Computations in Riemannian Geometry and Geometric Analysis

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These notes are intended as a practical reference when doing basic calculations in Riemannian geometry. I hope they will be helpful to the reader who is familiar with the concepts of Riemannian geometry but isn't an expert when it comes to calculations; they may provide a useful supplement to a more expository text on Riemannian geometry (I recommend, in addition to the standard texts, the lecture notes of Ben Andrews, which can be found online, and the book by Andrews and Hopper).

Proofs are placed at the end of each section, and are numbered based on the right-hand side numbering. The numbers on the right hand side are ~~all~~ mostly links that go back and forth between an equation and its proof (actually this doesn't work at the moment). Because this is intended more as a reference than as something to be read from start to finish, I've only made a little effort to keep concepts in order of dependence.

This is a slow but steady work in progress, with still much to be done, possibly including some major reorganization. Last updated: March 8, 2025

TODO: a section on Hodge theory. References: HRF 1.5

TODO: finish all TODOs

TODO: read Jost Ch. 2-3 and probably also the rest of the book

TODO: fix all links and make sure every equation has a proof

0 Notation and Conventions

The convention I use for the curvature of a connection agrees with Chow and Lee, meaning that it is the opposite of Andrews. My convention for the factor in front of the wedge product of alternating tensors agrees with Andrews (and Lee?) and opposes Chow. I take Andrews's and Chow's convention for the way to write the rank of a mixed tensor (opposite to Lee), which says that a (k, ℓ) tensor takes as input k vector fields and ℓ covector fields. For example, we'll say that the metric is a $(2, 0)$-tensor. See Section 3 for more about this. Throughout, unless otherwise stated, we will be considering a Riemannian n-manifold $M = (M^n, g)$.

$\Gamma(E)$	the set of sections of the bundle E over M
$\mathcal{T}^k_\ell(M)$	the set of (k, ℓ)-tensors; that is, sections of $(T^*M)^{\otimes k} \otimes (TM)^{\otimes \ell}$
$\wedge^k T^*M$	the k-form bundle on M
$\Omega^k(T^*M)$	the set of sections of $\wedge^k T^*M$, i.e. the set of k-forms on M; $\Gamma(\wedge^k T^*M)$
$d\,\mathrm{Vol}$	the volume form of a Riemannian manifold

Contents

2 Basic notions

This section contains constructions that don't depend on a Riemannian metric, but also contains some Riemannian-metric based identities.

2.1 Vector fields

By construction (see any book on Riemannian geometry), a vector field $X \in \Gamma(TM)$ satisfies the Leibniz rule

$$X(fg) = fX(g) + gX(f)$$

for $f, g \in C^\infty(M)$. From this it follows that

$$X(fY) = X(f) \cdot Y + fX(Y). \tag{1}$$

If $f \in C^\infty(M)$ and $r \colon \mathbb{R} \to \mathbb{R}$,

$$X(r \circ f) = (r' \circ f)X(f) \tag{2}$$

2.2 The differential and gradient

The **differential** df of a function $f \in C^\infty(M)$ is the 1-form defined by

$$(df)(X) = X(f)$$

for $X \in \mathscr{X}(M)$. Let $\operatorname{grad} f$ denote the vector field dual to df. That is, $g(\operatorname{grad} f, X) = (df)(X) = X(f)$. Sometimes ∇f is used to denote either df or $\operatorname{grad} f$ (or both). It is also used to denote the total covariant derivative of f (see below), but this is not really an abuse of notation since the total covariant derivative of f is equal to df.

$$\operatorname{grad}(fh) = f \operatorname{grad} h + h \operatorname{grad} f \tag{3}$$

In coordinates:

$$df = (\partial_i f)dx^i \tag{4}$$

$$\operatorname{grad} f = g^{ij}(\partial_j f)\partial_i \tag{5}$$

2.3 Differential forms

Given a vector space V, a $(k, 0)$-tensor $\omega \in \bigotimes^k V^*$ is said to be alternating if it is antisymmetric under interchange of any two of its arguments. The set of alternating $(k, 0)$-tensors on V is denoted $\wedge^k V^*$. In particular, we are interested in $\wedge^k T_p^* M$; the space of alternating $(k, 0)$-tensors at p. Two special cases are 0- and 1-tensors, which are functions and covectors respectively. These are trivially alternating, so we have $\wedge^0 T^* M = C^\infty(M)$ and $\wedge^1 T^* M = T^* M$.

The **wedge product** of an alternating k-tensor and an alternating ℓ-tensor is a $(k+\ell)$-tensor, defined by

$$S \wedge T = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma)(S \otimes T) \circ \sigma,$$

where the composition with σ denotes applying the permutation σ to the $k+\ell$ inputs to $S \otimes T$. There is another convention (used by Chow, for example) for this that involves a different factor in front. A useful special case occurs with a multiple wedge product of covectors

The wedge product satisfies the following properties, for forms ω, η, μ and $c \in C^\infty(M)$,:

(i) $\omega \wedge (\eta \wedge \mu) = (\omega \wedge \eta) \wedge \mu$

(ii) $(c\omega) \wedge \eta = \omega \wedge (c\eta) = c(\omega \wedge \eta)$

(iii) If $\omega, \eta \in \wedge^k T_p M$, then

$$(\omega + \eta) \wedge \mu = \omega \wedge \mu + \eta \wedge \mu.$$

(iv) If $\omega \in \wedge^k T_p M$ and $\eta \in \wedge^\ell T_p M$, then

$$\eta \wedge \omega = (-1)^{k\ell} \omega \wedge \eta.$$

The space $\wedge^k T_p M$ is in fact a vector space: given a basis $\{\omega^i\}_{i=1}^n$ for $T_p^* M$, the set

$$\left\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} : 0 < i_1 < \cdots < i_k < n\right\}$$

is a basis for $\wedge^k T_p M$. Thus $\wedge^k T_p M$ has dimension $\binom{n}{k}$. Moreover, the set $\{\wedge^k T_p M : p \in M\}$ of all alternating k-tensors at points of M has a bundle structure. A **k-form** is a smooth section of the bundle $\wedge^k T^* M$. The set of all k-forms on M is denoted $\Omega^k(M)$.

2.3.1 The volume form

If M is oriented, there is a unique n-form $d\mu = d\mu_g$ called the **volume form**, defined in local coordinates by

$$d\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

If $\{\omega^i\}_{i=1}^n$ is an oriented orthonormal coframe for $T^* M$, then

$$d\mu = \omega^1 \wedge \cdots \wedge \omega^n. \tag{6}$$

Despite the notation, the volume form $d\mu$ is generally not the exterior derivative of some $(n-1)$-form μ.

2.3.2 Exterior derivative

The exterior derivative is the unique linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying

(i) If $f \in \Omega^0(M) = C^\infty(M)$, then df is the same as the differential of f.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d^2 = 0$.

Using these axioms, we can determine the following expression for d. Suppose we have coordinate covector fields dx^i. If we have the k-form ω given by (??? sums are taken over increasing k-tuples)

$$\omega = \sum_{i_1,\dots,i_k} \omega_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then (I think this sum is just taken over all tuples).

$$d\omega = \sum_{i,i_1,\dots,i_k} \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{7}$$

Strictly speaking, there is still some more work to be done to make sure everything works here, even though it seems like we have a nice expression for d. One needs to show that this doesn't depend on the coordinates, and justify the claim that this operator is unique. For arguments of these facts, see Ben Andrews's lecture notes on differential geometry, or one of many other books on geometry.

If ω is a 1-form, we have the following useful expression

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{8}$$

The previous expression generalizes: if ω is a k-form, the exterior derivative satisfies (here the hat notation means we are removing an argument)

$$\begin{aligned}(d\omega)(X_0,\dots,X_k) = &\sum_{j=0}^{k}(-1)^j X_j\omega(X_0,\dots,\hat{X}_j,\dots,X_k) \\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k).\end{aligned} \tag{9}$$

According to some conventions for the wedge product, this expression may differ by a factor of $\frac{1}{k+1}$ (e.g. in [2]). This expression can also be used to define the exterior derivative in a way that is explicitly independent of coordinates.

See section 2.4.1 for information about the formal adjoint to the exterior derivative.

2.3.3 Interior product

The interior product is, for each $X \in T_pM$, a linear map $\iota_X\colon \wedge^k T_p^*M \to \wedge^{k-1}T_p^*M$. If $\omega \in \wedge^0 T_p^*M$ (so that ω is a number), we define $\iota_X\omega = 0$. Otherwise, the interior product is the unique linear operator $\iota_X\colon \Omega^k(M) \to \Omega^{k-1}(M)$ satisfying similar properties to the exterior derivative:

(i) When $\omega \in \Omega^1(M) = \Gamma(T^*M)$, then $\iota_X\omega = \omega(X)$.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$\iota_X(\omega\wedge\eta) = (\iota_X\omega)\wedge\eta + (-1)^k\omega\wedge(\iota_X\eta)$$

(iii) $\iota_X^2 = 0$.

From (ii) it follows that

$$\iota_X(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} (-1)^{i+1} \omega_1 \wedge \cdots \wedge \iota_X(\omega_i) \wedge \cdots \wedge \omega_k.$$

We can determine that ι_X has the following expression:

$$\iota_X(\omega)(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}). \tag{10}$$

In particular, for covectors $\omega^1, \ldots, \omega^k$, we have

$$\iota_X(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(X) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k. \tag{11}$$

Differential forms are exactly the objects that we integrate over a manifold. For more about integration, see Section 7.1.

2.3.4 The Hodge star operator

Before defining this operator, we need to define an inner product of forms. We could construct k-forms as alternating tensors on any vector space. Then an inner product on the vector space induces an inner product on the k-forms. We define this on basis k-forms by (taking $\{\omega^i\}_{i=1}^k$ as a basis for the vector space)

$$\left\langle \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}, \omega^{j_i} \wedge \cdots \wedge \omega^{j_k} \right\rangle = \det(\left\langle \omega^{i_a}, \omega^{j_b} \right\rangle),$$

where on the right hand side we are using whatever metric we are given on the underlying vector space. In our case, the underlying vector space is the cotangent space, and the inner product is of course the Riemannian metric, so the product becomes (asssuming that $\{\omega^i\}$ is orthonormal)

$$\left\langle \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}, \omega^{j_i} \wedge \cdots \wedge \omega^{j_k} \right\rangle = \det(\delta^{i_a j_b}).$$

The Hodge star operator $* : \wedge^k T^* M \to \wedge^{n-k} T^* M$ is the linear operator defined by

$$\langle \omega, \eta \rangle \; d\,\mathrm{Vol} = \omega \wedge * \eta.$$

For example, if $\{\omega^i\}_{i=1}^n$ is a positively oriented orthonormal coframe, then for any k,

$$(\omega^1 \wedge \cdots \wedge \omega^k) \wedge *(\omega^1 \wedge \cdots \wedge \omega^k) = d\,\mathrm{Vol},$$

since the inner product of $\omega^1 \wedge \cdots \wedge \omega^k$ with itself is clearly 1.

$$*(\omega^1 \wedge \cdots \wedge \omega^k) = \omega^{k+1} \wedge \cdots \wedge \omega^n. \tag{12}$$

As an operator on $\wedge^k T^* M$,

$$*^2 = (-1)^{k(n-k)}. \tag{13}$$

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Proof. It suffices to prove it on a basis for $\wedge^k T_pM$. In particular, let $\{\omega^i\}_{i=1}^n$ be a positively oriented orthonormal coframe (basis for T_p^*M). Then recall that

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} : 0 < i_1 < \cdots < i_k < n\}$$

is a basis for $\wedge^k T_pM$. For a multi-index $I = (i_1, \ldots, i_k)$, let $\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$. Since

$$* * (\omega^1 \wedge \cdots \wedge \omega^k) = *(\omega^{k+1} \wedge \cdots \wedge \omega^n),$$

and

$$\omega^{k+1} \wedge \cdots \wedge \omega^n \wedge *(\omega^{k+1} \wedge \cdots \wedge \omega^n) = \omega^1 \wedge \cdots \wedge \omega^n,$$

it is not hard to see that the result holds when $I = (1, \ldots, k)$. It is also not hard to see that $* * \omega^I = \pm\omega^I$; the point is just to figure out what the sign must be. It suffices to show that the sign only depends on n and k (and not I).

□

The Hodge star satisfies the following commutation relations, where $*_k$ denotes $*$ acting on k-forms.

$$*d = (-1)^{k+1}\delta, \qquad *_{k+1}d^k = (-1)^{k+1}\delta^{n-k}*_k \tag{14}$$

$$*d = (-1)^{k+1}\delta, \qquad *_{k+1}d^k = (-1)^{k+1}\delta^{n-k}*_k \tag{14}$$

Proof.

$$\delta^{n-k}*_k = (-1)^{(n-k)n+n+1} * d * * = (-1)^{(n-k)n+n+1}(-1)^{k(n-k)} * d = (-1)^{k+1} * d$$

□

$$*\delta = (-1)^k d*, \qquad *_{k-1}\delta^k = (-1)^k d^{n-k}*_k \tag{15}$$

$$*\delta = (-1)^k d*, \qquad *_{k-1}\delta^k = (-1)^k d^{n-k}*_k \tag{15}$$

Proof.

$$*\delta = (-1)^{kn+n+1} * *d* = (-1)^{nk+n+1}(-1)^{(n-k+1)(k-1)}d* = (-1)^k d * .$$

□

$$*\Delta^k = \Delta^{n-k}* \tag{16}$$

$$*\Delta = \Delta* \tag{16}$$

Proof. □

2.4 Divergence

Note that for a vector field X, $d(\iota_X(d\mu))$ is an n-form, so it is $f d\mu$ for some smooth function f. We call this function the **divergence** of X, so that

$$d(\iota_X\, d\mu) = \operatorname{div} X\, d\mu.$$

We could also have defined the divergence as the trace of the covariant derivative:

$$\operatorname{div} X = \operatorname{tr} \nabla X = (\nabla X)(\partial_i, dx^i) = (\nabla_i X)(dx^i). \tag{17}$$

In local coordinates, we have the expression

$$\operatorname{div}(X^i \partial_i) = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}). \tag{18}$$

The product of a function f and a vector field X satisfies

$$\operatorname{div}(fX) = X(f) + f \operatorname{div} X. \tag{19}$$

The characterization of divergence as the trace of the covariant derivative allows us to define the divergence of a (k, ℓ)-tensor as the $(k, \ell - 1)$-tensor

I'm pretty sure it's this:

$$(\operatorname{div} T)(\omega^2, \ldots, \omega^\ell, X_1, \ldots, X_k) = \operatorname{tr}[(\nabla T)(\cdot, \cdot, \omega^2, \ldots, \omega^\ell, X_1, \ldots, X_k)].$$

$$(\operatorname{div} T)(X_1, \ldots, X_{\ell-1}) = \operatorname{tr}(\nabla_T())$$

In particular

This is from LeeRM, p. 149:

If F is any smooth k-tensor field (covariant, contravariant, or mixed), we define the divergence of F by

$$\operatorname{div} F = \operatorname{tr}_g(\nabla F),$$

where the trace is taken over the last two indices of the $(k + 1)$-tensor field ∇F. (If F is purely contravariant, then tr_g can be replaced with tr, because the next-to-last index of ∇F is already an upper index.)

In Lee's notation, the last two indices are

2.4.1 The codifferential

A useful operator in Hodge theory is δ, the formal adjoint to d on one-forms. In particular, for a function f and a form ω, if we define $\delta(\omega) = -\operatorname{div}(\omega^\sharp)$, we have

$$\langle \delta\omega, f \rangle = \langle \omega, df \rangle, \tag{20}$$

where $\langle \delta\omega, f \rangle = \int_M (\delta\omega) f\, d\mu$, and $\langle \omega, df \rangle = \int_M g(\omega, df)\, d\mu$. Another standard way to define δ is in terms of the Hodge star operator and the exterior derivative:

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha \tag{21}$$

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha \tag{21}$$

Proof. Here we'll prove the equivalence of the adjoint definition of δ and the definition above. Check for a missing factor of p. TODO □

2.5 The Laplacian(s)

The simplest version of the Laplacian is defined for functions $f \in C^\infty(M)$ by

$$\Delta f = \operatorname{div} \operatorname{grad} f.$$

The sign here is a matter of convention; our choice results in negative eigenvalues of the Laplacian. This can be extended to act on tensor bundles. This operator is called the **connection Laplacian**, the **rough Laplacian**, or the **Laplace-Beltrami operator** (especially when applied to functions); there are other second order linear elliptic operators referred to as the Laplacian as well. We define the rough Laplacian on tensors by $\Delta \colon \Gamma(\mathcal{T}^k_\ell(M)) \to \Gamma(\mathcal{T}^k_\ell(M))$ by

$$\Delta T = \operatorname{div} \nabla T = \operatorname{tr}_g \nabla^2 T = g^{ij} \nabla_i \nabla_j T,$$

where the trace is taken over the two new indices introduced by ∇^2. In coordinates,

$$\Delta = g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right) \tag{22}$$

For functions, this has the coordinate expression

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^j} \right). \tag{23}$$

$$\Delta(fh) = f \Delta h + h \Delta f + 2 \langle \operatorname{grad} f, \operatorname{grad} h \rangle \tag{24}$$

From this it follows that the heat operator $\partial_t - \Delta$ satisfies the product rule

$$(\partial_t - \Delta)(fh) = f(\partial_t - \Delta)(h) + h(\partial_t - \Delta)(f) - 2 \langle \nabla f, \nabla h \rangle. \tag{25}$$

If $f \in C^\infty(M)$ and $r \colon \mathbb{R} \to \mathbb{R}$, then

$$\Delta(r \circ f) = (r' \circ f) \Delta f + (r'' \circ f) |\nabla f|^2. \tag{26}$$

A frequently useful special case of this is when $r = \log$, so then we have

$$\Delta \log f = \frac{\Delta f}{f} - \frac{|\nabla f|^2}{f^2}.$$

There is also the **Lichnerowicz Laplacian**, see [1] Appendix A.4, and the **Hodge(-de Rham) Laplacian** on forms, and the **harmonic map Laplacian**, see p. 85 of [1]. The Lichnerowicz laplacian on 2-tensors is formally the same as the Hodge-de Rham laplacian acting on 2-forms.

The Hodge Laplacian is a family of maps $-\Delta_d : \Omega^p(T^*M) \to \Omega^p(T^*M)$ defined by

$$-\Delta_d = d\delta + \delta d$$

On a 1-form ω, we have the **Bochner formula**

$$\Delta \omega = \Delta_d \omega + \operatorname{Rc}(\omega). \tag{27}$$

The Lichnerowicz Laplacian commutes with the heat operator under the Ricci flow. See p. 110 of Chow HRF.

See Lemma 3.1 of Patodi: curvature and eigenforms of the laplace operator for a formula for the laplacian on forms (and some references).

2.6 Computations in special coordinates

Proofs of various identities can be simplified by choosing particular coordinate systems at a point. The idea is that essentially all quantities we are interested in are independent of coordinates, so we only need to prove an identity involving such quantities in a particular coordinate system, and it will hold in general. Thus we are free to choose the simplest coordinate system for the problem.

Most often the simplest coordinate system is **normal coordinates**, in which the metric becomes the identity matrix at a given point p. We define these coordinates by taking an orthonormal basis $\{e_i\}$ for T_pM, and letting $\exp_p^{-1} : U \to B_\epsilon(0)$ be the chart map, where $U \ni p$ and ϵ are chosen to make this a diffeomorphism. In normal coordinates at p, we have the following:

$$g_{ij}(p) = \delta_{ij} \tag{28}$$

$$\Gamma^k_{ij}(p) = 0 \tag{29}$$

$$\partial_k g_{ij}(p) = 0. \tag{30}$$

It follows from the local integrability of vector fields that, given a vector field $X = X^i \partial_i$, we can choose coordinates so that $X^i = 0$ for $i > 1$. The common practice of not bothering to raise or lower indices but still sum over repeated indices as in the Einstein convention reflects the fact that we can perform computations using normal coordinates. For example, we might write $\nabla_i X_i$ to mean $\sum_i \nabla_i X_i$, since, in normal coordinates,

$$g^{ij}\nabla_j X_i = \delta^{ij}\nabla_j X_i = \sum_{i=1}^n \nabla_i X_i.$$

TODO: more details on this, and other types of useful coordinates; harmonic, geodesic, isothermal, Fermi, others (I guess some of these are the same)?

2.7 Cartan's moving frames

See also some exposition in Volume 2 of Spivak and in Chow's Lectures on Differential Geometry.

We use generalized Einstein notation frequently throughout this section. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on an open subset of M. Let $\{\omega^i\}_{i=1}^n$ be the dual orthonormal basis for T^*M, defined by $\omega^i(e_j) = \delta^i_j$. We define the **connection 1-forms** $\omega^{i,j}$ (corresponding to $\{e_i\}$) to be the components of the Levi-Civita connection with respect to $\{e_i\}$. That is,

$$\nabla_X e_i = \omega^{i,j}(X) e_j.$$

Equivalently, we could define

$$\omega^{i,j} = g(\nabla_{e_k} e_i, e_j)\omega^k, \tag{31}$$

or

$$\omega^{i,j}(X) = g(\nabla_X e_i, e_j). \tag{32}$$

These are antisymmetric in i and j,

$$\omega^{i,j} = -\omega^{j,i}, \tag{33}$$

and satisfy the **first Cartan structure equation**:

$$d\omega^i = \omega^j \wedge \omega^{j,i} = \omega^{i,j} \wedge \omega^j. \tag{34}$$

We also have

$$d\omega^j(e_i, e_j) = \omega^{i,j}(e_j) - \omega^{j,k}(e_i). \tag{35}$$

From this it follows that

$$\omega^{i,k}(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{36}$$

Now we define the **curvature 2-forms** $\Omega^{i,j}$ by

$$\Omega^{i,j}(X,Y)e_j = \frac{1}{2}\,\mathrm{Rm}(X,Y)e_i.$$

These measure the noncommutativity of taking two covariant derivatives. We could also define these by (TODO: check the constant?)

$$\Omega^{i,j} = \frac{1}{2}\,\mathrm{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell, \tag{37}$$

so that

$$\Omega^{i,j}(e_k, e_\ell) = \mathrm{Rm}_{ijk\ell}\,. \tag{38}$$

These satisfy the following, called the **second Cartan structure equation**:

$$\Omega^{i,j} = d\omega^{i,j} - \omega^{i,k} \wedge \omega^{k,j} \tag{39}$$

This gives us a way to compute curvatures. For example, on a surface M^2, we have

$$d\omega^1 = \omega^2 \wedge \omega^{2,1}, \quad d\omega^2 = \omega^1 \wedge \omega^{1,2}, \quad \Omega^{1,2} = d\omega^{1,2}$$

2.8 Proofs

$$X(fY) = X(f) \cdot Y + fX(Y). \tag{1}$$

Proof. For $g \in C^\infty(M)$,

$$\begin{aligned}
[X(fY)](g) &= X(f \cdot Y(g)) \\
&= X(f) \cdot Y(g) + fX(Y(g)) \\
&= [X(f) \cdot Y + fX(Y)](g).
\end{aligned}$$

□

$$X(r \circ f) = (r' \circ f)X(f) \tag{2}$$

Proof. This follows from the chain rule on $\mathbb{R}^n$. First consider the case where $X = \partial_i$. Let ψ be a chart about $p \in M$.

$$\begin{aligned}
\partial_i|_p(r \circ f) &\coloneqq \left.\frac{\partial}{\partial x^i}\right|_{\psi(p)} (r \circ f \circ \psi^{-1}) \\
&= r'(f(p)) \cdot \left.\frac{\partial}{\partial x^i}\right|_{\psi(p)} \frac{(f \circ \psi^{-1})}{\partial x^i} \\
&= r'(f(p))\partial_i f.
\end{aligned}$$

Then the general case follows by linearity. □

$$d\mu = \omega^1 \wedge \cdots \wedge \omega^n \tag{6}$$

Proof. Nothing to prove here. $\square$

$$d\omega = \sum_{i,i_1,\dots,i_k} \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{7}$$

Proof. We use first the linearity of d and next its product rule.

$$\begin{aligned}
d\omega &= d\left(\sum_{i_1,\dots,i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\
&= \sum_{i_1,\dots,i_k} d(\omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&= \sum_{i_1,\dots,i_k} d\omega_{i_1\cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + (-1)^k \omega_{i_1\cdots i_k} d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}).
\end{aligned}$$

But now since $d^2 = 0$, the second term is 0, and by the expression (4) for the differential, the result follows. $\square$

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{8}$$

Proof. By definition,

$$(dx^i \wedge dx^j)(X,Y) = \left(\sum_{\sigma \in S_2} \operatorname{sgn}(\sigma)(dx^i \otimes dx^j) \circ \sigma\right)(X,Y) = X^i Y^j - X^j Y^i$$

the left hand side is

$$\begin{aligned}
d\omega(X,Y) &= \left(\sum_{i,j \le n} \partial_i \omega_j dx^i \wedge dx^j\right)(X,Y) \\
&= \sum_{i,j} \partial_i \omega_j (X^i Y^j - X^j Y^i)
\end{aligned}$$

On the other hand, note that

$$\begin{aligned}
X\omega(Y) &= X^i \partial_i (\omega_j dx^j (Y^k \partial_k)) \\
&= X^i \partial_i (\omega_j Y^j) \\
&= X^i Y^j \partial_i(\omega_j) + X^i \omega_j \partial_i (Y^j),
\end{aligned}$$

and

$$\begin{aligned}
\omega([X,Y]) &= \omega(X^i \partial_i (Y^j \partial_j) - Y^k \partial_k (X^\ell \partial_\ell)) \\
&= \omega(X^i \partial_i (Y^j) \partial_j - Y^k \partial_k (X^\ell) \partial_\ell) \\
&= X^i \omega_j \partial_i (Y^j) - Y^k \omega_\ell \partial_k (X^\ell).
\end{aligned}$$

so the right hand side becomes

$$\begin{aligned}
X\omega(Y) - Y\omega(X) - \omega([X,Y]) &= X^i Y^j \partial_i(\omega_j) + X^i \omega_j \partial_i (Y^j) - Y^k X^\ell \partial_k (\omega_\ell) - X^k \omega_\ell \partial_k (Y^\ell) \\
&\qquad - X^p \omega_q \partial_p (Y^q) + Y^t \omega_s \partial_t (X^s) \\
&= X^i Y^j \partial_i(\omega_j) - Y^k X^\ell \partial_k (\omega_\ell).
\end{aligned}$$

$\square$

$$(d\omega)(X_0,\ldots,X_k) = \sum_{j=0}^{k}(-1)^j X_j \omega(X_0,\ldots,\hat{X}_j,\ldots,X_k) + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k). \tag{9}$$

Proof. TODO □

$$\iota_X(\omega)(X_1,\ldots,X_{k-1}) = \omega(X,X_1,\ldots,X_{k-1}) \tag{10}$$

Proof. In coordinates,

$$\begin{aligned}\iota_X(\omega)(X_1,\ldots,X_{k-1}) &= \iota_X(\omega_{i_1\ldots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k})(X_1,\ldots,X_{k-1})\\ &= \end{aligned}$$

TODO □

$$\iota_X(\omega^1\wedge\cdots\wedge\omega^k) = \sum_{i=1}^{k}(-1)^{i-1}\omega^i(X)\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^k. \tag{11}$$

Proof. Note that by property (ii) defining the interior product,

$$\iota_X(\omega^1\wedge\cdots\wedge\omega^k) = \iota_X(\omega^1)\wedge\omega^2\wedge\cdots\wedge\omega^k + (-1)\omega^1\wedge\iota_X(\omega^2\wedge\cdots\wedge\omega^k),$$

and the result follows by induction (and by property (i), which says $\iota_X(\omega)=\omega(X)$ for 1-forms ω). □

$$*(\omega^1\wedge\cdots\wedge\omega^k) = \omega^{k+1}\wedge\cdots\wedge\omega^n. \tag{12}$$

Proof. In this case we have

$$\begin{aligned}\left\langle\omega^1\wedge\cdots\wedge\omega^k,\omega^1\wedge\cdots\wedge\omega^k\right\rangle d\,\mathrm{Vol} &= d\,\mathrm{Vol}\\ &= (\omega^1\wedge\cdots\wedge\omega^k)\wedge(\omega^{k+1}\wedge\cdots\wedge\omega^n),\end{aligned}$$

which establishes the result. □

$$\operatorname{grad}(fh) = f\operatorname{grad}h + h\operatorname{grad}f \tag{3}$$

Proof. Recall that $\operatorname{grad} f$ is defined to be the vector field so that for all vector fields X,

$$\langle\operatorname{grad}f, X\rangle = X(f).$$

Now

$$\begin{aligned}\langle\operatorname{grad}(fh),X\rangle &= X(fh)\\ &= fX(h)+hX(f)\\ &= f\langle\operatorname{grad}h,X\rangle + h\langle\operatorname{grad}f,X\rangle\\ &= \langle f\operatorname{grad}h + h\operatorname{grad}f, X\rangle.\end{aligned}$$

□

$$df = (\partial_i f) dx^i \tag{4}$$

Proof. This follows immediately, since

$$df(\partial_i) = \partial_i(f).$$

□

$$\operatorname{grad} f = g^{ij}(\partial_j f)\partial_i \tag{5}$$

Proof. Recall that for any vector field X,

$$X = dx^i(X)\partial_i = X^i\partial_i.$$

So, writing in coordinates

$$\begin{aligned}
g(\operatorname{grad} f, X) &= df(X) \\
g_{ij}dx^i(\operatorname{grad} f)X^j &= (\partial_k f)dx^k(X) \\
g_{ij}dx^i(\operatorname{grad} f)X^j &= (\partial_k f)X^k \\
g_{ij}dx^i(\operatorname{grad} f) &= (\partial_j f) \\
dx^i(\operatorname{grad} f) &= g^{ij}(\partial_j f) \\
\operatorname{grad} f &= g^{ij}(\partial_j f)\partial_i.
\end{aligned}$$

□

$$\operatorname{div} X = \operatorname{tr} \nabla X = (\nabla X)(\partial_i, dx^i) = (\nabla_i X)(dx^i). \tag{17}$$

Proof. TODO □

$$\operatorname{div}(X^i\partial_i) = \frac{1}{\sqrt{\det g}}\partial_i(X^i\sqrt{\det g}). \tag{18}$$

Proof. Apply Cartan's formula (44) and the definition of the divergence as the quantity satisfying $d(\iota_X d\mu) = \operatorname{div} X \, d\mu$.

TODO: idk what this is supposed to prove but it's not right □

$$\operatorname{div}(fX) = X(f) + f \operatorname{div} X \tag{19}$$

Proof. One can prove this using coordinates, but there is a nicer way.

$$\begin{aligned}
\operatorname{div}(fX) &= \operatorname{tr}(\nabla fX) \\
&= \operatorname{tr}(\cdot(f)X + f\nabla.X) \\
&= \operatorname{tr}(\cdot(f)X) + f\operatorname{tr}(\nabla.X) \\
&= (\partial_i(f)X)(dx^i) + f \operatorname{div} X \\
&= dx^i(\partial_i(f)X^k\partial_k) + f \operatorname{div} X \\
&= \partial_i(f)X^i + f \operatorname{div} X \\
&= X(f) + f \operatorname{div} X.
\end{aligned}$$

□

$$\langle \delta\omega, f\rangle = \langle \omega, df\rangle. \tag{20}$$

Proof. First observe that from (19), we get

$$-\delta(f\omega) = \omega^\sharp(f) - f\delta\omega.$$

Using the definition of divergence of $\omega^\sharp$ as the quantity satisfying $d(\iota_{\omega^\sharp} d\mu) = \operatorname{div} \omega^\sharp d\mu = -\delta\omega d\mu$, we have

$$\begin{aligned}
\langle f, \delta\omega \rangle &= \int_M f\delta\omega \, d\mu \\
&= \int_M (\delta(f\omega) + \omega^\sharp(f)) \, d\mu \\
&= \int_M \delta(f\omega) \, d\mu + \int_M \omega^\sharp(f) \, d\mu \\
&= \int_M \omega^\sharp(f) \, d\mu \\
&= \int_M g(df, \omega) \\
&= \langle df, \omega \rangle .
\end{aligned}$$

□

$$\Delta = g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right) \tag{22}$$

Proof. This follows immediately from (59). □

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial f}{\partial x^j} \right). \tag{23}$$

Proof. This follows immediately from the coordinate expression for the divergence and $\operatorname{grad} f$. □

$$\Delta(fh) = f\Delta h + h\Delta f + 2 \langle \nabla f, \nabla h \rangle \tag{24}$$

Proof.

$$\begin{aligned}
\Delta(fh) &= \operatorname{div} \operatorname{grad}(fh) \\
&= \operatorname{div}(f \operatorname{grad} h + h \operatorname{grad} f) \\
&= \operatorname{div}(f \operatorname{grad} h) + \operatorname{div}(h \operatorname{grad} f) \\
&= (\operatorname{grad} h)(f) + f \operatorname{div}(\operatorname{grad} h) + (f \leftrightarrow h) \\
&= \langle \operatorname{grad} h, \operatorname{grad} f \rangle + f\Delta h + (f \leftrightarrow h) \\
&= f\Delta h + h\Delta f + 2 \langle \operatorname{grad} f, \operatorname{grad} h \rangle .
\end{aligned}$$

□

$$(\partial_t - \Delta)(fh) = f(\partial_t - \Delta)(h) + h(\partial_t - \Delta)(f) - 2 \langle \nabla f, \nabla h \rangle . \tag{25}$$

Proof. TODO □

$$\Delta(r \circ f) = (r' \circ f)\Delta f + (r'' \circ f) \left| \nabla f \right|^2 \tag{26}$$

Proof. By definition, and using (2) to evaluate terms like $\partial_i(r \circ f)$,

$$\begin{aligned}
\Delta(r \circ f) &= g^{ij}\nabla^2_{ij}(r \circ f) \\
&= g^{ij}(\nabla_i(\nabla_j(r \circ f)) - \nabla_{\nabla_i \partial_j}(r \circ f)) \\
&= g^{ij}(\partial_i\partial_j(r \circ f) - \Gamma^k_{ij}\partial_k(r \circ f)) \\
&= g^{ij}(\partial_i((r' \circ f)\partial_j f) - \Gamma^k_{ij}(r' \circ f)\partial_k f) \\
&= g^{ij}((r'' \circ f)\partial_i f \partial_j f + (r' \circ f)\partial_i\partial_j f - (r' \circ f)\Gamma^k_{ij}\partial_k f) \\
&= g^{ij}(r'' \circ f)\partial_i f \partial_j f + (r' \circ f)\Delta f \\
&= (r'' \circ f)\,|\nabla f|^2 + (r' \circ f)\Delta f.
\end{aligned}$$

□

$$\Delta\omega = \Delta_d\omega + \mathrm{Rc}(\omega) \tag{27}$$

Proof. hello TODO

□

$$g_{ij}(p) = \delta_{ij} \tag{28}$$

Proof. Recall that $d(\exp_p)_0 = \mathrm{Id}$.

$$\frac{\partial}{\partial x^i} = d(\exp_p)_0\left(\left.\frac{\partial}{\partial e^i}\right|_0\right) = \frac{\partial}{\partial e^i},$$

from which (28) follows. □

$$\Gamma^k_{ij}(p) = 0 \tag{29}$$

Proof. This follows immediately from (28) and the definition of Γ. □

$$\partial_k g_{ij}(p) = 0 \tag{30}$$

Proof. We have

$$\begin{aligned}
\partial_k g_{ij} &= \partial_k g(\partial_i, \partial_j) \\
&= g(\partial_k\partial_i, \partial_j) + g(\partial_i, \partial_k\partial_j).
\end{aligned}$$

Since ∂_i are coordinate vector fields, $\partial_i\partial_j = 0$, so the proof is done. □

$$\omega^{i,j} = g(\nabla_{e_k} e_i, e_j)\omega^k \tag{31}$$

Proof. The first definition tells us that $\nabla_{\partial_k} e_i = (\omega^{i,j})_k e_j$, and so

$$g(\nabla_{\partial_k}, e_i, e_\ell) = (\omega^{i,\ell})_k.$$

But this is equivalent to the second definition. □

$$\omega^{i,j}(X) = g(\nabla_X e_i, e_j). \tag{32}$$

Proof. TODO □

$$\omega^{i,j} = -\omega^{j,i} \tag{33}$$

Proof.
$$\omega^{i,j} = \Gamma^j_{ki} = -\Gamma^i_{kj} = -\omega^{j,i}.$$ □

$$d\omega^i = \omega^{i,j} \wedge \omega^j \tag{34}$$

Proof. Recall the identity (8):

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Applying this with $\omega = \omega^i$, $X = e_k$, and $Y = e_\ell$, we get

$$\begin{aligned} d\omega(e_k, e_\ell) &= e_k(\omega^i(e_\ell)) - e_\ell(\omega^i(e_k)) - \omega^i([e_k, e_\ell]) \\ &= e_k(\delta^i_\ell) - e_\ell(\delta^i_k) - \omega^i(\nabla_{e_k} e_\ell - \nabla_{e_\ell} e_k) \\ &= -\omega^i(\Gamma^j_{k\ell} e_j - \Gamma^j_{\ell k} e_j) \\ &= -\Gamma^i_{k\ell} + \Gamma^i_{\ell k} \\ &= \Gamma^\ell_{ki} - \Gamma^k_{\ell i}, \end{aligned}$$

where in the last line we used 51 On the other hand, the right hand side becomes

$$\begin{aligned} (\omega^{i,j} \wedge \omega^j)(e_k, e_\ell) &= \omega^{i,j}(e_k)\omega^j(e_\ell) - \omega^{i,j}(e_\ell)\omega^j(e_k) \\ &= g(\nabla_{e_k} e_i, e_j)\delta^j_\ell - g(\nabla_{e_\ell} e_i, e_j)\delta^j_k \\ &= g(\nabla_{e_k} e_i, e_\ell) - g(\nabla_{e_\ell} e_i, e_k) \\ &= \Gamma^\ell_{ki} - \Gamma^k_{\ell i}. \end{aligned}$$ □

$$d\omega^j(e_i, e_j) = \omega^{i,j}(e_j) - \omega^{j,k}(e_i). \tag{35}$$

Proof. First observe that since $\nabla_{e_k} e_i = \omega^{i,j}(e_k)e_j$, we have that $\omega^\ell(\nabla_{e_k} e_i) = \omega^{i,\ell}(e_k)$. Using this fact and the definition of the exterior derivative,

$$\begin{aligned} d\omega^k(e_i, e_j) &= e_i\omega^k(e_j) - e_j\omega^k(e_i) - \omega^k([e_i, e_j]) \\ &= -\omega^k(\nabla_{e_i} e_j - \nabla_{e_j} e_i) \\ &= -\omega^{j,k}(e_i) + \omega^{i,k}(e_j). \end{aligned}$$ □

$$\omega^{i,k}(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{36}$$

Proof. Write out each term of the right hand side using (35), and use antisymmetry to cancel/combine terms. □

$$\Omega_{i,j} = -\frac{1}{2}\,\mathrm{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell \tag{37}$$

Proof. TODO □

$$\Omega_{i,j}(e_k, e_\ell) = \mathrm{Rm}_{ijk\ell}\,. \tag{38}$$

Proof. Simply note that, since $\mathrm{Rm}_{ijk\ell} = -\mathrm{Rm}_{ij\ell k}$, we have

$$\begin{aligned}
\Omega_{i,j}(e_k, e_\ell) &= \frac{1}{2}(\mathrm{Rm}_{ijpq}\,\omega_p \wedge \omega_q)(e_k, e_\ell) \\
&= \frac{1}{2}\mathrm{Rm}_{ijpq}(\delta_{pk}\delta_{q\ell} - \delta_{p\ell}\delta_{qk}) \\
&= \frac{1}{2}\mathrm{Rm}_{ijk\ell} - \frac{1}{2}\mathrm{Rm}_{ij\ell k} \\
&= \mathrm{Rm}_{ijk\ell}\,.
\end{aligned}$$

□

$$\Omega^{i,j} = d\omega^{i,j} - \omega^{i,k} \wedge \omega^{k,j} \tag{39}$$

Proof. We give two proofs, respectively using the different definitions of $\Omega^{i,j}$. If we know that $\Omega^{i,j} = -\frac{1}{2}\mathrm{Rm}_{ijk\ell}\,\omega_k \wedge \omega_\ell$, we proceed as follows. From (31), we have that $\omega_{i,j}(e_k) = g(\nabla_{e_k} e_i, e_j)$. By taking the exterior derivative of both sides (thinking of the right hand side as the covector $X \mapsto g(\nabla_X e_i, e_j)$), and using (8), we have

$$\begin{aligned}
d\omega^{i,j}(e_k, e_\ell) &= e_k\omega^{i,j}(e_\ell) - e_\ell\omega^{i,j}(e_k) - \omega^{i,j}([e_\ell, e_k]) \\
&= dg(\nabla e_i, e_j)(e_k, e_\ell) \\
&= e_k g(\nabla_{e_\ell} e_i, e_j) - e_\ell g(\nabla_{e_k} e_i, e_j) - g(\nabla_{[e_\ell, e_k]} e_i, e_j) \\
&= g(\nabla_{e_k}\nabla_{e_\ell} e_i, e_j) + g(\nabla_{e_\ell} e_i, \nabla_{e_k} e_j) - g(\nabla_{e_\ell}\nabla_{e_k} e_i, e_j) \\
&\quad - g(\nabla_{e_k} e_i, \nabla_{e_\ell} e_j) - g(\nabla_{[e_\ell, e_k]} e_i, e_j).
\end{aligned}$$

Observing that $\omega^{i,p}(e_\ell)\omega^{j,p}(e_k) = g(\nabla_{e_\ell} e_i, e_p) g(\nabla_{e_k} e_j, e_p) = g(\nabla_{e_\ell} e_i, \nabla_{e_k} e_j)$, we continue

$$\begin{aligned}
&= g(\nabla_{e_k}\nabla_{e_\ell} e_i - \nabla_{e_\ell}\nabla_{e_k} e_i - \nabla_{[e_\ell, e_k]} e_i, e_j) + \omega^{i,p}(e_\ell)\omega^{j,p}(e_k) - \omega^{i,p}(e_k)\omega^{j,p}(e_\ell) \\
&= \mathrm{Rm}_{k\ell ij} + (\omega^{i,p} \wedge \omega^{j,p})(e_\ell, e_k) \\
&= \mathrm{Rm}_{ijk\ell} + (\omega^{i,p} \wedge \omega^{p,j})(e_k, e_\ell) \\
&= (\Omega^{i,j} + \omega^{i,p} \wedge \omega^{p,j})(e_k, e_\ell)
\end{aligned}$$

TODO: the other version □

---

3 Tensors

3.1 The induced metric on tensor bundles

Suppose we have a metric g on some bundle $\pi : E \to M$. That is, we have a section of $E^* \otimes E^*$ such that at each point $p \in M$, g_p is an inner product on the fiber E_p. Then the metric on E defines a bundle isomorphism $\iota_g \colon E \to E^*$ by

$$\iota_g(\xi) : \eta \mapsto g_p(\xi, \eta), \quad \xi, \eta \in E_p.$$

Moreover, there is a unique metric g on E^* such that ι_g is a bundle isometry:

$$g(\iota_g(\xi), \iota_g(\eta)) = g(\xi, \eta), \quad \xi, \eta \in E_p.$$

So we see that a metric extends to tensor duals. It also extends to tensor products. Given bundles E_1, E_2 with metrics g_1, g_2,

$$g = g_1 \otimes g_2 \in \Gamma((E_1^* \otimes E_1^*) \otimes (E_2^* \otimes E_2^*)) \equiv \Gamma((E_1 \otimes E_2)^* \otimes (E_1 \otimes E_2)^*)$$

is the unique metric such that

$$g(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) = g_1(\xi_1, \xi_2) g_2(\eta_1, \eta_2).$$

The induced metric on a tensor product of dual bundles agrees with the induced metric on a dual bundle of a tensor product. So, starting with a metric on the tangent bundle TM, we get metrics (all denoted g) on all the tensor bundles $\mathcal{T}^k_\ell$. We can write this in coordinates in the following way. For $S, T \in \mathcal{T}^k_\ell(M)$, we have (at a point p),

$$g(S,T) = g^{a_1 b_1} g^{a_2 b_2} \cdots g^{a_k b_k} g_{i_1 j_1} \cdots g_{i_\ell j_\ell} S^{i_1 \ldots i_\ell}_{a_1 \ldots a_k} T^{j_1 \ldots j_\ell}_{b_1 \ldots b_k}.$$

3.2 Type changing with the metric

An $(k, 0)$-tensor is called **covariant**, and a $(0, \ell)$-tensor is called **contravariant**. For example, forms are covariant, and vectors are contravariant. The terminology relates to how the components change under a change of basis. If we scale some basis vectors by a factor of C, then the components of a vector with respect to that basis scale by a factor of C^{-1}; hence contravariant.

An (s,t)-tensor T is a section of $(TM)^{\otimes t} \otimes (T^*M)^{\otimes s}$. That is, it is a product of t vectors and s covectors, meaning that it takes t covectors and s vectors as input, so it has s lower (covariant) indices, and t upper (contravariant) indices. To add to the confusion, recall that we can think of a $(1,1)$-tensor either as an object that takes a vector and a covector and returns a scalar, or as an object that takes a vector and returns a vector. This generalizes; a (k,ℓ)-tensor can also be thought of as an object that takes k vectors and returns (a tensor product of) ℓ vectors. Sickeningly, not all authors agree on the convention for the roles of k and ℓ; for example some would call a (k,ℓ)-tensor what others would call (ℓ, k).

For any $|k| \leq \min\{s,t\}$, we can make T into a $(s-k, t+k)$-tensor by using the natural isomorphism (provided by the Riemannian metric) between TM and T^*M given by $v \mapsto g(v, \cdot) \in T^*M$. So in the tensor product above, we can replace TM's by T^*M's arbitrarily, and thereby get any sort of tensor we want with rank $s+t$.

In coordinates, we can write (given a frame E_i and the coframe ξ^i),

$$T^{i_1 \cdots i_t}{}_{j_1 \ldots j_s} E_{i_1} \otimes \cdots \otimes E_{i_t} \otimes \xi^{j_1} \otimes \cdots \otimes \xi^{j_s}.$$

TODO: check/fix the s and t's in the section.

Then to make T a $(s+1, t-1)$-tensor, replace some E_{i_k} by $g_{i_k j} \xi^j$ to get

$$T^{i_1 \cdots i_{k-1}}{}_j{}^{i_{k+1} \cdots i_s}{}_{j_1 \cdots j_t} E_{i_1} \otimes \cdots \otimes \xi^j \otimes \cdots \otimes E_{i_s} \otimes \xi^{j_1} \otimes \cdots \otimes \xi^{j_t},$$

where

$$T^{i_1 \cdots i_{k-1}}{}_j{}^{i_{k+1} \cdots i_s}{}_{j_1 \cdots j_t} := g_{i_k j} T^{i_1 \cdots i_s}{}_{j_1 \ldots j_t}$$

TODO: I think the s and t are wrong here

3.3 Contractions and traces

TODO: this section needs help; see Andrews-Hopper p. 22 and Lee p. 395. Don't forget that these have different notation for tensor indices.

Given a (k,ℓ)-tensor T, where $k, \ell \geq 1$, we can form various $(k-1, \ell-1)$-tensors by **tracing** T; that is, by evaluating one of the covector factors of T at one of the vector factors. Specifically, there are $k\ell$ different traces we can take, since we can evaluate any of the covector fields at any of the vector fields. In the case where T is a $(1,1)$-tensor,

$$\operatorname{tr}(T) = \operatorname{tr}(T^i{}_j\, E_i \otimes \xi^j) = T^i{}_j\, \xi^j(E_i) = T^i{}_j\, \delta^j_i = T^i{}_i\,.$$

More generally, if T is a (k,ℓ)-tensor, and we evaluate the a^{th} factor of T at the b^{th} factor of T, we have, for vector fields $X_1, \ldots, X_{k-1}$, and covector fields $\omega_1, \ldots, \omega_{\ell-1}$,

$$\begin{aligned}
&(\operatorname{tr}_{ab} T)(\omega_1, \ldots, \omega_{\ell-1}, X_1, \ldots, X_{k-1}) \\
&= \operatorname{tr}[(\omega, X) \mapsto T(\omega_1, \ldots, \omega_{a-1}, \omega, \omega_{a+1}, \ldots, \omega_{\ell-1}, X_1, \ldots, X_{b-\ell-1}, X, X_{b-\ell+1}, \ldots, X_{k+1})]
\end{aligned}$$

where on the right hand side we are now just taking the trace over a $(1,1)$ tensor again. In coordinates, this is just

$$\operatorname{tr}_{ab} T = T^{i_1 \cdots p \cdots i_\ell}{}_{j_1 \cdots p \cdots j_k}\, \partial_{i_1} \cdots \partial_{i_{a-1}} \partial_{i_{a+1}} \cdots \partial_{i_\ell} dx^{j_1} \cdots dx^{j_{b-\ell-1}} dx^{j_{b-\ell+1}} \cdots dx^{j_k}.$$

Using the isomorphism induced by g between TM and T^*M (see the previous section), we can take the trace over any pair of indices of any tensor (with rank at least 2). The idea is, given a pair of indices, if they are both covariant or both contravariant, we can isomorphically replace one or the other by its covariant or contravariant dual under the metric. Then we have reduced to one of the cases discussed previously. A useful example is the divergence of a 1-form ω, which can be defined as the trace of the $(2,0)$-tensor $\nabla\omega$.

TODO

The norm squared of a tensor can be compared to the square of the trace. In particular, if T is a (1,1)-tensor, then

$$n\,|T|^2 \geq \operatorname{tr}(T)^2. \tag{40}$$

$$n\,|T|^2 \geq \operatorname{tr}(T)^2. \tag{40}$$

Proof. This follows from (a corollary to) the Cauchy-Schwarz inequality:

$$\sum_{i \leq n} x_i^2 \geq \frac{1}{n} \left(\sum_{i \leq n} x_i \right)^2.$$

$$\begin{aligned}
n\,|T|^2 &= n\,\langle T, T \rangle \\
&= n \sum_i T_i^i \\
&\geq (\sum T_i^i)^2 \\
&= \operatorname{tr}(T)^2.
\end{aligned}$$

□

4 Lie Derivatives

Let X, Y be vector fields, and let $\Psi_{X,t}$ be the flow of X, so that $\Psi_{X,t}(x)$ is the point that x is "flowed to" by X after time t. Then $D\Psi_{X,t}|_x$ is an isomorphism between T_xM and $T_{\Psi_{X,t}(x)}M$. (Note that in this case the pullback is the inverse of the differential, so it does not matter if we use the pullback or the inverse of the pushforward.) Now the point is that $(D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})$ is an element of T_xM for each t, so we can differentiate this at $t = 0$. With these remarks in mind, we define the **Lie derivative** of Y along the flow of X by

$$\mathcal{L}_X Y|_x = \frac{d}{dt}\Big|_{t=0} \left((D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})\right).$$

There are many different notations for the quantities in this equation. It is usually written more concisely as

$$\mathcal{L}_X Y|_x = \frac{d}{dt}\Big|_{t=0} (\Psi_t^* Y).$$

Define the **Lie bracket** of vector fields by

$$[X, Y](f) \coloneqq X(Y(f)) - Y(X(f)),$$

then

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X. \tag{41}$$

For vector fields $V, X_1, \ldots, X_k$, and a tensor field A in $\mathcal{T}_0^k(M)$,

$$(\mathcal{L}_V A(X_1, \ldots, X_k)) = (\mathcal{L}_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, \mathcal{L}_V X_i, \ldots, X_k), \tag{42}$$

where TODO: TAGS

$$\begin{aligned}(\mathcal{L}_V A)(X_1, \ldots, X_k) &= V(A(X_1, \ldots, X_k)) - \sum_{i=1}^{k} A(X_1, \ldots, [V, X_i], \ldots, X_k) \qquad (43)\\ &= (\nabla_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, X_{i-1}, \nabla_{X_i} V, X_{i+1}, \ldots, X_k)\end{aligned}$$

Cartan's formula states that for any differential form ω and any (smooth) vector field V,

$$\mathcal{L}_V \omega = \iota_V(d\omega) + d(\iota_V \omega). \tag{44}$$

We have the following product rule for a Lie derivative of a wedge product

$$\mathcal{L}_V(\alpha \wedge \beta) = (\mathcal{L}_V \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_V \beta). \tag{45}$$

There is also the following identity for the interior product. For a differential form ω and vector fields V, W,

$$\mathcal{L}_W(\iota_V \omega) = \iota_V(\mathcal{L}_W \omega) + \iota_{[W,V]}\omega. \tag{46}$$

4.1 Proofs

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X. \tag{41}$$

Proof. Consider the action of $\mathcal{L}_X Y$ on a smooth function f. Note that by the chain rule and the fact that $\mathrm{Id}_{T_xM} = \Psi_{-X,t} \circ \Psi_{X,t}$

$$D\Psi_{-X,t}|_{\Psi_{X,t}(x)} \circ D\Psi_{X,t}|_x = \mathrm{Id}_{T_xM},$$

so $(D\Psi_{X,t}|_x)^{-1} = D\Psi_{-X,t}|_{\Psi_{X,t}(x)}$. By definition of the derivative,

$$(D\Psi_{-X,t}|_x(Y))f = Y|_{\Psi_{X,t}(x)}(f \circ \Psi_{-X,t}).$$

Then

$$\begin{aligned}\mathcal{L}_X Y|_x(f) &= \frac{d}{dt}\left((D\Psi_{X,t}|_x)^{-1}(Y_{\Psi_{X,t}(x)})\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(Y|_{\Psi_{X,t}(x)}(f \circ \Psi_{-X,t})\right)\end{aligned}$$

TODO □

$$(\mathcal{L}_V A(X_1, \ldots, X_k)) = (\mathcal{L}_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, \mathcal{L}_V X_i, \ldots, X_k) \tag{42}$$

Proof. TODO □

$$\begin{aligned}(\mathcal{L}_V A)(X_1, \ldots, X_k) &= V(A(X_1, \ldots, X_k)) - \sum_{i=1}^{k} A(X_1, \ldots, [V, X_i], \ldots, X_k) \\ &= (\nabla_V A)(X_1, \ldots, X_k) + \sum_{i=1}^{k} A(X_1, \ldots, X_{i-1}, \nabla_{X_i} V, X_{i+1}, \ldots, X_k)\end{aligned} \tag{43}$$

Proof. TODO □

$$\mathcal{L}_V \omega = \iota_V(d\omega) + d(\iota_V \omega). \tag{44}$$

Proof. The proof is by direct computation. We can simplify by choosing coordinates for which TODO □

$$\mathcal{L}_V(\alpha \wedge \beta) = (\mathcal{L}_V \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_V \beta). \tag{45}$$

Proof. TODO □

$$\mathcal{L}_W(\iota_V \omega) = \iota_V(\mathcal{L}_W \omega) + \iota_{[W,V]}\omega. \tag{46}$$

Proof. TODO □

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5 Connections on vector bundles

We begin this section in some generality. There is a concept of a connection on a vector bundle over a manifold that allows a coordinate invariant way of taking directional derivatives. The special case that arises most often in Riemannian geometry is the case of the Levi-Civita connection, which is a connection on the tangent bundle satisfying certain compatibility properties with the Riemannian metric.

Let E be a vector bundle over M. A **connection** ∇ on E is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ (written as $(X, \xi) \mapsto \nabla_X \xi$, and we think of the first entry as specifying the direction in which to take the derivative of the second entry) that satisfies the following properties.

(1) C^∞ linearity in X:
$$\nabla_{X+fY}\xi = \nabla_X \xi + f\nabla_Y \xi.$$

(2) $\mathbb{R}$-linearity in ξ:
$$\nabla_X(r\xi) = r\nabla_X \xi.$$

(3) A product/Leibniz rule in ξ:
$$\nabla_X(f\xi) = X(f) \cdot \xi + f\nabla_X \xi.$$

The connection coefficients of such a connection can be defined with respect to a given local frame $\{\xi_i\}$ for E by the equation
$$\nabla_i \xi_j = \Gamma_{ij}^k \xi_k.$$
Here we use the same notation as for the Christoffel symbols, but I think the term Christoffel symbols refers specifically to the coefficients of the Levi-Civita connection.

5.1 Connections on product structures

A useful special case is when we are given a connection on the tangent bundle TM. Then by requiring a few properties to hold, this connection extends to a unique connection, denoted ∇, on the tensor bundle of M (which can be thought of as formed by taking arbitrary duals and products of TM). The properties are:

(1) On TM, ∇ agrees with the given connection.

(2) On $C^\infty(M) = T^0M$, ∇ is the action of a vector as a derivation:
$$\nabla_X f = Xf,$$
for any smooth function f.

(3) ∇ obeys the product rule with respect to tensor products:
$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G),$$
for any tensors F and G.

(4) ∇ commutes with all contractions:
$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F),$$
for any tensor F.

We say that a connection is **compatible** with the metric if it additionally satisfies

(4) metric compatibility:

$$X(g(\xi, \eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta),$$

which can also be stated as

$$\nabla g = 0,$$

where the left hand side (and the right hand side) is a tensor field in $\mathcal{T}_0^3(M)$. The proof that these conditions are equivalent follows from the definition of ∇g below.

5.2 The Levi-Civita Connection

The Levi-Civita connection for a given Riemannian metric g is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ (written as $(X, \xi) \mapsto \nabla_X \xi$) that satisfies the preceding 4 properties, and is

(5) torsion-free (also known as symmetric):

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

It satisfies **Koszul's formula**:

$$2 \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle - Z \langle X, Y \rangle + Y \langle Z, X \rangle - \langle [Y, Z], X \rangle + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle . \tag{47}$$

The metric compatibility condition tells us that

$$\nabla(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y), \tag{48}$$

where we interpret the right hand side as the covector $Z \mapsto g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$. The fundamental theorem of Riemannian geometry states that, given a Riemannian manifold (M, g), there is a unique connection ∇ on TM that is symmetric and compatible with g. So we are justified when we say *the* Levi-Civita connection. To paraphrase Andrews-Hopper we mention that this connection is canonical because the symmetry and compatibility conditions are invariantly defined natural properties that force the connection to coincide with the tangential connection, whenever M is realized as a submanifold of $\mathbb{R}^n$ with the induced metric (which is always possible by the Nash embedding).

5.3 Christoffel symbols

Given some coordinate basis $\{\partial_i\}_{i=1}^n$, the **Christoffel symbols** (of the Levi-Civita connection) are the unique coefficients, (i.e. smooth functions) satisfying

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

It follows from this and the above properties that

$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma_{ij}^k) \partial_k. \tag{49}$$

In particular,

$$\nabla_i X = (\partial_i X^\ell + X^j \Gamma_{ij}^\ell) \partial_\ell.$$

For the Levi-Civita connection, we can calculate these coefficients in coordinates by

$$\Gamma^k_{ji} = \Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}). \tag{50}$$

The symbols also satisfy the antisymmetry relation

$$\Gamma^k_{ij} = -\Gamma^j_{ik}. \tag{51}$$

Despite using the same notation, we cannot think of the Christoffel symbols as a $(1, 2)$-tensor. However, given two metrics $g, \widetilde{g}$ the difference of the coefficients of the two corresponding Levi-Civita connections does form a tensor:

$$\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}$$

5.4 Covariant derivatives

We can take directional derivatives of functions using only the differentiable structure of a manifold. The covariant derivative is defined using the metric, and allows us to differentiate vector fields and other tensors. We already know how to take the covariant derivative of a vector field, which is a $(0, 1)$-tensor. As mentioned above, this now extends to arbitary (k, ℓ)-tensors. First generalize to $(0, \ell)$-tensors: $\nabla_X : \mathcal{T}^0_\ell(M) \to \mathcal{T}^0_\ell(M)$ is defined by

$$\nabla_X(X_1 \otimes \cdots \otimes X_\ell) := \sum_{i=1}^{\ell} X_1 \otimes \cdots \otimes \nabla_X X_i \otimes \cdots \otimes X_\ell.$$

By requiring the covariant derivative to satisfy a product rule (the following is purely symbolic the first time we write it)

$$\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y),$$

we can define the first term on the right hand side, since all other terms we understand. Note that this is just property (3) from the above requirements for extending to tensor bundles. Thus we have the covariant derivative of 1-forms:

$$(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y).$$

In coordinates (here we are referring to the total covariant derivative of ω, see below),

$$\nabla_i \omega_j := (\nabla \omega)_{ij} = \partial_i \omega_j - \Gamma^k_{ij} \omega_k \tag{52}$$

If $F \in \mathcal{T}^k_\ell(M)$ is a tensor field, and X, Y_k are vector fields and ω^j are 1-forms, then we have the following, which is sometimes used as a definition, but in fact it follows from the requirements for a covariant derivative to extend from the tangent bundle to the tensor bundle.

$$\begin{aligned}(\nabla_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) = {} & X(F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k)) \\ & - \sum_{j=1}^{\ell} F(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^\ell, Y_1, \ldots, Y_k) \\ & - \sum_{i=1}^{k} F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k).\end{aligned} \tag{53}$$

$$(\nabla_X F)(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k) = X(F(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k)) - \sum_{j=1}^{\ell} F(\omega^1, \dots, \nabla_X \omega^j, \dots, \omega^\ell, Y_1, \dots, Y_k) - \sum_{i=1}^{k} F(\omega^1, \dots, \omega^\ell, Y_1, \dots, \nabla_X Y_i, \dots, Y_k). \tag{53}$$

Proof. TODO; see Andrews-Hopper p. 26 □

We can think of ∇F as a $(k+1, \ell)$-tensor field, called the **total covariant derivative** of F, by

$$(\nabla F)(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k, X) = (\nabla_X F)(\omega^1, \dots, \omega^\ell, Y_1, \dots, Y_k).$$

There are different conventions about where to put the X in this definition; I'm not sure if it matters. An important property of covariant derivatives is that they "commute with contractions," a property that follows from the fact that $\nabla g \equiv 0$. TODO

There is also a horrible expression for the covariant derivative in coordinates TODO

See LeeRM for stuff about covariant derivative in coordinates, as well as the semicolon notation, which may actually be kind of good TODO.

The following formula for commuting covariant derivatives at the expense of introducing a Riemann curvature term is quite useful, although it's probably better to just look at the more common special cases below.

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\alpha^{\ell_1 \dots \ell_s}_{k_1 \dots k_r} = -\sum_{h=1}^{r} \mathrm{Rm}^p_{ijk_h}\, \alpha^{\ell_1 \dots \ell_s}_{k_1 \dots k_{h-1} p k_{h+1} \dots k_r} - \sum_{h=1}^{s} \mathrm{Rm}^{\ell_h}_{ijp}\, \alpha^{\ell_1 \dots \ell_{h-1} p \ell_{h+1} \dots \ell_s}_{k_1 \dots k_r} \tag{54}$$

For example, if ω is a 1-form,

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\omega_k = -\mathrm{Rm}^\ell_{ijk}\, \omega_\ell. \tag{55}$$

5.5 The Hessian

We can then take the total covariant derivative of F to get the **Hessian** of F, sometimes denoted $\nabla^2 F$, which is of course a $(k+2, \ell)$-tensor field.

It follows from the torsion-free property of the Levi-Civita connection that the Hessian is symmetric:

$$(\nabla^2 f)(X, Y) = (\nabla^2 f)(Y, X). \tag{56}$$

We have

$$\nabla^2_{X,Y} F \coloneqq (\nabla^2 F)(X, Y) = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F. \tag{57}$$

The proof makes it more clear how the tensors on the right hand side actually work. In the case of a function f, we have that (the first equality follows immediately from (57))

$$\begin{aligned} (\nabla^2 f)(X, Y) &= X(Y(f)) - (\nabla_X Y)(f) \\ &= g(\nabla_X \operatorname{grad} f, Y) \\ &= \frac{1}{2}(\mathcal{L}_{\operatorname{grad} f}\, g)(X, Y). \end{aligned} \tag{58}$$

In coordinates, we can write

$$\nabla_i \nabla_j f = \partial_i(\partial_j f) - \Gamma_{ij}^k \partial_k f. \tag{59}$$

In particular, since $(\nabla^2 f)(X, Y) = g(\nabla_X \operatorname{grad} f, Y)$, the $(1,1)$-tensor associated to ∇^2 is given by $(\nabla^2 f)(X) = \nabla_X \operatorname{grad} f$. The Hessian satisfies the following product rule for functions.

$$\nabla^2(fh) = f\nabla^2 h + h\nabla^2 f + \nabla f \otimes \nabla h + \nabla h \otimes \nabla f. \tag{60}$$

5.5.1 Bochner Formulas

For any function u on a Riemannian manifold,

$$\Delta \left|\nabla u\right|^2 = 2\left\langle \Delta\nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2. \tag{61}$$

We also have the formula for the commutator of the Laplacian and the covariant derivative:

$$\Delta(du) = d(\Delta u) + \operatorname{Rc}(\nabla u), \tag{62}$$

sometimes also written as $\Delta\nabla u = \nabla\Delta u + \operatorname{Rc}(\nabla u)$.

$$\Delta(du) = d(\Delta u) + \operatorname{Rc}(\nabla u), \tag{62}$$

Proof. By (55),

$$\nabla_i \nabla_j \nabla_k u = \nabla_j \nabla_i \nabla_k u - \operatorname{Rm}_{ijk\ell} \nabla_\ell u.$$

Note that on the left-hand side we can commute ∇_j and ∇_k. From this equation,

$$\begin{aligned} g^{ik}\nabla_i \nabla_k \nabla_j u &= g^{ik}(\nabla_j \nabla_i \nabla_k u + \operatorname{Rm}_{jik\ell} \nabla_\ell u) \\ \Delta\nabla_j u &= \nabla_j \Delta u + \operatorname{Rc}_{j\ell} \nabla_\ell u. \end{aligned}$$

□

5.6 Proofs

$$2\left\langle \nabla_X Y, Z\right\rangle = X\left\langle Y, Z\right\rangle - Z\left\langle X, Y\right\rangle + Y\left\langle Z, X\right\rangle - \left\langle [Y,Z], X\right\rangle + \left\langle [X,Y], Z\right\rangle - \left\langle [X,Z], Y\right\rangle. \tag{47}$$

Proof. The metric compatibility condition says

$$\begin{aligned} X\left\langle Y, Z\right\rangle &= \left\langle \nabla_X Y, Z\right\rangle + \left\langle Y, \nabla_X Z\right\rangle \\ Y\left\langle Z, X\right\rangle &= \left\langle \nabla_Y Z, X\right\rangle + \left\langle Z, \nabla_Y X\right\rangle \\ Z\left\langle X, Y\right\rangle &= \left\langle \nabla_Z X, Y\right\rangle + \left\langle X, \nabla_Z Y\right\rangle. \end{aligned}$$

By adding/subtracting these expressions, using symmetry and linearity of the metric, and the torsion-free property ($\nabla_X Y - \nabla_Y X = [X, Y]$), we obtain

$$X\left\langle Y, Z\right\rangle + Y\left\langle Z, X\right\rangle - Z\left\langle X, Y\right\rangle = 2\left\langle \nabla_X Y, Z\right\rangle - \left\langle [X,Y], Z\right\rangle + \left\langle Y, [X,Z]\right\rangle + \left\langle X, [Y,Z]\right\rangle.$$

□

$$\nabla(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y). \tag{48}$$

Proof. This really does follow immediately from the metric compatibility condition. □

$$\Gamma^k_{ji} = \Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \tag{50}$$

Proof. Apply the Koszul formula (47) to coordinate basis vectors:

$$\begin{aligned} 2\Gamma^\ell_{ij} g_{\ell k} &= 2\left\langle \Gamma^\ell_{ij}\partial_\ell, \partial_k \right\rangle \\ &= 2\left\langle \nabla_{\partial_i}\partial_j, \partial_k \right\rangle \\ &= \partial_i \left\langle \partial_j, \partial_k \right\rangle - \partial_k \left\langle \partial_i, \partial_j \right\rangle + \partial_j \left\langle \partial_k, \partial_i \right\rangle \\ &= \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}. \end{aligned}$$

Now multiply both sides by g^{km} and the result follows. □

$$\Gamma^k_{ij} = -\Gamma^j_{ik} \tag{51}$$

Proof. The metric compatibility condition applied to the correct basis vectors says

$$\begin{aligned} 0 &= (\nabla g)(\partial_j, \partial_k, \partial_i) \\ &= \nabla_{\partial_i} g(\partial_j, \partial_k) \\ &= g(\nabla_i e_j, e_k) + g(\nabla_i e_k, e_j) \\ &= \Gamma^k_{ij} + \Gamma^j_{ik}. \end{aligned}$$

□

$$\nabla_i \omega_j \coloneqq (\nabla\omega)_{ij} = \partial_i \omega_j - \Gamma^k_{ij}\omega_k \tag{52}$$

Proof.

$$\begin{aligned} (\nabla\omega)(\partial_i, \partial_j) &= (\nabla_i \omega)(\partial_j) \\ &= \nabla_i(\omega_j) - \omega(\nabla_i \partial_j) \\ &= \partial_i \omega_j - \omega(\Gamma^k_{ij}\partial_k) \\ &= \partial_i \omega_j - \Gamma^k_{ij}\omega_k. \end{aligned}$$

□

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\alpha^{\ell_1\ldots\ell_s}_{k_1\ldots k_r} = -\sum_{\ell=1}^{r} \mathrm{Rm}^{p}_{ijk_h}\, \alpha^{\ell_1\ldots\ell_s}_{k_1\ldots k_{h-1}pk_{h+1}\ldots k_r} - \sum_{h=1}^{s} \mathrm{Rm}^{\ell_h}_{ijp}\, \alpha^{\ell_1\ldots\ell_{h-1}p\ell_{h+1}\ldots\ell_s}_{k_1\ldots k_r}. \tag{54}$$

Proof. TODO □

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_k = -\,\mathrm{Rm}^{\ell}_{ijk}\,\omega_\ell. \tag{55}$$

Proof. This follows immediately from (54), but I guess there should be a direct way to prove it as well.

TODO □

$$(\nabla^2 f)(X, Y) = (\nabla^2 f)(Y, X). \tag{56}$$

Proof. The torsion free property of the L-C connection says that

$$(\nabla_X Y)(f) - (\nabla_Y X)(f) = [X,Y](f) \coloneqq X(Y(f)) - Y(X(f)),$$

and by rearranging this we get that

$$(X(Y(f)) - (\nabla_X Y)(f) = Y(X(f)) - (\nabla_Y X)(f),$$

which is exactly the desired equality. □

$$\nabla^2_{X,Y} F \coloneqq (\nabla^2 F)(X,Y) = \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F. \tag{57}$$

Proof. According to the general definition of covariant derivative (and the definition of total covariant derivative) above,

$$\begin{aligned}(\nabla(\nabla F))(Y,X) &= (\nabla_X(\nabla F))(Y) \\ &= \nabla_X[(\nabla F)(Y)] - \nabla F(\nabla_X Y) \\ &= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.\end{aligned}$$

To make more explicit what is actually going on here, we can write, supposing F is a (k,ℓ)-tensor,

$$\begin{aligned}\nabla^2 F(X,Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) &= \nabla_X(\nabla F)(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) \\ &= X(\nabla F(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k)) \\ &\qquad - (\nabla F)(\nabla_X Y,\omega^1,\dots,\omega^\ell,W_1,\dots,W_k) \\ &\qquad - \sum_{i=1}^{\ell}(\nabla F)(Y,\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^\ell,W_1,\dots,W_k) \\ &\qquad - \sum_{i=1}^{k}(\nabla F)(Y,\omega^1,\dots,\omega^\ell,W_1,\dots,\nabla_X W_i,\dots,W_k) \\ &= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.\end{aligned}$$

There is also a proof on page 99 of Lee-RM. □

$$(\nabla^2 f)(X,Y) = X(Y(f)) - (\nabla_X Y)(f) = g(\nabla_X \operatorname{grad} f, Y) = \frac{1}{2}(\mathcal{L}_{\operatorname{grad} f}\, g)(X,Y) \tag{58}$$

Proof. The second equality:

$$\begin{aligned}X(Y(f)) - (\nabla_X Y)(f) &= X(g(\operatorname{grad} f, Y)) - g(\operatorname{grad} f, \nabla_X Y) \\ &= g(\nabla_X \operatorname{grad} f, Y) + g(\operatorname{grad} f, \nabla_X Y) - g(\operatorname{grad} f, \nabla_X Y) \\ &= g(\nabla_X \operatorname{grad} f, Y).\end{aligned}$$

The last equality: using (43) (TODO: check this reference; should it be 2a or 2b?) for the Lie derivative of the metric, metric compatibility, (41), and denoting $\operatorname{grad} f$ by ∇f, we calculate

$$\begin{aligned}(\mathcal{L}_{\nabla f}\, g)(X,Y) &= (\nabla f)(g(X,Y)) - g([\nabla f, X], Y) - g(X, [\nabla f, Y]) \\ &= g(\nabla_{\nabla f} X, Y) + g(X, \nabla_{\nabla f} Y) - g(\nabla_{\nabla f} X - \nabla_X(\nabla f), Y) - g(X, \nabla_{\nabla f} Y - \nabla_Y(\nabla f)) \\ &= g(\nabla_X(\nabla f), Y) + g(X, \nabla_Y(\nabla f)) \\ &= X(g(\nabla f, Y)) - g(\nabla f, \nabla_X Y) + Y(g(X, \nabla f)) - g(\nabla_Y X, \nabla f) \\ &= (\nabla^2 f)(X,Y) + (\nabla^2 f)(Y,X) \\ &= 2(\nabla^2 f)(X,Y).\end{aligned}$$

□

$$\nabla_i \nabla_j f = \partial_i(\partial_j f) - \Gamma^k_{ij}\partial_k f. \tag{59}$$

Proof. Recalling that, by definition $\nabla_i \partial_j = \Gamma^k_{ij}\partial_k$,

$$\begin{aligned}
\nabla_i \nabla_j f &= \nabla_i(\nabla_j f) - \nabla_{\nabla_i \partial_j} f \\
&= \partial_i \partial_j f - \nabla_{\Gamma^k_{ij}\partial_k} f \\
&= \partial_i \partial_j f - \Gamma^k_{ij}\nabla_k f \\
&= \partial_i \partial_j f - \Gamma^k_{ij}\partial_k f
\end{aligned}$$

This also follows from □

$$\nabla^2(fh) = f\nabla^2 h + h\nabla^2 f + \nabla f \otimes \nabla h + \nabla h \otimes \nabla f \tag{60}$$

Proof. First note that we are using ∇f to denote the gradient of f. Then

$$\begin{aligned}
\nabla^2(hf)(X,Y) &= \nabla_X(\nabla_Y(fh)) - \nabla_{\nabla_X Y}(fh) \\
&= \nabla_X(fY(h) + hY(f)) - (\nabla_X Y)(fh) \\
&= X(f)Y(h) + fX(Y(h)) + X(h)Y(f) + hX(Y(f)) - h(\nabla_X Y)(f) - f(\nabla_X Y)(h) \\
&= f(X(Y(h)) - (\nabla_X Y)(h)) + h(X(Y(f)) - (\nabla_X Y)(f)) \\
&\qquad + (\nabla f \otimes \nabla h)(X,Y) + (\nabla h \otimes \nabla f)(X,Y) \\
&= f(\nabla^2 h)(X,Y) + h(\nabla^2 f)(X,Y) + (\nabla f \otimes \nabla h)(X,Y) + (\nabla h \otimes \nabla f)(X,Y).
\end{aligned}$$

□

$$\Delta \left|\nabla u\right|^2 = 2\left\langle \Delta \nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2. \tag{61}$$

Proof. The coordinate-free way to do this goes as follows. I use dots to keep track of the entries over which the trace is taken.

$$\begin{aligned}
\Delta \left\langle \nabla u, \nabla u\right\rangle &= \operatorname{tr} \nabla^2 \left\langle \nabla u, \nabla u\right\rangle \\
&= \operatorname{tr}(\nabla.(2\left\langle \nabla.\nabla u, \nabla u\right\rangle) \\
&= 2\operatorname{tr}(\left\langle \nabla.\nabla.\nabla u, \nabla u\right\rangle + \left\langle \nabla.\nabla u, \nabla.\nabla u\right\rangle) \\
&= 2\left\langle \Delta \nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2,
\end{aligned}$$

as desired.

In normal coordinates, we can calculate

$$\begin{aligned}
\Delta \left|\nabla u\right|^2 &= \Delta(\nabla_i u \nabla_i u) \\
&= \nabla_j \nabla_j(\nabla_i u \nabla_i u)) \\
&= 2\nabla_j \nabla_j \nabla_i u \nabla_i u + 2\nabla_j \nabla_i u \nabla_j \nabla_i u \\
&= 2\left\langle \Delta \nabla u, \nabla u\right\rangle + 2\left|\nabla^2 u\right|^2.
\end{aligned}$$

□

6 Curvature

6.1 Curvature of a connection on a vector bundle

Reference: Andrews-Hopper Section 2.7.1. If ∇ is a connection on a vector bundle E over M, the curvature of ∇ on E is the section $R_\nabla \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ defined by

$$R_\nabla(X,Y)\xi = \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X,Y]}\xi.$$

6.2 Riemann curvature

Riemann curvature is the special case of the previous construction where the connection is the Levi-Civita connection on the tangent bundle over M. In particular, the $(3,1)$-tensor (field) version of the Riemann curvature tensor is a $C^\infty(M)$-multilinear map $\Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= (\nabla^2)(X,Y,Z) - (\nabla^2)(Y,X,Z) \\ &= \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z. \end{aligned}$$

In coordinates, we can write

$$R = R_{ijk}{}^{\ell} dx^i \otimes dx^j \otimes dx^k \otimes \partial_\ell,$$

so that

$$R(X,Y)Z = R_{ijk}{}^{\ell} X^i Y^j Z^k \partial_\ell.$$

where

$$R_{ijk}{}^{\ell} \partial_\ell = R(\partial_i, \partial_j)\partial_k.$$

We can get a $(4,0)$-tensor version of R by defining

$$R_{ijkl} = R(\partial_i, \partial_j, \partial_k, \partial_\ell) \coloneqq \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell \rangle .$$

Then $R_{ijk\ell} = g_{\ell m} R_{ijk}{}^{m}$. This tensor satisfies the symmetries

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \tag{63}$$

and the 1st and 2nd Bianchi identities: TODO: TAGS

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{1}$$

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{2}$$

The once contracted 2nd Bianchi identity:

$$g^{im} \nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{66}$$

We can calculate the coefficients in terms of the Christoffel symbols as well:

$$R_{ijk}{}^{\ell} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm} \tag{3}$$

$$R_{ijk\ell} = \frac{1}{2}(\partial_j \partial_k g_{i\ell} + \partial_i \partial_\ell g_{jk} - \partial_i \partial_k g_{j\ell} - \partial_j \partial_\ell g_{ik}) + g_{\ell p}(\Gamma^m_{ik}\Gamma^p_{jm} - \Gamma^m_{jk}\Gamma^p_{im}). \tag{4}$$

In calculations, we frequently get Riemann curvature terms appearing from commuting covariant derivatives, following from rearranging the formula that defines the Riemann tensor. See (54).

6.3 Ricci curvature

The Ricci tensor, denoted Rc or R, is defined to be the trace of the Riemann tensor:

$$\operatorname{Rc}(Y,Z) \coloneqq \operatorname{tr}(X \mapsto R(X,Y)Z),$$

or in coordinates

$$R_{ij} = R_{kij}{}^{k} = g^{km} R_{kijm}.$$

The Ricci tensor satisfies the twice contracted second Bianchi identity:

$$2g^{ij}\nabla_i \operatorname{Rc}_{jk} = \nabla_k R. \tag{67}$$

The Ricci tensor can be expressed in terms of the metric:

$$-2\operatorname{Rc}_{ij} = g^{k\ell}(\partial_k\partial_\ell g_{ij} + \partial_i\partial_j g_{k\ell} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_k g_{i\ell}) + \text{lower order terms}, \tag{68}$$

where the lower order terms involve only one derivative of g. The Ricci tensor is invariant under diffeomorphisms; that is, if ϕ is a diffeomorphism of M, then

$$\operatorname{Rc}_{\phi^* g} = \phi^* \operatorname{Rc}_g .$$

6.4 Scalar curvature

The scalar curvature is defined to be the trace (with respect to the metric) of the Ricci curvature:

$$R = \operatorname{tr}_g \operatorname{Rc} = \operatorname{Rc}_i{}^{i} = g^{ij}\operatorname{Rc}_{ij} .$$

Proofs

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \tag{63}$$

Proof. Using the fact that the Lie bracket is antisymmetric, it is clear from the definition that $R(X,Y)Z = -R(Y,X)Z$, from which the equality $R_{ijk\ell} = -R_{jik\ell}$ follows. To show the second equality, we begin by showing that $R(X,Y,Z,Z) = 0$ for any Z. First note that by metric compatibility,

$$\begin{aligned} X(Y(|W|^2)) &= X(Y\langle W,W\rangle) \\ &= X(2\langle \nabla_Y W, W\rangle) \\ &= 2\langle \nabla_X\nabla_Y W, W\rangle + 2\langle \nabla_Y W, \nabla_X W\rangle . \end{aligned}$$

Similarly,

$$Y(X(|W|^2)) = 2\langle \nabla_Y\nabla_X W, W\rangle + 2\langle \nabla_X W, \nabla_Y W\rangle ,$$

and

$$[X,Y]\,|W|^2 = 2\left\langle \nabla_{[X,Y]} W, W\right\rangle .$$

Now, subtracting the second and third of these two equations from the first and cancelling terms, we have

$$\begin{aligned}
0 &= X(Y(|W|^2)) - Y(X(|W|^2)) - [X,Y]\,|W|^2 \\
&= 2\,\langle \nabla_X \nabla_Y W, W\rangle - 2\,\langle \nabla_Y \nabla_X W, W\rangle - 2\left\langle \nabla_{[X,Y]} W, W\right\rangle \\
&= 2\,\langle R(X,Y)W, W\rangle \\
&= R(X,Y,W,W).
\end{aligned}$$

Applying this,

$$\begin{aligned}
0 &= \langle R(\partial_i, \partial_j)\partial_k + \partial_\ell, \partial_k + \partial_\ell\rangle \\
&= R_{ijkk} + R_{ijk\ell} + R_{ij\ell k} + R_{ij\ell\ell} \\
&= R_{ijk\ell} + R_{ij\ell k}.
\end{aligned}$$

To prove the last equality, we use the first (algebraic) Bianchi identity. □

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{64}$$

Proof. This will follow from

$$R(X,Y)Z + R(Z,Y)X + R(Y,X) = 0.$$

Expand using the definition of R, and then apply symmetry of the connection:

$$\begin{aligned}
0 = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + & \\
+ \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X - \nabla_{[Z,Y]} X + & \\
+ \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z & \\
= \nabla_X (\nabla_Y Z - &
\end{aligned}$$

□

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{65}$$

Proof. TODO □

$$g^{im} \nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{66}$$

Proof.

$$\begin{aligned}
\nabla_i R_{jk\ell m} &= \nabla_j R_{ik\ell m} - \nabla_k R_{ij\ell m} \\
g^{im} \nabla_i R_{jk\ell m} &= \nabla_j g^{im} R_{ik\ell m} - \nabla_k g^{im} R_{ij\ell m} \\
&= \nabla_j R_{k\ell} - \nabla_k R_{j\ell}
\end{aligned}$$

□

$$2 g^{ij} \nabla_i R_{jk} = \nabla_k R. \tag{67}$$

Proof. Start with the 2nd Bianchi identity, contract twice, and apply some symmetries of the Riemann tensor:

$$\begin{aligned}
g^{im}g^{j\ell}(\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}) &= 0 \\
g^{im}\nabla_i g^{j\ell} R_{jk\ell m} + g^{j\ell}\nabla_j g^{im} R_{ki\ell m} + g^{im}\nabla_k g^{j\ell} R_{ij\ell m} &= 0 \\
-g^{im}\nabla_i g^{j\ell} R_{jkm\ell} - g^{j\ell}\nabla_j g^{im} R_{ik\ell m} + g^{im}\nabla_k g^{j\ell} R_{jim\ell} &= 0 \\
-g^{im}\nabla_i R_{km} - g^{j\ell}\nabla_j R_{k\ell} + \nabla_k g^{im} R_{im} &= 0 \\
-2g^{ij}\nabla_i R_{jk} + \nabla_k R &= 0.
\end{aligned}$$

□

$$-2R_{ij} = g^{k\ell}(\partial_k\partial_\ell g_{ij} + \partial_i\partial_j g_{k\ell} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_k g_{i\ell}) + \text{lower order terms}, \tag{68}$$

Proof. I don't want to type this, but it just involves writing the Ricci tensor in terms of the Riemann tensor, the Riemann tensor in terms of the Christoffel symbols, and the Christoffel symbols in terms of the metric. □

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7 Geometric Analysis

7.1 Integration

7.1.1 Stokes's theorem

Suppose M is an oriented n-manifold with boundary, and suppose ω is a compactly supported $(n-1)$-form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega. \tag{69}$$

From this we can obtain several very useful special cases. The **divergence theorem** says that for any smooth 1-form α on a compact manifold with boundary,

$$\int_M \operatorname{div}(\alpha)\, d\mu = \int_{\partial M} \alpha(\nu)\, d\sigma, \tag{70}$$

where ν is the outward unit normal to the boundary, and $d\sigma$ is the volume form of the boundary.

In the case of a vector field, we have

$$\int_M \operatorname{div} X = \int_{\partial M} g(X,\nu)\, d\sigma. \tag{71}$$

7.1.2 Integration by parts

Suppose $u, v \in C^\infty(M)$. If M is closed,

$$\int_M \Delta u\, d\mu = 0, \tag{72}$$

If M is compact,

$$\int_M (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma. \tag{73}$$

Where ν and σ are the outward (?) unit normal and the volume form of ∂M.

In particular, on a closed manifold, the right hand side is 0, so $\int u\Delta v = \int v\Delta u$. If M is compact, TODO: LABEL

$$\int_M u\Delta v\, d\mu + \int_M \langle \nabla u, \nabla v\rangle\, d\mu = \int_{\partial M} \frac{\partial v}{\partial \nu} u\, d\sigma. \tag{74}$$

In particular, on a closed manifold, $\int \langle \nabla u, \nabla v\rangle = -\int u\Delta v$.

We also have

$$\int_M g(\operatorname{grad} f, X)\, d\mu = \int_{\partial M} f g(X, \nu)\, d\sigma - \int_M f \operatorname{div} X\, d\mu.$$

Right now I'm just copying this from LeeRM, p 149. TODO: change the notation and add some exposition

The integration by parts formula generalizes to covariant tensor fields. If F is a smooth covariant k-tensor field and G is a smooth covariant $(k+1)$-tensor field on a compact smooth Riemannian manifold with boundary, then

$$\int_M \langle \nabla F, G\rangle\, dV_g = \int_{\partial M} \left\langle F \otimes N^\flat, G\right\rangle dV_{\hat{g}} - \int_M \langle F, \operatorname{div} G\rangle\, dV_g, \tag{75}$$

where $\hat{g}$ is the induced metric on ∂M.

$$\int_M \langle \nabla F, G\rangle\, dV_g = \int_{\partial M} \left\langle F \otimes N^\flat, G\right\rangle dV_{\hat{g}} - \int_M \langle F, \operatorname{div} G\rangle\, dV_g, \tag{75}$$

Proof. TODO □

In coordinates we have the more suggestive

$$\int_M F_{i_1\ldots i_k;j} G^{i_1\ldots i_k j}\, dV_g = \int_{\partial M} F_{i_1\ldots i_k} G^{i_1\ldots i_k j} N_j\, dV_{\hat{g}} - \int_M F_{i_1\ldots i_k} {G^{i_1\ldots i_k j}}_{;j}\, dV_g. \tag{76}$$

$${T_a}^b$$

$$\int_M F_{i_1\ldots i_k;j} G^{i_1\ldots i_k j}\, dV_g = \int_{\partial M} F_{i_1\ldots i_k} G^{i_1\ldots i_k j} N_j\, dV_{\hat{g}} - \int_M F_{i_1\ldots i_k} {G^{i_1\ldots i_k j}}_{;j}\, dV_g. \tag{76}$$

Proof. TODO □

7.2 Miscellaneous

If $A(s)$ is a 1-parameter family of invertible square matrices, then

$$\frac{d}{ds}\log(\det A) = (A^{-1})^{ij} \frac{d}{ds} A_{ij}. \tag{77}$$

7.3 Variation formulae

References: Sheridan's notes and Andrew-Hopper Chapter 4.

Suppose that $g(t)$ is a time-dependent Riemannian metric, and

$$\frac{\partial}{\partial s} g_{ij}(s) = h_{ij}(s).$$

Then we have the following evolution equations for various geometric objects. Some cases are only states for the special case of the Ricci flow, i.e. when $h = -2\,\mathrm{Rc}$. In this case I use ∂_t rather than ∂_s to distinguish.

Metric inverse:

$$\frac{\partial}{\partial s} g^{ij} = -h^{ij} = -g^{ik} g^{jl} h_{kl} \tag{78}$$

For time-independent vector fields, and an evolving metric $g(s)$, we define $\dot{\nabla} = \partial_s \nabla$ by $\dot{\nabla}_X Y = \partial_s(\nabla_X Y)$. Then

$$\left\langle \dot{\nabla}_X Y, Z \right\rangle = -(\nabla_X \mathrm{Rc})(Y,Z) + (\nabla_Z \mathrm{Rc})(X,Y) - (\nabla_Y \mathrm{Rc})(X,Z). \tag{79}$$

The Laplacian on functions evolves by

$$\frac{\partial}{\partial s} \Delta_{g(s)} = -h_{ij} \nabla^i \nabla^j - g^{k\ell} \left(g^{ij} \nabla_i h_{j\ell} - \frac{1}{2} \nabla_\ell \left(g^{ij} h_{ij} \right) \right) \nabla_k. \tag{80}$$

$$\frac{\partial}{\partial s} \Delta_{g(s)} = -h_{ij} \nabla^i \nabla^j - g^{k\ell} \left(g^{ij} \nabla_i h_{j\ell} - \frac{1}{2} \nabla_\ell \left(g^{ij} h_{ij} \right) \right) \nabla_k. \tag{80}$$

Proof. For a smooth function u, we have (TODO: I'm not entirely comfortable with the second equality)

$$\begin{aligned}
\partial_t(\Delta u) &= \partial_t \left[g^{ij} \left(\partial_i \partial_j - \Gamma^k_{ij} \partial_k \right) u \right] \\
&= (\partial_t g^{ij}) \nabla_i \nabla_j u - g^{ij} (\partial_t \Gamma^k_{ij}) \nabla_k u + \Delta(\partial_t u) \\
&= -h^{ij} \nabla_i \nabla_j u - g^{ij} \left(\frac{1}{2} g^{k\ell} \left(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij} \right) \right) \nabla_k u \\
&= -h^{ij} \nabla_i \nabla_j u - g^{k\ell} \left(g^{ij} \nabla_i h_{j\ell} - \frac{1}{2} \nabla_\ell (g^{ij} h_{ij}) \right) \nabla_k u.
\end{aligned}$$

□

In particular if $g(t)$ is a solution to Ricci flow, the function Laplacian $\Delta_{g(t)}$ evolves by

$$\partial_t \Delta_{g(t)} = 2\,\mathrm{Rc}_{ij}\, \nabla_i \nabla_j. \tag{81}$$

Another special case of this is when g changes by a conformal factor. If $\partial_t g = fg$ for some smooth function f, then

$$\partial_t \Delta = -f\Delta + \left(\frac{n}{2} - 1 \right) \nabla f \cdot \nabla. \tag{82}$$

$$\partial_t \Delta = -f\Delta + \left(\frac{n}{2} - 1 \right) \nabla f \cdot \nabla. \tag{82}$$

Proof. It suffices to calculate

$$g^{k\ell}\left(g^{ij}\nabla_i(fg) - \frac{1}{2}\nabla_\ell(g^{ij}fg_{ij})\right) = \left(1-\frac{n}{2}\right)\nabla^k f.$$

□

Christoffel symbols:

$$\frac{\partial}{\partial s}\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \tag{83}$$

Riemann curvature tensor:

$$\partial_s R_{ijk}{}^{\ell} = \frac{1}{2}g^{\ell p}\left(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\right) \tag{84}$$

For the Ricci tensor, we mention 3 different expressions. The first gives the variation of the Ricci tensor in terms of the variation of the connection.

$$\partial_s \operatorname{Rc}_{ij} = \nabla_p(\partial_s\Gamma^p_{ij}) - \nabla_i(\partial_s\Gamma^p_{pj}). \tag{85}$$

We also have

$$\frac{\partial}{\partial s}\operatorname{Rc}_{ij} = -\frac{1}{2}\left(\Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i(\operatorname{div} v)_j - \nabla_j(\operatorname{div} v)_i\right), \tag{86}$$

where Δ_L denotes the Lichnerowicz Laplacian. This can be written as

$$\frac{\partial}{\partial s}(-2R_{ij}) = \Delta_L v_{ij} + \nabla_i X_j + \nabla_j X_i, \tag{87}$$

where $X = \frac{1}{2}\nabla V - \operatorname{div} v$.

$$\frac{\partial}{\partial s}(-2R_{ij}) = \Delta_L v_{ij} + \nabla_i X_j + \nabla_j X_i, \tag{87}$$

Proof. TODO □

$$\frac{\partial}{\partial s}\operatorname{Rc}_{ij} = -\frac{1}{2}\left(\Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i(\operatorname{div} v)_j - \nabla_j(\operatorname{div} v)_i\right), \tag{86}$$

Proof. TODO □

7.4 Proofs

$$\int_{\partial M}\omega = \int_M d\omega. \tag{69}$$

Proof. TODO; a bit involved. See Guillemin and Pollack p. 183. □

$$\int_M \operatorname{div}(\alpha)\,d\mu = \int_{\partial M}\alpha(\nu)\,d\sigma, \tag{70}$$

Proof. Some issues with orientability here? See Problem 2-22 p. 51 of Lee-RM. Recall that the divergence of a vector field X is defined to satisfy $d(\iota_X\,d\mu) = \operatorname{div} X\,d\mu$.

TODO □

$$\int_M \operatorname{div} X = \int_{\partial M} g(X, \nu)\, d\sigma. \tag{71}$$

Proof. The idea is to apply Stokes's theorem (69) with $\alpha = \iota_X(d\mu)$. With this definition,

$$d\alpha = d\iota_X(d\mu) = \operatorname{div} X\, d\mu,$$

by definition of the divergence.

$$\begin{aligned} d\alpha &= d\iota_X(d\mu) \\ &= \end{aligned}$$

See HRF Theorem 1.47
TODO $\square$

$$\int_M \Delta u\, d\mu = 0, \tag{72}$$

Proof. Recall that $\nabla u = \operatorname{div} \operatorname{grad} u$. Then apply the divergence theorem (70) and the fact that M has no boundary. $\square$

$$\int_M (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right)\, d\sigma. \tag{73}$$

Proof. Take $X = u\nabla v - v\nabla u$ in the divergence theorem (for vector fields) (71). $\square$

$$\int_M u\Delta v\, d\mu + \int_M \langle \nabla u, \nabla v\rangle\, d\mu = \int_{\partial M} \frac{\partial v}{\partial \nu} u\, d\sigma. \tag{74}$$

Proof. TODO $\square$

$$\frac{d}{ds} \log(\det A) = (A^{-1})^{ij} \frac{d}{ds} A_{ij}. \tag{77}$$

Proof. First, a lemma:

Lemma 7.1. $\det(I + tA) = 1 + \operatorname{tr}(A)t + \mathcal{O}(t^2)$.

Proof. Recall that $\det(tA) = t^n \det(A)$. Also recall that the characteristic polynomial of A is the product

$$(x - \lambda_1) \cdots (x - \lambda_n),$$

where λ_i's are eigenvalues of A. In particular, the characteristic polynomial is also $\det(xI - A)$.

$$\begin{aligned} \det(I + tA) &= t^n \det(t^{-1}I - (-A)) \\ &= t^n (t^{-1} + \lambda_1)(t^{-1} + \lambda_2) \cdots (t^{-1} + \lambda_n) \\ &= t^n \left(t^{-n} + t^{-(n-1)} \sum_{i \le n} \lambda_i + t^{-(n-2)} a_{n-2} + \cdots + t^{-1} a_1 + a_0 \right) \\ &= 1 + \operatorname{tr}(A)t + \mathcal{O}(t^2). \end{aligned}$$

$\square$

Lemma 7.2. $D_A \det(X) = \det(A)\operatorname{tr}(XA^{-1})$, where $D_A \det$ is the differential of $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$ at A. In particular, the differential of det at the identity matrix is just the trace.

Proof. Note that det is a smooth function $\mathrm{GL}_m(\mathbb{R}) \to \mathbb{R}$ (because the determinant is a polynomial expression of the components of the matrix). First we consider the differential of det at I. By definition and using the previous lemma,

$$\begin{aligned} D_I \det(X) &= \lim_{h\to 0} \frac{\det(I+hX) - \det(I)}{h} \\ &= \lim_{h\to 0} \frac{1 + \operatorname{tr}(X)h + \mathcal{O}(h^2) - 1}{h} \\ &= \operatorname{tr}(X). \end{aligned}$$

Now,

$$\begin{aligned} D_A \det(X) &= \lim_{h\to 0} \frac{\det(A+hX) - \det(A)}{h} \\ &= \det(A) \lim_{h\to 0} \frac{\det(I+hXA^{-1}) - \det(I)}{h} \\ &= \det(A) D_I(XA^{-1}) \\ &= \det(A)\operatorname{tr}(XA^{-1}). \end{aligned}$$

□

Note that $A(s)$ defines a curve in $\mathrm{GL}_m(\mathbb{R})$. We can think of $\frac{d}{ds}\det(A(s))$ either as the derivative of a function $\mathbb{R} \to \mathbb{R}$, or as the derivative of $\det : \mathrm{GL}_m(\mathbb{R}) \to \mathbb{R}$ in the direction of $\frac{d}{ds}A(s)$. From the second point of view we get

$$\begin{aligned} \frac{d}{ds}\det(A(s)) &= D_A \det\left(\frac{d}{ds}A(t)\right) \\ &= \det(A)\operatorname{tr}\left(A^{-1}\frac{d}{ds}A(s)\right), \end{aligned}$$

finally proving (77). □

$$\frac{\partial}{\partial t} g^{ij} = -h^{ij} = -g^{ik}g^{jl}h_{kl} \tag{78}$$

Proof.

$$\begin{aligned} 0 &= \partial_t \delta^i_k \\ &= \partial_t (g^{ij} g_{jk}) \\ &= (\partial_t g^{ij}) g_{jk} + g^{ij}(\partial_t g_{jk}) \\ (\partial_t g^{ij}) g_{jk} g^{k\ell} &= -g^{k\ell} g^{ij} (\partial_t g_{jk}) \\ (\partial_t g^{ij}) \delta^\ell_j &= -g^{k\ell} g^{ij} (\partial_t g_{jk}), \end{aligned}$$

and the result follows. □

$$\left\langle \dot{\nabla}_X Y, Z \right\rangle = -(\nabla_X \operatorname{Rc})(Y,Z) + (\nabla_Z \operatorname{Rc})(X,Y) - (\nabla_Y \operatorname{Rc})(X,Z). \tag{79}$$

Proof. To derive this, we take the derivative of Koszul's formula (47), which says

$$2g(\nabla_X Y, Z) = Xg(Y,Z) - Zg(X,Y) + Yg(Z,X) - g([Y,Z],X) + g([X,Y],Z) - g([X,Z],Y).$$

For the first term (here we are thinking of g as time-dependent, and X, Y, Z as time-independent),

$$\begin{aligned}\frac{\partial}{\partial s} 2g(\nabla_X Y, Z) &= 2(\partial_s g)(\nabla_X Y, Z) + 2g(\dot{\nabla}_X Y, Z) \\ &= -4\,\mathrm{Rc}(\nabla_X Y, Z) + 2g(\dot{\nabla}_X Y, Z).\end{aligned}$$

For the next term,

$$\begin{aligned}\frac{\partial}{\partial s} Xg(Y,Z) &= X((\partial_s g)(Y,Z)) \\ &= -2X\,\mathrm{Rc}(Y,Z),\end{aligned}$$

and the two terms after that give something similar. For the fifth term,

$$\begin{aligned}\frac{\partial}{\partial s} g([Y,Z],X) &= (\partial_s g)([Y,Z],X) \\ &= -2\,\mathrm{Rc}(\nabla_Y Z - \nabla_Z Y, X),\end{aligned}$$

and the sixth and seventh terms are similar. Putting this together, we get

$$\begin{aligned}-4\,\mathrm{Rc}(\nabla_X Y, Z) + 2g(\dot{\nabla}_X Y, Z) &= -2X\,\mathrm{Rc}(Y,Z) + 2Z\,\mathrm{Rc}(X,Y) - 2Y\,\mathrm{Rc}(Z,X) \\ &\quad - [-2\,\mathrm{Rc}(\nabla_Y Z, X) + 2\,\mathrm{Rc}(\nabla_Z Y, X)] \\ &\quad + [-2\,\mathrm{Rc}(\nabla_X Y, Z) + 2\,\mathrm{Rc}(\nabla_Y X, Z)] \\ &\quad - [-2\,\mathrm{Rc}(\nabla_X Z, Y) + 2\,\mathrm{Rc}(\nabla_Z X, Y)] \\ g(\dot{\nabla}_X Y, Z) &= -X\,\mathrm{Rc}(Y,Z) + Z\,\mathrm{Rc}(X,Y) - Y\,\mathrm{Rc}(Z,X) \\ &\quad + \mathrm{Rc}(\nabla_Y Z, X) - \mathrm{Rc}(\nabla_Z Y, X) + \mathrm{Rc}(\nabla_X Y, Z) \\ &\quad + \mathrm{Rc}(\nabla_Y X, Z) + \mathrm{Rc}(\nabla_X Z, Y) - \mathrm{Rc}(\nabla_Z X, Y) \\ &= -(\nabla_X \mathrm{Rc})(Y,Z) - (\nabla_Y \mathrm{Rc})(X,Z) + (\nabla_Z \mathrm{Rc})(X,Y).\end{aligned}$$

We used that $(\nabla_X \mathrm{Rc})(Y,Z) = X(\mathrm{Rc}(Y,Z)) - \mathrm{Rc}(\nabla_X Y, Z) - \mathrm{Rc}(Y, \nabla_X Z)$. □

$$\partial_t \Delta_{g(t)} = 2\,\mathrm{Rc}_{ij} \nabla_i \nabla_j. \tag{81}$$

Proof. We give two proofs. For $f \in C^\infty(M)$, using the coordinate expression (59) for the Hessian,

$$\begin{aligned}(\partial_t \Delta_{g(t)}) f &:= \partial_t (g^{ij} \nabla_i \nabla_j) f \\ &= (\partial_t g^{ij}) \nabla_i \nabla_j f + g^{ij} (\partial_t \nabla_i \nabla_j f) \\ &= 2\,\mathrm{Rc}^{ij} \nabla_i \nabla_j f + g^{ij} (\partial_t (\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f)) \\ &= 2\,\mathrm{Rc}^{ij} \nabla_i \nabla_j f + g^{ij} (\partial_t \Gamma_{ij}^k) \partial_k f\end{aligned}$$

Now, we calculate, using the contracted second Bianchi identity $2g^{ij}\nabla_i \operatorname{Rc}_{jk} = \nabla_k R$.

$$\begin{aligned}
g^{ij}\partial_t \Gamma^k_{ij} &= g^{ij}\left[\frac{1}{2}g^{k\ell}(\nabla_i(-2\operatorname{Rc}_{j\ell}) + \nabla_j(-2\operatorname{Rc}_{i\ell}) - \nabla_\ell(-2\operatorname{Rc}_{ij}))\right] \\
&= -g^{j\ell}g^{ij}\nabla_i \operatorname{Rc}_{j\ell} - g^{k\ell}g^{ij}\nabla_j \operatorname{Rc}_{i\ell} + g^{ij}g^{k\ell}\nabla_\ell \operatorname{Rc}_{ij} \\
&= -\frac{1}{2}g^{k\ell}\nabla_\ell R - \frac{1}{2}g^{k\ell}\nabla_\ell R + g^{k\ell}\nabla_\ell g^{ij}\operatorname{Rc}_{ij} \\
&= 0,
\end{aligned}$$

from which the result follows. □

Proof. The second proof is slightly less coordinate-dependent. Let $f, h \in C^\infty(M)$. Then

$$\begin{aligned}
\int_M h\Delta f \, d\mu &= -\int_M \langle \nabla h, \nabla f\rangle \; d\mu \\
&= -\int_M g^{ij}\nabla_i h \nabla_j f \, d\mu.
\end{aligned}$$

Differentiating both sides with respect to t gives

$$\int_M [h\dot{\Delta} f\, d\mu + h\Delta f(\partial_t \, d\mu)] = -\int_M [(\partial_t g^{ij})\nabla_i h \nabla_j f \, d\mu + g^{ij}\nabla_i h \nabla_j f(\partial_t \, d\mu)].$$

Now use the fact that $\partial_t d\mu = -R\, d\mu$, and $\partial_t g^{ij} = 2R^{ij}$ to get

$$\begin{aligned}
\int_M [\dot{\Delta} f - R\Delta f] h \, d\mu &= -\int_M [2\operatorname{Rc}_{ij}\nabla_j f - Rg^{ij}\nabla_j f](\nabla_i h)\, d\mu \\
&= \int_M \nabla_i [2\operatorname{Rc}_{ij}\nabla_j f - Rg^{ij}\nabla_j f] h \, d\mu.
\end{aligned}$$

Since $h \in C^\infty(M)$ was arbitrary,

$$\begin{aligned}
\dot{\Delta} f - R\Delta f &= \nabla_i(2\operatorname{Rc}_{ij}\nabla_j f - Rg^{ij}\nabla_j f) \\
&= (2\nabla_i \operatorname{Rc}_{ij})\nabla_j f + 2\operatorname{Rc}_{ij}\nabla_i\nabla_j f - g^{ij}\nabla_i R \nabla_j f - Rg^{ij}\nabla_i\nabla_j f \\
&= \nabla_j R \nabla_j f + 2\operatorname{Rc}_{ij}\nabla_i\nabla_j f - \nabla_j R\nabla_j f - R\Delta f \\
&= 2\operatorname{Rc}_{ij}\nabla_i\nabla_j f - R\Delta f,
\end{aligned}$$

and the result follows. □

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \tag{83}$$

Proof. By the coordinate expression (50) for the Christoffel symbols, we have

$$\partial_t \Gamma^k_{ij} = \frac{1}{2}(\partial_t g^{k\ell})(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) + \frac{1}{2}g^{k\ell}(\partial_i\partial_t g_{j\ell} + \partial_j\partial_t g_{i\ell} - \partial_\ell\partial_t g_{ij}).$$

Now we work in normal coordinates at some point p, so $\partial_i g_{ij} = 0$, and $\partial_i A = \nabla_i A$ at p for any tensor A.

TODO □

$$\partial_t R_{ijk}{}^{\ell} = \frac{1}{2} g^{\ell p} \left(\nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \right) \tag{84}$$

Proof. TODO □

$$\partial_t \operatorname{Rc}_{ij} = \nabla_p(\partial_t \Gamma^p_{ij}) - \nabla_i(\partial_t \Gamma^p_{pj}). \tag{85}$$

Proof. hi TODO □

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8 Submanifolds

Many of these ideas have are special cases of things discussed previously, in the case where the metric is induced by some immersion or embedding into a higher dimensional Riemannian manifold.

As the notation for this section is quite painful, here is a seperate notation glossary just for submanifolds, although it should essentially overlap with notation from the rest of the document. In most cases I am following Mat Langford's notation; see https://suppiluliuma.neocities.org/RG.pdf

Let M^n and N^{n+k} be smooth manifolds, and $X : M \to N$ a smooth immersion. Then we denote

$dX : TM \to TN$	the derivative of X
X^*TN	the pullback bundle (over M)
$dX(TM)$	the subbundle of X^*TN from the embedding $(p,u) \mapsto (p, dX(u))$ of TM into X^*TN
$\langle \cdot, \cdot \rangle, g$	the metrics on N, M respectively
$X^* \langle \cdot, \cdot \rangle$	the pullback metric on X^*TN: $X^* \langle (p,u), (p,v) \rangle$
N_pM	the normal space to M at $p \in M$, i.e. $N_pM = \{\nu \in T_X(p)N : \langle u, \nu \rangle = 0 \text{ for all } u \in dX_p(T_pM)\}$
NM	the normal subbundle of TN in the case where X is an embedding
NM	the normal subbundle of X^*TN (over M), i.e. $NM = \{\nu \in X^*TN : {}^X \langle u, \nu \rangle = 0 \text{ for all } u \in dX(TM)_{\pi(\nu)}\}$
D	the connection on N
${}^XD : TM \times \Gamma(X^*TN) \to X^*TN$	the pullback connection on X^*TN, defined by ${}^XD_u X^*V := (\pi(u), D_{dX(u)}V)$
∇	connection on TM
$\nabla^\perp$	connection on NM
II	second fundamental form; $\mathrm{II} \in \Gamma(T^*M \otimes T^*M \otimes NM)$, i.e. $\mathrm{II}(u,v) = ({}^XD_u(dX(V)))^\perp$, for an extension V of v
W	Weingarten tensor; $W \in \Gamma(T^*M \otimes TM \otimes N^*M)$

8.1 Second fundamental form

Roughly, $\mathrm{II}(u,v)$ is the normal (to the image of the immersion) component of how the vector field V is changing in the direction of u.

References

[1] Bennett Chow and Dan Knopf. The Ricci Flow: An Introduction. Vol. 1. American Mathematical Soc., 2004.

[2] Bennett Chow, Peng Lu, and Lei Ni. Hamilton's Ricci flow. Vol. 77. American Mathematical Society, Science Press, 2023.