Unit-1
Integral Calculus

Tracing of curves.
(Cartesian form)

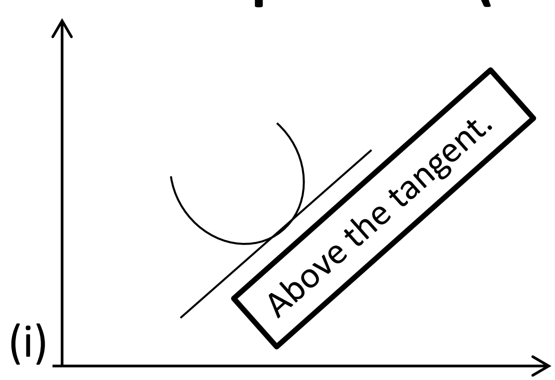
Tracing of curves means finding approximate shape of the curves using different properties like symmetry, Intercepts, tangents, asymptotes, region of existence etc.

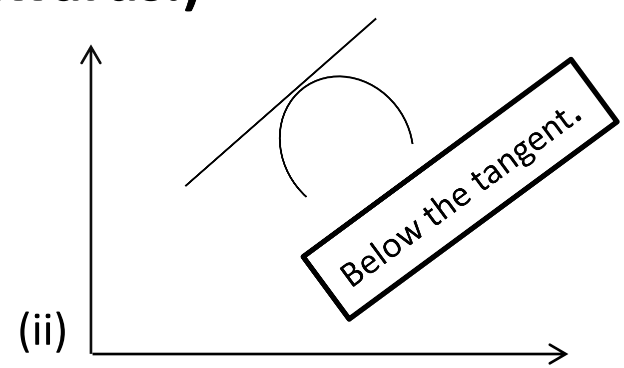
The knowledge of tracing of curves is useful in applications of integration in finding area, mass, center of gravity, volume etc.

Concavity.

(i) Concave upwards (Convex downwards.)

(ii) Convex upwards (Concave downwards.)



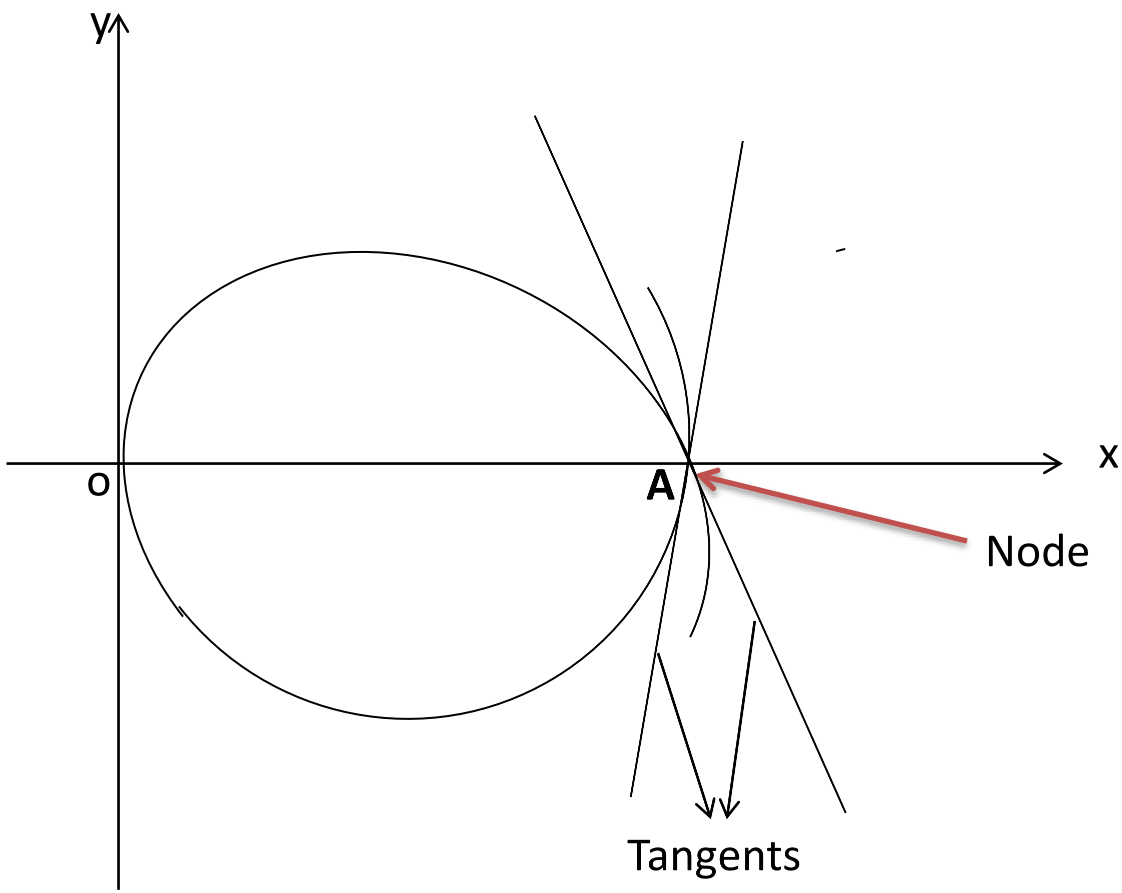


2. Singular Points. Following points are called as singular points.

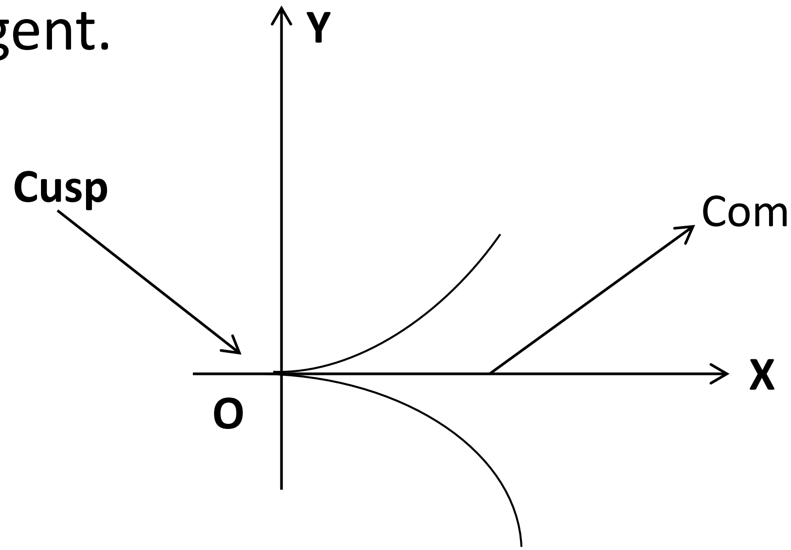
(i) Double Point – through which two branches of curve pass.

(ii) Multiple Point - through which more than one branches of curve pass.

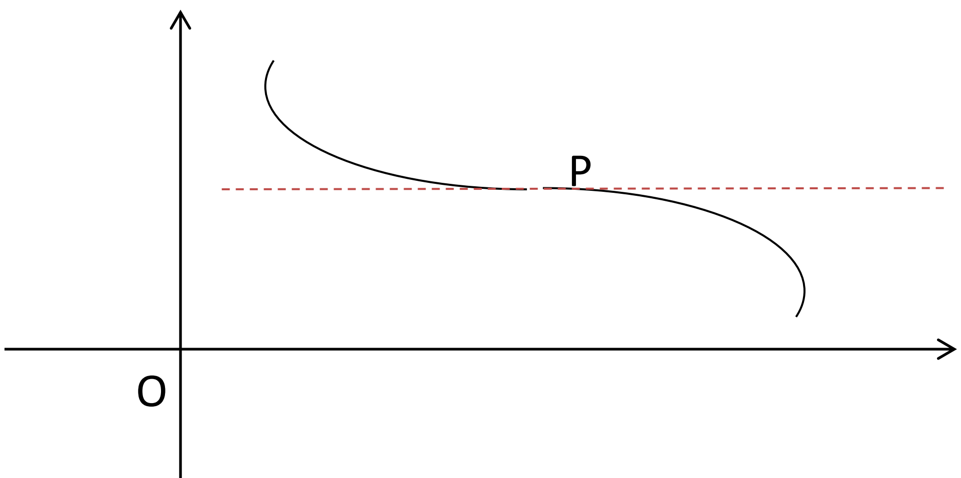
(iii) Node – A double point is called a node if distinct branches have distinct tangents.



IV. Cusp: A double point is called cusp if two branches have a common tangent.



V. Point Of inflexion : A curve has inflexion at P if it changes from concavity upwards to concavity downwards or vice versa.



VI. Isolated Point : A point is called isolated point or conjugate point if there no real points on the cure in the vicinity of the point P.

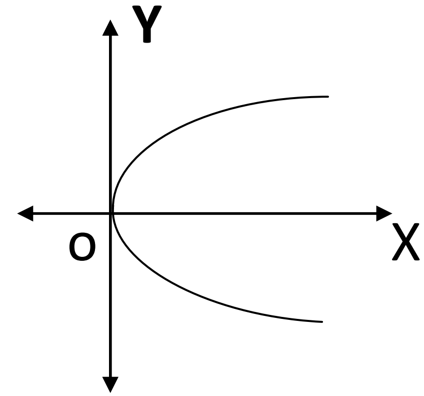
The curve y = f(x)

a	Is increasing in [a , b] If f ’(x) > 0 , ∀ x ∈ [a , b]	Concave upwards in [a , b] If f ‘’ (x) > 0 , ∀ x ∈ [a , b]
b	Is decreasing in [a , b] If f ’(x) < 0 , ∀ x ∈ [a , b]	Concave downwards in [a , b] If f ‘’ (x) < 0 , ∀ x ∈ [a , b]
c	Has extreme point If f ’(x) = 0 , for some x ∈ [a , b]	Has a point of inflexion If f’’(x) = 0 , for some x ∈ [a , b]

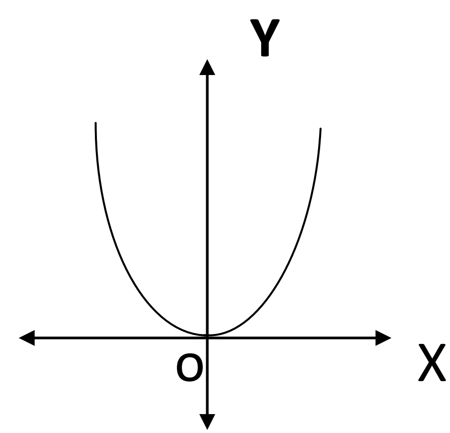
Rules For Tracing Of Cartesian Curves.

Rule 1 : Symmetry :

(a) Symmetry about X- axis: If equation of the curve remains unchanged by changing y to – y or all the powers of y in the equation are even. e.g. $y^2 = 4ax$.

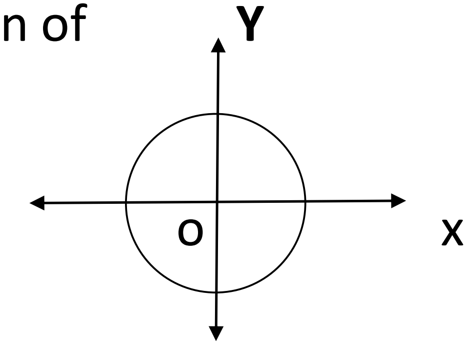


(b) Symmetry about Y- axis: If equation of the curve remains unchanged by changing x to – x **or** all the powers of x in the equation are even. e.g. $x^2 = 4ay$.



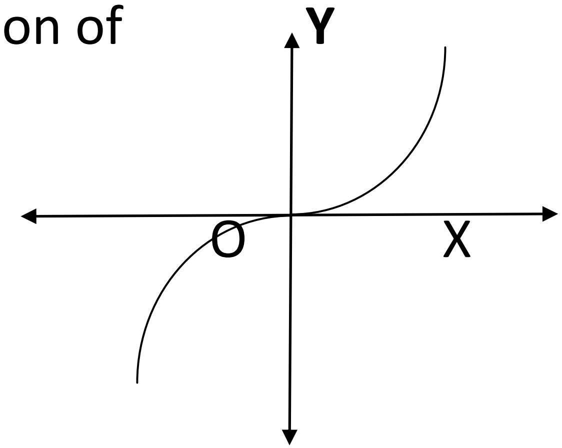
(c) Symmetry about both X and Y axes: If equation of the curve contains all even powers of x and y .

e.g. $x^2 + y^2 = r^2$.



(d) Symmetry in opposite quadrants: If equation of the curve remains unchanged by changing x to – x and y to – y simultaneously.

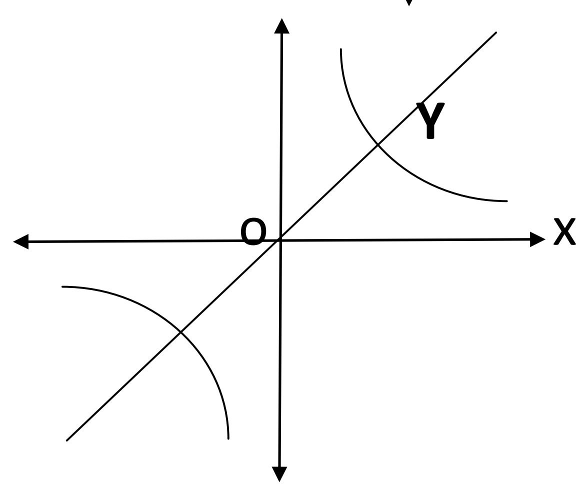
e.g. $y = x^3$



(e) Symmetry about the line y = x:

If equation of the curve remains unchanged by changing x to y and y to x .

e.g. $xy = c^2$

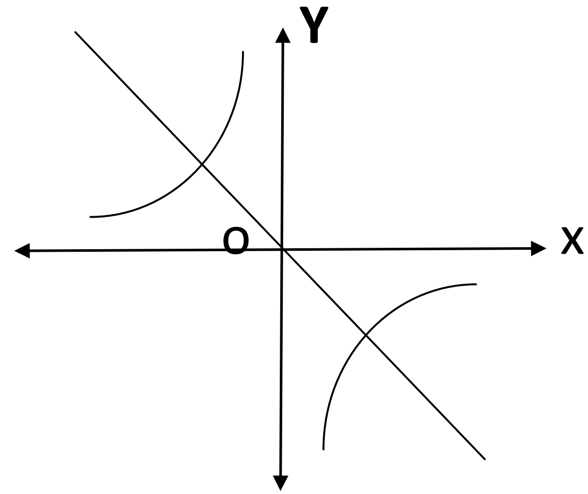


(e) Symmetry about the line $y = -x$:

If equation of the curve remains unchanged

by changing x to $-y$ and y to $-x$

e.g. $xy = -c^2$



Rule 2 : Points Of Intersection :

(a) **Origin :** If the equation of the curve does not contain any arbitrary constant then the curve passes through origin.

(b) **Intersections with the co-ordinate axes :** If possible express the equation in the explicit form, $y = f(x)$ or $x = f(y)$.

Intersection with X-axis; put $y = 0$ and Intersection with Y-axis; put $x = 0$.

Find the tangents at these points, if necessary and position of the curve relative to these lines

(c) If a curve is symmetric about the line $y = x$ or $y = -x$ find the points of intersections of the curve with these lines and also the tangents at that point because ***tangent leads the curve.***

Rule 3 :Tangents:

(a) Tangents at the origin : If a curve is given by a rational integral algebraic equation and passes through origin : **the equation of the tangent or tangents at origin can be obtained by equating to zero, the lowest degree terms taken together in the equation of the curve.**

(b) Tangents at any other points : To find nature of tangent at any point P find $\frac{dy}{dx}$ at that point.

(i) If $\left(\frac{dy}{dx}\right)_P = 0 \Longrightarrow$ Tangent at P is parallel to X- axis.

(ii) If $\left(\frac{dy}{dx}\right)_P = \infty \Longrightarrow$ Tangent at P is parallel to Y- axis.

(iii) If $\left(\frac{dy}{dx}\right)_P > 0 \Longrightarrow$ Tangent at P makes acute angle with X- axis.

(iv) If $\left(\frac{dy}{dx}\right)_P < 0 \Longrightarrow$ Tangent at P makes obtuse angle with X- axis.

Rule 4 : Asymptotes : Asymptotes are tangents at infinity.

(a) **Asymptotes parallel to X - axis** are obtained by equating to zero the coefficients of highest degree term in x.

(b) **Asymptotes parallel to Y - axis** are obtained by equating to zero the coefficients of highest degree term in y.

(c) **Oblique asymptotes:** Asymptotes not parallel to co – ordinate axes are called oblique asymptotes. If curve is not symmetric about X or Y – axis then we check for oblique asymptotes. Equation of oblique asymptote can be obtained by two methods.

(i) **Method 1 :** Let $y = mx + c$ be the asymptote. To find m and c substitute this y in the given equation $f(x, y)$ so we get the

points of intersection with the curve i.e. $f(x, mx + c) = 0$.

Equating to zero two successive highest powers of x we find m and c.

(ii) Method 2 : Let y = mx + c be the asymptote. To find m and c

- Consider the highest degree (n) term of the equation substitute x = 1 ,and y = m in that term call it $\emptyset_n(m)$
- Similarly find $\emptyset_{n-1}(m)$.
- Solve $\emptyset_n(m) = 0$ to find m.
- From $\emptyset_n(m)$ find $\emptyset'_n(m)$.
- To find c , use the formula $c = -\frac{\emptyset_{n-1}(m)}{\emptyset'_n(m)}$.

Rule 5 : Region of absence of The curve :

(a) If possible express the equation in the explicit form, y = f (x) And examine how y varies as x varies continuously.

(b) For y = f(x), if y becomes imaginary for some value of x > a (say) Then no part of the curve exists beyond x = a.

(c) For x = f(y), if x becomes imaginary for some value of y > b (say) Then no part of the curve exists beyond y = b.

Some Useful Remarks :

(a) When we have to solve for $y = f(x)$, put $x = 0$ see what is y.

Observe how y varies as x increases from 0 to $+\infty$ with special attention to the values of y for which $y = 0$ or $y \to +\infty$.

Also observe how y behaves as x becomes negative and $x \to -\infty$. with special attention for y becoming zero or $y \to -\infty$.

(b) If $y \to \infty$ as $x \to a$ then $x = a$ must be an asymptote ∥ to Y – axis.

If $x \to \infty$ as $y \to b$ then $y = b$ must be an asymptote ∥ to x – axis.

(c) If $y \to \infty$ as $x \to \infty$ and there is approximately linear relation between x and y for larger values of x, we may expect an oblique asymptote.

(d) If the curve is symmetric about X – axis or in the opposite quadrants then only positive values of y may be considered. We may draw the curve for negative values of y by symmetry.

Similarly, for symmetry about Y – axis only positive values of x may be considered.

Note :

(a) If there are two points of the curve on any axis and curve does not exists on either side of the points then there is always a loop between the two points.

(b) Use only as many steps necessary.

Type I : Curves Given by Cartesian Co-ordinates , (Explicit Forms.)

Ex 1 : Trace the curve $y\,(\,x^2 + \;4a^2\,) = 8\,a^3$

Sol : 1. The given equation can be written as y = $\frac{8\,a^3}{(\,x^2 + 4a^2\,)}$

2. Symmetry : Equation contains even powers of x $\Longrightarrow$ curve is symmetric about Y – axis.

3. Points of intersections : x = 0 $\rightarrow y = 2a \therefore \mathrm{P}(0\,,2a)$ is on the curve.

4. Tangents at origin: Equating lowest degree term to zero, but curve does not pass through origin (0,0). So no tangent at origin.

5. Tangents at any other point: $\frac{dy}{dx} = \frac{-16a^3x}{(x^2+4a^2\,)^2}\;, \left(\frac{dy}{dx}\right)_P = 0\,, \mathrm{P}(0{,}2a)$

So tangent at P(0,2a) is parallel to X – axis.

6. Asymptotes: Parallel to Y(or X – axis) is obtained by equating to zero the coefficient of highest degree term in y (or x).

(i) coefficient of highest degree term in y is $(4a^2 + x^2) = 0$,not possible

$\Longrightarrow$ no asymptote parallel to Y – axis.

(ii) coefficient of highest degree term in x is y and $y = 0$

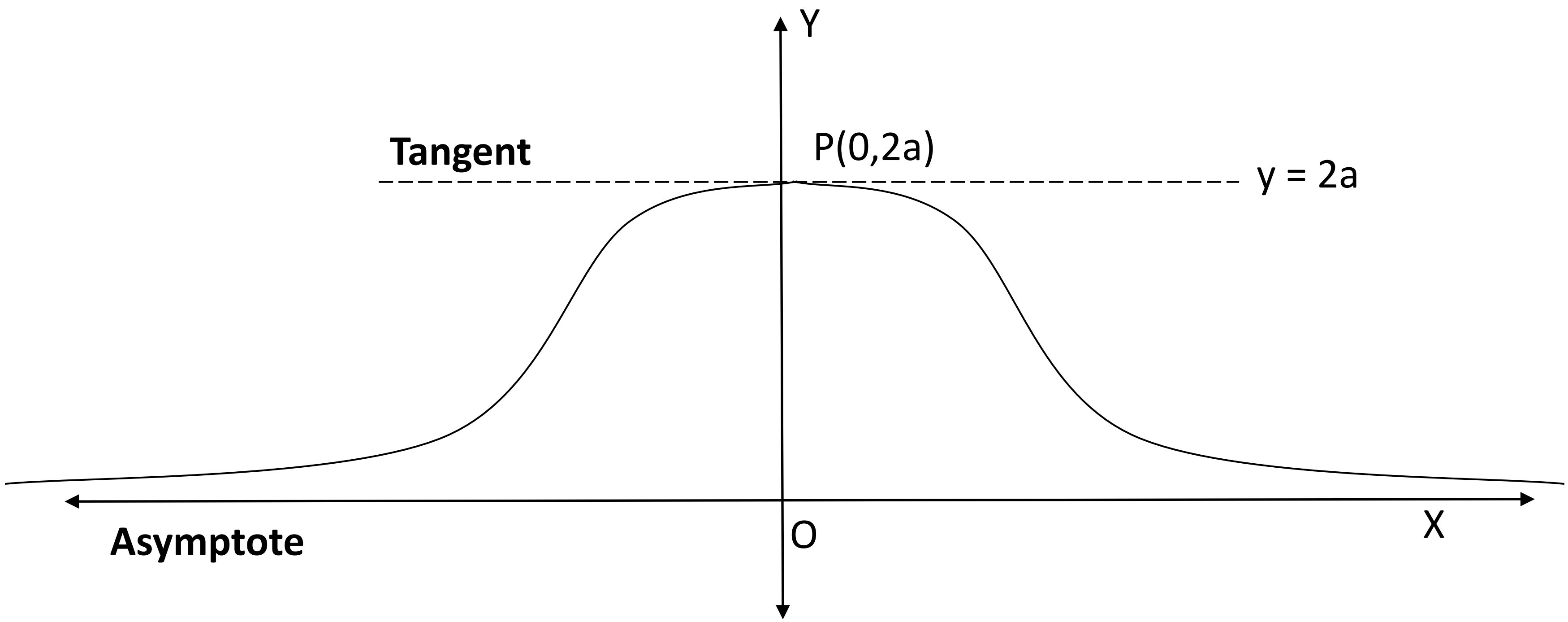
$\Longrightarrow$ y = 0 asymptote parallel to X – axis. i.e. X – axis is asymptote.

7. Region : $x^2 = \frac{8a^3}{y} - 4a^2$

(i) at y = - a , x^2 is - ve (ii) at y = 3a , x^2 is - ve.

$\therefore$ For y < 0 and y > 2a , x becomes imaginary. Therefore the curve does not exist for y < 0 and y > 2a.

i.e. curve exists for 0 < y $\leq$ 2a. for all values of x.


Y
Tangent
P(0,2a)
y = 2a
Asymptote
O
X

Ex 2 : Trace the curve $y^2\,(\,2a-x) = x^3$

Sol : 1. The given equation can be written as $y^2 = \dfrac{x^3}{(\,2a-x)}$

2. Symmetry : Equation contains even powers of y

$\Longrightarrow$ curve is symmetric about X - axis.

3. Points of intersections : x = 0 $\rightarrow y = 0$ *and* y $= 0 \rightarrow x = 0$

$\Longrightarrow O(0\,,0)$ is on the curve.

4. Tangents at origin: Equating lowest degree term to zero,

i.e. 2a y^2 = 0 ,i.e. $\therefore y^2 = 0$ $\Longrightarrow$ X – axis is tangent at origin.

5. Tangents at any other point: No other point.

6. Asymptotes: Parallel to Y(or X – axis) is obtained by equating to zero the coefficient of highest degree term in y (or x).

(i) coefficient of highest degree term in y is $(2a - x) = 0$,

$\Longrightarrow$ x = 2a , is asymptote parallel to Y – axis.

(ii) coefficient of highest degree term in x is -1

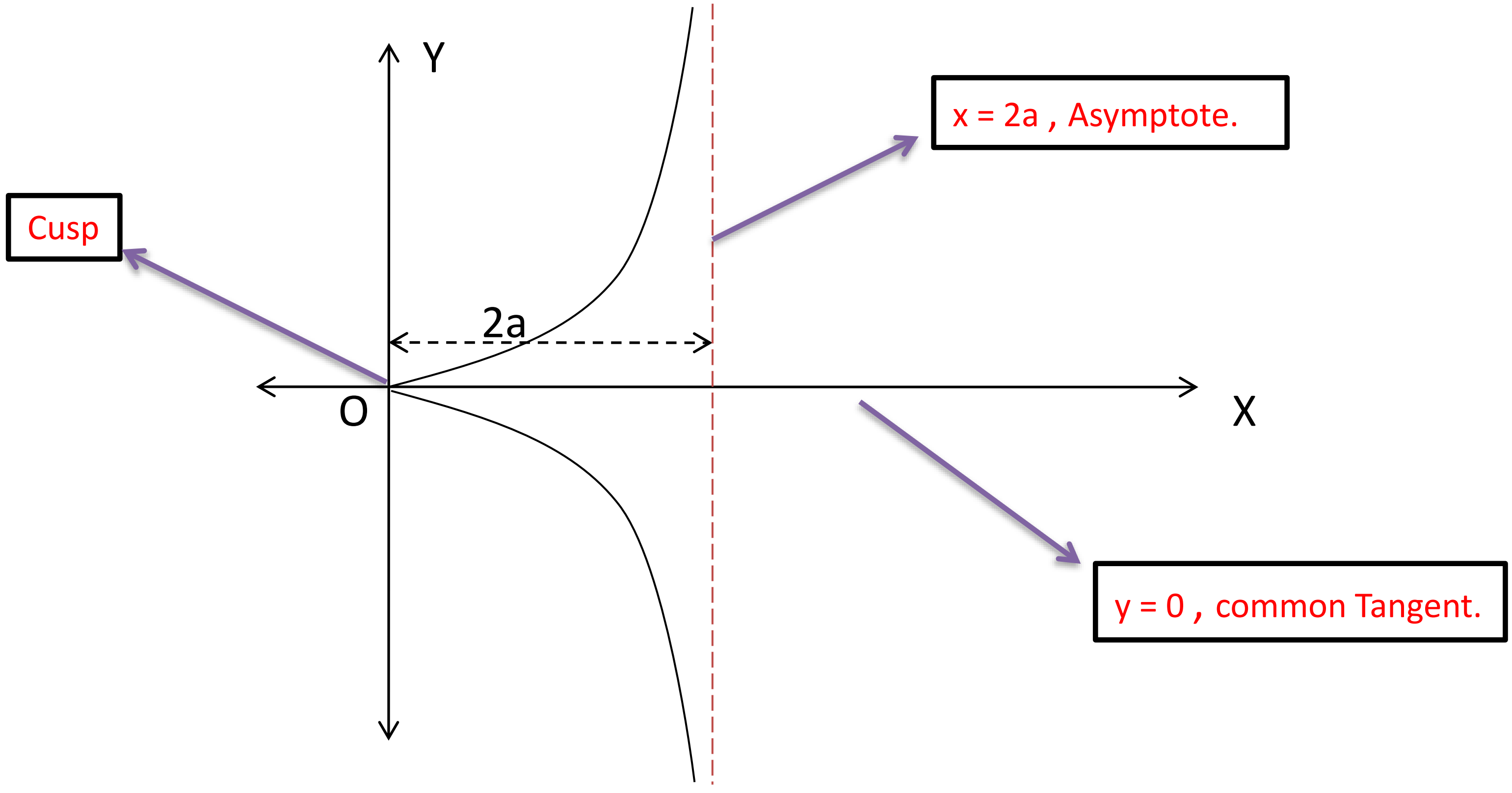
and $-1 = 0$ *not possible*. $\Longrightarrow$ no asymptote parallel to X – axis.

7. Region : The given equation can be written as $y^2 = \frac{x^3}{(2a-x)}$

At x = - a , y^2 is – ve ; At x = 3 a , y^2 is – ve.

For x < 0 and x > 2a , y is imaginary Therefore the curve does not exist for x < 0 and x > 2 a.

Thus the curve exists for 0 < x < 2a .



Y
x = 2a , Asymptote.
Cusp
2a
O
X
y = 0 , common Tangent.

Ex 3 : Trace the curve $x\,(x^2 + y^2) = a\,(x^2 - y^2)$

Sol : 1. The given equation can be written as $y^2 = \frac{x^2\,(a-x)}{(a+x)}$

2. Symmetry : Equation contains even powers of y

$\Longrightarrow$ curve is symmetric about X - axis.

3. Points of intersections : $y = 0 \rightarrow x = 0 \; and \; x = a$

$\Longrightarrow O(0\,,0)$ and A (a , 0) are on the curve.

$\therefore$ There is loop between O and A

4. Tangents at origin: Equating lowest degree term to zero,

i.e. $y^2 - x^2 = 0$,i.e. $\therefore\Longrightarrow y = \pm x$,

y = x and y = - x are tangents at origin.

5. Tangents at any other point:

$\frac{dy}{dx} = \infty$ at (a , 0) $\Longrightarrow$ tangent at (a , 0)is parallel to Y axis.

6. Asymptotes: Parallel to Y(or X – axis) is obtained by equating to zero the coefficient of highest degree term in y (or x).

(i) Highest degree term in y is y^2 and its coefficient $(a + x) = 0$,

$\Longrightarrow$ x = - a is asymptote parallel to Y – axis.

(ii) coefficient of highest degree term in x is 1 and $1 \neq 0$

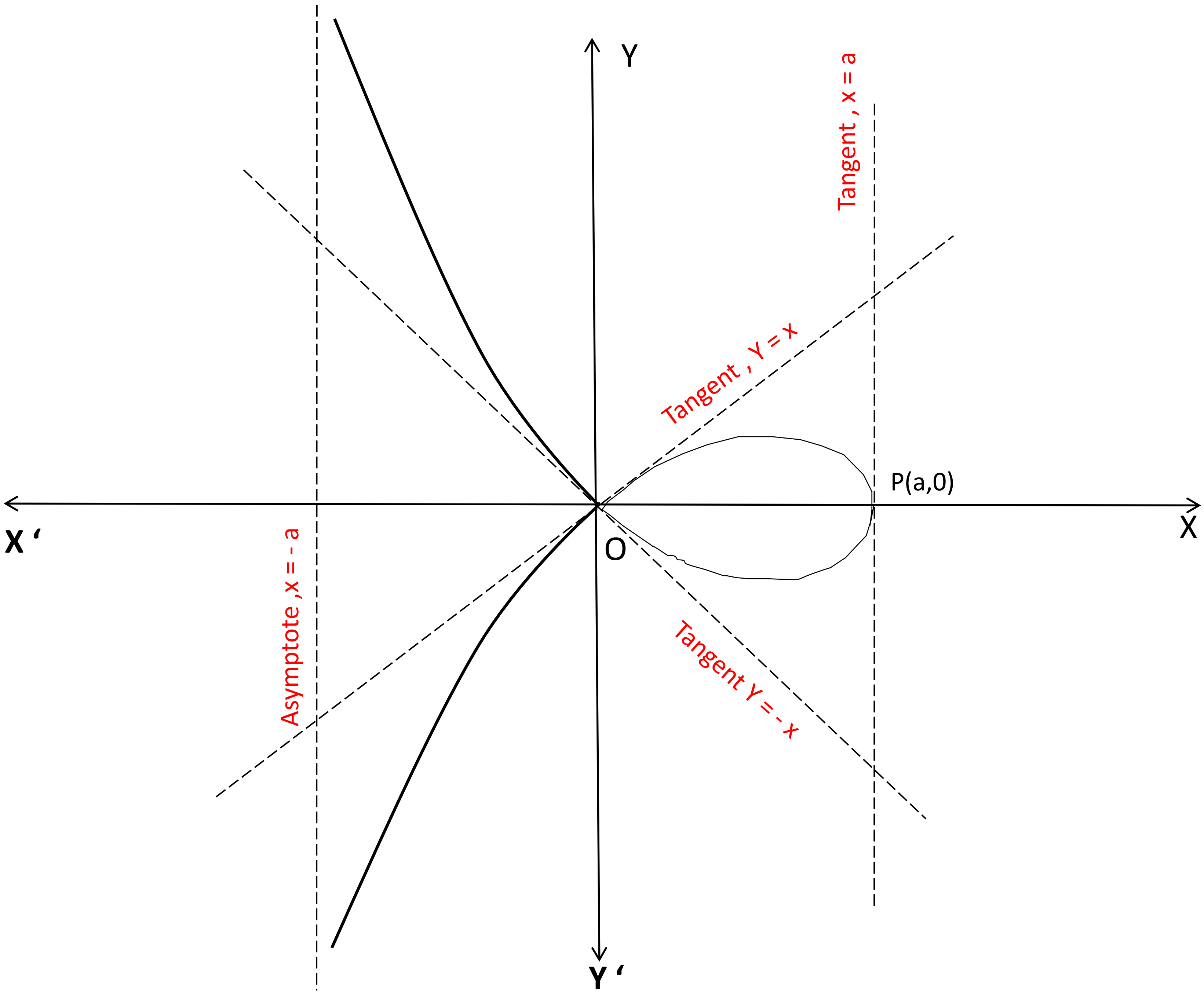
$\Longrightarrow$ no asymptote parallel to X – axis.

7. Region : $y^2 = \frac{x^2(a-x)}{(x+a)}$,

At x = - 2a ; x = 2 a , $y^2 = -ve$; for x < - a and x > a ; y is imaginary.

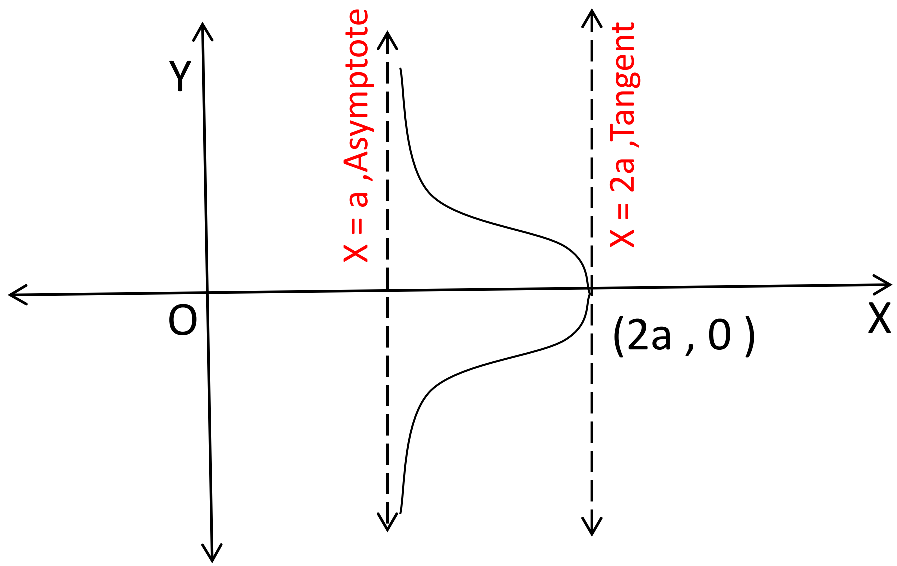
the curve does not exist for x <- a and x > a.

Therefore the curve exists for - a < x $\leq$ a.

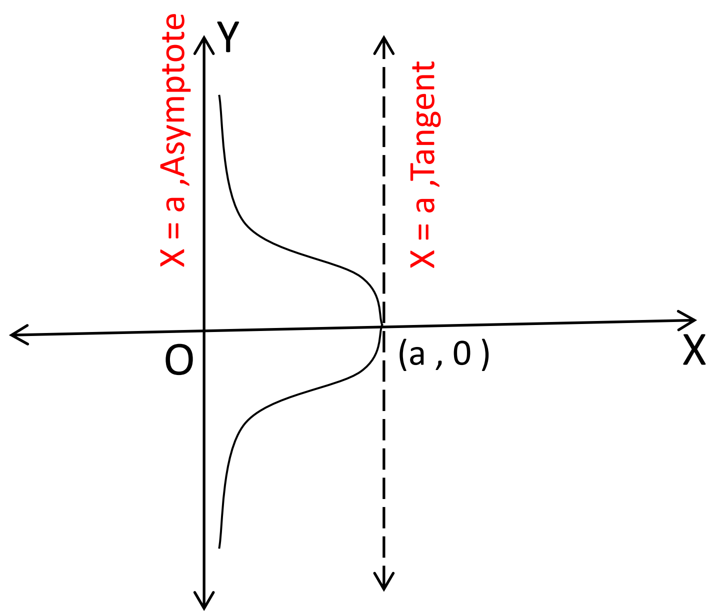


Y
Tangent , x = a
Tangent , Y = x
P(a,0)
X '
X
O
Asymptote ,x = - a
Tangent Y = - x
Y '

Ex 4 : Trace the curve $y^2\ (\mathrm{x} - \mathrm{a}) = x^2\ (2a\ - x)$



Ex 5 : Trace the curve $\mathrm{x}\,y^2\ = a^2\ (a - x)$



Type II : Curves Given by Cartesian Co-ordinates , (Implicit Forms.)

Ex 6 : Trace the curve $x^3 + y^3 = 3axy$ (*Folium of decarts*)

Sol : 1. Symmetry : Equation remains same after replacing (x , y) by

(y , x) $\Longrightarrow$ curve is symmetric about the line y = x.

2. Points of intersections : $y = 0 \rightarrow x = 0 \Longrightarrow O(0, 0)$ is on the curve.

Intersection with the line y = x $\Longrightarrow A(\frac{3a}{2}, \frac{3a}{2})$ is on the curve.

3. Tangents at origin: Equating lowest degree term to zero,

i.e. 3axy = 0 ,i.e. $y = 0$, $x = 0$, are tangents at origin.

4. Tangents at any other point: $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{ay - \boldsymbol{x^2}}{(\boldsymbol{y^2} - ax)}$

$\Longrightarrow \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)_{\mathrm{A}(\frac{3a}{2}, \frac{3a}{2})} = -1 \therefore$ tangent at $\mathrm{A}\left(\frac{3\mathrm{a}}{2}, \frac{3\mathrm{a}}{2}\right)$ makes angle $\frac{3\pi}{4}$ with X $-$ axis.

5. Oblique Asymptotes: The equation of the curve is $x^3 + y^3 = 3axy$

Let y = mx + c be the Oblique Asymptote.

$x^3 + (mx + c)^3 - 3ax(mx + c) = 0$

Equating the coefficients two successive highest powers of x to 0.

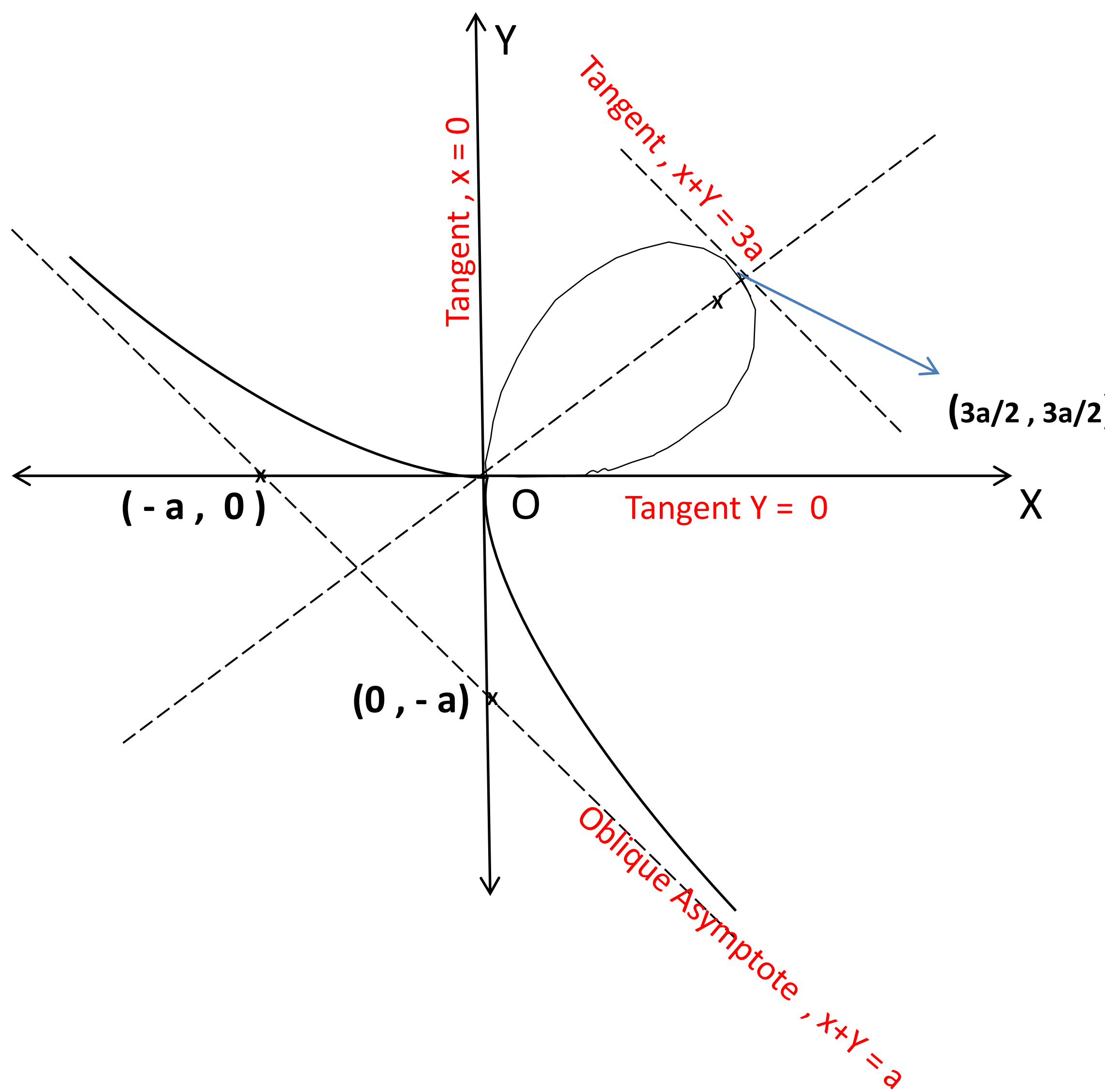
Coeff. Of $x^3 \Longrightarrow 1 + m^3 = 0$, $\therefore m = -1$

Coeff. Of $x^2 \Longrightarrow 3m^2c - 3am = 0$, $\therefore c = -a$

$\therefore y = -x - a$ or $\mathbf{x + y + a = 0}$ is equation of Oblique Asymptote.

7. Region :

The curve exists for all values of x .



Y
Tangent , x = 0
Tangent , x+Y = 3a
(3a/2 , 3a/2)
(- a , 0)
O
Tangent Y = 0
X
(0 , - a)
Oblique Asymptote , x+Y = a

Ex 7 : Trace the curve $x^4 + y^4 = a^2 (x^2 - y^2)$

Sol : 1. The given equation can be written as

$$x^2 (x^2 - a^2) + y^2 (y^2 + a^2) = 0$$

2. Symmetry : Equation contains even powers of x and y

$\Longrightarrow$ curve is symmetric about both X and Y - axis.

3. Points of intersections : $y = 0 \Longrightarrow x = 0, x = a, x = -a$

$x = 0 \Longrightarrow y = 0 \;\therefore O(0,0)$ *is a double point.*

and the curve intersects $X - $ *axis in points* $A(a, 0);\; B(-a, 0)$.

There is loop between 0and A ; also between 0 and B

4. Tangents at origin: Equating lowest degree term to zero,

i.e. $y^2 - x^2 = 0$,i.e. $\therefore \Longrightarrow y = \pm x$,

$y = x$ and $y = -x$ are tangents at origin.

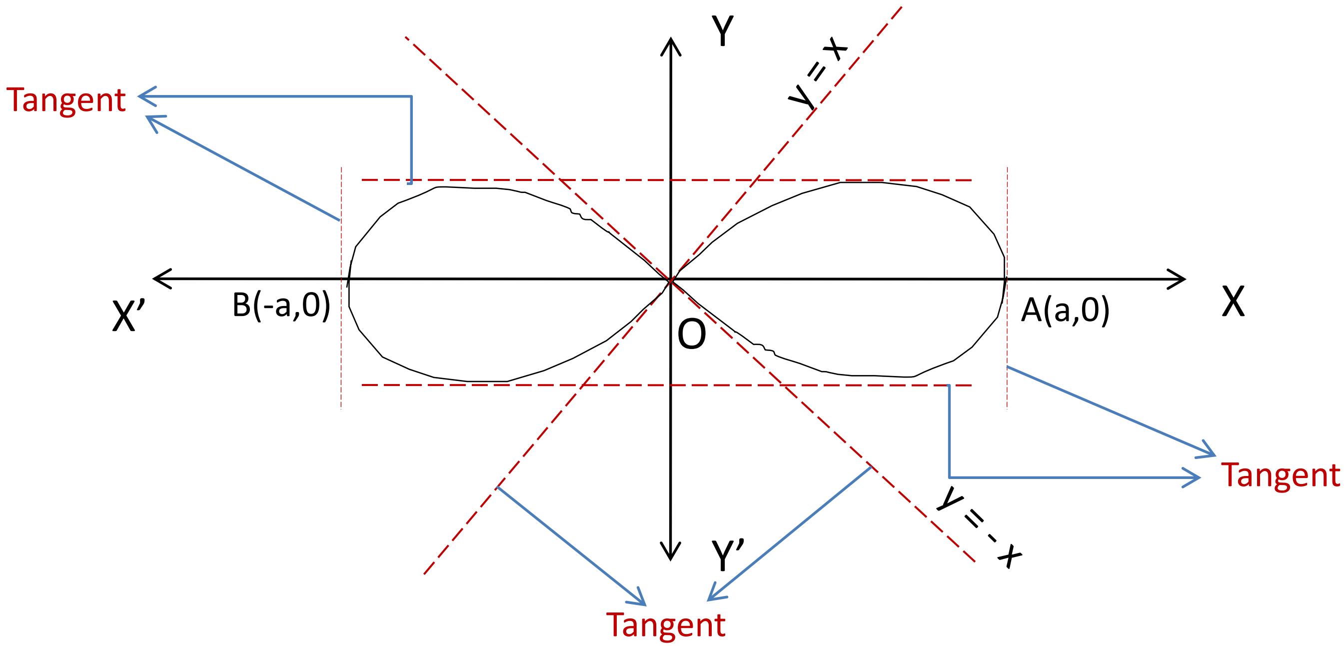
5. Tangents at any other point: $\frac{dy}{dx} = \frac{x(2x^2-a^2)}{y(2y^2+a^2)}$

$\frac{dy}{dx} = \infty$ at $(a , 0)$ and $(-a, 0)$

$\Longrightarrow$ tangents at $(a , 0)$and $(- a, 0)$ are parallel to Y axis.

6.Asymptotes : There no asymptote.

7. Region : The curve exists for $a \leq x \leq a$.



Tangent
Y
y = x
X'
B(-a,0)
O
A(a,0)
X
Tangent
Y'
y = - x
Tangent