ALGEBRAIC GEOMETRY

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ABSTRACT. These are terse "notes" in algebraic geometry which I've made for fun. I don't intend for these to be useful to anyone but myself. These are mostly drawn from a combination of [Vak23] through self-teaching along with some discussions with other students. I do not prove all results and use quite a few blackboxes from commutative algebra. Most results can be found with proof in [AM69] or [Lan02] as I shall reference them throughout the text.

CONTENTS

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1. SHEAVES

1.1. **The germ of a smooth function.** Let X be a topological space and $p \in X$. We shall consider elements of $C(X)$, the set (ring) of continuous functions from X to $\mathbb{R}$. Define the relation $\sim$ on $C(X)$ by $f \sim g$ if and only if there is a neighborhood U of p on which $f = g$. That this relation is an equivalence relation is not hard to show. The set of equivalence classes is denoted by $\mathscr{O}_p$. It is not hard to see that this has the structure of a ring with pointwise addition and multiplication, which is also well defined.

We contend that the ring $\mathscr{O}_p$ is local. Indeed, consider the ideal $\mathfrak{m}_p$ of all germs that vanish at p (that this is an ideal is trivial to check). Further, if $f \in \mathscr{O}_p \backslash \mathfrak{m}_p$, then there is a neighborhood U of p on which f is nonzero, whereby f is a unit in $\mathscr{O}_p$, implying the desired conclusion. The quotient ring is a field, $\mathbb{R}$.

1.2. **Presheaves.**

Definition 1.1 (Presheaf). Let X be a topological space and $\mathfrak{Top}(X)$ denote the poset category of all open sets in X along with inclusion maps. A *presheaf* on X is a contravariant functor $\mathscr{F}$ from $\mathfrak{Top}(X)$ to **CRing**.

Definition 1.2 (Stalk). Define the *stalk* of a presheaf $\mathscr{F}$ at a point $p \in X$ to be the colimit of the diagram induced by $\mathscr{F}$. The index category in this case, $\mathfrak{Top}^{\mathrm{op}}(X)$ is a filtered category since given any two open sets, there is an open set contained in both.

If $p \in U$ and $f \in \mathscr{F}(U)$, then the image of f in $\mathscr{F}_p$ is called the *germ of f at p*.

Definition 1.3 (Sheaf). A presehaf $\mathscr{F}$ is a *sheaf* if it satisfies the following two axioms.

Gluability axiom.: If $\{U_i\}_{i \in I}$ is an open cover of U, then given $f_i \in \mathscr{F}(U_i)$ for all $i \in I$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is some $f \in \mathscr{F}(U)$ such that $\rho_{U,U_i}(f) = f_i$ for all $i \in I$.

Identity axiom.: If $\{U_i\}_{i \in I}$ is an open cover of U and $f_1, f_2 \in \mathscr{F}(U)$, and $f_1|_{U_i} = f_2|_{U_i}$ for all $i \in I$, then $f_1 = f_2$.

A presheaf satisfying the identity axiom is called a *separated presheaf*.

Definition 1.4 (Morphisms). A *morphism of presheaves* $\mathscr{F}, \mathscr{G} : \mathfrak{Top}(X)^{\mathrm{op}} \to$ **CRing** is a natural transformation between the functors $\mathscr{F}$ and $\mathscr{G}$, that is, a collection $\{\phi(U)\}$ of maps for each $U \in \mathfrak{Top}(X)$ such that for each $U \hookrightarrow V$, the diagram

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\rho_{V,U}"] \arrow[d, "\phi(V)"'] & \mathscr{F}(U) \arrow[d, "\phi(U)"] \\
\mathscr{G}(V) \arrow[r, "\rho_{V,U}"'] & \mathscr{G}(U)
\end{tikzcd}$$

commutes. Similarly, a *morphism of sheaves* is simply a morphism of presheaves between sheaves, since the category of sheaves on X is a full subcategory of the category of presheaves on X.

Proposition 1.5. *Let $\pi : X \to Y$ be a continuous map and $\mathscr{F}$ a presheaf on X. Define $\pi_* \mathscr{F}$ by*

$$\pi_* \mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$$

where $V \in \mathfrak{Top}(Y)$. Then, $\pi_* \mathscr{F}$ is a presheaf on X.

Proof. Straightforward, since π^{-1} is itself a functor from $\mathfrak{Top}(Y)$ to $\mathfrak{Top}(X)$ and the composition of a contravariant functor and a covariant functor is a contravariant functor. ■

Definition 1.6. Let $\pi : X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ be a presheaf on X. Then, $\pi_* \mathscr{F}$ is called the *pushforward of $\mathscr{F}$ by π* and is a presheaf on Y.

References

[AM69] Michael Atiyah and Ian MacDonald. *Introduction to Commutative Algebra*. CRC Press, 1969.
[Lan02] Serge Lang. *Algebra*. Springer Science & Business Media, 2002.
[Vak23] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. 2023.