Category Theory

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Chapter 1

Introduction and Elementary Definitions

1.1 Preliminary Definitions

Definition 1.1 (Category). A category $\mathscr{A}$ consists of

1. a collection $\text{ob}(\mathscr{A})$ of objects
2. for each $A, B \in \text{ob}(\mathscr{A})$ a collection $\mathscr{A}(A, B)$ of morphisms from A to B
3. for each $A, B, C \in \text{ob}(\mathscr{A})$, a composition function

$$\circ : \mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C)$$

   mapping $(g, f) \mapsto g \circ f$.

4. for each $A \in \text{ob}(\mathscr{A})$, an element $\mathbf{id}_A$ of $\mathscr{A}(A, A)$ called the identity on A.

satisfying the following:

associativity: for each $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$

identity: for each $f \in \mathscr{A}(A, B)$, we have $f \circ \mathbf{id}_A = f = \mathbf{id}_B \circ f$

Set is the category of sets with morphisms as set maps.

Definition 1.2 (Functor). Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A functor $F : \mathscr{A} \to \mathscr{B}$ consists of

- a function $\text{ob}(\mathscr{A}) \to \text{ob}(B)$ written as $A \mapsto F(A)$
- for each $A, A' \in \mathscr{A}$, a function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$, written as $f \mapsto F(f)$

satisfying the following axioms

covariancy: $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$

identity consistency: $F(\mathbf{id}_A) = \mathbf{id}_{F(A)}$ whenever $A \in \mathscr{A}$

Such a functor is sometimes also called a **covariant functor**.

Let $\mathbf{Top}_*$ denote the category of topological spaces equipped with a basepoint. Let π be the map that maps a pointed topological space (X, x_0) to its fundamental group $\pi(X, x_0)$. We claim that this is a covariant functor. Let $\phi : (X, x_0) \to (Y, y_0)$ be a continuous function. One knows from algebraic topology that the above continuous map induces a homomorphism $\phi_* : \pi(X, x_0) \to \pi(Y, y_0)$ given by $[f] \mapsto [\phi \circ f]$. It is not hard to see that this is a covariant functor.

Definition 1.3 (Contravariant Functor). Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A contravariant functor from $\mathscr{A}$ to $\mathscr{B}$ is a functor $F : \mathscr{A}^{\text{op}} \to \mathscr{B}$.

Let **Top** be the category of topological spaces. For a topological space X, let $C(X)$ denote the ring of continuous functions $X \to \mathbb{R}$. That is, $C(X) \in$ **Ring**. We claim that $C(X)$ is a contravariant functor from **Top** to **Ring**. Indeed, let $f : X \to Y$ be a continuous function. Then, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

The continuous function f induces a map $f_* : C(Y) \to C(X)$ given by $g \mapsto g \circ f$. It is not hard to see now that the functor C is a contravariant functor from **Top** to **Ring** which maps a morphism f to a morphism f_*.

Definition 1.4 (Presheaf). A presheaf is a contravariant functor from $\mathscr{A}$ to **Set**. That is, a functor $F : \mathscr{A}^{\text{op}} \to$ **Set**.

Let X be a topological space and let $\mathscr{O}(X)$ denote the category of open subsets of X with inclusion morphisms. This gives $\mathscr{O}(X)$ the structure of a poset. Consider now the map $F : \mathscr{O}(X)^{\text{op}} \to$ **Set** given by

$$F(U) = \{\text{continuous functions } U \to \mathbb{R}\}$$

That this is a functor follows from the fact that if $U \subseteq V$, then the restriction of a continuous function $f : V \to \mathbb{R}$ to U is continuous.

Definition 1.5. A functor $F : \mathscr{A} \to \mathscr{B}$ is *faithful* if for each $A, A' \in \mathscr{A}$, the map $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$ given by $f \mapsto F(f)$ is injective.

Similarly, it is said to be *full* if the map is surjective.

Definition 1.6 (Natural Transformation). Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $F, G : \mathscr{A} \longrightarrow \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \to G$ is a family $\left(F(A) \xrightarrow{\eta_A} G(A)\right)_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(A') \arrow[d, "\eta_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

The maps η_A are called the *components* of η. When η_A is an isomorphism for all $A \in \mathscr{A}$, then η is said to be a natural isomorphism.

Consider **CRing**, the category of commutative rings and **Mon**, the category of monoids. Consider the covariant functor M_n : **CRing** $\to$ **Mon** that maps a commutative ring R to the monoid $M_n(R)$ of $n \times n$ matrices with entries from R.

Consider now the forgetful functor U : **CRing** $\to$ **Mon** that maps a ring R to its multiplicative monoid. It is not hard to see that $\det_n$ is a natural transformation from $M_n \to U$.

1.2 Adjoints

Definition 1.7. Let $\mathscr{A}$ and $\mathscr{B}$ be categories with $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. Then F is said to be *left adjoint* to G and G is said to be *right adjoint* to F if for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$, there is a natural isomorphism