Prof. Daniel J. Bodony
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Roe's Approximate Riemann Solver

Roe's (1981) method to improve the performance of Godunov's method (less cost and less dissipation) was to seek a 'smart' linearized Riemann problem that could be solved exactly using the exact initial data. That is, starting with the quasi-linear form

$$q_t + A(q)q_x = 0$$

find an $\hat{A}(q_L, q_R)$ such that

$$q_t + \hat{A}q_x = 0$$

was the PDE to be solved with the initial data

$$q(x,0) = \begin{cases} q_L, & x < 0 \\ q_R, & x > 0. \end{cases}$$

There are many ways in which a $\hat{A}(q_L, q_R)$ can be constructed, such as

$$\hat{A}(q_L, q_R) = \frac{1}{2}\left(A(q_L) + A(q_R)\right)$$

but these *may not* lead to the desired performance. (Recall: $f(q_L, q_R) = (f(q_L) + f(q_R))/2$ did not work well.) What Roe did was first set out a series of properties that $\hat{A}$ should have, and then find a way to construct it. He argued that having

Property (A) : Hyperbolicity. $\hat{A}$ should have purely real eigenvalues $\hat{\lambda}_i(q_L, q_R)$ with corresponding set of complete eigenvectors $\hat{x}_i$.

Property (B) : Consistency with the exact Jacobian:

$$\hat{A}(q, q) = A(q).$$

Property (C) : Conservation *across* discontinuities:

$$f(q_R) - f(q_L) = \hat{A}(q_L, q_R)\left(q_R - q_L\right).$$

Roe's ingenuity was how to find an $\hat{A}$ for the Euler equations; the properties were well established. He found that the matrix $\hat{A}$ is *equal to the original matrix A* when evaluated at the *Roe* average state, $\hat{q}$,

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-1)\hat{H} - \hat{u}^2 - \hat{c}^2 & (3-\gamma)\hat{u} & \gamma - 1 \\ (\hat{u}/2)[(\gamma-3)\hat{H} - \hat{c}^2] & \hat{H} - (\gamma-1)\hat{u}^2 & \gamma\hat{u} \end{bmatrix}$$

where

$$\hat{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$
$$\hat{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

where $\hat{H}$ is the total enthalpy. It is easy to show that the eigenvalues and eigenvectors of $\hat{A}$ are

$$\begin{aligned}\hat{\lambda}_1 = \hat{u} - \hat{c}, &\quad \hat{v}_1 = [1,\ \hat{u} - \hat{c},\ \hat{H} - \hat{u}\hat{c}]^T \\ \hat{\lambda}_2 = \hat{u}, &\quad \hat{v}_2 = [1,\ \hat{u},\ \hat{u}^2/2]^T \\ \hat{\lambda}_3 = \hat{u} + \hat{c}, &\quad \hat{v}_3 = [1,\ \hat{u} + \hat{c},\ \hat{H} + \hat{u}\hat{c}]^T\end{aligned}$$

where

$$\hat{c}^2 = (\gamma - 1)\left(\hat{H} - \frac{\hat{u}^2}{2}\right)$$

is the speed of sound.

To complete the method we need to evaluate the numerical flux $F_{i+1/2}(q_L, q_R)$ using this information. As before we solve the *linearized* Riemann problem using $\hat{A}$. We decompose the left and right states into their characteristic amplitudes

$$q_L = \sum_{p=1}^{3} \beta_p \hat{v}_p$$
$$q_R = \sum_{p=1}^{3} \gamma_p \hat{v}_p$$

such that the jump in q can be written

$$q_R - q_L = \sum_{p=1}^{3} (\gamma_p - \beta_p)\hat{v}_p = \sum_{p=1}^{3} \alpha_p \hat{v}_p$$

which gives the solution *along the ray* $x/t = 0$

$$\begin{aligned} q^* &= q_L + \sum_{\lambda_p < 0} \alpha_p \hat{v}_p \\ &= q_R - \sum_{\lambda_p > 0} \alpha_p \hat{v}_p \end{aligned}$$

which can be combined into a single expression

$$q^* = \frac{q_L + q_R}{2} + \frac{1}{2}\left[\sum_{\lambda_p<0} - \sum_{\lambda_p>0}\right] \alpha_p \hat{v}_p.$$

The numerical flux is then

$$F_{i+1/2} = f(q^*) = \hat{A}q^* = \frac{1}{2}\hat{A}(q_L + q_R) - \frac{1}{2}\sum_{p=1}^{3} |\lambda_p|\alpha_p\hat{v}_p$$

(This uses the fact that $\hat{A}\hat{v}_p = \lambda_p\hat{v}_p$.) One last step is to use homogeneity property of the Euler equations to get

$$F_{i+1/2} = \frac{1}{2}(f(q_L) + f(q_R)) - \frac{1}{2}\sum_{p=1}^{3} |\lambda_p|\alpha_p\hat{v}_p$$

is the desired numerical flux. Notice that it is made up of a central average (which has no dissipation) plus the influence of the *upwind* waves, which adds dissipation. With a little bit of algebra you can show that the jumps α_p are

$$\begin{aligned}
\alpha_1 &= \frac{1}{2\hat{c}^2}(\Delta p - \hat{c}\hat{\rho}\Delta u) \\
\alpha_2 &= \Delta\rho - \frac{1}{\hat{c}^2}\Delta p \\
\alpha_3 &= \frac{1}{2\hat{c}^2}(\Delta p + \hat{c}\hat{\rho}\Delta u)
\end{aligned}$$

where $\hat{\rho} = \sqrt{\rho_L\rho_R}$ is the geometric average of the density and

$$\Delta\rho = \rho_R - \rho_L,\ \Delta p = p_R - p_L, \text{ and } \Delta u = u_R - u_L.$$

Entropy fix Through experience, users of the Roe scheme found that when $|\hat{u}| \approx \hat{c}$ the Roe solver would give the wrong answer. This is a result of linearizing the problem and it can be shown that the cause is an expansion solution that violates the second law of thermodynamics. To *fix* this, it is common to fudge the eigenvalues $\hat{\lambda}_1$ or $\hat{\lambda}_3$ by

$$\hat{\lambda}_p \to \frac{1}{2}\left(\frac{\hat{\lambda}_p^2}{\epsilon} + \epsilon\right), \text{ for } \epsilon \ll 1$$

according to whether $|\hat{\lambda}_1| < \epsilon$ or $|\hat{\lambda}_3| < \epsilon$. This is called an "entropy fix." It is necessary to implement it for the Roe scheme to be "well behaved." The value of ϵ is arbitrary so long as it is small relative to unity.