# Computations in Riemannian Geometry and Geometric Analysis

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These notes are intended as a practical reference when doing basic calculations in Riemannian geometry. I hope they will be helpful to the reader who is familiar with the concepts of Riemannian geometry but isn't an expert when it comes to calculations; they may provide a useful supplement to a more expository text on Riemannian geometry (I recommend, in addition to the standard texts, the lecture notes of Ben Andrews, which can be found online, and the book by Andrews and Hopper).

Proofs are placed at the end of each section, and are numbered based on the right-hand side numbering. The numbers on the right hand side are ~~all~~ mostly links that go back and forth between an equation and its proof (!). Because this is intended more as a reference than as something to be read from start to finish, I've only made a little effort to keep concepts in order of dependence.

This is a slow but steady work in progress, with still much to be done, possibly including some major reorganization. Last updated: October 31, 2024

## 0 Notation and Conventions

I believe my sign convention for the curvature of a connection agrees with Chow, meaning that it is the opposite of Andrews. (Lee?) My convention for the factor in front of the wedge product of alternating tensors agrees with Andrews (and Lee?) and opposes Chow.

Throughout, unless otherwise stated, we will be considering a Riemannian $n$-manifold $M = (M^n, g)$.

| | |
|---|---|
| $\Gamma(E)$ | the set of sections of the bundle $E$ over $M$ |
| $\mathcal{T}^k_\ell(M)$ | the set of $(k,\ell)$-tensors; that is, sections of $(T^*M)^{\otimes k} \otimes (TM)^{\otimes \ell}$ |
| $\wedge^k T^*M$ | the $k$-form bundle on $M$ |
| $\Omega^k(T^*M)$ | the set of sections of $\wedge^k T^*M$, i.e. the set of $k$-forms on $M$; $\Gamma(\wedge^k T^*M)$ |

## Contents

# 1 Basic notions

This section contains constructions that don't depend on a Riemannian metric, but also contains some Riemannian-metric based identities.

## 1.1 Vector fields

By construction (see any book on Riemannian geometry), a vector field $X \in \Gamma(TM)$ satisfies the Leibniz rule

$$X(fg) = fX(g) + gX(f)$$

for $f, g \in C^\infty(M)$. From this it follows that

$$X(fY) = X(f) \cdot Y + fX(Y). \tag{1}$$

If $f \in C^\infty(M)$ and $r\colon \mathbb{R} \to \mathbb{R}$,

$$X(r \circ f) = (r' \circ f)X(f) \tag{2}$$

## 1.2 Differential forms

A $k$-tensor $\omega \in \otimes^k T_p^* M$ is said to be alternating if it is antisymmetric under interchange of any two of its arguments. We denote by $\wedge^k T_p M$ the space of alternating $k$-tensors at $p$. The **wedge product** of an alternating $k$-tensor and an alternating $\ell$-tensor is a $(k+\ell)$-tensor, defined by

$$S \wedge T = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)(S \otimes T) \circ \sigma,$$

where the composition with $\sigma$ denotes applying the permutation $\sigma$ to the $k+\ell$ inputs to $S \otimes T$. There is another convention (used by Chow for example) for this that involves a different factor in front.

The wedge product satisfies the properties

(i) $f \wedge (g \wedge h) = (f \wedge g) \wedge h$

(ii) $(cf) \wedge g = f \wedge (cg) = c(f \wedge g)$

(iii) If $f, g \in \wedge^k T_p M$, then
$$(f+g) \wedge h = f \wedge h + g \wedge h.$$

(iv) If $f \in \wedge^k T_p M$, $g \in \wedge^\ell T_p M$, then
$$g \wedge f = (-1)^{k\ell} f \wedge g.$$

TODO: it should be the case that a wedge product with a function is the same as multiplication but I'm not sure how that follows.

0- and 1-tensors (which are functions and covectors respectively) are trivially alternating, so we have $C^\infty(M) = \wedge^0 TM$ and $T^*M = \wedge^1 T^*M$.

See Ben's notes.

### 1.2.1 Exterior derivative

A $k$-form is a smooth section of the bundle $\wedge^k T^*M$ of alternating $k$-tensors on $M$. The set of all $k$-forms on $M$ is denoted $\Omega^k(M)$. The exterior derivative is the unique linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying

(i) If $f \in \Omega^0(M) = C^\infty(M)$, then $df$ is the same as the differential of $f$.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then
$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d^2 = 0$.

Using these axioms, we can determine the following expression for $d$. Suppose we have coordinate covector fields $dx^i$. If we have the $k$-form $\omega$ given by (??? sums are taken over increasing $k$-tuples)

$$\omega = \sum_{i_1,\ldots,i_k} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then (I think this sum is just taken over all tuples).

$$d\omega = \sum_{i,i_1,\ldots,i_k} \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{3}$$

Strictly speaking, there is still some more work to be done to make sure everything works here, even though it seems like we have a nice expression for $d$. One needs to show that this doesn't depend on the coordinates, and justify the claim that this operator is unique. For arguments of these facts, see Ben Andrews's lecture notes on differential geometry, or one of many other books on geometry.

If $\omega$ is a 2-form, we have the following useful expression

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{4}$$

The previous expression generalizes: $\omega$ is a $k$-form, the exterior derivative satisfies (here the hat notation means we are removing an argument)

$$\begin{aligned}(d\omega)(X_0,\ldots,X_k) = &\sum_{j=0}^{k}(-1)^j X_j \omega(X_0,\ldots,\hat{X}_j,\ldots,X_k) \\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).\end{aligned} \tag{5}$$

According to some conventions for the wedge product, this expression may differ by a factor of $\frac{1}{k+1}$ (e.g. in [3]). This expression can also be used to define the exterior derivative in a way that is explicitly independent of coordinates.

$$\begin{aligned}(d\omega)(X_0,\ldots,X_k) = &\sum_{j=0}^{k}(-1)^j X_j \omega(X_0,\ldots,\hat{X}_j,\ldots,X_k) \\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).\end{aligned} \tag{5}$$

*Proof.* TODO □

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#### 1.2.2 Interior product

The interior product is, for each $X \in T_pM$, a linear map $\iota_X : \wedge^k T_p^*M \to \wedge^{k-1}T_p^*M$. If $\omega \in \wedge^0 T_p^*M$ (so that $\omega$ is a number), we define $\iota_X\omega = 0$. Otherwise, the interior product is the unique linear operator $\iota_X \Omega^k(M) \to \Omega^{k-1}(M)$ satisfying similar properties to the exterior derivative:

(i) When $\omega \in \Omega^1(M) = \Gamma(T^*M)$, then $\iota_X\omega = \omega(X)$.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$\iota_X(\omega\wedge\eta) = (\iota_X\omega)\wedge\eta + (-1)^k\omega\wedge(\iota_X\eta)$$

(iii) $\iota_X^2 = 0$.

We can determine that $\iota_X$ has the following expression:

$$\iota_X(\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}). \tag{6}$$

$$\iota_X(\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}) \tag{6}$$

*Proof.* TODO □

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In particular, for covectors $\omega^1, \dots, \omega^k$, we have

$$\iota_X(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(X) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k(X). \tag{7}$$

Differential forms are exactly the objects that we integrate over a manifold. For more about integration, see Section 6.1.

## 1.3 The differential and gradient

The **differential** $df$ of a function $f \in C^\infty(M)$ is the 1-form defined by

$$(df)(X) = X(f)$$

for $X \in \mathscr{X}(M)$. Let $\operatorname{grad} f$ denote the vector field dual to $df$. That is, $g(\operatorname{grad} f, X) = (df)(X) = X(f)$. Sometimes $\nabla f$ is used to denote either $df$ or $\operatorname{grad} f$ (or both). It is also used to denote the total covariant derivative of $f$ (see below), but this is not really an abuse of notation since the total covariant derivative of $f$ is equal to $df$.

$$\operatorname{grad}(fh) = f \operatorname{grad} h + h \operatorname{grad} f \tag{8}$$

In coordinates:

$$df = (\partial_i f) dx^i \tag{9}$$

$$\operatorname{grad} f = g^{ij} (\partial_j f) \partial_i \tag{10}$$

## 1.4 The volume form

If $M$ is oriented, there is a unique $n$-form $d\mu = d\mu_g$ called the **volume form**, defined in local coordinates by

$$d\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

If $\{\omega^i\}_{i=1}^n$ is an oriented orthonormal coframe for $T^*M$, then

$$d\mu = \omega^1 \wedge \cdots \wedge \omega^n. \tag{11}$$

$$d\mu = \omega^1 \wedge \cdots \wedge \omega^n \tag{11}$$

*Proof.* TODO □

## 1.5 Divergence

Note that for a vector field $X$, $d(\iota_X(d\mu))$ is an $(n-1)$-form, so it is $f d\mu$ for some smooth function $f$. We call this function the **divergence** of $X$, so that

$$d(\iota_X \, d\mu) = \operatorname{div} X \, d\mu.$$

We could also have defined the divergence as the trace of the covariant derivative:

$$\operatorname{div} X = \operatorname{tr} \nabla X = (\nabla X)(\partial_i, dx^i) = (\nabla_i X)(dx^i). \tag{12}$$

In local coordinates, we have the expression

$$\operatorname{div}(X^i \partial_i) = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}). \tag{13}$$

The product of a function $f$ and a vector field $X$ satisfies

$$\operatorname{div}(fX) = X(f) + f \operatorname{div} X. \tag{14}$$

The characterization of divergence as the trace of the covariant derivative allows us to define the divergence of a $(k, \ell)$-tensor as the $(k, \ell - 1)$-tensor

$$(\operatorname{div} T)(X_1, \ldots, X_{\ell-1}) = \operatorname{tr}(\nabla_T())$$

## 1.6 The Laplacian(s)

The simplest version of the Laplacian is defined for functions $f \in C^\infty(M)$ by

$$\Delta f = \operatorname{div} \operatorname{grad} f.$$

This can be extended to act on tensor bundles. This operator is called the **connection Laplacian**, the **rough Laplacian**, or the **Laplace-Beltrami operator**; there are other second order linear elliptic operators referred to as the Laplacian as well. We define the rough Laplacian on tensors by $\Delta \colon \Gamma(\mathcal{T}^k_\ell(M)) \to \Gamma(\mathcal{T}^k_\ell(M))$ by

$$\Delta T = \operatorname{div} \nabla T = \operatorname{tr}_g \nabla^2 T = g^{ij} \nabla_i \nabla_j T,$$

where the trace is taken over the two new indices introduced by $\nabla^2$. For functions, this has the coordinate expression

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^j} \right). \tag{15}$$


$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^j} \right). \tag{15}$$


*Proof.* TODO □

$$\Delta(fh) = f\Delta h + h\Delta f + 2 \langle \operatorname{grad} f, \operatorname{grad} h \rangle \tag{16}$$

From this it follows that the heat operator $\partial_t - \Delta$ satisfies the product rule

$$(\partial_t - \Delta)(fh) = f(\partial_t - \Delta)(h) + h(\partial_t - \Delta)(f) - 2 \langle \nabla f, \nabla h \rangle. \tag{17}$$

If $f \in C^\infty(M)$ and $r \colon \mathbb{R} \to \mathbb{R}$, then

$$\Delta(r \circ f) = (r' \circ f)\Delta f + (r'' \circ f) |\nabla f|^2 \tag{18}$$

There is also the **Lichnerowicz Laplacian**, see [2] Appendix A.4.

## 1.7 Computations in normal coordinates

At any given point, we can choose a coordinate system called **normal coordinates** that frequently makes calculations simpler. The idea is that essentially all quantities we are interested in are independent of coordinates, so we only need to prove an identity involving such quantities in a particular coordinate system, and it will hold in general. Thus we are free to choose the simplest coordinate system for the problem, which often turns out to be this one.

We define these coordinates by taking an orthonormal basis $\{e_i\}$ for $T_pM$, and letting $\exp_p^{-1} : U \to B_\epsilon(0)$ be the chart map, where $U \ni p$ and $\epsilon$ are chosen to make this a diffeomorphism. In normal coordinates at $p$, we have the following:

$$g_{ij}(p) = \delta_{ij} \tag{19}$$

$$\Gamma_{ij}^k(p) = 0 \tag{20}$$

$$\partial_k g_{ij}(p) = 0. \tag{21}$$

## 1.8 Cartan's moving frames

See also some exposition in Volume 2 of Spivak and in Chow's Lectures on Differential Geometry.

We use generalized Einstein notation frequently throughout this section. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on an open subset of $M$. Let $\{\omega^i\}_{i=1}^n$ be the dual orthonormal basis for $T^*M$, defined by $\omega^i(e_j) = \delta^i_j$. We define the **connection 1-forms** $\omega^{i,j}$ (corresponding to $\{e_i\}$) to be the components of the Levi-Civita connection with respect to $\{e_i\}$. That is,

$$\nabla_X e_i = \omega^{i,j}(X) e_j.$$

Equivalently, we could define

$$\omega^{i,j} = g(\nabla_{e_k} e_i, e_j)\omega^k, \tag{22}$$

or

$$\omega^{i,j}(X) = g(\nabla_X e_i, e_j). \tag{23}$$

These are antisymmetric in $i$ and $j$, and satisfy the **first Cartan structure equation**:

$$\omega^{i,j} = -\omega^{j,i} \tag{24}$$

$$d\omega^i = \omega^j \wedge \omega^{j,i} = \omega^{i,j} \wedge \omega^j. \tag{25}$$

We also have

$$d\omega^j(e_i, e_j) = \omega^{i,j}(e_j) - \omega^{j,k}(e_i). \tag{26}$$

From this it follows that

$$\omega^{i,k}(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{27}$$

Now we define the **curvature 2-forms** $\Omega^{i,j}$ by

$$\Omega^{i,j}(X, Y)e_j = \frac{1}{2}\operatorname{Rm}(X, Y)e_i.$$

These measure the noncommutativity of taking two covariant derivatives. We could also define these by (TODO: check the constant?)

$$\Omega^{i,j} = \frac{1}{2}\operatorname{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell, \tag{28}$$

so that

$$\Omega^{i,j}(e_k, e_\ell) = \mathrm{Rm}_{ijk\ell}\,. \tag{29}$$

These satisfy the following, called the **second Cartan structure equation**:

$$\Omega^{i,j} = d\omega^{i,j} - \omega^{i,k} \wedge \omega^{k,j} \tag{30}$$

This gives us a way to compute curvatures. For example, on a surface $M^2$, we have

$$d\omega^1 = \omega^2 \wedge \omega^{2,1}, \quad d\omega^2 = \omega^1 \wedge \omega^{1,2}, \quad \Omega^{1,2} = d\omega^{1,2}$$

## 1.9 Proofs

$$X(fY) = X(f) \cdot Y + fX(Y). \tag{1}$$

*Proof.* For $g \in C^\infty(M)$,

$$\begin{aligned} [X(fY)](g) &= X(f \cdot Y(g)) \\ &= X(f) \cdot Y(g) + fX(Y(g)) \\ &= [X(f) \cdot Y + fX(Y)](g). \end{aligned}$$

□

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$$X(r \circ f) = (r' \circ f)X(f) \tag{2}$$

*Proof.* This follows from the chain rule on $\mathbb{R}^n$. First consider the case where $X = \partial_i$. Let $\psi$ be a chart about $p \in M$.

$$\begin{aligned} \partial_i|_p(r \circ f) &\coloneqq \left.\frac{\partial}{\partial x^i}\right|_{\psi(p)} (r \circ f \circ \psi^{-1}) \\ &= r'(f(p)) \cdot \left.\frac{\partial}{\partial x^i}\right|_{\psi(p)} \frac{(f \circ \psi^{-1})}{\partial x^i} \\ &= r'(f(p))\partial_i f. \end{aligned}$$

Then the general case follows by linearity. □

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$$d\omega = \sum_{i,i_1,\ldots,i_k} \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{3}$$

*Proof.* We use first the linearity of $d$ and next its product rule.

$$\begin{aligned} d\omega &= d\left(\sum_{i_1,\ldots,i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\ &= \sum_{i_1,\ldots,i_k} d(\omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= \sum_{i_1,\ldots,i_k} d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + (-1)^k \omega_{i_1\cdots i_k} d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}). \end{aligned}$$

But now since $d^2 = 0$, the second term is 0, and by the expression (9) for the differential, the result follows. □

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{4}$$

*Proof.* By definition,

$$(dx^i \wedge dx^j)(X,Y) = \left(\sum_{\sigma\in S_2} \mathrm{sgn}(\sigma)(dx^i \otimes dx^j)\circ\sigma\right)(X,Y) = X^iY^j - X^jY^i$$

the left hand side is

$$\begin{aligned} d\omega(X,Y) &= \left(\sum_{i,j\le n} \partial_i\omega_j dx^i \wedge dx^j\right)(X,Y) \\ &= \sum_{i,j} \partial_i\omega_j(X^iY^j - X^jY^i) \end{aligned}$$

On the other hand, note that

$$X\omega(Y) = X^i\partial_i(\omega_j dx^j(Y^k\partial_k)) = X^i\partial_i(\omega_jY^j) = X^iY^j\partial_i(\omega_j) + X^i\omega_j\partial_i(Y^j),$$

and

$$\begin{aligned} \omega([X,Y]) &= \omega(X^i\partial_i(Y^j\partial_j) - Y^k\partial_k(X^\ell\partial_\ell)) \\ &= \omega(X^i\partial_i(Y^j)\partial_j - Y^k\partial_k(X^\ell)\partial_\ell) \\ &= X^i\omega_j\partial_i(Y^j) - Y^k\omega_\ell\partial_k(X^\ell). \end{aligned}$$

so the right hand side becomes

$$\begin{aligned} X\omega(Y) - Y\omega(X) - \omega([X,Y]) &= X^iY^j\partial_i(\omega_j) + X^i\omega_j\partial_i(Y^j) - Y^kX^\ell\partial_k(\omega_\ell) - X^k\omega_\ell\partial_k(Y^\ell) \\ &\quad - X^p\omega_q\partial_p(Y^q) + Y^t\omega_s\partial_t(X^s) \\ &= X^iY^j\partial_i(\omega_j) - Y^kX^\ell\partial_k(\omega_\ell). \end{aligned}$$

□

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$$\iota_X(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k}(-1)^{i-1}\omega^i(X)\omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k(X). \tag{7}$$

*Proof.* Note that by property (ii) defining the interior product,

$$\iota_X(\omega^1 \wedge \cdots \wedge \omega^k) = \iota_X(\omega^1) \wedge \omega^2 \wedge \cdots \wedge \omega^k + (-1)\omega^1 \wedge \iota_X(\omega^2 \wedge \cdots \wedge \omega^k),$$

and the result follows by induction (and by property (i), which says $\iota_X(\omega) = \omega(X)$ for 1-forms $\omega$). □

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$$\mathrm{grad}(fh) = f\,\mathrm{grad}\,h + h\,\mathrm{grad}\,f \tag{8}$$

*Proof.* Recall that $\mathrm{grad}\,f$ is defined to be the vector field so that for all vector fields $X$,

$$\langle \mathrm{grad}\,f, X\rangle = X(f).$$

Now

$$\begin{aligned}\langle \operatorname{grad}(fh), X\rangle &= X(fh)\\ &= fX(h) + hX(f)\\ &= f\langle \operatorname{grad} h, X\rangle + h\langle \operatorname{grad} f, X\rangle\\ &= \langle f \operatorname{grad} h + h \operatorname{grad} f, X\rangle .\end{aligned}$$

□

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$$df = (\partial_i f)dx^i \tag{9}$$

*Proof.* This follows immediately, since

$$df(\partial_i) = \partial_i(f).$$

□

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$$\operatorname{grad} f = g^{ij}(\partial_j f)\partial_i \tag{10}$$

*Proof.* Recall that for any vector field $X$,

$$X = dx^i(X)\partial_i = X^i\partial_i.$$

So, writing in coordinates

$$\begin{aligned}g(\operatorname{grad} f, X) &= df(X)\\ g_{ij}dx^i(\operatorname{grad} f)X^j &= (\partial_k f)dx^k(X)\\ g_{ij}dx^i(\operatorname{grad} f)X^j &= (\partial_k f)X^k\\ g_{ij}dx^i(\operatorname{grad} f) &= (\partial_j f)\\ dx^i(\operatorname{grad} f) &= g^{ij}(\partial_j f)\\ \operatorname{grad} f &= g^{ij}(\partial_j f)\partial_i.\end{aligned}$$

□

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(12)

*Proof.* TODO □

(13)

*Proof.* TODO □

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$$\operatorname{div}(fX) = X(f) + f \operatorname{div} X \tag{14}$$

*Proof.* One can prove this using coordinates, but there is a nicer way.

$$\begin{aligned}
\operatorname{div}(fX) &= \operatorname{tr}(\nabla fX) \\
&= \operatorname{tr}(\,\cdot\,(f)X + f\nabla.X) \\
&= \operatorname{tr}(\,\cdot\,(f)X) + f\operatorname{tr}(\nabla.X) \\
&= (\partial_i(f)X)(dx^i) + f\operatorname{div}X \\
&= dx^i(\partial_i(f)X^k\partial_k) + f\operatorname{div}X \\
&= \partial_i(f)X^i + f\operatorname{div}X \\
&= X(f) + f\operatorname{div}X.
\end{aligned}$$

□

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$$\Delta(fh) = f\Delta h + h\Delta f + 2\langle\nabla f, \nabla h\rangle \tag{16}$$

*Proof.*

$$\begin{aligned}
\Delta(fh) &= \operatorname{div}\operatorname{grad}(fh) \\
&= \operatorname{div}(f\operatorname{grad}h + h\operatorname{grad}f) \\
&= \operatorname{div}(f\operatorname{grad}h) + \operatorname{div}(h\operatorname{grad}f) \\
&= (\operatorname{grad}h)(f) + f\operatorname{div}(\operatorname{grad}h) + (f \leftrightarrow h) \\
&= \langle\operatorname{grad}h, \operatorname{grad}f\rangle + f\Delta h + (f \leftrightarrow h) \\
&= f\Delta h + h\Delta f + 2\langle\operatorname{grad}f, \operatorname{grad}h\rangle\,.
\end{aligned}$$

□

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$$\Delta(r\circ f) = (r'\circ f)\Delta f + (r''\circ f)\,|\nabla f|^2 \tag{18}$$

*Proof.* By definition, and using (2) to evaluate terms like $\partial_i(r\circ f)$,

$$\begin{aligned}
\Delta(r\circ f) &= g^{ij}\nabla^2_{ij}(r\circ f) \\
&= g^{ij}(\nabla_i(\nabla_j(r\circ f)) - \nabla_{\nabla_i\partial_j}(r\circ f)) \\
&= g^{ij}(\partial_i\partial_j(r\circ f) - \Gamma^k_{ij}\partial_k(r\circ f)) \\
&= g^{ij}(\partial_i((r'\circ f)\partial_j f) - \Gamma^k_{ij}(r'\circ f)\partial_k f) \\
&= g^{ij}((r''\circ f)\partial_i f\partial_j f + (r'\circ f)\partial_i\partial_j f - (r'\circ f)\Gamma^k_{ij}\partial_k f) \\
&= g^{ij}(r''\circ f)\partial_i f\partial_j f + (r'\circ f)\Delta f \\
&= (r''\circ f)\,|\nabla f|^2 + (r'\circ f)\Delta f.
\end{aligned}$$

□

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$$g_{ij}(p) = \delta_{ij} \tag{19}$$

*Proof.* Recall that $d(\exp_p)_0 = \mathrm{Id}$.

$$\frac{\partial}{\partial x^i} = d(\exp_p)_0\left(\left.\frac{\partial}{\partial e^i}\right|_0\right) = \frac{\partial}{\partial e^i},$$

from which (19) follows. □

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$$\Gamma^k_{ij}(p) = 0 \tag{20}$$

*Proof.* This follows immediately from (19) and the definition of $\Gamma$. □

$$\partial_k g_{ij}(p) = 0 \tag{21}$$

*Proof.* We have

$$\begin{aligned}\partial_k g_{ij} &= \partial_k g(\partial_i, \partial_j) \\ &= g(\partial_k \partial_i, \partial_j) + g(\partial_i, \partial_k \partial_j).\end{aligned}$$

Since $\partial_i$ are coordinate vector fields, $\partial_i \partial_j = 0$, so the proof is done. □

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$$\omega^{i,j} = g(\nabla_{e_k} e_i, e_j)\omega^k \tag{22}$$

*Proof.* The first definition tells us that $\nabla_{\partial_k} e_i = (\omega^{i,j})_k e_j$, and so

$$g(\nabla_{\partial_k}, e_i, e_\ell) = (\omega^{i,\ell})_k.$$

But this is equivalent to the second definition. □

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$$\omega^{i,j}(X) = g(\nabla_X e_i, e_j). \tag{23}$$

*Proof.* TODO □

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$$\omega^{i,j} = -\omega^{j,i} \tag{24}$$

*Proof.*

$$\omega^{i,j} = \Gamma^j_{ki} = -\Gamma^i_{kj} = -\omega^{j,i}.$$ □

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$$d\omega^i = \omega^{i,j} \wedge \omega^j \tag{25}$$

*Proof.* Recall the identity (4):

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Applying this with $\omega = \omega^i$, $X = e_k$, and $Y = e_\ell$, we get

$$\begin{aligned}d\omega(e_k, e_\ell) &= e_k(\omega^i(e_\ell)) - e_\ell(\omega^i(e_k)) - \omega^i([e_k, e_\ell]) \\ &= e_k(\delta^i_\ell) - e_\ell(\delta^i_k) - \omega^i(\nabla_{e_k} e_\ell - \nabla_{e_\ell} e_k) \\ &= -\omega^i(\Gamma^j_{k\ell} e_j - \Gamma^j_{\ell k} e_j) \\ &= -\Gamma^i_{k\ell} + \Gamma^i_{\ell k}.\end{aligned}$$

On the other hand, the right hand side becomes

$$\begin{aligned}(\omega^{i,j} \wedge \omega^j)(e_k, e_\ell) &= \omega^{i,j}(e_k)\omega^j(e_\ell) - \omega^{i,j}(e_\ell)\omega^j(e_k) \\ &= g(\nabla_{e_k} e_i, e_j)\delta^j_\ell - g(\nabla_{e_\ell} e_i, e_j)\delta^j_k \\ &= g(\nabla_{e_k} e_i, e_\ell) - g(\nabla_{e_\ell} e_i, e_k) \\ &= \Gamma^\ell_{ki} - \Gamma^k_{\ell i}.\end{aligned}$$ □

---

$$d\omega^j(e_i, e_j) = \omega^{i,j}(e_j) - \omega^{j,k}(e_i). \tag{26}$$

*Proof.* First observe that since $\nabla_{e_k} e_i = \omega^{i,j}(e_k) e_j$, we have that $\omega^\ell(\nabla_{e_k} e_i) = \omega^{i,\ell}(e_k)$. Using this fact and the definition of the exterior derivative,

$$\begin{aligned} d\omega^k(e_i, e_j) &= e_i \omega^k(e_j) - e_j \omega^k(e_i) - \omega^k([e_i, e_j]) \\ &= -\omega^k(\nabla_{e_i} e_j - \nabla_{e_j} e_i) \\ &= -\omega^{j,k}(e_i) + \omega^{i,k}(e_j). \end{aligned}$$

□

---

$$\omega^{i,k}(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{27}$$

*Proof.* Write out each term of the right hand side using (26), and use antisymmetry to cancel/combine terms. □

---

$$\Omega_{i,j} = -\frac{1}{2} \operatorname{Rm}_{ijk\ell} \omega_k \wedge \omega_\ell \tag{28}$$

*Proof.* TODO □

---

$$\Omega_{i,j}(e_k, e_\ell) = \operatorname{Rm}_{ijk\ell}. \tag{29}$$

*Proof.* Simply note that, since $\operatorname{Rm}_{ijk\ell} = -\operatorname{Rm}_{ij\ell k}$, we have

$$\begin{aligned} \Omega_{i,j}(e_k, e_\ell) &= \frac{1}{2}(\operatorname{Rm}_{ijpq} \omega_p \wedge \omega_q)(e_k, e_\ell) \\ &= \frac{1}{2} \operatorname{Rm}_{ijpq}(\delta_{pk}\delta_{q\ell} - \delta_{p\ell}\delta_{qk}) \\ &= \frac{1}{2} \operatorname{Rm}_{ijk\ell} - \frac{1}{2} \operatorname{Rm}_{ij\ell k} \\ &= \operatorname{Rm}_{ijk\ell}. \end{aligned}$$

□

---

$$\Omega^{i,j} = d\omega^{i,j} - \omega^{i,k} \wedge \omega^{k,j} \tag{30}$$

*Proof.* We give two proofs, respectively using the different definitions of $\Omega^{i,j}$. If we know that $\Omega^{i,j} = -\frac{1}{2} \operatorname{Rm}_{ijk\ell} \omega_k \wedge \omega_\ell$, we proceed as follows. From (22), we have that $\omega_{i,j}(e_k) = g(\nabla_{e_k} e_i, e_j)$. By taking the exterior derivative of both sides (thinking of the right hand side as the covector $X \mapsto g(\nabla_X e_i, e_j)$), and using (4), we have

$$\begin{aligned} d\omega^{i,j}(e_k, e_\ell) &= e_k \omega^{i,j}(e_\ell) - e_\ell \omega^{i,j}(e_k) - \omega^{i,j}([e_\ell, e_k]) \\ &= dg(\nabla e_i, e_j)(e_k, e_\ell) \\ &= e_k g(\nabla_{e_\ell} e_i, e_j) - e_\ell g(\nabla_{e_k} e_i, e_j) - g(\nabla_{[e_\ell, e_k]} e_i, e_j) \\ &= g(\nabla_{e_k} \nabla_{e_\ell} e_i, e_j) + g(\nabla_{e_\ell} e_i, \nabla_{e_k} e_j) - g(\nabla_{e_\ell} \nabla_{e_k} e_i, e_j) \\ &\quad - g(\nabla_{e_k} e_i, \nabla_{e_\ell} e_j) - g(\nabla_{[e_\ell, e_k]} e_i, e_j). \end{aligned}$$

Observing that $\omega^{i,p}(e_\ell)\omega^{j,p}(e_k) = g(\nabla_{e_\ell} e_i, e_p) g(\nabla_{e_k} e_j, e_p) = g(\nabla_{e_\ell} e_i, \nabla_{e_k} e_j)$, we continue

$$\begin{aligned} &= g(\nabla_{e_k} \nabla_{e_\ell} e_i - \nabla_{e_\ell} \nabla_{e_k} e_i - \nabla_{[e_\ell, e_k]} e_i, e_j) + \omega^{i,p}(e_\ell)\omega^{j,p}(e_k) - \omega^{i,p}(e_k)\omega^{j,p}(e_\ell) \\ &= \operatorname{Rm}_{k\ell ij} + (\omega^{i,p} \wedge \omega^{j,p})(e_\ell, e_k) \\ &= \operatorname{Rm}_{ijk\ell} + (\omega^{i,p} \wedge \omega^{p,j})(e_k, e_\ell) \\ &= (\Omega^{i,j} + \omega^{i,p} \wedge \omega^{p,j})(e_k, e_\ell) \end{aligned}$$

TODO: the other version □

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# 2 Tensors

## 2.1 Type changing with the metric

An $(k, 0)$-tensor is called **covariant**, and a $(0, \ell)$-tensor is called **contravariant**. For example, forms are covariant, and vectors are contravariant. The terminology relates to how the components change under a change of basis. If we scale some basis vectors by a factor of $C$, then the components of a vector with respect to that basis scale by a factor of $C^{-1}$; hence contravariant.

An $(s, t)$-tensor $T$ is a section of $(TM)^{\otimes t} \otimes (T^*M)^{\otimes s}$. That is, it is a product of $t$ vectors and $s$ covectors, meaning that it takes $t$ covectors and $s$ vectors as input, so it has $s$ lower (covariant, I think) indices, and $t$ upper (contravariant, I think) indices. To add to the confusion, recall that we can think of a $(1, 1)$-tensor either as an object that takes a vector and a covector and returns a scalar, or as an object that takes a vector and returns a vector. This generalizes; a $(k, \ell)$-tensor can also be thought of as an object that takes $k$ vectors and returns (a tensor product of) $\ell$ vectors.

For any $|k| \leq \min\{s, t\}$, we can make $T$ into a $(s-k, t+k)$-tensor by using the natural isomorphism (provided by the Riemannian metric) between $TM$ and $T^*M$ given by $v \mapsto g(v, \cdot) \in T^*M$. So in the tensor product above, we can replace $TM$'s by $T^*M$'s arbitrarily, and thereby get any sort of tensor we want with rank $s + t$.

In coordinates, we can write (given a frame $E_i$ and the coframe $\xi^i$),

$$T^{i_1 \cdots i_t}{}_{j_1 \cdots j_s} E_{i_1} \otimes \cdots \otimes E_{i_t} \otimes \xi^{j_1} \otimes \cdots \otimes \xi^{j_s}.$$

TODO: check/fix the s and t's in the section.

Then to make $T$ a $(s+1, t-1)$-tensor, replace some $E_{i_k}$ by $g_{i_k j}\xi^j$ to get

$$T^{i_1 \cdots i_{k-1}}{}_j{}^{i_{k+1} \cdots i_s}{}_{j_1 \cdots j_t} E_{i_1} \otimes \cdots \otimes \xi^j \otimes \cdots \otimes E_{i_s} \otimes \xi^{j_1} \otimes \cdots \otimes \xi^{j_t},$$

where

$$T^{i_1 \cdots i_{k-1}}{}_j{}^{i_{k+1} \cdots i_s}{}_{j_1 \cdots j_t} \coloneqq g_{i_k j} T^{i_1 \cdots i_s}{}_{j_1 \dots j_t}$$

## 2.2 Contractions and traces

TODO: this section needs help; see Andrews-Hopper p. 22 and Lee p. 395.

Given a $(k, \ell)$-tensor $T$, where $k, \ell \geq 1$, we can form various $(k-1, \ell-1)$-tensors (traces of $T$) by evaluating one of the covector field factors of $T$ at one of the vector field factors. Specifically, there are $k\ell$ different traces we can take, since we can evaluate any of the covector fields at any of the vector fields. In the case where $T$ is a $(1, 1)$-tensor,

$$\operatorname{tr}(T) = \operatorname{tr}(T^i{}_j \, E_i \otimes \xi^j) = T^i{}_j \, \xi^j(E_i) = T^i{}_j \, \delta^j_i = T^i{}_i \, .$$

More generally, if $T$ is a $(k, \ell)$-tensor, and we evaluate the $a^{\text{th}}$ factor of $T$ at the $b^{\text{th}}$ factor of $T$, we have, for vector fields $X_1, \dots, X_{\ell-1}$, and covector fields $\omega_1, \dots, \omega_{k-1}$,

$$\begin{aligned}
&(\operatorname{tr}_{ab} T)(\omega_1, \dots, \omega_{k-1}, X_1, \dots, X_{\ell-1}) \\
&= \operatorname{tr}[(\omega, X) \mapsto T(\omega_1, \dots, \omega_{a-1}, \omega, \omega_{a+1}, \dots, \omega_{k-1}, X_1, \dots, X_{b-k-1}, X, X_{b-k+1}, \dots, X_{\ell+1})]
\end{aligned}$$

where on the right hand side we are now just taking the trace over a $(1, 1)$ tensor again. In coordinates, this is just

$$\operatorname{tr}_{ab} T = T^{i_1 \cdots k \cdots i_k}{}_{j_1 \cdots k \cdots j_\ell} \partial_{i_1} \cdots \partial_{i_{a-1}} \partial_{i_{a+1}} \cdots \partial_{i_k} dx^{j_1} \cdots dx^{j_{b-k-1}} dx^{j_{b-k+1}} \cdots dx^{j_\ell}.$$

Using the isomorphism induced by $g$ between $TM$ and $T^*M$ (see the previous section), we can TODO

# 3 Lie Derivatives

Let $X, Y$ be vector fields, and let $\Psi_{X,t}$ be the flow of $X$, so that $D\Psi_{X,t}|_x$ is an isomorphism between $T_xM$ and $T_{\Psi_{X,t}(x)}M$. (Note that in this case the pullback is the inverse of the differential, so it does not matter if we use the pullback or the inverse of the pushforward.) So $(D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})$ is an element of $T_xM$ for each $t$, so we can differentiate this at $t = 0$. With these remarks in mind, we define the **Lie derivative** of $Y$ along the flow of $X$ by

$$\mathcal{L}_X Y|_x = \frac{d}{dt}\bigg|_{t=0} \left((D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})\right).$$

Define the **Lie bracket** of vector fields by

$$[X, Y](f) := X(Y(f)) - Y(X(f)),$$

then

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X. \tag{31}$$

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X. \tag{31}$$

*Proof.* Consider the action of $\mathcal{L}_X Y$ on a smooth function $f$. Note that by the chain rule and the fact that $\mathrm{Id}_{T_xM} = \Psi_{-X,t} \circ \Psi_{X,t}$

$$D\Psi_{-X,t}|_{\Psi_{X,t}(x)} \circ D\Psi_{X,t}|_x = \mathrm{Id}_{T_xM},$$

so $(D\Psi_{X,t}|_x)^{-1} = D\Psi_{-X,t}|_{\Psi_{X,t}(x)}$. By definition of the derivative,

$$(D\Psi_{-X,t}|_x(Y))f = Y|_{\Psi_{X,t}(x)}(f \circ \Psi_{-X,t}).$$

Then

$$\begin{aligned}\mathcal{L}_X Y|_x(f) &= \frac{d}{dt}\left((D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})\right)\big|_{t=0} \\ &= \frac{d}{dt}\left(Y|_{\Psi_{X,t}(x)}(f \circ \Psi_{-X,t})\right)\end{aligned}$$

TODO ∎

---

For vector fields $V, X_1, \ldots, X_k$, and a tensor field $A$ in $\mathcal{T}_0^k(M)$,

$$(\mathcal{L}_V A(X_1, \ldots, X_k)) = (\mathcal{L}_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, \mathcal{L}_V X_i, \ldots, X_k) \tag{32}$$

$$(\mathcal{L}_V A)(X_1, \ldots, X_k) = V(A(X_1, \ldots, X_k)) - \sum_{i=1}^{k} A(X_1, \ldots, [V, X_i], \ldots, X_k) \tag{33}$$

**Cartan's formula** states that for any differential form $\omega$ and any (smooth) vector field $V$,

$$\mathcal{L}_V \omega = \iota_V(d\omega) + d(\iota_V \omega). \tag{34}$$

$$\mathcal{L}_V \omega = \iota_V(d\omega) + d(\iota_V \omega). \tag{34}$$

*Proof.* TODO: finish

Using the definitions of $\mathcal{L}$, $d$, and $\iota$, we calculate each term:

$$\begin{aligned}(\mathcal{L}_V\omega)(X_1,\ldots,X_k) &= V(\omega(X_1,\ldots,X_k)) - \sum_{i=1}^{k}\omega(X_1,\ldots,[V,X_i],\ldots,X_k)\\ &\quad+\sum_{i=1}^{k}\omega(X_1,\ldots,\mathcal{L}_V X_i,\ldots,X_k),\\ (\iota_V(d\omega))(X_1,\ldots,X_k) &= \end{aligned}$$

□

We have the following product rule for a Lie derivative of a wedge product

$$\mathcal{L}_V(\alpha\wedge\beta) = (\mathcal{L}_V\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_V\beta). \tag{35}$$

$$\mathcal{L}_V(\alpha\wedge\beta) = (\mathcal{L}_V\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_V\beta). \tag{35}$$

*Proof.* TODO □

There is also the following identity for the interior product. For a differential form $\omega$ and vector fields $V, W$,

$$\mathcal{L}_W(\iota_V\omega) = \iota_V(\mathcal{L}_W\omega) + \iota_{[W,V]}\omega. \tag{36}$$

$$\mathcal{L}_W(\iota_V\omega) = \iota_V(\mathcal{L}_W\omega) + \iota_{[W,V]}\omega. \tag{36}$$

*Proof.* TODO □

# 4 Levi-Civita connection

TODO: this section should be rewritten; the stuff about covariant derivatives applies to more general connections on vector bundles over a manifold. See 2.5 of Andrews-Hopper.

Let $E$ be a vector bundle over $M$. The Levi-Civita connection for a given Riemannian metric $g$ is a map $\nabla : \Gamma(TM)\times\Gamma(E)\to\Gamma(E)$ (written as $(X,\xi)\mapsto\nabla_X\xi$) that satisfies the following properties. Note that the first three are defining properties of a general connection, and the last two properties make $\nabla$ the Levi-Civita connection with respect to $g$. For $X, Y\in\Gamma(TM)$, $f\in C^\infty(M)$, $r\in\mathbb{R}$, and $\xi\in\Gamma(E)$,

(1) $C^\infty$ linearity in $X$:
$$\nabla_{X+fY}\xi = \nabla_X\xi + f\nabla_Y\xi.$$

(2) $\mathbb{R}$-linearity in $\xi$:
$$\nabla_X(r\xi) = r\nabla_X\xi.$$

(3) A product/Leibniz rule in $\xi$:
$$\nabla_X(f\xi) = X(f)\cdot\xi + f\nabla_X\xi.$$

(4) Metric compatibility:
$$X(g(\xi,\eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta),$$
which can also be stated as
$$\nabla g = 0,$$
where the left hand side (and the right hand side) is a tensor field in $\mathcal{T}^3_0(M)$. The proof that these conditions are equivalent follows from the definition of $\nabla g$ below.

(5) Torsion-free (also known as symmetry):
$$\nabla_X Y - \nabla_Y X = [X,Y]$$

It satisfies **Koszul's formula**:
$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle - Z\langle X,Y\rangle + Y\langle Z,X\rangle - \langle [Y,Z],X\rangle + \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle . \tag{37}$$

The metric compatibility condition tells us that
$$\nabla(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y), \tag{38}$$
where we interpret the right hand side as the covector $Z \mapsto g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

## 4.1 Christoffel symbols

Given some coordinate basis $\{\partial_i\}_{i=1}^n$, the **Christoffel symbols** (of the Levi-Civita connection) are the unique coefficients, (i.e. smooth functions) satisfying
$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k.$$
It follows from this and the above properties that
$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma^k_{ij})\partial_k. \tag{39}$$
In particular,
$$\nabla_i X = (\partial_i X^\ell + X^j \Gamma^\ell_{ij})\partial_\ell \tag{40}$$
For the Levi-Civita connection, we can calculate these coefficients in coordinates by
$$\Gamma^k_{ji} = \Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}). \tag{41}$$
The symbols also satisfy the antisymmetry relation
$$\Gamma^k_{ij} = -\Gamma^j_{ik}. \tag{42}$$
Despite using the same notation, we cannot think of the Christoffel symbols as a $(1,2)$-tensor. However, given two metrics $g, \widetilde{g}$ the difference of the coefficients of the two corresponding Levi-Civita connections does form a tensor:
$$\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}$$

## 4.2 Covariant derivatives

We can take directional derivatives of functions using only the differentiable structure of a manifold. The covariant derivative is defined using the metric, and allows us to differentiate vector fields and other tensors. If $F \in \mathcal{T}^k_\ell(M)$ is a tensor field, and $X, Y_k$ are vector fields and $\omega^j$ are 1-forms, then

$$\begin{aligned}(\nabla_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) = X(F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k)) \\ - \sum_{j=1}^{\ell} F(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^\ell, Y_1, \ldots, Y_k) \\ - \sum_{i=1}^{k} F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k).\end{aligned}$$

We can think of $\nabla F$ as a $(k+1, \ell)$-tensor field, called the **total covariant derivative** of $F$, by

$$(\nabla F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k, X) = (\nabla_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k).$$

There are different conventions about where to put the $X$ in this definition; I'm not sure if it matters. An important property of covariant derivatives is that they "commute with contractions," a property that follows from the fact that $\nabla g \equiv 0$. TODO

There is also a horrible expression for the covariant derivative in coordinates TODO

The following formula for commuting covariant derivatives at the expense of introducing a Riemann curvature term is quite useful, although it's probably better to just look at the more common special cases below.

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha^{\ell_1 \ldots \ell_s}_{k_1 \ldots k_r} = - \sum_{h=1}^{r} \mathrm{Rm}^p_{ijk_h} \, \alpha^{\ell_1 \ldots \ell_s}_{k_1 \ldots k_{h-1} p k_{h+1} \ldots k_r} - \sum_{h=1}^{s} \mathrm{Rm}^{\ell_h}_{ijp} \, \alpha^{\ell_1 \ldots \ell_{h-1} p \ell_{h+1} \ldots \ell_s}_{k_1 \ldots k_r} \tag{43}$$

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha^{\ell_1 \ldots \ell_s}_{k_1 \ldots k_r} = - \sum_{\ell=1}^{r} \mathrm{Rm}^p_{ijk_h} \, \alpha^{\ell_1 \ldots \ell_s}_{k_1 \ldots k_{h-1} p k_{h+1} \ldots k_r} - \sum_{h=1}^{s} \mathrm{Rm}^{\ell_h}_{ijp} \, \alpha^{\ell_1 \ldots \ell_{h-1} p \ell_{h+1} \ldots \ell_s}_{k_1 \ldots k_r}. \tag{43}$$

*Proof.* TODO □

For example, if $\omega$ is a 1-form,

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \omega_k = - \mathrm{Rm}^\ell_{ijk} \, \omega_\ell. \tag{44}$$

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \omega_k = - \mathrm{Rm}^\ell_{ijk} \, \omega_\ell. \tag{44}$$

*Proof.* This follows immediately from (43), but I guess there should be a direct way to prove it as well.

TODO □

## 4.3 The Hessian

We can then take the total covariant derivative of $F$ to get the **Hessian** of $F$, sometimes denoted $\nabla^2 F$, which is of course a $(k+2, \ell)$-tensor field.

It follows from the torsion-free property of the Levi-Civita connection that the Hessian is symmetric:

$$(\nabla^2 f)(X, Y) = (\nabla^2 f)(Y, X). \tag{45}$$

We have

$$\nabla^2_{X,Y} F \coloneqq (\nabla^2 F)(X,Y) = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F. \tag{46}$$

The proof makes it more clear how the tensors on the right hand side actually work. In the case of a function $f$, we have that (the first equality follows immediately from (46))

$$\begin{aligned} (\nabla^2 f)(X,Y) &= X(Y(f)) - (\nabla_X Y)(f) \\ &= g(\nabla_X \operatorname{grad} f, Y) \\ &= \frac{1}{2}(\mathcal{L}_{\operatorname{grad} f} g)(X,Y). \end{aligned} \tag{47}$$

In coordinates, we can write

$$\nabla_i \nabla_j f = \partial_i(\partial_j f) - \Gamma^k_{ij}\partial_k f. \tag{48}$$

In particular, since $(\nabla^2 f)(X,Y) = g(\nabla_X \operatorname{grad} f, Y)$, the $(1,1)$-tensor associated to $\nabla^2$ is given by $(\nabla^2 f)(X) = \nabla_X \operatorname{grad} f$. The Hessian satisfies the following product rule for functions.

$$\nabla^2(fh) = f\nabla^2 h + h\nabla^2 f + \nabla f \otimes \nabla h + \nabla h \otimes \nabla f. \tag{49}$$

#### 4.3.1 Bochner Formulas

For any function $u$ on a Riemannian manifold,

$$\Delta \left|\nabla u\right|^2 = 2\left\langle \Delta\nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2. \tag{50}$$

We also have the formula for the commutator of the Laplacian and the covariant derivative:

$$\Delta(du) = d(\Delta u) + \operatorname{Rc}(\nabla u), \tag{51}$$

sometimes also written as $\Delta\nabla u = \nabla\Delta u + \operatorname{Rc}(\nabla u)$.

$$\Delta(du) = d(\Delta u) + \operatorname{Rc}(\nabla u), \tag{51}$$

*Proof.* By (44),

$$\nabla_i\nabla_j\nabla_k u = \nabla_j\nabla_i\nabla_k u - \operatorname{Rm}_{ijk\ell}\nabla_\ell u.$$

Note that on the left-hand side we can commute $\nabla_j$ and $\nabla_k$. From this equation,

$$\begin{aligned} g^{ik}\nabla_i\nabla_k\nabla_j u &= g^{ik}(\nabla_j\nabla_i\nabla_k u + \operatorname{Rm}_{jik\ell}\nabla_\ell u) \\ \Delta\nabla_j u &= \nabla_j \Delta u + \operatorname{Rc}_{j\ell}\nabla_\ell u. \end{aligned}$$

□

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# Proofs

$$2\left\langle \nabla_X Y, Z\right\rangle = X\left\langle Y,Z\right\rangle - Z\left\langle X,Y\right\rangle + Y\left\langle Z,X\right\rangle - \left\langle [Y,Z],X\right\rangle + \left\langle [X,Y],Z\right\rangle - \left\langle [X,Z],Y\right\rangle. \tag{37}$$

*Proof.* The metric compatibility condition says

$$\begin{aligned} X\left\langle Y,Z\right\rangle &= \left\langle \nabla_X Y, Z\right\rangle + \left\langle Y, \nabla_X Z\right\rangle \\ Y\left\langle Z,X\right\rangle &= \left\langle \nabla_Y Z, X\right\rangle + \left\langle Z, \nabla_Y X\right\rangle \\ Z\left\langle X,Y\right\rangle &= \left\langle \nabla_Z X, Y\right\rangle + \left\langle X, \nabla_Z Y\right\rangle. \end{aligned}$$

By adding/subtracting these expressions, using symmetry and linearity of the metric, and the torsion-free property ($\nabla_X Y - \nabla_Y X = [X, Y]$), we obtain

$$X \langle Y, Z\rangle + Y \langle Z, X\rangle - Z \langle X, Y\rangle = 2 \langle \nabla_X Y, Z\rangle - \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle .$$

$\square$

---

$$\nabla(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y). \tag{38}$$

*Proof.* This really does follow immediately from the metric compatibility condition. $\square$

---

$$\Gamma^k_{ji} = \Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \tag{41}$$

*Proof.* Apply the Koszul formula (37) to coordinate basis vectors:

$$\begin{aligned}
2\Gamma^\ell_{ij} g_{\ell k} &= 2 \left\langle \Gamma^\ell_{ij} \partial_\ell, \partial_k \right\rangle \\
&= 2 \langle \nabla_{\partial_i} \partial_j, \partial_k\rangle \\
&= \partial_i \langle \partial_j, \partial_k\rangle - \partial_k \langle \partial_i, \partial_j\rangle + \partial_j \langle \partial_k, \partial_i\rangle \\
&= \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}.
\end{aligned}$$

Now multiply both sides by $g^{km}$ and the result follows. $\square$

---

$$\Gamma^k_{ij} = -\Gamma^j_{ik} \tag{42}$$

*Proof.* The metric compatibility condition applied to the correct basis vectors says

$$\begin{aligned}
0 &= (\nabla g)(\partial_j, \partial_k, \partial_i) \\
&= \nabla_{\partial_i} g(\partial_j, \partial_k) \\
&= g(\nabla_i e_j, e_k) + g(\nabla_i e_k, e_j) \\
&= \Gamma^k_{ij} + \Gamma^j_{ik}.
\end{aligned}$$

$\square$

---

$$(\nabla^2 f)(X, Y) = (\nabla^2 f)(Y, X). \tag{45}$$

*Proof.* The torsion free property of the L-C connection says that

$$(\nabla_X Y)(f) - (\nabla_Y X)(f) = [X, Y](f) \coloneqq X(Y(f)) - Y(X(f)),$$

and by rearranging this we get that

$$(X(Y(f)) - (\nabla_X Y)(f) = Y(X(f)) - (\nabla_Y X)(f),$$

which is exactly the desired equality. $\square$

---

$$\nabla^2_{X,Y} F \coloneqq (\nabla^2 F)(X, Y) = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F. \tag{46}$$

*Proof.* According to the general definition of covariant derivative (and the definition of total covariant derivative) above,

$$\begin{aligned}(\nabla(\nabla F))(Y,X) &= (\nabla_X(\nabla F))(Y)\\ &= \nabla_X[(\nabla F)(Y)] - \nabla F(\nabla_X Y)\\ &= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.\end{aligned}$$

To make more explicit what is actually going on here, we can write, supposing $F$ is a $(k,\ell)$-tensor,

$$\begin{aligned}\nabla^2 F(X,Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) &= \nabla_X(\nabla F)(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k)\\ &= X(\nabla F(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k))\\ &\quad - (\nabla F)(\nabla_X Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k)\\ &\quad - \sum_{i=1}^{\ell}(\nabla F)(Y,\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^\ell,W_1,\dots,W_k)\\ &\quad - \sum_{i=1}^{k}(\nabla F)(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,\nabla_X W_i,\dots,W_k)\\ &= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.\end{aligned}$$

There is also a proof on page 99 of Lee-RM. $\square$

---

$$(\nabla^2 f)(X,Y) = X(Y(f)) - (\nabla_X Y)(f) = g(\nabla_X \operatorname{grad} f, Y) = \frac{1}{2}(\mathcal{L}_{\operatorname{grad} f}\, g)(X,Y) \tag{47}$$

*Proof.* The second equality:

$$\begin{aligned}X(Y(f)) - (\nabla_X Y)(f) &= X(g(\operatorname{grad} f, Y)) - g(\operatorname{grad} f, \nabla_X Y)\\ &= g(\nabla_X \operatorname{grad} f, Y) + g(\operatorname{grad} f, \nabla_X Y) - g(\operatorname{grad} f, \nabla_X Y)\\ &= g(\nabla_X \operatorname{grad} f, Y).\end{aligned}$$

The last equality: using (**??**) for the Lie derivative of the metric, metric compatibility, (31), and denoting $\operatorname{grad} f$ by $\nabla f$, we calculate

$$\begin{aligned}(\mathcal{L}_{\nabla f}\, g)(X,Y) &= (\nabla f)(g(X,Y)) - g([\nabla f, X], Y) - g(X, [\nabla f, Y])\\ &= g(\nabla_{\nabla f} X, Y) + g(X, \nabla_{\nabla f} Y) - g(\nabla_{\nabla f} X - \nabla_X(\nabla f), Y) - g(X, \nabla_{\nabla f} Y - \nabla_Y(\nabla f))\\ &= g(\nabla_X(\nabla f), Y) + g(X, \nabla_Y(\nabla f))\\ &= X(g(\nabla f, Y)) - g(\nabla f, \nabla_X Y) + Y(g(X, \nabla f)) - g(\nabla_Y X, \nabla f)\\ &= (\nabla^2 f)(X,Y) + (\nabla^2 f)(Y,X)\\ &= 2(\nabla^2 f)(X,Y).\end{aligned}$$

$\square$

---

$$\nabla_i \nabla_j f = \partial_i(\partial_j f) - \Gamma^k_{ij}\partial_k f. \tag{48}$$

*Proof.* Recalling that, by definition $\nabla_i \partial_j = \Gamma^k_{ij}\partial_k$,

$$\begin{aligned}\nabla_i \nabla_j f &= \nabla_i(\nabla_j f) - \nabla_{\nabla_i \partial_j} f \\ &= \partial_i \partial_j f - \nabla_{\Gamma^k_{ij}\partial_k} f \\ &= \partial_i \partial_j f - \Gamma^k_{ij} \nabla_k f \\ &= \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f\end{aligned}$$

□

---

$$\nabla^2(fh) = f\nabla^2 h + h\nabla^2 f + \nabla f \otimes \nabla h + \nabla h \otimes \nabla f \tag{49}$$

*Proof.* First note that we are using $\nabla f$ to denote the gradient of $f$. Then

$$\begin{aligned}\nabla^2(hf)(X,Y) &= \nabla_X(\nabla_Y(fh)) - \nabla_{\nabla_X Y}(fh) \\ &= \nabla_X(fY(h) + hY(f)) - (\nabla_X Y)(fh) \\ &= X(f)Y(h) + fX(Y(h)) + X(h)Y(f) + hX(Y(f)) - h(\nabla_X Y)(f) - f(\nabla_X Y)(h) \\ &= f(X(Y(h)) - (\nabla_X Y)(h)) + h(X(Y(f)) - (\nabla_X Y)(f)) \\ &\quad + (\nabla f \otimes \nabla h)(X,Y) + (\nabla h \otimes \nabla f)(X,Y) \\ &= f(\nabla^2 h)(X,Y) + h(\nabla^2 f)(X,Y) + (\nabla f \otimes \nabla h)(X,Y) + (\nabla h \otimes \nabla f)(X,Y).\end{aligned}$$

□

---

$$\Delta \left|\nabla u\right|^2 = 2\left\langle \Delta \nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2. \tag{50}$$

*Proof.* The coordinate-free way to do this goes as follows. I use dots to keep track of the entries over which the trace is taken.

$$\begin{aligned}\Delta \left\langle \nabla u, \nabla u\right\rangle &= \operatorname{tr} \nabla^2 \left\langle \nabla u, \nabla u\right\rangle \\ &= \operatorname{tr}(\nabla.(2\left\langle \nabla.\nabla u, \nabla u\right\rangle)) \\ &= 2\operatorname{tr}(\left\langle \nabla.\nabla.\nabla u, \nabla u\right\rangle + \left\langle \nabla.\nabla u, \nabla.\nabla u\right\rangle) \\ &= 2\left\langle \Delta \nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2,\end{aligned}$$

as desired.

In normal coordinates, we can calculate

$$\begin{aligned}\Delta \left|\nabla u\right|^2 &= \Delta(\nabla_i u \nabla_i u) \\ &= \nabla_j \nabla_j(\nabla_i u \nabla_i u)) \\ &= 2\nabla_j \nabla_j \nabla_i u \nabla_i u + 2\nabla_j \nabla_i u \nabla_j \nabla_i u \\ &= 2\left\langle \Delta \nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2.\end{aligned}$$

□

# 5 Curvature

## 5.1 Curvature of a connection on a vector bundle

Reference: Andrews-Hopper Section 2.7.1. If $\nabla$ is a connection on a vector bundle $E$ over $M$, the curvature of $\nabla$ on $E$ is the section $R_\nabla \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ defined by

$$R_\nabla(X,Y)\xi = \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X,Y]}\xi.$$

## 5.2 Riemann curvature

Riemann curvature is the special case of the previous construction where the connection is the Levi-Civita connection on the tangent bundle over $M$. In particular, the $(3,1)$-tensor (field) version of the Riemann curvature tensor is a $C^\infty(M)$-multilinear map $\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ defined by

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &= (\nabla^2)(X,Y,Z) - (\nabla^2)(Y,X,Z) \\ &= \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z. \end{aligned}$$

In coordinates, we can write

$$R = R_{ijk}{}^{\ell} dx^i \otimes dx^j \otimes dx^k \otimes \partial_\ell,$$

so that

$$R(X,Y)Z = R_{ijk}{}^{\ell} X^i Y^j Z^k \partial_\ell.$$

where

$$R_{ijk}{}^{\ell}\partial_\ell = R(\partial_i,\partial_j)\partial_k.$$

We can get a $(4,0)$-tensor version of $R$ by defining

$$R_{ijkl} = R(\partial_i,\partial_j,\partial_k,\partial_\ell) := \langle R(\partial_i,\partial_j)\partial_k,\partial_\ell\rangle.$$

Then $R_{ijk\ell} = g_{\ell m}R_{ijk}{}^{m}$. This tensor satisfies the symmetries

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \tag{52}$$

and the 1st and 2nd Bianchi identities:

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{53}$$

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{54}$$

The once contracted 2nd Bianchi identity:

$$g^{im}\nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{55}$$

We can calculate the coefficients in terms of the Christoffel symbols as well:

$$R_{ijk}{}^{\ell} = \partial_i\Gamma^{\ell}_{jk} - \partial_j\Gamma^{\ell}_{ik} + \Gamma^{m}_{jk}\Gamma^{\ell}_{im} - \Gamma^{m}_{ik}\Gamma^{\ell}_{jm} \tag{56}$$

$$R_{ijk\ell} = \frac{1}{2}(\partial_j\partial_k g_{i\ell} + \partial_i\partial_\ell g_{jk} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_\ell g_{ik}) + g_{\ell p}(\Gamma^{m}_{ik}\Gamma^{p}_{jm} - \Gamma^{m}_{jk}\Gamma^{p}_{im}). \tag{57}$$

In calculations, we frequently get Riemann curvature terms appearing from commuting covariant derivatives, following from rearranging the formula that defines the Riemann tensor. See (43).

## 5.3 Ricci curvature

The Ricci tensor, denoted Rc or $R$, is defined to be the trace of the Riemann tensor:

$$\mathrm{Rc}(Y,Z) := \mathrm{tr}(X \mapsto R(X,Y)Z),$$

or in coordinates

$$R_{ij} = R_{kij}{}^{k} = g^{km} R_{kijm}.$$

The Ricci tensor satisfies the twice contracted second Bianchi identity:

$$2g^{ij} \nabla_i \operatorname{Rc}_{jk} = \nabla_k R. \tag{58}$$

The Ricci tensor can be expressed in terms of the metric:

$$-2\operatorname{Rc}_{ij} = g^{k\ell}(\partial_k \partial_\ell g_{ij} + \partial_i \partial_j g_{k\ell} - \partial_i \partial_k g_{j\ell} - \partial_j \partial_k g_{i\ell}) + \text{lower order terms}, \tag{59}$$

where the lower order terms involve only one derivative of $g$. The Ricci tensor is invariant under diffeomorphisms; that is, if $\phi$ is a diffeomorphism of $M$, then

$$\operatorname{Rc}_{\phi^* g} = \phi^* \operatorname{Rc}_g .$$

### 5.4 Scalar curvature

The scalar curvature is defined to be the trace (with respect to the metric) of the Ricci curvature:

$$R = \operatorname{tr}_g \operatorname{Rc} = \operatorname{Rc}_i{}^{i} = g^{ij} \operatorname{Rc}_{ij} .$$

# Proofs

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \tag{52}$$

*Proof.* Using the fact that the Lie bracket is antisymmetric, it is clear from the definition that $R(X,Y)Z = -R(Y,X)Z$, from which the equality $R_{ijk\ell} = -R_{jik\ell}$ follows. To show the third equality, we show that $R(X,Y,Z,Z) = 0$ for any $Z$. First note that, by metric compatibility,

$$\begin{aligned} X(Y(|W|^2)) &= X(Y\langle W, W\rangle) \\ &= X(2\langle \nabla_Y W, W\rangle) \\ &= 2\langle \nabla_X \nabla_Y W, W\rangle + 2\langle \nabla_Y W, \nabla_X W\rangle . \end{aligned}$$

Similarly,

$$\begin{aligned} Y(X(|W|^2)) &= 2\langle \nabla_Y \nabla_X W, W\rangle + 2\langle \nabla_X W, \nabla_Y W\rangle , \\ [X,Y]\,|W|^2 &= 2\left\langle \nabla_{[X,Y]} W, W\right\rangle . \end{aligned}$$

Now

$$\begin{aligned} 0 &= X(Y(|W|^2)) - Y(X(|W|^2)) - [X,Y]\,|W|^2 \\ &= 2\langle \nabla_X \nabla_Y W, W\rangle - 2\langle \nabla_Y \nabla_X W, W\rangle - 2\left\langle \nabla_{[X,Y]} W, W\right\rangle \\ &= 2\langle R(X,Y)W, W\rangle \\ &= R(X,Y,W,W). \end{aligned}$$

Applying this,

$$\begin{aligned} 0 &= \langle R(\partial_i, \partial_j)\partial_k + \partial_\ell, \partial_k + \partial_\ell\rangle \\ &= R_{ijkk} + R_{ijk\ell} + R_{ij\ell k} + R_{ij\ell\ell} \\ &= R_{ijkl} + R_{ij\ell k}. \end{aligned}$$

To prove the last equality, we use the first (algebraic) Bianchi identity. ☐

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{53}$$

*Proof.* This will follow from

$$R(X,Y)Z + R(Z,Y)X + R(Y,X) = 0.$$

Expand using the definition of $R$, and then apply symmetry of the connection:

$$\begin{aligned} 0 &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \\ &\quad + \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X - \nabla_{[Z,Y]} X + \\ &\quad + \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \\ &= \nabla_X (\nabla_Y Z - \end{aligned}$$

□

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{54}$$

*Proof.* □

$$2g^{ij} \nabla_i R_{jk} = \nabla_k R. \tag{58}$$

*Proof.* Start with the 2nd Bianchi identity, contract twice, and apply some symmetries of the Riemann tensor:

$$\begin{aligned} g^{im} g^{j\ell} (\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}) &= 0 \\ g^{im} \nabla_i g^{j\ell} R_{jk\ell m} + g^{j\ell} \nabla_j g^{im} R_{ki\ell m} + g^{im} \nabla_k g^{j\ell} R_{ij\ell m} &= 0 \\ -g^{im} \nabla_i g^{j\ell} R_{jkm\ell} - g^{j\ell} \nabla_j g^{im} R_{ik\ell m} + g^{im} \nabla_k g^{j\ell} R_{jim\ell} &= 0 \\ -g^{im} \nabla_i R_{km} - g^{j\ell} \nabla_j R_{k\ell} + \nabla_k g^{im} R_{im} &= 0 \\ -2g^{ij} \nabla_i R_{jk} + \nabla_k R &= 0. \end{aligned}$$

□

$$-2R_{ij} = g^{k\ell}(\partial_k \partial_\ell g_{ij} + \partial_i \partial_j g_{k\ell} - \partial_i \partial_k g_{j\ell} - \partial_j \partial_k g_{i\ell}) + \text{lower order terms}, \tag{59}$$

*Proof.* I don't want to type this, but it just involves writing the Ricci tensor in terms of the Riemann tensor, the Riemann tensor in terms of the Christoffel symbols, and the Christoffel symbols in terms of the metric. □

*Proof.* From the 2nd Bianchi identity,

$$\begin{aligned} \nabla_i R_{jk\ell m} &= \nabla_j R_{ik\ell m} - \nabla_k R_{ij\ell m} \\ g^{im} \nabla_i R_{jk\ell m} &= \nabla_j g^{im} R_{ik\ell m} - \nabla_k g^{im} R_{ij\ell m} \\ &= \nabla_j R_{k\ell} - \nabla_k R_{j\ell} \end{aligned}$$

□

# 6 Geometric Analysis

## 6.1 Integration

### 6.1.1 Stokes's theorem

Suppose $M$ is an oriented $n$-manifold with boundary, and suppose $\omega$ is a compactly supported $(n-1)$-form on $M$. Then

$$\int_{\partial M} \omega = \int_M d\omega. \tag{60}$$


$$\int_{\partial M} \omega = \int_M d\omega. \tag{60}$$


*Proof.* TODO

See Guillemin and Pollack p. 183. □

From this we can obtain several very useful special cases. The **divergence theorem** says that for any smooth 1-form $\alpha$ on a compact manifold with boundary,

$$\int_M \operatorname{div}(\alpha)\, d\mu = \int_{\partial M} \alpha(\nu)\, d\sigma, \tag{61}$$


$$\int_M \operatorname{div}(\alpha)\, d\mu = \int_{\partial M} \alpha(\nu)\, d\sigma, \tag{61}$$


*Proof.* TODO □

where $\nu$ is the outward unit normal to the boundary, and $d\sigma$ is the volume form of the boundary.

### 6.1.2 Integration by parts

Suppose $u, v \in C^\infty(M)$. If $M$ is closed,

$$\int_M \Delta u\, d\mu = 0 \tag{62}$$

If $M$ is compact,

$$\int_M (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M} \left( u\frac{\partial u}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma. \tag{63}$$

Where $\nu$ and $\sigma$ are TODO. In particular, on a closed manifold, the right hand side is 0, so $\int u\Delta v = \int v\Delta u$. If $M$ is compact,

$$\int_M u\Delta v\, d\mu + \int_M \langle \nabla u, \nabla v \rangle\, d\mu = \int_{\partial M} \frac{\partial v}{\partial \nu} u\, d\sigma. \tag{64}$$

In particular, on a closed manifold, $\int \langle \nabla u, \nabla v \rangle = -\int u\Delta v$.

I haven't really thought about this but I happened to run across this while I was looking at something else; see p. 437 of [1].

**Lemma 6.1.** *A sufficient condition for integration by parts on noncompact hypersurfaces. Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a complete hypersurface. If $u, v \in C^\infty(\mathcal{M})$ satisfy*

$$\int_{\mathcal{M}} (|u\nabla v| + |\nabla u|\,|\nabla v| + |u\Delta_f v|)\, dm < \infty,$$

*then*

$$\int_{\mathcal{M}} u\Delta_f v\, dm = -\int_{\mathcal{M}} \langle \nabla u, \nabla v\rangle\ dm.$$

*Proof.* See [1] p. 437. □

---

## 6.2 Miscellaneous

If $A(s)$ is a 1-parameter family of invertible square matrices, then

$$\frac{d}{ds}\log(\det A) = (A^{-1})^{ij}\frac{d}{ds}A_{ij}. \tag{65}$$


$$\frac{d}{ds}\log(\det A) = (A^{-1})^{ij}\frac{d}{ds}A_{ij}. \quad (65)$$


*Proof.* First

**Lemma 6.2.** $\det(I + tA) = 1 + \mathrm{tr}(A)t + \mathcal{O}(t^2)$.

*Proof.* Recall that $\det(tA) = t^n \det(A)$. Also recall that the characteristic polynomial of $A$ is the product

$$(x - \lambda_1)\cdots(x - \lambda_n),$$

where $\lambda_i$'s are eigenvalues of $A$. In particular, the characteristic polynomial is also $\det(xI - A)$.

$$\begin{aligned}
\det(I + tA) &= t^n \det(t^{-1}I - (-A)) \\
&= t^n(t^{-1} + \lambda_1)(t^{-1} + \lambda_2)\cdots(t^{-1} + \lambda_n) \\
&= t^n\left(t^{-n} + t^{-(n-1)}\sum_{i\le n}\lambda_i + t^{-(n-2)}a_{n-2} + \cdots + t^{-1}a_1 + a_0\right) \\
&= 1 + \mathrm{tr}(A)t + \mathcal{O}(t^2).
\end{aligned}$$

□

**Lemma 6.3.** $D_A \det(X) = \det(A)\,\mathrm{tr}(XA^{-1})$, *where* $D_A \det$ *is the differential of* $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ *at $A$. In particular, the differential of* $\det$ *at the identity matrix is just the trace.*

*Proof.* Note that det is a smooth function $\mathrm{GL}_m(\mathbb{R}) \to \mathbb{R}$ (because the determinant is a polynomial expression of the components of the matrix). First we consider the differential of det at $I$. By definition and using the previous lemma,

$$\begin{aligned}
D_I \det(X) &= \lim_{h\to 0}\frac{\det(I + hX) - \det(I)}{h} \\
&= \lim_{h\to 0}\frac{1 + \mathrm{tr}(X)h + \mathcal{O}(h^2) - 1}{h} \\
&= \mathrm{tr}(X).
\end{aligned}$$

Now,

$$
\begin{aligned}
D_A \det(X) &= \lim_{h\to 0} \frac{\det(A+hX)-\det(A)}{h} \\
&= \det(A) \lim_{h\to 0} \frac{\det(I+hXA^{-1})-\det(I)}{h} \\
&= \det(A) D_I(XA^{-1}) \\
&= \det(A)\operatorname{tr}(XA^{-1}). \qquad \square
\end{aligned}
$$

Note that $A(s)$ defines a curve in $\mathrm{GL}_m(\mathbb{R})$. We can think of $\frac{d}{ds}\det(A(s))$ either as the derivative of a function $\mathbb{R}\to\mathbb{R}$, or as the derivative of $\det : \mathrm{GL}_m(\mathbb{R}) \to \mathbb{R}$ in the direction of $\frac{d}{ds}A(s)$. From the second point of view we get

$$
\begin{aligned}
\frac{d}{ds}\det(A(s)) &= D_A \det\left(\frac{d}{ds}A(t)\right) \\
&= \det(A)\operatorname{tr}\left(A^{-1}\frac{d}{ds}A(s)\right),
\end{aligned}
$$

finally proving (65).

$\square$

## 6.3 Variation formulae

See Sheridan's notes for this section

Suppose that $g(t)$ is a time-dependent Riemannian metric, and

$$\frac{\partial}{\partial t} g_{ij}(t) = h_{ij}(t).$$

Then we have the following evolution equations for various geometric objects (note in some cases the result is only stated for the Ricci flow, i.e. when $h_{ij} = -2\operatorname{Rc}_{ij}$). Metric inverse:

$$\frac{\partial}{\partial t} g^{ij} = -h^{ij} = -g^{ik}g^{jl}h_{kl} \tag{66}$$

For time-independent vector fields, and an evolving metric $g(t)$, we define $\dot{\nabla} = \partial_t \nabla$ by $\dot{\nabla}_X Y = \partial_t(\nabla_X Y)$. Then

$$\left\langle \dot{\nabla}_X Y, Z \right\rangle = -(\nabla_X \operatorname{Rc})(Y,Z) + (\nabla_Z \operatorname{Rc})(X,Y) - (\nabla_Y \operatorname{Rc})(X,Z). \tag{67}$$

If $g(t)$ is a solution to Ricci flow, the function Laplacian $\Delta_{g(t)}$ evolves by

$$\partial_t \Delta_{g(t)} = 2\operatorname{Rc}_{ij} \nabla_i \nabla_j. \tag{68}$$

Christoffel symbols:

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \tag{69}$$

Riemann curvature tensor:

$$\partial_t R_{ijk}{}^{\ell} = \frac{1}{2} g^{\ell p}\left(\nabla_i\nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik}\right) \tag{70}$$

Ricci tensor

$$\partial_t \operatorname{Rc}_{ij} = \nabla_p(\partial_t \Gamma^p_{ij}) - \nabla_i(\partial_t \Gamma^p_{pj}). \tag{71}$$

$$\partial_t \operatorname{Rc}_{ij} = \nabla_p(\partial_t \Gamma^p_{ij}) - \nabla_i(\partial_t \Gamma^p_{pj}). \tag{71}$$

*Proof.* hi □

---

### 6.4 Proofs

$$\frac{\partial}{\partial t} g^{ij} = -h^{ij} = -g^{ik} g^{jl} h_{kl} \tag{66}$$

*Proof.*

$$\begin{aligned}
0 &= \partial_t \delta^i_k \\
&= \partial_t (g^{ij} g_{jk}) \\
&= (\partial_t g^{ij}) g_{jk} + g^{ij}(\partial_t g_{jk}) \\
(\partial_t g^{ij}) g_{jk} g^{k\ell} &= -g^{k\ell} g^{ij}(\partial_t g_{jk}) \\
(\partial_t g^{ij}) \delta^\ell_j &= -g^{k\ell} g^{ij}(\partial_t g_{jk}),
\end{aligned}$$

and the result follows. □

---

$$\left\langle \dot{\nabla}_X Y, Z \right\rangle = -(\nabla_X \operatorname{Rc})(Y, Z) + (\nabla_Z \operatorname{Rc})(X, Y) - (\nabla_Y \operatorname{Rc})(X, Z). \tag{67}$$

*Proof.* □

---

$$\partial_t \Delta_{g(t)} = 2 \operatorname{Rc}_{ij} \nabla_i \nabla_j. \tag{68}$$

*Proof.* We give two proofs. For $f \in C^\infty(M)$, using the coordinate expression (48) for the Hessian,

$$\begin{aligned}
(\partial_t \Delta_{g(t)}) f &:= \partial_t (g^{ij} \nabla_i \nabla_j) f \\
&= (\partial_t g^{ij}) \nabla_i \nabla_j f + g^{ij} (\partial_t \nabla_i \nabla_j f) \\
&= 2 \operatorname{Rc}^{ij} \nabla_i \nabla_j f + g^{ij} (\partial_t (\partial_i \partial_j f - \Gamma^k_{ij} \partial_k f)) \\
&= 2 \operatorname{Rc}^{ij} \nabla_i \nabla_j f + g^{ij} (\partial_t \Gamma^k_{ij}) \partial_k f
\end{aligned}$$

Now, we calculate, using the contracted second Bianchi identity $2 g^{ij} \nabla_i \operatorname{Rc}_{jk} = \nabla_k R$.

$$\begin{aligned}
g^{ij} \partial_t \Gamma^k_{ij} &= g^{ij} \left[ \frac{1}{2} g^{k\ell} (\nabla_i (-2 \operatorname{Rc}_{j\ell}) + \nabla_j (-2 \operatorname{Rc}_{i\ell}) - \nabla_\ell (-2 \operatorname{Rc}_{ij})) \right] \\
&= -g^{j\ell} g^{ij} \nabla_i \operatorname{Rc}_{j\ell} - g^{k\ell} g^{ij} \nabla_j \operatorname{Rc}_{i\ell} + g^{ij} g^{k\ell} \nabla_\ell \operatorname{Rc}_{ij} \\
&= -\frac{1}{2} g^{k\ell} \nabla_\ell R - \frac{1}{2} g^{k\ell} \nabla_\ell R + g^{k\ell} \nabla_\ell g^{ij} \operatorname{Rc}_{ij} \\
&= 0,
\end{aligned}$$

from which the result follows. □

*Proof.* The second proof is slightly less coordinate-dependent. Let $f, h \in C^\infty(M)$. Then

$$\begin{aligned}\int_M h\Delta f\, d\mu &= -\int_M \langle \nabla h, \nabla f\rangle\, d\mu \\ &= -\int_M g^{ij}\nabla_i h \nabla_j f\, d\mu.\end{aligned}$$

Differentiating both sides with respect to $t$ gives

$$\int_M [h\dot{\Delta} f\, d\mu + h\Delta f(\partial_t\, d\mu)] = -\int_M [(\partial_t g^{ij})\nabla_i h\nabla_j f\, d\mu + g^{ij}\nabla_i h\nabla_j f(\partial_t\, d\mu)].$$

Now use the fact that $\partial_t d\mu = -R\, d\mu$, and $\partial_t g^{ij} = 2R^{ij}$ to get

$$\begin{aligned}\int_M [\dot{\Delta} f - R\Delta f] h\, d\mu &= -\int_M [2\,\mathrm{Rc}_{ij}\,\nabla_j f - Rg^{ij}\nabla_j f](\nabla_i h)\, d\mu \\ &= \int_M \nabla_i[2\,\mathrm{Rc}_{ij}\,\nabla_j f - Rg^{ij}\nabla_j f] h\, d\mu.\end{aligned}$$

Since $h \in C^\infty(M)$ was arbitrary,

$$\begin{aligned}\dot{\Delta} f - R\Delta f &= \nabla_i(2\,\mathrm{Rc}_{ij}\,\nabla_j f - Rg^{ij}\nabla_j f) \\ &= (2\nabla_i\,\mathrm{Rc}_{ij})\nabla_j f + 2\,\mathrm{Rc}_{ij}\,\nabla_i\nabla_j f - g^{ij}\nabla_i R\nabla_j f - Rg^{ij}\nabla_i\nabla_j f \\ &= \nabla_j R\nabla_j f + 2\,\mathrm{Rc}_{ij}\,\nabla_i\nabla_j f - \nabla_j R\nabla_j f - R\Delta f \\ &= 2\,\mathrm{Rc}_{ij}\,\nabla_i\nabla_j f - R\Delta f,\end{aligned}$$

and the result follows. □

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$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \tag{69}$$

*Proof.* By the coordinate expression (41) for the Christoffel symbols, we have

$$\partial_t\Gamma^k_{ij} = \frac{1}{2}(\partial_t g^{k\ell})(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) + \frac{1}{2}g^{k\ell}(\partial_i\partial_t g_{j\ell} + \partial_j\partial_t g_{i\ell} - \partial_\ell\partial_t g_{ij}).$$

Now we work in normal coordinates at some point $p$, so $\partial_i g_{ij} = 0$, and $\partial_i A = \nabla_i A$ at $p$ for any tensor $A$.

□

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# 7 Submanifolds

Many of these ideas have are special cases of things discussed previously, in the case where the metric is induced by some immersion or embedding into a higher dimensional Riemannian manifold.

As the notation for this section is quite painful, here is a seperate notation glossary just for submanifolds, although it should essentially overlap with notation from the rest of the document. In most cases I am following Mat Langford's notation; see https://suppiluliuma.neocities.org/RG.pdf.

Let $M^n$ and $N^{n+k}$ be smooth manifolds, and $X : M \to N$ a smooth immersion. Then we denote

| | |
|---|---|
| $dX\colon TM \to TN$ | the derivative of $X$ |
| $X^*TN$ | the pullback bundle (over $M$) |
| $dX(TM)$ | the subbundle of $X^*TN$ from the embedding $(p,u) \mapsto (p, dX(u))$ of $TM$ into $X^*TN$ |
| $\langle \cdot, \cdot \rangle, g$ | the metrics on $N, M$ respectively |
| $X^* \langle \cdot, \cdot \rangle$ | the pullback metric on $X^*TN$: $X^* \langle (p,u), (p,v) \rangle$ |
| $N_pM$ | the normal space to $M$ at $p \in M$, i.e. $N_pM = \{\nu \in T_X(p)N : \langle u, \nu \rangle = 0 \text{ for all } u \in dX_p(T_pM)\}$ |
| $NM$ | the normal subbundle of $TN$ in the case where $X$ is an embedding |
| $NM$ | the normal subbundle of $X^*TN$ (over $M$), i.e. $NM = \{\nu \in X^*TN : {}^X \langle u, \nu \rangle = 0 \text{ for all } u \in dX(TM)_{\pi(\nu)}\}$ |
| $D$ | the connection on $N$ |
| ${}^XD\colon TM \times \Gamma(X^*TN) \to X^*TN$ | the pullback connection on $X^*TN$, defined by ${}^XD_u X^*V \coloneqq (\pi(u), D_{dX(u)}V)$ |
| $\nabla$ | connection on $TM$ |
| $\nabla^\perp$ | connection on $NM$ |
| $\mathrm{I\!I}$ | second fundamental form; $\mathrm{I\!I} \in \Gamma(T^*M \otimes T^*M \otimes NM)$, i.e. $\mathrm{I\!I}(u,v) = ({}^XD_u(dX(V)))^\perp$, for an extension $V$ of $v$ |
| $W$ | Weingarten tensor; $W \in \Gamma(T^*M \otimes TM \otimes N^*M)$ |

## 7.1 Second fundamental form

Roughly, $\mathrm{I\!I}(u,v)$ is the normal (to the image of the immersion) component of how the vector field $V$ is changing in the direction of $u$.

# References


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