Unit- 2. (continued).

→ Eigen value & Eigen vectors.
→ Diagonalization of a matrix.
→ symmetric matrices.

Now →

* **Linear Transformations :**

→ **co-ordinate vector :**

Let $V(F)$ be vector space over field F. Let $B = \{u_1, u_2, \cdots u_n\}$ be basis of V. Consider $u \in V$. As B is basis & $u \in V$ then $u \in B$ and can be written as linear combination.

$u = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$.

where, $a_1, a_2, \cdots a_n$ are scalars.

Then $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [u, B]$ is co-ordinate vector.

→ **Linear Transformation :**

Let U & V be two vector spaces over same field F. Then a function $T : U \rightarrow V$ is called Linear Transformation if.

i) $T(u+v) = T(u) + T(v) \quad \forall\ u, v \in U$

ii). $T(au) = a \cdot T(u) \quad \forall\ a \in F, u \in U$

* Matrix of linear Transformation Relative to ordered Basis :

Let $T : U \longrightarrow V$.

Let $B = \{u_1, u_2, \cdots u_n\}$ basis of U

$B' = \{v_1, v_2, \cdots v_m\}$ basis of V.

Thus, $T(u_1), T(u_2), \cdots T(u_n) \in V$.

as they are images of $u_1, u_2, \cdots u_n \in U$.

$$\therefore \; T(u_1) = a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m$$
$$T(u_2) = a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$T(u_n) = a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m.$$

The coefficient mtx is :

$$A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{m \times n}.$$

Transpose is.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}.$$

which is req. mtx of linear Transformaton relative to given ordered basis.

Ex-1 find linear Transformation $T : R^2 \longrightarrow R^3$ whose mtx is $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$ where,

$B = \{ (1,1), (0,2) \}$, $B' = \{ (0,1,1), (1,0,1), (1,1,0) \}$.

$\Rightarrow$ Here, $u_1 = (1,1)$, $u_2 = (0,2)$

$v_1 = (0,1,1)$, $v_2 = (1,0,1)$, $v_3 = (1,1,0)$.

Now,

$T(u_1) = T(1,1)$ & $T(u_2) = T(0,2)$

Thus,

$T(1,1) = 1(0,1,1) + (-2)(1,0,1) + 0(1,1,0)$

$= (0,1,1) + (-2,0,-2) + (0,0,0)$

$T(1,1) = (-2,1,-1)$.

Also,

$T(u_2) = T(0,2) \longrightarrow (1)$

$= -1(0,1,1) + 3(1,0,1) + 1(1,1,0)$

$= (0,-1,-1) + (3,0,3) + (1,1,0)$

$T(0,2) = (4,0,2) \longrightarrow (2)$

Let $(x,y) \in R^2$ be any vector. Thus it can be written as L.C. of basis vectors.

$\therefore (x,y) = a(1,1) + b(0,2) \longrightarrow (3)$

$\therefore T(x,y) = T[a(1,1) + b(0,2)]$

$= a\,T(1,1) + b\,T(0,2)$.

$\therefore T(x,y) = a(-2,1,-1) + b(4,0,2) \longrightarrow (4)$

Now,

from (3)

$(x, y) = (a, a+2b)$

$\therefore x = a,$

$y = a + 2b.$

$2b = y - a$

$b = \frac{y-a}{2}.$

$\therefore x = a, \ b = \frac{y-x}{2}. \rightarrow (4).$

Thus (*) becomes

$T(x, y) = x(-2, 1, -1) + \frac{y-x}{2}(4, 0, 2)$

$= (-2x, x, -x) + (2y-x, 0, y-x).$

$$\boxed{T(x,y) = (2y-4x, \ x, \ y-2x)} \in \mathbb{R}^3.$$

[Ex] Find matrix representing $T: R^3 \to R^4$ such that $T(x,y,z) = (x+y+z, 2x+z, 2y-z, 6y)$ relative to the standard basis of R^3 & R^4.

$\Rightarrow$ To find $[T: B, B']$.

Standard ordered basis for given V.S. R^3 & R^4 are

$R^3 \Rightarrow B = \{ \overset{u_1}{(1,0,0)}, \overset{u_2}{(0,1,0)}, \overset{u_3}{(0,0,1)} \}$

$R^4 \Rightarrow B' = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$.

Now,

$T(u_1) = T(1,0,0) = (1,2,0,0)$

$T(u_2) = T(0,1,0) = (1,0,2,6)$

$T(u_3) = T(0,0,1) = (1,1,-1,0)$.

which gives the coordinates.

Matrix form is,

$$A' = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 6 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

Taking Transpose

$$(A')^T = A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix} = [T: B, B']$$

which is required standard mtx representation.

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

std. ordered Basis elmt of $\mathbb{R}^3$.

(3) Consider Linear Transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$.

If $T(e_1) = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 9 \\ 4 \\ 5 \\ 0 \end{bmatrix}$ & $T(e_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

Find -

i) Matrix prepresentation of T.

ii). If $x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ find $T(x)$.

$\Rightarrow$ Given $T(e_1)$, $T(e_2)$ & $T(e_3)$ already.

The standard matrix representation for $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ is,

$A = [\, T(e_1) \quad T(e_2) \quad T(e_3) \,]$.

(1^{st} column 2^{nd} column 3^{rd} column)

$\therefore A = \begin{bmatrix} 3 & 9 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

{ cross-verify al,

$T(e_1) = A \cdot e_1.$

$= \begin{bmatrix} 3 & 9 & 1 \\ 2 & 4 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

iii) Given $x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

To find $T(x)$.

Now,

$$x = 2e_1 - e_2 + 3e_3.$$

Taking Linear Transformⁿ B.S. we get.

$$T(x) = T(2e_1 - e_2 + 3e_3)$$

as T is Linear it is closed under vector addⁿ & scalar multiplication.

$$\therefore T(x) = 2T(e_1) - T(e_2) + 3T(e_3).$$

$$= 2\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 9 \\ 4 \\ 5 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 4 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 9 \\ 4 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \\ -6 \end{bmatrix}$$

$$\boxed{\therefore T(x) = \begin{bmatrix} 0 \\ 0 \\ -7 \\ -6 \end{bmatrix}}$$