* Differentiation under Integral sign:
(DUIS)

If $I = \int_a^b f(x, \alpha)\,dx$ then

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx.$$

① Prove that $\int_0^1 \frac{x^a - 1}{\log x}\,dx = \log(1+a)$; $a \geq 0$.

$\Rightarrow$ Let $\phi(a) = \int_0^1 \frac{x^a - 1}{\log x}\,dx. \longrightarrow ①$

Diff both side w.r.to a.

$$\frac{d\phi}{da} = \int_0^1 \frac{\partial}{\partial a}\frac{(x^a - 1)}{\log x}\,dx = \int_0^1 \frac{1}{\cancel{\log x}} \cdot x^a \cdot \cancel{\log x} \cdot dx.$$

$$= \int_0^1 x^a\,dx.$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}.$$

$\therefore \frac{d\phi}{da} = \frac{1}{a+1}$

Int. B.S.

$\phi(a) = \log(a+1) + c \ldots$ $\left\{ \int \frac{1}{x}\,dx = \log x. \right.$

put $a = 0$.

$\therefore \phi(0) = 0 + c$;

But $\phi(0) = \int_0^1 \frac{x^0 - 1}{\log x}\,dx = 0$.

$\therefore \boxed{c = 0}$. from ①.

$\therefore$ $\phi(a) = \log(a+1)$. Thus. $\int_0^1 \frac{x^a - 1}{\log x}\,dx = \log(a+1)$

② Evaluate $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$ $(a>-1)$.

$\Rightarrow$ Let $\phi(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$. $\longrightarrow$ ①.

Diff. w.r.to a.

$$\frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}\left[1-e^{-ax}\right]\right]dx.$$

$$= \int_0^\infty \frac{e^{-x}}{x}(-x.\,e^{-ax})\,dx.$$

$$= \int_0^\infty e^{-(a+1)x}\,dx.$$

$$= \left[\frac{e^{-(a+1).x}}{-(a+1)}\right]_0^\infty.$$

$$= \frac{e^{-\infty}}{-(a+1)} + \frac{e^{0}}{+(a+1)}.$$

$$\frac{d\phi}{da} = \frac{1}{a+1}$$

Int B.S. w.r.to a.

$$\phi(a) = \int \frac{1}{a+1}\,da = \log(a+1) + c.$$

put $a=0$,

$$\phi(0) = 0 + c.$$

$$\int_0^\infty \frac{e^{-x}}{x}(1-1).\,dx = c.$$

$$\therefore\ 0 = c \quad \Rightarrow \boxed{c=0}$$

Thus from ①

$$\boxed{\int_0^\infty \frac{e^{-x}(1-e^{-ax})}{x}\,dx = \log(a+1)}$$

Ex. 5 : Show that $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)} \cdot dx = \frac{\pi}{2}\log(1+a)$.

Sol. : Let

$$\phi(a) = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx$$

$$\frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx = \int_0^\infty \frac{1\cdot(x)}{1+a^2x^2}\cdot\frac{1}{x(1+x^2)}\, dx$$

$$= \int_0^\infty \frac{dx}{(1+a^2x^2)(1+x^2)} = \int_0^\infty \left(\frac{\frac{1}{(1-1/a^2)}}{1+a^2x^2} + \frac{\frac{1}{1-a^2}}{1+x^2}\right) dx$$

$$= \frac{1}{1-a^2}\left[\int_0^\infty \frac{dx}{1+x^2} - \int_0^\infty \frac{a^2}{1+a^2x^2}\, dx\right] = \frac{1}{1-a^2}\left[\tan^{-1}x - a\tan^{-1}(ax)\right]_0^\infty$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a\cdot\frac{\pi}{2}\right] = \frac{\pi}{2}\frac{(1-a)}{(1-a)(1+a)} = \frac{\pi}{2}\cdot\frac{1}{1+a}$$

$$d\phi = \frac{\pi}{2}\cdot\frac{da}{1+a}$$

$\therefore$ $$\phi(a) = \frac{\pi}{2}\log(1+a) + C$$

To determine C, we put $a = 0$ $\therefore$ $\phi(0) = C$.

But $$\phi(0) = \int_0^\infty \frac{\tan^{-1}0}{x(1+x^2)}\, dx = 0 \quad \therefore C = 0$$

Hence $$\boxed{\phi(a) = \frac{\pi}{2}\log(1+a)}$$