Topology

Swayam Chube

July 9, 2022

Contents

Chapter 1

Topological Spaces

Definition 1.1 (Topology, Topological Space). A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

- $\emptyset$ and X are in $\mathcal{T}$
- The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$
- The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a *topological space*.

Definition 1.2 (Open Set). Let X be a topological space with associated topology $\mathcal{T}$. A subset U of X is said to be open if it is an element of $\mathcal{T}$.

This immediately implies that both $\emptyset$ and X are open. In fact, we shall see that they are also closed. The topology $\mathcal{T}$ of all subsets of X is called the **discrete topology** while the topology $\mathcal{T} = \{\emptyset, X\}$ is called the **indiscrete topology** or the **trivial topology**.

Definition 1.3. Let X be a set and $\mathcal{T}, \mathcal{T}'$ be two topologies defined on X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$. Further, if $\mathcal{T}' \subsetneq \mathcal{T}$, then $\mathcal{T}'$ is said to be *strictly finer* than $\mathcal{T}$.

Definition 1.4 (Basis). If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

- For each $x \in X$, there is at least one basis element B containing x
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Definition 1.5 (Generated Topology). Let $\mathcal{B}$ be a basis for a topology on X. The *topology generated by* $\mathcal{B}$ is defined as follows: A subset U of X is said to be open in X if for each $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 1.6. The collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ is indeed a topology on X.

Proof. Obviously $\emptyset, X \in \mathcal{T}$. Suppose $\{U_\alpha\}$ is a J indexed collection of sets in $\mathcal{T}$. Let $U = \bigcup_{\alpha \in J} U_\alpha$. Then, for each $x \in U$, there is an $\alpha \in J$ such that $x \in U_\alpha$ and thus, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq U$ and thus $U \in \mathcal{T}$. Let $U_1, U_2 \in \mathcal{T}$ and $x \in U_1 \cap U_2$. Then, there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$ and thus, $x \in B_1 \cap B_2 \subseteq U_1 \cap U_2$. But, by definition, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$ and consequently $U_1 \cap U_2 \in \mathcal{T}$. This finishes the proof. ■

Lemma 1.7. Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. Trivially note that all elements of $\mathcal{B}$ must be in $\mathcal{T}$ and thus, their unions too. Conversely, let $U \in \mathcal{T}$, then for all $x \in U$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. It is not hard to see that $U = \bigcup_{x \in U} B_x$ and we have the desired conclusion. ■

Lemma 1.8. Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof. We first show that $\mathcal{B}$ is a basis. Since X is an open set, for each $x \in X$, there is $C \in \mathcal{C}$ such that $x \in C$. Let $C_1, C_2 \in \mathcal{C}$. Since both C_1 and C_2 are given to be open, so is their intersection. Thus, for each $x \in C_1 \cap C_2$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq C_1 \cap C_2$. Therefore, $\mathcal{B}$ is a basis.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}$ be the topology associated with X. Let $U \in \mathcal{T}$, then for each $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$, and thus $U \in \mathcal{T}'$ by definition. Conversely, let $W \in \mathcal{T}'$. Since W can be written as a union of a collection of sets in $\mathcal{C}$, all of which are open, W must be open too and thus $W \in \mathcal{T}$. This finishes the proof. ■

Lemma 1.9. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then, the following are equivalent:

- $\mathcal{T}'$ is finer than $\mathcal{T}$
- For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Then $B \in \mathcal{T}$ and thus $B \in \mathcal{T}'$. As a result, there is, by definition $B' \in \mathcal{T}'$ such that $x \in B' \subseteq B$.

Conversely, let $U \in \mathcal{T}$. Since $\mathcal{B}$ generates $\mathcal{T}$, for each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. But due to the second condition, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$, implying that U is in the topology generated by $\mathcal{B}'$, that is $\mathcal{T}'$. This finishes the proof. ■

Definition 1.10 (Subbasis). A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of S.

Proposition 1.11. The topology generated by $\mathcal{S}$ is indeed a topology.

Proof. For this, it suffices to show that the set $\mathcal{B}$ of all finite intersections of elements of S forms a basis. Since the union of all elements of $\mathcal{S}$ equals X, for each $x \in X$, there is $S \in \mathcal{S}$ such that $x \in S$ and note that S must be an element of $\mathcal{B}$. Finally, since the intersection of any two elements of $\mathcal{B}$ can trivially be written as a finite intersection of elements of $\mathcal{S}$, it must be an element of $\mathcal{B}$ and we are done. ■

Definition 1.12 (Order Topology). Let X be a set with a simple order relation an dassume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X
2. All intervals of the form $[a_0, b)$ where a_0 is the smalest element (if any) of X
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X

The collection $\mathcal{B}$ is a basis for a topology on X which is called the *order topology*.

Definition 1.13 (Product Topology). Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$ where U and V are open sets in X and Y respectively.

Proposition 1.14. The collection $\mathcal{B}$ is indeed a basis.

Proof. The first condition is trivially satisfied since $X \times Y \in \mathcal{B}$. Suppose $x \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = U_3 \times V_3$ for some open sets U_3 and V_3 in X and Y respectively. This finishes the proof. ■

Proposition 1.15. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B},\ C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. Let W be an open set in $X \times Y$ and $(x, y) \in W$. Then, by definition, there is $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $(x, y) \in B \times C \subseteq W$ and we are done due to a preceeding lemma. ■

Definition 1.16. Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$ and let $\pi_2 : X \times Y \to Y$ be defined by the equation $\pi_2(x, y) = y$. The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Then, by definition if U is an open subset of X, then $\pi_1^{-1}(U) = U \times Y$ and similarly, if V is an open subset of Y, then $\pi_2^{-1}(V) = X \times V$.

Proposition 1.17. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Since $X \times Y \in \mathcal{S}$, the union of all elements of $\mathcal{S}$ is $X \times Y$ and thus $\mathcal{S}$ is a subbasis. Let $\mathcal{B}$ be the basis generated by all finite intersections of S. It suffices to show that $\mathcal{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$. For any U and V open in X and Y respectively, we may write $U \times V = (U \times Y) \cap (X \times V)$ and is therefore a member of $\mathcal{B}$. Conversely, the finite intersection of elements of S is of the form $(U_1 \cap \ldots \cap U_m) \times (V_1 \cap \ldots \cap V_m)$, which is a product of two open sets and is an element of $\mathcal{B}$, which finishes the proof. ∎

Definition 1.18. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the *subspace topology*. With this topology, the topological space Y is called a *subspace* of X. Its open sets consist of all intersections of open sets of X with Y.

Proposition 1.19. $\mathcal{T}_Y$ is a topology on Y.

Proof. Since $\varnothing \in \mathcal{T}$, $\varnothing = Y \cap \varnothing \in \mathcal{T}_Y$ and since $X \in \mathcal{T}$, $Y = Y \cap X \in \mathcal{T}_Y$. Further,

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = Y \cap \bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_Y$$

And finally, $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2) \in \mathcal{T}_Y$. This finishes the proof. ∎

Lemma 1.20. If $\mathcal{B}$ is a basis for the topology of X and $Y \subseteq X$. Then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let V be an open set in Y. Then, there is U in X such that $V = U \cap Y$. Since each $x \in V$ is an element of U, there is, by definition $B \in \mathcal{B}$ such that $x \in B \subseteq U$, consequently, $x \in B \cap Y \subseteq V$ and we are done due to a preceeding lemma. ∎

Proposition 1.21. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Follows from the fact that $U = V \cap Y$ for some V that is open in X. ∎

1.1 Closed Sets and Limit Points

Definition 1.22 (Closed Set). A subset A of a topological space X is said to be *closed* if the set $X \backslash A$ is open.

Theorem 1.23. Let X be a topological space. Then the following conditions hold:

1. $\varnothing$ and X are closed
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

Proof. All follow from De Morgan's laws. ∎

Proposition 1.24. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof. If A is closed in Y then $Y \setminus A$ is open and thus, there is an open set B in X such that $Y \setminus A = Y \cap B$. Then,

$$A = Y \setminus (Y \cap B) = Y \cap (X \setminus B)$$

which finishes the proof. ■

Corollary. Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Trivial. ■

Definition 1.25 (Interior, Closure). Let X be a topological space and $A \subseteq X$. The *interior* of A is defined as the union of all open sets contained in A and the *closure* of A is defined as the intersection of all closed sets containing A. The interior of A is denoted by $\operatorname{Int} A$ and the closure of A is denoted by $\overline{A}$.

Then, by definition, we have that

$$\operatorname{Int} A \subseteq A \subseteq \overline{A}$$

Theorem 1.26. Let Y be a subspace of X and A be a subset of Y. Let $\overline{A}$ denote the closure of A in X. Then, the closure of A in Y is given by $\overline{A} \cap Y$.

Proof. Let $\mathcal{F}$ be the collection of all closed sets in X containing A. Then, by a preceeding theorem, we know that the set of all closed sets in Y containing A is given by $Y \cap \mathcal{F}$. And thus,

$$\bigcup_{C \in Y \cap \mathcal{F}} C = Y \cap \bigcup_{C \in \mathcal{F}} C = Y \cap \overline{A}$$

This finishes the proof. ■

Theorem 1.27. Let A be a subset of the topological space X.

- Then $x \in \overline{A}$ if and only if every open set U containing x intersects A
- Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only

if every basis element B containing x intersects A

Proof.

- Suppose $x \in \overline{A}$ and U be an open set containing x. Suppose for the sake of contradiction, there is an open set U in X that contains x but does not intersect A, in which case $X \backslash U$ is a closed set containing A and not containing x. By definition, since $\overline{A} \subseteq X \backslash U$, x may not be an element of $\overline{A}$, a contradiction. Conversely, suppose every open set U containing x intersects A and that $x \notin \overline{A}$. But then, the set $X \backslash \overline{A}$ is open and contains x but does not intersect A, a contradiction.

- Suppose $x \in \overline{A}$, then every open set containing x intersects A. Since all elements of $\mathcal{B}$ are open, they intersect A. Conversely, since every open set U containing x has a basis subset B that contains x and therefore intersects A, U must intersect A. This finishes the proof.

■

The stetement "U is an open set containing x" is often shortened to "U is a **neighborhood** of x".

Definition 1.28. If A is a subset of the topological space X and if $x \in X$, we say that x is a *limit point* or *cluster point* or *accumulation point* of A if every neighborhood of x intersects A in some point other than x itself.

For example every element of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

Theorem 1.29. Let A be a subset of the topological space X and let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'$$

Proof. If $x \in A'$, due to the preceeding theorem, $x \in \overline{A}$ but since by definition, $A \subseteq \overline{A}$, we have that $A \cup A' \subseteq \overline{A}$.

Conversely let $x \in \overline{A}$. If $x \in A$, we are done. If not, then x is such that every open set containing x intersects A. But since $x \notin A$, the intersection must contain at least one point distinct from x, implying that $x \in A'$. This finishes the proof. ■

Corollary. A subset of a topological space is closed if and only if it contains all its limit points.

Definition 1.30 (Hausdorff Spaces). A topological space X is called a *Hausdorff space* if for each pair x_1 and x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Theorem 1.31. Every finite point set in a Hausdorff space X is closed.

Proof. It suffices to show this for a single point set, say $\{x_0\}$. For any $x \in X$ different from x_0, there are open sets U and V such that $x_0 \in U$ and $x \in V$ and $U \cap V = \emptyset$. And thus, x may not be in the closure of $\{x_0\}$. This finishes the proof. ■

The condition that finite point sets be closed has been given its own name, the T_1 **axiom**.

Theorem 1.32. Let X be a space satisfying the T_1 axiom and $A \subseteq X$. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. If every neighborhood of x intersects A at infinitely many points, then it intersects it in at least one point other than x and thus $x \in A'$.

Conversely, suppose x is a limit point of x but there is a neighborhood U of x that intersecs A in only finitely many points. Let $U \cap (A \backslash \{x\}) = \{x_1, \ldots, x_m\}$. Then, the open set $U \cap (X \backslash \{x_1, \ldots, x_m\})$ contains x but does not intersect A, which is contradictory to the fact that x is a limit point of A. ■

Theorem 1.33. If X is a Hausdorff space, then a sequence of points of X convertes to at most one point of X.

Proof. Suppose the sequence $\{x_n\}$ converges to two distinct points x and y. Then, by definition, there exist disjoint neighborhoods U and V of x and y respectively. Since x_n converges to x, U contains all but finitely many elements of the sequence but that means V cannot, a contradiction. ■

1.2 Continuous Functions

Definition 1.34. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is open in X.

Theorem 1.35. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent

1. f is continuous
2. for every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$
3. for every closed set B of Y, the set $f^{-1}(B)$ is closed in X
4. for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$

Proof. (1) $\Rightarrow$ (2). Let $x \in \overline{A}$ and V be an open set containing $f(x)$. We know by definition that $f^{-1}(V)$ is open and therefore intersects A. As a consequence, V intersects $f(A)$, implying that $f(x) \in \overline{f(A)}$.

(2) $\Rightarrow$ (3). Let $A = f^{-1}(B)$. Let $x \in \overline{A}$. Then,

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$$

and thus $x \in f^{-1}(B) = A$, implying that $A \subseteq \overline{A} \subseteq A$, finishing the proof.

(3) $\Rightarrow$ (1). Let V be an open set in Y and let $U = f^{-1}(V)$. Since $Y \backslash V$ is closed, so is $f^{-1}(Y \backslash V) = f^{-1}(Y) \backslash U = X \backslash U$. Then, by definition, U must be open.

(1) $\Leftrightarrow$ (4). The forward direction is trivial. Conversely, let V be an open set in Y and $U = f^{-1}(V)$. For each $x \in U$, there is an open set U_x such that $U_x \subseteq U$. Then, $U = \bigcup_{x \in U} U_x$ is open. This finishes the proof. ■

Definition 1.36 (Homeomorphism). Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is a *homeomorphism*.

As a result, any property of X that is entirely expressed in terms of the topology of X yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called a **topological property**.

If $f : X \to Y$ is an injective, continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological imbedding** or simply an **imbedding** of X in Y.

Theorem 1.37. Let X, Y and Z be topological spaces

1. (Constant) If $f : X \to Y$ maps all of X to a single point of Y, then it is continuous
2. (Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous
3. (Composites) If $f : X \to Y$ and $g : Y \to Z$ are ocontinuous, then the map $g \circ f : X \to Z$ is continuous
4. (Domain Restriction) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.
5. (Range Restriction/Expansion) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y ias a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.
6. (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets $\{U_\alpha\}$ such that $f|_{U_\alpha}$ is continuous for each α.

Proof.

1. Trivial
2. Trivial
3. Let V be an open set in Z. Then, $g^{-1}(V)$ is open in Y and $f^{-1} \circ g^{-1}(V)$ is open in X and thus $g \circ f$ is continuous
4. Notice that $f|_A \equiv f \circ j$

5. Let V be an open set in Z. Then, there is an open set W in Y such that $V = Z \cap W$. Since the range of f is a subset of Z, we have
$$g^{-1}(V) = g^{-1}(Z \cap W) = f^{-1}(Z \cap W) = f^{-1}(W)$$
which is open in X and thus, g is continuous. A similar argument can be applied in the second case.

6. Let V be an open set in Y, then we may write
$$f^{-1}(V) = \bigcup_{\alpha} f|_{U_\alpha}^{-1}(V \cup U_\alpha)$$
which is a union of a collection of open sets and is therefore open. This finishes the proof.

■

Lemma 1.38 (Pasting Lemma). Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous If $f(x) = g(x)$ for every $x \in A \cap B$ then f and g combine to give a continuous function $h : X \to Y$ defined as
$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. We then have $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f is continuous, we know that $f^{-1}(C)$ is closed in A and therefore in X similarly, so is $g^{-1}(C)$, which finishes the proof. ■

Theorem 1.39. Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ then f is continuous if and only if the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. The maps f_1 and f_2 are called the *coordinate maps* of f.

Proof. We know that the projection maps π_1, π_2 are continuous. We note that $f_1(a) = \pi_1(f(a))$ and $f_2(a) = \pi_2(f_2(a))$. If f is continuous, then so are f_1 and f_2.

Conversely, suppose f_1 and f_2 are continuous and $U \times V$ be a basis element for the product topology on $X \times Y$. We know due to a preceeding result that both U and V are open in X and Y respectively. Then
$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$
which is an intersection of two open sets and is therefore open. ■

1.3 Metric Topology

Definition 1.40 (Metric). A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x,y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$
2. $d(x,y) = d(y,x)$ for all $x, y \in X$
3. (Triangle Inequality) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$

For $\epsilon > 0$, define the set

$$B_d(x,\epsilon) = \{y \mid d(x,y) < \epsilon\}$$

Definition 1.41 (Metric Topology). If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x,\epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X, called the *metric topology* induced by d.

Proposition 1.42. The collection of all ϵ-balls $B_d(x,\epsilon)$ for all $x \in X$ and $\epsilon > 0$ is a basis.

Proof. The first condition is trivially satisfied. Suppose $z \in B(x,\epsilon) \cap B(y,\epsilon)$. Let $r = \frac{1}{2}\min\{\epsilon - d(x,z), \epsilon - d(y,z)\}$. It is obvious, due to the triangle inequality, that $B(z,r) \subseteq B(x,\epsilon) \cap B(y,\epsilon)$. ■

Definition 1.43 (Metrizable). If X is a topological space, X is said to be *metrizable* if there exists a metric d on the set X that induces the topology of X.

A **metric space** is a metrizable space X together with a specific metric d that gives the topology of X.

Definition 1.44. Let X be a metric space with metric d. A subset A of X is said to be *bounded* if there is some number M such that $d(a_1, a_2) \leq M$ for every pair a_1, a_2 of points of A. if A is bounded and non-empty, the *diameter* of A is

defined to be
$$\operatorname{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Proposition 1.45. Every metric space is Hausdorff.

Proof. Trivial. ■

Theorem 1.46. Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by the equation
$$\overline{d}(x, y) = \min\{d(x, y), 1\}$$
Then $\overline{d}$ is a metric that induces the same topology as d.

Proof. We need only check the triangle inequality. This is euqivalent to
$$\overline{d}(x, y) + \overline{d}(y, z) \geq \overline{d}(x, z)$$
Obviously if either one of $\overline{d}(x, y)$ or $\overline{d}(y, z)$ is greater than or equal to 1, then we are done. If not, then
$$\overline{d}(x, y) + \overline{d}(y, z) = d(x, y) + d(y, z) \geq \overline{d}(x, z) \geq \min\{d(x, z), 1\}$$

Let $\mathcal{T}$ be the topology on X induced by d, having basis $\mathcal{B}$. Let $\overline{\mathcal{B}}$ be the set of all balls induced by $\overline{d}$ having radius strictly less than 1. Let U be an open set in $\mathcal{T}$ and $x \in U$, then, by definition, there is $B_d(x, \epsilon)$ in $\mathcal{B}$ such that $x \in B_d(x, \epsilon) \subseteq U$. The ball $B_{\overline{d}}(x, \frac{1}{2}\min\{\epsilon, 1\})$ is contained in $B_d(x, \epsilon)$ and also contains x. Thus, $\overline{\mathcal{B}}$ is a basis for $\mathcal{T}$. This finishes the proof. ■