DERIVED CATEGORIES

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1. Lecture 1

Definition 1.1. $f : X \to Y$ is a *quasi-isomorphism* if $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.

Definition 1.2. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is an *additive category* if it satisfies the following:

(1) For all $A, B \in \mathrm{ob}(\mathcal{C})$, the sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group.
(2) If $f : A \to B$, $g_1, g_2 : B \to C$ and $h : C \to D$ are morphisms. Then,

$$h \circ (g_1 + g_2) \circ f = h \circ g_1 \circ f + h \circ g_2 \circ f.$$

(3) There is a zero object in $\mathcal{C}$, that is, there is an object which is both the initial and terminal object.
(4) Finite products and finite coproducts exist in this category.

Definition 1.3 (R-additive category). Let R be a commutative ring and $\mathcal{C}$ an additive category. We say that $\mathcal{C}$ is an R-category if

(1) for all $A, B \in \mathrm{ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an R-module.
(2) For $\in \mathrm{Hom}(A, B)$, $g_1, g_2 \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$,

$$\begin{aligned} (r_1 g_1 + r_2 g_2) \circ (sf) &= (r_1 s) g_1 \circ f + (r_2 s) g_2 \circ f \\ sh \circ (r_1 g_1 + r_2 g_2) &= (sr_1) h \circ g_1 + (sr_2) h \circ g_2. \end{aligned}$$

Definition 1.4. Let $\mathcal{C}, \mathcal{D}$ be R-additive categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. We say F is R-linear if the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is R-linear.

Definition 1.5. Let $\mathcal{A}$ be an abelian category and $\mathcal{C} = \mathrm{ch}(\mathcal{A})$, the category of chain complexs on $\mathcal{A}$. Let $C, D \in \mathrm{ob}(\mathcal{C})$ be chain complexes and $f, g : C \to D$ be co-chain maps. We say that f is homotopic to g, written $f \simeq g$ if if there are maps $s_n : C^n \to D^{n-1}$ such that

$$f_n - g_n = s_{n+1} \circ d_n^C + d_{n-1}^D \circ s_n.$$

If $f \simeq 0$, then we say f is nulhomotopic. Denote by $I(C, D)$ the collection of nulhomotpic maps.

Remark 1.1. *If $u : B \to C$ and $v : D \to E$ are co-chain maps and $f : C \to D$ is nulhomotopic, then $f \circ u$ and $v \circ f$ are also nulhomotopic.*

Definition 1.6 (Homotopy Category).

Let R be a commutative ring and C an additive R-category with $\Sigma : \mathcal{C} \to \mathcal{C}$ is an invertible functor, that is, an equivalence of categories

Definition 1.7. A candidate triangle in $\mathcal{C}$ with respect to Σ is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that $v \circ u = w \circ v = 0$.

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A morphism $\eta : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ is a triple f, g, h such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

Definition 1.8 (Pre-triangulated Category). Let $\mathcal{C}$ be an additive R-category, Σ an equivalence of categories $\mathcal{C} \to \mathcal{C}$. A **class** of candidate triangles with respect to Σ, called *distinguished triangles*.

(1) The candidate triangle $X \xrightarrow{\mathbf{id}} X \to 0 \to \Sigma X$ is distinguished.
(2) If (X, Y, Z, u, v, w) is distinguished and (X', Y', Z', u', v', w') is a candidate triangle such that there is an isomorphism between them, then the latter is also a distinguished triangle.
(3) Let $u : X \to Y$ be any morphism. Then, there exists a distinguished triangle fo the form

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow \Sigma X.$$

(4) (Rotation of Triangles) If $X \overset{u}{to} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle, then so are

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{u} Y \xrightarrow{v} Z$$

(5) For any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, dotted, "\exists h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

Proposition 1.9. *Let $\mathcal{C}$ be a pre triangulated category and let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle and $U \in \mathrm{ob}(\mathcal{C})$. Then, the sequence

$$\mathrm{Hom}(U, X) \xrightarrow{f_*} \mathrm{Hom}(U, Y) \xrightarrow{g_*} \mathrm{Hom}(U, Z)$$

is exact.

Proof. That $\mathrm{im}\, f_* \subseteq \ker g_*$ is straightforward. Conversely, suppose $t \in \ker g_*$, that is, $g \circ t = 0$. Consider now the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Sigma^{-1} U \arrow[r] \arrow[d, dotted] & \Sigma^{-1} U \arrow[d, "\Sigma^{-1} t"] \\ Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & \Sigma^{-1} X \arrow[r, "\Sigma^{-1} f"'] & \Sigma^{-1} Y \end{tikzcd}$$

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A similar result holds for $\mathrm{Hom}(-, U)$.

Proposition 1.10. *Consider the commutative diagram:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

If f and g are isomorphisms, then so is h.

Proof. Hom the entire thing using $\mathrm{Hom}(Z', -)$ and then use the five lemma to see that $h_* : \mathrm{Hom}(Z', Z) \to \mathrm{Hom}(Z', Z')$ is an isomorphism and thus, there is $\theta : Z' \to Z'$ such that $h \circ \theta = \mathbf{id}_{Z'}$.

Similarly, using $\mathrm{Hom}(-, Z)$, there is $\delta : Z' \to Z$ such that $\delta \circ h = \mathbf{id}_Z$ whence h is an isomorphism. ∎

Definition 1.11. Let $\mathcal{A}$ be an abelian category and $K(A)$ be the *homotopy category*. Let $f : B \to C$ be a chain map. The *cone of f* is defined as

$$\mathrm{cone}(f)^n := B^{n+1} \oplus C^n$$

and the boundary maps are given by $\partial^n(b, c) = \left(-\partial^B_{n+1}(b), \partial^C_n(c) - f(b)\right)$. Show that this is a complex.

Definition 1.12 (Shift of a Complex). Let C be a co-chain complex and m an integer. Define the complex $C[m]$ by $C[m]^n = C^{m+n}$ and the map $d : C[m]^n \to C[m]^{n+1}$ is given by $(-1)^m d^{n+m}$.