Computations in Riemannian Geometry, Geometric Analysis, and the Ricci Flow

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Contents

These notes are intended as a practical reference when doing basic calculations in Riemannian geometry. I hope they will be helpful to the reader who is familiar with the concepts of Riemannian geometry but isn't an expert when it comes to calculations; they may provide a useful supplement to a more expository text on Riemannian geometry (I recommend, in addition to the standard texts, the lecture notes of Ben Andrews, which can be found online, and the book by Andrews and Hopper).

Proofs are placed at the end of each section, and are numbered based on the right-hand side numbering. The numbers on the right hand side are links that go back and forth between an equation and its proof.

This is a slow but steady work in progress, with still much to be done, possibly including some major reorganization. Last updated: September 20, 2025

TODO: finish all proofs

TODO: check out `https://www.math.ucla.edu/~petersen/Papers/44.pdf`

TODO: a section on Hodge theory. References: HRF 1.5

TODO: geodesics and jacobi fields?

TODO: distance functions?

TODO: Uhlenbeck's trick and more Ricci flow specific things

TODO: add bibliography and improve citations everywhere

TODO: I think I need to reconsider the overall organization; in particular wouldn't it be better for the Lapalcian and Hessian sections to be close together?

TODO: stuff about Weyl, Schouten, Cotton tensors?

TODO: finish all TODOs

0 Notation and Conventions

The convention I use for the curvature of a connection agrees with Chow and Lee, meaning that it is the opposite of Andrews. My convention for the factor in front of the wedge product of alternating tensors agrees with Andrews (and Lee?) and opposes Chow. I take Andrews's and Chow's convention for the way to write the rank of a mixed tensor (opposite to Lee), which says that a (k, ℓ) tensor takes as input k vector fields and ℓ covector fields. For example, we'll say that the metric is a $(2, 0)$-tensor. See Section 3 for more about this. Throughout, unless otherwise stated, we will be considering a Riemannian n-manifold $M = (M^n, g)$.

$\Gamma(E)$	the set of sections of the bundle E over M
$\mathcal{T}_k^\ell(M)$	the set of (k, ℓ)-tensors; that is, sections of $(T^*M)^{\otimes k} \otimes (TM)^{\otimes \ell}$
$\wedge^k T^*M$	the k-form bundle on M
$\Omega^k(T^*M)$	the set of sections of $\wedge^k T^*M$, i.e. the set of k-forms on M; $\Gamma(\wedge^k T^*M)$
$d\,\mathrm{Vol},\ d\mu$	the volume form of a Riemannian manifold

1 Basic notions

This section contains a few constructions from the theory of smooth manifolds that don't depend on a Riemannian metric, but also contains some Riemannian-metric based identities.

1.1 Coordinates

By definition, a smooth manifold has coordinate charts at every point. A coordinate chart is a pair (U, φ), where U is an open subset of the manifold, and $\varphi : U \to \hat{U}$ is a diffeomorphism, where $\hat{U} \subset \mathbb{R}^n$. This gives us a way to talk about a point on our manifold (within the coordinate chart U) by specifying n numbers. In particular, if $\varphi(p) = (x^1, \dots, x^n)$, we often think of the point p just as the n-tuple $(x^1, \dots, x^n)$.

There is a canonical frame $\left\{\frac{\partial}{\partial x^i}\right\}$ for $\mathbb{R}^n$, which pulls back along the chart map to give a frame (also denoted $\left\{\frac{\partial}{\partial x^i}\right\}$, or just $\{\partial_i\}$) on M. In this way, coordinates (x^i) are associated to a (local) frame for the tangent bundle.

1.2 Vector fields

By construction (see any book on Riemannian geometry), a vector field $X \in \Gamma(TM)$ satisfies the Leibniz rule

$$X(fg) = fX(g) + gX(f)$$

for $f, g \in C^\infty(M)$. From this we get the Leibniz rule:

$$X(fY) = X(f) \cdot Y + fX(Y). \tag{1}$$

If $f \in C^\infty(M)$ and $r \colon \mathbb{R} \to \mathbb{R}$, we have the chain rule

$$X(r \circ f) = (r' \circ f)X(f) \tag{2}$$

1.3 The differential and gradient

The **differential** df of a function $f \in C^\infty(M)$ is the 1-form defined by

$$(df)(X) = X(f)$$

for $X \in \Gamma(TM)$. Let $\operatorname{grad} f$ denote the vector field dual to df. That is, $g(\operatorname{grad} f, X) = (df)(X) = X(f)$. Sometimes ∇f is used to denote either df or $\operatorname{grad} f$ (or both). It is also used to denote the total covariant derivative of f (see below), but this is not really an abuse of notation since the total covariant derivative of f is equal to df.

$$\operatorname{grad}(fh) = f \operatorname{grad} h + h \operatorname{grad} f \tag{3}$$

In coordinates:

$$df = (\partial_i f) dx^i \tag{4}$$

$$\operatorname{grad} f = g^{ij} (\partial_j f) \partial_i \tag{5}$$

This differential is a special case of the derivative of a map $\phi : M \to N$ between manifolds. For this sort of map, if $\{x^i\}$ are coordinates on M, with frame $\{\partial_i\}$ and coframe $\{dx^i\}$, and $\{y^i\}$, $\{\partial_{y^i}\}$, and $\{dy^i\}$ are the corresponding objects on N, then, at some point p,

$$d\phi(\partial_i) = \frac{\partial(y^j \circ \phi)}{\partial x^i} \partial_{y^j} \tag{6}$$

$$d\phi(\partial_i) = \frac{\partial(y^j \circ \phi)}{\partial x^i} \partial_{y^j} \tag{6}$$

Proof. TODO; see p. 62 of [5]. □

From this it follows that

$$d\phi = \frac{\partial \phi^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\alpha}. \tag{7}$$

$$d\phi = \frac{\partial \phi^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\alpha}. \tag{7}$$

Proof. TODO □

1.4 Differential forms

Given a vector space V, a $(k, 0)$-tensor $\omega \in \bigotimes^k V^*$ is said to be alternating if it is antisymmetric under interchange of any two of its arguments. The set of alternating $(k, 0)$-tensors on V is denoted $\wedge^k V^*$. In particular, we are interested in $\wedge^k T_p^* M$; the space of alternating $(k, 0)$-tensors at p. Two special cases are 0- and 1-tensors, which are functions and covectors respectively. These are trivially alternating, so we have $\wedge^0 T^*M = C^\infty(M)$ and $\wedge^1 T^*M = T^*M$.

The **wedge product** of an alternating k-tensor and an alternating ℓ-tensor is a $(k + \ell)$-tensor can be defined explicitly by

$$S \wedge T = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)(S \otimes T) \circ \sigma,$$

where the composition with σ denotes applying the permutation σ to the $k + \ell$ inputs to $S \otimes T$. There is another convention (used by Chow, for example) for this that involves a different factor in front. A useful special case occurs with a multiple wedge product of covectors TODO.

We can also define the wedge product implicitly to be the linear map satisfying the following properties, for forms ω, η, μ and $c \in C^\infty(M)$,:

(i) $\omega \wedge (\eta \wedge \mu) = (\omega \wedge \eta) \wedge \mu$

(ii) $(c\omega) \wedge \eta = \omega \wedge (c\eta) = c(\omega \wedge \eta)$

(iii) If $\omega, \eta \in \wedge^k T_p M$, then
$$(\omega + \eta) \wedge \mu = \omega \wedge \mu + \eta \wedge \mu.$$

(iv) If $\omega \in \wedge^k T_p M$ and $\eta \in \wedge^\ell T_p M$, then
$$\eta \wedge \omega = (-1)^{k\ell} \omega \wedge \eta.$$

The space $\wedge^k T_p M$ is in fact a vector space: given a basis $\{\omega^i\}_{i=1}^n$ for $T_p^* M$, the set

$$\left\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} : 0 < i_1 < \cdots < i_k < n\right\}$$

is a basis for $\wedge^k T_p M$. Thus $\wedge^k T_p M$ has dimension $\binom{n}{k}$. Moreover, the set $\{\wedge^k T_p M : p \in M\}$ of all alternating k-tensors at points of M has a bundle structure. A **k-form** is a smooth section of the bundle $\wedge^k T^* M$. The set of all k-forms on M is denoted $\Omega^k(M)$.

1.4.1 The volume form

If M is oriented, there is a unique n-form $d\mu = d\mu_g$ called the **volume form**, defined in local coordinates by

$$d\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

If $\{\omega^i\}_{i=1}^n$ is an oriented orthonormal coframe for $T^* M$, then

$$d\mu = \omega^1 \wedge \cdots \wedge \omega^n. \tag{8}$$

Despite the notation, the volume form $d\mu$ is generally not the exterior derivative of some $(n-1)$-form μ.

1.4.2 Exterior derivative

The exterior derivative is the unique linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying

(i) If $f \in \Omega^0(M) = C^\infty(M)$, then df is the same as the differential of f.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d^2 = 0$.

Using these axioms, we can determine the following expression for d. Suppose we have coordinate covector fields dx^i. If we have the k-form ω given by (??? sums are taken over increasing k-tuples)

$$\omega = \sum_{i_1, \ldots, i_k} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then (I think this sum is just taken over all tuples).

$$d\omega = \sum_{i, i_1, \ldots, i_k} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{9}$$

Strictly speaking, there is still some more work to be done to make sure everything works here, even though it seems like we have a nice expression for d. One needs to show that this doesn't depend on the coordinates, and justify the claim that this operator is unique. For arguments of these facts, see Ben Andrews's lecture notes on differential geometry, or one of many other books on geometry.

If ω is a 1-form, we have the following useful expression

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]). \tag{10}$$

The previous expression generalizes: if ω is a k-form, the exterior derivative satisfies (here the hat notation means we are removing an argument)

$$\begin{aligned}(k+1)(d\omega)(X_0,\ldots,X_k) = &\sum_{j=0}^{k}(-1)^j X_j\omega(X_0,\ldots,\hat{X}_j,\ldots,X_k)\\ &+\sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).\end{aligned} \tag{11}$$

Some authors prefer to leave out the factor of $\frac{1}{k+1}$ in this definition. This expression can also be used to define the exterior derivative in a way that is explicitly independent of coordinates.

See section 1.5.1 for information about the formal adjoint to the exterior derivative.

1.4.3 Interior product

The interior product is, for each $X \in T_pM$, a linear map $\iota_X\colon \wedge^k T_p^*M \to \wedge^{k-1}T_p^*M$. If $\omega \in \wedge^0 T_p^*M$ (so that ω is a number), we define $\iota_X\omega = 0$. Otherwise, the interior product is the unique linear operator $\iota_X\colon \Omega^k(M) \to \Omega^{k-1}(M)$ satisfying similar properties to the exterior derivative:

(i) When $\omega \in \Omega^1(M) = \Gamma(T^*M)$, then $\iota_X\omega = \omega(X)$.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$\iota_X(\omega\wedge\eta) = (\iota_X\omega)\wedge\eta + (-1)^k\omega\wedge(\iota_X\eta)$$

(iii) $\iota_X^2 = 0$.

From (ii) it follows that

$$\iota_X(\omega_1\wedge\cdots\wedge\omega_k) = \sum_{i=1}^{k}(-1)^{i+1}\omega_1\wedge\cdots\wedge\iota_X(\omega_i)\wedge\cdots\wedge\omega_k.$$

We can determine that ι_X has the following expression:

$$\iota_X(\omega)(X_1,\ldots,X_{k-1}) = \omega(X,X_1,\ldots,X_{k-1}). \tag{12}$$

In particular, for covectors $\omega^1,\ldots,\omega^k$, we have

$$\iota_X(\omega^1\wedge\cdots\wedge\omega^k) = \sum_{i=1}^{k}(-1)^{i-1}\omega^i(X)\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^k. \tag{13}$$

Differential forms are exactly the objects that we integrate over a manifold. For more about integration, see Section 7.2.

1.4.4 The Hodge star operator

Before defining this operator, we need to define an inner product of forms. We could construct k-forms as alternating tensors on any vector space. Then an inner product on the vector space induces an inner product on the k-forms. We define this on basis k-forms by (taking $\{\omega^i\}_{i=1}^k$ as a basis for the vector space)

$$\left\langle \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}, \omega^{j_i} \wedge \cdots \wedge \omega^{j_k} \right\rangle = \det(\left\langle \omega^{i_a}, \omega^{j_b} \right\rangle),$$

where on the right hand side we are using whatever metric we are given on the underlying vector space. In our case, the underlying vector space is the cotangent space, and the inner product is of course the Riemannian metric, so the product becomes (asssuming that $\{\omega^i\}$ is orthonormal)

$$\left\langle \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}, \omega^{j_i} \wedge \cdots \wedge \omega^{j_k} \right\rangle = \det(\delta^{i_a j_b}).$$

The Hodge star operator $* : \wedge^k T^*M \to \wedge^{n-k} T^*M$ is the linear operator defined by

$$\langle \omega, \eta \rangle \; d\,\mathrm{Vol} = \omega \wedge *\eta.$$

For example, if $\{\omega^i\}_{i=1}^n$ is a positively oriented orthonormal coframe, then for any k,

$$(\omega^1 \wedge \cdots \wedge \omega^k) \wedge *(\omega^1 \wedge \cdots \wedge \omega^k) = d\,\mathrm{Vol},$$

since the inner product of $\omega^1 \wedge \cdots \wedge \omega^k$ with itself is clearly 1.

$$*(\omega^1 \wedge \cdots \wedge \omega^k) = \omega^{k+1} \wedge \cdots \wedge \omega^n. \tag{14}$$

As an operator on $\wedge^k T^*M$,

$$*^2 = (-1)^{k(n-k)}. \tag{15}$$

The Hodge star satisfies the following commutation relations, where $*_k$ denotes $*$ acting on k-forms.

$$*d = (-1)^{k+1}\delta, \qquad *_{k+1} d^k = (-1)^{k+1} \delta^{n-k} *_k \tag{16}$$

$$*\delta = (-1)^k d*, \qquad *_{k-1} \delta^k = (-1)^k d^{n-k} *_k \tag{17}$$

$$*\Delta^k = \Delta^{n-k} * \tag{18}$$

1.5 Divergence

Note that for a vector field X, $d(\iota_X(d\mu))$ is an n-form, so it is $f d\mu$ for some smooth function f. We call this function the **divergence** of X, so that

$$d(\iota_X \, d\mu) = \operatorname{div} X \, d\mu.$$

We could also have defined the divergence as the trace of the covariant derivative:

$$\operatorname{div} X = \operatorname{tr} \nabla X = (\nabla X)(\partial_i, dx^i) = (\nabla_i X)(dx^i). \tag{19}$$

In local coordinates, we have the expression

$$\operatorname{div}(X^i \partial_i) = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}). \tag{20}$$

The product of a function f and a vector field X satisfies

$$\operatorname{div}(fX) = X(f) + f \operatorname{div} X. \tag{21}$$

The characterization of divergence as the trace of the covariant derivative allows us to define the divergence of a (k, ℓ)-tensor as the $(k, \ell-1)$-tensor

I'm pretty sure it's this:

$$(\operatorname{div} T)(\omega^2, \dots, \omega^\ell, X_1, \dots, X_k) = \operatorname{tr}[(\nabla T)(\cdot, \cdot, \omega^2, \dots, \omega^\ell, X_1, \dots, X_k)].$$

$$(\operatorname{div} T)(X_1, \dots, X_{\ell-1}) = \operatorname{tr}(\nabla_T())$$

In particular TODO

This is from LeeRM, p. 149:

If F is any smooth k-tensor field (covariant, contravariant, or mixed), we define the divergence of F by

$$\operatorname{div} F = \operatorname{tr}_g(\nabla F),$$

where the trace is taken over the last two indices of the $(k+1)$-tensor field ∇F. (If F is purely contravariant, then tr_g can be replaced with tr, because the next-to-last index of ∇F is already an upper index.)

In Lee's notation, the last two indices are TODO

1.5.1 The codifferential

A useful operator in Hodge theory is δ, the formal adjoint to d on one-forms. In particular, for a function f and a form ω, if we define $\delta(\omega) = -\operatorname{div}(\omega^\sharp)$, we have

$$\langle \delta\omega, f \rangle = \langle \omega, df \rangle, \tag{22}$$

where $\langle \delta\omega, f \rangle = \int_M (\delta\omega) f \, d\mu$, and $\langle \omega, df \rangle = \int_M g(\omega, df) \, d\mu$. Another standard way to define δ is in terms of the Hodge star operator and the exterior derivative:

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha \tag{23}$$

1.6 The Laplacian(s)

The simplest version of the Laplacian is defined for functions $f \in C^\infty(M)$ by

$$\Delta f = \operatorname{div} \operatorname{grad} f.$$

The sign here is a matter of convention; our choice results in negative eigenvalues of the Laplacian. This can be extended to act on tensor bundles. This operator is called the **connection Laplacian**, the **rough Laplacian**, or the **Laplace-Beltrami operator** (especially when applied to functions); there are other second order linear elliptic operators referred to as the Laplacian as well. We define the rough Laplacian on tensors by $\Delta \colon \Gamma(\mathcal{T}^k_\ell(M)) \to \Gamma(\mathcal{T}^k_\ell(M))$ by

$$\Delta T = \operatorname{div} \nabla T = \operatorname{tr}_g \nabla^2 T = g^{ij} \nabla_i \nabla_j T,$$

where the trace is taken over the two new indices introduced by ∇^2. In coordinates,

$$\Delta = g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right) \tag{24}$$

For functions, this has the coordinate expression

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^j} \right). \tag{25}$$

$$\Delta(fh) = f\Delta h + h\Delta f + 2 \langle \operatorname{grad} f, \operatorname{grad} h \rangle \tag{26}$$

From this it follows that the heat operator $\partial_t - \Delta$ satisfies the product rule

$$(\partial_t - \Delta)(fh) = f(\partial_t - \Delta)(h) + h(\partial_t - \Delta)(f) - 2 \langle \nabla f, \nabla h \rangle . \tag{27}$$

If $f \in C^\infty(M)$ and $r \colon \mathbb{R} \to \mathbb{R}$, then

$$\Delta(r \circ f) = (r' \circ f)\Delta f + (r'' \circ f) |\nabla f|^2 . \tag{28}$$

A frequently useful special case of this is when $r = \log$, so then we have

$$\Delta \log f = \frac{\Delta f}{f} - \frac{|\nabla f|^2}{f^2}.$$

We can get several Bochner-type formulas as a consequence of this next formula:

$$\Delta(g(X, Y)) = g(\Delta X, Y) + g(\Delta Y, X) + 2g(\nabla X, \nabla Y). \tag{29}$$

$$\Delta g(X, Y) = \operatorname{div} \nabla$$

There is also the **Lichnerowicz Laplacian**, which is defined for symmetric 2-tensors by (using a local orthonormal frame $\{e_i\}$)

$$(\Delta_L T)(X, Y) = (\Delta T)(X, Y) + 2 \sum_i h(R(e_i, X)Y, e_i) - h(\mathrm{Rc}(X), Y) - h(X, \mathrm{Rc}(Y)).$$

In coordinates,

$$\Delta_L v_{ij} = \Delta v_{ij} + 2R_{kij\ell} v_{k\ell} - R_{ik} v_{jk} - R_{jk} v_{ik}. \tag{30}$$

see [3] Appendix A.4, and the **Hodge(-de Rham) Laplacian** on forms, and the **harmonic map Laplacian**, see p. 85 of [3]. The Lichnerowicz laplacian on 2-tensors is formally the same as the Hodge-de Rham laplacian acting on 2-forms.

The Lichnerowicz Laplacian commutes with the heat operator under the Ricci flow. See p. 110 of Chow HRF.

The Hodge Laplacian is a family of maps $-\Delta_d : \Omega^p(T^*M) \to \Omega^p(T^*M)$ defined by

$$-\Delta_d = d\delta + \delta d.$$

If ω is a 2-form, then

$$(\Delta_d \omega)_{ij} = \Delta \omega_{ij} + 2R_{ik\ell j} \omega_{k\ell} - R_{ik} \omega_{kj} - R_{jk} \omega_{ik}. \tag{31}$$

On a 1-form ω, we have a **Bochner formula**

$$\Delta \omega = \Delta_d \omega + \mathrm{Rc}(\omega). \tag{32}$$

Given a map $\phi : (M^n, g) \to (N^m, h)$, the **map Laplacian** of ϕ is defined by

$$\begin{aligned}(\Delta_{g,h}\phi)^\gamma &= \Delta_g(\phi^\gamma) + g^{ij}(\Gamma(h)^\gamma_{\alpha\beta}\circ\phi)\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\\ &= g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k} + (\Gamma(h)^\gamma_{\alpha\beta}\circ\phi)\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right),\end{aligned}$$

where $\{x^i\}$, $\{y^i\}$ are coordinates on M and N respectively. Note that $\Delta_{g,h}\phi \in C^\infty(\phi^*TN)$.

Remark 1.1. Recall[1] that there is a canonical isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$. Recall also[2] that we can think of the derivative $D\phi$ of a map $\phi : M^n \to N^m$ as a section of the vector bundle $E \coloneqq T^*M \otimes f^*TN$. On E there is a natural metric and compatible connection $\nabla^{g,h}$ defined by (the dual of) the Riemannian metric g, its associated Levi-Civita connection on T^*M, the pullback by f of the metric h, and its associated Levi-Civita connection on TN. So $\nabla^{g,h}D\phi$ is a section of the bundle $T^*M \otimes_S T^*M \otimes f^*TN$. The map Laplacian is the trace with respect to f of $\nabla^{g,h}D\phi$:

$$\Delta_{g,h}\phi = \mathrm{tr}_g\left(\nabla^{g,h}D\phi\right). \tag{33}$$

An important special case is when $M = N$, and $\phi = \mathrm{Id}$. Then the above equation simplifies to

$$(\Delta_{g,h}\,\mathrm{Id})^k = g^{ij}\left(-\Gamma(g)^k_{ij} + \Gamma(h)^k_{ij}\right) \tag{34}$$

See Lemma 3.1 of Patodi: curvature and eigenforms of the laplace operator for a formula for the laplacian on forms (and some references).

1.7 Computations in special coordinates

Proofs of various identities can be simplified by choosing particular coordinate systems at a point. The idea is that essentially all quantities we are interested in are independent of coordinates, so we only need to prove an identity involving such quantities in a particular coordinate system, and it will hold in general. Thus we are free to choose the simplest coordinate system for the problem.

1.7.1 Normal coordinates

The most common simplifying coordinates are **normal coordinates**, in which the metric becomes the identity matrix at a given point p. We define these coordinates by taking an orthonormal basis $\{e_i\}$ for T_pM, and letting $\exp_p^{-1} : U \to B_\epsilon(0)$ be the chart map, where $U \ni p$ and ϵ are chosen to make this a diffeomorphism. In normal coordinates at p, we have the following:

$$g_{ij}(p) = \delta_{ij} \tag{35}$$
$$\Gamma^k_{ij}(p) = 0 \tag{36}$$
$$\partial_k g_{ij}(p) = 0. \tag{37}$$

The common practice of not bothering to raise or lower indices but still sum over repeated indices as in the Einstein convention reflects the fact that we can perform computations using normal coordinates. For example, we might write $\nabla_i X_i$ to mean $\sum_i \nabla_i X_i$, since, in normal coordinates,

$$g^{ij}\nabla_j X_i = \delta^{ij}\nabla_j X_i = \sum_{i=1}^n \nabla_i X_i.$$

[1] From Section 3.1, for example.
[2] From section 2.2, for example.

1.7.2 Polar normal coordinates

Suppose we have a normal coordinate chart $\phi : V \to \mathbb{R}^n$ defined on a normal neighborhood V of p. For every choice of polar coordinates $\hat{\Theta} : \phi(V) \to \mathbb{R}^{n-1} \times \mathbb{R}^+$, we get a smooth coordinate map $\Theta = \hat{\Theta} \circ \phi$ on an open subset of $V \setminus \{p\}$. Such coordinates are called **polar normal coordinates**. They have the property that the last coordinate function r is the radial distance function on V, and the other coordinates are constant along the integral curves of $\operatorname{grad} r$.

1.7.3 Fermi coordinates

Fermi coordinates generalize normal coordinates to tubular neighborhoods of a submanifold (the model case being a geodesic). Given a k-dimensional submanifold S, let $\exp^{\perp}$ denote the restriction of the exponential map to the normal bundle of S (called the normal exponential map). Given some coordinates ψ for S, let $\{E_1, \ldots, E_{n-k}\}$ be a local orthonormal frame for NS. The coordinate map ψ and frame $\{E_i\}$ yield a diffeomorphism from a subset of $\mathbb{R}^n$ to a subset of the normal bundle given by[3]

$$B(x^1, \ldots, x^k, v^1, \ldots, v^{n-1}) = (q, v^1 E_1|_q + \cdots + v^{n-k} E_{n-k}|_q),$$

where $\psi(q) = (x^1, \ldots, x^k)$. By composing the inverse of the normal exponential map with the inverse of this map, we get the Fermi coordinates.

Given an embedded k-dimensional submanifold S, let U be a normal neighborhood of S in M, and let $(x^1, \ldots, x^k, v^1, \ldots, v^{n-p})$ be Fermi coordinates on an open subset $U_0 \subset U$. For convenience, also write $x^{k+j} = v^j$. Then the Fermi coordinates have the following properties, for any $q \in P \cap U_0$,

$$q \in P \cap U_0 \iff q = (x^1, \ldots, x^k, 0, \ldots, 0), \tag{38}$$

$$g_{ij} = \begin{cases} 0 & 1 \leq i \leq k \text{ and } k+1 \leq j \leq n \\ \delta_{ij} & p+1 \leq i, j \leq n. \end{cases} \tag{39}$$

$$\Gamma^k_{ij} = 0, \quad p+1 \leq i, j \leq n. \tag{40}$$

$$\partial_i g_{jk}(q) = 0, \quad p+1 \leq i, j, k \leq n \tag{41}$$

1.7.4 Coordinates adapted to a vector field

It follows from the local integrability of vector fields that, given a vector field $X = X^i \partial_i$, we can choose coordinates so that $X^1 = 1$ and $X^i = 0$ for $i > 1$.

1.7.5 Harmonic coordinates

A coordinate chart $(x^1, \ldots, x^n)$ is called **harmonic** if each coordinate function is harmonic with respect to the Laplace-Beltrami operator:

$$\Delta_g x^i = 0.$$

[3] I'm actually leaving out a few details to the effect that we can without loss of generality define this and that on the same neighborhood and so on. See the section on Fermi coordinates in [LeeRM] if you feel like you can't live without these sorts of technicalities.

1.7.6 Isothermal coordinates

These are a special case of harmonic coordinates on a surface, where the coordinate functions satisfy the Cauchy-Riemann equations. In particular, the coordinate chart provides a conformal equivalence between an open subset of the manifold and an open subset of $\mathbb{R}^2$. Such coordinate always exist locally.

1.8 Cartan's moving frames

See also some exposition in Volume 2 of Spivak and in [2].

We use generalized Einstein notation frequently throughout this section. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on an open subset of M. Let $\{\omega^i\}_{i=1}^n$ be the dual orthonormal basis for T^*M, defined by $\omega^i(e_j) = \delta^i_j$. For each i, we define the **connection 1-forms** $\{\omega^{i,j}\}_{j=1}^n$ (corresponding to e_i) to be the components of the Levi-Civita connection with respect to e_i. That is,

$$\nabla_X e_i = \omega^{i,j}(X) e_j.$$

Equivalently, we could define

$$\omega^{i,j} = g(\nabla_{e_k} e_i, e_j)\omega^k, \tag{42}$$

or

$$\omega^{i,j}(X) = g(\nabla_X e_i, e_j). \tag{43}$$

These are antisymmetric in i and j,

$$\omega^{i,j} = -\omega^{j,i}, \tag{44}$$

and satisfy the **first Cartan structure equation**:

$$d\omega^i = \omega^j \wedge \omega^{j,i} = \omega^{i,j} \wedge \omega^j. \tag{45}$$

By examining the proof of this equation, one can see that the first Cartan structure equation is just the torsion-free property of the Levi-Civita connection expressed in the language of moving frames. In particular, the torsion 2-form T^i defined by $T^i = d\omega^i - \omega^j \wedge \omega^{j,i}$ is the ith component of the usual torson tensor for the Levi-Civita connection.

We also have

$$d\omega^j(e_i, e_j) = \omega^{i,j}(e_j) - \omega^{j,k}(e_i). \tag{46}$$

From this it follows that

$$\omega^{i,k}(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{47}$$

Now we define the **curvature 2-forms** $\Omega^{i,j}$ by

$$\Omega^{i,j}(X,Y)e_j = \frac{1}{2}\operatorname{Rm}(X,Y)e_i.$$

These measure the noncommutativity of taking two covariant derivatives. We could also define these by (TODO: check the constant?)

$$\Omega^{i,j} = \frac{1}{2}\operatorname{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell, \tag{48}$$

so that

$$\Omega^{i,j}(e_k, e_\ell) = \mathrm{Rm}_{ijk\ell}\,. \tag{49}$$

These satisfy the following, called the **second Cartan structure equation**:

$$\Omega^{i,j} = d\omega^{i,j} - \omega^{i,k} \wedge \omega^{k,j} \tag{50}$$

This gives us a way to compute curvatures. For example, on a surface M^2, we have

$$d\omega^1 = \omega^2 \wedge \omega^{2,1}, \quad d\omega^2 = \omega^1 \wedge \omega^{1,2}, \quad \Omega^{1,2} = d\omega^{1,2}$$

1.9 Proofs

$$X(fY) = X(f) \cdot Y + fX(Y).$$

Proof. For $g \in C^\infty(M)$,

$$\begin{aligned} [X(fY)](g) &= X(f \cdot Y(g)) \\ &= X(f) \cdot Y(g) + fX(Y(g)) \\ &= [X(f) \cdot Y + fX(Y)](g). \end{aligned}$$

□

$$X(r \circ f) = (r' \circ f)X(f)$$

Proof. This follows from the chain rule on $\mathbb{R}^n$. First consider the case where $X = \partial_i$. Let ψ be a chart about $p \in M$.

$$\begin{aligned} \partial_i|_p(r \circ f) &\coloneqq \left.\frac{\partial}{\partial x^i}\right|_{\psi(p)} (r \circ f \circ \psi^{-1}) \\ &= r'(f(p)) \cdot \left.\frac{\partial}{\partial x^i}\right|_{\psi(p)} \frac{(f \circ \psi^{-1})}{\partial x^i} \\ &= r'(f(p))\partial_i f. \end{aligned}$$

Then the general case follows by linearity. □

$$\operatorname{grad}(fh) = f \operatorname{grad} h + h \operatorname{grad} f \tag{3}$$

Proof. Recall that $\operatorname{grad} f$ is defined to be the vector field so that for all vector fields X,

$$\langle \operatorname{grad} f, X \rangle = X(f).$$

Now

$$\begin{aligned} \langle \operatorname{grad}(fh), X \rangle &= X(fh) \\ &= fX(h) + hX(f) \\ &= f \langle \operatorname{grad} h, X \rangle + h \langle \operatorname{grad} f, X \rangle \\ &= \langle f \operatorname{grad} h + h \operatorname{grad} f, X \rangle\,. \end{aligned}$$

□

$$df = (\partial_i f)dx^i \tag{4}$$

Proof. This follows immediately, since

$$df(\partial_i) = \partial_i(f).$$

□

$$\operatorname{grad} f = g^{ij}(\partial_j f)\partial_i \tag{5}$$

Proof. Recall that for any vector field X,

$$X = dx^i(X)\partial_i = X^i\partial_i.$$

So, writing in coordinates

$$\begin{aligned}
g(\operatorname{grad} f, X) &= df(X)\\
g_{ij}dx^i(\operatorname{grad} f)X^j &= (\partial_k f)dx^k(X)\\
g_{ij}dx^i(\operatorname{grad} f)X^j &= (\partial_k f)X^k\\
g_{ij}dx^i(\operatorname{grad} f) &= (\partial_j f)\\
dx^i(\operatorname{grad} f) &= g^{ij}(\partial_j f)\\
\operatorname{grad} f &= g^{ij}(\partial_j f)\partial_i.
\end{aligned}$$

□

$$d\mu = \omega^1 \wedge \cdots \wedge \omega^n \tag{8}$$

Proof. Nothing to prove here. □

$$d\omega = \sum_{i,i_1,\ldots,i_k} \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{9}$$

Proof. We use first the linearity of d and next its product rule.

$$\begin{aligned}
d\omega &= d\left(\sum_{i_1,\ldots,i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\\
&= \sum_{i_1,\ldots,i_k} d(\omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k})\\
&= \sum_{i_1,\ldots,i_k} d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + (-1)^k \omega_{i_1\cdots i_k} d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}).
\end{aligned}$$

But now since $d^2 = 0$, the second term is 0, and by the expression (4) for the differential, the result follows. □

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{10}$$

Proof. By definition,

$$(dx^i \wedge dx^j)(X, Y) = \left(\sum_{\sigma \in S_2} \operatorname{sgn}(\sigma)(dx^i \otimes dx^j) \circ \sigma \right)(X, Y) = X^i Y^j - X^j Y^i$$

the left hand side is

$$\begin{aligned} d\omega(X, Y) &= \left(\sum_{i,j \le n} \partial_i \omega_j dx^i \wedge dx^j \right)(X, Y) \\ &= \sum_{i,j} \partial_i \omega_j (X^i Y^j - X^j Y^i) \end{aligned}$$

On the other hand, note that

$$\begin{aligned} X\omega(Y) &= X^i \partial_i(\omega_j dx^j(Y^k \partial_k)) \\ &= X^i \partial_i(\omega_j Y^j) \\ &= X^i Y^j \partial_i(\omega_j) + X^i \omega_j \partial_i(Y^j), \end{aligned}$$

and

$$\begin{aligned} \omega([X, Y]) &= \omega(X^i \partial_i(Y^j \partial_j) - Y^k \partial_k(X^\ell \partial_\ell)) \\ &= \omega(X^i \partial_i(Y^j)\partial_j - Y^k \partial_k(X^\ell)\partial_\ell) \\ &= X^i \omega_j \partial_i(Y^j) - Y^k \omega_\ell \partial_k(X^\ell). \end{aligned}$$

so the right hand side becomes

$$\begin{aligned} X\omega(Y) - Y\omega(X) - \omega([X, Y]) &= X^i Y^j \partial_i(\omega_j) + X^i \omega_j \partial_i(Y^j) - Y^k X^\ell \partial_k(\omega_\ell) - X^k \omega_\ell \partial_k(Y^\ell) \\ &\quad - X^p \omega_q \partial_p(Y^q) + Y^t \omega_s \partial_t(X^s) \\ &= X^i Y^j \partial_i(\omega_j) - Y^k X^\ell \partial_k(\omega_\ell). \end{aligned}$$

□

$$\begin{aligned} (d\omega)(X_0, \ldots, X_k) &= \sum_{j=0}^{k} (-1)^j X_j \omega(X_0, \ldots, \hat{X}_j, \ldots, X_k) \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k). \end{aligned} \tag{11}$$

Proof. TODO □

$$\iota_X(\omega)(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}) \tag{12}$$

Proof. In coordinates,

$$\begin{aligned} \iota_X(\omega)(X_1, \ldots, X_{k-1}) &= \iota_X(\omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k})(X_1, \ldots, X_{k-1}) \\ &= \end{aligned}$$

TODO □

$$\iota_X(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(X) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k. \tag{13}$$

Proof. Note that by property (ii) defining the interior product,

$$\iota_X(\omega^1 \wedge \cdots \wedge \omega^k) = \iota_X(\omega^1) \wedge \omega^2 \wedge \cdots \wedge \omega^k + (-1)\omega^1 \wedge \iota_X(\omega^2 \wedge \cdots \wedge \omega^k),$$

and the result follows by induction (and by property (i), which says $\iota_X(\omega) = \omega(X)$ for 1-forms ω). □

$$*(\omega^1 \wedge \cdots \wedge \omega^k) = \omega^{k+1} \wedge \cdots \wedge \omega^n. \tag{14}$$

Proof. In this case we have

$$\begin{aligned}\left\langle \omega^1 \wedge \cdots \wedge \omega^k, \omega^1 \wedge \cdots \wedge \omega^k \right\rangle d\,\mathrm{Vol} = d\,\mathrm{Vol}& \\ = (\omega^1 \wedge \cdots \wedge \omega^k) \wedge (\omega^{k+1} \wedge \cdots \wedge \omega^n)&,\end{aligned}$$

which establishes the result. □

$$*^2 = (-1)^{k(n-k)}. \tag{15}$$

Proof. It suffices to prove it on a basis for $\wedge^k T_pM$. In particular, let $\{\omega^i\}_{i=1}^n$ be a positively oriented orthonormal coframe (basis for T_p^*M). Then recall that

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} : 0 < i_1 < \cdots < i_k < n\}$$

is a basis for $\wedge^k T_pM$. For a multi-index $I = (i_1, \ldots, i_k)$, let $\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$. Since

$$**(\omega^1 \wedge \cdots \wedge \omega^k) = *(\omega^{k+1} \wedge \cdots \wedge \omega^n),$$

and

$$\omega^{k+1} \wedge \cdots \wedge \omega^n \wedge *(\omega^{k+1} \wedge \cdots \wedge \omega^n) = \omega^1 \wedge \cdots \wedge \omega^n,$$

it is not hard to see that the result holds when $I = (1, \ldots, k)$. It is also not hard to see that $**\omega^I = \pm\omega^I$; the point is just to figure out what the sign must be. It suffices to show that the sign only depends on n and k (and not I).

TODO □

$$*d = (-1)^{k+1}\delta, \qquad *_{k+1}d^k = (-1)^{k+1}\delta^{n-k}*_k \tag{16}$$

Proof.

$$\delta^{n-k}*_k = (-1)^{(n-k)n+n+1} * d ** = (-1)^{(n-k)n+n+1}(-1)^{k(n-k)} * d = (-1)^{k+1} * d$$

□

$$*\delta = (-1)^k d*, \qquad *_{k-1}\delta^k = (-1)^k d^{n-k} *_k \tag{17}$$

Proof.
$$*\delta = (-1)^{kn+n+1} **d* = (-1)^{nk+n+1}(-1)^{(n-k+1)(k-1)}d* = (-1)^k d * .$$
□

$$*\Delta = \Delta * \tag{18}$$

Proof. TODO □

$$\operatorname{div} X = \operatorname{tr} \nabla X = (\nabla X)(\partial_i, dx^i) = (\nabla_i X)(dx^i). \tag{19}$$

Proof. TODO □

$$\operatorname{div}(X^i \partial_i) = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}). \tag{20}$$

Proof. Apply Cartan's formula (56) and the definition of the divergence as the quantity satisfying $d(\iota_X d\mu) = \operatorname{div} X \, d\mu$.

TODO: idk what this is supposed to prove but it's not right □

$$\operatorname{div}(fX) = X(f) + f \operatorname{div} X \tag{21}$$

Proof. One can prove this using coordinates, but there is a nicer way.

$$\begin{aligned}
\operatorname{div}(fX) &= \operatorname{tr}[Y \mapsto \nabla_Y (fX)] \\
&= \operatorname{tr}[Y \mapsto (Y(f)X + f\nabla_Y X)] \\
&= \operatorname{tr}[Y \mapsto Y(f)X] + f \operatorname{tr}[Y \mapsto \nabla_Y X] \\
&= (\partial_i(f)X)(dx^i) + f \operatorname{div} X \\
&= dx^i(\partial_i(f)X^k \partial_k) + f \operatorname{div} X \\
&= \partial_i(f)X^i + f \operatorname{div} X \\
&= X(f) + f \operatorname{div} X.
\end{aligned}$$
□

$$\langle \delta\omega, f \rangle = \langle \omega, df \rangle . \tag{22}$$

Proof. First observe that from (21), we get

$$-\delta(f\omega) = \omega^\sharp(f) - f\delta\omega.$$

Using the definition of divergence of $\omega^\sharp$ as the quantity satisfying $d(\iota_{\omega^\sharp} d\mu) = \operatorname{div}\omega^\sharp d\mu = -\delta\omega d\mu$, we have

$$\begin{aligned}
\langle f, \delta\omega \rangle &= \int_M f\delta\omega \, d\mu \\
&= \int_M (\delta(f\omega) + \omega^\sharp(f)) \, d\mu \\
&= \int_M \delta(f\omega) \, d\mu + \int_M \omega^\sharp(f) \, d\mu \\
&= \int_M \omega^\sharp(f) \, d\mu \\
&= \int_M g(df, \omega) \\
&= \langle df, \omega \rangle \, .
\end{aligned}$$

□

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha \tag{23}$$

Proof. Here we'll prove the equivalence of the adjoint definition of δ and the definition above. Check for a missing factor of p. TODO □

$$\Delta = g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right) \tag{24}$$

Proof. This follows immediately from (78). □

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^j} \right) . \tag{25}$$

Proof. This follows immediately from the coordinate expression for the divergence and $\operatorname{grad} f$. □

$$\Delta(fh) = f\Delta h + h\Delta f + 2 \langle \nabla f, \nabla h \rangle \tag{26}$$

Proof.

$$\begin{aligned}
\Delta(fh) &= \operatorname{div}\operatorname{grad}(fh) \\
&= \operatorname{div}(f \operatorname{grad} h + h \operatorname{grad} f) \\
&= \operatorname{div}(f \operatorname{grad} h) + \operatorname{div}(h \operatorname{grad} f) \\
&= (\operatorname{grad} h)(f) + f \operatorname{div}(\operatorname{grad} h) + (f \leftrightarrow h) \\
&= \langle \operatorname{grad} h, \operatorname{grad} f \rangle + f\Delta h + (f \leftrightarrow h) \\
&= f\Delta h + h\Delta f + 2 \langle \operatorname{grad} f, \operatorname{grad} h \rangle \, .
\end{aligned}$$

□

$$(\partial_t - \Delta)(fh) = f(\partial_t - \Delta)(h) + h(\partial_t - \Delta)(f) - 2 \langle \nabla f, \nabla h \rangle \, . \tag{27}$$

Proof. TODO □

$$\Delta(r \circ f) = (r' \circ f)\Delta f + (r'' \circ f)\left|\nabla f\right|^2 \tag{28}$$

Proof. By definition, and using (2) to evaluate terms like $\partial_i(r \circ f)$,

$$\begin{aligned}
\Delta(r \circ f) &= g^{ij}\nabla^2_{ij}(r \circ f) \\
&= g^{ij}(\nabla_i(\nabla_j(r \circ f)) - \nabla_{\nabla_i \partial_j}(r \circ f)) \\
&= g^{ij}(\partial_i\partial_j(r \circ f) - \Gamma^k_{ij}\partial_k(r \circ f)) \\
&= g^{ij}(\partial_i((r' \circ f)\partial_j f) - \Gamma^k_{ij}(r' \circ f)\partial_k f) \\
&= g^{ij}((r'' \circ f)\partial_i f \partial_j f + (r' \circ f)\partial_i\partial_j f - (r' \circ f)\Gamma^k_{ij}\partial_k f) \\
&= g^{ij}(r'' \circ f)\partial_i f \partial_j f + (r' \circ f)\Delta f \\
&= (r'' \circ f)\left|\nabla f\right|^2 + (r' \circ f)\Delta f.
\end{aligned}$$

□

$$\Delta(g(X, Y)) = g(\Delta X, Y) + g(\Delta Y, X) + 2g(\nabla X, \nabla Y). \tag{29}$$

Proof. This can be done in normal coordinates. A slightly more invariant approach (that still uses an orthonormal frame) is to observe that, using the formula for the Hessian of a function,

$$\begin{aligned}
\nabla^2(g(X, Y))(\alpha, \beta) &= \alpha(\beta(g(X, Y))) - (\nabla_\alpha \beta)(g(X, Y)) \\
&= \alpha(g(\nabla_\beta X, Y) + g(X, \nabla_\beta Y)) - g(\nabla_{\nabla_\alpha \beta} X, Y) - g(X, \nabla_{\nabla_\alpha \beta} Y) \\
&= g(\nabla_\alpha \nabla_\beta X, Y) + g(\nabla_\beta X, \nabla_\alpha Y) + g(\nabla_\alpha X, \nabla_\beta Y) + g(X, \nabla_\alpha \nabla_\beta Y) \\
&\quad - g(\nabla_{\nabla_\alpha \beta} X, Y) - g(X, \nabla_{\nabla_\alpha \beta} Y).
\end{aligned}$$

Now applying this to normal coordinate basis vectors gives the result. □

$$\Delta_L v_{ij} = \Delta v_{ij} + 2R_{kij\ell} v_{k\ell} - R_{ik} v_{jk} - R_{jk} v_{ik}. \tag{30}$$

Proof. In orthonormal coordinates,

$$\begin{aligned}
\Delta_L v_{ij} &\coloneqq (\Delta_L v)(\partial_i, \partial_j) \\
&= (\Delta v)(\partial_i, \partial_j) + 2\sum_k v(R(\partial_k, \partial_i)\partial_j, \partial_k) - h(\mathrm{Rc}(\partial_i), \partial_j) - h(\partial_i, \mathrm{Rc}(\partial_j)) \\
&= \Delta v_{ij} + 2v(R^\ell_{kij}\partial_\ell, \partial_k) - v(\mathrm{Rc}_i^k\, \partial_k, \partial_j) - v(\partial_i, \mathrm{Rc}_j^k\, \partial_k) \\
&= \Delta v_{ij} + 2R_{kij\ell} v_{k\ell} - R_{ik} v_{jk} - R_{jk} v_{ik}.
\end{aligned}$$

□

$$(\Delta_d \omega)_{ij} = \Delta \omega_{ij} + 2R_{ik\ell j}\omega_{k\ell} - R_{ik}\omega_{kj} - R_{jk}\omega_{ik}. \tag{31}$$

Proof. First observe that, by (11), in normal coordinates,

$$\begin{aligned}
(d\omega)_{ijk} &= (d\omega)(\partial_i, \partial_j, \partial_k) \\
&= \partial_i \omega(\partial_j, \partial_k) - \partial_j \omega(\partial_i, \partial_k) + \partial_k \omega(\partial_i, \partial_j) \\
&= \nabla_i \omega_{jk} - \nabla_j \omega_{ij} + \nabla_k \omega_{ij}
\end{aligned}$$

TODO: finish □

$$\Delta\omega = \Delta_d\omega + \mathrm{Rc}(\omega) \tag{32}$$

Proof. hello TODO

□

$$\Delta_{g,h}\phi = \mathrm{tr}_g\left(\nabla^{g,h} D\phi\right). \tag{33}$$

Proof. Let x^i be local coordinates on M, with frame $\{\partial_i\}$ and coframe $\{dx^i\}$, and let y^α be local coordinates on N with frame $\{\partial_{y^\alpha}\}$ and coframe $\{dy^\alpha\}$. If $\phi^\alpha = y^\alpha \circ \phi$, then

$$D\phi = \frac{\partial\phi^\alpha}{\partial x^j} dx^j \otimes \partial_{y^\alpha}.$$

Let ∇^M, ∇^N be the Levi-Civita connections of g on TM and h on TN, respectively. Then their Christoffel symbols are

$$\nabla^M_i \partial_j = \Gamma^k_{ij}\partial_k, \quad \nabla^N_\alpha \partial_{y^\beta} = \mathrm{H}^\gamma_{\alpha\beta}\,\partial_{y^\gamma}.$$

On T^*M we have $\nabla^M_i dx^j = -\Gamma^j_{ik}dx^k$ (TODO: check this). By (63), the pullback connection ${}^\phi\nabla$ has coefficients

$${}^\phi\nabla^N_{\partial_i}\partial_{y^\alpha} = \frac{\partial\phi^\ell}{\partial x^i}(\mathrm{H}^k_{\ell\alpha}\circ\phi)\partial_{y^k}$$

Now we have connections on T^*M and f^*TN; the product connection $\nabla^{g,h}$ on $T^*M \otimes f^*TN$ is defined by

$$\nabla^{g,h} = \nabla^M \otimes \mathrm{Id} + \mathrm{Id} \otimes ({}^\phi\nabla^N).$$

Now we can compute

$$\begin{aligned}
\nabla^{g,h}_{\partial_i} D\phi &= \nabla^{g,h}_{\partial_i}\left(\frac{\partial\phi^\alpha}{\partial x^j}dx^j \otimes \partial_{y^\alpha}\right) \\
&= \frac{\partial\phi^\alpha}{\partial x^i\partial x^j}(dx^j \otimes \partial_{y^\alpha}) + \frac{\partial\phi^\alpha}{\partial x^j}\left(\nabla^M_{\partial_i}dx^j\right)\otimes\partial_{y^\alpha} + \frac{\partial\phi^\alpha}{\partial x^j}dx^j \otimes \left({}^\phi\nabla^N_{\partial_i}\partial_{y^\alpha}\right) \\
&= \frac{\partial\phi^\alpha}{\partial x^i\partial x^j}(dx^j \otimes \partial_{y^\alpha}) + \frac{\partial\phi^\alpha}{\partial x^j}\left(-\Gamma^j_{ik}dx^k\right)\otimes\partial_{y^\alpha} + \frac{\partial\phi^\alpha}{\partial x^j}dx^j \otimes \left(\frac{\partial\phi^\ell}{\partial x^i}(\mathrm{H}^k_{\ell\alpha}\circ\phi)\partial_{y^k}\right) \\
&= \left(\frac{\partial\phi^\alpha}{\partial x^i\partial x^j} - \frac{\partial\phi^\alpha}{\partial x^\ell}\Gamma^\ell_{ij} + \frac{\partial\phi^m}{\partial x^j}\frac{\partial\phi^\ell}{\partial x^i}(\mathrm{H}^\alpha_{\ell m}\circ\phi)\right)(dx^j \otimes \partial_{y^\alpha})
\end{aligned}$$

In particular, we have shown that

$$(\nabla^{g,h}D\phi)^\alpha_{ij} = \frac{\partial^2\phi^\alpha}{\partial x^i\partial x^j} - \Gamma^\ell_{ij}\frac{\partial\phi^\alpha}{\partial x^\ell} + (\mathrm{H}^\alpha_{km}\circ\phi)\frac{\partial\phi^m}{\partial x^j}\frac{\partial\phi^k}{\partial x^i}.$$

One can check that this is symmetric in i and j. □

$$(\Delta_{g,h}\,\mathrm{Id})^k = g^{ij}\left(-\Gamma(g)^k_{ij} + \Gamma(h)^k_{ij}\right) \tag{34}$$

Proof. This follows immediately from the definition of the map Laplacian. □

$$g_{ij}(p) = \delta_{ij} \tag{35}$$

Proof. Recall that $d(\exp_p)_0 = \mathrm{Id}$.

$$\frac{\partial}{\partial x^i} = d(\exp_p)_0 \left(\left.\frac{\partial}{\partial e^i}\right|_0 \right) = \frac{\partial}{\partial e^i},$$

from which (35) follows. □

$$\Gamma^k_{ij}(p) = 0 \tag{36}$$

Proof. This follows immediately from (35) and the definition of Γ. □

$$\partial_k g_{ij}(p) = 0 \tag{37}$$

Proof. We have

$$\begin{aligned} \partial_k g_{ij} &= \partial_k g(\partial_i, \partial_j) \\ &= g(\partial_k \partial_i, \partial_j) + g(\partial_i, \partial_k \partial_j). \end{aligned}$$

Since ∂_i are coordinate vector fields, $\partial_i \partial_j = 0$, so the proof is done. □

$$q \in P \cap U_0 \iff q = (x^1, \ldots, x^k, 0, \ldots, 0), \tag{38}$$

Proof. TODO □

$$g_{ij} = \begin{cases} 0 & 1 \leq i \leq k \text{ and } k+1 \leq j \leq n \\ \delta_{ij} & p+1 \leq i, j \leq n. \end{cases} \tag{39}$$

Proof. TODO □

$$\Gamma^k_{ij} = 0, \quad p+1 \leq i, j \leq n. \tag{40}$$

Proof. TODO □

$$\partial_i g_{jk}(q) = 0, \quad p+1 \leq i, j, k \leq n \tag{41}$$

Proof. TODO □

$$\omega^{i,j} = g(\nabla_{e_k} e_i, e_j)\omega^k \tag{42}$$

Proof. The first definition tells us that $\nabla_{\partial_k} e_i = (\omega^{i,j})_k e_j$, and so

$$g(\nabla_{\partial_k}, e_i, e_\ell) = (\omega^{i,\ell})_k.$$

But this is equivalent to the second definition. □

$$\omega^{i,j}(X) = g(\nabla_X e_i, e_j). \tag{43}$$

Proof. TODO $\square$

$$\omega^{i,j} = -\omega^{j,i} \tag{44}$$

Proof.
$$\omega^{i,j} = \Gamma^j_{ki} = -\Gamma^i_{kj} = -\omega^{j,i}.$$ $\square$

$$d\omega^i = \omega^{i,j} \wedge \omega^j \tag{45}$$

Proof. Recall the identity (10):

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Applying this with $\omega = \omega^i$, $X = e_k$, and $Y = e_\ell$, we get

$$\begin{aligned}
d\omega(e_k, e_\ell) &= e_k(\omega^i(e_\ell)) - e_\ell(\omega^i(e_k)) - \omega^i([e_k, e_\ell]) \\
&= e_k(\delta^i_\ell) - e_\ell(\delta^i_k) - \omega^i(\nabla_{e_k} e_\ell - \nabla_{e_\ell} e_k) \\
&= -\omega^i(\Gamma^j_{k\ell} e_j - \Gamma^j_{\ell k} e_j) \\
&= -\Gamma^i_{k\ell} + \Gamma^i_{\ell k} \\
&= \Gamma^\ell_{ki} - \Gamma^k_{\ell i},
\end{aligned}$$

where in the last line we used 69 On the other hand, the right hand side becomes

$$\begin{aligned}
(\omega^{i,j} \wedge \omega^j)(e_k, e_\ell) &= \omega^{i,j}(e_k)\omega^j(e_\ell) - \omega^{i,j}(e_\ell)\omega^j(e_k) \\
&= g(\nabla_{e_k} e_i, e_j)\delta^j_\ell - g(\nabla_{e_\ell} e_i, e_j)\delta^j_k \\
&= g(\nabla_{e_k} e_i, e_\ell) - g(\nabla_{e_\ell} e_i, e_k) \\
&= \Gamma^\ell_{ki} - \Gamma^k_{\ell i}.
\end{aligned}$$ $\square$

$$d\omega^j(e_i, e_j) = \omega^{i,j}(e_j) - \omega^{j,k}(e_i). \tag{46}$$

Proof. First observe that since $\nabla_{e_k} e_i = \omega^{i,j}(e_k)e_j$, we have that $\omega^\ell(\nabla_{e_k} e_i) = \omega^{i,\ell}(e_k)$. Using this fact and the definition of the exterior derivative,

$$\begin{aligned}
d\omega^k(e_i, e_j) &= e_i\omega^k(e_j) - e_j\omega^k(e_i) - \omega^k([e_i, e_j]) \\
&= -\omega^k(\nabla_{e_i} e_j - \nabla_{e_j} e_i) \\
&= -\omega^{j,k}(e_i) + \omega^{i,k}(e_j).
\end{aligned}$$ $\square$

$$\omega^{i,k}(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{47}$$

Proof. Write out each term of the right hand side using (46), and use antisymmetry to cancel/combine terms. $\square$

$$\Omega_{i,j} = -\frac{1}{2}\,\mathrm{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell$$

Proof. TODO □

$$\Omega_{i,j}(e_k, e_\ell) = \mathrm{Rm}_{ijk\ell}\,. \tag{49}$$

Proof. Simply note that, since $\mathrm{Rm}_{ijk\ell} = -\mathrm{Rm}_{ij\ell k}$, we have

$$\begin{aligned}
\Omega_{i,j}(e_k, e_\ell) &= \frac{1}{2}(\mathrm{Rm}_{ijpq}\,\omega_p \wedge \omega_q)(e_k, e_\ell) \\
&= \frac{1}{2}\mathrm{Rm}_{ijpq}(\delta_{pk}\delta_{q\ell} - \delta_{p\ell}\delta_{qk}) \\
&= \frac{1}{2}\mathrm{Rm}_{ijk\ell} - \frac{1}{2}\mathrm{Rm}_{ij\ell k} \\
&= \mathrm{Rm}_{ijk\ell}\,.
\end{aligned}$$

□

$$\Omega^{i,j} = d\omega^{i,j} - \omega^{i,k} \wedge \omega^{k,j} \tag{50}$$

Proof. We give two proofs, respectively using the different definitions of $\Omega^{i,j}$. If we know that $\Omega^{i,j} = -\frac{1}{2}\mathrm{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell$, we proceed as follows. From (42), we have that $\omega_{i,j}(e_k) = g(\nabla_{e_k} e_i, e_j)$. By taking the exterior derivative of both sides (thinking of the right hand side as the covector $X \mapsto g(\nabla_X e_i, e_j)$), and using (10), we have

$$\begin{aligned}
d\omega^{i,j}(e_k, e_\ell) &= e_k\omega^{i,j}(e_\ell) - e_\ell\omega^{i,j}(e_k) - \omega^{i,j}([e_\ell, e_k]) \\
&= dg(\nabla e_i, e_j)(e_k, e_\ell) \\
&= e_k g(\nabla_{e_\ell} e_i, e_j) - e_\ell g(\nabla_{e_k} e_i, e_j) - g(\nabla_{[e_\ell, e_k]} e_i, e_j) \\
&= g(\nabla_{e_k}\nabla_{e_\ell} e_i, e_j) + g(\nabla_{e_\ell} e_i, \nabla_{e_k} e_j) - g(\nabla_{e_\ell}\nabla_{e_k} e_i, e_j) \\
&\quad - g(\nabla_{e_k} e_i, \nabla_{e_\ell} e_j) - g(\nabla_{[e_\ell, e_k]} e_i, e_j).
\end{aligned}$$

Observing that $\omega^{i,p}(e_\ell)\omega^{j,p}(e_k) = g(\nabla_{e_\ell} e_i, e_p)g(\nabla_{e_k} e_j, e_p) = g(\nabla_{e_\ell} e_i, \nabla_{e_k} e_j)$, we continue

$$\begin{aligned}
&= g(\nabla_{e_k}\nabla_{e_\ell} e_i - \nabla_{e_\ell}\nabla_{e_k} e_i - \nabla_{[e_\ell, e_k]} e_i, e_j) + \omega^{i,p}(e_\ell)\omega^{j,p}(e_k) - \omega^{i,p}(e_k)\omega^{j,p}(e_\ell) \\
&= \mathrm{Rm}_{k\ell ij} + (\omega^{i,p} \wedge \omega^{j,p})(e_\ell, e_k) \\
&= \mathrm{Rm}_{ijk\ell} + (\omega^{i,p} \wedge \omega^{p,j})(e_k, e_\ell) \\
&= (\Omega^{i,j} + \omega^{i,p} \wedge \omega^{p,j})(e_k, e_\ell)
\end{aligned}$$

TODO: the other version □

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2 Bundles

2.1 Fiber bundles

A fiber bundle over a topological space X is a way of associating a topological space to each point of X. Put another way, it is a "larger" topological space that locally looks like a product $X \times F$ for some space F. In particular, a fiber bundle is described by data (E, X, π, F). We call E the **total space** of the bundle, $\pi : E \to X$ the **bundle projection**, and the space $\pi^{-1}(x) \approx F$ the **fiber** (at x). We require the following local triviality condition: for every $x \in X$, there is a neighborhood U

of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that $\mathrm{proj}_U \circ \varphi = \pi$. We call (U, π) a **local trivialization** around x.

A **section** of a bundle maps each point of the base space to some element of the fiber at that point.

2.1.1 Pullback bundle

If (E, Y, π_Y, F) is a fiber bundle, and $f : X \to Y$ is a continuous map, then there is a bundle over X induced by f, called the **pullback bundle**. The fiber over a point x is just the fiber $\pi^{-1}(f(x))$. In particular, the pullback bundle f^*E is the space

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\} \subset X \times E$$

with the subspace topology and the projection map $\pi_X : f^*E \to X$ given by projection onto the first factor. If (U, φ) is a local trivialization for E, then $(f^{-1}(U), \psi)$ is a local trivialization for f^*E, where $\psi(x, e) = (x, \mathrm{proj}_2(\varphi(e)))$.

Moreover, a section σ of E over Y induces a section of f^*E, called the **pullback** of σ, by defining

$$f^*\sigma(x) \coloneqq (x, \sigma(f(x))).$$

2.2 Vector bundles

In Riemiannian geometry we very frequently deal with a special case called a vector bundle (over a manifold). This is when the base space X is actually a manifold, and the space at each point is a vector space of fixed dimension. The prototypical example of this is the tangent bundle.

Definition 2.1. A (smooth) vector bundle of rank n has data (E, M, π), where E (called the total space) and M (the base space) are smooth manifolds, and $\pi : E \to M$ is smooth. Moreover, each fiber $\pi^{-1}(p)$ is an n-dimensional real vector space, and the following condition, called local triviality, is satisified: for each $p \in M$, there is a neighborhood U and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for every $q \in U$, $\pi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n$ is a vector space isomorphism.

There are two notions of a bundle map that are closely related but worth distinguishing. The first is when we have two bundles over a common base.

Definition 2.2. If $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are bundles over a common space M, then a vector bundle homomorphism from E_1 to E_2 over M is a continuous map $\varphi : E_1 \to E_2$ such that

(1) $\pi_2 \circ \varphi = \pi_1$, and

(2) for any point $x \in M$, the map $\phi : \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ is a vector space homomorphism.

More generally, if we have $\pi_M : E_M \to M$ and $\pi_N : E_N \to N$, then a continuous map φ is a vector bundle homomorphism if there is a continuous map $f : M \to N$ such that

(1) $f \circ \pi_M = \pi_N \circ \phi$, i.e. that the following diagram commutes.

$$\begin{array}{ccc} E_M & \xrightarrow{\varphi} & E_N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N \end{array}$$

(2) for any point $x \in M$, the map $\phi : \pi_1^{-1}(x) \to \pi_2^{-1}(f(x))$ is a vector space homomorphism.

We call φ a bundle map covering f.

It follows immediately from the definitions that a bundle map over M in the first sense is a bundle map covering the identity map on M in the second sense. Conversely, general bundle maps can be thought of as bundle maps over a fixed base space using pullback bundles. A bundle map from $E_M \to E_N$ covering f is equivalent to a bundle map from $E_M \to f^*E_N$ over M.

2.2.1 Tangent bundle

At each point p of a n-dimensional manifold M, the tangent space T_pM is a vector space of rank n. These vector spaces can be put together to form a vector bundle, called the tangent bundle, denoted TM. Two things that make this bundle interesting are that smooth sections of the tangent bundle are vector fields, and if $f : M \to N$ is a smooth map between manifolds, then its derivative Df is a map $Df : TM \to TN$. We can also consider it as a map $Df : TM \to f^*TN$ (see the remarks on bundle homomorphisms).

2.3 Principal bundles

Another useful construction is a principal bundle; in this case we have a fiber bundle and a group G with a continuous right action on the bundle that preserves the fibers and acts freely and transitively. In this case the fibers are homeomorphic to G. A common example here is the frame bundle of a vector bundle.

Principal bundles may not have global sections. In particular, a principal bundle is trivial (i.e. has a global product structure) if and only if it admits a global section.

2.3.1 Tangent frame bundle

The frame bundle is a principal bundle where the fiber at a point $p \in M$ is the set of all ordered bases (frames) for T_pM, and the group that acts is the general linear group $\mathrm{GL}(n, \mathbb{R})$. The action is by change of basis. More generally one can construct a frame bundle corresponding to any vector bundle.

2.3.2 Orthonormal frame bundle

3 Tensors

3.1 Tensor products

Given two vector spaces V and W over $\mathbb{R}$, their **tensor product** is another vector space $V \otimes W$, with the characterizing property that there is a bilinear map

$$\otimes : V \times W \to V \otimes W$$

such that for any bilinear map $B : V \times W \to U$ (where U is a vector space), there is a unique linear map $\tilde{B} : V \otimes W \to U$ so that the diagram commutes:

$$\begin{tikzcd}
V \otimes W \arrow[dr, "\tilde{B}"] & \\
V \times W \arrow[u, "\otimes"] \arrow[r, "B"'] & U
\end{tikzcd}$$

In particular, if $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^m$ are bases for V, W respectively, then $\{e_i \otimes f_j\}_{ij \leq mn}$ is a basis for $V \otimes W$.

If $A : V \to V'$ and $B : W \to W'$ are linear maps, we can define their tensor product by

$$A \otimes B : V \otimes W \to V' \otimes W', \quad (A \otimes B)(v \otimes w) = A(v) \otimes B(w).$$

(Technically we have only defined it for tensors of the form $v \otimes w$ for some $v, w \in V, W$, but this definition extends linearly.)

An important isomorphism. There is a canonical isomorphism

$$V^* \otimes W \cong \mathrm{Hom}(V, W).$$

This is given by (the linear extension of) the map

$$\alpha \otimes w \mapsto [v \mapsto \alpha(v)w].$$

We are also often interested in a related product, the **symmetric tensor product**.

3.2 Tensor types and indices

In general, one can construct tensors on any (finite-dimensional) vector space (and its dual and tensor products of these), but we'll focus on the case where the underlying spaces are tangent spaces and their duals.

An (s,t)-tensor T is a section of $(TM)^{\otimes t} \otimes (T^*M)^{\otimes s}$. That is, it is a product of t vectors and s covectors, meaning that it takes t covectors and s vectors as input, so it has s lower (covariant) indices, and t upper (contravariant) indices. To add to the confusion, recall that we can think of a $(1,1)$-tensor either as an object that takes a vector and a covector and returns a scalar, or as an object that takes a vector and returns a vector. This generalizes: a (k,ℓ)-tensor can also be thought of as an object that takes k vectors and returns (a tensor product of) ℓ vectors. Sickeningly, not all authors agree on the convention for the roles of k and ℓ; for example some would call a (k,ℓ)-tensor what others would call (ℓ, k).

An $(k,0)$-tensor is called **covariant**, and a $(0,\ell)$-tensor is called **contravariant**. For example, forms are covariant, and vectors are contravariant. The terminology relates to how the components change under a change of basis. If we scale some basis vectors by a factor of C, then the components of a vector with respect to that basis scale by a factor of C^{-1}; hence contravariant.

In the presence of a basis, a tensor can be specified by its coordinates. For example, we could specify a form on $\mathbb{R}^2$ by $(x,y) \mapsto 2x - y$. This is the form $A_i dx^i$, where $i = 1,2$ and $A_1 = 2$, $A_2 = -1$. A linear endomorphism on a vector space has the form $A^i_j \partial_i \otimes dx^j$; this is commonly represented by the matrix of values A^i_j. In general, if $(e_i)_{i \leq n}$ is a basis for a vector space, and $(\epsilon^i)_{i \leq n}$ is a basis for its dual, a tensor (of type (p,q)) might look like

$$A = A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_q}.$$

Most of the time the horizontal position of the indices does not matter, but in general it can have some significance. We usually say that for a tensor, either all the "covector inputs" should come first, and then the "vector inputs" (or possibly the other way around for some authors), so we don't need to worry. However, we could have a tensor with nonstandard order of inputs, like some linear combination of $e_i \otimes \epsilon^j \otimes e_k$. In this case, we would write

$$A^i{}_j{}^k e_i \otimes \epsilon^j \otimes e_k.$$

We denote the set of (k,ℓ)-tensors by

$$\mathcal{T}_k^\ell = \Gamma(\otimes^\ell TM \otimes \otimes^k T^*M),$$

since, as mentioned above, the components of such a tensor have k lower indices and ℓ upper ones.

3.2.1 Type changing with the metric

The Riemannian metric provides an isomorphism between TM and T^*M. In particular, if X is a vector in T_pM, then defining $X^\flat$ by $X^\flat(Y) = g(X,Y)$, then $X^\flat$ is a covector in T_p^*M. To go backwards, if we have some covector $\omega \in T_p^*M$, if we define $\omega^\sharp$ to be the unique vector field that satisfies $g(\omega^\sharp, X) = \omega(X)$, we get the inverse mapping. For those not familiar with music theory, the notation for these objects (which, after it's been defined, is hardly ever used) comes from the fact that flat ($\flat$) means lower in pitch, and turning a vector $X = X^i\partial_i$ into a covector $\omega = \omega_i dx^i$ involves lowering its indices (and of course the opposite holds for the sharp ($\sharp$) notation which turns covectors into vectors).

For any $|k| \leq \min\{s,t\}$, we can make T into a $(s-k, t+k)$-tensor by using these natural isomorphisms. So in the tensor product above, we can replace TM's by T^*M's arbitrarily, and thereby get any sort of tensor we want with rank $s+t$.

In coordinates, we can write (given a frame E_i and the coframe ξ^i),

$$T^{i_1\cdots i_t}{}_{j_1\cdots j_s} E_{i_1} \otimes \cdots \otimes E_{i_t} \otimes \xi^{j_1} \otimes \cdots \otimes \xi^{j_s}.$$

TODO: check/fix the s and t's in the section.

Then to make T a $(s+1, t-1)$-tensor, replace some E_{i_k} by $g_{i_k j}\xi^j$ to get

$$T^{i_1\cdots i_{k-1}}{}_j{}^{i_{k+1}\cdots i_s}{}_{j_1\cdots j_t} E_{i_1} \otimes \cdots \otimes \xi^j \otimes \cdots \otimes E_{i_s} \otimes \xi^{j_1} \otimes \cdots \otimes \xi^{j_t},$$

where

$$T^{i_1\cdots i_{k-1}}{}_j{}^{i_{k+1}\cdots i_s}{}_{j_1\cdots j_t} := g_{i_k j} T^{i_1\cdots i_s}{}_{j_1\cdots j_t}$$

TODO: I think the s and t are wrong here

For an explicit example, see the section on the Ricci tensor.

3.3 Contractions and traces

TODO: this section needs help; see Andrews-Hopper p. 22 and Lee p. 395. Don't forget that these have different notation for tensor indices.

Given a (k,ℓ)-tensor T, where $k,\ell \geq 1$, we can form various $(k-1,\ell-1)$-tensors by **tracing** T; that is, by evaluating one of the covector factors of T at one of the vector factors. Specifically, there are $k\ell$ different traces we can take, since we can evaluate any of the covector fields at any of the vector fields. In the case where T is a $(1,1)$-tensor,

$$\operatorname{tr}(T) = \operatorname{tr}(T^i{}_j\, E_i \otimes \xi^j) = T^i{}_j\, \xi^j(E_i) = T^i{}_j\, \delta_i^j = T^i{}_i\,.$$

More generally, if T is a (k,ℓ)-tensor, and we evaluate the a^{th} factor of T at the b^{th} factor of T, we have, for vector fields $X_1, \ldots, X_{k-1}$, and covector fields $\omega_1, \ldots, \omega_{\ell-1}$,

$$\begin{aligned}&(\operatorname{tr}_{ab} T)(\omega_1, \ldots, \omega_{\ell-1}, X_1, \ldots, X_{k-1})\\ &= \operatorname{tr}[(\omega, X) \mapsto T(\omega_1, \ldots, \omega_{a-1}, \omega, \omega_{a+1}, \ldots, \omega_{\ell-1}, X_1, \ldots, X_{b-\ell-1}, X, X_{b-\ell+1}, \ldots, X_{k+1})]\end{aligned}$$

where on the right hand side we are now just taking the trace over a $(1,1)$ tensor again. In coordinates, this is just

$$\operatorname{tr}_{ab} T = T^{i_1\cdots p\cdots i_\ell}{}_{j_1\cdots p\cdots j_k}\,\partial_{i_1}\cdots\partial_{i_{a-1}}\partial_{i_{a+1}}\cdots\partial_{i_\ell}dx^{j_1}\cdots dx^{j_{b-\ell-1}}dx^{j_{b-\ell+1}}\cdots dx^{j_k}.$$

Using the isomorphism induced by g between TM and T^*M (see the previous section), we can take the trace over any pair of indices of any tensor (with rank at least 2). The idea is, given a pair of indices, if they are both covariant or both contravariant, we can isomorphically replace one or the other by its covariant or contravariant dual under the metric. Then we have reduced to one of the cases discussed previously. A useful example is the divergence of a 1-form ω, which can be defined as the trace of the $(2,0)$-tensor $\nabla\omega$.

TODO

The norm squared of a tensor can be compared to the square of the trace. In particular, if T is a (1,1)-tensor, then

$$n\,|T|^2 \geq \operatorname{tr}(T)^2. \tag{51}$$

3.4 Pullbacks and pushforwards

3.4.1 Pushforward of a vector field

If $F : M \to N$ is a smooth map, then for each $p \in M$, the derivative F_* (also called the pushforward of F), is a smooth map $F_*|_p : T_pM \to T_{F(p)}N$. Thus, if we have a vector field X on M, there is a vector field F_*X on N defined pointwise by applying F_* to X.

3.4.2 Pullback of a $(k,0)$-tensor

If $F : M \to N$ is a smooth map between manifolds, and $\xi \in \mathcal{T}^0_k(M)$ is a tensor field on N, then there is a tensor field on M induced by F, called the **pullback** of ξ. This is defined by the following identity, for a point $p \in M$ and vector fields X_i (although on the right hand side we omit the p).

$$(F^*\xi)_p(X_1,\ldots,X_k) = \xi(F_*X_1,\ldots,F_*X_k),$$

3.5 The induced metric on tensor bundles

Suppose we have a metric g on some bundle $\pi : E \to M$. That is, we have a section of $E^* \otimes E^*$ such that at each point $p \in M$, g_p is an inner product on the fiber E_p. Then the metric on E defines a bundle isomorphism $\iota_g \colon E \to E^*$ by

$$\iota_g(\xi) : \eta \mapsto g_p(\xi,\eta), \quad \xi,\eta \in E_p.$$

Moreover, there is a unique metric g on E^* such that ι_g is a bundle isometry:

$$g(\iota_g(\xi),\iota_g(\eta)) = g(\xi,\eta), \quad \xi,\eta \in E_p.$$

So we see that a metric extends to tensor duals. It also extends to tensor products. Given bundles E_1, E_2 with metrics g_1, g_2,

$$g = g_1 \otimes g_2 \in \Gamma((E_1^* \otimes E_1^*) \otimes (E_2^* \otimes E_2^*)) \equiv \Gamma((E_1 \otimes E_2)^* \otimes (E_1 \otimes E_2)^*)$$

is the unique metric such that

$$g(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) = g_1(\xi_1,\xi_2)g_2(\eta_1,\eta_2).$$

The induced metric on a tensor product of dual bundles agrees with the induced metric on a dual bundle of a tensor product. So, starting with a metric on the tangent bundle TM, we get metrics (all denoted g) on all the tensor bundles $\mathcal{T}_k^\ell$. We can write this in coordinates in the following way. For $S, T \in \mathcal{T}_k^\ell(M)$, we have (at a point p),

$$g(S,T) = g^{a_1 b_1} g^{a_2 b_2} \cdots g^{a_k b_k} g_{i_1 j_1} \cdots g_{i_\ell j_\ell} S^{i_1 \ldots i_\ell}_{a_1 \ldots a_k} T^{j_1 \ldots j_\ell}_{b_1 \ldots b_k}.$$

3.6 Proofs

$$n\,|T|^2 \geq \operatorname{tr}(T)^2. \tag{51}$$

Proof. This follows from (a corollary to) the Cauchy-Schwarz inequality:

$$\sum_{i \leq n} x_i^2 \geq \frac{1}{n}\left(\sum_{i \leq n} x_i\right)^2.$$

$$\begin{aligned} n\,|T|^2 &= n\,\langle T, T\rangle \\ &= n T_i^i \\ &\geq \left(T_i^i\right)^2 \\ &= \operatorname{tr}(T)^2. \end{aligned}$$

□

4 Lie Derivatives

Let X, Y be vector fields, and let $\Psi_t = \Psi_{X,t}$ be the flow of X, so that $\Psi_t(x)$ is the point that x is "flowed to" by X after time t. Then $D\Psi_t|_x$ is an isomorphism between T_xM and $T_{\Psi_t(x)}M$. (Note that in this case the pullback is the inverse of the differential, so it does not matter if we use the pullback or the inverse of the pushforward.) What we get is that $(D\Psi_t|_x)^{-1}(Y_{\Psi_t(x)})$ is an element of T_xM for each t, so we can differentiate this at $t = 0$. With these remarks in mind, we define the **Lie derivative** of Y along the flow of X to be the vector field

$$(\mathcal{L}_X Y)_x = \left.\frac{d}{dt}\right|_{t=0} \left((D\Psi_t|_x)^{-1}(Y_{\Psi_t(x)})\right).$$

There are many different notations for the quantities in this equation. It is usually written more concisely, e.g. as

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} (\Psi_t^* Y).$$

Define the **Lie bracket** of vector fields by

$$[X, Y](f) := X(Y(f)) - Y(X(f)),$$

then

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X. \tag{52}$$

TODO: look at time-dependent families of diffeomorphisms and their derivatives a bit more generally

4.1 Lie derivatives of tensor fields

If X is a vector field and α is a tensor field, we define (TODO: make sure the notation is consistent; this came from Lee)

$$(\mathcal{L}_X\alpha)_p = \frac{d}{dt}\Big|_{t=0}(\Psi_t^*\alpha)_p = \lim_{t\to 0}\frac{d(\Psi_t)_p^*(\alpha_{\Psi_t(p)}) - \alpha_p}{t}.$$

Note that $d(\Psi_t)_p^*(\alpha_{\Psi_t(p)})$ is the value of the pullback tensor field $\Psi_t^*\alpha$ at p.

For vector fields $V, X_1, \ldots, X_k$, and a covariant tensor field A in $\mathcal{T}_k^0(M)$,

$$(\mathcal{L}_V A(X_1,\ldots,X_k)) = (\mathcal{L}_V A)(X_1,\ldots,X_k) + \sum_{i=1}^{k} A(X_1,\ldots,\mathcal{L}_V X_i,\ldots,X_k), \tag{53}$$

From this it follows that

$$(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \tag{54}$$

In particular, we have

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$$

and in more particular, for a function f,

$$\mathcal{L}_{\nabla f} g = 2\nabla^2 f, \qquad (\mathcal{L}_{\nabla f} g)_{ij} = 2\nabla_i\nabla_j f.$$

From this we get that

$$\mathcal{L}_V(df) = d(\mathcal{L}_V f). \tag{55}$$

Cartan's formula states that for any differential form ω and any (smooth) vector field V,

$$\mathcal{L}_V\omega = \iota_V(d\omega) + d(\iota_V\omega). \tag{56}$$

We have the following product rules for a Lie derivative of a wedge product or a tensor product. If V is a vector field and α, β are forms,

$$\mathcal{L}_V(\alpha\wedge\beta) = (\mathcal{L}_V\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_V\beta). \tag{57}$$

If V is a vector field and A, B are tensor fields,

$$\mathcal{L}_V(A\otimes B) = (\mathcal{L}_V A)\otimes B + A\otimes(\mathcal{L}_V B). \tag{58}$$

There is also the following identity for the interior product. For a differential form ω and vector fields V, W,

$$\mathcal{L}_W(\iota_V\omega) = \iota_V(\mathcal{L}_W\omega) + \iota_{[W,V]}\omega. \tag{59}$$

If $\varphi : N \to M$ is a diffeomorphism, X is a vector field, and α is any tensor,

$$\varphi^*(\mathcal{L}_X\alpha) = \mathcal{L}_{\varphi^* X}(\varphi^*\alpha) \tag{60}$$

4.2 Killing fields

A vector field X is called **Killing** if $\mathcal{L}_X g = 0$. This is the same as saying that the flow of X generates a 1-parameter family of isometries of (M, g). Moreover, the Lie bracket of any two Killing vector fields is Killing, and if (M, g) is complete, then any Killing field is complete. It follows that the Lie algebra of Killing vector fields is naturally isomorphic to the Lie algebra of the Lie group of isometries of (M, g).

The following identity holds for any vector field V,

$$\Delta V + \mathrm{Rc}(V) = \mathrm{div}(\mathcal{L}_V g) - \frac{1}{2}\nabla \,\mathrm{tr}(\mathcal{L}_V g), \tag{61}$$

$$\Delta V + \mathrm{Rc}(V) = \mathrm{div}(\mathcal{L}_V g) - \frac{1}{2}\nabla \,\mathrm{tr}(\mathcal{L}_V g), \tag{61}$$

Proof. TODO □

---

which for Killing fields implies that

$$\Delta V + \mathrm{Rc}(V) = 0 \tag{62}$$

$$\Delta V + \mathrm{Rc}(V) = 0 \tag{62}$$

Proof. TODO □

4.3 Proofs

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X. \tag{52}$$

Proof. Consider the action of $\mathcal{L}_X Y$ on a smooth function f. Note that by the chain rule and the fact that $\mathrm{Id}_{T_xM} = \Psi_{-X,t} \circ \Psi_{X,t}$

$$D\Psi_{-X,t}|_{\Psi_{X,t}(x)} \circ D\Psi_{X,t}|_x = \mathrm{Id}_{T_xM},$$

so $(D\Psi_{X,t}|_x)^{-1} = D\Psi_{-X,t}|_{\Psi_{X,t}(x)}$. By definition of the derivative,

$$(D\Psi_{-X,t}|_x(Y))f = Y|_{\Psi_{X,t}(x)}(f \circ \Psi_{-X,t}).$$

Then

$$\begin{aligned}\mathcal{L}_X Y|_x(f) &= \frac{d}{dt}\left((D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})\right)\big|_{t=0} \\ &= \frac{d}{dt}\left(Y|_{\Psi_{X,t}(x)}(f \circ \Psi_{-X,t})\right)\end{aligned}$$

TODO □

$$(\mathcal{L}_V A(X_1, \ldots, X_k)) = (\mathcal{L}_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, \mathcal{L}_V X_i, \ldots, X_k) \tag{53}$$

Proof. TODO $\square$

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \tag{54}$$

Proof. By (53) and (52),

$$\begin{aligned}
(\mathcal{L}_X g)(Y, Z) &= \mathcal{L}_X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z]) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y, Z) + g(\nabla_Y X, Z) - g(Y, \nabla_X Z) + g(Y, \nabla_Z X) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X).
\end{aligned}$$

$\square$

$$\mathcal{L}_V(df) = d(\mathcal{L}_V f) \tag{55}$$

Proof. By (**??**),

$$\begin{aligned}
(\mathcal{L}_V df)(X) &= V(df(X)) - df([V, X]) \\
&= V(X(f)) - df(V(X) - X(V)) \\
&= X(V(f)) \\
&= X(\mathcal{L}_V f) \\
&= d(\mathcal{L}_V f)(X).
\end{aligned}$$

$\square$

$$\mathcal{L}_V \omega = \iota_V(d\omega) + d(\iota_V \omega). \tag{56}$$

Proof. We'll prove that it holds for k-forms by induction on k. The base case is a 0-form $f \in C^\infty(M)$. Then the right hand side becomes $V(f)$ since $\iota_V f = 0$ by definition. So we see that the result holds in this case.

Now let $k \geq 1$ and suppose Cartan's formula has been proved for forms of degree less than k. Let ω be an arbitrary smooth k-form, written in local coordinates as

$$\omega = \sum_{I = i_1 < \cdots < i_k} \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Writing $u = x^{i_1}$ and $\beta = \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, we see that each term in this sum can be written in the form $du \wedge \beta$, where u is a smooth function and β is a smooth $(k-1)$-form. By (55), $\mathcal{L}_V(du) = d(\mathcal{L}_V u) = d(Vu)$. Using this fact and the product rule (for wedge products),

$$\begin{aligned}
\mathcal{L}_V(du \wedge \beta) &= (\mathcal{L}_V du) \wedge \beta + du \wedge (\mathcal{L}_V \beta) \\
&= d(Vu) \wedge \beta + du \wedge (\iota_V(d\beta) + d(\iota_V \beta)) \\
&= d(Vu) \wedge \beta + du \wedge (\iota_V(d\beta)) + du \wedge (d(\iota_V \beta)) \\
&= -(Vu)d\beta + d(Vu) \wedge \beta + du \wedge (\iota_V(d\beta)) + (Vu)d\beta + du \wedge (d(\iota_V \beta))
\end{aligned}$$

Now, using the definition of the interior product (see 1.4.3)

$$\iota_V(-du \wedge d\beta) = \iota_V(-du) \wedge d\beta - (-du) \wedge \iota_V(d\beta) = -\iota_V(du) \wedge d\beta + du \wedge \iota_V(d\beta)$$

TODO: finish $\square$

$$\mathcal{L}_V(\alpha \wedge \beta) = (\mathcal{L}_V\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_V\beta). \tag{57}$$

Proof. TODO ☐

$$\mathcal{L}_V(A \otimes B) = (\mathcal{L}_V A) \otimes B + A \otimes (\mathcal{L}_V B) \tag{58}$$

Proof. TODO: See LeeSM, p. 322. The idea is to pick the right coordinates for which this follows from the ordinary product rule. ☐

$$\mathcal{L}_W(\iota_V\omega) = \iota_V(\mathcal{L}_W\omega) + \iota_{[W,V]}\omega. \tag{59}$$

Proof. This would follow once we have shown (60), since the wedge product is a linear combination of tensor products.

I think maybe we can also do this straight from the definition of the Lie derivative TODO ☐

$$\varphi^*(\mathcal{L}_X\alpha) = \mathcal{L}_{\varphi^*X}(\varphi^*\alpha) \tag{60}$$

Proof. TODO ☐

5 Connections on vector bundles

A connection is exactly what is needed to differentiate sections of a vector bundle in the direction of some tangent vector (field) on the manifold. The special case that arises most often in Riemannian geometry is the Levi-Civita connection, which is the unique connection on the tangent bundle (meaning we can differentiate vector fields) which satisfies certain compatibility properties with the Riemannian metric.

Let E be a vector bundle over M. A **connection** ∇ on E is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ (written as $(X, \xi) \mapsto \nabla_X\xi$, and we think of the first entry as specifying the direction in which to take the derivative of the second entry) that satisfies the following properties.

(1) C^∞ linearity in X:
$$\nabla_{X+fY}\xi = \nabla_X\xi + f\nabla_Y\xi.$$

(2) $\mathbb{R}$-linearity in ξ:
$$\nabla_X(r\xi) = r\nabla_X\xi.$$

(3) A product/Leibniz rule in ξ:
$$\nabla_X(f\xi) = X(f) \cdot \xi + f\nabla_X\xi.$$

The connection coefficients of such a connection can be defined with respect to a given local frame $\{\xi_i\}$ for E by the equation
$$\nabla_i\xi_j = \mathbf{\Gamma}_{ij}^k\xi_k.$$
In the case where $E = TM$, these coefficients are just the Christoffel symbols Γ_{ij}^k.

5.1 The pullback connection

Let ∇ be a connection on E over N, and $f : M \to N$ a smooth map. For $\xi \in \Gamma(E)$, define the restriction $\xi_f \in \Gamma(f^*E)$ by

$$\xi_f(p) = \xi(f(p)) \in E_{f(p)} = (f^*E)_p,$$

for all $p \in M$.

There is a unique connection ${}^f\nabla$ on f^*E, called the pullback connection, such that

$${}^f\nabla_v(\xi_f) = \nabla_{f_*v}\xi$$

for any $v \in TM$ and $\xi \in \Gamma(E)$. This definition isn't quite complete; not every element of $\Gamma(f^*E)$ is the restriction of some section of E. For arbitrary $\xi \in \Gamma(f^*E)$, fix $p \in M$ and choose a local frame $\sigma_1, \ldots, \sigma_k$ about $f(p)$ for E. Then we can write $\xi = \xi^i(\sigma_i)_f$ with each ξ^i a smooth function defined near p, so the rules for a connection together with the definition of pullback connection give

$$\begin{aligned}
{}^f\nabla_v\xi &= {}^f\nabla_v(\xi^i(\sigma^i)_f) \\
&= \xi^i\, {}^f\nabla_v(\sigma_i)_f + v(\xi^i)(\sigma_i)_f \\
&= \xi^i\nabla_{f_*v}\sigma_i + v(\xi^i)(\sigma_i)_f.
\end{aligned}$$

If ∂_i is a local frame for TM and ξ_i is the pullback of some local frame for TN (so that ξ_i is a local frame for f^*TN), we can describe the pullback connection in coordinates by

$${}^f\nabla_{\partial_i}\xi_j = \frac{\partial f^\ell}{\partial x^i}(\Gamma^k_{\ell j} \circ f)\xi_k. \tag{63}$$

Note that for a metric g on E, if g is compatible with ∇, then ${}^f\nabla_v(g_f) = \nabla_{f_*v}g = 0$, meaning that ${}^f\nabla$ is compatible with the restriction metric g_f.

We also have that the curvature of the pullback connection is the pullback of the curvature of the original connection, i.e.

$$R_{{}^f\nabla}(X,Y)\xi_f = (f^*R_\nabla)(X,Y)\xi, \quad X, Y \in \Gamma(TM), \xi \in \Gamma(E). \tag{64}$$

5.2 Connections on product structures

A useful special case is when we are given a connection on the tangent bundle TM. Then by requiring a few properties to hold, this connection extends to a unique connection, denoted ∇, on the tensor bundle of M (which can be thought of as formed by taking arbitrary duals and products of TM). The properties are:

(1) On TM, ∇ agrees with the given connection.

(2) On $C^\infty(M) = T^0M$, ∇ is the action of a vector as a derivation:

$$\nabla_X f = Xf,$$

for any smooth function f.

(3) ∇ obeys the product rule with respect to tensor products:

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G),$$

for any tensors F and G.

(4) ∇ commutes with all contractions:

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F),$$

for any tensor F.

We say that a connection is **compatible** with the metric if it additionally satisfies

(4) metric compatibility:

$$X(g(\xi,\eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta),$$

which can also be stated as

$$\nabla g = 0,$$

where the left hand side (and the right hand side) is a tensor field in $\mathcal{T}^0_3(M)$. The proof that these conditions are equivalent follows from the definition of ∇g below.

5.3 The Levi-Civita Connection

The Levi-Civita connection for a given Riemannian metric g is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ (written as $(X, \xi) \mapsto \nabla_X \xi$) that satisfies the preceding 4 properties, and is

(5) torsion-free (also known as symmetric):

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

It satisfies **Koszul's formula**:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle. \tag{65}$$

The metric compatibility condition tells us that

$$\nabla(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y), \tag{66}$$

where we interpret the right hand side as the covector $Z \mapsto g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$. The fundamental theorem of Riemannian geometry states that, given a Riemannian manifold (M, g), there is a unique connection ∇ on TM that is symmetric and compatible with g. So we are justified when we say *the* Levi-Civita connection. To paraphrase Andrews-Hopper we mention that this connection is canonical because the symmetry and compatibility conditions are invariantly defined natural properties that force the connection to coincide with the tangential connection, whenever M is realized as a submanifold of $\mathbb{R}^n$ with the induced metric (which is always possible by the Nash embedding).

5.4 Christoffel symbols

Given some coordinate basis $\{\partial_i\}_{i=1}^n$, the **Christoffel symbols** (of the Levi-Civita connection) are the unique coefficients (i.e. smooth functions) satisfying

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k.$$

It follows from this and the above properties that

$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma^k_{ij})\partial_k. \tag{67}$$

In particular,

$$\nabla_i X = (\partial_i X^\ell + X^j \Gamma^\ell_{ij})\partial_\ell.$$

For the Levi-Civita connection, we can calculate these coefficients in coordinates by

$$\Gamma^k_{ji} = \Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}). \tag{68}$$

The symbols also satisfy the antisymmetry relation

$$\Gamma^k_{ij} = -\Gamma^j_{ik}. \tag{69}$$

Despite using the same notation, we cannot think of the Christoffel symbols as a $(1, 2)$-tensor. However, given two metrics $g, \widetilde{g}$ the difference of the coefficients of the two corresponding Levi-Civita connections does form a tensor:

$$\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}$$

TODO: write more about this.

5.5 Covariant derivatives

We can take directional derivatives of functions using only the differentiable structure of a manifold. The covariant derivative is defined using the metric, and allows us to differentiate vector fields and other tensors. We already know how to take the covariant derivative of a vector field, which is a $(0, 1)$-tensor. As mentioned above, this now extends to arbitary (k, ℓ)-tensors. First generalize to $(0, \ell)$-tensors: $\nabla_X : \mathcal{T}^\ell_0(M) \to \mathcal{T}^\ell_0(M)$ is defined by

$$\nabla_X(X_1 \otimes \cdots \otimes X_\ell) \coloneqq \sum_{i=1}^{\ell} X_1 \otimes \cdots \otimes \nabla_X X_i \otimes \cdots \otimes X_\ell.$$

By requiring the covariant derivative to satisfy a product rule (the following is purely symbolic the first time we write it)

$$\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y),$$

we can define the first term on the right hand side, since all other terms we understand. Note that this is just property (3) from the above requirements for extending to tensor bundles. Thus we have the covariant derivative of 1-forms:

$$(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y).$$

In coordinates (here we are referring to the total covariant derivative of ω, see below),

$$\nabla_i \omega_j \coloneqq (\nabla \omega)_{ij} = \partial_i \omega_j - \Gamma^k_{ij} \omega_k \tag{70}$$

If $F \in \mathcal{T}^k_\ell(M)$ is a tensor field, and X, Y_k are vector fields and ω^j are 1-forms, then we have the following, which is sometimes used as a definition, but in fact it follows from the requirements for a covariant derivative to extend from the tangent bundle to the tensor bundle.

$$\begin{aligned}(\nabla_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) = {} & X(F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k)) \\ & - \sum_{j=1}^{\ell} F(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^\ell, Y_1, \ldots, Y_k) \\ & - \sum_{i=1}^{k} F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k).\end{aligned} \tag{71}$$

We can think of ∇F as a $(k+1,\ell)$-tensor field, called the **total covariant derivative** of F, by

$$(\nabla F)(\omega^1,\ldots,\omega^\ell,Y_1,\ldots,Y_k,X) = (\nabla_X F)(\omega^1,\ldots,\omega^\ell,Y_1,\ldots,Y_k).$$

There are different conventions about where to put the X in this definition; I'm not sure if it matters. An important property of covariant derivatives is that they "commute with contractions," a property that follows from the fact that $\nabla g \equiv 0$. TODO

There is also a horrible expression for the covariant derivative in coordinates TODO

See LeeRM for stuff about covariant derivative in coordinates, as well as the semicolon notation, which may actually be kind of good TODO.

The following formula for commuting covariant derivatives at the expense of introducing a Riemann curvature term is quite useful, although it's probably better to just look at the more common special cases below.

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\alpha^{\ell_1\ldots\ell_s}_{k_1\ldots k_r} = -\sum_{h=1}^{r} \mathrm{Rm}^p_{ijk_h}\,\alpha^{\ell_1\ldots\ell_s}_{k_1\ldots k_{h-1}pk_{h+1}\ldots k_r} - \sum_{h=1}^{s} \mathrm{Rm}^{\ell_h}_{ijp}\,\alpha^{\ell_1\ldots\ell_{h-1}p\ell_{h+1}\ldots\ell_s}_{k_1\ldots k_r} \tag{72}$$

For example, if ω is a 1-form,

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_k = -\,\mathrm{Rm}^{\ell}_{ijk}\,\omega_\ell. \tag{73}$$

If T is a $(2,0)$-tensor, then

$$\nabla_i\nabla_j T_{k\ell} - \nabla_j\nabla_i T_{k\ell} = -R^p_{ijk}T_{p\ell} - R^p_{ij\ell}T_{kp}. \tag{74}$$

5.6 The Hessian

We can then take the total covariant derivative of F to get the **Hessian** of F, sometimes denoted $\nabla^2 F$, which is of course a $(k+2,\ell)$-tensor field. It follows from the torsion-free property of the Levi-Civita connection that the Hessian is symmetric:

$$(\nabla^2 f)(X,Y) = (\nabla^2 f)(Y,X). \tag{75}$$

We have

$$\nabla^2_{X,Y}F \coloneqq (\nabla^2 F)(X,Y) = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y}F. \tag{76}$$

The proof makes it more clear how the tensors on the right hand side actually work. In the case of a function f, we have that (the first equality follows immediately from (76))

$$\begin{aligned}(\nabla^2 f)(X,Y) &= X(Y(f)) - (\nabla_X Y)(f)\\ &= g(\nabla_X \operatorname{grad} f, Y)\\ &= \frac{1}{2}(\mathcal{L}_{\operatorname{grad} f}\, g)(X,Y).\end{aligned} \tag{77}$$

In coordinates, we can write

$$\nabla_i\nabla_j f = \partial_i(\partial_j f) - \Gamma^k_{ij}\partial_k f. \tag{78}$$

In particular, since $(\nabla^2 f)(X,Y) = g(\nabla_X \operatorname{grad} f, Y)$, the $(1,1)$-tensor associated to ∇^2 is given by $(\nabla^2 f)(X) = \nabla_X \operatorname{grad} f$. The Hessian satisfies the following product rule for functions.

$$\nabla^2(fh) = f\nabla^2 h + h\nabla^2 f + \nabla f\otimes\nabla h + \nabla h\otimes\nabla f. \tag{79}$$

The following is a commutator formula for the Laplacian and the Hessian on functions

$$\Delta(\nabla_i\nabla_j f) = \nabla_i\nabla_j\Delta f + R_{j\ell}\nabla_i\nabla_j f + R_{i\ell}\nabla_\ell\nabla_j f - 2R^\ell_{kij}\nabla_k\nabla_\ell f + (\nabla_j R_{\ell i} + \nabla_i R_{j\ell} - \nabla_\ell R_{ji})\nabla_\ell f. \quad (80)$$

In two dimensions, and applied to scalar curvature, this becomes

$$\Delta(\nabla_i\nabla_j R) = \nabla_i\nabla_j\Delta R + 2R\nabla_i\nabla_j R - (R\Delta R - \tfrac{1}{2}|\nabla R|^2)g_{ij} + \nabla_i R\nabla_j R. \quad (81)$$

5.6.1 Bochner Formulas

For any function u on a Riemannian manifold,

$$\tfrac{1}{2}\Delta|\nabla u|^2 = \langle \Delta\nabla u, \nabla u\rangle + \left|\nabla^2 u\right|^2 = \langle \nabla\Delta u, \nabla u\rangle + \mathrm{Rc}(\nabla u, \nabla u) + \left|\nabla^2 u\right|^2. \quad (82)$$

The second inequality follows from the commutator formula for the Laplacian and the covariant derivative:

$$\Delta(du) = d(\Delta u) + \mathrm{Rc}(\nabla u), \quad (83)$$

sometimes also written as $\Delta\nabla u = \nabla\Delta u + \mathrm{Rc}(\nabla u)$.

5.7 Proofs for Section 5

$$^f\nabla_{\partial_i}\xi_j = \frac{\partial f^\ell}{\partial x^i}(\Gamma^k_{\ell j}\circ f)\xi_k. \quad (63)$$

Proof. Let y_i be local coordinates on N. Recall that

$$f_*|_x(\partial_i|_x) = \frac{\partial f^\ell}{\partial x^i}(x)\partial_{y^\ell}|_{f(x)}$$

(see (6)), here ϕ^ℓ is the ℓth coordinate of ϕ in some local coordinates on N (or do I mean on f^*E). By definition,

$$\begin{aligned}
({}^f\nabla_{\partial_i}\xi_j)(x) &= f^*(\nabla_{f_*\partial_i}\partial_{y^j}) \\
&= f^*\left(\nabla_{\frac{\partial f^\ell}{\partial x^i}\partial_{y^\ell}}\partial_{y^j}\right) \\
&= f^*\left(\frac{\partial f^\ell}{\partial x^i}(x)\Gamma^k_{\ell j}(f(x))\partial_{y^k}\right) \\
&= \frac{\partial f^\ell}{\partial x^i}(x)\Gamma^k_{\ell j}(f(x))\xi_k(x)
\end{aligned}$$

□

$$R_{^f\nabla}(X,Y)\xi_f = (f^*R_\nabla)(X,Y)\xi, \quad X,Y\in\Gamma(TM), \xi\in\Gamma(E). \quad (64)$$

Proof. TODO; see [1] □

$$2\langle\nabla_X Y, Z\rangle = X\langle Y,Z\rangle - Z\langle X,Y\rangle + Y\langle Z,X\rangle - \langle [Y,Z],X\rangle + \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle. \quad (65)$$

Proof. The metric compatibility condition says

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$
$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$
$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

By adding/subtracting these expressions, using symmetry and linearity of the metric, and the torsion-free property ($\nabla_X Y - \nabla_Y X = [X,Y]$), we obtain

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = 2\langle \nabla_X Y, Z\rangle - \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle.$$

□

$$\nabla(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y). \tag{66}$$

Proof. This really does follow immediately from the metric compatibility condition (don't forget that we are interpreting the left-hand side as the covector $Z \mapsto g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$). □

$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma^k_{ij})\partial_k \tag{67}$$

Proof.

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^i\partial_i} Y^j \partial_j \\
&= X^i \nabla_{\partial_i}(Y^j \partial_j) \\
&= X^i[\partial_i(Y^j)\partial_j + Y^j \nabla_{\partial_i}\partial_j] \\
&= X^i[\partial_i(Y^j)\partial_j + Y^j \Gamma^k_{ij}\partial_k] \\
&= (X(Y^k) + X^i Y^j \Gamma^k_{ij})\partial_k.
\end{aligned}$$

□

$$\Gamma^k_{ji} = \Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \tag{68}$$

Proof. Apply the Koszul formula (65) to coordinate basis vectors:

$$\begin{aligned}
2\Gamma^\ell_{ij} g_{\ell k} &= 2\left\langle \Gamma^\ell_{ij}\partial_\ell, \partial_k\right\rangle \\
&= 2\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle \\
&= \partial_i\langle \partial_j, \partial_k\rangle - \partial_k\langle \partial_i, \partial_j\rangle + \partial_j\langle \partial_k, \partial_i\rangle \\
&= \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}.
\end{aligned}$$

Now multiply both sides by g^{km} and the result follows. □

$$\Gamma^k_{ij} = -\Gamma^j_{ik} \tag{69}$$

Proof. The metric compatibility condition applied to the correct basis vectors says

$$\begin{aligned}
0 &= (\nabla g)(\partial_j, \partial_k, \partial_i) \\
&= \nabla_{\partial_i} g(\partial_j, \partial_k) \\
&= g(\nabla_i e_j, e_k) + g(\nabla_i e_k, e_j) \\
&= \Gamma^k_{ij} + \Gamma^j_{ik}.
\end{aligned}$$

□

$$\nabla_i \omega_j \coloneqq (\nabla\omega)_{ij} = \partial_i \omega_j - \Gamma_{ij}^k \omega_k \tag{70}$$

Proof.

$$\begin{aligned}
(\nabla\omega)(\partial_i, \partial_j) &= (\nabla_i \omega)(\partial_j) \\
&= \nabla_i(\omega_j) - \omega(\nabla_i \partial_j) \\
&= \partial_i \omega_j - \omega(\Gamma_{ij}^k \partial_k) \\
&= \partial_i \omega_j - \Gamma_{ij}^k \omega_k.
\end{aligned}$$

□

$$\begin{aligned}
(\nabla_X F)(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k) = X(F(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k)) \\
- \sum_{j=1}^{\ell} F(\omega^1, \dots, \nabla_X \omega^j, \dots, \omega^\ell, Y_1, \dots, Y_k) \\
- \sum_{i=1}^{k} F(\omega^1, \dots, \omega^\ell, Y_1, \dots, \nabla_X Y_i, \dots, Y_k).
\end{aligned} \tag{71}$$

Proof. TODO; see Andrews-Hopper p. 26 □

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_{k_1 \dots k_r}^{\ell_1 \dots \ell_s} = - \sum_{\ell=1}^{r} \mathrm{Rm}_{ijk_h}^{p} \, \alpha_{k_1 \dots k_{h-1} p k_{h+1} \dots k_r}^{\ell_1 \dots \ell_s} - \sum_{h=1}^{s} \mathrm{Rm}_{ijp}^{\ell_h} \, \alpha_{k_1 \dots k_r}^{\ell_1 \dots \ell_{h-1} p \ell_{h+1} \dots \ell_s}. \tag{72}$$

Proof. TODO □

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \omega_k = - \mathrm{Rm}_{ijk}^{\ell} \, \omega_\ell. \tag{73}$$

Proof. This follows immediately from (72), but I guess there should be a direct way to prove it as well.

TODO □

$$\nabla_i \nabla_j T_{k\ell} - \nabla_j \nabla_i T_{k\ell} = -R_{ijk}^p T_{p\ell} - R_{ij\ell}^p T_{kp}. \tag{74}$$

Proof. This is a corollary to (72). □

$$(\nabla^2 f)(X, Y) = (\nabla^2 f)(Y, X). \tag{75}$$

Proof. The torsion free property of the L-C connection says that

$$(\nabla_X Y)(f) - (\nabla_Y X)(f) = [X, Y](f) \coloneqq X(Y(f)) - Y(X(f)),$$

and by rearranging this we get that

$$(X(Y(f)) - (\nabla_X Y)(f) = Y(X(f)) - (\nabla_Y X)(f),$$

which is exactly the desired equality. □

$$\nabla^2_{X,Y} F \coloneqq (\nabla^2 F)(X,Y) = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F. \tag{76}$$

Proof. According to the general definition of covariant derivative (and the definition of total covariant derivative) above,

$$\begin{aligned}
(\nabla(\nabla F))(Y,X) &= (\nabla_X(\nabla F))(Y) \\
&= \nabla_X[(\nabla F)(Y)] - \nabla F(\nabla_X Y) \\
&= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.
\end{aligned}$$

To make more explicit what is actually going on here, we can write, supposing F is a (k,ℓ)-tensor,

$$\begin{aligned}
\nabla^2 F(X,Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) &= \nabla_X(\nabla F)(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) \\
&= X(\nabla F(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k)) \\
&\qquad - (\nabla F)(\nabla_X Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) \\
&\qquad - \sum_{i=1}^{\ell}(\nabla F)(Y,\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^\ell,W_1,\dots,W_k) \\
&\qquad - \sum_{i=1}^{k}(\nabla F)(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,\nabla_X W_i,\dots,W_k) \\
&= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.
\end{aligned}$$

There is also a proof on page 99 of Lee-RM. □

$$(\nabla^2 f)(X,Y) = X(Y(f)) - (\nabla_X Y)(f) = g(\nabla_X \operatorname{grad} f, Y) = \frac{1}{2}(\mathcal{L}_{\operatorname{grad} f}\, g)(X,Y) \tag{77}$$

Proof. The second equality:

$$\begin{aligned}
X(Y(f)) - (\nabla_X Y)(f) &= X(g(\operatorname{grad} f, Y)) - g(\operatorname{grad} f, \nabla_X Y) \\
&= g(\nabla_X \operatorname{grad} f, Y) + g(\operatorname{grad} f, \nabla_X Y) - g(\operatorname{grad} f, \nabla_X Y) \\
&= g(\nabla_X \operatorname{grad} f, Y).
\end{aligned}$$

The last equality: using (**??**) (TODO: check this reference; should it be 2a or 2b?) for the Lie derivative of the metric, metric compatibility, (52), and denoting $\operatorname{grad} f$ by ∇f, we calculate

$$\begin{aligned}
(\mathcal{L}_{\nabla f}\, g)(X,Y) &= (\nabla f)(g(X,Y)) - g([\nabla f, X], Y) - g(X, [\nabla f, Y]) \\
&= g(\nabla_{\nabla f} X, Y) + g(X, \nabla_{\nabla f} Y) - g(\nabla_{\nabla f} X - \nabla_X(\nabla f), Y) - g(X, \nabla_{\nabla f} Y - \nabla_Y(\nabla f)) \\
&= g(\nabla_X(\nabla f), Y) + g(X, \nabla_Y(\nabla f)) \\
&= X(g(\nabla f, Y)) - g(\nabla f, \nabla_X Y) + Y(g(X, \nabla f)) - g(\nabla_Y X, \nabla f) \\
&= (\nabla^2 f)(X,Y) + (\nabla^2 f)(Y,X) \\
&= 2(\nabla^2 f)(X,Y).
\end{aligned}$$

□

$$\nabla_i \nabla_j f = \partial_i(\partial_j f) - \Gamma^k_{ij}\partial_k f. \tag{78}$$

Proof. Recalling that, by definition $\nabla_i \partial_j = \Gamma^k_{ij}\partial_k$,

$$\begin{aligned}
\nabla_i \nabla_j f &= \nabla_i(\nabla_j f) - \nabla_{\nabla_i \partial_j} f \\
&= \partial_i \partial_j f - \nabla_{\Gamma^k_{ij}\partial_k} f \\
&= \partial_i \partial_j f - \Gamma^k_{ij} \nabla_k f \\
&= \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f
\end{aligned}$$

This also follows from □

$$\nabla^2(fh) = f\nabla^2 h + h\nabla^2 f + \nabla f \otimes \nabla h + \nabla h \otimes \nabla f \tag{79}$$

Proof. First note that we are using ∇f to denote the gradient of f. Then

$$\begin{aligned}
\nabla^2(hf)(X,Y) &= \nabla_X(\nabla_Y(fh)) - \nabla_{\nabla_X Y}(fh) \\
&= \nabla_X(fY(h) + hY(f)) - (\nabla_X Y)(fh) \\
&= X(f)Y(h) + fX(Y(h)) + X(h)Y(f) + hX(Y(f)) - h(\nabla_X Y)(f) - f(\nabla_X Y)(h) \\
&= f(X(Y(h)) - (\nabla_X Y)(h)) + h(X(Y(f)) - (\nabla_X Y)(f)) \\
&\quad + (\nabla f \otimes \nabla h)(X,Y) + (\nabla h \otimes \nabla f)(X,Y) \\
&= f(\nabla^2 h)(X,Y) + h(\nabla^2 f)(X,Y) + (\nabla f \otimes \nabla h)(X,Y) + (\nabla h \otimes \nabla f)(X,Y).
\end{aligned}$$

□

$$\Delta(\nabla_i\nabla_j f) = \nabla_i\nabla_j\Delta f + R_{j\ell}\nabla_i\nabla_j f + R_{i\ell}\nabla_\ell\nabla_j f - 2R^\ell_{kij}\nabla_k\nabla_\ell f + (\nabla_j R_{\ell i} + \nabla_i R_{j\ell} - \nabla_\ell R_{ji})\nabla_\ell f. \tag{80}$$

Proof.

$$\begin{aligned}
\Delta(\nabla_i\nabla_j f) &= \nabla_k\nabla_k(\nabla_i\nabla_j f) \\
&= \nabla_k(\nabla_k(\nabla_i\nabla_j f))
\end{aligned}$$

Use (73), which says that $\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = -R^\ell_{ijk}\nabla_\ell f$.

$$\begin{aligned}
&= \nabla_k(\nabla_i\nabla_k\nabla_j f - R^\ell_{kij}\nabla_\ell f) \\
&= \nabla_k\nabla_i\nabla_k\nabla_j f - (\nabla_k R^\ell_{kij})\nabla_\ell f - R^\ell_{kij}(\nabla_k\nabla_\ell f)
\end{aligned}$$

Use (74), which says that for a $(2,0)$-tensor T, $\nabla_i\nabla_j T_{k\ell} - \nabla_j\nabla_i T_{k\ell} = -R^p_{ijk}T_{p\ell} - R^p_{ij\ell}T_{kp}$.

$$= \nabla_i\nabla_k\nabla_k\nabla_j f - R^\ell_{kik}\nabla_\ell\nabla_j f - R^\ell_{kij}\nabla_k\nabla_\ell f - (\nabla_k R^\ell_{kij})\nabla_\ell f - R^\ell_{kij}\nabla_k\nabla_\ell f$$

On the first term, apply (83), which is the commutator formula for Δ and ∇. On the second term, use the fact that $R^\ell_{kik} = R_{kik\ell} = -R_{ki\ell k} = -R_{i\ell}$. On the fourth term, apply the once contracted second Bianchi identity $\nabla_m R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}$, in the form $\nabla_k R_{kij\ell} = -\nabla_k R_{j\ell ik} = -(\nabla_j R_{\ell i} - \nabla_\ell R_{ji})$.

$$\begin{aligned}
&= \nabla_i(\nabla_j\nabla_k\nabla_k f + R_{j\ell}\nabla_\ell f) + R_{i\ell}\nabla_\ell\nabla_j f - 2R^\ell_{kij}\nabla_k\nabla_\ell f + (\nabla_j R_{\ell i} - \nabla_\ell R_{ji})\nabla_\ell f \\
&= \nabla_i\nabla_j\Delta f + \nabla_i R_{j\ell}\nabla_\ell f + R_{j\ell}\nabla_i\nabla_\ell f + R_{i\ell}\nabla_\ell\nabla_j f - 2R^\ell_{kij}\nabla_k\nabla_\ell f + (\nabla_j R_{\ell i} - \nabla_\ell R_{ji})\nabla_\ell f \\
&= \nabla_i\nabla_j\Delta f + R_{j\ell}\nabla_i\nabla_\ell f + R_{i\ell}\nabla_\ell\nabla_j f - 2R^\ell_{kij}\nabla_k\nabla_\ell f + (\nabla_j R_{\ell i} + \nabla_i R_{j\ell} - \nabla_\ell R_{ji})\nabla_\ell f
\end{aligned}$$

□

$$\Delta(\nabla_i \nabla_j R) = \nabla_i \nabla_j \Delta R + 2R\nabla_i \nabla_j R - (R\Delta R - \tfrac{1}{2}\left|\nabla R\right|^2)g_{ij} + \nabla_i R \nabla_j R. \tag{81}$$

Proof. Use the fact that in 2d we have (90), which in this case says $R_{kij\ell} = \frac{1}{2}R(g_{k\ell}g_{ij} - g_{kj}g_{i\ell})$. Also in 2d, $R_{ij} = \frac{1}{2}Rg_{ij}$, and so $\nabla_i R_{jk} = \frac{1}{2}\nabla_i R g_{jk}$. Substituting these in and simplifying gives the result. □

$$\Delta\left|\nabla u\right|^2 = 2\left\langle \Delta\nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2. \tag{82}$$

Proof. The coordinate-free way to do this goes as follows. I use dots to keep track of the entries over which the trace is taken.

$$\begin{aligned}
\Delta\left\langle \nabla u, \nabla u\right\rangle &= \operatorname{tr}\nabla^2\left\langle \nabla u, \nabla u\right\rangle \\
&= \operatorname{tr}(\nabla.(2\left\langle \nabla.\nabla u, \nabla u\right\rangle)) \\
&= 2\operatorname{tr}(\left\langle \nabla.\nabla.\nabla u, \nabla u\right\rangle + \left\langle \nabla.\nabla u, \nabla.\nabla u\right\rangle) \\
&= 2\left\langle \Delta\nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2,
\end{aligned}$$

as desired.

In normal coordinates, we can calculate

$$\begin{aligned}
\Delta\left|\nabla u\right|^2 &= \Delta(\nabla_i u \nabla_i u) \\
&= \nabla_j \nabla_j(\nabla_i u \nabla_i u)) \\
&= 2\nabla_j \nabla_j \nabla_i u \nabla_i u + 2\nabla_j \nabla_i u \nabla_j \nabla_i u \\
&= 2\left\langle \Delta\nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2.
\end{aligned}$$

□

$$\Delta(du) = d(\Delta u) + \mathrm{Rc}(\nabla u), \tag{83}$$

Proof. By (73),

$$\nabla_i \nabla_j \nabla_k u = \nabla_j \nabla_i \nabla_k u - \mathrm{Rm}_{ijk\ell}\nabla_\ell u.$$

Note that on the left-hand side we can commute ∇_j and ∇_k. From this equation,

$$\begin{aligned}
g^{ik}\nabla_i \nabla_k \nabla_j u &= g^{ik}(\nabla_j \nabla_i \nabla_k u + \mathrm{Rm}_{jik\ell}\nabla_\ell u) \\
\Delta\nabla_j u &= \nabla_j \Delta u + \mathrm{Rc}_{j\ell}\nabla_\ell u.
\end{aligned}$$

□

6 Curvature

6.1 Curvature of a connection on a vector bundle

Reference: Andrews-Hopper Section 2.7.1. If ∇ is a connection on a vector bundle E over M, the curvature of ∇ on E is the section $R_\nabla \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ defined by

$$R_\nabla(X,Y)\xi = \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X,Y]}\xi.$$

6.2 Riemann curvature

Riemann curvature is the special case of the previous construction where the connection is the Levi-Civita connection on the tangent bundle over M. In particular, the $(3,1)$-tensor (field) version of the Riemann curvature tensor is a $C^\infty(M)$-multilinear map $\Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z. \end{aligned}$$

In coordinates, we can write

$$R = R_{ijk}{}^{\ell} dx^i \otimes dx^j \otimes dx^k \otimes \partial_\ell,$$

so that

$$R(X,Y)Z = R_{ijk}{}^{\ell} X^i Y^j Z^k \partial_\ell.$$

where

$$R_{ijk}{}^{\ell} \partial_\ell = R(\partial_i, \partial_j)\partial_k.$$

We can get a $(4,0)$-tensor version of R by lowering an index:

$$R_{ijkl} = R(\partial_i, \partial_j, \partial_k, \partial_\ell) \coloneqq \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell \rangle .$$

Then $R_{ijk\ell} = g_{\ell m} R_{ijk}{}^m$. This tensor satisfies the symmetries

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \tag{84}$$

and the 1st and 2nd Bianchi identities:

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{85}$$

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{86}$$

The once contracted 2nd Bianchi identity:

$$g^{im} \nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{87}$$

We can calculate the coefficients in terms of the Christoffel symbols as well:

$$R_{ijk}{}^{\ell} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk} \Gamma^\ell_{im} - \Gamma^m_{ik} \Gamma^\ell_{jm} \tag{88}$$

$$R_{ijk\ell} = \frac{1}{2}(\partial_j \partial_k g_{i\ell} + \partial_i \partial_\ell g_{jk} - \partial_i \partial_k g_{j\ell} - \partial_j \partial_\ell g_{ik}) + g_{\ell p}(\Gamma^m_{ik} \Gamma^p_{jm} - \Gamma^m_{jk} \Gamma^p_{im}). \tag{89}$$

In calculations, we frequently get Riemann curvature terms appearing from commuting covariant derivatives, following from rearranging the formula that defines the Riemann tensor. See (72).

TODO: expand on this.

In two and three dimensions, the Riemann curvature tensor simplifies. In particular, in dimension 2,

$$R_{ijk\ell} = \frac{1}{2} R (g_{i\ell} g_{jk} - g_{ik} g_{j\ell}). \tag{90}$$

$$R_{ijk\ell} = \frac{1}{2} R (g_{i\ell} g_{jk} - g_{ik} g_{j\ell}). \tag{90}$$

Proof. TODO □

6.3 Ricci curvature

The Ricci tensor, denoted Rc or R, is defined to be the trace of the Riemann tensor:

$$\mathrm{Rc}(Y,Z) \coloneqq \mathrm{tr}(X \mapsto R(X,Y)Z),$$

or in coordinates

$$R_{ij} = R_{kij}{}^{k} = g^{km} R_{kijm}.$$

The Ricci tensor satisfies the twice contracted second Bianchi identity:

$$\operatorname{div}\mathrm{Rc} = \frac{1}{2}\nabla R,$$

which in coordinates is

$$2g^{ij}\nabla_i \mathrm{Rc}_{jk} = \nabla_k R. \tag{91}$$

The Ricci tensor can be expressed in terms of Christoffel symbols:

$$R_{jk} = R^i_{ijk} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^p_{jk}\Gamma^i_{ip} - \Gamma^p_{ik}\Gamma^i_{jp} \tag{92}$$

or in terms of the metric:

$$-2\,\mathrm{Rc}_{ij} = g^{k\ell}(\partial_k\partial_\ell g_{ij} + \partial_i\partial_j g_{k\ell} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_k g_{i\ell}) + \text{lower order terms}, \tag{93}$$

where the lower order terms involve only one derivative of g. The Ricci tensor is invariant under diffeomorphisms; that is, if ϕ is a diffeomorphism of M, then

$$\mathrm{Rc}_{\phi^* g} = \phi^* \mathrm{Rc}_g \,.$$

The standard form of the Ricci tensor is as a $(2,0)$-tensor, but this means that it also has a $(1,1)$ version. That is, the Ricci tensor is normally an object that takes two vectors and gives a number. If we consider the map $X \mapsto [Y \mapsto \mathrm{Rc}(X,Y)]$, then we see that we can think of Rc as an object that takes a vector and gives a covector. By the musical isomorphisms, the covector $\mathrm{Rc}(X,\cdot)$ corresponds to the vector Y that satisfies $\mathrm{Rc}(X,\cdot) = g(Y,\cdot)$. That is, $\mathrm{Rc}(X,\cdot) = g(\mathrm{Rc}(X),\cdot)$.

which by the musical isomorphisms is equivalent to an object that takes a vector and gives a vector. This satisfies

$$\mathrm{Rc}(X) = (\mathrm{Rc}^i_j\, X^j)\partial_i, \qquad \mathrm{Rc}^k_j = g^{ik}\,\mathrm{Rc}_{ij} \tag{94}$$

6.4 Scalar curvature

The scalar curvature is defined to be the trace (with respect to the metric) of the Ricci curvature:

$$R = \mathrm{tr}_g \mathrm{Rc} = \mathrm{Rc}_i{}^i = g^{ij}\,\mathrm{Rc}_{ij}\,.$$

In local coordinates,

$$R = g^{jk}R_{jk} = g^{jk}(\partial_i\Gamma^i_{jk} - \partial_j\Gamma^i_{ik} + \Gamma^p_{ij}\Gamma^i_{ip} - \Gamma^p_{ij}\Gamma^i_{jp}) \tag{95}$$

$$R = g^{jk}R_{jk} = g^{jk}(\partial_i\Gamma^i_{jk} - \partial_j\Gamma^i_{ik} + \Gamma^p_{ij}\Gamma^i_{ip} - \Gamma^p_{ij}\Gamma^i_{jp}) \tag{95}$$

Proof. TODO ∎

6.5 Algebraic perspective on the curvature tensor

There is also an algebraic perspective on the Riemannian curvature tensor.

6.5.1 The space of algebraic curvature tensors

6.5.2 Kulkarni-Nomizu product

Given two $(2,0)$-tensors we want to build a

Let $S^2(M)$ denote the bundle of symmetric $(2,0)$-tensors over M and let

$$\owedge : S^2(M) \times S^2(M) \to$$

TODO

See Bamler's lecture 1, also a bit about this in ChowHRF. Also

6.6 Proofs

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \tag{84}$$

Proof. Using the fact that the Lie bracket is antisymmetric, it is clear from the definition that $R(X,Y)Z = -R(Y,X)Z$, from which the equality $R_{ijk\ell} = -R_{jik\ell}$ follows. To show the second equality, we begin by showing that $R(X,Y,Z,Z) = 0$ for any Z. First note that by metric compatibility,

$$\begin{aligned} X(Y(|W|^2)) &= X(Y\langle W, W\rangle) \\ &= X(2\langle \nabla_Y W, W\rangle) \\ &= 2\langle \nabla_X \nabla_Y W, W\rangle + 2\langle \nabla_Y W, \nabla_X W\rangle . \end{aligned}$$

Similarly,

$$Y(X(|W|^2)) = 2\langle \nabla_Y \nabla_X W, W\rangle + 2\langle \nabla_X W, \nabla_Y W\rangle ,$$

and

$$[X,Y]\,|W|^2 = 2\left\langle \nabla_{[X,Y]} W, W\right\rangle .$$

Now, subtracting the second and third of these two equations from the first and cancelling terms, we have

$$\begin{aligned} 0 &= X(Y(|W|^2)) - Y(X(|W|^2)) - [X,Y]\,|W|^2 \\ &= 2\langle \nabla_X \nabla_Y W, W\rangle - 2\langle \nabla_Y \nabla_X W, W\rangle - 2\left\langle \nabla_{[X,Y]} W, W\right\rangle \\ &= 2\langle R(X,Y)W, W\rangle \\ &= R(X,Y,W,W). \end{aligned}$$

Applying this,

$$\begin{aligned} 0 &= \langle R(\partial_i, \partial_j)\partial_k + \partial_\ell, \partial_k + \partial_\ell\rangle \\ &= R_{ijkk} + R_{ijk\ell} + R_{ij\ell k} + R_{ij\ell\ell} \\ &= R_{ijk\ell} + R_{ij\ell k}. \end{aligned}$$

To prove the last equality, we use the first (algebraic) Bianchi identity. ∎

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{85}$$

Proof. This will follow from

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Expand using the definition of R, apply symmetry of the connection several times, and then finally the Jacobi identity:

$$\begin{aligned}
0 &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \\
&\quad + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X + \\
&\quad + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y \\
&= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) \\
&\quad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
&= \nabla_X [Y,Z] - \nabla_{[Y,Z]} X + \nabla_Y [Z,X] - \nabla_{[Z,X]} Y + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\
&= 0.
\end{aligned}$$

□

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{86}$$

Proof. TODO See, e.g. here `https://math.stackexchange.com/questions/1494262/direct-proof-of-the-second-` or LeeRM p. 204 □

$$g^{im} \nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{87}$$

Proof. By the second Bianchi identity (86) and the fact that $R_{ki\ell m} = -R_{ik\ell m}$,

$$\nabla_i R_{jk\ell m} = \nabla_j R_{ik\ell m} - \nabla_k R_{ij\ell m},$$

so

$$\begin{aligned}
g^{im} \nabla_i R_{jk\ell m} &= \nabla_j g^{im} R_{ik\ell m} - \nabla_k g^{im} R_{ij\ell m} \\
&= \nabla_j R_{k\ell} - \nabla_k R_{j\ell}
\end{aligned}$$

□

$$R_{ijk}{}^{\ell} = \partial_i \Gamma^{\ell}_{jk} - \partial_j \Gamma^{\ell}_{ik} + \Gamma^m_{jk} \Gamma^{\ell}_{im} - \Gamma^m_{ik} \Gamma^{\ell}_{jm} \tag{88}$$

Proof. Note that

$$\begin{aligned}
R_{ijk}{}^{\ell} \partial_\ell &= R(\partial_i, \partial_j) \partial_k \\
&= \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k - \nabla_{[\partial_i, \partial_j]} \partial_k \\
&= \nabla_i (\Gamma^a_{jk} \partial_a) - \nabla_j (\Gamma^b_{ik} \partial_b) \\
&= \partial_i \Gamma^a_{jk} \partial_a + \Gamma^b_{jk} \nabla_i \partial_b - \partial_j \Gamma^c_{ik} \partial_c - \Gamma^d_{ik} \nabla_j \partial_d \\
&= \partial_i \Gamma^a_{jk} \partial_a + \Gamma^b_{jk} \Gamma^e_{ib} \partial_e - \partial_j \Gamma^c_{ik} \partial_c - \Gamma^d_{ik} \Gamma^f_{jd} \partial_f,
\end{aligned}$$

from which the result follows. □

$$R_{ijk\ell} = \frac{1}{2}(\partial_j\partial_k g_{i\ell} + \partial_i\partial_\ell g_{jk} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_\ell g_{ik}) + g_{\ell p}(\Gamma^m_{ik}\Gamma^p_{jm} - \Gamma^m_{jk}\Gamma^p_{im}). \tag{89}$$

Proof. TODO ☐

$$2g^{ij}\nabla_i R_{jk} = \nabla_k R. \tag{91}$$

Proof. Start with the 2nd Bianchi identity, contract twice, and apply some symmetries of the Riemann tensor:

$$\begin{aligned}
g^{im}g^{j\ell}(\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}) &= 0\\
g^{im}\nabla_i g^{j\ell} R_{jk\ell m} + g^{j\ell}\nabla_j g^{im} R_{ki\ell m} + g^{im}\nabla_k g^{j\ell} R_{ij\ell m} &= 0\\
-g^{im}\nabla_i g^{j\ell} R_{jkm\ell} - g^{j\ell}\nabla_j g^{im} R_{ik\ell m} + g^{im}\nabla_k g^{j\ell} R_{jim\ell} &= 0\\
-g^{im}\nabla_i R_{km} - g^{j\ell}\nabla_j R_{k\ell} + \nabla_k g^{im} R_{im} &= 0\\
-2g^{ij}\nabla_i R_{jk} + \nabla_k R &= 0.
\end{aligned}$$

☐

$$R_{jk} = R^i_{ijk} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^p_{jk}\Gamma^i_{ip} - \Gamma^p_{ik}\Gamma^i_{jp} \tag{92}$$

Proof. TODO ☐

$$-2R_{ij} = g^{k\ell}(\partial_k\partial_\ell g_{ij} + \partial_i\partial_j g_{k\ell} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_k g_{i\ell}) + \text{lower order terms}, \tag{93}$$

Proof. I don't want to type this, but it just involves writing the Ricci tensor in terms of the Riemann tensor, the Riemann tensor in terms of the Christoffel symbols, and the Christoffel symbols in terms of the metric. ☐

$$\mathrm{Rc}(X) = (\mathrm{Rc}^i_j\, X^j)\partial_i, \qquad \mathrm{Rc}^k_j = g^{ik}\,\mathrm{Rc}_{ij} \tag{94}$$

Proof. This is of course just an exercise in working out the definitions (like most exercises). From a $(2,0)$ tensor, we have a mapping from vectors to covectors $X \mapsto \mathrm{Rc}(X,\cdot)$. Then by the musical isomorphisms, the vector confusingly denoted $\mathrm{Rc}(X)$ associated to the covector $\mathrm{Rc}(X,\cdot)$ must satisfy

$$g(\mathrm{Rc}(X), Y) = \mathrm{Rc}(X,\cdot)(Y) = \mathrm{Rc}(X,Y).$$

That is,

$$g_{k\ell}\,\mathrm{Rc}(X)^k Y^\ell = \mathrm{Rc}_{ij}\,X^iY^j \implies g_{kj}\,\mathrm{Rc}(X)^k = \mathrm{Rc}_{ij}\,X^i \implies \mathrm{Rc}(X)^k = g^{kj}\,\mathrm{Rc}_{ij}\,X^i = \mathrm{Rc}^k_i\,X^i,$$

which is what we wanted to show. ☐

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7 Geometric Analysis

7.1 Distance functions

The growth of Jacobi fields in a normal neighborhood is controlled by the Hessian of the radial distance function, which satisfies the Riccati equation. Most of the comparison theorems use this machinery.

Let (M, g) be an n-dimensional Riemannian manifold, and let U be a normal neighborhood of a point $p \in M$, and let $r : U \to \mathbb{R}$ be the radial distance function, defined in normal coordinates (x^i) at p by

$$r(x) = \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

Define the radial vector field on $U \setminus \{p\}$ by

$$\partial_r = \frac{x^i}{r(x)} \frac{\partial}{\partial x^i}.$$

For a proof of the following theorem, see Theorem 6.9 of [LeeRM], as well as most of the standard texts on Riemannian geometry.

Theorem 7.1 (The Gauss Lemma). *Let ∂_r denote the radial vector field on $U \setminus \{p\}$. Then ∂_r is a unit vector field orthogonal to the geodesic spheres in $U \setminus \{p\}$.*

We can form two tensors of total rank 2 from r: the covariant Hessian of r: $\nabla^2 r$, and the covariant derivative of ∂_r: $\nabla \partial_r$. As we have already seen, these turn out to be the essentially the same thing: $\nabla \partial_r$ is obtained from $\nabla^2 r$ by raising one of its indices. We can also interpret the $(1, 1)$-tensor field $\nabla \partial_r$ as a field of endomorphisms of TM over $U \setminus \{p\}$ by defining

$$\mathcal{H}_r(w) = \nabla_w \partial_r, \quad w \in TM|_{U \setminus \{p\}}.$$

This is called the Hessian operator of r, and it is related to the $(0, 2)$-Hessian by

$$g(\mathcal{H}_r(v), w)) = \left(\nabla^2 r\right)(v, w), \quad q \in U \setminus \{p\}, v, w \in T_q M.$$

This has the following properties:

$$g(\mathcal{H}_r(v), w) = g(v, \mathcal{H}_r(w)v) \tag{96}$$

$$\mathcal{H}_r(\partial_r) = 0. \tag{97}$$

The restriction of $\mathcal{H}_r$ to vectors tangent to a level set of r is equal to the shape operator of the level set associated with the normal vector field $N = -\partial_r$.

(98)

We also have that in every choice of polar normal coordinates $(\theta^1, \ldots, \theta^{n-1}, r)$ on a subset of $U \setminus \{p\}$, the covariant Hessian of r is given by

$$\nabla^2 r = \frac{1}{2} \sum_{\alpha, \beta = 1}^{n-1} (\partial_r g_{\alpha\beta}) d\theta^\alpha d\theta^\beta. \tag{99}$$

7.2 Integration

7.2.1 Stokes's theorem

Suppose M is an oriented n-manifold with boundary, and suppose ω is a compactly supported $(n-1)$-form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega. \tag{100}$$

From this we can obtain several very useful special cases. The **divergence theorem** says that for any smooth 1-form α on a compact manifold with boundary,

$$\int_M \operatorname{div}(\alpha)\, d\mu = \int_{\partial M} \alpha(\nu)\, d\sigma, \tag{101}$$

where ν is the outward unit normal to the boundary, and $d\sigma$ is the volume form of the boundary.

In the case of a vector field, we have

$$\int_M \operatorname{div} X = \int_{\partial M} g(X, \nu)\, d\sigma. \tag{102}$$

7.2.2 Integration by parts

Suppose $u, v \in C^\infty(M)$. There are a few useful forms of the same idea. First, if M is closed,

$$\int_M \Delta u\, d\mu = 0, \tag{103}$$

$$\int_M (u\Delta v - v\Delta u)\, d\mu = 0 \tag{104}$$

More generally, if M is compact but possibly has a boundary,

$$\int_M (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma. \tag{105}$$

Where ν and σ are the outward (?) unit normal and the volume form of ∂M.

In particular, on a closed manifold, the right hand side is 0, so $\int u\Delta v = \int v\Delta u$. If M is compact,

$$\int_M u\Delta v\, d\mu + \int_M \langle \nabla u, \nabla v \rangle\, d\mu = \int_{\partial M} \frac{\partial v}{\partial \nu} u\, d\sigma. \tag{106}$$

In particular, on a closed manifold, $\int \langle \nabla u, \nabla v \rangle = -\int u\Delta v$.

We also have

$$\int_M g(\operatorname{grad} f, X)\, d\mu = \int_{\partial M} f g(X, \nu)\, d\sigma - \int_M f \operatorname{div} X\, d\mu.$$

Right now I'm just copying this from LeeRM, p 149. TODO: change the notation and add some exposition

The integration by parts formula generalizes to covariant tensor fields. If F is a smooth covariant k-tensor field and G is a smooth covariant $(k+1)$-tensor field on a compact smooth Riemannian manifold with boundary, then

$$\int_M \langle \nabla F, G \rangle\, d\mu = \int_{\partial M} \left\langle F \otimes N^\flat, G \right\rangle d\sigma - \int_M \langle F, \operatorname{div} G \rangle\, d\mu, \tag{107}$$

In coordinates we have the more suggestive

$$\int_M F_{i_1 \dots i_k; j} G^{i_1 \dots i_k j}\, dV_g = \int_{\partial M} F_{i_1 \dots i_k} G^{i_1 \dots i_k j} N_j\, dV_{\hat{g}} - \int_M F_{i_1 \dots i_k} G^{i_1 \dots i_k j}{}_{;j}\, dV_g. \tag{108}$$

7.3 Miscellaneous

7.3.1 Rescaling manifolds

Consider a time-dependent family of metrics g_t, and a parabolic rescaling $\lambda^2 g_{\lambda^{-2}t}$. Say that the scale of some quantity is k if it changes by λ^k under the rescaling. For any function f and vector field X, we have the following scales.

$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$
g^{-1}		Rc	$\lvert X\rvert$	g
R_{ijkl}		Γ_{ij}^k	$\sqrt{t}$	$\lvert X\rvert^2$
$\lvert\mathrm{Rm}\rvert, \lvert\mathrm{Rc}\rvert, R$		R_{ijk}^{ℓ}		t
$\lvert\nabla^2 f\rvert$		$\nabla^2 f, df$		
∇f		$\nabla^k f$		

7.3.2 Something else

If $A(s)$ is a 1-parameter family of invertible square matrices, then

$$\frac{d}{ds}\det(A(s)) = \det(A)\operatorname{tr}\left(A^{-1}\frac{d}{ds}A(s)\right) \tag{109}$$

The following special case is occasionally useful.

$$\frac{d}{ds}\log(\det A) = (A^{-1})^{ij}\frac{d}{ds}A_{ij}. \tag{110}$$

7.4 Variation formulae

References: Sheridan's notes and Andrew-Hopper Chapter 4.

Suppose that $g(t)$ is a family of Riemannian metrics, and

$$\frac{\partial}{\partial s}g_{ij}(s) = h_{ij}(s).$$

Then we have the following evolution equations for various geometric objects. Denote by H the trace of h, i.e. $H = g^{ij}h_{ij}$. Some cases are only stated for the special case of the Ricci flow, i.e. when $h = -2\,\mathrm{Rc}$. We distinguish such cases with the use of ∂_t rather than ∂_s.

Metric inverse:

$$\frac{\partial}{\partial s}g^{ij} = -h^{ij} = -g^{ik}g^{jl}h_{kl} \tag{111}$$

Christoffel symbols:

$$\frac{\partial}{\partial s}\Gamma_{ij}^k = \frac{1}{2}g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \tag{112}$$

Volume element:

$$\frac{\partial}{\partial s}d\mu_s = \frac{H}{2}d\mu_s \tag{113}$$

Scalar curvature:

$$\frac{\partial}{\partial s}R = -\Delta H + \operatorname{div}(\operatorname{div} h) - \langle h, \mathrm{Rc}\rangle\,. \tag{114}$$

Ricci curvature: For the Ricci tensor, we mention several different expressions. The first gives the variation of the Ricci tensor in terms of the variation of the connection.

$$\partial_s \operatorname{Rc}_{ij} = \nabla_p(\partial_s \Gamma^p_{ij}) - \nabla_i(\partial_s \Gamma^p_{pj}). \tag{115}$$

From this we get

$$\frac{\partial}{\partial s} R_{ij} = \frac{1}{2}\nabla_\ell(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) - \frac{1}{2}\nabla_i \nabla_j H. \tag{116}$$

We also have

$$\frac{\partial}{\partial s} \operatorname{Rc}_{ij} = -\frac{1}{2}\left(\Delta_L h_{ij} + \nabla_i \nabla_j H - \nabla_i (\operatorname{div} h)_j - \nabla_j (\operatorname{div} h)_i\right), \tag{117}$$

where Δ_L denotes the Lichnerowicz Laplacian. This can be written as

$$\frac{\partial}{\partial s}(-2R_{ij}) = \Delta_L h_{ij} + \nabla_i X_j + \nabla_j X_i, \tag{118}$$

where $X = \frac{1}{2}\nabla H - \operatorname{div} h$.

The Levi-Civita connection: For time-independent vector fields, and an evolving metric $g(s)$, we define $\dot{\nabla} = \partial_s \nabla$ by $\dot{\nabla}_X Y = \partial_s(\nabla_X Y)$. Then, under the Ricci flow,

$$\left\langle \dot{\nabla}_X Y, Z \right\rangle = -(\nabla_X \operatorname{Rc})(Y, Z) + (\nabla_Z \operatorname{Rc})(X, Y) - (\nabla_Y \operatorname{Rc})(X, Z). \tag{119}$$

The Laplacian (on functions):

$$\frac{\partial}{\partial s}\Delta_{g(s)} = -h_{ij}\nabla^i \nabla^j - g^{k\ell}\left(g^{ij}\nabla_i h_{j\ell} - \frac{1}{2}\nabla_\ell\left(g^{ij}h_{ij}\right)\right)\nabla_k. \tag{120}$$

In particular if $g(t)$ is a solution to Ricci flow, the function Laplacian $\Delta_{g(t)}$ evolves by

$$\partial_t \Delta_{g(t)} = 2\operatorname{Rc}_{ij} \nabla_i \nabla_j. \tag{121}$$

Another special case of this is when g changes by a conformal factor. If $\partial_t g = fg$ for some smooth function f, then

$$\partial_t \Delta = -f\Delta + \left(\frac{n}{2} - 1\right)\nabla f \cdot \nabla. \tag{122}$$

The Hessian: for a general variation, we have the following, which is not very useful. For any function $f = f(x, s)$,

$$\partial_s(\nabla_i \nabla_j f) = \nabla_i \nabla_j(\partial_s f) - (\partial_s \Gamma^k_{ij})\partial_k f. \tag{123}$$

The Ricci flow case is

$$\partial_t(\nabla_i \nabla_j f) = \nabla_i \nabla_j(\partial_t f) + (\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij})\nabla^\ell f. \tag{124}$$

Riemann curvature tensor:

$$\partial_s {R_{ijk}}^\ell = \frac{1}{2}g^{\ell p}\left(\nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik}\right) \tag{125}$$

7.5 Proofs

$$\int_{\partial M} \omega = \int_M d\omega. \tag{100}$$

Proof. First note that both sides of the equation are linear in ω, so we can assume that ω has compact support contained in the image of a local parametrization $U \to X$, where U is an open subset of $\mathbb{R}^n$ or H^n, where H^n is a half-space in $\mathbb{R}^n$. (a general ω can be written as a sum of objects like this).

First assume that U is open in $\mathbb{R}^n$, meaning that $h(U)$ does not intersect the boundary. Then

$$\int_{\partial X} \omega = 0 \quad \text{and} \quad \int_X d\omega = \int_U h^*(d\omega) = \int_U d\nu,$$

where $\nu = h^*\omega$. The fact that the integral behaves well under pullbacks is a fundamental property of the machinery of forms and their integration. See Chapter 4, §4 of [4] for more about integration of forms on manifolds.

Since ν is a $(n-1)$-form in n-space, we may write it as

$$\nu = \sum_{i=1}^{n} (-1)^{i+1} f_1 dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where the hat represents a term that is omitted from the product. Then

$$d\nu = \left(\sum_i \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge d_n,$$

and

$$\int_{R^n} d\nu = \sum_i \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_n.$$

The integral over $\mathbb{R}^n$ is computed as usual by an iterated sequence of integrals over $\mathbb{R}$, which by Fubini's theorem can be take in any order. Integrating the ith term first with respect to x_i gives

$$\int_{\mathbb{R}^n} \frac{\partial f_i}{\partial dx_i} dx_1 \cdots dx_n = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial dx_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n.$$

But by the fact that ν has compact support and the fundamental theorem of calculus, we have that $\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i = 0$. Thus $\int_X d\omega = 0$, as desired.

Now suppose that $h(U)$ intersects the boundary of H^n. The preceding analysis applies, except for the last term. Since the boundary of H^n is the set where $x_n = 0$, the last integral is

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial f_n}{\partial x_n} dx_n \right) dx_1 \cdots dx_n$$

The compact support assumption implies that f_n vanishes outside of some large interval $(0, a)$, but we can't guarantee that $f_n(0) = 0$, which is what we would need to get the result directly from the fundamental theorem of calculus.

If we apply the fundamental theorem of calculus in the same way as above, we get

$$\int_X d\omega = \int_{\mathbb{R}^n} -f_k(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1}.$$

On the other hand,

$$\int_{\partial X} \omega = \int_{\partial H^k} \nu.$$

Since $x_k = 0$ on ∂H^n, $dx_n = 0$ on ∂H^n as well. Consequently, if $i < n$, the form $(-1)^{i+1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx^n$ restricts to 0 on ∂H^k. So the restriction of ν to ∂H^k is $(-1)^{n-1} f(x_1, \ldots, x_{n-1}, 0) dx_1 \wedge \cdots \wedge dx_{n-1}$, whose integral over ∂H^k is therefore $\int_{\partial X} \omega$.

Now ∂H^n is naturally diffeomorphic to $\mathbb{R}^{n-1}$ under the obvious map

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0),$$

but this diffeomorphism does not always carry the usual orientation of $\mathbb{R}^{n-1}$ to the boundary orientation of ∂H^k. In particular, let $e_1, \ldots, e_n$ be the standard ordered basis for $\mathbb{R}^n$, so that $e_1, \ldots, e_{n-1}$ is the standard ordered basis for $\mathbb{R}^{n-1}$. Since H^k is the upper half-space, the outward unit normal to ∂H^k is $-e_k = (0, \ldots, 0, -1)$. Thus in the boundary orientation of ∂H^n, the sign of the ordered basis $(e_1, \ldots, e_{n-1})$ is defined to be the sign of the ordered basis $(e_n, e_1, \ldots, e_{n-1})$ in the standard orientation of H^k. The latter sign is easily seen to be $(-1)^k$, so the usual diffeomorphism $\mathbb{R}^k \to \partial H^k$ alters orientation by this factor $(-1)^k$.

TODO: finish this. See Guillemin and Pollack p. 183. □

$$\int_M \operatorname{div}(\alpha)\, d\mu = \int_{\partial M} \alpha(\nu)\, d\sigma, \tag{101}$$

Proof. Some issues with orientability here? See Problem 2-22 p. 51 of Lee-RM. Recall that the divergence of a vector field X is defined to satisfy $d(\iota_X\, d\mu) = \operatorname{div} X\, d\mu$.

TODO □

$$\int_M \operatorname{div} X = \int_{\partial M} g(X, \nu)\, d\sigma. \tag{102}$$

Proof. The idea is to apply Stokes's theorem (100) with $\alpha = \iota_X(d\mu)$. With this definition,

$$d\alpha = d\iota_X(d\mu) = \operatorname{div} X\, d\mu,$$

by definition of the divergence.

$$\begin{aligned} d\alpha &= d\iota_X(d\mu) \\ &= \end{aligned}$$

See HRF Theorem 1.47

TODO □

$$\int_M \Delta u\, d\mu = 0, \tag{103}$$

Proof. Recall that $\nabla u = \operatorname{div} \operatorname{grad} u$. Then apply the divergence theorem (101) and the fact that M has no boundary. □

$$\int_M (u\Delta v - v\Delta u)\, d\mu = 0 \tag{104}$$

Proof. This follows immediately from (105) and the fact that M has no boundary. ☐

$$\int_M (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)\, d\sigma. \tag{105}$$

Proof. Take $X = u\nabla v - v\nabla u$ in the divergence theorem (for vector fields) (102). ☐

$$\int_M u\Delta v\, d\mu + \int_M \langle \nabla u, \nabla v\rangle\, d\mu = \int_{\partial M} \frac{\partial v}{\partial \nu} u\, d\sigma. \tag{106}$$

Proof. Take $X = u\nabla v$ in (102). We are using the product rule for the divergence (21), which can be written as

$$\operatorname{div}(u\nabla v) = \langle \nabla u, \nabla v\rangle + u\Delta v.$$

☐

$$\int_M \langle \nabla F, G\rangle\, d\mu = \int_{\partial M} \left\langle F \otimes N^\flat, G\right\rangle\, d\sigma - \int_M \langle F, \operatorname{div} G\rangle\, d\mu, \tag{107}$$

Proof. TODO ☐

$$\int_M F_{i_1\dots i_k;j} G^{i_1\dots i_k j}\, dV_g = \int_{\partial M} F_{i_1\dots i_k} G^{i_1\dots i_k j} N_j\, dV_{\hat{g}} - \int_M F_{i_1\dots i_k} G^{i_1\dots i_k j}{}_{;j}\, dV_g. \tag{108}$$

Proof. TODO ☐

$$\frac{d}{ds}\det(A(s)) = \det(A(s))\operatorname{tr}\left(A(s)^{-1}\frac{d}{ds}A(s)\right) \tag{109}$$

Proof. First, a lemma:

Lemma 7.2. $\det(I + tA) = 1 + \operatorname{tr}(A)t + \mathcal{O}(t^2)$.

Proof. Recall that $\det(tA) = t^n \det(A)$. Also recall that the characteristic polynomial of A is the product

$$(x - \lambda_1)\cdots(x - \lambda_n),$$

where λ_i's are eigenvalues of A. In particular, the characteristic polynomial is also $\det(xI - A)$.

$$\begin{aligned}
\det(I + tA) &= t^n \det(t^{-1}I - (-A)) \\
&= t^n (t^{-1} + \lambda_1)(t^{-1} + \lambda_2)\cdots(t^{-1} + \lambda_n) \\
&= t^n \left(t^{-n} + t^{-(n-1)}\sum_{i\le n} \lambda_i + t^{-(n-2)} a_{n-2} + \cdots + t^{-1}a_1 + a_0 \right) \\
&= 1 + \operatorname{tr}(A)t + \mathcal{O}(t^2).
\end{aligned}$$

☐

Lemma 7.3. $(D_A \det)(X) = \det(A)\operatorname{tr}(XA^{-1})$, where $(D_A \det)(X)$ is the differential of $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ at A in the direction of X. In particular, the differential of det at the identity matrix is just the trace.

Proof. Note that det is a smooth function $\mathrm{GL}_m(\mathbb{R}) \to \mathbb{R}$ (because the determinant is a polynomial expression of the components of the matrix). First we consider the differential of det at I. By definition and using the previous lemma,

$$\begin{aligned}(D_I \det)(X) &= \lim_{h\to 0} \frac{\det(I + hX) - \det(I)}{h} \\ &= \lim_{h\to 0} \frac{1 + \operatorname{tr}(X)h + \mathcal{O}(h^2) - 1}{h} \\ &= \operatorname{tr}(X).\end{aligned}$$

Now,

$$\begin{aligned}(D_A \det)(X) &= \lim_{h\to 0} \frac{\det(A + hX) - \det(A)}{h} \\ &= \det(A) \lim_{h\to 0} \frac{\det(I + hXA^{-1}) - \det(I)}{h} \\ &= \det(A) D_I(XA^{-1}) \\ &= \det(A) \operatorname{tr}(XA^{-1}).\end{aligned}$$

□

Note that $A(s)$ defines a curve in $\mathrm{GL}_m(\mathbb{R})$. We can think of $\frac{d}{ds}\det(A(s))$ either as the derivative of a function $\mathbb{R} \to \mathbb{R}$, or as the derivative of $\det : \mathrm{GL}_m(\mathbb{R}) \to \mathbb{R}$ at $A(s)$ in the direction of $\frac{d}{ds}A(s)$, which is a vector field along $A(s)$. From the second point of view we get

$$\begin{aligned}\frac{d}{ds}\det(A(s)) &= (D_{A(s)} \det)\left(\frac{d}{ds}A(s)\right) \\ &= \det(A(s)) \operatorname{tr}\left(A(s)^{-1}\frac{d}{ds}A(s)\right),\end{aligned}$$

finally proving (109). □

$$\frac{d}{ds}\log(\det A) = (A^{-1})^{ij}\frac{d}{ds}A_{ij}. \tag{110}$$

Proof. This follows immediately from (109). □

$$\frac{\partial}{\partial t}g^{ij} = -h^{ij} = -g^{ik}g^{jl}h_{kl} \tag{111}$$

Proof.

$$\begin{aligned}0 &= \partial_t \delta^i_k \\ &= \partial_t(g^{ij}g_{jk}) \\ &= (\partial_t g^{ij})g_{jk} + g^{ij}(\partial_t g_{jk}) \\ (\partial_t g^{ij})g_{jk}g^{k\ell} &= -g^{k\ell}g^{ij}(\partial_t g_{jk}) \\ (\partial_t g^{ij})\delta^\ell_j &= -g^{k\ell}g^{ij}(\partial_t g_{jk}),\end{aligned}$$

and the result follows. □

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \tag{112}$$

Proof. By the coordinate expression (68) for the Christoffel symbols, we have

$$\partial_t \Gamma^k_{ij} = \frac{1}{2}(\partial_t g^{k\ell})(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) + \frac{1}{2}g^{k\ell}(\partial_i\partial_t g_{j\ell} + \partial_j\partial_t g_{i\ell} - \partial_\ell\partial_t g_{ij}).$$

The result follows in normal coordinates at some point p, so $\partial_i g_{ij} = 0$, and $\partial_i A = \nabla_i A$ at p for any tensor A. In particular, the first term becomes 0, and the second term is exactly what we want, since $\partial_t g_{ij} = h_{ij}$. □

$$\frac{\partial}{\partial s}d\mu_s = \frac{H}{2}d\mu_s \tag{113}$$

Proof. Note that, in coordinates,

$$d\mu_s = \sqrt{\det g_{ij}(s)}dx^1 \wedge \cdots \wedge dx^n.$$

Denote $\omega = dx^1 \wedge \cdots \wedge dx^n$. Then, using (109),

$$\begin{aligned}
\frac{\partial}{\partial s}d\mu_s &= \frac{\partial}{\partial s}\sqrt{\det g_{ij}(s)}\,\omega \\
&= \frac{1}{2\sqrt{\det g_{ij}(s)}}\frac{\partial}{\partial s}\det g_{ij}(s)\,\omega \\
&= \frac{1}{2\sqrt{\det g_{ij}(s)}}(\det g_{ij}(s))g^{k\ell}(s)h_{k\ell}(s)\,\omega \\
&= \frac{H}{2}\sqrt{\det g_{ij}(s)}\,\omega \\
&= \frac{H}{2}d\mu_s.
\end{aligned}$$

□

$$\frac{\partial}{\partial s}R = -\Delta H + \operatorname{div}(\operatorname{div} h) - \langle h, \operatorname{Rc}\rangle\,. \tag{114}$$

Proof. In coordinates, this is

$$\frac{\partial}{\partial s}R = -\Delta H + \nabla_i\nabla_j h_{ji} - h_{ij}R_{ij}.$$

TODO □

$$\partial_s \operatorname{Rc}_{ij} = \nabla_p(\partial_s\Gamma^p_{ij}) - \nabla_i(\partial_s\Gamma^p_{pj}). \tag{115}$$

Proof. Recall that $R_{ij} = R^p_{pij}$ and

$$R^\ell_{ijk} = \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \Gamma^p_{jk}\Gamma^\ell_{ip} - \Gamma^p_{ik}\Gamma^\ell_{jp}.$$

Then

$$\partial_s R_{ij} = \partial_s(\partial_p\Gamma^p_{ij} - \partial_i\Gamma^p_{pj} + \Gamma^m_{ij}\Gamma^p_{pm} - \Gamma^m_{pj}\Gamma^p_{im}),$$

and in normal coordinates this gives the result. □

$$\frac{\partial}{\partial s}R_{ij} = \frac{1}{2}\nabla_\ell(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) - \frac{1}{2}\nabla_i\nabla_j H. \tag{116}$$

Proof. Using (115),

$$\begin{aligned}
\partial_s R_{ij} &= \partial_p \partial_s \Gamma^p_{ij} - \partial_i \partial_s \Gamma^k_{kj} \\
&= \partial_p(\tfrac{1}{2}g^{p\ell}(\partial_i h_{j\ell} + \partial_j h_{i\ell} - \partial_\ell h_{ij}) - \partial_i(\tfrac{1}{2}g^{k\ell}(\partial_k h_{j\ell} + \partial_j h_{k\ell} - \partial_\ell h_{kj})) \\
&= \tfrac{1}{2}g^{p\ell}(\partial_p\partial_i h_{j\ell} + \partial_p\partial_j h_{i\ell} - \partial_p\partial_\ell h_{ij}) - \tfrac{1}{2}g^{k\ell}(\partial_i\partial_k h_{j\ell} + \partial_i\partial_j h_{k\ell} - \partial_i\partial_\ell h_{kj}) \\
&= \tfrac{1}{2}g^{p\ell}\partial_p\partial_j h_{i\ell} - \tfrac{1}{2}g^{p\ell}\partial_p\partial_\ell h_{ij} - \tfrac{1}{2}\partial_i\partial_j H + \tfrac{1}{2}g^{k\ell}\partial_i\partial_\ell h_{kj} \\
&= \frac{1}{2}\partial_\ell\partial_j h_{i\ell} - \frac{1}{2}\partial_\ell\partial_\ell h_{ij} + \frac{1}{2}\partial_i\partial_\ell h_{\ell j} - \frac{1}{2}\partial_i\partial_j H
\end{aligned}$$

and the result follows after relabeling indices. □

$$\frac{\partial}{\partial s}R_{ij} = -\frac{1}{2}\left(\Delta_L h_{ij} + \nabla_i\nabla_j H - \nabla_i(\operatorname{div} h)_j - \nabla_j(\operatorname{div} h)_i\right), \tag{117}$$

Proof. Recall that

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{kij\ell}h_{k\ell} - R_{ik}h_{jk} - R_{jk}h_{ik}.$$

Now start from (116). The main thing we use is the commutator formula for a 2-tensor, see (74).

$$\begin{aligned}
\frac{\partial}{\partial s}R_{ij} &= -\frac{1}{2}(\Delta h_{ij} + \nabla_i\nabla_j H - \nabla_\ell\nabla_i h_{j\ell} - \nabla_\ell\nabla_j h_{i\ell}) \\
&= -\frac{1}{2}(\Delta h_{ij} + \nabla_i\nabla_j H - \nabla_i\nabla_\ell h_{j\ell} + R^p_{\ell ij}h_{p\ell} + R^p_{\ell i\ell}h_{pj} - \nabla_j\nabla_\ell h_{i\ell} + R^p_{\ell ji}h_{p\ell} + R^p_{\ell j\ell}h_{pi}) \\
&= -\frac{1}{2}(\Delta h_{ij} + \nabla_i\nabla_j H - \nabla_i\nabla_\ell h_{j\ell} + R^p_{\ell ij}h_{p\ell} - R^p_i h_{pj} - \nabla_j\nabla_\ell h_{i\ell} + R^p_{\ell ji}h_{p\ell} - R^p_j h_{pi}) \\
&= -\frac{1}{2}(\Delta_L h_{ij} + \nabla_i\nabla_j - \nabla_i\nabla_\ell h_{j\ell} - \nabla_j\nabla_\ell h_{i\ell}).
\end{aligned}$$
□

$$\frac{\partial}{\partial s}(-2R_{ij}) = \Delta_L h_{ij} + \nabla_i X_j + \nabla_j X_i, \tag{118}$$

Proof. There's nothing to show here; recall that in the identity above we have $X = \frac{1}{2}\nabla H - \operatorname{div} h$. Then the identity follows immediately from (117). □

$$\left\langle \dot{\nabla}_X Y, Z \right\rangle = -(\nabla_X \operatorname{Rc})(Y,Z) + (\nabla_Z \operatorname{Rc})(X,Y) - (\nabla_Y \operatorname{Rc})(X,Z). \tag{119}$$

Proof. To derive this, we take the derivative of Koszul's formula (65), which says

$$2g(\nabla_X Y, Z) = Xg(Y,Z) - Zg(X,Y) + Yg(Z,X) - g([Y,Z],X) + g([X,Y],Z) - g([X,Z],Y).$$

For the first term (here we are thinking of g as time-dependent, and X, Y, Z as time-independent),

$$\begin{aligned}
\frac{\partial}{\partial s}2g(\nabla_X Y, Z) &= 2(\partial_s g)(\nabla_X Y, Z) + 2g(\dot{\nabla}_X Y, Z) \\
&= -4\operatorname{Rc}(\nabla_X Y, Z) + 2g(\dot{\nabla}_X Y, Z).
\end{aligned}$$

For the next term,

$$\begin{aligned}\frac{\partial}{\partial s} X g(Y,Z) &= X((\partial_s g)(Y,Z))\\ &= -2X \operatorname{Rc}(Y,Z),\end{aligned}$$

and the two terms after that give something similar. For the fifth term,

$$\begin{aligned}\frac{\partial}{\partial s} g([Y,Z],X) &= (\partial_s g)([Y,Z],X)\\ &= -2\operatorname{Rc}(\nabla_Y Z - \nabla_Z Y, X),\end{aligned}$$

and the sixth and seventh terms are similar. Putting this together, we get

$$\begin{aligned}-4\operatorname{Rc}(\nabla_X Y, Z) + 2g(\dot{\nabla}_X Y, Z) &= -2X\operatorname{Rc}(Y,Z) + 2Z\operatorname{Rc}(X,Y) - 2Y\operatorname{Rc}(Z,X)\\ &\quad - [-2\operatorname{Rc}(\nabla_Y Z, X) + 2\operatorname{Rc}(\nabla_Z Y, X)]\\ &\quad + [-2\operatorname{Rc}(\nabla_X Y, Z) + 2\operatorname{Rc}(\nabla_Y X, Z)]\\ &\quad - [-2\operatorname{Rc}(\nabla_X Z, Y) + 2\operatorname{Rc}(\nabla_Z X, Y)]\\ g(\dot{\nabla}_X Y, Z) &= -X\operatorname{Rc}(Y,Z) + Z\operatorname{Rc}(X,Y) - Y\operatorname{Rc}(Z,X)\\ &\quad + \operatorname{Rc}(\nabla_Y Z, X) - \operatorname{Rc}(\nabla_Z Y, X) + \operatorname{Rc}(\nabla_X Y, Z)\\ &\quad + \operatorname{Rc}(\nabla_Y X, Z) + \operatorname{Rc}(\nabla_X Z, Y) - \operatorname{Rc}(\nabla_Z X, Y)\\ &= -(\nabla_X \operatorname{Rc})(Y,Z) - (\nabla_Y \operatorname{Rc})(X,Z) + (\nabla_Z \operatorname{Rc})(X,Y).\end{aligned}$$

We used that $(\nabla_X \operatorname{Rc})(Y,Z) = X(\operatorname{Rc}(Y,Z)) - \operatorname{Rc}(\nabla_X Y, Z) - \operatorname{Rc}(Y, \nabla_X Z)$. □

$$\frac{\partial}{\partial s}\Delta_{g(s)} = -h_{ij}\nabla^i\nabla^j - g^{k\ell}\left(g^{ij}\nabla_i h_{j\ell} - \frac{1}{2}\nabla_\ell\left(g^{ij}h_{ij}\right)\right)\nabla_k. \tag{120}$$

Proof. For a smooth function u, we have

$$\begin{aligned}\partial_t(\Delta u) &= \partial_t\left[g^{ij}\left(\partial_i\partial_j - \Gamma_{ij}^k\partial_k\right)u\right]\\ &= (\partial_t g^{ij})\nabla_i\nabla_j u - g^{ij}(\partial_t\Gamma_{ij}^k)\nabla_k u + \Delta(\partial_t u)\\ &= -h^{ij}\nabla_i\nabla_j u - g^{ij}\left(\frac{1}{2}g^{k\ell}\left(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}\right)\right)\nabla_k u\\ &= -h^{ij}\nabla_i\nabla_j u - g^{k\ell}\left(g^{ij}\nabla_i h_{j\ell} - \frac{1}{2}\nabla_\ell(g^{ij}h_{ij})\right)\nabla_k u.\end{aligned}$$

□

$$\partial_t\Delta_{g(t)} = 2\operatorname{Rc}_{ij}\nabla_i\nabla_j. \tag{121}$$

Proof. We give two proofs. For $f \in C^\infty(M)$, using the coordinate expression (78) for the Hessian,

$$\begin{aligned}(\partial_t\Delta_{g(t)})f &:= \partial_t(g^{ij}\nabla_i\nabla_j)f\\ &= (\partial_t g^{ij})\nabla_i\nabla_j f + g^{ij}(\partial_t\nabla_i\nabla_j f)\\ &= 2\operatorname{Rc}^{ij}\nabla_i\nabla_j f + g^{ij}(\partial_t(\partial_i\partial_j f - \Gamma_{ij}^k\partial_k f))\\ &= 2\operatorname{Rc}^{ij}\nabla_i\nabla_j f + g^{ij}(\partial_t\Gamma_{ij}^k)\partial_k f\end{aligned}$$

Now, we calculate, using the contracted second Bianchi identity $2g^{ij}\nabla_i \operatorname{Rc}_{jk} = \nabla_k R$.

$$\begin{aligned}
g^{ij}\partial_t \Gamma^k_{ij} &= g^{ij}\left[\frac{1}{2}g^{k\ell}(\nabla_i(-2\operatorname{Rc}_{j\ell}) + \nabla_j(-2\operatorname{Rc}_{i\ell}) - \nabla_\ell(-2\operatorname{Rc}_{ij}))\right] \\
&= -g^{j\ell}g^{ij}\nabla_i \operatorname{Rc}_{j\ell} - g^{k\ell}g^{ij}\nabla_j \operatorname{Rc}_{i\ell} + g^{ij}g^{k\ell}\nabla_\ell \operatorname{Rc}_{ij} \\
&= -\frac{1}{2}g^{k\ell}\nabla_\ell R - \frac{1}{2}g^{k\ell}\nabla_\ell R + g^{k\ell}\nabla_\ell g^{ij}\operatorname{Rc}_{ij} \\
&= 0,
\end{aligned}$$

from which the result follows. □

Proof. The second proof is slightly less coordinate-dependent. Let $f, h \in C^\infty(M)$. Then

$$\begin{aligned}
\int_M h\Delta f \, d\mu &= -\int_M \langle \nabla h, \nabla f\rangle \, d\mu \\
&= -\int_M g^{ij}\nabla_i h \nabla_j f \, d\mu.
\end{aligned}$$

Differentiating both sides with respect to t gives

$$\int_M [h\dot{\Delta} f \, d\mu + h\Delta f(\partial_t \, d\mu)] = -\int_M [(\partial_t g^{ij})\nabla_i h \nabla_j f \, d\mu + g^{ij}\nabla_i h\nabla_j f(\partial_t \, d\mu)].$$

Now use the fact that $\partial_t d\mu = -R\, d\mu$, and $\partial_t g^{ij} = 2R^{ij}$ to get

$$\begin{aligned}
\int_M [\dot{\Delta} f - R\Delta f]h \, d\mu &= -\int_M [2\operatorname{Rc}_{ij}\nabla_j f - Rg^{ij}\nabla_j f](\nabla_i h)\, d\mu \\
&= \int_M \nabla_i[2\operatorname{Rc}_{ij}\nabla_j f - Rg^{ij}\nabla_j f]h \, d\mu.
\end{aligned}$$

Since $h \in C^\infty(M)$ was arbitrary,

$$\begin{aligned}
\dot{\Delta} f - R\Delta f &= \nabla_i(2\operatorname{Rc}_{ij}\nabla_j f - Rg^{ij}\nabla_j f) \\
&= (2\nabla_i \operatorname{Rc}_{ij})\nabla_j f + 2\operatorname{Rc}_{ij}\nabla_i\nabla_j f - g^{ij}\nabla_i R\nabla_j f - Rg^{ij}\nabla_i\nabla_j f \\
&= \nabla_j R\nabla_j f + 2\operatorname{Rc}_{ij}\nabla_i\nabla_j f - \nabla_j R\nabla_j f - R\Delta f \\
&= 2\operatorname{Rc}_{ij}\nabla_i\nabla_j f - R\Delta f,
\end{aligned}$$

and the result follows. □

$$\partial_t \Delta = -f\Delta + \left(\frac{n}{2} - 1\right)\nabla f \cdot \nabla. \tag{122}$$

Proof. It suffices to calculate

$$g^{k\ell}\left(g^{ij}\nabla_i(fg) - \frac{1}{2}\nabla_\ell(g^{ij}fg_{ij})\right) = \left(1 - \frac{n}{2}\right)\nabla^k f,$$

which is not too hard. □

$$\partial_s(\nabla_i\nabla_j f) = \nabla_i\nabla_j(\partial_s f) - (\partial_s\Gamma^k_{ij})\partial_k f. \tag{123}$$

Proof.

$$\begin{aligned}\partial_s(\nabla_i\nabla_j f) &= \partial_s(\partial_i\partial_j f - \Gamma_{ij}^k\partial_k f)\\ &= \partial_i\partial_j(\partial_s f) - (\partial_s\Gamma_{ij}^k)\partial_k f - \Gamma_{ij}^k\partial_k(\partial_s f)\\ &= \nabla_i\nabla_j(\partial_s f) - (\partial_s\Gamma_{ij}^k)\partial_k f.\end{aligned}$$ □

$$\partial_t(\nabla_i\nabla_j f) = \nabla_i\nabla_j(\partial_t f) + (\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij})\nabla^\ell f. \tag{124}$$

Proof. This follows from (123) with the variation of the Christoffel symbols (112) in the Ricci flow case □

$$\partial_t R_{ijk}{}^\ell = \frac{1}{2}g^{\ell p}\left(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\right) \tag{125}$$

Proof. TODO □

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8 Submanifolds

Many of these ideas have are special cases of things discussed previously, in the case where the metric is induced by some immersion or embedding into a higher dimensional Riemannian manifold.

As the notation for this section is quite painful, here is a seperate notation glossary just for submanifolds, although it should essentially overlap with notation from the rest of the document. In most cases I am following Mat Langford's notation; see `https://suppiluliuma.neocities.org/RG.pdf`.

Let M^n and N^{n+k} be smooth manifolds, and $X : M \to N$ a smooth immersion. Then we denote

$dX\colon TM \to TN$	the derivative of X
X^*TN	the pullback bundle (over M)
$dX(TM)$	the subbundle of X^*TN from the embedding $(p,u) \mapsto (p, dX(u))$ of TM into X^*TN
$\langle \cdot, \cdot \rangle, g$	the metrics on N, M respectively
$X^* \langle \cdot, \cdot \rangle$	the pullback metric on X^*TN: $X^* \langle (p,u), (p,v) \rangle$
N_pM	the normal space to M at $p \in M$, i.e. $N_pM = \{\nu \in T_X(p)N : \langle u, \nu \rangle = 0 \text{ for all } u \in dX_p(T_pM)\}$
NM	the normal subbundle of TN in the case where X is an embedding
NM	the normal subbundle of X^*TN (over M), i.e. $NM = \{\nu \in X^*TN : {}^X \langle u, \nu \rangle = 0 \text{ for all } u \in dX(TM)_{\pi(\nu)}\}$
D	the connection on N
${}^XD\colon TM \times \Gamma(X^*TN) \to X^*TN$	the pullback connection on X^*TN, defined by ${}^XD_u X^*V \coloneqq (\pi(u), D_{dX(u)}V)$
∇^M, ∇^N	connections on TM, TN, respectively
II	second fundamental form; $\mathrm{II} \in \Gamma(T^*M \otimes T^*M \otimes NM)$, i.e. $\mathrm{II}(u,v) = ({}^XD_u(dX(V)))^\perp$, for an extension V of v
W	Weingarten tensor; $W \in \Gamma(T^*M \otimes TM \otimes N^*M)$

If M is a submanifold of N, then the metric on N induces a metric on M. We can express the Levi-civita connection ∇^M of M from the one on N, ∇^N by

$$\nabla^M_X Y = (\nabla^N_X Y)^\top, \quad \text{for } X, Y \in \Gamma(TM), \tag{126}$$

where $\top : T_xN \to T_xM$ is orthogonal projection.

$$\nabla^M_X Y = (\nabla^N_X Y)^\top, \quad \text{for } X, Y \in \Gamma(TM). \tag{126}$$

Proof. In order to make sense of the right-hand side, we need to extend X and Y locally to a neighborhood of M in N. This can be done, for instance, in local coordinates around $x \in M$ that locally map M to $\mathbb{R}^m \subset \mathbb{R}^n$. The extension of $X = X^i(x)\frac{\partial}{\partial x^i}$ then for example is

$$\widetilde{X}(x^1, \dots, x^n) = \sum_{i=1}^m X^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}.$$

We then have $g_N(\widetilde{X}, \widetilde{Y})(x) = g_M(X,Y)(x)$, and $[\widetilde{X}, \widetilde{Y}](x) = [X,Y](x)$. Due to the equality of these quantities together with Koszul's formula that expresses the Levi-Civita connection only in terms of these quantities, we get the equality of the two connections. We should also argue that this does not depend on the choice of extension. If we take two different extensions $\widetilde{X}, \widetilde{\widetilde{X}}$, then we would like to show that their difference $Z = \widetilde{X} - \widetilde{\widetilde{X}}$ satisfies $(\nabla^N_Z Y)^\top = 0$, and similarly for different extensions of

Y. By considering the expression of the connection in terms of Christoffel symbols, we see that if Z vanishes at a point, then the Christoffel symbol term goes away. But the remaining term is tangent to M, and thus goes away when we take the orthogonal projection onto M. □

8.1 Second fundamental form

Roughly, $\mathrm{II}(u, v)$ is the normal (to the image of the immersion) component of how the vector field V is changing in the direction of u.

8.2 Riemannian submersions

reference: LeeRM

Let $\tilde{M}, M$ be Riemannian manifolds and $\pi : \tilde{M} \to M$ a smooth submersion. By the submersion level set theorem, each fiber $\tilde{M}_y = \pi^{-1}(y)$ is a properly embedded smooth submanifold of $\tilde{M}$. At each point $x \in \tilde{M}$, we define two subspaces of the tangent space $T_x\tilde{M}$ as follows: the **vertical tangent space** at x is

$$V_x = \ker(d\pi_x) = T_x(\tilde{M}_{\pi(x)})$$

(that is, the tangent space to the fiber containing x), and the **horizontal tangent space** at x is its orthogonal complement:

$$H_x = (V_x)^\perp.$$

Then the tangent space $T_x\tilde{M}$ decomposes as an orthogonal direct sum $T_x\tilde{M} = H_x \oplus V_x$. Note that the vertical space is well defined for every submersion, because it does not refer to the metric, but the horizontal space depends on the metric. In particular, a Riemannian submersion has the property that it preserved the length of horizontal tangent vectors.

A vector field on $\tilde{M}$ is said to be **horizontal** if its value at each point lies in the horizontal space at that point; a **vertical** vector field is defined similarly. Given a vector field X on M, a vector field $\tilde{X}$ on $\tilde{M}$ is called a **horizontal lift** of X if $\tilde{X}$ is horizontal and π-related to X (that is, $d\pi_x(\tilde{X}_x) = X_{\pi(x)}$ for each $x \in \tilde{M}$.

The following is Proposition 2.25 in [LeeRM]. If $\pi : \tilde{M} \to M$ is a smooth submersion and $\tilde{g}$ is a Riemannian metric on $\tilde{M}$,

(a) Every smooth vector field W on $\tilde{M}$ can be expressed uniquely in the form $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and both W^H and W^V are smooth.

(b) Every smooth vector field on M has a unique smooth horizontal lift to $\tilde{M}$.

(c) For every $x \in \tilde{M}$ and $v \in H_x$, there is a vector field $X \in \Gamma(TM)$ whose horizontal lift $\tilde{X}$ satisfies $\tilde{X}_x = v$.

This is problem 5-6 of LeeRM. Suppose $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ is a Riemannian submersion. If Z is any vector field on M, we let $\tilde{Z}$ denote its horizontal lift to $\tilde{M}$. For every pair of vector fields X, Y on M, we have

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(X, Y) \circ \pi, \tag{127}$$

$$[\tilde{X}, \tilde{Y}]^H = \widetilde{[X, Y]}, \tag{128}$$

$$[\tilde{X}, W] \text{ is vertical if } W \in \Gamma(T\tilde{M}) \text{ is vertical.} \tag{129}$$

Let $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of $\tilde{g}$ and g, respectively. Show that for every pair of vector fields $X, Y \in \Gamma(TM)$, we have

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \tfrac{1}{2}[\tilde{X}, \tilde{Y}]^V. \tag{130}$$

8.2.1 O'Neill's formula

$$\begin{aligned} \mathrm{Rm}(W, X, Y, Z) \circ \pi = \widetilde{\mathrm{Rm}}(\tilde{W}, \tilde{X}, \tilde{Y}, \tilde{Z}) - \frac{1}{2} \left\langle [\tilde{W}, \tilde{X}]^V, [\tilde{Y}, \tilde{Z}]^V \right\rangle \\ - \frac{1}{4} \left\langle [\tilde{W}, \tilde{Y}]^V, [\tilde{X}, \tilde{Z}]^V \right\rangle + \frac{1}{4} \left\langle [\tilde{W}, \tilde{Z}]^V, [\tilde{X}, \tilde{Y}]^V \right\rangle. \end{aligned} \tag{131}$$

8.3 Proofs

$$g(\mathcal{H}_r(v), w) = g(v, \mathcal{H}_r(w)v) \tag{96}$$

Proof. TODO □

$$\mathcal{H}_r(\partial_r) = 0. \tag{97}$$

Proof. TODO □

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(98)

The restriction of $\mathcal{H}_r$ to vectors tangent to a level set of r is equal to the shape operator of the level set associated with the normal vector field $N = -\partial_r$.

Proof. TODO □

$$\nabla^2 r = \frac{1}{2} \sum_{\alpha,\beta=1}^{n-1} (\partial_r g_{\alpha\beta}) d\theta^\alpha d\theta^\beta. \tag{99}$$

Proof. TODO □

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(X, Y) \circ \pi, \tag{127}$$

Proof. TODO □

$$[\tilde{X}, \tilde{Y}]^H = \widetilde{[X, Y]}, \tag{128}$$

Proof. TODO □

$$[\tilde{X}, W] \text{ is vertical if } W \in \Gamma(T\tilde{M}) \text{ is vertical.} \tag{129}$$

Proof. TODO □

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \tfrac{1}{2}[\tilde{X}, \tilde{Y}]^V. \tag{130}$$

Proof. TODO

Hint: let $\tilde{Z}$ be a horizontal lift and W a vertical vector field on $\tilde{M}$, and compute $\left\langle \tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z} \right\rangle$ and $\left\langle \tilde{\nabla}_{\tilde{X}} \tilde{Y}, W \right\rangle$ using Koszul's formula. ☐

$$\begin{aligned} \mathrm{Rm}(W,X,Y,Z) \circ \pi = \widetilde{\mathrm{Rm}}(\tilde{W},\tilde{X},\tilde{Y},\tilde{Z}) - \frac{1}{2} \left\langle [\tilde{W},\tilde{X}]^V, [\tilde{Y},\tilde{Z}]^V \right\rangle \\ -\frac{1}{4} \left\langle [\tilde{W},\tilde{Y}]^V, [\tilde{X},\tilde{Z}]^V \right\rangle + \frac{1}{4} \left\langle [\tilde{W},\tilde{Z}]^V, [\tilde{X},\tilde{Y}]^V \right\rangle . \end{aligned} \tag{131}$$

Proof. TODO ☐

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9 Ricci Flow

9.1 Variation formulae

For completeness, we reconsider some of the general variation formulae above in the special case of the Ricci flow.

$$\partial_t R = \Delta R + 2 \left|\mathrm{Rc}\right|^2 \tag{132}$$

$$\partial_t R = \Delta R + 2 \left|\mathrm{Rc}\right|^2 \tag{132}$$

Proof. Previously we showed that if $\partial_t g = h$, then

$$\partial_t R = -\Delta H + \mathrm{div}(\mathrm{div}\, h) - \langle h, \mathrm{Rc} \rangle .$$

For the Ricci flow, $h = -2\,\mathrm{Rc}$, so $H = -2R$. The twice contracted second Bianchi identity says that $\mathrm{div}\,\mathrm{Rc} = \frac{1}{2}\nabla R$, and then the divergence of this gives a Laplacian term. So the calculation is

$$\begin{aligned} \partial_t R &= -\Delta(-2R) + \mathrm{div}(\mathrm{div}(-2\,\mathrm{Rc})) - \langle -2\,\mathrm{Rc}, \mathrm{Rc} \rangle \\ &= 2\Delta R - 2\,\mathrm{div}(\tfrac{1}{2}\nabla R) + 2 \left|Rc\right|^2 \\ &= 2\Delta R - \Delta R + 2 \left|\mathrm{Rc}\right|^2 \\ &= \Delta R + 2 \left|\mathrm{Rc}\right|^2 . \end{aligned}$$

☐

9.2 DeTurck's trick

Suppose we have some possibly time-dependent family of vector fields X_t, and a metric g flowing by the Lie derivative $\mathcal{L}_X g$. That is, g is a solution to the equation

$$\partial_t g = \mathcal{L}_X g.$$

If we define ψ_t to be the Then by the computation

$$\frac{d}{dt} \psi_t^* g_t$$

9.3 Uhlenbeck's trick

The idea is that computations are easy to do in an orthonormal basis. However, in this time-dependent situation, a basis that is orthonormal at one time will not in general be orthonormal at later times. However, there is a simple ODE we can solve to find an orthonormal basis at all times.

Let $\{e_i(t_0)\}_{i=1}^n \subset T_pM$ be an orthonormal basis for g_{t_0} at p. Observe that if we solve

$$\frac{d}{dt}e_i(t) = \mathrm{Rc}_t(e_i(t)),$$

then

$$\begin{aligned}\frac{d}{dt}g_t(e_i(t), e_j(t)) &= -2\,\mathrm{Rc}_t(e_i(t), e_j(t)) + g_t(\mathrm{Rc}_t(e_i(t)), e_j(t)) + g_t(e_i(t), \mathrm{Rc}(e_j(t)))\\ &= 0,\end{aligned}$$

by definition of the (1,1)-version of Rc. Thus the basis remains orthonormal. This has an interpretation in "spacetime." Given a Ricci flow $(M, g_t)_{t\in I}$, there is an associated spacetime manifold $M \times I$. We can look at the tangent space to a point (p, t) in this manifold, but for some reason it's more useful to look at the **spatial tangent bundle**, which we define by

$$T^{\mathrm{spat}}(M \times I) = \mathrm{proj}_M^*(TM).$$

We can view this as embedded in the tangent bundle of $M \times I$, since $T^{\mathrm{spat}}(M \times I) = \ker dt$, where $t = \mathrm{proj}_I$. We also have the time vector field $\partial_t \coloneqq (\mathrm{proj}_I)_*$.

With this setup, there is a 1-1 correspondence between time-dependent vector fields $(X_t)_{t\in I}$ on M and sections of $T^{\mathrm{spat}}(M \times I)$. The family of metrics $(g_t)_{t\in I}$ induces a metric on $T^{\mathrm{spat}}(M \times I)$.

Now we want to phrase Uhlenbeck's trick using this machinery, which we do by constructing a connection on the spatial tangent bundle. We define the space-time connection $\tilde\nabla$ by

$$\begin{cases}\tilde\nabla_V X = \nabla_V X & V \in T^{\mathrm{spat}}_{(p,t)}(M \times I)\\ \tilde\nabla_{\partial_t} X = \partial_t X - \mathrm{Rc}_t(X), & \end{cases}$$

where ∇ is the Levi-Civita connection with respect to g_t. It is easy to show that this connection is metric.

A Examples

In this section we present some concrete examples of functions, manifolds, flows, and so on.

Example A.1 (Curvature of the 2-sphere)**.**

Example A.2 (Constant curvature spaces)**.** A Riemannian metric g has constant sectional curvature c if and only if its curvature tensor satisfies

$$\mathrm{Rm} = \frac{1}{2}cg \owedge g. \tag{1}$$

In this case, the Ricci and scalar curvature of g are given by the formulas

$$\mathrm{Rc} = (n-1)cg, \qquad R = n(n-1)c, \tag{2}$$

and the Riemann curvature endomorphism is

$$R(X,Y)Z = c\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right). \tag{3}$$

That is, in terms of any basis,

$$R_{ijk\ell} = c(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}), \qquad R_{ij} = (n-1)cg_{ij}. \tag{4}$$

Example A.3 (Pullback of a tensor field)**.** See example 12.29 from Lee

References

[1] Ben Andrews and Christopher A Hopper. The Ricci flow in Riemannian geometry. en. 2011th ed. Lecture notes in mathematics. Berlin, Germany: Springer, Nov. 2010.

[2] Bennett Chow and Yutze Chow. "Lectures on Differential Geometry". In: Graduate Studies in Mathematics (2024). URL: https://api.semanticscholar.org/CorpusID:120912336.

[3] Bennett Chow and Dan Knopf. The Ricci Flow: An Introduction. Vol. 1. American Mathematical Soc., 2004.

[4] Victor Guillemin and Alan Pollack. Differential Topology. en. Upper Saddle River, NJ: Pearson, Aug. 1974.

[LeeRM] John M Lee. Introduction to Riemannian manifolds. en. 2nd ed. Graduate texts in mathematics. Cham, Switzerland: Springer International Publishing, Jan. 2019.

[5] John M Lee. Introduction to Smooth Manifolds. en. 2nd ed. Graduate texts in mathematics. New York, NY: Springer, Aug. 2012.