# PAWS 2024: SYMMETRIES OF ROOT SYSTEMS

MELISSA EMORY

## 1. Introduction

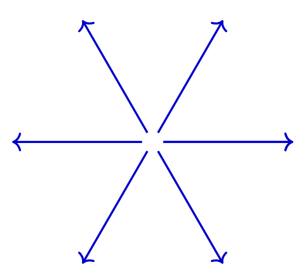
These lecture notes are in draft form and are created for the Preliminary Arizona Winter School 2024: Symmetries of root systems course. The hope for these lecture notes is to make the subject accessible to advanced undergraduate and early graduate students most of whom have a semester in Abstract Algebra. The main sources for the notes are [Hum90], [Car89], [Hal15].

## 2. Intuitive Definition

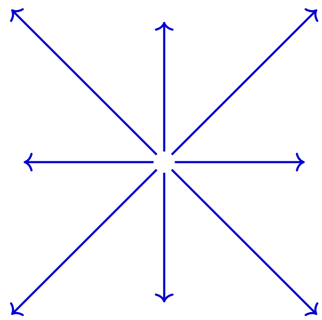
A root system is a "very symmetrical" set of vectors in n-dimensional Euclidean space. The classical motivation for studying root systems is their role in the classification of semi-simple Lie algebras, i.e. classical and exceptional groups, over the complex numbers. The root datum has connections to the Langlands dual group as well as $L$-functions. Thus, root systems have connections to representation theory, number theory, algebra, geometry, and physics.

Each of the following examples are in $\mathbb{R}^2$. How would one describe the symmetries that you see in the pictures? What properties can you derive?

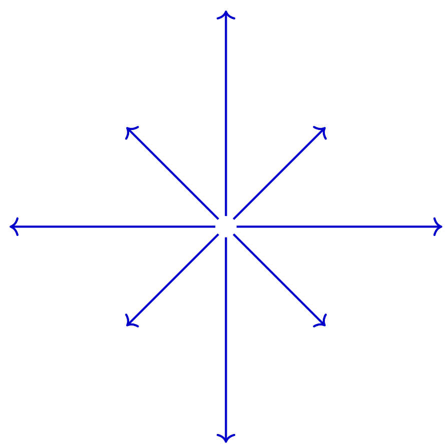
Type $A_2$



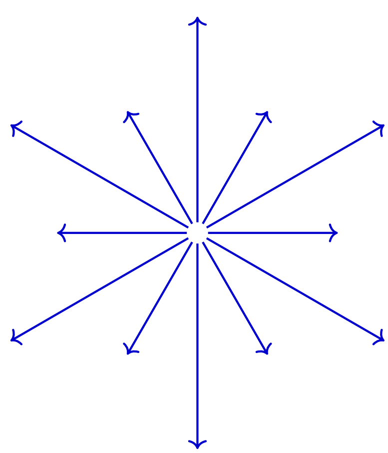
Type $B_2$



Type $C_2$



Type $G_2$



We will do an actual definition in a minute, but from these pictures, we see that

(1) For any vector $v$, $-v$ is also in the root system

(2) If you take any vector $v$ and look at the in the perpendicular to $v$, reflect he whole picture across that line, the reflection corresponds to the original picture.

Let $V$ be a real Euclidean space endowed with a positive definite symmetric bilinear form $(\lambda, \mu)$. A reflection $s$ is a linear operator on $V$ on which it sends a nonzero vector $\alpha$ to $-\alpha$ and fixes pointwise the hyperplane $H_\alpha$ orthogonal to $\alpha$. We typically write $s = s_\alpha$ understanding that for any $c \in \mathbb{R}, s_{c\alpha} = s_\alpha$. A finite subgroup of reflections is an interesting type of finite subgroup of $\mathrm{O}(V)$ and there is such a classification of all such groups!

Note that

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \tag{2.1}$$

Once can easily compute that $s_\alpha^2 = 1$ and so $s_\alpha$ has order 2 in the group of orthogonal transformations $\mathrm{O}(V)$.

## 3. Basic Examples of Types of Root Systems

In our intuitive description of root systems, we only dealt with Rank 2. But these generalize to higher rank in a natural way.

3.1. **($A_{n-1}, n \geqslant 2$).** Consider the symmetric group $\mathcal{S}_n$. $\mathcal{S}_n$ can be though of as a subgroup of $\mathrm{O}(n, \mathbb{R})$. We can see this by noting that $\mathbb{R}^n$ has a standard basis of vectors $\{e_1, e_2, \ldots, e_n\}$. We can make a permutation act on $V$ by permuting the standard basis vectors ( permuting the subscripts). Recall that a transposition is a permutation which exchanges two elements and keeps all other fixed. Moreover, every permutation can be written as a product of transpositions. So what happens with the transposition $(ij)$? Let's consider $\mathbb{R}^4$. We can see that the trasnposition (23) as as a reflection sending $e_2 - e_3$ to it negative and fixing pointwise the orthogonal compliment. Not that this is all the vectors having equal $i$th and $j$th component). As we stated before and was in Problem set 0,, $\mathcal{S}_n$ is generated by transpositions so it is a reflection group. Indeed, $\mathcal{S}_n$ is generated by the transpositions $(i, i+1), 1 \leqslant i \leqslant n-1$.

3.2. **($B_n, n \geqslant 2$).** Let $V$ be $\mathbb{R}^n$, so $\mathcal{S}_n$ acts on $V$ by permuting the basis vectors as before. There are other reflections sending $e_i$ to $-e_i$ and fixing the other $e_j$. Since the dimension of $V$ is $n$, these sign changes generate of group of order $2^n$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. This group of sign changes is

- normalized by $\mathcal{S}_n$
- intersects $\mathcal{S}_n$ trivially.

Conjugating the sign change $e_1$ to $-e_1$ by a transpositions gives another sign change so the semidirect product with $\mathcal{S}_n$ is a group of order $2^n n!$ and is a reflection group that we call $W$. In general this will normally be a Weyl group which we will discuss later. The Weyl group can be thought of as the skeleton of the corresponding group of Lie type.

3.3. **($D_n, n \geqslant 4$).** There is another reflection group acting on $\mathbb{R}^n$ by permuting the standard basis vectors, it is a subgroup of index 2 in the group of type $B_n$ above. Note that $\mathcal{S}_n$ normalizes the subgroup consisting of sign changes which involve an even number of signs, generated by the reflections sending $e_i + e_j$ to $-(e_i + e_j), i \neq j$. So the semidirect product is also a reflection group.

## 4. Root Systems

The origins of why these are called roots, come from the roots of characteristic polynomials. One of the reasons that we study root systems is that many groups have associated root data and this root data determines this group, up to isomorphism. Around the time period of 1880-1900 Cartan and Killing proved the following.

**Theorem 4.1.** *Every semisimple Lie algebra over the complex numbers has an associated root system and the root system determines the Lie algebra (up to isomorphism).*

The result was generalized by Chevalley in the 1940s and 1950s for reductive groups. That is, he showed that

**Theorem 4.2.** *Every reductive algebraic group has associated root data, and up to isomorphism, this root data determines the group.*

Next we define roots systems more formally. We fix a finite dimensional real vector space $V := \mathbb{R}^l$ with the standard Euclidean inner product, which is also known as the dot product.

**Definition 4.3.** *For $\alpha \in V, H_\alpha$ denotes the hyperplane or subspace perpendicular to $\alpha$, i.e.*

$$H_\alpha = \{\beta \in V : (\alpha, \beta) = 0\}.$$

**Definition 4.4.** *A subset $\Phi$ of $V$ is called a root system of $V$ if the following axioms are satisfied*

1. *$\Phi$ is a finite set of non-zero vectors*
2. *$\Phi$ spans $V$.*
3. *if $r, \lambda \in \Phi$, then*

$$s_r(\lambda) = \lambda - \frac{2(r,\lambda)}{(r,r)} r \in \Phi.$$

*Another way to say this is that every root is closed under reflection through the hyperplane perpendicular to $r$*

4. *(Integrability)If $r, \lambda \in \Phi$ then $\langle \lambda, r \rangle = \dfrac{2(r,\lambda)}{(r,r)}$ is an integer.*

*An element $\alpha \in \Phi$ is a* ***root****.*

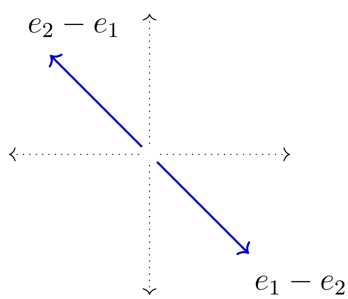
*We will also assume in the definition for our root system, that it is* ***reduced****, that is to say that the only scalar multiples of a root $\lambda \in \Phi$ are $\lambda$ and $-\lambda$.*

Not all definitions have item (4) - this is also called a Crystallographic root system. Since $\Phi$ spans $V$, the dimension of $V$ over the real numbers is $l$ and is an invariant of $\Phi$ called the **rank** of $\Phi$, and denote it by $rk(\Phi)$.

**Definition 4.5.** *Let $\Phi$ be a root system in $V$. For $\alpha \in \Phi$, the hyperplanes $H_\alpha$ divide $V$ into connected components. These connected components of $V \setminus \bigcup_\alpha H_\alpha$ are the* ***Weyl Chambers*** *of $\Phi$ (We will see pictures in a minute, and will make sense then.)*

4.1. **Examples.** We will work in $\mathbb{R}^l$ with standard basis $e_1, e_2, \dots e_l$ with the standard Euclidean inner product, also known as the dot product.

**Example 4.6.** *The type $A_1$ root system. For this example we will work in $\mathbb{R}^2$ with the standard basis $e_1, e_2$. Let $\Phi = \{e_1 - e_2, e_2 - e_1\}$. Geometrically we can represent this with the following picture*

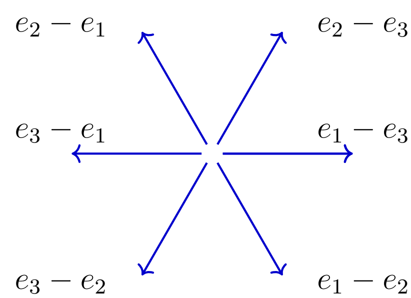



*Let $V$ be the span of $(-1, 1)$. Then $\Phi$ is a root system in $E$. Let's check the integrability property (iv) in Def 4.4.*

$$\frac{2(e_1 - e_2, e_2 - e_1)}{(e_2 - e_1, e_2 - e_1)} = \frac{2(-1-1)}{(1+1)} = -2.$$

*This is the root system of type $A_1$ which corresponds to the dimension of $V$, but we drew this in $\mathbb{R}^2$ to generalize this to the rank 2 examples below.*

**Example 4.7.** *The type $A_2$ root system. Consider $\mathbb{R}^3$ with the standard basis vectors $e_1, e_2, e_3$, Let $\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}$. The span of $\Phi$ is the plane with normal vector $e_1 + e_2 + e_3$. Let $V$ be this subspace. We claim that $\Phi$ is a root system in $V$. Geometrically, we can represent this with the following picture.*



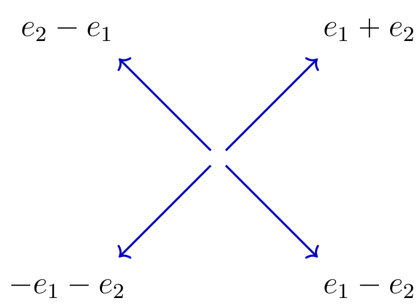

*To check integrability, one must check all the cases such as*

$$\frac{2(e_1 - e_2, e_2 - e_3)}{(e_1 - e_2, e_1 - 2_2)} = \frac{2(-1)}{2} = -1.$$

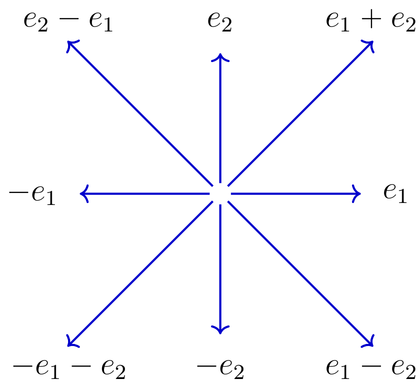
*Bot notice there are a lot of cases to check!*

**Example 4.8.** *The $A_l$ root system. Let $e_1, e_2, \ldots, e_{l+1}$ be the standard basis of $\mathbb{R}^{l+1}$. Let $\Phi = \{\pm(e_i - e_j) : 1 \leqslant i|j \leqslant l+1\}$. Let $V \subset \mathbb{R}^{l+1}$ be the span of $\Phi$ with the inner form being the dot product.*

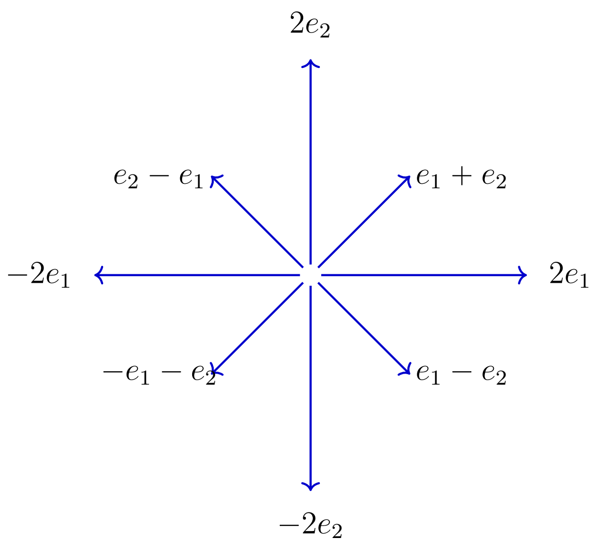
**Example 4.9.** *$A_1 \times A_1$ root system. Let $e_1, e_2$ be the standard basis. Consider $\mathbb{R}^2$ with the dot product. We have two copies of the $A_1$ root system. One given as before with $\{e_1 - e_2, e_2 - e_1\}$ and the other given by $\{e_1 + e_2, -e_1 - e_2\}$. Let $\Phi = \{e_1 - e_2, e_2 - e_1, e_1 + e_2, -e_1 - e_2\}$ and as before we can represent geometrically in the following picture.*



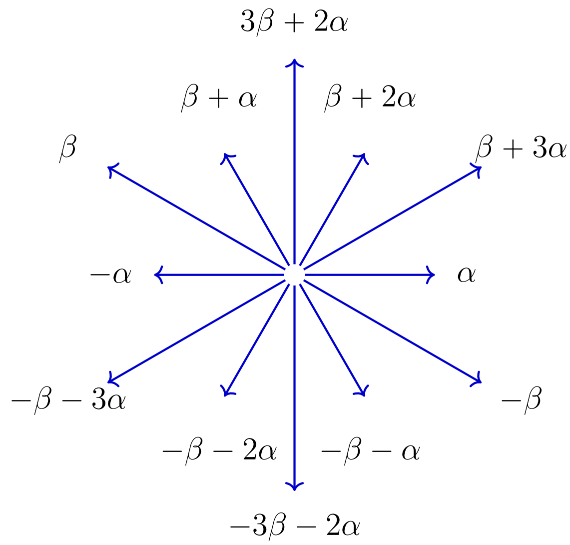

**Example 4.10.** *The $B_2$ root system. Consider $\mathbb{R}^2$ with the dot product with basis $e_1, e_2$. Let $\Phi = \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\}$. Then we can show that this is a root system, and can be represented geometrically as follows: Type $B_2$*




**Example 4.11.** *The Type $C_2$ root system. In $\mathbb{R}^2$ with the dot product and $\Phi = \{\pm 2e_1, \pm 2e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\}$*




**Example 4.12.** *The Type $G_2$ root system. Now consider $\mathbb{R}^3$ with basis elements $e_1, e_2, e_3$. Let $\Phi = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$. Note that the fist six vectors are the same as for $A_2$ and these are in the hyperplane perpendicular to $e_1 + e_2 + e_3$. The other vectors are in the same plane, so let $V$ be the plane. Let $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$.*




As it turns out, we have now described all of the irreducible root systems of rank 2.

This is a nice place to give the following definitions that will be needed for Problem set 2. Let $\Phi$ be a root system. The set of **positive roots**, which we denote by $\Phi^+$ is a subset of $\Phi$ such that

- For each $\alpha \in \Phi$ exactly one of $\alpha$ or $-\alpha$ is contained in $\Phi^+$.
- For any two distince roots $\alpha, \beta \in \Phi^+$ such that $\alpha+\beta$ is a root, $\alpha+\beta \in \Phi^+$.

An element in $\Phi^+$ is called a **simple root** if it can not be written as the sum of two elements in $\Phi^+$. The set of simple roots is referred to as a base for $\Phi$.

## 5. Classification of Root Systems

The integrability condition in Def 4.4 restricts what angles are possible. We define the symbol,

$$\langle \beta, \alpha \rangle := 2\frac{(\alpha,\beta)}{(\beta,\beta)}.$$

Since $\langle \beta, \alpha \rangle$ and $\langle \alpha, \beta \rangle$ are integers we have that

$$\begin{aligned}\langle \beta, \alpha \rangle\langle \alpha, \beta \rangle &= 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \cdot 2\frac{(\alpha,\beta)}{(\beta,\beta)} \\ &= 4\frac{(\alpha,\beta)^2}{|\alpha|^2|\beta|^2} \\ &= 4\cos^2\theta = (2\cos\theta)^2.\end{aligned}$$

Since $2\cos\theta \in [-2,2]$ and $(2\cos\theta)^2$ is an integer,

$$\cos\theta \in \left\{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\}$$

which corresponds to the angles

$$\left\{\frac{\pi}{2}, \frac{\pi}{3} \text{ or } \frac{5\pi}{3}, \frac{\pi}{4} \text{ or } \frac{7\pi}{4}, \frac{\pi}{6} \text{ or } \frac{11\pi}{6}, 0 \text{ or } \pi\right\}.$$

We are wanting a reduced system, so no scalar multiples of a root $\alpha$ other than $\alpha$ and $-\alpha$ so 0 and $\pi$ are not included, these correspond to $2\alpha$ and $-2\alpha$. Another way to look at this is if $4\cos\theta = 4$ and then $\theta = \pi$ which would make $\beta = \pm\alpha$. The gives the following possibilities:

| $\langle\alpha,\beta\rangle$ | $\langle\beta,\alpha\rangle$ | $\theta$ | $\left(\frac{\lvert\beta\rvert}{\lvert\alpha\rvert}\right)^2$ |
|---|---|---|---|
| 0 | 0 | $\frac{\pi}{2}$ | unrestricted |
| 1 | 1 | $\frac{\pi}{3}$ | 1 |
| -1 | -1 | $\frac{2\pi}{3}$ | 1 |
| 1 | 2 | $\frac{\pi}{4}$ | 2 |
| -1 | -2 | $\frac{3\pi}{4}$ | 2 |
| 1 | 3 | $\frac{\pi}{6}$ | 3 |
| -1 | -3 | $\frac{5\pi}{6}$ | 3 |

In the $A_l$ root system, all roots have equal length. In the $B_2$ and $C_2$ root systems, two roots have length corresponding to 2 (the square of the ratio of the lengths). In the $G_2$ root system, two roots have length corresponding to 3 and all the angles are multiples of $\frac{\pi}{6}$. This ends up leading to a classification of root systems.

We begin with a the following proposition which can be found in [Hal15], and is also left as an exercise in Problem Set 3.

**Proposition 5.1.** *Suppose $\alpha$ and $\beta$ are roots, $\alpha$ is not a multiple of $\beta$ and $(\alpha, \alpha\rangle \geqslant \langle\beta, \beta)$. Then one of the following holds.*

*(1)* $(\alpha, \beta) = 0$
*(2)* $(\alpha, \alpha) = (\beta, \beta)$ *and the angle between $\alpha$ and $\beta$ is $\pi/3$ or $2\pi/3$.*
*(3)* $(\alpha, \alpha) = 2(\beta, \beta)$ *and the angle between $\alpha$ and $\beta$ is $\pi/4$ or $3\pi/4$.*
*(4)* $(\alpha, \alpha) = 3(\beta, \beta)$ *and the angle between $\alpha$ and $\beta$ is $\pi/6$ or $5\pi/6$.*

**Proposition 5.2.** *Every rank two root system is isomorphic to $A_1 \times A_1, A_2, B_2$, or $G_2$.*

*Proof.* We assume that $V = \mathbb{R}^2$; and let $\Phi \subset \mathbb{R}^2$ be a root system and let $\theta$ be the smallest angle occurring between any two vectors in $\Phi$. Since the element in $\Phi$ spans $\mathbb{R}^2$, we can find two linearly independent vectors $\alpha$ and $\beta$ in $\Phi$. If the angle between $\alpha$ and $\beta$ is greater than $\pi/2$, then the angle between $\alpha$ and $-\beta$ is less than $\pi/2$; thus the minimum angle is at most $\pi/2$. From 5.1, $\theta \in \{\pi/2, \pi/3, \pi/4, \pi/6\}$. Let $\alpha$ and $\beta$ be two elements of $\Phi$ such that the angle between them is the minimum angle $\theta$. Then the vector $-s + \beta(\alpha)$ will be a vector that is at angle $\theta$ to $\beta$ but on the opposite side of $\beta$ from $\alpha$. Thus, $-s_\beta(\alpha)$ is at angle $2\theta$ to $\alpha$. Then $-s_{s_\beta \cdot \alpha}\beta$ is at most $3\theta$ to $\alpha$. Continuing we get vectors at angle $n\theta$ to $\alpha$ for all $n$. Since a nontrivial positive multiple of a roots is not a root, these vectors are unique. Each of the allowed values of $\theta$ eveny divides $2\pi$, we will eventually get to $\alpha$ again (else there would be an angle smaller than $\theta$).

Thus $\phi$ must consist of $n$ equally spaced vectors with consecutive vectors separated by angle $\theta$, where $\theta$ is one of the acute angles in Proposition 5.1. If say, $\theta = \pi/4$ then in order to satisfy the length requires of Proposition 5.1, the roots must alternate between shorter length and a second length that is greater by a factor of $\sqrt{2}$. Thus our root system must be isomorphic to $B_2$. Similar reasoning shows that all remaining values of $\theta$ yield one of the root systems $A_1 \times A_1, A_2, B_2, G_2$. $\square$

## 6. A base of a root system and Weyl Chambers

**Definition 6.1.** *A nonempty root system $\Phi$ us **irreducible** if it is not the direct sum of two nonempty root systems.*

**Definition 6.2.** *A nonempty root system $\Phi$ is **reducible** if it can be written as a disjoint union of nonempty root systems, i.e.*

$$\Phi = \Phi_1 \bigsqcup \Phi_2$$

*where $\Phi_1$ and $\Phi_2$ are root systems.*

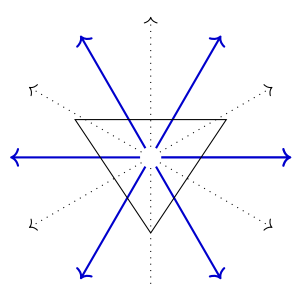
Every root system is the direct sum of some set of irreducible root systems, and this composition is unique up to the order of the terms. This is why for classification purposes, we only need to consider irreducible root systems.

**Definition 6.3.** *Two root systems $(V_1, \Phi_1)$ and $(V_2, \Phi_2)$ are **isomorphic** if there is an invertible linear transformation $V_1$ to $V_2$ that maps $\Phi_1$ to $\Phi_2$ such that for each pair of roots, the number $\langle x, y\rangle$ is preserved.*

6.1. **Weyl group.** The group $W$ generated by all reflections $s_\alpha$, where $\alpha \in \Phi$, is known as the Weyl group of $\Phi$.

Recall that $s_\alpha$ is the reflection through the hyperplane of $\alpha$. Since $\Phi$ is a root system, each $s_\alpha$ preserves $\Phi$ so $W(\Phi)$ may also be viewed as a subgroup of the permutation group of $\Phi$. Note that $W(\Phi)$ is then finite.

**Example 6.4.** *The Weyl group of $A_2$. Recall that the root system of $A_2$ is*



*The Weyl group of this system is the subgroup of the symmetry group of $A_2$ generated by reflections. This is the symmetry group of an equilateral triangle, which is $\mathcal{S}_3$. (Think of as the vertices of an equilateral triangle).*

Note that $W$ is not the full symmetry group of the root system. If we rotate by 60 degrees, $\Phi$ is preserved, but not as an element of $W$.

6.2. **Weyl Chambers.** If $\Phi \subset V$ is a root system, consider the hyperplane $H_\alpha$ perpendicular to each root $\alpha$ and recall that $s_\alpha$ denotes the reflection about the hyperplane. From before, the Weyl group is the group of reflections generated by the $s'_\alpha s$. Note that the complement of this set of hyperplanes is disconnected and each component is called a **Weyl chamber**.

If $\Phi$ is a root system, a subset $\Delta$ of $\Phi$ is called a base if the following conditions hold"

(1) $\Delta$ is a basis for $V$ as a vector space.
(2) Each root $\alpha$ can be expressed as a linear combination of elements of $\Delta$ with linear coefficients in such a way that these coefficients are all non-positive or all non-negative.

The non-negative roots are called **positive roots** and the non-positive roots are called **negative roots**. The elements of $\Delta$ are called **simple roots**.

Fix a set $\Delta$ of simple roots, the fundamental Weyl chamber associated to $\Delta$ are the set of points $v \in V$ such that $(\alpha, v) > 0$ for all $\alpha \in \Delta$.

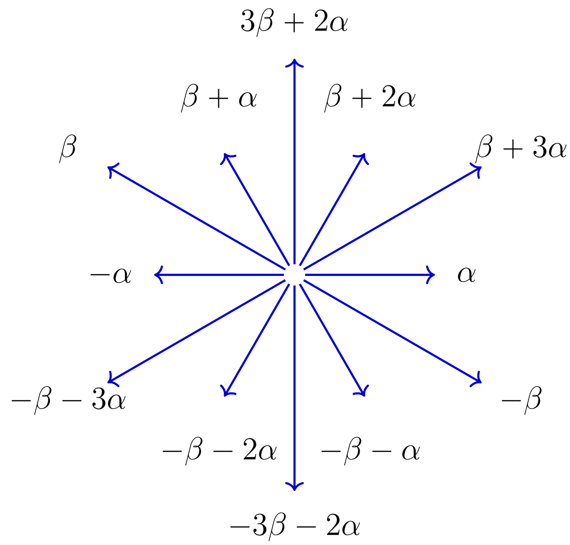
**Example 6.5.** *Recall that for $A_2$,*

$$\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}.$$

*And as it turns out,*

$$\Delta = \{e_1 - e_2, e_3 - e_1\}.$$

**Example 6.6.** *Base for $G_2$. Recall from, before that we had $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$ and the roots were linear combinations of $\alpha$ and $\beta$. Then $\Delta = \{\alpha, \beta\}$ for $G_2$. Which makes sense when we look at the geometric interpresation of the root system for $G_2$.*




,

**Theorem 6.7.** *The order of the Weyl group is equal to the number of Weyl chambers.*

*Proof.* See Hall, Proposition 8.23 and Proposition 8.27. □

Recall from our $A_2$ example, the Weyl group has 6 elements and there are 6 Weyl chambers.

## 7. Cartan Matrices and Dynkin Diagrams

**Theorem 7.1.** *(Existence Theorem) Let $\Phi$ be an irreducible root system. Then there exists a simple Lie algebra over $\mathbb{C}$ which has a root system equivalent to $\Phi$.*

A proof of the existence theorem can be found in [Tit66].

**Theorem 7.2.** *(Isomorphism theorem) Any two simple Lie algebras over $\mathbb{C}$ with equivalent root systems are isomorphic.*

A proof of this theorem is in Jacobson[1]. Note that since any root system is uniquely a union of irreducible root system, it suffices to classify irreducible root systems. This is done using Dynkin diagrams. Is is more convenient to encode them by the Cartan matrix which we define now. Let $\Phi$ be a root system with base $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Recall that $\langle \alpha, \beta \rangle = \dfrac{2(\beta, \alpha)}{(\beta, \beta)}$ and the **Cartan** matrix is given by

$$A = (a_{ij}) = (\langle \alpha_i, \alpha_j \rangle)$$

which is an $l \times l$ matrix with entries in the integers thanks to property (4) of Definition 4.4 and this does not depend on the base that we choose. The following properties of the Cartan matrix follow from what we have done.

**Proposition 7.3.**
- *The diagaonal elements of the Cartan matrix are $a_{ii} = 2$.*
- *The entries $a_{ij}$ is a nonpositive integer for any $\imath neq j$*
- *The entry $a_{ij}a_{ji} = 4\cos^2\theta \in \{0, 1, 2, 3, \}$ where $\theta$ is the angle between $\alpha_i$ and $\alpha_j$.*
- *Let $d_i = |\alpha_i|^2$. Then the matrix $d_i A is$ symmetric and positive definite.*

So every root sytem defines a Cartan matrix.

**Example 7.4.** *Cartan matrix for $G_2$. We use the base for $G_2$ of $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$ for $G_2$. Then since we already know that the diagonal elements are 2 from the Proposition we just need to compute the following*

$$\langle \alpha, \beta \rangle = \frac{2(2e_2 - e_1 - e_3, e_1 - e_2)}{(2e_2 - e_1 - e_3, 2e_2 - e_1 - e_3)} = \frac{2(-1-2)}{1+4+1} = -1$$
$$\langle \beta, \alpha \rangle = \frac{2(e_1 - e_2, 2e_2 - e_1 - e_2)}{(e_1 - e_2, e_1 - e_2)} = \frac{2(-1-2)}{2} = -3$$

*Thus the Cartan matrix for $G_2$ is*

$$G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

The Cartan matrices of the other individual simple algebras are shown below:

$$A_l : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & 0 & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & & & & & \\ & & & & & & & \\ & & & & & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & 0 & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix},$$

$$B_l : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & 0 & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & & & & & \\ & & & & & & & \\ & & & & & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & 0 & & & -1 & 2 & -1 \\ & & & & & & -2 & 2 \end{pmatrix},$$

$$C_l : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & 0 & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & & & & & \\ & & & & & & & \\ & & & & & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & 0 & & & -1 & 2 & -2 \\ & & & & & & -1 & 2 \end{pmatrix},$$

$$D_l : \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & 0 & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & & & & & \\ & & & & & & & \\ & & & & & -1 & & \\ & & & & -1 & 2 & -1 & -1 \\ & & 0 & & & -1 & 2 & 0 \\ & & & & & -1 & 0 & 2 \end{pmatrix},$$

$$F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_6 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_7 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_8 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

In the last lecture we will see an example of using these matrices!

## 8. Dynkin Diagrams

Let $\Phi$ be a root system with base $\Delta$, the associated Dynkin diagram corresponds to the roots in $\Delta$. We can either write the Dynkind diagram directly from the roots, or from the Cartan matrix. Edges are drawn between the vertices as follows:

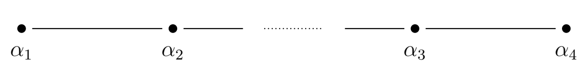
- If the vectors are orthogonal, there is no edge.
- If the vectors form an angle of 120 degrees, there is an undirected single edge
- If the vectors form an angle of 135 degrees, there is a directed double edge
- If the vectors form an angle of 150 degrees, there is a directed triple edge.

We can also form the Dynkin Diagrams by reading off the Cartan matrices.

- Indices $i$ are vertices.
- Vertices $i$ and $j$ are connected by $a_{ij}a_{ji}$ lines (for $\imath \neq j$).
- if $a_{ij} \neq a_{ji}$ (i.e. $|\alpha_i|^2 \neq |\alpha_j|^2$), then the arrow on the lines goes from the long root to the short root.

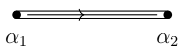
**Example 8.1.** *Dynkin diagram for type $A_2$. There are only two roots in the base of $A_2$ so the Dynkin diagram has two vertices , call them $\alpha_1, \alpha_2$. Between them is just one edge and they have the same length so the graph is not directed. So the Dynkin diagram for $A_2$ is*


**Example 8.2.** *Dynkin diagram for type $A_l$. We use the same base as before for tyoe $A_l$ and use the Cartan matrix to read off the edges. Note that all of the edges have the same length.*




**Example 8.3.** *Dynkin diagram for type $G_2$. We have the simple roots, we will call them $\alpha_1, \alpha_2$ with $\alpha_2$ being the longer root. From the Cartan matrix we have the $a_{12}a_{21} = (-1)(-3) = 3$, so there are three edges between the two vertices. We*

*also have an arrow pointing to the longer root. Hence, we have*




The Dynkin diagram completely determines the Cartan matrix if we fix the labeling of the vertices and vice versa. It is also clear that the root system is irreducible if and only if the Dynkin diagram is connected.

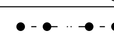
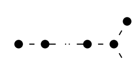
8.1. **Classification of Dynkin diagrams.** The following theorem gives a complete classification of irreducible root systems.

**Theorem 8.4.** *Connected Dynkin diagrams are classified by the following pictures which are $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$. Moreover, every Cartan matrix satisfying Proposition 7.3 is a Cartan matrix of some root system.*

First we need to discuss these other root systems. Note that Types $A_n, B_n, C_n, D_n$ we call classical groups because they can be thought of as subgroups of $GL_n$. The groups $G_2, F_4, E_6, E_7$, and $E_8$ are called exceptional groups. We have discussed $G_2$ already, so let's look at the other exceptional groups.

The $F_4$ root system is the union of the $B_4$ root system with the vectors $\sum_{i=1}^{4}\left(\pm\frac{1}{2}e_i\right)$. Recall that $B_4$ has the roots $\pm e_i \pm e_j$ (24 of them with squared length 2) and $\pm\frac{1}{2}e_i$ (8 of them with squared length 1). We then have 16 more roots $\sum_{i=1}^{4}\left(\pm\frac{1}{2}e_i\right)$ (squared length 1); this gives a total of 48 roots. The $F_4$ system has 3 simple roots.

Follows is the description of the simple Lie algebras over the complex numbers. For each algebra, we have the dimension, the rank, the number $N$ of positive roots, the order of the Weyl group $W$ and the Dynkin diagram.

| Type | Dim | Rank | N | —W— | Dynkin diagram |
|---|---|---|---|---|---|
| $A_l (l \geqslant 1)$ | $l(l+2)$ | $l$ | $\frac{1}{2}l(l+1)$ | $(l+1)!$ |  |
| $B_l (l \geqslant 2)$ | $l(2l+1)$ | $l$ | $l^2$ | $2^l \cdot l!$ |  |
| $C_l (l \geqslant 3)$ | $l)2l+1)$ | $l$ | $l^2$ | $2^l \cdot l!$ |  |
| $D_l (l \geqslant 4)$ | l(2l-1) | $l$ | $l^2$ | $2^{l-1} \cdot l!$ |  |
| $G_2$ | 14 | 2 | 6 | 12 |  |
| $F_4$ | 52 | 4 | 24 | $2^7 \cdot 3^2$ |  |
| $E_6$ | 78 | 6 | 36 | $2^7 \cdot 3^4 \cdot 5$ |  |
| $E_7$ | 133 | 7 | 63 | $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ |  |
| $E_8$ | 248 | 8 | 120 | $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ |  |

## 9. Lie groups and Lie algebras

At this point, I think it is a good idea to go back and do some light definitions on Lie groups and Lie algebras.

**Definition 9.1.** *A **Lie group** is a group that is also a differential manifold, such that the group multiplication and taking inverses is differentiable.*

For example, a real Lie group $G$ is a group such that the map $\mu : G \times G \to G$ sending $(x, y)$ to $x^{-1}y$ is a smooth mapping of the product manifold into $G$. The **Lie bracket**, which is also known as the commutator, is defined to be

$$[X, Y] = [XY - YX]$$

. Basically it measures how far something is from being commutative. With this operator, we can now define a Lie algebra.

**Definition 9.2.** *A **Lie algebra** $\mathfrak{g}$ is a vector space over a field $F$ together with a Lie bracket such that the Lie bracket has the following properties:*

- *Bilinear:*
  $[ax + by, z] = a[x, z] + b[y, z]$
  $[z, ax + by] = a[z, x] + b[z, y]$
  *for all $a, b \in F$ and $x, y, z \in \mathfrak{g}$*
- *The Alternating Property*
  $[x, x]$*=0 for all $x \in \mathfrak{g}$.*
- *Jacobi identity*
  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all $x, y, z \in \mathfrak{g}$.*

A Lie algebra is called **semisimple** if it is a direct sum of nonabelian Lie algebras with no non-zero proper ideal.

For a finite dimensional Lie algebra over a field of characteristic 0, the following are equivalent:

- $\mathfrak{g}$ is semi-simple
- the killing form $\kappa(x, y) = \mathrm{tr}(ad(x), ad(y))$ is nondegenerate.
- $\mathfrak{g}$ has no nonzero abelian ideals.
- $\mathfrak{g}$ has no nonzero solvable ideals.
- the radical (the maximal solvable ideal) of $\mathfrak{g}$ is zero.

## 10. Root systems and the Langlands program

Here we give an introduction to the Langlands program. The Langlands program makes bridges between different areas of mathematics, such as Harmonic Analysis, Algebraic Geometry, Number Theory. When proposing these lectures, he wrote a letter to Andrew Weil,

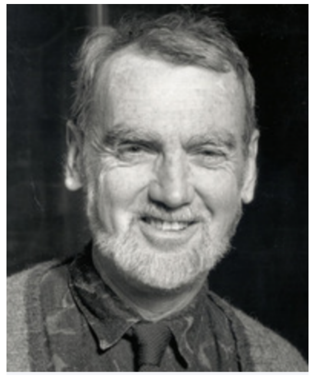


FIGURE 1. Robert Langlands

Letter to Professor Weil: "In response to your invitation to come and talk, I wrote the following letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation, I would appreciate that ; if not- I am sure you have a waste basket handy."

First we, need to define the Weil group. But before we do, we need to define p-adic groups. First we start with the $p$-adic integers.

**Definition 10.1.** *The $p$-**adic integers** $\mathbb{Z}_p$ are the completion of the integers $\mathbb{Z}$ with respect to the $p$-adic absolute $|\cdot|_p$, where $|p^s \cdot r|_p = \left(\frac{1}{p}\right)^s$ where $r$ and $p$ are relatively prime. A $p$-adic integer is of the form*

$$a_0 + a_1 p + a_2 p^2 + \cdots$$

*for some $a_i, 0 \leqslant a_i < p$.*

**Definition 10.2.** *The $p$-**adic numbers** $\mathbb{Q}_p$ are the fraction field of the $p$-adic integers $\mathbb{Z}_p$ and they are the completion of the rational numbers $\mathbb{Q}$ with respect to the $p$-adic absolute value $|\cdot|_p$. A $p$- adic number is of the form*

$$a_{-n}p^{-n} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$$

*for some coefficients $a_i$, $0 \leqslant a_i < p$.*

The $p$-adic numbers $\mathbb{Q}_p$, we have the completion of $\mathbb{Q}$ with respect to the $p$-adic absolute value. They area also totally disconnected and have infinitely many compact subgroups under addition. Note that the characteristic of $\mathbb{Q}_p$ is zero.

10.1. **The Weil group.** Let $F$ be a $p$-adic field ( a finite extension of $\mathbb{Q}_p$. The absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ comes with a surjection

$$\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \widehat{\mathbb{Z}} := \varprojlim_n (\mathbb{Z}/n\mathbb{Z}).$$

The kernel of this surjection is the **inertia group** denoted by $\mathcal{I}_F$. Consider the short exact sequence of topological groups

$$1 \longrightarrow \mathcal{I}_F \hookrightarrow \mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \widehat{\mathbb{Z}} \longrightarrow 0.$$

Let $W_F^{\diamond}$ be the pre-image of $\mathbb{Z}$ so that we have the s.e.s. of topological groups

$$1 \longrightarrow \mathcal{I}_F \hookrightarrow W_F^{\diamond} \twoheadrightarrow \mathbb{Z} \longrightarrow 0.$$

We enlarge the collection of open sets in $W_F^{\diamond}$ so that the map $W_F^{\diamond} \twoheadrightarrow \mathbb{Z}$ is continuous in the discrete topology. $W_F^{\diamond}$ with the discrete topology is the **Weil group** $W_F$. We now have the short exact sequence of topological groups

$$1 \longrightarrow \mathcal{I}_F \hookrightarrow W_F \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}$ has the discrete topology. Denote the abelianization of $W_F$ by $W_F^{ab} := W_F/W_F^{der}$. Local class field theory asserts that there is a canonical surjective homomorphism of topological groups

$$\alpha_F : W_F \to F^{\times}$$

with the commutative diagram

$$\begin{tikzcd}
\mathscr{W}_F \arrow[r, "\alpha_F"] \arrow[d, "\varepsilon"'] & F^{\times} \arrow[d, "val_F"] \\
\mathbb{Z} \arrow[r, "I_{\mathbb{Z}}"'] & \mathbb{Z}
\end{tikzcd}$$

such that $\alpha_F$ induces an isomorphism

$$W_F^{ab} \cong F^{\times}.$$

Now let $G$ be a reductive p-adic group and $V$ a vector space over $\mathbb{C}$.

**Definition 10.3.** *A* ***representation*** *of $G$ is a group homomorphism*

$$\phi : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$$

*and is* ***smooth*** *if for every $v \in V$ the stabilizer group*

$$G_v = \{g \in G | \phi(g)v = v\}$$

*is an open subgroup of $G$.*

The smooth dual of $G$ is defined as

$$\mathbf{Irr}(G) := \{\text{Smooth irreducible representations of} G\}/\sim.$$

The $L$-group of $G$ is denoted ${}^L G$ and ${}^L G := \widehat{G} \rtimes W_F$

A **Langlands parameter** for $G$ is a group homomorphism

$$\phi : W_F \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L G$$

| G | $\widehat{G}$ | G | $\widehat{G}$ |
|---|---|---|---|
| $\mathrm{GL}_m$ | $\mathrm{GL}_m(\mathbb{C})$ | $G_2$ | $G_2(\mathbb{C})$ |
| $\mathrm{SO}_{2m+1}$ | $\mathrm{Sp}_{2m}(\mathbb{C})$ | $\mathrm{GSp}_{2m}$ | $\mathrm{GSpin}_{2m+1}(\mathbb{C})$ |
| $\mathrm{SO}_{2m}$ | $\mathrm{SO}_{2m}(\mathbb{C})$ | $\mathrm{GSO}_{2m}$ | $\mathrm{GSpin}_{2m}(\mathbb{C})$ |

such that

- $\phi$ is continuous
- $\phi|_{\mathrm{SL}(2,\mathbb{C})}$ is a morphism of complex algebraic groups
$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \widehat{G}$$
- The following diagram commutes

$$\begin{tikzcd}
\mathscr{W}_F \times \mathrm{SL}(2,\mathbb{C}) \arrow[r, "\varphi"] \arrow[d] & {}^L G \arrow[d] \\
\mathrm{Gal}(\overline{F}\,|F) \arrow[r, "I"'] & \mathrm{Gal}(\overline{F}\,|F)
\end{tikzcd}$$

- $\phi(W_F) \subset ({}^L G)_{\text{semi-simple}}$

Recall, a Langlands parameter
$$\phi : W_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^L G$$
We can form the set
$$\Phi(G) := \{\text{Langlands parameters for} G\}/\widehat{G}$$
The **local Langlands conjecture (LLC)** asserts that there is a map of sets
$$\alpha_G : \mathbf{Irr}(G) \to \Phi(G)$$
which is finite-to-one and has certain properties.

- $G$ quasi-split, $\alpha_G$ is expected to be surjective
- The fibers of $\alpha_G$ are called ***L*-packets**

Let's see an example of the Local Langlands Conjecture for $\mathrm{GL}(1,F) = F^\times$. Then $\widehat{G} = \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^\times$. Let $\phi$ be a Langlands parameter for $\mathrm{GL}(1,F)$. Then $\phi|_{\mathrm{SL}(2,\mathbb{C})}$ is a morphism of complex algebraic groups
$$\mathrm{SL}(2,\mathbb{C}) \to \widehat{GL}(1,\mathbb{C}) = \mathbb{C}^\times.$$
Then only one is trivial, so
$$\begin{aligned}\Phi(G) &= \{\text{Langlands parameters for} G\}/\widehat{G} \\ &= \{\text{ continuous homomorphisms } W_F \to \mathbb{C}^\times\}\end{aligned}$$

From local class field theory, $W_F^{ab} \cong F^\times$ gives a bijection

$$\{\text{continuous homomorphisms } W_F \to \mathbb{C}^\times\} \longleftrightarrow \mathbf{Irr}(F^\times)$$

Thus the LLC holds for $\mathrm{GL}(1, F)$. The goal of the LLC is to extend local class field theory to non-abelian connected reductive p-adic groups.

Our goal for this day of lectures is to utilize a root system in the Langlands program. We will do this with a group called the general Spin group.

10.2. **GSpin.** Let $(V, q)$ be a (nondegenerate) quadratic space over a nonarchimedean local field $F$ of characteristic 0 with dimension $n$. The general Spin group associated with $(V, q)$, which we denote by $\mathrm{GSpin}(V)$ is a reductive groups over $F$ such that we have the following short exact sequence

$$1 \longrightarrow Z^\circ \longrightarrow \mathrm{GSpin}(V) \xrightarrow{P} \mathrm{SO}(V) \longrightarrow 1 ,$$

where $Z^\circ \simeq \mathrm{GL}_1$ is the connected component of the center of $\mathrm{GSpin}(V)$. Here we call the surjection

$$P : \mathrm{GPin}(V) \longrightarrow \mathrm{O}(V).$$

Let

$$T(V) := \bigoplus_{\ell=0}^{\infty} V^{\otimes \ell} = F \oplus V \oplus V^{\otimes 2} \oplus \cdots$$

be the tensor algebra of $V$. We have the natural inclusion $V \hookrightarrow T(V)$. We define the Clifford algebra $C(V)$ by

$$C(V) := T(V)/\langle v \otimes v - q(v) \cdot 1 \ : \ v \in V \rangle,$$

which is an associative $F$-algebra. The natural inclusion $V \hookrightarrow T(V)$ gives the natural inclusion $V \hookrightarrow C(V)$. Note that in $C(V)$ we have

$$v \cdot v = q(v) \in F$$

for all $v \in V$.

We denote by $C^\ell(V)$ the image of $V^{\otimes \ell}$ in $C(V)$. Though the Clifford algebra is not a direct sum of $C^\ell(V)$'s, it is a direct sum of even terms and odd terms; namely

$$C(V) = C^+(V) \oplus C^-(V),$$

where

$$C^+(V) = \sum_{\ell \text{ even}} C^\ell(V) \quad C^-(V) = \sum_{\ell \text{ odd}} C^\ell(V).$$

Also note that we actually have $C = \sum_{\ell=0}^{n} C^n(V)$ because for $\ell > n$ any element in $C^\ell(V)$ is written as a sum of lower degree terms.

It is known that $\dim_F C(V) = 2^n$ and $\dim_F C^\pm(V) = 2^{n-1}$. Note that $C^+(V)$ is a subalgebra of $C(V)$, called the even Clifford algebra. Both $C(V)$ and $C^+(V)$ are central simple algebras central over $F$ or over the quadratic etale algebra $F[x]/(x^2 - d_V)$ where $d_V$ is the discriminant of $V$. (See [Sch85, 2.10 Theorem, p.332] or

[Shi04, Theorem 2.8, p.19].) We can now define the groups $\mathrm{GPin}(V)$ and $\mathrm{GSpin}(V)$ as follows.

**Definition 10.4.** *We define*

$$\mathrm{GSpin}(V) := \{g \in C^+(V)^\times \ : \ gVg^{-1} = V\},$$

*and call* $\mathrm{GSpin}(V)$ *the general Spin group of* $V$*.*

**Remark 10.5.** *Sometimes in the literature, the group* $\mathrm{GSpin}(V)$ *the special Clifford group, and denoted by* $S\Gamma(V)$ *(or some other symbols). But we avoid this terminology because in representation theory of* $p$*-adic groups or in automorphic forms it seems to be more common to call* $\mathrm{GSpin}(V)$ *the general Spin group.*

In this section, we describe split $\mathrm{GSpin}_n$ in terms of its root datum.

**Proposition 10.6.** *The root datum of* $\mathrm{GSpin}_n, n = 2m+1$ *or* $2m$*, can be described as the following. Let*

$$X = \mathbb{Z}e_o \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m,$$
$$X^\vee = \mathbb{Z}e_o^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_m^*,$$

*and let*

$$\langle\ ,\ \rangle : X \times X^\vee \to \mathbb{Z}$$

*be the standard pairing. The root datum for* $\mathrm{GSpin}_n$ *is* $(X, R, X^\vee, R^\vee)$ *with* $R$ *and and* $R^\vee$ *generated by the simple roots and coroots given by, respectively,*

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{m-1} = e_{m-1} - e_m, \alpha_m = e_m\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{m-1}^\vee = e_{m-1}^* - e_m^*, \alpha_m^\vee = 2e_m^* - e_0^*\},$$

*if* $n = 2m+1$ *and by*

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{m-1} = e_{m-1} - e_m, \alpha_m = e_{m-1} + e_m\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{m-1}^\vee = e_{m-1}^* - e_m^*, \alpha_m^\vee = e_{m-1}^* + e_m^* - e_0^*\},$$

*if* $n = 2m$*.*

*Proof.* See [Asg02, Section 2]. ☐

The root datum listed in Proposition 10.6 corresponds with the root datum listed in Section 2.1. For example, $\alpha_1(x) = (e_1 - e_2)(x) = x_1x_2^{-1}$. The root datum of Proposition 10.6 determines the split group $\mathrm{GSpin}_n$ uniquely, equipped with a Borel subgroup containing a maximal torus $T$. Note that $\alpha_i : T \to GL_1$ and $\alpha_i^\vee : GL_1 \to T$ is such that $\alpha_i \circ \alpha_j^\vee(x) = x^{\langle\alpha_i,\alpha_j^\vee\rangle}$. When $n$ is odd, $\mathrm{GSpin}_n$ has a Dynkin diagram of type $B_m$, which was given earlier.

Now that the root structure for GSpin has been described explicitly, we can define GSpin as follows:

**Definition 10.7.** *For $n \geqslant 3$, we can define*

$$\mathrm{G}_n \cong \frac{\mathrm{GL}_1 \times \mathrm{Spin}_n}{\{(1,1),(-1,c)\}}$$

*where $c = \alpha_m^\vee(-1)$ if $n = 2m+1$ and $c = \alpha_{m-1}^\vee(-1)\alpha_m^\vee(-1)$ if $n = 2m$. We define* $\mathrm{GSpin}_0 = \mathrm{GSpin}_1 = \mathrm{GL}_1$*. For* $\mathrm{GSpin}_1, \mathrm{GSpin}_2$ *see* [Emo20].

**Proposition 10.8.** *The center of* $\mathrm{GSpin}_n$ *is given by*

$$Z(\mathrm{G}_n) = \begin{cases} \mathrm{A}_0 & \text{if } n = 2m+1, \\ \mathrm{A}_0 \cup (\zeta_0 A_0) & \text{if } n = 2m \end{cases}$$

*where* $\mathrm{A}_0 = \{e_0^*(\lambda) : \lambda \in \mathrm{GL}_1\}$ *and* $\zeta_0 = e_1^*(-1)e_2^*(-1)\cdots e_m^*(-1)$.

*Proof.* See [AS06, Prop. 2.3]. □

$W_F$ is a subgroup of $Gal(\overline{F}/F)$. By a representation of $W_F$ we mean a continuous homomorphism from $W_F \to GL(V)$, where $V$ is a finite-dimensional complex vector space. A representation of $W_F$, call it $\sigma$, is unramified if $\sigma$ restricted to $I_F$ is trivial, and ramified otherwise.

Now let $G_v = \mathrm{GSpin}_{2m+1}(F_v)$. Assume that $G_v$ is split. Then $\widehat{G}_v = \mathrm{GSp}_{2m}(\mathbb{C})$. The Local Langlands Conjecture gives

$$\mathrm{Irr}(G_v) \twoheadrightarrow \{\phi : WD_{F_v} \to \mathrm{GSp}_{2m}\}$$

sending $\pi_v$ to $\phi_{\pi_v}$.

Let $\pi_v$ be unramified. Conjecturally, $\phi_{\pi_v}$ is trivial on $\mathrm{SL}_2(\mathbb{C})$ and on $I_{F_v}$. Then we have that

$$\phi_{\pi_v} : I_{F_v}\backslash W_{F_v} \to \mathrm{GSp}_{2m}(\mathbb{C}).$$

From the exact sequence (**??**), $I_{F_v}\backslash W_{F_v} \cong \langle Frob\rangle \cong \mathbb{Z}$. Denote $\varpi_F$ as the uniformizer of $F_v$. Since $\pi_v$ is unramified, it is given by an unramified character $\chi$ of the maximal torus $T(F_v)$. This means that there are unramified characters $\chi_0, \chi_1, \ldots, \chi_m$ of $F_v^\times$ such that we have the following ([AS06, proof of Thm 1.1]):

$$\phi_{\pi_v}(Frob) = \begin{pmatrix} \chi_1(\varpi_F) & & & & & \\ & \ddots & & & & \\ & & \chi_m(\varpi_F) & & & \\ & & & \chi_1^{-1}(\overline{\varpi_F})\chi_0(\varpi_F) & & \\ & & & & \ddots & \\ & & & & & \chi_m^{-1}(\varpi_F)\chi_0(\varpi_F) \end{pmatrix}$$

which is a subset of $\mathrm{GSp}_{2m}(\mathbb{C})$.

Let $q$ be the cardinality of the residue field. The standard local $L$-function for $\pi_v$ is

$$\begin{aligned} L(s,\pi_v) = L(s,\phi_{\pi_v}) &= det(I - q^{-s}\phi_{\pi_v}(Frob))^{-1} \\ &= \prod_{i=1}^{m} \frac{1}{1-q^{-s}\chi_i(\varpi_F)} \frac{1}{1-q^{-s}\chi_i^{-1}\chi_0(\varpi_F)} \\ &= \prod_{i=1}^{m} L(s,\chi_i)L(s,\chi_i^{-1}\chi_0). \end{aligned}$$

## References

[AS06] Mahdi Asgari and Freydoon Shahidi. Generic transfer for general spin groups. *Duke Math. J.*, 132(1):137–190, 2006.

[Asg02] Mahdi Asgari. Local $L$-functions for split spinor groups. *Canad. J. Math.*, 54(4):673–693, 2002.

[Car89] Roger W. Carter. *Simple groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.

[Emo20] Melissa Emory. On the global Gan-Gross-Prasad conjecture for general spin groups. *Pacific J. Math.*, 306(1):115–151, 2020.

[Hal15] Brian Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[Sch85] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[Shi04] Goro Shimura. *Arithmetic and analytic theories of quadratic forms and Clifford groups*, volume 109 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[Tit66] J. Tits. Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples. *Inst. Hautes Études Sci. Publ. Math.*, (31):21–58, 1966.

Department of Mathematics, Oklahoma State University; 401 Mathematical Sciences, Oklahoma State University Stillwater, OK 74078

*Email address*: `melissa.emory@okstate.edu`