Differentiation Under Integral Sign (DUIS)

1. Introduction.

- In addition to variables, additional parameters

$$I(\alpha) = \int_a^b f(x,\alpha)dx \longrightarrow$$ α = Parameter , x = Variable.

Rule 1 : Integrals with constant limits.

$$If \;\; I(\alpha) = \int_a^b f(x,\alpha)dx \quad then \quad \boxed{\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx}$$

a & b constants $\Rightarrow$ LHS derivative $\Rightarrow$ Partial derivative RHS

EXAMPLES

$$Ex\,1: Show\ \ that \int_0^1 \frac{x^a - 1}{\log x}\,dx = \log\ (a+1)\ \ ,(a \geq 0)$$

$$Solution: \quad Let \quad I(a) = \int_0^1 \frac{x^a - 1}{\log x}\,dx$$

$$I^{'}(a) = \frac{d}{da}\int_0^1 \frac{x^a - 1}{\log x}\,dx = \int_0^1 \frac{\partial}{\partial a}\,\frac{x^a - 1}{\log x}\,dx \qquad applying\ \ DUIS$$

$$I^{'}(a) = \int_0^1 \frac{x^a \log x}{\log x}\,dx \quad = \int_0^1 x^a\,dx$$

Integrating w.r.t. x $\quad = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$

$$I^{'}(a) = \frac{1}{a+1}$$

Integrating w.r.t. a $I(a) = \log(a+1) + c$

$$Put\; a = 0, \quad I(0) = \log(0+1) + c \rightarrow c = 0$$

$$\therefore I(a) = \log(a+1), a \geq 0$$

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$Ex\ 2\text{:}\ Prove\ that\ \int_0^\infty e^{-x^2} \cos 2\lambda x dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$

$Solution:\ Let\ I(\lambda) = \int_0^\infty e^{-x^2} \cos 2\lambda\, x\, dx\ ;\quad Where\ \lambda\ is\ a\ parameter$

$By\ DUIS\ I^{'}(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} e^{-x^2} \cos 2\lambda\, x dx$

$$I'(\lambda) = \int_0^\infty e^{-x^2}(-2x).\sin 2\lambda x\,dx \quad = \int_0^\infty (\sin 2\lambda x).(e^{-x^2}(-2x))\,dx$$

$\because$ Integratio n by parts & using $\int e^{f(x)} f'(x)dx = e^{f(x)}$

$$= \left[\sin 2\lambda x\, e^{-x^2}\right]_0^\infty - \int_0^\infty 2\lambda e^{-x^2} \cos 2\lambda x\,dx$$

$$= (0 - 0) - 2\lambda\,\mathrm{I}(\lambda) \qquad \therefore\ \frac{I'(\lambda)}{I(\lambda)} = -2\lambda$$

Integratin g w.r.t. λ , $\log \mathrm{I}(\lambda) = -\lambda^2 + c$ *i.e.* $I(\lambda) = e^{-\lambda^2}.e^c$

$$and\ \ for\ \lambda = 0,\ I(0) = \int_0^\infty e^{-x^2}.dx = \frac{\sqrt{\pi}}{2} = e^c \qquad \therefore I(\lambda) = \frac{\sqrt{\pi}}{2}e^{-\lambda^2}$$

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$$Ex\ 3:\ Evaluate\ \ I(m) = \int_0^\infty e^{-ax}\frac{\sin mx}{x}\,dx.\ Hence\ \ find\ \int_0^\infty \frac{\sin x}{x}\,dx$$

$$Solution:\ Let\ \ I(m) = \int_0^\infty e^{-ax}\frac{\sin mx}{x}dx\ ;\ \ By\ DUIS\ \ I^{'}(m) = \int_0^\infty \frac{\partial}{\partial m}e^{-ax}\frac{\sin mx}{x}dx$$

$$I^{'}(m) = \int_0^\infty e^{-ax}\frac{x\cos mx}{x}dx$$

$$= \int_0^\infty e^{-ax}\cos mx\,dx$$

Integrating w.r.t. x

$$\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos ax + b\sin bx\right]$$

$$= \left\{\frac{e^{-ax}}{a^2+m^2}\left[-a\cos mx + m\sin mx\right]\right\}_0^\infty$$

$$\therefore I^{'}(m) = \frac{a}{a^2+m^2}$$

Integrating w.r.t. m ; $I(m)=\tan^{-1}\dfrac{m}{a}+c$

$$for\ \ m=0,c=0\ ;\ \ Thus\ I(m)=\int_0^\infty e^{-ax}\frac{\sin mx}{x}dx=\tan^{-1}\frac{m}{a}$$

$$Put\ \ m=1,a=0\ \ \therefore\int_0^\infty\frac{\sin x}{x}dx=\tan^{-1}\frac{1}{0}=\frac{\pi}{2}$$

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$$Exercise: Evaluate\int_0^\infty e^{-ax}\frac{\sin x}{x}dx. Hence\ find\int_0^\infty\frac{\sin x}{x}dx \qquad Answer: \frac{\pi}{2}-\tan^{-1}a\ \ ;\ \ \frac{\pi}{2}$$

Ex 4: Prove that $\int_0^\infty \frac{e^{-\propto x}.Sin\ x}{x} dx = Cot^{-1}\alpha$;

Hence deduce that $\int_0^\infty \frac{Sin\ x}{x} = \frac{\pi}{2}$

Solution : Let $\emptyset(\alpha) = \int_0^\infty \frac{e^{-\propto x}.Sin\ x}{x} dx$

Applying **DUIS** $\frac{d\emptyset}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha} \frac{e^{-\propto x}.Sin\ x}{x} dx$

$$= -\int_0^\infty \frac{e^{-\propto x}(x).Sin\ x}{x} dx$$

$$= -\int_0^\infty e^{-\propto x}. Sin\, x\, dx$$

$$= - \left[\frac{e^{-ax}}{a^2+1}\left(-a\, sinx - \cos x\right)\right]_0^{\infty}$$

$$\therefore\ I'\,(\,a\,) = \frac{-1}{x^2+1}$$

Integrating w.r.t. a I (a) = - tan^{-1} a + c

Let a$\longrightarrow$ ∞ , $\lim_{a\to\infty} I\,(\,a\,) = 0 = -\frac{\pi}{2} + c \implies c = \frac{\pi}{2}$

I (a) = $\frac{\pi}{2}$ - tan^{-1} a = $\cot^{-1} a$

Put a = 0 , I (a) =$\int_0^{\infty} \frac{Sin\ x}{x}\, dx = \frac{\pi}{2}$

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PROBLEMS INVOLVING TWO PARAMETERS.

Procedure is exactly same as that of problems involving one parameter.

$$Ex5: Show\ \ that \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx = \log\frac{b}{a}\ \ ;\ \ a\ and\ b\ are\ two\ parameters.$$

$$Solution:\ Let\ I(a) = \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx\ \ Differentiating\ \ w.r.t.a\ and\ applying\ \textbf{DUIS}$$

$$I^{'}(a) = \int_0^\infty \frac{\partial}{\partial a}\frac{e^{-ax}-e^{-bx}}{x}dx = \int_0^\infty \frac{(-x)e^{-ax}-0}{x}dx = -\int_0^\infty e^{-ax}dx = -\left[\frac{e^{-ax}}{-a}\right]_0^\infty = -\frac{1}{a}$$

$$\therefore I^{'}(a) = -\frac{1}{a}$$

Integrating w.r.t. a ; I(a) = - log a + c

Put a = b ; I(b) = - log b + c , c = log b since I(b) = 0

Thus I(a) = - log a + logb , $I(a) = \log\dfrac{b}{a}$ ===============================

$$Ex\ 7 : Show\ that \int_0^1 \frac{x^a - x^b}{\log x} dx = \log\left[\frac{a+1}{b+1}\right] \ ; a > 0\ ,\ b>0$$

$$Solution : Let\ I(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx$$

Applying **DUIS** $\frac{d}{da} I(a) = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - x^b}{\log x} dx$ $\frac{d}{da} I(a) = \int_0^1 \frac{x^a . \log x - 0}{\log x} dx$

$\therefore\ I^{'}(a) = \int_0^1 x^a dx$ Integrating w.r.t. x $\therefore\ I^{'}(a) = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$

Integrating w.r.t. a $\therefore\ I(a) = \log(a+1) + c$ ––––––––(1)

Replace a by b $\therefore\ I(b) = \log(b+1) + c\ ; but\ I(b) = 0$ $\therefore\ c = -\log(b+1)$

Putting the value in (1) $I(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx = \log\left[\frac{a+1}{b+1}\right]$

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LEIBNITZ RULE : Integrals with limits as Functions of the Parameter.

$$If\ a\ and\ b\ are\ functions\ of\ parameter\ \alpha$$

$$i.e.\ I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx\ Then$$

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b,\alpha).\frac{db}{d\alpha} - f(a,\alpha).\frac{da}{d\alpha}$$

$$Ex\ 6 : Verify\ Leibnitz\ rule\ of\ DUIS\ for\ the\ integral \int_a^{a^2} \frac{dx}{x+a}$$

Solution: Part I:

Let $I(a) = \int_a^{a^2} \frac{dx}{x+a}$

$= [\log(x+a)]_a^{a^2} = \log(a + a^2) - \log 2a = \log \frac{a(a+1)}{2a}$

$= \log \frac{(a+1)}{2} \qquad \therefore\ I'(a) = \frac{1}{a+1}$ ---------------------------- (1)

Part II : *Again Let* $I(a) = \int_a^{a^2} \frac{dx}{x+a}$

By DUIS Leibnitz rule $I'(a) = \int_a^{a^2} \frac{\partial}{\partial x} \frac{1}{x+a}\, dx + (\frac{d}{da} a^2).(\frac{1}{a+a^2}) - (\frac{d}{da} a).(\frac{1}{a+a})$

$$= \int_a^{a^2} \frac{-1}{(x+a)^2}\,dx + \left(\frac{2a}{a+a^2}\right) - \frac{1}{2a} = \left[\frac{1}{(x+a)}\right]_a^{a^2} + \left(\frac{2}{a+1}\right) - \frac{1}{2a}$$

$$= \frac{1}{(a^2+a)} - \frac{1}{2a} + \left(\frac{2}{a+1}\right) - \frac{1}{2a}$$

$$\therefore \quad I'(a) = \frac{1}{a+1} \quad -------------(2)$$

From equation (1) and (2) DUIS Leibnitz Rule is verified.

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$$Ex\ \ 7 : If\ f(x) = \int_0^x (x-t)^2 G(t)dt$$

$$then\ prove\ that\ \ \frac{d^3f}{dx^3} = 2G(x)$$

Solution: f(x) = $\int_0^x (x-t)^2\, G(t)dt$

$$\frac{df}{dx} = \frac{d}{dx}\int_0^x (x-t)^2\, G(t)dt$$

By DUIS = $\int_0^x \frac{\partial}{\partial x}(x-t)^2\ G(t)\ dt\ + \frac{dx}{dx}.(0) - \frac{d}{dx}(0).t^2G(t)$

$$\therefore \frac{df}{dx} = \int_0^x 2(x-t)\ \ G(t)dt$$

Again applying DUIS

$$\frac{d^2f}{dx^2} = \int_0^x \frac{\partial}{\partial x}2(x-t)\ \ G(t)dt\ + 0 - 0$$

$$\frac{d^2f}{dx^2} = \int_0^x 2G(t)dt$$

Again applying DUIS,

$$\frac{d^3 f}{dx^3} = \int_0^x \frac{\partial}{\partial x} 2G(t)dt + \frac{dx}{dx} . 2G(t) - 0$$

$= 0 + 2G(t) - 0$

$$\therefore \frac{d^3 f}{dx^3} = 2G(t)$$

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Ex 8 : Show that $\displaystyle \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x}\,dx$ ***is independent of a***

Solution: To show that $I'(a) = 0$, $\displaystyle I'(a) = \frac{d}{da}\int_{\pi/6a}^{\pi/2a} \frac{\sin\ ax}{x}\,dx$

Appling DUIS

$$I'(a) = \int_{\pi/6a}^{\pi/2a} \frac{\partial}{\partial a}\frac{\sin\ ax}{x}\,dx + \left(\frac{-\pi}{2a^2}\right).\frac{\sin\frac{\pi}{2}}{\left(\frac{\pi}{2a}\right)} - \left(\frac{-\pi}{6a^2}\right).\frac{\sin\frac{\pi}{6}}{\left(\frac{\pi}{6a}\right)}$$

$$= \int_{\pi/6a}^{\pi/2a} \cos ax\,dx - \frac{1}{a} + \frac{1}{2a} = \left[\frac{\sin\ ax}{a}\right]_{\pi/6a}^{\pi/2a} - \frac{1}{a} + \frac{1}{2a}$$

$$= \frac{1}{a} - \frac{1}{2a} - \frac{1}{a} + \frac{1}{2a} = 0$$

$$\therefore\ I'(a) = 0$$

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