Lecture 12: Diagonalization

A square matrix D is called diagonal if all but diagonal entries are zero:

$$D = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}_{n\times n}. \tag{1}$$

Diagonal matrices are the simplest matrices that are basically equivalent to vectors in R^n. Obviously, D has eigenvalues $a_1,\ a_2, ..., a_n$, and for each $i = 1, 2, ..., n$,

$$D\vec{e}_i = \begin{bmatrix} a_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ a_i \\ \vdots \\ 0 \end{bmatrix} = a_i\vec{e}_i.$$

We thus conclude that for any diagonal matrix, eigenvalues are all diagonal entries, and $\vec{e}_i$ is an eigenvector associated with a_i, the ith diagonal entry, i.e.,

$$\vec{e}_i \text{ is an eigenvector associated with eigenvalue } a_i,\ \text{ for } i = 1, 2, ..., n.$$

Due to the simplicity of diagonal matrices, one likes to know whether any matrix can be similar to a diagonal matrix. Diagonalization is a process of finding a diagonal matrix that is similar to a given non-diagonal matrix.

Definition 12.1. An $n \times n$ matrix A is called diagonalizable if A is similar to a diagonal matrix D.

Example 12.1. Consider

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix},\ D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix},\ P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}.$$

(a) Verify $A = PDP^{-1}$
(b) Find D^k and A^k
(c) Find eigenvalues and eigenvectors for A.

Solution: (a) It suffices to show that $AP = PD$ and that P is invertible. Direct calculations lead to

$$\det P = -1 \neq 0 \Longrightarrow P \text{ is invertible}$$

Again, direct computation leads to

$$AP = \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix},\quad PD = \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix}.$$

Therefore $AP = PD$.

(b)

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}, \quad D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}.$$

$$\begin{aligned} A^2 &= PDP^{-1}\left(PDP^{-1}\right) = PDP^{-1}PDP^{-1} = PD^2P^{-1} \\ A^k &= PD^kP^{-1} \end{aligned}$$

(c) Eigenvalues of A = Eigenvalues of D, which are $\lambda_1 = 5$, $\lambda_2 = 3$. For D, we know that

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is an eigenvectors for } \lambda_1 = 5: \; D\vec{e}_1 = 5\vec{e}_1 \tag{2}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is an eigenvectors for } \lambda_2 = 3: \; D\vec{e}_2 = 3\vec{e}_2. \tag{3}$$

Since $AP = PD$, from (2) and (3), respectively, we see

$$\begin{aligned} AP\vec{e}_1 &= PD\vec{e}_1 = P\left(5\vec{e}_1\right) = 5P\vec{e}_1, \\ AP\vec{e}_2 &= PD\vec{e}_2 = P\left(3\vec{e}_2\right) = 3P\vec{e}_2. \end{aligned}$$

Therefore,

$$P\vec{e}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an eigenvector associated with eigenvalue $\lambda = 5$ for matrix A, and

$$P\vec{e}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is an eigenvector of A associated with eigenvalue $\lambda = 3$.

From this example, we observation that if A is diagonalizable and A is similar to a diagonal matrix D (as in (1)) through an invertible matrix P,

$$AP = PD.$$

Then

$$P\vec{e}_i \text{ is an eigenvector associated with } a_i, \text{ for } i = 1, 2, ..., n.$$

This generalization can be easily verified in the manner analogous to Example 12.1. Moreover, these n eigenvectors $P\vec{e}_1, P\vec{e}_2, ..., P\vec{e}_n$ are linear independent. To see independence, we consider the linear system

$$\sum_{i=1}^{n} \lambda_i P\vec{e}_i = \vec{0}.$$

Suppose the linear system admits a solution $(\lambda_1, ..., \lambda_n)$. Then

$$P\left(\sum_{i=1}^{n} \lambda_i \vec{e}_i\right) = \vec{0}.$$

Since P is invertible, the above equation leads to

$$\sum_{i=1}^{n} \lambda_i \vec{e}_i = \vec{0}.$$

Since has $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n$ are linear independent, it follows that the last linear system has only the trivial solution $\lambda_i = 0$ for all $i = 1, 2, ..., n$. Therefore, $P\vec{e}_1, P\vec{e}_2, ..., P\vec{e}_n$ are linear independent. This observation leads to

Theorem 12.1. (Diagonalization).A $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors. Furthermore, suppose that A has n linearly independent eigenvectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. Set

$$P = [\vec{v}_1, \vec{v}_2, ..., \vec{v}_n], \; D = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$$

where a_i is the eigenvalue associated with $\vec{v}_i$, i.e., $A\vec{v}_i = a_i\vec{v}_i$.Then, P is invertible and

$$P^{-1}AP = D.$$

Proof. We have demonstrated that if A is diagonalizable, then A has n linearly independent eigenvectors. We next show that the reverse is also true: if A has n linearly independent eigenvectors, then A must be diagonalizable. To this end, we only need to verify that $AP = PD$ with P and D described above, since P is obviously invertible due to independence of eigenvectors. The proof of $AP = PD$ is straightforward by calculation as follows:

$$\begin{aligned} AP &= A[\vec{v}_1, \vec{v}_2, ..., \vec{v}_n] \\ &= [A\vec{v}_1, A\vec{v}_2, ..., A\vec{v}_n] \\ &= [a_1\vec{v}_1, a_2\vec{v}_2, ..., a_n\vec{v}_n], \end{aligned}$$

$$\begin{aligned} PD &= P[a_1\vec{e}_1, a_2\vec{e}_2, ..., a_n\vec{e}_n] \\ &= [a_1P\vec{e}_1, a_2P\vec{e}_2, ..., a_nP\vec{e}_n]. \end{aligned}$$

Now,

$$P\vec{e}_1 = [\vec{v}_1, \vec{v}_2, ..., \vec{v}_n]\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix} = \vec{v}_1, \quad P\vec{e}_2 = [\vec{v}_1, \vec{v}_2, ..., \vec{v}_n]\begin{bmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{bmatrix} = \vec{v}_2, ...$$

This shows $AP = PD$. ■

Example 12.2. Diagonalize

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution: Diagonalization means finding a diagonal matrix D and an invertible matrix P such that $AP = PD$. We shall follow Theorem 12.1 step by step.

Step 1. Find all eigenvalues. From computations

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} \\ &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -\lambda^3 + \left(\lambda^2 - 4\lambda^2\right) + 4 \\ &= \left(-\lambda^3 + \lambda^2\right) + \left(-4\lambda^2 + 4\right) \\ &= -\lambda^2(\lambda - 1) - 4\left(\lambda^2 - 1\right) \\ &= -(\lambda - 1)\left[\lambda^2 + 4(\lambda + 1)\right] \\ &= -(\lambda - 1)(\lambda + 2)^2. \end{aligned}$$

we see that A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$ (this is a double root).

Step 2. Find all eigenvalues – find a basis for each eigenspace $Null(A - \lambda I_i)$.

For $\lambda_1 = 1$,

$$\begin{aligned} A - \lambda_1 I &= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \overset{R_1 \to R_2}{\longrightarrow} \begin{bmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{bmatrix} \\ &\overset{R_2 + R_1 \to R_2}{\longrightarrow} \begin{bmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, linear system

$$(A - I)\vec{x} = 0$$

reduces to

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + x_3 &= 0, \end{aligned}$$

where x_3 is the free variable. Choose $x_3 = 1$, we obtain an eigenvector

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -2$,

$$\begin{aligned} A - \lambda_2 I &= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The corresponding linear system is

$$x_1 + x_2 + x_3 = 0$$

It follows that x_2 and x_3 are free variables. As we did before, we need to select (x_2, x_3) to be $(1, 0)$ and $(0, 1)$. Choose $x_2 = 1$, $x_3 = 0 \Longrightarrow x_1 = -x_2 - x_3 = -1$; choose $x_2 = 0$, $x_3 = 1 \Longrightarrow x_1 = -x_2 - x_3 = -1$. We thus got two independent eigenvectors for $\lambda_2 = -2$:

$$\vec{v}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Step 3. Assemble orderly D and P as follows: there are several choices to pair D and P.

$$\begin{aligned} Choice\#1: \ D &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad P = [\vec{v}_1, \vec{v}_2, \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ Choice\#2: \ D &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad P = [\vec{v}_1, \vec{v}_3, \vec{v}_2] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ Choice\#3: \ D &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = [\vec{v}_2, \vec{v}_3, \vec{v}_1] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Remark. Not every matrix is diagonalizable. For instance,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \det(A - \lambda I) = (\lambda - 1)^2.$$

The only eigenvalue is $\lambda = 1$; it has the multiplicity $m = 2$. From

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we see that $(A - I)$ has only one pivot. Thus $r(A - I) = 1$. By Dimension Theorem, we have

$$\dim Null\,(A - I) = 2 - r(A) = 2 - 1 = 1.$$

In other words, the basis consists of $Null\,(A - I)$ consists of only one vector. For instance, one may choose

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

as a basis. According to the discussion above, if A is diagonalizable, i.e., $AP = PD$ for a diagonal matrix

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

then

$$P\vec{e}_1 \text{ is an eigenvector of } A \text{ associated with } a,$$
$$P\vec{e}_2 \text{ is an eigenvector of } A \text{ associated with } b.$$

Furthermore, since P is invertible,

$$P\vec{e}_1 \text{ and } P\vec{e}_2 \text{ are linearly independent (why?).}$$

Since we have already demonstrated that there is no more than two linearly independent eigenvectors, it is impossible to diagonalize A.

In general, if λ_0 is an eigenvalue of multiplicity m (i.e., the characteristic polynomial

$$\det(A - \lambda I) = (\lambda - \lambda_0)^m Q(\lambda),\ \ Q(\lambda_0) \neq 0\ ),$$

then

$$\dim(Null\,(A - \lambda I)) \leq m.$$

Theorem 12.2. Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$ with multiplicity $m_1, m_2, ..., m_p$, respectively. Then,

1. $n_i = \dim(Null\,(A - \lambda_i I)) \leq m_i$ and $m_1 + m_2 + ... + m_p \leq n$.

2. A is diagonalizable iff $n_i = m_i$ for all $i = 1, 2, ..., p$, and

$$m_1 + m_2 + ... + m_p = n.$$

In this case, we have $P^{-1}AP = D$, where P and D can be obtained as follows. Let $\mathcal{B}_i$ be a basis of $Null\,(A - \lambda_i I)$ for each i. Then

$$P = [\mathcal{B}_1, ..., \mathcal{B}_p],\quad D = \begin{bmatrix} \lambda_1 I_{m_1} & ... & 0 \\ ... & ... & ... \\ 0 & ... & \lambda_p I_{m_p} \end{bmatrix},\ I_{m_i} = (m_i \times m_i) \text{ identity}$$

i.e., the first m_1 columns of P form $\mathcal{B}_1$, the eigenvectors for λ_1, the next m_2 columns of P are $\mathcal{B}_2$, then $\mathcal{B}_3$, etc. The last m_p columns of P are from $\mathcal{B}_p$; the first m_1 diagonal entries of D are λ_1, the next m_2 diagonal entries of D are λ_2, and so on.

3. In particular, if A has n distinct real eigenvalues, then A is diagonalizable.

4. Any symmetric matrix is diagonalizable.

Note that, as we saw before, there are multiple choices for assembling P. For instance, if A is 5×5, and A has two eigenvalues $\lambda_1 = 2, \lambda_2 = 3$ with a basis $\{\vec{a}_1, \vec{a}_2\}$ for $Null\,(A - 2I)$ and a basis $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ for $Null\,(A - 3I)$, respectively, then, we have several choices to select pairs of (P, D) :

$$choice\#1 : P = \left[\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2, \vec{b}_3\right], \ D = \begin{bmatrix} 2I_2 & 0 \\ 0 & 3I_3 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$choice\#2 : P = \left[\vec{a}_1, \vec{b}_1, \vec{b}_3, \vec{a}_2, \vec{b}_2\right], \ D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Example 12.3. Diagonalize A

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Solution: We need to find all eigenvalues and eigenvectors. Since A is an upper triangle matrix, we see that eigenvalues are $\lambda_1 = 5$, $m_1 = 2$, $\lambda_2 = -3$, $m_2 = 2$.

For $\lambda_1 = 5$,

$$A - 5I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ -1 & -2 & 0 & -8 \end{bmatrix} \xrightarrow{R_4 + R_3 \to R_4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ 0 & 2 & -8 & -8 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ 0 & 1 & -4 & -4 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 8 & 16 \\ 0 & 1 & -4 & -4 \end{bmatrix}$$

Therefore, x_3 and x_4 are free variable, and

$$\begin{aligned} x_1 &= -8x_3 - 16x_4 \\ x_2 &= 4x_3 + 4x_4. \end{aligned}$$

Choose $(x_3, x_4) = (1, 0) \Longrightarrow x_1 = -8, x_2 = 4$; Choose $(x_3, x_4) = (0, 1) \Longrightarrow x_1 = -16,\ x_2 = 4.$ We obtain two independent eigenvectors

$$\begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \quad (\text{for } \lambda_1 = 5).$$

For $\lambda_2 = -3$,

$$A - (-3)\, I = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix} \overset{R_4 + R_3 \to R_4}{\longrightarrow} \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\overset{\substack{R_3 - 2R_4 \to R_3 \\ R_2 - 4R_4 \to R_2}}{\longrightarrow} \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \overset{R_1 - 8R_3 \to R_1}{\longrightarrow} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

Hence

$$x_1 = 0,\ x_2 = 0.$$

Choose $(x_3, x_4) = (1, 0)$ and $(x_3, x_4) = (0, 1)$, respectively, we have eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{for } \lambda_2 = -3).$$

Assemble pairs (P, D) : There are several choice. For instance,

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

or

$$P = \begin{bmatrix} -8 & 0 & -16 & 0 \\ 4 & 0 & 4 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

- **Homework 12.**

- 1. Suppose that A is similar to

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

   (a) Find the characteristic polynomial and eigenvalues of A.

   (b) Let $P^{-1}AP = D$, where

$$P = \begin{bmatrix} 2 & 1 & 0 & 5 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 4 \end{bmatrix}.$$

   Find a basis for each eigenspace of A.

2. Diagonalize the following matrices.

   (a) $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$, $\lambda = 1, 5$.

   (b) $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

3. Show that if A is diagonalizable, so is A^2.

4. Suppose that $n \times n$ matrices A and B have exact the same linearly independent eigenvectors that span R^n. Show that (a) both A and B can be simultaneously diagonalized (i.e., there are the same P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices), and (b) AB is also diagonalizable.

5. for each statement, determine and explain true or false.

   (a) A is diagonalizable if $A = P^{-1}DP$ for some diagonal matrix D.

   (b) If A is diagonalizable, then A is invertible.

   (c) If $AP = PD$ with D diagonal matrix, then nonzero columns of P are eigenvectors of A.

   (d) Let A be a symmetric matrix and λ be an eigenvalue of A with multiplicity 5. Then the eigenspace $Null\,(A - \lambda I)$ has dimension 5.