Representation Theory of Finite Groups

Swayam Chube

June 9, 2023

Contents

Abstract

Throughout this report, unless mentioned otherwise, all vector spaces are finite dimensional over $\mathbb{C}$.

Chapter 1

Representations of Finite Groups

Definition 1.1 (Representation). A *representation* of a group G is a homomorphism

$$\varphi : G \to \mathrm{Aut}_{\mathbf{Vec}}(V) = \mathrm{GL}(V)$$

for some finite-dimensional non-zero vector space V. The dimension of V is called the *degree* of φ.

In particular, from the above definition, we note that G acts on V and the action is compatible with the vector space structure of V. In this case, V is called a *G-module*. We shall use φ_g to denote $\varphi(g)$ and the action of g on v is denoted by $\varphi_g(v)$ or sometimes $g \cdot v$. **Henceforth, a representation refers to a representation $\varphi : G \to \mathrm{GL}(V)$ where V is a finite-dimensional nonzero $\mathbb{C}$-vector space and G is a finite group.**

Definition 1.2 (Direct Sum of Representations). Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations. Then, the map

$$\varphi \oplus \psi : G \to \mathrm{GL}(V \oplus W)$$

given by

$$(\varphi \oplus \psi)_g (v, w) = \left(\varphi_g(v), \psi_g(w)\right)$$

for all $g \in G$ and $(v, w) \in V \oplus W$.

Note, for subspaces V_1 and V_2 of V, when we write $V = V_1 \oplus V_2$, we mean there is an isomorphism $V_1 \oplus V_2 \to V$ given by $(v_1, v_2) \mapsto v_1 + v_2$. This is known as the internal direct sum.

Definition 1.3 (Representation Homomorphism). Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations of a finite group G. A *homomorphism of representations* φ and ψ is a linear transformation $T : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"'] & W \end{tikzcd}$$

commutes for all $g \in G$. The set of all representation homomorphisms from φ to ψ is denoted by $\mathrm{Hom}_G(\varphi, \psi)$ and is a $\mathbb{C}$-vector space.

An *equivalence of representations* is a homomorphism of representations which is also an isomorphism of vector spaces.

Proposition 1.4. $\mathrm{Hom}_G(\varphi, \psi)$ *is a vector subspace of* $\mathrm{Hom}(V, W)$.

Proof. Indeed, if $S, T \in \mathrm{Hom}_G(\varphi, \psi)$ and $a \in \mathbb{C}$, then for all $v \in V$ and $g \in G$,

$$(S + aT)(\varphi_g(v)) = S \circ \varphi_g(v) + aS \circ \varphi_g(v) = \varphi_g(S(v)) + \varphi_g(aT(v)) = \varphi_g((S + aT)(v))$$

and the conclusion follows. ■

Definition 1.5 (G-invariant subspace). Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \leq V$ is said to be *G-invariant* if for all $g \in G$ and $w \in W$, $\varphi_g(w) \in W$. Or more succinctly, for each $g \in G$, $\varphi_g(W) \leq W$. A representation is said to be *irreducible* if has no nonzero proper G-invariant subspaces. It is said to be *reducible* otherwise.

Proposition 1.6. *Let* $\varphi : G \to \mathrm{GL}(V)$ *be reducible and* $\psi : G \to \mathrm{GL}(W)$ *be equivalent to* φ. *Then* ψ *is reducible.*

Proof. Let $T \in \mathrm{Hom}_G(V, W)$ be a linear isomorphism and $U \leq V$ be a nonzero proper G-invariant subspace. It is not hard to argue that $T(U)$ is G-invariant, consequently W is reducible. ■

Corollary. If a representation is equivalent to an irreducible representation, then it is irreducible.

Lemma 1.7. *Let* $\varphi : G \to \mathrm{GL}(V)$ *be a representation and* $W \leq V$ *be a* G*-invariant subspace. Then, the restriction* $\varphi|_W : G \to \mathrm{GL}(W)$ *is also a representation. This is called a* ***subrepresentation*** *of* φ.

Proof. Since $\varphi_g(w) \in W$ for each $w \in W$, we see that $\varphi_g|_W$ is a linear transformation $W \to W$ (as it descended from φ_g). Since $\varphi_g : V \to V$ has a trivial kernel, so does $\varphi_g|_W$, whereby it is a linear isomorphism. ■

Definition 1.8 (Decomposable Representation). A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *decomposable* if there are nonzero G-invariant subspaces V_1, V_2 of V such that $V = V_1 \oplus V_2$.

Obviously, every decomposable representation is reducible and equivalently, every irreducible representation is indecomposable.

Proposition 1.9. *If* $\varphi : G \to \mathrm{GL}(V)$ *is a decomposable representation with* $V = V_1 \oplus V_2$, *further, if* $\varphi_1 = \varphi|_{V_1}$ *and* $\varphi_2 = \varphi|_{V_2}$, *then* $\varphi \sim \varphi_1 \oplus \varphi_2$.

Proof. The map $T : V_1 \oplus V_2 \to V$ given by $T(v_1, v_2) = v_1 + v_2$ is a linear isomorphism. Therefore, for all $g \in G$,

$$T((\varphi_1 \oplus \varphi_2)_g(v_1, v_2)) = (\varphi_1)_g(v_1) + (\varphi_2)_g(v_2) = \varphi_g(v_1 + v_2) = \varphi_g(T(v_1, v_2))$$

implying the desired conclusion. ■

Remark 1.0.1. *Inductively, if* $V = V_1 \oplus \cdots \oplus V_n$ *and* $\varphi_i = \varphi|_{V_i}$, *then* $\varphi \sim \bigoplus_{i=1}^n \varphi_i$.

Proposition 1.10. *Let $\varphi : G \to \mathrm{GL}(V)$ be decomposable and $\psi : G \to \mathrm{GL}(W)$ a representation equivalent to φ. Then ψ is decomposable.*

Proof. Let $T \in \mathrm{Hom}_G(\varphi, \psi)$ be a linear isomorphism. Further, let $V_1, V_2 \leq V$ be nonzero proper G-invariant subspaces such that $V = V_1 \oplus V_2$. Let $W_1 = T(V_1)$ and $W_2 = T(V_2)$. Since T is an isomorphism, $W_1 \cap W_2 = 0$ and $W = W_1 + W_2$, whereby $W = W_1 \oplus W_2$. Further, for all $g \in G$ and $w_1 \in W_1$, there is a unique $v_1 \in V_1$ such that $T(v_1) = w_1$ and

$$\psi_g(w_1) = \psi_g(T(v_1)) = T(\varphi_g(v_1)) \in W_1$$

similarly, W_2 is also G-invariant and ψ is decomposable. ∎

1.1 Schur's Lemma

Proposition 1.11. *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations and $T \in \mathrm{Hom}_G(\varphi, \psi)$. Then,* $\ker T$ *and* $\operatorname{im} T$ *are both G-invariant subspaces of V and W respectively.*

Proof. Indeed, for all $g \in G$, $v \in \ker T$ and $w \in \operatorname{im} T$, there is a corresponding $u \in V$ such that $T(u) = w$ and we have

$$T(\varphi_g(v)) = \psi_g(T(v)) = 0 \qquad \psi_g(w) = \psi_g(T(u)) = T(\varphi_g(u)) \in \operatorname{im} T$$

implying the desired conclusion. ∎

Lemma 1.12 (Schur). *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be irreducible representations and $T \in \mathrm{Hom}_G(\varphi, \psi)$. Then,*

(a) T is invertible or $T = 0$.

(b) if $\varphi \not\sim \psi$, then $T = 0$.

(c) if $V = W$, then $T = \lambda \mathbf{id}_V$ for some $\lambda \in \mathbb{C}$.

Proof. (a) Since $\ker T$ is G-invariant, we must have $\ker T \in \{0, V\}$. In the latter case, $T = 0$. In the former case, we must have $\operatorname{im} T \in \{0, W\}$ obviously the former may not hold since V is nonzero, consequently, $\operatorname{im} T = W$ and T is a linear isomorphism.

(b) Immediate from (a).

(c) Since we are working over an algebraically closed field, $\mathbb{C}$, there is $\lambda \in \mathbb{C}$ which is an eigenvalue of T. Note that $\widetilde{T} = T - \lambda \mathbf{id}_V \in \mathrm{Hom}_G(V, V)$ but since $\ker \widetilde{T} \neq 0$, we must have $\widetilde{T} = 0$ and $T = \lambda \mathbf{id}_V$. ∎

Corollary. An irreducible representation of an abelian group has degree 1, consequently, is a character.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation with G an abelian group. Fix some $g \in G$, then for all $h \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "\rho_h"] \arrow[d, "\rho_g"'] & V \arrow[d, "\rho_g"] \\ V \arrow[r, "\rho_h"'] & V \end{tikzcd}$$

commutes. Consequently, $\rho_g \in \mathrm{Hom}_G(\rho, \rho)$. From Lemma 1.12, $\rho_g = \lambda_g \mathbf{id}_V$. Due to the irreducibility of the representation, we must have $\dim V = 1$. ∎

1.2 Maschke's Theorem

Definition 1.13 (Completely Reducible). A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *completely reducible* if there are nonzero proper G-invariant subspaces $\{V_i\}_{i=1}^n$ such that $V = V_1 \oplus \cdots \oplus V_n$ and $\varphi|_{V_i}$ is irreducible for all $1 \leq i \leq n$.

From Remark 1.0.1, we have $\varphi \sim \varphi_{V_1} \oplus \cdots \oplus \varphi_{V_n}$.

Definition 1.14 (Unitary Representation). Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. A representation $\rho : G \to \mathrm{GL}(V)$ is said to be *unitary* if for all $g \in G$ and $u, v \in V$,

$$\langle u, v \rangle = \langle \rho_g(u), \rho_g(v) \rangle$$

Remark 1.2.1. *If V is a finite dimensional $\mathbb{C}$ vector space, then there is a non trivial inner product on V. Indeed, pick any basis $\{v_i\}_{i=1}^n$ for V and define*

$$\left\langle \sum_{i=1}^n a_i v_i, \sum_{i=1}^n b_i v_i \right\rangle = \sum_{i=1}^n \overline{a_i} b_i$$

where $\overline{z}$ is the complex conjugate of z.

Proposition 1.15. *Let $\varphi : G \to \mathrm{GL}(V)$ be a unitary representation, where V is a finite dimensional inner product space. Then, there is an equivalent unitary representation $\psi : G \to \mathrm{GL}(n, \mathbb{C})$.*

Proof. Due to Gram-Schmidt Orthonormalization, there is an orthonormal basis $\{v_1, \ldots, v_n\}$. Consider the linear isomorphism $T : V \to \mathbb{C}^n$ given by $T(e_i) = v_i$ for $1 \leq i \leq n$. Next, define $\psi : G \to \mathrm{GL}(n, \mathbb{C})$ by $\psi_g(e_i) = T(\varphi_g(v_i))$ for $1 \leq i \leq n$ and extend linearly. First, we must show that ψ is a group homomorphism. Indeed, let $g, h \in G$ and let $\varphi_h(v_i) = \sum_{j=1}^n a_j v_j$. Then,

$$\begin{aligned}
\psi_g(\psi_h(v_i)) &= \psi_g(T(\varphi_h(v_i))) \\
&= \psi_g\left(T\left(\sum_{j=1}^n a_j v_j\right)\right) \\
&= \sum_{j=1}^n a_j \psi_g(e_j) \\
&= \sum_{j=1}^n a_j T(\varphi_g(e_j)) \\
&= T\left(\varphi_g\left(\sum_{j=1}^n a_j e_j\right)\right) \\
&= T(\varphi_g(\varphi_h(v_i))) = \psi_{gh}(v_i).
\end{aligned}$$

Next, we must show that ψ is a unitary representation. For this, it suffices to show that ψ_g conserves the inner product for the standard basis. Indeed,

$$\langle \psi_g(e_i), \psi_g(e_j) \rangle = \langle T(\varphi_g(v_i)), T(\varphi_g(v_j)) \rangle = \langle \varphi_g(v_i), \varphi_g(v_j) \rangle = \langle v_i, v_j \rangle = \delta_{ij}$$

This completes the proof. ∎

Lemma 1.16. *Let $\varphi : G \to \mathrm{GL}(V)$ be a unitary representation. If φ is reducible, then it is decomposable.*

Proof. Let $W \leq V$ be a nonzero proper G-invariant subspace and $W^\perp$ its orthogonal complement. We contend that $W^\perp$ is G-invariant. This coupled with $V = W \oplus W^\perp$ would immediately imply the desired conclusion. Indeed, let $w^\perp \in W^\perp$. Then, for all $w \in W$ and $g \in G$, there is $w' \in W$ such that $\rho_g(w') = w$ and

$$\langle w, \rho_g(w^\perp)\rangle = \langle \rho_g(w'), \rho_g(w^\perp)\rangle = \langle w', w^\perp\rangle = 0$$

which completes the proof. ■

Proposition 1.17. *Every reducible representation of a finite group G is decomposable.*

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be a reducible representation. As observed in Remark 1.2.1, there is an inner product $\langle \cdot, \cdot \rangle$ associated with V. We shall construct a G-invariant inner product using this. Define, for $u, v \in V$,

$$(u, v) = \frac{1}{|G|} \sum_{g \in G} \langle \varphi_g(u), \varphi_g(v)\rangle$$

Obviously, $(u, u) \geq 0$, $(u, v) = \overline{(v, u)}$ and $(\alpha u + \beta v, w) = \overline{\alpha}(u, w) + \overline{\beta}(v, w)$ whereby $(\cdot, \cdot)$ is an inner product. Now, for any $g \in G$, we have

$$(\varphi_g(u), \varphi_g(v)) = \frac{1}{|G|} \sum_{h \in G} \langle \varphi_{hg}(u), \varphi_{hg}(v)\rangle = (u, v)$$

Upon equipping V with this inner product, φ is a unitary representation, and we are done due to Lemma 1.16. ■

Corollary. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. Then φ is either irreducible or decomposable.

Theorem 1.18 (Maschke). *Every representation of a finite group is completely reducible.*

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. We shall prove this statement by induction on the degree of φ. The base case with $\deg \varphi = 1$ is trivial. Now suppose $\deg \varphi = n > 1$. If φ is irreducible, then we are done. Else, φ is reducible and there are nonzero proper G-invariant subspaces U and W of V such that $V = U \oplus W$. Now, $\varphi|_U$ and $\varphi|_W$ are subrepresentations with degree strictly less than n, and hence the induction hypothesis applies. Consequently, we have decompositions:

$$U = U_1 \oplus \cdots \oplus U_m \qquad W = W_1 \oplus \cdots \oplus W_n$$

such that each subrepresentation $\varphi|_{U_i}$ and $\varphi|_{W_i}$ is irreducible. Since

$$V = U \oplus W = U_1 \oplus \cdots \oplus U_m \oplus W_1 \oplus \cdots \oplus W_n$$

we see that φ is completely reducible. This completes the proof. ■

Theorem 1.19. *Uniqueness of decomposition.*

Proof. Suppose there are equivalent decompositions $V_1 \oplus \cdots \oplus V_n$ and $W_1 \oplus \cdots \oplus W_m$ of a representation $\varphi : G \to \mathrm{GL}(V)$. Consider the composition $V_i \hookrightarrow V_1 \oplus \cdots \oplus V_n \xrightarrow{\mathbf{id}_V} W_1 \oplus \cdots \oplus W_m \twoheadrightarrow W_j$ and denote it by T_{ij}. We contend that $T_{ij} \in \mathrm{Hom}_G(\varphi|_{V_i}, \varphi|_{W_j})$. Indeed, for all $g \in G$ and $v_i \in V_i$, we have

$$T_{ij}(\varphi_g(v_i)) = \pi_j(\varphi_g(v_i)) = \varphi_g(\pi_j(v_i)) = \varphi_g(T_{ij}(v_i))$$

but since both $\varphi|_{V_i}$ and $\varphi|_{W_j}$ are irreducible representations, due to Lemma 1.12, T_{ij} is either 0 or an isomorphism and the latter is possible if and only if $V_i = W_j$. This implies the desired conclusion, since now we have a bijection between the sets $\{V_i\}_{i=1}^n$ and $\{W_j\}_{j=1}^n$. ■

Chapter 2

Character Theory

Again, throughout this chapter, G denotes a finite group and all vector spaces V are finite dimensional, nonzero and over $\mathbb{C}$.

2.1 Schur's Orthogonality Relations

Definition 2.1 (Group Algebra). Let k be a field and G a finite group. Define

$$k[G] = \{f \mid f : G \to k \text{ is a morphism in } \mathbf{Set}\}$$

Further, define addition and multiplication as

$$\begin{aligned} (f_1 + f_2)(g) &= f_1(g) + f_2(g) \\ (cf)(g) &= cf(g) \\ (f_1 \cdot f_2)(g) &= \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2) \end{aligned}$$

Further, if $k = \mathbb{C}$, then we may define an inner product as

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

Finally, there is a natural inclusion $\iota : k \hookrightarrow k[G]$ where

$$\iota(c)(g) = \begin{cases} c & g = 1_G \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.2. *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be representations and suppose $T \in \mathrm{Hom}_{\mathbb{C}}(V, W)$. Then,*

(a) $T^\sharp = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \psi_{g^{-1}} \circ T \circ \varphi_g \in \mathrm{Hom}_G(\varphi, \psi)$.

(b) if $T \in \mathrm{Hom}_G(\varphi, \psi)$, then $T^\sharp = T$.

(c) the map $P : \mathrm{Hom}_{\mathbb{C}}(V, W) \to \mathrm{Hom}_G(\varphi, \psi)$ defined by $P(T) = T^\sharp$ is a surjective linear transformation.

Proof. The proof of (a) follows from elementary computation. Indeed, for all $v \in V$, we have

$$\begin{aligned}
T^\sharp(\varphi_g(v)) &= \frac{1}{|G|}\sum_{h\in G}\psi_{h^{-1}}\circ T\circ\varphi_{hg}(v)\\
&= \frac{1}{|G|}\sum_{h\in G}\psi_{gh^{-1}}\circ T\circ\varphi_{h}(v)\\
&= \frac{1}{|G|}\sum_{h\in G}\psi_g\circ\psi_{h^{-1}}\circ T\circ\varphi_{h}(v)\\
&= \psi_g\circ T^\sharp(v)
\end{aligned}$$

whence $T^\sharp \in \mathrm{Hom}_G(\varphi,\psi)$.

Now, if $T \in \mathrm{Hom}_G(\varphi,\psi)$, then $T\circ\varphi_g(v) = \psi_g\circ T(v)$, whereby for all $g \in G$ and $v \in V$,

$$T^\sharp(v) = \frac{1}{|G|}\sum_{g\in G}\psi_{g^{-1}}\circ\psi_g\circ T(v) = T(v)$$

That P is surjective is obvious from (b). It remains to show that it is a linear transformation. Indeed, if $S, T \in \mathrm{Hom}_{\mathbb{C}}(V,W)$ and $a \in \mathbb{C}$, then

$$\begin{aligned}
(S+aT)^\sharp(v) &= \frac{1}{|G|}\sum_{g\in G}\psi_{g^{-1}}\circ(S+aT)\circ\varphi_g(v)\\
&= \frac{1}{|G|}\sum_{g\in G}\psi_{g^{-1}}\circ S\circ\varphi_g(v) + \frac{1}{|G|}\sum_{g\in G}a\psi_{g^{-1}}\circ T\circ\varphi_g(v)\\
&= S^\sharp + aT^\sharp
\end{aligned}$$

implying the desired conclusion. ■

Proposition 2.3. *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be irreducible representations of G and let $T : V \to W$ be a linear map. Then,*

(a) if $\varphi \not\sim \psi$, then $T^\sharp = 0$

(b) if $\varphi = \psi$, then $T^\sharp = \dfrac{\mathrm{tr}(T)}{\deg\varphi}\mathbf{id}_V$

Recall from linear algebra that the *trace* of a linear operator between finite dimensional vector spaces is independent of the choice of a basis, and thus the quantity $\mathrm{tr}(T)$ is unambiguous.

Proof. Since $T^\sharp \in \mathrm{Hom}_G(\varphi,\psi)$, due to Lemma 1.12, we must have $T^\sharp = 0$. On the other hand, if $\varphi = \psi$, then $T^\sharp = \lambda\mathbf{id}_V$ for some $\lambda \in \mathbb{C}$. We have

$$\mathrm{tr}(T^\sharp) = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}\left(\varphi_{g^{-1}}\circ T\circ\varphi_g\right) = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}\left(T\circ\varphi_g\circ\varphi_{g^{-1}}\right) = \mathrm{tr}(T).$$

But we also have

$$\mathrm{tr}(T) = \mathrm{tr}(T^\sharp) = \lambda\,\mathrm{tr}(\mathbf{id}_V) = \lambda\deg\varphi$$

This completes the proof. ■

If $\varphi : G \to \mathrm{GL}(n,\mathbb{C})$ is a representation, for $1 \le i,j \le n$, define $\varphi_{ij} : G \to \mathbb{C}$, a set map by $\varphi_{ij}(g) = (\varphi(g))_{ij}$, which is the (i,j)-th entry of the matrix φ_g. Note that $\varphi_{ij} \in \mathbb{C}[G]$ and we shall treat it as such while talking about inner products.

Within the ring $\mathcal{M}_{mn}(\mathbb{C})$, let E_{ij} denote the matrix with the (i,j)-th entry as 1 and the others as 0.

Lemma 2.4. *Let $\varphi : G \to U(n, \mathbb{C})$ and $\psi : G \to U(n, \mathbb{C})$ be unitary representations. Let $A = E_{kl} \in \mathcal{M}_{mn}(\mathbb{C})$. Then, $A^\sharp = (\langle \psi_{ki}, \varphi_{lj} \rangle)_{ij}$.*

Note that $\mathcal{M}_{mn}(\mathbb{C})$ is precisely $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m)$ and thus A represents a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ and hence it makes sense to define $A^\sharp$.

Proof. We have

$$A^\sharp = \frac{1}{|G|} \sum_{g \in G} \psi(g^{-1}) E_{kl} \varphi(g)$$

In particular,

$$(A^\sharp)_{ij} = \frac{1}{|G|} \sum_{g \in G} (\psi(g^{-1}))_{ik} (\varphi(g))_{lj} = \sum_{g \in G} \overline{\psi_{ik}(g)} \varphi_{lj}(g)$$

where the last equality follows since the matrix $\psi(g)$ is unitary, consequently, $\psi(g^{-1}) = \psi(g)^*$. ∎

Theorem 2.5 (Schur's Orthogonality Relations). *Let $\varphi : G \to U(n, \mathbb{C})$ and $\psi : G \to U(m, \mathbb{C})$ be inequivalent irreducible unitary representations. Then,*

(a) $\langle \psi_{kl}, \varphi_{ij} \rangle = 0$ *for all* $1 \leq i, j \leq n$ *and* $1 \leq k, l \leq m$.

(b) $\langle \varphi_{kl}, \varphi_{ij} \rangle = \begin{cases} 1/n & i = k \wedge j = l \\ 0 & \textit{otherwise} \end{cases}$

Proof. From Lemma 2.4, $\langle \psi_{kl}, \varphi_{ij} \rangle = (E_{ki}^\sharp)_{lj}$. But due to Proposition 2.3, $E_{ki}^\sharp = 0$ whence (a) follows. Similarly,

$$\langle \varphi_{kl}, \varphi_{ij} \rangle = (E_{ki}^\sharp)_{lj} = \left(\frac{\mathrm{tr}(E_{ki})}{n} \mathbf{id}_n \right)_{lj} = \frac{1}{n} \delta_{ki} \delta_{lj}$$

and the conclusion follows. ∎

2.2 Characters

Definition 2.6 (Character of a Representation). Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. The *character* of φ is a function $\chi_\varphi : G \to \mathbb{C}$ given by $\chi_\varphi(g) = \mathrm{tr}(\varphi_g)$.

In particular, if $\varphi : G \to \mathrm{GL}(n, \mathbb{C})$ is a representation, then

$$\chi_\varphi(g) = \sum_{i=1}^{n} \varphi_{ii}(g)$$

Remark 2.2.1. *Since $\varphi(1_G) = \mathbf{id}_V$, $\chi_\varphi(1_G) = \mathrm{tr}(\mathbf{id}_V) = \deg \varphi$. Thus, the character encodes information about the degree of a representation.*

Proposition 2.7. *If $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are equivalent representations, then $\chi_\varphi = \chi_\psi$.*

Proof. There is a linear isomorphism $T : V \to W$ such that for all $g \in G$, $\psi_g = T \circ \varphi_g T^{-1}$ and thus

$$\chi_\psi(g) = \mathrm{tr}(T \circ \varphi_g T^{-1}) = \mathrm{tr}(\varphi_g) = \chi_\varphi(g)$$

∎

Lemma 2.8. *Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(V)$ be two representations of G. Then,*

$$\chi_{\varphi\oplus\psi} = \chi_\varphi + \chi_\psi$$

Proof. We may suppose without loss of generality that $\varphi : G \to \mathrm{GL}(n,\mathbb{C})$ and $\psi : G \to \mathrm{GL}(m,\mathbb{C})$ for some positive integers m and n. Then, $\varphi \oplus \psi : G \to \mathrm{GL}(n+m,\mathbb{C})$ is given by

$$(\varphi\oplus\psi)_g = \begin{pmatrix} [\varphi_g] & 0 \\ 0 & [\psi_g] \end{pmatrix}$$

The conclusion is obvious. ■

Proposition 2.9. *Let $\varphi : G \to \mathrm{GL}(V)$ be a representation of G. Then, for all $g, h \in G$,*

$$\chi_\varphi(g) = \chi_\varphi(hgh^{-1})$$

In particular, χ_φ is a function on the conjugacy classes of G.

Proof. We have

$$\chi_\varphi(hgh^{-1}) = \mathrm{tr}(\varphi_{hgh^{-1}}) = \mathrm{tr}(\varphi_h \circ \varphi_g \circ \varphi_{h^{-1}}) = \mathrm{tr}(\varphi_g) = \chi_\varphi(g)$$

■

Definition 2.10 (Class Function). Let k be a field. A function $f : G \to k$ is called a *class function* if $f(g) = f(hgh^{-1})$ for all $g, h \in G$. That is, f is constant on the conjugacy classes of G. The space of such functions is denoted by $Z(k[G])$.

Note that every character χ_ρ is an element of $\mathbb{C}[G]$, in particular, it lies in $Z(\mathbb{C}[G])$.

Proposition 2.11. *$Z(k[G])$ is the center of the algebra $k[G]$. Consequently, $Z(k[G])$ is a subspace of $k[G]$.*

Proof. Straightforward computation. ■

Corollary. $\dim Z(k[G]) = |\,\mathrm{cl}(G)|$.

Proof. Let $C_1, \ldots, C_k$ denote the conjugacy classes of G. The functions $\{f_i\}_{i=1}^k$ form a basis for $Z(k[G])$, where

$$f_i(g) = \begin{cases} 1 & g \in C_i \\ 0 & \text{otherwise} \end{cases}$$

since they are orthonormal and span $Z(k[G])$. ■

Theorem 2.12 (First Orthogonality Relations). *If $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are irreducible representations, then*

$$\langle \chi_\varphi, \chi_\psi \rangle = \begin{cases} 1 & \varphi \sim \psi \\ 0 & \varphi \not\sim \psi \end{cases}$$

Proof. Without loss of generality, due to Proposition 1.15, we may suppose that $\varphi : G \to U(n, \mathbb{C})$ and $\psi : G \to U(m, \mathbb{C})$ are unitary representations. Note that this does not change the value of the character. We have

$$\begin{aligned}
\langle \chi_\varphi, \chi_\psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\varphi(g)} \chi_\psi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \overline{\varphi_{ii}(g)} \sum_{j=1}^{m} \psi_{jj}(g) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{|G|} \sum_{g \in G} \overline{\varphi_{ii}(g)} \psi_{jj}(g) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \varphi_{ii}(g), \psi_{jj}(g) \rangle
\end{aligned}$$

If $\varphi \not\sim \psi$, then every term in the sum is zero, whereby $\langle \chi_\varphi, \chi_\psi \rangle = 0$. On the other hand, if $\varphi \sim \psi$, then we may suppose without loss of generality that $\varphi = \psi$, in which case, we have

$$\langle \chi_\varphi, \chi_\varphi \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{|G|} \delta_{ij} = n/|G| = 1$$

This completes the proof. ∎

Corollary. There are at most $|\mathbf{cl}(G)|$ equivalence classes of irreducible representations.

We have now established that there are finitely many equivalence classes of irreducible representations. Pick a representative from each equivalence class, say $\rho^{(1)}, \ldots, \rho^{(s)}$ such that each $\rho^{(i)} : G \to U(\deg \rho^{(i)}, \mathbb{C})$ is a unitary representation[1]. Further, introduce the notation

$$n\rho^{(i)} = \underbrace{\rho^{(i)} \oplus \cdots \oplus \rho^{(i)}}_{n-\text{times}}$$

Then, every representation φ of G is equivalent to

$$n_1 \rho^{(1)} \oplus \cdots \oplus n_s \rho^{(s)}$$

and thus

$$\chi_\varphi = n_1 \chi_{\rho^{(1)}} + \cdots + n_s \chi_{\rho^{(s)}}$$

The integers n_i may be recovered from χ_φ as

$$n_i = \langle \chi_\varphi, \chi_{\rho^{(i)}} \rangle$$

2.3 Regular Representation

Recall that $\mathbb{C}[G]$ is a $\mathbb{C}$-vector space with elements of the form

$$\sum_{g \in G} c_g g$$

[1] Recall that this can be done since every representation is equivalent to a unitary representation

Define the action of G on $\mathbb{C}[G]$ by

$$g \cdot \left(\sum_{h \in G} c_g h \right) = \sum_{h \in G} c_h gh = \sum_{h \in G} c_{g^{-1}h} h$$

It is not hard to verify that G acts through linear isomorphisms, whereby, we have a homomorphism $\Phi : G \to \mathrm{GL}(\mathbb{C}[G])$. This representation is called the **regular representation** of G. The degree of this representation is $|G|$.

Proposition 2.13. *We have*

$$\chi_\Phi(g) = \begin{cases} |G| & g = 1_G \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Consider $\mathbb{C}[G]$ with the basis $\{g \mid g \in G\}$. If $g \neq 1_G$, then the matrix of the linear transformation Φ_g with respect to this basis has no elements on the diagonal, since left multiplication by an element of a group has no fixed points.

On the other hand, if $g = 1_G$, then the diagonal of the matrix representation of Φ_g is composed of only 1's, whereby $\mathrm{tr}(\Phi_{1_G}) = |G|$. ∎

We shall now represent Φ as the direct sum of the irreducible representations $\rho^{(1)}, \ldots, \rho^{(s)}$. The multiplicity d_i of each $\rho^{(i)}$ may be recovered as

$$d_i = \langle \chi_\varphi, \chi_{\rho^{(i)}} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\varphi(g)} \chi_{\rho^{(i)}}(g) = \frac{1}{|G|} \overline{\chi_\varphi(1_G)} \chi_{\rho^{(i)}}(1_G) = \deg \rho^{(i)}$$

Thus, we have

$$|G| = \deg \Phi = \sum_{i=1}^{s} d_i \deg \rho^{(i)} = \sum_{i=1}^{s} (\deg \rho^{(i)})^2 = \sum_{i=}^{s} d_i^2$$

Lemma 2.14. *The set*

$$\mathcal{B} = \{ \sqrt{d_k} \rho_{ij}^{(k)} \mid 1 \leq k \leq s,\ 1 \leq i, j \leq d_k \}$$

forms an orthonormal basis for $\mathbb{C}[G]$.

Proof. Orthonormality follows from Theorem 2.5. On the other hand, since

$$|\mathcal{B}| = \sum_{i=1}^{s} d_i^2 = |G| = \dim \mathbb{C}[G]$$

we have that $\mathcal{B}$ is a basis. ∎

Theorem 2.15. $\{\chi_{\rho^{(1)}}, \ldots, \chi_{\rho^{(s)}}\}$ *forms a basis for* $Z(\mathbb{C}[G])$.

Proof. Obviously, the aforementioned characters are orthonormal, and thus linearly independent. We shall show they span $Z(\mathbb{C}[G])$. Let $f \in Z(\mathbb{C}[G])$ be a class function. Then, there are $c_{ij}^{(k)}$ such that

$$f = \sum_{i,j,k} c_{ij}^{(k)} \rho_{ij}^{(k)}$$

Since f is a class function, we may write, for all $x \in G$,

$$\begin{aligned} f(x) &= \frac{1}{|G|} \sum_{g \in G} f(gxg^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} c_{ij}^{(k)} \rho_{ij}^{(k)}(gxg^{-1}) \\ &= \sum_{i,j,k} c_{ij}^{(k)} \frac{1}{|G|} \left[\sum_{g \in G} \rho_g^{(k)} \rho_x^{(k)} \rho_{g^{-1}}^{(k)} \right]_{ij} \\ &= \sum_{i,j,k} c_{ij}^{(k)} \left[(\rho_x^{(k)})^\sharp \right]_{ij} \\ &= \sum_{i,j,k} c_{ij}^{(k)} \frac{\operatorname{tr}(\rho_x^{(k)})}{\deg \rho^{(k)}} \delta_{ij} \\ &= \sum_{i,j,k} c_{ij}^{(k)} \chi_{\rho^{(k)}}(x) \delta_{ij} \end{aligned}$$

whereby f is a linear combination of $\{\chi_{\rho^{(1)}}, \ldots, \chi_{\rho^{(s)}}\}$. This completes the proof. ■

Corollary. There are exactly $|\mathbf{cl}(G)| = \dim Z(\mathbb{C}[G])$ inequivalent irreducible representations of G.

Definition 2.16 (Character Table). Let G be a finite group and $C_1, \ldots, C_s$ the conjugacy classes of G and $\chi_1, \ldots, \chi_s$ the corresponding irreducible characters[a]. The *character table* of G is the $s \times s$ matrix $\mathbf{X}$ with $X_{ij} = \chi_i(C_j)$.

[a] That is, characters of inequivalent irreducible representations

We contend that the character table has orthogonal columns. This would imply that the character table $\mathbf{X}$ is invertible.

Theorem 2.17 (Second Orthogonality Theorem). *Let C and C' be conjugacy classes of G, further, let $g \in C$ and $g' \in C'$. Then,*

$$\sum_{i=1}^{s} \overline{\chi_i(g)} \chi_i(g') = \begin{cases} |G|/|C| & C = C' \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let δ_C be the indicator function for the conjugacy class C. Then, we have

$$\begin{aligned} \delta_C(g') &= \sum_{i=1}^{s} \langle \chi_i, \delta_C \rangle \chi_i(g') \\ &= \sum_{i=1}^{s} \frac{1}{|G|} \sum_{x \in G} \overline{\chi_i(x)} \delta_C(x) \chi_i(g') \\ &= \sum_{i=1}^{s} \frac{1}{|G|} \sum_{x \in C} \overline{\chi_i(x)} \chi_i(g') \\ &= \sum_{i=1}^{s} \frac{|C|}{|G|} \overline{\chi_i(g)} \chi_i(g') \end{aligned}$$

The conclusion is now obvious. ■

2.4 Dimension Theorem

Recall from Commutative Algebra that if $A \subseteq B$ are commutative rings, then the integral closure C of A in B is a subring of B containing A.

Definition 2.18 (Algebraic Integer). An *algebraic integer* is an element in the integral closure of $\mathbb{Z}$ in $\mathbb{C}$, that is, an element in $\mathbb{C}$ which is integral over $\mathbb{Z}$. Denote by $\mathcal{A}$ the ring of algebraic integers.

Lemma 2.19. *An element $y \in \mathbb{C}$ is an algebraic integer if and only if there exist $y_1, \ldots, y_n \in \mathbb{C}$, not all zero, such that for all $1 \leq i \leq n$,*

$$yy_i = \sum_{j=1}^{n} a_{ij} y_j$$

for some $a_{ij} \in \mathbb{Z}$.

Proof. We prove the converse first. Let A denote the matrix $[a_{ij}]_{1 \leq i,j \leq n}$ and y the column vector $\begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T$. According to our assumptions, $AY = yY$, whereby $\det(A - yI) = 0$ whence y is an algebraic integer.

On the other hand, if y is an algebraic integer, then $y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0$ for integers $a_0, \ldots, a_{n-1}$. Let $y_i = y^{i-1}$. Then, we have the relations $yy_i = y_{i+1}$ for $1 \leq i \leq n-2$ and

$$yy_{n-1} = -a_0 - \cdots - a_{n-1}y^{n-1}$$

This completes the proof. ■

Proposition 2.20. *Let χ be a character of a group G. Then, $\chi(g)$ is an algebraic integer for all $g \in G$.*

Proof. Let χ be the character associated with a represetnation $\varphi : G \to \mathrm{GL}(n, \mathbb{C})$ for some positive integer n. Then, $\varphi_g^{|G|} = \mathbf{id}_{n \times n}$, whereby the eigenvalues of φ_g satisfy $\lambda^{|G|} - 1 = 0$, and thus are algebraic integers. Since $\mathcal{A}$ is a ring, it contains

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(\varphi_g) = \chi(g)$$

■

Lemma 2.21. *Let $\varphi : G \to \mathrm{GL}(V)$ be an irreducible representation of degree d. Let $g \in G$ and m the size of the conjugacy class containing g. Then, $\frac{m}{d}\chi_\varphi(g)$ is an algebraic integer.*

Proof. ■

Theorem 2.22 (Dimension Theorem). *Let $\varphi : G \to \mathrm{GL}(V)$ be an irreducible representation of degree d. Then d divides $|G|$.*

The main idea is to show that $|G|/d$ is an algebraic integer, and thus lies in $\mathcal{A} \cap \mathbb{Z} = \mathbb{Z}$. This would finish the proof.

Proof. We have

$$1 = \langle \chi_\varphi, \chi_\varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\varphi(g)} \chi_\varphi(g)$$

Let $C_1, \ldots, C_s$ be the conjugacy classes of G with sizes $m_1, \ldots, m_s$ respectively. Let χ_i denote the value of χ_φ on C_i. Then,

$$|G| = \sum_{i=1}^{s} m_i \overline{\chi_i} \chi_i \implies \frac{|G|}{d} = \sum_{i=1}^{s} \overline{\chi_i} \left(\frac{m_i}{d} \chi_i \right)$$

and thus $|G|/d$ is an algebraic integer, whence an integer. ■

Chapter 3

Burnside's Theorems

3.1 Burnside's Theorem on Solvability

This section requires the reader to have some knowledge of finite Galois Theory, in particular, that of the Galois Group and the Norm map.

Lemma 3.1. *Let $\varphi : G \to \mathrm{GL}(d, \mathbb{C})$ be an irreducible representation and $C \subseteq G$ be a conjugacy class such that $\gcd(|C|, d) = 1$. Then, either*

(a) For all $g \in C$, there is $\lambda_g \in \mathbb{C}^\times$ such that $\varphi_g = \lambda_g \mathbf{id}_{d\times d}$

(b) χ_φ vanishes on C.

Proof. Suppose (a) does not hold, we shall show that (b) holds. For all $g \in G$, $\varphi_g^{|C|} = \mathbf{id}_{d\times d}$ and thus the minimal polynomial of φ_g is separable whence it is diagonalizable. Pick some $g \in C$. Since φ_g is not a scalar matrix by assumption, it must have distinct eigenvalues[1]. Let $\alpha = \chi_\varphi(g)/d$. Due to Lemma 2.21, $m\chi_\varphi(g)/d$ is an algebraic integer. Since $\gcd(m, d) = 1$, there are positive integers p, q such that $pm + qd = 1$ and thus

$$\frac{\chi_\varphi(g)}{d} = \frac{(pm+qd)\chi_\varphi(g)}{d} = p\frac{m\chi_\varphi(g)}{d} + q\chi_\varphi(g) \in \mathcal{A}.$$

Further,

$$\alpha = \frac{1}{d}(\lambda_1 + \cdots + \lambda_d)$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of φ_g, with multiplicity. According to our hypothesis, not all the λ_i's are equal. Let $n = |G|$.

We shall show that $|N_{\mathbb{Q}}^{\mathbb{Q}(\zeta_n)}(\alpha)| < 1$. Since every λ_i is an n-th root of unity, any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ maps it to another root of unity, and thus, $d\sigma(\alpha)$ is a sum of roots of unity, not all equal, and thus, $|d\sigma(\alpha)| < d$, that is, $|\sigma(\alpha)| < 1$. This immediately gives us that $|N_{\mathbb{Q}}^{\mathbb{Q}(\zeta_n)}(\alpha)| < 1$.

Since $N^{\mathbb{Q}(\zeta_n)\mathbb{Q}(\alpha)}$ is an algebraic integer, owing to it being a product of algebraic integers, and is also a rational number[2], it must be an integer. But due to the constraint $|N_{\mathbb{Q}}^{\mathbb{Q}(\zeta_n)}(\alpha)| < 1$, we must have $N_{\mathbb{Q}}^{\mathbb{Q}(\zeta_n)}(\alpha) = 0$ and thus $\alpha = 0$. This completes the proof. ∎

[1] Else the diagonalization of φ_g would be a scalar matrix, forcing φ_g to be scalar
[2] Since N_k^K is a multiplicative function from K to k

Lemma 3.2. *Let G be a finite non-abelian group. Suppose there is a nontrivial conjugacy class $C \neq \{1_G\}$ of prime power order, p^n, then G is not simple.*

Proof. ■