Set Theory

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Abstract

The main reference for this was [Kun80]. The SLP covered Chapters I through VII of the same. There are many proofs that have been omitted throughout this report. The reader is encouraged to read the source instead of these notes to gain a better understanding of the elements of Set Theory.

Contents

Chapter 1

The Zermelo-Fraenkel Axioms

1.1 Axioms of Set Theory

We shall discuss Zermelo-Fraenkel Set Theory, which is a first order theory, with signature $\mathsf{ZF} = (\varnothing, \{\in\})$. That is, there are no function symbols and the only predicate is the "belongs to" relation.

ZF0 (Nonempty Domain) There is at least one set.

$$\exists x(x = x)$$

This axiom is redundunt since **ZF7** guarantees the existence of an infinite set and thus the domain of discourse must be nonempty.

ZF1 (Extensionality) Informally speaking, a set is determined uniquely by its elements.

$$\forall x \forall y (\forall z (z \in x \iff z \in y) \implies x = y)$$

ZF2 (Foundation/Regularity) This states that any nonempty set contains an element that is disjoint from it.

$$\forall x \left[\exists y (y \in x) \implies \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))\right]$$

ZF3 (Comprehension) Informally speaking, this axiom allows us to define sets in the set-builder notation. Let ϕ be a valid first order formula with free variables $w_1, \ldots, w_n, x, z$. Then

$$\forall z \forall w_1, \ldots, w_n \exists y \forall x \left(x \in y \iff x \in z \wedge \phi\right)$$

Notice how this is the same as writing

$$y = \{x \in z \mid \phi\}$$

ZF4 (Pairing) Informally, this states that given two sets x and y, there is a set $z = \{x, y\}$.

$$\forall x \forall y \exists z \forall w (w \in z \iff (w = x \vee w = y))$$

ZF5 (Union) This axiom allows us to take a union of a collection of sets.

$$\forall \mathscr{F} \exists A \forall y (x \in y \wedge y \in \mathscr{F} \implies x \in A)$$

ZF6 (Replacement Scheme) Let ϕ be a valid formula without Y as a free variable. Then,

$$\forall A (\forall x \in A\ \exists! y\ \phi(x, y) \implies \exists Y\ \forall x \in A\ \exists y \in Y \phi(x, y))$$

Informally speaking, this allows us to replace the elements of a set to obtain a new set.

ZF7 (Infinity) There is an infinite inductive set.

$$\exists x (\varnothing \in x \wedge \forall y \in x (S(y) \in x))$$

ZF8 (Power Set) Every set has a set containing all its subsets. It is important to note that this need not be **the** power set.

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

ZF9 (Choice) Informally, given a collection of nonempty sets X, there is a choice function that chooses one element from each set in X.

$$\forall X \left(\varnothing \notin X \implies \exists f : X \to \bigcup X,\ \forall x \in X (f(x) \in x) \right).$$

We have been a bit sloppy in stating the axioms. Notice that our signature does not contain a predicate $\subseteq$ or the successor function S, neither do we know, a priori, of the existence of **the** empty set.

To define the formula $\subseteq (x, y)$, use

$$\subseteq (x, y) := \forall z (z \in x \implies z \in y)$$

As for the successor function, given any set x, using **ZF4**, there is a set $y = \{x\}$. Using **ZF5**, we may define $S(y) := x \cup y$. Finally, using **ZF0** and **ZF3**, we know of the existence of the empty set as

$$\exists x (x = x \wedge \exists y \forall z (z \in x \iff z \in y \wedge z \neq z))$$

Further, due to **ZF1**, the empty set is unique.

1.2 Consequences of the Axioms

Theorem 1.1. *There is no universal set. That is,*

$$\neg \exists z \forall x (x \in z)$$

Proof. If there were a universal set, then using **ZF3**, we may construct the set $y = \{x \in z \mid x \notin x\}$. Then, it is not hard to argue that

$$y \in y \iff y \notin y,$$

a contradiction. ■

Definition 1.2 (Power Set). Let x be a set. Due to **ZF8**, there is a set z containing all the subsets of x. Using Comprehension, we may construct

$$\mathscr{P}(x) := \{y \in z \mid y \subseteq x\}.$$

This is known as the **power set** of x.

Definition 1.3. Let $\mathscr{F}$ denote a set. Let A be a set satisfying **ZF5**. Define

$$\bigcup \mathscr{F} := \{x \in A \mid \exists y \in \mathscr{F} (x \in y)\}$$

and

$$\bigcap \mathscr{F} := \{x \in A \mid \forall y \in \mathscr{F} (x \in y)\}.$$

1.3 Relations, Functions and Well Ordering

Definition 1.4 (Ordered Pair). For sets x, y, define the ordered pair $\langle x, y \rangle$ by

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

The set on the right is constructed by using the pairing axiom twice.

Definition 1.5 (Cartesian Product). Let A and B be sets. Using Replacement, we may define, for each $y \in B$,

$$A \times \{y\} := \{z \mid \exists x \in A(z = \langle x, y \rangle)\}.$$

Again, by Replacement, define the set

$$\mathscr{F} := \{z \mid \exists y \in B(z = A \times \{y\})\}.$$

Finally, define

$$A \times B := \bigcup \mathscr{F}.$$

Definition 1.6 (Relation, Function). Let A be a set. A relation R on A is a subset of $A \times A$. Define the domain and range of a relation as

$$\mathrm{dom}(R) := \{x \in A \mid \exists y(\langle x, y \rangle \in R)\} \qquad \mathrm{ran}(R) := \{y \mid \exists x(\langle x, y \rangle \in R)\}.$$

We write xRy to denote $\langle x, y \rangle \in R$.

A relation f is said to be a function if

$$\forall x \in \mathrm{dom}(f) \exists! y \in \mathrm{ran}(f)(\langle x, y \rangle \in f).$$

We use $f : A \to B$ to denote a function f with $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.

Definition 1.7 (Total Ordering, Well Ordering). A *total ordering* is a pair $\langle A, R \rangle$ where A is a set and R is a relation that is irreflexive, transitive and satisfies trichotomy.

We say R *well-orders* A if $\langle A, R \rangle$ is a total ordering and every non empty subset of A has an R-least element.

We use $\mathrm{pred}(A, x, R)$ to denote the set $\{y \in A \mid yRx\}$.

Lemma 1.8. *Let $\langle A, R \rangle$ be a well-ordering. Then for all $x \in A$, $\langle A, R \rangle \ncong \langle \mathrm{pred}(A, x, R), R \rangle$.*

Proof. Suppose $\langle A, R \rangle \cong \langle \mathrm{pred}(A, x, R), R \rangle$ and let $f : A \to \mathrm{pred}(A, x, R)$ be the order isomorphism. Let x be the R-least element of the set

$$\{y \in A \mid f(y) \neq y\},$$

which obviously exists since the aforementioned set is nonempty. If $xRf(x)$, there is some $y \in A$ with yRx and $f(y) = x \neq y$ a contradiction to the choice of x. On the other hand, if $f(x)Rx$, then $f(f(x)) \neq f(x)$ since f is injective, a contradiction to the choice of x. This completes the proof. ∎

Theorem 1.9. *Let $\langle A, R \rangle$ and $\langle B, S \rangle$ be two well-orderings. Then exactly one of the following holds:*

(a) $\langle A, R \rangle \cong \langle B, S \rangle$.

(b) $\exists y \in B \left(\langle A, R \rangle \cong \langle \mathrm{pred}(B, y, S), S \rangle\right)$.

(c) $\exists x \in A \left(\langle \mathrm{pred}(A, x, R), R \rangle \cong \langle B, S \rangle\right)$.

Proof. Let

$$f := \{\langle v, w \rangle \mid v \in A,\ w \in B,\ \langle \mathrm{pred}(A, v, R), R \rangle \cong \langle \mathrm{pred}(B, w, S), S \rangle\}.$$

Due to the preceeding lemma, if $\langle v_1, w\rangle, \langle v_2, w\rangle \in f$, then $v_1 = v_2$. Similarly, if $\langle v, w_1\rangle, \langle v, w_2\rangle \in f$, then $w_1 = w_2$. Hence, f is an injective function.

It is not hard to argue that f is an order isomorphism from an initial segment of A to an initial segment of B. Both these segments may not be proper else we could find another isomorphism from an initial segment of A to an initial segment of B by extending one of the isomorphisms in f. This completes the proof. ■

Chapter 2

Ordinal Numbers

2.1 Transitive Sets

Definition 2.1. A set x is said to be *transitive* if

$$\forall y \forall z (z \in y \wedge y \in x \implies z \in x).$$

Proposition 2.2. *A set x is transitive if and only if*

$$\forall y (y \in x \implies y \subseteq x).$$

Proof. Suppose x is transitive and $y \in x$. Since for all $z \in y$, $z \in x$, we must have $y \subseteq x$. The converse is trivial. ■

Proposition 2.3. *If x is a transitive set, then so is $x \cup \{x\}$.*

Proof. ■

Proposition 2.4. *If x is a transitive set, then so is $\mathscr{P}(x)$.*

Proof. ■

Proposition 2.5. *If $\mathscr{F}$ is a family of transitive sets, then so is $\bigcup \mathscr{F}$.*

Proof. ■

Proposition 2.6. *If x is a transitive set, then so is every $z \in x$.*

Proof. ■

2.2 Ordinals

Definition 2.7 (Ordinal). A set x is said to be an *ordinal* if it is transitive and well ordered by $\in$. That is, the pair $\langle x, \in_x \rangle$ is a well ordering, where

$$\in_x := \{\langle v, w \rangle \in x \times x \mid v \in w\}.$$

Theorem 2.8 (Properties of Ordinals).

(a) *If x is an ordinal and $y \in x$, then y is an ordinal and $y = \mathrm{pred}(x, y)$.*

(b) *If $x \cong y$ are ordinals, then $x = y$.*

(c) *If x, y are ordinals, then exactly one of the following is true: $x = y$, $x \in y$ or $y \in x$.*

(d) *If C is a nonempty set of ordinals, then $\exists x \in C\ \forall y \in C(x \in y \vee x = y)$. That is, every nonempty set of ordinals has a minimum element.*

Proof. (a) Due to Proposition 2.6, y is a transitive and owing to it being the subset of a well ordered set, it is well ordered too, hence an ordinal.

(b) Let $f : x \to y$ be an isomorphism. Let

$$A := \{z \in x \mid f(z) \neq z\}.$$

Suppose A is nonempty, then it has a least element, say $w \in x$. If $v \in w$, then $v = f(v) \in f(w)$ whence $w \subseteq f(w)$. On the other hand, if $v \in f(w)$, then there is some $u \in w$ such that $v = f(u) = u \in w$ and thus $f(w) = w$, a contradiction.

(c) Follows from Theorem 1.9.

(d) First note that it suffices to find $x \in C$ with $x \cap C = \varnothing$ for if $y \in C$ is another ordinal with $x \neq y$, then $y \notin x$ lest $x \cap C \neq \varnothing$.

Pick any $x \in C$. If $x \cap C = \varnothing$, then we are done. Else, let $x' \in x \cap C$ be the $\in$-least element. It is not hard to argue that $x' \cap C = \varnothing$ and we are done. ∎

Lemma 2.9. *If A is a transitive set of ordinals, then A is an ordinal.*

Proof. We must first show that the membership relation $\in_A$ is a linear order. This follows from Theorem 2.8 (c) and the fact that A is a transitive set. Lastly, to see that A is well ordered, simply invoke Theorem 2.8 (d). ∎

Theorem 2.10. *If $\langle A, R \rangle$ is a well ordering, then there is a unique ordinal C such that $\langle A, R \rangle \cong C$.*

Proof. Let

$$B := \{a \in A \mid \exists x_a(x_a \text{ is an ordinal } \wedge \langle \mathrm{pred}(A, a, R), R \rangle \cong x_a)\},$$
$$f := \{\langle b, x_b \rangle \mid b \in B\}.$$

First, note that for all $b \in B$, x_b, since it exists must be unique and thus f is a well defined function with $\mathrm{dom}(f) = B$.

Let $C = \mathrm{ran}(f)$. We contend that C is an ordinal. Let $y \in x \in C$ and $a \in B$ be such that $g : \mathrm{pred}(A, a, R) \to x$ is an isomorphism. Then, there is some $b \in \mathrm{pred}(A, a, R)$ with $g(b) = y$. It is not hard to see that the restriction $g : \mathrm{pred}(A, b, R) \to y$ is an isomorphism whence $y \in C$ and thus C is an ordinal due to the preceeding lemma.

The function $f : B \to C$ is obviously a surjection. We contend that it is an isomorphism. Indeed, let $a, b \in B$ with aRb and $g : \mathrm{pred}(A, b, R) \to x_b$ be *the* isomorphism. If $y = g(a)$, then the restriction $g : \mathrm{pred}(A, a, R) \to y$ is an isomorphism whence $f(a) = y \in x = f(b)$ and f is an order isomorphism.

Suppose $B \neq A$. Let $b \in A \backslash B$ be the R-least element. Then, $\operatorname{pred}(A, b, R) \subseteq B$. Now suppose $B \neq \operatorname{pred}(A, b, R)$, consequently, there is some $b' \in B \backslash \operatorname{pred}(A, b, R)$, then bRb' and if there is an order isomorphism from $\operatorname{pred}(A, b', R)$ to some ordinal x, then there must be one from $\operatorname{pred}(A, b, R)$ as we have argued earlier, a contradiction.

Thus, either $B = A$ or $B = \operatorname{pred}(A, b, R)$ for some $b \in A$. In the latter case, the function f is an order isomorphism between $\operatorname{pred}(A, b, R)$ and an ordinal C whence $b \in B$, a contradiction. Thus $B = A$ and the proof is complete. ■

Definition 2.11 (Type of a Well Ordering). If $\langle A, R \rangle$ is a well ordering, then $\operatorname{type}(A, R)$ is the unique ordinal C such that $\langle A, R \rangle \cong C$.

Henceforth, we use Greek letters $\alpha, \beta, \gamma, \ldots$ to vary over ordinals. That is, saying $\forall \alpha(\ldots)$ is equivalent to saying $\forall x(x \text{ is an ordinal } \ldots)$. Further, since the ordinals are well ordered, we write $\alpha < \beta$ to denote $\alpha \in \beta$ and similarly, $\alpha \leq \beta$ means $\alpha \in \beta \vee \alpha = \beta$.

Definition 2.12. Let X be a set of ordinals. Define

$$\sup(X) := \bigcup X \quad \text{and} \quad \min(X) := \bigcap X.$$

Further, for an ordinal α, let $S(\alpha)$ denote the set $\alpha \cup \{\alpha\}$.

Lemma 2.13. *(a) $\forall \alpha, \beta(\alpha \leq \beta \iff \alpha \subseteq \beta)$.*

(b) If X is a set of ordinals, $\sup(X)$ is the least ordinal $\geq$ all elements of X and if $X \neq \varnothing$, $\min(X)$ is the least ordinal in X.

Proof. (a) The forward direction is obvious. Suppose $\alpha \subseteq \beta$. If $\alpha = \beta$, then we are done. If not, let γ be the $<$-least element of $\beta \backslash \alpha$. We contend that $\gamma = \alpha$. Indeed, if $x \in \gamma$, then $x \notin \beta \backslash \alpha$ lest we contradict the minimality of γ consequently, $x \in \alpha$ whence $\gamma \subseteq \alpha$. On the other hand, since $\alpha = \operatorname{pred}(\beta, \alpha)$, we have $\alpha \leq \gamma$ and thus $\alpha \subseteq \gamma$. This shows that $\alpha = \gamma \in \beta$ and the conclusion follows.

(b) ■

Lemma 2.14. *For an ordinal α, $S(\alpha)$ is an ordinal, $\alpha < S(\alpha)$ and*

$$\forall \beta(\beta < S(\alpha) \iff \beta \leq \alpha).$$

Definition 2.15 (Successor, Limit Ordinal). An ordinal α is said to be a *successor ordinal* if there is an ordinal β such that $\alpha = S(\beta)$. On the other hand, α is said to be a *limit ordinal* if $\alpha \neq \varnothing$ and α is not a successor ordinal.

2.3 Transfinite Induction and Recursion

2.3.1 Classes but informally

Informally speaking, a class is any collection of the form

$$\{x \mid \phi(x)\}$$

where $\phi(x)$ is a well defined first order formula. As we have seen earlier, the class

$$\{x \mid x = x\}$$

is not a set. A *proper class* is a class which is not a set. One uses boldface letters to denote classes.

Definition 2.16. Denote

$$\mathbf{V} := \{x \mid x = x\} \qquad \mathbf{ON} := \{x \mid x \text{ is an ordinal}\}.$$

To be completely formal, a class is simply a first order formula with one or more free variables. For example, the class of all ordinals can be thought of as the formula

$$\mathbf{ON}(x) = x \text{ is an ordinal.}$$

We can extend this to define functions between classes $\mathbf{A}$ and $\mathbf{B}$. A function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is given by a first order logic formula in two variables $\mathbf{F}(x, y)$ such that

$$\forall x\ \mathbf{A}(x) \implies \exists! y\ \left(\mathbf{B}(y) \wedge \mathbf{F}(x, y)\right).$$

Theorem 2.17 (Transfinite Induction on ON). *If $\mathbf{C} \subseteq \mathbf{ON}$ and $\mathbf{C} \neq \varnothing$, then $\mathbf{C}$ has a least element.*

Proof. The proof is exactly like Theorem 2.8 (d). ∎

One must note that there is a significant difference between Theorem 2.8 (d) and Theorem 2.17. The former is a single provable statement in ZFC while the latter is a theorem schema which represents an infinite collection of theorems. In particular, suppose the class $\mathbf{C}$ corresponded to a formula $\mathbf{C}(x, z_1, \ldots, z_n)$, then Theorem 2.17 in this case says the following:

$$\begin{aligned}\forall z_1, \ldots, z_n \Big\{ &\left[\forall x (\mathbf{C}(x, z_1, \ldots, z_n) \implies x \text{ is an ordinal}) \wedge \exists x \mathbf{C}(x, z_1, \ldots, z_n)\right] \\ &\implies \Big[\exists x\ (\mathbf{C}(x, z_1, \ldots, z_n) \wedge \forall y (\mathbf{C}(y, z_1, \ldots, z_n) \implies y \geq x))\Big] \Big\}.\end{aligned}$$

And Theorem 2.17 specifies one such formula for each well-formed sentence $\mathbf{C}$.

Theorem 2.18 (Transfinite Recursion on ON). *If $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, then there is a unique $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ such that*

$$\forall \alpha\ \left(\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)\right).$$

The formal restatement of the above in terms of first order logic is the following:

$$\forall x \exists! y\ \mathbf{F}(x, y) \implies \Big[\forall \alpha \exists! y\ \mathbf{G}(\alpha, y) \wedge \forall \alpha \exists x \exists y\ (\mathbf{G}(\alpha, y) \wedge \mathbf{F}(x, y) \wedge x = \mathbf{G} \restriction \alpha)\Big]$$

where

$$(x = \mathbf{G} \restriction \alpha) := \mathsf{function}(x) \wedge \mathsf{dom}(x) = \alpha \wedge \left(\forall \beta \in \mathsf{dom}(x)\ \mathbf{G}(\beta, x(\beta))\right).$$

Similarly, one can encode the uniqueness condition.

Proof. ∎

2.4 Ordinal Arithmetic

Addition

Definition 2.19 (Ordinal Addition). If α, β are ordinals, then define $\alpha + \beta = \mathsf{type}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$ where

$$R = \{\langle\langle\xi, 0\rangle, \langle\eta, 0\rangle\rangle \mid \xi < \eta < \alpha\} \cup \{\langle\langle\xi, 0\rangle, \langle\eta, 1\rangle\rangle \mid \xi < \eta < \beta\} \cup \left[(\alpha \times \{0\}) \times (\beta \times \{1\})\right].$$

Informally speaking, we construct a new ordinal $\alpha + \beta$ by first "placing" α is a line and then placing β after it linearly. This is best visualized when α and β are finite ordinals.

To see that R indeed gives $\alpha \times \{0\} \cup \beta \times \{1\}$ the structure of a well order, let S be a nonempty subset. If $S \cap \alpha \times \{0\}$ is nonempty, then the minimal element of S exists and is the minimal element of $S \cap \alpha \times \{0\}$. On the other hand, if $S \cap \alpha \times \{0\} = \varnothing$, the minimal element of S exists and is the minimal element of $S \cap \beta \times \{1\}$.

Lemma 2.20. *For ordinals α, β, γ,*

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(b) $\alpha + 0 = \alpha$.

(c) $\alpha + 1 = S(\alpha)$.

(d) $\alpha + S(\beta) = S(\alpha + \beta)$.

(e) If β is a limit ordinal, then $\alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}$.

Proof. We shall only prove (e) since the others are straightforward. First, note that $\alpha + \beta \geq \alpha + \xi$ for every $\xi < \beta$, which is easy to see by setting up an obvious order preserving injection. ∎

Remark 2.4.1. *One must note that ordinal addition is* ***not commutative****. Indeed,*

$$1 + \omega = \sup\{1 + n \mid n < \omega\} = \omega$$

while

$$\omega + 1 = S(\omega) \neq \omega$$

where the last "non-equality" follows from the axiom of foundation. Thus, $1 + \omega \ncong \omega + 1$.

Multiplication

Definition 2.21. If α, β are ordinals, define $\alpha \cdot \beta = \operatorname{type}(\beta \times \alpha, R)$ where R is the dictionary order, given by

$$R = \left\{ \langle\langle \xi, \eta\rangle, \langle \xi', \eta'\rangle\rangle \;\middle|\; \xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta') \right\}.$$

We must first check that R is indeed a well ordering. That it is a strict linear order is clear. Let $S \subseteq \beta \times \alpha$ be a nonempty subset. Let S_1 be the projection of S onto β. This has a minimum element, say ξ. Consider now the set of all $\eta \in \alpha$ such that $\langle \xi, \eta\rangle \in S$. This is a nonempty subset of α and thus has a minimum element, say δ. Then, $\langle \xi, \delta\rangle$ is a minimum element of S.

Lemma 2.22. *For ordinals α, β, γ,*

(a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(b) $\alpha \cdot 0 = 0$.

(c) $\alpha \cdot 1 = \alpha$.

(d) $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$.

(e) If β is a limit ordinal, then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$.

(f) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Exponentiation

Definition 2.23. For ordinals α, β, we define α^β by recursion on β as

- $\alpha^0 = 1$.
- $\alpha^{\beta+1} = \alpha^\beta \cdot \beta$.
- If β is a limit ordinal, $\alpha^\beta = \sup\{\alpha^\xi \mid \xi < \beta\}$.

Remark 2.4.2. *Interestingly,*

$$2^\omega = \sup\{2^n \mid n < \omega\} = \omega.$$

2.5 Equivalent forms of the Axiom of Choice

Theorem 2.24 (Well Ordering Theorem). *For every nonempty set A, there is a relation $R \subsetneq A \times A$ such that R well orders A.*

AC $\implies$ WO

Let A be a set. We shall explicitly construct a well ordering on X using the Axiom of Choice. First, let $f : \mathcal{P}(A) \backslash \{\varnothing\} \to A$ be a choice function and extend it to $f : \mathcal{P}(A) \to A \coprod \{\varnothing\}$ by defining $f(\varnothing) = \varnothing$. We shall now use transfinite recursion to define a function F on the ordinals as follows:

$$\begin{aligned} F(0) &:= f(A) \\ F(\alpha) &:= f\left(\{x \in A \mid \forall \beta \in \alpha(F(\beta) \neq x)\}\right). \end{aligned}$$

First, note that if $F(\alpha) = F(\beta) \neq \varnothing$, then $\alpha = \beta$. Next, we contend that there must be an ordinal α with $F(\alpha) = \varnothing$. For if not, then we may apply the axiom of replacement and that of comprehension to obtain a set of all ordinals, a contradiction to the Burali-Forti paradox.

Let $\mathbf{C}$ denote the class of all ordinals α with $F(\alpha) = \varnothing$. Due to Theorem 2.17, there is a minimal such ordinal, say α_0, then

$$f\left(\{x \in A \mid \forall \beta \in \alpha_0(F(\beta) \neq x)\}\right) = \varnothing \implies \{x \in A \mid \forall \beta \in \alpha_0(F(\beta) \neq x)\} = \varnothing.$$

Let $G : A \to \alpha_0$ denote the inverse function of F. Define the relation $R \subsetneq A \times A$ by

$$R := \{\langle x, y\rangle \mid G(x) \in G(y)\}.$$

That this is a well ordering is easy to see.

WO $\implies$ AC

This direction, on the other hand, is much easier. Let X denote a collection of sets and let $Y = \bigcup X$. Let R be a well ordering on Y. Define the function $f : X \to Y$ by $f(x) = \min(x)$, the R-least element, which can be chosen since Y has been well ordered and $x \subsetneq Y$.

AC $\implies$ Zorn

Let X be a set and $P = (X, \leqq)$ be a poset on it such that every chain in P has an upper bound. Let $f : \mathscr{P}(X) \backslash \{\varnothing\} \to X$ be a choice function.

Suppose P has no maximal element. Then, every chain in P must have a strict upper bound. Let $\mathscr{C}$ be the set of all chains in P. Let $g : \mathscr{C} \to \mathscr{P}(X)$ map a chain in P to the set of all *strict* upper bounds. Consequently, $g(C) \neq \varnothing$ for every chain C in P.

We shall define a class function $F : \mathbf{ON} \to X$ using transfinite recursion. Begin with $F(0) = F(X)$. Now, for any ordinal $\alpha \in \mathbf{ON}$, let C_α denote the chain $\{F(\beta) \mid \beta < \alpha\}$ and define

$$F(\alpha) := f(g(C_\alpha)).$$

It is not hard to see that $F(\alpha) = F(\beta)$ if and only if $\alpha = \beta$ whence we may use Replacement to obtain a *set* of all ordinals, which is absurd.

Zorn $\implies$ AC

Let X be a collection of sets and $Y = \bigcup X$. Let P be the poset of pairs (S, f) where $S \subseteq X$ and $f : S \to Y$ is a function with $f(s) \in s$ for each $s \in S$. We say $(S, f) \leqq (S', f')$ if $S \subseteq S'$ and $f' \restriction_S = f$.

Let $C = \{(S_\alpha, f_\alpha)\}$ be a chain in P. Define the function $f : S := \bigcup_\alpha S_\alpha \to Y$ by $f(x) := f_\alpha(x)$ if $x \in S_\alpha$. Then, (S, f) is an upper bound for the chain C. Thus, due to Zorn's Lemma, P contains a maximal element, say $(\widetilde{S}, F)$. We contend that $\widetilde{S} = X$. For if not, then there is $x \in X \backslash \widetilde{S}$ and the function F can be extended to $\widetilde{S} \cup \{x\}$ by simply choosing an element of x and assigning it to x under F. This contradicts the maximality of $(\widetilde{S}, F)$ and hence, F is the desired choice function.

Chapter 3

Cardinal Numbers

Definition 3.1. Sets A and B are said to be *equinumerous* if there is a bijection $f : A \to B$. This is denoted by $A \approx B$. On the other hand, if there is an injection $f : A \to B$, it is denoted by $A \preceq B$. We write $A \prec B$ if $A \preceq B$ and $B \not\preceq A$.

Theorem 3.2 (Cantor-Schröder-Bernstein). $A \preceq B \wedge B \preceq A \implies A \approx B$.

Definition 3.3. For a set A, $|A|$ is the least α such that $\alpha \approx A$. α is a *cardinal* if and only if $\alpha = |\alpha|$.

From Theorem 2.24, there is a well ordering R on A and thus an ordinal α with an order preserving bijection between $\langle A, R\rangle$ and α, in particular, $A \approx \alpha$. Thus, $|A|$ is defined for every set. Further, note that α is a cardinal if and only if $\forall \beta < \alpha (\beta \not\approx \alpha)$ and for any ordinal α, $|\alpha| \leq \alpha$.

Lemma 3.4. *If* $|\alpha| \leq \beta \leq \alpha$, *then* $|\beta| = |\alpha|$.

Proof. Since $\beta \leq \alpha$, we have $\beta \subseteq \alpha$ and thus $\beta \preceq \alpha$. On the other hand, $|\alpha| \subseteq \beta$. Composing this inclusion with the bijection $\alpha \approx |\alpha|$, we have $\alpha \preceq \beta$. We are done due to Theorem 3.2. ■

Lemma 3.5. *If* $n \in \omega$, *then*

(a) $n \not\approx n + 1$.

(b) $\forall \alpha (\alpha \approx n \implies \alpha = n)$.

Proof. (a) Suppose not. Pick the smallest $n \in \omega$ such that $n \approx n + 1$. Note that $n \neq 0$. We have an injective function $f : n + 1 \to n$. Composing appropriately, we may suppose that $f(n) = n - 1$ where $n \in n + 1$ and $n - 1 \in n$. The restriction $f \upharpoonright_n$ is an injective function from n to $n - 1$ whence by Theorem 3.2, $n - 1 \approx n$, a contradiction.

(b) If $n < \alpha$, then $n + 1 \leq \alpha$ whence $n + 1 \preceq \alpha$. On the other hand, $\alpha \approx n < n + 1$, consequently $\alpha \approx n + 1$, a contradiction to (a).

Now suppose $\alpha < n$. Then, $|n| = |\alpha| \leq \alpha \leq \alpha + 1 \leq n$, consequently, $|\alpha + 1| = |n|$. But since $\alpha + 1 \approx n + 1$, we have $n + 1 \approx n$, a contradiction to (a). Thus $\alpha = n$. ■

Corollary 3.6. ω is a cardinal and so is every ordinal $n < \omega$.

Definition 3.7. A is *finite* if and only if $|A| < \omega$. A is *countable* if and only if $|A| \leq \omega$. We use the shorthand *infinite* to mean "not finite" and *uncountable* to mean "not countable".

Definition 3.8 (Cardinal Arithmetic). For cardinals κ and λ, define

$$\kappa \oplus \lambda := |\kappa \times \{0\} \cup \lambda \times \{1\}|, \quad \kappa \otimes \lambda := |\kappa \times \lambda|.$$

Unlike ordinal arithmetic, the operations $\oplus$ and $\otimes$ are commutative, which is obvious from the definition above. Furthermore, note that

$$|\kappa + \lambda| = |\lambda + \kappa| = \kappa \oplus \lambda \quad \text{and} \quad |\kappa \cdot \lambda| = |\lambda \cdot \kappa| = \kappa \otimes \lambda.$$

Lemma 3.9. *For $m, n \in \omega$, $n \oplus m = n + m < \omega$ and $n \otimes m = n \cdot m < \omega$.*

Proof. ■

Proposition 3.10. *Every infinite cardinal is a limit ordinal.*

Proof. Suppose $\kappa = \alpha + 1$ is a cardinal. Then, α is not a finite ordinal, that is, $\omega < \alpha$ and thus there is an ordinal β such that $\alpha = \omega + \beta$. Consequently, $1 + \alpha = 1 + \omega + \beta = \omega + \beta$ as we have seen previously that $1 + \omega = \omega$. Consequently,

$$|\kappa| = |\alpha + 1| = |1 + \alpha| = |\alpha|,$$

a contradiction to the fact that κ is a cardinal. ■

Theorem 3.11 (Tarski). *If κ is an infinite cardinal, then $\kappa \otimes \kappa = \kappa$.*

Proof. We shall prove this statement by transfinite induction on κ. That this statement holds for $\kappa = \omega$ is well known. Suppose now that $\kappa > \omega$ and the statement holds for each cardinal $\lambda < \kappa$.

Note that for an infinite ordinal $\alpha < \kappa$, we have $|\alpha| < \kappa$ and thus

$$|\alpha \times \alpha| = |\alpha| \otimes |\alpha| = |\alpha| < \kappa.$$

Let $\prec$ denote the strict lexicographic ordering on $\kappa \times \kappa$. Define the relation $\trianglelefteq$ on $\kappa \times \kappa$ by $\langle \alpha, \beta \rangle \trianglelefteq \langle \gamma, \delta \rangle$ if and only if

$$\max\{\alpha, \beta\} < \max\{\gamma, \delta\} \text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \langle \alpha, \beta \rangle \prec \langle \gamma, \delta \rangle.$$

That this relation is an ordering is immediate from the definition. We shall now show that this is a well ordering. Let $S \subseteq \kappa \times \kappa$ be nonempty. Using Replacement, construct the set S' which consists of $\max\{\alpha, \beta\}$ for all $\langle \alpha, \beta \rangle \in S$. Since $S' \subseteq \kappa$, it contains a minimum element, say α_0. Using Comprehension, construct the set S'' consisting of all pairs $\langle \alpha, \beta \rangle$ such that $\max\{\alpha, \beta\} = \alpha_0$. Now, $S'' \subseteq \kappa \times \kappa$, and under the lexicographic order, it has a minimum element, which is also the minimum element of S under the ordering $\trianglelefteq$.

Given any $\langle \alpha, \beta \rangle \in \kappa \times \kappa$, the set of all pairs preceeding it in $\langle \kappa \times \kappa, \trianglelefteq \rangle$ is a subset of

$$(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)$$

Since κ is a limit ordinal, we have $\max\{\alpha, \beta\} + 1 < \kappa$ and due to the induction hypothesis, the cardinality of the above set is strictly smaller than κ whence $|\kappa \times \kappa| \leq \kappa$. There is an obvious injection from κ into $\kappa \times \kappa$, forcing $|\kappa \times \kappa| = \kappa$ due to Theorem 3.2. ■

Corollary 3.12. Let κ, λ be infinite cardinals. Then,

(a) $\kappa \oplus \lambda = \kappa \otimes \lambda = \max\{\kappa, \lambda\}$,

(b) $|\kappa^{<\omega}| = \kappa$.

Proof. ■

Theorem 3.13 (Cantor). $\forall X \left(X \prec \mathscr{P}(X)\right)$.

Proof. Suppose not, then $X \approx \mathscr{P}(X)$ for some X, which follows from Theorem 3.2 and the fact that there is a canonical injection from X to $\mathscr{P}(X)$. Let $f : X \to \mathscr{P}(X) \to X$ be a bijection. Using Comprehension, construct the set

$$S := \{x \in X \mid x \notin f(x)\} \subseteq X.$$

Let $s \in X$ be the unique element such that $f(s) = S$. Then,

$$s \in S \iff s \notin S,$$

a contradiction. ■

Theorem 3.14. *$\forall \alpha \exists \kappa \left(\kappa > \alpha \text{ is a cardinal}\right)$ is true in ZF.*

If we were to work in ZFC then we could just well order $\mathscr{P}(\alpha)$ and consider its cardinality.

Proof. The statement is obvious for finite cardinals. Suppose now that $\alpha \geq \omega$. Let

$$W := \{R \in \mathscr{P}(\alpha \times \alpha) \mid R \text{ well orders } \alpha\} S := \{\text{type}(\langle \alpha, R \rangle) \mid R \in W\}.$$

Let $\beta = \sup(S)$. We contend that β is a cardinal and $\beta > \alpha$. First, note that if $\delta \in W$, then $S(\delta) \in W$, consequently, $\beta \notin W$. Further, $\beta \not\approx \alpha$ lest one could find a well ordering on α which is in order preserving bijection with β. Suppose β were not a cardinal. Then, there is some $\gamma < \beta$ with $\gamma \approx \beta$. By definition, there is η such that $\gamma \leq \eta < \beta$ with $\eta \in W$, consequently, $\eta \approx \beta$ but $\alpha \approx \eta$, a contradiction. This completes the proof. ■

Definition 3.15 (Successor, Limit Cardinals). Let α be an ordinal. Denote by α^+ the smallest *cardinal* strictly greater than α. A cardinal κ is said to be a *successor cardinal* if $\kappa = \alpha^+$ for some α. On the other hand, if $\kappa > \omega$ and is not a successor cardinal, then κ is said to be a *limit cardinal*.

Definition 3.16 (Aleph Numbers). Define the numbers $\aleph_\alpha$ by transfinite recursion on α.

(a) $\aleph_0 := \omega$.

(b) $\aleph_{\alpha+1} = (\aleph_\alpha)^+$.

(c) For a limit ordinal λ, define $\aleph_\lambda := \sup\{\aleph_\alpha \mid \alpha < \lambda\}$.

Theorem 3.17. *(a) Each $\aleph_\alpha$ is a cardinal.*

(b) Every infinite cardinal is equal to $\aleph_\alpha$ for some α.

(c) If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.

(d) $\aleph_\alpha$ is a limit cardinal ifa nd only if α is a limit ordinal.

(e) $\aleph_\alpha$ *is a successor cardinal if and only if* α *is a successor ordinal.*

Proof. All of these follow immediately from the definition above. ∎

Remark 3.0.1. *One often writes* ω_α *in place of* $\aleph_\alpha$*. We adopt both conventions and use them interchangeably.*

Lemma 3.18. *If there is a surjective function* $f : X \to Y$*, then* $|Y| \leq |X|$.

Proof. Consider the set

$$S = \{f^{-1}(y) \mid y \in Y\},$$

which can be constructed using Replacement. Let $g : Y \to S$ be given by $g(y) = f^{-1}(y)$ and F be a choice function on S. Then, the composition $F \circ g$ is an injective function from Y to X, implying the desired conclusion. ∎

Definition 3.19 (Cardinal Exponentiation). For sets A and B, define

$$A^B := {}^B A := \{f \subseteq \mathscr{P}(B \times A) \mid f \text{ is a function}\}.$$

For cardinals κ and λ, define $\kappa^\lambda := |{}^\lambda \kappa|$.

Theorem 3.20. *Let* $2 \leq \kappa \leq \lambda$ *and* λ *an infinite cardinal. Then,* $\kappa^\lambda = 2^\lambda$.

Proof. Obviously, ${}^\lambda 2 \approx \mathscr{P}(\lambda)$ which can be seen by looking at the characteristic function of each subset of λ. Then, we have

$${}^\lambda k \preceq {}^\lambda \lambda \preceq \mathscr{P}(\lambda \times \lambda) \preceq \mathscr{P}(\lambda) \preceq {}^\lambda 2.$$

The conclusion follows from Theorem 3.2. ∎

Theorem 3.21. *Let* $\mathscr{B}(\mathbb{R})$ *denote the Borel* σ*-algebra on* $\mathbb{R}$ *with the standard topology. Then,* $|\mathscr{B}(\mathbb{R})| = 2^{\aleph_0}$*, the cardinality of the continuum.*

Proof. That $\mathscr{B}(\mathbb{R})$ has cardinality at least that of the continuum is straightforward since it contains all singletons. Showing the reverse direction is a bit involved and requires transfinite recursion.

First, note that $\mathbb{R}$ is second countable and thus has a countable abse for its topology, denote this by S_0. For an ordinal $\alpha < \omega_1$, let $S_{\alpha+1}$ denote the collection of all unions of the form

$$\bigcup_i A_i \cup \bigcup_j (\mathbb{R} \backslash B_j)$$

where A_i and B_j are chosen from S_α. Note that if $|S_\alpha| \leq 2^{\aleph_0}$, then the number of these unions that can be formed is at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ since there is a surjection from the set of all functions $\aleph_0 \to S_\alpha$ onto $S_{\alpha+1}$.

On the other hand, if α is a limit ordinal, define

$$S_\alpha = \bigcup_{\lambda < \alpha} S_\lambda.$$

We contend that $S = \bigcup_{\alpha < \omega_1} S_\alpha$ is a σ-algebra. Obviously, S contains $\varnothing$ and $\mathbb{R}$, and is closed under complementation. Let $\{A_n\}_{n=1}^\infty$ be a sequence in S. For each positive integer n, let $\alpha(n)$ denote the minimal ordinal λ such that $A_n \in S_\lambda$. Note that for each n, the cardinality $|\alpha(n)| \leq \omega$. Hence, if $\beta = \sup_{n<\omega} \alpha(n)$, then $|\beta| \leq \omega$, consequently, $\beta < \omega_1$ and $\{A_n \mid n < \omega\} \subseteq S_\beta$ implying that $\bigcup_{n=1}^\infty A_n \in S_{\beta+1} \subseteq S$.

As a result, S contains $\mathscr{B}(\mathbb{R})$ but the cardinality of S is at most

$$|\omega_1| \otimes 2^{\aleph_0} = \aleph_1 \otimes 2^{\aleph_0} \leq 2^{\aleph_0} \otimes 2^{\aleph_0} = 2^{\aleph_0}.$$

This completes the proof. ∎

Theorem 3.22 (Mazurkiewicz, 1914). *There is a subset $A \subseteq \mathbb{R}^2$ which meets* every *line in the plane at* exactly 2 *points.*

Proof. Let $\mathscr{L}$ denote the set of all possible lines in the plane. The cardinality of $\mathscr{L}$ is at least $2^{\aleph_0}$ and atmost $2^{\aleph_0} \otimes 2^{\aleph_0} = 2^{\aleph_0}$. Since this is in bijection with $2^{\aleph_0}$, it has an induced well ordering, which we denote by $\mathscr{L} = \{L_\alpha \mid \alpha < 2^{\aleph_0}\}$.

We shall, using transfinite recursion construct a chain X_α of subsets of $\mathbb{R}^2$ for $\alpha < 2^{\aleph_0}$ such that $|X_\alpha| < 2^{\aleph_0}$ and $|X_\alpha \cap L_\beta| \leq 2$ for each $\beta < 2^{\aleph_0}$.

Begin with $X_0 = \{x_0\}$ for any $x_0 \in \mathbb{R}^2$. Suppose now that the sequence has been constructed for each $\beta < \alpha$ where $\alpha > 0$. Let $Y_\alpha := \bigcup_{\beta<\alpha} X_\beta$. Let S_α denote the set of all lines between two points in Y_α. Note that the cardinality of S_α is strictly smaller than $2^{\aleph_0}$.

Let γ be the smallest ordinal such that $|L_\gamma \cap Y_\alpha| \leq 1$. If no such ordinal exists, then Y_α is the desired set. Suppose such a γ does exist. Then, the set

$$L_\gamma \setminus \underbrace{\left(\bigcup_{L \in S_\alpha} L \cup \bigcup_{\beta<\gamma} L_\beta \cup Y_\alpha \right)}_{T}$$

which is non empty, since the intersection of L_γ with T has cardinality strictly smaller than $2^{\aleph_0}$. Let x_α be one such element in the above set and define $X_\alpha = Y_\alpha \cup \{x_\alpha\}$.

It is not hard to see that X_α satisfies the desired properties and thus we may continue this procedure and obtain $\{X_\alpha \mid \alpha < 2^{\aleph_0}\}$. Let $X = \bigcup_{\alpha<2^{\aleph}_0} X_\alpha$. This is the required set. ■

Chapter 4

Well Founded Sets

Throughout this chapter, we shall work in ZF^-, which is ZF without the axiom of foundation.

Definition 4.1. By transfinite recursion, deifne $R(\alpha)$ for each $\alpha \in \mathbf{ON}$ by

(a) $R(0) = \varnothing$,

(b) $R(\alpha + 1) = \mathscr{P}(R(\alpha))$,

(c) $R(\alpha) = \bigcup_{\lambda<\alpha} R(\lambda)$ when λ is a limit ordinal.

Finally, define the first order formula

$$\mathbf{WF}(x) := \exists\alpha(x \in R(\alpha)).$$

We denote by **WF** the class corresponding to the above formula.

Lemma 4.2. *For each α,*

1. *$R(\alpha)$ is transitive.*
2. *$\forall\xi \leq \alpha(R(\xi) \subseteq R(\alpha))$.*

Proof. We prove both statements by transfinite induction on α. The base case with $\alpha = 0$ is trivial. Suppose $\alpha = \beta + 1$. Since $R(\beta)$ is transitive, so is its power set as we have seen earlier and obviously $R(\beta) \subseteq R(\alpha)$ since $R(\beta) \in R(\alpha)$. Finally, suppose α is a limit ordinal. Then, (b) is immediate and (a) follows from the fact taht the union of transitive sets is transitive. ■

Remark 4.0.1. *As a consequence of the definition of* **WF***, for any $x \in$* **WF***, the least α for which $x \in R(\alpha)$ must be a successor ordinal.*

Definition 4.3. If $x \in$ **WF**, then $\mathrm{rank}(\alpha)$ is the *least* β such tdhat $x \in R(\beta + 1)$.

Lemma 4.4. *For any α,*

$$R(\alpha) = \{x \in \mathbf{WF} \mid \mathrm{rank}(x) < \alpha\}.$$

Proof. Trivial. ■

Lemma 4.5. *If $y \in$* **WF***, then*

(a) $\forall x \in y(x \in \mathbf{WF} \wedge \mathrm{rank}(x) < \mathrm{rank}(y))$, *and*

(b) $\operatorname{rank}(y) = \sup\{\operatorname{rank}(x) + 1 \mid x \in y\}$.

Proof. Let $\alpha = \operatorname{rank}(y)$. Then, $y \in R(\alpha + 1) = \mathscr{P}(R(\alpha))$ and thus $y \subseteq \mathscr{R}(\alpha)$, consequently, $x \in R(\alpha)$ and $\operatorname{rank}(x) < \alpha$.

As for the second part, let $\alpha = \sup\{\operatorname{rank}(x) + 1 \mid x \in y\}$. From (a), we know that $\alpha \leq \operatorname{rank}(y)$. Further, each $x \in y$ has rank $< \alpha$ and thus $y \subseteq R(\alpha)$ whence $y \in R(\alpha + 1)$, consequently, $\operatorname{rank}(y) \leq \alpha$. ■

Corollary 4.6. There is no $x \in \mathbf{WF}$ such that $x \in x$.

Proof. If this were true, then $\operatorname{rank}(x) < \operatorname{rank}(x)$, a contradiction. ■

Lemma 4.7. *(a)* $\forall \alpha \in \mathbf{ON}(\alpha \in \mathbf{WF} \wedge \operatorname{rank}(\alpha) = \alpha)$.

(b) $\forall \alpha \in \mathbf{ON}(R(\alpha) \cap \mathbf{ON} = \alpha)$.

Proof. We shall prove (a) using transfinite induction on α. That (a) holds for $\alpha = 0$ is trivial. Now suppose (a) holds for each $\beta < \alpha$. Then, we have

$$\operatorname{rank}(\alpha) = \sup\{\operatorname{rank}(\beta) + 1 \mid \beta < \alpha\} = \sup\{\beta \mid \beta < \alpha\} = \alpha$$

which proves (a). It is easy to see that (b) is immediate from (a). ■

Lemma 4.8. $\forall x(x \in \mathbf{WF} \iff x \subseteq \mathbf{WF})$.

Proof. The forward direction follows from the transitivity of **WF**. As for the reverse direction, let $x \subseteq$ **WF** and let

$$\alpha = \sup\{\operatorname{rank}(y) + 1 \mid y \in x\}.$$

Then, $x \subseteq R(\alpha)$, consequently, $x \in R(\alpha + 1)$. ■

Lemma 4.9. *(a)* $\forall n \in \omega(|R(n)| < \omega)$.

(b) $|R(\omega)| = \omega$.

Proof. (a) is immediate from induction on n. Obviously, $\omega \subseteq R(\omega)$. On the other hand, note that $R(\omega)$ is a countable union of countable sets and is thus countable. ■

4.1 Well Founded Relations

Definition 4.10. A relation R is *well-founded* on a set A if

$$\forall X \subseteq A\left[X \neq \varnothing \implies \exists y \in X\left(\neg \exists z \in X\left(zRy\right)\right)\right].$$

For example, if $\langle A, R\rangle$ is a well-ordering, then R is well-founded on A.

Lemma 4.11. *If $A \in \mathbf{WF}$, then $\in$ is well-founded on A.*

Proof. Let $X \subseteq A$ be nonempty and $\alpha = \min\{\operatorname{rank}(y) \mid y \in X\}$. Choose some $y \in X$ with $\operatorname{rank}(y) = \alpha$. Then y is $\in$-minimal in X. ■

Lemma 4.12. *If A is transitive and $\in$ is well-founded on A, then $A \in \mathbf{WF}$.*

Proof. Suppose not. Then equivalently, $A \not\subseteq \mathbf{WF}$, whence $A \backslash \mathbf{WF}$ is nonempty. Let $y \in A \backslash \mathbf{WF}$ be the $\in$-least element of $A \backslash \mathbf{WF}$. If $z \in y$, then $z \in A$ due to the transitivity of A but on the other hand, $z \notin A \backslash \mathbf{WF}$ lest one contradicts the minimality of y. Therefore, $z \in \mathbf{WF}$. Consequently, $y \subseteq \mathbf{WF}$ whence $y \in \mathbf{WF}$, a contradiction. ∎

Definition 4.13. Let

$$\bigcup^{0} A = A,$$

and for each $0 < n < \omega$, define, recursively,

$$\bigcup^{n+1} A = \bigcup \left(\bigcup^{n} A \right).$$

Finally, set

$$\operatorname{trcl}(A) := \bigcup \left\{ \bigcup^{n} A \mid n \in \omega \right\}.$$

Lemma 4.14. *Let A be a set. Then,*

(a) $A \subseteq \operatorname{trcl}(A)$.

(b) $\operatorname{trcl}(A)$ *is transitive.*

(c) If $A \subseteq T$, and T is transitive, then $\operatorname{trcl}(A) \subseteq T$.

(d) If A is transitive, then $\operatorname{trcl}(A) = A$.

(e) $\operatorname{trcl}(A) = A \cup \bigcup \{\operatorname{trcl}(x) \mid x \in A\}$.

Proof. (a) Trivial.

(b) If $x \in \operatorname{trcl}(A)$, then there is some n such that $x \in \bigcup^n A$, therefore $x \subseteq \bigcup^{n+1} A$, whence $x \subseteq \operatorname{trcl}(A)$. Thus $\operatorname{trcl}(A)$ is transitive.

(c) We shall show by induction on n that $\bigcup^n A \subseteq T$. The base case is given to begin with. Suppose $\bigcup^n A \subseteq T$. Then, due to transitivity, $\bigcup^{n+1} A = \bigcup \left(\bigcup^n A\right) \subseteq T$. The conclusion follows.

(d) Follows from (a) and (c) by taking $T = A$.

(e) First, note that if $x \in A$, then $x \in \operatorname{trcl}(A)$ and due to transitivity $x \subseteq \operatorname{trcl}(A)$. From (c), we have $\operatorname{trcl}(x) \subseteq \operatorname{trcl}(A)$. Let

$$T = A \cup \bigcup \{\operatorname{trcl}(x) \mid x \in A\}.$$

Then, it is easy to see that T must be transitive and from what we concluded earlier, $T \subseteq \operatorname{trcl}(A)$ but since $A \subseteq T$, from (c), we must have $\operatorname{trcl}(A) \subseteq T$ whence $\operatorname{trcl}(A) = T$. ∎

Theorem 4.15. *For any set A, the following are equivalent:*

(a) $A \in \mathbf{WF}$.

(b) $\operatorname{trcl}(A) \in \mathbf{WF}$.

(c) $\in$ *is well-founded on* $\operatorname{trcl}(A)$.

Proof. ∎

4.2 The Axiom of foundation

Recall the axiom of foundation

$$\forall x \left(x \neq \varnothing \implies \exists y \left(y \in x \wedge \neg \exists z \left(z \in x \wedge z \in y \right) \right) \right).$$

or equivalently,

$$\forall x \left(x \neq \varnothing \implies \exists y \left(y \in x \wedge y \cap x = \varnothing \right) \right).$$

Theorem 4.16. *The following are equivalent:*

(a) the Axiom of Foundation.

(b) $\forall A (\in$ *is well-founded on* $A)$

(c) $\mathbf{V} = \mathbf{WF}$.

Proof. That (a) and (b) are equivalent is immediate from the definition of well-foundedness. Let $A \in \mathbf{V}$. Then, $\in$ is well founded on A and thus on $\operatorname{trcl}(A)$, consequently, $A \in \mathbf{WF}$ and thus $\mathbf{V} = \mathbf{WF}$. The converse is trivial. ■

4.3 Induction and Recursion on Well-Founded Relations

We extend the notion of well-foundedness to classes as follows.

Definition 4.17. $\mathbf{R}$ is well founded on $\mathbf{A}$ if and only if

$$\forall X \subseteq \mathbf{A} \left[X \neq \varnothing \implies \exists y \in X \left(\neg \exists z \in X (z \mathbf{R} y) \right) \right].$$

Definition 4.18. $\mathbf{R}$ is *set-like* on $\mathbf{A}$ if for all $x \in \mathbf{A}$, $\{y \in \mathbf{A} \mid y \mathbf{R} x\}$ is a set. If $\mathbf{R}$ is set-like on $\mathbf{A}$, then

(a) $\operatorname{pred}(\mathbf{A}, x, \mathbf{R}) = \{y \in \mathbf{A} \mid y \mathbf{R} x\}$.

(b) $\operatorname{pred}^0(\mathbf{A}, x, \mathbf{R}) = \operatorname{pred}(\mathbf{A}, x, \mathbf{R})$.

(c) $\operatorname{pred}^{n+1}(\mathbf{A}, x, \mathbf{R}) = \bigcup \{\operatorname{pred}(\mathbf{A}, y, \mathbf{R}) \mid y \in \operatorname{pred}^n(\mathbf{A}, x, \mathbf{R})\}$.

(d) $\operatorname{cl}(\mathbf{A}, x, \mathbf{R}) = \bigcup \{\operatorname{pred}^n(\mathbf{A}, x, \mathbf{R}) \mid n \in \omega\}$.

Lemma 4.19. *If* $\mathbf{R}$ *is set-like on* A *and* $x \in \mathbf{A}$, *then for each* $y \in \operatorname{cl}(\mathbf{A}, x, \mathbf{R})$, $\operatorname{pred}(\mathbf{A}, y, \mathbf{R}) \subseteq \operatorname{cl}(\mathbf{A}, x, \mathbf{R})$.

Proof. There is some nonnegative integer n such that $y \in \operatorname{pred}^n(\mathbf{A}, y, \mathbf{R})$. Then,

$$\operatorname{pred}(\mathbf{A}, y, \mathbf{R}) \subseteq \operatorname{pred}^{n+1}(\mathbf{A}, x, \mathbf{R}).$$

The conclusion follows. ■

Theorem 4.20. *If* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$, *then every non-empty subclass* $\mathbf{X}$ *of* $\mathbf{A}$ *has an* $\mathbf{R}$-*minimal element.*

Proof. Pick some $x \in \mathbf{X}$. If this $\mathbf{R}$-minimal, then we are done. If not, then consider $\mathbf{X} \cap \operatorname{cl}(\mathbf{A}, x, \mathbf{R})$ is a nonempty *subset* of A, since $\operatorname{cl}(\mathbf{A}, x, \mathbf{R})$ is a set. This means that it has an $\mathbf{R}$-minimal element, say y. From the previous lemma, y must be $\mathbf{R}$-minimal in $\mathbf{X}$. ■

Remark 4.3.1. *Notice the similarity of the above with Theorem 2.17. This in particular means that we can apply transfinite induction on well-founded set-like relations.*

Theorem 4.21 (Well-Founded Transfinite Recursion). *Assume* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{A}$*. If* $\mathbf{F} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$*, then there is a unique* $\mathbf{G} : \mathbf{A} \to \mathbf{V}$ *such that*

$$\forall x \in \mathbf{A} \left[\mathbf{G}(x) = \mathbf{F}\left(x, \mathbf{G} \restriction \operatorname{pred}(\mathbf{A}, x, \mathbf{R})\right) \right].$$

Definition 4.22. If $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$, define

$$\operatorname{rank}(x, \mathbf{A}, \mathbf{R}) = \sup\{\operatorname{rank}(y, \mathbf{A}, \mathbf{R}) + 1 \mid y\mathbf{R}x \wedge y \in \mathbf{A}\}.$$

Lemma 4.23. *If* $\mathbf{A}$ *is transitive and* $\in$ *is well-founded on* $\mathbf{A}$*, then* $\mathbf{A} \subseteq \mathbf{WF}$ *and* $\operatorname{rank}(x, \mathbf{A}, \in) = \operatorname{rank}(x)$ *for all* $x \in A$*.*

Proof. We have already seen the proof of the first half. Now, consider an $\in$-minimal element of $\{x \in \mathbf{A} \mid \operatorname{rank}(x, \mathbf{A}, \mathbf{R}) \neq \operatorname{rank}(x)\}$. This would give an immediate contradiction since we have shown

$$\operatorname{rank}(y) = \sup\{\operatorname{rank}(x) + 1 \mid x \in y\}.$$

This completes the proof. ∎

Definition 4.24. Let $\mathbf{R}$ be well-founded and set-like on $\mathbf{A}$. Define the *Mostowski collapsing function*, $\mathbf{G}$ of $\mathbf{A}, \mathbf{R}$ by

$$\mathbf{G}(x) = \{\mathbf{G}(y) \mid y \in \mathbf{A} \wedge y\mathbf{R}x\}.$$

The *Mostowski collapse*, $\mathbf{M}$ of $\mathbf{A}, \mathbf{R}$ is defined to be the range of $\mathbf{G}$.

Lemma 4.25. *With notation from the above definition.*

(a) $\forall x, y \in \mathbf{A}(x\mathbf{R}y \implies \mathbf{G}(x) \in \mathbf{G}(y))$.

(b) $\mathbf{M}$ *is transitive.*

(c) $\mathbf{M} \subseteq \mathbf{WF}$.

(d) If $x \in \mathbf{A}$*, then* $\operatorname{rank}(x, \mathbf{A}, \mathbf{R}) = \operatorname{rank}(\mathbf{G}(x))$.

Proof. Both (a) and (b) are obvious. For (c), we contend that $\forall x \in \mathbf{A}(\mathbf{G}(x) \in \mathbf{WF})$. Consider the set $S = \{x \in \mathbf{A} \mid \mathbf{G}(x) \in \mathbf{WF}\}$. Suppose S is nonempty, then it has a $\mathbf{R}$-minimal element, say z. For every $y \in \mathbf{A}$ such that $y\mathbf{R}x$, $\mathbf{G}(y) \in \mathbf{WF}$, consequently, $\mathbf{G}(z) \subseteq \mathbf{WF}$ whence $\mathbf{G}(z) \in \mathbf{WF}$, a contradiction.

For (d), first note that

$$\operatorname{rank}(\mathbf{G}(x)) = \sup\{\operatorname{rank}(v) + 1 \mid v \in \mathbf{G}(x)\} = \sup\{\operatorname{rank}(\mathbf{G}(y)) + 1 \mid y\mathbf{R}x\}.$$

Using well-founded induction on x, one can now conclude. ∎

Definition 4.26. $\mathbf{R}$ is said to be *extensional* on $\mathbf{A}$ if

$$\forall x, y \in \mathbf{A} \left(\forall z \in \mathbf{A} \left(z\mathbf{R}x \iff z\mathbf{R}y \right) \implies x = y \right).$$

Informally, this is equivalent to saying that the Axiom of Extensionality is true in $\mathbf{A}$ if $\in$ is interpreted as $\mathbf{R}$.

Theorem 4.27 (Mostowski Collapsing Theorem). *Suppose* $\mathbf{R}$ *is well-founded, set-like, and extensional on* $\mathbf{A}$*, then there is a transitive class* $\mathbf{M}$ *and a bijective map* $\mathbf{G} : \mathbf{A} \to \mathbf{R}$ *such that* $\mathbf{G}$ *is an isomorphism between* $(\mathbf{A}, \mathbf{R})$ *and* $(\mathbf{M}, \in)$*. Furthermore,* $\mathbf{M}$ *and* $\mathbf{G}$ *are unique.*

Proof. First, we shall show that $\mathbf{G}$ is injective. Suppose not. Let x be the $\mathbf{R}$-minimal element of

$$\{x \in \mathbf{A} \mid \exists y \in \mathbf{A}(x \neq y \wedge \mathbf{G}(x) = \mathbf{G}(y))\}.$$

Then, there is some $y \in \mathbf{A}$ with $x \neq y$ and $\mathbf{G}(x) = \mathbf{G}(y)$. Recall that $\mathbf{R}$ is extensional.

Suppose first that there is some $z \in \mathbf{A}$ with $z\mathbf{R}x$ but $\neg(z\mathbf{R}y)$. Since $\mathbf{G}(z) \in \mathbf{G}(x) = \mathbf{G}(y)$, there is some $w\mathbf{R}y$ such that $\mathbf{G}(w) = \mathbf{G}(z)$ and $z \neq w$. This contradicts the minimality of x. A similar argument works for the other case. Thus, $\mathbf{G}$ is injective.

That $\mathbf{G}$ is an isomorphism now follows from the fact that it is a surjection and maps $\mathbf{R}$ to $\in$. Lastly, we must show uniqueness of $\mathbf{G}$ and $\mathbf{M}$. This follows from a simple well-founded induction on x to show that $\mathbf{G}(x) = \mathbf{G}'(x)$ for any other isomorphism $\mathbf{G}'$. ∎

Chapter 5

Easy Consistency Proofs

5.1 Relativization

Definition 5.1. Let $\mathbf{M}$ be any class. For any formula ϕ, define $\phi^{\mathbf{M}}$, the *relativization* of ϕ to $\mathbf{M}$ by induction on ϕ, by

(a) $(x = y)^{\mathbf{M}}$ is $x = y$.

(b) $(x \in y)^{\mathbf{M}}$ is $x \in y$.

(c) $(\phi \wedge \psi)^{\mathbf{M}}$ is $\phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}}$.

(d) $(\neg\phi)^{\mathbf{M}}$ is $\neg\phi^{\mathbf{M}}$

(e) $(\exists\phi)^{\mathbf{M}}$ is $\exists x(x \in \mathbf{M} \wedge \phi^{\mathbf{M}})$.

Lemma 5.2. *Let S and T be two sets of sentences in the language of set theory under consideration, and suppose for some class $\mathbf{M}$, we can prove from T that $\mathbf{M} \neq 0$ and $\mathbf{M}$ is a model for S. Then,* $\mathrm{Con}(T) \implies \mathrm{Con}(S)$.

Proof. Quite straightforward. Suppose S were inconsistent. Then, there is a sentence χ such that we could prove $\chi \wedge \neg\chi$ from S. Then, we could begin a formal proof from T and prove that S is true in $\mathbf{M}$ and hence, $\chi^{\mathbf{M}} \wedge \neg\chi^{\mathbf{M}}$, a contradiction. Thus, T is inconsistent. ∎

Recall the Axiom of Extensionality,

$$\forall x, y \left(\forall z(z \in x \iff z \in y) \implies x = y\right).$$

Relativized to $\mathbf{M}$, it looks like

$$\forall x, y \in \mathbf{M} \left(\forall z \in \mathbf{M}(z \in x \iff z \in y) \implies x = y\right).$$

Lemma 5.3. *If $\mathbf{M}$ is transitive, the Axiom of Extensionality is true in M.*

Proof. We have seen this in the previous chapter. ∎

Lemma 5.4. *If for each formula $\phi(x, z, w_1, \ldots, w_n)$ with only the displayed variables free,*

$$\forall z, w_1, \ldots, w_n \in \mathbf{M} \left(\{x \in z \mid \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n) \in \mathbf{M}\}\right),$$

then the Axiom of Comprehension is true in $\mathbf{M}$.

Proof. Since the relativized instance of Comprehension is given by

$$\forall z, w_1, \ldots, w_n \in \mathbf{M} \exists y \in \mathbf{M} \left(x \in y \iff x \in z \wedge \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n) \right).$$

The conclusion is now obvious from the hypothesis. ■

Remark 5.1.1. *In particular, if* $\forall z \in \mathbf{M}(\mathscr{P}(z) \subseteq \mathbf{M})$, *then the Comprehension Axiom is true in* $\mathbf{M}$. *This shall turn out to be useful later on.*

Theorem 5.5. *Assume the consistency of* ZF^-. *If* $\mathbf{M} = \{0\}$, *then Extensionality + Comprehension +* $\forall y(y = 0)$ *is consistent.*

Proof. Extensionality is true in $\mathbf{M}$ since it is transitive while Comprehension is true in $\mathbf{M}$ from the preceeding remark. ■

The above can be written as

$$\mathsf{Con}(\mathsf{ZF}^-) \implies \mathsf{Con}\left(\mathsf{Extensionality} + \mathsf{Comprehension} + \forall y(y = 0)\right).$$

Let us now look at the Power Set Axiom. Recall that this is

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y).$$

We shall first see what $z \subseteq x$ becomes upon relativizing to $\mathbf{M}$. Note that $z \subseteq x$ is shorthand for $\forall w(w \in z \implies w \in x)$. This relativized to $\mathbf{M}$ is $\forall w \in \mathbf{M}(w \in z \implies w \in x)$ which is equivalent to writing $z \cap \mathbf{M} \subseteq x$. We may now write down the relativized Power Set Axiom as

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M}(z \cap \mathbf{M} \subseteq x \implies z \in y).$$

Suppose $\mathbf{M}$ is transitive. Then, $z \in \mathbf{M}$ implies $z \subseteq \mathbf{M}$ whence $z \cap \mathbf{M} = z$ nad hence, the Power Set Axiom holds in $\mathbf{M}$ if and only if

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M}(z \subseteq x \implies z \in y).$$

In particular, we have the following.

Lemma 5.6. *If* $\mathbf{M}$ *is transitive, the Power Set Axiom holds in* $\mathbf{M}$ *if and only if*

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M}(\mathscr{P}(x) \cap \mathbf{M} \subseteq y).$$

Lemma 5.7. *If*

$$\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}(x \in z \wedge y \in z), \quad \text{and} \quad \forall x \in \mathbf{M} \exists z \in \mathbf{M}(\bigcup x \subseteq z),$$

then the Pairing and Union Axioms are true in $\mathbf{M}$.

Proof. Obvious. ■

In particular, the Pairing and Union Axioms are true in $R(\omega)$ and $\mathbf{WF}$, which is trivial to see. Next, we shall show that Replacement is also true in $R(\omega)$ and $\mathbf{WF}$. First, a lemma.

Lemma 5.8. *Suppose for each formula* $\phi(x, y, A, w_1, \ldots, w_n)$ *and each* $A, w_1, \ldots, w_n \in \mathbf{M}$, *if*

$$\forall x \in A \exists! y \in \mathbf{M}\, \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n),$$

then

$$\exists Y \in \mathbf{M} \left(\{y \mid \exists x \in A\, \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y \right).$$

Then the Replacement Scheme is true in **M**.

Proof. Again, obvious. ■

Consider now the case $\mathbf{M} = R(\omega)$ or **WF** and let Y be as defined in the lemma above. If $\mathbf{M} = \mathbf{WF}$, $Y \subseteq \mathbf{WF}$ and thus, $Y \in \mathbf{WF}$. On the other hand, if $\mathbf{M} = R(\omega)$, then $|Y|$ is finte and hence, $Y \subseteq R(n)$ for some n, whence $Y \in R(n+1)$. This shows that replacement is true.

We now move onto foundation. The relativization of foundation to **M** is

$$\forall x \in \mathbf{M} \left(\exists y \in \mathbf{M}(y \in x) \implies \exists y \in \mathbf{M}(y \in x \wedge \neq \exists z \in \mathbf{M}(z \in x \wedge z \in y))\right).$$

Now, if $\mathbf{M} \subseteq \mathbf{WF}$ and $x \in \mathbf{M}$, pick $y \in \mathbf{M} \cap x$ with the smallest rank. It is routine to show that this satisfies the statement of the relativized axiom. This shows that Foundation is true in any $\mathbf{M} \subseteq \mathbf{WF}$. We now have the following.

Theorem 5.9. **WF** *and* $R(\omega)$ *are models of* $\mathsf{ZF} - \mathsf{Inf}$.

5.2 Absoluteness

Definition 5.10. Let ϕ be a formula with $x_1, \dots, x_n$ free.

(a) If $\mathbf{M} \subseteq \mathbf{N}$, then ϕ is *absolute* for $\mathbf{M}, \mathbf{N}$ if and only if

$$\forall x_1, \dots, x_n \in \mathbf{M} \left(\phi^{\mathbf{M}}(x_1, \dots, x_n) \iff \phi^{\mathbf{N}}(x_1, \dots, x_n)\right).$$

(b) ϕ is said to be *absolute* for **M** if it is absolute for $\mathbf{M}, \mathbf{V}$.

Obviously, if ϕ is absolute for **M** and **N**, and if $\mathbf{M} \subseteq \mathbf{N}$, then ϕ is absolute for $\mathbf{M}, \mathbf{N}$.

Remark 5.2.1. *If* $\mathbf{M} \subseteq \mathbf{N}$ *and* ϕ, ψ *are both absolute for* $\mathbf{M}, \mathbf{N}$*, so are* $\neg\phi$ *and* $\phi \wedge \psi$*. This is trivial to prove. Further, note that* $x = y$ *and* $x \in y$ *are absolute for all* **M***, and any formula without quantifiers is built using the aforementioned atomic formulae and* $\neg$ *and* $\wedge$*. Therefore, a quantifier-free formula is absolute for any* **M**.

Lemma 5.11. *If* $\mathbf{M} \subseteq \mathbf{N}$ *are both transitive and* ϕ *is absolute for* $\mathbf{M}, \mathbf{N}$*, then so is* $\exists x \in y \phi$.

Proof. Trivial. ■

Definition 5.12. The collection Δ_0 of formulas are those built up inductively by:

(a) $x \in y$ and $x = y$ are Δ_0.

(b) If ϕ, ψ are Δ_0, so are $\neg\phi$ and $\phi \wedge \psi$.

(c) If ϕ is Δ_0, so is $\exists x(x \in y \wedge \phi)$.

Corollary 5.13. If **M** is transitive and ϕ is Δ_0, then ϕ is absolute for **M**.

Proof. This follows from the previous lemma applied to $\mathbf{M}, \mathbf{V}$ and the above definition. ■

Lemma 5.14. *Suppose* $\mathbf{M} \subseteq \mathbf{N}$ *are models for a set of sentences* S*, such that*

$$S \vdash \forall x_1, \ldots, x_n \left(\phi(x_1, \ldots, x_n) \iff \psi(x_1, \ldots, x_n) \right)$$

then ϕ *is absolute for* $\mathbf{M}, \mathbf{N}$ *if and only if* ψ *is.*

Proof. Follows immediately from the definition of absoluteness. ■

Note that a function $F(x_1, \ldots, x_n)$ can be defined through a formula in $\mathbf{M}$ if the following is true:

$$\forall x_1, \ldots, x_n \exists! y \ \phi(x_1, \ldots, x_n, y).$$

Definition 5.15. If $\mathbf{M} \subseteq \mathbf{N}$ and $F(x_1, \ldots, x_n)$ is a defined function, we say that F is *absolute* for $\mathbf{M}, \mathbf{N}$ if ϕ is.

Remark 5.2.2. *There is now a tedious verification that almost all the operations that we use in set theory are absolute for a transitive model* $\mathbf{M}$ *that is a model for* $\mathsf{ZF}^- - \mathsf{P} - \mathsf{Inf}$.

Lemma 5.16. *Let* $\mathbf{M}$ *be a transitive model for* $\mathsf{ZF}^- - \mathsf{P} - \mathsf{Inf}$. *If* $\omega \in \mathbf{M}$*, then the Axiom of Infinity is true in* $\mathbf{M}$.

Proof. Due to the remark above, 0 and S are absolute in $\mathbf{M}$. The Axiom of Infinity relativized to $\mathbf{M}$ now looks like

$$\exists x \in \mathbf{M}(0 \in x \wedge \forall y \in x(S(y) \in x)),$$

which is seen to be true by taking $x = \omega$. ■

The same argument yields that the Axiom of Infinity is false in $R(\omega)$, since any $x \in \mathbf{WF}$ containing 0 and closed under S has infinite rank. Thus, we have:

Theorem 5.17. $R(\omega)$ *is a model of* $\mathsf{ZFC} - \mathsf{Inf} + (\neg \mathsf{Inf})$.

Proof. It only remains to check that AC holds, for which we need to come up with a well-ordering for any $A \in R(\omega)$. But A is finite and thus can be well ordered. Le $R \subseteq A \times A$ be a well-order. Then, R must lie in $R(\omega)$ due to transitivity. The proof is complete due to the following lemma. ■

Lemma 5.18. *Suppose* $\mathbf{M}$ *is a transitive model of* $\mathsf{ZF}^- - \mathsf{P} - \mathsf{Inf}$. *Let* $A, R \in \mathbf{M}$ *and suppose that* R *well-orders* A. *Then* $(R \text{ well-orders } A)^{\mathbf{M}}$.

5.3 $H(\kappa)$

Definition 5.19. For an infinite cardinal κ, $H(\kappa) = \{x \mid |\mathrm{trcl}(x)| < \kappa\}$. Note that choice is not required for this definition, since we implicitly assume that $|y| < \kappa$ means that y is well-orderable and $|y| < \kappa$.

That each $H(\kappa)$ is a set and not a proper class follows from the following:

Theorem 5.20. *For any infinite* κ, $H(\kappa) \subseteq R(\kappa)$.

Chapter 6

Forcing

6.1 Some Infinitary Combinatorics

Definition 6.1 (Cofinal). If $f : \alpha \to \beta$, then f maps α *cofinally* if and only if $\operatorname{ran}(f)$ is unbounded in β. The *cofinality* of β is the least α that can be cofinally mapped into β.

Lemma 6.2. *There is a cofinal map $f : \mathrm{cf}(\beta) \to \beta$ which is strictly increasing.*

Proof. Let $\alpha = \mathrm{cf}(\beta)$ and $f : \alpha \to \beta$ be a cofinal map. Define the map $g : \alpha \to \beta$ by

$$g(\eta) = max\{f(\eta), \sup\{f(\xi) + 1 \colon \xi < \eta\}\}.$$

It is not hard to argue that $g : \alpha \to \beta$ is cofinal. ■

Lemma 6.3. *If α is a limit ordinal and $f : \alpha \to \beta$ is a strictly increasing cofinal map, then* $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

Proof. Upon composing f with a strictly increasing cofinal map $\mathrm{cf}(\alpha) \to \alpha$, we see that $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$. Next, we would like to show that $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$. Let $g : \mathrm{cf}(\beta) \to \beta$ be a strictly increasing cofinal map. Define $h : \mathrm{cf}(\beta) \to \alpha$ by

$$h(\eta) = \inf_{\xi \in \alpha} f(\xi) > g(\eta).$$

It is not hard to see that h is a cofinal map. Therefore, $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$ and this completes the proof. ■

Definition 6.4. β is said to be *regular* if β is a limit ordinal and $\mathrm{cf}(\beta) = \beta$.

Lemma 6.5. *If β is regular, then β is a cardinal.*

Proof. Cofinality is a cardinal. ■

Lemma 6.6. *All successor cardinals are regular.*

Proof. Suppose there is a strictly increasing cofinal map $f : \kappa \to \kappa^+$. For each $\alpha \in \kappa$, note that $|f(\alpha)| \leq \kappa$ and due to cofinality, $\kappa^+ = \bigcup_{\alpha \in \kappa} f(\alpha)$. This is a cardinality contradiction. ■

Lemma 6.7 (Δ-system Lemma). *Let $\mathscr{A}$ be an uncountable collection of finite sets. Then, there is an uncountable subcollection $\mathscr{B}$ of $\mathscr{A}$ and a set r such that for all $x, y \in \mathscr{B}$ with $x \neq y$, $x \cap y = r$.*

Proof. There is a minimum cardinal $n < \omega$ such that there are uncountably many elements of $\mathscr{A}$ with cardinality n. Let $\mathscr{A}'$ denote the subcollection of these elements. We shall induct on the aforementioned unique n for $\mathscr{A}$.

If $n = 1$, then there is nothing to prove since the elements of $\mathscr{A}'$ are disjoint. Now, suppose $n > 1$. If there is some a that is in uncountably many elements of $\mathscr{A}'$, then consider $\mathscr{A}'' = \{x \backslash \{a\} : a \in x \in \mathscr{A}'\}$. We may now apply the induction hypothesis to find a subcollection $\mathscr{B}''$ of $\mathscr{A}''$ with the required property. Consider then the collection $\mathscr{B} = \{x \cup \{a\} : x \in \mathscr{B}''\}$.

On the other hand, suppose there is no such a that is in uncountably many elements of $\mathscr{A}'$. Begin with any $x_0 \in \mathscr{A}'$. Let $\alpha < \omega_1$ be an ordinal and suppose the sequence $(x_i)_{i<\alpha}$ has been constructed. The union $x = \bigcup_{i<\alpha} x_i$ is a countable set (being the countable union of finite sets). Now, every element of x is in countably many elements of $\mathscr{A}'$, consequently, there are uncountably many elements of $\mathscr{A}'$ that are disjoint from x. Define x_α to be any one of these and continue this process.

The sequence $(x_i)_{i<\omega_1}$ is now an uncountable collection of pairwise disjoint elements of $\mathscr{A}'$ and is our desired $\mathscr{B}$ with $r = \varnothing$. ■

Definition 6.8. Let $(\mathbb{P}, \leq)$ be a partial order. Elements $p, q \in \mathbb{P}$ are said to be *compatible* if there is an r with $r \leq p$ and $r \leq q$. An *antichain* is a collection of pairwise incompatible elements in $\mathbb{P}$.

6.2 Some Technicalities about Forcing

Throughout this section, let $\mathbb{P}$ be a poset in M, a countable transitive model for ZFC.

Definition 6.9 (Forcing). Let $p \in \mathbb{P}$ and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown. Then, we write $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ if

$$\forall G \subseteq \mathbb{P} \left((G \text{ is generic} \wedge p \in G) \implies \phi^{M[G]}(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G)) \right).$$

Definition 6.10. If $E \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, then E is said to be *dense below* p if $\forall q \leq p \left(\exists r \in E (r \leq q) \right)$.

We now come to the ugliest definition of this article.

Definition 6.11. Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown, $p \in \mathbb{P}$ and $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$.

- $p \Vdash^* \tau_1 = \tau_2$ if and only if for all $\langle \pi_1, s_1 \rangle \in \tau_1$,

$$\{q \leq p : q \leq s_1 \implies \exists \langle \pi_2, s_2 \rangle \in \tau_2 \left(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2 \right)\}$$

  is dense below p. And for all $\langle \pi_2, s_2 \rangle \in \tau_2$,

$$\{q \leq p : q \leq s_2 \implies \exists \langle \pi_1, s_1 \rangle \in \tau_1 \left(q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2 \right)\}$$

  is dense below p.

- $p \Vdash^* \tau_1 \in \tau_2$ if and only if

$$\{q : \exists \langle \pi, s \rangle \in \tau_2 \left(q \leq s \wedge q \Vdash^* \pi = \tau_1 \right)\}$$

  is dense below p.

- $p \Vdash^* (\phi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n))$ if and only if

$$p \Vdash^* \phi(\tau_1, \ldots, \tau_n) \text{ and } p \Vdash^* \psi(\tau_1, \ldots, \tau_n).$$

- $p \Vdash^* \neg\phi(\tau_1, \ldots, \tau_n)$ if and only if there is no $q \leq p$ such that $q \Vdash^* \phi(\tau_1, \ldots, \tau_n)$.
- $p \Vdash^* \exists x \phi(x, \tau_1, \ldots, \tau_n)$ if and only if

$$\{r \colon \exists \sigma \in V^{\mathbb{P}} \left(r \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n) \right) \}$$

is dense below p.

Lemma 6.12. *Let phi$(x_1, \ldots, x_n)$ be a formula with all free variables shown and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$.*

1. *If $p \in G$ and $(p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$, then $\left(\phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G))\right)^{M[G]}$.*
2. *If $\phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G))^{M[G]}$, then $\exists p \in G(p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$.*

Proof. Omitted due to length. ■

Theorem 6.13. *Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Then,*

1. *for all $p \in \mathbb{P}$,*
$$p \Vdash \phi(\tau_1, \ldots, \tau_n) \iff \left(p \Vdash^* \phi(\tau_1, \ldots, \tau_n)\right)^M.$$
2. *for all G that are $\mathbb{P}$-generic over M,*
$$\phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G))^{M[G]} \iff \exists p \in G \left(p \Vdash \phi(\tau_1, \ldots, \tau_n)\right).$$

6.3 Breaking CH

Throughout this section, let M be a countable transitive model of ZFC.

Definition 6.14. Let $I, J \in M$. Define

$$\mathrm{Fn}(I, J) = \{p \colon p \text{ is a function } \wedge |p| < \omega \wedge \mathrm{dom}(p) \subseteq I \wedge \mathrm{ran}(p) \subseteq J\}.$$

Order $\mathrm{Fn}(I, J)$ by $p \leq q$ if and only if $p \supseteq q$. Then, $\mathrm{Fn}(I, J)$ has a maximum element, 0, the empty function.

Lemma 6.15. *If $I, J \in M$ and I is infinite, $J \neq \varnothing$, and G is $\mathrm{Fn}(I, J)$-generic over M, then $\bigcup G$ is a surjective function $I \to J$.*

Proof. That $\bigcup G$ is a function is trivial. For $i \in I$, let

$$D_i := \{p \in \mathrm{Fn}(I, J) \colon i \in \mathrm{dom}(p)\}.$$

This is a dense set in $\mathrm{Fn}(I, J)$ and hence, intersects G. Thus, $i \in \mathrm{dom}\left(\bigcup G\right)$ and hence, $I = \mathrm{dom}\left(\bigcup G\right)$.
Let $j \in J$ and consider the set

$$D_j := \{p \in \mathrm{Fn}(I, J) \colon j \in \mathrm{ran}(p)\}.$$

This is dense in $\mathrm{Fn}(I, J)$ and hence, intersects G. Consequently, $J = \mathrm{ran}\left(\bigcup G\right)$. ■

Lemma 6.16. *If $\kappa \in M$ is a cardinal and G is* $\mathrm{Fn}(\kappa \times \omega, 2)$*-generic over M, then $(2^\omega \geq |\kappa|)^{M[G]}$.*

Note that we must use $|\kappa|$ instead of κ since the extension may not preserve cardinals.

Proof. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Then, $f = \bigcup G$ is a function $\kappa \times \omega \to 2$. Define for any $\alpha, \beta \in \kappa$,

$$D_{\alpha,\beta} = \{p \in \mathbb{P} \colon \exists n \in \omega\left(\langle \alpha, n\rangle \in \mathrm{dom}(p) \wedge \langle \beta, n\rangle \in \mathrm{dom}(p) \wedge p(\alpha, n) \neq \mathrm{dom}(\beta, n)\right)\}.$$

This is dense in $\mathbb{P}$ and hence, $G \cap D_{\alpha,\beta}$ is non empty.

Now, let $g_\alpha : \omega \to 2$ by $g_\alpha(n) = f(\alpha, n)$. Due to the above argument, $g_\alpha \neq g_\beta$ whenever $\alpha \neq \beta$. Note that all the g_α's are elements of $M[G]$. Consequently, there are at least κ many distinct functions from $\omega \to 2$ in $M[G]$. The conclusion follows. ∎

Lemma 6.17. *If I is any set and J is countable, then* $\mathrm{Fn}(I, J)$ *has the countable chain condition.*

Proof. Let $\{p_\alpha\}$ be a collection in $\mathrm{Fn}(I, J)$. Let $a_\alpha = \mathrm{dom}(p_\alpha)$. Using the Δ-system lemma, there is a subcollection $X \subseteq \omega_1$ such that $a_\alpha \cap a_\beta = r$ for every $\alpha \neq \beta$ in X. Note that r must be a finite set.

Note that J^r is countable and hence, there is an uncountable subcollection $Y \subseteq X$ such that for all $\alpha \in Y$, $p_\alpha \restriction r$ is the same. In particular, this meanas that all the p_αs for $\alpha \in Y$ are compatible. Thus, we cannot have an uncountable antichain in $\mathrm{Fn}(I, J)$. ∎

Lemma 6.18. *Let $\mathbb{P} \in M$, $(\mathbb{P}$ has c.c.c$)^M$, and $A, B \in M$. Let G be $\mathbb{P}$-generic over M and let $f : A \to B$ be in $M[G]$. Then there is a map $F : A \to \mathscr{P}(B)$ with $F \in M$ such that*

$$\forall a \in A(f(a) \in F(a)) \quad \text{and} \quad \forall a \in A\left(|F(a)| \leq \omega\right)^M.$$

Proof. Let τ be a $\mathbb{P}$-name in $M^{\mathbb{P}}$ such that $f = \tau_G$. Note that "τ is a function from $\hat{A} \to \hat{B}$" is a true statement in $M[G]$. Therefore, there is a $p \in G$ that forces the above statement. Define

$$F(a) := \{b \in B \colon \exists q \leq p\left(q \Vdash \tau(\hat{a}) = \hat{b}\right)\}.$$

Since $\Vdash$ is definable, $F \in M$.

Let $a \in A$ and $b = f(a)$. Then, there is $r \in G$ such that $r \Vdash \tau(\hat{a}) = \hat{b}$. Since G is a filter, there is a $q \in G$ with $q \leq r$ and $q \leq p$. Consequently, $q \Vdash \tau(\hat{a}) = \hat{b}$, so $b \in F(a)$.

Lastly, we must show that $\left(|F(a)| \leq \omega\right)^M$. ∎

Definition 6.19. If $\mathbb{P} \in M$, then $\mathbb{P}$ *preserves cardinals* if whenever G is $\mathbb{P}$-generic over M

$$\forall \beta \in o(M)\left((\beta \text{ is a cardinal})^M \iff (\beta \text{ is a cardinal})^{M[G]}\right).$$

Note that ω is absolute and hence, we need only worry about preservation of cardinals $\beta > \omega$.

Corollary 6.20. If $\mathbb{P} \in M$ and $(\mathbb{P}$ has c.c.c$)^M$, then $\mathbb{P}$ preserves cardinals.

Proof. ∎

Definition 6.21. If $\mathbb{P} \in M$, then $\mathbb{P}$ *preserves cofinalities* if whenever G is $\mathbb{P}$-generic over M and γ is a limit ordinal in M,

$$\mathrm{cf}(\gamma)^M = \mathrm{cf}(\gamma)^{M[G]}.$$

Lemma 6.22. *If $\mathbb{P} \in M$ and preserves cofinalities, then it preserves cardinals.*

Proof. If $\alpha \geq \omega$ is a regular cardinal of M, then

$$\mathrm{cf}(\alpha)^{M[G]} = \mathrm{cf}(\alpha)^M = \alpha,$$

whence α is a regular cardinal (in particular, a cardinal) of $M[G]$.

Now, suppose β is a limit cardinal in M. Then, every successor cardinal smaller than β is a regular cardinal and hence, remains a regular cardinal in $M[G]$. Consequently, β is also a limit cardinal in $M[G]$. This completes the proof. ■

Lemma 6.23. *Let $\mathbb{P} \in M$. Suppose whenever G is $\mathbb{P}$-generic over M and κ is a regular uncountable cardinal of M, $(\kappa$ is regular$)^{M[G]}$. Then $\mathbb{P}$ preserves cofinalities.*

Proof. Let γ be a limit ordinal in M, and let $(\kappa = \mathrm{cf}(\gamma))^M$. Then, there is an $f : \kappa \to \gamma$ in M that is cofinal and strictly increasing. If $(\kappa = \omega)^M$, then it is absolute and hence $(\kappa = \omega)^{M[G]}$. On the other hand, if $\kappa > \omega$, then $(\kappa \text{ is regular})^{M[G]}$ according to the given hypothesis. Since $f \in M[G]$, we have $(\kappa = \mathrm{cf}(\gamma))^{M[G]}$. This completes the proof. ■

Theorem 6.24. *If $\mathbb{P} \in M$ and $(\mathbb{P}$ has c.c.c$)^M$, then $\mathbb{P}$ preserves cofinalities.*

Proof. If not, then due to the previous lemma, there is a $\kappa \in M$ with $\kappa > \omega$, that is regular in M but not $M[G]$. Hence, there is an $\alpha < \kappa$ and a map $f \in M[G]$ $f : \alpha \to \kappa$ that is cofinal. Hence, there is a map $F : \alpha \to \mathscr{P}(\kappa)$ such that for all $\xi < \alpha$, $f(\xi) \in F(\xi)$ and $|F(\xi)| \leq \omega$ in M.

Define $S = \bigcup_{\xi<\alpha} F(\xi)$. Then, $S \in M$ and is unbounded subset of κ, further, has cardinality $|\alpha|$. This is a contradiction to κ being regular in M. ■

6.3.1 ZFC $+ \neg$CH is consistent

Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, then $\mathbb{P}$ has c.c.c in M and thus preserves cardinals. As we have seen earlier, if G is $\mathbb{P}$-generic, then $(2^\omega \geq \omega_2)^{M[G]}$ due to all the absoluteness results we have shown above.

Bibliography

[Kun80] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. North-Holland, 1980.