Boolean Algebra

Boolean Algebra Summary

- We can interpret high or low voltage as representing true or false.
- A variable whose value can be either 1 or 0 is called a Boolean variable.
- AND, OR, and NOT are the basic Boolean operations.
- We can express Boolean functions with either an expression or a truth table.
- Every Boolean expression can be converted to a circuit.
- Now, we'll look at how Boolean algebra can help simplify expressions, which in turn will lead to simpler circuits.

Boolean Algebra Summary

- Recall that the two binary values have different names:
  - True/False
  - On/Off
  - Yes/No
  - 1/0
- We use 1 and 0 to denote the two values.
- The three basic logical operations are:
  - AND
  - OR
  - NOT
- AND is denoted by a dot (·).
- OR is denoted by a plus (+).
- NOT is denoted by an overbar (¯), a single quote mark (') after, or (~) before the variable

Boolean Algebra Summary

- Examples:
  - is read "Y is equal to A AND B."
  - is read "z is equal to x OR y."
  - is read "X is equal to NOT A."

Tabular listing of the values of a function for all possible combinations of values on its arguments
Example: Truth tables for the basic logic operations:

AND		
X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

OR		
X	Y	$Z = X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT	
X	$Z=\overline{X}$
0	1
1	0

Boolean Operator Precedence

- **The order of evaluation is:**
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- **Consequence: Parentheses appear around OR expressions**
- **Example:** $F = A(B + C)(C + D)$

Boolean Algebra Postulates

- Commutative Law

$$x \cdot y = y \cdot x \qquad x + y = y + x$$

- Identity Element

$$x \cdot 1 = x \qquad x + 0 = x$$

$$x' \bullet 1 = x' \qquad x' + 0 = x'$$

- Complement

$$x \cdot x' = 0 \qquad x + x' = 1$$

- Theorem 1
  - $x \cdot x = x$ $\quad$ $x + x = x$
- Theorem 2
  - $x \cdot 0 = 0$ $\quad$ $x + 1 = 1$
- Theorem 3: *Involution*
  - $(x')' = x$ $\quad$ $(\overline{\overline{x}}) = x$

Boolean Algebra Theorems

- Theorem 4:
  - *Associative:* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
    $(x + y) + z = x + (y + z)$
  - *Distributive* :
    $$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$
    $$x + (y \cdot z) = (x + y) \cdot (x + z)$$

- Theorem 5: *DeMorgan*
  - $(x \cdot y)' = x' + y'$ $\quad$ $(x + y)' = x' \cdot y'$
  - $\overline{x \cdot y}) = \overline{x} + \overline{y}$ $\quad$ $(\overline{x + y}) = \overline{x} \cdot \overline{y}$

- Theorem 6: *Absorption*
  - $x \cdot (x + y) = x$ $\quad$ $x + (x \cdot y) = x$

Truth Table to Verify DeMorgan's

$$\overline{X + Y} = \overline{X} \cdot \overline{Y} \qquad \overline{X \cdot Y} = \overline{X} + \overline{Y}$$

X	Y	$X \cdot Y$	$X+Y$	$\overline{X}$	$\overline{Y}$	$\overline{X+Y}$	$\overline{X} \cdot \overline{Y}$	$\overline{X \cdot Y}$	$\overline{X}+\overline{Y}$
0	0	0	0	1	1	1	1	1	1
0	1	0	1	1	0	0	0	1	1
1	0	0	1	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0	0

- Generalized DeMorgan's Theorem:

$$\overline{X_1 + X_2 + \ldots + X_n} = \overline{X_1} \cdot \overline{X_2} \cdot \ldots \cdot \overline{X_n}$$

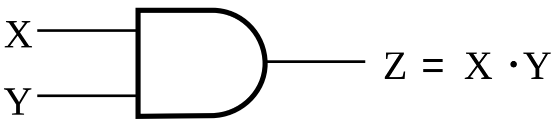
$$\overline{X_1 \cdot X_2 \cdot \ldots \cdot X_n} = \overline{X_1} + \overline{X_2} + \ldots + \overline{X_n}$$

Logic Gates

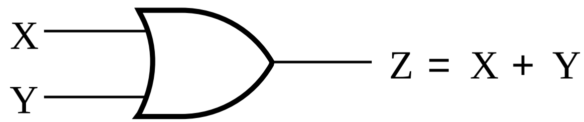
- In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in *relays*. The switches in turn opened and closed the current paths.
- Later, *vacuum tubes* that open and close current paths electronically replaced relays.
- Today, *transistors* are used as electronic switches that open and close current paths.

Logic Gate Symbols

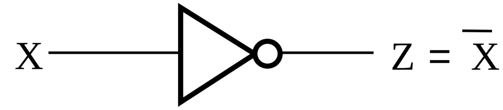
- **Logic gates have special symbols:**



AND gate



OR gate

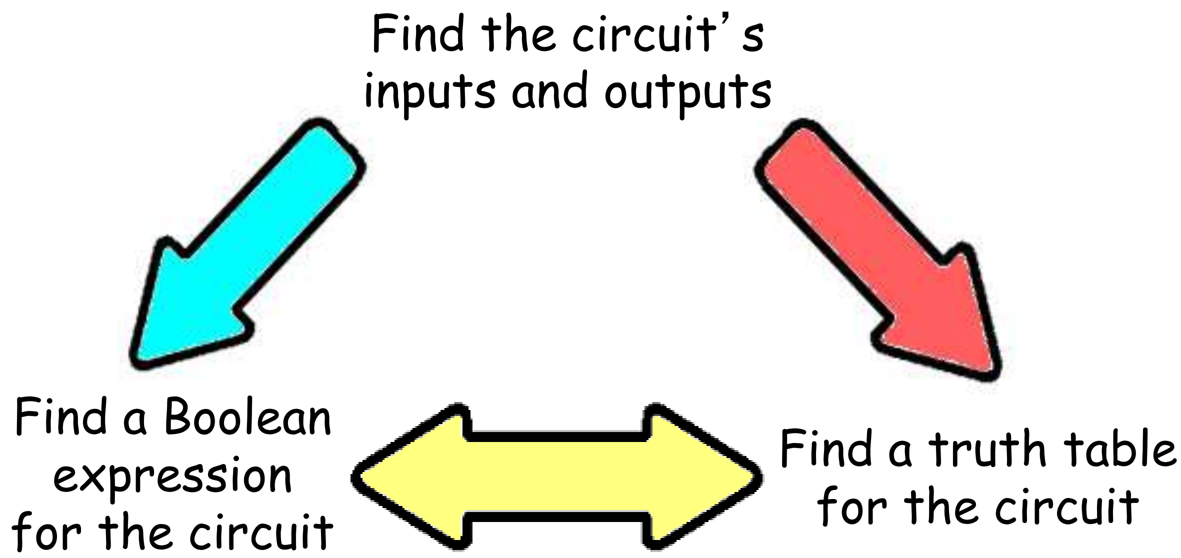


NOT gate or inverter

Boolean Functions

- A ***Boolean function*** is a function whose arguments, as well as the function itself, assume values from a two-element set ({0, 1}).
- ***Example***: $F(x, y) = x'y' + xyz + x'y$

- After finding the circuit inputs and outputs, you can come up with either an expression or a truth table to describe what the circuit does.
- You can easily convert between expressions and truth tables.



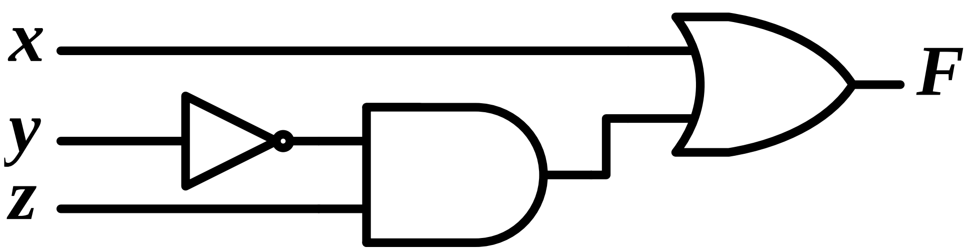
- **Boolean Expression/Function**

  Example: $F(x, y) = x + y' z$

- **Truth Table**

  All possible combinations of input variables

- **Logic Circuit**



x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Logic Diagrams and Expressions

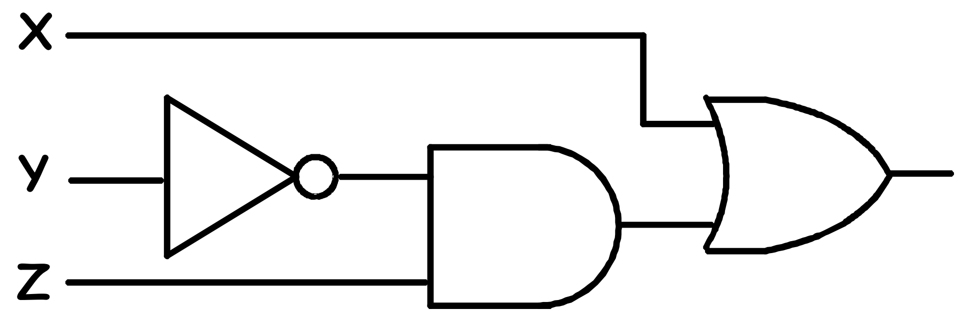
Truth Table

X Y Z	$F = X + \bar{Y} \times Z$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Logic Equation/Boolean Function

$$F = X + \bar{Y}\, Z$$

Logic Circuit



- Boolean equations, truth tables and logic diagrams describe the same function!
- Truth tables are unique, but expressions and logic diagrams are not. This gives flexibility in implementing functions.

Boolean Functions Exercise

- The truth table for the function:

  $F(X, Y, Z) = X\,Y + \overline{Y}\,Z$ *is:*

X	Y	Z	$X\,Y$	$\overline{Y}$	$\overline{Y}\,Z$	$F = X\,Y + \overline{Y}\,Z$
0	0	0	0	1	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	1	0	0
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

Draw the logic circuit for the boolean function above.

Converting from Truth Table to Boolean Function

- In designing digital circuits, the designer often begins with a truth table describing what the circuit should do.
- The design task is largely to determine what type of circuit will perform the function described in the truth table.
- While some people seem to have a natural ability to look at a truth table and immediately envision the necessary logic gate or relay logic circuitry for the task, there are procedural techniques available for the rest of us.
- Here, Boolean algebra proves its utility in a most dramatic way!

Converting from Truth Table to Boolean Function

- This problem will be solved by showing that any Boolean function can be represented by a Boolean sum of Boolean products of the variables and their complements or the product of sums.

- There are two ways to convert from truth tables to Boolean functions:
  1. Using Sum of Products /Minterms
  2. Using Product of Sums /Maxterms

Converting from Truth Table to Boolean Function

- **Minterm**
  - Product (*AND* function)
  - Contains all variables
  - Evaluates to '1' for a specific combination

Example

$$\left.\begin{array}{l} A = 0 \\ B = 0 \\ C = 0 \end{array}\right\} \quad \begin{array}{ccccc} \bar{A} & & \bar{B} & & \bar{C} \\ (\bar{0}) & \cdot & (\bar{0}) & \cdot & (\bar{0}) \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \cdot & 1 & \cdot & 1 = 1 \end{array}$$

	A B C	Minterm	
0	0 0 0	m_0	$\bar{A}\bar{B}\bar{C}$
1	0 0 1	m_1	$\bar{A}\bar{B}C$
2	0 1 0	m_2	$\bar{A}B\bar{C}$
3	0 1 1	m_3	$\bar{A}BC$
4	1 0 0	m_4	$A\bar{B}\bar{C}$
5	1 0 1	m_5	$A\bar{B}C$
6	1 1 0	m_6	$AB\bar{C}$
7	1 1 1	m_7	ABC

- **Maxterm**
  - Sum (*OR* function)
  - Contains all variables
  - Evaluates to '0' for a specific combination

Example

$A = 1$, $B = 1$, $C = 1$

$$\overline{A} \quad \overline{B} \quad \overline{C}$$

$$(\overline{1}) \cdot (\overline{1}) \cdot (\overline{1})$$

$$0 \cdot 0 \cdot 0 = 0$$

	A B C	Maxterm	
0	0 0 0	M_0	$A+B+C$
1	0 0 1	M_1	$A+B+\overline{C}$
2	0 1 0	M_2	$A+\overline{B}+C$
3	0 1 1	M_3	$A+\overline{B}+\overline{C}$
4	1 0 0	M_4	$\overline{A}+B+C$
5	1 0 1	M_5	$\overline{A}+B+\overline{C}$
6	1 1 0	M_6	$\overline{A}+\overline{B}+C$
7	1 1 1	M_7	$\overline{A}+\overline{B}+\overline{C}$

Converting from Truth Table to Boolean Function

- Truth Table to Boolean Function

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

$$F = \bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C + ABC$$

Using Minterms

Converting from Truth Table to Boolean Function

- Truth Table to Boolean Function

$$F = (A+B+C)(A+\overline{B}+C)(A+\overline{B}+\overline{C})(\overline{A}+\overline{B}+C)$$

A B C	$\overline{F}$
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	0
1 1 0	1
1 1 1	0

Using Maxterms

Converting from Truth Table to Boolean Function

- Sum of *Minterms*

$$F = \bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C + ABC$$

$$F = m_1 + m_4 + m_5 + m_7$$

$$F = \sum(1,4,5,7)$$

- *Product of Maxterms*

$$\bar{F} = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + AB\bar{C}$$

$$\bar{\bar{F}} = \overline{\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + AB\bar{C}}$$

$$F = \overline{\bar{A}\bar{B}\bar{C}} \cdot \overline{\bar{A}B\bar{C}} \cdot \overline{\bar{A}BC} \cdot \overline{AB\bar{C}}$$

$$F = (A+B+C)(A+\bar{B}+C)(A+\bar{B}+\bar{C})(\bar{A}+\bar{B}+C)$$

$$F = M_0 \qquad M_2 \qquad M_3 \qquad M_6$$

$$F = \prod(0,2,3,6)$$

	A B C	F	$\bar{F}$
0	0 0 0	0	1
1	0 0 1	1	0
2	0 1 0	0	1
3	0 1 1	0	1
4	1 0 0	1	0
5	1 0 1	1	0
6	1 1 0	0	1
7	1 1 1	1	0