MA5106: Fourier Analysis

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Contents

Chapter 1

Lecture 1

The origins of Fourier analysis lie in solving the heat equation:

$$\Delta u = \partial_t u$$

where Δ denotes the Laplacian.

In order to solve this, Fourier believed for a long time that one could expand a function as a series

$$f \sim \sum_k a_k \sin kx + \sum b_k \cos kx.$$

This is not true. In 1876, Paul Du Bois-Reymond gave an example of a continuous function whose Fourier series does not converge. But in 1966, Carleson showed that given an L^2 function on $[0, 1]$, the points at which the Fourier series does not converge has measure 0.

There are many applications to PDEs, in solving the

Laplace Equation: $\Delta u = 0$,

Heat Equation: $\partial_t u = \Delta u$,

Wave Equation: $\partial_{tt} u = \Delta u$.

Definition 1.1 (Fourier Series). Given $f \in L^1[a, b]$, its k-th *Fourier coefficient* is defined as

$$\widehat{f}(k) := \frac{1}{L} \int_a^b f(x) \exp\left(-\frac{2\pi i k}{L} x\right) \, dx.$$

where $L = b - a$.

The *Fourier series* of f is given formally by

$$f \sim \sum_{k \in \mathbb{Z}} \widehat{f}(k) \exp\left(\frac{2\pi i k}{L} x\right).$$

The question is whether

$$\lim_{n \to \infty} \sum_{k=-n}^{n} \widehat{f}(k) \exp\left(\frac{2\pi i k}{L} x\right) = f(x)$$

in the following cases:

- if $f \in L^1[a, b]$. Here we cannot expect pointwise convergence because one can just change the value of f at a single point without affecting its Fourier series.

- if $f \in C[a,b]$. This is not true because of an example by Paul Du Bois-Reymond.
- if $f \in C^1[a,b]$ then this is true.
- if $f \in L^2[a,b]$, then there may not be pointwise convergence but there is convergence in the L^2-norm.

There are notions of convergence other than pointwise and uniform. For example Cesàro and Abel. Fejér had proved that for continuous functions, the Cesàro sums converge uniformly to the function, whatever that means.

Example 1.2. Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x$. Then,

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \exp(-ikx)\, dx = \begin{cases} 0 & k = 0 \\ \frac{(-1)^k i}{k} & k \neq 0. \end{cases}$$

The Fourier series is then given by

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} \frac{\sin kx}{k}.$$

Chapter 2

Lecture 2

2.1 Functions on the Unit Circle

Denote
$$S^1 := \{z \in \mathbb{C} \colon |z| = 1\},$$
the unit circle. We parametrize points on the circle as $e^{i\theta}$.

Given a function $F : S^1 \to \mathbb{C}$, using the above remark, we can define a function $f : [-\pi \to \pi] \to \mathbb{C}$ by
$$f(\theta) = F(e^{i\theta}).$$
The continuity and differentiability properties of F correspond to those of f.

Conversely, given a function on an interval on the real line that agree on the endpoints, we can simply push it to the unit circle using something similar. Indeed, if $f : \mathbb{R} \to \mathbb{C}$ is a periodic function on $\mathbb{R}$ of period T, we define a function $F : S^1 \to \mathbb{C}$ by
$$F\left(\exp\left(\frac{2\pi i}{T}\theta\right)\right) = f(\theta).$$

Example 2.1 (Dirichlet Kernel). We define
$$D_N(x) := \sum_{n=-N}^{N} e^{ikx}$$
in $[-\pi \to \pi]$. Simple manipulation shows
$$D_N(x) = \begin{cases} 2N+1 & x = 0 \\ \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} & x \neq 0. \end{cases}$$
This is obviously a smooth function on $[-\pi, \pi]$ and represents a smooth function on the circle because it is a trigonometric polynomial.

Example 2.2 (Poisson Kernel). We define
$$P(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

where $0 \leq r < 1$ and $-\pi \leq \theta \leq \pi$. This obviously converges due to the comparison test. A closed form for this is

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

This is handy in solving the Dirichlet Problem on the unit disk as is seen in a complex analysis course.

2.2 Convergence of Fourier Series I

Let $f \in L^1(S^1)$. Define

$$S_N(f)(\theta) = \sum_{k=-N}^{N} \widehat{f}(k) e^{-ik\theta},$$

the partial sums of the Fourier series.

Theorem 2.3 (Uniqueness of Fourier Series). *Suppose $f \in L^1(S^1)$ and $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$. Then, $f(\omega) = 0$ for all points of continuity ω.*

Proof. Without loss of generality, suppose that $\omega = 0$, f is a real valued function on $[-\pi, \pi]$ and $f(0) > 0$. Since $\cos^k(x)$ is a polynomial in e^{imx}'s, the integral $\int_{-\pi}^{\pi} f(x) \cos^m x \, dx = 0$ for all $m \geq 0$.

Since f is continuous at 0, we can pick a $\pi/2 > \delta > 0$ such that $f(x) \geq f(0)/2$ in $[-\delta, \delta]$. Let $\varepsilon > 0$ be such that $\cos\delta = 1 - 2\varepsilon$. Then, $|\varepsilon + \cos\theta| \leq 1 - \varepsilon$ on $[-\pi, \pi] \backslash [-\delta, \delta]$. Now, choose $\eta < \delta$ such that $\varepsilon + \cos\theta \geq 1 + \varepsilon/2$ on $[-\eta, \eta]$. Let $p(\theta) = \varepsilon + \cos\theta$ on $[-\pi, \pi]$.

We have

$$0 = \int_{-\pi}^{\pi} f(\theta) p(\theta)^n \, d\theta = \int_{|\theta| \leq \delta} f(\theta) p(\theta)^n \, d\theta + \int_{\delta \leq |\theta| \leq \pi} f(\theta) p(\theta)^n \, d\theta,$$

whence

$$\left| \int_{\delta \leq |\theta| \leq \pi} f(\theta) p(\theta)^n \, d\theta \right| = \left| \int_{|\theta| \leq \delta} f(\theta) p(\theta)^n \, d\theta \right|.$$

The left hand side is bounded above by

$$\int_{\delta \leq |\theta| \leq \pi} |f(\theta)| |p(\theta)|^n \, d\theta \leq (1 - \varepsilon)^n \|f\|_1.$$

while the right hand side is bounded below by

$$\delta f(0) \left(1 + \frac{\varepsilon}{2}\right)^n.$$

It is not hard to see that this is not possible for sufficiently large n.

Next, we must argue this for complex valued functions f. Indeed, one can capture the real and complex parts as $(f + \overline{f})/2$ and $(f - \overline{f})/2i$, and the conclusion would follow. ■

Proposition 2.4. *Let $f \in C(S^1)$. Suppose $\sum_{k \in \mathbb{Z}} |\widehat{f}(k)|$ converges. Then the Fourier series of f converges uniformly to f. That is, $S_n(f) \rightrightarrows f$.*

Proof. Using the triangle inequality, we have

$$|S_n(f)(\theta) - S_m(f)(\theta)| \leq \sum_{m < |k| \leq n} |\widehat{f}(k)|,$$

whence, $\{S_n(f)\}$ is a Cauchy sequence in $C(S^1)$ and hence, the partial sums converge to some continuous function F on the circle.

It remains to show that $F = f$. Note that the Fourier coefficients of F and f are the same. Indeed, due to uniform convergence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)e^{-ik\theta}\,d\theta = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} S_n(f)(\theta)e^{-ik\theta}\,d\theta = \widehat{f}(k).$$

The conclusion follows from the fact that if a continuous function has all Fourier coefficients as 0, then it must be the zero function. ■

Chapter 3

Lecture 3

Proposition 3.1. *Let $f \in C^2(S^1)$. Then, there is a constant $C > 0$ such that*

$$|\widehat{f}(n)| \leq \frac{C}{n^2}$$

for all $n \in \mathbb{Z}\backslash\{0\}$.

Proof. This is a standard integration by parts application. Indeed,

$$\begin{aligned} 2\pi\widehat{f}(n) &= \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta \\ &= -(-in)^{-1}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\, d\theta \\ &= (-in)^{-2}\int_{-\pi}^{\pi} f''(\theta)e^{-in\theta}\, d\theta. \end{aligned}$$

We know that f'' is bounded by M on $[-\pi, \pi]$ (owing to it being continuous). Then,

$$2\pi|\widehat{f}(n)| \leq \frac{1}{n^2} \times 2\pi \times M.$$

This completes the proof. ■

Corollary 3.2. If $f \in C^2(S^1)$, the Fourier series of f converges uniformly to f.

Definition 3.3. A function f on the circle is said to be Hölder continuous of class $\alpha > 0.$ if there is a $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all $x, y \in S^1$. This is denoted by $f \in C^{0,\alpha}(S^1)$.

Remark 3.0.1. *The uniform convergence of $S_N f$ to f holds for $f \in C^{0,\alpha}(S^1)$ where $\alpha > 1/2$. This is due to Bernstein.*

3.1 Convolutions

We have

$$
\begin{aligned}
S_N f(x) &= \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx} \\
&= \sum_{k=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{ik(x-y)}\, dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-N}^{N} e^{ik(x-y)\, dy} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy \\
&= (f * D_N)(x).
\end{aligned}
$$

This is called a convolution.

Definition 3.4. Given $f, g \in L^1(S^1)$, define

$$
(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y)\, dy.
$$

It is not hard to argue that $f * g \in L^1(S^1)$ using Fubini's Theorem.

Proposition 3.5. *Let $u, v, w \in L^1(S^1)$. Then,*

(a) $u * v = v * u$.

(b) $u * (v + w) = u * v + u * w$.

(c) $(\lambda u) * v = \lambda (u * v)$.

(d) $u * (v * w) = (u * v) * w$.

(e) $\widehat{u * v}(k) = \widehat{u}(k) \widehat{v}(k)$.

(f) *$u * v$ is continuous if $u, v \in L^1(S^1)$ and bounded.*

Proof. We have

$$
2\pi (v * u)(x) = \int_{-\pi}^{\pi} v(y) u(x-y)\, dy = \int_{-\pi+x}^{\pi+x} v(x-z) u(z)\, dz = \int_{-\pi}^{\pi} v(x-z) u(z)\, dz = 2\pi (u * v)(x).
$$

For (e),

$$
\begin{aligned}
\widehat{u * v}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (u * v)(x) e^{-ikx}\, dx \\
&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u(y) v(x-y) e^{-ikx}\, dy dx \\
&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u(y) e^{-iky} v(x-y) e^{-ik(x-y)}\, dy dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{u}(k) v(x-y) e^{-ik(x-y)}\, dx \\
&= \widehat{u}(k) \widehat{v}(k).
\end{aligned}
$$

■

Chapter 4

Lecture 4

Definition 4.1. A sequence of continuous functions $K_n : S^1 \to \mathbb{C}$ are said to be *a family of good kernels* if:

(a)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\, dx = 1$$
for all $n \in \mathbb{N}$.

(b) There is a positive real number $M > 0$ such that
$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$$
for all $n \in \mathbb{N}$.

(c) For every $\delta > 0$,
$$\lim_{n\to\infty} \int_{|x|\geq\delta} |K_n(x)|\, dx = 0$$

Theorem 4.2. *Suppose $\{K_m\}_{m\geq 1}$ is a family of good kernels and $f \in L^1(S^1) \cap L^\infty(S^1)$. Then,*
$$(f * K_m)(x) \to f(x) \quad \text{as } m \to \infty$$
at all points of continuity of f. Moreover, if $f \in C(S^1)$, then the convergence is uniform.

Proof. We have
$$\left|(f * K_n)(x) - f(x)\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} (f(x-y) - f(x))K_n(y)\, dy\right|.$$
Using continuity of f at x, one can pick a small enough δ such that $f(x-y) - f(x)$ is small for all $y \in [-\delta, \delta]$. Then break the integral into two parts and use part (c) of the definition of a good kernel. ■

Proposition 4.3. *The Dirichlet kernel is not a good kernel.*

Proof. We have
$$\int_{-\pi}^{\pi} |D_N(x)|\, dx = \int_{-\pi}^{\pi} \left|\frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\frac{x}{2}}\right| dx.$$

The integral on the right can be broken down into segments $\left[\frac{k\pi}{N+\frac{1}{2}}, \frac{(k+1)\pi}{N+\frac{1}{2}}\right]$. Integrate over each of them and find a $\Omega(\log N)$ bound on the integral. This shows that it diverges. $\blacksquare$

Recall that L^2 is a Hilbert space with the inner product given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

There is the Hölder Inequality

$$|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2.$$

Chapter 5

Lecture 5

Let $f \in L^2(-\pi, \pi)$ and denote $e_k = e^{ik\theta}$ which is a function on the unit circle. The n-th Fourier coefficient of f is given by

$$\widehat{f}(k) = \langle f, e_k \rangle \qquad \forall k \in \mathbb{Z}.$$

We would show that the $\{e_k\}$'s form an orthonormal basis for $L^2(S^1)$.

Theorem 5.1. *Given $f \in L^2(S^1)$, we have*

1. *$S_n(f) \to f$ in the L^2-norm.*
2. *$\|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2$.*
3. *$\widehat{f}(k) \to 0$ as $|k| \to \infty$.*

Proof. ■

Theorem 5.2. *Let f be a Lipschitz function on S^1. Then, $S_n(f)$ converges pointwise to f.*

Proof. We have

$$S_N(f)(\omega) - f(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(\omega - t) - f(\omega) \right] D_N(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\omega - t) - f(\omega)}{t} \frac{t \sin\left(N + \frac{1}{2}\right) t}{\sin \frac{t}{2}}\, dt.$$

Note that

$$\sin\left(N + \frac{1}{2}\right) t = \sin(Nt) \cos\left(\frac{t}{2}\right) + \cos(Nt) \sin\left(\frac{t}{2}\right).$$

Let

$$F(t) = \frac{f(\omega - t) - f(t)}{t},$$

whence $|F(t)| \leq M$ on S^1, as a result, $F \in L^2$.

Then,

$$S_N(f)(\omega) - f(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(t) t \cos(t/2)}{\sin(t/2)} \sin(Nt) + F(t) t \cos(Nt)\, dt.$$

Note that $\frac{t\cos(t/2)}{\sin(t/2)}$ is continuous and hence, $F(t)\frac{t\cos(t/2)}{\sin(t/2)}$ is again in L^2. Also, $tF(t)$ is in L^2. Invoking the Riemann Lebesgue Lemma, we have the desired conclusion. ■

Remark 5.0.1. *If $f \in C^{0,1}(S^1)$, then $\widehat{f}(k) = \mathcal{O}(1/|k|)$.*

Chapter 6

Lecture 6

Remark 6.0.1. *If $\alpha > 1/2$ and $f \in C^{0,\alpha}(S^1)$, then $S_N(f)(x) \to f(x)$ for all $x \in S^1$.*

Definition 6.1. Define the Cesàro Sums to be

$$\sigma_N(f)(x) := \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N}.$$

Convergence in the sense of Cesáro means $\sigma_N(f)(x) \to f(x)$.

Example 6.2. Consider the series $\sum_{k=0}^{\infty}(-1)^k$. This obviously does not converge, as the individual terms do not tend to 0. The partial sums are given by

$$s_n = \begin{cases} 1 & k \text{ is even} \\ 0 & k \text{ is odd.} \end{cases}$$

The n-th Cesàro sum is

$$\sigma_n = \frac{s_0 + \cdots + s_{n-1}}{n} = \frac{1}{n}\left\lfloor \frac{n+1}{2} \right\rfloor.$$

Consequently, $\sigma_n \to 1/2$ as $n \to \infty$.

Proposition 6.3. *If $\sum c_k = s$ then $\sigma_n \to s$.*

Proof. We have

$$s - \sigma_n = \frac{(s - s_0) + \cdots + (s - s_{n-1})}{n}.$$

Let $\varepsilon > 0$ be given. Then, there is a sufficiently large $N > 0$ such that for all $n \geq N$, $|s - s_n| < \varepsilon/2$. Now, for $M > N$,

$$|s - \sigma_M| \leq \sum_{n=0}^{N-1} \frac{|s - s_n|}{M} + \sum_{n=N}^{M-1} \frac{|s - s_n|}{M}.$$

For sufficiently large M, the first sum on the right can be made smaller than $\varepsilon/2$. The conclusion would follow. ■

Example 6.4. Suppose $c_k = o(1/k)$ as $k \to \infty$. Then, if $\sum c_k$ is Cesàro summable, then the series converges.

We can now compute the n-th Cesàro means of the Fourier series. These would be given by convolution with the Cesàro means of the Dirichlet kernel,

$$\frac{\sum_{n=0}^{N-1} D_n(x)}{N} = \frac{1}{N}\sum_{n=0}^{N-1} \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}} = \frac{1}{N}\sum_{n=0}^{N-1} \frac{\cos nx - \cos(n+1)x}{2\sin^2\frac{x}{2}} = \frac{1}{N}\frac{1-\cos Nx}{2\sin^2\frac{x}{2}}.$$

This is called the Fejér Kernel,

$$F_N(x) = \frac{1}{N}\frac{\sin^2\frac{N}{2}x}{\sin^2\frac{x}{2}}.$$

We contend that the Fejér kernel is a good kernel. In which case, we only need to show that

$$\lim_{N\to\infty}\int_{|x|\geq\delta} F_N(x)\,dx \to 0$$

as $N \to \infty$. To see this, note that

$$\int_{|x|\geq\delta} F_N(x)\,dx \leq 2\pi \times \frac{1}{N} \times \frac{1}{\sin^2\frac{\delta}{2}}.$$

The conclusion follows. We immediately obtain the following:

Theorem 6.5. *If $f \in C(S^1)$, then the Fourier series of f converges to it uniformly in the sense of Cesàro.*

Definition 6.6. A series $\sum c_k$ is said to be Abel summable if

$$\lim_{r\to 1^-}\sum_{k=0}^{\infty} c_k r^k$$

exists.

It is a theorem due to Abel that a convergent series is Abel summable. The converse is obviously not true.

Definition 6.7. A Fourier series is said to be Abel summable if $\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{-ikx}$ converges in the sense of Abel.

This gives a kernel

$$\sum_{k\in\mathbb{Z}} r^{|k|}e^{ik\theta} = \frac{1-r^2}{1-2r\cos\theta+r^2},$$

which is just the Poisson kernel.

Chapter 7

Lecture 7

The Abel sums for a Fourier series are given by

$$A(r,f)(\theta) = \sum_{k\in\mathbb{Z}} r^{|k|}\widehat{f}(k)e^{ik\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)P_r(\theta-\omega)\,d\omega.$$

Theorem 7.1. *Let $f \in C^1(S^1)$. Then,*

$$\lim_{r\to 1^-} A(r,\theta) = f(\theta)$$

uniformly on S^1.

To see this, first note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1,$$

using the series expansion of $P_r(\theta)$ and the fact that it is uniformly convergent as a function of θ when r is fixed.

$$\begin{aligned}|A(r,\theta) - f(\theta)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(\theta-\omega) - f(\theta)\big)\,P_r(\omega)\,d\omega\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta-\omega) - f(\theta)|P_r(\omega)\,d\omega.\end{aligned}$$

Let $\varepsilon > 0$ be given. Using the fact that f is uniformly continuous on S^1, choose δ small enough such that $|f(\theta-\omega) - f(\theta)| < \varepsilon/2$ for $|\omega| < \delta$.

Now, break the integral into two parts

$$\le \frac{1}{2\pi}\int_{|\omega|\le\delta} |f(\theta-\omega) - f(\theta)|P_r(\omega)\,d\omega + \frac{1}{2\pi}\int_{\pi\ge|\omega|>\delta} |f(\theta-\omega) - f(\theta)|P_r(\omega)\,d\omega.$$

The first half is smaller than $\varepsilon/2$. The second one is bounded above by $\frac{M}{\pi}\int_{\delta\le|\omega|\le\pi} P_r(\omega)\,d\omega$. Now using the property of good kernels, this can be made small enough. This completes the proof. ■

Theorem 7.2 (A Tauberian Theorem). *Suppose $\sum c_n$ is Cesàro summable and $c_n = o(1/n)$, then $\sum c_n$ is summable.*

Proof. Let s_n for $n \ge 0$ denote the partial sums and

$$\sigma_n = \frac{s_0 + \cdots + s_{n-1}}{n} = \frac{nc_0 + (n-1)c_1 + \cdots + c_{n-1}}{n}.$$

As a result,

$$s_n - \sigma_n = \frac{1}{n}\left(\sum_{k=0}^{n} kc_k\right).$$

As a result,

$$|s_n - \sigma_n| \leq \frac{1}{n}\sum_{k=0}^{n}|kc_k|.$$

Since $kc_k \to 0$, there is an $N >> 0$ such that $|kc_k| < \varepsilon/2$ whenever $k \geq N$. Let $M >> 0$ be such that

$$\frac{1}{M}\sum_{k=0}^{N-1}|kc_k| < \varepsilon/2.$$

Then, one can conclude that for all $m \geq M$, $|s_m - \sigma_m| < \varepsilon$. This completes the proof. ■

Example 7.3. Consider $\sum_{n=0}^{\infty}(-1)^n(n+1)$. This is obviously not convergent and is not Cesàro summable either. On the other hand, the abel sums are

$$A(r) = \sum_{n=0}^{\infty}(-1)^n(n+1)r^n\ dr = \frac{1}{(1+r)^2}.$$

Therefore, this series is Abel convergent.

Chapter 8

Lecture 8

Theorem 8.1 (Banach-Steinhaus Theorem). *Let X be a Banach space and Y a normed linear space and $\{\Lambda_\alpha\}$ is a collection of bounded linear transformations. Then, either there is $M < \infty$ such that $\|\Lambda_\alpha\| \leq M$ for all α or there is a dense G_δ set subset G of X such that $\sup_\alpha \|\Lambda_\alpha x\| = \infty$ for all $x \in G$.*

Proof. See Rudin's RCA. ■

Theorem 8.2. *The set of function $f \in C(S^1)$ whose Fourier series diverges at 0 forms a dense G_δ subset of $C(S^1)$.*

Proof. For any $f \in L^1$, we have

$$S_N(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)D(\omega)\, d\omega.$$

Note that $D_N(-\omega) = D_N(\omega)$ for all $\omega \in S^1$. The operator $T_N : C(S^1) \to \mathbb{C}$ given by $T_N(f) = S_N(f)(0)$. This is a bounded linear functional (not hard to argue).

Fix some N. Define the function $g : [-\pi, \pi] \to \mathbb{C}$ by

$$g(x) = \begin{cases} 1 & D_N(x) > 0 \\ -1 & D_N(x) < 0 \\ 0 & D_N(x) = 0. \end{cases}$$

Then, $g_N D_N = |D_N|$. Define the functions $f_n : [-\pi, \pi] \to \mathbb{C}$ given by

$$f_n(x) = \max\{\min\{nD_N(x), 1\}, -1\}.$$

Then, f_n converges pointwise to g and $|f_n|$ is dominated by the constant function 1 on $[-\pi, \pi]$. Further, $\|f_n\| = 1$ for all $n \in \mathbb{N}$. As a result, $\|T_N\| \geq |T_N(f_n)|$ for all n. In the limit $n \to \infty$, we have $\|T_N\| \geq \|D_N\|_1$. But the right hand side diverges as $N \to \infty$. Therefore, from the Banach Steinhaus Theorem, the conclusion follows. ■

Chapter 9

Lecture 10

Heat equation

$u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ which satisfies

$$\begin{cases} \partial_t u = \Delta u = \sum_{i=1}^n \partial_{ii} u \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

The heat equation has a regularization effect, that is, even if u_0 is a terrible function, the solution u for positive time will be smooth. It is not symmetric with respect to time.

9.1 Fourier Transform

Definition 9.1. Let $f \in L^1(\mathbb{R}^n)$, define, for $\xi \in \mathbb{R}^n$,

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x)\, dx.$$

This yields a function $\widehat{f} : \mathbb{R}^n \to \mathbb{R}$.

Note that we shall restrict our attention to Schwarz Space instead of L^1.

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \colon \lim_{x \to \infty} (1 + |x|^2)^{m/2} |\partial^\alpha f(x)| = 0,\ \forall m \in \mathbb{N}\}$$

Obviously, $C_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$. Note that the Schwartz class is equivalent to the class of rapidly decreasing functions

$$\lim_{|x| \to \infty} |x|^m |\partial^\alpha f| = 0$$

Example 9.2. $f(x) = e^{-|x|^2} \in \mathscr{S}(\mathbb{R}^n)$. Note that the partial derivatives of this function are of the form $p(x)f(x)$ where p is a multivariate polynomial. Therefore, it suffices to show that $|p(x)f(x)| \to 0$ as $x \to \infty$ for all polynomials p, whence, it suffices to do this for monomials. Consider a monomial $x_1^{m_1} \ldots x_r^{m_n}$. Then,

$$|x_1^{m_1} \ldots x_n^{m_n} e^{-|x|^2}| = \prod_{i=1}^n |x_i e^{-x_i^2}|.$$

This completes the proof.

We now contend that the Fourier Transform of a function in the Schwartz space lies in the Schwatz space.

First, we show that $\mathscr{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Indeed, note that there is a positive constant $M > 0$ such that

$$|f(x)| \leq \frac{M}{(1+|x|^2)^n} \leq \frac{M}{x_1^2 \dots x_n^2}$$

Choose some $R > 0$ and let A denote the closed cube of length R centered at the origin. Then,

$$\int_{\mathbb{R}^n} |f(x)|\, dx = \int_A |f(x)|\, dx + \int_{\mathbb{R}^n \setminus A} f(x)\, dx.$$

The first term is bounded. For the second, use Fubini's theorem and integrate $M/x_1^2 \dots x_n^2$ instead. Another way to see this is to integrate over the product manifold $(1, \infty) \times S^{n-1}$, which is diffeomorphic to $\mathbb{R}^n \setminus \overline{B}(0,1)$ and invoke Fubini's Theorem for differential forms.

Now, we examine the differentiation properties of the fourier transform.

$$\widehat{\partial_j f}(\xi) = i\xi_j \widehat{f}(\xi) \qquad 1 \leq j \leq n$$
$$\partial_j \widehat{f}(\xi) = -i\widehat{x_j f}(\xi) \qquad 1 \leq j \leq n$$

Chapter 10

Lecture 11

We have seen in the previous class that the fourier trnasform is a linear operator on the schwartz space and also derived the various properties under differentiation.

Translation, we have

$$f\widehat{(x-x_0)}(\xi) = e^{-i\langle x_0,\xi\rangle}\widehat{f}(\xi).$$

Upon scaling, we have

$$\widehat{f(\lambda x)}(\xi) = \lambda^{-n}\widehat{f}(\xi/\lambda).$$

Proposition 10.1. *If f, g are in the Schwartz space, then*

$$\widehat{f*g}(\xi) = (2\pi)^{n/2}\widehat{f}(\xi)\widehat{g}(\xi).$$

We now compute the Fourier Transform of the Gaussian $f(x) = e^{-|x|^2}$. We have

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}} e^{-\left(\sum_i x_i^2 + \sum_i x_i\xi_i\right)} = \frac{1}{2^{n/2}}\exp\left(-\frac{1}{4}\|\xi\|^2\right).$$

Therefore, $G(x) = \exp(-\|x\|^2/2)$ is invariant under the Fourier Transform.

Chapter 11

Lecture 12

Theorem 11.1. *Let $f \in \mathscr{S}$. Then,*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{f}(\xi)\, d\xi.$$

Proof. Let

$$K_t(z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle} e^{-t|\xi|^2}\, d\xi.$$

Now,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} e^{-t|\xi|^2} f(y)\, dy d\xi = (f * K_t)(x).$$

Note that

$$K_t(z) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|z|^2}{4t}).$$

This is the Heat Kernel. Of course, we are assuming $t > 0$. ∎

Chapter 12

Lecture 13

Theorem 12.1 (Parseval's Identity). *If $u, v \in \mathscr{S}$, then $\langle u, v\rangle_{L^2} = \langle \widehat{u}, \widehat{v}\rangle_{L^2}$.*

Theorem 12.2 (Plancherel's Formula). $\|u\|_{L^2} = \|\widehat{u}\|_{L^2}$.

Note that the Schwarz space is dense in L^2 and hence, one can extend the Fourier transform to all of L^2. This would give that the Fourier transform is an isometry from L^2 to L^2 and is also a bijection.

Distributions and Weak Derivatives

We work with locally integrable functions. A function f is said to be locally integrable if $f \in L^1(K)$ for all compact $K \subseteq \mathbb{R}^n$. This class of functions is denoted by L^1_{loc}.

Weak Derivative

Definition 12.3. If $f \in L^1_{loc}(\mathbb{R}^n)$. We say that f is weakly differentiable with respect x_i if there exists $g_i \in L^1_{loc}$ such that for any "test function" $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f \partial_i \varphi = -\int_{\mathbb{R}^n} g_i \varphi.$$

The function g_i is the i-th weak partial derivative of f.

Definition 12.4. Let α be a multiindex and $f \in L^1_{loc}(\mathbb{R}^n)$. We say that f has weak derivative of order $|\alpha|$ if there is a function $g \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f \partial^\alpha \varphi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g \varphi.$$

For all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Note that the weak derivative is unique up to measure 0 sets. One can also show that mixed weak derivatives are equal upto measure 0 sets.

Example 12.5.

$$f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Consider

$$g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

We contend that g is the weak derivative of f. Note that f is differentiable on $(0, \infty)$ and hence, we can use integration by parts to conclude that g is the required function. I guess this can be used to prove a similar statement for piecewise smooth functions.

Example 12.6. The function $f = \chi_{[0,\infty)}$ is not weakly differentiable. If it were, then there would exist $g \in L^1_{loc}(\mathbb{R})$ such that

$$\int_0^\infty \varphi' = -\int_{\mathbb{R}} g\varphi,$$

for every test function φ on $\mathbb{R}$. The left hand side is $-\varphi(0)$.

Therefore,

$$\int_{\mathbb{R}} g\varphi = \varphi(0)$$

for all test functions φ. A standard argument shows that this is not possible. Ezpz nimbu squeezy.

Distributions

Let T be a linear functional on $C_c^\infty(\mathbb{R}^n)$. Let $D(\mathbb{R}^n)$ be the same space with the inductive limit topology. This is also called the space of "test functions".

$D(\mathbb{R}^n)$ is a complete metric space consisting of C_c^∞-functions.

Definition 12.7. In $D(\mathbb{R}^n)$, $\varphi_m \to \varphi$ if Supp $\varphi_m \subseteq K$, compact in $\mathbb{R}^n$. And $\text{Supp}(\varphi) \subseteq K$. On this compact set, $\varphi_m^{(n)} \to \varphi^{(n)}$ for all $n \geq 0$ in the sup norm.

Definition 12.8 (Distributions). Let $T \in D'(\mathbb{R}^n)$, where $D'(\mathbb{R}^n)$ is the dual of $D(\mathbb{R}^n)$. We write $T(\varphi)$.

It suffices to check

$$|\langle T, \varphi \rangle| \leq C_K \sum_{|\alpha| \leq i_K} \sup_K |\partial^\alpha \varphi|$$

for every compact K and for all φ such that $\text{Supp}(\varphi) \subseteq K$ and $i_K < \infty$.

Chapter 13

Lecture 14

Fix a compact set K. Let φ, ψ be elements of $D(\mathbb{R}^n)$. Then, we define

$$d_K(\varphi, \psi) = \max_{i \in \mathbb{N}} \frac{1}{2^i} \frac{\|\varphi - \psi\|_{K,i}}{1 + \|\varphi - \psi\|_{K,i}}.$$

where

$$\|\varphi\|_{K,i} = \sum_{j=0}^{i} \|\varphi^{(j)}\|_K.$$

In this metric, $D(K)$ is a complete metric space. This is called a Fréchet space.

A local base for this topology is $\|\varphi\|_{K,l} \leq \frac{1}{l}$ for all $l \in \mathbb{N}$. Therefore, $D(K)$ forms a locally convex topological vector space.

Take an exhaustion of $\mathbb{R}^n$, the natural choice is closed balls of integer radius. Let us denote this sequence by $\{K_n\}$ which are compact sets in $\mathbb{R}^n$. Further, $K_n \subseteq \operatorname{int} K_{n+1}$ and $\bigcup K_n = \mathbb{R}^n$.

$$p_{l,a}(\varphi) = \sup_{m \geq 1} \sup_{\mathbb{R} \setminus K_m} \frac{1}{a_m} \sum_{|\alpha| < l_m} |\partial^\alpha \varphi|$$

where $a_m \to 0$ and $l_m \to \infty$. Note that l and a are sequences. This gives a family of seminorms. These generate a topology on $D(\mathbb{R}^n)$, with which this is a complete metric space.

Distributions are dual of $D(\mathbb{R}^n)$ with the weak-$*$ topology. $T \in D'(\mathbb{R}^n)$ is a linear functional such that

1. T is continuous.
2. If $\varphi_m \to \varphi$ in $D(\mathbb{R}^n)$ then $T(\varphi_m) \to T(\varphi)$.
3. For every compact $K \subseteq \mathbb{R}^n$, there is a positive integer $i_K > 0$ and a constant C_K such that

$$|T(\varphi)| \leq C_K \sum_{|\alpha| \leq i_K} |\partial^\alpha \varphi|.$$

   For every φ such that $\operatorname{Supp} \varphi \subseteq K$.

Example 13.1. The Dirac Delta is an example of a distribution, δ_ξ where $\xi \in \mathbb{R}^n$. The action is given by $\delta_\xi(\varphi) = \varphi(\xi)$. It is not hard to see that this is indeed a distribution.

Example 13.2. Take $f \in L^1_{loc}(\mathbb{R}^n)$, then f generates a distribution $T_f \in D'(\mathbb{R}^n)$ whose action is defined as

$$T_f(\varphi) = \int_{\mathbb{R}^n} \varphi f.$$

This is also a zeroth order distribution, that is $i_K = 0$ works.

Example 13.3. Fix $\xi \in \mathbb{R}$. Then, T defined as

$$T(\varphi) = \varphi''(\xi)$$

Show that $T \in D'(\mathbb{R})$, that is, it is a distribution.

Now, we define the derivative of a distribution. The idea is to pass the derivative on the action.

Theorem 13.4 (Also a Definition). *Let $T \in D'(\mathbb{R}^n)$. We define the distributional derivative $\partial^\alpha T$ of order $|\alpha|$ of T as*

$$(\partial^\alpha T)(\varphi) = (-1)^\alpha T(\partial^\alpha \varphi).$$

This is obviously a linear functional on $D(\mathbb{R}^n)$. Also, the local boundedness thing is quite obvious.

We have $C_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$ where all the inclusions are strict. We care about the dual of the middle object. The dual of the Schwarz space is the space of Tempered Distributions.

Taking duals, we have

$$\mathcal{E}'(\mathbb{R}^n) \subseteq \mathscr{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n).$$

The first class is called the space of distributions with compact support.

Let $\Omega \subseteq \mathbb{R}^n$ be open. Then, we say that T vanishes on Ω if $T(\varphi) = 0$ for all $\varphi \in D(\Omega)$.

We define the support of T to be $\mathbb{R}^n \setminus \bigcup \Omega$ where the union is taken over all open sets on which T vanishes. A compactly supported distribution is one whose support is a compact set (quite obvious).

Note that if T has support K and $\psi \in D(\mathbb{R}^n)$, we can define

$$\langle \psi T, \varphi \rangle = T(\varphi\psi).$$

Theorem 13.5. *Let $T \in D'(\mathbb{R}^n)$. Then for any multi-index α, there are continuous functions $g_\alpha \in C(\mathbb{R}^n)$ such that any compact set K intersects the suport of g_α's.*

$$\langle T, \varphi \rangle = \sum_\alpha (-1)^\alpha \int_{\mathbb{R}^n} g_\alpha \partial^\alpha \varphi.$$

Chapter 14

Lecture 15

Definition 14.1 (Convolution of Distributions). Let $T \in D'(\mathbb{R}^n)$ and $\phi \in D(\mathbb{R}^n)$. Their convolution is defined as a function in $C^\infty(\mathbb{R}^n)$, given by

$$(T * \phi)(x) = \langle T, \phi(x - \cdot) \rangle.$$

We have

$$\partial^\alpha (T * \phi)(x) = \langle T, \partial^\alpha(\phi(x - \cdot)) \rangle = T * \partial^\alpha \phi(x) = \partial^\alpha T * \phi(x).$$

Be a bit careful here, the derivative is with respect to x instead of y. As a result, $T * \phi$ is a smooth function on $\mathbb{R}^n$.

Lemma 14.2. *Let $T \in D'(\mathbb{R}^n)$. Consider the standard mollifiers $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $\eta_\varepsilon > 0$, $\operatorname{Supp} \eta_\varepsilon \subseteq B(0, \varepsilon)$ and $\int_{\mathbb{R}^n} \eta_\varepsilon = 1$.*

Then, $T * \eta_\varepsilon \to T$ as $\eta \to 0$ in $D'(\mathbb{R}^n)$. That is, $\langle T * \eta_\varepsilon, \phi \rangle \to \langle T, \phi \rangle$ as $\varepsilon \to 0$.

Equivalently,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} (T * \eta_\varepsilon)(x) \phi(x) \, dx = \langle T, \phi \rangle.$$

Proof. Let $\widetilde{\phi}(x) = \phi(-x)$. Then,

$$\langle T, \phi \rangle = (T * \widetilde{\phi})(0).$$

Using this,

$$\langle T * \eta_\varepsilon, \phi \rangle = (T * \eta_\varepsilon) * \widetilde{\eta}(0) = T * (\eta_\varepsilon * \widetilde{\phi})(0) = \langle T, \widetilde{\eta_\varepsilon * \widetilde{\phi}} \rangle = \langle T, \widetilde{\eta}_\varepsilon * \phi \rangle = \langle T, \eta_\varepsilon * \phi \rangle.$$

Now, note that $\eta_\varepsilon * \phi \to \phi$ as $\varepsilon \to 0$. This completes the proof. We have cheated a lot in this proof btw. ∎

Example 14.3. We define the translation operator τ_ξ as

$$\tau_\xi \phi(x) = \phi(x - \xi).$$

Then,

$$\tau_\xi \phi(x) = (\delta_\xi * \phi)(x)$$

where δ_ξ is the dirac distribution centered at ξ. In particular, $\phi = \delta_0 * \phi$.

Also,

$$\partial^\alpha \phi = \partial^\alpha \delta_0 * \phi.$$

Therefore, derivatives can also be treated as convolutions with certain distributions (derivatives of dirac in fact).

Tempered Distributions

Let

$$\|\phi\|_{\alpha,\beta} := \sup_{\mathbb{R}^n} \left| x^\alpha \partial^\beta \phi \right|.$$

In the Schwarz class, this supremum is always finite (the definition). The Topology on the Schwarz class is the Frechet topology generated by the above seminorms.

There is a more workable definition of convergence (which determines the topology). $\phi_m \to \phi$ if for every pair of α, β, the seminorms tend to 0 blah blah.

Denote $\mathscr{S}'(\mathbb{R}^n)$ as the space of Tempered Distributions, as we saw earlier.

Definition 14.4. Let $T \in \mathscr{S}'(\mathbb{R}^n)$. We define the Fourier Transform of T as $\widehat{T} \in \mathscr{S}'(\mathbb{R}^n)$ as

$$\langle \widehat{T}, \phi \rangle = \langle T, \widehat{\phi} \rangle.$$

The right hand side is well defined since the Fourier transform of an element in the Schwarz class is in the Schwarz class.

Definition 14.5. The inverse Fourier transform of $T \in \mathscr{S}'(\mathbb{R}^n)$ is defined as

$$\langle \widetilde{T}, \phi \rangle = \langle T, \widetilde{\phi} \rangle$$

where $\widetilde{\phi}(x) = \phi(-x)$.

Example 14.6. Let $\mathbf{1}$ denote the constant distribution in the Tempered Class. Then,

$$\widehat{\mathbf{1}} = \sqrt{2}\delta_0,$$

where, we are working over $\mathbb{R}$ instead of $\mathbb{R}^n$.

Chapter 15

Lecture 16

Consider the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{on } (0,\infty) \times \mathbb{R}^n \\ u = u_0 & t = 0,\ x \in \mathbb{R}^n. \end{cases}$$

Take the Fourier transform in the x-variable to get

$$\partial_t \widehat{u}(\xi, t) = -\left(\sum_{j=1}^{n} \xi_j^2 \right) \widehat{u}(\xi, t).$$

This is an ordinary differential equation with solution

$$\widehat{u}(\xi, t) = A(\xi) \exp\left(-\|\xi\|^2 t \right)$$

with initial conditions $\widehat{u} = \widehat{u}_0$ for $t = 0$. This gives

$$A(\xi) = \widehat{u}_0(\xi).$$

Therefore,

$$\widehat{u}(\xi, t) = \widehat{u}_0(\xi) \exp\left(-\|\xi\|^2 t \right).$$

Being the product of two Fourier transforms, the inverse fourier transform of the above function is the convolution of the individual inverse fourier transforms.

The Inverse Fourier Transform of the second term is what we have already calculated,

$$K_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t} \right),$$

which is called the Heat Kernel.

Therefore,

$$u(x,t) = u_0(x) * K_t(x).$$

That is,

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t} \right) u_0(y)\, dy$$

for $t > 0$ and $x \in \mathbb{R}^n$.

Properties of the Heat Kernel

1. $K_t(x)$ is smooth on $(0, \infty) \times \mathbb{R}^n$.
2. $K_t(x) > 0$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$.
3. $(\partial_t - \Delta)K_t = 0$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$. Take this as an exercise.
4. $\int_{\mathbb{R}^n} K_t(x)\, dx = 1$ for all $t > 0$.
5. For all $\delta > 0$, we have that
$$\lim_{t \to 0} \int_{|x| \geq \delta} K_t(x)\, dx = 0.$$

Theorem 15.1. *Let $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then,*

$$u(x,t) := K_t * f = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)\, dy$$

satisfies the heat equation.

Proof. It is easy to see that u satisfies the differential equation. It remains to show continuity. Indeed,

$$u(x,t) - f(x) = \int_{\mathbb{R}^n} K_t(x-y)\left(f(y) - f(x)\right)\, dy.$$

Pick $\varepsilon > 0$. There is a $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $|x - y| < \delta$. Then, we have

$$|u(x,t) - f(x)| \leq \int_{|x-y|<\delta} + \int_{|x-y|\geq\delta}.$$

The first quantity is bounded above by ε while the second tends to 0 as $t \to \infty$. Conclude. ■

Chapter 16

Lecture 17

The Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Using integration by parts, we get

$$\Gamma(x+1) = x\Gamma(x).$$

Using this characterization, one can extend Γ on $\mathbb{R}\backslash\{0,-1,-2,\dots\}$. It also follows that $\Gamma(n) = n!$.

Theorem 16.1. *Consider the $(n-1)$-dimensional unit sphere, S^{n-1}. The surface area of it is given by*

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and we denote this by ω_{n-1}.

Proof. Recall that we had

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{n/2}.$$

Shifting to polar coordinates, we have

$$\int_{r=0}^\infty r^{n-1}\int_{S^{n-1}} f(r\sigma)d\sigma dr.$$

Since f is a radial function,

$$\pi^{n/2} = \int_{r=0}^\infty r^{n-1}f(r)\int_{S^{n-1}} d\sigma dr = \omega_{n-1}\int_{r=0}^\infty r^{n-1}e^{-r^2}\,dr.$$

Let $s = r^2$. Then, the integral on the right becomes $\frac{1}{2}\Gamma(n/2)$. This completes the proof. ∎

Laplace equation and Newtonian Potential

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n.$$

Taking Fourier transforms, we have

$$|\xi|^2\widehat{u}(\xi) = \widehat{f}(\xi).$$

Using the Plancherel formula, we have

$$\langle -\Delta\varphi, \varphi\rangle = \langle -\widehat{\Delta\varphi}, \widehat{\varphi}\rangle = \langle |\xi|^2\widehat{\varphi}(\xi), \widehat{\varphi}(\xi)\rangle \geq 0.$$

Define

$$(-\Delta)^{-1}f := \int_0^\infty (K_t * f)(x,t)\, dt$$
$$= \int_0^\infty \int_{\mathbb{R}^n} K_t(x-y) f(y)\, dydt$$

We shall calculate

$$\int_0^\infty K_t(x)\, dt = \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\, dt.$$

Make the substitution $s = 1/t$ to get

$$\frac{1}{(4\pi)^{n/2}} \int_0^\infty s^{(n-4)/2} e^{-|x|^2 s/4}\, ds.$$

Now set $r = |x|^2 s/4$. Then, we have

$$\frac{|x|^{2-n}}{(4\pi)^{n/2}} \int_0^\infty e^{-r} r^{(n-2)/2-1}\, dr = \frac{|x|^{2-n}}{4\pi^{n/2}} \Gamma\left(\frac{n-2}{2}\right)$$

for $n \geq 2$.

Then,

$$(-\Delta)^{-1}f = \int_{\mathbb{R}^n} \frac{d_n}{|x-y|^{n-2}} f(y)\, dy.$$

where

$$d_n = \frac{\Gamma\left(\frac{n}{2}-1\right)}{4\pi^{n/2}} = \frac{1}{4\pi^{n/2}} \frac{\Gamma(n/2)}{n/2-1} = \frac{1}{n-2} \frac{1}{\omega_{n-1}}.$$

Chapter 17

Lecture 18

Recall on $\mathbb{R}^n$, we have Green's Functions

$$G(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{(n-2)\omega_{n-1}}\frac{1}{|x|^{n-2}} & n \geq 3. \end{cases}$$

Then, $u(x) = (G * f)(x)$.

A harmonic function satisfies $\Delta u = 0$. Multiply by u and integrate by parts to obtain

$$0 = \int_{\mathbb{R}^n} u\Delta u = \sum_{i=1}^{n}\int_{\mathbb{R}^n} u\partial_{ii}u.$$

This gives (check this!)

$$\int_{\mathbb{R}^n} |\nabla u|^2 = 0.$$

This shows that u must be constant.

Let $E(u) = \int_{\mathbb{R}^n} |\nabla u|^2$. Consider

$$\begin{cases} -\Delta u = 0 & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Solutions of this equation can be obtained as

$$\min_{u\in C^2(B(0,1))}\int_{B(0,1)} |\nabla u|^2.$$

Let $E(u)$ denote the functional $\int_{B(0,1)} |\nabla u|^2$. Consider a minimizing sequence (u_α).

The Sobolev space $W^{1,2}$ is given by the set of all u that satisfy

$$\int_{B(0,1)} |\nabla u|^2 + u^2 < \infty.$$

which is equivalent to stating (using the Poincare inequality),

$$\int_{B(0,1)} |\nabla u|^2$$

Then, u_α weakly converges to u_0 in $W^{1,2}(B(0,1))$ and u_0 solves the equation. In this case it is just the trivial solution but this gives a general method to find solutions to such initial value problems. This is called the Dirichlet principle.

Wave Equation

$$\begin{cases} \partial_{tt} u = \Delta u & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = f(x) \\ \partial_t u(x,0) = g(x). \end{cases}$$

Take fourier transform with respect to x to obtain

$$\partial_{tt}\widehat{u}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t) = 0.$$

The initial conditions become

$$\widehat{u}(\xi,0) = \widehat{f}(\xi) \qquad \partial_t \widehat{u}(\xi,0) = \widehat{g}(\xi).$$

We get

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(|\xi|t) + \frac{\widehat{g}(\xi)}{|\xi|}\sin(|\xi|t).$$

Now, take the Inverse Fourier transform and obtain the following

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \cos(|\xi|t)\widehat{f}(\xi)\,d\xi + \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sin(|\xi|t)\frac{\widehat{g}(\xi)}{|\xi|}\,d\xi.$$

Now write this as a convolution of the inverse fourier transforms of $\sin/x$ and cos respectively. This should give you the kernel corresponding to the wave equation. This is not beautiful.

Define

$$E(t) = \int_{\mathbb{R}^n} \partial_t(u)^2 + \int_{\mathbb{R}^n} |\nabla u(t,\cdot)|^2.$$

Then,

$$E'(t) = 0 \implies E(t) = E(0) = \int_{\mathbb{R}^n} f^2 + \int_{\mathbb{R}^n} g^2.$$

that is, energy is conserved for the wave equation.

For $n = 1$, we have

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds.$$

For $n = 3$,

$$u(x,T) = \partial_t\left(\frac{t}{4\pi}\int_{S^2} f(x - t\sigma)\,d\sigma\right) + \frac{t}{4\pi}\int_{S^2} g(x - t\sigma)\,d\sigma.$$

For $n = 2$, we have

$$u(x,t) = \partial_t\left(\frac{t}{4\pi}\int_{|z|\le 1} f(x - tz)\frac{1}{\sqrt{1-|z|^2}}\,dz\right) + \frac{t}{4\pi}\int_{|z|\le 1} g(x - tz)\frac{1}{\sqrt{1-|z|^2}}\,dz.$$

Chapter 18

Lecture 19