# Differential Topology

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# Chapter 0

# Preliminaries

The coordinates on $\mathbb{R}^n$ are denoted by $x^1, \dots, x^n$. Let $U \subseteq \mathbb{R}^n$ be open. A real valued function $f : U \to \mathbb{R}$ is said to be $C^k$ at $p \in U$ if its partial derivatives

$$\frac{\partial^i f}{\partial x^{i_1} \cdots \partial x^{i_j}}$$

of all orders $j \leq k$ exist and are continuous at $p$. In particular, $f$ is $C^\infty$ if it is $C^k$ for all $k \geq 0$. Henceforth, we use the term *smooth function* to mean $C^\infty$ function.

**Definition 0.1 (Diffeomorphism).** A smooth map $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ is said to be a *diffeomorphism* if it is bijective and has a smooth inverse.

In other words, diffeomorphisms are the isomorphisms in the category of smooth manifolds, $\mathbf{Man}^\infty$.

## 0.1 Exterior Algebra

**Definition 0.2.** A $k$-linear function $f : V^n \to \mathbb{R}$ is said to be

**symmetric** if for each $\sigma \in \mathfrak{S}_n$,

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$

**alternating** if for each $\sigma \in \mathfrak{S}_n$,

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (\operatorname{sgn} \sigma) f(v_1, \dots, v_n)$$

We denote by $A_n(V)$, the $\mathbb{R}$-vector space of alternating $n$-linear functions on a vector space $V$. These are also called *alternating $n$-tensors* or *multicovectors of degree $n$*.

We can symmetrize and alternate operators. Let $f : V^n \to \mathbb{R}$ be an $n$-linear function. Define

$$\begin{aligned}
\operatorname{Sym}(f)(v_1, \dots, v_k) &= \sum_{\sigma \in \mathfrak{S}_n} f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) \\
\operatorname{Alt}(f)(v_1, \dots, v_k) &= \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(n)})
\end{aligned}$$

It is not hard to see that $\operatorname{Sym}(f)$ is symmetric and $\operatorname{Alt}(f)$ is alternating.

## 0.2 Differential Forms on $\mathbb{R}^n$

The concept of a differential form is dual to the concept of a vector field. For an open subset $U \subseteq \mathbb{R}^n$, a vector field assigns, to each point in $U$, a vector in the tangent space of the point, similarly, a differential form assigns a covector on the tangent space of the point.

# Chapter 1

# Manifolds

**Definition 1.1 (Locally Euclidean).** A topological space $M$ is said to be *locally Euclidean of dimension $n$* if for every point $p \in M$ there is a neighborhood $U$ of $p$, an open subset $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \to V$.

The pair $(U, \phi : U \to \mathbb{R}^n)$ is called a *chart*, $U$ is called a *coordinate neighborhood* and $\phi$ a *coordinate map*. We say that a chart $(U, \phi)$ is *centered* at $p \in U$ if $\phi(p) = \mathbf{0}$.

Recall that for $m \neq n$, nonempty open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic, consequently, the local dimension is well defined.

**Definition 1.2 (Manifold).** A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension $n$ if it is locally Euclidean of dimension $n$.

**Definition 1.3 (Compatibility of Charts).** Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are said to be *$C^\infty$-compatible* if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are smooth. These two maps are called the *transition functions* between the charts. If $U$ and $V$ are disjoint then the two charts are automatically $C^\infty$-compatible.

Henceforth, by compatible charts, we mean $C^\infty$-compatible charts.

**Definition 1.4 (Atlas).** A *$C^\infty$-atlas* on a locally Euclidean space $M$ is a collection $\mathfrak{U} = \{(U_\alpha, \psi_\alpha)\}_{\alpha \in J}$ of pairwise compatible charts that cover $M$.

An atlas $\mathfrak{M}$ on a locally Euclidean spacce is sadi to be *maximal* if it is not contained in a larger atlas.

**Lemma 1.5.** *Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts $(V, \psi)$ and $(W, \sigma)$ are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.*