Differential Topology

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Contents

Part I

Multivariable Calculus

Chapter 1

Differentiation

Definition 1.1. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be *differentiable* at $a \in U$ if there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to \mathbf{0}} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0$$

The linear transformation T is called the *derivative* of f at a and is denoted by $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$.

The following proposition establishes the uniqueness of the derivative at a point, if it exists.

Proposition 1.2. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in U$. Then, there is a unique linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to \mathbf{0}} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0$$

Proof. Let $\mu, \lambda : \mathbb{R}^n \to \mathbb{R}^m$ be two linear transformations satisfying the requirements. Then, we have

$$\|\lambda(h) - \mu(h)\| \leq \|f(a+h) - f(a) - \mu(h)\| + \|f(a+h) - f(a) - \lambda(h)\|$$

Consequently,

$$\lim_{h \to \mathbf{0}} \frac{\|\lambda(h) - \mu(h)\|}{\|h\|} \leq \lim_{h \to \mathbf{0}} \frac{\|\lambda(h) - \mu(h)\| \leq \|f(a+h) - f(a) - \mu(h)\|}{\|h\|} + \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0$$

Now, let $x \in \mathbb{R}^n$. Then,

$$0 = \lim_{t \to 0} \frac{\|\mu(tx) - \lambda(tx)\|}{\|tx\|} = \frac{\|\mu(x) - \lambda(x)\|}{\|x\|}$$

This completes the proof. ∎

Theorem 1.3 (Chain Rule). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be functions differentiable at a and $b = f(a)$ respectively. Then, the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a) = Dg(b) \circ Df(a)$$

Proof. ∎

Proposition 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given by $f = (f_1, \ldots, f_m)$. Then f is differentiable if and only if each $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable and*

$$Df(a) = \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_m(a) \end{bmatrix}$$

Proof. Suppose f is differentiable and π_i denote the projection on the i-th coordinate. Since π_i is differentiable, so is $f_i = \pi_i \circ f$. Conversely suppose each f_i is differentiable and let

$$A = \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_m(a) \end{bmatrix}$$

Then, for $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, we have

$$\begin{aligned} \frac{\|f(a+h) - f(a) - Ah\|}{\|h\|} &= \frac{\left\| \begin{bmatrix} f_1(a+h) - f_1(a) - Df_1(a)h \\ \vdots \\ f_m(a+h) - f_m(a) - Df_m(a)h \end{bmatrix} \right\|}{\|h\|} \\ &\leq \sum_{i=1}^{m} \frac{\|f_i(a+h) - f(a) - Df_i(a)(h)\|}{\|h\|} \end{aligned}$$

whence the limit tends to 0 as $h \to \mathbf{0}$ which completes the proof. ■

Definition 1.5 (Partial Derivatives). Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. The limit

$$\lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

if it exists is called the *i-th partial derivative of f at a* and is denoted by $D_i f(a)$. We also define *mixed partial derivatives* of f at a by

$$D_{i,j} f(a) = D_i(D_j f)(a).$$

Theorem 1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}$. If $D_{i,j} f$ and $D_{j,i} f$ are continuous in an open set containing a, then*

$$D_{i,j} f(a) = D_{j,i} f(a)$$

Proof. The proof uses Fubini's Theorem and is therefore postponed. ■

Lemma 1.7. *Let $A \subseteq \mathbb{R}^n$ be a closed rectangle. If the maximum (resp. minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof. Let $a = (a_1, \ldots, a_n)$ and $h_i(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$. Then h_i has a maximum (resp. minimum) at a_i, is defined in an open interval containing a_i and is differentiable at a_i, whence from the calculus of a single variable, we see that $0 = h_i'(a_i) = D_i f(a)$, which completes the proof. ■

Theorem 1.8. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ and is given by $f = (f_1, \ldots, f_m)$, then $D_j f_i(a)$ exists for $1 \leq i \leq m$ and $1 \leq j \leq n$ and $Df(a)$ is the $m \times n$ matrix $\left[D_j f_i(a)\right]_{i,j}$.*

Proof. Since f is differentiable, $Df(a)$ is the matrix obtained by stackig $Df_i(a)$ as rows. Therefore, it suffices to prove the statement of the theorem in the case $m = 1$, that is $f : \mathbb{R}^n \to \mathbb{R}$ is given to be differentiable.

Consider the map $h : \mathbb{R} \to \mathbb{R}^n$ given by

$$h(x) = (a_1, \ldots, x, \ldots, a_n).$$

Then, due to Theorem 1.3,

$$D_j f(a) = D(f \circ h)(a_j) = Df(h(a_j))Dh(a_j) = Df(a) \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

This completes the proof. ■

Theorem 1.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$ with $f = (f_1, \ldots, f_m)$. If there is an open set U containing a on which $D_j f_i$ exists and is continuous at a for $1 \leq i \leq m$ and $1 \leq j \leq n$, then f is differentiable at a.*

Proof. Due to Proposition 1.4, we may suppose that $m = 1$. Let $r > 0$ such that $B(a, r) \subseteq U$ and h be sufficiently small such that $a + h \in B(a, r)$. Then,

$$f(a+h) - f(a) = f(a_1+h_1, \ldots, a_n) - f(a_1, \ldots, a_n) + \cdots + f(a_1+h_1, \ldots, a_n+h_n) - f(a_1+h_1, \ldots, a_{n-1}+h_{n-1}, a_n)$$

Using the mean value theorem, we have

$$f(a_1 + h_1, \ldots, a_i + h_i, \ldots, a_n) - f(a_1 + h_1, \ldots, a_{i-1} + h_{i-1}, \ldots, a_n) = h_i D_i f(c_i)$$

where $c_i = (a_1 + h_1, \ldots, b_i, a_{i+1}, \ldots, a_n) \in B(a, r)$ for some $b_i \in (a_i, a_i + h_i)$.

Let $\varepsilon > 0$ be given. Using uniform continuity on some (bounded) closed (and therefore compact) rectangle contained in U, we may choose an $r > 0$ such that whenever $|x - y| \leq r$, $|D_i f(x) - D_i f(y)| < \varepsilon/n$ for each $1 \leq i \leq n$. Note that this can be done because all the D_i's are continuous on U. Then, we have, for any $\|h\| < r$,

$$\begin{aligned} \frac{\left\|f(a+h) - f(a) - \sum_{i=1}^n h_i D_i f(a)\right\|}{\|h\|} &= \frac{\left\|\sum_{i=1}^n h_i D(c_i) - \sum_{i=1}^n h_i D_i f(a)\right\|}{\|h\|} \\ &\leq \sum_{i=1}^n \frac{\|h_i(D_i f(c_i) - D_i f(a))\|}{\|h\|} \\ &\leq \sum_{i=1}^n \|D_i f(c_i) - D_i f(a)\| < \varepsilon \end{aligned}$$

This completes the proof. ■

1.1 Inverse and Implicit Function Theorem

Lemma 1.10.

Theorem 1.11 (Inverse Function Theorem). *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $a \in U$ such that $\det(Df(a)) \neq 0$. Then, there is an open set V containing a and an open set W containing $f(a)$ such that the restriction $f : V \to W$ is a diffeomorphism.*

Proof. Upon composing f with a suitable linear transformation[1], we may suppose, without loss of generality that $Df(a) = \mathbf{id}_{n\times n}$. Then, we have

$$0 = \lim_{h\to\mathbf{0}} \frac{\|f(a+h) - f(a) - h\|}{\|h\|}$$

and thus, we may shrink U to a small enough open set such that $f(x) \neq f(a)$ for all $x \in U$. Since f is continuously differentiable, the function $\det Df(x)$ is a continuous function, and since $\det Df(a) \neq 0$, we may shrink U further such that $\det Df(x) \neq 0$ for all $x \in U$.

Using the continuity and therefore uniform continuity of $D_j f_i$ for each pair i, j, we may choose a closed rectangle A in U such that for all $x, y \in A$,

$$|D_j f_i(x) - D_j f_i(y)| < \frac{1}{2n^2}.$$

Consider now the function $g(x) = f(x) - x$. This is also continuously differentiable and for $x, y \in A$,

$$|D_j g_i(x) - D_j g_i(y)| = |D_j f_i(x) - D_j f_i(y)| < \frac{1}{2n^2}$$

Thus, using Lemma 1.10, we have for $x_1, x_2 \in A$,

$$\|(f(x_1) - f(x_2)) - (x_1 - x_2)\| = \|(f(x_1) - x_1) - (f(x_2) - x_2)\| = \|g(x_1) - g(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|,$$

consequently,

$$\|f(x_1) - f(x_2)\| \geq \frac{1}{2}\|x_1 - x_2\|$$

Thus f restricted to A is an injective map.

Let γ denote the boundary of A. Since $f(\gamma)$ is a compact set not containing $f(a)$, there is $d > 0$ such that for all $x \in \gamma$, $|f(a) - f(x)| \geq d$. Let $W = B(f(a), d/2)$. We contend that for every $y \in W$, there is a *unique* $x \in A$ such that $f(x) = y$.

Indeed, consider the function

$$h(x) = \|f(x) - y\|^2 = \sum_{i=1}^{n} |f_i(x) - y_i|^2.$$

Since f is a continuous function, so is h and since A is compact, there is a point $x_0 \in A$ at which h attains its minimum. First, notice that x_0 may not lie on γ since for all $x \in \gamma$, by construction, we have $\|f(a) - y\| < d/2 < \|f(x) - y\|$.

Since x_0 lies in the interior of A and the partials $D_j h$ exist for all j, we have

$$0 = D_j h(x_0) = 2\sum_{i=1}^{n} (f_i(x_0) - y_i) D_j f_i(x_0).$$

[1] We may do this as $\det Df(a) \neq 0$.

Equivalently, we may write this in matrix form as

$$0 = \begin{bmatrix} D_1 f_1(x_0) & \cdots & D_1 f_n(x_0) \\ \vdots & \ddots & \vdots \\ D_n f_1(x_0) & \cdots & D_n f_n(x_0) \end{bmatrix} \begin{bmatrix} f_1(x_0) - y_1 \\ \vdots \\ f_n(x_0) - y_n \end{bmatrix} = Df(x_0) \begin{bmatrix} f_1(x_0) - y_1 \\ \vdots \\ f_n(x_0) - y_n \end{bmatrix}.$$

We have $\det Df(x_0) \neq 0$ since $x_0 \in A \subseteq U$, and thus $f_i(x_0) = y_i$, equivalently, $f(x_0) = y$. The uniqueness follows from the injectivity of f on A.

Let $V = f^{-1}(W) \cap \operatorname{int}(A)$. Henceforth, we work with the restriction $f : V \to W$, which we have shown to be a continuously differentiable bijection. It remains to show that the inverse is continuously differentiable. Let $p : W \to V$ denote the inverse of f. Then, we have

$$\|p(y_1) - p(y_2)\| \leq 2\|y_1 - y_2\|$$

for all $y_1, y_2 \in W$ whence continuity of p follows. It remains to show the differentiability of p. ■

p is differentiable

We note that the condition on the continuity of the derivative cannot be dropped from the hypothesis of Theorem 1.11. Indeed, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This is differentiable on $\mathbb{R}$ with $f'(0) \neq 0$, but the derivative,

$$f'(x) = \begin{cases} \frac{1}{2} - \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

is not continuous at $x = 0$. For sufficiently large N, consider the point $x_N = 2/(2N+1)\pi$. It is not hard to argue that $f'(x_N) < 0$ whence f is not injective in any neighborhood containing 0. Thus it may not have an inverse, let alone a differentiable one.

Theorem 1.12 (Implicit Function Theorem).

Add in later

Chapter 2

Integration

Definition 2.1 (Oscillation). Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be bounded and $a \in A$. For every $\delta > 0$, define

$$M(a, f, \delta) = \sup\{f(x) \mid x \in A,\ \|x - a\| < \delta\} \qquad m(a, f, \delta) = \inf\{f(x) \mid x \in A,\ \|x - a\| < \delta\}$$

The *oscillation* of f at a is defined by

$$o(f, a) = \lim_{\delta \to 0}(M(a, f, \delta) - m(a, f, \delta))$$

We impose the boundedness condition on f to make sure that both $M(a, f, \delta)$ and $m(a, f, \delta)$ are well defined real numbers. Note that upon fixing a, the function $M(a, f, \cdot)$ is a decreasing function of $\delta > 0$ and $m(a, f, \cdot)$ is an increasing function of $\delta > 0$ whereby, the limit exists, since $M(a, f, \cdot) - m(a, f, \cdot)$ is a decreasing function of δ and is bounded below by 0.

Proposition 2.2. *A bounded function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous at $a \in A$ if and only if $o(f, a) = 0$.*

Proof. Suppose f is continuous at a. Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $\|x - a\| < \delta$ and $x \in A$. Then, for all such x, $M(a, f, \delta) - m(a, f, \delta) < 2\varepsilon$, consequently, $o(f, a) = 0$.

Conversely, suppose $o(f, a) = 0$. Let $\varepsilon > 0$ be given. Then, there is a $\delta > 0$ such that $M(a, f, \delta) - m(a, f, \delta) < \varepsilon$. Then, for all $x \in A$ with $\|x - a\| < \delta$, we have

$$-\varepsilon < -(M(a, f, \delta) - m(a, f, \delta)) \leq f(x) - f(a) \leq M(a, f, \delta) - m(a, f, \delta) < \varepsilon.$$

This completes the proof. ■

Theorem 2.3. *Let $A \subseteq \mathbb{R}^n$ be closed. If $f : A \to \mathbb{R}$ is a bounded function and $\varepsilon > 0$, then the set $B = \{x \in A \mid o(f, x) \geq \varepsilon\}$ is closed.*

Proof. We shall show that $\mathbb{R}^n \backslash B$ is open. If $x \in \mathbb{R}^n \backslash B$ and $x \notin A$, then there is trivially an open rectangle containing x disjoint from A and thus from B. On the other hand, if $x \in A$, then there is a $\delta > 0$ such that $M(x, f, \delta) - m(x, f, \delta) < \varepsilon$. Let C be an open rectangle contained in the open ball $B(x, \delta)$ in $\mathbb{R}^n$ (this may contain points not in A). Let $y \in C \cap A$. Choose δ' such that $B(y, \delta') \subseteq C$. Then, $M(y, f, \delta') < M(x, f, \delta)$ and $m(y, f, \delta') \geq m(x, f, \delta)$ whence $M(y, f, \delta') - m(y, f, \delta') < \varepsilon$ and $y \notin B$. This completes the proof. ■

2.1 The Setup

We borrow the idea of partitions from the Riemann Integral of a function of one variable.

Definition 2.4 (Partition). Let $A \subseteq \mathbb{R}^n$ be a closed rectangle, i.e. $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$. A partition of A is a collection $P = (P_1, \ldots, P_n)$ where each P_i given by $a = t_0^{(i)} < t_1^{(i)} < \cdots < t_{m_i}^{(i)} = b$ is a partition of the interval $[a_i, b_i]$.

Rectangles of the form

$$[t_{r_i}^{(1)}, t_{r_i+1}^{(1)}] \times \cdots \times [t_{r_n}^{(n)}, t_{r_n+1}^{(n)}]$$

are called *subrectangles of the partition P*. The collection of subrectangles of P is denoted by $\mathscr{S}(P)$. A partition $P' = (P'_1, \ldots, P'_n)$ is said to *refine* P if each P'_i refines P_i.

Definition 2.5 (Integral). Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a closed rectangle A and let P be a partition of A. For each $S \in \mathscr{S}(P)$ define

$$m_S(f) := \inf\{f(x) \mid x \in S\} \quad \text{and} \quad M_S(f) := \sup\{f(x) \mid x \in S\}.$$

Using this, we define the *upper and lower sums of f for the partition P* as

$$L(f, P) := \sum_{S \in \mathscr{S}(P)} m_S(f) v(S) \quad \text{and} \quad U(f, P) := \sum_{S \in \mathscr{S}(P)} M_S(f) v(S).$$

The function f is said to be *integrable over A* if

$$\mathbf{L}\int_A f := \sup_{P \in \mathscr{P}(A)} L(f, P) = \inf_{P \in \mathscr{P}(A)} U(f, P) =: \mathbf{U}\int_A f.$$

This common value is called the *integral of f over A* and is denoted by either

$$\int_A f \quad \text{or} \quad \int_A f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n.$$

Lemma 2.6. *Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ is a closed rectangle and $P, P' \in \mathscr{P}(A)$.*

(a) If P' refines P, then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

(b) $L(f, P') \leq U(f, P)$.

Proof. (a) Straightforward computation.

(b) Let $P'' = P \cup P' := (P_1 \cup P'_1, \ldots, P_n \cup P'_n)$. Then P'' refines both P and P' whence

$$L(f, P') \leq L(f, P'') \leq U(f, P'') \leq U(f, P).$$

∎

Proposition 2.7. *Let $A \subseteq \mathbb{R}^n$ be a closed rectangle and $f : A \to \mathbb{R}$ a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$, there is $P \in \mathscr{P}(A)$ such that $U(f, P) - L(f, P) < \varepsilon$.*

Proof. Suppose f is integrable. Then, there are partitions $P, P' \in \mathscr{P}(A)$ such that

$$\int_A f - \frac{\varepsilon}{2} < L(f, P) \leq U(f, P') < \int_A f + \frac{\varepsilon}{2}.$$

Let $P'' \in \mathscr{P}$ refine both P and P'. Then,

$$\int_A f - \frac{\varepsilon}{2} < L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P') < \int_A f + \frac{\varepsilon}{2}$$

whence $U(f, P'') - L(f, P'') < \varepsilon$. The converse is trivial to prove. ■

Lemma 2.8. *Let $A \subseteq \mathbb{R}$ be a closed rectangle, $f : A \to \mathbb{R}$ a bounded function and $\varepsilon > 0$ such that $o(f, x) < \varepsilon$ for all $x \in A$. Then there is a partition $P \in \mathscr{P}(A)$ such that $U(f, P) - L(f, P) < \varepsilon v(A)$.*

Proof. ■

Definition 2.9 (Integration over Jordan measurable sets). Let $C \subseteq \mathbb{R}^n$ be a Jordan measurable set and $f : A \to \mathbb{R}^n$ a bounded function on a closed rectangle A containing C. Then, we *define*

$$\int_C f = \int_A \chi_C \cdot f.$$

2.2 Fubini's Theorem

Theorem 2.10 (Fubini). *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed rectangles and $f : A \times B \to R$ be a bounded integrable function. Denote by g_x the function $f(x, \cdot) : B \to \mathbb{R}$ and let*

$$\mathfrak{L}(x) = \mathbf{L} \int_B g_x \quad \textit{and} \quad \mathfrak{U}(x) = \mathbf{U} \int_B g_x.$$

Then $\mathfrak{L}, \mathfrak{U} : A \to \mathbb{R}$ are integrable and

$$\int_A \mathfrak{L} = \int_{A \times B} f = \int_A \mathfrak{U}.$$

Proof. Let P be a partition of $A \times B$. Then, P is of the form (P_A, P_B) where P_A is a partition of A and P_B is a partition of B. Then, every subrectangle in $\mathscr{S}(P)$ is of the form $S_A \times S_B$ where $S_A \in \mathscr{S}(P_A)$ and $S_B \in \mathscr{S}(P_B)$.

$$\begin{aligned}
L(f, P) &= \sum_{S \in \mathscr{S}(P)} m_S(f) v(S) \\
&= \sum_{S_A \in \mathscr{S}(P_A)} \sum_{S_B \in \mathscr{S}(P_B)} m_{S_A \times S_B}(f) v(S_A \times S_B) \\
&= \sum_{S_A \in \mathscr{S}(P_A)} \left(\sum_{S_B \in \mathscr{S}(P_B)} m_{S_A \times S_B}(f) v(S_B) \right) v(S_A) \\
&\leq \sum_{S_A \in \mathscr{S}(P_A)} \left(\sum_{S_B \in \mathscr{S}(P_B)} m_{S_B}(g_x) v(S_B) \right) v(S_A) \\
&\leq \sum_{S_A \in \mathscr{S}(P_A)} \left(\mathbf{L} \int_B g_x \right) v(S_A) = \sum_{S_A \in \mathscr{S}(P_A)} \mathfrak{L}(x) v(S_A)
\end{aligned}$$

for all $x \in S_A$. Therefore,

$$L(f, P) \leq \sum_{S_A \in \mathscr{S}(P_A)} m_{S_A} \left(\mathfrak{L}(x) \right) v(S_A) = L(\mathfrak{L}, P_A).$$

Using a similar argument, we obtain $U(f,P) \geq U(\mathfrak{U}, P_A)$ whence

$$L(f,P) \leq L(\mathfrak{L}, P_A) \leq \underbrace{U(\mathfrak{L}, P_A) \leq U(\mathfrak{U}, P_A)}_{\mathfrak{L} \leq \mathfrak{U} \text{ for all } x \in A} \leq U(f,P).$$

Since f is integrable, for every $\varepsilon > 0$, there is a partition P of $A \times B$ such that $U(f,P) - L(f,P) < \varepsilon$ whence $U(\mathfrak{L}, P_A) - L(\mathfrak{L}, P_A) < \varepsilon$, implying that $\mathfrak{L}$ is integrable over A and

$$\int_{A \times B} f = \int_A \mathfrak{L}.$$

A similar argument can be applied for $\mathfrak{U}$. This completes the proof. ∎

2.3 Partitions of Unity

2.3.1 The Bump Function

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

It is not hard to argue that $f \in C^\infty(\mathbb{R})$. Consider now $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = f(1-x)f(1+x).$$

Then $g \in C^\infty(\mathbb{R})$ and g is nonzero only on $(-1,1)$ and is positive there. Define

$$h(x) = \frac{\int_0^x g\left(\frac{x+1}{2}\right)\, dx}{\int_0^1 g\left(\frac{x+1}{2}\right)\, dx}.$$

Then $h \in C^\infty(\mathbb{R})$ such that $h(x) = 0$ for all $x \leq 0$ and $h(x) = 1$ for all $x \geq 1$.

Let now $U \subseteq \mathbb{R}^n$ be open and $C \subseteq U$ a compact subset. For each $a \in C$, there is an $\varepsilon_a > 0$ such that the cube

$$a \in \underbrace{[a_1 - \varepsilon_a, a_1 + \varepsilon_a] \times \cdots \times [a_n - \varepsilon_a, a_n + \varepsilon_a]}_{Q_a} \subseteq U.$$

Consider the function $F_a : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F_a(x) = \prod_{i=1}^n f\left(\frac{x_i - a_i}{\varepsilon_a}\right).$$

Then, $F_a(x) > 0$ for all $x \in \operatorname{Int} Q_a$ and $F_a(x) = 0$ for all $x \notin Q_a$. The collection $\{\operatorname{Int} Q_a\}_{a \in C}$ forms an open cover of C whence has a finite subcover, say $\{Q_{a_1}, \ldots, Q_{a_m}\}$. Let

$$F(x) = \sum_{i=1}^m F_{a_i}(x).$$

Then, $F(x) > 0$ for all $x \in C$ and $F(x) = 0$ for all $x \notin Q := \bigcup_{i=1}^m Q_{a_i}$, which is a closed (in fact, compact) set contained in U.

Let $\delta := \inf_{x \in C} F(x)$. Since C is compact, this minimum is achieved somewhere in C and thus is nonzero. Consider the composition $G : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$G(x) := h\left(F(x)/\delta\right).$$

Then $G(x)$ is a C^∞ function such that

- $G(x) = 1$ for all $x \in C$,
- $G(x) = 0$ for all $x \notin Q$,
- and thus $\mathrm{Supp}(G) \subseteq Q \subseteq U$ is a compact set.

This is called the *bump function*.

2.3.2 Constructing Partitions of Unity

Lemma 2.11. *Let $U \subseteq \mathbb{R}^n$ be an open set. Then, there is an ascending chain of compact sets $K_1 \subseteq K_2 \subseteq \cdots$ such that $K_i \subseteq \operatorname{Int} K_{i+1}$ and $U \subseteq \bigcup_{i=1}^{\infty} K_i$.*

Proof. ■

Theorem 2.12. *Let $A \subseteq \mathbb{R}^n$ and $\mathscr{U}$ be an open cover of A. Then, there is a collection Φ of $C^\infty(\mathbb{R})$ functions with the following properties:*

(a) For each $x \in A$ and $\varphi \in \Phi$, $0 \leq \varphi(x) \leq 1$.

(b) For each $\varphi \in \Phi$, there is an open set $U \in \mathscr{U}$ such that $\mathrm{Supp}(\varphi) \subseteq U$.

(c) The collection $\{\mathrm{Supp}(\varphi) \mid \varphi \in \Phi\}$ is a locally finite collection of compact sets.

(d) For each $x \in A$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$. This makes sense since only finitely many of the φ are nonzero for any $x \in A$.

Such a collection is called a partition of unity for A subordinate to $\mathscr{U}$.

Proof. There are three steps in this proof. First, we construct a partition of unity in the case when A is compact. Then, for an open A, we use the compact exhaustion of A to construct a partition of unity and finally, the case for an arbitrary A follows immediately, as we shall see.

Case 1. A is compact.

Case 2. A is open.

Case 3. A is arbitrary. ■

Remark 2.3.1. *Since $\mathbb{R}^n$ is second countable, every open cover of A can be reduced to a countable open cover of A whence we may choose our partition of unity to contain only countably many terms.*

Definition 2.13 (Extended Integral). An open cover $\mathscr{U}$ of an open set $A \subseteq \mathbb{R}^n$ is said to be *admissible* if each $U \in \mathscr{U}$ is contained in A. Let $f : A \to \mathbb{R}$ be such that for all $x \in A$, f is bounded in some open set containing x and the set of discontinuities of f in A has measure 0, then, f is said to be *integrable in the extended sense* if the sum

$$\int_{\varphi \in \Phi} \varphi \cdot |f|$$

converges for some countable partition of unity subordinate to $\mathscr{U}$. The integral of f is now defined as

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f.$$

Recall that due to Remark 2.3.1, we know that every admissible open cover admits a countable partition of unity subordinate to it.

Theorem 2.14. *Let $A \subseteq \mathbb{R}^n$ be open, $f : A \to \mathbb{R}$ be a function.*

(a) Let Ψ be another partition of unity subordinate to an admissible cover $\mathscr{V}$ of A, then $\sum_{\psi \in \Psi} \int_A \psi \cdot |f|$ also converges and

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\psi \in \Psi} \int_A \psi \cdot F.$$

(b) If A and f are bounded, then f is integrable in the extended sense.

(c) If A is Jordan-measurable and f is bounded, then this definition of $\int_A f$ agrees with the old one.

Proof. ■

2.4 Change of Variables

Theorem 2.15. *Let $A \subseteq \mathbb{R}^n$ be an open set and $g : A \to \mathbb{R}^n$ an injective, continuously differentiable function such that $\det(Dg(x)) \neq 0$ for all $x \in A$. If $f : g(A) \to \mathbb{R}$ is integrable, then*

$$\int_{g(A)} f = \int_A (f \circ g) |\det Dg|$$

Proof. ■ Add in later

Part II

Manifolds

Chapter 3

Smooth Manifolds

3.1 Topological manifolds

Definition 3.1 (Locally Euclidean). A topological space X is said to be *locally Euclidean of dimension n* if every $x \in X$ has a neighborhood $U \subseteq X$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 3.2 (Manifold). A *topological manifold of dimension n* is a topological space which is Hausdorff, second countable and locally Euclidean of dimension n.

Remark 3.1.1. *Recall from Algebraic Topology, that an open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$ only if $m = n$. This is proved using the* Excision Theorem. *Therefore, the dimension of a manifold is unambiguously defined.*

Definition 3.3 (Chart). Let M be a topological n-manifold. A *coordinate chart* on M is a pair (U, φ) where U is an open set of M and $\varphi : U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The chart (U, φ) is said to be *centered* at $p \in M$ if $\varphi(p) = 0$.

If $\widehat{U}$ is an open ball in $\mathbb{R}^n$ then U is said to be a *coordinate ball*, and similarly, if $\widehat{U}$ is an open cube in $\mathbb{R}^n$, then U is said to be a *coordinate cube*.

The map φ is called a *local coordinate map* and the component functions $(x^1, \dots, x^n)$ of φ defined by $\varphi(p) = (x^1(p), \dots, x^n(p))$ are called *local coordinates* on U.

An *atlas* $\mathscr{A}$ is a collection $\{(U_i, \varphi_i)\}_{i \in I}$ such that $\{U_i\}_{i \in I}$ forms an open cover of M.

Remark 3.1.2. *Instead of specifying a chart (U, φ) we might as well specify the coordiante functions $(U, (x^1, \dots, x^n))$. Both these formulations are equivalent since one can be recovered from the other.*

Example 3.4 (The Product Manifold). Let $M_1, \dots, M_k$ be topological manifolds of dimensions $n_1, \dots, n_k$ respectively. Then, the topological space $M_1 \times \cdots \times M_k$ is Hausdorff and second countable. Further, let $\mathbf{p} = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$. Then, for each index j, there is a coordinate chart (U_j, φ_j) containing p_j. It is not hard to argue that

$$\varphi_1 \times \cdots \times \varphi_k : M_1 \times \cdots \times M_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is an embedding and thus $M_1 \times \cdots \times M_k$ is a manifold of dimension $n_1 + \cdots + n_k$.

From the above example, we see that the n-dimensional torus $\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n\text{-times}}$ is an n-dimensional topological manifold.

3.1.1 Some Topological Properties

Lemma 3.5. *Manifolds are locally compact Hausdorff.*

Proof. Straightforward. ∎

Lemma 3.6. *Manifolds are paracompact.*

Proof. Every regular Lindelöf space is paracompact. ∎

3.2 Smooth Structure

Definition 3.7 (Smooth Atlas). Let M be a topological n-manifold. The charts (U, φ) and (V, ψ) on M are said to be *smoothly compatible* if

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^n$$

is a diffeomorphism. An atlas $\mathscr{A}$ is said to be a *smooth atlas* if any two charts in $\mathscr{A}$ are smoothly compatible. An atlas $\mathscr{A}$ is said to be *maximal* if it is a maximal element in the poset of all atlases on M.

Definition 3.8 (Smooth Manifold). Let M be a topological n-manifold. A *smooth structure* on M is a maximal smooth atlas. A *smooth manifold* is a pair $(M, \mathscr{A})$ where M is a topological manifold and $\mathscr{A}$ is a smooth structure on M.

Remark 3.2.1. *There exist topological manifolds that admit no smooth structures at all. There is one such compact* 10*-dimensional manifold due to Kervaire.*

Proposition 3.9. *Let M be a topological manifold.*

(a) Every smooth atlas $\mathscr{A}$ for M is contained in a unique maximal smooth atlas.

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas[a]*.*

[a] This is equivalent to requiring the charts in both the atlases to be compatible with one another.

Proof. (a) Let $\overline{\mathscr{A}}$ denote the set of all charts on M which are smoothly compatible with every chart in $\mathscr{A}$. This obviously contains $\mathscr{A}$. We contend that this is a smooth structure on M.

Let (U, φ) and (V, ψ) be two elements of $\overline{\mathscr{A}}$ we shall show that they are smoothly compatible. We need only check this when both are not in $\mathscr{A}$. We shall show that $\psi \circ \varphi^{-1}$ is smooth. The same proof would show that $\varphi \circ \psi^{-1}$ is smooth whereby both are diffeomorphisms.

Let $x \in \varphi(U \cap V)$. Then there is a unique $p \in U \cap V$ with $\varphi(p) = x$. Let (W, θ) be a chart in $\mathscr{A}$ with $x \in W$. Since this chart is smoothly compatible with (U, φ) and (V, ψ), the maps

$$\psi \circ \theta^{-1} : \theta(W \cap V) \to \psi(W \cap V) \quad \text{and} \quad \theta \circ \varphi^{-1} : \varphi(W \cap U) \to \theta(W \cap U)$$

are smooth, whence the composition

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$$

is smooth on a neighborhood of x. Since this is true for all $x \in \varphi(U \cap V)$, we have that $\overline{\mathscr{A}}$ is a smooth atlas.

Now, if $\mathscr{B}$ is any other smooth atlas containing $\mathscr{A}$, then every chart in $\mathscr{B}$ is smoothly compatible with every chart in $\mathscr{A}$ whence $\mathscr{B} \subseteq \overline{\mathscr{A}}$. This proves both uniqueness and maximality.

(b) Let $\mathscr{A}$ and $\mathscr{B}$ be the two atlases on M. Due to (a), $\overline{\mathscr{A} \cup \mathscr{B}}$ is a smooth structure containing $\mathscr{A}$ and $\mathscr{B}$. Due to uniqueness of the smooth structure, we are done.

■

Remark 3.2.2. *It is* not necessary *that a topological manifold admits exactly one smooth structure. Take for example the topological manifold $\mathbb{R}$ and two homeomorphisms $\mathbf{id}_{\mathbb{R}}$ and $\psi : \mathbb{R} \to \mathbb{R}$ given by $\psi(x) = x^3$. We have two atlases $\{\mathbf{id}_{\mathbb{R}}\}$ and $\{\psi\}$ on $\mathbb{R}$, and thus they give rise to two smooth structures on $\mathbb{R}$. We note that these structures are not the same since $\mathbf{id}_{\mathbb{R}}$ and ψ are not smoothly compatible. Indeed, $\mathbf{id}_{\mathbb{R}} \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}$ is the map $x \mapsto x^{1/3}$ which is not smooth.*

Definition 3.10. If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a *smooth chart* and the corresponding coordinate map φ is called a *smooth coordinate map*.

A *smooth coordinate domain* is the domain of some smooth coordinate chart. A *smooth coordinate ball* is a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space.

A set $B \subseteq M$ is called a *regular coordinate ball* if there is a smooth coordinate ball $B' \supseteq \overline{B}$ and a smooth coordinate map $\varphi : B' \to \mathbb{R}^n$ such that or some positive reals $r < r'$,

$$\varphi(B) = B(0, r), \quad \varphi(\overline{B}) = \overline{B(0, r)}, \quad \varphi(B') = B(0, r').$$

In particular, every regular coordinate ball is *precompact* in M.

Proposition 3.11. *Every smooth manifold has a countable basis of regular coordinate balls.*

Proof. It suffices to ind a basis of regular coordinate balls since a countable basis can then be extracted from it, as is well known. For any $p \in M$, let $\varphi_p : U_p \to \widehat{U}_p$ be a smooth coordinate map with $p \in U_p$. Let $r_p > 0$ be such that $B(\varphi_p(p), r_p) \subseteq \widehat{U}_p$. It is not hard to see that the collection

$$\left\{ \varphi_p^{-1}(B(\varphi_p(p), r)) \mid 0 < r < r_p,\ p \in M \right\}$$

forms a basis and each element is a regular coordinate ball. This completes the proof. ■

Lemma 3.12 (Smooth Manifold Chart Lemma). *Let M be a set, and suppose a collection $\{(U_\alpha, \varphi_\alpha)\}$ is given such that*

(a) Each $\varphi_\alpha : U_\alpha \to \widehat{U}_\alpha \subseteq \mathbb{R}^n$ is a bijection where $\widehat{U}_\alpha$ is an open subset of $\mathbb{R}^n$.

(b) For each α, β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.

(c) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.

(d) A countable subcollection of $\{U_\alpha\}$ covers M.

(e) *Whenever $p \neq q$ are distinct points in M, either there is some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.*

Then M has a unique smooth manifiold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. We begin by first topologizing M. Let

$$\mathcal{B} := \{\varphi_\alpha^{-1}(V) \mid V \subseteq \widehat{U}_\alpha \text{ is open}\}.$$

We contend that $\mathcal{B}$ forms a basis for some topology on M. Indeed, let $V \subseteq \widehat{U}_\alpha$ and $W \subseteq \widehat{U}_\beta$ be open sets and $p \in \varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$. We have

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W))$$

which is an element of $\mathcal{B}$ since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth and thus continuous map.

Along with this topology, each φ_α is an open continuous map which is also a bijection on $\widehat{U}_\alpha$ whence it is a homeomorphism. That M is Hausdorff follows almost immediately from (e). Indeed, let $p \neq q \in M$. If there are disjoint U_α, U_β containing p and q respectively, then we have a separation. If not, then $p, q \in U_\alpha$ for some α. Let V, W be a separation of $\varphi_\alpha(p), \varphi_\alpha(q)$ in $\widehat{U}_\alpha$, then $\varphi_\alpha^{-1}(V)$ and $\varphi_\alpha^{-1}(W)$ forms a separation of p, q in U_α.

Next, we must show that M is second countable. Note that each U_α is second countable, owing to it being homeomorphic to an open subset of $\mathbb{R}^n$, and since a countable number of U_α's cover M, we have that M is second countable.

Finally, (c) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas and is therefore contained in a unique smooth structure. This completes the proof. ■

The above lemma will be useful in defining the tangent bundle on a smooth manifold.

3.3 Manifolds with Boundary

Definition 3.13 (Manifold with Boundary). An *n-dimensional manifold with boundary* is a second countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or an open subset of $\mathbb{H}^n$ in the subspace topology.

A *chart on* M is a pair (U, φ) where $U \subseteq M$ is an open subset and $\varphi : U \to \widehat{U}$ is a homeomorphism onto either an open subset of $\mathbb{R}^n$ or an open subset of $\mathbb{H}^n$. In the former case, the chart is called an *interior chart* and in the latter case, it is called a *boundary chart*.

A point $p \in M$ is called an *interior point of* M if it is in the domain of some interior chart and similarly, it is called a *boundary point of* M if it is in the domain of a boundary chart (U, φ) such that $\varphi(p) \in \partial\mathbb{H}^n$.

The set of all interior points in M is denoted by $\operatorname{Int} M$ and the set of all boundary pionts in M is denoted by ∂M.

Remark 3.3.1. *From the above definitions, it is obvious that every manifold is a manifold with boundary but the converse is not true. This is illustrated in the following theorem, whose proof we postpone. In particular, if the boundary of a manifold with boundary is nontrivial, then it is not a manifold.*

Add link to proof

Theorem 3.14 (Topological Invariance of Boundary). *Let M be a topological manifold with boundary. Then, $M = \partial M \sqcup \operatorname{Int} M$. That is, the boundary and interior of M are disjoint sets whose union is all of M.*

Proposition 3.15. *Let M be a topological n-manifold with boundary. Then*

(a) Int M *is an open subset of M and a topological n-manifold.*

(b) ∂M *is a closed subset of M and a topological $(n-1)$-manifold.*

(c) M *is a topological manifold if and only if* $\partial M = \varnothing$.

(d) *If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.*

3.4 Smooth Maps

Definition 3.16. If M is a smooth n-manifold (without boundary),

Definition 3.17. Let M and N be smooth manifolds with or without boundary and $A \subseteq M$. A map $F : A \to N$ is said to be *smooth on A* if for every $p \in A$ there is an open neighborhood $W \subseteq M$ and a smooth map $\widetilde{F} : W \to N$ whose restriction to $W \cap A$ agrees with F.

3.5 Partition of Unity

Definition 3.18 (Partition of Unity). Let M be a topological space and $\mathscr{U}$ an open cover of M indexed by a set J. A *partition of unity subordinate to $\mathscr{U}$* is an indexed family (ψ_α) of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

1. $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in J$ and $x \in M$.
2. $\mathrm{Supp}(\psi_\alpha) \subseteq U_\alpha$ or each $\alpha \in J$
3. The set $\{\mathrm{Supp}(\psi_\alpha)\}$ is locally finite.
4. $\sum_{\alpha \in J} \psi_\alpha(x) = 1$ for all $x \in M$.

A partition of unity is said to be *smooth* if each ψ_α is a smooth function.

Theorem 3.19. *Let M be a smooth manifold with or without boundary and $\mathscr{U} = (U_\alpha)_{\alpha \in J}$ be an indexed open cover of M. Then there is a smooth partition of unity subordinate to $\mathscr{U}$.*

Definition 3.20 (Bump function). Let M be a topological space, $A \subseteq M$ a closed subset and $U \subseteq M$ an open subset containing A. A continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for A supported in U* if $0 \leq \psi \leq 1$ on M, $\psi|_A = 1$ and $\mathrm{Supp}\,\psi \subseteq U$.

Proposition 3.21. *Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset $U \subseteq M$ containing A, there is a smooth bump function or A supported in U.*

Proof. The collection $\{U, M\backslash A\}$ is an open cover of M and thus has a smooth partition of unity $\{\psi_1, \psi_2\}$ subordinate to it with $\operatorname{Supp}(\psi_1) \subseteq U$ and $\psi_1 + \psi_2 = 1$ on M. Since $\operatorname{Supp}(\psi_2) \subseteq M\backslash A$, we have $\psi_2|_A = 0$ whence $\psi_1|_A = 1$ and thus ψ_1 is the desired smooth bump function. ■

Lemma 3.22 (Extension Lemma for Smooth Maps into $\mathbb{R}^k$). *Let M be a smooth manifold with or without boundary, $A \subseteq M$ a closed subset and $f : A \to \mathbb{R}^k$ a smooth function. For any open subset U containing A, there is a smooth function $\widetilde{f} : M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and* $\operatorname{Supp} f \subseteq U$.

Proof. For each $a \in A$, by definition, there is an open neighborhood W_a of a and a function $\widetilde{f}_p$: ■

Complete proof. Simple application of POU

Chapter 4

Tangent Spaces

4.1 Tangent Vectors

4.1.1 On $\mathbb{R}^n$

Definition 4.1. Let $a \in \mathbb{R}^n$. A map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is said to be a *derivation at a* if it is linear over $\mathbb{R}$ and satisfies the product rule:

$$w(fg) = f(a)w(g) + g(a)w(f).$$

Denote by $T_a\mathbb{R}^n$ the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. This is obviously a vector space under the operations:

$$(w_1 + w_2)(f) = w_1(f) + w_2(f) \quad \text{and} \quad (cw)(f) = cw(f)$$

for all $w_1, w_2 \in T_a\mathbb{R}^n$ and $c \in \mathbb{R}$.

Lemma 4.2. *Suppose $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$.*

(a) If f is a constant function, then $w(f) = 0$.

(b) If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Proof. (a) Let $f \equiv c \in \mathbb{R}$. First, consider the constant function $g \equiv 1 \in \mathbb{R}$. Note that $g = g^2$ and thus

$$w(g) = w(g^2) = g(a)w(g) + g(a)w(g) = 2w(g)$$

whence $w(g) = 0$ and $w(f) = cw(g) = 0$.

(b) Trivial. ■

For a vector $v \in \mathbb{R}^n$ and a point $a \in \mathbb{R}^n$, let $D_v|_a$ denote the *directional derivative* at a in the direction of v, which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(a + vt)\Big|_{t=0}$$

It is not hard to see that $D_v|_a \in T_a\mathbb{R}^n$. Indeed, if $f, g \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} D_v|_a(fg) &= \frac{\mathrm{d}}{\mathrm{d}t} \big(f(a+vt)g(a+vt)\big)\Big|_{t=0} \\ &= f(a)\frac{\mathrm{d}}{\mathrm{d}t} g(a+vt)\Big|_{t=0} + g(a)\frac{\mathrm{d}}{\mathrm{d}t} f(a+vt)\Big|_{t=0} \\ &= f(a)D_v|_a(g) + g(a)D_v|_a(f). \end{aligned}$$

Proposition 4.3. *The map $v_a \mapsto D_v|_a$ is an isomorphism of vector spaces from $\mathbb{R}^n_a \to T_a\mathbb{R}^n$.*

Proof. Call this map Φ. The fact that Φ is a linear transformation follows from

$$D_v|_a(f) = v \cdot \nabla(f)(a).$$

Next, we shall show that it is injective. Let $v_a \in \mathbb{R}^n_a$ such that $D_v|_a \equiv 0$. Consider the function $\pi_j : \mathbb{R}^n \to \mathbb{R}$ which is the projection on the j-th coordinate. This is obviously in $C^\infty(\mathbb{R}^n)$. Then, we have

$$0 = D_v|_a(\pi_j) = \frac{\mathrm{d}}{\mathrm{d}t}(a^j + v^j t) = v^j$$

whence $v = 0$ and the kernel of Φ is trivial.

Lastly, we must show that Φ is a surjection. Let $w \in T_a\mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$. Due to , we may write

Reference Taylor's Theorem

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{j=1}^{n}\sum_{i=1}^{n}(x^i - a^i)(x^j - a^j)\int_0^1 (1-t)\frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\ \mathrm{d}t.$$

Evaluating this at $x = a$, we have that

$$w(f)(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) w(x^i) = D_v|_a(f)$$

where $v = (w(x^1), \ldots, w(x^n))$. This completes the proof. ■

Corollary 4.4. For $a \in \mathbb{R}^n$, the n derivations

$$\left.\frac{\partial}{\partial x^1}\right|_a, \ldots, \left.\frac{\partial}{\partial x^n}\right|_a$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

4.1.2 On a Manifold

Definition 4.5. Let M be a smooth manifold with or without boundary and let $p \in M$. A linear map $w : C^\infty(M) \to \mathbb{R}$ is said to be a *derivation at* p if it obeys the product rule:

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M).$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM is a vector space called the *tangent space to* M *at* p. An element of T_pM is called a *tangent vector to* M *at* p.

Lemma 4.6. *Let M be a smooth manifold with or without boundary, $p \in M$, $w \in T_pM$and $f, g \in C^\infty(M)$.*

(a) *If f is a constant function then $w(f) = 0$.*

(b) *If $f(p) = g(p) = 0$, then $w(fg) = 0$.*

Proof. Same as the proof for $\mathbb{R}^n$. ■

Lemma 4.7. *Let $p \in M$ and $f, g \in C^\infty(M)$ such that $f = g$ in some open neighborhood of p. Then, for any $v \in T_pM$, $v(f) = v(g)$.*

Proof. Let $h = f - g \in C^\infty(M)$ and $p \subseteq U$ be a neighborhood on which h vanishes. The collection $\{M \backslash \{p\}, U\}$ is an open cover of M whence there is a smooth partition of unity $\{\psi, \psi'\}$ subordinate to it.

Note that for all $x \in M \backslash U$, $\psi(x) = 1$ and $\psi'(x) = 0$ whence $\psi \cdot h = h$ on all of M and $\psi(p) = 0 = h(p)$ and thus
$$0 = v(\psi \cdot h) = v(h) = v(f) - v(g).$$
■

4.2 Differential of a Smooth Map

Definition 4.8 (Differential). Let M and N be smooth manifolds with or without boundary and $F : M \to N$ a smooth map. For each $p \in M$, the *differential of F at p* is the map
$$dF_p : T_pM \to T_{F(p)}N$$
given by
$$dF_p(v)(f) = v(f \circ F)$$
for all $f \in C^\infty(N)$.

Proposition 4.9. *The map $dF_p : T_pM \to T_{F(p)}N$ is a linear transformation.*

Proof. Let $v \in T_pM$. Then, for $f, g \in C^\infty(N)$ and $c \in \mathbb{R}$, we have
$$dF_p(v)(f + cg) = v((f + cg) \circ F) = v(f \circ F + cg \circ F) = v(f \circ F) + v(cg \circ F) = v(f \circ F) + cv(g \circ F),$$
and
$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F). \end{aligned}$$
Thus $dF_p(v)$ is indeed a derivation on N at p.

Next, we must show that dF_p is a linear transformation. Indeed, if $v, w \in T_pM$ and $c \in \mathbb{R}$, we have for all $f \in C^\infty(N)$,
$$dF_p(v + cw)(f) = (v + cw)(f \circ F) = v(f \circ F) + cw(f \circ F) = dF_p(v) + cdF_p(w).$$
This completes the proof. ■

Proposition 4.10. *Let M, N and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$.*

(a) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$.

(b) $d(\mathbf{id}_M)_p = \mathbf{id}_{T_pM} : T_pM \to T_pM$.

(c) *If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

Proof. (a) This is almost *by definition*. Let $f \in C^\infty(P)$ and $v \in T_pM$. Then,

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = v((f \circ G) \circ F)$$

and

$$dG_{F(p)}(dF_p(v))(f) = dF_p(v)(f \circ G) = v(f \circ G \circ F).$$

(b) For any $v \in T_pM$ and $f \in C^\infty(M)$,

$$d(\mathbf{id}_M)_p(v)(f) = v(f \circ \mathbf{id}_M) = v(f).$$

(c) Let $G = F^{-1}$. From (a) and (b),

$$\mathbf{id}_{T_pM} = dG_{F(p)} \circ dF_p,$$

whence the conclusion follows. ∎

In particular, Proposition 4.10 shows that the map $T : \mathbf{Diff}_* \to \mathbf{Vec}$ which maps

$$(M, p) \mapsto T_pM \quad \text{and} \quad [F : (M, p) \to (N, F(p))] \mapsto \left[dF_p : T_pM \to T_{F(p)}N\right]$$

is a *covariant functor*. We shall see a similar functor from **Diff** to **Diff** in an upcoming section.

Lemma 4.11. *Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset (and thus a manifold in its own right) and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism of vector spaces.*

Proof. Suppose $v \in T_pU$ such that $d\iota_p(v) = 0 \in T_pM$. Let B be a regular coordinate ball contained in U, so that is, $\overline{B} \subseteq U$. ∎

Complete this

Proposition 4.12. *Let M be a smooth n-manifold (without boundary). Then for any $p \in M$, T_pM is an n-dimensional vector space.*

Proof. Let (U, φ) be a smooth chart containing p. Due to the preceeding lemma, T_pU is isomorphic to T_pM as vector spaces. Thus, it suffices to show that T_pU is an n-dimensional vector space. We have a diffeomorphism $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ whence $d\varphi_p : T_pU \to T_{\varphi(p)}\widehat{U}$ is an isomorphism of vector spaces but since the latter is isomorphic to $\mathbb{R}^n$ (as a vector space) as we have seen earlier, we are done. ∎

Add analogous result for manifolds with boundary.

4.2.1 Computation in coordinates

Let M be a smooth manifold (without boundary) and $p \in M$. Let (U, φ) be a chart whose domain contains p with coordinate functions $x^1, \ldots, x^n$. We have seen in this section that $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ as vector spaces. If we denote by

$$\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \ldots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$$

a basis of $T_{\varphi(p)}M$, then their preimages under $d\varphi_p$ form a basis for T_pM. We denote these by

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Thus, any $v \in T_pM$ can be represented as

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p$$

where the v^i's are caled the *components of* v with respect to the coordinate basis. It is important to note that this representation is dependent on the coordinate basis and thus the chart chosen.

4.3 The Tangent Bundle

Definition 4.13 (Tangent Bundle). Let M be a smooth manifold with or without boundary. The *tangent bundle of* M, denoted by TM is defined as

$$TM = \coprod_{p \in M} T_pM.$$

This is equipped with the natural projection $\pi : TM \to M$ which maps every vector in T_pM to $p \in M$.

TM is a smooth $2n$-manifold

Let M be a smooth manifold (without boundary). We shall use Lemma 3.12 to construct a smooth structure on TM. Let (U, φ) be a smooth chart for M with coordinate functions $x^1, \ldots, x^n$. Define the map

$$\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$$

by

$$\widetilde{\varphi}\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\right).$$

It is not hard to see that $\operatorname{im} \widetilde{\varphi} = \varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. This map is a bijection since it has an explicit inverse given by

$$\widetilde{\varphi}^{-1}\left(\mathbf{x}, v^1, \ldots, v^n\right) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(\mathbf{x})}.$$

Definition 4.14 (Global Differential). Let M and N be smooth manifold with or without boundary and Let $F : M \to N$ be a smooth map. The *global differential* or *global tangent map* is a map $dF : TM \to TN$ which maps $v \in T_pM$ to $dF_p(v) \in T_{F(p)}N$.

In other words, the global differential obtained by simply stitching together the dF_p's for all $p \in M$.

Chapter 5

Submersions and Immersions

5.1 Maps of Constant Rank

Definition 5.1. Let $F : M \to N$ be a map between smooth manifolds with or without boundary. For a point $p \in M$, define the *rank of F at p* to be the rank of the linear transformation $dF_p : T_pM \to T_{F(p)}N$. If F has the same rank r at all points in M, then it is said to have *constant rank* and we write $\operatorname{rank} F = r$.

The map F is called a *smooth submersion* if $\operatorname{rank} F = \dim N$ and a *smooth immersion* if $\operatorname{rank} F = \dim M$.

Proposition 5.2. *Let $F : M \to N$ be a smooth map between smooth manifolds with or without boundary and $p \in M$. If dF_p is a surjection, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is an injection, then p has a neighborhood U such that $F|_U$ is an immersion.*

Definition 5.3 (Local Diffeomorphism). A map $F : M \to N$ between smooth manifolds with or without boundary is called a *local diffeomorphism* if every $p \in M$ has a neighborhood U such that $F(U)$ is open in N and the restriction $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 5.4 (Inverse Function Theorem for Manifolds). *Let $F : M \to N$ be a smooth map between smooth manifolds (without boundary). If $p \in M$ is a point such that dF_p is invertible, then there are neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof. Let (U, φ) and (V, ψ) be smooth charts for M and N centered at p and $F(p)$ respectively. Then,

$$\widehat{F} := \psi \circ F \circ \varphi^{-1} : \widehat{U} = \varphi(U) \subseteq \mathbb{R}^n \to \widehat{V} = \psi(V) \subseteq \mathbb{R}^n$$

is a smooth map with $\widehat{F}(0) = 0$. Since φ and ψ are diffeomorphisms, the linear transformations $d(\varphi^{-1})_0$ and $d\psi_{F(p)}$ are invertible and thus the composition

$$d\widehat{F}_p = d\psi_{F(p)} \circ dF_p \circ d(\varphi^{-1})_0$$

is invertible. Thus, due to Theorem 1.11, there are open subsets $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ such that the restriction $\widehat{F}|_{\widehat{V}_0} : \widehat{U}_0 \to \widehat{V}_0$ is a diffeomorphism. Let $U_0 := \varphi^{-1}(\widehat{U}_0)$ and $V_0 := \psi^{-1}(\widehat{V}_0)$. Then, F restricts to a diffeomorphism of U_0 to V_0. This completes the proof. ∎

5.1.1 The Rank Theorems

Theorem 5.5 (Local Rank Theorem). *Let $F : M \to N$ be a smooth map between smooth manifolds (without boundary) of dimensions m and n respectively with constant rank r. For each $p \in M$, there exist smooth charts (U, φ) for M centered at p and (V, ψ) or N centered at $F(p)$ such that $F(U) \subseteq V$ in which F has a coordinate representation of the form*

$$\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

Proof. ■

Theorem 5.6. *Let $F : M \to N$ be a smooth map between manifolds (without boundary) of constant rank.*

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

Proof. ■

Chapter 6

Vector Fields

Definition 6.1 (Vector Field). Let M be a smooth manifold with or without boundary. A *vector field on* M is a continuous section $X : M \to TM$ of the map $\pi : TM \twoheadrightarrow M$.

A vector field is said to be *smooth* if the section $X : M \to TM$ is a smooth map of manifolds. A *rough* vector field is simply a section $X : M \to TM$ which need not be continuous. The value of a vector field at a point is denoted by either $X(p)$ or X_p.

The support of X is defined to be

$$\operatorname{Supp} X = \overline{\{p \in M \mid X_p \neq 0\}}.$$

A vector field is said to be *compactly supported* if $\operatorname{Supp} X \subseteq M$ is compact.

In other words, a section $X : M \to TM$ is said to be a

vector field if it is a morphism in **Top**.

smooth vector field if it is a morphism in **Diff**.

rough vector field if it is a morphism in **Set**.

Chapter 7

Vector Bundles

7.1 Vector Bundles

Definition 7.1. Let M be a topological space. A *real vector bundle of rank k over M* is a topological space E together with a continuous surjection $\pi : E \twoheadrightarrow M$ satisfying the following conditions:

(a) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

(b) For every $p \in M$, there is a neighborhood U of p in M and a homeomorophism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a *local trivialization of E over U* satisfying the following additional conditions:

- If $\pi_U : U \times \mathbb{R}^k \twoheadrightarrow U$ is the natural projection, then $\pi_U \circ \Phi = \pi$.
- For each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

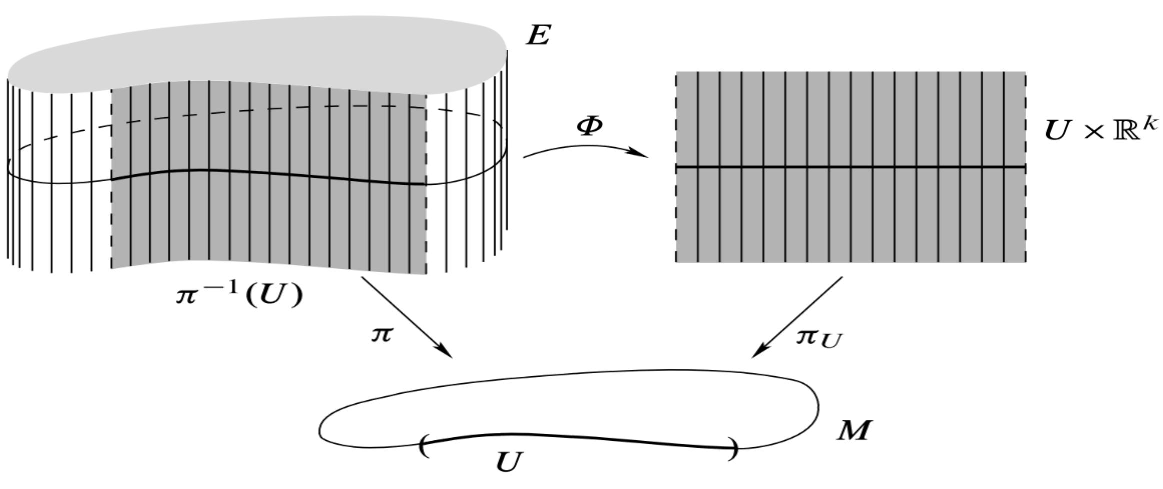

Figure 7.1: A local trivialization of a vector Bundle

Lemma 7.2. *Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \varnothing$. There exists a smooth map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form*

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v).$$

Lemma 7.3 (Vector Bundle Chart Lemma). *Let M be a smooth manifold with or without boundary, and suppose that for each $p \in M$, there is a real vector space of fixed dimension k. Let $E = \coprod_{p \in M} E_p$ and let $\pi : E \to M$ be the obvious projection map. Suppose that we have*

(a) an open cover $\{U_\alpha\}_{\alpha \in J}$ of M,

(b) for each $\alpha \in J$, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism $E_p \to \{p\} \times \mathbb{R}^k$.

(c) for each $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth map

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$$

such that the map $\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = \left(p, \tau_{\alpha\beta}(p)v\right).$$

Then E has a unique topology and smooth structure making it into a smooth manifold with or without boundary and a smooth rank k vector bundle over M, with $\pi : E \to M$ as a projection and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.

7.2 Local and Global Frames

Definition 7.4 (Local Section). Let $\pi : E \to M$ be a vector bundle and $U \subseteq M$ be open. A *local section* of E over U is a continuous map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathbf{id}_U$.

Definition 7.5. Let $\pi : E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, a k-tuple of (continuous) local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be *linearly independent* if their values $(\sigma_1(p), \dots, \sigma_k(p))$ orm a linearly independent k-tuple in E_p for each $p \in U$. Similarly, the k-tuple is said to *span E* if their values span E_p for each $p \in U$.

A *local frame for E over U* is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections over U that span E. A local frame is called a *global frame* if $U = M$.

7.3 The Cotangent Bundle

Definition 7.6. Let V be a finite-dimensional vector space. A *covector on V* is an element of the dual space V^*.

Chapter 8

Tensors

Throughout this chapter, all vector spaces are assumed to be over $\mathbb{R}$. They will usually be finite dimensional but we shall explicitly mention this in order to avoid confusion.

8.1 Tensors

Definition 8.1. Let $V_1, \ldots, V_k$ and W be vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is said to be alternating if for each $1 \le i \le k$,

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = a_i F(v_1, \ldots, v_i, \ldots, v_k) + a_i' F(v_1, \ldots, v_i', \ldots, v_k).$$

We denote by $\mathscr{L}(V_1, \ldots, V_k; W)$ the set of all multilinear maps from $V_1 \times \cdots \times V_k \to W$.

For linear map $f_i : V_i \to \mathbb{R}$ for $1 \le i \le k$, define the multilinear map

$$f_1 \otimes \cdots \otimes f_k : V_1 \times \cdots \times V_k \to \mathbb{R}$$

by $(f_1 \otimes \cdots \otimes f_k)(v_1, \ldots, v_k) = f_1(v_1) \cdots f_k(v_k)$. This notation is a consequence of the forthcoming Theorem 8.2.

Remark 8.1.1 (Constructing the Tensor Product). *In this remark we recall a construction from module theory. Let $V_1, \ldots, V_k$ be vector spaces. Let $\mathfrak{F}(V_1 \times \cdots \times V_k)$ denote the free vector space on $V_1 \times \cdots \times V_k$. Let W denote the subspace spanned by elements*

$$\mathbf{e}_{(v_1, \ldots, v_i + v_i', \ldots, v_k)} - \mathbf{e}_{(v_1, \ldots, v_k)} - \mathbf{e}_{(v_1, \ldots, v_i', \ldots, v_k)}$$
$$\mathbf{e}_{(v_1, \ldots, av_i, \ldots, v_k)} - a\mathbf{e}_{(v_1, \ldots, v_k)}$$

for $1 \le i \le k$ and $a \in \mathbb{R}$. Then, the vector space $\mathfrak{F}(V_1 \times \cdots \times V_k)/W$ is called the tensor product of $V_1, \ldots, V_k$ *and is denoted by $V_1 \otimes \cdots \otimes V_k$. The tensor product has the property that every multilinear map from $V_1 \times \cdots \times V_k$ factors through it.*

Theorem 8.2. *Let $V_1, \ldots, V_k$ be finite dimensional vector spaces. Then, there is a natural isomorphism*

$$V_1^* \otimes \cdots \otimes V_k^* \cong \mathscr{L}(V_1, \ldots, V_k; \mathbb{R}).$$

Proof. Consider the map $\Phi : V_1^* \times \cdots \times V_k^* \to \mathscr{L}(V_1, \ldots, V_k; \mathbb{R})$ given by

$$\Phi(f_1, \ldots, f_k)(v_1, \ldots, v_k) = f_1(v_1) \cdots f_k(v_k).$$

It is not hard to see that Φ is a multilinear map. Thus, this induces a map

$$\varphi : V_1^* \otimes \cdots \otimes V_k^* \to \mathscr{L}(V_1, \ldots, V_k; \mathbb{R}).$$

The fact that φ is an isomorphism follows from the fact that it maps the basis of $V_1^* \otimes \cdots \otimes V_k^*$ to the basis of $\mathscr{L}(V_1 \times \cdots \times V_k, \mathbb{R})$. ■

8.1.1 Covariant and Contravariant Tensors

Definition 8.3 (Covariant, Contravariant Tensor). Let V be a finite dimensional vector space. If k is a positive integer, a *covariant k-tensor on V* is an element of the k-fold tensor product

$$T^k(V^*) := \underbrace{V^* \otimes \cdots \otimes V^*}_{k\text{-times}}.$$

The number k is called the *rank* of the aforementioned covariant tensor.

Similarly, a *contravariant k-tensor on V* is an element of the k-fold tensor product

$$T^k(V) := \underbrace{V \otimes \cdots \otimes V}_{k\text{-times}}.$$

Again, the number k is called the *rank* of the contravariant tensor.

There are also the *mixed tensors on V of type (k,l)* which are elements of

$$T^{(k,l)}(V) := \underbrace{V \otimes \cdots \otimes V}_{k\text{-times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l\text{-times}}.$$

Due to the natural isomorphism of Theorem 8.2, we may identify a covariant k-tensor as a multilinear map $V_1 \times \cdots \times V_k \to \mathbb{R}$ and similarly, we may identify a contravariant k-tensor as a multilinear map $V_1^* \times \cdots \times V_k^* \to \mathbb{R}$. We shall switch between these identifications to suit our needs.

8.1.2 Symmetric and Alternating Tensors

Definition 8.4 (Symmetric, Alternating Tensor). Let V be a finite dimensional vector space. A covariant k-tensor α on V is said to be *symmetric* if for every $\sigma \in \mathfrak{S}_k$,

$$\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Similarly, it is said to be *alternating* if

$$\alpha(v_1, \ldots, v_k) = \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

The vector space of all alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*)$.

Next, we define the symmetrization and alternation operators. Indeed, if α is a covariant k-tensor on V, then define

$$\operatorname{Sym}(\alpha)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

$$\operatorname{Alt}(\alpha)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Then, it is not hard to argue that $\operatorname{Sym}(\alpha)$ is a symmetric covariant k-tensor on V while $\operatorname{Alt}(\alpha)$ is an alternating covariant k-tensor on V.

8.1.3 Tensor Fields on a Manifold

Definition 8.5. Let M be a smooth manifold with or without boundary. Define the *bundle of covariant k-tensors on M* by

$$T^k\left(T^*M\right) := \coprod_{p\in M} T^k(T_p^*M).$$

Analogously, define the *bundle of contravariant k-tensors on M* by

$$T^k\left(TM\right) := \coprod_{p\in M} T^k(T_pM)$$

and finally, the *bundle of mixed tensors of type* (k, l) *on M* by

$$T^{(k,l)}\left(TM\right) := \coprod_{p\in M} T^{(k,l)}(T_pM).$$

Any one of these bundles is called a *tensor bundle over M*.

Theorem 8.6. *Let k and l be nonnegative integers and M a smooth n-manifold with or without boundary. Then,* $T^k(T^*M)$*,* $T^k(TM)$ *and* $T^{(k,l)}(TM)$ *have natural structures as smooth vector bundles over M.*

Proof. We shall provide a proof of the statement only for $T^k(TM)$. We contend that $T^k(TM)$ is a smooth vector bundle of rank n^k over M. ■

use Lemma 7.3

Chapter 9

Differential Forms

We have defined alternating tensors and the alternation operator in the chapter on tensors. From our identification, if V is a finite dimensional vector space, then $\Lambda^k(V^*)$ is simply the set of all alternating covariant k-tensors on V.

Given a positive integer k, an ordered k-tuple $I = (i_1, \ldots, i_k)$ of positive integers is called a *multi-index of length k*. If $\varepsilon^1, \ldots, \varepsilon^n$ is a basis or V^*, for each multi-index $I = (i_1, \ldots, i_k)$ of length k with $1 \leq i_1, \ldots, i_k \leq n$, define a covariant k-tensor ε^I by

$$\varepsilon^I(v_1, \ldots, v_k) := \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix}.$$

Since $\det : \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \mathbb{R}$ is an alternating function, so is ε^I.

Proposition 9.1. *Let V be an n-dimensional vector space. If (ε^i) is a basis for V^*, then for each positive integer $k \leq n$, the collection of k-covectors*

$$\mathscr{E} := \{\varepsilon^I \mid I \textit{ isa n increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k}.$$

Proof. ■

Definition 9.2 (Wedge Product). Let V be a finite dimensional vector space and $\omega \in \Lambda^k(V^*), \eta \in \Lambda^l(V^*)$. Define their *wedge product* or *exterior product* to be

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta) \in \Lambda^{k+l}(V^*).$$

Consequently,

$$\Lambda(V^*) := \bigoplus_{k=0}^{\infty} \Lambda^k(V^*)$$

forms a graded algebra under the wedge product.

Proposition 9.3. *Let V be an n-dimensional vector space and $(\varepsilon^1, \dots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$$

where $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$, their concatenation.

9.1 Differential Forms on Manifolds

Let M be a smooth n-manifold with or without boundary. Define

$$\Lambda^k(T^*M) := \coprod_{p \in M} \Lambda^k(T_p^*M).$$

Note that $\Lambda^k(T^*M) \subseteq T^k(T^*M)$.

Proposition 9.4. *$\Lambda^k(T^*M)$ is a smooth subbundle of $T^k(T^*M)$ and therefore is a smooth vector bundle of rank $\binom{n}{k}$ over M.*

Proof. ∎ Add in later

Definition 9.5. Let M be a smooth n-manifold with or without boundary. A section of $\Lambda^k(T^*M)$ is called a *differential k-form* or just a *k-form*. The integer k is called the *degree of the form*. The vector space of smooth k-forms on M is denoted by $\Omega^k(M)$.

Chapter 10

Orientations

10.1 Orientation of Vector Spaces

Definition 10.1. Let V be a finite dimensional vector space. We say that two ordered bases $(E_1, \dots, E_n)$ and $(\widetilde{E}_1, \dots, \widetilde{E}_n)$ are *consistently oriented* if the linear transformation $T : V \to V$ given by $T(E_i) = \widetilde{E}_i$ has positive determinant.

Let $\mathscr{B}$ be the set of all basis of a finite dimensional vector space V. Define the relation $\sim$ on $\mathscr{B}$ by $B \sim \widetilde{B}$ if and only if both are consistently oriented. It is not hard to see that this is an equivalence relation and the number of equivalence classes is precisely 2.

Definition 10.2. An *orientation* for V is defined by specifying either one of the two equivalence classes. A vector space together with a choice of orientation is called an *oriented vector space*. If V is oriented, then any ordered basis $(E_1, \dots, E_n)$ that is in the given orientation is said to be *positively oriented* and any basis that is not in the given orientation is said to be *negatively oriented*.

10.2 Orientation of Manifolds

Definition 10.3. For a smooth manifold with or without boundary, we define a *pointwise orientation* on M to be a choice of orientation of each tangent space.

If $(E_i)_{i=1}^n$ is a local frame for TM, we say that $(E_i)_{i=1}^n$ is positively oriented if $(E_1|_p, \dots, E_n|_p)$ is a positively oriented basis for T_pM for each $p \in U$. A negatively oriented frame is defined analogously.

A pointwise orientation of M is said to be *continuous* if every point of M is in the domain of an oriented local frame. An *orientation of M* is a continuous pointwise orientation and M is said to be *orientable* if there exists such an orientation.

Theorem 10.4. *Let M be a smooth n-manifold with or without boundary. Any non-vanishing n-form ω on M determines a unique orientation of M for which $\omega(p)$ is positively oriented for each $p \in M$. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*