 (Product Category). For every pair of categories $\mathscr{A}$ and $\mathscr{B}$, there is the product category $\mathscr{A} \times \mathscr{B}$ where

(a) $\mathrm{ob}(\mathscr{A} \times \mathscr{B}) = \mathrm{ob}(\mathscr{A}) \times \mathrm{ob}(\mathscr{B})$

(b) $(\mathscr{A} \times \mathscr{B})((A_1, B_1), (A_2, B_2)) = \mathscr{A}(A_1, A_2) \times \mathscr{B}(B_1, B_2)$ for all $A_1, A_2 \in \mathscr{A}$ and $B_1, B_2 \in \mathscr{B}$

(c) For $(f_1, g_1) \in (\mathscr{A} \times \mathscr{B})((A_1, B_1), (A_2, B_2))$ and $(f_2, g_2) \in (\mathscr{A} \times \mathscr{B})((A_2, B_2), (A_3, B_3))$, $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$

(d) $\mathbf{id}_{(A,B)} = (\mathbf{id}_A, \mathbf{id}_B)$ for all $(A, B) \in \mathscr{A} \times \mathscr{B}$

Definition 1.3 (Isomorphism). A morphism $f \in \mathscr{A}(A, B)$ is said to be an *isomorphism* if there is $g \in \mathscr{A}(B, A)$ such that $g \circ f = \mathbf{id}_A$ and $f \circ g = \mathbf{id}_B$.

An isomorphism in **Set** is simply a bijection while an isomorphism in **Top** is a homeomorphism.

Definition 1.4 (Small, Locally Small). A category $\mathscr{A}$ is said to be *small* $\mathrm{ob}(\mathscr{A})$ is a set and $\mathscr{A}(A, B)$ is a set for all $A, B \in \mathscr{A}$. Similarly, $\mathscr{A}$ is said to be locally small if $\mathscr{A}(A, B)$ is a set for all $A, B \in \mathscr{A}$.

Definition 1.5 (Mono, Epi). In a category $\mathscr{A}$, an arrow $f : A \to B$ is called a/an:

mono if for any $C \in \mathscr{A}$ given any $g, h : C \to A$, $f \circ g = f \circ h$ implies $g = h$

epi if for any $C \in \mathscr{A}$ given any $g, h : B \to C$, $g \circ f = h \circ f$ implies $g = h$

It is important to note that mono + epi $\neq$ iso. For example, let **P** be a poset category. Then, every arrow $p \leq q$ is a mono and an epi but not all arrows are isos. Similarly, in **CRing**, the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a mono and an epi but not an iso.

Proposition 1.6. *Epis in* **Grp** *are precisely surjective group homomorphisms.*

Proof. ∎

Definition 1.7 (Initial, Terminal). In a category $\mathscr{A}$, an object $\mathbf{0}$ is said to be *initial* if for each $A \in \mathscr{A}$, $\mathscr{A}(\mathbf{0}, A)$ is a singleton. Similarly, an object $\mathbf{1}$ is said to be *terminal* if for each $A \in \mathscr{A}$, $\mathscr{A}(A, \mathbf{1})$ is a singleton.

In **Set**, the empty set is the initial object while every singleton is a terminal object. In **CRing**, $\mathbb{Z}$ is an initial object while the zero ring is a terminal object.

Proposition 1.8. *Initial (terminal) objects are unique upto a unique isomorphism.*

Proof. Let $\mathbf{0}$ and $\mathbf{0}'$ be initial objects in $\mathscr{A}$. Then, there are unique morphisms $f : \mathbf{0} \to \mathbf{0}'$ and $g : \mathbf{0} \to \mathbf{0}'$. Since $g \circ f \in \mathscr{A}(\mathbf{0}, \mathbf{0}')$, we must have $g \circ f = \mathbf{id}_{\mathbf{0}}$ and similarly, $f \circ g = \mathbf{id}_{\mathbf{0}'}$. Hence f and g are isomorphisms. Uniqueness follows from the definition of initial objects.

An analogous proof works for terminal objects. ∎

1.1.2 Functors

Definition 1.9 (Functor). Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ consists of

- a function $\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(B)$ written as $A \mapsto F(A)$
- for each $A, A' \in \mathscr{A}$, a function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$, written as $f \mapsto F(f)$

satisfying the following axioms

covariancy: $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$

identity consistency: $F(\mathbf{id}_A) = \mathbf{id}_{F(A)}$ whenever $A \in \mathscr{A}$

Such a functor is sometimes also called a **covariant functor**.

Let $\mathbf{Top}_*$ denote the category of topological spaces equipped with a basepoint. Let π be the map that maps a pointed topological space (X, x_0) to its fundamental group $\pi(X, x_0)$. We claim that this is a covariant functor. Let $\phi : (X, x_0) \to (Y, y_0)$ be a continuous function. One knows from algebraic topology that the above continuous map induces a homomorphism $\phi_* : \pi(X, x_0) \to \pi(Y, y_0)$ given by $[f] \mapsto [\phi \circ f]$. It is not hard to see that this is a covariant functor.

Definition 1.10 (Contravariant Functor). Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A contravariant functor from $\mathscr{A}$ to $\mathscr{B}$ is a functor $F : \mathscr{A}^{\text{op}} \to \mathscr{B}$.

Let **Top** be the category of topological spaces. For a topological space X, let $C(X)$ denote the ring of continuous functions $X \to \mathbb{R}$. That is, $C(X) \in$ **Ring**. We claim that $C(X)$ is a contravariant functor from **Top** to **Ring**. Indeed, let $f : X \to Y$ be a continuous function. Then, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

The continuous function f induces a map $f_* : C(Y) \to C(X)$ given by $g \mapsto g \circ f$. It is not hard to see now that the functor C is a contravariant functor from **Top** to **Ring** which maps a morphism f to a morphism f_*.

Definition 1.11 (Presheaf). A presheaf is a contravariant functor from $\mathscr{A}$ to **Set**. That is, a functor $F : \mathscr{A}^{\text{op}} \to \mathbf{Set}$.

Let X be a topological space and let $\mathscr{O}(X)$ denote the category of open subsets of X with inclusion morphisms. This gives $\mathscr{O}(X)$ the structure of a poset. Consider now the map $F : \mathscr{O}(X)^{\text{op}} \to \mathbf{Set}$ given by

$$F(U) = \{\text{continuous functions } U \to \mathbb{R}\}$$

That this is a functor follows from the fact that if $U \subseteq V$, then the restriction of a continuous function $f : V \to \mathbb{R}$ to U is continuous.

Definition 1.12. A functor $F : \mathscr{A} \to \mathscr{B}$ is *faithful* if for each $A, A' \in \mathscr{A}$, the map $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$ given by $f \mapsto F(f)$ is injective.

Similarly, it is said to be *full* if the map is surjective.

1.1.3 Natural Transformations

Definition 1.13 (Natural Transformation). Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $F, G : \mathscr{A} \longrightarrow \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \to G$ is a family $\left(F(A) \xrightarrow{\eta_A} G(A)\right)_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(A') \arrow[d, "\eta_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

The maps η_A are called the *components* of η. When η_A is an isomorphism for all $A \in \mathscr{A}$, then η is said to be a natural isomorphism.

Consider **CRing**, the category of commutative rings and **Mon**, the category of monoids. Consider the covariant functor $M_n : \mathbf{CRing} \to \mathbf{Mon}$ that maps a commutative ring R to the monoid $M_n(R)$ of $n \times n$ matrices with entries from R.

Consider now the forgetful functor $U : \mathbf{CRing} \to \mathbf{Mon}$ that maps a ring R to its multiplicative monoid. It is not hard to see that $\det_n$ is a natural transformation from $M_n \to U$.

1.1.4 The Category of Functors

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Denote by $[\mathscr{A}, \mathscr{B}]$, the category with objects as functors from $\mathscr{A}$ to $\mathscr{B}$ and morphisms as natural transformations between functors. That this is a category follows from the fact that the composition of two natural transformations is a natural transformation.

In particular, the category $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ is called the *presheaf category* on $\mathscr{A}$.

Chapter 2

Limits and Colimits

Definition 2.1 (Diagram). A *diagram* is a functor $D : \mathscr{I} \to \mathscr{A}$ where $\mathscr{I}$ is some indexing category. The category $\mathscr{I}$ is sometimes called the *shape category*.

For example, $\mathscr{I}$ could be given by

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The corresponding diagram is called a *pullback diagram*. The dual to this is the *pushout diagram* given by the indexing category:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd}$$

Definition 2.2 (Cone). Let $D : \mathscr{I} \to \mathscr{A}$ be a diagram. A *cone* on D is an object $A \in \mathscr{A}$, the *vertex* of the cone, together with a family

$$\left(A \xrightarrow{f_I} D(I) \right)_{I \in \mathscr{I}}$$

of maps in $\mathscr{A}$ such that for each $I, J \in \mathrm{ob}(\mathscr{I})$, and $u \in \mathscr{I}(I, J)$, the following diagram commutes.

$$\begin{tikzcd} & D(I) \arrow[dd, "D(u)"] \\ A \arrow[ur, "f_I"] \arrow[dr, "f_J"'] & \\ & D(J) \end{tikzcd}$$

We shall denote such a cone by the shorthand $(A, \{f_I\}_{I \in \mathscr{I}})$.

For example, a cone over the a pullback diagram is

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, dashed] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

2.1 Limits

Definition 2.3 (Limit). Let $D : \mathscr{I} \to \mathscr{A}$ be a diagram. A *limit* of D is a cone $(L, \{p_I\}_{I \in \mathscr{I}})$ such that for any other cone $(A, \{f_I\}_{I \in \mathscr{I}})$, there is a unique map $\widetilde{f} : A \to L$ such that for all $I \in \mathscr{I}$, $p_I \circ \widetilde{f} = f_I$. The maps p_I are called the *projections* of the limit.

For example, a limit over a pullback diagram is

$$\begin{tikzcd}
A \arrow[drr, "f_1"] \arrow[dr, dotted, "\widetilde{f}"] \arrow[ddr, "f_2"'] & & \\
& L \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

Definition 2.4 (Product). Let $\mathscr{I}$ be a shape category with no morphisms other than the identity morphisms. Then, a *product* in a category $\mathscr{A}$ is a limit over a diagram $D : \mathscr{I} \to \mathscr{A}$.

In particular if $\mathscr{I}$ is empty, then a limit over a diagram $D : \mathscr{I} \to \mathscr{A}$ is simply the *final object*.

Chapter 3

Adjoints

Definition 3.1. Let $\mathscr{A}$ and $\mathscr{B}$ be locally small categories and $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. We say that F is *left adjoint* to G and G is *right adjoint* to F if there is a natural isomorphism between (bi)functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$.

$$\mathrm{Hom}_{\mathscr{B}}(F-, -) \cong \mathrm{Hom}_{\mathscr{A}}(-, G-)$$

We denote this adjunction by $F \dashv G$.

Upon unraveling the definition, we see that for every morphism $(f, g) : (A, B) \to (A', B')$ in $\mathscr{A} \times \mathscr{B}$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(FA, B) \arrow[r] \arrow[d, "\eta_{(A,B)}"'] & \mathrm{Hom}_{\mathscr{B}}(FA', B') \arrow[d, "\eta_{(A',B')}"] \\
\mathrm{Hom}_{\mathscr{A}}(A, GB) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(A', GB')
\end{tikzcd}$$

In particular, we require that for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$, there is a set bijection

$$\mathrm{Hom}_{\mathscr{B}}(FA, B) \longleftrightarrow \mathrm{Hom}_{\mathscr{A}}(A, GB)$$

Chapter 4

Representables

4.1 Representable Functors

Definition 4.1. Let $\mathscr{A}$ be a locally small category. Then, for each $A \in \mathscr{A}$, define the functor $H_A : \mathscr{A}^{\text{op}} \to \mathbf{Set}$ given by $H_A(B) = \mathscr{A}(B, A)$. Similarly, define $H^A : \mathscr{A} \to \mathbf{Set}$ by $H^A(B) = \mathscr{A}(A, B)$.

Notice that $H_\bullet : \mathscr{A}^{\text{op}} \to [\mathscr{A}^{\text{op}}, \mathbf{Set}]$ is a functor. This is known as the *Yoneda Embedding* of $\mathscr{A}$.

4.2 Yoneda's Lemma

Theorem 4.2. *Let $\mathscr{A}$ be a locally small category and $X \in \text{ob}\left([\mathscr{A}^{op}, \mathbf{Set}]\right)$. Then,*

$$[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \cong X(A)$$

naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \mathbf{Set}]$.

In particular, we are looking at the functors

$$\begin{aligned} [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_{\bullet_1}, \bullet_2) &: \mathscr{A}^{\text{op}} \times [\mathscr{A}^{\text{op}}, \mathbf{Set}] \to \mathbf{Set} \\ \bullet_2(\bullet_1) &: \mathscr{A}^{\text{op}} \times [\mathscr{A}^{\text{op}}, \mathbf{Set}] \to \mathbf{Set} \end{aligned}$$

and would like to show that they are naturally isomorphic.

Proof. In other words, for each pair $(A, X) \in \mathscr{A}^{\text{op}} \times [\mathscr{A}^{\text{op}}, \mathbf{Set}]$, we must define a map

$$\theta_{A,X} : [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X) \to X(A)$$

and show that this is a natural bijection. The most natural way to define this map is

$$\theta_{A,X}(\alpha) = \alpha_A(\mathbf{id}_A) \qquad \alpha \in [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X)$$

In order to show that $\theta_{A,X}$ is a bijection, we must construct its inverse. Let

$$\phi_{A,X} : X(A) \to [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X)$$

be defined by constructing a natural transformation $\widetilde{x} \in [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X)$ for each $x \in X(A)$. In particular, we must define, for each $B \in \mathscr{A}$, a map $\widetilde{x}_B : H_A(B) = \mathscr{A}(B, A) \to X(B)$. We do this by simply choosing $\widetilde{x}_B(f) = (X(f))(x) \in X(B)$ for each $f \in \mathscr{A}(B, A) = \mathscr{A}^{\text{op}}(A, B)$. Because $X(f)$ is a map from $X(A)$ to $X(B)$ in $\mathbf{Set}$.

We shall now show that $\widetilde{x}$ is a natural transformation. To do so, we must show that the following square commmutes for each $B, B' \in \mathrm{ob}(\mathscr{A})$ and $g \in \mathscr{A}^{\mathrm{op}}(B, B') = \mathscr{A}(B', B)$

$$\begin{tikzcd}
H_A(B) = \mathscr{A}(B,A) \arrow[r, "H_A(g)"] \arrow[d, "\widetilde{x}_B"'] & H_A(B') = \mathscr{A}(B',A) \arrow[d, "\widetilde{x}_{B'}"] \\
X(B) \arrow[r, "X(g)"'] & X(B')
\end{tikzcd}$$

For any $h \in \mathscr{A}(B, A)$, we have $(H_A(g))(f) = f \circ g \in \mathscr{A}(B', A)$. Then, $\widetilde{x}_{B'}(f \circ g) = (X(f \circ g))(x)$. On the other hand, we have

$$(X(g))(\widetilde{x}_B(f)) = (X(g))((X(f))(x))$$

But by (contravariant) functoriality of X, we see that $X(f \circ g) = X(g) \circ X(f)$ and therefore, the square commutes.

Next, we must show that $\theta_{A,X}$ and $\phi_{A,X}$ are inverses to one another, which would establish that they are bijections. Pick some $x \in X(A)$. We must show that $\theta_{A,X}(\phi_{A,X}(x)) = x$. Let us, for the ease of notation, denote $\theta_{A,X} = \widehat{(\cdot)}$ and we already have the notation $\phi_{A,X}(x) = \widetilde{x}$.

Pick any $\alpha \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$. Then, $\widetilde{\widehat{\alpha}}$ is an element of $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$, therefore, it suffices to show, for all $B \in \mathrm{ob}(\mathscr{A})$ that $(\widetilde{\widehat{\alpha}}) = \alpha_B$. Indeed, let $f \in H_A(B) = \mathscr{A}^{\mathrm{op}}(A, B) = \mathscr{A}(B, A)$

$$(\widetilde{\widehat{\alpha}}_B)(f) = (X(f))(\widehat{\alpha}) = (X(f))(\alpha_A(\mathbf{id}_A))$$

Now, since α is a natural transformation between the functors H_A and X, we have the following commutative diagram:

$$\begin{tikzcd}
H_A(A) \arrow[r, "H_A(f)"] \arrow[d, "\alpha_A"'] & H_A(B) \arrow[d, "\alpha_B"] \\
X(A) \arrow[r, "X(f)"'] & X(B)
\end{tikzcd}$$

Therefore, for $\mathbf{id}_A \in H_A(A)$, we have

$$(X(f))(\alpha_A(\mathbf{id}_A)) = \alpha_B(\mathbf{id}_A \circ f) = \alpha_B(f)$$

and the conclusion follows.

We have now shown that $\theta_{A,X}$ and $\phi_{A,X}$ are bijections. Lastly, we must show that they are natural. TODO: Add in later ■