Algebraic Topology

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Contents

Chapter 0

Homological Algebra

This chapter is mainly taken from [Wei94].

0.1 Basic Definitions

Definition 0.1. A *chain complex* C of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with R-module homomorphisms $d_n : C_n \to C_{n-1}$ such that the composition $d_n \circ d_{n-1} = 0$ for each $n \in \mathbb{Z}$. Define the *n-th homology module* of C to be

$$H_n(C) := \ker(d_n)/\operatorname{im}(d_{n+1}).$$

A *morphism* of chain complexes $u : C \to D$ is a collection of R-module homomorphisms $u_n : C_n \to D_n$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & \cdots . \\
\cdots \arrow[r, "d_{n+2}"'] & D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & \cdots
\end{tikzcd}$$

We denote the category of chain complexes of R-modules by $\mathbf{Ch}(R-\mathbf{Mod})$.

Proposition 0.2. *A morphism $u : C \to D$ of chain complexes induces a sequence of R-module homomorphisms between the homology modules, denoted by u_*.*

Proof. ■

Definition 0.3. An **Ab**-category is a locally small category $\mathscr{A}$ in which $\operatorname{Hom}(A,B)$ has the structure of an abelian group for all $A, B \in \mathscr{A}$ and composition of morphisms distributes over addition. That is, given a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "g"'] & C \arrow[r, "h"] & D ,
\end{tikzcd}$$

we have $h \circ (g + g') \circ f = h \circ g \circ f + h \circ g' \circ f$ in $\operatorname{Hom}(A, D)$.

An *additive functor* $F : \mathscr{A} \to \mathscr{B}$ is a functor between **Ab**-categories such that the induced map

$\text{Hom}(A, A') \to \text{Hom}(FA, FA')$ is a group homomorphism.

An *additive category* is an **Ab**-category $\mathscr{A}$ with a null (zero) object and a product $A \times B$ for every pair $A, B \in \mathscr{A}$.

Proposition 0.4. *In an additive category, finite products are the same as finite coproducts.*

Proof. Let $A, B \in \mathscr{A}$ have a product $A \times B \in \mathscr{A}$ with maps $p : A \times B \to A$ and $q : A \times B \to B$. Consider the pair of maps $\mathbf{id}_A : A \to A$ and $0 : A \to B$. This induces a unique map $i : A \to A \times B$ such that $p \circ i = \mathbf{id}_A$ and $q \circ i = 0$. Similarly, there is a map $j : B \to A \times B$ such that $p \circ j = 0$ and $q \circ j = \mathbf{id}_B$. Note that

$$p \circ (i \circ p + j \circ q) = p \qquad q \circ (i \circ p + j \circ q) = q$$

whence $i \circ p + j \circ q = \mathbf{id}_{A \times B}$.

We contend that the pair (i, j) defines a coproduct of A, B. Indeed, if $D \in \mathscr{A}$ with maps $f : A \to D$ and $g : B \to D$, set $d = f \circ p + g \circ q : A \times B \to D$. We have

$$d \circ i = (f \circ p + g \circ q) \circ i = f \circ p \circ i + g \circ q \circ i = f$$

and similarly, $d \circ j = g$. It now remains to show the uniqueness of d. Suppose $d' : A \times B \to D$ is a morphism, then

$$(d - d') \circ \mathbf{id}_{A \times B} = (d - d') \circ (i \circ p + j \circ q) = 0$$

whereby completing the proof. ■

Definition 0.5 (Kernel, Cokernel). Let $\mathscr{A}$ be an additive category. A *kernel* of a morphism $f : B \to C$ is defined to be a map $i : A \to B$ such that $f \circ i = 0$ and for every other morphism $j : D \to B$ with $f \circ j = 0$, there is a unique morphism $u : D \to A$ such that $j = i \circ u$. This is expressed in the following diagram.

$$\begin{tikzcd}
 & B \arrow[dr, "f"] & \\
 & A \arrow[u, "i"] \arrow[r, "0"] & C \\
D \arrow[uur, "j"] \arrow[ur, dotted, "\exists! u"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

Similarly, a *cokernel* of $f : B \to C$ is defined to be a map $p : C \to A$ such that $p \circ f = 0$ and for any morphism $q : C \to D$ with $q \circ f = 0$, there is a unique map $u : A \to D$ such that $q = u \circ p$. This is expressed in the following diagram.

$$\begin{tikzcd}
 & B \arrow[d, "0"] \arrow[dr, "f"] \arrow[ddl, "0"'] & \\
 & A \arrow[dl, dotted, "\exists! u"'] & C \arrow[l, "p"] \arrow[dll, "q"] \\
D & &
\end{tikzcd}$$

Proposition 0.6. *A kernel is always monic and a cokernel is always epic.*

Proof. ■

Definition 0.7. An *abelian category* is an additive category $\mathscr{A}$ such that

1. every morphism in $\mathscr{A}$ has a kernel and a cokernel,
2. every monic in $\mathscr{A}$ is the kernel of its cokernel and
3. every epi in $\mathscr{A}$ is the cokernel of its kernel.

Theorem 0.8 (Freyd-Mitchell Embedding Theorem). *Let $\mathscr{A}$ be a small abelian category. Then, there is a ring R and a full, faithful and exact functor $F : \mathscr{A} \to R - \mathbf{Mod}$.*

In particular, what this means is that in all diagram chases involving objects in a general abelian category, we may treat the objects as elements in $R - \mathbf{Mod}$ for some ring R, which makes our life much easier.

Definition 0.9. Let C and D be chain complexes. Two chain maps $f, g : C \to D$ are said to be *chain homotopic* if there are R-module homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$f_n - g_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n.$$

Proposition 0.10. *If $f, g : C \to D$ are chain homotopic, then $f_* = g_*$.*

Proof. ■

0.1.1 Some Diagram Chasing

Theorem 0.11 (Snake Lemma). *Let A, B, C, A', B', C' be R-modules that fit into the following commutative diagram*

$$\begin{tikzcd}
 & \ker\alpha \arrow[r, dotted] \arrow[d] & \ker\beta \arrow[r, dotted] \arrow[d] & \ker\gamma \arrow[d] & \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}\alpha \arrow[r, dotted] & \operatorname{coker}\beta \arrow[r, dotted] & \operatorname{coker}\gamma &
\end{tikzcd}$$

with exact rows. Then, there is a map $\partial : \ker\gamma \to \operatorname{coker}\alpha$ which makes the induced sequence

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\partial} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$$

exact. Further, if f is injective, then so is the induced map $\ker\alpha \to \ker\beta$ and if g' is surjective, then so is the induced map $\operatorname{coker}\beta \to \operatorname{coker}\gamma$.

Proof. ■

Corollary 0.12 (Five Lemma). Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\eta"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

with exact rows.

(a) If β, δ are injective and α is surjective, then γ is injective.

(b) If β, δ are surjective and η is injective, then γ is surjective.

Theorem 0.13. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of chain complexes. Then, there is a long exact sequence of homology groups given by*

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$$

Proof. Note that the kernel of the map $d : A_n \to Z_{n-1}A$ contains $d(A_{n-1})$, therefore, we have an induced map $\widetilde{d} : A_n/d(A_{n+1}) \to Z_{n-1}A$ given by $d(a_n + d(A_{n+1})) = d(a_n)$ for all $a_n \in A_n$. Note that $\ker \widetilde{d} = H_n(A)$ and $\operatorname{coker} \widetilde{d} = H_{n-1}(A)$. Similarly, define $\widetilde{d}$ for the chain complexes B and C.

We now have a commutative diagram

$$\begin{tikzcd} & A_n/d(A_{n+1}) \arrow[r] \arrow[d, "\widetilde{d}"'] & B_n/d(B_{n+1}) \arrow[r] \arrow[d, "\widetilde{d}"'] & C_n/d(C_{n+1}) \arrow[r] \arrow[d, "\widetilde{d}"'] & 0 \\ 0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) & \end{tikzcd}$$

with exact rows. The conclusion now follows from Theorem 0.11 ∎

0.2 Derived Functors

Chapter 1

Topological Preliminaries

1.1 Cell Complexes

Definition 1.1 (Cell Complex). Cell complexes are constructed using an inductive procedure.

(a) Begin with a discrete set X^0, whose points are regarded as 0-cells.

(b) Inductively, form the *n-skeleton* X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} = \partial e^n_\alpha \to X^{n-1}$.

(c) This inductive process can either be stopped at a finite stage or continued indefinitely, setting $X = \bigcup_{n=1}^{\infty} X^n$. In the latter case, X is given the *weak topology*.

Example 1.2 (Real Projective Space $\mathbb{R}\mathrm{P}^n$). Recall that $\mathbb{R}\mathrm{P}^n$ is defined as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the identification $x \sim \lambda x$. This can equivalently be thought of as the hemisphere D^n with the identification $x \sim -x$ for $\partial D^n = S^{n-1}$. Under this identification, S^{n-1} quotients to $\mathbb{R}\mathrm{P}^{n-1}$ whereby, $\mathbb{R}\mathrm{P}^n$ is obtained by simply attaching an n-cell to $\mathbb{R}\mathrm{P}^{n-1}$ through the quotient map $\varphi : S^{n-1} = \partial D^n \to \mathbb{R}\mathrm{P}^{n-1}$. Thus, the cell complex structure of $\mathbb{R}\mathrm{P}^n$ is $e^0 \cup e^1 \cup \cdots \cup e^n$, i.e. one cell in each dimension $0 \leq i \leq n$.

Example 1.3 (Complex Projective Plane $\mathbb{C}\mathrm{P}^n$).

Definition 1.4. A *subcomplex* of a cell complex X is a closed subspace $A \subseteq X$ that is a union of cells of x. A pair (X, A) consisting of a cell complex X and a subcomplex A is called a *CW pair*.

Remark 1.1.1. *The property of A being a subcomplex depends on the CW structure of X. For example, S^{n-1} is not a subcomplex of S^n with the natural CW structure obtained by gluing two D^n's. But, we may choose a different CW structure for S^n wherein we begin with the equitorial S^{n-1} and attach two D^n's to it, via the obvious boundary map. In this case, S^{n-1} is indeed a subcomplex of S^n.*

1.2 Homotopy Extension Property

Definition 1.5 (Homotopy Extension Property). A pair (X, A) with $A \subseteq X$ is said to have the *homotopy extension property* if for any topological space Y, a map $f_0 : X \to Y$ and a homotopy $H : A \times I \to Y$ such that $H|_{A\times\{0\}} = f_0|_A$, there is an extension of H, $\widetilde{H} : X \times I \to Y$ with $H|_{X\times\{0\}} = f_0$.

Proposition 1.6. *A pair (X, A) with A closed*[a] *in X has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a* retract *of $X \times I$.*

[a]This is superfluous

Proof. Suppose (X, A) has the homotopy extension property. Consider the identity map $\mathbf{id} : X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$. This may be extended to a map $f : X \times I \to X \times \{0\} \cup A \times I$ which restricts to the identity map on $X \times \{0\} \cup A \times I$. This shows that the latter is a retract of the former. ■

Proposition 1.7. *If (X, A) is a CW-pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$ whereby (X, A) has the homotopy extension property.*

Proof. ■

(X, A) CW-pair has homotopy ext property

Proposition 1.8. *If the pair (X, A) has the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Proof. Since A is contractible, there is a homotopy between the inclusion $A \hookrightarrow X$ and the constant map on A. Due to the homotopy extension property, this can be extended to a homotopy $F : X \times I \to X$ such that $F|_{X\times\{0\}} = \mathbf{id}_X$. Let $q : X \twoheadrightarrow X/A$ denote the quotient map and $\widetilde{q} : X \times I \to X/A \times I$ denote the quotient map with the obvious identification.

Consider the composition $q \circ F$. Then, for $a, a' \in A$, $q \circ F(a, t) = q \circ F(a', t)$ for all t whereby this induces a continuous map $\widetilde{F} : X/A \times I \to X/A$. Let $f_1 := F|_{X\times\{1\}}$ and $\widetilde{f}_1 = \widetilde{F}|_{X\times\{1\}}$. Then, f_1 maps all of A to a single point whence it induces a map $g : X/A \to X$ such that $f_1 = g \circ q$.

We contend that $\widetilde{f}_1 = q \circ g$. Indeed, for any $\overline{x} \in X/A$, there is $x \in X$ such that

$$q \circ g(\overline{x}) = q \circ g \circ q(x) = q \circ f_1(x) = \widetilde{f}_1 \circ q(x) = \widetilde{f}_1(\overline{x}).$$

This shows that $g \circ q = f_1 \simeq \mathbf{id}_X$ through F while $q \circ g = \widetilde{f}_1 \simeq \mathbf{id}_{X/A}$ through $\widetilde{F}$ whence the conclusion follows. ■

Corollary 1.9. If (X, A) is a CW-pair with A contractible, then the quotient map $q : X \to A/A$ is a homotopy equivalence.

Example 1.10.

Chapter 2

The Fundamental Group

2.1 Fundamental Groupoid and Group

Definition 2.1 (Homotopy). Let X and Y be topological spaces. A homotopy is a continuous function $H : X \times I \to Y$. A *homotopy* between two functions $f, g : X \to Y$ is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Definition 2.2 (Homotopy of Paths). Let X be a topological space and $f, g : I \to X$ be paths. Then, f and g are said to be *path homotopic* if there is a continuous function $H : I \times I \to X$ such that $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ for all $s \in I$. We denote this by $f \simeq_p g$.

Proposition 2.3. *The relation $\simeq$ on the set of all paths in X is an equivalence relation.*

Proposition 2.4. *Let $f : I \to X$ be a path and $\varphi : I \to I$ be a continuous function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then, $f \simeq_p f \circ \varphi$.*

Proof. Define the function $\Phi : I \times I \to X$ by

$$\Phi(s, t) = f(t\varphi(s) + (1 - t)s)$$

It is not hard to see that Φ is a path homotopy between f and $f \circ \varphi$. ∎

Consider the set of all equivalence classes of paths in X under the equivalence relation $\simeq_p$. Define the operation $*$ on pairs of equivalence classes $[f]$ and $[g]$ where $f(1) = g(0)$ by

$$[f] * [g] = [f * g]$$

where

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t - 1) & 1/2 < t \leq 1 \end{cases}$$

Proposition 2.5. *The operation $*$ is associative. That is,*

$$[f]*([g]*[h]) = ([f]*[g])*h$$

Proof. Note that $[f]*([g]*[h])$ is the equivalence class containing the path:

$$\alpha(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(4t-2) & 1/2 < t \le 3/4 \\ h(4t-3) & 3/4 < t \le 1 \end{cases}$$

Consider the piecewise linear function $\varphi : [0,1] \to [0,1]$ that maps $[0,1/2]$ to $[0,1/4]$, $[1/2,3/4]$ to $[1/4,1/2]$ and $[1/2,1]$ to $[3/4,1]$, then through $\alpha \circ \varphi$, the conclusion follows. ∎

Proposition 2.6. *Let $f : I \to X$ be a path and $\overline{f} : I \to X$ be given by $\overline{f}(t) = f(1-t)$ for $t \in I$. Then $f * \overline{f}$ is path homotopic to the constant path at x_0.*

Proof. Define the homotopy

$$H(s,t) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2}(1-t) \\ f(1-t) & \frac{1}{2}(1-t) \le s \le \frac{1}{2}(1+t) \\ \overline{f}(2s-1) & \frac{1}{2}(1+t) \le s \le 1 \end{cases}.$$

Then, $H(\cdot,0) = f * \overline{f}$ and $H(\cdot,1)$ is the constant path at x_0 which completes the proof. ∎

Definition 2.7 (Fundamental Group). Let $\pi_1(X,x_0)$ be the set of equivalence classes of paths $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$. Then, $\pi_1(X,x_0)$ forms a group under the operation $*$. This is known as the *fundamental group* of X based at x_0.

Let $\mathbf{Top}_*$ denote the category of pointed topological spaces, that is, the category wherein objects are pairs (X,x_0) where $x_0 \in X$ and a morphism $f : (X,x_0) \to (Y,y_0)$ is a continuous map $f : X \to Y$ with $f(x_0) = y_0$.

Proposition 2.8. *Let $f : (X,x_0) \to (Y,y_0)$ be a morphism in $\mathbf{Top}_*$. Then, the map $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ given by $[\alpha] \mapsto [f \circ \alpha]$ is a homomorphism of groups. Further, if*

$$(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (Z,z_0)$$

then $(g \circ f)_* = g_* \circ f_*$.

Proof. If H is a path homotopy between α_1 and α_2 in X, then $f \circ H$ is a homotopy between $f \circ \alpha_1$ and $f \circ \alpha_2$ in Y. Thus, the map f_* is well defined. Next, suppose $[\alpha], [\beta] \in \pi_1(X,x_0)$, then, it is not hard to see that $(f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta)$, consequently, f_* is a homomorphism of groups. The final assertion is obvious from the definition. ∎

As a result, we see that π_1 is a (covariant) functor from $\mathbf{Top}_*$ to $\mathbf{Grp}$.

Theorem 2.9. *Let X be path connected and $x_0, x_1 \in X$. Let $\alpha : I \to X$ be a path from x_0 to x_1. Then, the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $[f] \mapsto [\overline{\alpha} * f * \alpha]$ is a group isomorphism.*

Proof. It is easy to see that $\hat{\alpha}$ is a homomorphism. The surjectivity and injectivity of this map are obvious. ∎

Proposition 2.10. *Let X be path connected and $h : X \to Y$ be a continuous map. If $x_0, x_1 \in X$ with $\alpha : I \to X$ a path between them and $\beta = h \circ \alpha$, then we have the following commutative diagram:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(h_{x_0})_*"] \arrow[d, "\hat{\alpha}"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \\
\pi_1(X, x_1) \arrow[r, "(h_{x_1})_*"] & \pi_1(Y, y_1)
\end{tikzcd}$$

Proof. Let $[f] \in \pi_1(X, x_0)$. Then,

$$\hat{\beta} \circ (h_{x_0})_*([f]) = \hat{\beta}([h \circ f]) = [\overline{\beta} * h \circ f * \beta]$$

and

$$(h_{x_1})_* \circ \hat{\alpha}([f]) = (h_{x_1})_*([\overline{\alpha} * f * \alpha]) = [\overline{\beta} * h \circ f * \beta]$$

This completes the proof. ∎

2.2 Retracts and Deformation Retracts

Definition 2.11. A *retraction* of a space X onto a subspace A is a map $r : X \to X$ such that $\operatorname{im}(r) = A$ and $r|_A = \mathbf{id}_A$. A *deformation retraction* is a homotopy $H : X \times I \to X$ between $\mathbf{id}_X$ and a retraction r from X onto A. That is, $H|_{X \times \{0\}} = \mathbf{id}_X$ and $H|_{X \times \{1\}} = r$. A deformation retract is said to be *strong* if $H|_{A \times \{t\}} = \mathbf{id}_A$ for all $t \in I$.

Proposition 2.12. *If a space X retracts onto a subspace A and $x_0 \in A$, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \to X$ is injective. If A isa deformation retract of X, then i_* is an isomorphism.*

Proof. Let $r : X \to A$ denote the (restriction of the) retraction of X onto A. Then, $r \circ i = \mathbf{id}_A$ whence $r_* \circ i_* = \mathbf{id}_{\pi_1(A, x_0)}$ whence i_* must be injective.

Now, let $H : X \times I \to X$ be a deformation retraction of X onto A. It suffices to show that i_* is surjective. Indeed, let $f : I \to X$ be a loopt based at x_0. Then, the map $\Phi : I \times I \to X$ given by $\Phi(s, t) = H(f(s), t)$ is a path homotopy between f and $g = H|_{I \times \{1\}}$. Since $g \in \pi_1(A, x_0)$, we see that i_* must be surjective. ∎

Definition 2.13 (Homotopy Equivalence). A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there is a continuous map $g : Y \to X$ such that $g \circ f$ is homotopic to $\mathbf{id}_X$ and $f \circ g$ is homotopic to $\mathbf{id}_Y$.

2.3 van Kampen's Theorem

The following formulation has been taken from [Mun02]

Theorem 2.14 (van Kampen). *Let $X = U \cup V$ where U and V are open in X. Further, suppose U, V and $U \cap V$ are nonempty and path connected. Let H be a group, $x_0 \in U \cap V$ and*

$$\phi_1 : \pi_1(U, x_0) \to H \qquad \phi_2 : \pi_1(V, x_0) \to H$$

be homomorphisms. Finally, let i_1, i_2, j_1, j_2 be the homomorphisms of fundamental groups induced by inclusion maps. Then, there is a unique map $\Phi : \pi_1(X, x_0) \to H$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dotted, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Notice how the diagram resembles that of a pushout in a general category and hence, has the universal property and hence, the object, if it exists is unique up to a unique isomorphism. In the special case that $U \cap V$ is simply connected, that is, has a trivial fundamental group, the commutative diagram reduces to that of a coproduct. And it is well known that the coproduct in the category of groups is the free product.

Proof. Let $\mathscr{L}(U, x_0), \mathscr{L}(V, x_0), \mathscr{L}(U \cap V, x_0)$ denote the set of loops in U, V and $U \cap V$. The path homotopy class of a path f in X, U, V and $U \cap V$ is denoted by $[f]$, $[f]_U$, $[f]_V$ and $[f]_{U\cap V}$ respectively. The proof proceeds in multiple steps. The main idea is to first define a set map ρ on the set of loops contained completely in either U or V, then extend it to a set map σ on the set of paths contained completely in either U or V and finally extend it to a set map τ on the set of all paths in X.

Once the map τ is defined, we shall show that $\tau(f) = \tau(g)$ whenever $f \simeq_p g$ and therefore, τ would descend to a group homomorphism from $\pi_1(X, x_0)$ to H.

Step 1: Defining the set map $\rho : \mathscr{L}(U, x_0) \cup \mathscr{L}(V, x_0) \to H$.

This has quite a natural definition:

$$\rho(f) = \begin{cases} \phi_1([f]_U) & f \text{ is contained completely in } U \\ \phi_2([f]_V) & f \text{ is contained completely in } V \end{cases}$$

For a loop contained in $U \cap V$, the map ρ is well defined due to the commutativity of the diagram. It is not hard to see that if $f, g \in \mathscr{L}(U, x_0)$, then $\rho(f * g) = \rho(f)\rho(g)$.

Step 2: Extend the map ρ to a map $\sigma : \mathscr{P}(U) \cup \mathscr{P}(V) \to H$.

For each $x \in X$, fix a path α_x from x_0 to x such that whenever x lies in U, V or $U \cap V$, α_x lies completely in U, V or $U \cap V$ respectively.

Let f be a path from x_1 to x_2 that lies completely in U or completely in V. Define

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})$$

Now, let f and g be paths completely contained in U. If $f \simeq_p g$ in U, then $\alpha_{x_1} * f * \alpha_{x_2}^{-1} \simeq_p \alpha_{x_1} * g * \alpha_{x_2}^{-1}$ in U and from the definition of ρ, we see that

$$\sigma(f) = \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1}) = \rho(\alpha_{x_1} * g * \alpha_{x_2}^{-1}) = \sigma(g)$$

Next, if f is a path from x_1 to x_2 and g is a path from x_2 to x_3 (both contained in U), then

$$\begin{aligned}\sigma(f * g) &= \rho(\alpha_{x_1} * f * g * \alpha_{x_3}^{-1}) \\ &= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1} * \alpha_{x_2} * g * \alpha_{x_3}^{-1}) \\ &= \rho(\alpha_{x_1} * f * \alpha_{x_2}^{-1})\rho(\alpha_{x_2} * g * \alpha_{x_3}^{-1}) = \sigma(f)\sigma(g)\end{aligned}$$

Step 3: Extend the map σ to a map $\tau : \mathscr{P}(X) \to H$

Let $f : I \to X$ be a path. It is not hard to argue, using Lebesgue's Number Lemma, that there is a mesh δ such that for every partition $0 = s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ of $[0, 1]$ with mesh less than δ, $f([s_i, s_{i+1}])$ is completely contained in either U or V for $0 \le i \le n - 1$.

Denote by f_i, the restriction of f to $[s_i, s_{i+1}]$. Define

$$\tau(f, P) = \sigma(f_0) \cdots \sigma(f_{n-1})$$

We contend that the map $\tau(f, P)$ is independent of the partition chosen, so long as its mesh is less than δ. To do so, we first show that refining a partition with mesh less than δ does not change the image under τ, for which, it suffices to show that adding a single point to the partition does not change the image. Indeed, let $c \in (s_i, s_{i+1})$ be added to the partition. But since $f([s_i, c])$ and $f([c, s_{i+1}])$ lie completely either in U or in V, we have that $\sigma(f|_{[s_i,c]})\sigma(f|_{[c,s_{i+1}]}) = \sigma(f|_{[s_i,s_{i+1}]})$ whence the conclusion follows.

Now, let P_1 and P_2 be two partitions of $[0, 1]$ with mesh less than δ. Then $P_1 \cup P_2$ is a partition that refines both P_1 and P_2, consequently,

$$\tau(f, P_1) = \tau(f, P_1 \cup P_2) = \tau(f, P_2)$$

which establishes our claim.

Step 4: If $f \simeq_p g$ in X, then $\tau(f) = \tau(g)$.

Let $F : I \times I \to X$ be a path homotopy between f and g. Using the Lebesgue Number Lemma, there are partitions $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $f([s_i, s_{i+1}] \times [t_i, t_{i+1}])$ is completely contained in either U or V.

Step 5: $\tau(f * g) = \tau(f)\tau(g)$

Let P be a partition of $f * g$ such that $(f * g)([s_i, s_{i+1}])$ is completely contained in either U or V. Define $P^* = P \cup \{1/2\}$. It is not hard to see, using P^* that τ is multiplicative.

Step 6: Constructing the homomorphism Φ.

Restrict the map τ to $\tau : \mathscr{L}(X, x_0) \to H$. From **Step 4**, it follows that there is a map $\Phi : \pi_1(X, x_0) \to H$ and from **Step 5**, we get that Φ is a homomorphism.

The above argument establishes the existence of a group homomorphism $\Phi : \pi_1(X, x_0) \to H$ making the diagram commute. We must now show that the map Φ is unique. But this follows from the fact that the generators of Φ are precisely the images of the generators of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ under the homomorphisms j_1 and j_2 respectively. ■

2.3.1 Alternate Formulation of van Kampen's Theorem

The following formulation and proof has been taken from [Hat00]. The upshot of this formulation is that it gives a recipe for computing the presentation of the fundamental group which is hard to see from the previous formulation.

Let X be a topological space and $\{A_\alpha\}_{\alpha \in J}$ be an open cover of path connected subspaces of X. Let $x_0 \in X$ be a basepoint such that $x_0 \in A_\alpha$ for each $\alpha \in J$. The inclusion $A_\alpha \hookrightarrow X$ induces a group homomorphism

$j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ where we have dropped the basepoint to avoid clutter. Similarly, the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ induces a group homomorphism $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$.

Due to the Universal Property of Free Products, the group homomorphisms j_α induce a group homomorphism

$$\Phi : *_{\alpha\in J}\pi_1(A_\alpha) \to \pi_1(X).$$

Proposition 2.15. *If each intersection $A_\alpha \cap A_\beta$ is path connected, then Φ is surjective.*

Sketch of Proof. The proof of surjectivity follows the same proof of **Step2** in the proof of Theorem 2.14. It suffices to show that any element in $\pi_1(X)$ can be represented as the product of finitely many elements of $j_\alpha\left(\pi_1(A_\alpha)\right)$.

Take any loop $f : I \to X$ based at x_0 and then using the Lebesgue Number Lemma, find a partition $0 \le t_0 < \cdots < t_n = 1$ of I such that the image $f([t_i, t_{i+1}])$ is completely contained in some A_α for each $0 \le i \le n-1$. Now, join the endpoints of each such path to x_0, which can be done since each $A_\alpha \cap A_\beta$ is path connected. This immediately gives us a decomposition of $[f]$. ■

Proposition 2.16. *If in addition to the hypothesis of , each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of the surjection Φ is generated by the set*

$$\left\{ i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \middle|\, \omega \in \pi_1(A_\alpha, A_\beta) \text{ for all } \alpha, \beta \in J \right\}.$$

Proof. ■

Proof of general van kampen

Chapter 3

Covering Spaces

Definition 3.1 (Covering Space). A covering space of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the condition that there is an open cover $\{U_\alpha\}$ of X such that for each $\alpha \in J$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically by p to U_α.

Notice that for each $x \in X$, the subspace $p^{-1}(x)$ of $\widetilde{X}$ has the discrete topology.

Proposition 3.2. *Let $p : \widetilde{X} \to X$ be a covering map where X is connected. If for some $x \in X$, $|p^{-1}(x)| = n \in \mathbb{N}$, then for all $x' \in X$, $|p^{-1}(x')| = n$.*

Proof. The map $x \mapsto |p^{-1}(x)|$ is locally constant and thus continuous. Owing to X being connected and $\mathbb{N}$ having the discrete topology, the aforementioned map must be constant. ∎

3.1 Lifting Properties

Definition 3.3 (Lift). Let $f : Y \to X$ be a continuous and $p : \widetilde{X} \to X$ be a covering map. A *lift* of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p \circ \widetilde{f}$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 3.4. *Let Y be connected and $p : \widetilde{X} \to X$ a covering map. If $f : Y \to X$ is a continuous map having two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$, that agree at some point in Y, then they agree on all of Y.*

Proof. Let
$$A = \{y \in Y \mid \widetilde{f}_1(y) = \widetilde{f}_2(y)\}$$
We shall show that A is clopen in Y, whence we would be done owing to A being nonempty. Let $y \in A$ and $x = f(y)$. There is a neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of $\{V_\alpha\}$ which are homeomorphically mapped to U. Let V_β be the one containing $\widetilde{x} = \widetilde{f}_1(y) = \widetilde{f}_2(y)$. Then, due to continuity,

there is a neighborhood N of y that is mapped into V_β by both $\widetilde{f}_1$ and $\widetilde{f}_2$. Then, for all $z \in N$, $p \circ \widetilde{f}_1(z) = p \circ \widetilde{f}_2(z)$ but since p is injective on V_β, we must have $\widetilde{f}_1(z) = \widetilde{f}_2(z)$, consequently, $N \subseteq A$ and A is open.

On the other hand, if $y \notin A$, then $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$ lie in distinct open sets V_{β_1} and V_{β_2}, consequently, for all $z \in N = \widetilde{f}_1^{-1}(V_{\beta_1}) \cap \widetilde{f}_2^{-1}(V_{\beta_2})$, $\widetilde{f}_1(z) \neq \widetilde{f}_2(z)$, thereby completing the proof. ■

Theorem 3.5 (Homotopy Lifting Property). *Let $p : \widetilde{X} \to X$ be a covering map and $F : Y \times I \to X$ a continuous map. Let $\widetilde{F}_0 : Y \to \widetilde{X}$ be a lift of $F|_{X \times \{0\}}$. Then, there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}|_{X \times \{0\}} = \widetilde{F}_0$.*

Proof. The first step is to define a lift $\widetilde{F}$ on the strip $N \times I$ where N is a neighborhood of some point $y \in Y$.

Fix some $y_0 \in Y$. Each point $y_0 \times t$ has a neighborhood $N_t \times (a_t, b_t)$ which maps maps into an evenly covered neighborhood of $F(y_0 \times t)$. Note that the strip $\{y_0\} \times I$ is compact and is thus covered by finitely many of the N_t's, whence we may choose a neighborhood N of y_0 in Y and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each $N \times [t_i, t_{i+1}]$ is contained in $N_t \times (a_t, b_t)$ for some $t \in I$.

Now suppose $\widetilde{F}$ has been constructed on $N \times [0, t_i]$. We already have a lift for $N \times \{0\}$ and this is our base case. The space $N \times [t_i, t_{i+1}]$ is mapped into an evenly covered neighborhood U by F. Let $\widetilde{U} \subseteq \widetilde{X}$ be the unique open set in $\widetilde{X}$ containing the point $\widetilde{F}(y_0 \times t_i)$. There is a neighborhood N' of y_0 such that $N' \times \{t_i\}$ is mapped into $\widetilde{U}$ by $\widetilde{F}$. Replace N by N' henceforth. The composition $p^{-1} \circ F$ now lifts F on $N' \times [t_i, t_{i+1}]$ and since it agrees with $\widetilde{F}$ on $N' \times \{t_i\}$, we have an extension to $\widetilde{F}$ on $N \times [0, t_{i+1}]$, which is continuous due to the Pasting Lemma.

Now, for each $y \in Y$, we have constructed a lift $\widetilde{F}_y$ on $N_y \times I$ where N_y is some neighborhood of y. We must now argue that we can indeed paste these lifts together. Let $y \in N_{y'} \cap N_{y''}$. Since $\{y\} \times I$ is connected and $\widetilde{F}_{y'}$ and $\widetilde{F}_{y''}$ are two lifts which agree at $y \times 0 \in \{y\} \times I$, both the lifts must agree throughout due to Theorem 3.4. This also establishes the uniqueness of the lift $\widetilde{F}$ whereby completing the proof. ■

Corollary 3.6 (Path Lifting). Let $f : I \to X$ be a path and let $x_0 = f(0)$. For any $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x}_0$.

Corollary 3.7 (Path Homotopy Lifting). Let $H : I \times I \to X$ be a path homotopy. Then, the unique lift $\widetilde{H} : I \times I \to \widetilde{X}$, is also a path homotopy.

Proof. Since the image $\widetilde{H}(\{0\} \times I)$ is connected and a subset of the discrete fiber of $p^{-1}(\widetilde{H}(0 \times 0))$, it must be a single point. Similarly argue for the image $\widetilde{H}(\{1\} \times I)$. This completes the proof. ■

Proposition 3.8. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Proof. It suffices to show that $\ker p_*$ is trivial. Indeed, let $f : I \to \widetilde{X}$ be such that $p_*([f]) = 1_{\pi_1(X, x_0)}$. Thus, there is a path homotopy $F : I \times I \to X$ such that $F|_{I \times \{1\}}$ is the constant map at x_0 while $F|_{I \times \{0\}}$ is the map $p \circ f$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ I \times I \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

We have a lift $\widetilde{F} : I \times \{0\} \to \widetilde{X}$ of the bottom edge given by $\widetilde{F}(t \times 0) = f(t)$ and due to Theorem 3.5, this can be extended to a lift $\widetilde{F} : I \times I \to \widetilde{X}$. Consider the connected subspace $Y = \{0\} \times I \cup I \times \{1\} \cup \{1\} \times I$ of $I \times I$. The restriction $\widetilde{F}|_Y$ maps into $p^{-1}(x_0)$, which has the discrete topology, whereby the restriction must be a constant map equal to $\tilde{x}_0$ since $\widetilde{F}(0 \times 0) = \tilde{x}_0$. Thus, $\widetilde{F}$ must be a path homotopy between f and the constant path $\tilde{x}_0$, thereby completing the proof. ■

Proposition 3.9. *Let $\widetilde{X}$ and X be path connected spaces with a covering map $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$. Then, there is a bijection between the right cosets of $H = p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and $p^{-1}(x_0)$.*

Proof. ■

right coset to p^{-1} bijection

Theorem 3.10 (Lifting Criterion). *Let Y be path connected and locally path connected and $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Then, for any continuous map $f : (Y, y_0) \to (X, x_0)$, a lift $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.*

Proof. The forward direction is trivial, we shall prove the converse. Let $y \in Y$ and γ denote a path from y_0 to y in Y. The image $f \circ \gamma$ is a path in X beginning at x_0 and has a lift $\widetilde{f \circ \gamma}$ to a path in $\widetilde{X}$ beginning at $\tilde{x}_0$. Define the map $\tilde{f} : Y \to \widetilde{X}$ by $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$.

First, we must show that this is a well defined map, independent of the choice of γ. Indeed, let γ' be another path in Y from y_0 to y. Then, $\gamma' * \overline{\gamma}$ is a loop in γ' based at y_0, whence $f \circ (\gamma' * \overline{\gamma})$ is a loop in X based at x_0. We must now show that this can be lifted to a *loop*[1] in $\widetilde{X}$ based at $\tilde{x}_0$.

According to our hypothesis, $\left[f \circ \gamma' * \overline{f \circ \gamma}\right] \in p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ whence there is a path homotopy $H : I \times I \to X$ between $p \circ \tilde{h}$ and $f \circ \gamma' * \overline{f \circ \gamma}$ where $\tilde{h}$ is a loop in $\widetilde{X}$. Due to Corollary 3.7, this lifts to a homotpy of paths in $\widetilde{X}$. Due to the uniqueness of path lifting, $\widetilde{H}$ is a path homotopy between the loop $\tilde{h}$ and some other loop $\tilde{\alpha}$ in $\widetilde{X}$ that maps to $f \circ \gamma' * \overline{f \circ \gamma}$ under p.

Considering the first and second halves of $\tilde{\alpha}$, we see that they are $\widetilde{f \circ \gamma'}$ and $\overline{\widetilde{f \circ \gamma'}}$ respectively. Therefore, $\widetilde{f \circ \gamma'}(1) = \widetilde{f \circ \gamma}(1)$ and the map $\tilde{f}$ is well defined.

Finally, we must show that $\tilde{f}$ is continuous. Let $y \in Y$ and $U \subseteq X$ be an evenly covered open neighborhood of $f(y)$. Choose an open path connected neighborhood V of y that is contained in $p^{-1}(U)$. Let $\widetilde{U}$ be the open set in $\widetilde{X}$ that is homeomorphically mapped to U by p and contains $\tilde{f}(y)$. We shall show that $\tilde{f}(V) \subseteq \widetilde{U}$, which would imply $\tilde{f}|_V = p^{-1} \circ f$, thereby implying local continuity and thus the continuity of $\tilde{f}$.

First, fix some path γ from y_0 to y in Y. Let $y' \in V$ and choose some path η from y to y' contained in V. The path $\gamma * \eta$ is a path from y_0 to y'. The composition $p^{-1} \circ \eta$ is a path from $\tilde{f}(y)$ in $\widetilde{U}$, moreover, the composition $\widetilde{f \circ \gamma} * p^{-1} \circ \eta$ is a path from $\tilde{x}_0$ lifting $\gamma * \eta$, whence $p^{-1} \circ \eta(1) = \tilde{f}(1)$ and $\tilde{f}(V) \subseteq \widetilde{U}$. This completes the proof. ■

3.2 The Universal Cover

Definition 3.11 (Semilocally Simply-Connected). A topological space X is said to be *semilocally simply-connected* if each point $x \in X$ has a neighborhood U such that the inclusion induced homomorphism $i_* : \pi(U, x) \to \pi(X, x)$ is trivial.

[1] We can always lift this to a path but that will not suffice in this case

Henceforth, a topological space is said to be *nice* if it is path-connected, locally path-connected and semilocally simply-connected.

Theorem 3.12. *If X is nice, then there is a simply connected space $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$.*

Proof. Pick a basepoint $x_0 \in X$. Define

$$\widetilde{X} = \{[\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0\}$$

and the function $p : \widetilde{X} \to X$ by $p([\gamma]) = \gamma(1)$.

Let $\mathscr{U}$ denote the set of all path connected open sets $U \subseteq X$ such that the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial. Indeed, if $V \subseteq U \in \mathscr{U}$ is path connected and open, then the homomorphism induced by the inclusion $V \hookrightarrow X$ is the composition of the homomorphisms induced by $V \hookrightarrow U \hookrightarrow X$ and since the latter is trivial, the composition is trivial, consequently, $V \in \mathscr{U}$.

We contend that $\mathscr{U}$ forms a basis for the topology on X. Indeed, let W be a neighborhood of x, then there is a neighhborhood U of x such that the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial. Since X is locally path connected, there is a path connected neighborhood V of x that is contained in $U \cap W$, whence the conclusion follows.

We shall now topologize $\widetilde{X}$. Let γ be a path in X from x_0 and $U \in \mathscr{U}$ contain $\gamma(1)$. Define the set

$$U_{[\gamma]} = \{[\gamma * \eta] \mid \eta : I \to U,\ \eta(0) = \gamma(1)\}$$

where the equivalence classes are in X. Since U is path connected, $p : U_{[\gamma]} \to U$ is surjective. Moreover, since the homomorphism induced by the inclusion $U \hookrightarrow X$ is trivial, any two paths from $\gamma(1)$ to any point $x \in U$ are homotopic in X.

We contend that if $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$. Obviously, there is a path $\eta : I \to U$ such that $\gamma' = \gamma * \eta$, whence it follows that $\gamma' * \mu = \gamma * \eta * \mu$ and thus, $U_{[\gamma']} \subseteq U_{[\gamma]}$. On the other hand, $[\gamma * \mu] = [\gamma * \eta * \overline{\eta} * \mu]$ whereby the conclusion follows.

Next, we claim that the collection $\{U_\gamma\}$ forms a basis for a topology on $\widetilde{X}$. Suppose $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ where $U, V \in \mathscr{U}$, then $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Since $\mathscr{U}$ forms a basis, there is $W \in \mathscr{U}$ such that $W \subseteq U \cap V$, consequently, $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$. This proves our claim.

Consider the bijection $p : U_{[\gamma]} \to U$, we contend that this is a homeomorphism. For any basis element $V_{[\gamma']} \subseteq U_{[\gamma]}$, we have $p(V_{[\gamma']}) = V$, consequently, p is an open map. On the other hand, if $V \subseteq U$ is an open set, then $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for some $[\gamma'] \in U_{[\gamma]}$ with $\gamma'(1) \in V$. Since $V_{[\gamma']} \subseteq U_{[\gamma']} = U_{[\gamma]}$, we see that the restriction of p is continuous and therefore a homeomorphism.

Using the local formulation of continuity, we have that $p : \widetilde{X} \to X$ is a continuous map. Any $x \in X$ has a neighborhood $U \in \mathscr{U}$, consequently, $p^{-1}(U) = \bigcup U_{[\gamma]}$ where $[\gamma]$ ranges over all paths from x_0 to some point in U. It is not hard to argue that the sets $U_{[\gamma]}$ must partition $p^{-1}(U)$, whereby p is a covering map.

Finally, we must show that $\widetilde{X}$ is simply connected. Pick the base point $[x_0] \in \widetilde{X}$. First, we show that $\widetilde{X}$ is path connected. Let $[\gamma] \in \widetilde{X}$. Define $\gamma_t : I \to X$ by

$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \le s \le t \\ \gamma(t) & t < s \le 1 \end{cases}$$

It suffices to show that the map $\varphi : I \to \widetilde{X}$ given by $\varphi(t) = [\gamma_t]$ is continuous. Using the Lebesgue Number Lemma, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_{i-1}, t_i]) \subseteq U_i \in \mathscr{U}$. Let $p_i : U_{i[\gamma_{t_i}]} \to U_i$ be the restriction of p, which is a homeomorphism. Then, for all $t \in [t_{i-1}, t_i]$, $\varphi(t) = p_i^{-1}(\gamma(t))$ and continuity follows from the Pasting Lemma.

Next, we show $\pi_1(\widetilde{X}, [x_0]) = 0$. Since p_* is injective, it suffices to show that the image of p_* is trivial. Let γ be a loop in the image of p_*. Then, the map $t \mapsto [\gamma_t]$ is a lift of γ as we have seen earlier and is unique due to Theorem 3.5. Now, since the lift is a loop, we must have

$$[x_0] = [\gamma_1] = [\gamma]$$

consequently, γ is nulhomotopic. This completes the proof. ∎

Theorem 3.13. *Suppose X is nice. Then for every subgroup $H \subseteq \pi_1(X, x_0)$, there is a covering space $p : (X_H, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$.*

Proof. For $[\gamma], [\gamma'] \in \widetilde{X}$, define the relation $[\gamma] \sim_H [\gamma']$ to mean $\gamma(1) = \gamma'(1)$ and $[\gamma * \overline{\gamma'}] \in H$. This is obviously an equivalence relation. Let X_H denote the quotient space $X/\sim_H$ with $q : \widetilde{X} \to X_H$ the quotient map. Consider now the map $p : X_H \to X$ which is induced as shown in the following diagram.

$$\begin{tikzcd} \widetilde{X} \arrow[r, two heads, "\pi"] \arrow[d, two heads, "q"'] & X \\ X_H \arrow[ur, dotted, "\exists! p"'] & \end{tikzcd}$$

Let $U \subseteq X$ be an open neighborhood. Then, $p^{-1}(U)$ is a disjoint union $\bigsqcup U_{[\gamma]}$ where $[\gamma]$ is an equivalence class of paths with $\gamma(1) \in U$. Note that $[\gamma] \sim_H [\gamma']$ if and only if $[\gamma * \eta] \sim_H [\gamma' * \eta]$. Hence, if any two points in distinct neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified, then so are the entire neighborhoods. Hence, $p : X_H \to X$ is a covering map.

Choose the basepoint $\widetilde{x}_0 \in X_H$ the equivalence class under $\sim_H$ containing the point $[e_{x_0}]$ where e_{x_0} is the constant path at x_0. Let γ be a loop in X based at x_0. This lifts to a path from $[e_{x_0}]$ to $[\gamma]$ in $\widetilde{X}$. This lift maps to a loop in X_H if and only if $[e_{x_0}] \sim_H [\gamma]$ or equivalently, $[\gamma] \in H$. In particular, this means that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$. This completes the proof. ∎

Definition 3.14. If $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are covering spaces, then an *isomorphism between* them is a homeomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ f$.

Theorem 3.15. *Let (X, x_0) be path connected and locally path connected and $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be covering spaces. Then, for $\widetilde{x}_1 \in p_1^{-1}(x_0)$ and $\widetilde{x}_2 \in p_2^{-1}(x_0)$, there is an isomorphism $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

Proof. We prove the converse, since the forward direction is trivial. Using Theorem 3.10, there are lifts $\widetilde{p}_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ and $\widetilde{p}_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ of p_1 and p_2 respectively. This give us $p_1 = p_2 \circ \widetilde{p}_1$ and $p_2 = p_1 \circ \widetilde{p}_2$, whereby $p_1 \circ (\widetilde{p}_2 \circ \widetilde{p}_1) = p_1$. Note that this implies $\widetilde{p}_2 \circ \widetilde{p}_1$ is a lift of the map p_1, but since $\mathbf{id}_{(\widetilde{X}_1, \widetilde{x}_1)}$ is also a lift, and agree on $\widetilde{x}_1$, we must have that $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathbf{id}_{(\widetilde{X}_1, \widetilde{x}_1)}$ and similarly, $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathbf{id}_{(\widetilde{X}_2, \widetilde{x}_2)}$. This implies the desired conclusion. ∎

Theorem 3.16. *Let X be path connected and locally path connected. Then, there is a bijection between the isomorphism classes of path connected covering spaces $p : \widetilde{X} \to X$ (ignoring basepoints) and conjugacy classes of subgroups of $\pi_1(X)$[a].*

[a] The basepoint can be ignored since X is path connected.

Proof. Fix, once and for all, a basepoint $x_0 \in X$ and let $G = \pi_1(X, x_0)$. Suppose we have an isomorphism of covering spaces (ignoring basepoints)

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

given by the above diagram with the basepoints $\widetilde{x}_1 \in p_1^{-1}(x_0)$ and $\widetilde{x}_2 \in p_2^{-1}(x_0)$. Then, due to the commutativity of the diagram, we have

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_* \circ f_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$

We now contend that changing the basepoint in a covering space $p : \widetilde{X} \to X$ conjugates the image under p_*. Indeed, let $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$ where x_0 is the chosen basepoint of X and let $\widetilde{\gamma}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Let $H_i = p_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$ and let $g = p_*([\widetilde{\gamma}])$.

If $\widetilde{f}$ is a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, then

$$g^{-1} p_*([\widetilde{f}]) g = p_*([\overline{\widetilde{\gamma}} * \widetilde{f} * \widetilde{\gamma}]) \in H_1$$

and thus, $g^{-1} H_0 g \subseteq H_1$. Similarly, one can show that $g H_1 g^{-1} \subseteq H_0$, implying that H_0 and H_1 are conjugate subgroups of G.

Conversely, suppose $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are covering maps with $H_i = (p_1)_*(\pi_1(\widetilde{X}_i, \widetilde{x}_i))$ and there is $g \in G$ such that $H_2 = g^{-1} H_1 g$. Let γ be a loop in the equivalence class corresponding to g, then this has a lift $\widetilde{\gamma}$ in $\widetilde{X}_1$ and let $\widetilde{y}_1 = \widetilde{\gamma}(1) \in p_1^{-1}(x_0)$.

It is not hard to see that $(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{y}_1)) = H_2$ whence there is a basepoint preserving isomorphism (since the covering spaces are path connected) $f : (\widetilde{X}_1, \widetilde{y}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ thereby proving the theorem. ■

In conclusion, we have proved the following classification theorem.

Theorem 3.17. *Let X be nice and $x_0 \in X$ a chosen basepoint. Then there is a bijection between the set of basepoint preserving isomorphism classes of path connected covering spaces and the set of subgroups of $\pi_1(X, x_0)$.*

On the other hand, if basepoints are ignored, then there is a bijecion between the isomorphism classes of covering spaces of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

3.2.1 Action of π_1 on a fiber

Let $p : \widetilde{X} \to X$ be a covering map and $x_0 \in X$. We shall first define an action of the group $\pi_1(X, x_0)$ on $p^{-1}(x_0)$. For a loop γ based at x_0, define the function $L_\gamma : p^{-1}(x_0) \to p^{-1}(x_0)$ as follows: Choose some $\widetilde{x}_0 \in p^{-1}(x_0)$ and let $\widetilde{\gamma}$ denote the *unique* lift of γ to a path in $\widetilde{X}$ that begins at $\widetilde{x}_0$. Define $L_\gamma(\widetilde{x}_0) := \widetilde{\gamma}(1)$.

First, note that L_γ is a bijection since it has an inverse given by $L_{\overline{\gamma}}$. Now, suppose γ and γ' are path homotopic loops, that is, $[\gamma] = [\gamma']$. Choose some $\widetilde{x}_0 \in p^{-1}(x_0)$. Then, due to Corollary 3.7, the lifts $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are path homotopic too, whence $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$ and thus $L_\gamma = L_{\gamma'}$.

Finally, suppose γ and η are two loops based at x_0 and $\widetilde{x}_0 \in p^{-1}(x_0)$. Let $\widetilde{x}_1 = \widetilde{\gamma}(1)$ and $\widetilde{\eta}$ be the unique lift of η to a path in $\widetilde{X}$ beginning at $\widetilde{x}_1$. Then, $\widetilde{\gamma} * \widetilde{\eta}$ is the *unique* lift of $\gamma * \eta$ to $\widetilde{X}$ whence, $L_{\gamma * \eta} = L_\eta \circ L_\gamma$.

Consider now the map $\Phi : \pi_1(X, x_0) \to \mathfrak{S}\left(p^{-1}(x_0)\right)$ given by $[\gamma] \mapsto L_{\overline{\gamma}} = L_\gamma^{-1}$. Then,

$$\Phi([\gamma] * [\eta]) = L_{\overline{\gamma * \eta}} = L_{\overline{\eta} * \overline{\gamma}} = L_{\overline{\gamma}} \circ L_{\overline{\eta}} = \Phi([\gamma]) \circ \Phi([\eta]),$$

whence Φ is a group homomorphism and defines an *action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.*

3.3 Deck Transformations and Covering Space Actions

3.3.1 Deck Transformations

Definition 3.18 (Deck Transformations, Normal Coverings). For a covering space $p : \widetilde{X} \to X$, the isomorphisms $f : X \to X$ are called *deck transformations*. These form a group $G(\widetilde{X})$ under composition.

A covering space $p : \widetilde{X} \to X$ is said to be *normal* if for all $x \in X$ and each pair $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$, there is a deck transformation that maps $\widetilde{x} \mapsto \widetilde{x}'$.

Proposition 3.19. *Let $\widetilde{X}$ be connected. Then, $G(\widetilde{X})$ acts freely on $\widetilde{X}$. In particular, a deck transformation, in this case, is completely determined by where it sends a single point.*

Proof. Let $f \in G(\widetilde{X})$ have a fixed point $\widetilde{x}_0$. Then, this is a lift for $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ as seen from the following diagram.

$$\begin{tikzcd}
& (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\
(\widetilde{X}, \widetilde{x}_0) \arrow[ur, "{f,\mathbf{id}}"] \arrow[r, "p"'] & (X, x_0)
\end{tikzcd}$$

But since f and $\mathbf{id}$ agree at $\widetilde{x}_0$, they must agree everywhere due to Theorem 3.4. ∎

Theorem 3.20. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the <u>path-connected, locally path-connected</u> space X, and let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X, x_0)$. Then,*

(a) the covering space is normal if and only if H is normal in $\pi_1(X, x_0)$

(b) $G(\widetilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

Proof. Suppose the covering is normal, let $g^{-1}Hg$ be a conjugate of H in $\pi_1(X, x_0)$. Then, there is correspondingly $\widetilde{x}_1 \in p^{-1}(x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = g^{-1}Hg$. Since the covering is normal, there is a deck transformation $f : \widetilde{X} \to \widetilde{X}$ taking $\widetilde{x}_0$ to $\widetilde{x}_1$. From Theorem 3.15, we must have that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$, whereby $g^{-1}Hg = H$ and $H \trianglelefteq \pi_1(X, x_0)$.

Conversely, suppose $H \trianglelefteq \pi_1(X, x_0)$ and let $\widetilde{x}_1 \in p^{-1}(x_0)$. From Theorem 3.17, we have that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1))$ is conjugate to H but since H is normal, the former is equal to H. As a result, from Theorem 3.15, there is a deck transformation taking x_0 to x_1, consequently, the covering space is normal.

Note that given $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$, there is a unique deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$. Now, given some $[\gamma] \in N(H)$, there is a lift $\widetilde{\gamma} : I \to \widetilde{X}$ such that $\widetilde{\gamma}(0) = \widetilde{x}_0$. Define now the function $\phi : N(H) \to G(\widetilde{X})$ by $\phi([\gamma]) = \widetilde{\gamma}(1)$. Let $[\gamma], [\delta] \in N(H)$ with $\sigma = \phi([\gamma])$ and $\tau = \phi([\delta])$. Then, it is not hard to see that $\gamma * \delta$ lifts to $\widetilde{\gamma} * \sigma(\widetilde{\delta})$, which corresponds to the deck transformation $\sigma \circ \tau$, implying that ϕ is a homomorphism. Moreover, ϕ is also surjective, for if there is a deck transformation σ taking $\widetilde{x}_0$ to $\widetilde{x}_1$, then $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = H$. Now, let $\widetilde{\gamma}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ with $\gamma = p \circ \widetilde{\gamma}$. This implies $[\gamma] \in N(H)$, consequently, $\phi([\gamma]]) = \sigma$.

We now contend that $\ker \phi = H$. Obviously $H \subseteq \ker \phi$. On the other hand, if $[\gamma] \in \ker \phi$, then γ lifts to a loop based at $\widetilde{x}_0$, whereby, $[\gamma] \in H$. The proof is finished by invoking the first isomorphism theorem. ∎

3.3.2 Covering Space Actions

Definition 3.21 (Covering Space Action). A *group action* of G on a topological space Y is a homomorphism $\varphi : G \to \mathrm{Aut}_{\mathbf{Top}}(Y)$. A *covering space action* is a group action of G on Y such that for each $y \in Y$, there is a neighborhood U of y such that for all $g_1, g_2 \in G$, $g_1U \cap g_2U \neq \varnothing$, if and only if $g_1 = g_2$.

We may rephrase the definition of a covering space action as:

> A *covering space action* of G on Y is a group action such that for each $y \in Y$, there is a neighborhood U of y such that for all $g \in G$, $U \cap gU \neq \varnothing$ if and only if $g = 1_G$.

Theorem 3.22. *Let G act on Y through a covering space action.*

(a) The quotient map $p : Y \to Y/G$ given by $p(y) = Gy$ is a normal covering space.[a].

(b) If Y is path connected, then G is the group of deck transformations of the covering space $p : Y \to Y/G$

(c) If Y is path connected and locally path connected, then $G \cong \pi_1(Y/G, Gy_0)/p_*(\pi_1(Y, y_0))$.

[a] Hence the nomenclature

Proof. (a) Let $Gy \in Y/G$. Since G acts through a covering space action, there is a neighborhood U of Y such that the collection $\{gU \mid g \in G\}$ is that of disjoint open sets. Obviously, $V = \bigsqcup_{g \in G} gU$ is a saturated open set, whereby, $p(V)$ is open in Y/G and a neighborhood of Gy. We contend that the restriction $p : U \to p(V)$ is a homeomorphism. Indeed, if $W \subseteq U$ is open, then $p(W) \subseteq p(V)$ is open, since $p(W) = p\left(\bigsqcup_{g \in G} gW\right)$ and the term within the brackets is a saturated open set. This immediately implies that p is a covering map.

Furthermore, for any $g_1 y, g_2 y \in Gy$, there is the action $g_2 g_1^{-1}$ taking $g_1 y$ to $g_2 y$ whereby, the covering space is normal.

(b) Obviously, each element of G is a deck transformation. On the other hand, if $f : Y \to Y$ is a deck transformation, then for any $y \in Y$, $f(y) \in Gy$, whereby, there is $g \in G$ such that $g(y) = f(y)$. From Proposition 3.19, we have that $g = f$, implying the desired conclusion.

(c) This follows from Theorem 3.20. ∎

Example 3.23 (Fundamental group of S^1). Let the additive group $\mathbb{Z}$ act on $\mathbb{R}$ by translations. Then, $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle S^1. The action of $\mathbb{Z}$ is properly discontinuous and $\mathbb{R}$ is simply connected and thus

$$\mathbb{Z} \cong \pi_1(S^1, s_0)/p_*(\pi_1(\mathbb{R}, x_0)) \cong \pi_1(S^1, s_0).$$

Remark 3.3.1. *Similarly, one can obtain the fundamental group of the torus $S^1 \times S^1$ by considering the additive action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{C}$ which is also properly discontinuous. Note that this also gives the torus the structure of a Riemann surface.*

Example 3.24 (Fundamental group of $\mathbb{RP}^n$). Let $n \geq 2$. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on S^n wherein the nontrivial action is given by $x \mapsto -x$. Obviously this is a covering space action, whereby

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^n, x_0)/p_*(\pi_1(S^n, s_0)) \cong \pi_1(\mathbb{RP}^n, x_0)$$

where x_0 is the orbit of s_0 and we're done.

Now, consider the case $n = 1$. We know that $\mathbb{RP}^1$ is homeomorphic to the circle and thus has fundamental group isomorphic to $\mathbb{Z}$.

Chapter 4

Homology

4.1 The Setup

Definition 4.1 (Standard and Singular n-simplices). The standard n-simplex, denoted $\Delta^n \subseteq \mathbb{R}^{n+1}$ is given by
$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}.$$
Denote the $n+1$ vertices of Δ^n by $v_0^n, \ldots, v_n^n$ where $v_i^n = (0, \ldots, 1, \ldots, 0)$ in which 1 occurs at the i-th position. Further, define the i-th face map $\mathfrak{f}_i^n : \Delta^n \to \Delta^{n+1}$ for $0 \leq i \leq n+1$, first on the vertices of Δ^n by
$$\mathfrak{f}_i^n(v_j^n) = \begin{cases} v_j^{n+1} & j < i \\ v_{j+1}^{n+1} & j \geq i \end{cases}$$
and then extend linearly to all of Δ^n.

Given a topological space X, a *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$. Denote by $S_n(X)$, the set of all singular n-simplices in X and let $C_n(X)$ denote the *free abelian group* on $S_n(X)$.

Definition 4.2 (The Singular Complex). Let X be a topological space. Define the map $\partial_n : C_n(X) \to C_{n-1}(X)$ by first defining it on $S_n(X)$,
$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \mathfrak{f}_i^{n-1}$$
and then extending to all of $C_n(X)$ using the universal property of free modules.

Remark 4.1.1. *One can check that if $i \leq j \leq n$, then $\mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} = \mathfrak{f}_{j+1}^n \circ \mathfrak{f}_i^{n-1}$.*

Proposition 4.3. $\partial_n \circ \partial_{n+1} = 0$ *for $n \geq 1$.*

Proof. It suffices to check this on $S_{n+1}(X)$, the generator of $C_{n+1}(X)$. Indeed,

$$\begin{aligned}
\partial_n \circ \partial_{n+1}(\sigma) &= \sum_{i=0}^{n+1} (-1)^i \partial_n(\sigma \circ \mathfrak{f}_i^n) \\
&= \sum_{i=0}^{n+1} (-1)^i \sum_{j=0}^{n} (-1)^j \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} \\
&= \sum_{i=0}^{n+1} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} + \sum_{i=0}^{n+1} \sum_{j=i}^{n} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} \\
&= \sum_{i=0}^{n+1} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} + \sum_{i=0}^{n} \sum_{j=i}^{n} \sigma \circ f_{j+1}^n \circ f_i^{n-1} = 0.
\end{aligned}$$

■

Definition 4.4 (Singular Homology Groups). For a topological space X, the homology groups corresponding to the singular chain complex $C_\bullet(X)$ are called the *singular homology groups*.

Let X be a topological space. The standard 0-simplex is just the point $x = 1$ in $\mathbb{R}^1$. Thus, $S_0(X)$ can be identified with the underlying set of X, consequently, $C_0(X)$ can be identified with the free abelian group on X. Define the map $\varepsilon : S_0(X) \to \mathbb{Z}$ by $\varepsilon(x) = 1$ for each $x \in X$ and extend this to $C_0(X)$ through the universal property. It is evident that the map $\varepsilon : C_0(X) \to \mathbb{Z}$ is a surjection. Furthermore, $\varepsilon \circ \partial_1 = 0$, and thus, we may augment the singular chain complex as follows:

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

which we denote by $\widetilde{C}_\bullet(X)$ and the corresponding homology groups by $\widetilde{H}_n(X)$ which are called the *reduced homology groups*. Note that $H_n(X) = \widetilde{H}_n(X)$ for $n > 0$ therefore, the only difference observed is in $\widetilde{H}_0(X)$.

4.2 Some Functorial Properties

Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. There is an induced map $f_n : S_n(X) \to S_n(Y)$ given by $\sigma \mapsto f \circ \sigma$. This can be extended to a map $f_n : C_n(X) \to C_n(Y)$ through the universal property of free modules as follows.

$$\begin{tikzcd}
S_n(X) \arrow[r, "f_n"] \arrow[d, hook] & S_n(Y) \arrow[d, hook] \\
C_n(X) \arrow[r, dotted, "\exists! f_n"'] & C_n(Y)
\end{tikzcd}$$

We denote the sequence of maps $\{f_n\}_{n=0}^{\infty}$ by $f_\sharp$.

Proposition 4.5. *Given the setup as above,*

(a) $f_\sharp : C_\bullet(X) \to C_\bullet(Y)$ *is a chain map.*

(b) $g_\sharp \circ f_\sharp = (g \circ f)_\sharp$.

(c) *If* $\mathbf{id} : X \to X$ *is the identity map, then* $\mathbf{id}_\sharp$ *is a collection of identity maps on* $C_n(X)$ *for each nonnegative integer* n*.*

Proof. (a) We need to show that $\partial^Y_{n+1} \circ f_{n+1} = f_n \circ \partial^X_n : C_{n+1}(X) \to C_n(Y)$. It suffices to check the equality on elements of $S_{n+1}(X)$, owing to the universal property. Indeed, for $\sigma \in S_{n+1}(X)$, we have

$$\partial^Y_{n+1}(f \circ \sigma) = \sum_{i=0}^{n+1} (-1)^i f \circ \sigma \circ \mathfrak{f}^n_i$$

$$f_n \circ \partial^X_{n+1}(\sigma) = f_n \left(\sum_{i=0}^{n+1} (-1)^i \sigma \circ \mathfrak{f}^n_i \right) = \sum_{i=0}^{n+1} (-1)^i f \circ \sigma \circ \mathfrak{f}^n_i.$$

(b) Since $g_n \circ f_n = (g \circ f)_n$ on the elements of $S_n(X)$, the equality must hold on all of $C_n(X)$. We are implicitly using the universal property here.

(c) Trivial. ∎

Since $f_\sharp$ is a chain map, it induces a group homomorphism $H_n(X) \to H_n(Y)$ on the homology groups, which we denote by f_* or $(f_*)_n$. We shall try to avoid the latter for the sake of brevity.

f_* is functorial

.

We shall establish some notation to make our life easier. If $p_0, \dots, p_k \in \mathbb{R}^k$ are points, then we denote by $[p_0, \dots, p_k]$ the unique linear map $\tau : \Delta^k \to \mathbb{R}^k$ that maps $v^k_i \mapsto p_i$. In particular, this map is given by

$$\alpha_0 v^k_0 + \cdots + \alpha_k v^k_k \mapsto \alpha_0 p_0 + \cdots + \alpha_k p_k.$$

Now, let $A \subseteq \mathbb{R}^n$ be a convex subset. Given a map $\sigma : A \to X$ and $p_0, \dots, p_k \in A$, we denote by $\sigma|_{[p_0,\dots,p_k]}$ the composition $\sigma \circ [p_0, \dots, p_k]$.

Theorem 4.6. *Let* $f, g : X \to Y$ *be homotopic maps. Then,* $f_* = g_*$*.*

Proof. We have a map $F : X \times I \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. We shall construct a chain homotopy $P : C_\bullet(X) \to C_\bullet(Y)$ between the maps f and g. ∎

chain homotopy. only computation remains

Corollary 4.7. Let $f : X \to Y$ be a homotopy equivalence. Then, f_* is an isomorphism of groups.

Proof. There is a continuous map $g : Y \to X$ such that $g \circ f \simeq \mathbf{id}_X$ and $f \circ g \simeq \mathbf{id}_Y$. Thus, $g_* \circ f_* = \mathbf{id}_*$ and $f_* \circ g_* = \mathbf{id}_*$. The conclusion follows. ∎

Definition 4.8 (Relative Homology Groups). Let X be a topological space and $A \subseteq X$ a subspace. There is a canonical inclusion $\iota_n : C_n(A) \hookrightarrow C_n(X)$. Denote by $C_n(X, A)$ the abelian group $\operatorname{coker} \iota_n$. There is an induced map $\partial_n : \operatorname{coker} \iota_n \to \operatorname{coker} \iota_{n-1}$ giving us a chain complex $C_\bullet(X, A)$. The homology groups corresponding to this chain complex are called *relative homology groups* and denoted by $H_n(X, A)$.

We now have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A) \xrightarrow{\iota} C_\bullet(X) \longrightarrow C_\bullet(X, A) \longrightarrow 0$$

which, due to Theorem 0.13 gives us a long exact sequence of homology groups

$$\begin{aligned} \cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots \\ \cdots \longrightarrow H_0(X, A) \longrightarrow 0 \end{aligned}$$

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Todo list