Unit-V: VECTOR INTEGRATION

Vector integration

1.1 LINE INTEGRAL:

Line integral $= \int_c \left(\bar{F}.\frac{\overrightarrow{dr}}{ds}\right) ds = \int_c \bar{F}.\,\overline{dr}$

Note:

1) **Work:** If $\bar{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \bar{F}.\,\overline{dr}$
2) **Circulation:** If $\bar{V}$ represents the velocity of a liquid then $\oint_c \bar{V}.\,\overline{dr}$ is called the circulation of V round the closed curve c.
   If the circulation of V round every closed curve is zero then V is said to be irrotational there.
3) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

Note: If $\int_A^B \bar{F}.\,\overline{dr}$ is to be proved to be independent of path, then $\bar{F} = \nabla\emptyset$

here F is called **Conservative** (irrotational) vector field and $\emptyset$ is called the **Scalar potential.** And $\nabla \times \bar{F} = \nabla \times \nabla\emptyset = 0$

Example 1: Evaluate $\int_c \bar{F}.\,\overline{dr}$ where $\bar{F} = x^2\hat{\imath} + xy\hat{\jmath}$ and C is the boundary of the square in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a\ and$

$y = a.$

Solution: $\int_c \bar{F}.\,\overline{dr} = \int_{OA} \bar{F}.\,\overline{dr} + \int_{AB} \bar{F}.\,\overline{dr} + \int_{BC} \bar{F}.\,\overline{dr} + \int_{CO} \bar{F}.\,\overline{dr}$

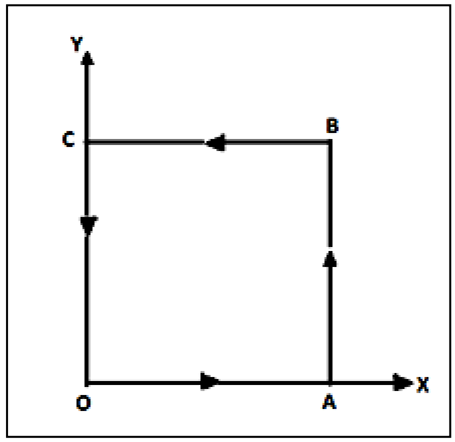
Here $\bar{r} = x\hat{\imath} + y\hat{\jmath},\ \overline{dr} = dx\hat{\imath} + dy\hat{\jmath},\ \bar{F} = x^2\hat{\imath} + xy\hat{\jmath}$

$\bar{F}.\,\overline{dr} = x^2dx + xydy$ ________(i)

⇨ On $OA,\ y = 0$

$\therefore\ \bar{F}.\ \overline{dr} = x^2 dx$ (From (i))

$\int_{OA} \bar{F}.\ \overline{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{a^3}{3}$ ________(ii)



⇨ On $AB,\ x = a$

$\therefore\ dx = 0$

$\therefore\ \bar{F}.\ \overline{dr} = ay\ dy$ (From (i))

$\int_{AB} \bar{F}.\ \overline{dr} = \int_0^a ay\ dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$ ________(iii)

⇨ On $BC,\ y = a$

$\therefore\ dy = 0$

$\therefore\ \bar{F}.\ \overline{dr} = x^2 dx$ (From (i))

$\int_{BC} \bar{F}.\ \overline{dr} = \int_a^0 x^2\ dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}$ ________(iv)

⇨ On $CO,\ x = 0$

$\therefore\ \bar{F}.\ \overline{dr} = 0$ (From (i))

$\int_{CO} \bar{F}.\ \overline{dr} = 0$ ________(v)

On adding (ii), (iii), (iv) and (v), we get

$\int_C \bar{F}.\ \overline{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$ ________**Ans.**

Example 2: A vector field is given by

$\bar{F} = (2y+3)\hat{\imath} + (xz)\hat{\jmath} + (yz-x)\hat{k}$. Evaluate $\int_C \bar{F}.\ \overline{dr}$ along the path c is $x = 2t,\ y = t,\ z = t^3\ from\ t = 0\ to\ t = 1.$

Solution:

$$\int_C \bar{F}.\ \overline{dr} = \int_C (2y+3)dx + (xz)dy + (yz-x)dz$$

$$\begin{bmatrix} since\ x = 2t & y = t & z = t^3 \\ \therefore\quad \frac{dx}{dt} = 2 & \frac{dy}{dt} = 1 & \frac{dz}{dt} = 3t^2 \end{bmatrix}$$

$$= \int_0^1 (2t+3)(2\,dt) + (2t)(t^3)dt + (t^4-2t)(3t^2dt)$$

$$= \int_0^1 (4t+6+2t^4+3t^6-6t^3)dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1$$

$$= \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}$$

$= 7.32857$ __________**Ans.**

Example 3: Suppose $\vec{F}(x,y,z) = x^3\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the force field. Find the work done by $\vec{F}$ along the line from the (1, 2, 3) to (3, 5, 7).

Solution: Work done $= \int_c \bar{F}.\overline{dr}$

$$= \int_{(1,2,3)}^{(3,5,7)} \left(x^3\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).d\left(\hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz\right)$$

$$= \int_{(1,2,3)}^{(3,5,7)} (x^3dx + ydy + zdz)$$

$$= \int_1^3 x^3dx + \int_2^5 y\,dy + \int_3^7 z\,dz$$

$$= \left[\frac{x^4}{4}\right]_1^3 + \left[\frac{y^2}{2}\right]_2^5 + \left[\frac{z^2}{2}\right]_3^7$$

$$= \left[\frac{81}{4} - \frac{1}{4}\right] + \left[\frac{25}{2} - \frac{4}{2}\right] + \left[\frac{49}{2} - \frac{9}{2}\right]$$

$$= \frac{80}{4} + \frac{21}{2} + \frac{40}{2}$$

$$= \frac{202}{4}$$

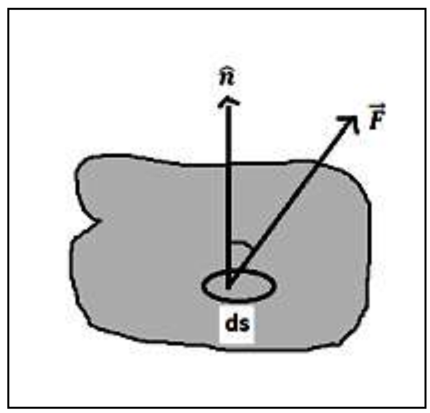
$= 50.5$ units ________**Ans.**

1.2 **Exercise**:

1) If a force $\bar{F} = 2x^2y\hat{\imath} + 3xy\hat{\jmath}$ displaces a particle in the xy-plane from (0, 0) to (1, 4) along a curve $y = 4x^2$. Find the work done.
2) If $\vec{A} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, evaluate the line integral $\oint \vec{A}\,\overrightarrow{dr}$ from (0, 0, 0) to (1, 1, 1) along the curve C.
3) Show that the integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (x^2y + 3xy^2)dy$ is independent of the path joining the points (1, 2) and (3, 4). Hence, evaluate the integral.

2.1 **SURFACE INTEGRAL**:

Let $\bar{F}$ be a vector function and S be the given surface.
Surface integral of a vector function $\bar{F}$ over the surface S is defined as the integral of the components of $\bar{F}$ along the normal to the surface.



Component of $\bar{F}$ along the normal$= \bar{F}.\hat{n}$
Where n = unit normal vector to an element ds and

$$\hat{n} = \frac{grad\ f}{|grad\ f|} \qquad ds = \frac{dx\,dy}{(\hat{n}.\hat{k})}$$

Surface integral of F over S

$$= \sum \bar{F}.\hat{n} \qquad = \iint_S (\bar{F}.\hat{n})ds$$

Note:

1) Flux $= \iint_S (\bar{F}.\hat{n})ds$ where, $\bar{F}$ represents the velocity of a liquid.

If $\iint_S (\bar{F}.\hat{n})ds = 0$, then $\bar{F}$ is said to be a Solenoidal vector point function.

3.1 VOLUME INTEGRAL:

Let $\bar{F}$ be a vector point function and volume V enclosed by a closed surface.
The volume integral $= \iiint_V \bar{F}\, dv$

Example 1: Evaluate $\iint_S \left(yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}\right).ds$ where S the surface of the sphere is $x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution: Here, $\emptyset = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla\emptyset$

$$= \hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2)$$

$$= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{a} \qquad [\because\ x^2 + y^2 + z^2 = a^2]$$

Here, $$\vec{F} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$$

$$\vec{F}.\hat{n} = \left(yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}\right).\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

Now, $$\iint_S \vec{F}.\hat{n}\ ds = \iint_S \left(\vec{F}.\hat{n}\right)\frac{dx\ dy}{|\hat{k}.\hat{n}|}$$

$$= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{3xyz\ dx\ dy}{a\left(\frac{z}{a}\right)}$$

$$= 3\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\,(a^2-x^2)dx$$

$$= \frac{3}{2}\left(\frac{a^2x^2}{2}-\frac{x^4}{4}\right)_0^a$$

$$= \frac{3}{2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right)$$

$$= \frac{3a^4}{8}$$ ________**Ans.**

<u>Example 2</u>: If $\bar{F} = 2z\hat{\imath} - x\hat{\jmath} + y\hat{k}$, evaluate $\iiint_V \bar{F}\, dv$ where, v is the region bounded by the surfaces $x = 0,\ y = 0,\ x = 2,\ y = 4,\ z = x^2,\ z = 2.$

<u>Solution</u>: $\iiint_V \bar{F}\, dv = \iiint\left(2z\hat{\imath} - x\hat{\jmath} + y\hat{k}\right) dx\, dy\, dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 \left(2z\hat{\imath} - x\hat{\jmath} + y\hat{k}\right) dz$$

$$= \int_0^2 dx \int_0^4 dy \left[z^2\hat{\imath} - xz\hat{\jmath} + yz\hat{k}\right]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy \left[4\hat{\imath} - 2x\hat{\jmath} + 2y\hat{k} - x^4\hat{\imath} + x^3\hat{\jmath} - x^2y\hat{k}\right]$$

$$= \int_0^2 dx \left[4y\hat{\imath} - 2xy\hat{\jmath} + y^2\hat{k} - x^4y\hat{\imath} + x^3y\hat{\jmath} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 \left(16\hat{\imath} - 8x\hat{\jmath} + 16\hat{k} - 4x^4\hat{\imath} + 4x^3\hat{\jmath} - 8x^2\hat{k}\right) dx$$

$$= \left[16x\hat{\imath} - 4x^2\hat{\jmath} + 16x\hat{k} - \frac{4x^5}{5}\hat{\imath} + x^4\hat{\jmath} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{\imath} - 16\hat{\jmath} + 32\hat{k} - \frac{128}{5}\hat{\imath} + 16\hat{\jmath} - \frac{64}{3}\hat{k}$$

$$= \frac{32\hat{\imath}}{5} + \frac{32\hat{k}}{3}$$

$= \frac{32}{15}(3\hat{\imath} + 5\hat{k})$ __________ **Ans.**

3.2 <u>Exercise</u>:

1) Evaluate $\iint_S (\bar{F}.\hat{n})ds$, where, $\vec{F} = 18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.
2) If $\vec{F} = (2x^2 - 3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \vec{F}\, dv$, where V is bounded by the plane $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

4.1 <u>GREEN'S THEOREM</u>: (Without proof)

If $\emptyset(x, y), \Psi(x, y), \frac{\partial \phi}{\partial y}$ *and* $\frac{\partial \Psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in $x - y$ plane, then

$$\oint_C (\phi\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

<u>Note</u>: Green's theorem in vector form

$$\int_c \bar{F}.\overline{dr} = \iint_R (\nabla \times \bar{F}).\hat{k}\, dR$$

Where, $\bar{F} = \emptyset\hat{\imath} + \Psi\hat{\jmath}$, $\bar{r} = x\hat{\imath} + y\hat{\jmath}$, $\hat{k}$ is a unit vector along z-axis and $dR = dx\, dy$.

<u>Example 1</u>: Using green's theorem, evaluate $\int_c (x^2\, y\, dx + x^2\, dy)$, where c is the boundary described counter clockwise of the triangle with vertices $(0,0), (1,0), (1,1)$.

<u>Solution</u>: By green's theorem, we have

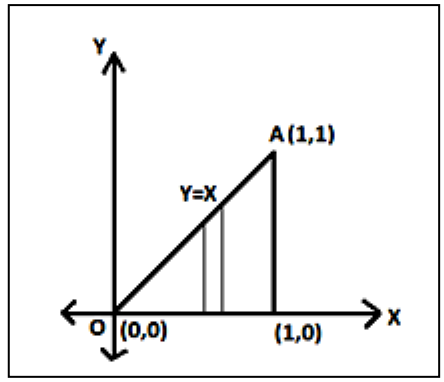
$$\oint_C (\phi\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$\int_c (x^2\, y\, dx + x^2\, dy) = \iint_R (2x - x^2) dx\, dy$$

$$= \int_0^1 (2x - x^2) dx \int_0^x dy$$

$$= \int_0^1 (2x - x^2) dx\, [y]_0^x$$

$$= \int_0^1 (2x^2 - x^3) dx$$



$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{5}{12}$$ ________ **Ans.**

Example 2: Use green's theorem to evaluate

$\int_c (x^2 + xy)dx + (x^2 + y^2)dy$, where c is the square formed by the lines $y = \pm 1,\ x = \pm 1$.

Solution: By green's theorem, we have

$$\oint_C (\phi\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(x^2 + xy)\right] dx\, dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} (2x - x) dxdy$$

$$= \int_{-1}^{1} \int_{-1}^{1} x\, dxdy$$

$$= \int_{-1}^{1} x\, dx\ \int_{-1}^{1} dy$$

$$= \int_{-1}^{1} x\, dx\, (y)_{-1}^{1}$$

$$= \int_{-1}^{1} x\, dx\, (1 + 1)$$

$$= \int_{-1}^{1} 2x\, dx$$

$$= (x^2)_{-1}^{1}$$

$$= 1 - 1$$

$$= 0$$ _________ **Ans.**

4.2 Exercise:

1) Apply Green's theorem to evaluate

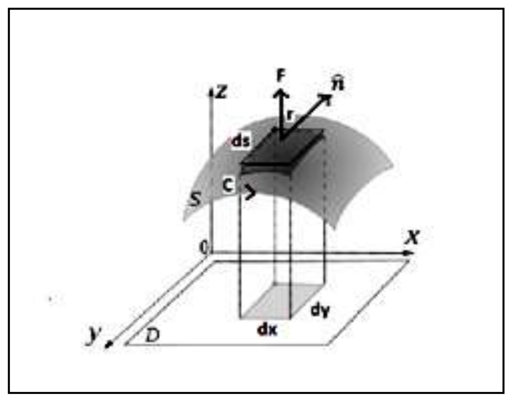
$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2 + y^2 = a^2$.

2) A vector field $\bar{F}$ is given by $\bar{F} = \sin y\, \hat{\imath} + x\,(1 + \cos y)\, \hat{\jmath}$.

Evaluate the line integral $\int_C \bar{F}.\overline{dr}$ where C is the circular path given by $x^2 + y^2 = a^2$.

5.1 STOKE'S THEOREM: (Relation between Line integral and Surface integral) (Without Proof)

Surface integral of the component of curl $\bar{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\bar{F}$ taken along the closed curve C.



Mathematically

$$\oint \bar{F}.\overline{dr} = \iint_S curl\ \bar{F}.\hat{n}\, ds$$

Where $\hat{n} = \cos\propto \hat{\imath} + \cos\beta\, \hat{\jmath} + \cos\gamma\, \hat{k}$

is a unit external normal to any surface ds.

OR

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint \bar{F}.\overline{dr} = \iint_S curl\ \bar{F}.\hat{n}\, ds = \iint_S curl\ \bar{F}.d\bar{S}$$

Example 1: Apply Stoke's theorem to find the value of

$\int_c (y\,dx + z\,dy + x\,dz)$

Where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

Solution: $\int_c (y\,dx + z\,dy + x\,dz)$

$= \int_c \left(y\,\hat{\imath} + z\,\hat{\jmath} + x\,\hat{k}\right).(\hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz)$

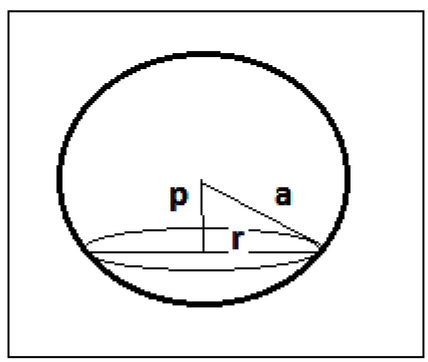
$= \int_c \left(y\,\hat{\imath} + z\,\hat{\jmath} + x\,\hat{k}\right).d\bar{r}$

$= \iint_S curl\left(y\,\hat{\imath} + z\,\hat{\jmath} + x\,\hat{k}\right).\hat{n}\,ds$ (By Stoke's theorem)

$= \iint_S \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(y\,\hat{\imath} + z\,\hat{\jmath} + x\,\hat{k}\right).\hat{n}\,ds$

$= \iint_S -(\hat{\imath} + \hat{\jmath} + \hat{k}).\hat{n}\,ds$ ________(i)

Where S is the circle formed by the integration of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.



$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}$

$= \frac{\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+z-a)}{|\nabla\emptyset|}$

$= \frac{\hat{\imath}+\hat{k}}{\sqrt{1+1}}$

$= \frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$

Putting the value of $\hat{n}$ in (i), we have

$= \iint_S -(\hat{\imath} + \hat{\jmath} + \hat{k}).\left(\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right) ds$

$= \iint_S -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) ds \quad \left[Use\ r^2 = R^2 - p^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}\right]$

$$= -\frac{2}{\sqrt{2}}\iint_S ds = -\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}$$ _______ **Ans.**

Example 2: Evaluate $\oint_C \bar{F}.\overline{dr}$ by stoke's theorem, where

$\bar{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at $(0,0,0), (1,0,0)$ and $(1,1,0)$.

Solution: We have, curl $\bar{F} = \nabla \times \bar{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= 0.\hat{\imath} + \hat{\jmath} + 2(x-y)\hat{k}$$

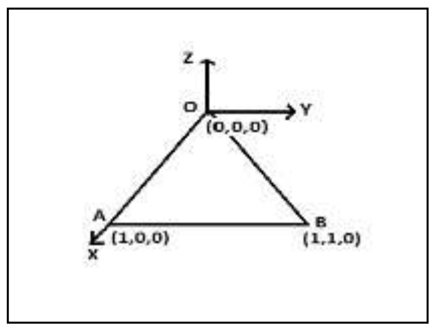
We observe that z co-ordinate of each vertex of the triangle is zero.

Therefore, the triangle lies in the xy-plane.

$$\therefore \quad \hat{n} = \hat{k}$$

$$\therefore \quad curl\,\bar{F}.\hat{n} = [\hat{\jmath} + 2(x-y)\hat{k}].\hat{k} = 2(x-y).$$

In the figure, only xy-plane is considered.



The equation of the line OB is $y = x$

By Stoke's theorem, we have

$$\oint_C \bar{F}.\overline{dr} = \iint_S (curl\,\bar{F}.\hat{n})ds$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)dxdy$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2}\right]dx$$

$$= 2\int_0^1 \frac{x^2}{2}dx$$

$$= \int_0^1 x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$ ________ **Ans.**

5.2 Exercise:

1) Use the Stoke's theorem to evaluate $\int_C [(x+2y)dx + (x-z)dy + (y-z)dz]$ where C is the boundary of the triangle with vertices $(2,0,0), (0,3,0) and\ (0,0,6)$ oriented in the anti-clockwise direction.
2) Apply Stoke's theorem to calculate $\int_c 4\ y\ dx + 2z\ dy + 6y\ dz$ Where c is the curve of intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$
3) Use the Stoke's theorem to evaluate $\int_C y^2\ dx + xy\ dy + xz\ dz$, where C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented in the positive direction.

6.1 GAUSS'S THEOREM OF DIVERGENCE: (Without Proof)

The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume Venclosed by the surface S.

Mathematically

$$\iint_S \vec{F}.\hat{n}\ ds = \iiint_V div\,\vec{F}\ dv$$

Example 1: Evaluate $\iint_S \vec{F}.\hat{n}\ ds$ where $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Solution: By Gauss's divergence theorem,

$$\iint_S \vec{F}.\hat{n}\ ds = \iiint_V (\nabla.\vec{F})\ dv$$

$$= \iiint_v \left(\hat{\imath}\ \frac{\partial}{\partial x} + \hat{\jmath}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right).\left(4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}\right) dv$$

$$= \iiint_v \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dx\ dy\ dz$$

$$= \iiint_v (4z - 2y + y)\ dx\ dy\ dz$$

$$= \iiint_v (4z - y)\, dx\, dy\, dz$$
$$= \int_0^1 \int_0^1 \left(\frac{4z^2}{2} - yz\right)_0^1 dx\, dy$$
$$= \int_0^1 \int_0^1 (2z^2 - yz)_0^1\ dx\, dy$$
$$= \int_0^1 \int_0^1 (2 - y)\ dx\, dy$$
$$= \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx$$
$$= \frac{3}{2} \int_0^1 dx$$
$$= \frac{3}{2} [x]_0^1$$
$$= \frac{3}{2} (1)$$
$$= \frac{3}{2}$$ ________ **Ans.**

Example 2: Evaluate surface integral $\iint \vec{F}.\hat{n}\, ds$, where $\vec{F} = (x^2 + y^2 + z^2)(\hat{\imath} + \hat{\jmath} + \hat{k})$, S is the surface of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 2$ and n is the unit normal in the outward direction to the closed surface S.

Solution: By gauss's divergence theorem,

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V div\, \vec{F}.\ dv$$

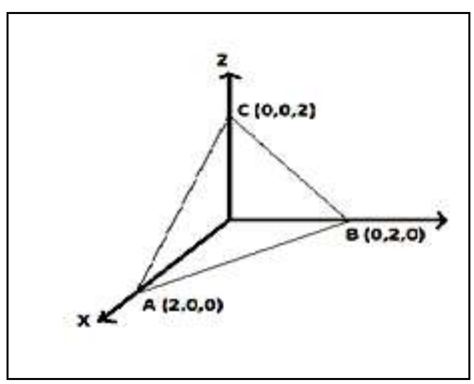
Where S is the surface of tetrahedron $x = 0, y = 0, z = 0, x + y + z = 2$

$$= \iiint_V \left(\hat{\imath}\ \frac{\partial}{\partial x} + \hat{\jmath}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right).(x^2 + y^2 + z^2)(\hat{\imath} + \hat{\jmath} + \hat{k})dv$$

$$= \iiint_V (2x + 2y + 2z)dv$$

$$= 2 \iiint_V (x + y + z)dx\, dy\, dz$$

$$= 2 \int_0^2 dx\ \int_0^{2-x} dy\ \int_0^{2-x-y} (x + y + z)\, dz$$



$$= 2\int_0^2 dx \int_0^{2-x} dy \left(xz + yz + \frac{z^2}{2}\right)_0^{2-x-y}$$

$$= 2\int_0^2 dx \int_0^{2-x} dy \left[2x - x^2 - xy + 2y - xy - y^2 + \frac{(2-x-y)^2}{2}\right]$$

$$= 2\int_0^2 dx \left[2xy - x^2y - xy^2 + y^2 - \frac{y^3}{3} - \frac{(2-x-y)^3}{6}\right]_0^{2-x}$$

$$= 2\int_0^2 dx \left[2x(2-x) - x^2(2-x) - x(2-x)^2 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\int_0^2 \left[4x - 2x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - 2x^2 + \frac{4x^3}{3} - \frac{x^4}{4} - \frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$= 2\left[-\frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$= 2\left[\frac{8}{3} - \frac{16}{12} + \frac{16}{24}\right]$$

$= 4$ ________ **Ans.**

6.2 <u>Exercise</u>:

1) Evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ and $\vec{F} = 3x\hat{\imath} + 4y\hat{\jmath} + 5z\hat{k}$.
2) Find $\iint_S \vec{F}.\hat{n}\, ds$, where $\vec{F} = (2x + 3z)\hat{\imath} - (xz + y)\hat{\jmath} + (y^2 + 2z)\hat{k}$ and S is the surface of the sphere having centre (3,-1, 2) and radius 3.
3) Use divergence theorem to evaluate $\iint_S \vec{A}.\overrightarrow{ds}$, where $\vec{A} = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

4) Use divergence theorem to show that $\iint_S \nabla\ (x^2 + y^2 + z^2).ds = 6V$, where S is any closed surface enclosing volume V.

7.1 REFERECE BOOKS:

1) Introduction to Engineering Mathematics
   By H. K. DASS. & Dr. RAMA VERMA
   S. CHAND
2) Higher Engineering Mathematics
   By B.V. RAMANA
   Mc Graw Hill Education
3) Higher Engineering Mathematics
   By Dr. B.S. GREWAL
   KHANNA PUBLISHERS
4) http://mecmath.net/calc3book.pdf