1. # Gamma function

Gamma function is defined as

$$\int_0^\infty x^{n-1} e^{-x}\, dx = \Gamma n$$

Now, $\Gamma n = (n-1)!$

eg. $\Gamma 4 = 3! = 3\times 2\times 1$

$\Gamma 9 = 8! = 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1.$

Mostly used to solve integrals of By part questions containing these funⁿ. as integrand.

Small Examples :

① $$\int_0^\infty x^8 e^{-x}\, dx = \Gamma 9 = 8!$$

② $$\int_0^\infty x^5 e^{-x}\, dx = \Gamma 6 = 5!$$

Examples containing e^{-ax} :

①. WKT.

$$\int_0^\infty x^{n-1} e^{-ax}\, dx = \frac{\Gamma n}{a^n}.$$

eg. $$\int_0^\infty x^6 e^{-5x}\, dx = \frac{\Gamma 7}{(5)^7}.$$

$$\int_0^\infty x^9 e^{-3x}\, dx = \frac{\Gamma 10}{(3)^{10}}$$

Q) Prove that $\Gamma(n+1) = n\Gamma(n)$.

$\Rightarrow$ $$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx.$$

By, By Parts

$$= x^n \int_0^\infty e^{-x}dx - \int_0^\infty \left(\frac{d}{dx}x^n \int e^{-x}dx\right)dx$$

$$= -x^n e^{-x} - \int_0^\infty n x^{n-1}\cdot(-e^{-x})\,dx.$$

$$= \left(-x^n e^{-x}\right)_0^\infty + \int_0^\infty n\cdot x^{n-1}\cdot e^{-x}\,dx$$

$$= (0-0) + n\int_0^\infty x^{n-1}e^{-x}\,dx$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$$ ... { by def^n

(2) Prove that $\int_0^1 \log\left(\frac{1}{y}\right)^{n-1} dy = \Gamma(n)$.

put $\log\frac{1}{y} = x \Rightarrow \frac{1}{y} = e^x$

$y = e^{-x}$.

$\therefore dy = -e^{-x}dx.$

as

$y \to 0$	$y \to 1$
$x \to \infty$	$x \to 0$.

$$\therefore \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \int_\infty^0 (x)^{n-1}(-e^{-x})\,dx.$$

$$= -\int_\infty^0 x^{n-1}e^{-x}\,dx.$$

$$= \int_0^\infty x^{n-1}e^{-x}\,dx \quad \cdots \left\{ \int_a^b f(x)dx = -\int_b^a f(x)dx \right.$$

$$= \Gamma(n).$$

(3) $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$.

$\Rightarrow$ put $x^2 = t$. $\Rightarrow x = \sqrt{t}$.

$$2x\,dx = dt$$

$$dx = \frac{dt}{2x}$$

$$dx = \frac{dt}{2\sqrt{t}}.$$

as

$x \to 0$	$t \to 0$
$x \to \infty$	$t \to \infty$

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \cdot \frac{1}{2} \cdot t^{-1/2} dt.$$

$$= \frac{1}{2} \cdot \int_0^\infty t^{-1/2} \cdot e^{-t}\, dt.$$

$$= \frac{1}{2} \cdot \Gamma\frac{1}{2}.$$

$$\boxed{\int_0^\infty e^{-x^2} dx = \frac{1}{2} \cdot \sqrt{\pi}.}$$

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(4) $\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx.$

$\Rightarrow$ put $3\sqrt{x} = t$

$9x = t^2.$

$dx = \frac{2t\,dt}{9}.$

$\therefore$ as $x \to 0$ $t \to 0$

$x \to \infty$ $t \to \infty$

$\therefore I = \int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx = \int_0^\infty \frac{t}{3} \cdot e^{-t} \frac{2t\,dt}{9}.$

$= \frac{2}{27} \int_0^\infty t^2 \cdot e^{-t} dt.$

$= \frac{2}{27} \times \sqrt{3} = \frac{2}{27} \times 2! = \frac{2}{27} \times 2 \times 1$

$= \frac{4}{27}.$

(5). $\int_0^\infty x^n\, e^{-k^2 x^2}\, dx.$

$\left\{ \frac{n-1}{2} + 1 = \frac{n+1}{2}. \right.$

$\Rightarrow$ put $k^2 x^2 = t.$

$2k^2 x\, dx = dt.$

$dx = \frac{dt}{2k^2 x.} = \frac{dt}{2k^2 \frac{\sqrt{t}}{k}} = \frac{dt}{2k\sqrt{t}}.$

$\therefore \int_0^\infty x^n e^{-k^2x^2} dx = \int_0^\infty \left(\frac{\sqrt{t}}{k}\right)^n \cdot e^{-t} \frac{dt}{2k\sqrt{t}}.$

$= \frac{1}{2k^{n+1}} \int_0^\infty (\sqrt{t})^{n-1} \cdot e^{-t} dt. = \frac{1}{2k^{n+1}} \int_0^\infty t^{\left(\frac{n-1}{2}\right)} \cdot e^{-t} dt.$

$= \frac{1}{2k^{n+1}} \cdot \sqrt{\frac{n+1}{2}}.$

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3.5 GAMMA FUNCTIONS

Consider the definite integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$, it is denoted by the symbol Γn (we read it as Gamma 'n') and is called as *Gamma function of n*. Thus,

$$\boxed{\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx \quad (n > 0)} \qquad ...(1)$$

Gamma function is also called as Euler's integral of the second kind.

3.6 PROPERTIES OF GAMMA FUNCTIONS

1. $$\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx$$

Proof : We have,

$$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx. \qquad \text{Put } x = t^2,\ dx = 2t\, dt.$$

x	0	∞
t	0	∞

$$= \int_0^\infty e^{-t^2}\, t^{2n-2}\, 2t\, dt = 2\int_0^\infty e^{-t^2}\, t^{2n-1}\, dt$$

$$\boxed{\Gamma n = 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx} \qquad ...(2)$$

[It may be borne in mind that variable of integration is immaterial in a definite integral.]

Relations (1) and (2) are both considered as definitions of Gamma functions.

2. $$\Gamma 1 = 1$$

Proof : By def. $$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

Put $n = 1$ $$\Gamma 1 = \int_0^\infty e^{-x}\, x^0\, dx = [-e^{-x}]_0^\infty = (-e^{-\infty} + e^0) = 0 + 1 = 1$$

$$\boxed{\Gamma 1 = 1}$$

3. Reduction Formula for Gamma Functions :

$$\Gamma(n+1) = n\,\Gamma n$$

Proof : By definition

$$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx,$$

Replace n by n + 1. $$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n\, dx.$$

Now, integrating by parts

$$\Gamma(n+1) = \left\{x^n(-e^{-x})\right\}_0^{\infty} - \int_0^{\infty} nx^{n-1}(-e^{-x})\,dx.$$

Now, $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$. Also if $n > 0$, $\dfrac{x^n}{e^x} = 0$ for $x = 0$ $\therefore \left[\dfrac{x^n}{e^x}\right]_0^{\infty} = 0$

$$\therefore \quad \Gamma(n+1) = 0 + n\int_0^{\infty} e^{-x}x^{n-1}\,dx = n\Gamma n$$

$$\therefore \quad \boxed{\Gamma(n+1) = n\Gamma n} \quad \text{...(4)}$$

If n is a positive integer,

$$\Gamma(n+1) = n(n-1)\Gamma(n-1) \qquad \because \Gamma n = (n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2) = n(n-1)(n-2)(n-3)(n-4)\ldots 3.2.1\,\Gamma 1$$

$$= n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots 3.2.1 \qquad \{\because \Gamma 1 = 1$$

$$\Gamma(n+1) = n! \quad \text{if n is a positive integer.}$$

Hence

$$\boxed{\begin{aligned}\Gamma(n+1) &= n\Gamma n \text{, in general}\\ &= n! \text{ if n is a positive integer.}\end{aligned}}$$

4. $\boxed{\Gamma 0 = \infty}$ $\qquad \because \Gamma n = \dfrac{\Gamma(n+1)}{n} \quad \therefore \Gamma 0 = \dfrac{\Gamma 1}{0} = \dfrac{1}{0} = \infty$

5. $\Gamma\dfrac{1}{2} = \sqrt{\pi}$

6. Since $\Gamma(n+1) = n!$

$\therefore$

$$\Gamma 6 = 5!, \qquad \Gamma 8 = 7!, \qquad \Gamma 2 = 1! = 1$$

$$\Gamma\frac{5}{2} = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$\Gamma\frac{11}{2} = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

7. For negative fraction n, we use $\Gamma n = \dfrac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\frac{5}{3}\right) = \left(-\frac{3}{5}\right)\Gamma\left(-\frac{2}{3}\right) = \left(-\frac{3}{5}\right)\left(-\frac{3}{2}\right)\Gamma\frac{1}{3} = \frac{9}{10}\Gamma\frac{1}{3}$$

3.7 TRANSFORMATION OF GAMMA FUNCTIONS

1. We know that

$$\Gamma n = \int_0^{\infty} e^{-x}x^{n-1}\,dx \qquad \text{put } x = ky \quad \therefore dx = k\,dy$$

x	0	∞
y	0	∞

$$= \int_0^{\infty} e^{-ky}k^{n-1}\cdot y^{n-1}\cdot k\cdot dy = k^n\int_0^{\infty} e^{-ky}\cdot y^{n-1}\,dy$$

$$\therefore \quad \boxed{\int_0^{\infty} e^{-ky}y^{n-1}\,dy = \frac{\Gamma n}{k^n}}$$

Ex. 11 : *Evaluate* $\int_0^\infty x^9 e^{-2x^2}\,dx.$ **(Dec. 2009, 2018; May 2006)**

Sol. :

$$I = \int_0^\infty x^9 e^{-2x^2}\,dx$$

Put $2x^2 = t$ or $x = \left(\frac{t}{2}\right)^{1/2}$, $4x\,dx = dt$

x	0	∞
t	0	∞

$$I = \int_0^\infty x^8 e^{-2x^2}\cdot x\,dx = \int_0^\infty \left(\frac{t}{2}\right)^4 e^{-t}\,\frac{dt}{4}$$

$$= \frac{1}{64}\int_0^\infty t^4 e^{-t}\,dt = \frac{1}{64}\int_0^\infty e^{-t}\,t^{5-1}\,dt$$

$$= \frac{1}{64}\overline{|5} = \frac{4!}{64} = \frac{24}{64}$$

$$\boxed{I = \frac{3}{8}}$$

Ex. 12 : *Show that* $\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \overline{|n}$

Sol. : Let

$$I = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$$ Put $\log\frac{1}{y} = t$ or $\frac{1}{y} = e^t$ or $y = e^{-t}$ $dy = -e^{-t}\,dt$

y	0	1
t	∞	0

$$= \int_\infty^0 t^{n-1}\,(-e^{-t})\,dt = \int_0^\infty e^{-t}\,t^{n-1}\,dt$$

$$\boxed{I = \overline{|n}}$$

Ex. 13 : *Evaluate* $\int_0^1 (x\log x)^4\,dx.$ **(May 2005; Dec. 2011, 2007)**

Sol. : Let

$$I = \int_0^1 (x\log x)^4\,dx$$ Put $\log x = -t$ or $x = e^{-t}$, $dx = -e^{-t}\,dt$

x	0	1
t	∞	0

$$= \int_\infty^0 (e^{-t})^4 (-t)^4 (-e^{-t}\,dt) = \int_0^\infty e^{-5t}\,t^4\,dt = \frac{\overline{|5}}{5^5}$$

$$\boxed{I = \frac{4!}{5^5}}$$

$\left(\text{Put } k = 5,\ n = 5 \text{ in } \int_0^\infty e^{-ky}\,y^{n-1}\,dy = \frac{\overline{|n}}{k^n}\right)$

$\int^\infty$ ... $\overline{|m}$ ... (M...