* Beta function :

Beta function is defined as,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; m>0 \quad n>0.$$

it is also called Euler's Integral.

Properties.

(1) P.T. $B(m,n) = B(n,m)$.

$\Rightarrow$ WKT, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$

But $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

$$\therefore \quad B(m,n) = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}(x)^{n-1}\,dx.$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$\boxed{B(m,n) = B(n,m)}$$

* Other form of Beta function :

1) $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx.$

(2) $B(m,n) = 2 \int_0^\infty \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta.$

* Relationship between Beta & Gamma function.

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Examples on Relation between Beta & Gamma

(1). $\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx.$

$\rightarrow$ put $x^2 = t.$

$$2x\,dx = dt$$

$$dx = \frac{dt}{2x}.$$

$$dx = \frac{dt}{2\sqrt{t}}$$

$$\therefore I = \int_0^\infty \frac{t}{(1+t)^{7/2}} \frac{dt}{2\sqrt{t}}.$$

$$= \frac{1}{2} \int_0^\infty \frac{t^{1/2}}{(1+t)^{7/2}} dt.$$

$$= \frac{1}{2} \cdot B\left(\frac{3}{2}, \frac{4}{2}\right) \quad \cdots \left\{ B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx. \right.$$

$$I = \frac{1}{2} \cdot \frac{\Gamma 3/2 \cdot \Gamma 2}{\Gamma 7/2}.$$

② $\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}}\,dx$.

$\Rightarrow I = \int_0^\infty \frac{x^2+x^6}{(1+x)^{10}}\,dx$

$= \int_0^\infty \frac{x^2}{(1+x)^{10}}\,dx + \int_0^\infty \frac{x^6}{(1+x)^{10}}\,dx.$

$= B(3,7) + B(7,3)$

$= B(3,7) + B(3,7) \quad \cdots \{B(m,n) = B(n,m)$

$= 2\,B(3,7)$

$= 2 \cdot \frac{\Gamma 3\,\Gamma 7}{\Gamma 10}.$

$= 2 \times \frac{(2!)(6!)}{9!}$

$= \cancel{2 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= \frac{2 \times 2 \times 1 \times \cancel{6!}}{9 \times \cancel{8} \times 7 \times \cancel{6!}} = \frac{1}{126}.$

$$\boxed{\therefore \int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}}\,dx = \frac{1}{126}.}$$

* Standard formulae :-

WKT,
$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta .$$

put $2m-1 = p$, $2n-1 = q$.

$$m = \frac{p+1}{2}, \quad n = \frac{q+1}{2}.$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2} \cdot B\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

$$= \frac{1}{2} \cdot \frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}.$$

* Examples :-

① Evaluate $\int_0^{\pi/2} \sin^5\theta \, d\theta$.

$$\Rightarrow \int_0^{\pi/2} \sin^5\theta \, d\theta = \int_0^{\pi/2} \sin^5\theta \cdot \cos^0\theta \, d\theta.$$

$$= \frac{1}{2} \cdot \frac{\Gamma 6/2 \cdot \Gamma 1/2}{\Gamma 7/2}.$$

$$= \frac{1}{2} \cdot \frac{\Gamma 3 \cdot \Gamma 1/2}{\cancel{\Gamma 5/2}\left(\frac{5}{2}\right)!}$$

$$= \frac{\cancel{3}+2\times 1 \; \Gamma 1/2}{\cancel{2}\times\left(\frac{5}{\cancel{2}}\times\frac{\cancel{3}}{2}\times\frac{1}{2}\right)\Gamma 1/2}$$

$$= \frac{8}{5}.$$

③ $\int_0^{\pi/6} \cos^4 3\theta \cdot \sin^2 6\theta \cdot d\theta$.

$\Rightarrow$ Let $3\theta = t$.

$3\, d\theta = dt$

$d\theta = \frac{dt}{3}$. as $\theta \to 0$ $t \to 0$.

$\theta \to \frac{\pi}{6}$ $t \to \pi/2$

$\therefore \quad I = \int_0^{\pi/2} \cos^4 t \cdot (\sin^2 2t) \cdot \frac{dt}{3}$.

$= \frac{1}{3} \cdot \int_0^{\pi/2} \cos^4 t \cdot (2\sin t \cdot \cos t)^2 \cdot dt$.

$= \frac{1}{3} \int_0^{\pi/2} 4\cos^6 t \cdot \sin^2 t \cdot dt$.

$= \frac{4}{3} \cdot \int_0^{\pi/2} \sin^2 t \cdot \cos^6 t \cdot dt$.

$= \frac{4}{3} \cdot \frac{\Gamma 3/2 \;\; \Gamma 7/2}{2\, \Gamma 10/2}$.

$= \frac{\overset{2}{\cancel{4}}}{3} \cdot \frac{\left(\frac{1}{2} \cdot \Gamma 1/2\right) \cdot \left(5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2\right)}{\cancel{2} \cdot \cancel{\Gamma}\; 4!}$.

$= \frac{15\pi}{24 \times 8}$.

* Result:

$$\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$$

Q) Evaluate $\int_0^\infty \frac{dx}{1+x^4}$.

OR.

P.T. $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

$\Rightarrow$ $I = \int_0^\infty \frac{dx}{1+x^4}$.

put $x^2 = \tan\theta$. $\Rightarrow x = \sqrt{\tan\theta}$.

$\therefore 2x\,dx = \sec^2\theta \cdot d\theta$.

$\therefore dx = \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}$.

when

$x \to 0$	$x \to \infty$
$\theta \to 0$	$\theta \to \pi/2$

$\therefore I = \int_0^{\pi/2} \frac{\sec^2\theta \cdot d\theta}{(1+\tan^2\theta)2\sqrt{\tan\theta}}$.

$= \frac{1}{2}\int_0^{\pi/2} \frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} \times \sqrt{\cot\theta}\,.\,d\theta \quad \ldots \{1+\tan^2\theta = \sec^2\theta.$

$= \frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\,.\,d\theta \,. = \frac{1}{2}\int_0^{\pi/2} \cos^{1/2}\theta \cdot \sin^{-1/2}\theta\, d\theta$

$\therefore I = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta\, d\theta$.

$$\therefore\ I = \frac{1}{2}\cdot\left[\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\cdot\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\cdot\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}\right]$$

$$= \frac{1}{4}\cdot\left[\frac{\Gamma\frac{1}{4}\ \Gamma\frac{3}{4}}{\Gamma 1}\right]$$

$$= \frac{1}{4}\cdot\Gamma\tfrac{1}{4}\cdot\Gamma\tfrac{3}{4}$$

put $n = \frac{1}{4}$ ... $\{$ to use $\Gamma n\ \Gamma 1-n = \frac{\pi}{\sin n\pi}$

$$\Gamma\tfrac{1}{4}\ \Gamma\left(1-\tfrac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\Gamma\tfrac{1}{4}\ \Gamma\tfrac{3}{4} = \frac{\pi}{1/\sqrt{2}} \quad \ldots \left\{ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \right.$$

$$= \sqrt{2}\,\pi.$$

$\therefore$ Thus

$$I = \frac{1}{4}\sqrt{2}\,\pi. = \frac{1}{2\times 2}\times\sqrt{2}\,\pi$$

$$= \frac{1}{2\cdot\sqrt{2}\cdot\cancel{\sqrt{2}}}\cdot\cancel{\sqrt{2}}\,\pi$$

$$\boxed{I = \frac{\pi}{2\sqrt{2}}}$$

H.W. ① $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$ ② $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta.$

3.9 BETA FUNCTION

Definition : Consider the definite integral $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \; m > 0, \; n > 0.$

It is denoted by the symbol B(m, n) (we read it as Beta (m, n)) and is called *Beta Function.*

$$\boxed{\mathbf{B\,(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \quad m > 0 \quad n > 0}}$$

The Beta function is also called as *Euler's integral of the first kind.*

For example, (1) $\quad B\left(3, \frac{3}{2}\right) = \int_0^1 x^2 (1-x)^{1/2} \, dx$

(2) $\int_0^1 t^4 (1-t)^{3/2} \, dt = B\left(5, \frac{5}{2}\right)$

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3.10 PROPERTIES OF BETA FUNCTIONS

1. $B(m, n) = B(n, m)$

Proof : $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}\,dx$

$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore B(m, n) = \int_0^1 (1-x)^{m-1}\cdot x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n, m)$

$\therefore \boxed{B(m, n) = B(n, m)}$

2. $\int_0^1 x^m (1-x)^n\,dx = B(m+1, n+1)$

3. $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$

Proof : $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$

x	0	1
θ	0	π/2

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$\boxed{B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta}$$

We consider this as a definition of Beta function.

Further, let $2m-1 = p$, $2n-1 = q$ $\therefore m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$ then

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta$$

Standard Formula :

$$\boxed{\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

4. Alternating definition :

5. Relation between Beta and Gamma Functions :

We have $$\boxed{B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$ (For the proof, refer page 9.36)

6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

We know, $$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

Put $p = q = 0$

$$\int_0^{\pi/2} d\theta = \frac{1}{2} \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} \Rightarrow \frac{\pi}{2} = \left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

$\therefore$ $$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Ex. 15 : *Prove that $B(m, n) = B(m, n+1) + B(m+1, n)$*

Sol. :

$$\text{R.H.S.} = B(m, n+1) + B(m+1, n)$$

$$= \frac{\Gamma m \,\Gamma (n+1)}{\Gamma (m+n+1)} + \frac{\Gamma (m+1)\,\Gamma n}{\Gamma (m+1+n)}$$

$$= \frac{\Gamma m \; n\Gamma n}{(m+n)\Gamma (m+n)} + \frac{m\Gamma m \,\Gamma n}{(m+n)\Gamma (m+n)}$$

$$= \frac{\Gamma m \,\Gamma n \,(n+m)}{(m+n)\Gamma (m+n)} = \frac{\Gamma m \,\Gamma n}{\Gamma (m+n)} = B(m, n) = \text{L.H.S.}$$

Ex. 16 : *Show that $B(m, n)$* $B(m+n, p) = \dfrac{\Gamma m \,\Gamma n \,\Gamma p}{\Gamma (m+n+p)}$

Sol. :

$$\text{L.H.S.} = B(m, n)\; B(m+n, p) = \frac{\Gamma m \,\Gamma n}{\Gamma (m+n)} \frac{\Gamma (m+n)\,\Gamma p}{\Gamma (m+n+p)} = \frac{\Gamma m \,\Gamma n \,\Gamma p}{\Gamma (m+n+p)} = \text{R.H.S.}$$