Representation Theory of Finite Groups

Swayam Chube

June 5, 2023

Contents

Abstract

Throughout this report, unless mentioned otherwise, all vector spaces are finite dimensional over $\mathbb{C}$.

Chapter 1

Representations of Finite Groups

Definition 1.1 (Representation). A *representation* of a group G is a homomorphism

$$\varphi : G \to \mathrm{Aut}_{\mathbf{Vec}}(V) = \mathrm{GL}(V)$$

for some finite-dimensional non-zero vector space V. The dimension of V is called the *degree* of φ.

In particular, from the above definition, we note that G acts on V and the action is compatible with the vector space structure of V. In this case, V is called a *G-module*. We shall use φ_g to denote $\varphi(g)$ and the action of g on v is denoted by $\varphi_g(v)$ or sometimes $g \cdot v$. **Henceforth, a representation refers to a representation $\varphi : G \to \mathrm{GL}(V)$ where V is a finite-dimensional nonzero $\mathbb{C}$-vector space.**

Definition 1.2 (Direct Sum of Representations). Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations. Then, the map

$$\varphi \oplus \psi : G \to \mathrm{GL}(V \oplus W)$$

given by

$$(\varphi \oplus \psi)_g (v, w) = \big(\varphi_g(v), \psi_g(w)\big)$$

for all $g \in G$ and $(v, w) \in V \oplus W$.

Note, for subspaces V_1 and V_2 of V, when we write $V = V_1 \oplus V_2$, we mean there is an isomorphism $V_1 \oplus V_2 \to V$ given by $(v_1, v_2) \mapsto v_1 + v_2$. This is known as the internal direct sum.

Definition 1.3 (Representation Homomorphism). Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations of a finite group G. A *homomorphism of representations* φ and ψ is a linear transformation $T : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"'] & W \end{tikzcd}$$

commutes for all $g \in G$. The set of all representation homomorphisms from φ to ψ is denoted by $\mathrm{Hom}_G(\varphi, \psi)$ and is a $\mathbb{C}$-vector space.

An *equivalence of representations* is a homomorphism of representations which is also an isomorphism of vector spaces.

Proposition 1.4. $\mathrm{Hom}_G(\varphi, \psi)$ *is a vector subspace of* $\mathrm{Hom}(V, W)$.

Proof. Indeed, if $S, T \in \mathrm{Hom}_G(\varphi, \psi)$ and $a \in \mathbb{C}$, then for all $v \in V$ and $g \in G$,

$$(S + aT)(\varphi_g(v)) = S \circ \varphi_g(v) + aS \circ \varphi_g(v) = \varphi_g(S(v)) + \varphi_g(aT(v)) = \varphi_g((S + aT)(v))$$

and the conclusion follows. ■

Definition 1.5 (G-invariant subspace). Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \leq V$ is said to be G-*invariant* if for all $g \in G$ and $w \in W$, $\varphi_g(w) \in W$. Or more succinctly, for each $g \in G$, $\varphi_g(W) \leq W$. A representation is said to be *irreducible* if has no nonzero proper G-invariant subspaces. It is said to be *reducible* otherwise.

Proposition 1.6. *Let* $\varphi : G \to \mathrm{GL}(V)$ *be reducible and* $\psi : G \to \mathrm{GL}(W)$ *be equivalent to* φ. *Then* ψ *is reducible.*

Proof. Let $T \in \mathrm{Hom}_G(V, W)$ be a linear isomorphism and $U \leq V$ be a nonzero proper G-invariant subspace. It is not hard to argue that $T(U)$ is G-invariant, consequently W is reducible. ■

Corollary. If a representation is equivalent to an irreducible representation, then it is irreducible.

Lemma 1.7. *Let* $\varphi : G \to \mathrm{GL}(V)$ *be a representation and* $W \leq V$ *be a* G*-invariant subspace. Then, the restriction* $\varphi|_W : G \to \mathrm{GL}(W)$ *is also a representation. This is called a* ***subrepresentation*** *of* φ.

Proof. Since $\varphi_g(w) \in W$ for each $w \in W$, we see that $\varphi_g|_W$ is a linear transformation $W \to W$ (as it descended from φ_g). Since $\varphi_g : V \to V$ has a trivial kernel, so does $\varphi_g|_W$, whereby it is a linear isomorphism. ■

Definition 1.8 (Decomposable Representation). A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *decomposable* if there are nonzero G-invariant subspaces V_1, V_2 of V such that $V = V_1 \oplus V_2$.

Obviously, every decomposable representation is reducible and equivalently, every irreducible representation is indecomposable.

Proposition 1.9. *If* $\varphi : G \to \mathrm{GL}(V)$ *is a decomposable representation with* $V = V_1 \oplus V_2$*, further, if* $\varphi_1 = \varphi|_{V_1}$ *and* $\varphi_2 = \varphi|_{V_2}$*, then* $\varphi \sim \varphi_1 \oplus \varphi_2$.

Proof. The map $T : V_1 \oplus V_2 \to V$ given by $T(v_1, v_2) = v_1 + v_2$ is a linear isomorphism. Therefore, for all $g \in G$,

$$T((\varphi_1 \oplus \varphi_2)_g(v_1, v_2)) = (\varphi_1)_g(v_1) + (\varphi_2)_g(v_2) = \varphi_g(v_1 + v_2) = \varphi_g(T(v_1, v_2))$$

implying the desired conclusion. ■

Remark 1.0.1. *Inductively, if* $V = V_1 \oplus \cdots \oplus V_n$ *and* $\varphi_i = \varphi|_{V_i}$*, then* $\varphi \sim \bigoplus_{i=1}^n \varphi_i$.

Proposition 1.10. *Let $\varphi : G \to \mathrm{GL}(V)$ be decomposable and $\psi : G \to \mathrm{GL}(W)$ a representation equivalent to φ. Then ψ is decomposable.*

Proof. Let $T \in \mathrm{Hom}_G(\varphi, \psi)$ be a linear isomorphism. Further, let $V_1, V_2 \leq V$ be nonzero proper G-invariant subspaces such that $V = V_1 \oplus V_2$. Let $W_1 = T(V_1)$ and $W_2 = T(V_2)$. Since T is an isomorphism, $W_1 \cap W_2 = 0$ and $W = W_1 + W_2$, whereby $W = W_1 \oplus W_2$. Further, for all $g \in G$ and $w_1 \in W_1$, there is a unique $v_1 \in V_1$ such that $T(v_1) = w_1$ and

$$\psi_g(w_1) = \psi_g(T(v_1)) = T(\varphi_g(v_1)) \in W_1$$

similarly, W_2 is also G-invariant and ψ is decomposable. ■

1.1 Schur's Lemma

Proposition 1.11. *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations and $T \in \mathrm{Hom}_G(\varphi, \psi)$. Then,* $\ker T$ *and* $\operatorname{im} T$ *are both G-invariant subspaces of V and W respectively.*

Proof. Indeed, for all $g \in G$, $v \in \ker T$ and $w \in \operatorname{im} T$, there is a corresponding $u \in V$ such that $T(u) = w$ and we have

$$T(\varphi_g(v)) = \psi_g(T(v)) = 0 \qquad \psi_g(w) = \psi_g(T(u)) = T(\varphi_g(u)) \in \operatorname{im} T$$

implying the desired conclusion. ■

Lemma 1.12 (Schur). *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be irreducible representations and $T \in \mathrm{Hom}_G(\varphi, \psi)$. Then,*

(a) T is invertible or $T = 0$.

(b) if $\varphi \not\sim \psi$, then $T = 0$.

(c) if $V = W$, then $T = \lambda \mathbf{id}_V$ for some $\lambda \in \mathbb{C}$.

Proof. (a) Since $\ker T$ is G-invariant, we must have $\ker T \in \{0, V\}$. In the latter case, $T = 0$. In the former case, we must have $\operatorname{im} T \in \{0, W\}$ obviously the former may not hold since V is nonzero, consequently, $\operatorname{im} T = W$ and T is a linear isomorphism.

(b) Immediate from (a).

(c) Since we are working over an algebraically closed field, $\mathbb{C}$, there is $\lambda \in \mathbb{C}$ which is an eigenvalue of T. Note that $\widetilde{T} = T - \lambda \mathbf{id}_V \in \mathrm{Hom}_G(V, V)$ but since $\ker \widetilde{T} \neq 0$, we must have $\widetilde{T} = 0$ and $T = \lambda \mathbf{id}_V$. ■

Corollary. An irreducible representation of an abelian group has degree 1, consequently, is a <u>character</u>.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation with G an abelian group. Fix some $g \in G$, then for all $h \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "\rho_h"] \arrow[d, "\rho_g"'] & V \arrow[d, "\rho_g"] \\ V \arrow[r, "\rho_h"'] & V \end{tikzcd}$$

commutes. Consequently, $\rho_g \in \mathrm{Hom}_G(\rho, \rho)$. From Lemma 1.12, $\rho_g = \lambda_g \mathbf{id}_V$. Due to the irreducibility of the representation, we must have $\dim V = 1$. ■

1.2 Maschke's Theorem

Definition 1.13 (Completely Reducible). A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *completely reducible* if there are nonzero proper G-invariant subspaces $\{V_i\}_{i=1}^n$ such that $V = V_1 \oplus \cdots \oplus V_n$ and $\varphi|_{V_i}$ is irreducible for all $1 \le i \le n$.

From Remark 1.0.1, we have $\varphi \sim \varphi_{V_1} \oplus \cdots \oplus \varphi_{V_n}$.

Definition 1.14 (Unitary Representation). Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. A representation $\rho : G \to \mathrm{GL}(V)$ is said to be *unitary* if for all $g \in G$ and $u, v \in V$,

$$\langle u, v \rangle = \langle \rho_g(u), \rho_g(v) \rangle$$

Remark 1.2.1. *If V is a finite dimensional $\mathbb{C}$ vector space, then there is a non trivial inner product on V. Indeed, pick any basis $\{v_i\}_{i=1}^n$ for V and define*

$$\left\langle \sum_{i=1}^n a_i v_i, \sum_{i=1}^n b_i v_i \right\rangle = \sum_{i=1}^n \overline{a_i} b_i$$

where $\overline{z}$ is the complex conjugate of z.

Lemma 1.15. *Let $\varphi : G \to \mathrm{GL}(V)$ be a unitary representation. If φ is reducible, then it is decomposable.*

Proof. Let $W \le V$ be a nonzero proper G-invariant subspace and $W^\perp$ its orthogonal complement. We contend that $W^\perp$ is G-invariant. This coupled with $V = W \oplus W^\perp$ would immediately imply the desired conclusion. Indeed, let $w^\perp \in W^\perp$. Then, for all $w \in W$ and $g \in G$, there is $w' \in W$ such that $\rho_g(w') = w$ and

$$\langle w, \rho_g(w^\perp) \rangle = \langle \rho_g(w'), \rho_g(w^\perp) \rangle = \langle w', w^\perp \rangle = 0$$

which completes the proof. ■

Proposition 1.16. *Every reducible representation of a finite group G is decomposable.*

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be a reducible representation. As observed in Remark 1.2.1, there is an inner product $\langle \cdot, \cdot \rangle$ associated with V. We shall construct a G-invariant inner product using this. Define, for $u, v \in V$,

$$(u, v) = \frac{1}{|G|} \sum_{g \in G} \langle \varphi_g(u), \varphi_g(v) \rangle$$

Obviously, $(u, u) \ge 0$, $(u, v) = \overline{(v, u)}$ and $(\alpha u + \beta v, w) = \overline{\alpha}(u, w) + \overline{\beta}(v, w)$ whereby $(\cdot, \cdot)$ is an inner product. Now, for any $g \in G$, we have

$$(\varphi_g(u), \varphi_g(v)) = \frac{1}{|G|} \sum_{h \in G} \langle \varphi_{hg}(u), \varphi_{hg}(v) \rangle = (u, v)$$

Upon equipping V with this inner product, φ is a unitary representation, and we are done due to Lemma 1.15. ■

Corollary. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. Then φ is either irreducible or decomposable.

Theorem 1.17 (Maschke). *Every representation of a finite group is completely reducible.*

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. We shall prove this statement by induction on the degree of φ. The base case with $\deg \varphi = 1$ is trivial. Now suppose $\deg \varphi = n > 1$. If φ is irreducible, then we are done. Else, φ is reducible and there are nonzero proper G-invariant subspaces U and W of V such that $V = U \oplus W$. Now, $\varphi|_U$ and $\varphi|_W$ are subrepresentations with degree strictly less than n, and hence the induction hypothesis applies. Consequently, we have decompositions:

$$U = U_1 \oplus \cdots \oplus U_m \qquad W = W_1 \oplus \cdots \oplus W_n$$

such that each subrepresentation $\varphi|_{U_i}$ and $\varphi|_{W_i}$ is irreducible. Since

$$V = U \oplus W = U_1 \oplus \cdots \oplus U_m \oplus W_1 \oplus \cdots \oplus W_n$$

we see that φ is completely reducible. This completes the proof. ■

Theorem 1.18. *Uniqueness of decomposition.*

Proof. Suppose there are equivalent decompositions $V_1 \oplus \cdots \oplus V_n$ and $W_1 \oplus \cdots \oplus W_m$ of a representation $\varphi : G \to \mathrm{GL}(V)$. Consider the composition $V_i \hookrightarrow V_1 \oplus \cdots \oplus V_n \xrightarrow{\mathbf{id}_V} W_1 \oplus \cdots \oplus W_m \twoheadrightarrow W_j$ and denote it by T_{ij}. We contend that $T_{ij} \in \mathrm{Hom}_G(\varphi|_{V_i}, \varphi|_{W_j})$. Indeed, for all $g \in G$ and $v_i \in V_i$, we have

$$T_{ij}(\varphi_g(v_i)) = \pi_j(\varphi_g(v_i)) = \varphi_g(\pi_j(v_i)) = \varphi_g(T_{ij}(v_i))$$

but since both $\varphi|_{V_i}$ and $\varphi|_{W_j}$ are irreducible representations, T_{ij} is either 0 or an isomorphism and the latter is possible if and only if $V_i = W_j$. This implies the desired conclusion, since now we have a bijection between the sets $\{V_i\}_{i=1}^n$ and $\{W_j\}_{j=1}^n$. ■