Set Theory

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August 17, 2023

Contents

Chapter 1

The Zermelo-Fraenkel Axioms

1.1 Axioms of Set Theory

We shall discuss Zermelo-Fraenkel Set Theory, which is a first order theory, with signature $\mathsf{ZF} = (\varnothing, \{\in\})$. That is, there are no function symbols and the only predicate is the "belongs to" relation.

ZF0 (Nonempty Domain) There is at least one set.

$$\exists x(x = x)$$

This axiom is redundant since **ZF7** guarantees the existence of an infinite set and thus the domain of discourse must be nonempty.

ZF1 (Extensionality) Informally speaking, a set is determined uniquely by its elements.

$$\forall x \forall y (\forall z(z \in x \iff z \in y) \implies x = y)$$

ZF2 (Foundation/Regularity) This states that any nonempty set contains an element that is disjoint from it.

$$\forall x \left[\exists y(y \in x) \implies \exists y(y \in x \land \neg \exists z(z \in x \land z \in y)) \right]$$

ZF3 (Comprehension) Informally speaking, this axiom allows us to define sets in the set-builder notation. Let ϕ be a valid first order formula with free variables $w_1, \ldots, w_n, x, z$. Then

$$\forall z \forall w_1, \ldots, w_n \exists y \forall x \left(x \in y \iff x \in z \land \phi \right)$$

Notice how this is the same as writing

$$y = \left\{ x \in z \mid \phi \right\}$$

ZF4 (Pairing) Informally, this states that given two sets x and y, there is a set $z = \{x, y\}$.

$$\forall x \forall y \exists z \forall w (w \in z \iff (w = x \lor w = y))$$

ZF5 (Union) This axiom allows us to take a union of a collection of sets.

$$\forall \mathscr{F} \exists A \forall y (x \in y \land y \in \mathscr{F} \implies x \in A)$$

ZF6 (Replacement Scheme) Let ϕ be a valid formula without Y as a free variable. Then,

$$\forall A(\forall x \in A \; \exists! y \; \phi(x, y) \implies \exists Y \; \forall x \in A \; \exists y \in Y \phi(x, y))$$

Informally speaking, this allows us to replace the elements of a set to obtain a new set.

ZF7 (Infinity) There is an infinite inductive set.

$$\exists x(\varnothing \in x \land \forall y \in x(S(y) \in x))$$

ZF8 (Power Set) Every set has a set containing all its subsets. It is important to note that this need not be **the** power set.

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

ZF9 (Choice) Informally, given a collection of nonempty sets X, there is a choice function that chooses one element from each set in X.

$$\forall X \left(\varnothing \notin X \implies \exists f : X \to \bigcup X, \ \forall x \in X (f(x) \in x) \right).$$

We have been a bit sloppy in stating the axioms. Notice that our signature does not contain a predicate $\subseteq$ or the successor function S, neither do we know, a priori, of the existence of **the** empty set.

To define the formula $\subseteq (x, y)$, use

$$\subseteq (x, y) := \forall z (z \in x \implies z \in y)$$

As for the successor function, given any set x, using **ZF4**, there is a set $y = \{x\}$. Using **ZF5**, we may define $S(y) := x \cup y$. Finally, using **ZF0** and **ZF3**, we know of the existence of the empty set as

$$\exists x (x = x \wedge \exists y \forall z (z \in x \iff z \in y \wedge z \neq z))$$

Further, due to **ZF1**, the empty set is unique.

1.2 Consequences of the Axioms

Theorem 1.1. *There is no universal set. That is,*

$$\neg \exists z \forall x (x \in z)$$

Proof. If there were a universal set, then using **ZF3**, we may construct the set $y = \{x \in z \mid x \notin x\}$. Then, it is not hard to argue that

$$y \in y \iff y \notin y,$$

a contradiction. ∎

Definition 1.2 (Power Set). Let x be a set. Due to **ZF8**, there is a set z containing all the subsets of x. Using Comprehension, we may construct

$$\mathscr{P}(x) := \{y \in z \mid y \subseteq x\}.$$

This is known as the **power set** of x.

Definition 1.3. Let $\mathscr{F}$ denote a set. Let A be a set satisfying **ZF5**. Define

$$\bigcup \mathscr{F} := \{x \in A \mid \exists y \in \mathscr{F} (x \in y)\}$$

and

$$\bigcap \mathscr{F} := \{x \in A \mid \forall y \in \mathscr{F} (x \in y)\}.$$

1.3 Relations, Functions and Well Ordering

Definition 1.4 (Ordered Pair). For sets x, y, define the ordered pair $\langle x, y \rangle$ by

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

The set on the right is constructed by using the pairing axiom twice.

Definition 1.5 (Cartesian Product). Let A and B be sets. Using Replacement, we may define, for each $y \in B$,

$$A \times \{y\} := \{z \mid \exists x \in A(z = \langle x, y \rangle)\}.$$

Again, by Replacement, define the set

$$\mathscr{F} := \{z \mid \exists y \in B(z = A \times \{y\})\}.$$

Finally, define

$$A \times B := \bigcup \mathscr{F}.$$

Definition 1.6 (Relation, Function). Let A be a set. A relation R on A is a subset of $A \times A$. Define the domain and range of a relation as

$$\text{dom}(R) := \{x \in A \mid \exists y(\langle x, y \rangle \in R)\} \qquad \text{ran}(R) := \{y \mid \exists x(\langle x, y \rangle \in R)\}.$$

We write xRy to denote $\langle x, y \rangle \in R$.

A relation f is said to be a function if

$$\forall x \in \text{dom}(f) \exists! y \in \text{ran}(f)(\langle x, y \rangle \in f).$$

We use $f : A \to B$ to denote a function f with $\text{dom}(f) = A$ and $\text{ran}(f) \subseteq B$.

Definition 1.7 (Total Ordering, Well Ordering). A *total ordering* is a pair $\langle A, R \rangle$ where A is a set and R is a relation that is irreflexive, transitive and satisfies trichotomy.

We say R *well-orders* A if $\langle A, R \rangle$ is a total ordering and every non empty subset of A has an R-least element.

We use $\text{pred}(A, x, R)$ to denote the set $\{y \in A \mid yRx\}$.

Lemma 1.8. *Let $\langle A, R \rangle$ be a well-ordering. Then for all $x \in A$, $\langle A, R \rangle \not\cong \langle \text{pred}(A, x, R), R \rangle$.*

Proof. Suppose $\langle A, R \rangle \cong \langle \text{pred}(A, x, R), R \rangle$ and let $f : A \to \text{pred}(A, x, R)$ be the order isomorphism. Let x be the R-least element of the set

$$\{y \in A \mid f(y) \neq y\},$$

which obviously exists since the aforementioned set is nonempty. If $xRf(x)$, there is some $y \in A$ with yRx and $f(y) = x \neq y$ a contradiction to the choice of x. On the other hand, if $f(x)Rx$, then $f(f(x)) \neq f(x)$ since f is injective, a contradiction to the choice of x. This completes the proof. ∎

Theorem 1.9. *Let $\langle A, R \rangle$ and $\langle B, S \rangle$ be two well-orderings. Then exactly one of the following holds:*

(a) $\langle A, R \rangle \cong \langle B, S \rangle$.

(b) $\exists y \in B \left(\langle A, R \rangle \cong \langle \text{pred}(B, y, S), S \rangle \right)$.

(c) $\exists x \in A \left(\langle \text{pred}(A, x, R), R \rangle \cong \langle B, S \rangle \right)$.

Proof. Let

$$f := \{\langle v, w \rangle \mid v \in A, \ w \in B, \ \langle \text{pred}(A, v, R), R \rangle \cong \langle \text{pred}(B, w, S), S \rangle\}.$$

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Due to the preceeding lemma, if $\langle v_1, w \rangle, \langle v_2, w \rangle \in f$, then $v_1 = v_2$. Similarly, if $\langle v, w_1 \rangle, \langle v, w_2 \rangle \in f$, then $w_1 = w_2$. Hence, f is an injective function.

It is not hard to argue that f is an order isomorphism from an initial segment of A to an initial segment of B. Both these segments may not be proper else we could find another isomorphism from an initial segment of A to an initial segment of B by extending one of the isomorphisms in f. This completes the proof. ∎

Chapter 2

Ordinal Numbers

2.1 Transitive Sets

Definition 2.1. A set x is said to be *transitive* if

$$\forall y \forall z (z \in y \wedge y \in x \implies z \in x).$$

Proposition 2.2. *A set x is transitive if and only if*

$$\forall y (y \in x \implies y \subseteq x).$$

Proof. Suppose x is transitive and $y \in x$. Since for all $z \in y$, $z \in x$, we must have $y \subseteq x$. The converse is trivial. ∎

Proposition 2.3. *If x is a transitive set, then so is $x \cup \{x\}$.*

Proof. ∎

Proposition 2.4. *If x is a transitive set, then so is $\mathscr{P}(x)$.*

Proof. ∎

Proposition 2.5. *If $\mathscr{F}$ is a family of transitive sets, then so is $\bigcup \mathscr{F}$.*

Proof. ∎

Proposition 2.6. *If x is a transitive set, then so is every $z \in x$.*

Proof. ∎

2.2 Ordinals

Definition 2.7 (Ordinal). A set x is said to be an *ordinal* if it is transitive and well ordered by $\in$. That is, the pair $\langle x, \in_x \rangle$ is a well ordering, where

$$\in_x := \{\langle v, w \rangle \in x \times x \mid v \in w\}.$$

Theorem 2.8 (Properties of Ordinals).

(a) *If x is an ordinal and $y \in x$, then y is an ordinal and $y = \mathrm{pred}(x, y)$.*

(b) *If $x \cong y$ are ordinals, then $x = y$.*

(c) *If x, y are ordinals, then exactly one of the following is true: $x = y$, $x \in y$ or $y \in x$.*

(d) *If C is a nonempty set of ordinals, then $\exists x \in C \, \forall y \in C (x \in y \vee x = y)$. That is, every nonempty set of ordinals has a minimum element.*

Proof.   (a) Due to Proposition 2.6, y is a transitive and owing to it being the subset of a well ordered set, it is well ordered too, hence an ordinal.

(b) Let $f : x \to y$ be an isomorphism. Let

$$A := \{z \in x \mid f(z) \neq z\}.$$

Suppose A is nonempty, then it has a least element, say $w \in x$. If $v \in w$, then $v = f(v) \in f(w)$ whence $w \subseteq f(w)$. On the other hand, if $v \in f(w)$, then there is some $u \in w$ such that $v = f(u) = u \in w$ and thus $f(w) = w$, a contradiction.

(c) Follows from Theorem 1.9.

(d) First note that it suffices to find $x \in C$ with $x \cap C = \varnothing$ for if $y \in C$ is another ordinal with $x \neq y$, then $y \notin x$ lest $x \cap C \neq \varnothing$.

Pick any $x \in C$. If $x \cap C = \varnothing$, then we are done. Else, let $x' \in x \cap C$ be the $\in$-least element. It is not hard to argue that $x' \cap C = \varnothing$ and we are done.   ∎

Lemma 2.9. *If A is a transitive set of ordinals, then A is an ordinal.*

Proof. We must first show that the membership relation $\in_A$ is a linear order. This follows from Theorem 2.8 (c) and the fact that A is a transitive set. Lastly, to see that A is well ordered, simply invoke Theorem 2.8 (d).   ∎

Theorem 2.10. *If $\langle A, R \rangle$ is a well ordering, then there is a <u>unique</u> ordinal C such that $\langle A, R \rangle \cong C$.*

Proof. Let

$$B := \{a \in A \mid \exists x_a (x_a \text{ is an ordinal} \wedge \langle \mathrm{pred}(A, a, R), R \rangle \cong x_a)\},$$
$$f := \{\langle b, x_b \rangle \mid b \in B\}.$$

First, note that for all $b \in B$, x_b, since it exists must be unique and thus f is a well defined function with $\mathrm{dom}(f) = B$.

Let $C = \mathrm{ran}(f)$. We contend that C is an ordinal. Let $y \in x \in C$ and $a \in B$ be such that $g : \mathrm{pred}(A, a, R) \to x$ is an isomorphism. Then, there is some $b \in \mathrm{pred}(A, a, R)$ with $g(b) = y$. It is not hard to see that the restriction $g : \mathrm{pred}(A, b, R) \to y$ is an isomorphism whence $y \in C$ and thus C is an ordinal due to the preceeding lemma.

The function $f : B \to C$ is obviously a surjection. We contend that it is an isomorphism. Indeed, let $a, b \in B$ with aRb and $g : \mathrm{pred}(A, b, R) \to x_b$ be *the* isomorphism. If $y = g(a)$, then the restriction $g : \mathrm{pred}(A, a, R) \to y$ is an isomorphism whence $f(a) = y \in x = f(b)$ and f is an order isomorphism.

Suppose $B \neq A$. Let $b \in A \setminus B$ be the R-least element. Then, $\text{pred}(A, b, R) \subseteq B$. Now suppose $B \neq \text{pred}(A, b, R)$, consequently, there is some $b' \in B \setminus \text{pred}(A, b, R)$, then bRb' and if there is an order isomorphism from $\text{pred}(A, b', R)$ to some ordinal x, then there must be one from $\text{pred}(A, b, R)$ as we have argued earlier, a contradiction.

Thus, either $B = A$ or $B = \text{pred}(A, b, R)$ for some $b \in A$. In the latter case, the function f is an order isomorphism between $\text{pred}(A, b, R)$ and an ordinal C whence $b \in B$, a contradiction. Thus $B = A$ and the proof is complete. ∎

Definition 2.11 (Type of a Well Ordering). If $\langle A, R \rangle$ is a well ordering, then $\text{type}(A, R)$ is the unique ordinal C such that $\langle A, R \rangle \cong C$.

Henceforth, we use Greek letters $\alpha, \beta, \gamma, \ldots$ to vary over ordinals. That is, saying $\forall \alpha(\ldots)$ is equivalent to saying $\forall x(x$ is an ordinal $\ldots)$. Further, since the ordinals are well ordered, we write $\alpha < \beta$ to denote $\alpha \in \beta$ and similarly, $\alpha \leq \beta$ means $\alpha \in \beta \vee \alpha = \beta$.

Definition 2.12. Let X be a set of ordinals. Define

$$\sup(X) := \bigcup X \quad \text{and} \quad \min(X) := \bigcap X.$$

Further, for an ordinal α, let $S(\alpha)$ denote the set $\alpha \cup \{\alpha\}$.

Lemma 2.13.   (a) $\forall \alpha, \beta(\alpha \leq \beta \iff \alpha \subseteq \beta)$.

   (b) If X is a set of ordinals, $\sup(X)$ is the least ordinal $\geq$ all elements of X and if $X \neq \varnothing$, $\min(X)$ is the least ordinal in X.

Proof.   (a) The forward direction is obvious. Suppose $\alpha \subseteq \beta$. If $\alpha = \beta$, then we are done. If not, let γ be the $<$-least element of $\beta \setminus \alpha$. We contend that $\gamma = \alpha$. Indeed, if $x \in \gamma$, then $x \notin \beta \setminus \alpha$ lest we contradict the minimality of γ consequently, $x \in \alpha$ whence $\gamma \subseteq \alpha$. On the other hand, since $\alpha = \text{pred}(\beta, \alpha)$, we have $\alpha \leq \gamma$ and thus $\alpha \subseteq \gamma$. This shows that $\alpha = \gamma \in \beta$ and the conclusion follows.

   (b) ∎

Lemma 2.14. *For an ordinal α, $S(\alpha)$ is an ordinal, $\alpha < S(\alpha)$ and*

$$\forall \beta(\beta < S(\alpha) \iff \beta \leq \alpha).$$

Definition 2.15 (Successor, Limit Ordinal). An ordinal α is said to be a *successor ordinal* if there is an ordinal β such that $\alpha = S(\beta)$. On the other hand, α is said to be a *limit ordinal* if $\alpha \neq \varnothing$ and α is not a successor ordinal.

2.3   Transfinite Induction and Recursion

2.3.1   Classes but informally

Informally speaking, a class is any collection of the form

$$\{x \mid \phi(x)\}$$

where $\phi(x)$ is a well defined first order formula. As we have seen earlier, the class

$$\{x \mid x = x\}$$

is not a set. A *proper class* is a class which is not a set. One uses boldface letters to denote classes.

Definition 2.16. Denote

$$\mathbf{V} := \{x \mid x = x\} \qquad \mathbf{ON} := \{x \mid x \text{ is an ordinal}\}.$$

To be completely formal, a class is simply a first order formula with one or more free variables. For example, the class of all ordinals can be thought of as the formula

$$\mathbf{ON}(x) = x \text{ is an ordinal}.$$

We can extend this to define functions between classes $\mathbf{A}$ and $\mathbf{B}$. A function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is given by a first order logic formula in two variables $\mathbf{F}(x, y)$ such that

$$\forall x\, \mathbf{A}(x) \implies \exists! y\, \left(\mathbf{B}(y) \wedge \mathbf{F}(x, y)\right).$$

Theorem 2.17 (Transfinite Induction on ON). *If $\mathbf{C} \subseteq \mathbf{ON}$ and $\mathbf{C} \neq \varnothing$, then $\mathbf{C}$ has a least element.*

Proof. The proof is exactly like Theorem 2.8 (d). ∎

One must note that there is a significant difference between Theorem 2.8 (d) and Theorem 2.17. The former is a single provable statement in ZFC while the latter is a theorem schema which represents an infinite collection of theorems. In particular, suppose the class $\mathbf{C}$ corresponded to a formula $\mathbf{C}(x, z_1, \ldots, z_n)$, then Theorem 2.17 in this case says the following:

$$\forall z_1, \ldots, z_n \Big\{ \left[\forall x(\mathbf{C}(x, z_1, \ldots, z_n) \implies x \text{ is an ordinal}) \wedge \exists x \mathbf{C}(x, z_1, \ldots, z_n)\right]$$
$$\implies \left[\exists x \left(\mathbf{C}(x, z_1, \ldots, z_n) \wedge \forall y(\mathbf{C}(y, z_1, \ldots, z_n) \implies y \geq x))\right)\right] \Big\}.$$

And Theorem 2.17 specifies one such formula for each well-formed sentence $\mathbf{C}$.

Theorem 2.18 (Transfinite Recursion on ON). *If $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, then there is a unique $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ such that*

$$\forall \alpha \left(\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)\right).$$

The formal restatement of the above in terms of first order logic is the following:

$$\forall x \exists! y\, \mathbf{F}(x, y) \implies \left[\forall \alpha \exists! y\, \mathbf{G}(\alpha, y) \wedge \forall \alpha \exists x \exists y \left(\mathbf{G}(\alpha, y) \wedge \mathbf{F}(x, y) \wedge x = \mathbf{G} \restriction \alpha\right)\right]$$

where

$$(x = \mathbf{G} \restriction \alpha) := \text{function}(x) \wedge \text{dom}(x) = \alpha \wedge \left(\forall \beta \in \text{dom}(x)\, \mathbf{G}(\beta, x(\beta))\right).$$

Similarly, one can encode the uniqueness condition.

Proof. ∎

2.4 Ordinal Arithmetic

Addition

Definition 2.19 (Ordinal Addition). If α, β are ordinals, then define $\alpha + \beta = \text{type}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$ where

$$R = \left\{\langle\langle \xi, 0\rangle, \langle \eta, 0\rangle\rangle \mid \xi < \eta < \alpha\right\} \cup \left\{\langle\langle \xi, 0\rangle, \langle \eta, 1\rangle\rangle \mid \xi < \eta < \beta\right\} \cup \left[(\alpha \times \{0\}) \times (\beta \times \{1\})\right].$$

Informally speaking, we construct a new ordinal $\alpha + \beta$ by first "placing" α is a line and then placing β after it linearly. This is best visualized when α and β are finite ordinals.

To see that R indeed gives $\alpha \times \{0\} \cup \beta \times \{1\}$ the structure of a well order, let S be a nonempty subset. If $S \cap \alpha \times \{0\}$ is nonempty, then the minimal element of S exists and is the minimal element of $S \cap \alpha \times \{0\}$. On the other hand, if $S \cap \alpha \times \{0\} = \varnothing$, the minimal element of S exists and is the minimal element of $S \cap \beta \times \{1\}$.

Lemma 2.20. *For ordinals α, β, γ,*

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(b) $\alpha + 0 = \alpha$.

(c) $\alpha + 1 = S(\alpha)$.

(d) $\alpha + S(\beta) = S(\alpha + \beta)$.

(e) If β is a limit ordinal, then $\alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}$.

Proof. We shall only prove (e) since the others are straightforward. First, note that $\alpha + \beta \geq \alpha + \xi$ for every $\xi < \beta$, which is easy to see by setting up an obvious order preserving injection. ∎

> complete this argument

Remark 2.4.1. *One must note that ordinal addition is **not commutative**. Indeed,*

$$1 + \omega = \sup\{1 + n \mid n < \omega\} = \omega$$

while

$$\omega + 1 = S(\omega) \neq \omega$$

where the last "non-equality" follows from the axiom of foundation. Thus, $1 + \omega \not\cong \omega + 1$.

Multiplication

Definition 2.21. If α, β are ordinals, define $\alpha \cdot \beta = \text{type}(\beta \times \alpha, R)$ where R is the dictionary order, given by
$$R = \left\{ \langle\langle \xi, \eta \rangle, \langle \xi', \eta' \rangle\rangle \; \middle| \; \xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta') \right\}.$$

We must first check that R is indeed a well ordering. That it is a strict linear order is clear. Let $S \subseteq \beta \times \alpha$ be a nonempty subset. Let S_1 be the projection of S onto β. This has a minimum element, say ξ. Consider now the set of all $\eta \in \alpha$ such that $\langle \xi, \eta \rangle \in S$. This is a nonempty subset of α and thus has a minimum element, say δ. Then, $\langle \xi, \delta \rangle$ is a minimum element of S.

Lemma 2.22. *For ordinals α, β, γ,*

(a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(b) $\alpha \cdot 0 = 0$.

(c) $\alpha \cdot 1 = \alpha$.

(d) $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$.

(e) If β is a limit ordinal, then $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$.

(f) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Proof. ∎

> Proof of ordinal multiplication

Exponentiation

Definition 2.23. For ordinals α, β, we define α^β by recursion on β as

- $\alpha^0 = 1$.

- $\alpha^{\beta+1} = \alpha^\beta \cdot \beta$.

- If β is a limit ordinal, $\alpha^\beta = \sup\{\alpha^\xi \mid \xi < \beta\}$.

Remark 2.4.2. *Interestingly,*
$$2^\omega = \sup\{2^n \mid n < \omega\} = \omega.$$

2.5 Equivalent forms of the Axiom of Choice

Theorem 2.24 (Well Ordering Theorem). *For every nonempty set A, there is a relation $R \subseteq A \times A$ such that R well orders A.*

AC $\implies$ WO

Let A be a set. We shall explicitly construct a well ordering on X using the Axiom of Choice. First, let $f : \mathcal{P}(A)\backslash\{\varnothing\} \to A$ be a choice function and extend it to $f : \mathcal{P}(A) \to A \coprod \{\varnothing\}$ by defining $f(\varnothing) = \varnothing$. We shall now use transfinite recursion to define a function F on the ordinals as follows:

$$F(0) := f(A)$$
$$F(\alpha) := f\left(\{x \in A \mid \forall \beta \in \alpha (F(\beta) \neq x)\}\right).$$

First, note that if $F(\alpha) = F(\beta) \neq \varnothing$, then $\alpha = \beta$. Next, we contend that there must be an ordinal α with $F(\alpha) = \varnothing$. For if not, then we may apply the axiom of replacement and that of comprehension to obtain a set of all ordinals, a contradiction to the Burali-Forti paradox.

Let $\mathbf{C}$ denote the class of all ordinals α with $F(\alpha) = \varnothing$. Due to Theorem 2.17, there is a minimal such ordinal, say α_0, then

$$f\left(\{x \in A \mid \forall \beta \in \alpha_0 (F(\beta) \neq x)\}\right) = \varnothing \implies \{x \in A \mid \forall \beta \in \alpha_0 (F(\beta) \neq x)\} = \varnothing.$$

Let $G : A \to \alpha_0$ denote the inverse function of F. Define the relation $R \subseteq A \times A$ by

$$R := \{\langle x, y \rangle \mid G(x) \in G(y)\}.$$

That this is a well ordering is easy to see.

WO $\implies$ AC

This direction, on the other hand, is much easier. Let X denote a collection of sets and let $Y = \bigcup X$. Let R be a well ordering on Y. Define the function $f : X \to Y$ by $f(x) = \min(x)$, the R-least element, which can be chosen since Y has been well ordered and $x \subseteq Y$.

AC $\implies$ Zorn

Let X be a set and $P = (X, \leq)$ be a poset on it such that every chain in P has an upper bound. Let $f : \mathscr{P}(X)\backslash\{\varnothing\} \to X$ be a choice function.

Suppose P has no maximal element. Then, every chain in P must have a strict upper bound. Let $\mathscr{C}$ be the set of all chains in P. Let $g : \mathscr{C} \to \mathscr{P}(X)$ map a chain in P to the set of all *strict* upper bounds. Consequently, $g(C) \neq \varnothing$ for every chain C in P.

We shall define a class function $F : \mathbf{ON} \to X$ using transfinite recursion. Begin with $F(0) = F(X)$. Now, for any ordinal $\alpha \in \mathbf{ON}$, let C_α denote the chain $\{F(\beta) \mid \beta < \alpha\}$ and define

$$F(\alpha) := f(g(C_\alpha)).$$

It is not hard to see that $F(\alpha) = F(\beta)$ if and only if $\alpha = \beta$ whence we may use Replacement to obtain a *set* of all ordinals, which is absurd.

Zorn $\implies$ AC

Let X be a collection of sets and $Y = \bigcup X$. Let P be the poset of pairs (S, f) where $S \subseteq X$ and $f : S \to Y$ is a function with $f(s) \in s$ for each $s \in S$. We say $(S, f) \leq (S', f')$ if $S \subseteq S'$ and $f' \restriction_S = f$.

Let $C = \{(S_\alpha, f_\alpha)\}$ be a chain in P. Define the function $f : S := \bigcup_\alpha S_\alpha \to Y$ by $f(x) := f_\alpha(x)$ if $x \in S_\alpha$. Then, (S, f) is an upper bound for the chain C. Thus, due to Zorn's Lemma, P contains a maximal element, say $(\widetilde{S}, F)$. We contend that $\widetilde{S} = X$. For if not, then there is $x \in X \backslash \widetilde{S}$ and the function F can be extended to $\widetilde{S} \cup \{x\}$ by simply choosing an element of x and assigning it to x under F. This contradicts the maximality of $(\widetilde{S}, F)$ and hence, F is the desired choice function.

Chapter 3

Cardinal Numbers

Definition 3.1. Sets A and B are said to be *equinumerous* if there is a bijection $f : A \to B$. This is denoted by $A \approx B$. On the other hand, if there is an injection $f : A \to B$, it is denoted by $A \preceq B$. We write $A \prec B$ if $A \preceq B$ and $B \npreceq A$.

Theorem 3.2 (Cantor-Schröder-Bernstein). $A \preceq B \wedge B \preceq A \implies A \approx B$.

Definition 3.3. For a set A, $|A|$ is the least α such that $\alpha \approx A$. α is a *cardinal* if and only if $\alpha = |\alpha|$.

From Theorem 2.24, there is a well ordering R on A and thus an ordinal α with an order preserving bijection between $\langle A, R \rangle$ and α, in particular, $A \approx \alpha$. Thus, $|A|$ is defined for every set. Further, note that α is a cardinal if and only if $\forall \beta < \alpha (\beta \napprox \alpha)$ and for any ordinal α, $|\alpha| \leq \alpha$.

Lemma 3.4. *If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.*

Proof. Since $\beta \leq \alpha$, we have $\beta \subseteq \alpha$ and thus $\beta \preceq \alpha$. On the other hand, $|\alpha| \subseteq \beta$. Composing this inclusion with the bijection $\alpha \approx |\alpha|$, we have $\alpha \preceq \beta$. We are done due to Theorem 3.2. ∎

Lemma 3.5. *If $n \in \omega$, then*

    (a) $n \napprox n + 1$.

    (b) $\forall \alpha (\alpha \approx n \implies \alpha = n)$.

Proof.   (a) Suppose not. Pick the smallest $n \in \omega$ such that $n \approx n + 1$. Note that $n \neq 0$. We have an injective function $f : n + 1 \to n$. Composing appropriately, we may suppose that $f(n) = n - 1$ where $n \in n + 1$ and $n - 1 \in n$. The restriction $f \restriction_n$ is an injective function from n to $n - 1$ whence by Theorem 3.2, $n - 1 \approx n$, a contradiction.

    (b) If $n < \alpha$, then $n + 1 \leq \alpha$ whence $n + 1 \preceq \alpha$. On the other hand, $\alpha \approx n < n + 1$, consequently $\alpha \approx n + 1$, a contradiction to (a).

    Now suppose $\alpha < n$. Then, $|n| = |\alpha| \leq \alpha \leq \alpha + 1 \leq n$, consequently, $|\alpha + 1| = |n|$. But since $\alpha + 1 \approx n + 1$, we have $n + 1 \approx n$, a contradiction to (a). Thus $\alpha = n$. ∎

Corollary 3.6. ω is a cardinal and so is every ordinal $n < \omega$.

Definition 3.7. *A* is *finite* if and only if $|A| < \omega$. *A* is *countable* if and only if $|A| \leq \omega$. We use the shorthand *infinite* to mean "not finite" and *uncountable* to mean "not countable".

Definition 3.8 (Cardinal Arithmetic). For cardinals κ and λ, define

$$\kappa \oplus \lambda := |\kappa \times \{0\} \cup \lambda \times \{1\}|, \quad \kappa \otimes \lambda := |\kappa \times \lambda|.$$

Unlike ordinal arithmetic, the operations $\oplus$ and $\otimes$ are commutative, which is obvious from the definition above. Furthermore, note that

$$|\kappa + \lambda| = |\lambda + \kappa| = \kappa \oplus \lambda \quad \text{and} \quad |\kappa \cdot \lambda| = |\lambda \cdot \kappa| = \kappa \otimes \lambda.$$

Lemma 3.9. *For $m, n \in \omega$, $n \oplus m = n + m < \omega$ and $n \otimes m = n \cdot m < \omega$.*

Proof. ∎

Proposition 3.10. *Every infinite cardinal is a limit ordinal.*

Proof. Suppose $\kappa = \alpha + 1$ is a cardinal. Then, α is not a finite ordinal, that is, $\omega < \alpha$ and thus there is an ordinal β such that $\alpha = \omega + \beta$. Consequently, $1 + \alpha = 1 + \omega + \beta = \omega + \beta$ as we have seen previously that $1 + \omega = \omega$. Consequently,

$$|\kappa| = |\alpha + 1| = |1 + \alpha| = |\alpha|,$$

a contradiction to the fact that κ is a cardinal. ∎

> if $\alpha \leq \beta$ there is an ordinal δ such that $\beta = \alpha + \delta$.

Theorem 3.11 (Tarski). *If κ is an infinite cardinal, then $\kappa \otimes \kappa = \kappa$.*

Proof. We shall prove this statement by transfinite induction on κ. That this statement holds for $\kappa = \omega$ is well known. Suppose now that $\kappa > \omega$ and the statement holds for each cardinal $\lambda < \kappa$.

Note that for an infinite ordinal $\alpha < \kappa$, we have $|\alpha| < \kappa$ and thus

$$|\alpha \times \alpha| = |\alpha| \otimes |\alpha| = |\alpha| < \kappa.$$

Let $\prec$ denote the strict lexicographic ordering on $\kappa \times \kappa$. Define the relation $\trianglelefteq$ on $\kappa \times \kappa$ by $\langle \alpha, \beta \rangle \trianglelefteq \langle \gamma, \delta \rangle$ if and only if

$$\max\{\alpha, \beta\} < \max\{\gamma, \delta\} \text{ or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \langle \alpha, \beta \rangle \prec \langle \gamma, \delta \rangle.$$

That this relation is an ordering is immediate from the definition. We shall now show that this is a well ordering. Let $S \subseteq \kappa \times \kappa$ be nonempty. Using Replacement, construct the set S' which consists of $\max\{\alpha, \beta\}$ for all $\langle \alpha, \beta \rangle \in S$. Since $S' \subseteq \kappa$, it contains a minimum element, say α_0. Using Comprehension, construct the set S'' consisting of all pairs $\langle \alpha, \beta \rangle$ such that $\max\{\alpha, \beta\} = \alpha_0$. Now, $S'' \subseteq \kappa \times \kappa$, and under the lexicographic order, it has a minimum element, which is also the minimum element of S under the ordering $\trianglelefteq$.

Given any $\langle \alpha, \beta \rangle \in \kappa \times \kappa$, the set of all pairs preceeding it in $\langle \kappa \times \kappa, \trianglelefteq \rangle$ is a subset of

$$(\max\{\alpha, \beta\} + 1) \times (\max\{\alpha, \beta\} + 1)$$

Since κ is a limit ordinal, we have $\max\{\alpha, \beta\} + 1 < \kappa$ and due to the induction hypothesis, the cardinality of the above set is strictly smaller than κ whence $|\kappa \times \kappa| \leq \kappa$. There is an obvious injection from κ into $\kappa \times \kappa$, forcing $|\kappa \times \kappa| = \kappa$ due to Theorem 3.2. ∎

Corollary 3.12. Let κ, λ be infinite cardinals. Then,

(a) $\kappa \oplus \lambda = \kappa \otimes \lambda = \max\{\kappa, \lambda\}$,

(b) $|\kappa^{<\omega}| = \kappa$.

Proof. ∎

Theorem 3.13 (Cantor). $\forall X \left(X \prec \mathscr{P}(X) \right)$.

Proof. Suppose not, then $X \approx \mathscr{P}(X)$ for some X, which follows from Theorem 3.2 and the fact that there is a canonical injection from X to $\mathscr{P}(X)$. Let $f : X \to \mathscr{P}(X) \to X$ be a bijection. Using Comprehension, construct the set
$$S := \{x \in X \mid x \notin f(x)\} \subseteq X.$$
Let $s \in X$ be the unique element such that $f(s) = S$. Then,
$$s \in S \iff s \notin S,$$
a contradiction. ∎

Theorem 3.14. $\forall \alpha \exists \kappa \, (\kappa > \alpha \text{ is a cardinal})$ *is true in* ZF.

If we were to work in ZFC then we could just well order $\mathscr{P}(\alpha)$ and consider its cardinality.

Proof. The statement is obvious for finite cardinals. Suppose now that $\alpha \geq \omega$. Let
$$W := \{R \in \mathscr{P}(\alpha \times \alpha) \mid R \text{ well orders } \alpha\} \quad S := \{\text{type}(\langle \alpha, R \rangle) \mid R \in W\}.$$
Let $\beta = \sup(S)$. We contend that β is a cardinal and $\beta > \alpha$. First, note that if $\delta \in W$, then $S(\delta) \in W$, consequently, $\beta \notin W$. Further, $\beta \not\approx \alpha$ lest one could find a well ordering on α which is in order preserving bijection with β. Suppose β were not a cardinal. Then, there is some $\gamma < \beta$ with $\gamma \approx \beta$. By definition, there is η such that $\gamma \leq \eta < \beta$ with $\eta \in W$, consequently, $\eta \approx \beta$ but $\alpha \approx \eta$, a contradiction. This completes the proof. ∎

Definition 3.15 (Successor, Limit Cardinals). Let α be an ordinal. Denote by α^+ the smallest *cardinal* strictly greater than α. A cardinal κ is said to be a *successor cardinal* if $\kappa = \alpha^+$ for some α. On the other hand, if $\kappa > \omega$ and is not a successor cardinal, then κ is said to be a *limit cardinal*.

Definition 3.16 (Aleph Numbers). Define the numbers $\aleph_\alpha$ by transfinite recursion on α.

(a) $\aleph_0 := \omega$.

(b) $\aleph_{\alpha+1} = (\aleph_\alpha)^+$.

(c) For a limit ordinal λ, define $\aleph_\lambda := \sup\{\aleph_\alpha \mid \alpha < \lambda\}$.

Theorem 3.17. (a) *Each $\aleph_\alpha$ is a cardinal.*

(b) *Every infinite cardinal is equal to $\aleph_\alpha$ for some α.*

(c) *If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.*

(d) *$\aleph_\alpha$ is a limit cardinal ifa nd only if α is a limit ordinal.*

(e) $\aleph_\alpha$ is a successor cardinal if and only if α is a successor ordinal.

Proof. All of these follow immediately from the definition above. ∎

Remark 3.0.1. *One often writes ω_α in place of $\aleph_\alpha$. We adopt both conventions and use them interchangeably.*

Lemma 3.18. *If there is a surjective function $f : X \to Y$, then $|Y| \leq |X|$.*

Proof. Consider the set
$$S = \{f^{-1}(y) \mid y \in Y\},$$
which can be constructed using Replacement. Let $g : Y \to S$ be given by $g(y) = f^{-1}(y)$ and F be a choice function on S. Then, the composition $F \circ g$ is an injective function from Y to X, implying the desired conclusion. ∎