1.14 CAYLEY–HAMILTON THEOREM

Theorem Every square matrix satisfies its own characteristic equation.

Proof Let A be n-rowed square matrix. Its characteristic equation is

$$|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n)$$

$$(A - \lambda I) \text{ adj } (A - \lambda I) = |A - \lambda I| I \qquad \ldots(1)$$

$$[\because A \text{ adj } (A) = |A| I]$$

Since adj $(A - \lambda I)$ has element as cofactors of elements of $|A - \lambda I|$, the elements of adj $(A - \lambda I)$ are polynomials in λ of degree $n - 1$or less. Hence, adj $(A - \lambda I)$ can be written as a matrix polynomial in λ.

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$$\text{adj}\,(A-\lambda I)=B_0\lambda^{n-1}+B_1\lambda^{n-2}+\ldots+B_{n-2}\lambda+B_{n-1}$$

where $B_0, B_1, \ldots, B_{n-1}$ are matrices of order n.

$$(A-\lambda I)\,\text{adj}\,(A-\lambda I)=(A-\lambda I)[B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}+\lambda+B_{n-1}]$$

$$|A-\lambda I|I=(A-\lambda I)[B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}\lambda+B_{n-1}]$$

$$(-1)^n[I\lambda^n+a_1I\lambda^{n-1}+a_2I\lambda^{n-2}+\cdots+a_{n-1}I\lambda+a_nI]$$

$$=(-IB_0)\lambda^n+(AB_0-IB_1)\lambda^{n-1}+(AB_1-IB_2)\lambda^{n-2}+\cdots+(AB_{n-2}-IB_{n-1})\lambda+AB_{n-1}$$

Equating corresponding coefficients,

$$\begin{aligned}
-IB_0 &= (-1)^n I\\
AB_0-IB_1 &= (-1)^n a_1 I\\
AB_1-IB_2 &= (-1)^n a_2 I\\
\vdots\quad & \quad\vdots\\
AB_{n-2}-IB_{n-1} &= (-1)^n a_{n-1} I\\
AB_{n-1} &= (-1)^n a_n I
\end{aligned}$$

Premultiplying the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots I$ and adding,

$$(-1)^n[A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots+a_nI]=0$$

Hence,
$$A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots+a_nI=0 \qquad \ldots(2)$$

Corollary If A is a nonsingular matrix, i.e., det $(A)\neq 0$ then premultiplying Eq.(2) by A^{-1}, we get

$$A^{n-1}+a_1A^{n-2}+a_2A^{n-3}+\cdots+a_nA^{n-1}=0$$

$$A^{-1}=-\frac{1}{a^n}[A^{n-1}+a_1A^{n-2}+\cdots+a_{n-1}I]$$

Example 1

Apply Cayley-Hamilton theorem to $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$ *and deduce that* $A^8=625I$

Solution

$$A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}1-\lambda & 2\\ 2 & -1-\lambda\end{vmatrix}=0$$

$$\lambda^2-S_1\lambda+S_2=0$$

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where S_1 = sum of the principal diagonal elements of $A = 1 - 1 = 0$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$
$$= -1 - 4$$
$$= -5$$

Hence, the characteristic equation is

$$\lambda^2 - 5 = 0$$

By the Cayley–Hamilton theorem, the matrix A satisfies its own characteristic equation.

$$A^2 - 5I = 0$$
$$A^2 = 5I$$
$$A^4 = 25I$$
$$A^8 = 625I$$

Example 2

Verify Cayley–Hamilton theorem for following matrix and hence, find A^{-1} and A^4.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ [May '04, Dec '06]

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = sum of the principal diagonal elements of $A = 2 + 2 + 2 = 6$

S_2 = sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$
$$= (4-1) + (4-1) + (4-1)$$
$$= 9$$

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$$S_3 = \det(A) = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) + 1(-2+1) + 1(1-2)$$

$$= 6 - 1 - 1$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0} \qquad \ldots(1)$$

The matrix A satisfies its own characteristic equation. Hence, the Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = \mathbf{0}$$

$$A^2 - 6A + 9I - 4A^{-1} = \mathbf{0}$$

$$4A^{-1} = (A^2 - 6A + 9I)$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

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Multiplying Eq. (1) by A,

$$A(A^3 - 6A^2 + 9A - 4I) = \mathbf{0}$$

$$A^4 - 6A^3 + 9A^2 - 4A = \mathbf{0}$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$= \begin{bmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{bmatrix} - \begin{bmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

Example 3

Show that Matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ *satisfied the Cayley–Hamilton theorem and hence find* A^{-1}, *if it exists.*

Solution

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = sum of the principal diagonal elements of $A = 0$

S_2 = sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} + \begin{vmatrix} 0 & -b \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$$

$$= (0 + a^2) + (0 + b^2) + (0 + c^2)$$

$$= a^2 + b^2 + c^2$$

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$$S_3 = \det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$

$$= 0 - c(0 - ab) - b(ac - 0)$$

$$= abc - abc$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 + (a^2 + b^2 + c^2)\lambda = 0$$

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3 - cb^2 - ca^2 & b^3 + bc^2 + ba^2 \\ c^3 + ca^2 + cb^2 & 0 & -ab^2 - ac^2 - a^3 \\ -bc^2 - b^3 - a^2 b & ac^2 + ab^2 + a^3 & 0 \end{bmatrix}$$

$$= -(a^2 + b^2 + c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= -(a^2 + b^2 + c^2)A$$

$$A^3 + (a^2 + b^2 + c^2)A = \mathbf{0}$$

The matrix A satisfies its own characteristic equation. Hence, the Cayley–Hamilton theorem is verified.

$$\det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$

$$= -c(0 - ab) - b(ac - 0)$$

$$= abc - abc = 0$$

Hence, A^{-1} does not exist.

Example 4

Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and verify the Cayley–Hamilton theorem for this matrix. Find* A^{-1} *and also express* $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as a linear polynomial in A.*

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Solution

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = sum of the principal diagonal elements of $A = 1 + 3 = 4$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3 - 8$$

$$= -5$$

Hence, the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = -1, 5$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \qquad \text{...(1)}$$

The matrix A satisfies its own characteristic equation. Hence, the Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^2 - 4A - 5) = \mathbf{0}$$

$$A - 4I - 5A^{-1} = \mathbf{0}$$

$$4A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$$= A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I) + 3(A^2 - 4A - 5I) + A + 5I$$

$$= (A^2 - 4A - 5I)(A^3 - 2A + 3I) + (A + 5I)$$

$$= \mathbf{0} + (A + 5I)$$

$$= A + 5I \qquad \text{[Using Eq. (1)]}$$

which is a linear polynomial in A.

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Example 5

Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and hence find the matrix represented by* $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. [May '05, Dec '05]

Solution

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = sum of the principal diagonal elements of $A = 2 + 1 + 2 = 5$

S_2 = sum of the minor of principal diagonal elements of A

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0) + (4-1) + (2-0)$$

$$= 7$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1)$$

$$= 4 - 0 - 1$$

$$= 3$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By the Cayley–Hamilton theorem,

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$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \text{...(1)}$$

Now,
$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$
$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$
$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$
$$= 0 + (A^2 + A + I) \qquad \text{[Using Eq. (1)]}$$
$$= A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 6

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *prove by induction that for every integer* $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. *Hence, find* A^{50}.

Solution

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

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where S_1 = sum of the principal diagonal elements of $A = 1 + 0 + 0 = 1$
S_2 = sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-0) + (0-0)$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1(0-1) + 0 + 0$$

$$= -1$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

By the Cayley–Hamilton theorem,

$$A^3 - A^2 - A + I = \mathbf{0}$$

$$A^3 = A^2 + A - I$$

$$= A^{3-2} + A^2 - I \qquad \text{...(1)}$$

Hence $A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.
Assuming that Eq. (1) is true for $n = k$,

$$A^k = A^{k-2} + A^2 - I$$

Multiplying both the sides by A,

$$A^{k+1} = A^{k-1} + A^3 - A$$

Substituting the value of A^3,

$$A^{k+1} = A^{k-1} + (A^2 + A - I) - A$$

$$A^{(k+1)-2} + A^2 - I$$

Hence $A^n = A^{n-2} + A^2 - I$ is true for $n = k + 1$

Thus, by mathematical induction, it is true for $n \geq 3$

We have,

$$A^n = A^{n-2} + A^2 - I$$

$$= (A^{n-4} + A^2 - I) + A^2 - I$$

$$= A^{n-4} + 2(A^2 - I)$$

$$= (A^{n-6} + A^2 - I) + 2(A^2 - I)$$

$$= A^{n-6} + 3(A^2 - I)$$

$$A^n = A^{n-2r} + r(A^2 - I)$$

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Putting $n = 50$ and $r = 24$,

$$A^{50} = A^{50-2(24)} + 24(A^2 - I)$$

$$= A^2 + 24A^2 - 24I$$

$$= 25A^2 - 24I$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

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