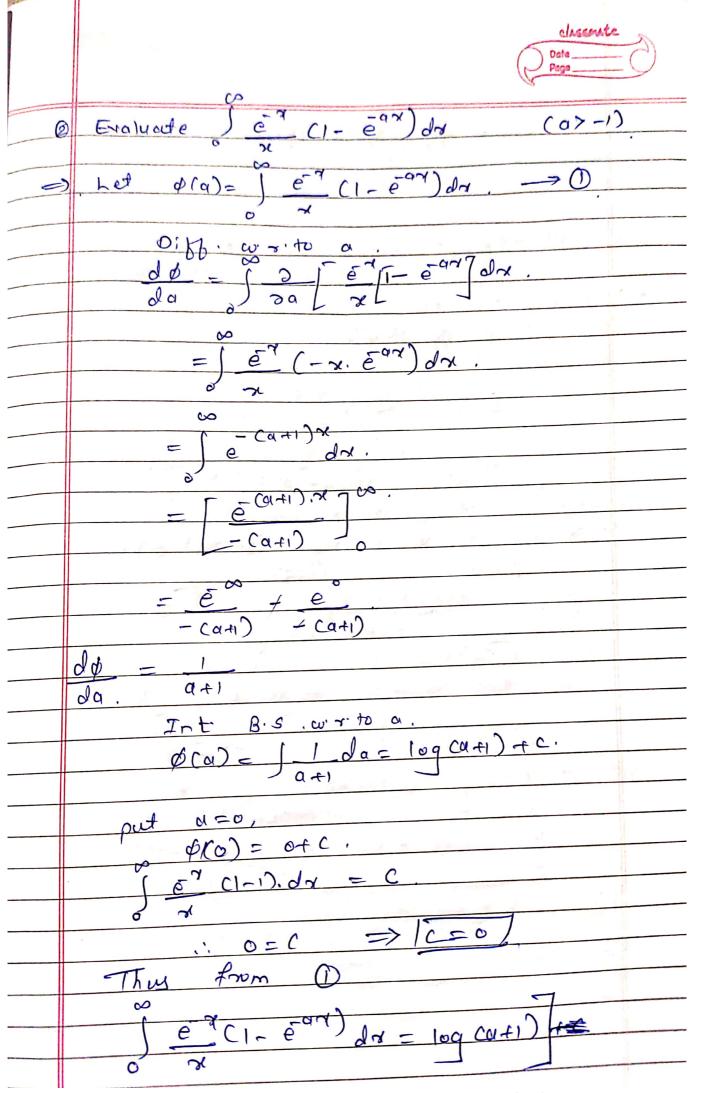


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Ex. 5: Show that 
$$\int_{0}^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^{2})} dx = \frac{\pi}{2} \log(1+a).$$

$$\phi(a) = \int_{0}^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^{2})} dx$$

$$\frac{d\phi}{da} = \int_{0}^{\infty} \frac{\partial}{\partial a} \frac{\tan^{-1}(ax)}{x(1+x^{2})} dx = \int_{0}^{\infty} \frac{1 \cdot (x)}{1+a^{2}x^{2}} \cdot \frac{1}{x(1+x^{2})} dx$$

$$= \int_{0}^{\infty} \frac{dx}{(1+a^{2}x^{2})(1+x^{2})} = \int_{0}^{\infty} \left(\frac{1}{(1-1/a^{2})} + \frac{1}{1-a^{2}}\right) dx$$

$$= \frac{1}{1-a^{2}} \left[\int_{0}^{\infty} \frac{dx}{1+x^{2}} - \int_{0}^{\infty} \frac{a^{2}}{1+a^{2}x^{2}} dx\right] = \frac{1}{1-a^{2}} \left[\tan^{-1}x - a\tan^{-1}(ax)\right]_{0}^{\infty}$$

$$= \frac{1}{1-a^{2}} \left[\frac{\pi}{2} - a \cdot \frac{\pi}{2}\right] = \frac{\pi}{2} \frac{(1-a)}{(1-a)(1+a)} = \frac{\pi}{2} \cdot \frac{1}{1+a}$$

 $d\phi = \frac{\pi}{2} \cdot \frac{da}{1+a}$ 

$$\phi (a) = \frac{\pi}{2} \log (1 + a) + C$$

To determine C, we put  $a = 0 : \phi(0) = C$ .

But

$$\phi (0) = \int_{0}^{\infty} \frac{\tan^{-1} 0}{x (1 + x^{2})} dx = 0 ... C = 0$$

Hence

$$\phi(a) = \frac{\pi}{2} \log (1 + a)$$