General Topology

Swayam Chube

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Contents

Chapter 1

Topological Spaces

Definition 1.1 (Topology, Topological Space). A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

- $\varnothing$ and X are in $\mathcal{T}$
- The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$
- The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a *topological space*.

Definition 1.2 (Open Set). Let X be a topological space with associated topology $\mathcal{T}$. A subset U of X is said to be open if it is an element of $\mathcal{T}$.

This immediately implies that both $\varnothing$ and X are open. In fact, we shall see that they are also closed. The topology $\mathcal{T}$ of all subsets of X is called the **discrete topology** while the topology $\mathcal{T} = \{\varnothing, X\}$ is called the **indiscrete topology** or the **trivial topology**.

Definition 1.3. Let X be a set and $\mathcal{T}, \mathcal{T}'$ be two topologies defined on X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$. Further, if $\mathcal{T}' \subsetneq \mathcal{T}$, then $\mathcal{T}'$ is said to be *strictly finer* than $\mathcal{T}$.

Definition 1.4 (Basis). If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

- For each $x \in X$, there is at least one basis element B containing x
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Definition 1.5 (Generated Topology). Let $\mathcal{B}$ be a basis for a topology on X. The *topology generated by* $\mathcal{B}$ is defined as follows: A subset U of X is said to be open in X if for each $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 1.6. The collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ is indeed a topology on X.

Proof. Obviously $\emptyset, X \in \mathcal{T}$. Suppose $\{U_\alpha\}$ is a J indexed collection of sets in $\mathcal{T}$. Let $U = \bigcup_{\alpha \in J} U_\alpha$. Then, for each $x \in U$, there is an $\alpha \in J$ such that $x \in U_\alpha$ and thus, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq U$ and thus $U \in \mathcal{T}$. Let $U_1, U_2 \in \mathcal{T}$ and $x \in U_1 \cap U_2$. Then, there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$ and thus, $x \in B_1 \cap B_2 \subseteq U_1 \cap U_2$. But, by definition, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$ and consequently $U_1 \cap U_2 \in \mathcal{T}$. This finishes the proof. $\blacksquare$

Lemma 1.7. Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. Trivially note that all elements of $\mathcal{B}$ must be in $\mathcal{T}$ and thus, their unions too. Conversely, let $U \in \mathcal{T}$, then for all $x \in U$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. It is not hard to see that $U = \bigcup_{x \in U} B_x$ and we have the desired conclusion. $\blacksquare$

Lemma 1.8. Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof. We first show that $\mathcal{B}$ is a basis. Since X is an open set, for each $x \in X$, there is $C \in \mathcal{C}$ such that $x \in C$. Let $C_1, C_2 \in \mathcal{C}$. Since both C_1 and C_2 are given to be open, so is their intersection. Thus, for each $x \in C_1 \cap C_2$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq C_1 \cap C_2$. Therefore, $\mathcal{B}$ is a basis.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}$ be the topology associated with X. Let $U \in \mathcal{T}$, then for each $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$, and thus $U \in \mathcal{T}'$ by definition. Conversely, let $W \in \mathcal{T}'$. Since W can be written as a union of a collection of sets in $\mathcal{C}$, all of which are open, W must be open too and thus $W \in \mathcal{T}$. This finishes the proof. ∎

Lemma 1.9. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then, the following are equivalent:

- $\mathcal{T}'$ is finer than $\mathcal{T}$
- For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Then $B \in \mathcal{T}$ and thus $B \in \mathcal{T}'$. As a result, there is, by definition $B' \in \mathcal{T}'$ such that $x \in B' \subseteq B$.

Conversely, let $U \in \mathcal{T}$. Since $\mathcal{B}$ generates $\mathcal{T}$, for each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. But due to the second condition, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$, implying that U is in the topology generated by $\mathcal{B}'$, that is $\mathcal{T}'$. This finishes the proof. ∎

Definition 1.10 (Subbasis). A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of S.

Proposition 1.11. The topology generated by $\mathcal{S}$ is indeed a topology.

Proof. For this, it suffices to show that the set $\mathcal{B}$ of all finite intersections of elements of S forms a basis. Since the union of all elements of $\mathcal{S}$ equals X, for each $x \in X$, there is $S \in \mathcal{S}$ such that $x \in S$ and note that S must be an element of $\mathcal{B}$. Finally, since the intersection of any two elements of $\mathcal{B}$ can trivially be written as a finite intersection of elements of $\mathcal{S}$, it must be an element of $\mathcal{B}$ and we are done. ∎

A *simple order* is a relation C such that

1. (Comparability) For all $x \neq y$, either xCy or yCx
2. (Non-reflexivity) For all x, it is not true that xCx
3. (Transitivity) For all x, y, z such that xCy and yCz, we have xCz

Suppose X is a set with a simple order relation, $<$. Suppose a and b are elements such that $a < b$, then there are four subsets of X that are called *intervals* determined by a and b:

$$\begin{aligned}(a, b) &= \{x \mid a < x < b\}\\ (a, b] &= \{x \mid a < x \leq b\}\\ [a, b) &= \{x \mid a \leq x < b\}\\ [a, b] &= \{x \mid a \leq x \leq b\}\end{aligned}$$

Definition 1.12 (Order Topology). Let X be a set with a simple order relation an dassume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X
2. All intervals of the form $[a_0, b)$ where a_0 is the smalest element (if any) of X
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X

The collection $\mathcal{B}$ is a basis for a topology on X which is called the *order topology*.

If X has no smallest element, there are no sets of type (2) and if X Has no largest element, then there are not sets of type (3).

Definition 1.13 (Product Topology). Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$ where U and V are open sets in X and Y respectively.

Proposition 1.14. The collection $\mathcal{B}$ is indeed a basis.

Proof. The first condition is trivially satisfied since $X \times Y \in \mathcal{B}$. Suppose $x \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = U_3 \times V_3$ for some open sets U_3 and V_3 in X and Y respectively. This finishes the proof. ■

It is important to note here that *every* open set in $X \times Y$ need not be of the form $U \times V$ where U is open in X and V is open in Y. For a counterexample, consider $\mathbb{R}^2$ equipped with the standard topology. The unit ball $x^2 + y^2 < 1$ is open in $\mathbb{R}^2$ but cannot be expressed in the form $U \times V$.

Proposition 1.15. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B},\ C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. Let W be an open set in $X \times Y$ and $(x, y) \in W$. Then, by definition, there is $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $(x, y) \in B \times C \subseteq W$, further, since B and C are open in X and Y respectively, $B \times C$ is also open in $X \times Y$ equipped with the product topology. Therefore, we are done due to a preceeding lemma. ■

Definition 1.16. Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$ and let $\pi_2 : X \times Y \to Y$ be defined by the equation $\pi_2(x, y) = y$. The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Then, by definition if U is an open subset of X, then $\pi_1^{-1}(U) = U \times Y$ and similarly, if V is an open subset of Y, then $\pi_2^{-1}(V) = X \times V$.

Proposition 1.17. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Since $X \times Y \in \mathcal{S}$, the union of all elements of $\mathcal{S}$ is $X \times Y$ and thus $\mathcal{S}$ is a subbasis. Let $\mathcal{B}$ be the basis generated by all finite intersections of S. It suffices to show that $\mathcal{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$. For any U and V open in X and Y respectively, we may write $U \times V = (U \times Y) \cap (X \times V)$ and is therefore a member of $\mathcal{B}$. Conversely, the finite intersection of elements of S is of the form $(U_1 \cap \ldots \cap U_m) \times (V_1 \cap \ldots \cap V_m)$, which is a product of two open sets and is an element of $\mathcal{B}$, which finishes the proof. ■

Definition 1.18. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the *subspace topology*. With this topology, the topological space Y is called a *subspace* of X. Its open sets consist of all intersections of open sets of X with Y.

Proposition 1.19. $\mathcal{T}_Y$ is a topology on Y.

Proof. Since $\emptyset \in \mathcal{T}$, $\emptyset = Y \cap \emptyset \in \mathcal{T}_Y$ and since $X \in \mathcal{T}$, $Y = Y \cap X \in \mathcal{T}_Y$. Further,

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = Y \cap \bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_Y$$

And finally, $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2) \in \mathcal{T}_Y$. This finishes the proof. ■

Lemma 1.20. If $\mathcal{B}$ is a basis for the topology of X and $Y \subseteq X$. Then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let V be an open set in Y. Then, there is U in X such that $V = U \cap Y$. Since each $x \in V$ is an element of U, there is, by definition $B \in \mathcal{B}$ such that $x \in B \subseteq U$, consequently, $x \in B \cap Y \subseteq V$ and we are done due to a preceeding lemma. ■

Proposition 1.21. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Follows from the fact that $U = V \cap Y$ for some V that is open in X. ■

1.1 Closed Sets and Limit Points

Definition 1.22 (Closed Set). A subset A of a topological space X is said to be *closed* if the set $X \backslash A$ is open.

Theorem 1.23. Let X be a topological space. Then the following conditions hold:

1. $\emptyset$ and X are closed
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

Proof. All follow from De Morgan's laws. ■

Proposition 1.24. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof. If A is closed in Y then $Y \backslash A$ is open and thus, there is an open set B in X such that $Y \backslash A = Y \cap B$. Then,

$$A = Y \backslash (Y \cap B) = Y \cap (X \backslash B)$$

which finishes the proof. ■

Corollary. Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Trivial. ■

Definition 1.25 (Interior, Closure). Let X be a topological space and $A \subseteq X$. The *interior* of A is defined as the union of all open sets contained in A and the *closure* of A is defined as the intersection of all closed sets containing A. The interior of A is denoted by $\operatorname{Int} A$ and the closure of A is denoted by $\overline{A}$.

Then, by definition, we have that

$$\operatorname{Int} A \subseteq A \subseteq \overline{A}$$

Corollary. Let X be a topological space. Then $A \subseteq X$ is closed if and only if $A = \overline{A}$.

Proof. Trivial. ■

Theorem 1.26. Let Y be a subspace of X and A be a subset of Y. Let $\overline{A}$ denote the closure of A in X. Then, the closure of A in Y is given by $\overline{A} \cap Y$.

Proof. Let $\mathcal{F}$ be the collection of all closed sets in X containing A. Then, by a preceeding theorem, we know that the set of all closed sets in Y containing A is given by $Y \cap \mathcal{F}$. And thus,

$$\bigcup_{C \in Y \cap \mathcal{F}} C = Y \cap \bigcup_{C \in \mathcal{F}} C = Y \cap \overline{A}$$

This finishes the proof. ■

Theorem 1.27. Let A be a subset of the topological space X.

- Then $x \in \overline{A}$ if and only if every open set U containing x intersects A
- Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A

Proof.

- Suppose $x \in \overline{A}$ and U be an open set containing x. Suppose for the sake of contradiction, there is an open set U in X that contains x but does not intersect A, in which case $X \backslash U$ is a closed set containing A and not containing x. By definition, since $\overline{A} \subseteq X \backslash U$, x may not be an element of $\overline{A}$, a contradiction. Conversely, suppose every open set U containing x intersects A and that $x \notin \overline{A}$. But then, the set $X \backslash \overline{A}$ is open and contains x but does not intersect A, a contradiction.

- Suppose $x \in \overline{A}$, then every open set containing x intersects A. Since all elements of $\mathcal{B}$ are open, they intersect A. Conversely, since every open set U containing x has a basis subset B that contains x and therefore intersects A, U must intersect A. This finishes the proof.

■

We shall see that it is more natural to use the first statement of the above theorem as a substitute for the definition of the closure.

The statement "U is an open set containing x" is often shortened to "U is a **neighborhood** of x".

Definition 1.28. If A is a subset of the topological space X and if $x \in X$, we say that x is a *limit point* or *cluster point* or *accumulation point* of A if every neighborhood of x intersects A in some point **other than** x **itself**.

For example every element of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

Theorem 1.29. Let A be a subset of the topological space X and let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'$$

Proof. If $x \in A'$, due to the preceeding theorem, $x \in \overline{A}$ but since by definition, $A \subseteq \overline{A}$, we have that $A \cup A' \subseteq \overline{A}$.

Conversely let $x \in \overline{A}$. If $x \in A$, we are done. If not, then x is such that every open set containing x intersects A. But since $x \notin A$, the intersection must contain at least one point distinct from x, implying that $x \in A'$. This finishes the proof. ■

Corollary. A subset of a topological space is closed if and only if it contains all its limit points.

Proof. Follows from the fact that a subset A of a topological space is closed if and only if $A = \overline{A}$. ∎

Definition 1.30 (Hausdorff Spaces). A topological space X is called a *Hausdorff space* if for each pair x_1 and x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Theorem 1.31. Every finite point set in a Hausdorff space X is closed.

Proof. It suffices to show this for a single point set, say $\{x_0\}$. For any $x \in X$ different from x_0, there are open sets U and V such that $x_0 \in U$ and $x \in V$ and $U \cap V = \emptyset$. And thus, x may not be in the closure of $\{x_0\}$. This finishes the proof. ∎

The condition that finite point sets be closed has been given its own name, the T_1 **axiom**. Note that there is a more standard version of T_1-spaces,

Definition 1.32 (T_1-space). Let X be a topological space. Then, X is said to be a T_1-space if for any two distinct points $x, y \in X$, there is a neighborhood containing x but not y.

Theorem 1.33. A space is T_1 if and only if it satisfies the T_1 axiom.

Proof. Let X be a T_1 topological space, $\{x_0\} \subseteq X$, and $x \in X \backslash \{x_0\}$. Then, there is a neighborhood of x not containing x_0, therefore, $x \notin \overline{\{x_0\}}$, as a result, $\{x_0\}$ is closed in X, consequently, every finite point set is closed in X.

Conversely, suppose X satisfies the T_1 axiom and $x, y \in X$ be distinct. Then, $x \notin \overline{\{y\}}$, then, using the above theorems, there is a neighborhood containing x that does not contain y, equivalently, X is a T_1-space. ∎

Corollary. Every Hausdorff space is T_1. *Note that Hausdorff spaces are also known as T_2 spaces.*

The converse is not true. That is, not every Fréchet space is Hausdorff. For example, let X be an infinite set and $\mathcal{T}$ be the *co-finite* topology on X, that is,

$$\mathcal{T} = \{U \mid U^c \text{ is finite}\}$$

That the above is a topology is trivially verified. Let $x, y \in X$, then $X \backslash \{y\}$ is an open set containing x but not y, therefore $(X, \mathcal{T})$ is T_1 (Fréchet).

Suppose there were disjoint open sets U and V such that $x \in U \subseteq X \backslash \{y\}$ and $y \in V \subseteq X \backslash \{x\}$. But then, $V \subseteq U^c$, contradicting the finiteness of U^c. As a result, $(X, \mathcal{T})$ is not T_2 (Hausdorff).

Theorem 1.34. Let X be a space satisfying the T_1 axiom and $A \subseteq X$. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. If every neighborhood of x intersects A at infinitely many points, then it intersects it in at least one point other than x and thus $x \in A'$.

Conversely, suppose x is a limit point of x but there is a neighborhood U of x that intersecs A in only finitely many points. Let $U \cap (A \backslash \{x\}) = \{x_1, \ldots, x_m\}$. Then, the open set $U \cap (X \backslash \{x_1, \ldots, x_m\})$ contains x but does not intersect A, which is contradictory to the fact that x is a limit point of A. ■

Theorem 1.35. If X is a Hausdorff space, then a sequence of points of X convertes to at most one point of X.

Proof. Suppose the sequence $\{x_n\}$ converges to two distinct points x and y. Then, by definition, there exist disjoint neighborhoods U and V of x and y respectively. Since x_n converges to x, U contains all but finitely many elements of the sequence but that means V cannot, a contradiction. ■

1.2 Continuous Functions

Definition 1.36 (Continuity). Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is open in X.

We note here that it suffices to check the above condition for just elements of either a *basis* or a *subbasis*.

Conversely, note that it need not be the case that an open set in X is mapped to an open set in Y. Simply consider any *constant function* from $\mathbb{R} \to \mathbb{R}$.

Theorem 1.37. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent

1. f is continuous
2. for every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$
3. for every closed set B of Y, the set $f^{-1}(B)$ is closed in X
4. for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$

Proof. $(1) \Rightarrow (2)$. Let $x \in \overline{A}$ and V be an open set containing $f(x)$. We know by definition that $f^{-1}(V)$ is open and therefore intersects A. As a consequence, V intersects $f(A)$, implying that $f(x) \in \overline{f(A)}$.

$(2) \Rightarrow (3)$. Let $A = f^{-1}(B)$. Let $x \in \overline{A}$. Then,

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$$

and thus $x \in f^{-1}(B) = A$, implying that $A \subseteq \overline{A} \subseteq A$, finishing the proof.

$(3) \Rightarrow (1)$. Let V be an open set in Y and let $U = f^{-1}(V)$. Since $Y \backslash V$ is closed, so is $f^{-1}(Y \backslash V) = f^{-1}(Y) \backslash U = X \backslash U$. Then, by definition, U must be open.

$(1) \Leftrightarrow (4)$. The forward direction is trivial. Conversely, let V be an open set in Y and $U = f^{-1}(V)$. For each $x \in U$, there is an open set U_x such that $U_x \subseteq U$. Then, $U = \bigcup_{x \in U} U_x$ is open. This finishes the proof. ∎

The converse of (3) is not true, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x^2}$. The range of said function is $(0, 1]$, which is obviously not closed in $\mathbb{R}$.

Further, notice that (4) is just the topological analogue of the epsilon-delta definition of continuity.

Definition 1.38 (Homeomorphism). Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is a *homeomorphism*.

As a result, any property of X that is entirely expressed in terms of the topology of X yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called a **topological property**.

If $f : X \to Y$ is an injective, continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological imbedding** or simply an **imbedding** of X in Y.

It is important to note that a bijection $f : X \to Y$ that is continuous need not have a continuous inverse. For example, consider $f : [0, 2\pi) \to \mathbb{S}^1$, given by $f(\theta) = e^{i\theta}$. Since the unit circle is compact, but $[0, 2\pi)$ is not, the inverse may not be continuous.

Theorem 1.39. Let X, Y and Z be topological spaces

1. (Constant) If $f : X \to Y$ maps all of X to a single point of Y, then it is continuous
2. (Inclusion) If A is a subspace of X, the inclusion function $j : A \hookrightarrow X$ is continuous
3. (Composites) If $f : X \to Y$ and $g : Y \to Z$ are ocontinuous, then the map $g \circ f : X \to Z$ is continuous
4. (Domain Restriction) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.

5. (Range Restriction/Expansion) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.
6. (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets $\{U_\alpha\}$ such that $f|_{U_\alpha}$ is continuous for each α.

Proof.

1. Trivial
2. Trivial
3. Let V be an open set in Z. Then, $g^{-1}(V)$ is open in Y and $f^{-1} \circ g^{-1}(V)$ is open in X and thus $g \circ f$ is continuous
4. Notice that $f|_A \equiv f \circ j$
5. Let V be an open set in Z. Then, there is an open set W in Y such that $V = Z \cap W$. Since the range of f is a subset of Z, we have

$$g^{-1}(V) = g^{-1}(Z \cap W) = f^{-1}(Z \cap W) = f^{-1}(W)$$

   which is open in X and thus, g is continuous. A similar argument can be applied in the second case.
6. Let V be an open set in Y, then we may write

$$f^{-1}(V) = \bigcup_\alpha f|_{U_\alpha}^{-1}(V \cap f(U_\alpha))$$

   which is a union of a collection of open sets and is therefore open. This finishes the proof.

■

Lemma 1.40 (Pasting Lemma). Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous If $f(x) = g(x)$ for every $x \in A \cap B$ then f and g combine to give a continuous function $h : X \to Y$

defined as

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. We then have $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f is continuous, we know that $f^{-1}(C)$ is closed in A and therefore in X similarly, so is $g^{-1}(C)$, which finishes the proof. ■

Theorem 1.41. Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ then f is continuous if and only if the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. The maps f_1 and f_2 are called the *coordinate maps* of f.

Proof. We know that the projection maps π_1, π_2 are continuous. We note that $f_1(a) = \pi_1(f(a))$ and $f_2(a) = \pi_2(f_2(a))$. If f is continuous, then so are f_1 and f_2.

Conversely, suppose f_1 and f_2 are continuous and $U \times V$ be a basis element for the product topology on $X \times Y$. We know due to a preceeding result that both U and V are open in X and Y respectively. Then

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

which is an intersection of two open sets and is therefore open. ■

1.3 Product Topology

Definition 1.42. Let J be an index set. Given a set X, we define a J-tuple of elements of X to be a function $x : J \to X$. If α is an element of J, we often denote the value of x at α by x_α rather than $x(\alpha)$ and call it the α-th coordinate of x. We often denote the function x itself by the symbol

$$(x_\alpha)_{\alpha \in J}$$

Definition 1.43 (Cartesian Product). Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets and let $X = \bigcup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples x of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, the set of all functions

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Definition 1.44 (Box Topology). Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the *box topology*.

Definition 1.45 (Product Topology). Let $\mathcal{S}_\beta$ denote the collection

$$S_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the *product topology*. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a *product space*.

It is not hard to see that $\mathcal{S}$ is indeed a subbasis and therefore defines a topology. Let $\mathcal{B}$ be the basis induced by $\mathcal{S}$. Then, any basis element is a finite intersection of elements of S and eventually would have the form

$$B = \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})$$

It is then obvious that the *box topology* is finer than the *product topology* since it has more open sets. In the case of finite products of topological spaces, obviously the two of them are equal, but this is not the case for infinite products of topological spaces, since the basis of the product topology are only finite intersections of the subbasis, implying that for any basis element of the form $B = \prod_{\alpha \in J} U_\alpha$, there exist infinitely many $\alpha \in J$ such that U_α is the entire space X_α and is therefore strictly coarser than the box topology.

As a rule of thumb:

> Whenever we consider the product $\prod_{\alpha \in J} X_\alpha$, we shall assume it is given the product topology unless we specifically state otherwise.

Theorem 1.46. Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$. The collection of all sets of the same form where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Proof. Straightforward. ■

Theorem 1.47. If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

Theorem 1.48. Let $\{X_\alpha\}$ be an indexed family of spaces and $A_\alpha \subseteq X_\alpha$ for each α. If $\prod X_\alpha$ is given by either the product or box topology, then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Proof. Let $x = (x_\alpha)$ be a point of $\prod \overline{A_\alpha}$ and let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains x. Since $x_\alpha \in \overline{A_\alpha}$, we know that there is $y_\alpha \in U_\alpha \cap A_\alpha$ and thus $y = (y_\alpha) \in U \cap \prod A_\alpha$.

Conversely, suppose $x = (x_\alpha) \in \overline{\prod A_\alpha}$, and let V_α be an arbitrary open set in X_α containing x_α. Since $\pi_\alpha^{-1}(V_\alpha)$ is an open set containing x, it must intersect $\prod A_\alpha$, thus, there is $y = (y_\alpha) \in \pi_\alpha^{-1}(V_\alpha) \cap \prod A_\alpha$, consequently, $y_\alpha \in V_\alpha \cap A_\alpha$, and it follows that $x_\alpha \in \overline{A_\alpha}$. This completes the proof. ■

Theorem 1.49. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each coordinate function f_α is continuous.

Proof. First, suppose f is continuous. Let U_β be open in X_β. The function π_β^{-1} maps U_β to an open set in $\prod X_\alpha$ and is therefore continuous. As a result, $f_\beta = \pi_\beta \circ f$ is continuous.

Conversely, suppose each coordinate function f_α is continuous. We remarked earlier that $\pi_\beta^{-1}(U_\beta)$ for some open set U_β is a subbasis for the product topology and it suffices to show that the inverse image under f of the same is open to imply continuity. Indeed,

$$f^{-1} \circ \pi_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$$

which is obviously open, since f_β is known to be continuous. This finishes the proof. ■

Caution. It is important to note that the above theorem **does not** hold for the box topology. As a simple counter example, consider the box topology on $\mathbb{R}^\omega$ and the function $f : \mathbb{R} \to \mathbb{R}^\omega$ given by $f(t) = (t, t, \ldots)$. Suppose f were continuous, then the inverse image of each basis element must be open in $\mathbb{R}^\omega$. Indeed, consider

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$$

the inverse image would have to contain some open interval $(-\delta, \delta)$ in the standard topology of $\mathbb{R}$, that is, $(-\delta, \delta) \subseteq f^{-1}(B)$, or equivalently, $f((-\delta, \delta)) \subseteq B$, which is absurd.

1.4 Metric Topology

Definition 1.50 (Metric). A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

For $\epsilon > 0$, define the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

Definition 1.51 (Metric Topology). If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X, called the *metric topology* induced by d.

Proposition 1.52. The collection of all ϵ-balls $B_d(x, \epsilon)$ for all $x \in X$ and $\epsilon > 0$ is a basis.

Proof. The first condition is trivially satisfied. Suppose $z \in B(x, \epsilon) \cap B(y, \epsilon)$. Let $r = \frac{1}{2} \min\{\epsilon - d(x, z), \epsilon - d(y, z)\}$. It is obvious, due to the triangle inequality, that $B(z, r) \subseteq B(x, \epsilon) \cap B(y, \epsilon)$. ■

Definition 1.53 (Metrizable). If X is a topological space, X is said to be *metrizable* if there exists a metric d on the set X that induces the topology of X.

A **metric space** is a metrizable space X together with a specific metric d that gives the topology of X.

Definition 1.54. Let X be a metric space with metric d. A subset A of X is said to be *bounded* if there is some number M such that $d(a_1, a_2) \leq M$ for every pair a_1, a_2 of points of A. if A is bounded and non-empty, the *diameter* of A is

defined to be
$$\operatorname{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Proposition 1.55. Every metric space is Hausdorff.

Proof. Trivial. ■

Theorem 1.56. Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by the equation
$$\overline{d}(x, y) = \min\{d(x, y), 1\}$$
Then $\overline{d}$ is a metric that induces the same topology as d.

Proof. We need only check the triangle inequality. This is euqivalent to

$$\overline{d}(x, y) + \overline{d}(y, z) \geq \overline{d}(x, z)$$

Obviously if either one of $\overline{d}(x, y)$ or $\overline{d}(y, z)$ is greater than or equal to 1, then we are done. If not, then

$$\overline{d}(x, y) + \overline{d}(y, z) = d(x, y) + d(y, z) \geq \overline{d}(x, z) \geq \min\{d(x, z), 1\}$$

Let $\mathcal{T}$ be the topology on X induced by d, having basis $\mathcal{B}$. Let $\overline{\mathcal{B}}$ be the set of all balls induced by $\overline{d}$ having radius strictly less than 1. Let U be an open set in $\mathcal{T}$ and $x \in U$, then, by definition, there is $B_d(x, \epsilon)$ in $\mathcal{B}$ such that $x \in B_d(x, \epsilon) \subseteq U$. The ball $B_{\overline{d}}(x, \frac{1}{2}\min\{\epsilon, 1\})$ is contained in $B_d(x, \epsilon)$ and also contains x. Thus, $\overline{\mathcal{B}}$ is a basis for $\mathcal{T}$. This finishes the proof. ■

Definition 1.57 (Euclidean, square Metric). Given $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we define the *Euclidean metric* on $\mathbb{R}^n$ by the equation

$$d(x, y) = \|x - y\| = \left((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right)^{1/2}$$

and the *square metric* ρ by the equation

$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

Lemma 1.58. Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \tag{1.1}$$

Proof. Suppose the $\epsilon - \delta$ condition holds. Let $B \in \mathcal{B}_d$ be a basis element for the topology induced by d and let x be an arbitrary element of B. Then, we can find ϵ such that $B_d(x, \epsilon) \subseteq B$ and thus, there exists δ such that $x \in B_{d'}(x, \delta) \subseteq B$. Taking the union of all such δ-balls for x, we have an open set in $\mathcal{T}'$ which corresponds to a basis element for $\mathcal{T}$, and thus $\mathcal{T}'$ is finer than $\mathcal{T}$.

Conversely, suppose $\mathcal{T}'$ is finer than $\mathcal{T}$, then the condition is trivially satisfied. ∎

Theorem 1.59. The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof. We shall first show that the topologies induced by d and ρ on $\mathbb{R}^n$ are identical. Indeed, we have, for any two points x and y that

$$\rho(x, y) \le d(x, y) \le \sqrt{n}\rho(x, y)$$

this immediately implies the conclusion due to the preceeding lemma.

Finally, we shall show that the topology induced by ρ is same as the product topology. Let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a basis element of the product topology and let $x \in B$, then for each i, there is an ϵ_i such that $(x - \epsilon_i, x + \epsilon_i) \subseteq (a_i, b_i)$. Choosing $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, we have that the topology induced by ρ is finer than the product topology. But since every basis element of the ρ-topology is inherently an element of the product topology, since it is a cartesian product of open intervals, it must be that the product topology is finer than the ρ-topology. This completes the proof. ∎

Definition 1.60 (Uniform Metric). Given an index set J and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\overline{\rho}$ given by

$$\overline{\rho}(x, y) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. This is called the *uniform metric* on $\mathbb{R}^J$ and the topology it induces is called the *uniform topology*.

Theorem 1.61 ($\epsilon - \delta$ Theorem). Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y respectively. Then continuity of f is equivalent to the requirement that tiven $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

Proof. Suppose f is continuous and let $\epsilon > 0$ be given. Consider the set $f^{-1}(B_Y(f(x), \epsilon))$, which is open in X and contains the point x. Therefore, there exists a δ-ball centered at x. If y is in this δ-ball, then $f(y)$ is in the ϵ-ball centered at $f(x)$ as desired.

Conversely, suppose the $\epsilon - \delta$ condition holds and let V be open in Y and $x \in f^{-1}(V)$. But since $f(x) \in V$, there exists ϵ such that $B_Y(f(x), \epsilon)$ is contained in V, consequently, there exists δ such that $B_X(x, \delta)$ is contained in $f^{-1}(V)$ and thus $f^{-1}(V)$ is open. This finishes the proof. ■

Lemma 1.62 (Sequence Lemma). Let X be a topological space; let $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.

Proof. Suppose there is a sequence of points of A converging to x, then each neighborhood of x contains a points of A and thus, by definition, $x \in \overline{A}$.

Conversely, suppose X is metrizable and $x \in \overline{A}$. For each $n \in \mathbb{N}$, consider $B_X(x, \frac{1}{n})$ which must contain at least one point of A, call it x_n. It is not hard to see that this sequence converges to x. This finishes the proof. ■

Note that the above lemma holds even after replacing *metrizable* by *Hausdorff*.

Theorem 1.63. Let $f : X \to Y$. If the function f is continuous, then for every $x \in X$ and every convergent sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Proof. Suppose f is continuous. Let V be an open set in Y containing $f(x)$. Then, $U = f^{-1}(V)$ is an open set in X containing x. By definition, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$ and as a result, $f(x_n) \in V$. This finishes the proof.

Conversely, suppose X is metrizable, with metric d and for each convergent sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$. Let A be a subset of X. We shall show that $f(\overline{A}) \subseteq \overline{f(A)}$, which would immediately imply continuity due to a preceeding theorem. Let $x \in \overline{A}$, and let x_n be a point of A within the ball $B(x, \frac{1}{n})$. The sequence x_n converges to x and so does $f(x_n)$ to $f(x)$, as a result, for each open set containing $f(x)$, there is a point of $f(A)$ in it. This finishes the proof. ■

Definition 1.64 (Uniform Convergence). Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence f_n *converges uniformly* to the function $f : X \to Y$ if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \epsilon$ for all $n > N$ and all $x \in X$.

Theorem 1.65 (Uniform Limit Theorem). Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to $f : X \to Y$, then f is continuous.

Proof. Let V be an open set in Y and $U = f^{-1}(V)$. Let $x_0 \in U$. We shall show that there is a neighborhood containing x_0, that is contained in U. Let $y_0 = f(x_0)$ and $\epsilon > 0$ be such that $B(y_0, \epsilon) \subseteq V$. We know there exists $N \in \mathbb{N}$ such that for all $x \in X$, $d(f_n(x), f(x)) < \epsilon/3$ for all $n \geq N$. Further, there is an open set W in X that contains x_0 such that $d(f_N(x_0), f_N(y)) < \epsilon/3$ for all $y \in W$, due to continuity of each f_n. Then, we have, for all $y \in W$

$$d(f(x), f(y)) \leq d(f(x_0), f_N(x_0)) + d(f_N(x_0), f_N(y)) + d(f_N(y), f(y)) < \epsilon$$

and thus, $f(y) \in V$. This finishes the proof. ■

It is not sufficient to replace *uniform convergence* with *pointwise convergence*. Take for example the sequence of functions $\{\cos^n x\}_{n=1}^{\infty}$ from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[0, 1]$. The limiting function is given by

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

which is obviously not continuous.

1.5 Quotient Topology

Definition 1.66 (Quotient Map). Let X and Y be topological spaces and $p : X \to Y$ be a surjection. The map p is said to be a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Obviously, p must be continuous. One notes that this condition is stronger than continuity, and is often called **strong continuity**.

Lemma 1.67. Let X and Y be topological spaces. Then $p : X \to Y$ is a quotient map if and only if it is surjective and for each $A \subseteq Y$, $p^{-1}(A)$ is closed in X if and only if A is closed in Y.

Proof. Suppose p is a quotient map. Then,

$$p^{-1}(Y \backslash A) = X \backslash p^{-1}(A)$$

If A were closed in Y, then $p^{-1}(A)$ is closed in X since p is continuous. On the other hand, if $p^{-1}(A)$ were closed in X, then $p^{-1}(Y \backslash A)$ would be open in X, and therefore, so would $Y \backslash A$, equivalently A is closed.

The converse is trivially evident, since it is equivalent to saying A is open in Y if and only if $p^{-1}(A)$ is open in X. ■

Definition 1.68 (Open, Closed Map). Let X and Y be topological spaces and $f : X \to Y$. Then f is said to be an *open map* if it maps open sets in X to open sets in Y and is said to be a *closed map* if it maps closed sets in X to closed sets in Y.

It immediately follows that $p : X \to Y$ is a quotient map if p is surjective, continuous and either open or closed.

We say a subset C of X is saturated with respect to the surjective map $p : X \to Y$ if it equals the complete inverse image of a subset of Y. Formally, there exists $A \subseteq Y$ such that $C = p^{-1}(A)$.

Definition 1.69 (Quotient Topology). Let X be a topological space, A a set and $p : X \to A$ be a surjective map. Then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the *quotient topology* induced by p.

Proposition 1.70. The above defined topology $\mathcal{T}$ is indeed a topology and is unique.

Proof. Let
$$\mathcal{T} = \{p(U) \mid U \in \mathcal{T}_X\}$$
That $\mathcal{T}$ is indeed a topology follows from

$$\bigcup_{V \in B \subseteq \mathcal{T}} V = \bigcup_{U \in p^{-1}(B)} p(U) \in \mathcal{T} = p\left(\bigcup_{U \in p^{-1}(B)} U\right) \in \mathcal{T}$$

Let $\{V_i\}_{i=1}^n$ be a collection of open sets in $\mathcal{T}$, then, there is a collection $\{U_i\}_{i=1}^n$ of open sets in $\mathcal{T}_X$ such that $U_i = p^{-1}(V_i)$. We then have

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n p(U_i) = p\left(\bigcap_{i=1}^n U_i\right) \in \mathcal{T}$$

We shall now show that $\mathcal{T}$ is unique. Suppose $\mathcal{T}'$ is another topology induced by p on A. It is obvious that $\mathcal{T} \subseteq \mathcal{T}'$. Further, for any $V \in \mathcal{T}'$, $U = p^{-1}(V) \in \mathcal{T}_X$, therefore, $V = p(U) \in \mathcal{T}$, consequently $\mathcal{T}' \subseteq \mathcal{T}$ and we have the desired conclusion. ■

Definition 1.71 (Quotient Space). Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a *quotient space* of X.

Theorem 1.72. Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p

1. if A is either open or closed in X, then q is a quotient map
2. if p is either an open map or a closed map, then q is a quotient map

Proof. ■

Chapter 2

Connectedness and Compactness

2.1 Connected Spaces

Definition 2.1 (Connected Space). Let X be a topological space. A *separation* of X is a pair U and V of disjoint nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X.

The above definition can be restated as follows

> A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

It isn't hard to show the equivalence of the two statements.

Lemma 2.2. If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof. Suppose A and B form a separation of Y, then A is both open and closed in Y, as a result, $A = \overline{A \cap Y} = \overline{A} \cap Y$, which immediately implies $\overline{A} \cap B = \emptyset$ and vice versa.

Conversely, suppose A and B are disjoint nonempty sets whose union is Y, such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. We may then conclude that $\overline{A} \cap Y = \emptyset$ and $\overline{B} \cap Y = B$. And thus, both A and B are closed in Y and since $A = Y \backslash B$ and $B = Y \backslash A$, they are open in Y as well and are therefore a separation of Y. This finishes the proof. ■

Lemma 2.3. If the sets C and D form a separation of X and Y is a connected subspace of X, then Y lies entirely within C or entirely within D.

Proof. Since C and D form a separation of X, both C and D are open in X and thus, $C \cap Y$ and $D \cap Y$ are both open in Y. If both are non-empty, then we have a separation for Y, contradicting the fact that it is connected. ∎

Theorem 2.4. The union of a collection of connected subspaces of X that have a point in common is connected.

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of X and $p \in \bigcup_\alpha A_\alpha$. Let $Y = \bigcup_\alpha A_\alpha$. Suppose $Y = C \cup D$ is a separation. Then, due to the preceeding lemma, each of the A_α must lie in either C or D, but since they have a point p in common, they must all lie in C or all in D, and as a result, either C or D must be empty, a contradiction. ∎

Theorem 2.5. Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is also connected.

Proof. Suppose $B = C \cup D$ is a separation of B. Then, without loss of generality, A lies completely in C. Then, $\overline{A} \subseteq \overline{C}$. But due to a preceeding lemma, $\overline{C}$ and D are disjoint. This implies, B is contained entirely in $\overline{C}$ and may not intersect D, a contradiction. ∎

Theorem 2.6. The image of a connected space under a continuous map is connected.

Proof. Let $f : X \to Y$ be a continuous map and $Z = f(X)$. Suppose $Z = A \cup B$ is a separation of Z. Then, the sets $f^{-1}(A)$ and $f^{-1}(B)$ are open in X and are non-empty, since A and B are both within the range of f, which is Z. This contradicts the fact that X is connected. ∎

Theorem 2.7. A finite cartesian product of connected spaces is connected.

Proof. It suffices to show the statement for the Cartesian Product of two connected spaces since the result in its generality follows due to induction. Let X and Y be connected topological spaces. Let $a \times b \in X \times Y$ be a "base point". Note that the sets $X \times b$ and $x \times Y$ are connected for all $x \in X$. Then, we have

$$X = \bigcup_{x \in X} \underbrace{(X \times b) \cup (x \times Y)}_{T_x}$$

Further, note that all the sets T_x have the point $a \times b$ in common, as a result, their union is also connected. ∎

Definition 2.8 (Linear Continuum). A simply ordered set L having more than one element is called a *linear continuum* if the following hold:

1. L has the least upper bound property
2. If $x < y$, there exists z such that $x < z < y$.

Theorem 2.9. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof. We shall show that every convex subspace of L is connected. Let Y be a convex subspace of L that is not connected and therefore has a separation $Y = A \cup B$. Choose $a \in A$ and $b \in B$ and let $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$, each of which is open and nonempty in $[a, b]$ due to the subspace topology, which is the same as the order topology. Let $c = \sup A_0$, we know this exists because of the least upper bound property. ∎

Theorem 2.10 (Intermediate Value Theorem). Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof. Consider the sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$, both of which are open in $f(X)$. Suppose there is no c such that $f(c) = r$, then $f(X) = A \cup B$, both of which are non-empty because $f(a) \in A$ and $f(b) \in B$ and is therefore a separation, a contradiction to the fact that a continuous function maps connected spaces to connected spaces. ∎

Definition 2.11 (Path, Path Connected). Given points x and y of the space X, a *path* in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be *path connected* if every pair of points of X can be joined by a path in X.

2.2 Compact Spaces

Definition 2.12 (Cover). A collection $\mathscr{A}$ of subsets of a space x is said to *cover* X or be a *covering* of X if the union of the elements of $\mathscr{A}$ is equal to X. It is called an *open covering* of X if its elements are open subsets of X.

Definition 2.13 (Compact). A space X is said to be compact if every open covering $\mathscr{A}$ of X contains a finite subcollection that also covers X.

This definition is extended to subspaces through the following lemma:

Lemma 2.14. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof. Suppose Y is compact and $\{A_\alpha\}$ is a covering of Y by sets open in X. Then, the collection $\{A_\alpha \cap Y\}$ is a covering of Y by sets open in Y and therefore has a finite subcollection $\{A_{\alpha_1} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$ that covers Y. As a result, $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a finite subcollection of open sets in X that cover Y. The converse follows similarly. ■

Theorem 2.15. Every closed subspace of a compact space is compact.

Proof. Let Y be closed in a compact space X and $\mathscr{A}$ be an open cover for Y. The collection $\mathscr{A} \cup \{X \backslash Y\}$ is an open cover for X and therefore has a finite subcover, say $\mathscr{B}$. In which case, $\mathscr{B} \backslash \{X \backslash Y\}$ is a finite subcover for Y, implying that it is compact. ■

Theorem 2.16. Every compact subspace of a Hausdorff space is closed.

Proof. Let Y be a compact subspace of a Hausdorff space X. Let $x_0 \in X \setminus Y$. Then, for each $y \in Y$, there exist disjoint open sets U_y and V_y such that $x_0 \in U_y$ and $y \in V_y$. The collection $\mathscr{A} = \{V_y \mid y \in Y\}$ forms an open cover for Y and thus, has a finite subcover, $\{V_{y_1}, \dots, V_{y_n}\}$. The corresponding open set $\bigcap_{i=1}^n U_{y_i}$ is open in X and disjoint from each V_{y_i} and thus, disjoint from Y. This implies that for each $x_0 \in X \setminus Y$, there is an open set containing it, that is contained in $X \setminus Y$. This implies that $X \setminus Y$ is open and thus Y is closed. ∎

Theorem 2.17. The image of a compact space under a continuous map is compact.

Proof. Let $f : X \to Y$ be continuous and $\mathscr{A}$ be an open cover for $f(X)$. Then $\mathscr{B} = \{f^{-1}(A) \mid A \in \mathscr{A}\}$ is an open cover for X and therefore has a finite subcover, $\{f^{-1}(A_1), \dots, f^{-1}(A_n)\}$. This immediately implies that the collection $\{A_1, \dots, A_n\}$ is a finite subcover for $f(X)$ and thus $f(X)$ is compact. ∎

Theorem 2.18. Let $f : X \to Y$ be bijective and continuous. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Due to a preceeding theorem, Y must be compact. Let U be an open set in X. It suffices to show that $f(U)$ is open in Y. Since $X \setminus U$ is closed in X, due to a preceeding theorem, it must be compact, as a result, $Y \setminus f(U) = f(X \setminus U)$ must be compact and thus closed (since Y is Hausdorff). Thus, $f(U)$ is open and f is a homeomorphism. ∎

Theorem 2.19. The product of finitely many compact spaces is compact.

The following proof was taken from here.

Proof. It suffices to show that the product of two compact sets is compact, since the theorem statement follows from induction. Let X and Y be compact spaces and $\mathscr{A}$ be an open cover for $X \times Y$. Then, for each point $(a, b) \in X \times Y$, there is an open set $A_{(a,b)}$ that contains it. Then, by definition, there is some basis element $U_{(a,b)} \times V_{(a,b)}$ that contains (a, b).

Let a be fixed and vary b over Y. The collection $\{V_{(a,b)}\}$ thus obtained is an open cover for Y and thus has a finite subcover $\{V_{(a,b_1(a))}, \dots, V_{(a,b_n(a))}\}$. Let now, $U_a = \bigcup_{i=1}^{n} U_{(a,b_i(a))}$ which is an open set that contains a. The collection $\{U_a\}$ forms an open cover of A and thus has a finite subcover $\{U_{a_1}, \dots, U_{a_m}\}$. It then follows that the collection $\{O_{(a_i,b_j(a_i))}\}$ where $1 \le i \le m$ and $1 \le j \le n$ forms a finite subcover for $X \times Y$, thus finishing the proof. ■

Definition 2.20 (Finite Intersection). A collection $\mathscr{C}$ of subsets of X is said to have the finite intersection property if for every finite subcollection $\{C_1, \dots, C_n\}$, the intersection $\bigcap_{i=1}^{n} C_i$ is nonempty.

Theorem 2.21. Let X be a topological space. Then X is compact if and only if for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is nonempty.

Proof. Suppose X is compact and $\mathscr{C}$ is a collection of closed sets in X having the finite intersection property. Then, the collection $\mathscr{A} = \{X \setminus C \mid C \in \mathscr{C}\}$ consists of open sets such that no finite subcollection may cover X, due to the finite intersection property. And thus, $\bigcup_{A \in \mathscr{A}} \subsetneq X$, and equivalently, $\bigcap_{C \in \mathscr{C}} C \neq \emptyset$.

Conversely, let $\mathscr{A}$ be an open cover for X and $\mathscr{C} = \{X \setminus A \mid A \in \mathscr{A}\}$. It is then obvious that $\bigcap_{C \in \mathscr{C}} C$ is empty and thus, $\mathscr{C}$ may not have the finite intersection property. As a result, there is a finite subcollection of $\mathscr{A}$ that covers X. This finishes the proof. ■

Theorem 2.22. Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.

Proof. **TODO: Find a clean approach** ■

Theorem 2.23. A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the Euclidean metric d or the square metric ρ.

Proof. It suffices to use only the ρ-metric since

$$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\rho(x,y)$$

Now, suppose A is compact. The collection $\{B_\rho(0,m) \mid m \in \mathbb{N}\}$ is an open cover for A and must contain a finite subcover. Let $B_\rho(0,M)$ be the largest ball in the subcover. Since all other balls are subsets of it, the set A must be too. This implies boundedness.

Conversely, suppose A is closed and bounded. Then there exists $N \in \mathbb{N}$ such that $\rho(x,y) \leq N$ for all $x,y \in A$. Equivalently, $\rho(x,0) \leq N$ for all $x \in A$. Thus, A is a closed subset of the compact set $[-N,N]^n$ and thus is compact due to a preceeding theorem. ■

Theorem 2.24 (Extreme Value Theorem). Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

Proof. Since f is continuous, $A = f(X)$ is compact. Suppose A does not have a maximum element. Then, the collection

$$\mathscr{A} = \{(-\infty, a) \mid a \in A\}$$

is an open cover for A and must have a finite subcover, say

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

Without loss of generality, let a_n be the maximum out of all the a_i's. Then, we note that a_n is never covered by the subcollection, a contradiction. A similar argument may be applied for the minimum element. ■

Definition 2.25. Let (X,d) be a metric space and A be a nonempty subset of X. For each $x \in X$, define the *distance from x to A* by

$$d(x,A) = \inf\{d(x,a) \mid a \in A\}$$

Lemma 2.26 (Lebesgue Number Lemma). Let $\mathscr{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathscr{A}$ containing it. The number δ is called a *Lebesgue number* for the covering $\mathscr{A}$.

Proof. Let $\mathscr{A}$ be an open covering of X. If X itself is an element of A then any value of δ works. Suppose now that $X \notin \mathscr{A}$ and $\{A_1, \dots, A_n\}$ be a finite subcollection of elements in $\mathscr{A}$ that cover X and $C_i = X \backslash A_i$ for all $1 \leq i \leq n$. Define the function

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

For any $x \in X$, not all of $d(x, C_i)$ may be 0, since they cannot all share a point. Thus, $f(x) > 0$. Since X is compact and f is continuous, due to the extreme value theorem, we know that f has a minimum value, say δ. We shall show that δ is a Lebesgue number for $\mathscr{A}$.

Let B be a subset of X having diameter less than δ. Let $x_0 \in B$; then $B \subseteq B_d(x_0, \delta)$, further, since $f(x_0) \geq \delta$, we must have an index m such that $d(x_0, C_m) \geq \delta$. Then, obviously, $B \cap C_m = \emptyset$ and consequently, $B \subseteq A_m$. ■

Definition 2.27. Let $f : (X, d_X) \to (Y, d_Y)$ be a function. f is said to be *uniformly continuous* if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points $x_0, x_1 \in X$,

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \epsilon$$

Theorem 2.28 (Uniform Continuity Theorem). Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map such that the metric space X is compact. Then f is uniformly continuous.

Proof. Let $\epsilon > 0$ be given. Consider the collection $\mathscr{B} = \{B_Y(y, \epsilon/2) \mid y \in Y\}$ which is an open cover of Y then $\mathscr{A} = \{f^{-1}(A) \mid A \in \mathscr{A}\}$ is an open cover of X and thus has a finite subcover, $\{A_1, \dots, A_n\}$. Let δ be the Lebesgue Number of $\mathscr{A}$. Then for any two points $x_0, x_1 \in X$ with $d_X(x_0, x_1) < \delta$, the two point subset $\{x_0, x_1\}$ has diameter δ and is therefore contained in some A_i. As a result, $f(x_0), f(x_1) \in B_Y(y, \epsilon/2)$ for some $y \in Y$. This immediately implies that $d_Y(f(x_0), f(x_1)) < \epsilon$. ■

2.3 Limit Point Compactness

Definition 2.29 (Limit Point Compact). A space X is said to be *limit point compact* if every infinite subset of X has a limit point.

Theorem 2.30. Compactness implies limit point compactness.

Proof. Let A be a set with no limit points. We shall show that A is finite. We see that A must be closed, since it trivially contains all its limit points. Since each $a \in A$ is not a limit point, we may choose an open set U_a such that $U_a \cap A = \{a\}$. Then, the collection $\mathscr{U} = \{U_a \mid a \in A\}$ is an open cover for A, consequently, $\mathscr{U} \cup \{X \backslash A\}$ is an open cover for X and has a finite subcover. Since the finite subcover can have only finitely many elements of $\mathscr{U}$, A must be finite. ■

Definition 2.31 (Sequentially Compact). Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < \cdots$$

is an increasing sequece of positive integers, then the sequence (x_{n_i}) is called a *subsequence* of (x_n). The space X is said to be *sequentially compact* if every sequence of points of X has a convergent subsequence.

Theorem 2.32. Let X be a metrizable space. Then the following are equivalent

1. X is compact
2. X is limit point compact
3. X is sequentially compact

Proof. We have already shown that $(1) \implies (2)$. Let us first show that $(2) \implies (3)$. Consider the set $A = \{x_n \mid n \in \mathbb{N}\}$. If A is finite, then there is some $x \in A$ such that $x_i = x$ for infinitely many indices i. This immediately gives us a convergent subsequence. If A is infinite, then there exists $x \in X$ that is a limit point of A. Then, for each $n \in \mathbb{N}$, choose $x_n \in B(x, 1/n) \cap A$. This sequence obviously converges to x and we are done.

Finally, we show that (3) $\implies$ (1). We first show that if X is sequentially compact, then the Lebesgue number lemma holds. Suppose not. Let $\mathscr{A}$ be an open covering of X. Then for every positive integer n, there is a set C_n of diameter less than $1/n$ that is not contained in any element of $\mathscr{A}$. Choose a point $x_n \in C_n$ for all positive integers n. Since X is sequentially compact, there must exist a convergent subsequence (x_{n_i}) that converges to some point $a \in A$. Since $\mathscr{A}$ covers X, there is some $A \in \mathscr{A}$ such that $a \in A$. Choose $\epsilon > 0$ such that $B(a, \epsilon) \subseteq A$. For sufficiently large i, we have $1/n_i < \epsilon/2$ and $d(x_{n_i}, a) < \epsilon/2$, then the set C_{n_i} lies in the $\epsilon/2$ neighborhood of x_{n_i} but since x_{n_i} lies in the $\epsilon/2$ neighborhood of a, C_{n_i} lies in the ϵ neighborhood of a, and thus $C_{n_i} \subseteq B(a, \epsilon) \subseteq A$.

Next, we show that if X is sequentially compact, then for every $\epsilon > 0$, there exists a finite covering of X by open ϵ-balls. Suppose not. Let $x_1 \in X$, then $B(x_1, \epsilon)$ may not cover X and thus, there is $x_2 \in X \backslash B(x_1, \epsilon)$. Keep choosing points in X this way, that is:

$$x_{n+1} \in X \backslash \bigcup_{i=1}^{n} B(x_i, \epsilon)$$

The sequence (x_n) is infinite and $d(x_i, x_j) \geq \epsilon$ whenever $i \neq j$. This obviously cannot have a convergent subsequence. A contradiction.

Coming back to the original proof. Let $\mathscr{A}$ be an open covering for X with Lebesgue number δ. Let $\epsilon = \delta/3$. Consider the finite covering of X with ϵ-balls. Each ball has a diameter of at most $2\delta/3$ and thus is contained in some element of $\mathscr{A}$. The collection of all such elements of $\mathscr{A}$ is a finite cover of X. Thus X is compact. This finishes the proof. ■

2.4 Local Compactness

Definition 2.33 (Local Compactness). A space X is said to be *locally compact* at x if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X itself is said to be *locally compact*.

One notes that a compact space is automatically locally compact. Conversely, it is not necessary that a locally compact space is compact. For example, the real line $\mathbb{R}$ with the standard topology is locally compact but not compact.

The space $\mathbb{R}^\omega$ is *not* locally compact; none of its basis elements are contained in compact subspaces, since all basis elements are of the form

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

whose closure is obviously not compact.

Theorem 2.34. Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:

1. X is a subspace of Y
2. The set $Y \backslash X$ consists of a single point
3. Y is a compact Hausdorff space

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof. We first show uniqueness. Let Y and Y' be two spaces satisfying these conditions. Define the function $h : Y \to Y'$ by letting h map the single point p of $Y \backslash X$ to the single point q of $Y' \backslash X$ and letting h equal the identity on X. Obviously, h is a bijection. It suffices to show that h maps open sets in Y to open sets in Y'. Let U be open in Y. If U does not contain p, it is contained in X and is open in X. Thus, $h(U) = U$ and is open in X. But since X is open in Y', $h(U)$ is open in Y'. Now, suppose $p \in U$. Then, $C = Y \backslash U$ is closed in Y and is thus compact in Y. Since C is contained in X, it is compact in X and thus $h(C) = C$ is compact in Y'. Since Y' is Hausdorff, C is also closed in Y' and thus $h(U) = Y' \backslash C$ is open in Y'. This establishes uniqueness.

Suppose now that X is locally compact and Hausdorff. Let $Y = X \cup \{\infty\}$. The topology on Y consists of the following sets:

1. all sets U that are open in X
2. all sets of the form $Y \backslash C$ where C is a compact subspace of X

We shall first show that this forms a topology on Y. The intersection of any two sets must be in the topology. If both sets are of the form (1), then we are trivially done. If both are of the form (2), then we have $Y \backslash C_1 \cap Y \backslash C_2 = Y \backslash (C_1 \cup C_2)$ which is obviously of the form (2). Consider an intersection of the form $U \cap (Y \backslash C) = U \cap (X \backslash C)$. Since X is Hausdorff and C is compact in X, C must also be closed in

X and thus $X\backslash C$ is open in X. Now, by induction it follows that finite intersections are also elements of the topology.

We now verify arbitrary unions. Obviously arbitrary unions of sets of type (1) form sets of type (1). Arbitrary unions of sets of type (2) are of the form

$$\bigcup (Y\backslash C_\alpha) = Y\backslash \bigcap C_\alpha = Y\backslash C$$

where C is some open set in X and is therefore of type (2). Finally, we need to verify the following:

$$\left(\bigcup U_\alpha\right) \cup \left(\bigcup (Y\backslash C_\beta)\right) = U \cup (Y\backslash C) = Y\backslash (C\backslash U)$$

one notes that if C is compact in X and U is open in X, then obviously $C\backslash U$ is compact in X (the proof is Straightforward). Thus this is also of type (2). And the collection is indeed a topology.

We now show that Y is compact Hausdorff. Let $x, y \in Y$. If both lie in X, then there exist disjoint open sets U, V in X that contain x and y respectively. Now suppose $x \in X$ and $y = \infty$. Consider a compact set C in X containing a neighborhood of x. Then $Y\backslash C$ contains Y and is disjoint from said neighborhood of X and thus Y is Hausdorff. Next, suppose $\mathscr{A}$ is an open cover of Y. Then, it must contain an element of the form $Y\backslash C$ where C is compact in X, since all the open sets in Y of type (1) do not contain ∞. Since $\mathscr{A}$ covers Y, $\mathscr{A}\backslash\{Y\backslash C\}$ covers C and therefore has a finite subcollection that covers C. This along with $Y\backslash C$ is a finite subcover for Y and thus Y is compact.

Finally, we show that if X is a subspace of Y satisfying all the conditions, then X is locally compact Hausdorff. The fact that X is Hausdorff follows from the fact that Y is Hausdorff. Let $x \in X$. We shall show that X is locally compact at x. Since Y is Hausdorff, there exist disjoint open sets in Y containing x and ∞ respectively. The set $Y\backslash V$ is closed in Y, but since Y is compact, $Y\backslash V$ is also compact in Y and is a subset of X that contains U. This implies local compactness and finishes the proof. ■

Definition 2.35 (Compactification). If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a *compactification* of X. If $Y\backslash X$ equals a single point, then Y is called the *one-point compactification* of X.

Chapter 3

Countability and Separation Axioms

3.1 Countability Axioms

Definition 3.1 (First Countable). A topological space X is said to have a *countable basis at* x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom* or to be *first-countable*.

Obviously, every metrizable space satisfies this axiom, since we may take

$$\mathcal{B}_x = \left\{ B\left(x, \frac{1}{n}\right) \mid n \in \mathbb{N} \right\}$$

Theorem 3.2. Let X be a first countable topological space.

1. Let A be a subset of X. If $x \in \overline{A}$, then there is a sequence of points of A converging to x.

2. Let $f : X \to Y$. If for every convergent sequence $\{x_n\}_n$ to x, the sequence $\{f(x_n)\}_n$ converges to $f(x)$, then f is continuous.

Note that the converses for both do not require the first-countable hypothesis.

Proof.

1. Suppose $x \in \overline{A}$. We shall construct a sequence of points in A converging to x. Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable basis at X. Let $a_1 \in A \cap B_1$. If there is no open set U_2 containing x that is contained in B_1, define $a_j = a_1$ for all $j \geq 2$. Otherwise, let B_j be an element of $\mathcal{B}$ that is contained in U_2, and choose $a_2 \in A \cap B_j$ and repeat. This gives rise to a convergent sequence.

2. **TODO: Fill this up**

■

Definition 3.3 (Second Countable). If a space X has a countable basis for its topology, then X is said to satisfy the *second countability axiom* or to be *second-countable*

From the definition of the topology generated by a basis, *second countability* implies *first countability*. Further, not every metric space is *second-countable*. To see this, consider the following lemma:

Lemma 3.4. Let X be a topological space with a countable basis $\mathcal{B}$, then any discrete subspace A of X must be countable.

Proof. The proof follows quite naturally. Since A is discrete, for each $a \in A$, there is $B_a \in \mathcal{B}$ such that $B_a \cap A = \{a\}$. Then, the function $f : A \to \mathcal{B}$ given by $f(a) = B_a$ is an injection and thus A is countable. ■

Now, consider $\mathbb{R}^\omega$ with the uniform topology. The set $A = \{0,1\}^\omega$ is uncountable and under the uniform topology, is discrete, since $\overline{\rho}(a,b) = 1$ for all $a, b \in A$ with $a \neq b$. This immediately implies that $\mathbb{R}^\omega$ under the uniform topology may not have a countable basis and cannot be second-countable.

We now show that the same isn't true for $\mathbb{R}^\omega$ equipped with the product topology. It is well known that the countable collection of all open intervals (a,b) with both $a, b \in \mathbb{Q}$ forms a basis for $\mathbb{R}$. Then, $\mathbb{R}^\omega$ has a **countable basis** of all open sets of the form $\prod_{n \in \mathbb{Z}^+} U_n$ where U_n is an open interval with rational end points for finitely many values of n and $U_n = \mathbb{R}$ for all others.

Theorem 3.5. A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of a second-countable spaces is second-countable.

Proof. The assertion about subspaces is trivially true in both cases. As for the second part note that a cross product of countable sets is countable. ■

Definition 3.6 (Lindelöf Space). A topological space X is said to be Lindelöf if every open cover has a countable subcover.
Obviously all compact spaces are Lindelöf

Definition 3.7 (Dense). A subset A of a space X is said to be *dense* in X if $\overline{A} = X$.

Theorem 3.8. Suppose that X is second countable. Then

1. X is Lindelöf
2. There exists a countable subset of X that is dense in X

Proof.

1. Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable basis for X and $\mathscr{A}$ be an open cover. For each $x \in X$, let A_x be an element in $\mathscr{A}$ containing x. By definition, there must exist a basis element B_x such that $x \in B_x \subseteq A_x$. Let $\mathscr{B} = \{B_x \mid x \in X\}$. Obviously $\mathscr{B} \subseteq \mathcal{B}$ and is therefore countable. Further, for each $B \in \mathscr{B}$, there is $A(B) \in \mathscr{A}$ containing B. Therefore, $\{A(B) \mid B \in \mathscr{B}\}$ forms a countable subcover.

2. Using the Axiom of Choice, choose a set $D = \{x_n \mid x_i \in B_i\}$. For each $x \in X \backslash D$, and an open set U containing x, then there is a basis element B_j containing x that is contained in U. Therefore, $x_j \in U$. This implies $x \in \overline{D}$.

■

3.2 Separation Axioms

Definition 3.9 (Regular Spaces). Suppose one-point sets are closed in X. Then X is said to be *regular* or a T_3*-space* if for each pair consisting of a point x and a closed set B disjoint from X, there exist disjoint open sets containing x and B, respectively.

Definition 3.10 (Normal Spaces). Suppose one-point sets are closed in X. Then X is said to be *noraml* or a T_4*-space* if for each pair A, B of disjoint closed sets in X, there exist disjoint open sets containing A and B.

It is not hard to see that

$$\text{Normal} \Longrightarrow \text{Regular} \Longrightarrow \text{Hausdorff}$$

Theorem 3.11. Let X be a topological space such that one point sets in X are closed.

1. X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$
2. X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subseteq U$.

Proof.

1. Suppose X is regular and $x \in U \in \mathcal{T}_X$. Since $X \backslash U$ is closed, there are disjoint open sets V and W such that $x \in V$ and $X \backslash U \subseteq W$. It is not hard to see that $\overline{V} \cap W = \varnothing$, therefore $\overline{V} \subseteq U$.

   Conversely, let $x \in U$ and $A \subseteq X$ be a closed set. Then, $X \backslash A$ is open and $x \in X \backslash A$. Therefore, there is an open set V containing x such that $\overline{V} \subseteq X \backslash A$. Then, $A \subseteq X \backslash \overline{V}$ and we are done.

2. Suppose X is normal. Then, $B = X \backslash U$ is a closed set disjoint from A. Therefore, there are open sets V, W containing A and B respectively such that $A \subseteq V$ and $B \subseteq W$. It is not hard to see that $\overline{V} \cap W = \varnothing$, therefore, $\overline{V} \subseteq U$.

   Conversely, let A be closed in X. Then, $B = X \backslash U$ is closed, and the sets V and $X \backslash \overline{V}$ contain A and B respectively, and are disjoint, therefore the space is normal.

■

Theorem 3.12.

1. A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff
2. A subspace of a regular space is regular; a product of regular spaces is regular.

Proof.

1. The subspace part is trivial. Let $(X_\alpha)_\alpha$ be a collection of Hausdorff spaces. Let $\mathbf{x}, \mathbf{y} \in \prod_\alpha X_\alpha$. Since $\mathbf{x} \neq \mathbf{y}$, there is an index β such that $x_\beta \neq y_\beta$. Therefore, there disjoint are open sets U, V in X_β such that $x_\beta \in U$ and $y_\beta \in V$. As a result, $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint and open in $\prod_\alpha X_\alpha$.
2. The subspace part is trivial. Let $\mathbf{x} \in \prod_\alpha X_\alpha$ where each X_α is regular and $U \subseteq \prod_\alpha X_\alpha$ be an open set containing $\mathbf{x}$. Let $\prod_\alpha U_\alpha$ be a basis element of $\prod_\alpha X_\alpha$ containing x that is also contained in U.. For each x_α, let V_α be an open set in X_α containing it such that $\overline{V_\alpha} \subseteq U_\alpha$. Note that if $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$ instead. As a result, $\prod_\alpha V_\alpha$ is in the product topology and its closure is contained in U. This completes the proof.

■

3.3 Normal Spaces

Theorem 3.13. Every regular space with a countable basis is normal.

Proof. Let X be a regular space with countable basis $\mathcal{B}$ and A, B be closed sets in X. For each $x \in X$, using regularity, there is an open set U_x containing x and disjoint from B. Further, using regularity, there is a neighborhood of x, V_x such that $\overline{V}_x \subseteq U_x$. Finally, choose a basis element B_x from $\mathcal{B}$ containing x that is contained in V.

We now have a countable cover $\{U_n\}$ for A, such that $\overline{U}_i \cap B = \emptyset$. Similarly, choose a countable open cover $\{V_n\}$ for B, such that $\overline{V}_i \cap A = \emptyset$. Let us now define

$$U_n' = U_n \setminus \bigcup_{i=1}^n \overline{V}_i \qquad V_n' = V_n \setminus \bigcup_{i=1}^n \overline{U}_i$$

We shall show that U_i' and V_j' are disjoint for any i, j. Without loss of generality, suppose $i \le j$. Suppose $x \in U_i' \cap V_j'$, therefore, $x \in U_i$ and $x \in V_j$, but using the definition of V_j', we must have that $x \notin V_j'$, a contradiction.

Finally, define

$$U = \bigcup_{i=1}^{\infty} U_i' \qquad V = \bigcup_{j=1}^{\infty} V_j'$$

These are disjoint open sets contain A and B respectively. This concludes the proof. ■

Theorem 3.14. Every compact Hausdorff space is normal.

Proof. Let X be a compact Hausdorff space. We shall first show that X is regular. Indeed, let $x \in X$ and $A \subseteq X$ be a closed set. Since X is compact so is A. For all $a \in A$, there are disjoint open sets U_a and V_a such that $x \in U_a$ and $a \in V_a$. Note that $\mathscr{A} = \{A_a \mid a \in A\}$ is an open cover for A and therefore has a finte subcover $\{V_{a_1}, \dots, V_{a_n}\}$. Let

$$U = \bigcap_{i=1}^{n} U_{a_i} \qquad V = \bigcup_{i=1}^{n} V_{a_i}$$

which are disjoint open sets containing x and A respectively.

Suppose A and B are disjoint closed sets in X. For each $a \in A$, there are disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Note that $\mathscr{A} = \{U_a \mid a \in A\}$ is an open cover for A, and therefore, has a finite subcover $\{A_{a_1}, \dots, A_{a_n}\}$. Choose

$$U = \bigcup_{i=1}^{n} U_{a_i} \qquad V = \bigcap_{i=1}^{n} V_{a_i}$$

which are disjoint open sets containing A and B respectively. ■

Theorem 3.15. Every metrizable space is normal.

Proof. Let (X, d) be a metric space and A, B be two disjoint closed subsets of X. ■

3.4 Urysohn's Lemma

Definition 3.16. Let X be a normal space and $A, B \subseteq X$ be two closed sets. Then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Let P be the countable set of all rational numbers in $[0, 1]$. First, define $U_1 = X \backslash B$, which is an open set containing A. Due to the normality of X, there is an open set U_0 containing A such that $\overline{U}_0 \subseteq U_1$.

We shall now define an open set U_p for all $p \in P$ such that

$$p < q \Longrightarrow \overline{U}_p \subseteq U_q$$

Let P_n be the set containing the first n rational numbers in some enumeration of P such that the first two enumerated rationals are 0 and 1. Let r be the $n + 1$-st rational in the enumeration. Obviously, since P_n is finite and $0, 1 \in P_n$, there are rationals $p, q \in P$ such that

$$p = \max\{x \in P_n \mid x < r\}$$
$$q = \min\{x \in P_n \mid x > r\}$$

Now, due to the induction hypothesis, $\overline{U}_p \subseteq U_q$ and therefore, using the normality of X, there is an open set U_r such that $\overline{U}_p \subseteq U_r$ and $\overline{U}_r \subseteq U_q$.

Let $s \in P_{n+1}$. If $s < p$, $\overline{U}_s \subseteq U_p \subseteq \overline{U}_p \subseteq U_r$ and if $q < s$, $\overline{U}_r \subseteq U_q \subseteq \overline{U}_q \subseteq U_s$. Therefore, the induction hypothesis holds.

Now that we have defined U_p for all $p \in P$, we shall define

$$U_p = \begin{cases} \varnothing & p < 0 \\ X & p > 1 \end{cases}$$

Now, for all $x \in X$, define the function $f : X \to [0, 1]$ as

$$f(x) = \inf\{p \mid x \in U_p\}$$

Note that since for all $p > 1$, $x \in U_p$ and the rationals are dense in the reals, $0 \leq f(x) \leq 1$. For all $a \in A$, note that $a \in U_0$, therefore $f(a) = 0$. Similarly, for all $b \in B$, note that $b \notin U_1$, as a result $b \notin U_p$ for all $p \in [0, 1]$, but $b \in U_q$ for all $q > 1$, therefore, $f(b) = \inf\{q \in \mathbb{Q} \mid q > 1\} = 1$

All that remains is to show that f is continuous. Let $x \in X$ and $(c, d) \in [0, 1]$ be an open interval containing $f(x)$. Choose any two rational numbers p, q such that $c < p < f(x) < q < d$. Let us consider the image of the set $Y = U_q \backslash \overline{U}_p$. For all $y \in Y$, $f(y) > p$, while $f(y) < q$, therefore, $f(y) \in (c, d)$, as a result, $f(Y) \subseteq (c, d)$ and f is continuous. This completes the proof. ■

Definition 3.17. If A and B are two subsets of a topological space X, and if there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B *can be separated by a continuous function*.

Definition 3.18 (Completely Regular). A space X is *completely regular* or $T_{3\frac{1}{2}}$ if one-point sets are closed in X and for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Theorem 3.19. A subspace of a completely regular space is regular. A product of completely regular spaces is completely regular.

Proof. **TODO: Add in later** ■

3.5 The Urysohn Metrization Theorem

Theorem 3.20 (Urysohn Metrization Theorem). Every regular space X with a countable basis is metrizable.

Proof. Recall first that every regular space with a countable basis is normal. We shall show that X is metrizable by constructing an imbedding of X into $\mathbb{R}^\omega$. We shall first show that there is a countable sequence of functions $\{f_n\}$ from X to $[0,1]$ such that for all $x_0 \in X$ and a neighborhood U of x_0, there is a function f_n such that $f(x_0) > 0$ and $f(x) = 0$ for all $x \in X \backslash U$.

Let $\mathcal{B} = \{B_n\}$ be a countable basis for X. Then, for all pairs (m,n) such that $\overline{B}_m \subseteq B_n$, define the function $g_{m,n} : X \to [0,1]$, using Urysohn's Lemma, such that $g_{m,n}(\overline{B}_m) = \{1\}$ and $g_{m,n}(B_n) = \{0\}$. Obviously, the set $\{g_{m,n} \mid m,n \in \mathbb{N}\}$ is countable and it is not hard to see that this is our desired sequence of functions $\{f_n\}$.

Define now the map $F : X \to \mathbb{R}^\omega$ given by

$$F(x) = (f_1(x), f_2(x), \dots)$$

We shall now show that F is an imbedding. First, we show that F is injective. Indeed, if $x \neq y$, then we know that there is an open set containing x but not y, therefore, there is a function f_n such that $f(x) > 0$ while $f(y) = 0$. As a result, $F(x) \neq F(y)$.

Since each of the functions f_i are continuous, so is F. We need only show now that F maps open sets in X to open sets in $\mathbb{R}^\omega$. Let $Z = F(X)$ and U be an open set in X. It suffices to show that for all $x_0 \in U$, there is an open set V in Z such that $f(x_0) \in V \subseteq F(U)$. There is $n \in \mathbb{N}$ such that $f_n(x_0) > 0$ and f_n vanishes outside U. Let $\pi_n : \mathbb{R}^\omega \to \mathbb{R}$ be the natural projection map. Let $V = \pi_n^{-1}((0, \infty)) \cap Z$. Obviously, note that $f(x_0) \in V$. Further, for all $z \in V$, note that there is $x \in X$ such that $z = F(x)$, but since $\pi_n(F(x)) > 0$, we must have that $x \in U$, therefore, $V \subseteq F(U)$. This shows that $F(U)$ is open in $\mathbb{R}^\omega$ and thus, F is an imbedding. This completes the proof. ■

3.6 Tietze Extension Theorem

Lemma 3.21. Let X be a normal space and A be a closed subspace of X. Let $f : A \to [-r, r]$ be a continuous map. Then, there is a continuous function $g : X \to [-r, r]$ such that

$$|f(a) - g(a)| \leq 2r/3 \qquad |g(x)| \leq r/3 \quad \text{for all } a \in A, x \in X$$

Proof. Define

$$I_1 = \left[-r, -\frac{r}{3}\right] \quad I_2 = \left[-\frac{r}{3}, \frac{r}{3}\right] \quad I_1 = \left[\frac{r}{3}, r\right]$$

and

$$B = f^{-1}(I_1) \qquad C = f^{-1}(I_3)$$

Since I_1 and I_3 are closed in $[-r, r]$, B and C must be disjoint and closed in X. Now, due to Urysohn's Lemma, there is a function $g : X \to [-r/3, r/3]$ such that $g(B) = \{-r/3\}$ and $g(C) = \{r/3\}$, which has a natural extension $g : X \to [-r, r]$.

Obviously, $|g(x)| \leq r/3$ for all $x \in X$. Further, for all $a \in A$, if $a \in B$, then $g(a) = -r/3$, and $f(a) \in I_1$, similarly, if $a \in C$, then $g(a) = r/3$ and $f(a) \in I_3$. This immediately implies the desired conclusion. ■

Theorem 3.22 (Tietze Extension Theorem). Let X be a normal space; let A be a closed subspace of X

1. Any continuous map of A into the closed interval $[-1, 1]$ of $\mathbb{R}$ may be extended to a continuous map of all X into $[-1, 1]$

2. Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$

Proof. The main idea of the proof is to construct a uniformly convergent sequence of continuous functions to f on A. This would immediately imply the continuity of the limiting function over X, due to the Uniform Limit Theorem.

1. Using the preceeding lemma, there is a fucntion $g_1 : X \to [-1, 1]$ such that $|f(a) - g_1(a)| \leq 2/3$, while $|g(x)| \leq 1/3$ for all $a \in A$ and $x \in X$. Let us define $f_1 : A \to [-2/3, 2/3]$ as

$$f_1(x) = f(x) - g_1(x)$$

which is a continuous function. Then, we may reuse the previous lemma to define a function $g_2(x) : X \to [-1, 1]$ such that $|f_1(a) - g_2(a)| \leq (2/3)^2$, while $|g(x)| \leq (2/3)(1/3)$ and so on. As a result, we define the function $g_n : X \to [-1, 1]$ satisfying

$$|f_{n-1}(a) - g_n(a)| \leq \left(\frac{2}{3}\right)^n \qquad |g(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$$

Finally, define the functions $s_n : X \to \mathbb{R}$

$$s_n(x) = \sum_{i=1}^{n} g_n(x)$$

We note that

$$-1 < -\frac{1}{3}\sum_{i=1}^{n}\left(\frac{2}{3}\right)^{i-1} \leq s_n(x) \leq \frac{1}{3}\sum_{i=1}^{n}\left(\frac{2}{3}\right)^{i-1} < 1$$

Hence, we may take the restriction of s_n to $[-1, 1]$, which would also be continuous since it is the range restriction of a sum of finitely many continuous functions. Now, due to the Weierstrass M-test, the sequence of functions s_n are uniformly convergent. Further, since

$$|f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^n$$

we know that the convergent function $s : X \to [-1, 1]$ agrees with f on A. This completes the proof.

2. Recall that the spaces $(-1,1)$ and $\mathbb{R}$ are homeomorphic. Therefore, it suffices to prove the statement for functions of the form $f : A \to (-1,1)$. Using the first part of this theorem, we know that there is a function $g : X \to [-1,1]$. We shall use this function to obtain an extension h of f from $X \to (-1,1)$. Let $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$. Since G is continuous, D is closed in X and must be disjoint from A. Then, using Urysohn's Lemma, there is a function $\phi : X \to [0,1]$ such that $\phi(A) = \{1\}$ and $\phi(D) = \{0\}$. Then, the function $h(x) = \phi(x) \cdot g(x)$ is a continuous function from X to $(-1,1)$ that agrees with f on A. This completes the proof.

■