1.2 Solution of System of Simultaneous Linear Equations

Matrices are very important themselves, but their greatest use lies in solving sets of linear equations. Note first that the system of simultaneous linear equations (n equations in n unknowns)

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array}$$

can be written $\mathbf{AX} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

There are four cases which arise depending on whether, or not, $|A| = 0$ and $b = 0$.

Case (I) $b \neq 0, |A| \neq 0$

In this case the matrix A is non-singular so A^{-1} exists and hence pre-multiplying both sides of the equation $AX = b$ by A^{-1} gives

$A^{-1}AX = A^{-1}b, \quad \rightarrow \quad IX = A^{-1}b, \quad \rightarrow \quad X = A^{-1}b.$

Thus we have a unique solution.

Case (ii) $b = 0, \ |A| \neq 0$

Again A^{-1} exists so the system $AX = 0$ has the unique solution $X = A^{-1}0 = 0$ (i.e. only the trivial (or zero) solution $X = 0$).

Case (iii) $b \neq 0, \ |A| = 0$

Since $|A| = 0$ the inverse matrix does not exist. This is the most complicated of the four cases and there are two possibilities:

Either we have no solution,

Or there are infinitely many solutions.

Consider the equations $\left.\begin{aligned} a1x + b1y + c1z &= \mathrm{d1} \\ a2x + b2y + c2z &= \mathrm{d2} \\ a3x + b3y + c3z &= \mathrm{d3} \end{aligned}\right\}$ …. (i)

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the system of equations (i) is equivalent to the matrix equations AX =D …..(ii)

Where A is the Coefficient matrix.

If the determinant of Coefficient be $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Multiplying both sides of (ii) by A^{-1}, we get

$$A^{-1}AX = A^{-1}D, \quad \rightarrow \quad IX = A^{-1}D, \quad \rightarrow \quad X = A^{-1}D$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad …….(iii)$$

Where A_1, B_1 etc. are cofactors of a_1, b_1 etc. and Adj.of A $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$, $(\Delta \neq 0)$

Hence equating the values of x, y and z to the corresponding elements in the product on the right side of (iii), we get the desired solution.

Example 1. Solve the equations 3x + y +2z = 3, 2x -3y - z = -3, x + 2y + z = 4 by adjoint method

Solution.

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(say)}.$$

Then $A_1 = -1, A_2 = 3, A_3 = 5$; $B_1 = -3, B_2 = 1, B_3 = 7$; $C_1 = 7, C_2 = -5, C_3 = -11$.

Also $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = 8$.

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x = 1, y = 2, z = -1$.

Ex.2. Solve the equations x + 2y – z = 6, 3x + 5y – z = 2, −2x – y −2z = 4 by adjoint method

Solution: Here A = $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, D = $\begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$

Then Δ = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ = 1(-10-1) – 2 (-6-2) -1 (-3+10) = -2

Adj. A = $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} -11 & 5 & 3 \\ 8 & -4 & -2 \\ 7 & -3 & -1 \end{bmatrix}$

.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} -11 & 5 & 3 \\ 8 & -4 & -2 \\ 7 & -3 & -1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 \\ -16 \\ -16 \end{bmatrix}$$

Hence x = 22, y = -16, z = -16.

Assignment 2

Q.1 Solve the following using the inverse matrix method:

(a) 2x − 3y = 1, 4x + 4y = 2

(b) 2x − 5y = 2 , −4x + 10y = 1

(c) 6x − y = 0, 2x − 4y = 1.

(d) 2x + 3y = 3 , 5x + 4y = 11

Q.2 Solve the following equations using matrix methods:

(a) $2x_1 + x_2 - x_3 = 0$, $x_1 + x_3 = 4$, $x_1 + x_2 + x_3 = 0$

(b) $x_1 - x_2 + x_3 = 1$, $-x_1 + x_3 = 1$, $x_1 + x_2 - x_3 = 0$

(c) $x_1 - 2x_2 + x_3 = 3$, $2x_1 + x_2 - x_3 = 5$, $3x_1 - x_2 + 2x_3 = 12$

(d) x + y + z = 3, 2x – y + z = 2, x – 2y + 3z = 2.