Abstract Algebra

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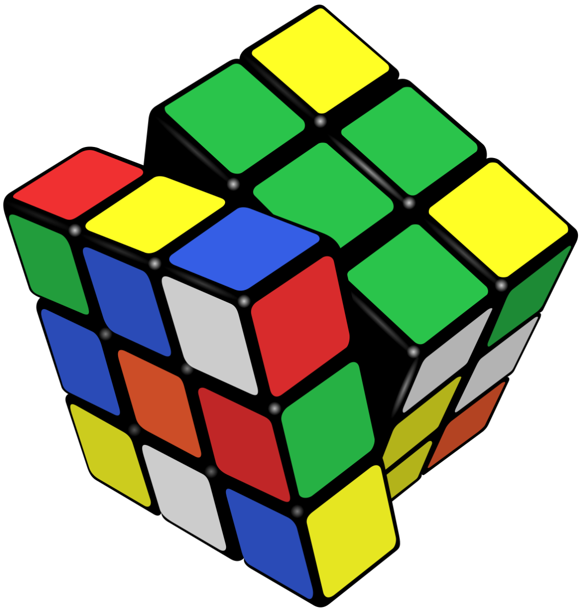

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Introduction

This is the report for the Summer of Science Reading Project for Abstract Algebra under the mentorship of Shourya Pandey. The reference text used for preparing the same was **Contemporary Abstract Algebra by Joseph A. Gallian**.

It is known to me that this report is notably long which is partly due to me getting carried away with the details of proofs during the first half of the project. In order to keep the reader engaged in the text, I have decided to omit long and routine proofs in the latter half of the report, but the theorems have still been cited in full detail.

The most important parts are Parts I, II and III which deal with Groups, Rings and Fields respectively. Part IV which deals with special topics has not been talked about in exceeding detail in order to refrain from making this report too technical and lackluster.

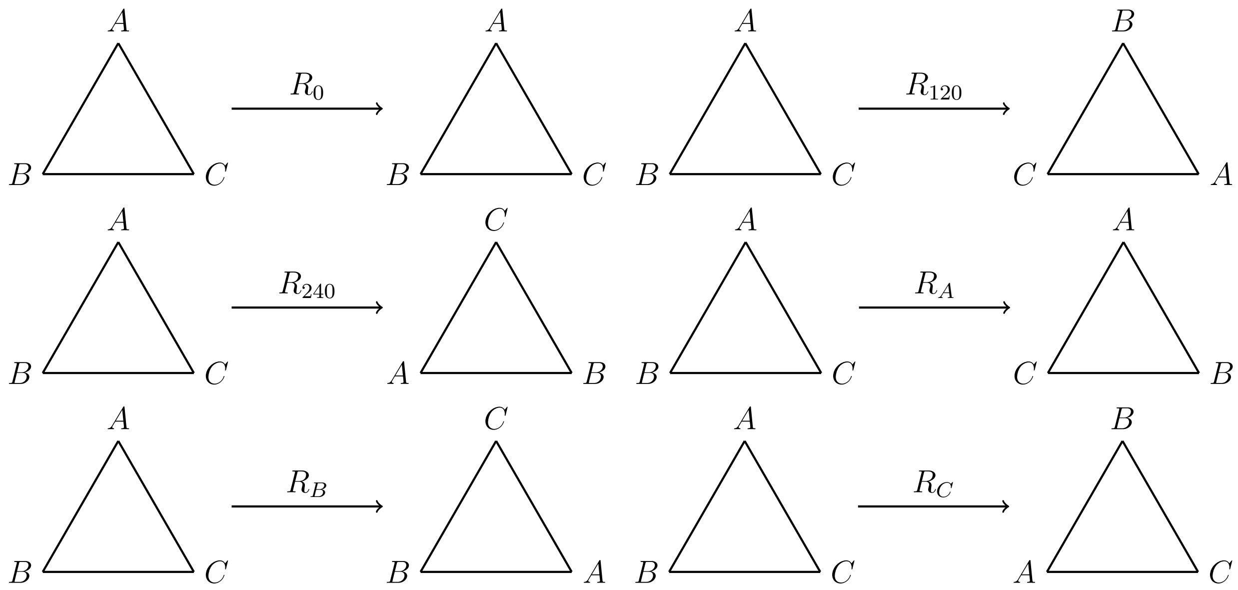
Part I

Groups

Chapter 1

An Introduction To Groups

We shall have a look at groups from the point of view of symmetry operations and then formalize the definition for abstract operations.
Say we want to study the symmetry elements of a planar equilateral triangle. First let us label the vertices of the traingle as A, B and C. Studying the symmetry operations is almost like studying the permissible permutations of A, B and C around the vertices of the given triangle. Pictorially, we can have the following symmetry operations:



Where R_θ denotes a clockwise rotation about the center of the triangle by θ degrees while R_A denotes the reflection through the plane of symmetry passing through A which (obviously) is perpendicular to $\overline{BC}$.
So, in total, we have six symmetry elements for a planar equilateral triangle. Let us now study the composition of these symmetry elements.
Obviously R_0 composed with any symmetry element $\mathcal{S}$ should result in $\mathcal{S}$ and R_θ composed with R_α would result in $R_{\theta+\alpha}$. But $R_A \circ R_{120}$ results in R_C whereas $R_{120} \circ R_A$ results in R_B. Representing all these in a tabular form, we obtain what is known as the Cayley table or more colloquially, a multiplication/operation table.
Note that for the table, the composition order is (column $\circ$ row).

	R_0	R_{120}	R_{240}	R_A	R_B	R_C
R_0	R_0	R_{120}	R_{240}	R_A	R_B	R_C
R_{120}	R_{120}	R_{240}	R_0	R_C	R_A	R_B
R_{240}	R_{240}	R_0	R_{120}	R_B	R_C	R_A
R_A	R_A	R_B	R_C	R_0	R_{120}	R_{240}
R_B	R_B	R_C	R_A	R_{240}	R_0	R_{120}
R_C	R_C	R_A	R_B	R_{120}	R_{240}	R_0

The above table shows us that the composition of any two symmetry elements gives another symmetry element. This property is called *closure* which is a necessary property for a group. Also, note that $a \circ b$ need not be the same as $b \circ a$. But, in the case when $a \circ b = b \circ a$ for all elements a and b, the group is said to be *commutative* or a scarier term for the same is *Abelian*[1]. Finally, one can check that $(a \circ b) \circ c = a \circ (b \circ c)$ for any three symmetry elements a, b and c.[2] This property is called *associativity* which is another necessary property for a group.

From the above discussion, we can conclude that the set of symmetries of a planar equilateral triangle forms a group.[3] This group is denoted by D_3 and is called "the dihedral group of order 6". Note that the order of the group is determined by the number of elements. In general, D_n is used to represent the set of symmetries of a planar n-gon. A natural question to ask at this point is whether the order of D_n is $2n$ (which you may or may not have conjectured while dealing with D_3).
The answer is **yes**. It is rather easy to prove. Let us consider a general n-gon given by $A_1 A_2 \cdots A_n$. We are interested in the "permissible permutations" of $A_1 A_2 \cdots A_n$. We note that any symmetry does not change the "adjacency" of the vertices that is, in particular, A_1 and A_2 must be adjacent in all such permutations. Now, for all the permutations, A_1 can be placed in n positions. Once the position of A_1 has been decided, there are only two valid placements of A_2. Once the positions of both A_1 and A_2 have been decided, the positions of all the vertices get determined. Thus, only $n \times 2 = 2n$ symmetry elements are possible.

[1] Named after Neils Henrik Abel

[2] I shall leave the verification of this as an exercise for the reader

[3] This is a slight abuse of language, since a group needs a binary operator associated with it but we shall deal with this in the next chapter.

Chapter 2

Formalizing Groups

In the previous chapter, we tried to develop the notion of a group in an informal way. In this chapter, we would like to formalize that same notion. For this, we begin with the definition of a binary operation

Definition 2.1 (Binary Operation). Let G be a set. A binary operation $\circ$ is simply a function $\circ : G \times G \to G$

Equipped with the above definition, we are now ready to define a group

Definition 2.2. Let G be a set and $\circ$ be a binary operation on G, then the ordered pair $(G, \circ)$ is said to be a group if the following conditions are satisfied:

1. *Associativity.* For any three (not necessarily distinct) elements $a, b, c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$.
2. *Identity.* There exists an element e in G such that $a \circ e = a = e \circ a$ for all elements $a \in G$.
3. *Inverses.* For all elements $a \in G$, there exists $b \in G$ such that $a \circ b = e = b \circ a$.

By the way, note that the definition of a binary operation itself implies that the group is closed, so we need not provide a separate axiom for *closure*.

Examples of Groups

1. $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$
2. $(\mathbb{R}\backslash\{0\}, \times)$ and $(\mathbb{Q}\backslash\{0\}, \times)$
3. $(D_n, \circ)$ where $\circ$ is the binary operator for composition of symmetries.
4. $(GL(n, \mathbb{R}), \times)$ where $GL(n, \mathbb{R})$ is the set of invertible square matrices of order n with real entries and $\times$ is matrix multiplication. This group is called the *general linear group* of $n \times n$ matrices over $\mathbb{R}$.

Examples of non-Groups

1. $(\mathbb{N}, +)$. Since there is no identity in $\mathbb{N}$.
2. $(\mathbb{R}, \times)$ and $(\mathbb{Q}, \times)$. Since 0 does not have an inverse.

Exercise. Let $U(n)$ denote the set of all natural numbers, less than or equal to n which are coprime to n. Show that $(U(n), \cdot)$ where $\cdot$ is multiplication modulo n forms a group.

Proof. Verifying closure is trivial. Further, it is easy to verify that $1 \in U(n)$ is the required identity. Finally, we are left with proving the existance of inverses. Let $a \in U(n)$. Recall, from Bezout's Lemma, there must exist integers x and y such that

$$ax + ny = \gcd(a, n) = 1$$

Reducing the above equation modulo n, we obtain (for some $x' > 0$)

$$ax' \equiv 1 \pmod{n}$$

The above equation also implies that $\gcd(x', n) = 1$, and thus $x' \in U(n)$ and is an inverse of a. Since $(U(n), \cdot)$ satisfies the group axioms, it is indeed a group. Further, since multiplication is commutative modulo n, the group is also *Abelian*. ■

Now that we have seen sufficient examples of groups, we shall have a look at some general properties of groups.

Here onwards, for some binary operation $\circ$, $a \circ b$ shall be represented simply by ab.

Proposition 2.3 (Uniqueness of Identity). Let $(G, \circ)$ be a group. Then the identity element of the group is unique.

Proof. Since $(G, \circ)$ is known to be a group, it has atleast one identity. Suppose, there exist two identies e_1 and e_2. Then, according to the definition of the identity,

$$e_1 = e_1 e_2 = e_2 e_1 = e_2$$

This implies that the identity is unique. ■

Proposition 2.4 (Left and Right Cancellation). Let $(G, \circ)$ be a group. Let $a, b, c \in G$.

- If $ba = ca$, then $b = c$
- If $ab = ac$, then $b = c$

Proof. Proving one of them is sufficient, since the proof of the other follows similarly. We have

$$ba = ca \implies \underbrace{(ba)a^{-1} = (ca)a^{-1} \implies b(aa^{-1}) = c(aa^{-1})}_{\text{due to associativity}} \implies b = c$$

■

Proposition 2.5 (Uniqueness of Inverse). Let $(G, \circ)$ be a group. For every $a \in G$, there exists a unique inverse $b \in G$ for the aforementioned a.

Proof. Since G is known to be a group, a has atleast one inverse. Suppose there exist two inverses b and c for a. Then,

$$b = be = b(ac) = (ba)c = c$$

This implies that the inverse of a is unique. ■

Proposition 2.6. Let $a, b \in G$. Then $(ab)^{-1} = b^{-1}a^{-1}$

Proof. Note that

$$e = aa^{-1} = (ae)a = (a(bb^{-1}))a^{-1} = ((ab)b^{-1})a^{-1} = (ab)(b^{-1}a^{-1})$$

and due to the uniqueness of the inverse, we have the desired conclusion. ■

Chapter 3

Finite Groups and Subgroups

We briefly talked about the orders of goups in **Chapter 1**, putting that formally, we have

Definition 3.7 (Order of a Group). Let $(G, \circ)$ be a group. If G has a finite number of elements, then we say the order of G, denoted by $|G|$ is equal to the number of elements. If G is not finite, then we say that the order of G is infinite.

Just as we define the order of a group, we can define the order of an element in a group.

Definition 3.8 (Order of an Element). Let $(G, \circ)$ be a group. Let $g \in G$. The order of g is defined to be the smallest posistive integer satisfying $g^n = e$ where e is the identity element in G. If no such n exists, then g is said to have infinite order. The order of g is denoted by $|g|$.

Proposition 3.9. Let $(G, \circ)$ be a finite group. Then, for every element $g \in G$, $|g|$ is finite.

Proof. Consider the set

$$\{g, g^2, \cdots, g^{|G|+1}\}$$

Since G is closed under $\circ$, the above set must be a subset of G but due to the Pigeon-hole Principle, there must exist two distinct indices $i > j$ such that $g^i = g^j$. This implies that $g^{(i-j)} = e$, implying that the order of g is $i - j \leq n$ which is obviously finite. ■

Definition 3.10 (Subgroup). Let $(G, \circ)$ be a group. If $H \subseteq G$ and $(H, \circ)$ forms a group, then H is said to be a subgroup of G.

In what follows, we shall discuss three propositions which are useful for determining whether or not a group is a subgroup of another.

Proposition 3.11. Let G be a group and H be a non-empty subset of G. If $ab^{-1} \in H$ whenever $a, b \in H$, then H is a subgroup of G.

Proof. Since $\circ$ is associative over G and $H \subseteq G$, we can conclude that $\circ$ is associative over H. Let $a \in H$, we know this exists since H is given to be non-empty. Then, according to the given hypothesis, $e = aa^{-1}$ also belongs to H, where e is the identity element of G. Further, since $H \subseteq G$, we know that $ea = a = ae$ for all $a \in H$ since $a \in G$. Now, since $e \in H$, according to the hypothesis, $a^{-1} = ea^{-1}$ must also belong to H, for every $a \in H$. Thus, H satisfies all the three group axioms, implying that H is a subgroup of G. ■

Proposition 3.12. Let G be a group and let H be a non-empty subset of G. If $ab \in H$ whenever $a, b \in H$ and $a^{-1} \in H$ whenever $a \in H$, then H is a subgroup of G.

Proof. Since $\circ$ is associative over G and $H \subseteq G$, we can conclude that $\circ$ is associative over H. Let $a \in H$, we know this element exists since H is known to be non-empty. Then, according to the hypothesis, $a^{-1} \in H$ and $aa^{-1} = e \in H$ where e is the identity element of G. For any element $a \in H$, note that $ae = a = ea$ since $a \in G$. Also, since the given hypothesis ensures the existance of inverses, we can conclude that H is infact a subgroup of G. ■

Proposition 3.13. Let H be a non-empty, finite subset of a group G. If H is closed under $\circ$, then H is a subgroup of G.

Proof. Let $a \in H$, we know this element exists since H is known to be non-empty. Conisder the following set
$$\{a, a^2, \cdots, a^{|H|+1}\}$$
Since H is closed under $\circ$ the above set must be a subset of H but due to the Pigeon-hole Principle, there must exist two indices $i > j$ such that $a^i = a^j$. Since $a \in G$, $a^{-1} \in G$, implying that $a^{i-j} = e$. But since $i - j > 0$, $a^{i-j} \in H$. Thus, H contains the identity element. It is now trivial that $ae = a = ea$ for all $a \in H$ since $a \in G$. We already had that $a^{i-j} = e$. Then, multiplying a^{-1} on both sides of the equality, we obtain $a^{i-j-1} = a^{-1}$. As long as $a \neq e$, $i - j > 1$, implying that $a^{i-j-1} \in H$ and thus for all $a \in H$, $a^{-1} \in H$. Since H satisfies all the group axioms, it is a subgroup of G. ■

Definition 3.14. Let G be a group. For any element $a \in G$, define
$$\langle a \rangle \stackrel{\text{def}}{=} \{a^n \mid n \in \mathbb{Z}\}$$

Proposition 3.15. Let G be a group and $a \in G$. Then $\langle a \rangle$ is a subgroup of G.

Proof. For all $m, n \in \mathbb{Z}$, we have $a^m, a^n \in G$ and also $a^{m-n} \in G$. Then, we are done due to **Proposition 3.11**. ■

Definition 3.16 (Center of a Group). Let G be a group. The center of G, denoted by $Z(G)$ is the set given by
$$Z(G) = \{a \mid a \in G; \quad ax = xa \quad \forall x \in G\}$$

In simpler terms, the Center of a Group is the set of all elements which commute with every element of G.

Proposition 3.17. The center of a group G is a subgroup of G.

Proof. Since $Z(G) \subseteq G$, it suffices to show that $Z(G)$ is a group. Since all the elements of $Z(G)$ are also elements of G, $\circ$ must be associative over $Z(G)$. Let $a, b \in Z(G)$ be not necessarily distinct elements. Then, for all $x \in G$, we can write $a = xax^{-1}$ and $b = xbx^{-1}$. Multiplying these two equalities, we obtain $ab = xabx^{-1}$, implying that $(ab)x = x(ab)$. Hence, $Z(G)$ is closed under $\circ$. Now, according to the defninition of the identity (of the group G), we know that it commutes with every element of G, implying that $e \in Z(G)$. Finally, if $ax = xa$ for all $x \in G$, we can write $xa^{-1} = a^{-1}x$ for all $x \in G$. Thus, $a^{-1} \in G$ whenever $a \in G$. Hence, $Z(G)$ is a group. ■

Similar to the center of a group, we define the centralizer for an element of a group.

Definition 3.18 (Centralizer). Let G be a group. Let $a \in G$, then the centralizer of a, denoted by $C(a)$ is the set of all elements in G which commute with a. That is,

$$C(a) \stackrel{\text{def}}{=} \{x \mid x \in G; \quad ax = xa\}$$

Proposition 3.19. For each $a \in G$, $C(a)$ is a subgroup of G.

Proof. Since $C(a) \subseteq G$, we know that $\circ$ is associative over $C(A)$. Let $x, y \in C(a)$ which are not necessarily distinct. Then

$$a(xy) = (ax)y = (xa)y = x(ay) = x(ya) = (xy)a$$

implying that $xy \in a$. That is, $C(a)$ is closed under $\circ$. Finally, note that if $ax = xa$, then simple rearrangement of terms gives us $x^{-1}a = ax^{-1}$, i.e. $x^{-1} \in G$ whenever $x \in G$. Then, due to **Proposition 3.12**, we can conclude that $C(a)$ is a subgroup of G. ■

Chapter 4

Cyclic Groups

We begin first by defining a cyclic group

Definition 4.20 (Cyclic Groups). A group G is called cyclic, if there is an element $a \in G$ such that $G = C(a)$

At this point, it may be tempting to conclude that a cyclic group is always finite. But, this is not the case, the simplest counterexample to this is $(\mathbb{Z}, +) = \langle 1 \rangle$.

Proposition 4.21. Let G be a group and $a \in G$. If a has infinite order then $a^i = a^j$ if and only if $i = j$. If a has finite order, say n, then $\langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\}$ and $a^i = a^j$ if and only if $n \mid i - j$.

Proof. Assume FTSOC, a has infinite order and there exist indices i and j such that $a^i = a^j$. But that implies $a^{i-j} = e$, contradicting the assumption that a has infinite order.
Now, assume that $|a| = n$. Let $a^i = a^j$ for some i and j. Then $a^{i-j} = e$. Let now $i - j = qn + r$. Note that the choice of q and r is guaranteed due to the division algorithm. This then implies that $(a^n)^q a^r = e$ or, equivalently, $a^r = e$. Since $r < n$, the only admissible value of r is zero, else it would contradict the fact that the order of a is n. And thus $n \mid i - j$. ∎

Corollary 4.1. Let G be a group and $a \in G$. Then $|a| = |\langle a \rangle|$.

Corollary 4.2. Let G be a group and let a be an element of order n in G. If $a^k = e$, then n divides k.

Corollary 4.3. Let G be a finite group and $a, b \in G$, then $|ab|$ divides $|a||b|$.

Proposition 4.22. Let G be a group and $a \in G$ be an element of order n and let k be a positive integer. Then $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and $|a^k| = n / \gcd(n, k)$.

Proof. For simplicity, let $d = \gcd(n, k)$. We need to show that $\langle a^k \rangle = \langle a^d \rangle$. One easily notes that $\langle a^d \rangle \subseteq \langle a^k \rangle$. But now, by Bezout's Lemma, there exist integers x and y such that $d = nx + ky$. Then $a^d = (a^k)y$, or equivalently, $\langle a^k \rangle \subseteq \langle a^d \rangle$. And thus, $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$.
Let $|a^k| = x$. Then by definition, $a^{kx} = e$. Using **Corollary 4.2**, we must have $n \mid kx$ or $n \mid dmx$. Since $\gcd(n, m) = 1$, $n \mid dx$ and it is now obvious that the smallest positive integer x satisfying the reduced equation is n/d and we have the desired conclusion. ■

Corollary 4.4. In a finite cyclic group, the order of an element divides the order of the group.

Proof. Since the group is given to be cyclic, there must exist an element a such that every element x can be written as a^k for some k. Then, using **Proposition 4.22**, $|x| = |a^k| = \gcd(n, k)$, where n is the order of a and hence that of the group. Hence, we have the desired conclusion. ■

Corollary 4.5. Let G be a group and $a \in G$ such that $|a| = n$. Then $\langle a^i \rangle = \langle a^j \rangle$ if and only if $\gcd(n, i) = \gcd(n, j)$, and $|a^i| = |a^j|$ if and only if $\gcd(n, i) = \gcd(n, j)$.

A better way to represent the above is

$$\langle a^i \rangle = \langle a^j \rangle \iff \gcd(n, i) = \gcd(n, j) \iff |a^i| = |a^j|$$

Corollary 4.6. Let G be a group and $a \in G$ with $|a| = n$. Then $\langle a \rangle = \langle a^j \rangle$ if and only if $\gcd(n, j) = 1$ and $|a| = |\langle a^j \rangle|$ if and only if $\gcd(n, j) = 1$.

Theorem 4.23 (Fundamental Theorem of Cyclic Groups). Every subgroup of a cyclic group is cyclic. Moreover if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a \rangle$ has exactly one subgroup of order k, namely $\langle a^{(n/k)} \rangle$.

Proof. Let G be a cyclic group such that $G = \langle a \rangle$ for some $a \in G$ and let the order of a be n. Let H be a subgroup of G. We would like to show that H is cyclic. Since H is finite and we can write every element in H as a^j for some $j \in \mathbb{N}$, we can find an element in H such that j is minimized for that element. Call this element x. We claim that $H = \langle x \rangle$. Assume for the sake of contradiction that there exists $b = a^k \in H$ such that $b \notin \langle x \rangle$, then, according to the division algorithm, there must exist non-negative integers q and r such that $k = qj + r$, where $r > 0$. And in that case, $a^r = b(x^{-1})^q \in H$, but this contradicts the minimality of j. Thus, $H = \langle x \rangle$ is a cyclic group.
Since we showed previously that any subgroup of a cyclic group G has a generator(is cyclic). If that generator is given by a^j, where a is the generator for G, then according to **Proposition 4.22**, the order of the subgroup must be $n/\gcd(n, j)$ which is obviously a divisor of n.
Let H_1 and H_2 be two subgroups of G of order k. Then, they each have a generator, say a^i and a^j respectively. Then, using **Corollary 4.5**, we can conclude that $\langle a^i \rangle = \langle a^j \rangle$ or

equivalently $H_1 = H_2$ which proves the uniqueness of the subgroup of order n. Since we proved in the previous result that all the subgroups of G must have order dividing n, the only permissible values of k are those that divide n. It is now easy to check that $\langle a^{n/k} \rangle$ is a subgroup of G and due to uniqueness, it is the only subgroup of G with order k. ■

Proposition 4.24. If d is a positive divisor of n, then the number of elements of order d in a cyclic group G of order n are $\varphi(d)$. Where φ is the Euler Totient Function.

Proof. Let a be the generator of G, then from **Proposition 4.22**, we know that all the elements of order d will be of the form a^k where $n/\gcd(n,k) = d$. Or, $n/d = \gcd(n,k)$, implying that $\gcd(d, k/(\frac{n}{d})) = 1$. Thus, k takes the form $q\frac{n}{d}$ where $\gcd(q,d) = 1$. Thus, we have exactly $\varphi(d)$ elements in G, having order d. ■

Exercise. Use the above result to show that

$$\sum_{d|n} \varphi(d) = n$$

Proposition 4.25. Let G be a finite group. Then the number of elements in G having order d is a multiple of $\varphi(d)$.

Proof. Let $a \in G$ have order d. Then, there are exactly $\varphi(d)$ elements in $\langle a \rangle$ which have order d. Let $b \in G$ be another element of order d but $b \notin \langle a \rangle$. We shall show that $\{x \mid x \in \langle a \rangle, |x| = d\} \cap \{x \mid x \in \langle b \rangle, |x| = d\} = \emptyset$. Suppose not, then there exists $c \in \{x \mid x \in \langle a \rangle, |x| = d\} \cap \{x \mid x \in \langle b \rangle, |x| = d\}$. But according to **Proposition 4.22**, $\langle c \rangle = \langle a \rangle$ and $\langle c \rangle = \langle b \rangle$ which contradicts the fact that $b \notin \langle a \rangle$. Thus $\{x \mid x \in \langle a \rangle, |x| = d\} \cap \{x \mid x \in \langle b \rangle, |x| = d\} = \emptyset$. Thus, the total number of elements with order d are now $2\varphi(d)$. Proceeding inductively, one can see that the number of elements with order d must be a multiple of $\varphi(d)$. ■

Chapter 5

Permutation Groups

Definition 5.26 (Permutation Group). A *permutation* of a set A is a bijective function $\sigma : A \to A$. A *permutation group* of a set A is the set of permutations of A that form a group under $\circ$, which is the binary operator for function composition.

It is important to note that a permutation group need not contain all the permutations of the set A. For example, consider the group D_4. This group can be viewed as a subset of the permutations of $\{1, 2, 3, 4\}$ where one can imagine the numbers to be the labels of the vertices of a square.
But, in the case when the permutation group contains all the possible permutations of a set with n elements, the group is termed *symmetric group of degree* n and is denoted by S_n.

In the subsequent discussion, we shall use the *cycle notation* to specify permutations. For example, conisder the permutation

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{bmatrix}$$

The above permutation can be broken into the following cycles

$$\begin{aligned} 1 \mapsto 2 \mapsto 1 \\ 3 \mapsto 4 \mapsto 6 \mapsto 3 \\ 5 \mapsto 5 \end{aligned}$$

The above cycles are represented as follows

$$(1, 2)(3, 4, 6)(5)$$

Note how we were able to breakdown the given permutation into a product of disjoint cycles, as you may have already conjectured:

Proposition 5.27. Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

Proof. Let σ be a permutation of $\{1, 2, \cdots, n\}$. Let $a_1 \in A$. Consider the set

$$\{a_1, \sigma(a_1), \sigma^2(a_1), \cdots, \sigma^n(a_1)\}$$

Due to the Pigeon-hole Principle, there must exist unequal non-negative indices i and j such that $\sigma^i(a_1) = \sigma^j(a_1)$. But, since σ is a bijective and hence invertible function, we can write $\sigma^{|i-j|}(a_1) = a_1$ where $|i - j| \neq 0$. Thus, we have found a cycle.
Continuing similarly in this fasion, we note that the number of elements which are not in a cycle after each iteration strictly decreases and hence, must end at some point. THus, we have divided the permutation into a product of disjoint cycles. ■

Proposition 5.28 (Disjoint Cycles Commute). If the pair of cycles α and β have no entries in common, then $\alpha\beta = \beta\alpha$.

Proof. Say both α and β are permutations of the set S. Let $a \in S$. If $a \notin (\alpha \cup \beta)$, then $\alpha\beta(a) = a = \beta\alpha(a)$. Else, if $a \in \alpha$, then $\alpha(\beta(a)) = \alpha(a)$ and $\beta(\alpha(a)) = \alpha(a)$. Similarly proceeding for the case when $a \in \beta$, we have the desired conclusion. ■

Lemma 5.29. The order of a cycle of length n is n.

Proof. Trivial. ■

Theorem 5.30 (Ruffini 1799). The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

Proof. Call the permutation of a set S, σ. Using **Proposition 5.27**, we can conclude that there exist disjoint cycles $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that

$$\sigma = \alpha_1\alpha_2 \cdots \alpha_k$$

Say the order of $\sigma = M$, then, using the fact that disjoint cycles commute, we can write

$$\varepsilon = \alpha_1^M \alpha_2^M \cdots \alpha_k^M$$

Let $a \in \bigcup_{i=1}^{k} \alpha_i$. Since a can be an element of exactly one of the $\alpha_i's$, then $a = \sigma(a) = \alpha_j^M(a)$ for some j. This obviously means that $\alpha_j^M = \varepsilon$ and equivalently, $\ell(\alpha_j) \mid M$. We now have the desired conclusion. ■

Lemma 5.31. Every (non-identity) cycle can be written as a product of 2-cycles.

Proof. Just notice that

$$(a_1a_2 \cdots a_n) = (a_1a_2)(a_2a_3) \cdots (a_{n-1}a_n)$$

■

As a corollary of the above lemma, we have the following result (which is more popular than the lemma)

Proposition 5.32. Every permutation in S_n is a product of 2-cycles.

Lemma 5.33. If $\varepsilon = \beta_1\beta_2\cdots\beta_r$ where the β's are 2-cycles, then r is even.

Proof due to Joseph A. Gallian. We proceed by induction on r. The base case, with $r = 2$ is trivial. Suppose now that $r > 2$. Assume that $\beta_r = ab$. Then, we may have the following cases for the value of $\beta_{r-1}\beta_r$.

$$\begin{aligned}\varepsilon &= (ab)(ab)\\ (ab)(bc) &= (ac)(ab)\\ (ac)(cb) &= (bc)(ab)\\ (ab)(cd) &= (cd)(ab)\end{aligned}$$

In the first case, we simply delete β_{r-1} and β_r from the original product and would be left with a product of $r-2$ cycles which give the identity. Thus, we should be done by strong induction.

Otherwise, note that the last cycle containing a has now shifted to the left by 1. Repeat the previous step again for $\beta_{r-2}\beta_{r-1}$ which should either give the product of the two as ε or shift a to the left once again. Note that a cannot be shifted to the left indefinitely, and hence we must reach a product of ε atleast once. Once that product is reached, apply strong induction on the remaining $r-2$ terms and we are done. ■

Proposition 5.34. If a permutation α can be expressed as a product of an even (odd) number of 2-cycles, then every decomposition of α into a product of 2-cycles must have an even (odd) number of 2-cycles. In symbols, if

$$\sigma = \beta_1\beta_2\cdots\beta_r = \gamma_1\gamma_2\cdots\gamma_s$$

Then, $r \equiv s \pmod 2$.

Proof. Note that

$$\begin{aligned}\beta_1\beta_2\cdots\beta_r &= \gamma_1\gamma_2\cdots\gamma_s\\ \varepsilon &= \gamma_1\gamma_2\cdots\gamma_s\beta_r\beta_{r-1}\cdots\beta_1\end{aligned}$$

Using **Lemma 5.33**, we can conclude that $r + s \equiv 0 \pmod 2$ and we have the desired conclusion. ■

Definition 5.35 (Even and Odd Permutations). A permutation that can be expressed as a product of an even number of 2-cycles is called an *even* permutation. A permutation that can be expressed as a product of an odd number of 2-cycles is called an *odd* permutation.

Proposition 5.36. The set of even permutations in S_n forms a subgroup of S_n.

Proof. Note first that the composition of any even permutation is also an even permutation due **Proposition 5.34**, this implies closure. Further, since the set of even permutations is a subset of the group of all permutations, we can conclude that $\circ$, which is the binary operator for function composition is associative on the set of even permutations. The identity element which is ε is also an even permutation, due to **Lemma 5.33** and must be an element in the set of even permutations. Let $\sigma = \beta_1\beta_2\cdots\beta_r$ then easily note that $\sigma^{-1} = \beta_r\beta_{r-1}\cdots\beta_1$ and hence, the set of all even permutations in S_n forms a subgroup of S_n. ■

Definition 5.37. The group of even permutations of $\{1, 2, \cdots, n\}$ is denoted by A_n and is called the *alternating group of degree n*.

Proposition 5.38. For $n > 1$, A_n has order $n!/2$.

Proof. Let c be any 2-cycle. Note that if $a \in A_n$, then $ca \notin A_n$, conversely, if $ca \notin \overline{A_n}$, then $a \in A_n$, since c is an invertible function. Thus, there is a bijection from A_n to $\overline{A_n}$. This completes the proof. ■

Chapter 6

Isomorphisms

Definition 6.39. An *isomorphism* from a group $(G, \circ)$ to a group $(\overline{G}, \cdot)$ is a bijective mapping (or function) $\phi : G \to \overline{G}$ that preserves the group operation. That is,

$$\phi(a \circ b) = \phi(a) \cdot \phi(b)$$

In the case there is an isomorphism from G to $\overline{G}$, we say that G and $\overline{G}$ are isomorphic and write $G \cong \overline{G}$.

Here onwards, we shall use the notation $\overline{g}$ to denote the element $\phi(g)$ for some $g \in G$.

Following are some examples of isomorphic groups:

- $(\mathbb{R}, +) \cong (\mathbb{R}^+, \times)$. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = e^x$.
- Any infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. Obviously, if $G = \langle a \rangle$, then, consider the mapping $\phi(a^k) = k$.
- Any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

Theorem 6.40 (Cayley 1854). Every group is isomorphic to a group of permutations.

Proof. Let G be the given group. We shall construct a permutation group $\overline{G}$ and show that $G \cong \overline{G}$. For any $g \in G$, define

$$T_g(x) = gx$$

Note due to closure, $T_g : G \to G$. Further, since left multiplication by g is invertibe, owing to the existence of g^{-1}, we conclude that T_g is infact bijective and hence a permutation on G. Consider now the set

$$\overline{G} = \{T_g \mid g \in G\}$$

We claim that $G \cong (\overline{G}, \circ)$ where $\circ$ is the binary operator for composition of functions. Indeed, consider the function $\phi(g) = T_g$, this is obviously closed, owing to the fact that $T_h(T_g(x)) = hgx = T_{hg}(x)$. Further, associativity is obvious, since $f(gh) = (fg)h$. The identity function, which is obviously given by T_e is also an element of $\overline{G}$. The inverse permutation for T_g is given by $T_{g^{-1}}$ which is trivial to verify. Thus, $\overline{G}$ satisfies all the axioms of being a group. Thus ϕ is an isomorphism from G to $\overline{G}$. ■

Proposition 6.41. Suppose $\phi : G \to \overline{G}$ is an isomorphism. Then,

1. ϕ carries the identity of G to that of $\overline{G}$.
2. For every integer n and every $a \in G$, $\phi(a^n) = \phi(a)^n$.
3. For any two $a, b \in G$, a and b commute if and only if $\phi(a)$ and $\phi(b)$ commute.
4. $G = \langle a \rangle$ if and only if $\overline{G} = \langle \phi(a) \rangle$.
5. $|a| = |\langle \phi(a) \rangle|$ for all $a \in G$.
6. For a fixed integer k and a fixed group element $b \in G$, the equation $x^k = b$ has as many solution as $x^k = \phi(b)$.
7. If G is finite, then both G and $\overline{G}$ have equal number of elements of the same order.

Proof. The proofs will not be detailed, since this proposition is rather trivial.

1. Let e be the identity element of G. Then note that for $g \in G$,
$$\overline{ge} = \phi(g)\phi(e) = \phi(ge) = \overline{g} = \phi(eg) = \phi(e)\phi(g) = \overline{eg}$$
This implies that $\overline{e}$ is the (unique) identity element of $\overline{G}$.
2. Trivial.
3. If a and b commute, then note that
$$\phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a)$$
Conversely if $\phi(b)$ and $\phi(a)$ commute, then note that
$$\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(ba)$$
But, since ϕ is a bijective function, we conclude that $ab = ba$.
4. If $G = \langle a \rangle$, then for every element $g \in G$, there exists $k \in \mathbb{Z}$ such that $g = a^k$. This implies, $\phi(g) = \phi(a)^k$. Since ϕ is bijective, we can safely conclude that $\phi(a)$ is the generator for G. The converse follows similarly.
5. Follows from the previous result.[1]
6. Again, follows from bijectivity.
7. Ofcourse, if $g \in G$ has order k, then $\phi(g)$ must have order k as well. Finish now using bijectivity of G.

■

[1] In other words, left as an exercise for the reader.

Proposition 6.42. Suppose ϕ is an isomorphism from a group G onto a group $\overline{G}$. Then

1. ϕ^{-1} is an isomorphism from $\overline{G}$ onto G.
2. G is Abelian if and only if $\overline{G}$ is Abelian.
3. G is cyclic if and only if $\overline{G}$ is cyclic.
4. If K is a subgroup of G, then $\phi(K)$ is a subgroup of $\overline{G}$.
5. $\phi(Z(G)) = Z(\overline{G})$.

Proof. Similar to the previous proposition, detailed proofs will not be provided, for the same reason

1. Trivial since ϕ is bijective.
2. Suppose G is Abelian. Let $\overline{a}$ and $\overline{b}$ be two elements of $\overline{G}$. Then
$$\overline{a}\overline{b} = \phi(ab) = \phi(ba) = \overline{b}\overline{a}$$
The converse follows similarly.
3. Follows from Property 4 of **Proposition 6.41**.
4. Note first that $\phi(K)$ is closed under whatever group operation is associated with $\overline{G}$. Further, assocaitivity follows obviously. From Property 1 and 2 of **Propsition 6.41**, we are guaranteed the existance of an identity and an inverse for each element. Thus $\phi(K)$ satisfies all the group axioms and is a subgroup of $\overline{G}$.
5. Follows from Property 3 of **Proposition 6.41**

■

Definition 6.43 (Automorphism). An isomorphism from a group G onto itself is called an *automorphism* of G. We use the notation $\mathrm{Aut}(G)$ to denote the set of all automorphisms of G.

Note that every group is isomorphic to itself under the identity function which is thus an automorphism.
Some examples of (non-trivial) automorphisms are:

- Let G be any group. Define $\phi : G \to G$ as $\phi(g) = g^{-1}$
- Let $\mathbb{C}$ be the group of complex numbers under addition. Define $\phi : \mathbb{C} \to \mathbb{C}$ as $\phi(z) = \overline{z}$

Definition 6.44 (Inner Automorphism Induced by a). Let G be a group and let $a \in G$. The function $\phi_a : G \to G$ defined as $\phi_a(x) = axa^{-1}$ for all $x \in G$ is called the *inner automorphism of G induced by a*. We use the notation $\mathrm{Inn}(G)$ to denote the set of all inner automorphisms of G.

Proposition 6.45. $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ are groups under $\circ$ which is the function composition operator.

Proof. Let $\phi_1, \phi_2 \in \mathrm{Aut}(G)$. Note that $\phi_1 \circ \phi_2$ is a bijective function as well and hence must be an element of $\mathrm{Aut}(G)$. This implies closure. The proof of associativity is trivial. As we discussed earlier, the identity function from G to itself is also an element of $\mathrm{Aut}(G)$, call this ε. Then note that $\varepsilon \circ \phi = \phi = \phi \circ \varepsilon$, implying that ε is the identity element in $\mathrm{Aut}(G)$. Finally, note that every bijective function has an inverse to conclude that $\mathrm{Aut}(G)$ is infact a group.
Let ϕ_a and ϕ_b be elements of $\mathrm{Inn}(G)$. Then, note that $\phi_a(\phi_b(x)) = (ab)x(ab)^{-1} = \phi_{ab}(x)$. This implies closure. Associativity is trivial. Note that $\phi_e(x) = x$ for all $x \in G$. Then $\phi_e(\phi_a(x)) = \phi_a(x) = \phi_a(\phi_e(x))$, implying that ϕ_e is the identity element of $\mathrm{Inn}(G)$. Now, $\phi_{a^{-1}}(\phi_a(x)) = x = \phi_a(\phi_{a^{-1}}(x)) = \phi_e(x)$. Thus $\phi_{a^{-1}}$ is the inverse of ϕ_a. Thus, $\mathrm{Inn}(G)$ is a group. Also, note that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$. ■

Proposition 6.46. For every positive integer$n$, $\mathrm{Aut}(Z_n) \cong U(n)$.

Proof. Recall that the group operation for $\mathbb{Z}_n$ is $+$ and that for $U(n)$ is $\times$. Let $f \in \mathrm{Aut}(\mathbb{Z}_n)$ and $u = f(1)$. We claim that $u \in U(n)$. Note first that $f(0) = 0$. Then, for all $k \geq 1$, $f(k) = \underbrace{u + u + \cdots + u}_{k \text{ times}}$ which can be proved via induction. Suppose $\gcd(n, u) = d > 1$. Let $m = \frac{n}{d}$. Then, $f(m) = \underbrace{u + u + \cdots + u}_{m \text{ times}} = 0 = e$, but this is contradictory to the fact that $m \neq 0$. Thus, $\gcd(u, n) = 1$, or $u \in U(n)$. Note that if $f(1)$ is specified to be u, then, one can find $f(k)$ for all k, implying that the map $f \mapsto u$ must be bijective. The rest of the proof is straightforward. ■

Corollary 6.7. There are exactly $\varphi(n)$ automorphisms for $\mathbb{Z}_n$.

Chapter 7

Cosets and Lagrange's Theorem

Definition 7.47 (Coset of H in G). Let G be a group and let H be a non-empty subset of G. For any $a \in G$, we define $aH = \{ah \mid h \in H\}$, $Ha = \{ha \mid h \in H\}$ and $aHa^{-1} = \{aha^{-1} \mid h \in H\}$. When H is a subgroup of G, the set aH is called the *left coset of H in G containing a*, whereas Ha is called the *right coset of H in G containing a*. In this case, the element a is called the *coset representative of aH or Ha*. We use $|aH|$ to denote the number of elements in the set aH and $|Ha|$ to denote the elements in Ha.

Lemma 7.48. Let H be a subgroup of G, and let a and b belong to G. Then,

1. $a \in aH$
2. $aH = H$ if and only if $a \in H$
3. $(ab)H = a(b(H))$ and $H(ab) = (Ha)b$
4. $aH = bH$ if and only if $a \in bH$
5. $aH = bH$ or $aH \cap bH = \emptyset$
6. $aH = bH$ if and only if $a^{-1}b \in H$
7. $|aH| = |bH|$
8. $aH = Ha$ if and only if $H = aHa^{-1}$
9. aH is a subgroup of G if and only if $a \in H$.

All the above lemmas are valid for right cosets of H in G.

Proof.

1. Since H is a subgroup of G, $e \in H$ where e is the identity element in G. Thus, $a = ae \in aH$
2. If $aH = H$, using (1), we can conclude that $a \in H$. Conversely, if $a \in H$, note that H is closed due to multiplication by a (since H is a group). Thus, the function $h \mapsto ah$ is bijective and hence an automorphism, thus $aH = H$.

3. Trivial

4. If $aH = bH$, using (1), it is trivial that $a \in bH$. On the other hand, if $a \in bH$, let $x = b^{-1}a \in H$. Note that due to (2), we can conclude that $xH = H$. But then, $bH = b(xH) = (bx)H = aH$. This completes the proof.

5. Follows from (4)

6. Follows from (4)

7. Since both aH and bH are bijective maps from H, we can conclude that $|aH| = |bH| = |H|$.

8. For all $x \in H$, there exists a unique $y \in H$ such that $ax = ya$, or $x = axa^{-1}$, which completes the proof.[1]

9. Suppose aH is a subgroup of G, then due to the group axioms, $e \in aH$, or equivalently, $a^{-1} \in H$. But since H is a group, it implies that $(a^{-1})^{-1} = a \in H$, which completes the proof.

■

Theorem 7.49 (Lagrange 1770[a]). If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left(right) cosets of H in G is $|G|/|H|$.

[a]Apparently, Lagrange stated this theorem in 1770, whereas it was proved by Pietro Abbati some 30 years later.

Proof. Let $a_1H, a_2H, \cdots, a_rH$ be the distinct cosets of H contained in G. Then, due to Property 5 of **Lemma 7.48**, one notes that the above cosets are disjoint. Further, for any $a \in G$, there exists a_i such that $a \in a_iH$. That means

$$G = \bigcup_{i=1}^{r} a_iH$$

or equivalently,

$$|G| = \sum_{i=1}^{r} |a_iH| \overset{(7)}{=} r|H|$$

This completes the proof. ■

I must remark that it is rather surprising that such an important theorem in Group Theory has such a short (an elegant) proof.

Definition 7.50 (Index of a Subgroup). The index of a subgroup H of G is the number of distince left/right cosets of H in G. It is denoted by $|G : H|$.

[1]Note that the uniqueness is due to the the fact that $x \mapsto ax$ is bijective.

Corollary 7.8. If G is a finite group, then for any $a \in G$, $|a|$ divides $|G|$.

Corollary 7.9. Groups of prime order are cyclic.

Proof. Let G be a group of prime order p. Then, for all $a \in G$, $a \neq p$, $|a| = p$. Thus, $G = \langle a \rangle$ for all $a \in G$. ■

Corollary 7.10. Let G be a finite group. Then for all $a \in G$, $a^{|G|} = e$.

Proposition 7.51. For two finite subgroups H and K of a group, define the set $HK = \{hk \mid h \in H, k \in K\}$. Then $|HK| = |H||K|/|H \cap K|$.

Proof. Although the set HK has $|H||K|$ products, not all the products are the same. We shall like to see the extent to which we can find two same products. Say we have $hk = h'k'$. Then, there must exist a unique group element $t \in G$ such that $t = h^{-1}h' = kk'^{-1}$. But note that $h^{-1}h' \in H$ and $kk'^{-1} \in K$ and thus, $t \in H \cap K$. Thus, there are atmost $|H \cap K$ ways to represent a product.
On the other hand, for all $t \in H \cap K$, we have $hk = (ht)(t^{-1}k)$. Thus, there are atleast $|H \cap K|$ ways to represent a product. In conclusion, there are exactly $|H \cap K$ ways to represent a product and we have the desired conclusion. ■

We shall see the power of the above result in proving the following result:

Proposition 7.52. Let G be a group of order $2p$, where p is a prime greater than 2. Then G is isomorphic to $\mathbb{Z}_{2p}$ or D_p.

Proof. According to Lagrange's Theorem, the orders of the elements of G must be divisors of $2p$ and hence can take values from the set $\{1, 2, p, 2p\}$. If there exists an element with order $2p$, then the group must be cyclic and must be isomorphic to $\mathbb{Z}_{2p}$.
Suppose now that there is no element with order $2p$, then the orders of each element, other than the identity must be from the set $\{2, p\}$. In the first case, we suppose that all elements (excluding the identity) have order 2. Then we can make unordered pairs of elements whose product is the identity. But note that the identity must be paired with itself, so in the case such a group did exist it would require an odd number of elements. Hence, there must exist an element $a \in G$ of order p. Consider $\langle a \rangle$, this has exactly p elements, then there must exist $b \in G \backslash \langle a \rangle$. There are now two choices, $|b| = 2$ or p. In the second case, note that due to **Proposition 7.51**, we can conclude that the $|\langle a \rangle \langle b \rangle| = p^2$ but since $\langle a \rangle \langle b \rangle \subseteq G$, we have a contradiction.
Thus, b must have order 2. Consider now the set $\langle a \rangle \langle b \rangle$, this must have cardinality $2p$, but since it is a subset of G, it must be exactly equal to G. Note now that $ab \in G$ and since $ab \notin \langle a \rangle$, ab must have order 2 or, equivalently $ab = ba^{-1}$. This now helps us uniquely determine the multiplication table for the group G. Thus, we can conclude that all non cyclic groups of order $2p$ are isomorphic to one another. But, since D_p is a non-cyclic group of order $2p$, we can conclude that G is isomorphic to D_p. ■

Definition 7.53 (Stabilizer of a Point). Let G be a group of permutations of a set S. For each i in S, let $\operatorname{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\operatorname{stab}_G(i)$ the *stabilizer of i in G.*

In other words, $\operatorname{stab}_G(i)$ is the set of permutations which fix i. Notice trivially, that $\operatorname{stab}_G(i)$ forms a subgroup of G.

Definition 7.54 (Orbit of a Point). Let G be a group of permutations of a set S. For each i in S, let $\operatorname{orb}_G(i) = \{\phi(i) \mid \phi \in G\}$. The set $\operatorname{orb}_G(i)$ is a subset of S called the *orbit of i under G*. We use $|\operatorname{orb}_G(i)|$ to denote the number of elements in $\operatorname{orb}_G(i)$.

In short, these are the set of images of i under all the possible permutations in G.

Theorem 7.55 (Orbit-Stabilizer-Theorem). Let G be a finite group of permutations of a set S. Then, for any i from S, $|G| = |\operatorname{orb}_G(i)||\operatorname{stab}_G(i)|$.

Proof. Let $j \in \operatorname{orb}_G(i)$. Let $\sigma(j)$ denote the permutation which simply swaps i with j. Consider now the right coset $\operatorname{stab}_G(i)\sigma(j)$ this is obviously a subset of all permutations in G which ■

Chapter 8

External Direct Products

Definition 8.56 (External Direct Product). Let $G_1, G_2, \cdots, G_n$ be a finite collection of groups. The *external direct product* of $G_1, G_2, \cdots, G_n$, written as $G_1 \oplus G_2 \oplus \cdots \oplus G_n$, is the set of all n-tuples for which the i-th component is an element of G_i and the operation is componentwise.

In symbols, we can write

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{(g_1, g_2, \cdots, g_n) \mid g_i \in G_i\}$$

Proposition 8.57. $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ is a group.

The proof is elementary and is left as an exercise for the reader.

Proposition 8.58. The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols,

$$|(g_1, g_2, \cdots, g_n)| = \operatorname{lcm}\{|g_1|, |g_2|, \cdots, |g_n|\}$$

Proof. Let e_i denote the identity element of the group G_i. Say the order of $(g_1, g_2, \cdots, g_n)$ is M. It is then trivial to show that $|g_i|$ must divide M for all permissible i. And in that case, the smallest positive value of M satisfying the same is defined as $\operatorname{lcm}\{|g_1|, |g_2|, \cdots, |g_n|\}$. ■

Proposition 8.59. Let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

Proof. Let $G = \langle g \rangle$ nad $H = \langle h \rangle$. If $|G|$ and $|H|$ are coprime, then we claim that $G \oplus H = \langle (g, h) \rangle$. Indeed, consider an element of $G \oplus H$ of the form (g^a, h^b). We would like to find a suitable x such that $(g^x, h^x) = (g^a, h^b)$. But, this is equivalent to

$$\begin{aligned} x &\equiv a \pmod{|G|} \\ x &\equiv b \pmod{|H|} \end{aligned}$$

Since $|G|$ and $|H|$ are coprime, we are done due to the Chinese Remainder Theorem. On the other hand, suppose $G \oplus H$ is cyclic and let $\gcd(|G|, |H|) = d$ and (g, h) be the generator for the same. Then, $(g, h)^{|G||H|/d} = (e, e)$ and hence

$$|G||H| = |(g, h)| \leq |G||H|/d$$

forcing $d = 1$ and $|G|$ and $|H|$ to be coprime. ■

Using simple induction, one can show

Corollary 8.11. An external direct product $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ of a finite number of finite cyclic groups is cyclic if and only if $|G_i|$ and $|G_j|$ are coprime whenever $i \neq j$.

Proposition 8.60. Suppose s and t are relatively prime. Then $U(st)$ is isomorphic to the external direct product of $U(s)$ and $U(t)$. That is, $U(st) \cong U(s) \oplus U(t)$.

Proof. Consider the mapping $\phi : U(st) \mapsto U(s) \oplus U(t)$ which is given by $x \overset{\phi}{\mapsto} \left(s\left\{\frac{x}{s}\right\}, t\left\{\frac{x}{t}\right\}\right)$. The map is surjective due to the Chinese Remainder Theorem. As for injectivity, suppose the images of $x, y \in U(st)$ are the same, then $s \mid x - y$ and $t \mid x - y$, or equivalently, $st \mid x - y$. But since both x and y are strictly smallar than st, this is possible if and only if $x = y$ and we have proved injectivity. Thus ϕ is bijective and is an isomorphism. ■

Chapter 9

Normal Subgroups and Factor Groups

Definition 9.61 (Normal Subgroups). A subgroup H of a group G is called a *normal subgroup* of G if $aH = Ha$ for all $a \in G$. We denote this by $H \lhd G$.

As an example, note trivially that every subgroup of an Abelian Group is normal.
The center of a group G, denoted by $Z(G)$ forms a normal subgroup of G. (Recall that we showed that $Z(G)$ is a subgroup in one of the previous chapters.)

Proposition 9.62. A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.

Proof. Suppose H is normal. Then, for all $x \in G$, and $h \in H$, there exists $h' \in H$ such that $xh = h'x$, or equivalently, $xhx^{-1} = h'$. Note that the map is injective and hence $xHx^{-1} \subseteq H$ for all $x \in G$.
Suppose now that $xHx^{-1} \subseteq H$ for all $x \in G$. Let $x = a$ for some $a \in G$, then $aHa^{-1} \subseteq H$, that is, $aH \subseteq Ha$. Taking now $x = a^{-1}$, we note that $a^{-1}Ha \subseteq H$, that is, $Ha \subseteq aH$. Thus, $aH = Ha$ for all $a \in G$ ■

Following here, we look at what are known as *Quotient Groups* or sometimes as *Factor Groups*[1]

Theorem 9.63 (Holder, 1889). Let G be a group and let H be a normal subgroup of G. The set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = (ab)H$.

Proof. First, we must show that the operation is well defined, that is, it is a valid function from $G/H \times G/H \to G/H$. Suppose that we have $a, a', b, b' \in G$ such that $aH = a'H$ and $bH = b'H$. We shall show that $(ab)H = (a'b')H$. According to the assumption, there exist $h_1, h_2 \in H$ such that $a' = ah_1$ and $b' = bh_2$. Then,

$$a'b'H = ah_1bh_2H = ah_1bH = ah_1Hb = aHb = abH$$

[1]Gallian likes to call them this for some reason.

Obviously, the identity element in G/H is $eH = H$. The inverse of an element aH is given obviously by $a^{-1}H$. Finally, we need to show associativity.

$$aH(bHcH) = aH((bc)H) = a(bc)H = (ab)cH = (ab)HcH = (aHbH)cH$$

This completes the proof that G/H is a group. ■

Theorem 9.64 (G/Z Theorem). Let G be a group and $Z(G)$ be its center. If the Quotient group $G/Z(G)$ is cyclic, then G is Abelian.

Proof. Let the generator for the Quotient group be $aZ(G)$ for some $a \in G$. Then, for all $g \in G$, there exists an index i such that $gZ(G) = a^iZ(G)$ and thus, there exists $z \in Z(G)$ such that $g = a^iz$. Then, for $g' \in G$, there exists $z' \in Z(G)$ such that $g' = a^jz'$ for some index j. Then,

$$gg' = a^iza^jz' = a^ia^jzz' = a^ja^iz'z = a^jz'a^iz = g'g$$

This completes the proof. ■

Proposition 9.65. Let G be a group. Then $G/Z(G) \cong \mathrm{Inn}(G)$.

Proof. For each $g \in G$, let ϕ_g denote the innermorphism induced by g. We shall show that the mapping $T : G/Z(G) \to \mathrm{Inn}(G)$ given by $gZ(G) \mapsto \phi_g$ is an isomorphism. First, we shall show that T is a well defined function. Asusme that $T(gZ(G)) = T(hZ(G))$. Then, there exists $z \in Z(G)$ such that $h = gz = zg$. Then the mapping

$$\phi_h(x) = (gz)x(z^{-1}g^{-1}) = gzz^{-1}xg^{-1} = \phi_g(x)$$

This shows that $\phi_g = \phi_h$.
It is clear that T is an onto function. We shall now show injectivity. Suppose $\phi_g = \phi_h$, then, for all $x \in G$, we have $gxg^{-1} = hxh^{-1}$. Equivalently, $(h^{-1}g)x = x(h^{-1}g)$, that means, $h^{-1}g = z$ for some $z \in G$, or $g = hz$. Now it is trivial to see that $gZ(G) = hZ(G)$. This completes the proof of injectivity. Note now that $T(ghZ(G)) = \phi_{gh} = \phi_g\phi_h = T(gZ(G))T(h(Z(G)))$. Now, we finally have that T is an isomorphism. ■

Theorem 9.66 (Cauchy's Theorem for Abelian Groups). Let G be a finite Abelian group. Let p be a prime dividing the order of G. Then, G has an element of order p.

Proof. We shall prove the statement by induction on $|G|$. The base case $|G| = 2$ is trivial, due to Lagrange's Theorem. Now note that there must exist an element in G with prime order. Since, there must exist an element $x \in G$ with order m which is divisible by some prime q, then the element $z = x^{m/q} \in G$ must have order q.
If $q = p$, then we are done. If not, consider now the Quotient group $\overline{G} = G/\langle z\rangle$. Due to Lagrange's Theorem, $|\overline{G}| = |G|/q$. Note that $\overline{G}$ must be Abelian and by induction, there must exist an element $y\langle z\rangle$ having order p. Thus, $y^p\langle z\rangle = \langle z\rangle$. Hence, $y^p \in \langle z\rangle$. If $y^p = e$, we are done. Else, y^p has order q and thus y^q has order p and this completes the proof. ■

Definition 9.67 (Internal Direct Product). We say that G is the *internal direct product* of H and K and write $G = H \times K$ if H and K are normal subgroups of G and

$$G = HK \qquad \text{and} \qquad H \cap K = \{e\}$$

Recall the definition that

$$HK = \{hk \mid h \in H, k \in K\}$$

We can extend the above definition further, as follows

Definition 9.68 (Extended Internal Direct Product). Let $H_1, H_2, \cdots, H_n$ be a finite collection of normal subgroups of G. We say that G is *internal direct product* of $H_1, H_2, \cdots, H_n$ and write $G = H_1 \times H_2 \times \cdots \times H_n$ if

- $G = H_1 H_2 \cdots H_n = \{h_1 h_2 \cdots h_n \mid h_i \in H_i\}$
- $H_1 H_2 \cdots H_i \cap H_{i+1} = \{e\}$ for $i = 1, 2, \cdots, n-1$.

Proposition 9.69. If G is the internal direct product of a finite number of subgroups $H_1, H_2, \cdots, H_n$, then G is isomorphic to the external direct product of $H_1, H_2, \cdots, H_n$. That is

$$H_1 \times H_2 \times \cdots \times H_n \cong H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

Before we prove the above, we shall prove the following lemma:

Lemma 9.70. Let G be a group which is the internal direct product of $H_1 \times H_2 \times \cdots \times H_n$. Then, for every element $g \in G$, there exist unique $h_i \in H_i$ for each permissible i such that $g = \prod_{i=1}^{n} h_i$. Further, the elements from different H_i's commute with one another.

Proof. Assume FTSOC, there exist two sets h_i and h_i' satisfying the requirements. Then,

$$h_1 h_2 \cdots h_{n-1} h_n = h_1' h_2' \cdots h_{n-1}' h_n'$$

Let us now denote $h_1 h_2 \cdots h_{n-1}$ as $x \in H_1 \times H_2 \times \cdots \times H_{n-1}$ and its counterpart by x'. Then, we have that $xh_n = x'h_n'$, and thus, $xx'^{-1} = h_n' h_n^{-1}$ but this means that $h_n = h_n'$. Now, working downwards from here, we have the desired conclusion.
First, we note that any pair of distinct H_i's only intersect at e. Let us now take h_1 and h_2 from H_1 and H_2 respectively. We shall show that they commute. (Note that we can do this without loss of generality since we showed that each of the H_i's mutually intersect only at e). We know, since H_1 and H_2 are normal that $h_1 H_2 = H_2 h_1$ and thus, there exists $y \in H_2$ such that $h_1 h_2 = y h_2$. Similarly, we know that $H_1 h_2 = h_2 H_1$ and thus, there exists $x \in H_1$ such that $h_1 h_2 = h_2 x$. This implies that $h_2 x = y h_1$, or $y^{-1} h_2 = h_1 x^{-1}$. But due to the condition on the H_i's they can intersect only at e and thus $x = h_1$ and $y = h_2$. ■

Proof of the Proposition. This now follows immediately from the above lemma that every element in G has a unique representation as a product of h_i's from each H_i. Then, we simply use the isomorphism

$$\phi(h_1 h_2 \cdots h_n) = (h_1, h_2, \cdots, h_n)$$

Note that due to the commutativity of the h_i's, this is actually an isomorphism. This completes the proof. ■

Proposition 9.71. Every group G of order p^2, where p is a prime, is isomorphic to Z_{p^2} or $Z_p \oplus Z_p$.

Proof. ■

We then instantly have the following corollary

Corollary 9.12. Every group of order p^2 is Abelian.

Chapter 10

Group Homomorphisms

Definition 10.72 (Homomorphism). A *homomorphism* $\phi : G \to \overline{G}$ is a mapping from G into $\overline{G}$ that preserves the group operation; that is, $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Definition 10.73 (Kernel). The *kernel* of a homomorphism $\phi : G \to \overline{G}$ is the set

$$\mathrm{Ker}(\phi) = \{x \in G \mid \phi(x) = e_{\overline{G}}\}$$

In simple words, the concept of a *homomorphism* generalizes the concept of an *isomorphism* to general functions that preserve group operations.

Proposition 10.74. Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let $g \in G$. Then

1. ϕ carries the identity of G to the identity of $\overline{G}$
2. $\phi(g^n) = \phi(g)^n$ for all $n \in \mathbb{Z}$
3. If $|g|$ is finite, then $|\phi(g)|$ is finite and divides $|g|$
4. $\mathrm{Ker}\,\phi$ is a subgroup of G.
5. $\phi(a) = \phi(b)$ if and only if $a \,\mathrm{Ker}\,\phi = b \,\mathrm{Ker}\,\phi$
6. If $\phi(g) = g'$, then $\phi^{-1}(g) = \{x \in G \mid \phi(x) = g'\} = g \,\mathrm{Ker}\,\phi$.

Proof.

1. Trivial
2. Trivial
3. Using (1) and (2), we have

$$e_{\overline{G}} = \phi(e_G) = \phi(g^{|g|}) = \phi(g)^{|g|}$$

the conclusion now follows.

4. It is easy to see that $\operatorname{Ker}\phi$ must be closed. Further, the group operation is associativer on $\operatorname{Ker}\phi$ since it is a subset of G. Due to (1), $e_G \in \operatorname{Ker}\phi$. Finally, note that if $\phi(a) = e_{\overline{G}}$, we must have $\phi(a^{-1}) = e_{\overline{G}}$. This shows that $\operatorname{Ker}\phi$ is a subgroup of G.

5. Suppose $\phi(a) = \phi(b)$. Then, let $x \in \operatorname{Ker}\phi$. We have

$$\phi(b^{-1}ax) = \phi(b^{-1})\phi(a)\phi(x) = e$$

Thus $b^{-1}ax \in \operatorname{Ker}\phi$ or $ax \in b\operatorname{Ker}\phi$ for all $x \in \operatorname{Ker}\phi$ which is equivalent to $a\operatorname{Ker}\phi \subseteq b\operatorname{Ker}\phi$. Similarly, one can obtain $b\operatorname{Ker}\phi \subseteq a\operatorname{Ker}\phi$.

6. Follows from (5).

■

Proposition 10.75. Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let H be a subgroup of G. Then

1. $\phi(H) = \{\phi(h) \mid h \in H\}$ is a subgroup of $\overline{G}$
2. If H is cyclic, then $\phi(H)$ is cyclic
3. If H is Abelian, then $\phi(H)$ is Abelian
4. If H is normal in G, then $\phi(H)$ is normal in $\phi(G)$
5. If $|\operatorname{Ker}\phi| = n$, then ϕ is an n-to-1 mapping from G onto $\phi(G)$.
6. If $|H| = n$, then $|\phi(H)|$ divides n.
7. If $\overline{K}$ is a subgroup of $\overline{G}$, then $\phi^{-1}(\overline{K})$ is a subgroup of G.
8. If $\overline{K}$ is a normal subgroup of $\overline{G}$, then $\phi^{-1}(\overline{K})$ is a normal subgroup of G.
9. If ϕ is onto and $\operatorname{Ker}\phi = \{e\}$, then ϕ is an isomorphism from G to $\overline{G}$.

Proof.

1. This is a routine proof
2. Routine, again
3. Trivial.
4. Let $h \in H$ and $g \in G$. Then, since H is normal in G, $ghg^{-1} \in H$. Thus, $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1} \in \phi(H)$. This completes the proof.
5. Trivial due to Property 6 of **Proposition 10.74**
6. Let ϕ_H denote the restriction of ϕ to H, that is, $\phi_H : H \to \phi(H)$. Then, according to the previous property, we trivially have that $|\phi(H)||\operatorname{Ker}\phi| = |H|$
7. Trivial using the first subgroup test which was discussed in **Chapter 3**.

8. Trivial using **Proposition 9.62**.

9. A consequence of (5).

■

Corollary 10.13. Let ϕ be a group homomorphism from G to $\overline{G}$. Then $\operatorname{Ker}\phi$ is a normal subgroup of G.

Theorem 10.76 (Jordan, 1870). Let ϕ be a group homomorphism from G to $\overline{G}$. Then, the mapping from $G/\operatorname{Ker}\phi$ to $\phi(G)$ given by $g\operatorname{Ker}\phi \mapsto \phi(g)$ is an isomorphism. In symbols, $G/\operatorname{Ker}\phi \cong \phi(G)$.

Proof. We shall use ψ to denote the mapping $g\operatorname{Ker}\phi \mapsto \phi(g)$. The fact that ψ is injective follows directly from Property (6) of **Proposition 10.74**. To show that ψ is operation preserving, note that

$$\psi(xy\operatorname{Ker}\phi) = \phi(xy) = \phi(x)\phi(y) = \psi(x\operatorname{Ker}\phi)\psi(y\operatorname{Ker}\phi)$$

The surjectivity is trivial and hence, we are done. ■

Proposition 10.77. Every normal subgroup of a group G is the kernel of a homomorphism of G. In particular, a normal subgroup N is the kernel of the mapping $\psi : G \to G/N$ which takes $g \mapsto gN$.

Proof. Let $x \in N$. Note trivially that $xN = N$, since N must be closed under multiplication by x and the multiplication is invertible. Say there exists $y \in G$ such that $\psi(y) = N$. We had already established that $e \in N$ and hence $ye = y \in N$ which completes the proof. ■

Chapter 11

Fundamental Theorem of Finite Abelian Groups

Theorem 11.78 (Fundamental Theorem of Finite Abelian Groups). Every finite Abelian group is a direct product of cyclic groups of prime-power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.

The proof of the above theorem requires the following four lemmas:

Lemma 11.79. Let G be a finite Abelian group of order $p^n m$ where p is a prime that does not divide m. Then, $G = H \times K$, where $H = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$. Moreover, $|H| = p^n$.

Proof. Since, we are given that $p \nmid m$, we know that $\gcd(m, p^n) = 1$, and due to Bézout's Lemma, there exist integers s and t such that $sm + tp^n = 1$. Thus, for any $x \in G$, we have

$$x = x^1 = x^{sm+tp^n} = x^{sm}x^{tp^n}$$

Now, due to Lagrange's theorem, $x^{sm} \in H$ and $x^{tp^n} \in K$. Thus, $G = HK$. Let $z \in H \cap K$, then, we know that $|z|$ divides $\gcd(m, p^n)$ or equivalently, $|z| = 1$ and thus, $z = e$. And hence, we can write $G = H \times K$.
As for the second part, we shall use **Proposition 7.51**, which immediately gives the desired conclusion. ∎

Lemma 11.80. Let G be an Abelian group of prime-power order and let a be an element of maximum order in G. Then G can be written in the form $\langle a \rangle \times K$.

Proof. Say $|G| = p^n$ where p is a prime. We shall induct on n. When $n = 1$, $G = \langle a \rangle \times \langle e \rangle$. Now, assume the hypothesis is true for all $k < n$. Among all the elements of G, choose the element of maximum order p^m. Then, we know that $g^{p^m} = e$ for all $g \in G$. We may now assume that $G \neq \langle a \rangle$, since this case is trivial. Let $b \in G \backslash \langle a \rangle$ such that $|b|$ is minimum. We further know that $|b^p| < |b|$ and hence $b^p \in \langle a \rangle$. Say $b^p = a^i$. Then, we can write

$$e = b^{p^m} = a^{ip^{m-1}}$$

and thus, $p \mid i$. Let $i = pj$. Then $b^p = a^{pj}$, or, $(a^{-j}b)^p = e$ (since G is abelian). Let $c = a^{-j}b$. Obviously $c \notin \langle a \rangle$. Since b was chosen to have minimum order, the order of b must be p, else c would have smaller order than b, an obvious contradiction. Now, if $\langle a \rangle \cap \langle b \rangle \neq \{e\}$ then there would exist an element z in both which generates $\langle b \rangle$ implying $b \in \langle a \rangle$ which is absurd.
Now consider the factor group $\overline{G} = G/\langle b \rangle$. Let $\overline{x}$ denote the coset $x\langle b \rangle$ in $\overline{G}$. If $|\overline{a}| < |a| = p^m$, then $\overline{a}^{p^{m-1}} = \overline{e}$. This means that $a^{p^{m-1}} \in \langle a \rangle \cap \langle b \rangle = \{e\}$ which is a contradiction. Thus, $|\overline{a}| = |a| = p^m$ and thus, $\overline{a}$ is the element with maximum order in $\overline{G}$ and hence, $\overline{G} = \langle \overline{a} \rangle \times \overline{K}$. Let K be the pullback of $\overline{K}$ under the natural homomorphism from G to $\overline{G}$. We claim that $\langle a \rangle \cap K = \{e\}$. If $x \in \langle a \rangle \cap K$, then $\overline{x} \in \langle \overline{a} \rangle \cap \overline{K} = \{\overline{e}\} = \langle b \rangle$ and $x \in \langle a \rangle \cap \langle b \rangle = \{e\}$. It is now trivial to conclude that $G = \langle a \rangle \times K$. ■

Lemma 11.81. A finite Abelian group of prime-power order is an internal direct product of cyclic groups.

Proof. This follows from the previous lemma. That is, we know $G = \langle a \rangle \times K$ then we can write $K = \langle a' \rangle \times K'$ and so on, and this cannot go on indefinitely and hence we are done. ■

Lemma 11.82. Suppose that G is a finite Abelian group of prime-power order. If $G = H_1 \times H_2 \times \cdots \times H_m$ and $G = K_1 \times K_2 \times \cdots \times K_n$, then $m = n$ and $|H_i| = |K_i|$ for all i.

Proof. We shall proceed by induction on $|G|$. The base case when $|G| = p$ is true. Now suppose that the statement is true for all Abelian groups of order less than $|G|$. For any abelian group L, the set $L^p = \{x^p \mid x \in L\}$ is a subgroup of L. Now, it follows that

$$G^p = H_1^p \times H_2^p \times \cdots \times H_{m'}^p = K_1^p \times K_2^p \times \cdots \times K_{n'}^p$$

where m' is the maximum index i such that $|H_i| > p$ and similarly define n'. Now, we know that $|G^p| < |G|$ and hence, $m' = n'$ by induction. Finally, note that we have

$$|G| = |H_1||H_2| \cdots |H_{m'}|p^{m-m'} = |K_1||K_2| \cdots |K_{n'}|p^{n-n'}$$

This now obviously implies that $m = n$ and we are done. ■

Finally, coming to the proof of the original theorem, from the first lemma in this chapter, we can write

$$G = H_1 \times H_2 \times \cdots \times H_n$$

where each H_i have prime-power order. And, finally the number of groups, namely n is uniquely determined due to the last lemma of the chapter. Thus, we have successfully proved the **Fundamental Theorem of Finite Abelian Groups**.

Part II

Rings

Chapter 12

Introduction to Rings

While studying groups, we talked about sets which were associated with one binary operation. In the case of rings, we shall talk about sets which are associated with two binary operations.

Definition 12.83 (Ring). A *ring* R is a set with two binary operations, addition (denoted by $a+b$) and multiplication (denoted by ab), such that for all $a, b, c \in R$:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. There is an additive identity 0. That is, there is an element 0 in R such that $a + 0 = a = 0 + a$ for all $a \in R$
4. There is an element $-a \in R$ such that $a + (-a) = 0$
5. $a(bc) = (ab)c$
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

Definition 12.84 (Commutative Ring). A ring is said to be *commutative* when multiplication is commutative on the elements of the ring.

Definition 12.85 (Unity). A unity (or identity, represented by 1) in a ring is a non-zero element that is an identity under multiplication. That is, for all $a \in R$, $a1 = 1a = a$. A non-zero element of a commutative ring with unity need not have a multiplicative inverse. When it does, we say that it is a unit of the ring. Thus, a is a unit if a^{-1} exists and vice versa.

Following are some examples of rings:

- The set $\mathbb{Z}$ of integers under ordinary addition and multiplication is a commutative ring with unity 1. The set of units is $\{-1, 1\}$.
- The set $\mathbb{Z}_n$ under addition and multiplication modulo n is a commutative ring with unity 1. The set of units is $U(n)$.

- The set $\mathbb{Z}[x]$ of integer polynomials under ordinary addition and multiplication is a commutative ring with unity $f \equiv 1$.

Following are some properties of rings:

Proposition 12.86. Let R be a ring and $a, b, c \in R$. Then,

1. 0 is unique in a ring.
2. $a0 = 0a = 0$
3. $a(-b) = (-a)b = -(ab)$
4. $(-a)(-b) = ab$
5. $a(b - c) = ab - ac$

Furthermore, if R has a unity element 1, then

5. $(-1)a = -a$
6. $(-1)(-1) = 1$

Proof.

1. Say there are two zeroes, 0_1 and 0_2. Then $0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$.
2. $a0 = a(0 + 0) = a0 + a0$. Thus implying that $a0 = 0$. Proving for $0a$ is similar.
3. Note that $0 = a(b - b) = ab + a(-b)$. This implies that $a(-b) = -(ab)$. Proving for $(-a)b$ is similar.
4. From the previous result, we know that $(-a)(-b) = (-(-a))b = ab$
5. Trivial
6. Using property 4, we have $(-1)a = 1(-a) = -a$
7. Follows from the above.

■

Proposition 12.87. If a ring R has a unity, it is unique. If a ring element has a multiplicative inverse, it is unique.

Proof. Follows the same proof method used while proving a similar statement in the case of a group. ■

Definition 12.88 (Subring). A subset S of a ring R is a *subring* of R if S is itself a ring with the operations of R.

Proposition 12.89 (Subring Test). A non-empty subset S of a ring R is a subring if S is closed under substraction and multiplication- that is, for all $a, b \in S$, $a - b, ab \in S$.

Proof. For all $a \in S$, $0 = a - a \in S$ and hence, $-a = 0 - a \in S$. Thus, for all $a, b \in S$, $a + b = a - (-b) \in S$ implying that S is closed under addition and multiplication. Also, we have found the additive identity 0 and shown the existence of an additive inverse for all elements of S. Finally, since $S \subseteq R$, multiplication will be associative on S and would also be left and right distributive. This implies that S is a subring of R. ■

Chapter 13

Integral Domains

Definition 13.90 (Zero-Divisor). A *zero-divisor* is a non-zero element a of a commutative ring R such that there is a non-zero element $b \in R$ with $ab = 0$(which obviously implies $ba = 0$).

Definition 13.91 (Integral Domain). An *integral domain* is a commutative ring with unity and no zero-divisors.

Some examples of integral domains are

- The ring of integers $\mathbb{Z}$ under ordinary addition and multiplication.
- The ring of Gaussian Integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ under ordinary addition and multiplication.
- The ring $\mathbb{Z}[x]$ of integral polynomials.

Proposition 13.92 (Cancellation). Let $a, b, c \in R$ which is an integral domain. If $a \neq 0$ and $ab = ac$, then $b = c$.

Proof. Using distributivity of multiplication, we can write $a(b - c) = 0$. But, since R is an integral domain, a cannot be a zero-divisor and hence $b - c = 0$, or equivalently $b = c$. ■

Definition 13.93. A *field* is a commutative ring with unity in which every non-zero element is a unit.

Proposition 13.94. A finite integral domain is a field.

Proof. Let F be a finite integral domain and $a \in F$ be a non-zero element. Consider the set

$$S = \{a, a^2, \cdots, a^{|F|+1}\}$$

Since F is closed under multiplication, there must exist indices i and j with $i > j$ such that $a^i = a^j$, or $a^{i-j} = 1$, due to the cancellation property. This immediately implies that a is a unit and hence F is a field. ■

Corollary 13.14. For every prime p, $\mathbb{Z}_p$ is a field.

For the simplicity of notation, for any positive integer n, we denote

$$n \cdot x \overset{\text{def}}{:=} \underbrace{x + x + \cdots + x}_{n \text{ times}}$$

Definition 13.95 (Characteristic of a Ring). The *characteristic* of a ring R is the least positive integer n such that $n \cdot x = 0$ for all $x \in R$. If no such integer exists, we say that R has characteristic 0. The characteristic of R is denoted by $\operatorname{char} R$.

Proposition 13.96. Let R be a ring with unity 1. If 1 has infinite order under addition, then $\operatorname{char} R = 0$. If 1 has order n under addition, then the characteristic of R is n.

Proof. The contrapositive of the first statement is obviously true, and hence the statement must be true as well. As for the second statement, note that for all $a \in R$,

$$\begin{aligned} n \cdot a &= \underbrace{a + a + \cdots + a}_{n \text{ times}} \\ &= \underbrace{1 \cdot a + 1 \cdot a + \cdots + 1 \cdot a}_{n \text{ times}} \\ &= (\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}})a \\ &= (n \cdot 1)a = 0a = 0 \end{aligned}$$

Thus, $\operatorname{char} R \leq n$. But since the order of 1 is exactly n, $\operatorname{char} R = n$. ■

Proposition 13.97. The characteristic of an integral domain is either 0 or a prime.

Proof. Using the previous result, we need only study the order of 1. Assume the order of 1 is given by st, where $s, t \in \mathbb{N}$. Then

$$0 = (st) \cdot 1 = s \cdot (t \cdot 1) = (s \cdot 1)(t \cdot 1)$$

But since we are working in an integral domain, we can conclude that either $s \cdot 1$ or $t \cdot 1$ must be 0. Now, if both of $s, t > 1$, then we have a contradiction to the minimality of the order st. Thus, either s or t must be 1 and st must be prime. ■

Chapter 14

Ideals and Factor Rings

Definition 14.98 (Ideal). A subring A of a ring R is called a (two-sided) *ideal* of R if for every $r \in R$ and every $a \in A$, both ra and ar are in A. That is, A is an ideal if $rA, Ar \subseteq A$ for all $r \in R$.
A is called a *proper* ideal of R if A is a proper subset of A.

It is obvious that every subring of a commutative ring is an ideal.

Proposition 14.99 (Ideal Test). A non-empty subset A of a ring R is an ideal of R if

1. $a - b \in A$ whenever $a, b \in A$
2. ra and ar are in A whenever $a \in A$ and $r \in R$.

Proof. The second proposition implies that A is closed under multiplication. Combined with the first, we can conclude by the "subring test" which we proved in the previous chapter that A is a subring of R. Finally, along with the second condition, A is an ideal of R. ■

Note that for any ring R, the set $\{0_R\}$ forms an ideal of R and is termed as the *trivial* ideal.

Definition 14.100 (Quotient Ring). Let R be a ring and let A be an ideal of R. The Quotient Ring R/A is defined as

$$R/A = \{r + A \mid r \in R\}$$

Proposition 14.101 (Existence of Factor Rings). Let R be a ring and let A be a subring of R. The set of cosets $\{r + A \mid r \in R\}$ is a ring under the operations $(s + A) + (t + A) = s + t + A$ and $(s + A)(t + A) = st + A$ if and only if A is an ideal of R.

Proof. One can trivially note that the set of cosets forms a group under addition. Furthermore, the above defined multiplication is a binary operation on the set of cosets furthermore, it is distributive over addition. We only need to show that multiplicaiton is

well-defined if and only if A is an ideal of R.
Suppose that A is an ideal. Then, suppose that $s+A = s'+A$ and $t+A = t'+A$ we only need to show hat $st + A = s't' + A$. According to the definition, $s' = s + a$ and $t' = t + b$ for some $a, b \in A$. Then

$$s't' = (s + a)(t + b) = st + at + sb + ab$$

And hence,

$$s't' + A = st + at + sb + ab + A = st + at + sb + A$$

now since $at + sb \in A$ we can conclude that multiplication is well defined.
Suppose now that A is a subring of R which is not an ideal but multiplication is we ll defined. Then, there exist $a \in A$ and $r \in R$ such that $ar \notin A$ but $ra \in A$. Now, since multiplication is well defined,

$$(0 + A)(r + A) = A$$

and hence, $ar \in A$ but this is absurd. Thus, A must be an ideal. ■

Definition 14.102 (Prime Ideal, Maximal Ideal). A *prime ideal* A of a commutative ring R is a proper ideal of R such that $a, b \in R$ and $ab \in A$ imply $a \in A$ or $b \in A$. A *maximal* ideal of a commutative ring R is a *proper* ideal of R such that, whenever B is an ideal of R and $A \subseteq B \subseteq R$, then $B = A$ or $B = R$.

In simpler words, the Maximal Ideal is the proper ideal which contains all the proper ideals. The only ideal that is not contained in the Maximal Ideal is the ring R.

Proposition 14.103. Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is an integral domain if and only if A is a prime ideal.

Proof. Suppose R/A is an integral domain and $ab \in A$ for some $a, b \in R$. Then, according to definition,

$$(a + A)(b + A) = ab + A = A$$

But, since A is the zero element of R/A, we must have either $a + A = A$ or $b + A = A$, that is, either $a \in A$ or $b \in A$.
Since R is a commutative ring and A is an ideal, we conclude that R/A is a commutative ring containing unity. Suppose now that A is prime and $(a + A)(b + A) = 0 + A = A$. Then, $ab \in A$ but since A is prime, it means $a \in A$ or $b \in A$, that is, $a + A = A$ or $b + A = A$. In other words, either of $a + A$ or $b + A$ is the zero coset in R/A. This completes the proof. ■

Lemma 14.104. Let R be a commutative ring with unity and let A be an ideal of R containing unity. Then, $A = R$.

Proof. Trivial ■

Proposition 14.105. Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is a field if and only if A is maximal.

Proof. Suppose that R/A is a field and B is an ideal of R that properly contains A. Let $b \in B$ be such that $b \notin A$. Then $b + A$ is a non-zero element of R/A. Therefore, there exists $c \in R$ such that $(b + A)(c + A) = 1 + A$, which is the multiplicative identity of R/A. But that means, $1 - bc \in A$. So $1 = 1 - bc + bc \in B$. Thus, using the previous lemma, we have $B = R$, that is, A is maximal.
Now suppose that A is maximal and let $b \in R$ but $b \notin A$. We shall show that $b + A$ has a multiplicative inverse. Consider $B = \{br + a \mid r \in R, a \in A\}$. This is an ideal of R that properly contains A. Since A is maximal, we must have $B = R$. Thus, $1 \in B$, say, $1 = bc + a'$ where $a' \in A$, then

$$1 + A = bc + a' + A = bc + A = (b + A)(c + A)$$

this completes the proof. ∎

Chapter 15

Ring Homomorphisms

Definition 15.106. A *ring homomorphism* ϕ from a ring R to a ring S is a mapping from R to S that preserves the two ring operations; that is, for all $a, b \in R$,

$$\phi(a+b) = \phi(a) + \phi(b) \qquad \text{and} \qquad \phi(ab) = \phi(a)\phi(b)$$

Definition 15.107 (Isomorphism). A ring homomorphism that is both one-to-one and onto is called a ring isomorphism.

Proposition 15.108 (Properties of Homomorphisms). Let ϕ be a ring homomorphism from a ring R to a ring S. Let A be a subring of R and let B be an ideal of S.

1. For any $r \in R$ and any positive integer n, $\phi(n \cdot r) = n \cdot \phi(r)$ and $\phi(r^n) = \phi(r)^n$.
2. $\phi(A)$ is a subring of S.
3. If A is an ideal and ϕ is onto S, then $\phi(A)$ is an ideal.
4. $\phi^{-1}(B)$ is an ideal of R.
5. If R is commutative, then $\phi(r)$ is commutative.
6. If R has a unity 1, $S \neq \{0\}$, and ϕ is onto, then $\phi(1)$ is the unity of S.
7. ϕ is an isomorphism if and only if ϕ is onto and $\operatorname{Ker}\phi = \{r \in R \mid \phi(r) = 0\} = \{0\}$.
8. If ϕ is an isomorphism from R onto S, then ϕ^{-1} is an isomorphism from S onto R.

Proof. I shall not go into much details.

1. Trivial
2. Trivial
3. This follows trivially from the fact that any element in S can be written as $\phi(r)$ for some $r \in R$ and any element in $\phi(A)$ can be written as $\phi(a)$ for some $a \in A$.

4. Let $r \in R$. Then, for any $x \in \phi^{-1}(B)$, we know that $\phi(rx) = \phi(r)\phi(x) \in B$ and we have the desired conclusion.

5. Trivial

6. Since ϕ is onto, for any $s \in S$, there exists $r \in R$ such that $\phi(r) = s$. Then, $\phi(1)s = \phi(1r) = \phi(r) = s$.

7. If ϕ is an isomorphism, it is obvious that ϕ is onto and $\operatorname{Ker}\phi = \{0\}$. Suppose now that ϕ is onto and $\operatorname{Ker}\phi = \{0\}$. We shall now show that ϕ is one-one. Suppose there exist $a, b \in R$ such that $\phi(a) = \phi(b)$. Then, $\phi(a - b) = 0$ and the conclusion now follows. Note that we must first prove that $\phi(-a) = -\phi(a)$ but that is easy enough to be omitted.

8. Trivial

■

Proposition 15.109. Let ϕ be a ring homomorphism from a ring R to a ring S. Then $\operatorname{Ker}\phi$ is an ideal of R.

Proof. Let $a, b \in \operatorname{Ker}\phi$. Then, $\phi(ab) = \phi(a)\phi(b) = 0$ and thus $ab \in \operatorname{Ker}\phi$. Furthermore, $0 = \phi(a) = \phi(b + a - b) = \phi(b) + \phi(a - b)$ and thus, $a - b \in \operatorname{Ker}\phi$. Then, from the Subring Test, we know that $\operatorname{Ker}\phi$ is a subring of R. Let $r \in R$, and $a \in \operatorname{Ker}\phi$, then $\phi(rk) = \phi(r)\phi(k) = 0$ and thus $rk \in \operatorname{Ker}\phi$. Hence, we have the desired conclusion. ■

Proposition 15.110. Let ϕ be a ring homomorphism from R to S. Then the mapping from $R/\operatorname{Ker}\phi$ to $\phi(R)$ given by $r + \operatorname{Ker}\phi \mapsto \phi(r)$ is an isomorphism. In symbols, $R/\operatorname{Ker}\phi \cong \phi(R)$

Proof. We shall first show that the mapping is well defined. Say $r + \operatorname{Ker}\phi = s + \operatorname{Ker}\phi$. Then, there exist $p, q \in \operatorname{Ker}\phi$ such that $r + p = s + q$ and equivalently,

$$\phi(r) = \phi(r) + \phi(p) = \phi(r + p) = \phi(s + q) = \phi(s) + \phi(q) = \phi(s)$$

We now obviously have that the mapping is surjective. We only need to show that the mapping is injective. For this purpose, suppose $\phi(r) = \phi(s)$. Then, as seen in the proof of the previous theorem, $r - s \in \operatorname{Ker}\phi$ and thus,

$$r + \operatorname{Ker}\phi = s + (r - s) + \operatorname{Ker}\phi = s + \operatorname{Ker}\phi$$

This completes the proof. ■

Proposition 15.111. Every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, and ideal A is the kernel of the mapping $r \mapsto r + A$ from R to R/A.

Proof. Trivial ■

Proposition 15.112. Let R be a ring with unity 1. The mapping $\phi : \mathbb{Z} \to R$ given by $n \mapsto n \cdot 1$ is a ring homomorphism.

Proof. Let $m, n \in \mathbb{Z}$. Then $\phi(m+n) = (m+n) \cdot 1 = m \cdot 1 + n \cdot 1$. Furthermore, $\phi(mn) = mn \cdot 1 = (m \cdot 1)(n \cdot 1)$. ■

Corollary 15.15. If R is a ring with unity and the characteristic of R is $n > 0$, then R contains a subring isomorphic to $\mathbb{Z}_n$. If the characteristic of R is zero, then R contains a subring isomorphic to $\mathbb{Z}$.

Proof. Let 1 be the unity of R and let $S = \{k \cdot 1 \mid k \in \mathbb{Z}\}$. Due to the previous result, we know that the mapping $k \mapsto k \cdot 1$ is a homomorphism. Then, we can write $\mathbb{Z}/\operatorname{Ker}\phi \cong S$. BUt, clearly $\operatorname{Ker}\phi = \langle n \rangle$, where n is the additive order of 1 and thus is the characteristic of R. So, when R has characteristic n, $S \cong \mathbb{Z}/\langle n \rangle \cong \mathbb{Z}_n$. Thus we have the desired conclusion. ■

Theorem 15.113 (Steinitz, 1910). If F is a field of characteristic p, then F contains a subfield isomorphic to $\mathbb{Z}_p$. If F is a field of characteristic 0, then F contains a subfield isomorphic to the rational numbers.

Proof. Due to the above corollary, R contains a subring isomorphic to $\mathbb{Z}_p$ if F has characteristic p, whereas R has a subring S isomorphic to $\mathbb{Z}$ if F has characteristic 0. In the latter case, let

$$T = \{ab^{-1} \mid a, b \in S, b \neq 0\}$$

Then, T is isomorphic to the rationals. ■

Proposition 15.114. Let D be an integral domain. Then, there exists a field F (called the field of quotients of D) that contains a subring isomorphic to D.

Proof. Let $S = \{(a, b) \mid a, b \in D, b \neq 0\}$. We define an equivalence relation on S by $(a, b) \sim (c, d)$ if $ad = bc$. Now, let F be the set of equivalence classes of S under the relation $\sim$ and we represent the equivalence class containing (x, y) by x/y. We define addition and multiplication on F by

$$a/b + c/d = (ad + bc)/(bd) \qquad \text{and} \qquad a/b \cdot c/d = (ac)/(bd)$$

We shall first show that the two operations are well defined. Suppose that $a/b = a'/b'$ and $c/d = c'/d'$. So that $ab' = a'b$ and $cd' = c'd$. It then follows that

$$\begin{aligned}(ad + bc)b'd' &= adb'd' + bcb'd' = (ab')dd' + (cd')bb' \\ &= (a'b)dd' + (c'd)bb' = a'd'bd + b'c'bd \\ &= (a'd' + b'c')bd\end{aligned}$$

As for multiplication, we have

$$acb'd' = (ab')(cd') = (a'b)(c'd) = a'c'bd$$

It is now easily verified that F is a field. Consider the mapping ϕ which takes $x \mapsto x/1$. We note that this mapping is injective. Now, for any equivalence class a/b, consider the pair $(ab^{-1}, 1)$ it is obvious that $ab^{-1} \in D$ and hence the map is surjective and finally, it is an isomorphism. ■

Chapter 16

Polynomial Rings

Definition 16.115 (Polynomial Rings). Let R be a commutative ring. The set of formal symbols

$$R[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in R, n \in \mathbb{N}_0\}$$

is called the ring of polynomials over R in the indeterminate x.
Two elements

$$a_n x^n + \cdots + a_1 x + a_0$$

and

$$b_m x^m + \cdots + b_1 x + b_0$$

of $R[x]$ are considered equal if and only if $a_i = b_i$ for all non-negative integers i.

We now define the 'multiplication' and 'addition' associated with the ring

Definition 16.116. Let R be a commutative ring and let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + \cdots + b_1 x + b_0$$

be elements of $R[x]$. Then

$$f(x) + g(x) = (a_s + b_s)x^s + \cdots + (a_1 + b_1)x + a_0 + b_0$$

where s is the maximum of m and n. Also

$$f(x)g(x) = c_{m+n}x^{m+n} + \cdots + c_1 x + c_0$$

where

$$c_k = a_k b_0 + a_{k-1}b_1 + \cdots + a_0 b_k$$

Definition 16.117. Let $f(x) \in R[x]$. If

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, we say that $f(x)$ has *degree* n. The term a_n is called the *leading coefficient* of $f(x)$. Polynomials of the form $g(x) = a_0$ are called *constant* polynomials.

Proposition 16.118. If D is an integral domain, then $D[x]$ is an integral domain.

Proof. Since D is commutative, it is clear that $D[x]$ is commutative as well. Furthermore, the constant polynomial $e(x) = 1$ serves as the unity of the ring. We now only need to show that the ring doesn't have any zero divisors. Indeed, consider

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + \cdots + b_1 x + b_0$$

where $a_n \neq 0$ and $b_m \neq 0$. Then the polynomial $f(x)g(x)$ shall have the leading coefficient as $a_n b_m$ which must be non-zero since D is an integral domain. This completes the proof. ∎

Proposition 16.119. Let F be a field and let $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Proof. We first show the existence of $q(x)$ and $r(x)$. If $f(x) = 0$ or $\deg f(x) < \deg g(x)$, we simply set $q(x) = 0$ and $r(x) = f(x)$. We shall now prove the statement by strong induction on the degree of $f(x)$. Say $\deg f(x) = n > m = \deg g(x)$. Then, consider $f_1(x) = f(x) - a_n b_n^{-1} x^{n-m} g(x)$. Then $\deg f_1(x) < \deg f(x)$ and by the induction hypothesis, there exist $q_1(x)$ and $r_1(x)$ such that $f_1(x) = g(x)q_1(x) + r_1(x)$. But then, we trivially have that $f(x) = (a_n b_m^{-1} x^{n-m} + q_1(x))g(x) + r_1(x)$. Thus, we have established the existence of q and r for some f and g. We shall now uniqueness. Assume now that

$$f(x) = g(x)q(x) + r(x) \qquad \text{and} \qquad f(x) = g(x)\overline{q}(x) + \overline{r}(x)$$

Substracting the two, we obtain

$$g(x)\left(q(x) - \overline{q}(x)\right) = \overline{r}(x) - r(x)$$

Suppose that $\overline{r}(x) - r(x) \neq 0$, then it must have degree atleast that of $g(x)$ which is absurd. Hence $\overline{r}(x) = r(x)$ and $\overline{q}(x) = q(x)$. This completes the proof. ∎

Corollary 16.16. Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

Proof. Easy enough by induction on the degree of $f(x)$. ∎

Corollary 16.17. Let F be a field, $a \in F$ and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

Proposition 16.120. A polynomial of degree n over a field has at most n zeros, counting multiplicity.

Proof. We shall proceed by induction on n. The base case with $n = 0$ and $n = 1$ are trivial. Assume now that the statement holds true for all $k < n$. Let $f(x)$ be a polynomial with degree n. Let a be a root of f multiplicity k, that is to say $f(x) = (x-a)^k q(x)$ such that $q(a) \neq 0$. It is obvious that $\deg q = n - k < n$ and hence q can have atmost $n - k$ zeroes due to the inductive hypothesis and hence f must have at most $n - k + k = n$ zeroes. This completes the proof. ■

Definition 16.121. A *principal ideal domain* is an integral domain R in which every ideal has the form $\langle a \rangle = \{ra \mid r \in R\}$ for some $a \in R$.

Proposition 16.122. Let F be a field. Then $F[x]$ is a principal ideal domain.

Proof. Due to the first result in this chapter we know that $F[x]$ must be an integral domain. Let I be an ideal in $F[x]$. If $I = \{0\}$, then $I = \langle 0 \rangle$. If $I \neq \{0\}$, then among all the elements of I, choose g to have minimal degree. We shall show that $I = \langle g(x) \rangle$. Say $f(x) \in I$, then due to the division algorithm, we can write $f(x) = g(x)q(x) + r(x)$. Since I is an ideal, $g(x)q(x) \in I$ and hence $r(x) \in I$. But we know that either $r(x) = 0$ or $\deg r(x) < \deg g(x)$ in which case, the second statement would contratict the minimality of the degree of $g(x)$ and hence $r(x) = 0$ and this completes the proof. ■

Proposition 16.123. Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F[x]$. Then $I = \langle g(x) \rangle$ if and only if $g(x)$ is a non-zero polynomial of minimum degree in I.

Proof. Corollary of the proof of the previous result. ■

Chapter 17

Factorization of Polynomials

Definition 17.124. Let D be an integral domain. A polynomial $f(x)$ from $D[x]$ that is neither the zero polynomial nor a unit in $D[x]$ is said to be *irreducible* over D if, whenever $f(x)$ is expressed as a product $f(x) = g(x)h(x)$, with $g(x)$ and $h(x)$ in $D[x]$, then either $g(x)$ or $h(x)$ is a unit in $D[x]$. A non-zero, non-unit element of $D[x]$ that is not irreducible over D is called *reducible* over D.

Proposition 17.125. Let F be a field. If $f(x) \in F[x]$ and $\deg f(x)$ is either 2 or 3, then $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F.

Proof. Assume now that $f(x) = g(x)h(x)$ where neither g nor h are the unity in D. Then, it is trivial that either $\deg g$ or $\deg h$ is 1 and we have a root of the polynoimal in F. Conversly, suppose that $f(a) = 0$, where $a \in F$. Then by the Factor Theorem, we know that $x - a$ is a factor of $f(x)$, therefore, $f(x)$ is reducible over F. ■

Definition 17.126. The content of a non-zero polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where the a_i's are integers, is the greatest common divisor of the integers $a_n, a_{n-1}, \cdots, a_0$. A *primitive polynomial* is an element of $\mathbb{Z}[x]$ with content 1.

Lemma 17.127. Let p be a prime and $f(x), g(x) \in \mathbb{Z}_p[x]$ such that $f(x)g(x) = 0$. Then either $f(x) = 0$ or $g(x) = 0$.

Proof. Proof is rather easy, start with the leading coefficients of both f and g and work your way down. ■

Theorem 17.128 (Gauß). The product of two primitive polynomials is primitive.

Proof. Let $f(x)$ and $g(x)$ be primitive polynomials. Suppose $f(x)g(x)$ is not primitive and let p be a prime divisor of the content of $f(x)g(x)$. Then, working in $\mathbb{Z}_p[x]$, we have that $\overline{f}\overline{g} = 0$ where $\overline{f}$ and $\overline{g}$ are the residues of f and g in $\mathbb{Z}_p[x]$. But due to the previous lemma, we know that either f or g must be 0 and hence, either f or g must be non-primitive. This is a contradiction! ■

Theorem 17.129 (Gauß[a]). Let $f(x) \in \mathbb{Z}[x]$. If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.

[a]Gallian doesn't attribute this theorem to Gauß but I'm pretty sure it's colloquially known as Gauß's Lemma.

Proof. Suppose that $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are in $\mathbb{Q}[x]$. We my assume that $f(x)$ is primitive. LEt a be the least common multiple of the denominators of the coefficients of $g(x)$ and b be the least common multiple of the denominators of the coeffieicnts $h(x)$. Then $abf(x) = ag(x)\cdot bh(x)$ where $ag(x)$ and $bh(x)$ in $\mathbb{Z}[x]$. Let c_1 be the content of $ag(x)$ and c_2 be the content of $bh(x)$. Then we can write $ag(x) = c_1g_1(x)$ and $bh(x) = c_2h_1(x)$. Then $abf(x) = c_1c_2g_1(x)h_2(x)$. Since $f(x)$ is primitive, the content of $abf(x)$ is ab also since the product of two primitive polynomials is primitive, it follows that the content of $c_1c_2g_1(x)h_1(x)$ is ab. Thus $ab = c_1c_2$ and we have the desired conclusion. ■

Proposition 17.130 (Mod p Irreducibility Test). Let p be a prime and suppose that $f(x) \in \mathbb{Z}[x]$ with $\deg f(x) \geq 1$. Let $\overline{f}(x)$ be the polynomial in $\mathbb{Z}_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $\overline{f}(x)$ is irreducible over $\mathbb{Z}_p$ and $\deg \overline{f}(x) = \deg f(x)$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Proof. From the previous theorem, we konw that if $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$. Then, reducing modulo p, $\deg f(x) = \deg \overline{f}(x)$, we have $\deg \overline{g}(x) \leq \deg g(x) < \overline{f}(x)$ and $\deg \overline{h}(x) \leq \deg h(x) < \deg \overline{f}(x)$. But, $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, and this contradicts our assumption that $\overline{f}(x)$ is irreducible over $\mathbb{Z}_p$. ■

Theorem 17.131 (Eisentein's Criterion, 1850). Let

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

IF there is a prime p such that $p \nmid a_n$, $p \mid a_{n-1}, \cdots, a_0$ and $p^2 \nmid a_0$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Proof. If $f(x)$ is reducible over $\mathbb{Q}$, we know by Gauß's Theorem that there exist integer polynomials $g(x)$ and $h(x)$ such that $f(x) = g(x)h(x)$ and $\deg g, \deg h \geq 1$. Say $g(x) = g_rx^r + \cdots + g_0$ and $h(x) = h_sx^s + \cdots + h_0$. Since $p \mid a_0$, p must divide either g_0 or h_0. Say without loss of generality, $p \mid g_0$ and $p \nmid h_0$. Since $p \nmid g_i$ for all i, since $p \nmid a_n$, we can say that there exists an index t such that $p \nmid g_t$ but $p \mid g_i$ for all $i < t$. Using this, we compute the coefficient $a_t = g_th_0 + \cdots + g_0h_t$. But since $t \leq \deg g < n$, we know that $p \mid a_t$ which forces $p \mid g_th_0$ a contradiction to $p^2 \nmid a_0$. This completes the proof. ■

Corollary 17.18. For any prime p, the the p-th cyclotomic polynomials:

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Q}$.

Proof. We have

$$f(x) = \Phi_p(x+1) = \frac{(x+1)^p - 1}{(x+1)-1} = x^{p-1} + \binom{p}{1}x^{p-2} + \cdots + \binom{p}{p-1}$$

now it is trivial due to Eisentein's Criterion that $f(x)$ is irreducible. ■

Proposition 17.132. Let F be a field and let $p(x) \in F[x]$. Then $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F.

Proof. Suppose that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$. Clearly $p(x)$ is neither zero nor a unit in $F[x]$. Suppose now $p(x) = g(x)h(x)$. Then, $\langle p(x) \rangle \subseteq \langle g(x) \rangle \subseteq F[x]$. Thus, $\langle p(x) \rangle = \langle g(x) \rangle$ or $F[x] = \langle g(x) \rangle$. In the first case, we would have $\deg p(x) = \deg g(x)$ which would force $\deg h(x) = 0$ which is absurd. In the second case, $\deg g(x) = 0$ which is absurd as well. Hence, $p(x)$ must be irreducible over F.
Suppose now that $p(x)$ is irreducible over F. Let I be any ideal of $F[x]$ such that $\langle p(x) \rangle \subseteq I \subseteq F[x]$. Because $F[x]$ is a Principal Ideal Domain, we know that $I = \langle g(x) \rangle$ for some $g(x) \in F[x]$. So $p(x) \in \langle g(x) \rangle$ and therefore, $p(x) = g(x)h(x)$ for some $h(x) \in F[x]$. Since $p(x)$ is irreducible over F, either $g(x)$ is constant or $h(x)$ is constant. In the first case, we would have $I = F[x]$ whereas in the second case, we would have $\langle p(x) \rangle = \langle g(x) \rangle$. And both cases imply that $\langle p(x) \rangle$ is maximal in $F[x]$. ■

Corollary 17.19. Let F be a field and $p(x)$ be an irreducible polynomial over F. Then $F[x]/\langle p(x) \rangle$ is a field.

Proof. From the above theorem, we have that $\langle p(x) \rangle$ is a maximal ideal. But then we are done due to **Proposition 14.105**. ■

Corollary 17.20. Let F be a field and let $p(x), a(x), b(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) \mid a(x)b(x)$, then $p(x) \mid a(x)$ or $p(x) \mid b(x)$.

Proof. Since $p(x)$ is irreducible, $F[x]/\langle p(x) \rangle$ is a field and, therefore, an integral domain and thus $\langle p(x) \rangle$ is a prime ideal due to **Proposition 14.103**. We now have that $a(x)b(x) \in \langle p(x) \rangle$. Thus, $a(x) \in \langle p(x) \rangle$ or $b(x) \in \langle p(x) \rangle$. This completes the proof. ■

Theorem 17.133. Every polynomial in $\mathbb{Z}[x]$ that is not the zero polynomial or a unit in $\mathbb{Z}[x]$ can be written in the form $b_1 b_2 \cdots b_s p_1(x) p_2(x) \cdots p_m(x)$, where the b_i's are irreducible polynomials of degree 0 and the p_i' sare irreducible polynomials of positive degree. Furthermore, if

$$b_1 b_2 \cdots b_s p_1(x) p_2(x) \cdots p_m(x) = c_1 c_2 \cdots c_s q_1(x) q_2(x) \cdots q_n(x)$$

where the c_i's and $q_i(x)$ satisfy the same hypothesis, then $s = t$, $m = n$ and the c_i's and q_i's are a permutation of b_i's and p_i's respectively.

Proof. ■

Chapter 18

Divisibility in Integral Domains

Definition 18.134. Let D be an integral domain. Elements $a, b \in D$ are said to be *associates* if $a = ub$, where u is a unit of D. A non-zero element a of an integral domain D is called an *irreducible* if a is not a unit and, wheneer $b, c \in D$, with $a = bc$, then b or c is a unit. A non-zero element a of an integral domain D is called a *prime* if a is not a unit and $a \mid bc$ implies $a \mid b$ or $a \mid c$.

Proposition 18.135. In an integral domain, every prime is an irreducible.

Proof. Let D be an integral domain and $a \in D$ be a prime. Let $b, c \in D$ such that $a = bc$. We know that $a \mid b$ or $a \mid c$. Suppose $b = at$. Then,

$$b = at = (bc)t = b(ct)$$

that is, $ct = 1$. Thus, c is a unit. ■

Proposition 18.136. In a principal ideal domain, an element is an irreducible if and only if it is a prime.

Proof. We know from the previous proposition that all primes are irreducible. We only need to show that all irreducible elements are primes.
Let a be an irreducible element of a principal ideal D. Suppose now that $a \mid bc$. We shall show that $a \mid b$ or $a \mid c$. Consider the ideal $I = \{ax + by \mid x, y \in D\}$ and let $\langle d \rangle = I$, we know d exists since D is a principle ideal domain. Since $a \in I$, we can write $a = dr$ and because a is irreducible, d is a unit or r is a unit. If d is a unit, then $I = D$ and we may write $1 = ax + by$. Then, $c = acx + bcy = acx + ay$ and since a divides both the terms on the right, it must divide c. On the other hand, if r is a unit, then $\langle a \rangle = \langle d \rangle = I$, and, because $b \in I$, there is an element t in D such that $at = b$. Thus, $a \mid b$. ■

Definition 18.137 (Unique Factorization Domain). An integral domain D is a *unique factorization domain* if

1. every non-zero element of D that is not a unit can be written as a product of irreducibles of D

2. the factorization into irreducibles is unique up to associates and the order in

which the factors appear.

,

Lemma 18.138. In a principal ideal domain D, any strictly increasing chain of ideals $I_1 \subset I_2 \subset \cdots$ must be finite in length.

Proof. Let $I_1 \subset I_2 \subset \cdots$ be a chain of strictly increasing ideals in an integral domain D, and let I be the union of all the ideals in this chain. For any $x \in I$, there must exist an index j such that $x \in I_j$ and hence for all $d \in D$, $xd, dx \in I_j \subseteq I$ and thus I is an ideal of D.
But since D is a principal ideal domain, there is an element a in D such that $I = \langle a \rangle$. Because $a \in I$, and $I = \bigcup I_i$, $a \in I_j$ for some index j. Then, we have $I_i \subseteq I = \langle a \rangle \subseteq I_j$, so that I_j must be the last member of the chain. This completes the proof. ■

Proposition 18.139 (PID$\Longrightarrow$UFD). Every principal ideal domain is a unique factorization domain.

Proof. Let D be a principal ideal domain and let a_0 be any non-zero non-unit in D. If a_0 is irreducible, we are done. We may assume that $a_0 = b_1a_1$ where neither b_1 nor a_1 is a unit and a_1 is non-zero. If a_1 is not irreducible then we can write $a_1 = b_2a_2$ and so on. Continuing in this fashion, we shall obtain a sequence $b_1, b_2, \cdots$ of elements that are not units in D and a sequence $a_0, a_1, a_2, \cdots$ of non-zero elements of D with $a_n = b_{n+1}a_{n+1}$ for each n. Hence, $\langle a_0 \rangle \subset \langle a_1 \rangle \subset \cdots$ but due to the previous lemma, it must be finite, implying that a_r must be irreducible. Thus, every $a \in D$ has atleast one irreducible factor.
Now write $a_0 = p_1c_1$, where p_1 is irreducible and c_1 is not a unit. If c_1 is not irreducible, then we can write $c_1 = p_2c_2$ and so on. Then, we obtain a strictly increasing sequence $\langle a_0 \rangle \subset \langle c_1 \rangle \subset \langle c_2 \rangle \subset \cdots$. Due to the previous lemma, this must terminate at some c_s where c_s is irreducible. Thus, we have been able to successfully factor a_0 as a product of irreducibles. **Thus every element of a principal ideal domain is a product of irreducibles.**
Suppose now that some element $a \in D$ can be written as

$$a = p_1p_2 \cdots p_r = q_1q_2 \cdots q_s$$

where the p's and the q's are irreducible and repetition is permitted. We shall induct on r. If $r = 1$, then a is irreducible and $s = 1$ and $p_1 = q_1$. Assume now the hypothesis is true for all $k < r$. We shall prove the same for r. Due to **Proposition 18.136** every p_i must be a prime. Then p_1 must divide some q_j, without loss of generality, let $p_1 \mid q_1$. Then, we can write $q_1 = up_1$ for some unit u (since q_1 is irreducible too). We now have that

$$p_2 \cdot p_r = (uq_2) \cdots q_s$$

but due to induction, we know that these factorizations are ideantical up to associates and the order in which the factors appear. Thus, the same is true about the two factorizations of a. ■

There is an alternate way to finish the proof, (assume $r \leq s$) note that every prime p_i divides q_j for some j and no two primes can divide the same q_j. So without loss of generality, we can let $p_i \mid q_i$ for each $i \leq r$. And, then we would have a product of units and some q_i's (assuming $r \neq s$) which is equal to unity, this is absurd since the primes are not unit. Thus $r = s$ and we are done.

Corollary 18.21. Let F be a field. Then $F[x]$ is a unique factorization domain.

Proof. Due to **Proposition 16.122** we have that $F[x]$ is a Principal Ideal Domain. This completes the proof :). ■

Definition 18.140 (Euclidean Domain). An integral domain D is called a *Euclidean Domain* if there is a functoin d called the *measure* from the nonzero elements of D to the nonnegative integers such that

- $d(a) \leq d(ab)$ for all non-zero $a, b \in D$
- if $a, b \in D$, $b \neq 0$, then there exist elements $q, r \in D$ such that $a = bq + r$, where $r = 0$ or $d(r) < d(b)$.

A few examples of Euclidean Domains are

- The ring $\mathbb{Z}$ equipped with the measure $d(a) = |a|$, which is the absolute value function is a Euclidean Domain.
- Let F be a field. Then $F[x]$ equipped with the measure function $d(f(x)) = \deg f(x)$ is a Euclidean Domain.
- The ring of Gaussian Integers equipped with the measure function $d(a+bi) = a^2+b^2$ is a Euclidean Domain.

Proposition 18.141 (ED$\Longrightarrow$PID). Every Euclidean Domain is a Primcipal Ideal Domain

Proof. Let D be a Euclidean Domain and I a non-zero ideal of D. Among all the non-zero elements of I, let a be such that $d(a)$ is a minimum. Then $I = \langle a \rangle$. For, if $b \in I$, there are elements q and r such that $b = aq + r$, where $r = 0$ or $d(r) < d(a)$, which contradicts the minimality of $d(a)$. Finally, the zero ideal is $\langle 0 \rangle$. ■

Corollary 18.22. Every Euclidean Domain is a Uniqe Factorization Domain.

Proposition 18.142. If D is a unique factorization domain, then $D[x]$ is a unique factorization domain.

Proof. ■

Part III

Fields

Chapter 19

Vector Spaces

This is basically MA 106.

Definition 19.143 (Vector Space). A set V is said to be a *vector space* over a field F if V is an Abelian group under addition (denoted by $+$) and , if for each $a \in F$ and $v \in V$, there is an element $av \in V$ such that the following conditions hold for all $a, b \in F$ and all $u, v \in V$.

1. $a(u + v) = au + av$
2. $(a + b)v = av + bv$
3. $a(bv) = (ab)v$
4. $1v = v$

Following are some examples of vector spaces:

- The set $\mathbb{R}^n = \{(a_1, a_2, \cdots, a_n) \mid a_i \in \mathbb{R}\}$ is a vector spcae over $\mathbb{R}$. Addition is component-wise while scalar multiplication is again component-wise.
- The set $\mathbb{Z}[x]$ of integer polynomials where addition and scalar multiplication are trivially defined.
- The set of complex numbers $\mathbb{C}$ is a vector space $\mathbb{R}$.

Definition 19.144 (Subspace). Let V be a vector space over a field F and let U be a subset of V. We say that U is a *subspace* of V if U if also a vector space over F under the operations of V.

Definition 19.145 (Linearly Dependant, Linearly Independant). A set S of vectors is said to be *linearly dependant* over the field F if there are vectors $v_1, v_2, \cdots, v_n$ from S and elements $a_1, a_2, \cdots, a_n$ from F, not all zero such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. A set of vectors that is not linearly dependant over F is called *linearly independant.*

Definition 19.146 (Basis). Let V be a vector space over F. A subset B of V is called a *basis* for V if B is linearly independant over F and every element of V is a linear combination of elements of B.

Proposition 19.147. If $\{u_1, u_2, \cdots, u_m\}$ and $\{w_1, w_2, \cdots, w_n\}$ are both bases of a vector space V over a field F, then $m = n$.

Proof. Suppose that $m \neq n$ and without loss of generality that $m < n$. Since the set of u_i's span V, we can write w_1 as a linear combination of the u_i's where not all weights are 0. WLOG say the first weight (that of u_1) is non-zero, then $\{w_1, u_2, \cdots, u_m\}$ spans V. Continuing in this fashion for $w_1, w_2, \cdots, w_m$, we notice that $\{w_1, w_2, \cdots, w_n\}$ span V which is a contradiction. This completes the proof. ■

Definition 19.148 (Dimension). A vector space that has a basis consisting of n elements is said to have *dimension* n. For completeness, the trivial vector space $\{0\}$ is said to be spanned by the empty set and to have dimension 0.

Chapter 20

Extension Fields

Definition 20.149 (Extension Field). A field E is an *extension field* of a field F if $F \subseteq E$ and the operations of F are those of E restricted to F.

Theorem 20.150 (Kroneker, 1887). Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Then, there is an extension field E of F in which $f(x)$ has a zero.

Proof. Since $F[x]$ is a Unique Factorization Domain, $f(x)$ has an irreducible factor, say, $p(x)$. We claim that $F[x]/\langle p(x)\rangle$ works. We already know that this is a field. Consider the homomorphism $\phi : F \to E$ which maps $a \mapsto a + \langle p(x)\rangle$, which is one-one. And, thus, E contains a subfield isomorphic to F. We may think of E as containing F if we simply identify the coset $a + \langle p(x)\rangle$ as just a and vice versa.
We now claim that $x + \langle p(x)\rangle$ is a root of $p(x)$. Note, we write $p(x) = a_n x^n + \cdots + a_0$. Then, we have

$$\begin{aligned} p(x + \langle p(x)\rangle) &= a_n(x + \langle p(x)\rangle)^n + \cdots + a_0 \\ &= a_n x^n + \cdots + a_0 + \langle p(x)\rangle = p(x) + \langle p(x)\rangle \\ &= 0 + \langle p(x)\rangle \end{aligned}$$

This completes the proof. ■

Definition 20.151. We use the notation $F(a_1, a_2, \cdots, a_n)$ to denote the smallest field of E which contains F and the set $\{a_1, a_2, \cdots, a_n\}$.

Definition 20.152 (Splitting Fields). Let E be an extension field of F and let $f(x) \in F[x]$ with degree atleast 1. We say that $f(x)$ *splits* in E if there are elements $a \in F$ and $a_1, a_2, \cdots, a_n \in E$ such that

$$f(x) = a(x - a_1)(x - a_2)\cdots(x - a_n)$$

We call E a splitting field for $f(x)$ over F if

$$E = F(a_1, a_2, \cdots, a_n)$$

Proposition 20.153. Let F be a field and let $f(x)$ be a non-constant polynomial in $F[x]$. Then, there exists a splitting field for $f(x)$ over F.

Proof. We shall induct on $\deg f(x)$. If $\deg f(x) = 1$, then $f(x)$ is linear and already splits over F. Now suppose that the statement is true for all $k < n$. We shall attempt to prove for $f(x)$ with $\deg f(x) = n$. Due to Kronecker, there exists a field E in which $f(x)$ has a zero, call it a_1. Then, we can write $f(x) = (x - a)g(x)$, by induction, there is a field K that contains E and all the zeros of $g(x)$, say, $a_2, \cdots, a_n$. Then, obviously, there exists a splitting field which is the subset of the aforementioned field. ■

Proposition 20.154. Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in come extension E of F, then $F(a)$ is isomorphic to $F[x]/\langle p(x)\rangle$. Furthermore, if $\deg p(x) = n$, then every member of $F(a)$ can be uniquely expressed in the form

$$c_{n-1}a^{n-1} + c_{n-2}a^{n-2} + \cdots + c_0$$

where $c_0, c_1, \cdots, c_{n-1} \in F$.

Proof. Consider the function ϕ from $F[x]$ to $F(a)$ given by $\phi(f(x)) = f(a)$. It is clear that ϕ is a ring homomorphism. Since $p(a) = 0$, we know that $\langle p(x)\rangle \subseteq \operatorname{Ker}\phi$. But, due to **Proposition 17.132**, $\langle p(x)\rangle$ is a maximal ideal and thus $\operatorname{Ker}\phi = \langle p(x)\rangle$ since $\operatorname{Ker}\phi \neq F[x]$. Due to the properties that we proved for ring homomorphisms, $\phi(F[x])$ is a subfield of $F(a)$. But since $F(a)$ is the minimum field which contains F and a, $\phi(F[x]) = F(a)$. And hence,

$$F[x]/\langle p(x)\rangle \cong \phi(F[x]) = F(a)$$

Now, note any element of $F[x]/\langle p(x)\rangle$ can be written as

$$c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_0 + \langle p(x)\rangle$$

where $c_0, \cdots, c_{n-1} \in F$ and this completes the proof. ■

Corollary 20.23. Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in some extension E of F and b is a zero of $p(x)$ in some extension E' of F, then $F(a) \cong F(b)$.

Lemma 20.155. Let F be a field, let $p(x) \in F[x]$ be irreducible over F, and let a be a zero of $p(x)$ in some extension of F. If ϕ is a field isomorphism from F to F' and b is a zero of $\phi(p(x))$ in some extension of F', then there is an isomorphism from $F(a)$ to $F'(b)$ that agrees with ϕ on F and carries a to b.

Proof. First observe that since $p(x)$ is irredicuble over F, $\phi(p(x))$ is irreducible over F'. Now note that the mapping $\Phi : F[x]/\langle p(x)\rangle \to F'[x]/\langle \phi(p(x))\rangle$ given by

$$f(x) + \langle p(x)\rangle \mapsto \phi(f(x)) + \langle \phi(p(x))\rangle$$

is a field isomorphism. We know that there is an isomorphism α from $F(a)$ to $F[x]/\langle p(x)\rangle$ and there is an isomorphism β from $F'[x]/\langle \phi(p(x))\rangle$ to $F'(b)$. Then, the isomorphism $\beta\Phi\alpha$ is the required mapping from $F(a)$ to $F'(b)$. ■

Proposition 20.156. Let ϕ be an isomorphism from a field F to a field F' and let $f(x) \in F[x]$. If E is a splitting field for $f(x)$ over F, and E' is a splitting field for $\phi(f(x))$ over F', then there is an isomorphism from E to E' that agrees with ϕ on F.

Proof. We use induction on $\deg f(x)$. If $\deg f(x) = 1$, then $E = F$ and $E' = F'$, so ϕ is the desired mapping. If $\deg f(x) > 1$, let $p(x)$ be an irreducible factor of $f(x)$, let a be a zero of $p(x)$ in E and let b be a zero of $\phi(p(x))$ in E'. Due to the previous lemma, there is an isomorphism α from $F(a)$ to $F'(b)$ that agrees with ϕ on F and carries a to b. We now write $f(x) = (x - a)g(x)$, where $g(x) \in F(a)[x]$. Then E is a splitting field for $g(x)$ over $F(a)$ and E' is a splitting field for $\alpha(g(x))$ over $F'(b)$. Since $\deg g(x) < \deg f(x)$, there is an isomorphism from E to E' that agrees with α on $F(a)$ and therefore with ϕ on F. ■

Corollary 20.24. Let F be a field and let $f(x) \in F[x]$. Then any two splitting fields of $f(x)$ over F are isomorphic.

Proof. Suppose that E and E' are splitting fields of $f(x)$ over F. Let now ϕ be the identity isomorphism. This completes the proof. ■

Definition 20.157. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ belong to $F[x]$. The derivative of $f(x)$, denoted by $f'(x)$, is the polynomial

$$na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

Lemma 20.158. Let $f(x)$ and $g(x)$ be elements of $F[x]$ and let $a \in F$. Then,

1. $(f(x) + g(x))' = f'(x) + g'(x)$
2. $(af(x))' = af'(x)$
3. $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$

The proof is omitted.

Proposition 20.159. A polynomial $f(x)$ over a field F has a multiple zero in some extension E if and only if $f(x)$ and $f'(x)$ have a common factor of positive degree in $F[x]$.

Proof. If a is a multiple root of $f(x)$ in some extension E, then there is a $g(x)$ in $E[x]$ such that $f(x) = (x-a)^2g(x)$. But then, we would have $f'(x) = (x-a)^2g'(x) + 2(x-a)g(x)$, implying that $f'(a) = 0$. Thus $x - a$ is a factor of both $f(x)$ and $f'(x)$ in the extension E of F. Now if $f(x)$ and $f'(x)$ have no common divisor of positive degree in $F[x]$, there are polynomials $h(x)$ and $k(x)$ in $F[x]$ such that $f(x)h(x) + f'(x)k(x) = 1$ (Bézout's Lemma) as an element of $E[x]$, but that would mean $x - a$ is a factor of 1, which is absurd.
Conversely, suppose that $f(x)$ and $f'(x)$ have a common factor of positive degree. Let a be a zero of the common factor. Then a is a zero of $f(x)$ and $f'(x)$. Since a is a zero of $f(x)$, there is a polynomial $q(x)$ such that $f(x) = (x-a)q(x)$. Then $f'(x) = q(x) + (x-a)q'(x)$ which obviously implies $x - a$ is a factor of $q(x)$. Thus $f(x)$ has a multiple zero. ■

Proposition 20.160. Let $f(x)$ be an irreducible polynomial over a field F. If F has characteristic 0, then $f(x)$ has no multiple zeros. If F has characteristic $p \neq 0$, then $f(x)$ has a multiple zero only if it is of the form $f(x) = g(x^p)$ for some $g(x)$ in $F[x]$.

Proof. If $f(x)$ has a multiple root, due to the previous proposition, $f(x)$ and $f'(x)$ have a common factor of positive degree in $F[x]$. Since $f(x)$ is irreducible, the only factor of $f(x)$ with positive degree is $f(x)$, which would imply $f(x)$ divides $f'(x)$ which is absurd since $\deg f(x) > \deg f'(x)$. Thus, we must have $f'(x) = 0$. Now, if $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1 = 0$, which is possible if and only if $ka_k = 0$ for all permissilble k. So, when $\operatorname{char} F = 0$, we would need to have $f(x)$ to be a constant polynomial which is reducible and hence, a contradiction.
Per On the other hand, when $\operatorname{char} F = p \neq 0$, we have $a_k = 0$ when p doesn't divide k. Thus, the only powers of x which appear in the expansion of $f(x)$ are the ones which are divisible by p. Thus, $f(x) = g(x^p)$ for some $g(x) \in F[x]$. ∎

Definition 20.161 (Perfect Field). A field F is called *perfect* if F has characteristic 0 or F has characteristic p and $F^p = \{a^p \mid a \in F\} = F$.

Theorem 20.162. Every finite field is perfect.

Proof. Let F be a finite field with characteristic p. Consider the mapping $\phi : F \to F$ given by $x \mapsto x^p$. Obviously, $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a+b) = \phi(a) + \phi(b)$. Furthermore, $\operatorname{Ker}\phi = \{0\}$ which is obvious. Thus, ϕ is one-one and, since F is finite, ϕ must be onto. Thus, $F^p = F$. ∎

Proposition 20.163. If $f(x)$ is an irreducible polynomial over a perfect field F, then $f(x)$ has no multiple zeros.

Proof. Suppose F has characteristic p and suppose that f has multiple zeros. Then, due to **Proposition 20.160**, $f(x) = g(x^p)$ for some $g(x) \in F[x]$. But since $F^p = F$, every coefficient of g can be written as b^p for some b. This is elucidated by the following

$$\begin{aligned} f(x) = g(x^p) &= b_n^p x^{pn} + \cdots + b_0^p \\ &= (b_nx^n + \cdots + b_0)^p \end{aligned}$$

but this is a contradiction, since we assumed that $f(x)$ was irreducible. ∎

Proposition 20.164. Let $f(x)$ be an irreducible polynomial over a field F and let E be a splitting field of $f(x)$ over F. Then, all the zeros of $f(x)$ in E have the same multiplicity.

Proof. Let a and b be distinct roots of $f(x)$ with a having multiplicity m. Then, we can write $f(x) = (x-a)^m g(x)$. Then, there exists a morphism ϕ which takes $a \mapsto b$ and is identity over F. Then, we have

$$f(x) = \phi(f(x)) = \phi((x-a)^m g(x)) = (x-b)^m \phi(g(x))$$

this completes the proof. ∎

Chapter 21

Algebraic Extensions

Definition 21.165 (Types of Extensions). Let E be an extension field of a field F and let $a \in E$. We call a *algebraic* over F, if a is the zero of some non-zero polynomial in $F[x]$. If a is not algebraic over F, it is called *transcendental* over F. An extension E of F is called an algebraic extension of F if every element of E is algebraic over F. If E is not an algebraic extension, it is called a *transcendental* extension of F. An extension of F of the form $F(a)$ is called a *simple* extension of F.

Proposition 21.166. Let E be an extension field of the field F and let $a \in E$. If a is transcendental over F, then $F(a) \cong F(x)$. If a is algebraic over F, then $F(a) \cong F[x]/\langle p(x)\rangle$, where $p(x)$ is a polynomial in $F[x]$ of minimum degree such that $p(a) = 0$ moreover $p(x)$ is irreducible over F.

Proof. Consider the homomorphism $\phi : F[x] \to F(a)$ given by $f(x) \mapsto f(a)$. If a is transcendental over F, then $\operatorname{Ker} \phi = \{0\}$. Thus, we may extend ϕ to an isomorphism $\Phi : F(x) \to F(a)$ by defining $\Phi(f(x)/g(x)) = f(a)/g(a)$.
There is a problem when a is algebraic, namely, $\operatorname{Ker} \phi = \langle p(x)\rangle$ where $p(x)$ has minimum degree among all elements of $\langle p(x)\rangle$. Thus, $p(a) = 0$ and it is hence irreducible over F. Now we are done by **Proposition 20.154**. ■

Proposition 21.167. If a is algebraic over a field F, then there is a unique monic irreducible polynomial $p(x)$ in $F[x]$ such that $p(a) = 0$.

Proof. Follows from the previous proof. ■

The next one follows too

Proposition 21.168. Let a be algebraic over F, and let $p(x)$ be the minimal polynomial for a over F. If $f(x) \in F[x]$ and $f(a) = 0$, then $p(x)$ divides $f(x)$ in $F[x]$.

Definition 21.169. Let E be the extension firld of a field F. We say that E has degree n over F and write $[E : F] = n$ if E has dimension n as a vector space over

F. If $[E : F]$ is finite, E is called a *finite extension* of F; otherwise, we say that E is an *infinite extension* of F.

Proposition 21.170. If E is a finite extension of F, then E is an algebraic extension of F.

Proof. Trivial enough. Consider any element $a \in E$, then the set $\{1, a, \cdots, a^n\}$ is linearly dependant and we have ourselves a polynomial of degree n which nullifies a and hence, we are done. ■

Proposition 21.171. Let K be a finite extension field of the field E and let E be a finite extensio nfield of the field F. Then K is a finite extension field of F and $[K : F] = [K : E][E : F]$.

Proof. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a basis for K over E and let $Y = \{y_1, y_2, \cdots, y_m\}$ be a basis for E over F, We shall show that

$$YX = \{y_j x_i \mid 1 \leq j \leq m,\ 1 \leq i \leq n\}$$

For any $a \in K$, it is not hard to see that a can be written as a linear combination of the elements in YX. Now, suppose that there exists a linear combination of YX which results in the zero element. Then

$$0 = \sum_{i,j} y_j x_i = \sum_i \left(\sum_j c_{ij} y_j \right) x_i$$

But, using the linear independance of x_i, we conclude that

$$0 = \sum_j c_{ij} y_j \implies c_i j = 0$$

This completes the proof. ■

Theorem 21.172 (Steinitz, 1910). If F is a field of characteristic 0, and a and b are algebraic over F, then there is an element c in $F(a, b)$ such that $F(a, b) = F(c)$.

Proposition 21.173. If K is an algebraic extension of E and E is an algebraic extension of F, then K is an algebraic extension of F.

Proof. Let $a \in K$. It suffices to show that a is in some finite extension of F. Since a is algebraic over E, there is an irreducible polynomial $p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$. Consider the following

$$\begin{aligned} F_0 &= F(b_0) \\ F_1 &= F_0(b_1) \\ &\vdots \\ F_n &= F_{n-1}(b_n) \end{aligned}$$

Now, $[F_n(a) : F_n] = n$ and hence:

$$[F_n(a) : F] = [F_n(a) : F_n][F_n : F_{n-1}] \cdots [F_0 : F]$$

which is obviously finite. ■

Corollary 21.25. Let E be an extension field of the field F. Then, the set of all elements of E that are algebraic over F is a subfield of E.

Chapter 22

Finite Fields

Proposition 22.174. For each prime p and each positive integer n, there is, up to isomorphism, a unique finite field of order p^n.

Proof. Consider the splitting field E of $f(x) = x^{p^n} - x$ over $\mathbb{Z}_p$. Since $f(x)$ splits in E, every zero of f must have multiplicity 1 in E and thus $f(x)$ has p^n distinct zeroes in E. Now, the set of zeroes of f in E is closed under addition, substraction, multiplication and division and hence the set of zeroes of $f(x)$ is an extension field of $\mathbb{Z}_p$. Thus the set of zeroes of f is E and thus $|E| = p^n$
As for uniqueness, say K is another field of the same order. Then K has a asubfield isomorphic to $\mathbb{Z}_p$ and since the non-zero elements of K form a multiplicative group of order $p^n - 1$, every element of K is a zero of $f(x)$ and hence K must be a splitting field over $\mathbb{Z}_p$ for f. But we know that there is only one such field upto isomorphism. ∎

Definition 22.175. Since there is only one field, upto isomorphism of order p^n, we shall denote it using $GF(p^n)$, called the *Galois Field of order p^n.*

Proposition 22.176. As a group under addition, $GF(p^n)$ is isomorphic to

$$\underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n-factors}$$

As a group under multiplication, the set of non-zero elements of $GF(p^n)$ is isomorphic to $\mathbb{Z}_{p^n-1}$ and hence, is cyclic.

Proof. Since $GF(p^n)$ has characteristic p, every non-zero element has additive order p. Then, by the Fundamental Theorem of Finite Abelian Groups, we have the desired conclusion.
We now come to the multiplicative group. First, due to the FUndamental Theorem of Finite Abelian Groups, GF is isomorphic to a direct product of the form $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_m}$. If the orders are relatively prime, then it follows that GF is cyclic. Suppose now that there exists d which divides all n_i., then each of these components must have a subgroup of order d. Thus, equivalently, there exist H and K as subgroups of GF of order d. Then each element of H and K is a zer oof $x^d - 1$, contradicting the fact that a polynomial of degree d over a field can have atmost d zeros. ∎

Corollary 22.26.
$$[GF(p^n) : GF(p^n)] = n$$

Proposition 22.177. For each divisor m of n, $GF(p^n)$ has a unique subfield of order p^m. Moreover, these are the only subfields of $GF(p^n)$.

Proof. If $m \mid n$, then $p^m - 1 \mid p^n - 1$. Let $K = \{x \in GF(p^n) \mid x^{p^m} = x\}$. It is not hard to show that K is a subfield of GF. Furthermore, since the polynomial $x^{p^m} - x$ has atmost p^m zeros in $GF(p^n)$, we must have that $|K| \leq p^m$. Now, let $\langle a \rangle = GF$, then $|a^{\frac{p^n-1}{p^m-1}}| = p^m - 1$, or equivalently, it follows that K is a subfield of GF with order p^m. Finally, now suppose that F is a subfield of GF. Then F must be isomorphic to $GF(p^m)$ for some m and hence
$$n = [GF(p^n) : GF(p^m)] \cdot m$$
And, thus, m must divide n. ■

Part IV

Special Topics

Chapter 23

Sylow Theorems

We come back to Group Theory in this chapter.

Definition 23.178 (Conjugacy Class). Let a and b be elements of a group G. We say that a and b are conjugate in G (and call b a conjugate of a) if $xax^{-1} = b$ for some $x \in G$. The *conjugacy* class of a is the set $\mathrm{cl}(a) = \{xax^{-1} \mid x \in G\}$.

Lemma 23.179. Say $a \sim b$ if a and b are conjugates in G. Then $\sim$ is an equivalence relation.

Proof.

Reflexivity: $eae^{-1} = a$

Symmetry: $xax^{-1} = b \implies x^{-1}bx = a$

Transitivity: $xax^{-1} = b$ and $yby^{-1} = c \implies (xy)a(xy)^{-1} = c$

This completes the proof. ■

Thus, we can divide the group G into equivalence classes.

Proposition 23.180. Let G be a group and let a be an element of G. Then,

$$|\,\mathrm{cl}(a)| = |G : C(a)|$$

Proof. Let ϕ be the mapping $xC(a) \mapsto xax^{-1}$. It is easy to verify that ϕ is well defined and injective. Thus, we have the desired conclusion. ■

Corollary 23.27. In a finite group, $|\,\mathrm{cl}(a)|$ divides $|G|$.

Proof. Recall from the chapter Cosets and Lagrange's Theorem, that the number of cosets of a subgroup divides $|G|$. ■

Definition 23.181 (Class Equation). For any finite group G,

$$|G| = \sum |G : C(a)|$$

where the sum runs over one element a from each conjugacy class of G.

Proposition 23.182. Let G be a non-trivial finite group whose order is a power of a prime p. Then $Z(G)$ is non-trivial.

Proof. If $a \in Z(G)$, then it is obvious that $a \sim a$ only. Thus $\mathrm{cl}(a) = \{a\}$
Then, we can rewrite the class equation as

$$|G| = |Z(G)| + \sum |G : C(a)|$$

where the second sum runs over the conjugacy classes with more than one element. Now, due to Lagrange's Theorem, we know that $|G : C(a)|$ must be a prime power. Now working modulo p, we have the desired conclusion that $p \mid Z(G)$. ■

Corollary 23.28. If $|G| = p^2$, where p is a prime, then G is Abelian.

Proof. Combining the previous theorem with Lagrange's theorem, we know that $Z(G)$ must have order p or p^2. If $Z(G)$ has order p^2, then we are done. If $Z(G)$ has order p, then $|G/Z(G)| = p$ and is thus cyclic and finally, we conclude that G is Abelian. ■

Theorem 23.183 (Sylow's First Theorem). Let G be a finite group and let p be a prime. If p^k divides $|G|$, then G has at least one subgroup of order p^k.

Proof. We shall induct on $|G|$. The base case with $|G| = 1$ is trivial. If G has a proper subgroup H such that p^k divides $|H|$, then according to the inductive hypothesis, we are done. Then, we may assume that p^k does not divide the order of any subgroup of G. Finally, consider the class equation for G in the form

$$|G| = |Z(G)| + \sum |G : C(a)|$$

Since p^k divides $|G| = |G : C(a)||C(a)|$ and p^k doesn't divide $|C(a)|$, we know that atleast p must divide $|G : C(a)|$ for all $a \notin Z(G)$. Then, obviously, from the class equation, we have that p must divide $|Z(G)|$. From the funcamental theorem of Finite Abelian Groups, we know that $Z(G)$ contains an element of order p. Call this x. Now note that $\langle x \rangle$ is a normal subgroup of G, since x commutes with all the elements of G. Then, we must have that p^{k-1} divides $|G/\langle x \rangle|$. Then, from the induction hypothesis, there must exist a subgroup of order p^{k-1} which will be of the form $H/\langle x \rangle$. But using the fact that $|\langle x \rangle| = p$, we must have that $|H| = p^k$. This contradicts the assumption that there is no subgroup with order divisible by p^k and hence we are done. ■

Definition 23.184 (Sylow p-Subgroup). Let G be a finite group and let p be a prime. If $v_p(|G|) = k$, then any subgroup of G of order p^k is called a Sylow p-subgroup of G.

Definition 23.185 (Conjugate Subgroups). Let H and K be subgroups of a group G. We say that H and K are *conjugate* in G if there is an element $g \in G$ such that $H = gKg^{-1}$.

Theorem 23.186 (Sylow's Second Theorem). If H is a subgroup of a finite group G and $|H|$ is a power of a prime p, then H is contained in some Sylow p-subgroup of G.

Proof. Let K be a Sylow p-subgroup of G and let C be the sets of all conjugates of K in G. Ofcourse, since conjugation is an automorphism, each element of C is a Sylow p subgroup of G. Now, let S_C denote the permutation group of C. Then, we can define the mapping ϕ_g which maps X in C to gXg^{-1}. Then, consider the mapping T which maps g to ϕ_g. It is easy to show that T is a homomorphism.
Now consider the image of H under T. Due to the Orbit-Stabilizer Theorem, we must have that $\mathrm{orb}_{T(H)}(K_i)|$ divides $|T(H)|$. and then $|\,\mathrm{orb}_{T(H)}(K_i)|$ must be a power of p. Ofcourse, if the orbit is 1, then it would mean that $gK_ig^{-1} = K_i$ for all $g \in H$. But that would mean that H is a subgroup of $N(K_i)$ but since all elements in H have prime power, ew also must have $H \leq K_i$.
Finally, all we need to show that there exists i such that the orbit of K_i is 1. Again, we have that $|C| = |G : N(K)|$. But since $|G : K|$ is not divisible by p, neither is $|C|$. Now, since the orbits partition C, we must have that $|C|$ is the sum of prime powers of p and if there is no orbit of size 1, then p would divide $|C|$ which is absurd. This completes the proof. ■

Theorem 23.187 (Sylow's Third Theorem). Let p be a prime and let G be a group of order $p^k m$ where p doesn't divide m. Then, the number n of Sylow p subgroups of G is congruent to 1 modulo p and divides m. Furthermore, any two Sylow p-subgroups of G are conjugate.

Proof. Similar to the previous theorem, we have S_C and T. Now, we have that $\mathrm{orb}_{T(K)}(K_i)$ is a power of p and orb is 1 if and only if $K \leq K_i$. Finally, since the orbits partition C, we can conclude that $n \equiv |C| \equiv 1 \pmod{p}$.
Now, suppose that H is a Sylow p subgroup not in C. Consider $T(H)$. Since $|C|$ is the sum of orbits sizes under the action of $T(H)$ and now orbit has size 1, since H is not in C, we must conclude that p divides C which is absurd.
Finally, $n = |G : N(K)|$ and n is relatively prime to p and hence must divide m. ■

Chapter 24

Finite Simple Groups

Definition 24.188. A group is *simple* if its only normal subgroups are the identity subgroup and the group itself.

Proposition 24.189. Let n be a positive integer that is not prime, and let p be a prime divisor of n. If 1 is the only divisor of n that is congruent to 1 modulo p, then there doesn't exist a simple group of order n.

Proof. If n is the power of a prime then a group of order n has a non-trivial cdnter and therefore is not simple. If n is not a prime power then every Sylow subgroup is proper and due to Sylow's Third Theorem, the number of Sylow p subgroups of a group of order n is congruent to 1 modulo p and divides n. Since 1 is the only such number, the Sylow p subgroup is unique and therefore is normal. ■

Proposition 24.190. An integerd (greater than 2) which is congruent to 2 modulo 4 is not the order of a simple group.

Proof. Let G be a group of order $2n$ where n is an odd integer greater than 1. Define the mapping T_g as $x \mapsto gx$ for $x \in G$. Then, it is easy to show that the mapping $g \mapsto T_g$ is well defined and an isomorphism. Cauchy's Theorem now guarantees the existence of $g \in G$ with order 2. In this case, T_g must contain only 2-cycles. Thus, there must be n disjoint 2 cycles in T_g which would make T_g an odd permutation. This would mean that the set of een permutations in the image of G is a normal subgroup of index 2. Thus, G is not simple. ■

Proposition 24.191. Let G be a group and let H be a subgroup of G. Let S be the group of all permutations of the left cosets of H in G. Then there is a homomorphism from G into S whose kernel lies in H and contains every normal subgroup of G that is contained in H.

Proof. We now define T_g as a permutation of the left coset of H by mapping xH to gxH. Verify now that the mapping $\alpha : g \mapsto T_g$ is a homomrophism.
Ofocurse, if $g \in \operatorname{Ker} \alpha$, then T_g Is the identity map and hence $H = gH$, and thus $g \in H$, which woudl imply $\operatorname{Ker} \alpha \subseteq H$.

Now let K be some normal subgroup contained in H. Then for some $k \in K$, and any $x \in G$, there must exist $k' \in K$ such that $kx = xk'$ or equivalently, one can conclude that T_k would be the identity permutation. Thus, $k \in \operatorname{Ker} \alpha$. This completes the proof. ■

Corollary 24.29. If G is a finite group and H is a proper subgroup of G such that $|G|$ does not divide $|G : H|!$, then H contains a non-trivial normal subgroup of G. In particular, G is not simple.

Proof. Follows from the previous proof. Basically, argue that $G/\operatorname{Ker} \alpha$ is isomorphic to some set of order $|G : H|!$ which would mean that $\operatorname{Ker} \alpha$ must be greater than 1. ■

Corollary 24.30. If a finite non-abelian, simple group G has a subgroup of index n, then G is isomorphic to a subgroup of A_n.

Chapter 25

Generators and Relations

We would first begin with some definitions and notations

Definition 25.192. For any set $S = \{a, b, \cdots\}$, we create a new set $S^{-1} = \{a^{-1}, b^{-1}, \cdots\}$ by replacing each x in S by x^{-1}. Define the set $W(S)$ to be the collection of all formal finite strings of the form $x_1 x_2 \cdots x_k$ where each $x_i \in S \bigcup S^{-1}$. The elements of $W(S)$ are called *words* from S. The empty word is denoted by e. For w_1 and w_2 in $W(S)$, define $w_1 w_2$ as the concatenated word.

Definition 25.193 (Equivalence Classes of Words). For any pair of elements u and v of $W(S)$, we say that u is related to v if v can be obtained from u by a finite sequence of insertions or deletions of words of the form xx^{-1} or $x^{-1}x$ where $x \in S$.

Proposition 25.194. Let S be a set of distinct symbols. For any word u in $W(S)$, let $\overline{u}$ denote the set of all words in $W(S)$ equivalent to u. Then, the set of all equivalence classes of elements of $W(S)$ is a group under the operation $\overline{u} \cdot \overline{v} = \overline{uv}$.

The above group is called a Free Group.

Proposition 25.195 (Universal Mapping Property). Every group is a homomorphic image of a free group.

Proof. Let G be a group and let S be a set of generators for G and take F to be the free group on S. Now consider the mapping

$$\phi(\overline{x_1 x_2 \cdots x_n}) = (x_1 x_2 \cdots x_n)_G$$

It is not hard to see that ϕ is well defined. As for operation preserving, we have

$$\begin{aligned} \phi(\overline{x_1 x_2 \cdots x_n})\phi(\overline{y_1 y_2 \cdots y_m}) &= (x_1 x_2 \cdots x_n)_G (y_1 y_2 \cdots y_m)_G \\ &= \phi(\overline{x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n}) \end{aligned}$$

Finally, we can say that ϕ is onto G since S generates G. ■

Corollary 25.31. Every group is isomorphic to a quotient group of a free group.

Definition 25.196. Let G be a group generated by some subset $A = \{a_1, a_2, \cdots, a_n\}$ and let F be the free group on A. Let $W = \{w_1, w_2, \cdots, w_m\}$ be a subset of F and let N be the smallest normal subgroup of F containing W. We say that G is given by the generators $a_1, a_2, \cdots, a_n$ and the relations $w_1 = w_2 = \cdots = w_m = e$ if there is an isomorphism from F/N onto G that carries a_iN to a_i.

The notation for the above situation is

$$G = \langle a_1, a_2, \cdots, a_n \mid w_1 = w_2 = \cdots = w_m = e\rangle$$

Theorem 25.197 (Dyck, 1882). Let

$$G = \langle a_1, a_2, \cdots, a_n \mid w_1 = w_2 = \cdots = w_m = e\rangle$$

and let

$$\overline{G} = \langle a_1, a_2, \cdots, a_n \mid w_1 = w_2 = \cdots = w_m = w_{m+1} = \cdots = w_{m+k} = e\rangle$$

Then $\overline{G}$ is a homomorphic image of G.

Proof. Let F be the free group on $\{a_1, a_2, \cdots, a_n\}$ and let N and M be the smallest normal subgroups containing $\{w_1, w_2, \cdots, w_m\}$ and $\{w_1, w_2, \cdots, w_{m+k}\}$. Then $F/N \cong G$ while $F/M \cong \overline{G}$. Finally if we take the obvious homomorphism $aN \mapsto aM$, it would induce a homomorphism from G to $\overline{G}$. ■

Proposition 25.198. Any group generated by a pair of elements of order 2 is dihedral.

Proof. Say G is generated by a and b, both with order 2. We divide this into two cases, first, when $|ab|$ is not finite. In this case, we would like to show that G is isomorphic to D_∞. We know due to Dyck that G is isomorphic to a factor group of D_∞, say D_∞/H. Let $h \in H$ and $h \neq e$. Without loss of generality, we may assume that h is of the form $(ab)^i$ or $(ab)^ia$.
If $h = (ab)^i$, then,

$$H = (ab)^iH = (abH)^i$$

$$(aH)(abH)(aH) = baH = (abH)^{-1}$$

implying that

$$D_\infty/H = \langle aH, bH\rangle = \langle aH, abH\rangle$$

Which would imply that G is finite which is not possible.
Next, if $h = (ab)^ia$,

$$H = (ab)^iaH = (ab)^iHaH$$

which implies that

$$(abH)^i = (ab)^iH = (aH)^{-1} = a^{-1}H = aH$$

Thus,

$$\langle aH, bH \rangle = \langle aH, abH \rangle \subseteq \langle abH \rangle$$

But since $|abH|$ is finite, it would mean D_∞/H is finite, which would force H to contain only the identity and G to be isomorphic to D_∞.
Finally, if $|ab|$ is finite, we would be able to write $G = \langle a, b \rangle = \langle a, ab \rangle$. And finally, one can show that a and ab satisfy the relations for D_n and we are done. ■

Chapter 26

Introduction to Algebraic Coding Theory

Definition 26.199. An (n, k) linear code over a finite field F is a $k-$dimensional subspace V of the vector space

$$F^n = \underbrace{F \oplus F \oplus \cdots \oplus F}_{n \text{ copies}}$$

over F. The members of V are called *code words*. When F is $\mathbb{Z}_2$, the code is called binary.

Definition 26.200 (Hamming Distance and Hamming Weight). The *Hamming Distance* between two vectors in F^n is the number of components in which they differ. The *Hamming Weight* of a vector is the number of nonzero components of the vector. The *Hamming Weight* of a linear code is the minimum weight of any nonzero vector in the code.

Proposition 26.201. For any vectors u, v and w, $d(u, v) \leq d(u, w) + d(w, v)$ and $d(u, v) = \text{wt}(u - v)$

Proof. The second equality is obvious. As for the first, note for the i-th position, if u and v disagree, and u agrees with w then v and w must disagree. ■

Proposition 26.202. If the *Hamming Weight* of a linear code is at least $2t + 1$, then the code can correct any t or fewer errors. Alternatively, the same code can detect any $2t$ or fewer errors.

Proof. We shall use nearest-neighbour decoding for this. Which means, for any vector v, we shall send a codeword v' such that the Hamming Distance $d(v, v')$ is minimum. If there is more than one v', we do not decode. Suppose the codeword u is received as v and had atmost t errors, implying that $d(u, v) \leq t$. Suppose now that w is some other code word not equal to u, then

$$2t + 1 \leq \text{wt}(w - u) = d(w, u) \leq d(w, v) + d(v, u) \leq d(w, v) + t$$

implying that $d(w,v) \geq t+1 > d(v,u)$. And hence v is correctly decoded as u.
As for the second prat, suppose the code word u is received as the vector v with atleast one error, but no more than $2t$ errors. That would imply $d(v,u) \leq 2t$. THen we would know that v cannot be a code word, since the Hamming Weight, that is the minimum distance between two code words is $2t+1$, a contradiction. Thus we have been able to detect that there are errors. ■

Generator Matrix

A generator matrix, G is basically a $k \times n$ matrix with linearly independant rows that carries F^k to a $k-$dimensional subset of F^n.
The Generator matrix is usually of the form

$$\left[I_k \mid A_{[k\times(n-k)]}\right]$$

Thus, the actual k information digits occur at the beginning of the code-word.
As an example, consider the $(6,3)$ linear code over $\mathbb{Z}_2$ given by the following generator matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which is responsible for the following encoding

$$\begin{array}{ccc} 000 & \mapsto & 000000 \\ 001 & \mapsto & 001111 \\ 010 & \mapsto & 010101 \\ 011 & \mapsto & 011010 \\ 100 & \mapsto & 100110 \\ 101 & \mapsto & 101001 \\ 110 & \mapsto & 110001 \\ 111 & \mapsto & 111100 \end{array}$$

Parity Check Matrix Decoding

Since we are able to encode messages with a generator matrix, w eneed a convenient method for decoding them. In the case where at most one error percode word has occurred, there is a simple method to get this done.
Now, suppose V is a systematic linear code over a field F due to the standard generator matrix $G = [I_k \mid A]$. Then, the matrix

$$H = \left[\frac{-A}{I_{n-k}}\right]$$

where all operations are done in F.
Consider now the following procedure

1. For any received word w, compute wH.

2. If wH is the zero vector, assume that no error was made.

3. If there is exactly one instance of a nonzero element $s \in F$, which is in the i-th row of wH, then we assume that the sent word was $w - (00 \cdots 0s00 \cdots 0)$ where s occurs as the i-th component.

4. If wH doesn't fit into any of the above categories, then the number of errors is greater than 1.

Lemma 26.203 (Orthogonality Relation). Let C be a systematic (n, k) linear code over F with a standard generator matrix G and parity-check matrix H. Then, for any vector v in F^n, we hae $vH = 0$ if and only if $v \in C$.

Proof. The proof is straightforward, with the method of contradiction. ■

Using the above lemma, it is possible to prove the following theorem.

Theorem 26.204 (Parity-Check Matrix Decoding). Parity-Check matrix decoding will correct any single erro if and only if the rows of the parity check-matrix are non-zero and no one row is a scalar multiple of any other row.

Coset Decoding

This method is attributed to David Slepian and is also referred to as "standard decoding". We create a table called the "standard array". Let $V = F^n$. The first row of the table is the set C of code words, beginning in column 1 with the identity $00 \cdots 0$. To form the next row, select an elemtn v of V which is not listed in the table so far. Among all the elements of $v + C$, select v' with minimum weight. Then add v' to the previous row and form the next row. Repeat this procedure until all the elements of V have been exhausted.
The decoding procedure is to take the element in the first row of the column in which the word w, which has been received is present.
The words in the first column are called the *coset leaders*.

Proposition 26.205. In coset decoding, a received word w is decoded as a code word c suc hthat $d(w, c)$ is minimum. That is, it is the same as nearest-neighbour decoding.

Proof. Let C be a linear code and w be the received word. Suppose that v is the coset leader for the coset $w + C$. Then, there must exist $c \in C$ such that $w = v + c$. Now, if c' is any code word then $w - c' \in w + C$ and that would mean that $\mathrm{wt}(w - c') \geq \mathrm{wt}(v)$. And thus,

$$d(w, c') = \mathrm{wt}(w - c') \geq \mathrm{wt}(v) = d(w, c)$$

■

Definition 26.206. If an (n, k) linear code over F has parity-check matrix H, then for any vector u in F^n, the vector uH is called the *syndrome* of u.

Proposition 26.207. Let C be an (n, k) linear code over F with a parity-check matrix H. Then, two vectors of F^n are in the same coset of C if and only if they have the same syndrome.

Proof. u and v are in the same coset of C if and only if $u - v$ is in C. But due to the Orthogonality Lemma, we would have that

$$0 = (u - v)H = uH - vH$$

■

Chapter 27

Introduction to Galois Theory

Some proofs in this chapter and the next are long and boring. I shall leave such proofs out in order to keep the reader engaged.

Definition 27.208 (Automorphism, Galois Group, Fixed Field). Let E be an extensionn field of the field F. An *automorphism* of E is a ring isomorphism from E onto E. The *Galois Group* of E over F, given by $\mathrm{Gal}(E/F)$, is the set of all automorphisms of E that take every element of F to itself. If H is a subgroup of $\mathrm{Gal}(E/F)$, the set

$$E_H = \{x \in E \mid \phi(x) = x \qquad \forall \phi \in H\}$$

The following theorem is the highlight of this chapter and is also quite a mouthful

Theorem 27.209 (Fundamental Theorem of Galois Theory). Let F be a field of characteristic 0 or a finite field. If E is the splitting field over F for some polynomial in $F[x]$, then the mapping from the set of subfields of E containing F to the set of subgroups of $\mathrm{Gal}(E/F)$ given by $K \to \mathrm{Gal}(E/K)$ is a one-to-one correspondence. Furthermore, for any subfield K of E containing F,

1. $[E : K] = |\,\mathrm{Gal}(E/K)|$ and $[K : F] = |\,\mathrm{Gal}(E/F)|/|\,\mathrm{Gal}(E/K)|$.
2. If K is the splitting field of some polynomial in $F[x]$, then $\mathrm{Gal}(E/K)$ is a normal subgroup of $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(K/F)$ is isomorphic to $\mathrm{Gal}(E/F)/\,\mathrm{Gal}(E/K)$.
3. $K = E_{\mathrm{Gal}(E/K)}$
4. If H is a subgroup of $\mathrm{Gal}(E/F)$, then $H = \mathrm{Gal}(E/E_H)$.

Definition 27.210 (Solvable by Radicals). Let F be a field, and let $f(x) \in F[x]$. We say that $f(x)$ is *solvable by radicals* over F if $f(x)$ splits in some extension $F(a_1, a_2, \cdots, a_n)$ of F and there exist positive integers $k_1, k_2, \cdots, k_n$ such that $a_1^{k_1} \in F$ and $a_i^{k_i} \in F(a_1, \cdots, a_{i-1})$.

Definition 27.211. We say that a group is *solvable* if G has a series of subgroups

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k = G$$

where, for each $0 \leq i < k$, H_i is normal in H_{i+1} and H_{i+1}/H_i is Abelian.

Again, the proof of the following theorem has been omitted for the aforementioned reason.

Proposition 27.212. Let F be a field of characteristic 0 and let $a \in F$. If E is the splitting field of $x^n - a$ over F, then the Galois Group $\text{Gal}(E/F)$ is solvable.

Proposition 27.213. Let G be a group and $N \triangleleft G$. If both N and G/N are solvable, then G is solvable.

Theorem 27.214 (Galois). Let F be a field of characteristic 0 and let $f(x) \in F[x]$. Supppose that $f(x)$ splits in $F(a_1, a_2, \cdots, a_t)$ where $a_1^{n_1} \in F$ and $a_i^{n_i} \in F(a_1, \cdots, a_{i-1})$. Let E be the splitting field for $f(x)$ over F in $F(a_1, a_2, \cdots, a_t)$. Then the Galois Group $\text{Gal}(E/F)$ is solvable.

Lemma 27.215. S_5 is not a solvable group.

The Quintic is NOT Solvable

Now that we have declared all our ammunition, it only remains to put it all together to solve this problem. The following proof is not that hard in the first place, but proving the previous theorems, especially **Theorem 27.209** is where all the difficulty lies.
Let $g(x) \in \mathbb{Z}[x]$ be an integer polynomial of degree 5 with the 5 zeroes being $a_1, a_2, \cdots, a_5$. Since any automorphism of $K = Q(a_1, \cdots, a_5)$ is completely determined by its action on the a_i's and must permute the a_i's we must have that $\text{Gal}(K/Q)$ is isomorphic to a subgroup of S_5. Furthermore we know that $[Q(a_1) : Q] = 5$ and hence 5 divides $[K : Q]$. And due to the Fundamental Theorem of Galois Theory, we know that 5 must divide $|\text{Gal}(K/Q)|$. Now due to Cauchy's Theorem, $\text{Gal}(K/Q)$ has an element of order 5. The only element in S_5 of order 5 is the cycle of length 5, $\text{Gal}(K/Q)$ must contain a 5 cycle. But since the transformation $a + bi \mapsto a - bi$ is also an element of $\text{Gal}(K/Q)$, we know that it contains both a 5-cycle and a 2-cycle implying that it is isomorphic to S_5. But due to the preceeding lemma, we are done.

Definition 27.216. This is a random definition