Commutative Algebra

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May 29, 2023

Abstract

Throughout this report, unless mentioned otherwise, all rings are assumed to be commutative. The term *noethering* is a portmanteau that is used in place of "noetherian ring" and is attributed to the accidental genius of Aryaman Maithani.

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Part I

Theory Building

Chapter 1

Rings and Ideals

Definition 1.1 (Krull Dimension). A sequence $\{\mathfrak{p}_0, \ldots, \mathfrak{p}_n\}$ of prime ideals in A is said to be strictly ascending of length n if $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$. The *Krull dimension* of A is defined to be the supremum of the lengths of all strictly ascending sequences of prime ideals in A and is denoted by $\dim A$.

Proposition 1.2. *Let A and B be rings. Then, every prime ideal in $A \times B$ is of the form $\mathfrak{p} \times B$ where $\mathfrak{p} \subseteq A$ is a prime ideal or $A \times \mathfrak{q}$ where $\mathfrak{q} \subseteq B$ is a prime ideal.*

Proof. It is known that ideals in $A \times B$ are of the form $\mathfrak{a} \times \mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A and B respectively. Consequently, the quotient

$$A \times B/\mathfrak{a} \times \mathfrak{b} \cong A/\mathfrak{a} \times B/\mathfrak{b}$$

For $\mathfrak{a} \times \mathfrak{b}$ we require $A/\mathfrak{a} \times B/\mathfrak{b}$ to be an integral domain. This is possible if and only if either $\mathfrak{a}$ is a prime and $\mathfrak{b} = B$ or $\mathfrak{a} = A$ and $\mathfrak{b}$ is a prime. This completes the proof. ■

1.1 Nilradical and Jacobson radical

Definition 1.3 (Multiplicatively Closed). A subset $S \subseteq A$ is said to be *multiplicatively closed* if

(a) $1 \in S$

(b) for all $x, y \in S$, $xy \in S$

Proposition 1.4. *Let $S \subsetneq A \backslash \{0\}$ be a multiplicatively closed subset. Then, there is a prime ideal $\mathfrak{p}$ disjoint from S.*

1.2 Local Rings

Definition 1.5. A commutative ring A is said to be local if it has a unique maximal ideal.

Proposition 1.6. *A is local if and only if the subset of non-units form an ideal.*

Obviously, a field k is a local ring. On the other hand, the polynomial ring $k[x]$ is not local, since both x and $1-x$ are non-units but their sum is a unit.

We contend that the ring $A = k[x_1, x_2, \ldots]/(x_1, x_2, \ldots)^2$ is local. Indeed, let π denote the canonical map $k[x_1, x_2, \ldots] \to A$ and $\mathfrak{m}$ be maximal in A. Then, $\pi^{-1}(\mathfrak{m})$ is maximal in $k[x_1, x_2, \ldots]$ and contains $(x_1, x_2, \ldots)^2$, therefore, contains $(x_1, x_2, \ldots)$. But the latter is maximal and therefore, $\pi^{-1}(\mathfrak{m}) = (x_1, x_2, \ldots)$ whence the maximal ideal is unique. Thus A is local.

1.3 Operations on Ideals

Obviously, the intersection $\mathfrak{a} \cap \mathfrak{b}$ of two ideals is an ideal. The sum of ideals is defined as the following collection

$$\sum_{i \in I} \mathfrak{a}_i = \left\{ \sum_{\text{finite } i \in I} a_i \,\middle|\, a_i \in \mathfrak{a}_i \right\}$$

It is not hard to argue that the sum is the smallest ideal containing the ideals $\{\mathfrak{a}_i\}_{i \in I}$. The product of two ideals is defined as

$$\mathfrak{ab} = \left\{ \sum_{\text{finite}} a_i b_i \,\middle|\, a_i \in \mathfrak{a},\ b_i \in \mathfrak{b} \right\}$$

Inductively, we may define powers of an ideal as $\mathfrak{a}^n = \mathfrak{a}\mathfrak{a}^{n-1}$ with the convention that $\mathfrak{a}^0 = (1) = A$.

Proposition 1.7. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subseteq A$ be ideals. Then,*

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{ab} + \mathfrak{ac}$$

Proof. Obviously, $\mathfrak{ab} \subseteq \mathfrak{a}(\mathfrak{b} + \mathfrak{c})$ and $\mathfrak{ac} \subseteq \mathfrak{a}(\mathfrak{b} + \mathfrak{c})$ and thus, $\mathfrak{ab} + \mathfrak{ac} \subseteq \mathfrak{a}(\mathfrak{b} + \mathfrak{c})$. On the other hand, any element of $\mathfrak{a}(\mathfrak{b} + \mathfrak{c})$ is a finite sum of the form $\sum_i a_i(b_i + c_i)$ which can be rearranged as $\sum_i a_i b_i + \sum_i a_i c_i \in \mathfrak{ab} + \mathfrak{ac}$. This completes the proof. ∎

Proposition 1.8. *(a) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals and let $\mathfrak{a}$ be an ideal contained in $\bigcup_{i=1}^n \mathfrak{p}_i$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some $1 \leq i \leq n$.*

(b) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $\mathfrak{p}$ be a prime ideal containing $\bigcap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some i.

For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$, define their ideal quotient as

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in A \mid x\mathfrak{b} \subseteq \mathfrak{a}\}$$

Proposition 1.9. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subseteq A$ be ideals. Then*

1. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$
2. $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{bc})$
3. $(\bigcap_{i \in I} \mathfrak{a}_i : \mathfrak{b}) = \bigcap_{i \in I} (\mathfrak{a}_i : \mathfrak{b})$

Proposition 1.10. *If every prime ideal in A is principal, then A is a principal ring.*

Proof. Suppose not. Let Σ be the poset of ideals in A that are not principal, ordered by inclusion and $\{\mathfrak{a}_i\}_{i\in I}$ be a chain in Σ. Let $\mathfrak{a} = \bigcup_{i\in I} \mathfrak{a}_i$. We claim that $\mathfrak{a}$ is not principal, for if it were, then $\mathfrak{a} = (a)$ for some $a \in A$. Then, $a \in \mathfrak{a}_i$ for some $i \in I$ whence $\mathfrak{a}_i = (a)$, a contradiction. Hence, every chain in Σ has an upper bound, therefore, Σ has a maximal element, say $\mathfrak{p}$.

We contend that $\mathfrak{p}$ is a prime ideal. Suppose not, then there are $a, b \notin \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Add in later ■

Proposition 1.11. *Let A be a UFD. Then A is a PID if and only if* $\dim A \leq 1$.

1.3.1 Radical Ideals

Definition 1.12 (Radical Ideal). For an ideal $\mathfrak{a} \subseteq A$, we define its *radical* as

$$\sqrt{\mathfrak{a}} = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$$

An ideal which is the radical of some ideal is called a *radical ideal*.

Obviously, $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$. From our definition, it is not hard to see that the radical is the smallest radical ideal that contains a certain ideal. As a result, if $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals, then $\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}}$.

Proposition 1.13. *Let $\mathfrak{a}, \mathfrak{b} \subseteq A$ be ideals. Then,*

(i) $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$

(ii) $\sqrt{\mathfrak{ab}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$

(iii) $\sqrt{\mathfrak{a}^n} = \sqrt{\mathfrak{a}}$ *for every* $n \in \mathbb{N}$

(iv) $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{a} + \sqrt{b}}$

Proof. (i) Trivial.

(ii) Since $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, we must have $\sqrt{\mathfrak{ab}} \subseteq \sqrt{\mathfrak{a} \cap \mathfrak{b}}$. On the other hand, if $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, there is a positive integer n such that $x^n \in \mathfrak{a} \cap \mathfrak{b}$, therefore, $x^{2n} \in \mathfrak{ab}$, and $x \in \sqrt{\mathfrak{ab}}$. This establishes the first equality.

As for the second inequality, if $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then there is a positive integer n such that $x^n \in \mathfrak{a} \cap \mathfrak{b}$, therefore, $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$. Conversely, if $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$, then there are positive integers m and n such that $x^m \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, consequently, $x^{m+n} \in \mathfrak{a} \cap \mathfrak{b}$, and the conclusion follows.

(iii) Immediate from (ii).

(iv) Obviously, $\sqrt{\mathfrak{a} + \mathfrak{b}} \subseteq \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$. On the other hand, note that $\sqrt{\mathfrak{a} + \mathfrak{b}}$ is a radical ideal containing $\sqrt{\mathfrak{a}}$ and $\sqrt{\mathfrak{b}}$, therefore, contains $\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$. Hence, $\sqrt{\mathfrak{a} + \mathfrak{b}} \supseteq \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ and the conclusion follows. ■

For a prime ideal $\mathfrak{p}$, note that $\sqrt{\mathfrak{p}} = \mathfrak{p}$ and due to (iii), $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for every positive integer n.

1.4 Extension and Contraction of Ideals

Definition 1.14. Let $\phi : A \to B$ be a ring homomorphism. If $\mathfrak{a} \subseteq A$ is an ideal, then we define its extension $\mathfrak{a}^e = \phi(\mathfrak{a})A$. If $\mathfrak{b} \subseteq B$ is an ideal, then we define its contraction $\mathfrak{b}^c = \phi^{-1}(\mathfrak{b})$.

Proposition 1.15. *(a)* $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ *and* $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$

(b) $\mathfrak{b}^c = \mathfrak{b}^{cec}$ *and* $\mathfrak{a}^e = \mathfrak{a}^{ece}$

(c) If C is the set of contracted ideals in A and E is the set of extended ideals in B, then $\mathfrak{a} \mapsto \mathfrak{a}^e$ is a bijection from C to E.

Proof. (a) Trivial.

(b) We have $\mathfrak{a}^e \subseteq (\mathfrak{a}^{ec})^e$ and $\mathfrak{a}^e \supseteq (\mathfrak{a}^e)^{ce}$. Similarly, $\mathfrak{b}^c \supseteq (\mathfrak{b}^c)^{ec}$ and $\mathfrak{b}^c \subseteq (\mathfrak{b}^c)^{ec}$ whence $\mathfrak{b}^c = \mathfrak{b}^{cec}$.

(c) Simply note that the maps $\mathfrak{a} \mapsto \mathfrak{a}^e$ and $\mathfrak{b} \mapsto \mathfrak{b}^c$ from C to E and E to C are inverses to one another. ■

1.5 The Zariski Topology

Definition 1.16 (Prime Spectrum). For a commutative ring A, define

$$\operatorname{spec} A = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal in } A\}$$

This is called the *prime spectrum* of the ring. Similarly, define

$$\text{m-spec } A = \{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal in } A\}$$

For each $E \subseteq A$, define

$$V(E) = \{\mathfrak{p} \in \operatorname{spec} A \mid E \subseteq \mathfrak{p}\}$$

Proposition 1.17. *(a) If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$*

(b) $V(0) = X$ *and* $V(1) = \varnothing$

(c) If $\{E_i\}_{i \in I}$ is a family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i)$$

It is not hard to see that the collection

$$\mathcal{T} = \{\operatorname{spec} A \backslash V(E) \mid E \subseteq A\}$$

is a topology on spec A. This is known as the *Zariski Topology*. In particular, $V(E)$ form closed subsets in spec A under the Zariski topology.

Proposition 1.18. *For each $f \in A$, let $D(f) = \operatorname{spec} A \backslash V(f)$. Then, the collection $\{D(f)\}_{f \in A}$ forms a basis for the Zariski topology on* spec A.

Proposition 1.19. *Let $f : A \to B$ be a ring homomorphism. Then, the map $f_* : \operatorname{spec} B \to \operatorname{spec} A$ given by $f_*(\mathfrak{q}) = f^{-1}(\mathfrak{p})$ is a continuous map. Further, if $g : B \to C$ is a ring homomorphism, then $(g \circ f)_* = f_* \circ g_*$.*

Proof. Let $\mathfrak{a} \subseteq A$ be an ideal. We shall show that $f_*^{-1}(V(\mathfrak{a}))$ is closed in B. Note that

$$\begin{aligned} f_*^{-1}(V_A(\mathfrak{a})) &= \{\mathfrak{p} \mid \mathfrak{a} \subseteq f_*(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \operatorname{spec} B \mid \mathfrak{a} \subseteq f^{-1}(\mathfrak{p})\} \\ &= V_B((f(\mathfrak{a}))) \end{aligned}$$

whence the conclusion follows.

Next, for any $\mathfrak{p} \in \operatorname{spec} C$, we have

$$(f_* \circ g_*)(\mathfrak{p}) = f_*(g^{-1}(\mathfrak{p})) = f^{-1}(g^{-1}(\mathfrak{p})) = (g \circ f)^{-1}(\mathfrak{p})$$

This completes the proof. ■

This shows that spec is a contravariant functor from **CRing** to **Top**.

1.5.1 On the Topological Properties

Proposition 1.20. spec A *is Hausdorff if and only if* $\dim A = 0$.

Proof. ($\Longrightarrow$) We shall show that if spec A is T_1, then $\dim A = 0$. Indeed, if spec A is T_1, then $\{\mathfrak{p}\}$ is a closed set for very prime ideal $\mathfrak{p}$, therefore, there is an ideal $I \subseteq A$ such that $V(I) = \{\mathfrak{p}\}$. As a result, $V(\mathfrak{p}) = \{\mathfrak{p}\}$ and $\mathfrak{p}$ is maximal.

($\Longleftarrow$) Suppose $\dim A = 0$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct ideals. We contend that there are $f \notin \mathfrak{p}$ and $g \notin \mathfrak{q}$ such that fg is contained in every prime ideal in A, equivalently, fg is contained in $\mathfrak{N}(A)$. Suppose not, that is, for every pair $f \notin \mathfrak{p}$ and $g \notin \mathfrak{q}$, there is a prime ideal $\mathfrak{p}$ disjoint from $\{f, g\}$.

Let $X = A \backslash (\mathfrak{p} \cap \mathfrak{q})$. Let Σ be the collection of ideals $\mathfrak{a}$ contained in $\mathfrak{p} \cap \mathfrak{q}$ such that for every finite subset $F \subseteq X$, there is a prime ideal $\mathfrak{P}$ containing $\mathfrak{a}$ that is disjoint from F. It is not hard to see that $(0) \in \Sigma$ and that every ascending chain has an upper bound given by the union of all elements in the chain.

Let J be a maximal element in Σ whose existence is guaranteed due to Zorn's Lemma. We shall show that J is prime. Indeed, let $xy \in J$ with $y \notin J$. Then, $J + (y) \notin \Sigma$, therefore, there is a finite subset $F_0 \subseteq X$ such that for each prime ideal $\mathfrak{P}$ containing $J + (y)$, $\mathfrak{P} \cap F_0 \neq \varnothing$.

Now, let $F \subseteq X$ be finite, then so is $F \cup F_0$, therefore, there is a prime ideal I containing J such that $I \cap (F \cup F_0) = \varnothing$, which implies that $y \notin I$, lest $J + (y) \subseteq I$. But since $xy \in J \subseteq I$, we must have that $x \in I$. This shows that $J + (x) \subseteq I$, therefore, $(J + (x)) \cap F = \varnothing$ whence $J + (x) \in \Sigma$ and $x \in J$ due to the maximality. This shows that J is prime.

Finally, we see that if there is a prime ideal J contained in $\mathfrak{p} \cap \mathfrak{q}$, contradicting $\dim A = 0$. Thus, there is $f \notin \mathfrak{p}$ and $g \notin \mathfrak{q}$ such that fg is contained in $\mathfrak{N}(A)$. Consider the basic open sets $D(f)$ and $D(g)$, which contain $\mathfrak{p}$ and $\mathfrak{q}$ respectively and their intersection $D(f) \cap D(g) = D(fg)$ is the empty set since fg is contained in ever prime ideal, thus, spec A is Hausdorff. ■

Corollary. If spec A is T_1, then spec A is Hasudorff.

Chapter 2

Modules

2.1 Introduction

Throughout this section, R denotes a general ring which need not be commutative.

Definition 2.1 (Module). A left R-module is an abelian group $(M, +)$ along with a ring action, that is, a ring homomorphism $\mu : R \to \mathrm{End}(M)$.

Henceforth, unless specified otherwise, an *R-module* refers to a *left R-module*. Trivially note that R is an R-module, so is any ideal in R and so is every quotient ring R/I where I is an ideal in R. When R is a field, an R-module is the same as a vector space.

Every abelian group G trivially forms a $\mathbb{Z}$-module. Using this and the forthcoming *Structure Theorem for Finitely Generated Modules over a PID*, we obtain the *Structure Theorem for Finitely Generated Abelian Groups*.

Definition 2.2 (Submodule). Let M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of R.

Proposition 2.3 (Submodule Criteria). *Let M be an R-module. Then $\varnothing \subsetneq N \subseteq M$ is a submodule if and only if for all $x, y \in N$ and $r \in R$, $x + ry \in N$.*

Proof. Straightforward definition pushing. ■

Definition 2.4 (Module Homomorphism). Let M, N be R-modules. A *module homomorphism* is a group homomorphism $\phi : M \to N$ such that for all $x \in M$ and $r \in R$, $\phi(rx) = r\phi(x)$.

In other words, a module homomorphism is simply an R-linear map.

Proposition 2.5 (Homomorphism Criteria). *Let M, N be R-modules. Then $\phi : M \to N$ is an R-module homomorphism if and only if for all $x, y \in M$ and $r \in R$, $\phi(x + ry) = \phi(x) + r\phi(y)$.*

Proof. Straightforward definition pushing. ■

It is not hard to see, using the above proposition and the submodule criteria that the image of an R-module under a homomorphism is a submodule.

Definition 2.6 (Kernel, Cokernel). Let $\phi : M \to N$ be an R-module homomorphism. We define

$$\ker \phi = \{x \in M \mid \phi(x) = 0\} \qquad \operatorname{coker} \phi = N/\phi(M)$$

For an R-module M, define the annihilator of M in R as

$$\operatorname{Ann}_R(M) = \{r \in R \mid rx = 0 \; \forall x \in M\}$$

It is trivial to check that $\operatorname{Ann}_R(M)$ is a left ideal in R, and if R were commutative, it would be an ideal. When $\operatorname{Ann}_A(M) = 0$, M is said to be a *faithful* A-module.

Proposition 2.7. *If I is an ideal contained in $\operatorname{Ann}_A(M)$, then M is naturally an A/I-module.*

Proof. Define the action $(a + I) \cdot m = a \cdot m$. It is easy to check that this action is well defined. Further,

$$(a + I) \cdot ((b + I) \cdot m) = (a + I) \cdot (bm) = (ab) \cdot m = ((a + I)(b + I)) \cdot m$$

This completes the proof. ■

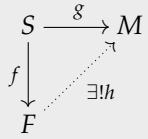
In particular, if $I = \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then M forms a vector space over $A/\mathfrak{m}$.

2.2 Free Modules

Throughout this section, R denotes a general ring which need not be commutative.

We define the free module using a universal property and then provide a construction for it. This should establish uniqueness.

Definition 2.8 (Universal Property of Free Modules). Let S be a non-empty set. A *free module on S* is an R-module F together with a mapping $f : S \to F$ such that for every R-module M and every set map $g : S \to M$, there is a unique R-module homomorphism $h : F \to M$ such that the following diagram commutes:



Let F be the set of all set functions $\phi : S \to R$ which takes nonzero values at finitely many elements of S. This has the structure of an R-module. Define the set map $f : S \to F$ by

$$f(s)(t) = \begin{cases} 1 & s = t \\ 0 & \text{otherwise} \end{cases}$$

We contend that (F, f) is a free module on S. Indeed, let $g : S \to M$ be a set map where M is an R-module. Define the linear map $h : F \to M$ by

$$h(f(s)) = g(s)$$

Since every element in F can uniquely be written as a linear combination of elements in $\{f(s)\}_{s \in S}$, we have successfully defined a module homomorphism $h : F \to M$ such that $g = h \circ f$. The uniqueness of this map is quite obvious. Hence, (F, f) is a free module on S.

Definition 2.9 (Basis). Let M be an R-module. Then $S \subseteq M$ is said to be a *basis* if it is linearly independent and generates M.

It is important to note that not every minimal generating set is a basis. Take for example the $\mathbb{Z}$-module $\mathbb{Z}$. Notice that $\{2, 3\}$ is a minimal generating set but is not a basis for it is not linearly independent.

2.2.1 Over a PID

Throughout this (sub)section, let R denote a PID.

Theorem 2.10. *Let F be a free R-module. If $H \leq F$ is a submodule, then H is free and* $\dim H \leq \dim F$.

Proof. Let $\{e_i\}_{i \in I}$ be a basis for F. Denote the projection map of the i-th coordinate by $p_i : F \to R$. Due to the Well Ordering Theorem, we can impose a well order $(I, \leqq)$ on I. Let F_i be the submodule generated by $\{e_j \mid j \leqq i\}$ and $H_i = H \cap F_i$. Now, $p_i(H_i)$ is an ideal in R, and therefore, is of the form $a_i R$ for some $a_i \in R$. Of course, it is possible that $a_i = 0$. If $a_i \neq 0$, then pick some $h_i \in H_i$ such that $p_i(h_i) = a_i$, on the other hand, if $a_i = 0$, then set $h_i = 0$. It is not hard to see from this definition that $p_i(h_j) = 0$ whenever $j < i$.

We contend that the set $S = \{h_i \neq 0 \mid i \in I\}$ forms a basis for H, this would immediately imply that $\dim H \leq \dim F$. First, we shall show that S is linearly independent. We shall do this by transfinite induction. The base case is trivial. Suppose the induction hypothsis holds for $S_i = \{h_j \in S \mid j < i\}$. If a linear combination of the elements of S_{i+1} is zero, then the coefficient of h_i must be nonzero. Therefore, we may write

$$bh_i = \sum_{k=1}^{n} a_{j_k} h_{j_k}$$

For some $a_{j_1}, \ldots, a_{j_n}, b \in R$. Upon projecting using p_i, we obtain $ba_i = 0$, consequently, $b = 0$, and S_{i+1} is linearly independent.

It is not hard to argue that the h_i's span H. Pick some $h \in H$. Note that only finitely many of the $p_i(h)$'s will be nonzero. Let them be $i_1 < \cdots < i_n$. Now work backwards from i_n to determine the coefficients of h_{i_k} for each $1 \leq k \leq n$. ∎

2.3 Finitely Generated Modules

Definition 2.11 (Finitely Generated Module). An R-module M is said to be finitely generated if there is a finite subset S of M which generates M. That is, there is no proper submodule N of M containing S.

A submodule of a finitely generated module need not be finitely generated, let $A = \mathbb{Z}[x_1, x_2, \ldots]$ and consider A as an A-module. The ideal $(x_1, x_2, \ldots)$ is not finitely generated.

Proposition 2.12. *An R-module M is finitely generated if and only if M is isomorphic to a quotient of $R^{\oplus n}$ for some positive integer n.*

Proof. We shall only prove the forward direction since the converse is trivial to prove. Suppose M is finitely generated. Then, it is generated by a finite subset $S = \{x_1, \ldots, x_m\}$. Define the R-module homomorphism $\phi : R^{\oplus n} \to M$ by $(r_1, \ldots, r_n) \mapsto r_1 x_1 + \cdots + r_n x_n$. From the first isomorphism theorem, we have $M \cong R^{\oplus n} / \ker \phi$. ∎

Proposition 2.13. *Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A. Let $\phi \in \mathrm{End}(M)$ such that $\phi(M) \subseteq \mathfrak{a}M$. Then, there are $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ such that*

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0$$

as an element of $\mathrm{End}(M)$, where a_k is treated as the homomorphism $x \mapsto a_k x$ in $\mathrm{End}(M)$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a generating set for M. Then, for all $1 \leq i \leq n$, there are coefficients $\{a_{i1}, \ldots, a_{in}\}$ in $\mathfrak{a}$ such that

$$\phi(x_i) = \sum_{j=1}^{n} a_{ij} x_j$$

We may rewrite this as

$$\sum_{j=1}^{n} (\phi\delta_{ij} - a_{ij}) x_j = 0$$

Let B denote the matrix $(\phi\delta_{ij} - a_{ij})_{1 \leq i,j \leq n}$. Then, multiplying by $\mathrm{adj}(B)$, we see that $\det(B)(x_j) = 0$ for all $1 \leq j \leq n$ where $\det(B)$ is viewed as an element in $\mathrm{End}(M)$ and thus, is the zero map in $\mathrm{End}(M)$. It is not hard to see that $\det(B)$ is in the required form. ■

Lemma 2.14 (Nakayama). *Let M be a finitely generated module and $\mathfrak{a} \subseteq \mathfrak{R}$ be an ideal such that $M = \mathfrak{a}M$. Then, $M = 0$.*

Proof. Let $\phi = \mathbf{id}$ be the identity homomorphism in $\mathrm{End}(M)$. Using Proposition 2.13, there are coefficients $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ satisfying the statement of the proposition. As a result, $x = 1 + a_{n-1} + \ldots + a_0$ is the zero endomorphism. But since $a_{n-1} + \ldots + a_0 \in \mathfrak{a} \subseteq \mathfrak{R}$, x is a unit and hence, $M = 0$. ■

Corollary. Let M be a finitely generated A-module, N a submodule of M and $\mathfrak{a} \subseteq \mathfrak{R}$ an ideal. If $M = \mathfrak{a}M + N$ then $M = N$.

Proof. We have $M/N = \mathfrak{a}M/N$, consequently, $M/N = 0$ and $M = N$ due to Lemma 2.14. ■

Lemma 2.15. *Let $(A, \mathfrak{m})$ be local and $k = A/\mathfrak{m}$. Let M be a finitely generated A-module. Let $\{\overline{x}_1, \ldots, \overline{x}_n\}$ be elements in $M/\mathfrak{m}M$ that form a basis for $M/\mathfrak{m}M$ as a k-vector space. Then, $\{x_1, \ldots, x_n\}$ generates M.*

Proof. Let N be the submodule generated by $\{x_1, \ldots, x_n\}$. Then, the composition $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is surjective, consequently, $M = N + \mathfrak{m}M$ whence, it follows that $M = N$. ■

2.4 Hom Modules and Functors

For R-modules M, N, we denote the set of all R-module homomorphisms from M to N by $\mathrm{Hom}_R(M, N)$. When the choice of the ring R is clear from the context, we shall denote this set by $\mathrm{Hom}(M, N)$.

Proposition 2.16. *Let M, N be A-modules. Then $\mathrm{Hom}(M, N)$ has the structure of an A-module.*

Proof. It is obvious that $\mathrm{Hom}(M, N)$ has the structure of an abelian group. Define the natural action by $(af)(x) = af(x)$. It is not hard to see that this action is well defined. ■

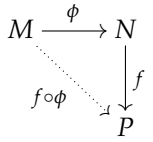
Proposition 2.17. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of A-modules. Then, for any A-module N, we have a natural isomorphism*

$$\mathrm{Hom}_A\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\right) = \prod_{\lambda\in\Lambda} \mathrm{Hom}_A(M_\lambda, N)$$

Proof. Since the direct sum is the product in $A-$ **Mod**, the conclusion follows from the universal property. ■

Theorem 2.18. *Let $\phi : M \to N$ be an A-module homomorphism. Then, for every R-module P, there is an induced A-module homomorphism $\overline{\phi} : \mathrm{Hom}(N,P) \to \mathrm{Hom}(M,P)$ and an induced A-module homomorphism $\widetilde{\phi} : \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N)$.*
Equivalently phrased, $\mathrm{Hom}(-,P)$ is a contravariant functor while $\mathrm{Hom}(P,-)$ is a covariant functor.

Proof. We shall prove only the first half of the assertion since the second half follows from a similar proof. Define $\overline{\phi}$ using the following commutative diagram:



To see that this is indeed an R-module homomorphism, we need only verify that for all $f, g \in \mathrm{Hom}(N,P)$ and all $r \in R$, $(f+rg)\circ\phi = f\circ\phi + rg\circ\phi$ which is trivial to check. ■

2.5 Exact Sequences

Definition 2.19. A sequence of module homomorphisms

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is said to be exact at N if $\mathrm{im}\, f = \ker g$. A short exact sequence is a sequence of module homomorphisms:

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

which is exact at M, N and P.

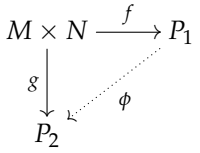
It is not hard to see that the sequence in the definition is short exact if and only if f is injective, g is surjective and $\mathrm{im}\, f = \ker g$.

2.5.1 Diagram Chasing Poster Children

2.6 Tensor Product

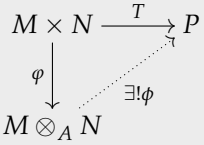
Definition 2.20 (Bilinear Map). Let M, N, P be A-modules. A map $T : M \times N \to P$ is said to be bilinear if for each $x \in M$, the mapping $T_x : N \to P$ given by $y \mapsto T(x,y)$ is A-linear and for each $y \in N$, the mapping $T_y : M \to P$ given by $x \mapsto T(x,y)$ is A-linear.

Fix two A-modules M and N. Let $\mathscr{C}$ denote the category of bilinear maps $T : M \times N \to P$ where P is any A-module. A morphism between two bilinear maps $f : M \times N \to P_1$ and $g : M \times N \to P_2$ in this category is a module homomorphism $\phi : P_1 \to P_2$ such that the following diagram commutes:



A universal object in $\mathscr{C}$ is called the tensor product of M and N and is denoted by $M \otimes N$. In other words, the tensor product is an initial object in the category $\mathscr{C}$.

Definition 2.21 (Universal Property of the Tensor Product). Let M, N, P be A-modules and $T : M \times N \to P$ be a bilinear map. Then, there is a unique A-module homomorphism $\phi : M \otimes N \to P$ such that the following diagram commutes:



Of course, having the universal property would imply that the tensor product, if it exists, is unique upto a unique isomorphism. We shall now construct a tensor product of M and N.

Constructing the Tensor Product

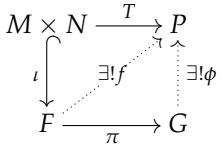
Let F be the free A-module on $M \times N$. Let us denote the basis elements of F by $e_{(x,y)}$ where $x \in M$ and $y \in N$. Now, for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $a \in A$, let D denote the submodule generated by elements of the form:

$$\begin{aligned}
&e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)} \\
&e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)} \\
&e_{(ax,y)} - ae_{(x,y)} \\
&e_{(x,ay)} - ae_{(x,y)}
\end{aligned}$$

Let $G = F/D$ and let $\varphi : M \times N \to G$ be the composition of the following maps:

$$M \times N \hookrightarrow F \twoheadrightarrow G$$

Let $T : M \times N \to P$ be a bilinear map. Consider the following commutative diagram:



To show that existence of ϕ, we must show that $D \subseteq \ker f$, since we can then finish using the universal property of the kernel. But this is trivial to check and follows from the fact that T is a bilinear map and completes the construction.

Similarly, we define the tensor product for a finite sequence of A-modules $\{M_i\}_{i=1}^n$. That is, given a multilinear map $T : \prod_{i=1}^{n} M_i \to P$, there is a unique A-module homomorphism ϕ such that the following diagram commutes:

$$\begin{array}{ccc} M_1 \times \cdots \times M_n & \xrightarrow{T} & P \\ {\scriptstyle\varphi}\downarrow & \nearrow_{\exists!\phi} & \\ M_1 \otimes \cdots \otimes M_n & & \end{array}$$

Proposition 2.22. *Let F and G be free A-modules with basis given by $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ respectively. Then, $F \otimes_A G$ is a free A-module with basis $\{f_i \otimes g_j\}_{i\in I,\, j\in J}$.*

Proof. It is not hard to see that the set $\{f_i \otimes g_j\}_{i\in I,\, j\in J}$ is generating for $F \otimes_A G$. Therefore, it suffices to show that this set is linearly independent. Suppose not, then there is a finite linear combination

$$\sum_{i\in I,\, j\in J} a_{ij} f_i \otimes g_j = 0$$

Pick some $i_0 \in I$ and $j_0 \in J$. Let $\phi : F \times G \to A$ be the bilinear map such that

$$\phi(f_i, g_j) = \begin{cases} 1 & i = i_0 \text{ and } j = j_0 \\ 0 & \text{otherwise} \end{cases}$$

This induces an A-module homomorphism $\varphi : F \otimes G \to A$ such that

$$\varphi(f_i \otimes g_j) = \begin{cases} 1 & i = i_0 \text{ and } j = j_0 \\ 0 & \text{otherwise} \end{cases}$$

whence, it follows that $a_{i_0 j_0} = 0$ and the collection $\{f_i \otimes g_j\}_{i\in I,\, j\in J}$ is linearly independent. ■

2.6.1 Properties of Tensor Product

Given two modules M and N with the canonical map $\varphi : M \times N \to M \otimes N$, we denote by $m \otimes n$, the element $\varphi(m, n)$ in $M \otimes N$.

Proposition 2.23. *Let M, N, P be A-modules and $\{M_i\}_{i\in I}$ a collection of A-modules. Then,*

(a) $M \otimes_A N \cong N \otimes_A M$

(b) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P) \cong M \otimes_A N \otimes_A P$

(c) $(\bigoplus_{i\in I} M_i) \otimes_A N \cong \bigoplus_{i\in I}(M_i \otimes_A N)$

(d) $A \otimes_A M \cong M$

Proof. (a) First, we shall show that there are well defined homomorphisms $M \otimes N \to N \otimes M$ and $N \otimes M \to M \otimes N$ mapping $m \otimes n \mapsto n \otimes m$ and $n \otimes m \mapsto m \otimes n$ respectively. This is best done using the universal property. Let $T : M \times N \to N \times M$ be the isomorphism $m \times n \mapsto n \times m$. Consider now the following commutative diagram:

$$\begin{array}{ccc} M \times N & \xrightarrow{T} & N \times M \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ M \otimes N & & N \otimes M \end{array}$$

Since both φ' and T are bilinear, so is $\varphi \circ T$, consequently, there is a unique induced homomorphism $f : M \otimes N \to N \otimes M$ making the diagram commute, consequently, $f(m \otimes n) = \varphi'(T(m \times n)) = n \otimes m$.

Similarly, there is a homomorphism $g : N \otimes M \to M \otimes N$ such that $g(n \otimes m) = m \otimes m$. It is not hard to see that $g \circ f = \mathbf{id}_{M \otimes N}$ and $f \circ g = \mathbf{id}_{N \otimes M}$, consequently, they are isomorphisms.

(b) We shall show $(M \otimes_A N) \otimes_A P \cong M \otimes_A N \otimes_A P$ since the proof of the other isomorphism follows analogously. Fix some $z \in P$ and consider the map $f_z : M \times N \to M \otimes_A N \otimes_A P$ given by $(x, y) \mapsto x \otimes y \otimes z$. This is an A-linear map and thus induces a map $g_z : M \otimes_A N \to M \otimes_A N \otimes_A P$ given by $g_z(x \otimes y) = x \otimes y \otimes z$. The map $G : (M \otimes_A N) \times P \to M \otimes_A N \otimes_A P$ given by $G(x \otimes y, z) = g_z(x \otimes y) = x \otimes y \otimes z$ is a well defined A-linear map which induces a map $h : (M \otimes_A N) \otimes_A P \to M \otimes_A N \otimes_A P$ given by $(x \otimes y) \otimes z \mapsto x \otimes y \otimes z$.

On the other hand, the map $F : M \times N \times P \to (M \otimes_A N) \otimes_A P$ given by $(x, y, z) \mapsto x \otimes y \otimes z$ is A-linear and thus induces a map $f : M \otimes_A N \otimes_A P \to (M \otimes_A N) \otimes_A P$ given by $x \otimes y \otimes z \mapsto (x \otimes y) \otimes z$. Since the maps f and h are inverses to one another for elementary tensors, they are inverses to one another over their respective domains, whereby both are isomorphisms.

(c) Define the map $f : (\bigoplus_{ß \in I} M_i) \times N \to \bigoplus (M_i \otimes_A N)$ by $f((m_i) \otimes n) = (m_i \otimes n)$, which is a bilinear map. This induces a map $\phi : (\bigoplus_{i \in I} M_i) \otimes_A N \to \bigoplus_{i \in I} (M_i \otimes_A N)$ such that $f((m_i) \otimes n) = (m_i \otimes n)$.

Now, consider the map $f_i : M_i \times N \to M \otimes N$ given by $f_i(m_i, n) = \iota_i(m_i) \otimes n$. This induces a map $g_i : M_i \otimes_A N \to M \otimes N$ such that $g_i(m_i \otimes n) = \iota_i(m_i) \otimes n$. We may now define a map $\psi : \bigoplus_{i \in I} (M_i \otimes_A N) \to (\bigoplus_{i \in I} M_i) \otimes_A N$ given by

$$\psi((m_i \otimes n_i)) = \sum g_i(m_i \otimes n_i)$$

Obviously the sum on the right is a finite sum. Further, since each each g_i is well defined, so is ψ.

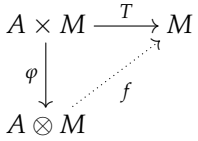
Lastly, we shall show that ϕ and ψ are inverses to one another. Indeed,

$$\psi \circ \phi((m_i) \otimes n) = \psi((m_i \otimes n)) = \sum \iota_i(m_i) \otimes n = (m_i) \otimes n$$

and

$$\phi \circ \psi((m_i \otimes n_i)) = \sum \phi(g_i(m_i \otimes n_i)) = (m_i \otimes n_i)$$

(d) Consider the map $T : A \times M \to M$ given by $(a, m) \mapsto am$. It is not hard to see that this map is bilinear, consequently, there is a map $f : A \otimes M \to M$ such that the following diagram commutes:



Note that $f(a \otimes m) = am$ by definition. Consider the map $g : M \to A \otimes M$ given by $g(m) = 1 \otimes m$. It is not hard to see that g is a well defined module homomorphism. Further, since $f \circ g$ and $g \circ f$ are the identity homomorphisms, they both must be isomorphisms.

■

Example 1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$ for all $m, n \in \mathbb{N}$. In particular, if m and n are coprime, then $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$.

Proof. Consider the module homomorphism $T : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. ■

Let $f : M \to M'$ and $g : N \to N'$ be A-module homomorphisms. Then, the map $\Phi : M \times N \to M' \otimes N'$ given by $\Phi(m, n) = f(m) \otimes g(n)$. It is not hard to see that Φ is bilinear. Consequently, it induces a map $f \otimes g : M \otimes N \to M' \otimes N'$ such that

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$$

Further, if $f' : M' \to M''$ and $g' : N' \to N''$ are A-module homomorphisms, then we have another map $f' \otimes g' : M' \otimes N' \to M'' \otimes N''$ such that

$$(f' \otimes g')(x \otimes y) = f'(x) \otimes g'(y)$$

Now, it is not hard to see that $(f' \circ f') \otimes (g' \circ g)$ and $(f' \otimes g') \circ (f \otimes g)$ agree on the elementary tensors, therefore, agree on all of $M \otimes N$.

2.7 Right Exactness

Proposition 2.24. *Let M, N, P be A-modules. Then, there is a natural isomorphism:*

$$\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \cong \mathrm{Hom}_A(M \otimes_A N, P)$$

Proof. Consider the map

$$\theta : \mathrm{Hom}_A(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$$

given by $\theta(\alpha)(m)(n) = \alpha(m \otimes n)$. Now, pick some $\eta \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$. Define the map $\zeta : M \times N \to P$ given by $\zeta(m, n) = \eta(m)(n)$. Obviously, ζ is bilinear and induces a map $\delta : M \otimes_A N \to P$ such that $\delta(m \otimes n) = \eta(m)(n)$. Call the map sending $\eta \mapsto \delta$ as β where

$$\beta : \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \to \mathrm{Hom}_A(M \otimes_A N, P)$$

and $\beta(\eta)(m \otimes n) = \eta(m)(n)$.

We contend that θ and β are inverses to one another. Indeed,

$$((\beta \circ \theta)(\alpha))(m \otimes n) = \theta(\alpha)(m)(n) = \alpha(m \otimes n)$$

and

$$((\theta \circ \beta)(\eta))(m)(n) = \beta(\eta)(m \otimes n) = \eta(m)(n)$$

whence the conclusion follows. ∎

In particular, we see that the functor $- \otimes_A N$ is the left adjoint of the functor $\mathrm{Hom}_A(N, -)$, consequently, $\mathrm{Hom}_A(N, -)$ is the right adjoint of $- \otimes_A N$.

Theorem 2.25. *The functor $- \otimes_A N$ is right exact. That is, given a exact sequence*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

the sequence

$$M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \longrightarrow 0$$

Proof. Since the given sequence is exact, so is

$$\mathrm{Hom}_A(M'', \mathrm{Hom}_A(N, P)) \xrightarrow{\overline{g}} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \xrightarrow{\overline{f}} \mathrm{Hom}_A(M', \mathrm{Hom}_A(N, P)) \longrightarrow 0$$

but from Proposition 2.24, so is

$$\mathrm{Hom}_A(M'' \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M' \otimes_A N, P) \longrightarrow 0$$

Since the above sequence is exact for all modules P, we have the desired conclusion. ∎

The tensor product is not left exact. Conider the sequence of $\mathbb{Z}$-modules

$$0 \hookrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

where $f(m) = 2m$. Upon tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f\otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

Note that

$$(f \otimes 1)(m \otimes \overline{n}) = 2m \otimes \overline{n} = m \otimes (2\overline{n}) = m \otimes 0 = 0$$

Therefore, the sequence cannot be exact.

2.8 Flat Modules

Definition 2.26 (Flat Module). An A-module M is sida to be flat if the functor $- \otimes_A N$ is exact.

We know that $- \otimes_A N$ is right exact, hence, it suffices to show that the functor is left exact.

Theorem 2.27. *Let N be a A-module. Then, the following are equivalent*

(a) N is flat

(b) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then the tensored sequence

$$0 \longrightarrow M' \otimes_A N \xrightarrow{f\otimes 1} M \otimes_A N \xrightarrow{g\otimes 1} M'' \otimes_A N \longrightarrow 0$$

is exact.

(c) If $f : M' \to M$ is injective, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective

(d) If $f : M' \to M$ is injective and M, M' are finitely generated, then $f \otimes_A 1 : M' \otimes_A N \to M \otimes_A N$ is injective.

Proof.

$(a) \iff (b)$**:** Is well known.

$(b) \implies (c)$**:** Immediate from considering the short exact sequence $0 \to M' \to M \to M/M' \to 0$.

$(c) \implies (b)$**:** Since $- \otimes_A N$ is known to be right exact as well.

TODO: Complete this later

■

Proposition 2.28. *Let $\{M_i\}_{i\in I}$ be a collection of A-modules. Then, $M = \bigoplus_{i\in I} M_i$ is flat if and only if M_i is flat for each $i \in I$.*

Proof. From the fact that

$$M \otimes_A N \cong \bigoplus_{i\in I} (M_i \otimes_A N)$$

and the isomorphism is natural. ■

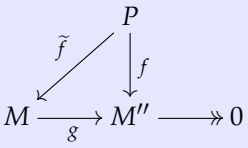
Corollary. Free modules are flat.

Proof. Obviously, A is a flat A-module, therefore, $\bigoplus_{\lambda\in\Lambda} A$ is free for every indexing set Λ. ■

2.9 Projective Modules

Theorem 2.29. *For an A-module P, the following are equivalent:*

(a) Every map $f : P \to M''$ can be lifted to $\widetilde{f} : P \to M$ in the following commutative diagram:



(b) Every short exact sequence $0 \to M' \to M \to P \to 0$ splits

(c) There is a module M such that $P \oplus M$ is free

(d) The functor $\mathrm{Hom}_A(P, -)$ is exact.

Proof.

$(a) \implies (b)$**:** Taking $M'' = P$ and $f = \mathbf{id}_P$, we have the desired conclusion.

$(b) \implies (c)$**:** Let F denote the free module on the set P. Then, the map $\Phi : F \to P$ given by $\Phi(e_x) = x$ for all $x \in P$ is a surjective A-module homomorphism. We have the following short exact sequence:

$$0 \to \ker\Phi \xrightarrow{\iota} F \xrightarrow{\Phi} P \to 0$$

This is known to split and thus, $F = \psi(P) \oplus \ker\Phi$ where $\psi : P \to F$ is the section.

$(c) \implies (d)$**:** Let $M' \to M \to M''$ be an exact sequence of modules and K be an A-module such that $P \oplus K = F \cong A^{\Lambda}$. Then, the induced sequence

$$\prod_{\lambda\in\Lambda} M' \to \prod_{\lambda\in\Lambda} M \to \prod_{\lambda\in\Lambda} M''$$

is exact. We have seen that there is a natural isomorphism $\mathrm{Hom}_A(A, M) \xrightarrow{\sim} M$, consequently, there is a natural isomorphism

$$\mathrm{Hom}_A(A^{\oplus\Lambda}, M) \xrightarrow{\sim} \prod_{\lambda\in\Lambda} M$$

whence it follows that the sequence

$$\mathrm{Hom}_A(A^{\oplus\Lambda}A, M') \to \mathrm{Hom}_A(A^{\oplus\Lambda}A, M) \to \mathrm{Hom}_A(A^{\oplus\Lambda}, M'')$$

But since $\mathrm{Hom}_A(A^{\oplus\Lambda}, M) \cong \mathrm{Hom}_A(P, M) \oplus \mathrm{Hom}_A(K, M)$, we have the desired conclusion.

$(d) \implies (a)$**:** Trivial.

■

Definition 2.30 (Projective Module). An A-module P satisfying any one of the four equivalent conditions of Theorem 2.29 is said to be a *projective A-module*.

In particular, from Theorem 2.29(c), we see that every free module is projective.

Lemma 2.31. *A finitely generated projective module P over a local ring $(A, \mathfrak{m})$ is free.*

Proof. Let $\{\overline{x}_1, \ldots, \overline{x}_n\}$ be a basis for $M/\mathfrak{m}M$ as a k-vector space where $k = A/\mathfrak{m}$. As we have seen earlier, $\{x_1, \ldots, x_n\}$ generates M. Let F be the free module with basis $\{e_1, \ldots, e_n\}$ and $\Phi : F \to M$ be the module homomorphism given by $\Phi(e_i) = x_i$ and $K = \ker \Phi$. Since M is projective, there is a module homomorphism $\psi : M \to F$ satisfying $\Phi \circ \psi = \mathbf{id}_M$ and $F = K \oplus \psi(M)$.

We contend that $K = \mathfrak{m}K$. Indeed, let $x \in K$, then $x = \sum r_i e_i$ for a unique choice $\{r_1, \ldots, r_n\}$. Then, $\sum r_i x_i = 0$, consequently, $r_i \in \mathfrak{m}$ for all i. Since $F = K \oplus \psi(M)$, we may write $e_i = u_i + v_i$ for some $u_i \in K$ and $v_i \in \psi(M)$. As a result,

$$x - \sum r_i u_i = \sum r_i v_i \in \ker \Phi \cap \psi(M) = \{0\}$$

and the conclusion follows.

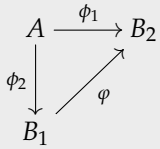
Finally due to Lemma 2.14, we must have that $K = 0$ whence M is free. ∎

Proposition 2.32. *Projective modules are flat.*

Proof. Follows from the fact that free modules are flat and projective modules are direct summands of free modules. ∎

2.10 Algebras

Definition 2.33. An *A-algebra* is a ring homomorphism $\phi : A \to B$. This endows B with the structure of an A-module. The algebra is said to be of *finite type* if B is finitely generated as an A-module. A homomorphism between algebras (ϕ_1, B_1) and (ϕ_2, B_2) is a map $\varphi : B_1 \to B_2$ making the following diagram commute.



This gives rise to a locally small category $A - \mathbf{Alg}$ with morphisms as defined above.

2.10.1 Tensor Product of Algebras

Consider the two A-algebras $f : A \to B$ and $f : A \to C$. Then, the map

$$\mu : B \times C \times B \times C \to B \otimes_A C$$

given by $\mu(b, c, b', c') = bb' \otimes cc'$ is A-multilinear, whereby it induces a map

$$\mu' : B \otimes_A C \otimes_A B \otimes_A C \to B \otimes_A C$$

given by $\mu'(b \otimes c \otimes b' \otimes c') = bb' \otimes cc'$. Let $D = B \otimes_A C$. Then, we have $\mu' : D \otimes_A D \to D$ given by $\mu'(b \otimes c, b' \otimes c') = bb' \otimes cc'$.

Let $\varphi : D \times D \to D \otimes_A D$ be the natural map. Then, the composition $\cdot = \mu' \circ \varphi : D \times D \to D$ is given by

$$(b \otimes c) \cdot (b' \otimes c') = bb' \otimes cc'$$

We contend that $(D \otimes_A D, +, \cdot, 0 \otimes 0, 1 \otimes 1)$ is a ring. To do this, we need only verify that multiplication distributes over addition. Indeed,

$$\begin{aligned}(b \otimes c) \cdot (b' \otimes c' + b'' \otimes c'') &= \mu' \left((b \otimes c) \otimes (b' \otimes c' + b'' \otimes c'')\right) \\ &= \mu'((b \otimes c \otimes b' \otimes c') + (b \otimes c \otimes b'' \otimes c'')) \\ &= bb' \otimes cc' + bb'' \otimes cc''\end{aligned}$$

Chapter 3

Localization

3.1 Rings of Fractions

Define the relation $\sim_S$ on $A \times S$ by $(a,s) \sim_S (a',s')$ if there is $t \in S$ such that $t(s'a - sa') = 0$. That this is an equivalence relation is easy to verify. We shall use a/s to denote the equivalence class $[(a,s)]$ in $A \times S/\sim_S$.

Consider the operations:

$$\frac{a}{s} + \frac{a'}{s'} = \frac{s'a + sa'}{ss'} \qquad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$$

It is not hard to see that these are well defined and endow $A \times S/\sim_S$ with a ring structure. We denote this ring by $S^{-1}A$ and is called the *ring of fractions* of A by S.

There is a natural ring homomorphism $\varphi : A \to S^{-1}A$ given by $\varphi(x) = x/1$. When A is an integral domain and $S = A\backslash\{0\}$, $S^{-1}A$ is precisely the field of fractions. Recall that if $\mathfrak{p}$ is a prime ideal in A, then $S = A\backslash\mathfrak{p}$ is a multiplicatively closed subset of A. We denote the ring $S^{-1}A$ by $A_\mathfrak{p}$.

Theorem 3.1. *The ring $A_\mathfrak{p}$ is local.*

Proof. Let $S = A\backslash\mathfrak{p}$ and define

$$\mathfrak{m} = \left\{ \frac{a}{s} \;\middle|\; a \in \mathfrak{p},\ s \in S \right\}$$

It is not hard to see that $\mathfrak{m}$ is an ideal in $A_\mathfrak{p}$. We contend that $\mathfrak{m}$ is the ideal of non-units in $A_\mathfrak{p}$. Indeed, if $a/s \in \mathfrak{m}$ is a unit, then there is $b/t \in A_\mathfrak{p}$ such that $(ab)/(st) = 1$, consequently, there is $w \in S$ such that $w(ab - st) = 0$, whence $wst \in \mathfrak{p}$, a contradiction.

On the other hand, if $a/s \notin \mathfrak{m}$, then a/s is a unit since $(a/s) \cdot (s/a) = 1$. Now, since the collection of all non-units forms an ideal, the ring must be local due to Proposition 1.6. ∎

Proposition 3.2. *Let $\mathfrak{m}$ be the unique maximal ideal of $A_\mathfrak{p}$. Then, $A_\mathfrak{p}/\mathfrak{m} \cong Q(A/\mathfrak{p})$ where the latter is the field of fractions of $A/\mathfrak{p}$.*

Proof. TODO: Add in later ∎

Similarly, when we let $S = \{a^n\}_{n\geq 0}$ for some $a \in A$, we denote $S^{-1}A$ by A_a.

There is a degenerate case, when we allow $0 \in S$, notice that the ring $S^{-1}A$ is the zero ring, since for all $a/s \in S^{-1}A$, we have $0(as) = 0$, therefore, $a/s = 0/s$.

Proposition 3.3. *Let $\{A_i\}_{i\in I}$ be a collection of commutative rings and $\{S_i \subseteq A_i\}$ be a collection of multiplica-*

tively closed sets. Then,

$$\left(\prod_{i\in I} S_i\right)^{-1}\left(\prod_{i\in I} A_i\right) \cong \prod_{i\in I}(S_i^{-1}A_i)$$

Proof. Define the map $\phi : \prod_{i\in I}(S_i^{-1}A_i) \to \left(\prod_{i\in I} S_i\right)^{-1}\left(\prod_{i\in I} A_i\right)$ given by

$$\phi\left(\left(\frac{a_i}{s_i}\right)_{i\in I}\right) = \frac{(a_i)_{i\in I}}{(s_i)_{i\in I}}$$

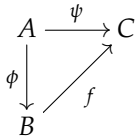
It is straightforward to argue that this map is well defined and surjective. We now contend that this is an isomorphism, for which it suffices to show that $\ker\phi$ is trivial. Indeed, if $(a_i/s_i)_{i\in I} \in \ker\phi$, then there is $(t_i)_{i\in I}$ such that $(t_i a_i)_{i\in I} = (0)_{i\in I}$ whereby, $t_i a_i = 0$ for each i and $a_i/s_i = 0$. This completes the proof. ■

Corollary. Let $\{A_i\}$ be a collection of rings then every localization of $\prod_{i\in I} A_i$ is of the form $(A_i)_{\mathfrak{p}_i}$ for some $i \in I$ where $\mathfrak{p}_i \subseteq A_i$ is a prime ideal.

Proof. Follows from the fact that prime ideals in $\prod_{i\in I} A_i$ are of the form $\pi_i^{-1}(\mathfrak{p}_i)$ where $\mathfrak{p}_i$ is a prime ideal in A_i and $\pi : \prod_{i\in I} A_i \to A_i$ is the natural projection map. ■

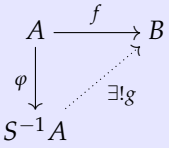
3.1.1 Universal Property

Fix a multiplicative subset $S \subseteq A$. Let $\mathscr{C}$ denote the category with objects as pairs (ϕ, B) where $\phi : A \to B$ is a ring homomorphism such that $\phi(s)$ is a unit in B for all $s \in S$. A morphism in this category is a map $f : (\phi, B) \to (\psi, C)$ making the following diagram commute.



The ring of fractions is an initial object in this category. Therefore, we have the following universal property. We shall verify in the "proof" that our construction of the field of fractions does satisfy this property and is therefore an initial object in $\mathscr{C}$.

Proposition 3.4. *Let $f : A \to B$ be a ring homomorphism such that $f(s)$ is a unit in B for all $s \in S$. Then there is a unique ring homomorphism $g : S^{-1}A \to B$ making the following diagram commute*



Proof. Define the map $g : S^{-1}A \to B$ by $g(a/s) = g(a)g(s)^{-1}$. To see that this map is well defined, note that if $a/s = a'/s'$, then there is $t \in S$ such that $t(s'a - sa') = 0$, consequently, $g(t)(g(s')g(a) - g(s)g(a')) = 0$. As a result, $g(a)g(s)^{-1} = g(a')g(s')^{-1}$. From this, it follows immediately that g is a ring homomorphism making the diagram commute.

As for uniqueness, note that for all $a/s \in S^{-1}A$,

$$g(a/s) = g(a/1)g(1/s) = g(a/1)g(s/1)^{-1} = f(a)f(s)^{-1}$$

which is fixed by the choice of f. This completes the proof. ■

3.2 Modules of Fractions

Let M be an A-module and $S \subseteq A$ be a multiplicatively closed subset. Define the relation $\sim_S$ on $M \times S$ by $(m,s) \sim_S (m',s')$ if and only if there is $t \in S$ such that $t(s'm - sm') = 0$. That this is an equivalence relation is easy to verify. We shall use m/s to denote the equivalence class $[(m,s)]$ in $M \times S/\sim_S$.

As in the previous section, there is a natural A-module homomorphism $\varphi : M \to S^{-1}M$ given by $\varphi(m) = m/1$. This map is called the *localization map*.

It is not hard to see that $S^{-1}M$ forms an A-module. Further, it also has the structure of an $S^{-1}A$ module under the action

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{a \cdot m}{st}$$

Let $f : M \to N$ be an A-module homomorphism. Consider the map $S^{-1}f : S^{-1}M \to S^{-1}N$ given by

$$S^{-1}f\left(\frac{m}{s}\right) = \frac{f(m)}{s}$$

We must first show that this is well defined. Indeed, if $m/s = m'/s'$, then there is $t \in S$ such that $t(s'm - sm') = 0$, consequently, $t(s'f(m) - sf(m')) = 0$, as a result, $f(m)/s = f(m')/s'$ in $S^{-1}M$.

We now contend that $S^{-1}f$ is an $S^{-1}A$ module homomorphism. Indeed, we have

$$S^{-1}f\left(\frac{m}{s} + \frac{a}{t}\frac{m'}{s'}\right) = S^{-1}f\left(\frac{ts'm + asm'}{sts'}\right) = \frac{f(ts'm + asm')}{sts'} = \frac{ts'f(m) + asf(m')}{sts'} = \frac{f(m)}{s} + \frac{f(m')}{s'}$$

Finally, let $f : M \to N$ and $g : N \to P$ be A-module homomorphisms. Then,

$$S^{-1}(g \circ f)\left(\frac{m}{s}\right) = \frac{g(f(m))}{s} \qquad S^{-1}g\left(S^{-1}f\left(\frac{m}{s}\right)\right) = S^{-1}g\left(\frac{f(m)}{s}\right) = \frac{g(f(m))}{s}$$

Theorem 3.5. $S^{-1} : A - \textbf{Mod} \to S^{-1}A - \textbf{Mod}$ *is an exact functor.*

Proof. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence. Then, for any $m'/s' \in S^{-1}M'$, we have

$$S^{-1}g\left(S^{-1}f\left(\frac{m'}{s'}\right)\right) = S^{-1}g\left(\frac{f(m')}{s'}\right) = \frac{g(f(m'))}{s'} = 0$$

As a result, $\operatorname{im}(S^{-1}f) \subseteq \ker(S^{-1}g)$. On the other hand, for $m/s \in \ker S^{-1}g$, we have $g(m)/s = 0$, consequently, there is $t \in S$ such that $tg(m) = 0$, equivalently, $g(tm) = 0$, whence, there is $m' \in M'$ such that $f(m') = tm$. Then, we have

$$f\left(\frac{m'}{st}\right) = \frac{f(m')}{st} = \frac{tm}{st} = \frac{m}{s}$$

whence, $\ker(S^{-1}g) \subseteq \operatorname{im}(S^{-1}f)$. This completes the proof. ∎

Proposition 3.6. *Let $N, P, \{M_i\}_{i\in I}$ be submodules of an A-module M. Then, for a multiplicatively closed $S \subseteq M$,*

(a) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$

(b) $S^{-1}\left(\sum_{i\in I} M_i\right) = \sum_{i\in I} S^{-1}M_i$

(c) $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$ *as $S^{-1}A$ modules.*

Proof. (a) We have the exact sequences $0 \to N \cap P \to N$ and $0 \to N \cap P \to P$. Due to Theorem 3.5, the sequences $0 \to S^{-1}(N \cap P) \to S^{-1}N$ and $0 \to S^{-1}(N \cap P) \to S^{-1}N$ are exact, consequently, $S^{-1}(N \cap P) \subseteq S^{-1}N \cap S^{-1}P$.

On the other hand, if $n/s = p/t$ for some $n \in N$, $p \in P$ and $s, t \in S$, there is some $u \in S$ such that $u(tn - sp) = 0$, equivalently, $m = utn = usp \in N \cap P$. Thus, $m/(stu) = n/s = p/t$, and the conclusion follows.

(b) Let $\overline{M} = \sum_{i \in I} M_i$. Then, there is the exact sequence $0 \to M_i \to \overline{M}$. Then, due to Theorem 3.5, the sequence $0 \to S^{-1}M_i \to S^{-1}\overline{M}$ is exact. Consequently, $\sum_{i \in I} S^{-1}M_i \subseteq S^{-1}\overline{M}$.

On the other hand, any element in $S^{-1}\overline{M}$ is of the form $(m_{i_1} + \cdots + m_{i_n})/s = m_{i_1}/s + \cdots + m_{i_n}/s$ for some $m_{i_n} \in M_{i_n}$ and $s \in S$. The conclusion follows.

(c) Consider the short exact sequence $0 \to N \to M \to M/N \to 0$. Due to Theorem 3.5, we obtain the short exact sequence of $S^{-1}A$-modules $0 \to S^{-1}N \to S^{-1}M \to S^{-1}(M/N) \to 0$ whereby the conclusion follows.

■

3.3 Local Properties

A property P defined on the class of modules is said to be local if for every A-module M,

M satisfies P if and only if $M_\mathfrak{p}$ satisfies P for each $\mathfrak{p} \in \operatorname{spec} A$.

Proposition 3.7. *Let M be an A-module. Then, the following are equivalent:*

1. $M = 0$
2. $M_\mathfrak{p} = 0$ *for each* $\mathfrak{p} \in \operatorname{spec} A$
3. $M_\mathfrak{m} = 0$ *for each* $\mathfrak{m} \in \text{m-spec } A$

Proof. That $(a) \implies (b) \implies (c)$ is obvious. We shall show $(c) \implies (a)$. ■

Proposition 3.8. *"Being an integral domain" is <u>not</u> a local property. Similarly, "being noetherian" is <u>not</u> a local property.*

Proof. Let A be a nonzero integral domain and consider the ring $R = A \times A$. This is not an integral domain. Due to Proposition 3.3, every localization of R is isomorphic to $A_\mathfrak{p}$ where $\mathfrak{p} \in \operatorname{spec} A$, consequently, is an integral domain.

As for the second assertion, consider the ring $R = k \times k \times \cdots$ where k is a nonzero field. This is obviously not noetherian due to the following ascending chain of ideals:

$$(0) \times (0) \times \cdots \subsetneq k \times (0) \times \cdots \subsetneq \cdots$$

But due to Proposition 3.3, every localization is isomorphic to k, consequently, is noetherian. ■

3.4 Extension and Contraction of Ideals

Definition 3.9. If $\mathfrak{a} \subseteq A$ is an ideal, $S \subseteq A$ a multiplicatively closed subset and $\varphi : A \to S^{-1}A$ the natural map. Define $S^{-1}\mathfrak{a}$ to be the extension of $\mathfrak{a}$ under the natural map φ.

Theorem 3.10. *Let $S \subseteq A$ be a multiplicatively closed set. Then,*

(a) Every ideal in $S^{-1}A$ is an extended ideal.

(b) If $\mathfrak{a} \subseteq A$ is an ideal, then

$$\mathfrak{a}^{ec} = \bigcup_{s \in S} (\mathfrak{a} : s)$$

Hence, $\mathfrak{a}^e = (1)$ if and only if $\mathfrak{a} \cap S \neq \varnothing$

(c) There is a bijection

$$\{\mathfrak{p} \in \operatorname{spec} A \mid S \cap \mathfrak{p} = \varnothing\} \leftrightarrow \operatorname{spec}(S^{-1}A)$$

given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$, which is just the extension map.

Proof. (a) Let $\mathfrak{a} \subseteq S^{-1}A$ be an ideal. We shall show that $\mathfrak{a}^{ce} = \mathfrak{a}$. We know that $\mathfrak{a}^{ce} \subseteq \mathfrak{a}$ therefore, it suffices to show the reverse inclusion. Let $x/s \in \mathfrak{a}$. Then, $x/1 \in \mathfrak{a}$, and $x \in \mathfrak{a}^c$. As a result, $x/1 \in \mathfrak{a}^{ce}$ and $x/s \in \mathfrak{a}^{ce}$, implying the desired conclusion.

(b)

(c) Let $\mathfrak{p}$ be a prime ideal in A that does not meet S. Let $a/s, b/t \in S^{-1}A$ such that $ab/st \in S^{-1}\mathfrak{p}$, whereby there is an element $p \in \mathfrak{p}$ and $r \in S$ such that $ab/st = p/r$ whence there is $u \in S$ such that $uabr = ustp$. Since $ur \notin \mathfrak{p}$, we must have $ab \in \mathfrak{p}$, consequently, either $a/s \in S^{-1}\mathfrak{p}$ or $b/t \in S^{-1}\mathfrak{p}$, implying the desired conclusion.

Conversely, since the contraction of any prime ideal in $S^{-1}\mathfrak{p}$ is also a prime ideal not meeting S, lest the prime ideal in $S^{-1}A$ contain a unit. Now, if $\mathfrak{p}$ is a prime ideal, then

$$\mathfrak{p} \subseteq \mathfrak{p}^{ec} = \bigcup_{s \in S} (\mathfrak{p} : s) \subseteq \mathfrak{p}$$

On the other hand, from (a), we see that if $\mathfrak{q}$ is a prime ideal in $S^{-1}A$, then $\mathfrak{q}^{ce} = \mathfrak{q}$, whereby the bijection is established.

■

Proposition 3.11. *The operation S^{-1} on ideals of A commutes with formation of finite sums, products, intersections and radicals.*

Corollary. $S^{-1}(\mathfrak{N}(A)) = \mathfrak{N}(S^{-1}A)$

Proof. Since $\mathfrak{N}(A) = \sqrt{(0)}$. ■

From the above proposition, we see that "$\mathfrak{N}(A) = (0)$" is a local property.

Proposition 3.12. *If M is finitely generated, then $S^{-1} \operatorname{Ann}_A(M) = \operatorname{Ann}_A(S^{-1}M)$.*

Proof. Induction on the number of generators. Sort of straightforward. Use the fact that

$$\operatorname{Ann}(N_1 + N_2) = \operatorname{Ann}(N_1) \cap \operatorname{Ann}(N_2)$$

■

Chapter 4

Primary Decomposition

A primary ideal is a generalization of the ideals $p^n\mathbb{Z}$ in $\mathbb{Z}$, as is evident from the following definition.

Definition 4.1 (Primary Ideals). An ideal $\mathfrak{q} \subseteq A$ is said to be *primary* if

$$xy \in \mathfrak{q} \implies x \in \mathfrak{q} \text{ or } y^n \in \mathfrak{q} \text{ for some } n > 0$$

From the definition, we see that every prime ideal is primary. It is not hard to see that

- $\mathfrak{q}$ is primary if and only if every zero divisor in $A/\mathfrak{q}$ is nilpotent.
- $\mathfrak{q}$ is primary if and only if (0) is primary in $A/\mathfrak{q}$.

Proposition 4.2. *If $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. Further, $\sqrt{\mathfrak{q}}$ is the smallest prime ideal containing $\mathfrak{q}$.*

Proof. Suppose $xy \in \sqrt{\mathfrak{q}}$, then there is $n > 0$ such that $x^n y^n \in \mathfrak{q}$, consequently, there is an $m > 0$ such that $x^n \in \mathfrak{q}$ or $y^{mn} \in \mathfrak{q}$, therefore, $x \in \sqrt{\mathfrak{q}}$ or $y \in \sqrt{\mathfrak{q}}$, whence $\sqrt{\mathfrak{q}}$ is prime. The second assertion is trivial. ■

If $\mathfrak{q}$ is a primary ideal, then $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is called the *associated prime ideal* of $\mathfrak{q}$ and $\mathfrak{q}$ is said to be $\mathfrak{p}$-*primary*.

Consider the ring $A = k[x, y]$ and the ideal $\mathfrak{q} = (x, y^2)$. The quotient ring $A/\mathfrak{q}$ is isomorphic to $k[y]/(y^2)$ where every zero divisor is nilpotent consequently, $\mathfrak{q}$ is primary. The radical ideal $\mathfrak{p} = \sqrt{\mathfrak{q}} = (x, y)$ is a prime ideal such that $\mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$, therefore, $\mathfrak{q}$ is not a prime power.

On the other hand, consider the ring $A = k[x, y, z]/(xy - z^2)$ and the prime ideal $\mathfrak{p} = (\overline{x}, \overline{z}) \subseteq A$. We contend that $\mathfrak{p}^2 \subseteq A$ is not primary. Indeed, note that $\overline{x}\overline{y} = \overline{z}^2 \in \mathfrak{p}^2$ but $\overline{x} \notin \mathfrak{p}^2$ and $\overline{y} \notin \mathfrak{p}^2$, and the conclusion follows.

Proposition 4.3. *If $\sqrt{\mathfrak{a}}$ is maximal, then $\mathfrak{a}$ is primary.*

Proof. Let $\mathfrak{m} = \sqrt{\mathfrak{a}}$ and $\phi : A \to A/\mathfrak{a}$ denote the natural map. Then, $\phi(\sqrt{\mathfrak{a}})$ is the maximal ideal in $A/\mathfrak{a}$ and is also the nilradical of $A/\mathfrak{a}$, consequently, $A/\mathfrak{a}$ is local and every non-unit is nilpotent. Hence, $\mathfrak{a}$ is primary. ■

Lemma 4.4. *If $\{\mathfrak{q}_i\}_{i=1}^n$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$.*

Proof. Obviously,

$$\sqrt{\mathfrak{q}} = \bigcap_{i=1}^{n} \sqrt{\mathfrak{q}_i} = \mathfrak{p}$$

Let $xy \in \mathfrak{q}$. If $y \in \mathfrak{p}$, then we are done, since $\mathfrak{p} = \sqrt{\mathfrak{q}}$. Else, $y^n \notin \mathfrak{q}_i$ for every positive integer n, since $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$ whereby $x \in \mathfrak{q}_i$ for each $1 \leq i \leq n$ and the conclusion follows. ■

Lemma 4.5. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and $x \in A$. Then*

(a) if $x \in \mathfrak{q}$, then $(\mathfrak{q} : x) = (1)$.

(b) if $x \notin \mathfrak{q}$, then $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary.

(c) if $x \notin \mathfrak{p}$, then $(\mathfrak{q} : x) = \mathfrak{q}$.

Proof. (a) Trivial.

(b) If $y \in (\mathfrak{q} : x)$, then $xy \in \mathfrak{q}$, therefore, $y \in \mathfrak{p}$. Thus, we have $\mathfrak{q} \subseteq (\mathfrak{q} : x) \subseteq \mathfrak{p}$. Taking radicals, $\mathfrak{p} \subseteq \sqrt{(\mathfrak{q} : x)} \subseteq \mathfrak{p}$, whereby $\sqrt{(\mathfrak{q} : x)} = \mathfrak{p}$.
On the other hand, if $yz \in (\mathfrak{q} : x)$, then $xyz \in \mathfrak{q}$. If $z \in \mathfrak{p}$, then we are done. Else, $xy \in \mathfrak{q}$ and $y \in (q : x)$ whence $(q : x)$ is $\mathfrak{p}$-primary.

(c) If $y \in (q : x)$, then $yx \in \mathfrak{q}$. Since $x \notin \mathfrak{p}$, we must have $y \in \mathfrak{q}$. This completes the proof.

■

Definition 4.6 (Primary Decomposition). A *primary decomposition* of an ideal $\mathfrak{a} \subseteq A$ is an expression of $\mathfrak{a}$ as a *finite* intersection of primary ideals.

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

The ideal $\mathfrak{a}$ is said to be *decomposable* if it has a primary decomposition. Moreover, if for all $1 \leq i \leq n$, $\sqrt{\mathfrak{q}_i}$ are distinct and

$$\bigcap_{j \neq i} \mathfrak{q}_j \not\subseteq \mathfrak{q}_i$$

then the primary decomposition is said to be *minimal*.

Using Lemma 4.4, it is not hard to see that every decomposable ideal has a minimal decomposition.

Theorem 4.7 (First Uniqueness Theorem). *Let $\mathfrak{a} \subseteq A$ be a decomposable ideal and*

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

be a minimal primary decomposition with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Then, the $\mathfrak{p}_i$'s are precisely the prime ideals the occur in the set $\{\sqrt{(\mathfrak{a} : x)} \mid x \in A\}$.

Proof. ■

Chapter 5

Integral Extensions

Definition 5.1 (Integral Extension). Let $A \subseteq B$ be a subring. Then, $\alpha \in B$ is said to be *integral* over A if it satisfies a monic polynomial in $A[x]$. The extension $A \hookrightarrow B$ is said to be integral if every element of B is integral over A.

Theorem 5.2. *Let $A \subseteq B$ be a subring and $\alpha \in B$. Then, the following are equivalent:*

(a) α is integral over A

(b) $A[\alpha]$ is a finitely generated A-module

(c) $A[\alpha]$ is contained in a subring C of B such that C is a finitely generated A-module

(d) There is a faithful $A[\alpha]$-module M which is finitely generated as an A-module.

Proof. $(a) \implies (b)$**:** If $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$. Then, it is not hard to argue that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ generated $A[\alpha]$ over A.

$(b) \implies (c)$**:** Take $C = A[\alpha]$

$(c) \implies (d)$**:** C is a faithful $A[\alpha]$ module which is a finitely generated A-module.

$(d) \implies (a)$**:** Let $\phi : M \to M$ be the map $m \mapsto \alpha \cdot m$. We have $\phi(M) \subseteq AM$, consequently, due to Proposition 2.13 (since $\mathfrak{a} = A$ is an ideal in A), there are $a_i \in A$ such that

$$(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0) \cdot m = 0$$

for each $m \in M$. But since M is a faithful $A[\alpha]$-module, we must have $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$, whereby α is integral over A.

■

In particular, from Theorem 5.2(c), we note that any element in a finite A-algebra is integral over A.

Proposition 5.3. *Let $\{\alpha_i\}_{i=1}^n$ be elements of B, each integral over A. Then the ring $A[\alpha_1, \ldots, \alpha_n]$ is a finitely generated A-module.*

Proof. Denote by A_k the subring $A[x_1, \ldots, x_k]$. We have that A_{k+1} is a finite A_k-algebra, whereby A_n is a finite A-algebra, thereby completing the proof. ■

Corollary. The set C of elements of B which are integral over A is a subring of B containing A.

Proof. Let $\alpha, \beta \in C$. Then, $A[\alpha, \beta]$ is a finite A-algebra. Now, $A \subseteq A[\alpha - \beta] \subseteq A[\alpha, \beta]$ and $A \subseteq A[\alpha\beta] \subseteq A[\alpha, \beta]$ whereby both $\alpha - \beta, \alpha\beta \in C$ and C is a ring. ■

The set C as defined above is called the *integral closure of A in B*. If $C = A$, then A is said to be *integrally closed in B*.

Theorem 5.4. *Let $A \subseteq B \subseteq C$ such that B/A and C/B are integral extensions. Then C/A is an integral extension.*

Proof. Let $\alpha \in C$. Then,

$$\alpha^n + b_{n-1}\alpha^{n-1} + \cdots + b_0 = 0$$

for some $b_i \in B$. Then, α is integral over $B' = A[b_0, \ldots, b_{n-1}]$, consequently, $B'[\alpha]$ is a finite B'-algebra. But since B' is a finite A-algebra, $B'[\alpha]$ is a finite A-algebra and α is integral over A. ■

Corollary. Let $A \subseteq B$ and C be the integral closure of A in B. Then, C is integrally closed in B.

Proof. Let $\alpha \in B$ be integral over C. Then, $C[\alpha]$ is integral over C, whereby $C[\alpha] = C$. ■

Proposition 5.5. *Let $A \subseteq B$ be an integral extension. Then,*

(a) if $\mathfrak{b} \subseteq B$ is an ideal and $\pi : B \to B/\mathfrak{b}$ is the canonical surjection, then $B/\mathfrak{b}$ is integral over $\pi(A)$. In particular, due to the First Isomorphism Theorem, we see that $B/\mathfrak{b}$ is integral over a copy of $A/\mathfrak{a}$ where $\mathfrak{a} = \mathfrak{b} \cap A$.

(b) if $S \subseteq A$ is multiplicatively closed, then $S^{-1}B$ is integral over $S^{-1}A$.

Proof. (a) Let $\beta \in B/\mathfrak{b}$, then there is some $\alpha \in B$ such that $\pi(\alpha) = \beta$. Then, there are $a_0, \ldots, a_{n-1} \in A$ such that

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$$

whereby

$$\beta^n + \pi(a_{n-1})\beta^{n-1} + \cdots + \pi(a_0) = 0$$

and the conclusion follows.

(b) Let $\alpha/s \in S^{-1}B$. Since α is integral over A, there are $a_0, \ldots, a_{n-1} \in A$ such that

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$$

then

$$(\alpha/s)^n + (a_{n-1}/s)(\alpha/s)^{n-1} + \cdots + a_0/s^n = 0$$

which completes the proof.

■

5.1 The Cohen-Seidenberg Theorems

5.1.1 Going Up Theorem

Proposition 5.6. *Let $A \subseteq B$ be an integral extension of integral domains. Then A is a field if and only if B is a field.*

Proof. ■

Proposition 5.7. *Let $A \subseteq B$ be an integral extension, $\mathfrak{q} \subseteq B$ a prime ideal and $\mathfrak{p} = \mathfrak{q}^c = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.*

Proof. Due to Proposition 5.5, $B/\mathfrak{q}$ is integral over a copy of $A/\mathfrak{p}$. The conclusion now follows from the above proposition. ■

Proposition 5.8. *Let $A \subseteq B$ be an integral extension. Let $\mathfrak{q}, \mathfrak{q}' \subseteq B$ be prime ideals of B such that $\mathfrak{q} \subseteq \mathfrak{q}'$. If $\mathfrak{q} \cap A = \mathfrak{q}' \cap A = \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{q}'$.*

Proof. Let $S = A \backslash \mathfrak{p}$ and treat all rings and ideals as A-modules. Then, $S^{-1}A \subseteq S^{-1}B$ is an integral extension and since $\mathfrak{q} \cap S = \mathfrak{q}' \cap S = \varnothing$, the ideals $S^{-1}\mathfrak{q}$ and $S^{-1}\mathfrak{q}'$ are prime ideals in B such that

$$S^{-1}\mathfrak{q} \cap S^{-1}A = S^{-1}(\mathfrak{q} \cap A) = S^{-1}\mathfrak{p} = S^{-1}(\mathfrak{q}' \cap A) = S^{-1}\mathfrak{q}' \cap S^{-1}A$$

where all the above equalities follow from treating $\mathfrak{p}, \mathfrak{q}, \mathfrak{q}', A$ as A-submodules of B, in particular, due to Proposition 3.6.

But note that $S^{-1}\mathfrak{p}$ is maximal in A whence $S^{-1}\mathfrak{q} = S^{-1}\mathfrak{q}'$ due to the previous proposition. But recall that under localization, the contraction after extension of prime ideals is the prime ideal itself, whereby the contraction of $S^{-1}\mathfrak{q}$ is $\mathfrak{q}$ whence $\mathfrak{q} = \mathfrak{q}'$. ■

Lemma 5.9. *Let $A \subseteq B$ be rings, B integral over A, and let $\mathfrak{p}$ be a prime ideal of A. Then there is a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$.*

5.1.2 Going Down Theorem

5.2 The Nullstellensatz

We require the following lemma due to Zariski. We shall see an alternate proof of this after looking at valuations.

Lemma 5.10. *Let B be a finite k-algebra. If B is a field, then it is a finite algebraic extension of k.*

Theorem 5.11 (Hilbert's Nullstellensatz Strong Form).

Chapter 6

Noetherian and Artinian Rings and Modules

6.1 Chain Conditions

Proposition 6.1. *Let M be an A-module and $\phi \in \mathrm{End}_A(M)$.*

(a) If M is noetherian and ϕ is surjective, then ϕ is injective.

(b) If M is artinian and ϕ is injective, then ϕ is surjective.

Proof. (a) Consider the ascending chain of submodules

$$\ker \phi \subseteq \ker \phi^2 \subseteq \cdots$$

Since M is noetherian, there is an index n such that $\ker \phi^n = \ker \phi^{n+1}$. Let $x \in \ker \phi^n$. Due to the surjectivity of ϕ, there is $y \in M$ such that $\phi(y) = x$, whence $\phi^{n+1}(y) = 0$ and $y \in \ker \phi^{n+1} = \ker \phi^n$. Therefore, $\ker \phi^n = 0$ and ϕ is injective.

(b) Consider the descending chain of submodules

$$\mathrm{im}\, \phi \supseteq \mathrm{im}\, \phi^2 \supseteq \cdots$$

Since M is artinian, there is an index n such that $\mathrm{im}\, \phi^n = \mathrm{im}\, \phi^{n+1}$. Then, for every $x \in M$, there is $y \in M$ such that $\phi^n(x) = \phi^{n+1}(y)$, whence $x = \phi(y)$, this establishes surjectivity.

■

6.2 Noetherian Rings

Lemma 6.2. *If A is Noetherian and $\phi : A \to B$ is a surjective ring homomorphism, then B is also Noetherian.*

Theorem 6.3 (Hilbert Basis Theorem). *If A is Noetherian, then so is $A[x]$.*

Note that the converse is also true since $A \cong A[x]/(x)$. The following proof is due to Sarges.

Proof. We shall show that every ideal in $A[x]$ is finitely generated. Suppose not and let $I \subseteq A[x]$ be an ideal that is not finitely generated. Choose $f_1 \in I$ with minimum degree. Now, inductively, choose $f_{k+1} \in I \backslash (f_1, \ldots, f_k)$ with the minimum degree. Obviously, this process goes on indefinitely, since we have assumed I to not be finitely generated. We now have

$$\begin{aligned} f_1 &= a_1 x^{d_1} + \text{lower degree terms} \\ f_2 &= a_2 x^{d_2} + \text{lower degree terms} \\ &\vdots \\ f_n &= a_n x^{d_n} + \text{lower degree terms} \\ &\vdots \end{aligned}$$

with $d_1 \leq d_2 \leq \cdots$. We also have the following ascending chain of ideals in A,

$$(a_1) \subseteq (a_1, a_2) \subseteq \cdots$$

Therefore, there is $n \in \mathbb{N}$ such that $(a_1, \ldots, a_n) = (a_1, \ldots, a_n, a_{n+1})$. Consequently, we may write a_{n+1} as a linear combination of $a_1, \ldots, a_n$, say

$$a_{n+1} = b_1 a_1 + \cdots + b_n a_n$$

for some $b_1, \ldots, b_n \in A$. Let

$$g = f_{n+1} - (b_1 x^{d_{n+1} - d_1} f_1 + \cdots + b_n x^{d_{n+1} - d_n} f_n)$$

It is not hard to argue that $g \in I \backslash (f_1, \ldots, f_n)$, but $\deg g \leq \deg f_{n+1}$, a contradiction. This completes the proof. ∎

An analogous theorem, with an analogous proof is true wherein $A[x]$ is replaced by $A[\![x]\!]$.

6.2.1 Primary Decomposition

Definition 6.4 (Irreducible). An ideal $\mathfrak{a} \subseteq A$ is said to be *irreducible* if for all ideals $\mathfrak{b}, \mathfrak{c} \subseteq A$,

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies \mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c}$$

Lemma 6.5. *In a noethering, every ideal can be expressed as a finite intersection of irreducible ideals.*

Proof. Let Σ be the poset of ideals that cannot be expressed as a finite intersection of irreducible ideals in A. Suppose Σ is nonempty, then every chain in Σ is finite (owing to noetherian-ness) whence has an upper bound, thus Σ has a maximal element (Zorn's Lemma), say $\mathfrak{a}$. Note that $\mathfrak{a}$ cannot be irreducible, therefore, there are ideals $\mathfrak{b}, \mathfrak{c}$ properly containing $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. Due to the maximality of $\mathfrak{a}$, both $\mathfrak{b}$ and $\mathfrak{c}$ can be expressed as a finite intersection of irreducible ideals in A, as a result, so can $\mathfrak{a}$, a contradiction. Thus Σ must be empty and the proof is complete. ∎

Lemma 6.6. *Every irreducible ideal in a noethering is primary.*

Proof. Let $\mathfrak{q} \subseteq A$ be an irreducible ideal. We shall show that (0) is primary in $A/\mathfrak{q}$, which is equivalent to $\mathfrak{q}$ being primary. Let $x, y \in A/\mathfrak{q}$ such that $xy = 0$. If $x \neq 0$, then consider the chain

$$\operatorname{Ann}(y) \subseteq \operatorname{Ann}(y^2) \subseteq \cdots$$

Since $A/\mathfrak{q}$ is a noethering, there is a positive integer n such that $\mathrm{Ann}(y^n) = \mathrm{Ann}(y^{n+1})$. We contend that $(x) \cap (y^n) = 0$. Indeed, if $z \in (x) \cap (y^n)$, then there are $u, v \in A/\mathfrak{q}$ such that $z = ux = vy^n$. Then,

$$vy^{n+1} = zy = uxy = 0$$

whence $v \in \mathrm{Ann}(y^{n+1}) = \mathrm{Ann}(y^n)$, whereby $z = 0$. But since (0) is irreducible and $x \neq 0$, we must have $y^n = 0$ and (0) is primary. This completes the proof. ■

6.3 Artinian Rings