Unit:1

Inverse Laplace transform

Definition: If the Laplace transform of f(t) is F(s), i.e. $L[f(t)] = F(s)$, then f(t) is called the inverse Laplace transform of F(s) and we write symbolically $L^{-1}[F(s)] = f(t)$, where L^{-1} is called inverse Laplace Transform operator.

Linearity Property: if c_1 and c_2 are any constants and $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, then

$$L^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1L^{-1}[F_1(s)] + c_2\, L^{-1}[\, F_2(s)]$$

$$= c_1f_1(t) + c_2f_2(t)$$

This result is easily extended to the addition of more than two functions.

Methods of finding inverse Laplace transform

1. Use of table of inverse Laplace transform
2. Use of theorems of Inverse Laplace transform
3. Use of partial fractions

1. **Use of table of inverse Laplace Transform**

From the table of Laplace transform of elementary functions and by using definition and linearity property, we can obtain corresponding Table of inverse Laplace Transform.

Function	**Laplace transform**	**Inverse Laplace Transform** $\mathbf{L^{-1}[F(s)] = f(t)}$
f(t)	**L[f(t)]= F(s)**	
1	$\frac{1}{s}, s > 0$	$L^{-1}\left[\frac{1}{s}\right] = 1$
e^{at}	$\frac{1}{s-a}, s > a$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
e^{-at}	$\frac{1}{s+a}, s > -a$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
Sin at	$\frac{a}{s^2+a^2}, s > 0$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
Cos at	$\frac{s}{s^2+a^2}, s > 0$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
Sinh at	$\frac{a}{s^2-a^2}, s > \lvert a\rvert$	$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
Cosh at	$\frac{s}{s^2-a^2}, s > \lvert a\rvert$	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0	$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$
t^{n-1}	$\frac{\Gamma(n)}{s^n}$, s > 0	$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$
If n is positive integer, $\Gamma(n+1) = n!$ and $\Gamma(n) = (n-1)!$ then we have $L[t^n] = \frac{n!}{s^{n+1}}$ and $L[t^{n-1}] = \frac{(n-1)!}{s^n}$		$\therefore\ L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!)}$ and $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

Note: For the unit step function $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$ we have $L[u(t)] = \frac{1}{s}$

$$\therefore L^{-1}\left[\frac{1}{s}\right] = u(t)$$

and for the displaced Unit step function $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$ we have $L[u(t-a)] = \frac{e^{-as}}{s}$

Therefore

$$L^{-1}\left[\frac{e^{-as}}{s}\right] = u(t-a)$$

Illustrations Using table of inverse Laplace Transform

Example 1: Find the inverse Laplace transform of the followings

i) $\frac{5}{s+3}$ ii) $\frac{1}{2s-3}$ iii) $\frac{4}{3s-1}$ iv) $\frac{2s+1}{s(s+1)}$

Solution: i) $L^{-1}\left[\frac{5}{s+3}\right] = 5L^{-1}\left[\frac{1}{s+3}\right] = 5e^{-3t}$ {from the result $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$ }

ii) $L^{-1}\left[\frac{1}{2s-3}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-3/2}\right]$ Since $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$$= \frac{1}{2}\, e^{\left(\frac{3}{2}\right)t}$$

iii) $L^{-1}\left[\frac{4}{3s-1}\right] = 4L^{-1}\left[\frac{1}{3s-1}\right] = \frac{4}{3}L^{-1}\left[\frac{1}{s-1/3}\right]$ Since $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$$= \frac{4}{3}\, e^{\left(\frac{1}{3}\right)t}$$

iv) $L^{-1}\left[\frac{2s+1}{s(s+1)}\right] = L^{-1}\left[\frac{s+(s+1)}{s(s+1)}\right] = L^{-1}\left[\frac{s}{s(s+1)} + \frac{s+1}{s(s+1)}\right] = L^{-1}\left[\frac{s}{s(s+1)}\right] + L^{-1}\left[\frac{s+1}{s(s+1)}\right]$

$$= L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{s)}\right] = e^{-t} + 1$$

Example 2: Obtain the inverse Laplace transform of the followings:

i) $\frac{2}{s^2+16}$ ii) $\frac{4s}{s^2-16}$ iii) $\frac{2s-5}{4s^2+25}$ iv) $\frac{3s-12}{s^2+8}$ v) $\frac{s-4}{s^2-4}$ iv) $\frac{s\cos\alpha + w\sin\alpha}{s^2+w^2}$

Solution: i) $L^{-1}\left[\frac{2}{s^2+16}\right] = 2L^{-1}\left[\frac{1}{s^2+16}\right] = 2L^{-1}\left[\frac{1}{s^2+4^2}\right]$ since $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

$$= 2\,\frac{1}{4}\sin 4t = \frac{1}{2}\sin 4t$$

ii) $L^{-1}\left[\frac{4s}{s^2-16}\right] = 4L^{-1}\left[\frac{s}{s^2-16}\right] = 4L^{-1}\left[\frac{s}{s^2-4^2}\right]$ since $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$$= 4\ \cosh 4t$$

iii) $L^{-1}\left[\frac{2s-5}{4s^2+25}\right] = \frac{1}{4}L^{-1}\left[\frac{2s-5}{s^2+\frac{25}{4}}\right] = \frac{1}{4}L^{-1}\left[\frac{2s-5}{s^2+(5/2)^2}\right]$ since $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$$= \frac{1}{4}L^{-1}\left[\frac{2s}{s^2+(5/2)^2}\right] + \frac{1}{4}L^{-1}\left[\frac{-5}{s^2+(5/2)^2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right] \qquad \left\{ L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right.$$

$$= \frac{1}{2}\cos\left(\frac{5}{2}\right)t - \frac{5}{4}\,\frac{1}{(5/2)}\sin\left(\frac{5}{2}\right)t \qquad \left.\& \; L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at \right\}$$

$$= \frac{1}{2}\cos\left(\frac{5}{2}\right)t - \frac{1}{2}\,\sin\left(\frac{5}{2}\right)t$$

iv) $$L^{-1}\left[\frac{3s-12}{s^2+8}\right] = L^{-1}\left[\frac{3s}{s^2+8} - \frac{12}{s^2+8}\right] = 3L^{-1}\left[\frac{s}{s^2+8}\right] - 12L^{-1}\left[\frac{1}{s^2+8}\right]$$

$$= 3L^{-1}\left[\frac{s}{s^2+(2\sqrt{2})^2}\right] - 12L^{-1}\left[\frac{1}{s^2+(2\sqrt{2})^2}\right] \qquad \left\{ L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right.$$

$$= 3\cos(2\sqrt{2})t - 12\,\frac{1}{2\sqrt{2}}\sin(2\sqrt{2})t \qquad \left. L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at \right\}$$

$$= 3\cos(2\sqrt{2})t - 3\sqrt{2}\sin(2\sqrt{2})t$$

v) $$L^{-1}\left[\frac{s-4}{s^2-4}\right] = L^{-1}\left[\frac{s}{s^2-4} - \frac{4}{s^2-4}\right] = L^{-1}\left[\frac{s}{s^2-4}\right] - 4L^{-1}\left[\frac{1}{s^2-4}\right]$$

Since $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$ and $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$ we have

$$L^{-1}\left[\frac{s-4}{s^2-4}\right] = \cosh 2t - 4\,\frac{1}{2}\sinh\, 2t$$
$$= \cosh 2t - 2\sinh\, 2t$$

vi) $$L^{-1}\left[\frac{s\cos\alpha + w\sin\alpha}{s^2+w^2}\right] = L^{-1}\left[\frac{s\cos\alpha}{s^2+w^2} + \frac{w\sin\alpha}{s^2+w^2}\right] = L^{-1}\left[\frac{s\cos\alpha}{s^2+w^2}\right] + L^{-1}\left[\frac{w\sin\alpha}{s^2+w^2}\right]$$

$$= \cos\alpha\; L^{-1}\left[\frac{s}{s^2+w^2}\right] + \sin\alpha\, L^{-1}\left[\frac{w}{s^2+w^2}\right]$$

Since $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$ and $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$ we have

$$L^{-1}\left[\frac{s\cos\alpha + w\sin\alpha}{s^2+w^2}\right] = \cos\alpha \,.\, \cos wt + \sin\alpha \,.\, \sin\; wt$$
$$= \cos(wt - \alpha)$$

Example 3: Find the inverse Laplace transform of the followings

i) $L^{-1}\left[\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right]$ ii) $L^{-1}\left[\frac{1}{s^4}\right]$ iii) $L^{-1}\left[\frac{s+1}{s^{4/3}}\right]$ iv) $L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right]$

Solution: i) Using the result $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ we have

$$L^{-1}\left[\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right] = L^{-1}\left[\frac{a_1}{s}\right] + L^{-1}\left[\frac{a_2}{s^2}\right] + L^{-1}\left[\frac{a_3}{s^3}\right]$$

$$= a_1\, L^{-1}\left[\frac{1}{s}\right] + a_2\, L^{-1}\left[\frac{1}{s^2}\right] + a_3 L^{-1}\left[\frac{1}{s^3}\right]$$ By linearity property

$$= a_1\,.1 + a_2 t + a_3\frac{t^2}{2!} = a_1\; + a_2 t + \frac{a_3 t^2}{2}$$

ii) Using the result $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$ we have

$$L^{-1}\left[\frac{1}{s^4}\right]=\frac{t^{4-1}}{(4-1)!}=\frac{t^3}{3!}=\frac{t^3}{6}$$

iii) $$L^{-1}\left[\frac{s+1}{s^{4/3}}\right]=L^{-1}\left[\frac{s}{s^{4/3}}\right]+L^{-1}\left[\frac{1}{s^{4/3}}\right]$$

$$=L^{-1}\left[\frac{s}{s.s^{1/3}}\right]+L^{-1}\left[\frac{1}{s^{4/3}}\right]$$

$$=L^{-1}\left[\frac{1}{s^{1/3}}\right]+L^{-1}\left[\frac{1}{s^{4/3}}\right]$$ Using $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{\Gamma(n)}$

$$=\frac{t^{\frac{1}{3}-1}}{\Gamma(\frac{1}{3})}+\frac{t^{\frac{4}{3}-1}}{\Gamma(\frac{4}{3})}=\frac{t^{-2/3}}{\Gamma(\frac{1}{3})}+\frac{t^{1/3}}{\frac{1}{3}\Gamma(\frac{1}{3})}=\frac{1}{\Gamma(\frac{1}{3}}(t^{-2/3}+3t^{1/3})$$

iv) $$L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right]=\frac{3}{2}L^{-1}\left[\frac{s^4-2s^2+1}{s^5}\right]=\frac{3}{2}\left\{L^{-1}\left[\frac{1}{s}\right]-2L^{-1}\left[\frac{1}{s^3}\right]+L^{-1}\left[\frac{1}{s^5}\right]\right\}$$

Now Using $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$

We have, $L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right]=\frac{3}{2}\left\{1-2\frac{t^2}{2!}+\frac{t^4}{4!}\right\}$ $=\frac{3}{2}\left\{1-t^2+\frac{t^4}{24}\right\}$

Example 4: Find the inverse Laplace transform of the followings

i) $L^{-1}\left[\frac{3}{s+2}-\frac{2s}{s^2+25}+\frac{3}{s^2+9}\right]$ ii) $L^{-1}\left[\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right]$ iii) $L^{-1}\left[\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right]$

Solution: i) $$L^{-1}\left[\frac{3}{s+2}-\frac{2s}{s^2+25}+\frac{3}{s^2+9}\right]=3L^{-1}\left[\frac{1}{s+2}\right]-2L^{-1}\left[\frac{s}{s^2+25}\right]+L^{-1}\left[\frac{3}{s^2+9}\right]$$

$$=3L^{-1}\left[\frac{1}{s+2}\right]-2L^{-1}\left[\frac{s}{s^2+5^2}\right]+3L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$=3e^{-2t}-2\cos 5t+3\,\frac{\sin 3t}{3}$$

$$=3e^{-2t}-2\cos 5t+\sin 3t$$

ii) $$L^{-1}\left[\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right]=L^{-1}\left[\frac{5}{s^2}+\frac{4}{s^3}-\frac{2s}{s^2+9}+\frac{18}{s^2+9}+\frac{24}{s^4}-\frac{30}{s^{\frac{7}{2}}}\right]$$

$$=5L^{-1}\left[\frac{1}{s^2}\right]+4L^{-1}\left[\frac{1}{s^3}\right]-2L^{-1}\left[\frac{s}{s^2+9}\right]+18\,L^{-1}\left[\frac{1}{s^2+9}\right]+24\,L^{-1}\left[\frac{1}{s^4}\right]-30\,L^{-1}\left[\frac{1}{s^{7/2}}\right]$$

$$=5\,t+4\left(\frac{t^2}{2!}\right)-2\cos t+18\left(\frac{\sin 3t}{3}\right)+24\left(\frac{t^3}{3!}\right)-30\left(\frac{t^{5/2}}{\Gamma(7/2)}\right)$$

$$=5\,t+2t^2-2\cos t+6\sin 3t+4\,t^3-\frac{16\,t^{5/2}}{\sqrt{\pi}}$$

$$\left\{\text{ using }\Gamma(n+1)=n\Gamma(n)\text{ and we have }\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\text{ we have, }\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}\right\}$$

iii) $$L^{-1}\left[\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right]=L^{-1}\left[\frac{6}{2s-3}\right]-L^{-1}\left[\frac{3}{9s^2-16}\right]-L^{-1}\left[\frac{4s}{9s^2-16}\right]$$

$$+L^{-1}\left[\frac{8}{16s^2+9}\right]-L^{-1}\left[\frac{6s}{16s^2+9}\right]$$

$$= L^{-1}\left[\frac{3}{s-\frac{3}{2}}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s^2-\frac{16}{9}}\right] - \frac{4}{9}L^{-1}\left[\frac{s}{s^2-\frac{16}{9}}\right] + \frac{8}{16}L^{-1}\left[\frac{1}{s^2+9/16}\right] - \frac{6}{16}L^{-1}\left[\frac{s}{s^2+9/16}\right]$$

$$= 3L^{-1}\left[\frac{1}{s-3/2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s^2-(4/3)^2}\right] - \frac{4}{9}L^{-1}\left[\frac{s}{s^2-(4/3)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+(3/4)^2}\right] - \frac{3}{8}L^{-1}\left[\frac{s}{s^2+(3/4)^2}\right]$$

$$= 3\,e^{3t/2} - \frac{1}{3}\,\frac{1}{4/3}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{1}{2}\,\frac{1}{3/4}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}$$

$$= 3\,e^{3t/2} - \frac{1}{4}\,\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{2}{3}\,\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}$$

2. Use of Theorems of Inverse Laplace Transform:

a) First shifting theorem:

Theorem: If $L^{-1}[F(s)] = f(t)$, then $L^{-1}[F(s+a)] = e^{-at}f(t)$

Proof: we have proved in the first shifting theorem of Laplace Transform that

"If $L[f(t)] = F(s)$ then$L[e^{-at}f(t)] = F(s+a)$", we have

i.e. $L^{-1}\{F(s+a)\} = e^{-at}f(t)$

Hence $\boxed{L^{-1}[F(s+a)] = e^{-at}f(t)}$

Remark 1: In words, this theorem states that the replacement of s by $s + a$ in $F(s)$ corresponds to multiplication of original function f(t) by e^{-at}.

Remark 2: In practice to obtain inverse Laplace transform of $F(s+a)$, obtain that $F(s)$ first **(i.e. first obtain $L^{-1}\{F(s)\}$)** and then multiply it by e^{-at}.

Example: obtain the inverse Laplace transform of the following functions:

i) $\frac{1}{(s+4)^6}$ ii) $\frac{s}{(s-3)^5}$ iii) $\frac{3s+1}{(s+1)^4}$ iv) $\frac{s}{s^2+6s+25}$

Solution: i) $L^{-1}\left[\frac{1}{(s+4)^6}\right] = e^{-4t}L^{-1}\left[\frac{1}{s^6}\right]$ by first shifting theorem $L^{-1}[F(s+a)] = e^{-at}f(t)$

$$= e^{-4t}\left[\frac{t^5}{5!}\right] \qquad \text{By } L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$= e^{-4t}\left[\frac{t^5}{120}\right]$$

ii) $L^{-1}\left[\frac{s}{(s-3)^5}\right] = L^{-1}\left[\frac{(s-3)+3}{(s-3)^5}\right] = L^{-1}\left[\frac{(s-3)}{(s-3)^5}\right] + L^{-1}\left[\frac{3}{(s-3)^5}\right]$

$$= L^{-1}\left[\frac{1}{(s-3)^4}\right] + 3L^{-1}\left[\frac{1}{(s-3)^5}\right]$$

$$= e^{3t}L^{-1}\left[\frac{1}{s^4}\right] + 3\,e^{3t}L^{-1}\left[\frac{1}{s^5}\right] \qquad \text{Using } L^{-1}[F(s+a)] = e^{-at}f(t)$$

$$= e^{-3t}\left[\frac{t^3}{3!}\right] + 3\,e^{-3t}\left[\frac{t^4}{4!}\right] \qquad \text{By } L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$= e^{-3t}\left[\frac{t^3}{6} + \frac{t^4}{8}\right]$$

iii) $L^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = L^{-1}\left[\frac{3(s+1)-2}{(s+1)^4}\right] = L^{-1}\left[\frac{3(s+1)}{(s+1)^4}\right] + L^{-1}\left[\frac{-2}{(s+1)^4}\right]$

$= 3L^{-1}\left[\frac{1}{(s+1)^3}\right] - 2L^{-1}\left[\frac{1}{(s+1)^4}\right]$

$= 3e^{-t}L^{-1}\left[\frac{1}{s^3}\right] - 2\,e^{-t}L^{-1}\left[\frac{1}{s^4}\right]$ Using $L^{-1}[F(s+a)] = e^{-at}f(t)$

$= 3e^{-t}\left[\frac{t^2}{2!}\right] - 2\,e^{-t}\left[\frac{t^3}{3!}\right]$ By $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

$= e^{-t}t^2\left[\frac{3}{2} - \frac{t}{3}\right]$

iv) $L^{-1}\left[\frac{s}{s^2+6s+25}\right] = L^{-1}\left[\frac{s}{s^2+6s++9+16}\right] = L^{-1}\left[\frac{s}{(s+3)^2+4^2}\right] = L^{-1}\left[\frac{(s+3)-3}{(s+3)^2+4^2}\right]$

$= L^{-1}\left[\frac{(s+3)}{(s+3)^2+4^2}\right] + L^{-1}\left[\frac{-3}{(s+3)^2+4^2}\right]$

$= L^{-1}\left[\frac{(s+3)}{(s+3)^2+4^2}\right] - 3L^{-1}\left[\frac{1}{(s+3)^2+4^2}\right]$

$= e^{-3t}L^{-1}\left[\frac{s}{s^2+4^2}\right] - 3\,e^{-3t}L^{-1}\left[\frac{1}{s^2+4^2}\right]$ Using $L^{-1}[F(s+a)] = e^{-at}f(t)$

$= e^{-3t}(\cos 4t) - 3\,e^{-3t}\left(\frac{\sin 4t}{4}\right)$ By $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

$= e^{-3t}\left\{\cos 4t - \frac{3}{4}\sin 4t\right\}$

b) Second shifting theorem:

Statement: If $L^{-1}[F(s)] = f(t)$, then $L^{-1}[e^{-as}F(s)] = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

Proof: We have proved that: IF $L[f(t)] = F(s)$ and $F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

then $L[F(t)] = e^{-as}F(s)$

Therefore $L^{-1}[e^{-as}F(s)] = F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

Hence $\boxed{L^{-1}[e^{-as}F(s)] = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}}$

Note: Since we can write F(t) in terms of Heaviside unit step function as $f(t-a)U(t-a)$, so we have the following equivalent result for the second shifting theorem

$$L^{-1}[e^{-as}F(s)] = f(t-a)U(t-a)$$

Remark 1: In words this theorem states that suppressing the factor e^{-as} in transform requires that the inverse of what remains be shifted a units to the right and cut-off to the left of the point $t = a$

Remark 2: In practice to obtain the inverse Laplace transform of $e^{-as}F(s)$, we first obtain inverse transform of F(s), say f(t) (i.e. factor e^{-as} is dropped initially), then to account for the factor e^{-as}, replace t by $t - a$ throughout in f(t) and multiply this result by $U(t-a)$.

Example: obtain the inverse Laplace transform of the following functions:

i) $\frac{e^{-\pi s}}{s+b}$ ii) $\frac{s\,e^{-4\pi s/5}}{s^2+25}$ iii) $\frac{e^{-3s}}{(s-2)^4}$ iv) $\frac{e^{-\pi s/2}+e^{-3\pi s/2}}{s^2+1}$

Solution: i) Comparing $\frac{e^{-\pi s}}{s+b}$ with $e^{-as}F(s)$, we have $F(s) = \frac{1}{s+b}$

Therefore $L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+b}\right] = e^{-bt} = f(t)$ (dropping $e^{-\pi s}$)

Hence by the second shifting theorem, with $a = \pi$, we get

$$L^{-1}[e^{-as}\,F(s)] = L^{-1}\left[\frac{e^{-\pi s}}{s+b}\right] = F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$= \begin{cases} f(t-\pi) & t > \pi \\ 0 & t < \pi \end{cases}$$

$$= \begin{cases} e^{-b(t-\pi)} & t > \pi \\ 0 & t < \pi \end{cases}$$

ii) Given example can be written as

$$\frac{e^{-3s}}{(s-2)^4} = e^{-3s}\left(\frac{1}{(s-2)^4}\right) = e^{-as}\,F(s)$$

Therefore $L^{-1}[F(s)] = L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t}\,L^{-1}\left[\frac{1}{s^4}\right] = e^{2t}\frac{t^3}{3!} = f(t)$ (dropping e^{-3s})

Hence by the second shifting theorem, with $a = 3$, we get

$$L^{-1}[e^{-as}\,F(s)] = L^{-1}\left[e^{-3s}\left(\frac{1}{(s-2)^4}\right)\right] = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$= \begin{cases} f(t-3) & t > 3 \\ 0 & t < 3 \end{cases} = \begin{cases} e^{2(t-3)}\,\dfrac{(t-3)^3}{3!}, & t > 3 \\ 0\,, & t < 3 \end{cases}$$

Or $$L^{-1}\left[\frac{e^{-3s}}{(s-2)^4}\right] = e^{2(t-3)}\,\frac{(t-3)^3}{3!}\,U(t-3)$$

iii) Given example can be written as

$$\frac{s\,e^{-4\pi s/5}}{s^2+25} = e^{-4\pi s/5}\left(\frac{s}{s^2+25}\right) = e^{-as}\,F(s)$$

Therefore $L^{-1}[F(s)] = L^{-1}\left[\frac{s}{s^2+25}\right] = \cos 5t = f(t)$ (dropping $e^{-4\pi s/5}$)

Hence by the second shifting theorem, with $a = \frac{4\pi}{5}$, we get

$$L^{-1}[e^{-as}\,F(s)] = L^{-1}\left[e^{-4\pi s/5}\left(\frac{s}{s^2+25}\right)\right] = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$= \begin{cases} f\left(t-\dfrac{4\pi}{5}\right) & t > \dfrac{4\pi}{5} \\ 0 & t < \dfrac{4\pi}{5} \end{cases}$$

$$= \begin{cases} \cos\left(t-\dfrac{4\pi}{5}\right) & t > \dfrac{4\pi}{5} \\ 0 & t < \dfrac{4\pi}{5} \end{cases}$$

Or $$L^{-1}\left[e^{-4\pi s/5}\left(\frac{s}{s^2+25}\right)\right] = \cos\left(t-\frac{4\pi}{5}\right)U\left(t-\frac{4\pi}{5}\right)$$

iv) we have $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = f(t)$

$$L^{-1}\left[\frac{e^{-\pi s/2}+e^{-3\pi s/2}}{s^2+1}\right] = L^{-1}\left[\frac{e^{-\pi s/2}}{s^2+1}\right] + L^{-1}\left[\frac{e^{-3\pi s/2}}{s^2+1}\right]$$

Hence by second shifting theorem, we get

$$L^{-1}\left[\frac{e^{-\frac{\pi s}{2}}+e^{-\frac{3\pi s}{2}}}{s^2+1}\right] = \sin(t-\pi/2)\,U(t-\pi/2)+\sin(t-3\pi/2)\,U(t-3\pi/2)$$

$$=\cos t\ U\left(t-\frac{\pi}{2}\right)+\cos t\ U\left(t-\frac{3\pi}{2}\right)$$

$$=\cos t\left\{U\left(t-\frac{\pi}{2}\right)+\ U\left(t-\frac{3\pi}{2}\right)\right\}$$

c) Change of scale theorem:

Statement: IF $L^{-1}[F(s)] = f(t)$, then $L^{-1}[\,F(k\,s)] = \frac{1}{k}\,f\left(\frac{t}{k}\right)$

Proof: We have proved in the change of scale of Laplace transform that:

IF $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}\,F\left(\frac{s}{a}\right)$

Therefore $L^{-1}\left[\frac{1}{a}\,F\left(\frac{s}{a}\right)\right] = f(at)$

Now putting $\frac{1}{a} = k$, we get, $L^{-1}[k\,F(ks)] = f\left(\frac{t}{k}\right)$

Hence $$\boxed{L^{-1}[\,F(k\,s)] = \frac{1}{k}\,f\left(\frac{t}{k}\right)}$$

Example 1: Prove that $L^{-1}\left[\frac{as}{a^2s^2+b^2}\right] = \frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$

Solution: Given $F(as) = \frac{as}{a^2s^2+b^2}$ We have, $F(s) = \frac{s}{s^2+b^2}$

Therefore $L^{-1}[F(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = f(t)$

Hence by chance of scale of inverse Laplace transform $L^{-1}[\,F(ks)] = \frac{1}{k}f\left(\frac{t}{k}\right)$we have

$$L^{-1}[F(as)] = L^{-1}\left[\frac{as}{a^2s^2+b^2}\right] = \frac{1}{a}\ f\left(\frac{t}{a}\right) = \frac{1}{a}\ \cos\left(\frac{bt}{a}\right)$$

Example 2: If $L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$ Prove that $L^{-1}\left[\frac{as}{a^2s^2+1}\right] = \frac{1}{a}\cos\frac{t}{a}$

Solution: Given $L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t = f(t)$ therefore $F(S) = \frac{s}{s^2+1}$

Hence by chance of scale of inverse Laplace transform $L^{-1}[\,F(ks)] = \frac{1}{k}f\left(\frac{t}{k}\right)$ we have

$$L^{-1}[F(as)] = L^{-1}\left[\frac{as}{a^2s^2+1}\right] = \frac{1}{a}f\left(\frac{t}{a}\right) = \frac{1}{a}\cos\left(\frac{t}{a}\right)$$

$$\Longrightarrow L^{-1}\left[\frac{as}{a^2s^2+1}\right] = \frac{1}{a}\cos\frac{t}{a}$$

d) Inverse Laplace transform of derivatives:

Statement: **If $L^{-1}[F(S) = f(t)]$, then $L^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,f(t)$**

Proof: since $L[t.\,f(t)] = -\frac{d}{ds}\,F(s)$ by Laplace transform of derivative

$\therefore$ $L^{-1}\left[-\frac{d}{ds}\,F(s)\right] = t\,f(t)$ or $L^{-1}\left[\frac{d}{ds}\,F(s)\right] = -\,t\,f(t)$ (1)

Hence $L^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,f(t)$

The generalization of higher order derivative is

$$L^{-1}\left[\frac{d^n}{ds^n}F(s)\right] = (-1)^n\,t^n\,f(t) \quad (2)$$

Remark 1: The result (1) can be interpreted as the differentiation of the transform corresponding to multiplication of the function by – t.

Remark 2: This theorem is often useful when the inverse of the transform cannot conveniently be found but the inverse of the derivative of the transform is known. In particular, when F(s) involves logarithmic or inverse circular functions.

Example: Find the inverse Laplace transform of each of the following functions:

(i) $\cot^{-1} s$ (ii) $\log\left(\frac{s+b}{s+a}\right)$ (iii) $\frac{s}{(s^2+a^2)^2}$

Solution: (i) Let $L^{-1}[\cot^{-1} s] = f(t)$

$$L^{-1}\left[\frac{d}{ds}\cot^{-1} s\right] = -t\,f(t) \quad \text{(Inverse Laplace transform of derivatives)}$$

$$L^{-1}\left[-\frac{1}{s^2+1}\right] = -t\,f(t) \quad \Rightarrow \sin t = t\,f(t) \quad \Rightarrow \frac{\sin t}{t} = f(t)$$

Hence $L^{-1}[\cot^{-1} s] = \frac{\sin t}{t} = f(t)$

Solution: (ii) Let $L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right] = f(t)$

$$L^{-1}[\log(s+b) - \log(s+a)] = f(t)$$

$$L^{-1}\left[\frac{d}{ds}\{\log(s+b) - \log(s+a)\}\right] = -t\,f(t) \quad \text{(by Inverse LT of derivatives)}$$

$$L^{-1}\left[\frac{1}{s+b} - \frac{1}{s+a}\right] = -t\,f(t) \quad \Rightarrow e^{-bt} - e^{-at} = t\,f(t)$$

$$\Rightarrow \frac{e^{-bt}-e^{-at}}{t} = -f(t) \quad \text{or} \quad \frac{e^{-at}-e^{-bt}}{t} = f(t)$$

Hence $L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right] = \frac{e^{-at}-e^{-bt}}{t} = f(t)$

Solution: (iii) since $\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) = \frac{-2s}{(s^2+a^2)^2}$ or $-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) = \frac{s}{(s^2+a^2)^2}$

we use result of inverse Laplace transform of derivative to obtain the required transform.

We know that $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

Therefore $L^{-1}\left[\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\}\right] = -t\,\frac{\sin at}{a}$

$$L^{-1}\left[\frac{-2s}{(s^2+a^2)^2}\right] = -\frac{t\,\sin at}{a} \quad \Rightarrow L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\,\sin at}{2a}$$

Hence $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\,\sin at}{2a}$

e) Inverse Laplace transform of Integrals:

If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\int_s^\infty F(s)ds\right] = \frac{f(t)}{t}$

Proof: we know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$

Taking inverse Laplace on both sides we get, $\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)ds\right]$

Hence $L^{-1}\left[\int_s^\infty F(s)ds\right] = \frac{f(t)}{t}$ (1)

We can generalize the above result as

$$L^{-1}\left[\int_s^\infty \int_s^\infty \ldots . \int_s^\infty F(s)ds \ldots dsds\right] = \frac{f(t)}{t^n} \qquad (2)$$

Remark 1: The result (1) can be interpreted as the integration of the transform of a function corresponds to division of the function by t.

Remark 2: The theorem is often useful when the integral of transform is simpler to work.

Examples: Obtain the inverse Laplace transform of each of the followings.

(i) $\frac{2s}{(s^2-4)^2}$ (ii) $\frac{2s+1}{(s^2+s+1)^2}$ (iii) $\frac{s}{(s^2+a^2)^2}$

Solution: (i) Let $L^{-1}\left[\frac{2s}{(s^2-4)^2}\right] = f(t)$

Therefore by inverse Laplace transform of integral we have

$$L^{-1}\left[\int_s^\infty \frac{2s}{(s^2-4)^2}ds\right] = \frac{f(t)}{t} \qquad \Rightarrow L^{-1}\left[\left\{-\frac{1}{s^2-4}\right\}_s^\infty\right] = \frac{f(t)}{t}$$

$\Rightarrow L^{-1}\left[\frac{1}{s^2-4}\right] = \frac{f(t)}{t}$ $\quad \Rightarrow \frac{1}{2}\sinh 3t = \frac{f(t)}{t}$ $\quad \Rightarrow f(t) = \frac{t}{2}\sinh 3t$

Hence $L^{-1}\left[\frac{2s}{(s^2-4)^2}\right] = f(t) = \frac{t}{2}\sinh 3t$

Solution: (ii) Let $L^{-1}\left[\frac{2s+1}{(s^2+s+1)^2}\right] = f(t)$

Therefore by inverse Laplace transform of integral we have

$$L^{-1}\left[\int_s^\infty \frac{2s+1}{(s^2+s+1)^2}ds\right] = \frac{f(t)}{t} \qquad \Rightarrow L^{-1}\left[\left\{-\frac{1}{s^2+s+1}\right\}_s^\infty\right] = \frac{f(t)}{t}$$

$\Rightarrow L^{-1}\left[\frac{1}{s^2+s+1}\right] = \frac{f(t)}{t}$ or $L^{-1}\left[\frac{1}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2}\right] = \frac{f(t)}{t}$

$\Rightarrow e^{-\frac{t}{2}} \quad L^{-1}\left[\frac{1}{s^2+\left(\sqrt{3}/2\right)^2}\right] = \frac{f(t)}{t}$ by first shifting theorem

$\Rightarrow e^{-\frac{t}{2}} \frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t = \frac{f(t)}{t}$ or $e^{-\frac{t}{2}} \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t = \frac{f(t)}{t}$

$\Rightarrow f(t) = \frac{2t}{\sqrt{3}}e^{-\frac{t}{2}} \sin\frac{\sqrt{3}}{2}t$

Hence $L^{-1}\left[\frac{2s+1}{(s^2+s+1)^2}\right] = f(t) = \frac{2t}{\sqrt{3}}e^{-\frac{t}{2}} \sin\frac{\sqrt{3}}{2}t$

Solution: (iii) Let $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = f(t)$

Therefore by inverse Laplace transform of integral we have

$$L^{-1}\left[\int_s^\infty \frac{s}{(s^2+a^2)^2}ds\right] = \frac{f(t)}{t} \qquad \Rightarrow L^{-1}\left[\frac{1}{2}\left\{-\frac{1}{s^2+a^2}\right\}_s^\infty\right] = \frac{f(t)}{t}$$

$\Rightarrow \frac{1}{2} L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{f(t)}{t}$ or $\frac{1}{2}.\frac{1}{a}\sin at = \frac{f(t)}{t}$

$\Rightarrow \frac{1}{2a}\sin at = \frac{f(t)}{t}$ or $f(t) = \frac{t}{2a}\sin at$

Hence $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = f(t)\frac{t}{2a}\sin at$

f) Multiplication by power of s:

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then $L^{-1}[s\,F(s)] = f'(t)$

Proof: we know that $L[f'(t)] = s\,F(s) - f(0)$ (by Laplace transform of derivative)

$$L[f'(t)] = s\,F(s) \qquad \left(\because f(0) = \lim_{t \to 0} f(t) = 0\right)$$

Therefore $L^{-1}[s\,F(s)] = f'(t)$

Hence $L^{-1}[s\,F(s)] = f'(t)$, if $f(0) = 0$ (1)

Generalizations to $L^{-1}[s^n\,F(s)]$, $n = 2, 3, 4$ .... are also possible

Remark 1: The result (1) can be interpreted as the multiplication of the transform by s corresponds to differentiation of a function of t w. r. to t.

Remark 2: If Laplace transform of unknown function f(t) contains the factor s, the inverse of that can be found by dropping the factor s, determining the inverse of the remaining portion of the transform and finally differentiating that inverse with respect to t.

Example: Use multiplication by power of s theorem to find

(i) $L^{-1}\left[\frac{s}{s^2+9}\right]$, given that $L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{\sin 3t}{3}$

(ii) $L^{-1}\left[\frac{s}{s^2-9}\right]$, given that $L^{-1}\left[\frac{1}{s^2-9}\right] = \frac{\sinh 3t}{3}$

Solution: (i) Given that $L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{\sin 3t}{3}$

Therefore $L^{-1}\left[\frac{s}{s^2+9}\right] = L^{-1}\left[s.\frac{1}{s^2+9}\right] = L^{-1}[s\,F(s)]$ ($\because$ sin(0)=0)

Hence by multiplication by s result $L^{-1}[s\,F(s)] = f'(t)$ we have

$$L^{-1}\left[\frac{s}{s^2+9}\right] = \frac{d}{dt}f(t) = \frac{d}{dt}\left(\frac{\sin 3t}{3}\right) = \frac{3\cos 3t}{3} = \cos 3t$$

Hence $L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$

Solution: (ii) Given that $L^{-1}\left[\frac{1}{s^2-9}\right] = \frac{\sinh 3t}{3} = f(t)$

Therefore $L^{-1}\left[\frac{s}{s^2-9}\right] = L^{-1}\left[s.\frac{1}{s^2-9}\right] = L^{-1}[s\,F(s)]$ ($\because$ sinh (0)=0)

Hence by multiplication by s result $L^{-1}[s\,F(s)] = f'(t)$ we have

$$L^{-1}\left[\frac{s}{s^2-9}\right] = \frac{d}{dt}f(t) = \frac{d}{dt}\left(\frac{\sinh 3t}{3}\right) = \frac{3\cosh 3t}{3} = \cosh 3t$$

Hence $L^{-1}\left[\frac{s}{s^2-9}\right] = \cosh 3t$

g) Division by s:

If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt$

Proof: we know that $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$ (by Laplace transform of integral)

Therefore $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt$ (1)

Generalizations to $L^{-1}\left[\frac{F(s)}{s^n}\right]$, $n = 2, 3, 4 \ldots$ are also possible

Remark 1: The result (1) can be interpreted as the division of the transform by s corresponds to integration of a function of t w. r. to t.

Remark 2: If Laplace transform of unknown function f(t) contains the factor $\frac{1}{s}$, the inverse of that can be found by dropping the factor $\frac{1}{s}$, determining the inverse of the remaining portion of the transform, and finally integrating that inverse with respect to t from 0 to t.

Example: Find the inverse Laplace transform of the followings

(I) $\frac{1}{s(s^2+9)}$ **(ii)** $\frac{1}{s(s+9)}$ **(iii)** $\frac{1}{s^3\,(s^2+1)}$

Solution: (i) Let $\frac{1}{s(s^2+9)} = \frac{F(s)}{s}$, we have $F(s) = \frac{1}{(s^2+9)}$ and

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{(s^2+9)}\right] = \frac{\sin 3t}{3} = f(t) \quad \text{(dropping the factor 1/s)}$$

$$L^{-1}\left[\frac{1}{s}\frac{1}{(s^2+9)}\right] = \int_0^t \frac{\sin 3t}{3}\,dt = \frac{1}{3}\left[\frac{-\cos 3t}{3}\right]_0^t = \frac{1-\cos 3t}{9}$$

$$\left\{\because \text{If } L^{-1}[F(s)] = f(t), \text{then } L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt\right\}$$

Solution: (ii) Let $\frac{1}{s(s+9)} = \frac{F(s)}{s}$, we have $F(s) = \frac{1}{(s+9)}$ and

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+9}\right] = e^{-9t} = f(t) \quad \text{(dropping the factor 1/s)}$$

$$L^{-1}\left[\frac{1}{s}\frac{1}{(S+9)}\right] = \int_0^t e^{-9t}\,dt = \left[\frac{e^{-9t}}{-9}\right]_0^t = \frac{1-e^{-9t}}{9}$$

Solution: (ii) We have $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$$L^{-1}\left[\frac{1}{s}\frac{1}{(s^2+1)}\right] = \int_0^t \sin t\,dt = [-\cos t]_0^t \qquad \text{\{by result (1) above\}}$$

$$= 1 - \cos t = f(t)\text{say} \qquad \text{Where } F(s) = \frac{1}{s}\frac{1}{(s^2+1)}$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t g(t)\,dt = \int_0^t (1-\cos t)\,dt$$

$$= [t - \sin t]_0^t = t - \sin t = g(t) \qquad \text{Where } G(s) = \frac{1}{s(s^2+1)}$$

$$\therefore\ L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\frac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}G(s)\right] = \int_0^t g(t)\,dt = \int_0^t (t-\sin t)\,dt$$

$$= \left[\frac{t^2}{2} + \cos t\right]_0^t = \frac{t^2}{2} + \cos t - 1$$

h) Use of convolution theorem:

If the functions H(s) can be expressed as product of two functions F(s) and G(s) whose inverses f(t) and g(t) respectively are known, then the inverse of the product $H(s) = F(s)G(s)$ can be obtained by using the convolution theorem.

Theorem: If $L^{-1}[F(s)] = f(t)$, $L^{-1}[G(s)] = g(t)$ and $H(s) = F(s)G(s)$, then

$$L^{-1}[H(s)] = L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

Proof: Since $L[f(t) * g(t)] = L\left[\int_0^t f(u)g(t-u)du\right] = F(s)G(s) = H(s)$

Hence

$$L^{-1}[H(s)] = L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

Note 1: since the convolution of f(t) and g(t) is commutative, f(t) and g(t) are interchangeable in the above result.

Hence

$$L^{-1}[H(s)] = L^{-1}[F(s)G(s)] = \int_0^t f(t-u)g(u)du = f(t) * g(t)$$

Note 2: If $L^{-1}[F(s)] = f(t)$, $L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s}\right] = 1$, then

$$L^{-1}[H(s)] = L^{-1}[F(s)G(s)] = L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\ .1du = f(t) * g(t)$$

Examples: Use convolution theorem to find the inverse Laplace of the following functions

i) $\frac{1}{s(s^2+a^2)}$ ii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}, a \neq b$ iii) $\frac{1}{(s+1)(s^2+1)}$ (iv) $\frac{1}{s\sqrt{s+4}}$

Solution: (i) We can write $\frac{1}{s(s^2+a^2)} = \frac{1}{s}\frac{1}{s^2+a^2} = F(s)G(s)$

Let $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{s^2+a^2}$

So that $f(t) = L^{-1}\left[\frac{1}{s}\right] = 1$ and $g(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

Hence by convolution theorem, we have

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = f(t) * g(t) = 1 * \frac{\sin at}{a} = \int_0^t f(t-u)g(u)du$$

$$= \int_0^t 1.\frac{\sin at}{a}du = \left[\frac{-\cos au}{a^2}\right]_0^t = \frac{1-\cos at}{a^2}$$

(ii) We can write $\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{s^2+a^2}.\frac{s}{s^2+b^2} = F(s)G(s)$

Let $F(s) = \frac{s}{s^2+a^2}$ and $G(s) = \frac{s}{s^2+b^2}$

So that $f(t) = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$ and $g(t) = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt$

Hence by convolution theorem, we have

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = f(t) * g(t) = \mathrm{coa}\ at * \cos bt = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \cos au \cos b(t-u)\, du = \frac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)]du$$

$$[\because 2\cos A\ \cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt)\} + \cos\{(a+b)u-bt)\}]du$$

$$=\frac{1}{2}\left[\frac{\sin\{(a-b)u+bt)\}}{a-b}+\frac{\sin\{(a+b)u-bt)\}}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{\sin at\ -\sin bt}{a-b}+\frac{\sin at-\sin bt}{a+b}\right]=\frac{1}{2}\left[\frac{\sin at\ -b\sin bt}{a^2-b^2}\right]$$

(iii) We can write $\frac{1}{(s+1)(s^2+1)}=\frac{1}{s+1}.\frac{1}{s^2+1}=F(s)G(s)$

Let $F(s)=\frac{1}{s+1}$ and $G(s)=\frac{1}{s^2+1}$

So that $f(t)=L^{-1}\left[\frac{1}{s+1}\right]=e^{-t}$ and $g(t)=L^{-1}\left[\frac{1}{s^2+1}\right]=\sin t$

Hence by convolution theorem, we have

$$L^{-1}\frac{1}{(s+1)(s^2+1)}=f(t)*g(t)=e^{-t}*\sin t=\int_0^t f(u)g(t-u)du$$

$$=\int_0^t e^{-u}\sin(t-u)\,du\ =\ \left[\frac{e^{-u}}{(-1)^2+(-1)^2}\{-\sin(t-u)+\cos(t-u)\}\right]_0^t$$

$$\because\int_0^t e^{at}\sin bt\,dt\ =\ \left[\frac{e^{at}}{a^2+b^2}\{a\sin bt-b\cos bt\}\right]_0^t$$

$$=\ \frac{e^{-t}}{2}\{(0+1)\}-\frac{1}{2}(-\sin t+\cos t)=\frac{1}{2}[\sin t-\cos t+e^{-t}]$$

(iv) We can write $\frac{1}{s\sqrt{s+4}}=\frac{1}{s}.\frac{1}{\sqrt{s+4}}=F(s)G(s)$

Let $F(s)=\frac{1}{s}$ and $G(s)=\frac{1}{\sqrt{s+4}}$

So that $f(t)=L^{-1}\left[\frac{1}{s}\right]=1$ and $g(t)=L^{-1}\left[\frac{1}{\sqrt{s+4}}\right]=\frac{e^{-4t}}{\sqrt{\pi t}}$

Hence by convolution theorem, we have

$$L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right]=f(t)*g(t)=1*\frac{e^{-4t}}{\sqrt{\pi t}}=\int_0^t f(t-u)g(u)du=\int_0^t 1.\frac{e^{-4u}}{\sqrt{\pi u}}du=\ \frac{1}{\sqrt{\pi}}\int_0^t\frac{e^{-4u}}{\sqrt{u}}du$$

Put $4u=y^2, \therefore du=\frac{1}{2}\,y\,dy$ and

u	0	t
y	0	$2\sqrt{t}$

Therefore $L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right]=\frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}}e^{-y^2}dy=\ \frac{1}{2}\text{erf}(2\sqrt{t})$

3. **Use of Partial fractions:**

In case where F(s) is a rational algebraic fractional, it is often convenient to find inverse Laplace transform by expressing F(s) in terms of partial fractions.

Consider a rational function $F(s)=\frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials with the degree of N(s) less than that of D(s) (i.e. proper fraction). Then $F(s)=\frac{N(s)}{D(s)}$ can be resolved into the sum of rational functions (called partial fractions) having the form $\frac{A}{(as+b)^r},\frac{As+B}{(as^2+bs+c)^r}$, where $r=1,2,3\ldots$ By finding the inverse Laplace transformation of the partial fractions, we can find $L^{-1}[F(s)]$.

Example 1: when the denominator has non –repeated linear factors, we write

$$\frac{s^2+10s+25}{(s-2)(2s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{2s-1} + \frac{C}{s+1}$$

Example 2: when the denominator has repeated linear factors, we write

$$\frac{s^2+10}{(s-2)(s+1)^3} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

Example 3: When the denominator has non-repeated quadratic factors, we write

$$\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{As+B}{s^2+2s+5} + \frac{Cs+D}{s^2+2s+2}$$

Example 4: When the denominator has repeated quadratic factors, we write

$$\frac{3s^2-4s+2}{(s^2+2s+5)^2(s-2)} = \frac{As+B}{s^2+2s+5} + \frac{Cs+D}{(s^2+2s+5)^2} + \frac{E}{s-2}$$

The constants A, B, C etc can be obtained by clearing of fractions (i.e. by multiplying both sides by the denominator of the given fraction) and equation coefficients of like powers of s on both sides or by using special methods.

Quadratic factor can also be written as product of linear factors with complex conjugate root and apply above method.

Example: Using partial fractions, find the inverse Laplace transform of following

(i) $\frac{3s+7}{s^2-2s-3}$ (ii) $\frac{11\,s^2-2s+5}{(s-2)(2s-1)(s+1)}$ (iii) $\frac{21s-9}{(s+1)(s-2)^3}$

(iv) $\frac{3s+1}{(s-1)(s^2+1)}$ (v) $\frac{2\,s^2-1}{(s^2+1)(s^2+4)}$ (vi) $\frac{s}{(s^4+4a^4)}$

Solution: (i) We have $\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)\,(s+1)}$

Here denominator has non-repeated linear factors

Let $\frac{3s+7}{(s-3)\,(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$ (1)

Multiplying both sided of (1) by $(s+1)(s-3)$, we obtain

$3s+7 = A(s+1) + B(s-3)$

Putting $s=-1$ we get, $3(-1)+7 = 0 + B(-`1-3)$ $\Rightarrow 4 = -4B$ $\Rightarrow B = -1$

Putting $s = 3$ we get, $3(3)+7 = A(3+1) + B(0)$ $\Rightarrow 16 = 4A$ $\Rightarrow A = 4$

Hence (1) can be written as $\frac{3s+7}{(s-3)\,(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$ and

$$L^{-1}\left[\frac{3s+7}{(s-3)\,(s+1)}\right] = L^{-1}\left[\frac{4}{s-3}\right] - L^{-1}\left[\frac{1}{s+1}\right] = 4e^{3t} - e^{-t}$$

(ii) We have $\frac{11\,s^2-2s+5}{(s-2)(2s-1)(s+1)}$

Here denominator has non-repeated linear factors

Let $\frac{11\,s^2-2s+5}{(s-2)(2s-1)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(2s-1)} + \frac{C}{(s+1)}$ (1)

Multiplying both sided of (1) by $(s-2)(2s-1)(s+1)$, we obtain

$$11\,s^2 - 2s + 5 = A(2s-1)(s+1) + B(s-2)(s+1) + C(s-2)(2s-1)$$

Putting $s = 2$, we get, $11(2) - 2(2) + 5 = A(4-1)(2+1)$ $\Longrightarrow A = 5$

Puttings $= 1/2$, we get, $11\left(\frac{1}{4}\right) - 2\left(\frac{1}{2}\right) + 5 = B\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+1\right)$ $\Longrightarrow B = -3$

Putting $s = -1$, we get, $11(-1) - 2(-1) + 5 = C(-3)(-3)$ $\Longrightarrow C = 2$

Hence $L^{-1}\left[\frac{11\,s^2-2s+5}{(s-2)(2s-1)(s+1)}\right] = L^{-1}\left[\frac{5}{s-2} + \frac{-3}{2s-1} + \frac{2}{s+1}\right]$

$$= 5L^{-1}\left[\frac{1}{s-2}\right] - 3L^{-1}\left[\frac{1}{2s-1}\right] + 2L^{-1}\left[\frac{1}{s+1)}\right]$$

$$= 5L^{-1}\left[\frac{1}{s-2}\right] - \frac{3}{2}L^{-1}\left[\frac{1}{s-1/2}\right] + 2L^{-1}\left[\frac{1}{s+1)}\right]$$

$$= 5e^{2t} - \frac{3}{2}e^{t/2} + 2e^{-t}$$

(iii) We have $\frac{21s-9}{(s+1)(s-2)^3}$

Here denominator has repeated linear factors

$$F(s) = \frac{21s-9}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3} \qquad (1)$$

$$21s - 9 = A(s-2)^3 + B\,(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

$$= A(s^3 - 2s^2 + 4s - 8) + B(s+1)(s^2 - 4s + 4) + C(s^2 - s - 2) + D(s+1)$$

$$21s - 9 = s^3(A+B) + s^2(-2A - 3B + C) + s(4A - C + D) + [-8A + 4B - 2C + D$$

Equation coefficients of identical powered terms of s from both sides we have

$$0 = A + B,\ 0 = -2A - 3B + C,\ \ 21 = 4A - C + D \text{ and } \ -9 = -8A + 4B - 2C + D$$

On solving simultaneously we get, $A = 2, B = -2, C = -2$ and $D = 11$

Hence from (1) $F(s) = \frac{2}{s+1} - \frac{2}{s-2} - \frac{2}{(s-2)^2} + \frac{11}{(s-2)^3}$

Taking inverse Laplace transform on both sides

$$L^{-1}[F(s)] = L^{-1}\left[\frac{2}{s+1}\right] - L^{-1}\left[\frac{2}{s-2}\right] - L^{-1}\left[\frac{2}{(s-2)^2}\right] + L^{-1}\left[\frac{11}{(s-2)^3}\right]$$

$$= 2L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{1}{s-2}\right] - 2L^{-1}\left[\frac{1}{(s-2)^2}\right] + 11\,L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

Since $L^{-1}[F(s+a)] = e^{-at}\ \ L^{-1}[F(s)] = e^{-at}f(t)$ and using $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ we have

$$L^{-1}[F(S)] = 2e^{-t} - 2e^{2t} - 2\,e^{2t}L^{-1}\left[\frac{1}{s^2}\right] + 11\,e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2e^{-t} - 2e^{2t} - 2\,e^{2t}\frac{t}{1!} + 11\,e^{2t}\frac{t^2}{2!} = 2e^{-t} + e^{2t}(-2 - 2t + 11\frac{t^2}{2}$$

(iv) We have $\frac{3s+1}{(s-1)(s^2+1)}$

Here denominator has one linear factor and one quadratic factors

$$F(s) = \frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \qquad (1)$$

Multiply both sides of (1) by $(s-1)(s^2+1)$ we obtain

$$3s + 1 = A(s^2 + 1) + (Bs + C)(s - 1)$$

$$= (A + B)s^2 + (C - B)s + A - C$$

Putting s=1, we get $3(1) + 1 = A(\,1 + 1) \Rightarrow A = 2$

To determine B and C equating coefficients of identical powered terms of s from both sides we have

$$0 = A + B, \text{ and } \quad 1 = A - C,$$

$$B = -A = -2 \text{ and } C = A - 1 = 1$$

Hence from $F(s) = \frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1} = \frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}$

Taking inverse Laplace transform on both sides

$$L^{-1}[F(s)] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2\,e^{t} - 2\cos t + \sin t$$

Applications to Differential equations

The Laplace transform is useful in solving differential equations and corresponding initial and boundary valued problems. The solution of differential equations involving functions of an impulsive type can also be solved by the use of Laplace transform in a very efficient manner. The general process of solution consists of three main steps.

1. The given differential equation is transformed into a simple algebraic equation (called subsidiary equation).
2. The subsidiary equation is solved by pure algebraic manipulations.
3. The solution of subsidiary equation is then transformed back to obtain the solution og the given differential equation.

In this way, the Laplace transform method reduces the problem of solving differential equation to an algebraic problem. Another advantage of this method over the classified method is that it solves the initial value problem directly without first finding general solution (complete solution) and then evaluating the arbitrary constants. We shall now illustrate this method in the following applications.

1. $L\left[\frac{dy}{dt}\right] = L[[\,y'] = s\,Y(s) - y(0)$
2. $L\left[\frac{d^2y}{dt^2}\right] = L[[\,y''] = s\,^2Y(s) - sy(0) - y\,'(0)$

3. $L\left[\frac{d^3y}{dt^3}\right] = L[[\,y'''] = s\,^3Y(s) - s^2y(0) - sy\,'(0) - y''(0)$

4. $L\left[\frac{d^4y}{dt^4}\right] = L[[\,y''''] = s\,^4Y(s) - s^3y(0) - s^2y\,'(0) - s\,y''(0) - y'''(0)$ etc.

Solution of ordinary differential equations with constant coefficients

Example: Find the solution of each of the following differential equations which satisfy the given conditions:

1. $y'' - 3y' + 2y = 12\,e^{-2t},\ y(0) = 2,\ y'(0) = 6$
2. $y'' + y = t, \qquad y(0) = 1,\ y'(0) = -2$
3. $y'' + 2y' + y = t\,e^{-t}, \qquad y(0) = 1,\ y'(0) = -2$
4. $y''' - 3y'' + 3y' - y = t^2\,e^t, \quad y(0) = 1,\ y'(0) = 0, y''(0) = -2$
5. $y''' - y = e^t, \qquad y(0) = y'(0) = y''(0) = 0$

Solution: (1) Given equation in actual notation can be written as

$$\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2y = 12\,e^{-2t}, \qquad y(0) = 2,\ \ y'(0) = 6$$

Taking Laplace transform on both sides we get

$$L\left[\frac{d^2y}{dt^2}\right] - 3\,L\left[\frac{dy}{dt}\right] + 2L[y] = 12\,L[e^{-2t}]$$

$$\therefore\ \{s^2Y(s) - sy(0) - y'(0)\} - 3\{sY(s) - y(0)\} + 2Y(s) = 12\frac{1}{s+2}$$

Substituting the given conditions $y(0) = 2,\ \ y'(0) = 6$ we have

$$\{s^2Y(s) - s(2) - 6\} - 3\{sY(s) - 2\} + 2Y(s) = 12\frac{1}{s+2}$$

$$(s^2 - 3s + 2)Y(s) - 2s = 12\frac{1}{s+2} \qquad \Rightarrow\ (s^2 - 3s + 2)Y(s) = 2s + 12\frac{1}{s+2}$$

⇨ $(s^2 - 3s + 2)Y(s) = \frac{2s^2+4s+12}{s+2}$

⇨ $Y(s) = \frac{2s^2+4s+12}{(s^2-3s+2)(s+2)} = \frac{2s^2+4s+12}{(s-1)(s-2)(s+2)}$

Using partial fractions, we can write

$$Y(s) = \frac{6}{s-1} + \frac{7}{s-2} + \frac{1}{s+2}$$

Taking inverse Laplace transform on both sides we get

$$y(t) = 6e^t + 7e^{2t} + e^{-2t}$$

Which is the required solution.

Solution: (2) Given equation $y'' + y = t, \qquad y(0) = 1,\ y'(0) = -2$

Taking Laplace transform on both sides we get

$$L[y''] + L[\,y] \;=\; L[t]$$

$$\therefore \;\{s^2Y(s) - sy(0) - y'(0)\} + Y(s) = \frac{1}{s^2}$$

Substituting the given conditions $y(0) = 1,\;\; y'(0) = -\,2$ we have

$$\{s^2Y(s) - s(1) - (-2)\} + Y(s) = \frac{1}{s^2}$$

$$(s^2 + 1)Y(s) - s + 2 = \frac{1}{s^2} \qquad \Longrightarrow\; (s^2 + 1)Y(s) = s - 2 + \frac{1}{s^2}$$

$$Y(s) = \frac{s-2}{(s^2+1)} + \frac{1}{s^2(s^2+1)} = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{s}{s^2+1} - \frac{3}{s^2+1} + \frac{1}{s^2}$$

Taking inverse Laplace transform on both sides we get

$$y(t) = \cos t - 3\sin 3t + t$$

This is the required solution of given the differential equation.

Solution: (3) Given equation

$$y'' + 2y' + y = t\,e^{-t}\,, \qquad y(0) = 1,\;\; y'(0) = \;-\,2$$

Taking Laplace transform on both sides we get

$$L[y''] + 2L[y'] + L\,[y] = L[t\,e^{-t}\,]$$

$$\therefore\;\{s^2Y(s) - sy(0) - y'(0)\} + 2\{sY(s) - y(0)\} + Y(s) = \frac{1}{(s+1)^2}$$

Substituting the given conditions $y(0) = 1,\;\; y'(0) = -2$ we have

$$\{s^2Y(s) - s(1) - (-2\_\} + 2\,\{sY(s) - 1\} + Y(s) = \frac{1}{(s+1)^2}$$

$$(s^2 + 2s + 1)Y(s) - s = \frac{1}{(s+1)^2} \qquad\qquad \Longrightarrow\; (s^2 + 2s + 1)Y(s) = s + \frac{1}{(s+1)^2}$$

$$\therefore Y(s) = \frac{s}{(s+1)^2} + \frac{1}{(s+1)^4} = \frac{s+1-1}{(s+1)^2} + \frac{1}{(s+1)^4} = \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^4}$$

Taking inverse Laplace transform on both sides we get

$$y(t) = e^{-t} - t\,e^{-t} + \frac{t^3e^{-t}}{3!}$$

Solution: (4) Given equation

$$y''' - 3y'' + 3y' - y = t^2\,e^t\,, \quad y(0) = 1, y'(0) = 0, y''(0) = -2$$

Taking Laplace transform on both sides we get

$$L[y'''] - 3\,L[y''] + 3L[y'] - L\,[y] = L[t^2\,e^t\,]$$

$$\therefore \{s^{\,3}Y(s) - s^2y(0) - sy^{\,'}(0) - y^{''}(0)\} - 3\{s^2Y(s) - sy(0) - y'(0)\}$$

$$+3\{sY(s) - y(0)\} - Y(s) = \tfrac{2}{(s-1)^3}$$

Substituting the given conditions $y(0) = 1, y'(0) = 0, y''(0) = -2$ we have

$$\{s^3 - 3s^2 + 3s - 1\}Y(s) - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

$$Y(s) = \frac{s^2 - 3s + 1}{s^3 - 3s^2 + 3s - 1} + \frac{2}{(s^3 - 3s^2 + 3s - 1)\,(s-1)^3} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\therefore Y(s) = \frac{s^2 - 3s + 1 - s}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$Y(s) = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

Taking inverse Laplace transform on both sides we get

$$y(t) = e^{-t} - t\,e^{-t} + \frac{t^2 e^{-t}}{2} + \frac{t^5 e^{-t}}{60}$$

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