Unit –I

Laplace Transform

Definition: Let f(t) be a function of t defined for all t >0. Then the Laplace Transform of f(t), denoted by **L[f(t)]** , is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \qquad (1)$$

Where s is a parameter, which may be real or complex.

The Laplace transform of $f(t)$ exists if the integral in (1) exists i.e. the integral in (1) converges for some value of s.

- The symbol L is Laplace transform operator.
- Generally, the transform will exist for more than one value of the parameter s, and hence $L[f(t)]$ defines a function of s, when it exists, and is denoted by F(s).
- There is one to one correspondence between f(t) and F(s), and the relation transforms f(t), a function of t, into a function of another variable s.
- The operation just described, which yields F(s) from a given function f(t) is called Laplace Transformation.

Linearity Property

If C_1 and C_2 are any constants and $f_1(t)$ and $f_2(t)$ are functions whose Laplace transform exist then

$$L[c_1F_1(s) + c_2F_2(s)] = c_1L[F_1(s)] + c_2\, L[\, F_2(s)]$$

Laplace transform of some elementary Functions

Using the fundamental definition of Laplace transform, we can obtain a table of Laplace transform of some elementary functions.

1 f(t)=1

By definition of Laplace transform: $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$, we have

$$L[1] = \int_0^\infty e^{-st}\, 1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, s > 0$$

Hence $\qquad L[1] == \frac{1}{s}, s > 0$

2 $\mathbf{f(t) = e^{at}}$

By definition of Laplace transform we obtain

$$L[e^{at}] = \int_0^\infty e^{-st}\, e^{at}dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \left[\lim_{t\to\infty} \frac{e^{-(s-a)t}}{s-a}\right] + \frac{1}{s-a}$$

The limits depends upon the sign of $(s-a)$. Under the restriction $s-a>0$ i.e. $s>a$, the limit will be zero

$$= 0 + \frac{1}{s-a} \quad \text{if } s > a = \frac{1}{s-a} \quad \text{if } s > a$$

Hence $\qquad L[e^{at}] = \frac{1}{s-a} \quad \text{if } s > a \qquad\qquad (2)$

Note 1: If we replace a by –a in the above result (2) we get

$$L[e^{-at}] = \frac{1}{s+a} \quad \text{if } s > -a$$

Note 2: If we take $a = 0$ in the result (2) then we get

$$L[e^{0t}] = L[1] = \frac{1}{s} \quad \text{if } s > 0$$

Note 3: If $f(t) = c^{at}$ then we obtain

$$L[c^{at}] = L\left[e^{at\log c}\right] = \frac{1}{s - a\log c} \quad \text{if } s > a\log c, c > 0$$

3. **If f(t) = sin at**

By definition of Laplace transform we obtain

$$L[\sin at] = \int_0^\infty e^{-st}\sin at\, dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$= \left[\lim_{s\to\infty}\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right] + \frac{1}{s^2+a^2}$$

$$= 0 + \frac{1}{s^2+a^2}, \text{if } s > 0 = \frac{1}{s^2+a^2}, \text{if } s > 0$$

$$\because \left\{\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}[a\sin bt - b\cos bt]\right\}$$

Hence $L[\sin at] = \frac{1}{s^2+a^2}, \text{if } s > 0$

4 If $\mathbf{f(t) = \cos at}$

By definition of Laplace transform we obtain

$$L[\cos at] = \int_0^\infty e^{-st}\cos at\, dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$= \left[\lim_{s\to\infty}\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right] + \frac{s}{s^2+a^2}$$

$$= 0 + \frac{s}{s^2+a^2}, \text{if } s > 0 = \frac{s}{s^2+a^2}, \text{if } s > 0$$

$$\because \left\{ \int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}[a\cos bt + b\sin bt] \right\}$$

Hence $L[\cos at] = \frac{s}{s^2+a^2}$, if $s > 0$

Laplace Transform of sin at and cos at using another method

We know that $L[e^{at}] = \frac{1}{s-a}$, $s > a$

Therefore

$$L[e^{iat}] = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)} = \frac{s+ia}{s^2-(ia)^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

That is $L[e^{iat}] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$ (i)

But $e^{iat} = \cos at + i\sin at$

$$L[e^{iat}] = L[\cos at + i\sin at] = L[\cos at] + i\,L[\sin at] \quad \text{(ii)}$$

From (i) and (ii) we get

$$L[\cos at] + i\,L[\sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating real and imaginary parts we get

$$L[\cos at] = \frac{s}{s^2+a^2} \quad \text{and} \quad L[\sin at] = \frac{a}{s^2+a^2}$$

5. Laplace Transform **if f(t) = Sinh at**

By definition of Laplace transform we obtain

$$L[\sinh at] = \int_0^\infty e^{-st}\sinh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\left\{\int_0^\infty e^{-st}\,e^{at}\,dt - \int_0^\infty e^{-st}\,e^{-at}\,d\right\}t$$

$$= \frac{1}{2}\{L[e^{at}] - L[e^{-at}]\} = \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} \text{ if } s > a \text{ and } s > -a$$

$$= \frac{a}{s^2-a^2}, \text{ if } s > |a|$$

Hence

$$L[\sinh at] = \frac{a}{s^2-a^2}, \text{ if } s > |a|$$

6 **Laplace Transform of f(t) = cosh at**

By definition of Laplace transform we obtain

$$L[\cosh at] = \int_0^\infty e^{-st}\cosh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\left\{\int_0^\infty e^{-st}\,e^{at}\,dt + \int_0^\infty e^{-st}\,e^{-at}\,d\right\}t$$

$$= \frac{1}{2}\{L[e^{at}] + L[e^{-at}]\} = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} \text{ if } s > a \text{ and } s > -a$$

$$= \frac{s}{s^2 - a^2}, \text{if } s > |a|$$

Hence

$$L\,[\cosh at] = \frac{s}{s^2 - a^2}, \text{if } s > |a|$$

Another method:

$$L[\sinh at\,] = L\left(\frac{e^{at} - e^{-at}}{2}\right) dt = \frac{1}{2}\,\{L[e^{at}] - L[e^{-at}]\} = \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2}$$

$$L[\cosh at\,] = L\left(\frac{e^{at} + e^{-at}}{2}\right) dt = \frac{1}{2}\,\{L[e^{at}] + L[e^{-at}]\} = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{s}{s^2 - a^2}$$

4 Laplace Transform if f(t) = t^n

By definition of Laplace transform we obtain

$$L[t^n\,] = \int_0^\infty e^{-st}\, t^n\, dt$$

Put $st = y$, $s\,dt = dy$ we have

t	0	∞
y	0	∞

$$L[t^n\,] = \int_0^\infty e^{-y}\left(\frac{y}{s}\right)^n \frac{dy}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-y}\, y^n\, dy = \frac{1}{s^{n+1}}\,\Gamma(n+1)$$

$$\Longrightarrow L[t^n\,] = \frac{1}{s^{n+1}}\,\Gamma(n+1)$$

$$\left\{\text{since } \Gamma(n) = \int_0^\infty e^{-y}\, y^{n-1}\, dy \quad \text{therefore } \Gamma(n+1) = \int_0^\infty e^{-y}\, y^n\, dy\right\}$$

If n is positive integer n, $\Gamma(n+1) = n\,\Gamma(n)$

$$\begin{aligned}
\Gamma(n+1) &= n\,\Gamma(n) \qquad \because \Gamma(n) = (n-1)\,\Gamma(n-1)\\
&= n\,(n-1)\Gamma(n-1)\\
&= n\,(n-1)(n-2)\Gamma(n-2)\\
&= n\,(n-1)(n-2)\Gamma(n-2) \ldots\ldots\ldots..1\,\Gamma1\\
&= n\,(n-1)(n-2)\Gamma(n-2) \ldots\ldots\ldots..1 \qquad \because \Gamma1 = 1\\
&= n\,!
\end{aligned}$$

$$\Gamma(n+1) = n\,!$$

Then $L[t^n\,] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

Hence $L[t^n\,] = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}} & \text{in general} \\ \frac{n!}{s^{n+1}} & \text{if n is positive integer} \end{cases}$

If $n = \frac{1}{2}$ in above result we get

$$L\left[t^{-1/2}\,\right] = \frac{\Gamma\left(-\frac{1}{2} + 1\right)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

Table of Elementary Laplace transforms

Function	Laplace transform
f(t)	**L[f(t)]= F(s)**
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
e^{-at}	$\frac{1}{s+a}, s > -a$
$\sin at$	$\frac{a}{s^2+a^2}, s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, s > \lvert a\rvert$
$t^n \quad n > -1$	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0
t^{n-1}	$\frac{\Gamma(n)}{s^n}$, s > 0
If n is positive integer, $\Gamma(n+1) = n!$ and $\Gamma(n) = (n-1)!$ then we have $L[t^n] = \frac{n!}{s^{n+1}}$ and $L[t^{n-1}] = \frac{(n-1)!}{s^n}$	

Illustrations on Laplace Transform of Elementary functions

Examples

We find Laplace Transform of followings

Example A: Obtain the Laplace transform of the followings functions:

1. $f(t) = 3e^{4t} + 6t^2 - 4\sin 3t + \cos 2t$

   Solution: Taking Laplace Transform on both sides

$$L[f(t)] = L[3e^{4t} + 6t^2 - 4\sin 3t + \cos 2t]$$

$$= 3L[e^{4t}] + 6L[t^2] - 4L[\sin 3t] + L[\cos 2t]$$

$$\left\{ \text{since} \quad L[e^{at}] = \frac{1}{s-a}, \; L[t^n] = \frac{n!}{s^{n+1}}, \; L[\sin at] = \frac{a}{s^2+a^2} \text{ and } L[\cos at] \right\}$$

$$= 3\frac{1}{s-4} + 6\frac{2!}{s^{2+1}} - 4\frac{3}{s^2+3^3} + \frac{s}{s^2+2^2}$$

$$= \frac{3}{s-4} + \frac{12}{s^3} - \frac{12}{s^2+27} + \frac{s}{s^2+4}, \quad s > 4$$

2. $4e^{2t} + 5e^{-3t}$

Solution: $L[4\,e^{2t}+5\,e^{-3t}] = : 4L[\,e^{2t}] + 5\,L[\,e^{-3t}] = 4\frac{1}{s-2} + 5\frac{1}{s+3}$

3. e^{at+b}

Solution: $L\left[e^{at+b}\right] = L\left[e^{at}\,e^{b}\right] = e^{b}\,L[e^{at}] = e^{b}\,\frac{1}{s-a} = \frac{e^{b}}{s-a},\ s > a$

4. $(e^{-2t}+e^{3t})^2$

Solution: $L(e^{-2t}+e^{3t})^2 = L[\,e^{-4t}+2e^{t}+e^{6t}] = L[e^{-4t}] + 2L[e^{t}] + L\,[e^{6t}]$

$$= \frac{1}{s+4} + 2\frac{1}{s-1} + \frac{1}{s-6},\ \ s > 6$$

5. 4^t

Solution: $L[4^t] = L\left[e^{t\log 4}\right] = \frac{1}{s-\log 4}\ ,\ s > \log 4$

$$L(= L[\,e^{-4t}+2e^{t}+e^{6t}] = L[e^{-4t}]$$

6. $5e^{-\frac{t}{2}} + t^{-\frac{1}{2}} + 7\sin\frac{t}{2}$

Solution: $L\left[5e^{-\frac{t}{2}} + t^{-\frac{1}{2}} + 7\sin\frac{t}{2}\right] = 5\,L\left[\,e^{-\frac{t}{2}}\right] + L\left[t^{-\frac{1}{2}}\right] + 7L\left[\sin\frac{t}{2}\right]$

$$= 5\,\frac{1}{s+\frac{1}{2}} + \sqrt{\frac{\pi}{s}} + 7\frac{1/2}{s^2+\left[\frac{1}{2}\right]^2} = 5\,\frac{1}{s+\frac{1}{2}} + \sqrt{\frac{\pi}{s}} + \frac{\frac{7}{2}}{s^2+\frac{1}{4}}$$

7. $2e^{3t}+3e^{-2t}$ **Ans.** $\frac{2}{s-3} + \frac{3}{s+2}$

8. $(e^{-at}-e^{-bt})^2$ **Ans.** $\frac{1}{s+2a} + \frac{2}{s+(a+b)} + \frac{1}{s+2b}$

9. $(2e^{3t}+5)^2$ **Ans.** $\frac{4}{s-6} + \frac{20}{s-3} + \frac{25}{s}$

10. C^{at+b} **Ans.** $\frac{c^b}{s-a\log c}$

Example B: Obtain the Laplace transform of the followings functions:

1. $f(t) = 2\sin 4t + 5\cos 2t$

Solution: Taking Laplace Transform on both sides

$$L[2\sin 4t + 5\cos 2t\,] = L\,[2\sin 4t\,] + L\,[\,5\cos 2t\,] = 2\ L[\sin 4t] + 5\ L\,[\cos 2t]$$

$$= 2\ \frac{4}{s^2+4^2} + 5\ \frac{s}{s^2+2^2} = \frac{8}{s^2+16} + \frac{5s}{s^2+4}$$

$$\left\{\text{since}\quad L[\sin at] = \frac{a}{s^2+a^2}\quad \text{and}\ \ L\,[\cos at\,] = \frac{s}{s^2+a^2}\right\}$$

2. $\sin 2t\ \cos 3t$

Solution: $L[\sin 2t\cos 3t\,] = \frac{1}{2}\,L[2\sin 2t\ \cos 3t] = \frac{1}{2}\,L[\,\sin 5t\ - \sin t]$

$$= \frac{1}{2}\left\{\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right\} = \frac{1}{2}\left\{\frac{2(s^2-5)}{(s^2+25)(s^2+1)}\right\}$$

3. $\cos t\cos 2t$

4. **Solution:** $L[\cos t\cos 2t] = \frac{1}{2}\,L[2\cos 2t\cos t] = \frac{1}{2}\,L[\cos 3t + \cos t]$

$$= \frac{1}{2}\left\{\frac{s}{s^2+3^2} + \frac{s}{s^2+1^2}\right\} = \frac{1}{2}\left\{\frac{s(s^2+5)}{(s^2+9)(s^2+1)}\right\}$$

5. $\cosh at - \cos bt$

Solution: $L[\,\cosh at - \cos bt\,] = L[\,\cosh at] - L[\cos bt] = \frac{s}{s^2-a^2} - \frac{s}{s^2-b^2}$

6. $\text{Cos}(wt + b)$ **Ans.** $\cos b\left(\frac{s}{s^2+w^2}\right) - \sin b\ \left(\frac{1}{s^2+w^2}\right), s > 0$

7. $\text{Sin}(wt + b)$ **Ans.** $\cos b\left(\frac{w}{s^2+w^2}\right) + \sin b\ \left(\frac{s}{s^2+w^2}\right), s > 0$

8. $3\cos(4t+7)$ **Ans.** $3\left[\cos 7\left(\frac{s}{s^2+4^2}\right) - \sin 7\ \left(\frac{4}{s^2+4^2}\right)\right], s > 0$

9. $5\sin(2t+3)$ **Ans.** $5\left\{\cos 3\left(\frac{2}{s^2+2^2}\right) + \sin 3\ \left(\frac{s}{s^2+2^2}\right)\right\}, s > 0$

10. $\sin^2 4t$ **Ans.** $\frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+8^2}\right]$ or $\frac{32}{s(s^2+64)}$ $s > 0$

11. $\cos^3 2t$

Solution: $L[\cos^3 2t] = L\left[\frac{\cos 6t + 3\cos 2t}{4}\right]$ $\because \cos 3t = 4\cos^3 t - 3\cos t$

$$= \frac{1}{4}\{L[\cos 6t] + L\,[3\cos 2t]\} = \frac{1}{4}\left\{\frac{s}{s^2+6^2} + \frac{3s}{s^2+2^2}\right\} = \frac{s(s^2+28)}{(s^2+4)(s^2+36)}, s > 0$$

12. $\cosh^3 2t$

Solution: $L[\cosh^3 2t] = L\left[\frac{\cosh 6t + 3\cosh 2t}{4}\right]$ $\because \cosh 3t = 4\cosh^3 t - 3\cosh t$

$$= \frac{1}{4}\{L[\cosh 6t] + L\,[3\cosh 2t]\} = \frac{1}{4}\left\{\frac{s}{s^2-6^2} + \frac{3s}{s^2-2^2}\right\} = \frac{s(s^2-28)}{(s^2-4)(s^2-36)}$$

13. $3\cos 2t - \sin 2t$ **Ans.** $\frac{3s}{s^2+4} - \frac{2}{s^2+4},\ s > 0$

14. $\cosh 5t + \cos 5t$ **Ans.** $\frac{s}{s^2-25} - \frac{s}{s^2+25},\ s > 0$

15. $\text{Cos}\ 3t\cos 2t$ **Ans.** $\frac{1}{2}\left\{\frac{s}{s^2+25} + \frac{s}{s^2+1}\right\}$ or $\frac{s(s^2+13)}{(s^2+1)(s^2+25)}$

16. $\sin 2t\cos 5t$ **Ans.** $\frac{1}{2}\left\{\frac{7}{s^2+49} - \frac{3}{s^2+9}\right\}$ or $\frac{s(s^2-21)}{(s^2+9)(s^2+49)}$

Example C: Obtain the Laplace transform of the followings functions:

1. $at + bt^2 + ct^3$

Solution: $L[\,at + bt^2 + ct^3\,] = a\,L\,[t] + b\,L\,[t^2] + cL\,[t^3]$ $\because\ L[t^n] = \frac{n!}{s^{n+1}}$

$$= a\frac{1}{s^2} + b\frac{2!}{s^3} + c\frac{3!}{3^4} = \frac{a}{s^2} + \frac{2b}{s^3} + \frac{6c}{3^4},\ \text{where } s > 0$$

2. $4t^3 + t^7 + t^{\frac{4}{3}}$

Solution: $L\,[4t^3 + t^7 + t^{\frac{4}{3}}] = 4\,L\,[t^3] + L\,[t^7] + L\left[t^{\frac{4}{3}}\right]$

$$\because L[t^n] = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}} & \text{in general} \\ \frac{n!}{s^{n+1}} & \text{if n is positive integer} \end{cases}$$

$$= 4\frac{3!}{s^4} + \frac{7!}{s^8} + \frac{\Gamma\left(\frac{4}{3}\right)}{3^{4/3+1}} = \frac{24}{s^4} + \frac{5040}{s^8} + \frac{\frac{4}{3}\cdot\frac{1}{3}\cdot\Gamma\left(\frac{1}{3}\right)}{3^{7/3}},\ \text{where } s > 0$$

3. $(2t+3)^3$

Solution: $L\,[(2t+3)^3] = L\,[\ (2t)^3 + 3\,(2t)^2(3) + 3\,(2t)(3)^2 + (3)^3]$

$$= 8\,L\,[t^3] + 36\,L\,[t^2\,] + 54\,L\,[\,t] + 27\,L\,[1\,]$$

$$= 8\ \frac{3!}{s^4} + 36\ \frac{2!}{s^3} + 54\ \frac{1!}{s^2} + 27\ \frac{1}{s} = \frac{48}{s^4} + \frac{72}{s^3} + \frac{54}{s^2} + \frac{27}{s},\qquad s > 0$$

4. $5t - 7e^{-6t} + t^{5/2}$

Solution: $5\,L[\,t\,] - 7\,L\,[e^{-6t}] + L\left[t^{\frac{5}{2}}\right] = 5\frac{1!}{s^2} - 7\frac{1}{s+6} + \frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{5}{2}+1}}$

$$= \frac{5}{s^2} - \frac{7}{s+6} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{7/2}} = \frac{5}{s^2} - \frac{7}{s+6} + \frac{15}{8}\sqrt{\frac{\pi}{s^7}}, \qquad s > 0$$

5. $t^2 - 3t + 5$ **Ans.** $\frac{5s^2-3s+2}{s^3}$
6. $4t^4 + 5t^3 + t^{1/2}$ **Ans.** $\frac{24}{s^5} + \frac{30}{s^4} + \frac{15}{8}\frac{\sqrt{\pi}}{2\, s^{3/2}}$
7. $a + \frac{b}{\sqrt{t}}$ **Ans.** $\frac{a}{s} + b\sqrt{\frac{\pi}{s}}$
8. $(t+2)^3 + (e^{2t}+3)^2$ **Ans.** $\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$
9. $(e^{2t}+3)^2$ **Ans.** $\frac{1}{s-4} + \frac{6}{s-2} + \frac{9}{s}$

Example D: Obtain the Laplace transform of the followings functions:

1. $f(t) = \begin{cases} a, & 0 < t < b \\ 0, & t > b \end{cases}$

Solution: Given function is discontinuous and we cannot get Laplace transform by using table of elementary Laplace transforms. Here we use fundamental definition of Laplace transform

By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^b e^{-st} f(t)dt + \int_b^\infty e^{-st} f(t)dt$

$$= \int_0^b e^{-st}\, a\, dt + \int_b^\infty e^{-st}\, 0\, dt = a\left[\frac{e^{-st}}{-s}\right]_0^b = a\left(\frac{e^{-sb}-1}{-s}\right) = \frac{a}{s}(1 - e^{-sb})$$

2. $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

Solution: By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^4 e^{-st} f(t)dt + \int_4^\infty e^{-st} f(t)dt$

$$= \int_0^b e^{-st}\, t\, dt + \int_b^\infty e^{-st}\, 5\, dt = \left\{t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right\}_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= \left[\left(\frac{4\,e^{-4s}}{-s} - \frac{e^{-4s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] + 5\left[0 + \frac{e^{-4s}}{s}\right] = \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

3. $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

Solution: By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st} f(t)dt + \int_\pi^\infty e^{-st} f(t)dt$

$$= \int_0^b e^{-st} \sin 2t\, dt + \int_b^\infty e^{-st}\, 0\, dt = \left[\frac{e^{-st}}{s^2+2^2}[-s\sin 2t - 2\cos 2t]\right]_0^\pi$$

$$\because \int e^{at} \sin bt = \frac{e^{at}}{a^2+b^2}[a \sin bt - b\cos bt]$$

$$= \frac{e^{-s\pi}}{s^2+2^2}[-s\sin 2\pi - 2\cos 2\pi] - \frac{e^{-0}}{s^2+2^2}[-s\sin 0 - 2\cos 0]$$

$$= \frac{1}{s^2+2^2}[e^{-s\pi}(-2) - (-2)] = \frac{2}{s^2+4}(1 - e^{-s\pi}), \qquad s > 0$$

4. $f(t) = \begin{cases} t/T, & 0 \le t < T \\ 1, & t > T \end{cases}$

Solution: By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt$

$$= \int_0^T e^{-st}\left(\frac{t}{T}\right) dt + \int_T^\infty e^{-st}\, 1\, dt = \frac{1}{T}\left\{t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right\}_0^T + \left[\frac{e^{-st}}{-s}\right]_T^\infty$$

$$= \frac{1}{T}\left[\left(\frac{T\,e^{-sT}}{-s} - \frac{e^{-sT}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] + \left[0 + \frac{e^{-sT}}{s}\right] = \frac{e^{-sT}}{-s} - \frac{e^{-sT}}{Ts^2} + \frac{1}{Ts^2} + \frac{e^{-sT}}{s} = \frac{1 - e^{-sT}}{Ts^2}$$

5. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$ **Ans.** $\frac{s+(s-1)e^{-s\pi}}{s^2+1}$

6. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2 - 2t + 2 & t \ge 1 \end{cases}$ **Ans.** $e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$

7. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$ **Ans.** $e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - \left(\frac{1}{s^2} + \frac{2}{s}\right)$

8. $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$ **Ans.** $\frac{s\,(1-e^{-2\pi s})}{s^2+1}$

B. General Theorems of Laplace Transforms

We shall now derive some theorems that will be used to find the Laplace transform of some functions not included in the table below.

Function	**Laplace transform**
f(t)	L[f(t)]= F(s)
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
e^{-at}	$\frac{1}{s+a}, s > -a$
$\sin at$	$\frac{a}{s^2+a^2}, s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}, s > \|a\|$
$t^n \quad n > -1$	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0
t^{n-1}	$\frac{\Gamma(n)}{s^n}$, s > 0
If n is positive integer, $\Gamma(n+1) = n!$ and $\Gamma(n) = (n-1)!$ then we have $L[t^n] = \frac{n!}{s^{n+1}}$ and $L[t^{n-1}] = \frac{(n-1)!}{s^n}$	

1 **First Shifting Theorem**:

If $L[f(t)] = F(s)$ then

$$L[e^{-at} f(t)] = \{L[f(t)]\}_{s\to s+a} = \{F(s)\}_{s\to s+a} = F(s+a)$$

That is $L[e^{-at} f(t)] = F(s+a)$

Proof: By definition of Laplace Transform

$$L\,[\,e^{-at}\,f(t)] = \int_0^{\infty} e^{-st}\,e^{-at}\,f(t)dt = \int_0^{\infty} e^{-(s+a)t}\;f(t)dt$$

$$= \int_0^{\infty} e^{-pt}\;f(t)dt \qquad \text{Where } p = s + a$$

$$= F(p), \qquad \text{by definition as } s\rightarrow p$$

$$= F(s + a)$$

Hence

$$L[e^{-at}\,f(t)] = F(s + a) \;=\; \{F(s)\}_{s\rightarrow s+a} =\; \{L[f(t)]\}_{\,s\rightarrow s+a}$$

Remark 1: In other words the Laplace transform of e^{-at} times a function of t is equal to the Laplace transform of the function f(t) with s replaced by $s + a$.

Procedure: To obtain Laplace transform of e^{-at} f(t), we first obtain Laplace transform of f(t) i.e. F(s), {by dropping the factor e^{-at} initially} and then replace s by $s + a$ in $L[f(t)]$ to account for the multiplying factor e^{-at}.

Examples based on first shifting theorem:

Ex. 1 Find the Laplace transform of each of the following functions:

(i) $\mathbf{e^{-4t}\cosh 5t}$

Solution: Here we use the first shifting theorem.{because given function is in the form of e^{-at} f(t)}

To obtain Laplace transform of e^{-4t} cosh 5t we first obtain Laplace transform of cosh 5t and then replace s by s+a in L[cosh 5t].

Let f(t) = cosh 5t with $a = 4$

Therefore L[f(t)]= L[cosh 5t] $= \frac{s}{s^2-5^2} = F(s), s > |5|$

i.e. $F(s) = \frac{s}{s^2-5^2},\;\; s > |5|$ …. (1)

Then by first shifting theorem

IF L[f(t)] = F(s) then L[e^{-at} f(t)]= F(s+a) that

$$L[e^{-4t}\cosh 5t] = F(s + a)$$

$$= F(s + 4)$$

To obtain F(s+4), we replace s by s+4 in equation (1) we get

$$L[e^{4t}\cosh 5t] = \frac{s}{(s+4)^2 - 5^2}$$

(ii) $\mathbf{e^{4t}\sin 5t}$

Solution: Here we also use the first shifting theorem.

To obtain Laplace transform of e^{4t}sin5t, we first obtain Laplace transform of sin5t and then replace s by s+a (where $a = -4$) in L[sin 5t].

Let f(t) = sin 5t with $a = -4$

Therefore L[f(t)]= L[sin 5t]$= \frac{5}{s^2+5^2} = F(s),\;\; s > 0$

i.e. $F(s) = \frac{5}{s^2+5^2},\;\; s > 0$ …. (1)

Then by first shifting theorem,"IF L[f(t)] = F(s) then L[e^{-at} f(t)]= F(s+a)" that

$$L[e^{4t}\sin 5t] = F(s + a) \quad \text{Where } a = -4$$

$$= F(s - 4)$$

To obtain $F(s - 4)$, we replace s by $s - 4$ in equation (1)

$$L[e^{4t}\sin 5t] = F(s-4) = \{F(s)\}_{s\to s-4} = \left\{\frac{5}{s^2+5^2}\right\}_{s\to s-4} = \frac{5}{(s-4)^2+5^2},\ \ s-4>0$$

(iii) $(t+5)^2e^{4t}$

Solution: given function can be written as

$$(t+5)^2e^{4t} = (t^2+10t+25)e^{4t} = f(t)e^{-at}$$

Here $f(t) = t^2+10t+25$ and $a=-4$

Now $L[f(t)] = L[t^2+10t+25]$

$$= L[t^2] + L[10t] + L[25]$$

$$= L[t^2] + 10\,L[t] + 25\,L[1] \quad \text{….. (1)}$$

We know that $L[1]=\frac{1}{s},\ s>0$ and $L[t^n] = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}} & \text{in general} \\ \frac{n!}{s^{n+1}} & \text{if n is positive integer} \end{cases}$

Hence $L[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$, and $L[t] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

Substituting these values in equation (1) we get

$$L[f(t)] = \frac{2}{s^3} + 10\frac{1}{s^2} + 25\frac{1}{s},\ \ s>0$$

$$= F(s) \quad \text{…(2)}$$

Now by first shifting theorem

$$L[(t+5)^2e^{4t}] = L[f(t)e^{4t}] = F(s+a) = F(s-4) = \{F(s)\}_{s\to s-4}$$

$$= \left\{\frac{2}{s^3} + 10\frac{1}{s^2} + 25\frac{1}{s}\right\}_{s\to s-4}$$

$$= \frac{2}{(s-4)^3} + 10\frac{1}{(s-4)^2} + 25\frac{1}{s-4},\quad s-4>0$$

Examples for Practice:

Ex. 1 Find the Laplace transform of each of the following functions:

(i) $e^{-at}\sin bt$

Solution: $L[\sin bt] = \frac{b}{s^2+a^2}$

$$L[e^{-at}\sin bt] = \{L[\sin bt]\}_{s\to s+a} = \left\{\frac{b}{s^2+b^2}\right\}_{s\to s+a} = \frac{b}{(s+a)^2+b^2}$$

(ii) $e^{-at}\cos bt$

Solution: $L[\cos bt] = \frac{s}{s^2+a^2}$

$$L[e^{-at}\cos bt] = \{L[\cos bt]\}_{s\to s+a} = \left\{\frac{s}{s^2+b^2}\right\}_{s\to s+a} = \frac{s+a}{(s+a)^2+b^2}$$

(iii) $e^{-at}\sinh bt$

Solution: $L[\sinh bt] = \frac{b}{s^2-b^2}$

$$L[e^{-at}\sinh bt] = \{L[\sinh bt]\}_{s\to s+a} = \left\{\frac{b}{s^2-b^2}\right\}_{s\to s+a} = \frac{b}{(s+a)^2-b^2}$$

(iv) $e^{-at}\cosh bt$

Solution: $L[\cosh bt] = \frac{s}{s^2-b^2}$

$$L[e^{-at}\cosh bt] = \{L[\cosh bt]\}_{s\to s+a} = \left\{\frac{s}{s^2-b^2}\right\}_{s\to s+a} = \frac{s+a}{(s+a)^2-b^2}$$

(v) $e^{4t}\cosh 5t$

Solution: $L[\cosh 5t] = \frac{s}{s^2-25}$

$$L[e^{4t}\cosh 5t] = \{L[\cosh 5t]\}_{s\to s-4} = \left\{\frac{s}{s^2-25}\right\}_{s\to s-4} = \frac{s-4}{(s-4)^2-25}$$

(vi) $e^{-at}t^n$

Solution: $L[\,t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

$$L[e^{-at}\,t^n] = \{L[t^n]\}_{s\to s+a} = \left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\}_{s\to s+a} = \frac{\Gamma(n+1)}{(s+a)^{n+1}}$$

(vii) $(t+2)^2e^{4t}$

Solution: $L\,(t+2)^2 = L[\,t^2+4t+4] = \frac{2!}{s^3} + 4\,\frac{1}{s^2} + 4\frac{1}{s} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

$$L[e^{4t}\,(t+2)^2\,] = \{L[(t+2)^2]\}_{s\to s-4} = \left\{\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right\}_{s\to s-4} = \frac{2}{(s-4)^3} + \frac{4}{(s-4)^2} + \frac{4}{s-4}$$

(viii) $.\,e^{-2t}(\,3\cos 6t - 5\sin 6t)$

Solution: $L[3\cos 6t - 5\sin 6t] = 3L[\cos 6t] - 5L[\sin 6t] = 3\frac{s}{s^2+36} - 5\frac{6}{s^2+36} = \frac{3s-30}{s^2+36}$

$$L[e^{-2t}(\,3\cos 6t - 5\sin 6t)\,] = \{L[3\cos 6t - 5\sin 6t]\}_{s\to s+2} = \left\{\frac{3s-30}{s^2+36}\right\}_{s\to s+2}$$

$$= \frac{3(s+2)-30}{(s+2)^2+36} = \frac{3s-24}{s^2+4s+40}$$

(ix) $e^{-3t}\sin^2 t$

Solution: $L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\{L[1] - L[\cos 2t]\} = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+4}\right\} = \frac{2}{s(s^2+4)}$

$$L[e^{-3t}\sin^2 t\,] = \{L\,[\sin^2 t]\}_{s\to s+3} = \left\{\frac{2}{s(s^2+4)}\right\}_{s\to s+3} = \frac{2}{(s+3)[(s+3)^2+4]}$$

$$= \frac{2}{(s+3)(s^2+6s+13)}$$

(x) $\cosh at \sin at$

Solution: $L[\,\cosh at\sin at\,] = L\,\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at\right\} = \frac{1}{2}\,\{L[\,e^{at}\sin at] + L[e^{-at}\sin at]\,\}$

$$= \frac{1}{2}\,[\{L[\sin at]\}_{s\to s-a} + \{L[\sin at]\}_{s\to s+a}] = \frac{1}{2}\left[\left\{\frac{a}{s^2+a^2}\right\}_{s\to s-a} + \left\{\frac{a}{s^2+a^2}]\right\}_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right] = \frac{a(s^2+2a^2)}{s^4+4a^4}$$

Ex. 2 Find the Laplace transform of each of the following functions:

(i) $e^{-at}(\,2\cos bt - 3\sin bt)$ **Ans.** $\frac{2s-2a-2b}{(s-a)^2+b^2}$

(ii) $2e^t\sin 4t\cos 2t$ **Ans.** $\frac{6}{s^2-2s+37} + \frac{2}{s^2-2s+5}$

(iii) $e^{-3t}\sin^2 t$ **Ans.** $\frac{2}{(s+3)(s^2+6s+13)}$

(iv) $e^{-3t}\,t^{3/2}$ **Ans.** $\frac{\Gamma(\frac{3}{2}+1)}{(s+3)^{\frac{7}{2}}}$

(v) $e^{-3t}t^3$ **Ans.** $\frac{6}{(s+3)^4}$

(vi) $e^{-t}\sin^3 t$ **Ans.** $\frac{6}{(s^2+2s+2)(s^2+2s+10)}$

2 Second shifting Theorem

Theorem: If $L[f(t)] = F(s)$ and $F(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ then $L[F(t)] = e^{-as}\,F(s)$

Proof: By definition of Laplace transform

$$L[F(t)] = \int_0^\infty e^{-st} F(t)dt = \int_0^a e^{-st} F(t)dt + \int_a^\infty e^{-st} F(t)dt$$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} f(t-a)dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

t	a	∞
u	0	∞

Put $t - a = u, dt = du$

$$L[F(t)] = \int_0^\infty e^{-s(a+u)} f(u)du = e^{-as} \int_0^\infty e^{-su} f(u)du = e^{-as} F(s)$$

Hence $L[F(t)] = e^{-as} F(s)$ where $F(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

The second shifting theorem concerns shifting on t-axis: the replacement of t in f(t) by (t-a) (i.e. shifting graph of f(t) to the right through distance a, corresponds to multiplication of the transform F(s) by e^{-as}.

To obtain the Laplace transform of F(t), we first obtain f(t) from f(t-a) and its Laplace transform F(s) and then required transform is written as e^{-as} F(s)..

Examples based on Second shifting theorem:

Ex. 1 Find L[F(t)] if $F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$

Solution: To obtain the Laplace transform of F(t), we first obtain f(t) from $f(t-a)$and its Laplace transform F(s) and then required transform is written as e^{-as} F(s).

Here $f(t-a) = \cos(t - 2\pi/3)$

Therefore $f(t) = \cos(t)$ with $a = 2\pi/3$

$$L[f(t)] = L[\cos t] = \frac{s}{s^2+1}, \ s > 0$$

$$\text{i.e. } F(s) = \frac{s}{s^2+1}, \ s > 0$$

Then by second shifting theorem

If $L[f(t)] = F(s)$ and $F(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ then $L[F(t)] = e^{-as} F(s)$ that

$$L[F(t)] = e^{-as} F(s) \quad \text{Where } a = 2\pi/3 \text{ and } F(s) = L[f(t)] = L[(\cos t)]$$

$$L[F(t)] = e^{(-2\pi/3)s}\left(\frac{s}{s^2+1}\right), \ s > 0$$

Ex.2 Find L[F(t)] if $F(t) = \begin{cases} 5\sin 3\left(t - \frac{\pi}{4}\right), & t > \frac{\pi}{4} \\ 0, & t < \frac{\pi}{4} \end{cases}$

Solution: To obtain the Laplace transform of F(t), we first obtain f(t) from $f(t-a)$and its Laplace transform F(s) and then required transform is written as e^{-as} F(s)..

Here $f(t-a) = 5\sin 3(t - \pi/4)$

Therefore $f(t) = 5\sin 3t$ where $a = \pi/4$

$$L[f(t)] = L[5\sin 3t] = 5\,L[\sin 3t] = 5\frac{3}{s^2+3^2}, \ s > 0$$

$$\text{i.e. } F(s) = \frac{5s}{s^2+9}, \ s > 0$$

Then by second shifting theorem:

$$L[F(t)] = e^{-as} F(s) \quad \text{Where } a = \pi/4 \text{ and } F(s) = \frac{5s}{s^2+9}, \ s > 0. \quad \ldots\ldots (2)$$

Hence from equation (2) we get

$$L[F(t)] = e^{(-\pi/4)s}\left(\frac{5s}{s^2+9}\right),\ s > 0$$

Ex. 3 Find the Laplace transform of the following functions by second shifting theorem:

i) $F(t) = \begin{cases}\cos(t-\alpha) & t > \alpha\\ 0, & t < \alpha\end{cases}$ **Ans.**: $e^{-\alpha s}\left(\frac{s}{s^2+1}\right)$

ii) $F(t) = \begin{cases}(t-1)^3, & t > 1\\ 0, & t < 1\end{cases}$ **Ans.**: $e^{-s}\left(\frac{6}{s^4}\right)$

iii) $F(t) = \begin{cases}e^{-4(t-3)}\sin 3(t-3), & t > 3\\ 0, & t < 3\end{cases}$ **Ans.**: $e^{-3s}\left(\frac{3}{(s+4)^2+9}\right)$

iv) $F(t) = \begin{cases}\sin 2(t-\pi), & t > \pi\\ 0, & t < \pi\end{cases}$ **Ans.**: $e^{-\pi s}\left(\frac{2}{s^2+4}\right)$

3 Change of scale theorem

Theorem: If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: By definition of Laplace transform

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

Put $at = u$, $adt = du$ We get

t	0	∞
u	0	∞

$$L[f(at)] = \int_0^\infty e^{-s(u/a)} f(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} f(u)du = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Ex. 1 If $L[\sin t] = \frac{1}{s^2+1}$ Find $L[\sin at]$

Solution: Given $L[f(t)] = L[\sin t] = \frac{1}{s^2+1} = F(s)$

Then by change of scale theorem we have

$$L[\sin at] = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}\frac{1}{\left(\frac{s}{a}\right)^2+1} = \frac{a}{s^2+a^2}$$

Ex. 2 If $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$ then, Find $L\left[\frac{\sin at}{t}\right]$

Solution: Given $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right) = F(s)$ so that $f(t) = \frac{\sin t}{t}$ and $f(at) = \frac{\sin at}{at}$

Therefore by change of scale theorem we have

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

⇨ $L\left(\frac{\sin at}{at}\right) = \frac{1}{a}F\left(\frac{s}{a}\right)$

⇨ $L\left(\frac{\sin at}{at}\right) = \frac{1}{a}\tan^{-1}\left(\frac{1}{s/a}\right)$

⇨ $\frac{1}{a}\ L\left(\frac{\sin at}{t}\right) = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right)$

⇨ $L\left(\frac{\sin at}{t}\right) = \tan^{-1}\left(\frac{a}{s}\right)$

Ex. 3 If $L[f(t)] = \frac{-2s^2+12s+8}{(s^2+4)^2}$ then obtain the Laplace transform of $f(2t)$

Solution: given $L[f(t)] = \frac{-2s^2+12s+8}{(s^2+4)^2} = F(s)$ (1)

Then by change of scale theorem we have

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right) \quad \text{Implies} \quad L[f(2t)] = \frac{1}{2}F\left(\frac{s}{2}\right)$$

$$L[f(t)] = \frac{1}{2}\frac{-2\left(\frac{s}{2}\right)^2 + 12\left(\frac{s}{2}\right) + 8}{\left(\left(\frac{s}{2}\right)^2 + 4\right)^2} = \frac{16[-s^2 + 12s + 16]}{(s^2 + 16)^2}$$

4 Laplace transform of Derivative

To solve differential equations by Laplace transform method Transform of derivative is required.

Theorem: If $L[f(t)] = F(s)$ then

$$L[f'(t)] = s\,L[\,f(t)] - f(0) = s\,F(s) - f(0)$$

is continuous for $t \geq 0$ and is of exponential of order α $\left[\text{i.e.} \lim_{b \to \infty} e^{-sb} f(b) = 0 \text{ for } s > \alpha\right]$.

Proof: Using integration by parts considering e^{-st} as first function we have

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt = \lim_{b\to\infty}\int_0^b e^{-st} f'(t)dt = \lim_{b\to\infty}\left\{[e^{-st} f(t)]_0^b + s\int_0^b e^{-st} f(t)dt\right\}$$

$$= \lim_{b\to\infty}\left\{\left[e^{-sb} f(b) - f(0)\right] + s\int_0^b e^{-st} f(t)dt\right\}$$

$$= s\int_0^\infty e^{-st} f(t)dt + e^{-\infty} f(\infty) - f(0) = sF(s) + f(0)$$

{Since f(t) is of exponential of order α}

Hence $L[f'(t)] = s\,L[\,f(t)] - f(0) = s\,F(s) - f(0)$

By applying above result to second order derivative $f''(t)$, we obtain

$$L[f''(t)] = s^2F(s) - s\,f(0) - f'(0)$$

Similarly

$$L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

By using mathematical induction we can obtain

$$L[f^n(t)] = s^n\,F(s) - s^{n-1}f(0) - s^{n-2}\,f'(0) - s^{n-3}\,f''(0) - \cdots - s\,f^{n-2}(0) - f^{n-1}(0)$$

In remembering the above remembering the above result following observations are quit useful.

- Except first term all terms are negative
- Power of n in first term is n (i.e. the order of the derivative whose Laplace transform is required) and goes on decreasing by one in subsequent terms up to zero.
- Multiplier of s^n is F(s)= L[f(t)] and that of subsequent terms $f(0), f'(0), f''(0)$ ... etc
- Sum of power of s and order of derivative of a function f(0) in every term (except first) is $(n - 1)$.

Laplace transform of derivative of a function f(t), corresponds to multiplication of transform F(s) by s.

Ex. 1: Find Laplace transform of **(i)** $\frac{d^2x}{dt^2}$ **(ii)** $\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 5y$, given that $y(0) = 2, y'(0) = -4$

Solution: (i) We Know that, $L[f''(t)] = s^2F(s) - s\,f(0) - f'(0)$

Taking Laplace Transform on both sides we get

$$L\left[\frac{d^2x}{dt^2}\right] = s^2\,X(s) - s\,x(0) - x'(0)$$

(ii) Given that, $\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 5y$ with $y(0) = 2$ and $y'(0) = -4$

Taking LT of $\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 5y$, we get

$$L\left[\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y\right] = \left[\frac{d^2y}{dt^2}\right] - 3\left[\frac{dy}{dt}\right] + 5y$$

$$= [s^2\,Y(s) - s\,y(0) - y'(0)] - 3[sY(s) - y(0)] + 5Y(s)$$

$$= \left[s^2\,Y(s) - 2s - (-4)\right] - 3[sY(s) - 2] + 5Y(s)$$

$$= [s^2 - 3s + 5]Y(s) - 2s + 4 + 6$$

$$= [s^2 - 3s + 5]Y(s) - 2s + 10$$

Ex. 2: Obtain the Laplace transform of y (t) if $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 4y = t$ and given that $y(0) = y'(0) = y''(0) = 1.$

Solution: Given $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 4y = t$ and given that $y(0) = y'(0) = y''(0) = 1.$

Taking Laplace Transform on both sides we get

$$L\left[\frac{d^3y}{dt^3}\right] - L\left[\frac{d^2y}{dt^2}\right] + 4\,L\left[\frac{dy}{dt}\right] - 4\,L[y] = L[t]$$

$$[s^3\,Y(s) - s^2\,y(0) - s\,y'(0) - y''(0)] - [s^2\,Y(s) - s\,y(0) - y'(0)]$$

$$+4[s\,Y(s) - y(0)] - 4Y(s) = \frac{1}{s^2}$$

$$\therefore [s^3 - s^2 + 4s - 4\,]Y(s) - [-s^2 - s - 1] - (-s - 1) - 4 = \frac{1}{s^2}$$

$$\{\because y(0) = y'(0) = y''(0) = 1\}$$

⇨ $(s^2 + 4)(s - 1)Y(s) = s^2 + 4 + \frac{1}{s^2}$

⇨ $Y(s) = \frac{1}{(s-1)} + \frac{1}{s^2(s^2+4)(s-1)}$

Ex. 3: Given that $4\,f''(t) + f'(t) = 0$, $f(0) = 0$ and $f'(0) = 2$ show that, $L[f(t)] = \frac{8}{4s^2+s}$.

Solution: Taking Laplace transform on both side of given equation, we get

$$L[4\,f''(t) + f'(t)] = 0$$

$$4L[f''(t)] + L[\,f'(t)] = 0$$

But we know that $L[f''(t)] = s^2F(s) - s\,f(0) - f'(0)$ and $L[f'(t)] = sF(s) - f(0)$

Therefore

$$4L[f''(t)] + L[f'(t)] = 0$$

⇨ $4[s^2F(s) - s\,f(0) - f'(0)] + sF(s) - f(0) = 0$

$$4[s^2F(s) - s\,.0 - 2] + sF(s) - 0 = 0$$

$$4s^2F(s) - 8 + sF(s) = 0$$

$$(4s^2 + s)F(s) = 8$$

⇨ $L[f(t)] = \frac{8}{4s^2+s}$

Ex. 4: Given that $4\,f''(t) + f(t) = 0$, $f(0) = 0$ and $f'(0) = 2$ show that, $L[f(t)] = \frac{8}{4s^2+1}$

5 Laplace Transform of integrals

Theorem: If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\,F(s) = \frac{1}{s}\,L[f(t)]$ or $\frac{1}{s}\,L[f(t)]$

Proof: Let $g(t) = \int_0^t f(u)\,du$ then $g'(t) = f(t)$ and $g(0) = 0$

Taking Laplace transform of both sides we get

$$L[g'(t)] = L[f(t)]$$

⇨ $s\,L[g(t)] - g(0) = F(s)$

⇨ $s\,L[g(t)] = F(s)$ $\qquad\because g(0) = 0$

⇨ $L[g(t)] = \frac{1}{s}F(s)$

⇨ $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}F(s)$

Hence $L\left[\int_0^t f(u)\,du\right] = L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\,L[f(t)] = \frac{1}{s}F(s)$

By using above result we can obtain

$$L\left[\int_0^t\int_0^t f(u)\,du\,du\right] = L\left[\int_0^t \varphi(t)\,dt\right] = \frac{1}{s}\,L[\varphi(t)] = \frac{1}{s}L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\left[\frac{1}{s}F(s)\right] = \frac{1}{s^2}F(s)$$

Remark 1: Laplace transform of integral of f(t) over (0,t) corresponds to division of transform F(s) by s

Remark 2: in general

$$\boxed{L\left[\int_0^t\int_0^t \ldots\ldots\int_0^t f(t)\,dt^n\right] = \frac{1}{s^n}\,L[F(s)] = \frac{1}{s^2}F(s)}$$

Ex. Obtain Laplace transform of $\int_0^t e^{-4t}\sin 3t\,\,dt$

Solution: Here $f(t) = e^{-4t}\sin 3t$

By first shifting theorem

$$L[e^{-at}f(t)] = [F(s)]_{s\to s+a} \text{ or } \{L[f(t)]\}_{s\to s+a} \text{ or } F(s+a) \text{ we have}$$

$$L[e^{-4t}\sin 3t\,] = \{L[\sin 3t]\}_{s\to s+4} = \left\{\frac{s}{s^2+3^2}\right\}_{s\to s+4} = \frac{s+3}{(s+3)^2+3^2} = F(s)$$

By L T of integration rule $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}F(s)$ we have

$$L\left[\int_0^t e^{-4t}\sin 3t\,\,dt\right] = \frac{1}{s}L[e^{-4t}\sin 3t\,] = \frac{1}{s}\,\frac{s+3}{(s+3)^2+3^2}$$

Ex. Find Laplace transform of $\int_0^t \sin 2t\,dt$

Sol. Here $f(t) = \sin 2t$ and $L[\sin 2t] = \frac{2}{s^2+4} = F(s)$

Hence by $\left[\int_0^t f(u)\,du\right] = L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\,L[f(t)] = \frac{1}{s}F(s)$ we have

$$L\left[\int_0^t \sin 2t\,\,dt\right] = \frac{1}{s}\,\frac{2}{s^2+4} = \frac{2}{s(s^2+4)}$$

Example: Verify $L\left[\int_0^t u^2e^{-u}du\right] = \frac{1}{s}\ L[t^2e^{-t}]$

Solution: LHS $=L\left[\int_0^t u^2e^{-u}du\right] = L[\{u^2(-e^{-u}) - (2u)(e^{-u}) + (2)(-e^{-u})\}_0^t]$

$$= L[\{(u^2 + 2u + 2)e^{-u}\}_0^t] = L[2 - (t^2 + 2t + 2)e^{-t}]$$

$$= 2\ L[1] - L[(t^2 + 2t + 2)e^{-t}] = \frac{2}{s} + \left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right]_{s\to s+1}$$

$$= \frac{2}{s} + \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{s+1} = \frac{2}{s(s+1)^3}$$

RHS $= \frac{1}{s}\ L[t^2e^{-t}] = \frac{1}{s}L[e^{-t}\ (t^2)] = \frac{1}{s}\ \left[\frac{2}{s^3}\right]_{s\to s+1} = \frac{2}{s(s+1)^3}$

Hence RHS= LHS ,

6 Multiplication by power of t

Theorem: If $L[f(t)] = F(s)$ then $L[t.f(t)] = (-1)\frac{d}{ds}L[f(t)] = (-1)\frac{d}{ds}\ F(s)$ and in general

$$L[t^n.f(t)] = (-1)^n\frac{d^n}{ds^n}L[f(t)] = (-1)^n\frac{d^n}{ds^n}[\ F(S)]$$

By definition, we have $F(s) = L[f(t)] = \int_0^\infty e^{-st}\ f(t)dt$

Diff. both side w. t. t. s , (by DUIS)we get

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st}\ f(t)dt = \int_0^\infty \frac{\partial}{\partial s}e^{-st}\ f(t)dt\ = \int_0^\infty -te^{-st}\ f(t)dt$$

$$= -\int_0^\infty e^{-st}\ [t\ f(t)]\ dt\ = -L[t\ f(t)] = (-1)L[t\ f(t)]$$

Hence $\qquad L[t.f(t)] = (-1)\frac{d}{ds}L[f(t)] = (-1)\frac{d}{ds}\ F(s) \qquad$ ..... (1)

By using result (1) above, we obtain

$$L[t^2.f(t)] = L\big[\ t\big(t\,f(t)\big)\big] = (-1)\frac{d}{ds}L[t\,f(t)] = (-1)\frac{d}{ds}\left\{(-1)\frac{d}{ds}\ L[t\ f(t)]\right\}$$

$$= (-1)^2\frac{d^2}{ds^2}L[t\ f(t)] = (-1)^2\frac{d^2}{ds^2}\ F(s)$$

Hence

$$L[t^2.f(t)] = (-1)^2\frac{d^2}{ds^2}\ F(s)$$

By mathematical induction we can obtain

$$L[t^n.f(t)] = (-1)^n\frac{d^n}{ds^n}L[f(t)] = (-1)^n\frac{d^n}{ds^n}[\ F(S)] \quad (2)$$

The above result (2) can be interpreted as the differentiation of the transformation of function f(t) corresponds to multiplication of the function f(t) by (-t).

Ex. Obtain Laplace transform of each of the following function:

1. $t.\frac{\sin at}{2a}$

Solution: Let's consider $t\frac{\sin at}{2a} = t\ f(t)$, where $f(t) = \frac{\sin at}{2a}$

Therefore

$L[f(t)] = L\left[\frac{\sin at}{2a}\right] = \frac{1}{2a}\ L[\sin at] = \frac{1}{2a}\ \frac{a}{s^2+a^2} = F(s)$

Hence by $L[t.f(t)] = (-1)\frac{d}{ds}L[f(t)]$ or $L[t.f(t)] = (-1)\frac{d}{ds}\ \{F(s)\}$ we have

$$L\left[t.\frac{\sin at}{2a}\right] = L[t.f(t)] = (-1)\frac{d}{ds}L[f(t)] = (-1)\frac{d}{ds}L\left(\frac{\sin at}{2a}\right)$$

$$= (-1)\frac{d}{ds}\left[\frac{1}{2a}\,\frac{a}{s^2+a^2}\right] = (-1)\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right] = (-1)\frac{1}{2}\left[\frac{-2s}{(s^2+a^2)^2}\right] = \frac{s}{(s^2+a^2)^2}$$

2. $t^2 \cos at$

Solution: consider $t^2 \cos at = t^2 f(t)$

Therefore $f(t) = \cos at$ and $L[f(t)] = L[\cos at] = \frac{s}{s^2+a^2} = F(s)$

Hence by $L[t^2.f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s)$ or $L[t^2.f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s)$ We have

$$L[t^2 \cos 4t] = L[\ t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}\ \left\{\frac{s}{s^2+a^2}\right\}$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right] = \frac{d}{ds}\left[\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \left[\frac{2s(s^2-3a^2)}{(s^2+a^2)^3}\right]$$

3. $\frac{1}{2a^3}$ (sin at − at cos at)

Solution: $L\left[\frac{1}{2a^3}\ (\sin at - at\cos at)\right] = \frac{1}{2a^3}\{L[\ \sin at] - a\,L[t\cos at]\}$

$$= \frac{1}{2a^3}\left\{\frac{a}{s^2+a^2} - a\,(-1)\frac{d}{ds}\,\frac{s}{s^2+a^2}\right\} = \frac{1}{2a^3}\left\{\frac{a}{s^2+a^2} + a\,\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{2a^3}\left\{\frac{a}{s^2+a^2} + a\,\frac{a^2-s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}\left\{\frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)^2}\right\}$$

4. $\frac{1}{2a}$ (sin at + at cos at

Solution: $L\left[\frac{1}{2a}\ (\sin at + at\cos at)\right] = \frac{1}{2a}\{L[\ \sin at] + a\,L[t\cos at]\}$

$$= a\left\{\frac{a}{s^2+a^2} + a\,(-1)\frac{d}{ds}\,\frac{s}{s^2+a^2}\right\} = \frac{1}{2a}\left\{\frac{a}{s^2+a^2} - a\,\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{2a}\left\{\frac{a}{s^2+a^2} - a\,\frac{a^2-s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\left\{\frac{a(s^2+a^2)-a(a^2-s^2)}{(s^2+a^2)^2} = \frac{s^2}{(s^2+a^2)^2}\right\}$$

5. $t^2 \sin 4t$

Solution: consider $t^2 \sin 4t = t^2 f(t)$

Therefore $f(t) = \sin 4t$ and $L[f(t)] = L[\sin 4t] = \frac{4}{s^2+4^2} = F(s)$

Hence by $L[t^2.f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s)$ or $L[t^2.f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s)$

We have $L[t^2 \sin 4t] = L[\ t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s)$

$$= \frac{d^2}{ds^2}\ \left\{\frac{4}{s^2+16}\right\} \qquad = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{4}{s^2+16}\right)\right]$$

$$= 4\frac{d}{ds}\left[-\frac{2s}{(s^2+16)^2}\right] = -8\frac{d}{ds}\left[\frac{s}{(s^2+16)^2}\right] = \left[\frac{24s^2-128}{(s^2+16)^3}\right]$$

6. $t^3\, e^{2t}$

Solution: consider $t^3 e^{2t} = t^2 f(t)$

Therefore $f(t) = e^{2t}$ and $L[f(t)] = L[e^{2t}] = \frac{1}{s-2} = F(s)$

Hence by $L[t^3.f(t)] = (-1)^3 \frac{d^2}{ds^2}F(s)$ or $L[t^3.f(t)] = (-1)^3 \frac{d^2}{ds^2}F(s)$

We have $L[t^3 e^{2t}] = L[t^3\ f(t)] = (-1)^3 \frac{d^3}{ds^3}F(s) = -\frac{d^3}{ds^3}\ \left\{\frac{1}{s-2}\right\}$

$$= -\frac{d^2}{ds^2}\ \left\{-\frac{1}{(s-2)^2}\right\} \qquad = \frac{d}{ds}\left[\frac{d}{ds}\frac{1}{(s-2)^2}\right]$$

$$= \frac{d}{ds}\left[-\frac{2}{(s-2)^3}\right] = \left[\frac{6}{(s-2)^4}\right]$$

7. $t^2 \sin at$

Solution: consider $t^2 \sin at = t^2 f(t)$

Therefore $f(t) = \sin at$ and $L[f(t)] = L[\sin at] = \frac{a}{s^2+4^2} = F(s)$

Hence by $L[t^2 . f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$ or $L[t^2 . f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$

We have $L[t^2 \sin at] = L[\ t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$

$$= \frac{d^2}{ds^2}\left\{\frac{a}{s^2+a^2}\right\} \qquad = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right]$$

$$= \frac{d}{ds}\left[-\frac{2as}{(s^2+a^2)^2}\right] = -\left[\frac{(s^2+a^2)^2(2a) - (2as)2(s^2+a^2)(2s)}{(s^2+a^2)^4}\right]$$

$$= -\left[\frac{(s^2+a^2)(2a) - (2as)2(2s)}{(s^2+a^2)^3}\right] = -\left[\frac{2as^2 + 2a^3 - 8as^2}{(s^2+a^2)^3}\right] = \frac{6as^2 - 2a^3}{(s^2+a^2)^3}$$

8. $t \cos at$ **Ans.** $\frac{s^2-a^2}{(s^2+a^2)^2}$

9. $\frac{t \sinh at}{2a}$ **Ans.** $\frac{s}{(s^2-a^2)^2}$

10. $t. \sin^3 t$

Solution: we have $L[\ \sin^3 t] = L\ \left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$ $\because \sin 3t = 3\sin t - 4\sin^3 t$

$$= \frac{3}{4}\left(\frac{1}{s^2+1} + \frac{1}{s^2+9}\right)$$

$$L[t \sin^3 t] = (-1)\frac{d}{ds}\left[\frac{3}{4}\left(\frac{1}{s^2+1} + \frac{1}{s^2+9}\right)\right] = -\frac{3}{4}\left(\frac{(-2s)}{(s^2+1)^2} + \frac{(-2s)}{(s^2+9)^2}\right)$$

$$= \frac{3s}{2}\left(\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right)$$

11. $t\, e^{3t} \sin 2t$

Solution: we have $L[\ \sin 2t] = \frac{2}{s^2+4}$

$$L[t \sin t] = (-1)\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = -\frac{2(-2s)}{(s^2+1)^2} = \frac{4s}{(s^2+1)^2}$$

$$\therefore\ L[e^{3t}\,(t \sin 2t)\ ] = \{L[t \sin 2t\ ]\}_{s\to s-3} = \left[\frac{4s}{(s^2+1)^2}\right]_{s\to s-3} = \frac{4(s-3)}{[(s-3)^2+1]^2}$$

12. $t\, e^{-2t}(2\cosh 3t - 4\sinh 2t)$

Solution: we have $L[\ 2\cosh 3t - 4\sinh 2t] = \frac{2s}{s^2-9} - \frac{8}{s^2-4}$

$$L[t\,(\cosh 3t - 4\sinh 2t] = (-1)\frac{d}{ds}\left(\frac{2s}{s^2-9} - \frac{8}{s^2-4}\right)$$

$$= -\left[\frac{(s^2-9)(2) - (2s)(2s)}{(s^2-9)^2} - \frac{8(2s)}{(s^2-4)^2}\right] = \left[\frac{2(s^2+9)}{(s^2-9)^2} - \frac{16s}{(s^2-4)^2}\right]$$

$$\therefore\ L[t\, e^{-2t}(2\cosh 3t - 4\sinh 2t)] = \{L[t\,(2\cosh 3t - 4\sinh 2t)]\}_{s\to s+2}$$

$$= \left[\frac{2(s^2+9)}{(s^2-9)^2} - \frac{16s}{(s^2-4)^2}\right]_{s\to s+2} = \frac{2(s+2)^2+18}{[(s+2)^2-9]^2} - \frac{16(s+2)}{[(s+2)^2-4]^2}$$

$$= \frac{2s^2 + +8s + 26}{[s^2 + 4s - 5]^2} - \frac{16(s + 2)}{[s^2 + 4s]^2}$$

7 Division by t

Theorem: If $L[f(t)] = F(s)$ then $\mathrm{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$ provided $\lim_{t\to 0+} \frac{f(t)}{t}$ exists

Proof: By definition, we have

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Integrate both side w. t. t. s, from s to ∞, we get

$$\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds = \int_0^\infty \left[f(t) \int_s^\infty e^{-st} ds\right] dt = \int_0^\infty f(t) \left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 + \frac{e^{-st}}{t}\right] dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = \mathrm{L}\left[\frac{f(t)}{t}\right]$$

$$\int_s^\infty F(s)ds = \mathrm{L}\left[\frac{f(t)}{t}\right]$$

Hence $\qquad \mathrm{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$

By using above result we obtain

$$\mathrm{L}\left[\frac{f(t)}{t^2}\right] = \mathrm{L}\left[\frac{f(t)}{t.t}\right] = \int_s^\infty \mathrm{L}\left[\frac{f(t)}{t}\right] ds = \int_s^\infty \int_s^\infty F(s)ds\ ds$$

Thus $\qquad \mathrm{L}\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_s^\infty F(s)ds\ ds$

Repeating the above procedure we can obtain

$$\boxed{\mathrm{L}\left[\frac{f(t)}{t^n}\right] = \int_s^\infty \int_s^\infty \int_s^\infty \dots \dots \int_s^\infty F(s)ds\, ds\, ds \dots \dots ds}$$

--n integrals--- ..... n times -----

The above result can be interpreted as the integration of the transform of function f(t) corresponds to division of the function f(t) by (t).

Ex. 1 Obtain the Laplace transforms of the functions $\frac{e^{-at} - e^{-bt}}{t}$ and evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$

Solution: In this example we use the result $\mathrm{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$

Here $\frac{f(t)}{t} = \frac{e^{-at} - e^{-bt}}{t}$ so that $f(t) = e^{-at} - e^{-bt}$ and

$$L[f(t)] = L[e^{-at} - e^{-bt}] = \frac{1}{s + a} - \frac{1}{s + b} = F(s)$$

Hence using Laplace transform of division by t we have

$$\mathrm{L}\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty L[e^{-at} - e^{-bt}]\, ds = \int_s^\infty \left[\frac{1}{s + a} - \frac{1}{s + b}\right] ds$$

$$= [\log(s+a) - \log(s+b)]_s^{\infty} = \left[\log\frac{s+a}{s+b}\right]_s^{\infty} = \log\infty - \log\frac{s+a}{s+b} = \log\frac{s+a}{s+b}$$

Hence $\quad L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \log\frac{s+a}{s+b}$

By definition of Laplace transform we have

$$\int_0^{\infty} e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\frac{s+a}{s+b}$$

Putting $s = 0$ we get

$$\int_0^{\infty}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\frac{0+a}{0+b} = \log\left(\frac{a}{b}\right)$$

Example 2: Find the Laplace transform of $\frac{\cos at - \cos bt}{t}$

Solution: Here $\frac{f(t)}{t} = \frac{\cos at - \cos bt}{t}$, so that $f(t) = \cos at - \cos bt$ and

$$L[f(t)] = L\,[\cos at - \cos bt] = \frac{s}{s^2+a^2} + \frac{s}{s^2+b^2} = F(s)$$

Hence by Laplace transform of division by t i.e $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)ds$ we have

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^{\infty}\left(\frac{s}{s^2+a^2} + \frac{s}{s^2+b^2}\right)ds = \frac{1}{2}\int_s^{\infty}\left(\frac{2s}{s^2+a^2} + \frac{2s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}[\log(s^2+a^2) - \log(s^2+b^2)]_s^{\infty} = \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

Example 3 Obtain the Laplace transforms of the functions $\frac{\sin at}{t}$ and prove that $\int_0^{\infty}\frac{\sin t}{t}dt = \frac{\pi}{2}$

Solution: In this example we use the result $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)ds$

Here $\frac{f(t)}{t} = \frac{\sin at}{t}$ so that $f(t) = \sin at$ and

$$L[f(t)] = L[\sin at] = \frac{a}{s^2+a^2} = F(s)$$

Hence using Laplace transform of division by t we have

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} L[\sin at]\,ds = \int_s^{\infty}\left[\frac{a}{s^2+a^2}\right]ds = \left[\tan^{-1}\frac{s}{a}\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

If a=1, we have

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s$$

By definition of Laplace transform we have

$$\int_0^{\infty} e^{-st}\left(\frac{\sin t}{t}\right)dt = \cot^{-1} s$$

Putting $s = 0$ we get

$$\int_0^{\infty}\frac{\sin t}{t}dt = \cot^{-1}(0) = \frac{\pi}{2}$$

Example 3 Obtain the Laplace transforms of the following functions

a. $\frac{1-\cos t}{t}$

b. $\frac{1-\cos t}{t^2}$

c. $\frac{\sin^2 t}{t^2}$

d. $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

e. $\int_0^t \frac{\sin t}{t}$

f. $\int_0^t \frac{1- e^{-x}}{x}\, dx$

g. $\int_0^t t\cosh t\, dt$

8 Convolution of two functions:

The convolution of functions f(t) and g(t) is denoted by $f(t) * g(t)$ and is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)du$$

<u>Note: convolution of two functions is commutative:</u> f(t)*g(t)= g(t)* f(t)

By definition of convolution $f(t) * g(t) = \int_0^t f(u)\, g(t-u)du$

Put $t - u = v$ or $u = t - v$ $\quad \therefore du = dv$ we get

u	0	t
v	t	0

$$f(t) * g(t) = \int_t^0 f(t-v)\, g(v)(-dv) = \int_0^t g(v)f(t-v)dv = g(t) * f(t)$$

This shows that the convolution of f(t) and g(t) obeys the commutative law of algebra.

Properties of convolution:

i) $f(t) * [g(t) + h(t)] = f(t) * g(t) + f(t) * h(t)$

ii) $f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$

Convolution Theorem: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then

$$L[f(t) * g(t)] = L\left[\int_0^t g(u)f(t-u)du\right] = F(s)G(s) \qquad (1)$$

- **Note 1:** In other words this theorem states that Laplace transform of convolution of two functions is equal to product of their Laplace Transforms.
- **Note 2:** since the convolution of f(t) and g(t) is commutative, we have from result (1)
$$L\left[\int_0^t f(t-u)g(u)du\right] = F(s)G(s)$$
- **Note 3:** the convolution theorem is useful to find the inverse Laplace transformation

Example 1: verify the convolution theorem for the pair of functions $f(t) = t$ and $g(t) = e^{at}$

Solution: To verify convolution theorem we have to prove

$$L\left[\int_0^t f(u)g(t-u)du\right] = F(s)G(s)$$

Here $\quad f(t) = t \quad \therefore F(s) = \frac{1}{s^2}$

and $\quad g(t) = e^{at} \quad \therefore G(s) = \frac{1}{s-a}$

Therefore $\quad F(s)G(s) = \frac{1}{s^2(s-a)} \quad (1)$

Now $\quad L[f(t) * f(t)] = L\left[\int_0^t f(u)g(t-u)du\right] = L\left[\int_0^t u\, e^{a(t-u)}du\right]$

$$= L\left[e^{at}\int_0^t u\, e^{-au}du\right] = L\left[e^{at}\left\{u\int e^{-au}du - \int(1)\frac{e^{-au}}{-a}du\right\}_0^t\right]$$

$$= L\left[e^{at}\left\{u\left(\frac{e^{-au}}{-a}\right) - \frac{e^{-au}}{a^2}\right\}_0^t\right] = L\left[e^{at}\left\{\left(t\frac{e^{-at}}{-a} - \frac{e^{-at}}{a^2}\right) - \left(0\frac{e^{-a0}}{-a} - \frac{e^{-a0}}{a^2}\right)\right\}\right]$$

$$= L\left[e^{at}\left\{t\frac{e^{-at}}{-a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right\}\right] = L\left[\frac{t}{-a} - \frac{1}{a^2} + \frac{e^{at}}{a^2}\right] = \frac{1}{a^2}L[e^{at} - at - 1]$$

$$= \frac{1}{a^2}\left[\frac{1}{s-a} - a\frac{1}{s^2} - \frac{1}{s}\right] = \frac{1}{a^2}\left[\frac{1}{s-a} - \frac{a+s}{s^2}\right] = \frac{1}{a^2}\left[\frac{s^2-(s-a)(s+a)}{s^2(s-a)}\right] = \frac{1}{a^2}\left[\frac{s^2-s^2+a^2}{s^2(s-a)}\right] = \frac{1}{s^2(s-a)}$$

(2)

From (1) and (2) convolution theorem is verified.

Example 2: Verify convolution theorem for pair of functions: $f(t) = t$ and $g(t) = \cos t$.

Example 3: Verify convolution theorem for pair of functions: $f(t) = t^2$ and $g(t) = e^{-at}$.

Example 4: Show that $1 * 1 = t$, and hence prove that

$$1 * 1 * 1 \ldots\ldots\ldots * 1 = \frac{t^{n-1}}{(n-1)!}$$

9 **Initial valued theorem**:

If $L[f(t) = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty}[s\,F(s)]$ provided these limits exist.

Proof: we have $L[f'(t)] = sF(s) - f(0)$ or $\int_0^\infty \mathbf{e^{-st}}f'(t)dt = sF(s) - f(0)$

Taking limit as $s \to \infty$, we get

$$\lim_{s\to\infty}\int_0^\infty \mathbf{e^{-st}}f'(t)dt = \lim_{s\to\infty}[sF(s) - f(0)]$$

{Since f'(t) is piecewise continuous & of exponential order, we have

$$\lim_{s\to\infty}\int_0^\infty \mathbf{e^{-st}}f'(t)dt = \mathbf{0}\}$$

$$\therefore 0 = \lim_{s\to\infty} sF(s) - f(0)$$

Or $$\lim_{s\to\infty}[s\,F(s)] = f(0) = \lim_{t\to 0} f(t)$$

Hence $$\lim_{t\to 0} f(t) = \lim_{s\to\infty}[s\,F(s)]$$

Example1: verify initial value theorem for the following functions

(i) $t + \sin 3t$ $\quad$ ii) $(3t+4)^2$ $\quad$ iii) $3 - 2\cos t$

Solution i) $f(t) = t + \sin 3t \quad \therefore F(s) = \frac{1}{s^2} + \frac{3}{s^2+9}$

Now $\lim_{t\to 0} f(t) = \lim_{t\to 0}(t + \sin 3t) = 0 + \sin 0 = 0 + 0 = 0$ $\quad$ and $\quad$ (1)

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{1}{s^2}+\frac{3}{s^2+9}\right) = \lim_{s\to\infty}\left(\frac{1}{s}+\frac{3s}{s^2+9}\right) = \lim_{s\to\infty}\left(\frac{1}{s}+\frac{3/s}{1+9/s^2}\right) = \frac{1}{\infty}+\frac{3/\infty}{1+9/\infty} = 0 \quad (2)$$

Since results (1) and (2) are same, the initial value theorem is verified.

Solution ii): here $f(t) = (3t+4)^2 = 9t^2 + 24t + 16$

$$\therefore\ F(s) = L[9t^2 + 24t + 16] = 9L[t^2] + 24L[t] + 16L[1]$$

$$\therefore\ F(s) = 9\frac{2}{s^3} + 24\frac{1}{s^2} + 16\frac{1}{s}$$

Now $\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[9t^2 + 24t + 16\right] = 0 + 0 + 16 = 16$ (1) and

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{18}{s^3}+\frac{24}{s^2}+\frac{16}{s}\right] = \lim_{s\to\infty}\left[\frac{18}{s^2}+\frac{24}{s}+16\right] = \frac{18}{\infty}+\frac{24}{\infty}+16 = 16 \quad (2)$$

Hence from (1) and (2) initial valued theorem is verified.

Solution iii): Here $f(t) = 3 - 2\cos t$

$$\therefore\ F(s) = L[3 - 2\cos t] = L[3] - 2L[\cos t] = \frac{3}{s} - 2\frac{s}{s^2+1} \qquad \therefore\ F(s) = \frac{3}{s} - \frac{2s}{s^2+1}$$

Now $\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[3 - 2\cos t\right] = 3 - 2\cos 0 = 3 - 2(1) = 1$ (1)

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{3}{s}-\frac{2s}{s^2+1}\right] = \lim_{s\to\infty}\left[3-\frac{2s^2}{s^2+1}\right] = \lim_{s\to\infty}\left[3-\frac{2}{1+1/s^2}\right] = 3-\frac{2}{1+1/\infty} = 3-2 = 1 \quad (2)$$

Hence from (1) and (2) initial valued theorem is verified

Example2: Verify initial value theorem for the following functions

(i) $3\,e^{-2t}$ (ii) $(2t-3)^2$

10 **Final valued theorem**: If $L[f(t) = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to 0}[s\,F(s)]$ provided these limits exist.

Proof: we have $L[f'(t)] = s\,F(s) - f(0)$ or $\int_0^\infty e^{-st} f'(t)dt = sF(s) - f(0)$

Taking limit as $s \to 0$, we get

$$\lim_{s\to 0}\int_0^\infty e^{-st} f'(t)dt = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\int_0^\infty e^{-0t} f'(t)dt = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\int_0^\infty f'(t)dt = \lim_{s\to 0}[sF(s)] - f(0)$$

$$[f(t)]_0^\infty = \lim_{s\to 0}[sF(s)] - f(0)$$

Or $$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0}[s\,F(s)] - f(0)$$

Hence $$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[s\,F(s)]$$

Remarks: The initial and final value theorems are useful in obtaining the initial and final values of a function from the limits of the transform function.

Example 1: Verify the final value theorem for the following functions.

i) $4e^{-3t}$ ii) $1 + e^{-t}(\sin 2t + \cos 2t)$ iii) t^2e^{-t}

Solution i) $f(t) = 4e^{-3t}$ $\therefore F(s) = L[4e^{-3t}] = 4\,\frac{1}{s+3}$

Now $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 4e^{-3t} = 4e^{-\infty} = 4(0) = 0$ and (1)

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{4}{s+3}\right) = \lim_{s\to 0}\left(\frac{4s}{s+9}\right) = 0 \quad (2)$$

Since results (1) and (2) are same, the final value theorem is verified.

Solution ii) $f(t) = 1 + e^{-t}(\sin 2t + \cos 2t)$

$\therefore F(s) = L[1 + e^{-t}(\sin 2t + \cos 2t)] = \frac{1}{s} + \left\{\frac{2}{s^2+4} + \frac{s}{s^2+4}\right\}_{s\to s+1}$ by first shifting theorem

$$= \frac{1}{s} + \left\{\frac{2+s}{s^2+4}\right\}_{s\to s+1} = \frac{1}{s} + \left\{\frac{2+s+1}{(s+1)^2+4}\right\}$$

$$F(s) = \frac{1}{s} + \left\{\frac{s+3}{s^2+2s+5}\right\}$$

Now $\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\{1 + e^{-t}(\sin 2t + \cos 2t)\} = 1 + \lim_{t\to\infty} e^{-t}(\sin 2t + \cos 2t) = 1 + 0 = 1$ (1)

And $\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{1}{s} + \left\{\frac{s+3}{s^2+2s+5}\right\}\right) = \lim_{s\to 0}\left(1 + \left\{\frac{s^2+3s}{s^2+2s+5}\right\}\right) = 1 + 0 = 1$ (2)

Since results (1) and (2) are same, the final value theorem is verified

Solution iii) $f(t) = t^2e^{-t}$

$\therefore F(s) = L[t^2e^{-t}] = \left\{\frac{2}{s^3}\right\}_{s\to s+1} = \frac{2}{(s+1)^3}$ by first shifting theorem

$$F(s) = \frac{2}{(s+1)^3}$$

Now $\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\{t^2 e^{-t}\} = \lim_{t\to\infty}\left(\frac{t^2}{e^t}\right)$ $\left(\frac{0}{0}\ \text{form}\right)$

Apply L Hospital rule $\lim_{t\to\infty} f(t) == \lim_{t\to\infty}\left(\frac{2t}{e^t}\right) = \lim_{t\to\infty}\left(\frac{2}{e^t}\right) = \frac{2}{\infty} = 0$ (1)

and $\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left(\frac{2}{(s+1)^3}\right) = \lim_{s\to 0}\left(\frac{2s}{(s+1)^3}\right) = \frac{0}{(0+1)^3} = 0$ (2)

Since results (1) and (2) are same, the final value theorem is verified

Example 2: Verify the final value theorem for the following functions

(i) $t^3\ e^{-4t}$ (ii) $2 + 3e^{-2t}\sin 4t$

11 **Unit Step function:** The unit step function is denoted by u(t) and is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Laplace transform of Unit step function: By definition of Laplace transform,

$$L[u(t)] = \int_0^\infty e^{-st}u(t)dt = \int_0^\infty e^{-st}1.dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s} = \frac{1}{s}$$

Hence $L[u(t)] = \frac{1}{s}$

Displaced Unit Step function: The unit step function is denoted by $u(t-a)$ and is defined as

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Laplace transform of displaced Unit step function: By definition of Laplace transform,

$$L[u(t-a)] = \int_0^\infty e^{-st}u(t-a)dt = \int_0^a e^{-st}\,u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt$$

$$= 0 + \int_a^\infty e^{-st}1.dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-\infty}}{-s} - \frac{e^{-as}}{-s} = 0 + \frac{e^{-as}}{s} = \frac{e^{-as}}{s}$$

Hence $L[u(t-a)] = \frac{e^{-as}}{s}$

12 Evaluation of Integrals Using Laplace transform

Laplace transformation is often useful in evaluating various integrals. This is illustrated in following examples

Ex. Evaluate each of the following integrals:

1. $\int_0^\infty te^{-3t}\sin\ tdt$
2. $\int_0^\infty t^2e^{-t}\sin\ tdt$

3. $\int_0^\infty t^3 e^{-t} \sin tdt$

4. $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$

5. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

6. $\int_0^\infty e^{-2t} \frac{\sinh t}{t} dt$

7. $\int_0^\infty e^{-2t} \sin^3 tdt$

Exercise on Laplace Transform

1. Find Laplace transform of each of the followings
   a. $t^3 e^{-3t}$
   b. $e^{at}(2\cos bt - 3\sin bt)$
   c. $e^{-t}(4t^3 + \cos(4t+7))$
2. Find L[f (t)] if
   a. $F(t) = \begin{cases} \sin 2(t-\pi), & t > \pi \\ 0, & t < \pi \end{cases}$
   b. $F(t) = \begin{cases} 5\cos(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$
3. Verify change of scale theorem for L[e^{2t}cos2t]
4. If $L[f(t)] = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$ Find L[f(2t)]
5. Find L[f ‘(t)] if
   a. f(t) =e^{-5t}sint
   b. f(t) =$\sin^2$ t
6. Given that $y''+2y'-8y = 0, \quad y(0) = 1, \; y'(0) = 8$, show that $L[y(t)] = \frac{2}{s-2} - \frac{1}{s+4}$
7. Obtain Laplace transform of
   i. $te^{-3t} \cos 2t$
   ii. $t(3\sin 2t - 2\cos 2t)$
   iii. $t\cos(4t+3)$
   iv. $t^2 \sin 2t$
8. Obtain Laplace transform of
   v. $t\int_0^t e^{-3t} \sin 2tdt$
   vi. $e^{-3t}\int_0^t t\sin 2tdt$
   vii. $\int_0^t te^{-3t} \sin 2tdt$
9. Find Laplace Transform of each of the following
   a. $\frac{\sinh t}{t}$
   b. $\frac{1-e^{-t}}{t}$

c. $\frac{e^{2t}-1}{t}$

d. $\frac{\cos 2t - \cos 3t}{t}$

10. Find the following convolution
    a. 1*1
    b. $1*e^t$
    c. $t*e^t$
    d. sin t*sin t
11. Using Laplace transform evaluate each of the following integrals

    a. $\int_0^\infty te^{-2t}\cos t\,dt$

    b. $\int_0^\infty t^2e^{-3t}\sinh 2t\,dt$

    c. $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$

    d. $\int_0^\infty \frac{\cos 3t - \cos 2t}{t}dt$

    e. $\int_0^\infty e^{-2t}\frac{\sinh t\sin t}{t}dt$

    f. $\int_0^\infty e^{3t}\cos^3 t\,dt$

    g. $\int_0^\infty \frac{\sin^3 t}{t}dt$

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