Commutative Algebra

Swayam Chube

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Abstract

This document mainly contains terse notes of commutative algebra and solutions to exercises from [1]. The three main references were [1], [3] and [4].

Except for in the chapter on modules, all rings are assumed to be commutative unless stated otherwise. We use a uniform convention to represent a commutative ring with A and a general ring with R. Similarly, we represent modules by one of M, N, P. A maximal ideal is generally denoted by $\mathfrak{m}$ while a prime ideal is denoted by $\mathfrak{p}$.

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Modules

2.1 Introduction

Throughout this section, R denotes a general ring which need not be commutative.

Definition 2.1 (Module). A left R-module is an abelian group $(M, +)$ along with a ring action, that is, a ring homomorphism $\mu : R \to \mathrm{End}(M)$.

Henceforth, unless specified otherwise, an *R-module* refers to a *left R-module*. Trivially note that R is an R-module, so is any ideal in R and so is every quotient ring R/I where I is an ideal in R. When R is a field, an R-module is the same as a vector space.

Every abelian group G trivially forms a $\mathbb{Z}$-module. Using this and the forthcoming *Structure Theorem for Finitely Generated Modules over a PID*, we obtain the *Structure Theorem for Finitely Generated Abelian Groups*.

Definition 2.2 (Submodule). Let M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of R.

Proposition 2.3 (Submodule Criteria). *Let M be an R-module. Then $\varnothing \subsetneq N \subseteq M$ is a submodule if and only if for all $x, y \in N$ and $r \in R$, $x + ry \in N$.*

Proof. Straightforward definition pushing. ■

Definition 2.4 (Module Homomorphism). Let M, N be R-modules. A *module homomorphism* is a group homomorphism $\phi : M \to N$ such that for all $x \in M$ and $r \in R$, $\phi(rx) = r\phi(x)$.

In other words, a module homomorphism is simply an R-linear map.

Proposition 2.5 (Homomorphism Criteria). *Let M, N be R-modules. Then $\phi : M \to N$ is an R-module homomorphism if and only if for all $x, y \in M$ and $r \in R$, $\phi(x + ry) = \phi(x) + r\phi(y)$.*

Proof. Straightforward definition pushing. ■

It is not hard to see, using the above proposition and the submodule criteria that the image of an R-module under a homomorphism is a submodule.

Definition 2.6 (Kernel, Cokernel). Let $\phi : M \to N$ be an R-module homomorphism. We define

$$\ker \phi = \{x \in M \mid \phi(x) = 0\} \qquad \operatorname{coker} \phi = N/\phi(M)$$

For an R-module M, define the annihilator of M in R as

$$\operatorname{Ann}_R(M) = \{r \in R \mid rx = 0 \ \forall x \in M\}$$

It is trivial to check that $\operatorname{Ann}(M)$ is a left ideal in R, and if R were commutative, it would be an ideal.

Proposition 2.7. *If I is an ideal contained in $\operatorname{Ann}_A(M)$, then M is naturally an A/I-module.*

Proof. Define the action $(a + I) \cdot m = a \cdot m$. It is easy to check that this action is well defined. Further,

$$(a + I) \cdot ((b + I) \cdot m) = (a + I) \cdot (bm) = (ab) \cdot m = ((a + I)(b + I)) \cdot m$$

This completes the proof. ■

In particular, if $I = \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then M forms a vector space over $A/\mathfrak{m}$.

2.2 Free Modules

Throughout this section, R denotes a general ring which need not be commutative. The content of this section is taken from [2].

We define the free module using a universal property and then provide a construction for it. This should establish uniqueness.

Definition 2.8 (Universal Property of Free Modules). Let S be a non-empty set. A *free module on S* is an R-module F together with a mapping $f : S \to F$ such that for every R-module M and every set map $g : S \to M$, there is a unique R-module homomorphism $h : F \to M$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f"'] & M \\ F \arrow[ur, dotted, "\exists! h"'] & \end{tikzcd}$$

Let F be the set of all set functions $\phi : S \to R$ which takes nonzero values at finitely many elements of S. This has the structure of an R-module. Define the set map $f : S \to F$ by

$$f(s)(t) = \begin{cases} 1 & s = t \\ 0 & \text{otherwise} \end{cases}$$

We contend that (F, f) is a free module on S. Indeed, let $g : S \to M$ be a set map where M is an R-module. Define the linear map $h : F \to M$ by

$$h(f(s)) = g(s)$$

Since every element in F can uniquely be written as a linear combination of elements in $\{f(s)\}_{s \in S}$, we have successfully defined a module homomorphism $h : F \to M$ such that $g = h \circ f$. The uniqueness of this map is quite obvious. Hence, (F, f) is a free module on S.

Definition 2.9 (Basis). Let M be an R-module. Then $S \subseteq M$ is said to be a *basis* if it is linearly independent and generates M.

It is important to note that not every minimal generating set is a basis. Take for example the $\mathbb{Z}$-module $\mathbb{Z}$. Notice that $\{2, 3\}$ is a minimal generating set but is not a basis for it is not linearly independent.

2.3 Finitely Generated Modules

Definition 2.10 (Finitely Generated Module). An R-module M is said to be finitely generated if there is a finite subset S of M which generates M. That is, there is no proper submodule N of M containing S.

Proposition 2.11. *An R-module M is finitely generated if M is isomorphic to a quotient of $R^{\oplus n}$ for some positive integer n.*

Proof. We shall only prove the forward direction since the converse is trivial to prove. Suppose M is finitely generated. Then, it is generated by a finite subset $S = \{x_1, \dots, x_m\}$. Define the R-module homomorphism $\phi : R^{\oplus n} \to M$ by $(r_1, \dots, r_n) \mapsto r_1x_1 + \dots + r_nx_n$. From the first isomorphism theorem, we have $M \cong R^{\oplus n} / \ker \phi$. ■

Proposition 2.12. *Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A. Let $\phi \in \mathrm{End}(M)$ such that $\phi(M) \subseteq \mathfrak{a}M$. Then, there are $a_0, \dots, a_{n-1} \in \mathfrak{a}$ such that*

$$\phi^n + a_{n-1}\phi^{n-1} + \dots + a_0 = 0$$

as an element of $\mathrm{End}(M)$, where a_k is treated as the homomorphism $x \mapsto a_kx$ in $\mathrm{End}(M)$.

Proof. Let $\{x_1, \dots, x_n\}$ be a generating set for M. Then, for all $1 \le i \le n$, there are coefficients $\{a_{i1}, \dots, a_{in}\}$ in $\mathfrak{a}$ such that

$$\phi(x_i) = \sum_{j=1}^{n} a_{ij}x_j$$

We may rewrite this as

$$\sum_{j=1}^{n} (\phi\delta_{ij} - a_{ij})x_j = 0$$

Let B denote the matrix $(\phi\delta_{ij} - a_{ij})_{1 \le i,j \le n}$. Then, multiplying by $\mathrm{adj}(B)$, we see that $\det(B)(x_j) = 0$ for all $1 \le j \le n$ where $\det(B)$ is viewed as an element in $\mathrm{End}(M)$ and thus, is the zero map in $\mathrm{End}(M)$. It is not hard to see that $\det(B)$ is in the required form. ■

Lemma 2.13 (Nakayama). *Let M be a finitely generated module and $\mathfrak{a} \subseteq \mathfrak{R}$ be an ideal such that $M = \mathfrak{a}M$. Then, $M = 0$.*

Proof. Let $\phi = \mathbf{id}$ be the identity homomorphism in $\mathrm{End}(M)$. Using Proposition 2.12, there are coefficients $a_0, \dots, a_{n-1} \in \mathfrak{a}$ satisfying the statement of the proposition. As a result, $x = 1 + a_{n-1} + \dots + a_0$ is the zero endomorphism. But since $a_{n-1} + \dots + a_0 \in \mathfrak{a} \subseteq \mathfrak{R}$, x is a unit and hence, $M = 0$. ■

Corollary. Let M be a finitely generated A-module, N a submodule of M and $\mathfrak{a} \subseteq \mathfrak{R}$ an ideal. If $M = \mathfrak{a}M + N$ then $M = N$.

Proof. We have $M/N = \mathfrak{a}M/N$, consequently, $M/N = 0$ and $M = N$ due to Lemma 2.13. ∎

Lemma 2.14. *Let $(A, \mathfrak{m})$ be local and $k = A/\mathfrak{m}$. Let M be a finitely generated A-module. Let $\{\overline{x}_1, \ldots, \overline{x}_n\}$ be elements in $M/\mathfrak{m}$ that form a basis for $M/\mathfrak{m}$ as a k-vector space. Then, $\{x_1, \ldots, x_n\}$ generates M.*

Proof. Let N be the submodule generated by $\{x_1, \ldots, x_n\}$. Then, the composition $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is surjective, consequently, $M = N + \mathfrak{m}M$ whence, it follows that $M = N$. ∎

Over a PID

Throughout this section, let R denote a principal ideal domain.

2.4 Hom Modules and Functors

For R-modules M, N, we denote the set of all R-module homomorphisms from M to N by $\operatorname{Hom}_R(M, N)$. When the choice of the ring R is clear from the context, we shall denote this set by $\operatorname{Hom}(M, N)$.

Proposition 2.15. *Let M, N be A-modules. Then $\operatorname{Hom}(M, N)$ has the structure of an A-module.*

Proof. It is obvious that $\operatorname{Hom}(M, N)$ has the structure of an abelian group. Define the natural action by $(af)(x) = af(x)$. It is not hard to see that this action is well defined. ∎

Proposition 2.16. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a collection of A-modules. Then, for any A-module N, we have a natural isomorphism*

$$\operatorname{Hom}_A\left(\bigoplus_{\lambda \in \Lambda} M_\lambda, N\right) = \prod_{\lambda \in \Lambda} \operatorname{Hom}_A(M_\lambda, N)$$

Proof. Since the direct sum is the product in $A - \mathbf{Mod}$, the conclusion follows from the universal property. ∎

Theorem 2.17. *Let $\phi : M \to N$ be an A-module homomorphism. Then, for every R-module P, there is an induced A-module homomorphism $\overline{\phi} : \operatorname{Hom}(N, P) \to \operatorname{Hom}(M, P)$ and an induced A-module homomorphism $\widetilde{\phi} : \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N)$.*

Equivalently phrased, $\operatorname{Hom}(-, P)$ is a contravariant functor while $\operatorname{Hom}(P, -)$ is a covariant functor.

Proof. We shall prove only the first half of the assertion since the second half follows from a similar proof. Define $\overline{\phi}$ using the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, dotted, "f \circ \phi"'] & N \arrow[d, "f"] \\ & P \end{tikzcd}$$

To see that this is indeed an R-module homomorphism, we need only verify that for all $f, g \in \operatorname{Hom}(N, P)$ and all $r \in R$, $(f + rg) \circ \phi = f \circ \phi + rg \circ \phi$ which is trivial to check. ∎

2.5 Exact Sequences

Definition 2.18. A sequence of module homomorphisms

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is said to be exact at N if $\operatorname{im} f = \ker g$. A short exact sequence is a sequence of module homomorphisms:

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

which is exact at M, N and P.

It is not hard to see that the sequence in the definition is short exact if and only if f is injective, g is surjective and $\operatorname{im} f = \ker g$.

2.5.1 Diagram Chasing Poster Children

2.6 Tensor Product

Definition 2.19 (Bilinear Map). Let M, N, P be A-modules. A map $T : M \times N \to P$ is said to be bilinear if for each $x \in M$, the mapping $T_x : N \to P$ given by $y \mapsto T(x, y)$ is A-linear and for each $y \in N$, the mapping $T_y : M \to P$ given by $x \mapsto T(x, y)$ is A-linear.

Fix two A-modules M and N. Let $\mathscr{C}$ denote the category of bilinear maps $T : M \times N \to P$ where P is any A-module. A morphism between two bilinear maps $f : M \times N \to P_1$ and $g : M \times N \to P_2$ in this category is a module homomorphism $\phi : P_1 \to P_2$ such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[d, "g"'] & P_1 \arrow[dl, dotted, "\phi"] \\
P_2 &
\end{tikzcd}$$

A universal object in $\mathscr{C}$ is called the tensor product of M and N and is denoted by $M \otimes N$. In other words, the tensor product is an initial object in the category $\mathscr{C}$.

Definition 2.20 (Universal Property of the Tensor Product). Let M, N, P be A-modules and $T : M \times N \to P$ be a bilinear map. Then, there is a unique A-module homomorphism $\phi : M \otimes N \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[r, "T"] \arrow[d, "\varphi"'] & P \\
M \otimes N \arrow[ur, dotted, "\exists!\phi"'] &
\end{tikzcd}$$

Of course, having the universal property would imply that the tensor product, if it exists, is unique upto a unique isomorphism. We shall now construct a tensor product of M and N.

Constructing the Tensor Product

Let F be the free A-module on $M \times N$. Let us denote the basis elements of F by $e_{(x,y)}$ where $x \in M$ and $y \in N$. Now, for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $a \in A$, let D denote the submodule generated by elements of the form:

$$
\begin{aligned}
&e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)} \\
&e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)} \\
&e_{(ax,y)} - ae_{(x,y)} \\
&e_{(x,ay)} - ae_{(x,y)}
\end{aligned}
$$

Let $G = F/D$ and let $\varphi : M \times N \to G$ be the composition of the following maps:

$$M \times N \hookrightarrow F \twoheadrightarrow G$$

Let $T : M \times N \to P$ be a bilinear map. Consider the following commutative diagram:

$$
\begin{tikzcd}
M \times N \arrow[r, "T"] \arrow[d, hook, "\iota"'] & P \\
F \arrow[r, "\pi"'] \arrow[ur, dotted, "\exists! f"] & G \arrow[u, dotted, "\exists! \phi"']
\end{tikzcd}
$$

To show that existence of ϕ, we must show that $D \subseteq \ker f$, since we can then finish using the universal property of the kernel. But this is trivial to check and follows from the fact that T is a bilinear map and completes the construction.

Similarly, we define the tensor product for a finite sequence of A-modules $\{M_i\}_{i=1}^n$. That is, given a multilinear map $T : \prod_{i=1}^{n} M_i \to P$, there is a unique A-module homomorphism ϕ such that the following diagram commutes:

$$
\begin{tikzcd}
M_1 \times \cdots \times M_n \arrow[r, "T"] \arrow[d, "\varphi"'] & P \\
M_1 \otimes \cdots \otimes M_n \arrow[ur, dotted, "\exists! \phi"']
\end{tikzcd}
$$

Properties of Tensor Product

Given two modules M and N with the canonical map $\varphi : M \times N \to M \otimes N$, we denote by $m \otimes n$, the element $\varphi(m, n)$ in $M \otimes N$.

Proposition 2.21. *Let M, N, P be A-modules. Then,*

(a) $M \otimes N \cong N \otimes M$

(b) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P) \cong M \otimes N \otimes P$

(c) $M \oplus N \otimes P \cong (M \otimes P) \oplus (N \otimes P)$

(d) $A \otimes M \cong M$

Proof. (a) First, we shall show that there are well defined homomorphisms $M \otimes N \to N \otimes M$ and $N \otimes M \to M \otimes N$ mapping $m \otimes n \mapsto n \otimes m$ and $n \otimes m \mapsto m \otimes n$ respectively. This is best done using the universal property. Let $T : M \times N \to N \times M$ be the isomorphism $m \times n \mapsto n \times m$. Consider now the following commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "T"] \arrow[d, "\varphi"'] & N \times M \arrow[d, "\varphi'"] \\ M \otimes N & N \otimes M \end{tikzcd}$$

Since both φ' and T are bilinear, so is $\varphi \circ T$, consequently, there is a unique induced homomorphism $f : M \otimes N \to N \otimes M$ making the diagram commute, consequently, $f(m \otimes n) = \varphi'(T(m \times n)) = n \otimes m$. Similarly, there is a homomorphism $g : N \otimes M \to M \otimes N$ such that $g(n \otimes m) = m \otimes m$. It is not hard to see that $g \circ f = \mathbf{id}_{M \otimes N}$ and $f \circ g = \mathbf{id}_{N \otimes M}$, consequently, they are isomorphisms.

(b)

(c)

(d) Consider the map $T : A \times M \to M$ given by $(a, m) \mapsto am$. It is not hard to see that this map is bilinear, consequently, there is a map $f : A \otimes M \to M$ such that the following diagram commutes:

$$\begin{tikzcd} A \times M \arrow[r, "T"] \arrow[d, "\varphi"'] & M \\ A \otimes M \arrow[ur, dotted, "f"'] & \end{tikzcd}$$

Note that $f(a \otimes m) = am$ by definition. Consider the map $g : M \to A \otimes M$ given by $g(m) = 1 \otimes m$. It is not hard to see that g is a well defined module homomorphism. Further, since $f \circ g$ and $g \circ f$ are the identity homomorphisms, they both must be isomorphisms.

■

Example 1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$ for all $m, n \in \mathbb{N}$. In particular, if m and n are coprime, then $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$.

Proof. Consider the module homomorphism $T : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. ■

Let $f : M \to M'$ and $g : N \to N'$ be A-module homomorphisms. Then, the map $\Phi : M \times N \to M' \otimes N'$ given by $\Phi(m, n) = f(m) \otimes g(n)$. It is not hard to see that Φ is bilinear. Consequently, it induces a map $f \otimes g : M \otimes N \to M' \otimes N'$ such that

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$$

Further, if $f' : M' \to M''$ and $g' : N' \to N''$ are A-module homomorphisms, then we have another map $f' \otimes g' : M' \otimes N' \to M'' \otimes N''$ such that

$$(f' \otimes g')(x \otimes y) = f'(x) \otimes g'(y)$$

Now, it is not hard to see that $(f' \circ f') \otimes (g' \circ g)$ and $(f' \otimes g') \circ (f \otimes g)$ agree on the elementary tensors, therefore, agree on all of $M \otimes N$.

2.7 Right Exactness

Proposition 2.22. *Let M, N, P be A-modules. Then, there is a natural isomorphism:*

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \cong \mathrm{Hom}_A(M \otimes_A N, P)$$

Proof. Consider the map

$$\theta : \mathrm{Hom}_A(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$$

given by $\theta(\alpha)(m)(n) = \alpha(m \otimes n)$. Now, pick some $\eta \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$. Define the map $\zeta : M \times N \to P$ given by $\zeta(m, n) = \eta(m)(n)$. Obviously, ζ is bilinear and induces a map $\delta : M \otimes_A N \to P$ such that $\delta(m \otimes n) = \eta(m)(n)$. Call the map sending $\eta \mapsto \delta$ as β where

$$\beta : \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \to \mathrm{Hom}_A(M \otimes_A N, P)$$

and $\beta(\eta)(m \otimes n) = \eta(m)(n)$.

We contend that θ and β are inverses to one another. Indeed,

$$((\beta \circ \theta)(\alpha))(m \otimes n) = \theta(\alpha)(m)(n) = \alpha(m \otimes n)$$

and

$$((\theta \circ \beta)(\eta))(m)(n) = \beta(\eta)(m \otimes n) = \eta(m)(n)$$

whence the conclusion follows. ■

In particular, we see that the functor $- \otimes_A N$ is the left adjoint of the functor $\mathrm{Hom}_A(N, -)$, consequently, $\mathrm{Hom}_A(N, -)$ is the right adjoint of $- \otimes_A N$.

Theorem 2.23. *The functor $- \otimes_A N$ is right exact. That is, given a exact sequence*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

the sequence

$$M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \longrightarrow 0$$

Proof. Since the given sequence is exact, so is

$$\mathrm{Hom}_A(M'', \mathrm{Hom}_A(N, P)) \xrightarrow{\overline{g}} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \xrightarrow{\overline{f}} \mathrm{Hom}_A(M', \mathrm{Hom}_A(N, P)) \longrightarrow 0$$

but from Proposition 2.22, so is

$$\mathrm{Hom}_A(M'' \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M' \otimes_A N, P) \longrightarrow 0$$

Since the above sequence is exact for all modules P, we have the desired conclusion. ■

The tensor product is not left exact. Conider the sequence of $\mathbb{Z}$-modules

$$0 \hookrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

where $f(m) = 2m$. Upon tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

Note that

$$(f \otimes 1)(m \otimes \overline{n}) = 2m \otimes \overline{n} = m \otimes (2\overline{n}) = m \otimes 0 = 0$$

Therefore, the sequence cannot be exact.

2.8 Flat Modules

Definition 2.24 (Flat Module).

Theorem 2.25. *Let N be a A-module. Then, the following are equivalent*

(a) *N is flat*

(b) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then the tensored sequence*

$$0 \longrightarrow M' \otimes_A N \xrightarrow{f\otimes 1} M \otimes_A N \xrightarrow{g\otimes 1} M'' \otimes_A N \longrightarrow 0$$

is exact.

(c) *If $f : M' \to M$ is injective, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective*

(d) *If $f : M' \to M$ is injective and M, M' are finitely generated, then $f \otimes_A 1 : M' \otimes_A N \to M \otimes_A N$ is injective.*

Proof.

(a)

■

2.9 Projective Modules

Theorem 2.26. *For an A-module P, the following are equivalent:*

(a) *Every map $f : P \to M''$ can be lifted to $\widetilde{f} : P \to M$ in the following commutative diagram:*

$$\begin{tikzcd} & P \arrow[dl, "\widetilde{f}"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & M'' \arrow[r, two heads] & 0 \end{tikzcd}$$

(b) *Every short exact sequence $0 \to M' \to M \to P \to 0$ splits*

(c) *There is a module M such that $P \oplus M$ is free*

(d) *The functor $\mathrm{Hom}_A(P, -)$ is exact.*

Proof.

$(a) \implies (b)$**:** Taking $M'' = P$ and $f = \mathbf{id}_P$, we have the desired conclusion.

$(b) \implies (c)$**:** Let F denote the free module on the set P. Then, the map $\Phi : F \to P$ given by $\Phi(e_x) = x$ for all $x \in P$ is a surjective A-module homomorphism. We have the following short exact sequence:

$$0 \to \ker \Phi \xrightarrow{\iota} F \xrightarrow{\Phi} P \to 0$$

This is known to split and thus, $F = \psi(P) \oplus \ker \Phi$ where $\psi : P \to F$ is the section.

$(c) \implies (d)$**:** Let $M' \to M \to M''$ be an exact sequence of modules and K be an A-module such that $P \oplus K = F \cong A^\Lambda$. Then, the induced sequence

$$\prod_{\lambda \in \Lambda} M' \to \prod_{\lambda \in \Lambda} M \to \prod_{\lambda \in \Lambda} M''$$

is exact. We have seen that there is a natural isomorphism $\mathrm{Hom}_A(A, M) \xrightarrow{\sim} M$, consequently, there is a natural isomorphism

$$\mathrm{Hom}_A(A^{\oplus\Lambda}, M) \xrightarrow{\sim} \prod_{\lambda \in \Lambda} M$$

whence it follows that the sequence

$$\mathrm{Hom}_A(A^{\oplus\Lambda}A, M') \to \mathrm{Hom}_A(A^{\oplus\Lambda}A, M) \to \mathrm{Hom}_A(A^{\oplus\Lambda}, M'')$$

But since $\mathrm{Hom}_A(A^{\oplus\Lambda}, M) \cong \mathrm{Hom}_A(P, M) \oplus \mathrm{Hom}_A(K, M)$, we have the desired conclusion.

$(d) \implies (a)$**:** Trivial.

■

Definition 2.27 (Projective Module). An A-module P satisfying any one of the four equivalent conditions of Theorem 2.26 is said to be a *projective A-module*.

In particular, from Theorem 2.26(c), we see that every free module is projective.

Lemma 2.28. *A finitely generated projective module P over a local ring $(A, \mathfrak{m})$ is free.*

Proof. Let $\{\overline{x}_1, \dots, \overline{x}_n\}$ be a basis for $M/\mathfrak{m}M$ as a k-vector space where $k = A/\mathfrak{m}$. As we have seen earlier, $\{x_1, \dots, x_n\}$ generates M. Let F be the free module with basis $\{e_1, \dots, e_n\}$ and $\Phi : F \to M$ be the module homomorphism given by $\Phi(e_i) = x_i$ and $K = \ker \Phi$. Since M is projective, there is a module homomorphism $\psi : M \to F$ satisfying $\Phi \circ \psi = \mathbf{id}_M$ and $F = K \oplus \psi(M)$.

We contend that $K = \mathfrak{m}K$. Indeed, let $x \in K$, then $x = \sum r_i e_i$ for a unique choice $\{r_1, \dots, r_n\}$. Then, $\sum r_i x_i = 0$, consequently, $r_i \in \mathfrak{m}$ for all i. Since $F = K \oplus \psi(M)$, we may write $e_i = u_i + v_i$ for some $u_i \in K$ and $v_i \in \psi(M)$. As a result,

$$x - \sum r_i u_i = \sum r_i v_i \in \ker \Phi \cap \psi(M) = \{0\}$$

and the conclusion follows.

Finally due to Lemma 2.13, we must have that $K = 0$ whence M is free. ■

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