TATE'S THESIS
VSRP NOTES

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1 Haar Measure

The main reference for this section is [DE09].

Definition 1.1. A *Radon measure* on a topological space X is a Borel measure $\mu : \mathscr{B} \to [0, \infty]$ that is

locally finite: every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$.

outer regular: every $S \in \mathscr{B}$ satisfies:

$$\mu(S) = \inf\{\mu(U) \mid S \subseteq U \text{ open}\}.$$

inner regular on open sets: every open $U \subseteq X$ satisfies

$$\mu(U) = \sup\{\mu(K) \mid U \supseteq K \text{ compact}\}.$$

We work under the assumption that all topological groups are Hausdorff.

Definition 1.2. A *left Haar measure* on a topological group G is a nonzero left-invariant Radon measure μ on G. That is, for each Borel set E, and $g \in G$,

$$\mu(E) = \mu(gE).$$

Analogously one defines a *right Haar measure*.

Theorem 1.3 (Existence and Uniqueness of Haar measure). Let G be a locally compact topological group.

(a) There is a left invariant Haar measure μ on G.

(b) Any other left Haar measure on G is a scalar multiple of the above.

An analogous statement holds for the right Haar measure. One must note that the left and right Haar measures need not coincide. Indeed, if μ_L is a left Haar measure, then μ_R given by $\mu_R(E) = \mu_L(E^{-1})$ is a right Haar measure.

Proposition 1.4. Let G be a locally compact topological group and μ a left Haar measure on G. G is compact if and only if $\mu(G) < \infty$. In this case, $\mu_L = \mu_R$.

Proof. Suppose G were compact. Then, there is a neighborhood U of e having finite measure. Note that $\{gU\}_{g \in G}$ forms an open cover of G that admits a finite subcover, whence it follows that $\mu(G) < \infty$.

Suppose G were not compact. Using local compactness, there is a neighborhood of the origin with compact closure. Call this closure K. Then, $0 < \mu(K) < \infty$. Set $s_1 = e$. We inductively construct a sequence $\{s_n\}$ such that the $s_n K$'s are pairwise disjoint. Note that $\bigcup_{i=1}^{n} s_i K K^{-1}$ is a finite union of compact sets and hence, is compact. Since G isn't compact, there is an s_{n+1} in G but not in the aforementioned union.

Finally, note that

$$\mu(G) > \mu\left(\bigcup_{i=1}^{n} s_i K\right) = n\mu(K).$$

The conclusion follows.

For the second part, we introduce the *modular function*. For each $x \in G$, define the measure $\mu_x : \mathscr{B} \to [0, \infty]$ by $\mu_x(E) = \mu(Ex)$. This is a left Haar measure and hence, $\mu_x = \Delta(x)\mu$. One can show that Δ is a continuous group homomorphism $G \to \mathbb{R}^+$. Therefore, the image of Δ must be a compact subgroup of $\mathbb{R}^+$. The only such subgroup is $\{1\}$. This completes the proof. ∎

Proposition 1.5. Let G be a locally compact group and H a closed subgroup. Then, G/H is a locally compact Hausdorff space.

Theorem 1.6 (Quotient Integration Theorem). Let G be a locally compact topological group and H a closed subgroup. Then, there is an invariant Radon measure $\nu \neq 0$ on G/H if and only if the modular functions Δ_G and Δ_H agree on H.

In this case, the measure ν is unique up to scaling. In particular, given Haar measures on G and H, there is a unique choice for ν. Finally, for ay $f \in L^1(G)$,

$$\int_G f(x)\, dx = \int_{G/H} \int_H f(xh)\, dh\, dx.$$

2 Pontryagin Duality

Henceforth, G is a locally compact abelian topological group.

Definition 2.1. A *character* of G is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$. A *unitary character* of G is a continuous group homomorphism $\chi : G \to S^1$.

We note that the nomenclature is not uniform. Some authors call these quasi-characters and characters, respectively.

Proposition 2.2. If G is compact, then every character of G is unitary.

Proof.

$$\begin{tikzcd} G \arrow[r, "\chi"] \arrow[dr, "\varphi"'] & \mathbb{C}^\times \arrow[d, "|\cdot|"] \\ & \mathbb{R}^+ \end{tikzcd}$$

Then, φ is a continuous group homomorphism. The image must be compact and hence, $\{1\}$. ∎

Definition 2.3 (Compact-Open Topology). Let X and Y be topological spaces. The *compact-open topology* on $C(X, Y)$, the set of continuous functions from X to Y is defined to be the topology generated by sets of the form

$$\{f \in C(X, Y) \mid f(K) \subseteq U\},$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

Definition 2.4 (Pontryagin Dual). The *Pontryagin dual* of G is the group

$$\widehat{G} := \mathrm{Hom}_{\mathrm{cts}}(G, S^1),$$

which is the group of unitary characters of G under pointwise multiplication. We equip $\widehat{G}$ with the *compact-open topology*.

Proposition 2.5. Let G be an abelian topological group.

(a) If G is discrete, then $\widehat{G}$ is compact.

(b) If G is compact, then $\widehat{G}$ is discrete.

(c) If G is locally compact, then $\widehat{G}$ is locally compact.

Example 2.6. We compute the Pontryagin dual of $\mathbb{R}$. Let $\chi : \mathbb{R} \to S^1$ be a continuous homomorphism. Since $\mathbb{R}$ is simply connected, this factors through the universal cover $\mathbb{R} \to S^1$ given by $x \mapsto \exp(2\pi i x)$. This gives us

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ \mathbb{R} \arrow[ur, dashed, "\widetilde{\chi}"] \arrow[r, "\chi"'] & S^1 \end{tikzcd}$$

where $\widetilde{\chi}$ is a continuous additive group homomorphism $\mathbb{R} \to \mathbb{R}$ and hence, is linear. Thus, $\widehat{\mathbb{R}} \cong \mathbb{R}$.

The Pontryagin dual is a contravariant functor from the category of locally compact abelian groups to itself.

Theorem 2.7 (Exactness of Pontryagin Dual). If $0 \to A \to B \to C \to 0$ is a short exact sequence of locally compact abelian groups, then so is $0 \to \widehat{C} \to \widehat{B} \to \widehat{A} \to 0$.

For each $g \in G$, there is the evaluation map $\mathrm{ev}_g : \widehat{G} \to S^1$ given by $\chi \mapsto \chi(g)$. This is a continuous group homomorphism and hence, an element of $\widehat{\widehat{G}}$.

Theorem 2.8 (Pontryagin Duality). The canonical map $G \to \widehat{\widehat{G}}$ given by $g \mapsto \mathrm{ev}_g$ is an isomorphism of locally compact abelian groups.

Definition 2.9 (Fourier Transform). If $f \in L^1(G)$, define the *Fourier Transform* $\widehat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\chi) = \int_G f(g)\overline{\chi(g)}\, dg.$$

Proposition 2.10. If $f \in L^1(G)$, then $\widehat{f} \in C_0(\widehat{G})$.

Theorem 2.11 (Fourier Inversion). Let G be a locally compact abelian group and dg a Haar measure on G. Then, there is a unique Haar measure $d\chi$ on $\widehat{G}$ called the *dual measure* such that if $f \in L^1(G)$ is such that $\widehat{f} \in L^1(G)$, then

$$f(g) = \int_{\widehat{G}} \widehat{f}(\chi)\, \mathrm{ev}_g(\chi)\, d\chi = \int_{\widehat{G}} \widehat{f}(\chi)\chi(g)\, d\chi$$

almost everywhere on G.

Remark 2.12. The measure $d\chi$ defined above on $\widehat{G}$ is called the *dual measure*.

Theorem 2.13 (Plancherel). Let dg and $d\chi$ be dual measures on G and $\widehat{G}$ respectively. For every $f \in L^1(G) \cap L^2(G)$, $\widehat{f} \in L^2(\widehat{G})$ and $\|f\|_2 = \|\widehat{f}\|_2$.

Therefore, the Fourier transform extends uniquely to an isometric automorphism of $L^2(G)$.

3 Restricted Direct Product

Definition 3.1. Let $\{(G_\alpha, H_\alpha)\}_{\alpha \in A}$ be a collection of topological groups where H_α is a subgroup of G_α. Define the restricted direct product

$$\widetilde{\prod}_{\alpha \in A} G_\alpha := \{(x_\alpha) \mid x_\alpha \in G_\alpha,\ x_\alpha \in H_\alpha \text{ almost everywhere}\}.$$

This obviously forms a group under pointwise multiplication.

References

[DE09] Anton Deitmar and Siegfried Echterhoff. *Principles of Harmonic Analysis*. Springer, 2009.