MA5106: HOMEWORK 2

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Problem 1

(a) For any $m \geq 0$, we have

$$\lim_{|x|\to\infty} (1+|x|^2)^{m/2}|x|^2 e^{-|x|^2} \leq \lim_{|x|\to\infty} (1+|x|^2)^{(m+2)/2} e^{-|x|^2} \leq \lim_{|x|\to\infty} (1+|x|^2)^n e^{-|x|^2}$$

where n is a positive integer greater than or equal to $(m+2)/2$. We shall show that the right-most limit is 0. Expanding $(1+|x|^2)^n$ using the binomial theorem, we see that it suffices to show that

$$\lim_{|x|\to\infty} |x|^r e^{-|x|^2} = 0.$$

Using the power series expansion of the exponential function, we have

$$e^{|x|^2} \geq 1 + |x|^2 + \cdots + \frac{1}{r!}|x|^{2r}.$$

Hence,

$$\lim_{|x|\to\infty} |x|^r e^{-|x|^2} \leq \lim_{|x|\to\infty} \frac{|x|^r}{1+|x|^2+\cdots+\frac{1}{r!}|x|^{2r}} = 0.$$

This completes the proof.

(b) This is immediate, because

$$\lim_{|x|\to\infty} (1+|x|^2)^3 \frac{1}{1+|x|^4} \leq \lim_{|x|\to\infty} \frac{|x|^6}{1+|x|^4},$$

which is infinity. Therefore, the function $1/(1+|x|^4)$ does not lie in $\mathcal{S}(\mathbb{R}^n)$.

(c) Let $f \in \mathcal{S}(\mathbb{R}^n)$. Since f is a continuous function, it is measurable. Note that there is a positive constant M such that

$$(1+|x|^2)^{n+1}|f(x)| \leq M$$

on $\mathbb{R}^n$. Let B denote the unit ball in $\mathbb{R}^n$. Then,

$$\int_{\mathbb{R}^n} |f| = \int_B |f| + \int_{\mathbb{R}^n \setminus B} |f|.$$

The first integral is obviously finite, since f is bounded on $\overline{B}$ and B has finite measure in $\mathbb{R}^n$.

As for the second one, note that

$$\int_{\mathbb{R}^n \setminus B} |f| \leq \int_{\mathbb{R}^n \setminus B} \frac{M}{(1+|x|^2)^{n+1}}\, dx.$$

The integrand on the right hand side is a radial function and $\mathbb{R}^n \setminus B$ can be identified with $S^{n-1} \times [1, \infty)$. Hence, using Fubini's Theorem,

$$\int_{\mathbb{R}^n \setminus B} \frac{M}{(1+|x|^2)^{n+1}}\, dx = \int_1^\infty \int_{S^{n-1}} \frac{Mr^{n-1}}{(1+r^2)^{n+1}}\, d\sigma dr$$

where σ parametrizes the sphere and r the interval $[1, \infty)$. Note that $(1 + r^2)^{n+1} \geq r^{2n+2}$ and hence, the integral on the right is bounded above by

$$\int_1^\infty \frac{M\omega_{n-1}}{r^{n+3}}\, dr < \infty.$$

This shows that $f \in L^1(\mathbb{R}^n)$.

Problem 2

We shall first show that the Fourier Transform of an L^1 function is continuous. Let $f \in L^1(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$ and $\xi_m \to \xi$ in $\mathbb{R}^n$ as $m \to \infty$. Then,

$$\widehat{f}(\xi_m) - \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \left(\exp(-i\xi_m \cdot x) - \exp(-i\xi \cdot x)\right)\, dx$$

Note that $|\exp(-i\xi_m \cdot x) - \exp(-i\xi \cdot x)| \leq 2$ due to the triangle inequality. Consequently, the integrand on the right is dominated by $2|f(x)|$, which is in L^1. Hence, the Dominated Convergence Theorem applies and we have

$$\lim_{m \to \infty} \widehat{f}(\xi_n) - \widehat{f}(\xi) = 0,$$

since as $m \to \infty$, the integrand on the right tends to 0 pointwise. Hence, $\widehat{f} : \mathbb{R}^n \to \mathbb{R}$ is continuous.

To see that the Fourier Transform is a bounded linear functional, note that for any $f \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, we have

$$\left|\widehat{f}(\xi)\right| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)||\exp(-i\xi \cdot x)|\, dx = \frac{1}{(2\pi)^{n/2}} \|f\|_1.$$

That is,

$$\|\widehat{f}\|_\infty \leq \frac{1}{(2\pi)^{n/2}} \|f\|_1.$$

This completes the proof.

Problem 3

For $x, y \in \mathbb{R}^n$, we have the inequality,

$$1 + |x| \leq 1 + |x-y| + |y| \leq (1 + |x-y|)(1 + |y|).$$

Therefore,

$$\begin{aligned}(1+|x|)^N|f * g(x)| &\leq \int_{\mathbb{R}^n} (1+|x-y|)^N(1+|y|)^N|f(y)||g(x-y)| \, dy \\ &\leq \int_{\mathbb{R}^n} \big((1+|y|)^{N+n+1}|f(y)|\big)\big((1+|x-y|)^N g(x-y)\big) \cdot \frac{1}{(1+|y|)^{n+1}} \, dy.\end{aligned}$$

Note that there are constants $C_N, C_{n+N+1} > 0$ such that

$$(1+|x-y|)^N|g(x-y)| \leq C_N \qquad \text{and } (1+|y|)^{N+n+1}|f(y)| \leq C_{N+n+1}$$

for all $x, y \in \mathbb{R}^n$. Therefore,

$$(1+|x|)^N|f * g(x)| \leq C_N C_{N+n+1} \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{n+1}} \, dy < \infty.$$

In conclusion, we have shown that if $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $(1+|x|)^N(f * g)(x)$ is bounded on $\mathbb{R}^n$. We shall use this result to show that $f * g$ lies in $\mathcal{S}(\mathbb{R}^n)$.

Let us first compute $\partial_i(f * g)(x)$. This is given by

$$\lim_{h \to 0} \frac{(f*g)(x+h_i) - (f*g)(x)}{h_i} = \lim_{h \to 0} \int_{\mathbb{R}^n} f(y) \frac{g(x+he_i-y) - g(x-y)}{h} \, dy.$$

Using the Mean Value Theorem, there is some $c \in (0, h)$ such that

$$\frac{g(x+he_i-y) - g(x-y)}{h} = \partial_i g(x + ce_i - y).$$

Note that $\partial_i g \in \mathcal{S}(\mathbb{R}^n)$ and hence is bounded in absolute value by some $M > 0$ on $\mathbb{R}^n$. Consequently,

$$\left| f(y) \frac{g(x+he_i-y) - g(x-y)}{h} \right| \leq M|f(y)|$$

for all $x, y \in \mathbb{R}^n$.

Take any sequence $h_n \to 0$ in $\mathbb{R}^n$. Because of what we discussed above, the Dominated Convergence Theorem applies. The integrand converges pointwise to $f(y)\partial_i g(x-y)$ and hence,

$$\lim_{n \to \infty} \frac{(f*g)(x+h_ne_i) - (f*g)(x)}{h_n} = \int_{\mathbb{R}^n} f(y)\partial_i g(x-y) \, dy = (f * \partial_i g)(x).$$

Hence, it follows that

$$\partial_\alpha(f * g) = f * \partial_\alpha g$$

for every multi-index α. This shows, in particular that $f * g$ is in $C^\infty(\mathbb{R}^n)$.

Hence, using what we proved at the start of this proof,

$$(1+|x|)^N \partial_\alpha (f * g)(x) = (1+|x|)^N (f * \partial_\alpha g)(x)$$

is bounded on $\mathbb{R}^n$, which completes the proof.

Problem 4

This is straightforward

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\ dy = \int_{\mathbb{R}^n} f(y)\ dy.$$

Thus, $f * g$ is a constant function taking the value $\int_{\mathbb{R}^n} f$.

Problem 5

We have,

$$\int_{\mathbb{R}^n} (f * g)(x)\ dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(x-y)\ dydx.$$

We contend that the function $F(x,y) = f(y)g(x-y)$ is in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Indeed, using Fubini's Theorem,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |F(x,y)| = \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x-y)|\ dxdy = \|f\|_1 \|g\|_1,$$

where the last equality follows from the translation invariance of the Lebesgue measure.

Using this, we can invoke Fubini's Theorem to evaluate the previous integral:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(x-y)\ dydx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(x-y)\ dxdy = \int_{\mathbb{R}^n} f(y)\ dy \int_{\mathbb{R}^n} g(x)\ dx,$$

where the last equality follows from the translation invariance of the Lebesgue measure.

Problem 6

For a function $f : \mathbb{R}^n \to \mathbb{R}$ and $t \in \mathbb{R}^n$, let $f_t : \mathbb{R}^n \to \mathbb{R}$ be given by $f_t(x) = f(x-t)$.

Lemma. *Let $f \in L^2(\mathbb{R}^n)$. For every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $|t| < \delta$, $\|f - f_t\|_2 < \varepsilon$.*

Proof. It is known that $C_c(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ is dense with respect to the L^2-norm. Therefore, there is a $g \in C_c(\mathbb{R}^n)$ such that $\|f-g\|_2 < \varepsilon/3$. Consequently, $\|f_t - g_t\|_2 < \varepsilon/3$ for every $t \in \mathbb{R}^n$. We shall find a δ such that $\|g - g_t\| < \varepsilon/3$. Since g has compact support, there is a $N > 0$ such that $\mathrm{Supp}(g) \subseteq \overline{B}(0, N)$. Further, since g has compact support, it is uniformly continuous whence, there is a $1 > \delta > 0$ such that $|g(x) - g(x-t)| < \sqrt{\dfrac{\varepsilon}{\mu(B(0, N+1))}}$ for all $x \in \mathbb{R}^n$ and $|t| < \delta$, where μ denotes the Lebesgue measure on $\mathbb{R}^n$. Note that the support of $g - g_t$ is contained in $\overline{B}(0, N+\delta) \subseteq \overline{B}(0, N+1)$.

For $|t| < \delta$, we have

$$\|g - g_t\|_2^2 = \int_{\mathbb{R}^n} |g(x) - g_t(x)|^2 \, dx \le \frac{\varepsilon}{\mu(B(0, N+1))} \mu(B(0, N+1)) = \varepsilon.$$

This completes the proof. $\square$

To see that $f * g$ is bounded, simply invoke Hölder's inequality:

$$\begin{aligned} |(f*g)(x)| &= \left| \int_{\mathbb{R}^n} f(y) g(x-y) \, dy \right| \\ &\le \|f \cdot g(x - \cdot)\|_1 \le \|f\|_2 \|g\|_2. \end{aligned}$$

Note that

$$|(f*g)(x+h) - (f*g)(x)| = \left| \int_{\mathbb{R}^n} f(y) \left(g(x+h-y) - g(x-y) \right) \, dy \right|.$$

Let $p : \mathbb{R}^n \to \mathbb{R}$ be given by $p(y) = g(x-y)$. Then, $g(x+h-y) = p(y-h) = p_h(y)$. As a result of Hölder's inequality, we obtain

$$|(f*g)(x+h) - (f*g)(x)| \le \|f \cdot (p_h - p)\| \le \|f\|_2 \|p_h - p\|_2,$$

which goes to 0 as $h \to 0$ due to the preceeding Lemma, implying the desired conclusion.

Problem 7

There are multiple ways to skin a cat. I chose the quickest one. We have seen in class that $\widehat{\varphi}(\xi) = \varphi(\xi)$. Therefore,

$$\widehat{\varphi * \varphi}(\xi) = (2\pi)^{n/2} \widehat{\varphi}(\xi) \widehat{\varphi}(\xi) = (2\pi)^{n/2} \exp(-|\xi|^2).$$

Let $\psi(x) = \exp(-|x|^2)$. Taking Fourier transform again and recalling that $\widehat{\widehat{f}}(x) = f(-x)$, we have

$$(\varphi * \varphi)(-x) = (2\pi)^{n/2} \widehat{\psi}(x) = (2\pi)^{n/2} \frac{1}{2^{n/2}} \widehat{\varphi}\left(\frac{x}{\sqrt{2}} \right)$$

Where we use the fact that

$$\widehat{f(\lambda x)}(\xi) = \frac{1}{\lambda^n}\widehat{f}\left(\frac{\xi}{\lambda}\right)$$

coupled with the fact that

$$\psi(x) = \varphi(\sqrt{2}x)$$

and that $\widehat{\varphi} = \varphi$.

Hence,

$$(\varphi * \varphi)(x) = \pi^{n/2}\exp\left(-\frac{|x|^2}{4}\right)$$

and we are done.

Problem 8

(a) From the definition, we have, for any test function φ,

$$\langle\widehat{1}, \varphi\rangle = \langle 1, \widehat{\varphi}\rangle = \int_{\mathbb{R}^n}\widehat{\varphi} = (2\pi)^{n/2}\varphi(0),$$

where the last equality follows from the Fourier Inversion Formula,

$$\varphi(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\widehat{\varphi}(\xi)\exp(i\xi\cdot x)\ d\xi.$$

Therefore, $\widehat{1} = (2\pi)^{n/2}\delta_0$.

(b) For any test function φ, we have

$$\langle\widehat{\delta_0}, \varphi\rangle = \langle\delta_0, \widehat{\varphi}\rangle = \widehat{\varphi}(0).$$

But note that

$$\widehat{\varphi}(0) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi(x)\ dx.$$

Hence, $\widehat{\delta_0} = \frac{1}{(2\pi)^{n/2}}$, the distribution induced by the constant function $1/(2\pi)^{n/2}$.

(c) For a test function φ, we have

$$\langle\widehat{x^\alpha}, \varphi\rangle = \langle x^\alpha, \widehat{\varphi}\rangle = \int_{\mathbb{R}^n} x^\alpha\widehat{\varphi}(x)\ dx.$$

Due to Fourier Inversion, we know

$$\varphi(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\widehat{\varphi}(x)\exp(i\xi\cdot x)\ dx.$$

Taking the partial derivative $\partial^\alpha/\partial\xi^\alpha$, we have

$$\partial_\alpha\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} i^\alpha \int_{\mathbb{R}^n} x^\alpha \widehat{\varphi}(x) \exp(i\xi \cdot x)\ dx.$$

Set $\xi = 0$ to obtain

$$(-i)^\alpha \partial_\alpha \varphi(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} x^\alpha \widehat{\varphi}(x)\ dx.$$

We have

$$\begin{aligned} \langle \widehat{x^\alpha}, \varphi \rangle &= \langle \delta_0, (-i)^\alpha \partial_\alpha (2\pi)^{n/2} \varphi \rangle \\ &= (-1)^\alpha \langle \delta_0, i^\alpha (2\pi)^{n/2} \varphi \rangle \\ &= \langle \partial_\alpha \delta_0, i^\alpha (2\pi)^{n/2} \varphi \rangle. \end{aligned}$$

Thus,

$$\widehat{x^\alpha} = (2\pi)^{n/2} i^\alpha \partial_\alpha \delta_0.$$

(d) For any test function φ, we have

$$\begin{aligned} \langle \widehat{\cos x}, \varphi \rangle = \langle \cos x, \widehat{\varphi} \rangle &= \left\langle \frac{e^{ix} + e^{-ix}}{2}, \widehat{\varphi} \right\rangle \\ &= \frac{1}{2} \int_{\mathbb{R}^n} (e^{ix} + e^{-ix}) \widehat{\varphi}(x)\ dx \\ &= \frac{1}{2} (2\pi)^{n/2} (\varphi(1) + \varphi(-1)), \end{aligned}$$

where the last equality follows from the Fourier Inversion Formula. Using this, we have

$$\widehat{\cos x} = \frac{1}{2} (2\pi)^{n/2} (\delta_1 + \delta_{-1})$$

and we are done.