Analytic Number Theory

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Chapter 1

Fundamentals

1.1 Arithmetic Functions

The main takeaway from this section will be the *Möbius Inversion Formula*.

Definition 1.1. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be an *arithmetic function* or a *number-theoretic function*.

Definition 1.2. A real, arithmetic function f is said to be *multiplicative* if for all $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$,

$$f(m)f(n) = f(mn)$$

On the other hand, if for all $m, n \in \mathbb{N}$,

$$f(m)f(n) = f(mn)$$

then f is said to be *completely multiplicative*.

Obviously, every completely multiplicative function is multiplicative.

Definition 1.3 (Dirichlet Product). Let f and g be arithmetic functions. Then, the *Dirichlet product*, or the *Dirichlet convolution* of f and g, denoted by $f * g : \mathbb{N} \to \mathbb{C}$ is defined as

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

or may be equivalently written as:

$$\sum_{d_1 d_2 = n} f(d_1)g(d_2)$$

Theorem 1.4. The *Dirichlet product* is associative and commutative. That is,

$$(f * g) * h = f * (g * h) \qquad \text{and} \qquad f * g = g * f$$

Proof. Trivial. ■

Theorem 1.5. If f is an arithmetic function with $f(1) \neq 0$, then there is a unique arithmetic function f^{-1}, called the Dirichlet inverse of f such that

$$f * f^{-1} = f^{-1} * f = \nu$$

Moreover, f^{-1} is given by the formulas

$$f^{-1}(1) = \frac{1}{f(1)} \qquad f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d \mid n \\ d < n}} f\left(\frac{n}{d}\right) f^{-1}(d)$$

Proof. Trivial ■

Theorem 1.6. If f is multiplicative and if g is given by

$$g(n) = \sum_{d \mid n} f(d)$$

then g is also multiplicative.

Proof. For $m, n \in \mathbb{N}$, such that $\gcd(m, n) = 1$, we have

$$\begin{aligned} g(m)g(n) &= \sum_{d|m} f(d) \sum_{d'|n} f(d') \\ &= \sum_{d|m} \sum_{d'|n} f(d)f(d') \\ &= \sum_{d|mn} f(d) \\ &= g(mn) \end{aligned}$$

Where the second last equality follows from the fact that any divisor of mn can be broken into two parts, one being a divisor of m and the other of n, since $\gcd(m, n) = 1$. ∎

Theorem 1.7. If f and g are multiplicative, then so is their *Dirichlet product*,

$$F(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

Proof. Similar to the previous proof and hence omitted. ∎

Theorem 1.8. If $f * g$ and g are multiplicative, then so is f.

Proof. ∎

As a corollary, we have that if g is multiplicative then so is g^{-1}.

Definition 1.9. Let $n \in \mathbb{N}$. Then the arithmetic functions $\tau(n)$ and $\sigma(n)$ are defined as follows:

$$\tau(n) = \sum_{d|n} 1 \qquad \sigma(n) = \sum_{d|n} d$$

In other words, $\tau(n)$ is the number of positive divisors of n and $\sigma(n)$ is the sum of all the positive divisors of n.

Theorem 1.10. Let n be a positive integer. Then,

1. $\tau(n)$ is multiplicative.
2. If n is a prime, say p, then $\tau(p) = 2$. If n is a prime power p^α, then $\tau(p^\alpha) = p^\alpha + 1$.
3. If n is a composite number of the form $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then
$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1)$$
4. The product of all divisors of a number n is
$$\prod_{d \mid n} d = n^{\tau(n)/2}$$

Proof.

1. Since the function $f(n) = 1$ is multiplicative, it follows that $\tau(n)$ is also multiplicative
2. Trivial
3. Trivial
4. Simply note that
$$\begin{aligned} n^{\tau(n)} &= \prod_{d \mid n} n \\ &= \prod_{d \mid n} d \left(\frac{n}{d}\right) \\ &= \prod_{d \mid n} d \prod_{d' \mid n} d' \\ &= \left(\prod_{d \mid n} d\right)^2 \end{aligned}$$
which gives us the desired conclusion.

■

Theorem 1.11. Let n be a positive integer. Then

1. $\sigma(n)$ is multiplicative.
2. If n is a prime, say p,, then $\sigma(p) = p + 1$. More generally, if n is a prime power p^α, then
$$\sigma(p^\alpha) = \frac{p^{\alpha+1} - 1}{p - 1}$$
3. If n is a composite number of the form $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then
$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$$

Proof.

1. Since the function $f(n) = n$ is multiplicative, it follows that $\sigma(n)$ is also multiplicative
2. Trivial
3. Trivial

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Definition 1.12. Let n be a positive integer. Eulers's totient ϕ-function is defined to be the number of positive integers k less than n which are relatively prime to n:
$$\phi(n) = \sum_{\substack{0 \le k < n \\ \gcd(k,n)=1}} 1$$

Lemma 1.13. For any positive integer n,
$$\sum_{d|n} \phi(d) = n$$

Proof. Let n_d denote the number of elements in $[n]$ having a greatest common divisor of d with n. Then

$$n = \sum_{d|n} n_d = \sum_{d|n} \phi\left(\frac{n}{d}\right) = \sum_{d|n} \phi(d)$$

■

Theorem 1.14. Let n be a positive integer. Then,

1. $\phi(n)$ is multiplicative
2. If n is a prime, say p, then $\phi(p) = p - 1$. Conversely, if p is a positive integer with $\phi(p) = p - 1$, then p is prime. Further, if n is a prime power p^α with $\alpha > 1$, then $\phi(p^\alpha) = p^\alpha - p^{\alpha-1}$
3. If n is a composite number of the form $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

Proof.

1. Find an elegant proof to this part
2. Trivial
3. Trivial

■

Definition 1.15. Let n be a positive integer. Then the *Möbius μ function $\mu(n)$* is defined as

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & n \text{ is not square free} \\ (-1)^k & n = p_1 \cdots p_k \text{ where } p_i\text{'s are primes} \end{cases}$$

Theorem 1.16. Let n be a positive integer. Then

1. $\mu(n)$ is multiplicative
2. Let
$$\nu(n) = \sum_{d|n} \mu(d)$$
then,
$$\nu(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

Proof.

1. Trivial
2. Note that for a prime p, and $\alpha \geq 1$, we have
$$\begin{aligned} \nu(p^\alpha) &= \sum_{d|p^\alpha} \mu(d) \\ &= \mu(1) + \mu(p) \\ &= 0 \end{aligned}$$
And we are done due to multiplicativity.

■

Theorem 1.17 (Möbius Inversion Formula). If f is any arithmetic function and if
$$g(n) = \sum_{d|n} f(d)$$
Then,
$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

Proof. We have

$$\begin{aligned}
\sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) &= \sum_{d|n} \mu(d) \sum_{a|n/d} f(a) \\
&= \sum_{d|n} \sum_{a|n/d} \mu(d) f(a) \\
&= \sum_{a|n} \sum_{d|n/a} \mu(d) f(a) \\
&= \sum_{a|n} f(a) \nu\left(\frac{n}{a}\right) \\
&= f(n)
\end{aligned}$$

■

Conversely, the following is also true:

Theorem 1.18 (Converse of Möbius Inversion). Let g be an arithmetic function and

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

then

$$g(n) = \sum_{d|n} f(d)$$

Proof. We have

$$\begin{aligned}
\sum_{d|n} f(d) &= \sum_{d|n} \sum_{a|d} \mu\left(\frac{d}{a}\right) g(a) \\
&= \sum_{a|n} \sum_{\lambda|n/a} \mu(\lambda) g(a) \\
&= \sum_{a|n} g(a) \nu\left(\frac{n}{a}\right) \\
&= g(n)
\end{aligned}$$

■

Theorem 1.19. Let f be multiplicative. Then f is *completely multiplicative* if and only if
$$f^{-1}(n) = \mu(n)f(n)$$

Proof. Suppose f is multiplicative.Obviously, $f(1) = 1$, and thus $f^{-1}(1) = 1 = \mu(1)f(1)$. We shall now induct on n with that as our base case. We have,
$$\begin{aligned} f^{-1}(n) &= -\sum_{\substack{d\mid n \\ d<n}} f\left(\frac{n}{d}\right)\mu(d)f(d) \\ &= -f(n)\sum_{\substack{d\mid n \\ d<n}} \mu(d) \\ &= (\mu(n) - \nu(n))f(n) \end{aligned}$$

Since we are given f is multiplicative, it suffices to show that $f(p^\alpha) = f(p)^\alpha$ for each prime p. Since we know that
$$\nu(n) = f * f^{-1} = \sum_{d\mid n} \mu(d)f(d)f\left(\frac{n}{d}\right)$$
taking $n = p^\alpha$ in the above equation, we obtain
$$f(p^\alpha) = f(p)f(p^{\alpha-1})$$
and the conclusion is obvious. ■

Theorem 1.20. For any positive integer n,
$$\phi(n) = n\sum_{d\mid n}\frac{\mu(d)}{d}$$

Proof. Let $f(n) = n$ for all positive integers n. Then
$$f(n) = \sum_{d\mid n}\phi(n)$$
and due to the Möbius inversion formula, we have
$$\phi(n) = \sum_{d\mid n}\mu(d)\frac{n}{d} = n\sum_{d\mid n}\frac{\mu(d)}{d}$$
■

Definition 1.21 (Von Mangoldt Function). Let n be a positive integer. Then, we define the *Von Mangoldt function* as

$$\Lambda(n) = \begin{cases} \log p & n = p^m \\ 0 & \text{otherwise} \end{cases}$$

It is not hard to show that

$$(\Lambda * 1)(n) = \sum_{d|n} \Lambda(d) = \log n$$

Theorem 1.22. For any positive integer n, we have

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d$$

Proof. Trivially follows from the Möbius inversion formula. ■

Definition 1.23 (Liouville Function). Let n be a positive integer. Then, we define the *Liouville function* as

$$\lambda(n) = \begin{cases} 1 & n = 1 \\ (-1)^{\alpha_1 + \ldots + \alpha_k} & n = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \end{cases}$$

It is evident from definition that the Liouville function is *completely multiplicative*.

Theorem 1.24. For any positive integer n, we have

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & n \text{ is a perfect square} \\ 0 & \text{otherwise} \end{cases}$$

Further, $\lambda^{-1}(n) = |\mu(n)|$.

Proof. We may trivially conclude that $\sum_{d|n} \lambda(d)$ is also multiplicative. Thus, it suffices to evaluate it at prime powers.

$$\sum_{d|p^\alpha} \lambda(d) = \begin{cases} 0 & \alpha \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

Conversely, we have that

$$\lambda^{-1}(n) = \mu(n)\lambda(n) =$$

■

1.2 Averages of Arithmetic Functions