

# Algebraic Geometry

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# Chapter 1

## Varieties

Throughout this chapter,  $k$  denotes an algebraically closed field,  $A$  denotes the polynomial ring  $k[x_1, \dots, x_n]$  in  $n$ -variables and  $R$  denotes the graded ring  $k[x_0, \dots, x_n]$  with the standard homogeneous polynomial grading.

### 1.1 Affine Varieties

**Definition 1.1.** For a subset  $T \subseteq A$ , define

$$Z(T) := \{p \in \mathbb{A}^n \mid f(p) = 0, \forall f \in T\}.$$

This is called the *zero-set* of  $T$ . Conversely, for a subset  $S \subseteq \mathbb{A}^n$ , define

$$\mathcal{I}(S) := \{f \in A \mid f(p) = 0, \forall p \in S\}.$$

This is called the *ideal generated* by  $S$ .

**Definition 1.2 (Algebraic Set).** A subset  $Y$  of  $\mathbb{A}^n$  is an *algebraic set* if there is a subset  $T \subseteq A$  such that  $Y = Z(T)$ .

**Theorem 1.3.** Let  $T_i \subseteq A$ ,  $\mathfrak{a} \trianglelefteq A$  an ideal and  $Y_i \subseteq \mathbb{A}^n$ .

- (a)  $Z(T) = Z(\langle T \rangle)$  where  $\langle T \rangle$  is the ideal generated by  $T$  in  $A$ .
- (b)  $Z(T_1 T_2) = Z(T_1) \cup Z(T_2)$ .
- (c)  $Z(\bigcup T_i) = \bigcap Z(T_i)$ . Hence, the collection of all algebraic sets in  $\mathbb{A}^n$  can be identified with the collection of closed sets in some topology on  $\mathbb{A}^n$ . This is called the Zariski Topology on  $\mathbb{A}^n$ .
- (d) If  $T_1 \subseteq T_2$ , then  $Z(T_1) \supseteq Z(T_2)$ .
- (e) If  $Y_1 \subseteq Y_2$ , then  $\mathcal{I}(Y_1) \supseteq \mathcal{I}(Y_2)$ .
- (f)  $\mathcal{I}(Y_1 \cup Y_2) = \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2)$ .
- (g)  $\mathcal{I}(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ .
- (h)  $Z(\mathcal{I}(Y)) = \overline{Y}$ , the closure of  $Y$  in the Zariski Topology.

(i) There is an inclusion reversing bijection between the radical ideals of  $A$  and the algebraic sets in  $\mathbb{A}^n$ .

**Definition 1.4 (Irreducible).** A topological space  $X$  is said to be *irreducible* if it cannot be written as the union of two proper closed subspaces.

**Proposition 1.5.** Let  $X$  be a topological space and  $Y \subseteq X$  an irreducible subspace. Then,  $\bar{Y}$  is irreducible.

*Proof.* Suppose  $\bar{Y} = Y_1 \cup Y_2$  where  $Y_1$  and  $Y_2$  are proper closed subspaces of  $\bar{Y}$ . Then,  $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$ . Since  $Y$  is irreducible, either  $Y \subseteq Y_1$  or  $Y \subseteq Y_2$ , consequently,  $\bar{Y} \subseteq Y_1$  or  $\bar{Y} \subseteq Y_2$ , a contradiction. ■

**Proposition 1.6.** Let  $X$  be an irreducible topological space and  $W \subseteq X$  a non-empty open subset. Then,  $W$  is dense and irreducible.

*Proof.* ■

**Definition 1.7 (Affine Variety, Quasi-Affine Variety).** An *affine algebraic variety* (or simply *affine variety*) is an irreducible closed subset of  $\mathbb{A}^n$ . An open subset of an affine variety is called a *quasi-affine variety*.

**Remark 1.1.1.** Let  $Y$  be a quasi-affine variety. Then, there is an affine variety  $X$  that contains  $Y$  and  $Y$  is open in  $X$ . Then,  $Y$  is dense in  $X$  and hence,  $\bar{Y} = X$ , where the closure is taken in  $\mathbb{A}^n$ .

**Proposition 1.8.** An algebraic set in  $\mathbb{A}^n$  is irreducible if and only if its corresponding ideal is prime in  $A$ .

*Proof.* ■

**Definition 1.9 (Coordinate Ring).** For an algebraic set  $Y \subseteq \mathbb{A}^n$ , we define the *affine coordinate ring*  $A[Y] := A/\mathcal{I}(Y)$ .

**Remark 1.1.2.** Note that  $A[Y]$  is always a reduced, finitely generated  $k$ -algebra and is an integral domain if and only if  $Y$  is irreducible.

**Definition 1.10 (Noetherian Space).** A topological space  $X$  is said to be *noetherian* if it has the ascending chain condition on open sets.

**Proposition 1.11.** A subspace of a noetherian topological space is noetherian.

*Proof.* Let  $X$  be noetherian and  $Y \subseteq X$ . Let  $U_1 \subseteq U_2 \subseteq \dots$  be an ascending chain of open subsets of  $Y$ , then there are open  $V_i$  in  $X$  such that  $V_i \cap Y = U_i$ . Let  $W_i = \bigcup_{1 \leq j \leq i} V_j$ . Then,  $W_1 \subseteq W_2 \subseteq \dots$  and  $W_i \cap Y = U_i$ . Since  $X$  is noetherian, the chain  $\{W_i\}$  stabilizes, consequently, so does the chain  $\{U_i\}$ . ■

**Proposition 1.12.** *A noetherian topological space is compact.*

*Proof.* Let  $\{U_\alpha\}$  be an open cover. Let  $\mathcal{A}$  be the collection of all finite unions of  $U_\alpha$ 's. Then,  $\mathcal{A}$  must contain a maximal element, which must be  $X$ . Thus,  $X$  is compact. ■

**Corollary 1.13.** *A Hausdorff noetherian space  $X$  is finite with the discrete topology.*

*Proof.* Due to the preceeding result, every subspace of  $X$  is compact (and therefore closed) and hence, every subspace of  $X$  is open. Thus,  $X$  has the discrete topology. A discrete compact set must be finite. ■

**Proposition 1.14.** *A noetherian topological space can be expressed as a finite union  $X = Y_1 \cup \dots \cup Y_n$  of irreducible closed subspaces. If we require that  $Y_i \not\subseteq Y_j$  for  $i \neq j$ , then the  $Y_i$ 's are uniquely determined and are called the **irreducible components** of  $X$ .*

*Proof.* Let  $\Sigma$  be the poset of closed subspaces of  $X$  that cannot be expressed as a finite union of irreducible subspaces. If this poset is non-empty, then it admits a minimal element, say  $Z$ . Note that  $Z$  cannot be irreducible, hence,  $Z = Z_1 \cup Z_2$  where  $Z_1$  and  $Z_2$  are proper closed subspaces of  $Z$ , whence, are a finite union of irreducible subspaces. Consequently,  $Z$  is a finite union of irreducible subspaces.

Suppose we have two minimal representations,  $X = Y_1 \cup \dots \cup Y_n = Y'_1 \cup \dots \cup Y'_m$ . Then,  $Y_i = (Y_i \cap Y'_1) \cup \dots \cup (Y_i \cap Y'_m)$ . Since  $Y_i$  is irreducible, there is an index  $j$  such that  $Y_i \subseteq Y'_j$ . Similarly, there is an index  $l$  such that  $Y'_j \subseteq Y_l$ . Therefore,  $Y_i \subseteq Y_l$ , consequently,  $i = l$ . This completes the proof. ■

**Definition 1.15 (Dimension).** Let  $X$  be a topological space. Then,

$$\dim X := \sup\{n \mid \exists Y_0 \subsetneq \dots \subsetneq Y_n \subseteq X, \text{ each } Y_i \text{ is irreducible}\}.$$

**Lemma 1.16.** *Let  $X$  be a topological space.*

(a) *If  $Y$  is a subspace of  $X$ , then  $\dim Y \leq \dim X$ .*

(b)  *$\{U_i\}_{i \in I}$  an open cover of  $X$ . Then,*

$$\dim X = \sup_{i \in I} \dim U_i.$$

(c) *If  $Y$  is a closed subspace of an irreducible finite-dimensional topological space  $X$ , and if  $\dim Y = \dim X$ , then  $Y = X$ .*

*Proof.* (a) Let  $Z_0 \subsetneq \dots \subsetneq Z_n \subseteq X$  be a chain of closed, irreducible subspaces of  $Y$ . Then, consider the chain of closures in  $X$ ,

$$\bar{Z}_0 \subseteq \dots \subseteq \bar{Z}_n \subseteq X.$$

We contend that the inclusion  $\bar{Z}_i \subseteq \bar{Z}_{i+1}$  is strict. Indeed, if  $\bar{Z}_i = \bar{Z}_{i+1}$ , then  $\bar{Z}_i \cap Y = \bar{Z}_{i+1} \cap Y$ , which is absurd, since the  $Z_j$ 's are closed in  $Y$ . Hence,  $\dim X \geq n$  and it follows that  $\dim X \geq \dim Y$ .

(b) From part (a), we know that  $\dim X \geq \sup_i \dim U_i$ . We shall show the inequality in the other direction.

Let  $Z_0 \subsetneq \dots \subsetneq Z_n \subseteq X$  be a chain of closed, irreducible subspaces of  $X$ . Pick some point  $x_0 \in Z_0$  and let  $U_i$  be an element of the open cover containing  $x_0$ . Consider the sequence of closed subspaces

$Z_0 \cap U_i \subseteq \cdots \subseteq Z_n \cap U_i$ . Each  $Z_j \cap U_i$  is an open subspace of  $Z_j$  and hence, is irreducible and dense in  $Z_j$ .

Next, we contend that the inclusions  $Z_j \cap U_i \subseteq Z_{j+1} \cap U_i$  are strict. Indeed, if  $Z_j \cap U_i = Z_{j+1} \cap U_i = Y$ , then,  $Y$  is dense in both  $Z_{j+1}$  and  $Z_j$  but  $Z_j$  is a proper closed subspace of  $Z_{j+1}$ , a contradiction. Thus,  $\dim U_i \geq n$ . That is,  $\sup_i \dim U_i \geq \dim X$ . The conclusion follows. ■

(c)

Add in later

**Proposition 1.17.** *If  $Y$  is an algebraic set, then  $\dim Y = \dim A[Y]$ , where the latter is the Krull dimension.*

*Proof.* Immediate from definition. ■

**Proposition 1.18.** *Let  $Y$  be a quasi-affine variety, then  $\dim Y = \dim \bar{Y}$ .*

*Proof.* Note that  $Y$  is open in  $\bar{Y}$  as we have argued in Remark 1.1.1. Suppose  $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq Y$  is a sequence of closed irreducible subsets of  $Y$ , then  $\bar{Z}_0 \subsetneq \cdots \subsetneq \bar{Z}_n \subseteq \bar{Y}$  is a sequence of closed irreducible subsets of  $\bar{Y}$ . Thus,  $\dim \bar{Y} \geq \dim Y$ .

Conversely, suppose  $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq \bar{Y}$  is a chain of closed irreducible subsets of  $\bar{Y}$ . Then, each  $Z_i \cap Y$  is an open subset of  $Z_i$  whence is irreducible. Further, if we have  $Z_i \cap Y = Z_{i+1} \cap Y$  for some  $i$ , then  $Z_{i+1} = (Z_{i+1} \setminus Y) \cup Z_i$ , both of which are closed, a contradiction. Thus,  $Z_{i+1} \cap Y \neq Z_i \cap Y$  and  $\dim Y \geq \dim \bar{Y}$ . This completes the proof. ■

**Proposition 1.19.** *The Zariski topology on  $\mathbb{A}^2$  is not the same as the product topology on  $\mathbb{A}^1 \times \mathbb{A}^1$ .*

*Proof.* Note that the diagonal  $\Delta$  in  $\mathbb{A}^2$  is  $Z((x - y))$  and hence, is closed. On the other hand,  $\mathbb{A}^1$  is not Hausdorff, whence, the diagonal  $\Delta$  in  $\mathbb{A}^1 \times \mathbb{A}^1$  is not closed. The conclusion follows. ■

We shall see how to form the product of two varieties in an upcoming section.

## 1.2 Projective Varieties

We recall a bit about homogeneous ideals first.

**Definition 1.20.** Let  $R = \bigoplus_{n \geq 0} R_n$  be a graded ring. An ideal  $\mathfrak{a} \subseteq R$  is said to be *homogeneous* if

$$\mathfrak{a} = \bigoplus_{n \geq 0} (\mathfrak{a} \cap R_n)$$

as an abelian group.

**Proposition 1.21.** *An ideal  $\mathfrak{a} \subseteq R$  is homogeneous if and only if  $\mathfrak{a}$  can be generated by homogeneous elements.*

*Proof.* The forward direction is trivial. Conversely, suppose  $\mathfrak{a}$  is generated by  $F = \bigcup F_i$  where each  $F_i \subseteq R_i$ . Obviously,  $\bigoplus_{n \geq 0} (\mathfrak{a} \cap R_n) \subseteq \mathfrak{a}$ . A generic element of  $\mathfrak{a}$  is of the form

$$a = \sum_{f \in F} r_f f = \sum_{f \in F} \sum_{i=0}^{\infty} r_{f,i} f$$

where  $r_{f,i} \in R_i$ . Consequently,  $r_{f,i} f$  is a homogeneous element in some  $R_j$  and also lies in  $\mathfrak{a}$ . Thus,  $\mathfrak{a} \subseteq \bigoplus_{n \geq 0} (\mathfrak{a} \cap R_n)$ . This completes the proof. ■

**Proposition 1.22.** *Homogeneous ideals are closed under sum, product, intersection and radicals.*

*Proof.* The first three are obvious. Let  $\mathfrak{a}$  be a homogeneous ideal,  $\mathfrak{b} = \sqrt{\mathfrak{a}}$  and let  $x^m \in \mathfrak{a}$  for some positive integer  $m$ . We can write  $x = x_{i_1} + \cdots + x_{i_k}$  where  $i_1 < \cdots < i_k$  and  $x_{i_j} \in R_{i_j}$ . Then,  $x^m$  has a non-zero component in  $R_{mi_k}$ , which is  $x_{i_k}^m$ . Thus,  $x_{i_k}^m \in \mathfrak{a}$ , consequently,  $x_{i_k} \in \mathfrak{b}$ . Then, we have that  $x - x_{i_k}$  also lies in  $\mathfrak{b}$ . Using this, we can argue that all the  $x_{i_j}$ 's lie in  $\mathfrak{b}$ , whence,  $\mathfrak{b} \subseteq \bigoplus_{n \geq 0} (\mathfrak{b} \cap R_n)$  and hence, equality holds. This completes the proof. ■

**Proposition 1.23.** *A homogeneous ideal  $\mathfrak{p} \subseteq R$  is prime if and only if for all homogeneous elements  $f, g \in R$ ,  $fg \in \mathfrak{p}$  implies  $f \in \mathfrak{p}$  or  $g \in \mathfrak{p}$ .*

*Proof.* We shall prove only the reverse direction. Suppose  $f, g \in R$ ,  $fg \in \mathfrak{p}$  but  $f, g \notin \mathfrak{p}$ . Let  $f = f_1 + \cdots + f_n$  and  $g = g_1 + \cdots + g_m$  where each  $f_i, g_j$  is homogeneous and are arranged according to increasing homogeneous degree. Let  $f_{n_0}$  be the largest such that  $f_{n_0} \notin \mathfrak{p}$ , similarly, choose  $g_{m_0}$ . Then,  $fg \in \mathfrak{p}$  implies

$$(f_1 + \cdots + f_{n_0})(g_1 + \cdots + g_{m_0}) \in \mathfrak{p}.$$

If we expand the left hand side,  $f_{n_0}g_{m_0}$  has the largest homogeneous degree among all the terms and hence, must lie in  $\mathfrak{p}$  (since the latter is a homogeneous ideal). Thus, either  $f_{n_0} \in \mathfrak{p}$  or  $g_{m_0} \in \mathfrak{p}$  according to our assumptions, a contradiction. This completes the proof. ■

**Definition 1.24.** The *projective  $n$ -space over  $k$* , denote  $\mathbb{P}^n$  is defined as the set of equivalence classes of the set

$$\underbrace{k \times \cdots \times k}_{n \text{ times}} \setminus \{(0, \dots, 0)\},$$

under the equivalence relation

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \iff \exists \lambda \in k^\times, y_i = \lambda x_i \text{ for every } 0 \leq i \leq n.$$

Let  $S = k[x_0, \dots, x_n]$  with the standard grading  $S = \bigoplus_{n \geq 0} S_n$  where  $S_n$  is the additive abelian subgroup consisting of homogeneous degree  $n$  polynomials in  $S$ .

**Definition 1.25 (Algebraic Set).** A subset  $Y$  of  $\mathbb{P}^n$  is said to be an *algebraic set* if there is a set  $T$  of homogeneous elements of  $S$  such that  $Y = Z(T)$ . Now, let  $Y \subseteq \mathbb{P}^n$ . Define

$$\mathcal{I}(Y) := \{f \in S^h \mid f(p) = 0, \forall p \in Y\},$$

where  $S^h = \bigcup_{n \geq 0} S_n$  is the set of all homogeneous polynomials in  $S$ .

Henceforth, we endow  $\mathbb{P}^n$  with the Zariski topology.

**Proposition 1.26.** *Algebraic sets are closed under finite unions and arbitrary intersections. Therefore, the Zariski topology is defined to be the collection of complements of algebraic sets in  $\mathbb{P}^n$ .*

*Proof.* Note that  $Z(T_1 T_2) = Z(T_1) \cup Z(T_2)$  and  $Z(\cup T_i) = \cap Z(T_i)$ . ■

**Example 1.27.** Let us consider  $\mathbb{P}^1$  with the Zariski topology. Note that every homogeneous ideal can be generated by finitely many homogeneous polynomials. It suffices to find  $Z(f)$  for a single homogeneous polynomial  $f(x, y)$ .

If  $[a_0 : a_1] \in Z(f)$ , then note that neither of the  $a_i$ 's can be zero. Thus,  $f(1, a_1/a_0) = 0$  and hence,  $a_1/a_0$  can take finitely many values. Hence, the Zariski topology on  $\mathbb{A}^1$  is precisely the cofinite topology.

**Proposition 1.28.** *Let  $T_i \subseteq S^h$ ,  $\mathfrak{a} \subseteq A$  and  $Y_i \subseteq \mathbb{P}^n$ .*

- (a) *If  $T_1 \subseteq T_2 \subseteq S^h$ , then  $Z(T_1) \supseteq Z(T_2)$ .*
- (b) *If  $Y_1 \subseteq Y_2 \subseteq \mathbb{P}^n$ , then  $\mathcal{J}(Y_1) \supseteq \mathcal{J}(Y_2)$ .*
- (c)  *$\mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$ .*
- (d) *If  $\mathfrak{a} \subseteq S$  is a homogeneous ideal, then  $\mathcal{J}(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ .*
- (e)  *$Z(\mathcal{J}(Y)) = \overline{Y}$ .*

*Proof.* (a), (b) and (c) are trivial and (e) follows easily. Consider (d). Let  $f \in \mathcal{J}(Z(\mathfrak{a}))$ . Note that  $f$  is a homogeneous polynomial in  $S$ . Consider the algebraic set  $V \subseteq \mathbb{A}^{n+1}$  generated by  $\mathfrak{a}$ . Since  $f$  vanishes on  $Z(\mathfrak{a}) \subseteq \mathbb{P}^n$ , it must vanish on  $V \subseteq \mathbb{A}^{n+1}$ . As a result, there is a positive integer  $q$  such that  $f^q \in \mathfrak{a}$ . The conclusion follows. ■

**Proposition 1.29.** *The following are equivalent:*

- (a)  $Z(\mathfrak{a}) = \emptyset$ .
- (b)  $\sqrt{\mathfrak{a}}$  is either  $S$  or  $S_+$ .
- (c)  $S_d \subseteq \mathfrak{a}$  for some  $d > 0$ .

*Proof.* (a)  $\implies$  (b) Let  $V \subseteq \mathbb{A}^{n+1}$  be the affine algebraic zero set of  $\mathfrak{a}$ . We have either  $V = \emptyset$  or  $V = \{(0, \dots, 0)\}$ . Then (b) follows from Hilbert's Nullstellensatz.

(b)  $\implies$  (c). Trivial.

(c)  $\implies$  (a). Note that  $\mathfrak{a}$  contains  $x_i^d$  for every  $0 \leq i \leq n$ . The conclusion follows. ■

**Corollary 1.30.** *There is a 1 – 1 inclusion-reversing bijection between projective algebraic subsets of  $\mathbb{P}^n$  and homogeneous radical ideals of  $S$  not equal to  $S_+$ .*



**Proposition 1.31.** A projective algebraic set  $Y \subseteq \mathbb{P}^n$  is irreducible if and only if  $\mathcal{I}(Y)$  is a prime ideal.

*Proof.* Suppose  $Y$  is irreducible. Let  $f, g$  be homogeneous elements such that  $fg \in \mathcal{I}(Y)$ . Then,  $Z(f) \cup Z(g) \supseteq Y$ , whence,  $Y$  is contained in either  $Z(f)$  or  $Z(g)$ , consequently, either  $f \in \mathcal{I}(Y)$  or  $g \in \mathcal{I}(Y)$  whence  $\mathcal{I}(Y)$  is prime.

Conversely, suppose  $Y = Y_1 \cup Y_2$  where  $Y_1, Y_2$  are closed subsets of  $\mathbb{P}^n$ . Then,  $\mathcal{I}(Y) = \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2)$ , consequently,  $\mathcal{I}(Y) = \mathcal{I}(Y_i)$  for some  $i \in \{1, 2\}$ , which follows from the fact that  $\mathcal{I}(Y)$  is prime. ■

**Corollary 1.32.**  $\mathbb{P}^n$  is irreducible.

**Definition 1.33 (Projective Variety, Quasi-Projective Variety).** A *projective variety* is an irreducible algebraic set in  $\mathbb{P}^n$ . A *quasi-projective variety* is an open subset of a projective variety.

**Theorem 1.34.** Let  $U_i$  denote the open set  $\mathbb{P}^n \setminus Z(\{x_i\})$ . The sets  $\{U_i\}_{i=0}^n$  cover  $\mathbb{P}^n$ . Consider the map  $\varphi_i : U_i \rightarrow \mathbb{A}^n$  given by

$$\varphi_i((a_0, \dots, a_n)) = \left( \frac{a_0}{a_i}, \dots, \frac{\hat{a}_i}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

Then,  $\varphi_i$  is a homeomorphism.

*Proof.* We shall prove this for  $i = 0$  and denote  $\varphi_0$  by  $\varphi : U_0 \rightarrow \mathbb{A}^n$ . ■

**Theorem 1.35.** Let  $Y$  be a projective  $n$ -variety with homogeneous coordinate ring  $S(Y)$ . Then,  $\dim S(Y) = \dim Y + 1$ .

*Proof.* Let  $U_i = \mathbb{P}^n \setminus Z(x_i)$ . We have seen that each  $U_i$  is homeomorphic to  $\mathbb{A}^n$  under the map  $\varphi_i : U_i \rightarrow \mathbb{A}^n$  as defined above. Let  $Y_i = \varphi_i(U_i \cap Y) \subseteq \mathbb{A}^n$ . Note further that  $Y_i$  is irreducible owing to  $U_i \cap Y$  being irreducible (since it is an open subset of  $Y$  which is irreducible). Thus,  $Y_i$  is an affine  $n$ -variety.

Note that we can identify  $A(Y_i)$  with the subring of degree 0 elements in  $S(Y)_{x_i}$ . Further, note that

$$S(Y)_{x_i} = (S(Y)_{x_i})_0[x_i, x_i^{-1}] \cong A(Y_i)[x_i, x_i^{-1}].$$

Note that  $\dim S(Y)_{x_i} = \dim S(Y)$  since they both have the same fraction fields. Then, it follows that

$$\dim S(Y) = \dim S(Y)_{x_i} = \dim A(Y_i)[x_i, x_i^{-1}] = \dim A(Y_i) + 1 = \dim Y_i + 1.$$

The equality  $\dim A(Y_i)[x_i, x_i^{-1}] = \dim A(Y_i) + 1$  follows by looking at the transcendence degree of the fraction fields.

Finally, note that  $\dim Y_i = \dim(Y \cap U_i)$  and hence,  $\dim Y = \sup \dim Y_i$ , consequently,  $\dim Y + 1 = \dim S(Y)$ . This completes the proof. ■

**Corollary 1.36.** Let  $Y$  be a quasi-projective variety. Then,  $\dim Y = \dim \bar{Y}$ .

*Proof.* Let  $U_i = \mathbb{P}^n \setminus Z(x_i)$  for  $0 \leq i \leq n$ . Note that  $Y \cap U_i$  is a quasi-affine variety whose closure is  $\bar{Y} \cap U_i$ . Thus,

$$\dim Y = \sup_i \dim(Y \cap U_i) = \sup_i \dim(\bar{Y} \cap U_i) = \dim \bar{Y}. \quad \blacksquare$$

**Definition 1.37 (Cone over a Projective Variety).** Let  $Y \subseteq \mathbb{P}^n$  be a projective algebraic set. Let  $\varphi : \mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\} \rightarrow \mathbb{P}^n$  be given by  $\theta((a_0, \dots, a_n)) = [(a_0, \dots, a_n)]$ . The *affine cone over  $Y$*  is defined to be

$$C(Y) := \theta^{-1}(Y) \cup \{(0, \dots, 0)\}.$$

**Definition 1.38 (Segre Embedding).** Define the map  $\psi : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^N$  where  $N = (m+1)(n+1) - 1$  by

$$\psi([a_0 : \dots : a_{n+1}], [b_0 : \dots : b_{m+1}]) = [[a_i b_j]_{i,j}].$$

This is called the *Segre embedding*.

**Proposition 1.39.** *With notation as above,  $\text{im } \psi$  is a subvariety of  $\mathbb{P}^N$ .*

*Proof.* Consider the  $k$ -algebra homomorphism  $\phi : k[\{z_{ij}\}] \rightarrow k[x_0, \dots, x_n, y_0, \dots, y_m]$  given by  $\phi(z_{ij}) = x_i y_j$ . Let  $\mathfrak{p} = \ker \phi$ . This is obviously a prime ideal in  $k[\{z_{ij}\}]$ .

Indeed, if  $f \in \mathfrak{p}$ , then  $f(\{x_i y_j\}) = 0$ . We can write  $f = \sum_{d \geq 0} f_d$ . Then,  $f_d(\{x_i y_j\}) = 0$  for every  $d \geq 0$ . Thus, every  $f_d \in \mathfrak{p}$  and it follows that  $\mathfrak{p}$  is homogeneous. We may now talk about  $Z(\mathfrak{p})$  as a projective variety in  $\mathbb{P}^N$ .

We contend that  $Z(\mathfrak{p}) = \text{im } \psi$ . The inclusion  $\text{im } \psi \subseteq Z(\mathfrak{p})$  is obvious. Now suppose  $[\{c_{ij}\}] \in Z(\mathfrak{p})$ . Without loss of generality, suppose  $c_{00} \neq 0$ . By normalizing coordinates, we may suppose that  $c_{00} = 1$ . Define  $[a_0 : \dots : a_n]$  and  $[b_0 : \dots : b_m]$  as follows.

$$a_i = \begin{cases} 1 & i = 0 \\ c_{i0} & i > 0 \end{cases} \quad \text{and} \quad b_j = \begin{cases} 1 & j = 0 \\ c_{0j} & j > 0 \end{cases}.$$

Now, note that  $z_{ij} z_{00} - z_{i0} z_{0j} \in \mathfrak{p}$  and hence,  $c_{ij} = c_{i0} c_{0j} = a_i b_j$ . Thus,

$$[\{c_{ij}\}] = \psi([a_0 : \dots : a_n], [b_0 : \dots : b_m]),$$

this completes the proof. ■

We shall use the Segre embedding to define the product of two (quasi-)projective varieties.

## 1.3 Morphisms

We begin by defining regular maps on varieties. The definitions will be different for affine and projective varieties.

**Definition 1.40 (Regular Map).** Let  $Y$  be a quasi-affine variety in  $\mathbb{A}^n$ . A function  $f : Y \rightarrow k$  is *regular at a point  $P \in Y$*  if there is an open neighborhood  $U$  of  $P$  in  $Y$  and polynomials  $g, h \in A$  such that  $h$  does not vanish on  $U$  and  $f = g/h$  on  $U$ . The map  $f$  is said to be *regular on  $Y$*  if it is regular at every point of  $Y$ .

Let  $Y$  be a quasi-projective variety in  $\mathbb{P}^n$ . A function  $f : Y \rightarrow k$  is *regular at a point  $P \in Y$*  if there is an open neighborhood  $U$  of  $P$  in  $Y$  and homogeneous polynomials  $g, h \in S^h$  of equal degree such that  $h$  does not vanish on  $U$  and  $f = g/h$  on  $U$ .

**Theorem 1.41.** *A regular function is continuous when  $k$  is identified with  $\mathbb{A}^1$  or  $\mathbb{P}^1$ .*

*Proof.* First, suppose  $Y$  is a quasi-affine variety. Identify  $k$  with  $\mathbb{A}^1$  and let  $\varphi : Y \rightarrow k$  be a regular map on  $Y$ . It suffices to show that the inverse image of a closed set in  $\mathbb{A}^1$  is closed in  $Y$ . But closed sets in  $\mathbb{A}^1$  are precisely finite subsets of  $k$ . Hence, it suffices to show that the inverse image of a singleton in  $\mathbb{A}^1$  is closed in  $Y$ . Let  $a \in k$ .

There is an open cover of  $Y$  such that on every open set of the cover,  $\varphi$  is of the form  $f/g$ . Pick such an open set  $U$ . Then,  $\varphi^{-1}(\{a\}) \cap U = Z(f - ag) \cap U$ , which is closed in  $U$ . Thus,  $\varphi^{-1}(\{a\})$  is closed in  $Y$ .

Now, suppose  $Y$  is a quasi-projective variety. Note that when  $k$  is identified with  $\mathbb{P}^1$ , it has the cofinite topology and a proof similar to the one in the preceding paragraphs works. ■

Henceforth, a *variety* refers to either a quasi-affine or quasi-projective variety. When a result explicitly depends on the type of variety, we shall mention it.

**Corollary 1.42.** Let  $Y$  be a variety. If  $f, g : Y \rightarrow k$  are regular functions that agree on an open subset of  $Y$ , then  $f = g$  on  $Y$ .

*Proof.* Let  $X = \{y \in Y \mid f(y) = g(y)\}$ . We know that  $X$  is closed and contains an open subset  $U$  of  $Y$ . But  $U$  is dense in  $Y$  (since  $Y$  is irreducible) and hence,  $X = Y$ . ■

**Definition 1.43 (Morphism).** Let  $X$  and  $Y$  be varieties over  $k$  (can be quasi-affine or quasi-projective). A *morphism*  $\varphi : X \rightarrow Y$  is a continuous map such that for every open set  $V \subseteq Y$  and for every regular function  $f : V \rightarrow k$ , the function  $f \circ \varphi : \varphi^{-1}(V) \rightarrow k$  is regular on  $\varphi^{-1}(V)$ .

$$\begin{array}{ccc} X \supseteq \varphi^{-1}(V) & \xrightarrow{\varphi} & V \\ & \searrow f \circ \varphi & \downarrow f \\ & & k \end{array}$$

**Proposition 1.44.** *The composition of two morphisms is a morphism. The identity map on a variety is a morphism.*

*Proof.* Let  $X, Y, Z$  be varieties and  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be morphisms. Let  $V \subseteq Z$  be an open set and  $f : V \rightarrow k$  be a regular function on  $V$ .

$$\begin{array}{ccccc} \varphi^{-1}\psi^{-1}V & \longrightarrow & \psi^{-1}V & \longrightarrow & V \\ & \searrow f \circ \psi \circ \varphi & \searrow f \circ \psi & \downarrow f & \\ & & & & k \end{array}$$

Applying the definition of a morphism twice, we see that  $f \circ \psi \circ \varphi$  is a regular function. This completes the proof. ■

**Definition 1.45 (Ring of Regular Functions).** Let  $Y$  be a variety. Then, the set of regular functions on  $Y$ , denoted  $\mathcal{O}(Y)$  forms a ring known as the *ring of regular functions on  $Y$* .

On the other hand, given any  $P \in Y$ , there is the *local ring of  $P$  on  $Y$* , which is the ring of germs of

regular functions at  $P$ . This is denoted by  $\mathcal{O}_{P,Y}$  or just  $\mathcal{O}_P$  if the variety is clear from the context.

**Remark 1.3.1.** Here, we explicitly define the ring of germs at  $P$ . Consider the set of all pairs  $\langle U, f \rangle$  where  $U$  is a neighborhood of  $P$  in  $Y$ . Next, define the relation  $\langle U, f \rangle \sim \langle V, g \rangle$  if  $f = g$  on  $U \cap V$ .

To see that this is an equivalence relation, we need only verify transitivity. Indeed, suppose  $\langle U, f \rangle \sim \langle V, g \rangle$  and  $\langle V, g \rangle \sim \langle W, h \rangle$ . Then,  $f = g = h$  on  $U \cap V \cap W$  which is a neighborhood of  $P$ . Since they agree on an open set,  $f = h$  on  $U \cap W$ .

Next, we must show that  $\mathcal{O}_P$  is local. Let  $\mathfrak{m}$  denote the collection of germs that vanish at  $P$ . This is obviously a maximal ideal in  $\mathcal{O}_P$ . If  $[\langle U, f \rangle] \in \mathcal{O}_P$  does not vanish at  $P$ , then there is a neighborhood  $V$  of  $P$  contained in  $U$  on which  $f$  does not vanish and  $f$  is a quotient of polynomials. Thus,  $1/f$  is also a quotient of polynomials on  $V$  and is a well-defined inverse of the germ  $[\langle V, f \rangle]$ . Hence,  $\mathcal{O}_P$  is local.

**Lemma 1.46.** Let  $X \subseteq \mathbb{A}^n$  be open and hence, a quasi-affine variety. If  $f : X \rightarrow k$  is a regular function then  $f = g/h$  for some  $g, h \in A = k[x_1, \dots, x_n]$ .

*Proof.* Let  $U \subseteq X$  be an open set such that  $f = g/h$  (reduced) on  $U$  for some polynomials  $g, h \in A$ . We contend that  $f(P) = g(P)/h(P)$  for all  $P \in X$ . Indeed, there is a neighborhood  $V$  of  $P$  in  $X$  such that  $f = g'/h'$  (reduced) on  $V$  for some  $g', h' \in A$ .

The intersection  $U \cap V$  is non-empty and  $g/h = g'/h'$  on  $U \cap V$ . Consequently,  $U \cap V \subseteq Z(gh' - g'h)$ . But  $U \cap V$  is open and dense in  $\mathbb{A}^n$ , whence  $gh = g'h$  as polynomials in  $A$ . Using the fact that  $A$  is a unique factorization domain and  $(g, h) = (g', h') = 1$ , we have that  $g = g'$  and  $h = h'$ . This completes the proof. ■

**Corollary 1.47.** Let  $X = \mathbb{A}^n \setminus \{(0, \dots, 0)\}$ . Then,  $\mathcal{O}(X) \cong k[x_1, \dots, x_n]$ .

*Proof.* Let  $f : X \rightarrow k$  be a regular function. Due to Lemma 1.46,  $f = g/h$  (reduced) on  $X$  for some  $g, h \in k[x_1, \dots, x_n]$ . Note that  $h$  does not vanish on  $X$  and hence, must be constant. Therefore,  $f \in k[x_1, \dots, x_n]$ . The conclusion follows. ■

**Definition 1.48 (Function Field).** Let  $Y$  be a variety. The *function field* of  $Y$ , denoted  $K(Y)$  is defined as the set of equivalence classes of the collection of pairs  $\langle U, f \rangle$  where  $U \subseteq Y$  is open and  $f : U \rightarrow k$  is a regular function on  $U$ . The equivalence relation is defined as:

$$\langle U, f \rangle \sim \langle V, g \rangle \iff f = g \text{ on } U \cap V.$$

The elements of  $K(Y)$  are called *rational functions* on  $Y$ .

**Remark 1.3.2.** That  $K(Y)$  is a well defined ring, one can argue as in Remark 1.3.1. We show that this ring is a field. Indeed, suppose  $\langle U, f \rangle$  is an element in  $K(Y)$  with  $f$  not identically 0 on  $U$ . Then, there is an open set  $V \subseteq U$  such that  $f$  is non-zero on  $V$ . Then,  $\langle V, 1/f \rangle$  is an inverse of  $\langle U, f \rangle$  in  $K(Y)$ .

**Remark 1.3.3.** Let  $Y$  be a variety. There is a natural map  $\mathcal{O}(Y) \rightarrow \mathcal{O}_P$  that sends a regular function on  $Y$  to the equivalence class of the pair  $\langle Y, f \rangle$ , which is obviously an injective map.

Next, there is also a map  $\mathcal{O}_P \rightarrow K(Y)$  that sends  $[\langle U, f \rangle] \mapsto [\langle U, f \rangle]$ . Again, this is obviously injective. Hence, we have an inclusion of rings

$$\mathcal{O}(Y) \subseteq \mathcal{O}_{P,Y} \subseteq K(Y)$$

and we shall often treat  $\mathcal{O}(Y)$  and  $\mathcal{O}_P$  as subrings of  $K(Y)$ .

**Theorem 1.49.** Let  $Y \subseteq \mathbb{A}^n$  be an affine variety. Then,

- (a)  $\mathcal{O}(Y) \cong A(Y)$ .
- (b) for each  $P \in Y$ ,  $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$  where  $\mathfrak{m}_P$  is the maximal ideal of functions in  $A(Y)$  vanishing at  $P$ . Consequently,  $\dim \mathcal{O}_P = \dim Y$ .
- (c)  $K(Y)$  is isomorphic to the fraction field of  $A(Y)$ . In particular, it is a finitely generated field extension of  $k$  of transcendence degree  $\dim Y$ .

*Proof.* First, note that every maximal ideal in  $A(Y)$  is of the form  $\mathfrak{m}_P$  for some  $P \in Y$ . There is a natural map  $A(Y)_{\mathfrak{m}_P} \rightarrow \mathcal{O}_P$  that sends  $f/g$  to  $[\langle Y \setminus Z(g), f/g \rangle]$ . This is obviously both injective and surjective and therefore, an isomorphism. This establishes (b).

Now, there is a natural map  $A \rightarrow \mathcal{O}(Y)$  that sends a polynomial to the regular function defined by it on  $Y$ . The kernel of this map is precisely  $\mathcal{I}(Y)$  and hence, induces an injective ring homomorphism  $\alpha : A(Y) \rightarrow \mathcal{O}(Y)$ . Upon identifying  $A(Y)$  with a subring of  $\mathcal{O}(Y)$  and using the fact that  $\mathcal{O}_P = A(Y)_{\mathfrak{m}_P}$ , we have

$$A(Y) \subseteq \mathcal{O}(Y) \subseteq \bigcap_{P \in Y} \mathcal{O}_P \subseteq \bigcap_{\mathfrak{m}_P} A(Y)_{\mathfrak{m}_P} = A(Y),$$

where the last equality follows from the fact that  $A$  is an integral domain. This establishes (a).

Finally, from (b) and the fact that  $\mathcal{O}_P \subseteq K(Y)$ , it would follow that  $K(Y)$  is precisely the fraction field of  $A(Y)$ . ■

**Remark 1.3.4 (The Functor  $\mathcal{O}$ ).** We shall quickly see that  $\mathcal{O}$  is a functor from the category of  $k$ -varieties,  $\mathfrak{Var}_k$  to the category of finitely generated  $k$ -algebras  $\mathbf{FinAlg}_k$ . Let  $\varphi : X \rightarrow Y$  be a morphism of varieties. Then, if  $f \in \mathcal{O}(Y)$ , then  $f \circ \varphi \in \mathcal{O}(X)$ . This induces a map  $\varphi_* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ . Thus,  $\mathcal{O}$  is a contravariant functor.

**Theorem 1.50.** Let  $X$  be any variety and  $Y$  an affine variety. Then, there is a natural bijection of sets

$$\alpha : \text{Hom}_{\mathfrak{Var}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\mathbf{Alg}}(\mathcal{O}(Y), \mathcal{O}(X)).$$

*Proof.* ■

**Corollary 1.51.** The functor  $X \mapsto \mathcal{O}(X) \equiv A(X)$  is a contravariant equivalence of categories between  $\mathfrak{Var}_k$  and  $\mathbf{FinAlg}_k$ .

**Example 1.52.** We contend that the quasi-affine variety  $\mathbb{A}^n \setminus \{(0, \dots, 0)\}$  is not isomorphic to an affine variety. Let  $X = \mathbb{A}^n \setminus \{(0, \dots, 0)\}$ , and suppose  $X$  is affine. The inclusion morphism  $\iota : X \hookrightarrow \mathbb{A}^n$  induces a restriction map  $\iota_* : \mathcal{O}(\mathbb{A}^n) \rightarrow \mathcal{O}(X) = k[x_1, \dots, x_n]$ , which is an isomorphism. From the equivalence of categories, it follows that  $\iota$  must be an isomorphism too, which is absurd, since it is not even surjective.

**Definition 1.53 (Locally Closed).** A subspace of a topological space is said to be *locally closed* if it is open in its closure. Equivalently, if it is the intersection of an open set and a closed set.

**Definition 1.54.** If  $X$  is a quasi-affine (resp. quasi-projective) variety and  $Y$  is an irreducible locally closed subset, then  $Y$  is also a quasi-affine (resp. quasi-projective) variety and is said to be a *subvariety* of  $X$ .

### 1.3.1 Product of Varieties

**Definition 1.55 (Product of Affine Varieties).** If  $X \subseteq \mathbb{A}^n$  and  $Y \subseteq \mathbb{A}^m$  are affine-varieties, their product is defined to be the topological space  $X \times Y \subseteq \mathbb{A}^{n+m}$  in the *subspace topology*.

**Proposition 1.56.** *With notation as above,*

- (a)  $X \times Y \subseteq \mathbb{A}^{n+m}$  is a closed irreducible subspace, and hence, an affine variety.
- (b)  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .
- (c)  $\dim X \times Y = \dim X + \dim Y$ .

*Proof.* (a) ■

**Definition 1.57 (Product of Quasi-Projective Varieties).**

## 1.4 Rational Maps

**Lemma 1.58.** *Let  $X, Y$  be varieties and  $\varphi, \psi : X \rightarrow Y$  be morphisms. If there is a non-empty open subset  $U \subseteq X$  such that  $\varphi|_U = \psi|_U$ , then  $\varphi = \psi$ .*

*Proof.* ■

**Definition 1.59 (Rational Map).**

# Chapter 2

## Schemes

### 2.1 Sheaves

Throughout this section, we shall work with sheaves of abelian groups. The definitions carry over to any abelian category with pretty much the same proofs.

**Definition 2.1 (Presheaf).** Let  $X$  be a topological space. A *presheaf of abelian groups on  $X$*  is a contravariant functor  $\mathcal{F} : \mathfrak{Top}(X) \rightarrow \mathbf{AbGrp}$  where  $\mathfrak{Top}(X)$  denotes the poset category of open sets in  $X$  ordered by inclusion.

For an open set  $U \subseteq X$ , we refer to  $\mathcal{F}(U)$  as the *sections* of the presheaf  $\mathcal{F}$  over the open set  $U$ . Sometimes, we use the notation  $\Gamma(U, \mathcal{F})$  to denote  $\mathcal{F}(U)$ .

**Definition 2.2 (Sheaf).** A presheaf  $\mathcal{F}$  on a topological space  $X$  is a *sheaf* if it satisfies the following additional conditions,

**Identity Axiom:** if  $U$  is an open set,  $\{V_i\}$  an open cover of  $U$ , and if  $s \in \mathcal{F}(U)$  is an element such that  $s|_{V_i} = 0$  for all  $i$ , then  $s = 0$ .

**Glueability Axiom:** if  $U$  is an open set,  $\{V_i\}$  an open cover of  $U$ , and if there are elements  $s_i \in \mathcal{F}(V_i)$  for each  $i$ , such that for each pair  $i, j$ ,  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ , then there is an  $s \in \mathcal{F}(U)$  such that  $s|_{V_i} = s_i$  for each  $i$ .

**Definition 2.3 (Direct Limit).** Let  $\mathcal{C}$  be a category,  $(I, \leq)$  a directed set and a collection  $\langle \{A_i\}, \{f_{ij} : A_i \rightarrow A_j\}_{i \leq j} \rangle$  with the following properties:

- (a)  $f_{ii}$  is the identity on  $A_i$ , and
- (b)  $f_{ik} = f_{jk} \circ f_{ij}$  for all  $i \leq j \leq k$ .

The *direct limit* of the above direct system (if it exists), denoted  $\varinjlim A_i$  is an object  $\langle X, \{\phi_i\} \rangle$  where  $\phi_i : A_i \rightarrow X$  are morphisms that satisfy, for all  $i \leq j$ ,  $\phi_i = \phi_j \circ f_{ij}$ . This object is universal in the sense that if  $\langle Y, \{\psi_i\} \rangle$  is another object with morphisms such that  $\psi_i = \psi_j \circ f_{ij}$  then there is a unique morphism  $u : X \rightarrow Y$  satisfying  $\psi_i = u \circ \phi_i$  for all  $i$ .

**Remark 2.1.1.** Now suppose  $\mathcal{C}$  is the category of abelian groups. We show that this category always admits a direct limit. Indeed, suppose  $\langle \{A_i\}, \{f_{ij}\} \rangle$  is a direct system. Let

$$X = \bigsqcup_{i \in I} A_i / \sim$$

where  $\langle i, a_i \rangle \sim \langle j, a_j \rangle$  if and only if there is some  $k \geq i, j$  such that  $f_{ik}(a_i) = f_{jk}(a_j)$ . The group operation is defined as,  $[\langle i, a_i \rangle] \cdot [\langle j, a_j \rangle] = [\langle k, a_k \rangle]$  where  $k \geq i, j$  and  $a_k = f_{ik}(a_i) f_{jk}(a_j)$ .

First, note that the group operation is well defined. Indeed, suppose  $[\langle i, a_i \rangle] = [\langle p, a_p \rangle]$ ,  $[\langle j, a_j \rangle] = [\langle q, a_q \rangle]$  and suppose that we chose  $[\langle r, a_r \rangle]$  to be  $[\langle p, a_p \rangle] \cdot [\langle q, a_q \rangle]$ . There is an index  $l \geq k, r$  sufficiently “large” such that  $f_{il}(a_i) = f_{pl}(a_p)$  and  $f_{jl}(a_j) = f_{ql}(a_q)$ . The conclusion now follows.

Consider maps  $\phi_i : A_i \rightarrow X$  given by  $\phi_i(a_i) = [\langle i, a_i \rangle]$ . This is obviously a morphism and  $\phi_j \circ f_{ij} = \phi_i$  for all  $i \leq j$ . Now, suppose  $\langle Y, \{\psi_i\} \rangle$  is another such object. Define  $u : X \rightarrow Y$  by

$$u([\langle i, a_i \rangle]) = \psi_i(a_i).$$

Again, it is not hard to see that this map is well defined and is the unique map making everything commute.

**Definition 2.4 (Stalk of a Presheaf).** Let  $\mathcal{F}$  be a presheaf on a topological space  $X$  and  $P \in X$ . Consider the directed set of neighborhoods of  $P$  in  $X$  ordered by reverse inclusion. The direct limit  $\varinjlim \mathcal{F}(U)$  over this directed system is called the *stalk* of  $\mathcal{F}$  at  $P$ , denoted  $\mathcal{F}_P$ .

**Definition 2.5 (Morphism of Presheaves/Sheaves).** Let  $\mathcal{F}$  and  $\mathcal{G}$  be presheaves on  $X$ . A *morphism*  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a *natural transformation* between these two functors. In other words, it is a collection of maps  $\{\varphi(U)\}_{U \in \mathcal{T}_{\text{op}}(X)}$  making the following diagram commute

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

for all  $U, V \in \mathcal{T}_{\text{op}}(X)$ . A morphism of presheaves is an *isomorphism* if and only if it admits an inverse.

**Proposition 2.6.** Let  $\mathcal{F}, \mathcal{G}$  be presheaves on  $X$  and  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  a morphism. Then, for each  $P \in X$ , there is an induced map  $\varphi : \mathcal{F}_P \rightarrow \mathcal{G}_P$  given by

$$\varphi_P([\langle U, s \rangle]) = [\langle U, \varphi_U(s) \rangle].$$

*Proof.* Straightforward. ■

**Proposition 2.7.** Let  $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$  be morphisms of presheaves. Then,  $(\psi \circ \varphi)_P = \psi_P \circ \varphi_P$ .

*Proof.* Let  $[\langle U, s \rangle] \in \mathcal{F}_P$ . Then,

$$\psi_P \circ \varphi_P([\langle U, s \rangle]) = [\langle U, \psi_U(\varphi_U(s)) \rangle] = [\langle U, (\psi \circ \varphi)_U(s) \rangle].$$

This completes the proof. ■

**Theorem 2.8.** Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves on a topological space  $X$ . Then,  $\varphi$  is an isomorphism if and only if the induced map on the stalk  $\varphi_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$  is an isomorphism for every  $P \in X$ .



*Proof.* The forward direction is trivial and we shall show the converse, for which it suffices to show that  $\varphi_U$  is an isomorphism for every  $U \in \mathfrak{Top}(X)$ .

First, we show injectivity. Let  $s \in \mathcal{F}(U)$  that maps to 0 in  $\mathcal{G}(U)$ . Let  $P \in U$ . Then,  $\varphi_P([\langle U, s \rangle]) = [\langle U, \varphi_U(s) \rangle] = 0$  whence  $[\langle U, s \rangle] = 0$  in  $\mathcal{F}_P$ . Hence, there is a neighborhood  $V_P$  of  $P$  in  $U$  on which  $s$  restricts to 0. Note that the  $V_P$ 's form an open cover of  $U$  and due to the identity axiom,  $s = 0$  on  $U$ , from which injectivity follows.

Next, we show surjectivity. Let  $t \in \mathcal{G}(U)$  and  $P \in U$ . Since  $\varphi_P$  is surjective, there is some  $[\langle V_P, s_P \rangle]$  with  $V_P \subseteq U$  in  $\mathcal{F}_P$  that maps to  $[\langle U, t \rangle]$  in  $\mathcal{G}_P$ . Thus,

$$[\langle V_P, \varphi_{V_P}(s_P) \rangle] = [\langle U, t \rangle].$$

Thus, we may shrink  $V_P$  such that  $\varphi_{V_P}(s_P) = t|_{V_P}$ , where we have redefined  $s_P$  to be its restriction to the shrunk  $V_P$ .

Now, let  $P, Q$  be points in  $U$ . We have

$$\varphi_{V_P \cap V_Q}(s_P|_{V_P \cap V_Q}) = \varphi_{V_P}(s_P)|_{V_P \cap V_Q} = t|_{V_P \cap V_Q}.$$

Similarly, we have  $\varphi_{V_P \cap V_Q}(s_Q|_{V_P \cap V_Q}) = t|_{V_P \cap V_Q}$ . Using the injectivity of  $\varphi_{V_P \cap V_Q}$  that we proved in the earlier paragraph, we have  $s_P|_{V_P \cap V_Q} = s_Q|_{V_P \cap V_Q}$ . Using the gluability axiom, we have an  $s \in \mathcal{F}(U)$  that restricts to  $s_P$  on  $V_P$  for every  $P$ .

Finally, we verify that  $\varphi_U(s) = t$ . Indeed,  $\{V_P\}$  is a cover of  $U$  and  $t|_{V_P} = \varphi_U(s)|_{V_P}$ . From the identity axiom, it follows that  $t = \varphi_U(s)$ . This completes the proof. ■

**Definition 2.9 (Sheafification).** Let  $\mathcal{F}$  be a presheaf on  $X$ . Then, there is a sheaf  $\mathcal{F}^+$  and a morphism of presheaves  $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$  with the property that for any other sheaf  $\mathcal{G}$  and morphism of presheaves  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ , there is a unique morphism of sheaves  $\psi : \mathcal{F}^+ \rightarrow \mathcal{G}$  making the following diagram commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ \theta \downarrow & \nearrow \exists! \psi & \\ \mathcal{F}^+ & & \end{array}$$

We have not yet constructed the sheafification map. In order to do so, we introduce another construction first.

**Definition 2.10 (Espace étalé).** Let  $\mathcal{F}$  be a presheaf on  $X$ . We define a topological space  $\text{Spé}(\mathcal{F})$  known as the *espace étalé* of the presheaf. The underlying set is

$$\text{Spé}(\mathcal{F}) = \bigsqcup_{P \in X} \mathcal{F}_P.$$

Let  $\pi : \text{Spé}(\mathcal{F}) \rightarrow X$  be the map that sends  $s \in \mathcal{F}_P$  to  $P$ . For every  $U \in \mathfrak{Top}(X)$  and  $s \in \mathcal{F}(U)$ , there is an induced map  $\bar{s} : U \rightarrow \text{Spé}(\mathcal{F})$  that sends  $P \mapsto s_P := [\langle U, s \rangle] \in \mathcal{F}_P$ . Note that  $\pi \circ \bar{s} = \text{id}_U$ .

The topology on  $\text{Spé}(\mathcal{F})$  is the finest topology such that for every  $U$  and every  $s \in \mathcal{F}(U)$ , the map  $\bar{s}$  is continuous.

**Proposition 2.11.** A basis for the topology on  $\text{Spé}(\mathcal{F})$  is given by the collection of sets of the form

$$\{(P, s_P) \mid P \in U\},$$

where  $U$  ranges over the open sets in  $X$  and  $s \in \mathcal{F}(U)$ .

*Proof.* Call the above collection of sets  $\mathcal{B}$ . First, we verify that this is a basis for a topology. Indeed, consider the intersection

$$\{(P, s_P) \mid P \in U\} \cap \{(Q, t_Q) \mid Q \in V\}.$$

If  $(p, \sigma)$  lies in the above intersection, then there is a neighborhood  $W$  of  $p$  contained in  $U \cap V$  such that  $s|_W = t|_W$ . Then, for any  $q \in W$ , we must have  $s_q = t_q$ . As a result,

$$\{(q, s_q) \mid q \in W\} \subseteq \{(P, s_P) \mid P \in U\} \cap \{(Q, t_Q) \mid Q \in V\}.$$

This establishes that the collection is indeed a basis.

Similarly, it is not hard to see that if  $\text{Spé}(\mathcal{F})$  is endowed with the topology generated by the above basis, then every map  $\bar{s}$  is continuous for every section  $s \in \mathcal{F}(U)$ . Hence, the topology on  $\text{Spé}(\mathcal{F})$  is finer than the topology generated by the above basis.

Let  $V \subseteq \text{Spé}(\mathcal{F})$  be an open set and let it contain a germ  $(P, s_P)$  where  $s \in \mathcal{F}(U)$  and  $U$  is a neighborhood of  $P$ . Let  $W = \bar{s}^{-1}(V)$ , which is a neighborhood of  $P$ . For every  $Q \in U \cap W$ ,  $\bar{s}(Q) \in V$ , and hence,  $(Q, s_Q) \in V$ . In particular,

$$\{(Q, s_Q) \mid Q \in U \cap W\} \subseteq V.$$

The conclusion follows. ■

**Corollary 2.12.** Let  $U$  be an open set in  $X$ . Then,  $\Gamma(U, \text{Spé}(\mathcal{F}))$ , the set of continuous sections, has the structure of an abelian group.

*Proof.* Let  $f, g : U \rightarrow \text{Spé}(\mathcal{F})$  be continuous maps. Define  $(f + g)(P) = f(P) + g(P) \in \mathcal{F}_P$ . Consider the basic open set  $B = \{(P, s_P) \mid P \in W\}$  where  $W \subseteq U$  is open and  $s \in \mathcal{F}(W)$ .

Now, let  $P \in (f + g)^{-1}(B)$ . Thus,  $f(P) + g(P) = s_P \in \mathcal{F}_P$ . Suppose  $f(P) = (P, t_P)$  and  $g(P) = (P, r_P)$ , where both  $t$  and  $r$  lie in  $\mathcal{F}(V)$  for a sufficiently small neighborhood  $V$  of  $P$  in  $U$ . Thus,  $t + r$  agrees with  $s$  on some neighborhood of  $P$  contained in  $V$ , whence,  $(f + g)^{-1}(B)$  contains a neighborhood around  $P$ . Thus, is open. This completes the proof. ■

**Remark 2.1.2.** From the above proof, we note that if  $f \in \Gamma(U, \text{Spé}(\mathcal{F}))$  and  $P \in U$ , then there is a neighborhood  $V$  of  $P$  contained in  $U$  and  $t \in \mathcal{F}(V)$  such that  $f(Q) = t_Q$  for every  $Q \in V$ . That is, every continuous section locally looks like the germs of a section in the presheaf.

**Proposition 2.13.** The sheafification map  $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$  exists and is unique up to a unique isomorphism.

*Proof.* Let  $\mathcal{F}^+(U) = \Gamma(U, \text{Spé}(\mathcal{F}))$ , the collection of all continuous sections  $U \rightarrow \text{Spé}(\mathcal{F})$ . If  $V \subseteq U$ , then there is an obvious map  $\Gamma(V, \text{Spé}(\mathcal{F})) \rightarrow \Gamma(U, \text{Spé}(\mathcal{F}))$  which restricts a section on  $V$  to that on  $U$ . It follows that  $\mathcal{F}^+$  is a presheaf. That the sheaf axioms are satisfied is trivial to check.

Define the map  $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$  by defining  $\theta_U : \mathcal{F}(U) \rightarrow \mathcal{F}^+(U)$  as  $s \mapsto \bar{s}$ , with the standard notation used previously. It is straightforward to check that this is indeed a morphism of presheaves.

It remains to verify that  $\theta$  has the universal property of sheafification. Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of presheaves where  $\mathcal{G}$  is a sheaf. Let  $U$  be open in  $X$  and  $f \in \mathcal{F}^+(U) = \Gamma(U, \text{Spé}(\mathcal{F}))$ . We must define  $\psi_U(f) \in \mathcal{G}(U)$ . We use the gluing axiom to do so.

For every point  $P \in U$ , there is a neighborhood  $V_P$  of  $P$  in  $U$  and a corresponding  $t^P \in \mathcal{F}(V_P)$  such that  $(t^P)_x = f(x)$  for all  $x \in V_P$ . Let  $Q \in U$ . Then, on  $W = V_P \cap V_Q$ , we see that  $\varphi_W(t^P) = \varphi_W(t^Q)$ . Hence, the gluing property applies and there is a  $t \in \mathcal{G}(U)$  such that  $t|_{V_P} = \varphi_{V_P}(t^P)$ . This defines a map of sheaves  $\psi : \mathcal{F}^+ \rightarrow \mathcal{G}$ . ■

**Proposition 2.14.** For all  $P \in X$ , there is an isomorphism  $\mathcal{F}_P \cong \mathcal{F}_P^+$ . That is, sheafification preserves stalks.

*Proof.* Define the map  $\alpha : \mathcal{F}_P \rightarrow \mathcal{F}_P^+$  that sends a germ  $[\langle U, s \rangle] \in \mathcal{F}_P$  to  $[\langle U, \bar{s} \rangle] \in \mathcal{F}_P^+$ . We first show that this is well defined. If  $[\langle U, s \rangle] = [\langle V, t \rangle]$ , then there is a neighborhood  $W$  of  $P$  contained in  $U \cap V$  such that  $s|_W = t|_W$ . Note that  $\bar{s} = \bar{t}$  on  $W$  and hence,  $[\langle W, \bar{s} \rangle] = [\langle W, \bar{t} \rangle]$ .

This map is obviously injective. To see surjectivity, simply recall that any section in  $\Gamma(U, \text{Spé}(\mathcal{F}))$  locally looks like  $\bar{s}$  on some neighborhood of  $P$  contained in  $U$ . This completes the proof. ■

**Definition 2.15 (Subsheaf).** A sheaf  $\mathcal{G}$  is said to be a *subsheaf* of a sheaf  $\mathcal{F}$  if for every open  $U \subseteq X$ ,  $\mathcal{G}(U) \subseteq \mathcal{F}(U)$  and the restriction maps of  $\mathcal{G}$  are induced by the restriction maps of  $\mathcal{F}$ .

**Proposition 2.16 (Kernel Sheaf).** Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves. Define the map  $\mathcal{F}' : \text{Top}(X) \rightarrow \mathbf{AbGrp}$  by  $\mathcal{F}'(U) = \ker \varphi_U \subseteq \mathcal{F}(U)$ . Then,  $\mathcal{F}'$  is a subsheaf of  $\mathcal{F}$  and is called the **kernel** of  $\varphi$  and is denoted by  $\ker \varphi$ .

*Proof.* Obviously,  $\mathcal{F}'$  is a presheaf with the restriction maps as those induced by  $\mathcal{F}$ . It remains to verify the sheaf axioms. The identity axiom is trivial to verify. Suppose now that  $\{V_i\}$  is an open cover of  $U$  and  $s_i \in \mathcal{F}'(V_i)$  such that  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ . Then, there is an  $s \in \mathcal{F}(U)$  such that  $s|_{V_i} = s_i$ . Note that  $\varphi_U(s)|_{V_i} = 0$  for every  $i$  and hence,  $\varphi_U(s) = 0$  from the identity axiom and it follows that  $s \in \mathcal{F}'(U)$ . This completes the proof. ■

**Proposition 2.17.**  $\ker \varphi_P \cong (\ker \varphi)_P$  for all  $P \in X$ . Therefore,  $\varphi$  is injective if and only if  $\varphi_P$  is injective for all  $P \in X$ .

*Proof.* ■

**Definition 2.18 (Image Sheaf).** Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves. Then, the map  $\mathcal{G}' : \text{Top}(X) \rightarrow \mathbf{AbGrp}$  given by  $\mathcal{G}'(U) = \text{im } \varphi_U$  is a presheaf on  $X$ . The *image sheaf* of  $\varphi$  is defined to be the sheafification of  $\mathcal{G}'$  and is denoted by  $\text{im } \varphi$ .

**Definition 2.19 (Injective/Surjective Morphisms).** A morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  of sheaves is said to be *injective* if  $\ker \varphi = 0$  and *surjective* if  $\text{im } \varphi = \mathcal{G}$ .

**Lemma 2.20.** Let  $\mathcal{F}, \mathcal{G}$  be presheaves on  $X$  and  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  a morphism such that  $\varphi_U$  is injective for every open subset  $U$  of  $X$ . Then,  $\varphi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}^+$  is also injective.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad} & \mathcal{G} \\ \downarrow & \searrow & \downarrow \\ \mathcal{F}^+ & \xrightarrow{\quad \exists! \quad} & \mathcal{G}^+ \end{array}$$

*Proof.* It suffices to check this at the level of stalks where it follows from the fact that sheafification preserves stalks. ■

**Remark 2.1.3.** From Lemma 2.20, we know that the map  $\text{im } \varphi \rightarrow \mathcal{G}$  is an injective morphism of sheaves and hence,  $\text{im } \varphi$  can be identified with a subsheaf of  $\mathcal{G}$ . Henceforth, we shall treat  $\text{im } \varphi$  as a subsheaf of  $\mathcal{G}$ .

**Definition 2.21 (Exact Sequence).** A sequence of morphisms

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

of sheaves is said to be *exact* at  $\mathcal{F}$  if  $\text{im } \varphi = \ker \psi$ . In general, a sequence of morphisms

$$\cdots \rightarrow \mathcal{F}_{i-1} \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_{i+1} \rightarrow \cdots$$

is said to be *exact* if it is exact at every  $\mathcal{F}_i$ .

**Proposition 2.22.**  $\text{im } \varphi_P = (\text{im } \varphi)_P$  for every  $P \in X$ . Therefore,  $\varphi$  is surjective if and only if  $\varphi_P$  is surjective for all  $P \in X$ .

*Proof.* Let  $\mathcal{H}$  denote the image presheaf of  $\varphi$  in  $\mathcal{G}$ . It is known that sheafification preserves stalks and thus,  $(\text{im } \varphi)_P = \mathcal{H}_P$ . It suffices to show that  $\mathcal{H}_P = \text{im } \varphi_P$ , which is obvious. Since any germ in  $\mathcal{H}_P$  is of the form  $[\langle U, t \rangle]$  where  $t$  is in the image of  $\varphi_U$ . On the other hand, an element in the image of  $\varphi_P$  is of the form  $[\langle U, \varphi_U(s) \rangle]$ . This completes the proof. ■

**Theorem 2.23.** A sequence of morphisms  $\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$  is exact if and only if for every  $P \in X$ , the induced sequence of abelian groups  $\mathcal{F}'_P \xrightarrow{\varphi_P} \mathcal{F}_P \xrightarrow{\psi_P} \mathcal{F}''_P$  is exact. That is, exactness can be checked at stalks.

*Proof.* The forward direction is obvious. Consider the converse. We have  $(\psi \circ \varphi)_P = \psi_P \circ \varphi_P = 0$  for every  $P \in X$ . Consequently,  $\text{im } \varphi_P \subseteq \ker \psi_P$  for all  $P \in X$  and hence,  $\text{im } \varphi \subseteq \ker \psi$ . Finally, note that since equality holds at each stalk, we must have  $\ker \psi = \text{im } \varphi$  and the sequence is exact. ■

**Lemma 2.24.** A morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  of sheaves is surjective if and only if for all  $U \subseteq X$  open and  $s \in \mathcal{G}(U)$ , there is a covering  $\{U_i\}$  of  $U$ , and there are elements  $t_i \in \mathcal{F}(U_i)$  such that  $\varphi_{U_i}(t_i) = s|_{U_i}$  for all  $i$ .

*Proof.* From Theorem 2.23,  $\varphi$  is surjective if and only if  $\varphi_P$  is surjective for all  $P \in X$ . Suppose  $\varphi$  is surjective. Let  $s \in \mathcal{F}(U)$ . Then, for every  $P \in U$ , there is a neighborhood  $U_P$  of  $P$  contained in  $U$  and  $t^P \in \mathcal{F}(U_P)$  such that  $\varphi_{U_P}(t^P) = s|_{U_P}$  (this follows from surjectivity of the stalk map).

Conversely, the given condition, precisely implies that the map  $\varphi_P$  is surjective at the level of stalks for all  $P \in X$ . This completes the proof. ■

**Example 2.25.** The following example shows that a surjective morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  of sheaves need not give rise to a surjective morphism  $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  of abelian groups for all open  $U \subseteq X$ .

Let  $X = \mathbb{C} \setminus \{0\}$ , and for an open  $U \subseteq X$ , let  $\mathcal{O}(U)$  denote the additive group of holomorphic functions on  $X$  and  $\mathcal{O}^*(X)$  the multiplicative group of nowhere vanishing holomorphic functions on  $X$ . Consider the map  $\exp : \mathcal{O} \rightarrow \mathcal{O}^*$  given by  $\exp_U(f) = e^{2\pi i f} \in \mathcal{O}^*(U)$ .

Note that the map on stalks  $\exp_P : \mathcal{O}_P \rightarrow \mathcal{O}^*_P$  is surjective since there is always a branch of the logarithm locally. Thus,  $\exp$  is a surjective morphism of sheaves. But note that setting  $U = X$  and taking  $\text{id}_X \in \mathcal{O}^*(X)$ , there is no function  $f \in \mathcal{O}(X)$  that maps to  $\text{id}_X$  under  $\exp_X$ .

**Definition 2.26 (Sheaf  $\mathcal{H}om$ ).** Let  $\mathcal{F}, \mathcal{G}$  be sheaves on  $X$ . The map  $U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$  is a sheaf on  $X$  known as the *sheaf of local morphisms of  $\mathcal{F}$  to  $\mathcal{G}$* , or “sheaf  $\mathcal{H}om$ ” for short and is denoted by  $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ .

**Remark 2.1.4.** We haven’t explicitly mentioned this but the restriction of a sheaf to an open subset is still a sheaf (the axioms can be verified easily).

**Proposition 2.27.**  $\mathcal{H}om(\mathcal{F}, \mathcal{G})$  is indeed a sheaf.

*Proof.* Obviously,  $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$  is an abelian group. ■

Complete this

**Definition 2.28 (Pushforward).** Let  $\mathcal{F}$  be a sheaf on  $X$  and  $f : X \rightarrow Y$  be a continuous function. Define the map  $f_*\mathcal{F} : \text{Top}(Y) \rightarrow \mathbf{AbGrp}$  by  $V \mapsto \mathcal{F}(f^{-1}V)$ .

**Proposition 2.29.** The pushforward is indeed a sheaf.

*Proof.* The pushforward is obviously a presheaf. Let  $\{V_i\}$  be an open cover of  $V$  and  $s \in f_*\mathcal{F}(V)$  such that  $s|_{V_i} = 0$  for every  $i$ . Note that  $s \in \mathcal{F}(f^{-1}V)$  and  $s|_{f^{-1}V_i} = 0$ , consequently,  $s = 0$ . Now suppose  $s_i \in f_*\mathcal{F}(V_i)$  such that  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ . Note that  $\{f^{-1}V_i\}$  forms an open cover of  $f^{-1}V$  and hence, there is  $s \in \mathcal{F}(f^{-1}V)$  such that  $s|_{f^{-1}V_i} = s_i$ . Hence,  $s \in f_*\mathcal{F}(V)$  such that  $s|_{V_i} = s_i$ . This completes the proof. ■

**Definition 2.30 (Section Functor).** Let  $U \subseteq X$  be an open set. Then, the map  $\mathcal{F} \mapsto \mathcal{F}(U)$  is functorial and is denoted by  $\Gamma(U, -)$ . This is the *section functor on  $U$* .

**Proposition 2.31.** Let  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  be a short exact sequence of sheaves. Then, the induced map

$$0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U)$$

is exact. That is,  $\Gamma(U, -)$  is a left exact functor.

*Proof.* The injectivity of  $\mathcal{F}'(U) \rightarrow \mathcal{F}(U)$  is immediate from the definition. Further, note that  $\varphi(\mathcal{F}')$  is already a subsheaf of  $\mathcal{F}$  since the map is injective. Therefore, exactness at  $\mathcal{F}(U)$  follows. This completes the proof. ■

**Definition 2.32 (Flasque Sheaves).** A sheaf  $\mathcal{F}$  is said to be *flasque* if whenever  $V \subseteq U$ , the restriction map  $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$  is surjective.

**Theorem 2.33.** Let  $\mathcal{F}, \mathcal{F}', \mathcal{F}''$  be sheaves on  $X$  and  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  an exact sequence of sheaves.

- (a) If  $\mathcal{F}'$  is flasque, then for any open  $U \subseteq X$ , the sequence  $0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$  is exact.
- (b) If  $\mathcal{F}'$  and  $\mathcal{F}$  are flasque, then so is  $\mathcal{F}''$ .
- (c) If  $f : X \rightarrow Y$  is a continuous map and  $\mathcal{F}$  is flasque, then  $f_*\mathcal{F}$  is a flasque sheaf on  $Y$ .

*Proof.* (a) First, note that the section functor  $\Gamma(U, -)$  is left exact and hence it suffices to show the surjectivity of the induced map  $\mathcal{F}(U) \rightarrow \mathcal{F}''(U)$ . Let  $s \in \mathcal{F}''(U)$ . Then, due to Lemma 2.24, there is an open cover  $\{U_i\}_{i \in I}$  of  $U$  and  $t_i \in \mathcal{F}(U_i)$  such that  $\psi_{U_i}(t_i) = s|_{U_i}$ . Consider the collection of all pairs  $\{(J, t_J)\}$  where  $J \subseteq I$  and  $t_J \in \mathcal{F}(\bigcup_{j \in J} U_j)$  such that  $t_J|_{U_j} = t_j$  for every  $j \in J$ . This has the structure of a poset with  $(J, t_J) \leq (J', t_{J'})$  if  $J \subseteq J'$  and  $t_{J'}$  restricted to  $\bigcup_{j \in J} U_j$  is equal to  $t_J$ .

Using the gluability axiom, it follows that every chain in the aforementioned poset has a maximal element, say  $(K, t_K)$ . Let  $V = \bigcup_{k \in K} U_k$ . We contend that  $V = U$ . Suppose not, then there is a  $U_i$  that is not contained in  $V$ . Let  $W = V \cap U_i$ . The images under  $\psi_W$  of  $t_K|_W$  and  $t_i|_W$  are the same. Therefore,  $t_K|_W - t_i|_W \in \ker g_W = \mathcal{F}'(W)$ . Since  $\mathcal{F}'$  is flasque, there is a  $z \in \mathcal{F}'(U_i)$  such that  $\varphi_{U_i}(z)|_W = t_K|_W - t_i|_W$ . Set  $y = \varphi_{U_i}(z)$  and  $t'_i = t_i + y \in \mathcal{F}(U_i)$ , then,  $t'_i|_W = t_K|_W$  and hence, there is a corresponding  $t^* \in \mathcal{F}(V \cup U_i)$  that restricts to  $t_K$  and  $t'_i$  on  $V$  and  $U_i$  respectively. Therefore,  $(K \cup \{i\}, t^*) \geq (K, t_K)$  thereby contradicting the maximality of  $(K, t_K)$  and hence,  $V = U$  and the conclusion follows.

(b) Let  $U \subseteq V$  be open subsets of  $X$ . Then, there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) \longrightarrow 0 \end{array}$$

where the first two vertical maps are surjective. It follows from the Snake Lemma that so is the third.

(c) Obvious. ■

## 2.2 Schemes

### 2.2.1 Spectrum of a Ring

**Definition 2.34 (Spectrum of a Ring).** Define

$$\text{Spec}(A) := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

This is equipped with the *Zariski Topology* in which the closed sets are given by

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The basic open sets for this topology are given by

$$D(f) := \{\mathfrak{p} \in \text{Spec}(A) \mid f \notin \mathfrak{p}\} = \text{Spec}(A) \setminus V(f).$$

For a subset  $S \subseteq \text{Spec}(A)$ , let

$$\mathcal{I}(S) := \bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq A.$$

Note that  $\mathcal{J}(S)$  is always a radical ideal.

**Proposition 2.35.** *Let  $S \subseteq \text{Spec}(A)$  and  $\mathfrak{a} \trianglelefteq A$  be a prime ideal.*

- (a)  $V(\mathcal{J}(S)) = \overline{S}$ .
- (b)  $\mathcal{J}(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ . This gives an inclusion-reversing bijection between the closed subsets of  $\text{Spec}(A)$  and radical ideals of  $A$ .
- (c) Under the aforementioned bijection, closed irreducible subsets correspond to prime ideals.

*Proof.* (a) Straightforward.

(b) Follows from the fact that

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}.$$

(c) Suppose  $S \subseteq \text{Spec}(A)$  is closed and irreducible. Let  $\mathfrak{a} = \mathcal{J}(S)$ , which is a radical ideal. Suppose  $f, g \in A \setminus \mathfrak{a}$  with  $fg \in \mathfrak{a}$ . Then,  $fg \in \mathfrak{p}$  for every  $\mathfrak{p} \in S$ . Then,  $D(fg) \cap S = \emptyset$  while  $D(f) \cap S$  and  $D(g) \cap S$  are not empty, a contradiction, since  $S$  is irreducible. Thus,  $\mathfrak{a}$  is prime.

Conversely, suppose  $S = V(\mathfrak{p})$ , where  $\mathfrak{p}$  is a prime ideal and  $S = S_1 \cup S_2$  where  $S_1 = V(\mathfrak{a})$  and  $S_2 = V(\mathfrak{b})$  where  $\mathfrak{a}, \mathfrak{b} \supseteq \mathfrak{p}$ . The equality gives us  $\mathfrak{a}\mathfrak{b} = \mathfrak{p}$ , implying that  $\mathfrak{p}$  contains either  $\mathfrak{a}$  or  $\mathfrak{b}$  whence  $S = S_1$  or  $S = S_2$ , whereby  $S$  is irreducible. ■

**Lemma 2.36.** *Let  $f, g \in A$ . Then,  $D(f) \subseteq D(g)$  if and only if  $f^n \in (g)$ , if and only if  $g$  is invertible in  $A_f$ .*

*Proof.* Suppose  $D(f) \subseteq D(g)$ . Then,  $V(f) \supseteq V(g)$ . Now,

$$\sqrt{(g)} = \bigcap_{\mathfrak{p} \in V(g)} \mathfrak{p} \ni f.$$

The converse is trivial. This proves the equivalence of the first two statements. The equivalence of the second and third statements is obvious. ■

**Definition 2.37 (Irreducible Component).** An *irreducible component* of a topological space is a maximal irreducible subset. These are obviously closed subspaces of the ambient topological space.

**Proposition 2.38.** *Every point  $p \in X$  is contained in an irreducible component of  $X$ .*

*Proof.* Standard application of Zorn's Lemma. ■

**Proposition 2.39.** *The irreducible components of  $\text{Spec}(A)$  are in bijection with the minimal primes of  $A$ .*

*Proof.* There is an inclusion-reversing bijection between the closed irreducible subsets of  $\text{Spec}(A)$  and the prime ideals of  $A$ . It is then obvious that the irreducible components of  $\text{Spec}(A)$  correspond to the minimal primes of  $A$ . ■

## 2.2.2 The Structure Sheaf