MA5106: MID-SEM EXAMINATION

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Problem 1

(a) This is Littlewood's Tauberian Theorem. I present the proof from Titchmarsh's Theory of Functions (pg. 233). The proof involves a usage of the Hardy-Littlewood Tauberian Theorem.

I use a_n to denote the series instead of c_n. We may suppose without loss of generality that the limit $s = 0$, which is justified by replacing a_0 by $a_0 - s$ if necessary.

In order to prove this theorem, we need to first prove Tauber's Theorem and then another lemma.

Lemma 1 (Kronecker). *If $b_n \to 0$ as $n \to \infty$, then*

$$\frac{\sum_{k=0}^{n} b_n}{n+1} \to 0$$

as $n \to \infty$.

Proof. Let $\varepsilon > 0$ be arbitrary. Since the series is convergent, there is a positive constant $M > 0$ such that $|b_n| < K$ for all $n \geq 0$. Also, there is $N > 0$ such that for all $n > N$, $|b_n| < \varepsilon/2$. Then, for any $M > N$,

$$\begin{aligned}\left|\frac{\sum_{n=0}^{M} b_n}{M+1}\right| &\leq \left|\frac{\sum_{n=0}^{N} b_n}{M+1}\right| + \left|\frac{\sum_{n=N+1}^{M} b_n}{M+1}\right| \\ &\leq \frac{(N+1)K}{M+1} + \frac{\sum_{n=N+1}^{M} |b_n|}{M+1} \\ &\leq \frac{(N+1)K}{M+1} + \frac{\varepsilon}{2}.\end{aligned}$$

One can choose M large enough so that the right hand side is smaller than ε and the conclusion follows. □

Theorem 2 (Tauber). *Let $\sum_{n=0}^{\infty} a_n$ be Able summable to a limit s and suppose $a_n = o(1/n)$. Then, $\sum_{n=0}^{\infty} a_n$ converges to s.*

Proof. Throughout this proof, $x \in (0, 1)$. Let $N > 0$. Then

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n = \underbrace{\sum_{n=N+1}^{\infty} a_n x^n}_{S_1} - \underbrace{\sum_{n=0}^{N} a_n (1 - x^n)}_{S_2}.$$

Note that

$$|S_2| \leq (1 - x) \sum_{n=0}^{N} n|a_n|,$$

since $1 + x + \cdots + x^{n-1} \leq n$. Let $\varepsilon > 0$ be arbitrary. Then, there is a sufficiently large N such that $|na_n| < \varepsilon$ for all $n > N$. Therefore,

$$|S_1| = \left|\sum_{n=N+1}^{\infty} na_n \frac{x^n}{n}\right| \leq \varepsilon \sum_{n=N+1} \frac{x^n}{n} \leq \varepsilon \sum_{n=N+1}^{\infty} \frac{x^n}{N+1} = \frac{\varepsilon x^{N+1}}{(N+1)(1-x)} < \frac{\varepsilon}{(N+1)(1-x)}.$$

Hence,

$$\left|\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n\right| < (1-x)\sum_{n=0}^{N} n|a_n| + \frac{\varepsilon}{(N+1)(1-x)}.$$

Now, for $1 - 1/(N+1) > x > 1 - 1/N$, we see that the right hand side is bounded above by

$$\frac{1}{N}\sum_{n=0}^{N} n|a_n| + \varepsilon.$$

Again, we may choose N larger so that the first term is smaller than ε. This is guaranteed by the previous lemma. Hence, for all sufficiently large N and $1-1/(N+1) > x > 1 - 1/N$, we have that the left hand side is smaller than 2ε.

Next,

$$\left|s - \sum_{n=0}^{N} a_n\right| \leq \left|s - \sum_{n=0}^{\infty} a_n x^n\right| + \left|\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n\right|.$$

Let $\varepsilon > 0$ be given and let N be large enough so that for all $x > 1 - 1/N$, the first term is smaller than ε and the second term is smaller than 2ε for all $1 - 1/N < x < 1 - 1/(N+1)$, which can be done due to the discussion above. As a result, for sufficiently large N, the left hand side of the above inequality is smaller than 3ε. Since ε was arbitrary positive, we see that $\sum_{n=0}^{\infty} a_n = s$. This completes the proof. $\square$

Next, we prove Hardy-Littlewood.

Lemma 3 (Karamata)**.** *Let $g : [0,1] \to \mathbb{R}$ and $0 < c < 1$. Suppose the restrictions of g to $[0,c)$ and $[c,1]$ are continuous and that*

$$\lim_{x \to c^-} g(x) \leq g(c).$$

For every $\varepsilon > 0$, there are polynomials $p(x)$ and $P(x)$ such that $p(x) \leq g(x) \leq P(x)$ for $0 \leq x \leq 1$ and

$$\|g - p\|_1 \leq \varepsilon, \quad \|g - P\|_1 \leq \varepsilon.$$

Proof. Using the definition of a limit, there is a $\delta > 0$ such that whenever $c - \delta \leq x < c$, we have

$$g(c^-) - \varepsilon/2 \leq g(x) \leq g(c^-) + \varepsilon/2.$$

Choose δ samll enough so that

$$\delta < \frac{\varepsilon}{g(c) - g(c^-)} \qquad \text{and} \qquad \delta < \frac{1}{2}.$$

Take L to be the linear function with

$$L(c-\delta) = g(c-\delta) + \varepsilon/2 \qquad L(c) = g(c) + \varepsilon/2.$$

For $c-\delta \le x < c$, we have

$$\begin{aligned}
L(x) - g(x) &= L(x) - g(c-\delta) + g(c-\delta) - g(c^-) + g(c^-) - g(c) \\
&= L(x) - L(c-\delta) + \varepsilon/2 + g(c-\delta) - g(c^-) + g(c^-) - g(x) \\
&\le L(c) - L(c-\delta) + \varepsilon/2 + \varepsilon/2 + \varepsilon/2 \\
&= g(c) - g(c-\delta) + 3\varepsilon/2 \\
&= g(c) - g(c^-) + g(c^-) - g(c-\delta) \\
&< \varepsilon/\delta + 2\varepsilon < 2\varepsilon/\delta.
\end{aligned}$$

Define $\Phi : [0,1] \to \mathbb{R}$ by

$$\Phi(x) = \begin{cases} g(x) + \varepsilon/2 & 0 \le x < c-\delta \\ \max\{L(x), g(x)+\varepsilon/2\} & c-\delta \le x \le c \\ g(x) + \varepsilon/2 & c < x \le 1. \end{cases}$$

Then,

$$\begin{aligned}
\|g - \Phi\|_1 & \int_0^1 \Phi(x) - g(x)\ dx \\
&= \int_0^{c-\delta} \varepsilon/2\ dx + \int_{c-\delta}^{c} \Phi(x) - g(x)\ dx + \int_c^1 \varepsilon/2 \\
&\le \varepsilon/2 + \int_{c-\delta}^{c} \Phi(x) - g(x)\ dx \\
&< \varepsilon/2 + \delta \cdot \frac{2\varepsilon}{\delta} = \frac{5\varepsilon}{2}.
\end{aligned}$$

Since Φ is continuous, there is a polynomial P such that $\|\Phi - P\|_\infty < \varepsilon/2$. Consequently, $g(x) \le P(x)$, since $\Phi(x) - g(x) \ge \varepsilon/2$. This also gives that $\|\Phi - P\|_1 \le \varepsilon/2$ and hence, using the triangle inequality, $\|g - P\| \le 3\varepsilon$.

Similar to the previous analysis, define the linear function l, taking the values $l(c-\delta) = g(c-\delta) - \varepsilon/2$ and $l(c) = g(c) - \varepsilon/2$. Again, it is not hard to see that $g(x) - l(x) < 2\varepsilon/\delta$.

Now, define the function $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} g(x) - \varepsilon/2 & 0 \le x < c-\delta \\ \min\{l(x), g(x)-\varepsilon/2\} & c-\delta \le x \le c \\ g(x) - \varepsilon/2 & c < x \le 1. \end{cases}$$

Using an argument similar to the previous case, there is a polynomial p such that $p \le g$ and $\|g - p\|_1 \le 3\varepsilon$. Replacing ε by $\varepsilon/3$, we have the desired conclusion. □

Theorem 4 (Hardy-Littlewood)**.** *If $a_n \ge 0$ for all $n \ge 0$ and*

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x},$$

then

$$s_n = \sum_{k=0}^{n} a_k \sim n.$$

Proof. For any $k \geq 0$, we have

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty} a_n x^n (x^k)^n &= \frac{1-x}{1-x^{k+1}}(1-x^{k+1})\sum_{n=0}^{\infty} a_n (x^{k+1})^n \\ &= \frac{1}{1+x+\cdots+x^k}(1-x^{k+1})\sum_{n=0}^{\infty} a_n (x^{k+1})^n.\end{aligned}$$

The term on the right tends to $1/(k+1)$ as $x \to 1^-$, which is also equal to $\int_0^1 t^k \, dt$. Using linearity of integrals, we may conclude that given any polynomial, then

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n P(x^n) = \int_0^1 P(t)\, dt.$$

Now, define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & 0 \leq t < e^{-1} \\ t^{-1} & e^{-1} \leq t \leq 1. \end{cases}$$

Let $\varepsilon > 0$. Using Karamata's Lemma, there are polynomials $p(x)$ and $P(x)$ such that $p(x) \leq g(x) \leq P(x)$ on $[0,1]$ and $\|g-p\|_1 \leq \varepsilon$ and $\|g-P\|_1 \leq \varepsilon$.

We now have

$$\begin{aligned}\limsup_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) &\leq \limsup_{x\to 1^-}\sum_{n=0}^{\infty} a_n x^n P(x^n) \\ &= \int_0^1 P(t)\, dt < \int_0^1 g(t)\, dt + \varepsilon.\end{aligned}$$

Similarly, we may argue that

$$\liminf_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) > \int_0^1 g(t)\, dt - \varepsilon.$$

Taking $\varepsilon \to 0$, we see that

$$1 = \int_0^1 g(t)\, dt = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n).$$

For any positive integer N, the evaluation at $x = e^{-1/N}$ is

$$\sum_{n=0}^{\infty} a_n x^n g(x^n) = \sum_{n=0}^{\infty} a_n e^{-n/N} g(e^{-n/N}) = \sum_{n=0}^{N} a_n e^{-n/N} e^{n/N} = s_N.$$

We have obtained that

$$1 = \lim_{N\to\infty}(1-e^{-1/N}) s_N$$

and hence,

$$s_N \sim \frac{1}{1-e^{-1/N}},$$

and the conclusion follows since

$$\lim_{N\to\infty} N\left(1-e^{-1/N}\right) = 1. \qquad \square$$

Lemma 5. *If $f(x)$ is a C^2 function on $(0,1)$ and $\lim_{x\to 1} f(x) = 0$ and there is $C > 0$ such that $|(1-x)^2 f''(x)| \le C$ on $(0,1)$, then*

$$\lim_{x\to 1}(1-x)f'(x) = 0.$$

Proof. Let $x' = x + \delta(1-x)$ where $0 < \delta < 1/2$. Then, using Taylor's Theorem,

$$f(x') = f(x) + \delta(1-x)f'(x) + \frac{1}{2}\delta^2(1-x)^2 f''(\xi)$$

for some $\xi \in (x, x')$. Hence,

$$(1-x)f'(x) = \frac{f(x') - f(x)}{\delta} - \frac{\delta}{2}(1-x)^2 f''(\xi) = (1-x)f'(\zeta) - \frac{\delta}{2}(1-x)^2 f''(\xi)$$

for some $\zeta \in (x, x')$, due to the Mean Value Theorem. Due to the conditions in the statement of the lemma, it is not hard to see that the right hand side is $O(\delta)$ and hence, the conclusion of the theorem follows by taking δ as small as desired. $\square$

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

on $(-1,1)$. According to the condition, we have $\lim_{x\to 1^-} f(x) = 0$. Further,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = O\left(\sum_{n=2}^{\infty}(n-1)x^{n-2}\right) = O\left(\frac{1}{(1-x)^2}\right).$$

Hence, due to the Lemma, $f'(x)$ is $o\left(\frac{1}{1-x}\right)$. Let $c > 0$ be a positive constant such that $|na_n| \le c$ for all $n \ge 0$. Then,

$$\sum_{n=1}^{\infty}\left(1 - \frac{na_n}{c}\right) = \frac{1}{1-x} - \frac{f'(x)}{c} \sim \frac{1}{1-x}.$$

Due to Hardy-Littlewood, we must have

$$\sum_{k=1}^{n} 1 - \frac{ka_k}{c} \sim n,$$

whence

$$\sum_{k=1}^{n} ka_k = o(n).$$

Let w_n denote the left hand side of the above. Then,

$$\begin{aligned}
f(x) - a_0 &= \sum_{n=1}^{\infty} \frac{w_n - w_{n-1}}{n} x^n = \sum_{n=1}^{\infty} w_n \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right) \\
&= \sum_{n=1}^{\infty} w_n \left(\frac{x^n - x^{n+1}}{n+1} + \frac{x^n}{n(n+1)}\right) \\
&= (1-x)\sum_{n=1}^{\infty} \frac{w_n}{n+1}x^n + \sum_{n=1}^{\infty} \frac{w_n}{n(n+1)}x^n.
\end{aligned}$$

Since $w_n = o(n)$, the first term on the right goes to 0 as $x \to 1$. But since $f(x) \to 0$ as $x \to 1$, we must have

$$\sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} x^n \to -a_0$$

as $x \to 1$. Now, note that $\frac{w_n}{n(n+1)} = o(1/n)$. Therefore, due to Tauber's Theorem,

$$\sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} = -a_0.$$

We get

$$\begin{aligned}
\lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} &= \lim_{N\to\infty} \sum_{n=1}^{N} w_n \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{w_n - w_{n-1}}{n} - \frac{w_N}{N+1} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} a_n - \lim_{N\to\infty} \frac{w_N}{N+1} \\
&= \sum_{n=1}^{N} a_n.
\end{aligned}$$

This shows that $\sum_{n=0}^{\infty} a_n = 0$ thereby completing the proof.

(b) This follows from (a). To see this, we shall show that if a sequence if Cesàro summable to s, then it is Abel summable to s. We may without loss of generality suppose that $s = 0$, this can be done by simply replacing c_0 by $c_0 - s$.

Let $s_N := \sum_{n=0}^{N} c_n$ and $\sigma_N := \frac{1}{N} \sum_{n=0}^{N-1} s_n$. For the sake of simplicity, set $s_{-1} = 0$ and $\sigma_0 = 0$.

We have

$$\sum_{n=0}^{N} c_n x^n = \sum_{n=0}^{N} (s_n - s_{n-1}) x^n = s_N x^N + \sum_{n=0}^{N-1} (x^n - x^{n+1}) s_n = s_N x^N + (1-x) \sum_{n=0}^{N-1} s_n x^n.$$

Now, note that $s_n = (n+1)\sigma_{n+1} - n\sigma_n$. Therefore,

$$\lim_{n\to\infty} s_n x^n = \lim_{n\to\infty} (n+1)\sigma_{n+1} x^n - n\sigma_n x^n = 0$$

since Cesàro summability implies that σ_n's are bounded and $nx^n \to 0$ as $n \to \infty$ when $|x| < 1$. Hence,

$$\sum_{n=0}^{\infty} c_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n.$$

Let $t_n = \sum_{k=0}^{n} s_n$ with the convention that $t_{-1} = 0$. Then,

$$\sum_{n=0}^{N} s_n x^n = t_N x^N + \sum_{n=0}^{N-1} t_n (x^n - x^{n+1}) = t_N x^N + (1-x) \sum_{n=0}^{N-1} t_n x^n.$$

Note that $t_n = (n+1)\sigma_{n+1}$ and hence, $t_n x^n = (n+1)\sigma_{n+1}x^n$, which goes to 0 as $n \to \infty$ since σ_n's form a bounded sequence and $(n+1)x^n \to 0$ as $n \to \infty$. As a consequence,

$$\sum_{n=0}^{\infty} s_n x^n = (1-x)\sum_{n=0}^{\infty} t_n x^n = (1-x)\sum_{n=0}^{\infty}(n+1)\sigma_{n+1}x^n.$$

Consequently,

$$\sum_{n=0}^{\infty} c_n x^n = (1-x)^2 \sum_{n=0}^{\infty}(n+1)\sigma_{n+1}x^n.$$

Let $M > 0$ be such that $|\sigma_n| \le M$ for all $n \ge 0$. Let $M > \varepsilon > 0$ be arbitrary. Then, there is an $N > 0$ such that for all $n > N$, $|\sigma_n| < \varepsilon$. Hence, for $x > 0$,

$$\begin{aligned}
\left|\sum_{n=0}^{\infty} c_n x^n\right| &\le (1-x)^2\left|\sum_{n=0}^{N}(n+1)\sigma_{n+1}x^n\right| + (1-x)^2\left|\sum_{n=N+1}^{\infty}(n+1)\sigma_{n+1}x^n\right| \\
&\le (1-x)^2 M\left|\sum_{n=0}^{N}(n+1)x^n\right| + (1-x)^2\varepsilon\left|\sum_{n=N+1}^{\infty}(n+1)x^n\right| \\
&\le (1-x)^2(M-\varepsilon)\sum_{n=0}^{N}(n+1)x^n + (1-x)^2\varepsilon\sum_{n=0}^{\infty}(n+1)x^n \\
&= (1-x)^2(M-\varepsilon)\sum_{n=0}^{N}(n+1)x^n + \varepsilon.
\end{aligned}$$

Note that in the limit $x \to 1^-$, the right hand side tends to ε^+. Hence, we may choose $\delta > 0$ such that for all $1-\delta < x < 1$, the first term on the right hand side is smaller than ε whence the left hand side is smaller than 2ε. This shows that the series is Abel summable to 0, which is what we intended to prove.

To finish (b), we simply invoke (a).

Problem 2

Note that

$$\begin{aligned}
|\widehat{f}(k) - \widehat{f_m}(k)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x)-f_m(x))\,e^{-ikx}\,dx\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-f_m(x)|\,dx = \|f-f_m\|_1.
\end{aligned}$$

But since $f_m \to f$ in L^1, we see that the right hand side goes to 0 as $m \to \infty$. Therefore,

$$\lim_{m\to\infty}\widehat{f_m}(k) = f(k)$$

for all $k \ge 1$.

Problem 3

(a) From Homework 1, we know

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]\, e^{-inx}\, dx.$$

Using Hölder continuity, there is $M > 0$ such that

$$|f(x) - f(x+\pi/n)| \le \frac{M\pi^\alpha}{|n|^\alpha},$$

for all $x \in \mathbb{R}$. Hence,

$$|\widehat{f}(n)| \le \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{M\pi^\alpha}{|n|^\alpha}\, dx = \mathcal{O}\left(\frac{1}{|n|^\alpha}\right).$$

(b) Since $\sum_{m=0}^\infty 2^{-m\alpha}$ converges, due to the Weierstrass M-test, the series for $f(x)$ converges absolutely on $\mathbb{R}$. Therefore, the Fourier coefficients can be computed by

$$\begin{aligned}\widehat{f}(n) &= \sum_{m=0}^\infty 2^{-m\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(2^m-n)x}\, dx \\ &= \begin{cases} 2^{-m\alpha}(= n^{-\alpha}) & n = 2^m \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Let $h \in \mathbb{R}$. Then,

$$\begin{aligned}|f(x+h) - f(x)| &= \left|\sum_{m=0}^\infty 2^{-m\alpha}\left(e^{i2^m(x+h)} - e^{i2^m}\right)\right| \\ &\le \sum_{m=0}^\infty 2^{-m\alpha}|e^{i2^m(x+h)} - e^{i2^m}| \\ &\le \sum_{2^m\le 1/|h|} 2^{-m\alpha}|e^{i2^m(x+h)} - e^{i2^m}| + \sum_{2^m>1/|h|} 2^{-m\alpha}|e^{i2^m(x+h)} - e^{i2^m}|.\end{aligned}$$

Trivially note that $|e^{i\theta} - 1| \le \theta$ and $|e^{i\theta} - e^{i\varphi}| \le 2$. Using the first inequality for the first term in the above sum and the second inequality for the second, we have

$$|f(x+h) - f(x)| \le \sum_{2^m\le 1/|h|} 2^{-m\alpha}\cdot 2^m|h| + \sum_{2^m>1/|h|} 2^{-m\alpha}\cdot 2.$$

Let N be the smallest non-negative integer such that $2^N > 1/|h|$. Then, the second term in the above sum is

$$\frac{2^{-N\alpha}}{1-2^{-\alpha}} = \mathcal{O}(|h|^\alpha).$$

If $|h| > 1$, then the first term is zero. Hence, suppose $|h| \le 1$. The first term is

$$\sum_{m=0}^{N-1} (2^m|h|)^{1-\alpha}\, |h|^\alpha \le |h|^\alpha \sum_{m=0}^{N-1}\left(2^{m-N+1}\right)^{1-\alpha} = |h|^\alpha\sum_{m=0}^{N-1} 2^{-m(1-\alpha)} \le \frac{|h|^\alpha}{1-2^{\alpha-1}},$$

which shows that the first term is also $\mathcal{O}(|h|^\alpha)$ thereby showing that f is of class $C^{0,\alpha}(S^1)$.

(c) I will essentially prove Bernstein's Theorem, that if $f \in C^{0,\alpha}(S^1)$ with $\alpha > 1/2$, then the Fourier series of f converges to f absolutely. First, note that there is $K > 0$ such that $|f(x) - f(y)| \leq K|x-y|^\alpha$ for all $x, y \in \mathbb{R}$.

Let $h \in \mathbb{R}$. Consider the function $g_h : \mathbb{R} \to \mathbb{R}$ given by $g_h(x) = f(x+h) - f(x-h)$. Note that

$$\begin{aligned}\widehat{g_h}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h)e^{-inx}\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-h)e^{-inx}\,dx \\ &= (e^{inh} - e^{-inh})\widehat{f}(n) = 2i\sin(nh)\widehat{f}(n).\end{aligned}$$

Using Parseval's Theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2 = \sum_{n\in\mathbb{Z}} 4\sin^2(nh)|\widehat{f}(n)|^2.$$

But note that $|g_h(x)| \leq K(2h)^\alpha$. Hence,

$$\sum_{n\in\mathbb{Z}} \sin^2(nh)|\widehat{f}(n)|^2 \leq \frac{K^2(2h)^{2\alpha}}{4}.$$

Let p be a positive integer and choose $h = \pi/2^{p+1}$. Then, for all $2^{p-1} < |n| \leq 2^p$, we have $|\sin(nh)| > 1/\sqrt{2}$. Hence,

$$\frac{1}{2}\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq \sum_{n\in\mathbb{Z}} \sin^2(nh)|\widehat{f}(n)|^2 \leq \frac{K^2(\pi/2^p)^{2\alpha}}{4} \leq \frac{K^2\pi^{2\alpha}}{2^{2+2\alpha p}},$$

that is,

$$\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq \frac{K^2\pi^{2\alpha}}{2^{2\alpha p+1}}.$$

Using the Cauchy Schwarz Inequality, we have

$$\frac{1}{2^{p-1}}\left(\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|\right)^2 \leq \sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq \frac{K^2\pi^{2\alpha}}{2^{2\alpha p+1}},$$

whence

$$\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)| \leq \frac{K\pi^\alpha}{2^{(\alpha-1/2)p+1}}$$

Hence,

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| \leq |\widehat{f}(0)| + \sum_{p=1}^{\infty} \frac{K\pi^\alpha}{2^{(\alpha-1/2)p+1}},$$

which converges since $\alpha > 1/2$. As a result, the Fourier series of f converges to f absolutely (we have seen this in class).

Problem 4

Consider the function $f : (-\pi, \pi] \to \mathbb{C}$ given by

$$f(x) = \begin{cases} e^{i\alpha x} & x \in (-\pi, \pi) \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier coefficients are

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\alpha-n)x}\, dx = \frac{1}{2\pi i(\alpha - n)} \left(e^{i(\alpha-n)x} - e^{-i(\alpha-n)x} \right) \\ &= \frac{\sin((\alpha - n)\pi)}{\pi(\alpha - n)} = (-1)^n \frac{\sin(\alpha\pi)}{\pi(\alpha - n)}. \end{aligned}$$

Note that f has a jump discontinuity at $\pi \sim -\pi$ on the circle, where

$$\lim_{x \to \pi^-} f(x) = e^{i\alpha\pi} \quad \text{and} \quad \lim_{x \to -\pi^+} f(x) = e^{-i\alpha\pi}.$$

Therefore, the Fourier series of f at π converges to

$$\frac{1}{2} \left(f(\pi^-) + f(-\pi^+) \right) = \cos(\pi\alpha).$$

This means

$$\cos(\pi\alpha) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{in\pi} \sin(\pi\alpha)}{\pi(\alpha - n)} = \sum_{n=-\infty}^{\infty} \frac{\sin(\pi\alpha)}{\pi(\alpha - n)}.$$

Therefore,

$$\frac{\pi}{\tan(\pi\alpha)} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha - n} + \frac{1}{\alpha + n} \right) = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2}.$$

Thus,

$$\frac{1}{2\alpha^2} - \sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{\pi}{2\alpha(\tan(\pi\alpha))}.$$

This proves (a).

To see (b), we evaluate the Fourier series at 0. Note that the function is differentiable in a neighborhood around 0 and hence,

$$\begin{aligned} 1 = f(0) &= \sum_{n=-\infty}^{\infty} (-1)^n \frac{\sin(\pi\alpha)}{\pi(\alpha - n)} \\ &= \frac{\sin(\pi\alpha)}{\pi} \left[\frac{1}{\alpha} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\alpha - n} + \frac{1}{\alpha + n} \right) \right] \\ &= \frac{\sin(\pi\alpha)}{\pi} \left[\frac{1}{\alpha} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2\alpha}{n^2 - \alpha^2} \right] \\ &= \frac{2\alpha \sin(\pi\alpha)}{\pi} \left[\frac{1}{2\alpha^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - \alpha^2} \right]. \end{aligned}$$

This proves (b).

Finally, we move on to (c). We have

$$\begin{aligned}\int_0^\infty \frac{t^{\alpha-1}}{1+t}\,dt &= \int_0^1 \frac{t^{\alpha-1}}{1+t}\,dt + \int_1^\infty \frac{t^{\alpha-1}}{1+t}\,dt \\ &= \int_0^1 \frac{t^{\alpha-1}}{1+t}\,dt + \int_0^1 \frac{u^{-\alpha}}{1+u}\,du\end{aligned}$$

where we have performed the substitution $t = 1/u$ in the second integral. We shall evaluate both integrals using the power series expansion of $1/(1+t)$. To justify this, we invoke the dominated convergence theorem. For $0 < r < 1$, note that the series

$$\sum_{n=0}^{\infty} (-1)^n t^n \tag{1}$$

converges absolutely on $[0, r]$. Define

$$A(r) := \int_0^r \frac{t^{\alpha-1}}{1+t}\,dt + \int_0^r \frac{t^{-\alpha}}{1+t}\,dt.$$

Using absolute convergence, we may interchange the integral and the summation to obtain

$$A(r) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\alpha+n} r^{\alpha+n} + \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1-\alpha} r^{n+1-\alpha}.$$

Using Abel's Theorem,

$$\begin{aligned}A(1) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha+n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1-\alpha} \\ &= \frac{1}{\alpha} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n-\alpha} - \frac{1}{n+\alpha} \right) \\ &= \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\alpha)}{n^2-\alpha^2} \\ &= \frac{\pi}{\sin(\pi\alpha)}.\end{aligned}$$

This completes the proof of (c).

Problem 5

First, suppose $m \geq n \geq 1$. Then, using integration by parts,

$$\int_{-1}^{1} \mathcal{L}_n(x)\mathcal{L}_m(x)\,dx = \left[\mathcal{L}_n(x) \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right]_{-1}^{1} - \int_{-1}^{1} \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m\,dx.$$

Note that $\dfrac{d^{m-1}}{dx^{m-1}}(x^2-1)^m$ vanishes at -1 and 1, since x^2-1 has multiplicity m at both -1 and 1. Hence, the first term in the above equation vanishes and we are left with

$$\int_{-1}^{1} \mathcal{L}_n(x)\mathcal{L}_m(x)\,dx = -\int_{-1}^{1} \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m\,dx.$$

We may keep repeating the step above to obtain

$$\int_{-1}^{1} \mathcal{L}_n(x)\mathcal{L}_m(x)\ dx = (-1)^l \int_{-1}^{1} \frac{d^{n+l}}{dx^{n+l}}(x^2-1)^n \frac{d^{m-l}}{dx^{m-l}}(x^2-1)^m\ dx.$$

for all $0 \le l \le n$.

Choosing $l = n$, we obtain

$$\langle \mathcal{L}_n, \mathcal{L}_m \rangle = (-1)^n \int_{-1}^{1} \frac{d^{2n}}{dx^{2n}}(x^2-1)^n \frac{d^{m-n}}{dx^{m-n}}(x^2-1)^m\ dx.$$

But $(x^2-1)^n$ is a $2n$-degree monic polynomial in x. Hence,

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!.$$

We get

$$\langle \mathcal{L}_n, \mathcal{L}_m \rangle = (-1)^n \int_{-1}^{1} (2n)! \frac{d^{m-n}}{dx^{m-n}}(x^2-1)^m\ dx.$$

If $m > n$, then

$$\langle \mathcal{L}_n, \mathcal{L}_m \rangle = (-1)^n (2n)! \left[\frac{d^{m-n-1}}{dx^{m-n-1}}(x^2-1)^m \right]_{-1}^{1} = 0,$$

since (x^2-1) has multiplicity m at -1 and 1.

Now, suppose $m = n$. Then,

$$\langle \mathcal{L}_n, \mathcal{L}_n \rangle = (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n\ dx = (2n)! \int_{-1}^{1} (1-x^2)^n\ dx.$$

It remains to compute

$$\begin{aligned}
\int_{-1}^{1} (1-x^2)^n\ dx &= 2\int_{0}^{1} (1-x^2)^n\ dx \\
&= 2\int_{0}^{1} (1-(1-y)^2)^n\ dy \\
&= 2\int_{0}^{1} (2y-y^2)^n\ dy \\
&= 2\int_{0}^{1} y^n(2-y)^n\ dy \\
&= \int_{0}^{1} y^n(2-y)^n\ dy + \int_{1}^{2} y^n(2-y)^n\ dy \\
&= \int_{0}^{2} y^n(2-y)^n\ dy.
\end{aligned}$$

Make the substitution $y = 2z$ to obtain

$$\int_{-1}^{1} (1-x^2)^n\ dx = \int_{0}^{2} 2^{2n+1} z^n (1-z)^n\ dz = 2^{2n+1} B(n+1, n+1) = 2^{2n+1} \frac{n!n!}{(2n+1)!}$$

where we have used the standard value of the Beta Function. This gives us

$$\langle \mathcal{L}_n, \mathcal{L}_n \rangle = \frac{(n!)^2 2^{2n+1}}{2n+1}.$$

It remains to deal with the case $m \geq n = 0$. Note that $\mathcal{L}_0(x) = 1$, the constant function. Therefore,

$$\langle \mathcal{L}_0, \mathcal{L}_m \rangle = \int_{-1}^{1} \mathcal{L}_m(x)\ dx.$$

If $m = 0$, the right hand side is 2. If $m > 0$, then the right hand side is

$$\int_{-1}^{1} \mathcal{L}_m(x)\ dx = \left[\frac{d^{m-1}}{dx^{m-1}} (x^2 - 1)^m \right]_{-1}^{1} = 0,$$

since $(x^2 - 1)^m$ has multiplicity m at -1 and 1. This proves (a) and (b).

Finally, we come to (c). First, we contend that the Lagrange polynomials are linearly independent (in the $\mathbb{C}$-vector space $\mathbb{C}[x]$). Indeed, if

$$\sum_{i=0}^{\infty} a_i \mathcal{L}_i = 0$$

where $a_i = 0$ almost everywhere. Then,

$$0 = \langle 0, \mathcal{L}_i \rangle = a_i \|\mathcal{L}_i\|_2^2,$$

whence each $a_i = 0$.

Further, note that $\deg \mathcal{L}_n = n$ and the $\mathbb{C}$ vector space spanned by $\{\mathcal{L}_0, \ldots, \mathcal{L}_n\}$ is a subspace of the vector space of polynomials of degree $\leq n$. But the latter has $\mathbb{C}$-dimension $n + 1$ and hence, $\{\mathcal{L}_0, \ldots, \mathcal{L}_n\}$ spans the latter. That is, every polynomial in $\mathbb{C}[x]$ can be uniquely written as a linear combination of the $\mathcal{L}_i$'s.

Let $S_N(f)$ denote the partial sum

$$S_N(f) := \sum_{n=0}^{N} \frac{\langle f, \mathcal{L}_n \rangle}{\|\mathcal{L}_n\|_2^2}$$

Let $\varepsilon > 0$ be given. By Weierstrass' Approximation Theorem, there is a polynomial $p(x) \in \mathbb{C}[x]$ such that $\|f - p\|_\infty < \varepsilon$. Let $N = \deg p$ and let $c_0, \ldots, c_N \in \mathbb{C}$ be such that

$$p = \sum_{n=0}^{N} c_n \mathcal{L}_n.$$

and set $c_n = 0$ for all $n > N$. Further, let $a_n = \frac{\langle f, \mathcal{L}_n \rangle}{\|\mathcal{L}_n\|_2^2}$. Now, for any $M \geq N$, we have

$$f - \sum_{n=0}^{M} c_n \mathcal{L}_n = (f - S_M(f)) + \sum_{n=0}^{M} b_n \mathcal{L}_n$$

where $b_n = a_n - c_n$. Using Pythagoras' Theorem,

$$\varepsilon^2 \geq \|f - \sum_{n=0}^{M} c_n \mathcal{L}_n\|_2^2 = \|f - S_M(f)\|_2^2 + \left\| \sum_{n=0}^{M} b_n \mathcal{L}_n \right\|_2^2 \geq \|f - S_M(f)\|_2^2$$

This shows that $S_M(f) \to f$ in $L^2[-1, 1]$, thereby completing the proof.