Solutions to Principles of Mathematical Analysis by Rudin

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2 Basic Topology

6. Let x be a limit point of E'. Let U be an open set containing x. By our assumption, there is $e' \in E$ such that $e' \in U$. Therefore, there is an open set V contained in U, containing e'. Since e' is a limit point of E, $V \cap E \neq \emptyset$, therefore, $U \cap E \neq \emptyset$ and x is a limit point of E, as a result, $x \in E'$. Hence, E' is closed.

   We have shown above that for any open set U,
   $$U \cap E' \neq \emptyset \implies U \cap E \neq \emptyset$$
   This immediately implies that $\overline{E}' \subseteq E'$. But since $E \subseteq \overline{E}$, we must have that $E' = \overline{E}'$.

7. Let (X, d) be the metric space

   (a) Obviously, if $x \in A_i'$ for some i, it must be an element of B_n'. Conversely, let $x \in B_n'$, if $x \notin \overline{A}_i$ for all i, then there are open sets U_i for $1 \leq i \leq n$ such that $x \in U_i$ and $U_i \cap A_i = \emptyset$. Let $U = \bigcap_{i=1}^n U_i$, which is open and is disjoint from $\bigcup_{i=1}^n A_i = B_n$, a contradiction. This argument works for any topological space, since we never used that X was metrizable.

   (b) The containment is trivial. Let $A_i = \left(\frac{1}{n}, 1\right)$. Then $B = (0, 1)$ and $1 \in \overline{B}$ while $1 \notin \overline{A}_i$ for all $i \in \mathbb{N}$.

8. Yes. Let $x \in E$, then, there is $\delta \in \mathbb{R}^+$ such that $B(x, \delta) \subseteq E$. Let U be an open set containing x, then $U \cap B(x, \delta)$ is an open set, therefore, there is $0 < \varepsilon < \delta$ such that $B(x, \varepsilon) \subseteq U \cap B(x, \delta)$, and thus, there is $e \in B(x, \varepsilon)$ with $e \neq x$, implying that $x \in E'$.

   No, the statement does not hold for closed sets. Take $E = \{(0, 0)\}$, which is obviously closed but does not have a limit point.

9. Let $(X, \mathcal{T}_X)$ be a topological space.

   (a) Let $x \in E^\circ$, then there is a neighborhood $U_x \in \mathcal{T}_X$ of x such that $x \in U_x \subseteq E$. Further, for any $y \in U_x$, it is clear that $y \in E^\circ$, since U_x is also a neighborhood for y. As a result, $U_x \subseteq E^\circ$. Finally, since we may write
   $$E^\circ = \bigcup_{x \in E^\circ} U_x$$
   we must have that E° is open.

   (b) If E is open, then for all $x \in E$, there is a neighborhood $U_x \in \mathcal{T}_X$ with $x \in U_x \subseteq E$, therefore $x \in E^\circ$, implying that $E^\circ = E$. The converse follows immediately from (a).

   (c) Straightforward.

   (d) Let $x \in (E^\circ)^c$. If $x \notin E$, then trivially, $x \in \overline{(E^c)}$. On the other hand, if $x \in E \backslash E^\circ$, for every neighborhood U of x, $U \cap E^c \neq \emptyset$, therefore, $x \in \overline{(E^c)}$. This implies $(E^\circ)^c \subseteq \overline{(E^c)}$.
   Conversely, if $x \in \overline{(E^c)}$, then for all neighborhoods U of x, $U \cap E^c \neq \emptyset$, hence, $U \subsetneq E$ and $x \notin E^\circ$. This completes the proof.

(e) No. Take $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean topology. Then, $\mathbb{Q}^\circ = \emptyset$, while $(\overline{\mathbb{Q}})^\circ = \mathbb{R}^\circ = \mathbb{R}$.

(f) No. Take $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean topology. Then, $\overline{\mathbb{Q}} = \mathbb{R}$ while $\overline{(\mathbb{Q}^\circ)} = \overline{\emptyset} = \emptyset$.

10. That the given function is a metric is straightforward. We have

$$B(x, \delta) = \begin{cases} \{x\} & \delta \leq 1 \\ X & \delta > 1 \end{cases}$$

Consequently, every subset of X is open, and as a result, every subset of X is also closed.

We shall show that only finite subsets of X are compact. Obviously, all finite subsets of X are compact. Let $A = \{x_1, \ldots, \}$ be an infinite subset of X. Then, $\mathscr{A} = \{B(x_i, 1) \mid x_i \in A\}$ is an open cover for A with no finite subcover.

11. (i) No. Take $x = 0$, $y = a > 0$ and $z = a/2$. Then, $d_1(x, y) = a^2 > a^2/2 = d_1(x, z) + d_1(z, y)$.

(ii) Yes. For any $x, y, z \in \mathbb{R}$, we have

$$\sqrt{|x-y|} \leq \sqrt{|x-z| + |y-z|} \leq \sqrt{|x-z|} + \sqrt{|y-z|}$$

(iii) No. Since $d_3(x, -x) = 0$ for all $x \in \mathbb{R}$

(iv) No. Since $d_4(2, 1) = 0$

(v) Yes. For $x, y, z \in \mathbb{R}$, we have

$$\begin{aligned} \frac{|x-z|}{1+|x-z|} + \frac{|y-z|}{1+|y-z|} &\geq \frac{|x-z|}{1+|x-z|+|y-z|} + \frac{|y-z|}{1+|x-z|+|y-z|} \\ &= \frac{|x-z|+|y-z|}{1+|x-z|+|y-z|} \\ &\geq \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Note that $|x-y|$ can be replaced by any valid metric.

12. Let $\mathscr{A}$ be an open cover for K. There is $U \in \mathscr{A}$ such that $0 \in U$. Let $B(0, \delta)$ be an open ball contained in U. Then, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{1}{n} \in U$. Consequently, only finitely many elements from $\mathscr{A}$ are required to cover K.

13. Consider $K = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. It is obviously compact and $K' = \{0\}$.

14. Let $\mathscr{A} = \{(\frac{1}{n}, 1) \mid n \in \mathbb{N}\}$. This obviously has no finite subcover.

15.
- Closed: Consider the collection of closed sets $\{A_\alpha\}_{\alpha \in \mathbb{R}}$ given by $A_\alpha = \{x \leq \alpha \mid \alpha \in \mathbb{R}\}$. It obviously has the finite intersection property, but $\bigcap_{\alpha \in \mathbb{R}} A_\alpha = \emptyset$.
- Bounded: Consider the collection of bounded sets $\{A_n\}$ where $A_n = \left(0, \frac{1}{n}\right)$. The collection obviously has the finite intersection property but $\bigcap_{i=1}^{n} A_n = \emptyset$.

16. That E is bounded is self-evident. Let $q \in E^c$, then $q^2 < 2$ or $q^2 > 3$, where the inequalities are strict since q is rational. It is now obvious that there is a neighborhood of q that is contained in E^c, as a result, E^c is open and E is closed. Finally, one also notes that E is open through a similar argument.

Consider the open cover $\{A_n\}_{n \in \mathbb{N}}$, where

$$A_n = \left\{2 + \frac{1}{n+1} < p^2 < 3 \mid p \in \mathbb{Q}\right\}$$

This obviously does not have a finite subcover and E is not compact.

17. Note that I assume the author is referring to an infinite decimal expansion. That E is not countable follows from a simple diagonalization argument.

    E is not dense in $[0,1]$ since there is a neighborhood of 0.5 that does not contain any point in E.

    Next, we shall show that E is closed. Define the sequence of sets $\{S_k\}_{k\in\mathbb{N}_0}$ inductively. $S_0 = [0,1]$ and S_{k+1} to be those points of S_k such that when

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

    we have $d_n \in \{4,7\}$ for all $1 \le n \le k+1$.

    Obviously, each S_k is closed and therefore, so is the intersection $E = \bigcap_{n=1}^{\infty} S_n$. Since E is closed in the compact set $[0,1]$, we must have that E is compact.

    Finally, we shall show that E is perfect. For this it suffices to show that every point of E is a limit point. Let $x \in E$ and $r > 0$. We shall show that $B(x,r) \cap (E\backslash\{x\}) \neq \emptyset$. Choose $N \in \mathbb{N}$ so large that $\frac{3}{10^N} < r$. Let

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

    choose $y = \overline{0.b_1b_2\cdots}$ such that $b_n = d_n$ for all $1 \le n \le N$ and $b_n \in \{4,7\}\backslash\{d_n\}$ for all $n \ge N+1$. As a result, we have

$$|x-y| \le \frac{3}{10^N}\left|\sum_{i=1}^{\infty} \frac{1}{10^N}\right| < \frac{3}{10^N} < r$$

    implying that $y \in B(x,r)$ and $x \in E'$.

18. Let S be the set of all irrational numbers $\theta = [a_0, a_1, \ldots]$ such that $a_0 = 0$ and $a_i \in \{1,2\}$ for all $i \in \mathbb{N}$. It is obvious that $S \subseteq \mathbb{R}\backslash\mathbb{Q}$.

    Let $x \in S$ be given by $[0, a_1, \ldots]$ and $\left\{\frac{p_n}{q_n}\right\}_{n\in\mathbb{N}_0}$ be its convergents. Let $r > 0$. Choose $N \in \mathbb{N}$ such that $\frac{2}{q_N^2} < r$. Define $y = [0, b_1, \ldots]$ such that $a_i = b_i$ for all $1 \le i \le N$, then $\frac{p_N}{q_N}$ is the N-th convergent of y. We now have

$$|x-y| \le \left|x - \frac{p_N}{q_N}\right| + \left|y - \frac{p_N}{q_N}\right| < \frac{2}{q_N^2} < r$$

    which implies that $x \in S'$ and $S \subseteq S'$. It now suffices to show that S is closed. It suffices to verify this for $x \in (0,1)\backslash S$. TODO: Not able to show that S is closed

19. (a) Trivial, since $\overline{A} = A$ and $\overline{B} = B$.

    (b) Suppose $\overline{A} \cap B \neq \emptyset$. Let $x \in \overline{A} \cap B$, therefore, $x \notin A$. There is a neighborhood U of x such that $U \subseteq B$ and thus, $U \cap A = \emptyset$, contradicting the fact that $x \in \overline{A}$.

    (c) Follows from the previous assertion.

    (d) Let $p, q \in X$ be any two distinct points. Due to part (c), we know that there must exist $r_\delta \in X$ for all $0 < \delta < d(p,q)$ such that $d(r_\delta, p) = \delta$. This implies the desired conclusion.

20.
    - Closure: We shall show that the closure of a connected set is indeed connected. Let $S \subseteq X$ be a connected set, where X is any topological space. Suppose $\overline{S}$ were not connected, then $\overline{S} = A \cup B$ for open sets A and B such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Define $A' = S \cap A$ and $B' = S \cap B$, which are open in the relative topology. Furthermore, $\overline{A'} \subseteq \overline{A}$ and it is now obvious that $\overline{A'} \cap B' = \emptyset$. This completes the proof.
    - Interior: Consider $([0,1] \times [0,1]) \cup ([1,2] \times [1,2])$.

21. (a) Suppose, without loss of generality that $\overline{A}_0 \cap B_0 = \emptyset$. If $x \in \overline{A}_0 \cap B_0$, then it is not hard to see that $p(x) \in A'$ and $p(x) \in B$, contradicting that A and B are separated.

(b) Since A_0 and B_0 are separated, they cannot cover $(0,1)$ since it is connected. As a result, there is $t_0 \in (0,1)$ such that $t_0 \notin A_0 \cup B_0$, equivalently, $p(t_0) \notin A \cup B$.

(c) Straightforward.

It follows from the above problem that the only clopen subsets of $\mathbb{R}^k$ are $\emptyset$ and $\mathbb{R}^k$.

22. Easy to see that $\overline{\mathbb{Q}^k} = \mathbb{R}^k$.

23. Let $\mathbb{Q}^+ = \{r_1, r_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be a countabe dense subset of X. We shall show that the collection $\{B(y_i, r_j)\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is a basis for the topology on (X,d). Let $x \in U \in \mathcal{T}_X$, then there is an open ball $B(x,r) \subseteq U$ for some $r > 0$. Since Y is dense in X, there is $y \in Y \cap B(x,r)$. Let q be a rational number such that $0 < q < r - d(x,y)$. We shall show that $B(y,q) \subseteq B(x,r)$. Indeed, for all $z \in B(y, r - d(x,y))$, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + r - d(x,y) = r$$

This completes the proof.

24. Suppose not, then we would have an infinite sequence $\{x_i\}_{i=1}^{\infty}$ such that $d(x_i, x_{i+1}) \geq \delta$, which is an immediate contradiction to the fact that the aforementioned sequence has a limit point.

This obviously implies that X can be covered by finitely many neighborhoods of radius δ, for any $\delta > 0$.

25. Fix some $n \in \mathbb{N}$. Then, the collection $\{B(x, \frac{1}{n})\}_{x\in X}$ is an open cover of X and owing to compactness, has a finite subcover $\{B(x_i, \frac{1}{n})\}_{i=1}^{N_n}$. It is not hard to see that

$$\bigcup_{n=1}^{\infty} \left\{ B(x_i, \frac{1}{n}) \right\}_{i=1}^{N_n}$$

is a countable basis for the topology on the metric (X,d).

26. Using (24), we know that X is separable and due to (23), it must have a countable basis. Let $\mathscr{A}$ be an open cover for X. Since X is second-countable, it must be Lindelöf. Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable subcover of $\mathscr{A}$. Define

$$F_n = \bigcap_{i=1}^{n} A_i^c$$

Obviously, $F_1 \supseteq F_2 \supseteq \cdots$, further, note that each F_i is closed. Suppose $\{A_n\}_{n\in\mathbb{N}}$ has no finite subcover. Then, $\bigcap_{i=1}^{\infty} F_n = \emptyset$. Let S be a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in F_n$. By our hypothesis, the above sequence has a limit point, say $x \in X$.

Now, we shall show that $x \in F_n$ for each $n \in \mathbb{N}$. Suppose not, say $x \notin F_n$ for some n. Then, due to the chain condition, $x \notin F_m$ for all $m \geq n$. Choose $r \in \mathbb{R}$ such that

$$0 < r < \min_{1\leq i\leq n} d(x, x_i)$$

Then, $B(x,r) \cap F_i = \emptyset$ for all $i \in \mathbb{N}$, a contradiction to the fact that x is a limit point of S.

As a result, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ which is again a contradiction to our initial assumption. Therefore, $\mathscr{A}$ has a finite subcover and X is compact.

3 Numerical Sequences and Series