AWS EXERCISES - REPRESENTATIONS AND CHARACTERS OF p-ADIC GROUPS

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Contents

0. Introduction/disclaimer

These are a set of exercises written for the 2025 Arizona Winter School about Representations of p-adic Groups. The presentation is intended to highlight the similarities and differences when working with finite groups of Lie type and p-adic groups, as well as what happens when we consider different coefficient fields.

Be advised that the exercises are not written in a linear order! There are many backwards and forwards references. Feel free to skip around to whatever topic seems interesting. There are various hints in the footnotes, and problems that are more difficult or tangiential to the main themes are marked with a $(*)$.

1. Representation theory of finite groups

Suppose G is a finite group and C is an algebraically closed field of arbitrary characteristic which serves as the coefficient field. (Many arguments carry over to the case where C is not algebraically closed.) We stress that $\mathrm{char}(C)$ is arbitrary, unless otherwise specified.

We use the notation (π, V) (or sometimes just V) to denote representations of G: V is a C-vector space and $\pi : G \longrightarrow \mathrm{GL}(V)$ is a homomorphism. Upon choosing a basis for V, we may realize π as landing in $\mathrm{GL}_n(C)$ for some n. We write $g \cdot v$ for $\pi(g)v$.

Note: many "standard" results and constructions (Schur's lemma, Maschke's theorem, etc.) are discussed in Section 4 in the more general context of smooth representations of profinite groups. If you get stuck, try looking there for ideas!

Exercise 1.1. Prove that a representation (π, V) is the same thing a left $C[G]$-module structure on V, where $C[G]$ denotes the group algebra of G over C.

Exercise 1.2 (Frobenius reciprocity)**.** Prove the two versions of Frobenius reciprocity[1]: suppose H is a subgroup of G, W a representation of H, and V a representation of G. Then we have isomorphisms of C-vector spaces

[1]Hint: for both versions, use an appropriate version of Tensor-Hom adjunction.

(a) $\mathrm{Hom}_G(C[G] \otimes_{C[H]} W, V) \cong \mathrm{Hom}_H(W, V|_H)$;
(b) $\mathrm{Hom}_G(V, \mathrm{Hom}_{C[H]}(C[G], W)) \cong \mathrm{Hom}_H(V|_H, W)$.

(In the first (resp., second) of these isomorphisms, we view $C[G]$ as a $(C[G], C[H])$-bimodule (resp., a $(C[H], C[G])$-bimodule).)

Exercise 1.3. Let $H \leq G$ be a subgroup of G, and let (τ, W) denote a representation of H.

(a) Prove that we have an isomorphism of G-representations
$$C[G] \otimes_{C[H]} W \cong \mathrm{Hom}_{C[H]}(C[G], W),$$
which is moreover natural in W.
(b) Show that the left $C[G]$-module $C[G]$ is both projective and injective as a G-representation. Deduce that the same is true for its contragredient representation $C[G]^\vee$.
(c) Suppose (π, V) is a finite-dimensional G-representation. Show that V is projective over $C[G]$ if and only if it is injective over $C[G]$.[2]

Thus, the above two exercises show that for finite groups, we can take $\mathrm{Ind}_H^G(W)$ to be either the "tensor version" or the "Hom version," and Frobenius reciprocity works "on both sides."

Exercise 1.4. Show that the 1-dimensional representations of G correspond bijectively to 1-dimensional representations of the abelianization $G^{\mathrm{ab}} := G/[G, G]$, where $[G, G]$ denotes the subgroup generated by all commutators $[g, h] := ghg^{-1}h^{-1}, g, h \in G$.

Exercise 1.5.

(a) Suppose $G = S_3$. Classify all irreducible representations of G over C. (Split the calculation according to prime factors of $|G|$, i.e., treat the cases $\mathrm{char}(C) = 2$, $\mathrm{char}(C) = 3$, and $\mathrm{char}(C) \neq 2, 3$ separately.)
(b) Suppose $G = S_4$. Classify all irreducible representations of G over C.
(c) Suppose $G = S_5$. Classify all irreducible representations of G over C.

Try to do this "by hand" from first principles, i.e., no character theory: find vector spaces on which G acts, and decompose them (a good place to start is the natural representation of S_n on C^n). Then prove that your putative list of irreducibles is complete.

Exercise 1.6. Suppose $\mathrm{char}(C) = p$ and $|G| = p^k$. Verify directly that any homomorphism
$$\chi : G \longrightarrow C^\times$$
must be trivial. Thus the only one-dimensional representation of G is the trivial character.

Exercise 1.7. Let $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle$, and suppose $\mathrm{char}(C) = p$. Let V denote the representation defined by homomorphism
$$\begin{array}{rcl} \pi : G & \longrightarrow & \mathrm{GL}_2(C) \\ g^k & \longmapsto & \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \end{array}$$
Prove that the G-stable subspace $W = \mathrm{span}\{e_1\} \subset V$ admits no G-stable complement. Thus, the representation V is not semisimple.

Exercise 1.8 (Mackey formula)**.** Let H, K be two subgroups of a finite group G, and let (τ, W) denote a representation of H. The Mackey formula says that we have an isomorphism of K-representations
$$\mathrm{Ind}_H^G(W)|_K \cong \bigoplus_{g \in H\backslash G/K} \mathrm{Ind}_{K \cap g^{-1}Hg}^K(W^g|_{K \cap g^{-1}Hg}),$$

[2]Hint: dualize.

where (τ^g, W^g) denotes the representation of $g^{-1}Hg$ given by $\tau^g(h') = \tau(gh'g^{-1})$ for $h' \in g^{-1}Hg$. The goal of this exercise will be to prove this decomposition. (For the purposes of this exercise, we will think of $\mathrm{Ind}_H^G(W)$ as the space of functions $f : G \longrightarrow W$ satisfying the appropriate equivariance condition.)

(a) Given $g \in H\backslash G/K$, let $\mathrm{Ind}_H^{HgK}(W)$ denotes the sub-vector space of $\mathrm{Ind}_H^G(W)$ of functions $f : G \longrightarrow W$ with support contained in the double coset HgK. Show that $\mathrm{Ind}_H^{HgK}(W)$ is a K-subrepsentation of $\mathrm{Ind}_H^G(W)|_K$. (That is, show that the subspace is stable by the K-action.)

(b) Show that we have an isomorphism of K-representations
$$\mathrm{Ind}_H^G(W)|_K \cong \bigoplus_{g\in H\backslash G/K} \mathrm{Ind}_H^{HgK}(W).$$

(c) Show that we have an isomorphism of K-representations
$$\mathrm{Ind}_H^{HgK}(W) \cong \mathrm{Ind}_{K\cap g^{-1}Hg}^K(W^g|_{K\cap g^{-1}Hg}).[3]$$

Exercise 1.9 (Regular representation)**.** In this exercise we will determine the structure of the regular representation, i.e., the group algebra $C[G]$ with G acting by left multiplication, using Exercise 4.10.

(a) Suppose (τ, W) is a finite-dimensional injective representation of G. Show that
$$W \cong \bigoplus_{U\in\mathrm{Irr}(G)} \mathrm{inj}_G(U)^{\oplus n_U}$$
for some $n_U \geq 0$ (depending on W).[4]

(b) By the previous part and Exercise 1.3(b), we can write $C[G] \cong \bigoplus_{U\in\mathrm{Irr}(G)} \mathrm{inj}_G(U)^{\oplus n_U}$ for some non-negative integers n_U. Show that $n_U = \dim_C(\mathrm{Hom}_G(U, C[G]))$.[5]

(c) Conclude that
$$C[G] \cong \bigoplus_{U\in\mathrm{Irr}(G)} \mathrm{inj}_G(U)^{\oplus \dim_C(U)}$$
and
$$|G| = \sum_{U\in\mathrm{Irr}(G)} \dim_C(U)\dim_C(\mathrm{inj}_G(U)).$$

(d) Suppose $G = S_3$. Use the previous part and Exercise 1.5 to determine the structure of injective envelopes of all irreducibles when $\mathrm{char}(C) = 2$ and $\mathrm{char}(C) = 3$.

Next, let us suppose that $C = \mathbb{C}$. Recall that in this case, any representation (π, V) of G has a **character** $\boldsymbol{\chi}_\pi$, which is a class function $\boldsymbol{\chi}_\pi : G \longrightarrow \mathbb{C}$ defined by
$$\boldsymbol{\chi}_\pi(g) := \mathrm{Tr}(\pi(g)).$$

Recall that we have:

- $\boldsymbol{\chi}_{\pi\oplus\pi'} = \boldsymbol{\chi}_\pi + \boldsymbol{\chi}_{\pi'}$;
- $\boldsymbol{\chi}_{\pi\otimes\pi'} = \boldsymbol{\chi}_\pi \cdot \boldsymbol{\chi}_{\pi'}$;
- $\boldsymbol{\chi}_{\pi^\vee} = \overline{\boldsymbol{\chi}_\pi}$;
- $\boldsymbol{\chi}_{\mathrm{Hom}(\pi,\pi')} = \overline{\boldsymbol{\chi}_\pi} \cdot \boldsymbol{\chi}_{\pi'}$.

[3] Hint: for $f \in \mathrm{Ind}_H^{HgK}(W)$, define a function f' by $f'(k) = f(gk)$ for $k \in K$. Show that $f' \in \mathrm{Ind}_{K\cap g^{-1}Hg}^K(W^g|_{K\cap g^{-1}Hg})$ and that $f \longmapsto f'$ is the desired K-equivariant isomorphism.

[4] Hint: use Exercise 4.10, properties $(*_3)$ and $(*_4)$.

[5] Hint: use Exercise 4.10, property $(*_6)$.

For the last point, recall that given two representations $(\pi, V), (\pi', V')$, we let $g \in G$ act on the vector space $\mathrm{Hom}(V, V')$ by $(g \cdot f)(v) = g \cdot (f(g^{-1} \cdot v))$.

If $f, f' : G \longrightarrow \mathbb{C}$ are two functions, we define their inner product by

$$\langle f, f' \rangle_G := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)}.$$

For a reference, see [Ser77, Part I].

Exercise 1.10. Suppose $(\pi, V), (\pi', V')$ are two finite-dimensional representations of G over $\mathbb{C}$.

(a) Show that

$$\dim_{\mathbb{C}}(V^G) = \frac{1}{|G|} \sum_{g \in G} \boldsymbol{\chi}_\pi(g) = \langle 1_G, \boldsymbol{\chi}_\pi \rangle_G,$$

where 1_G denotes the trivial representation of G.[6]

(b) Show that

$$\langle \boldsymbol{\chi}_\pi, \boldsymbol{\chi}_{\pi'} \rangle_G = \dim_{\mathbb{C}}(\mathrm{Hom}_G(V, V')).$$[7]

(c) Show that V is irreducible if and only if $\langle \boldsymbol{\chi}_\pi, \boldsymbol{\chi}_\pi \rangle_G = 1$.[8]

(d) Show that if V and V' are irreducible and inequivalent, then $\langle \boldsymbol{\chi}_\pi, \boldsymbol{\chi}_{\pi'} \rangle_G = 0$.

Exercise 1.11 (Heisenberg representation)**.** Suppose p is prime. Let G be a finite p-group with center $Z \cong \mathbb{F}_p$, and such that $\mathcal{V} := G/Z$ is an abelian group of exponent p. Fix a non-trivial character $\chi : Z \hookrightarrow \mathbb{C}^\times$.

(a) Define a function $\langle -, - \rangle : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{C}^\times$ by

$$\langle \overline{x}, \overline{y} \rangle = \chi(xyx^{-1}y^{-1}),$$

where $x, y \in G$ are lifts of $\overline{x}, \overline{y} \in \mathcal{V}$. Verify the following properties:

- $\langle \overline{x_1} \cdot \overline{x_2}, \overline{y} \rangle = \langle \overline{x_1}, \overline{y} \rangle \cdot \langle \overline{x_2}, \overline{y} \rangle$,
- $\langle \overline{x}, \overline{y_1} \cdot \overline{y_2} \rangle = \langle \overline{x}, \overline{y_1} \rangle \cdot \langle \overline{x}, \overline{y_2} \rangle$,
- $\langle \overline{x}, \overline{x} \rangle = 1$,
- For every $\overline{x} \neq 1$, there exists $\overline{y} \in \mathcal{V}$ such that $\langle \overline{x}, \overline{y} \rangle \neq 1$.

Therefore, if we choose an isomorphism $\iota : \mu_p(\mathbb{C}) \xrightarrow{\sim} \mathbb{F}_p$, we see that $\iota \circ \langle -, - \rangle : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{F}_p$ defines a **symplectic form** on the $\mathbb{F}_p$-vector space $\mathcal{V}$. In particular, $\dim_{\mathbb{F}_p}(\mathcal{V})$ is even, say $2d$.

(b) Recall that a subspace $\mathcal{H} \subset \mathcal{V}$ is called **isotropic** if $\iota \circ \langle \overline{h}, \overline{h}' \rangle = 0$ for all $\overline{h}, \overline{h}' \in \mathcal{H}$. Prove that if $\mathcal{H}$ is an isotropic subspace, then $\dim_{\mathbb{F}_p}(\mathcal{H}) \leq d$, and show furthermore that there exists a subspace achieving this maximum.

(c) Suppose $\mathcal{H}$ is a maximal isotropic subspace, and let H denote the preimage of $\mathcal{H}$ in G. Show that H is abelian, and that there exists a character $\chi_H : H \longrightarrow \mathbb{C}^\times$ such that $\chi_H|_Z = \chi$.

(d) Show that if $\chi_{H,1}$ and $\chi_{H,2}$ are two characters of H extending χ as above, then $\chi_{H,1}$ and $\chi_{H,2}$ are G-conjugate.[9]

(e) Show that $\{g \in G : \chi_H^g = \chi_H\} = H$.

(f) Consider now the G-representation $\mathrm{Ind}_H^G(\chi_H)$. Prove that $\mathrm{Ind}_H^G(\chi_H)$ is irreducible.[10]

(g) Prove that $\mathrm{Ind}_H^G(\chi_H)|_Z \cong \chi^{\oplus p^d} = \chi^{\oplus |\mathcal{V}|^{1/2}}$.

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[6] Hint: define $a := \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}[G]$. Show that the action of a on V is a projection, and then compute its trace in two ways.

[7] Hint: write $\mathrm{Hom}_G(V, V') = \mathrm{Hom}(V, V')^G$ and use the previous part.

[8] Hint: use Exercises 4.7 and 4.12.

[9] Hint: use the fact that any linear map $\mathcal{H} \longrightarrow \mathbb{F}_p$ is of the form $\iota \circ \langle -, \overline{x} \rangle$ for a uniquely determined $\overline{x} \in \mathcal{V}/\mathcal{H}$.

[10] Hint: use Frobenius reciprocity and the Mackey formula.

(h) Prove that, up to isomorphism, $\mathrm{Ind}_H^G(\chi_H)$ is the unique irreducible G-representation on which Z acts by χ.[11]

(i)* Calculate the character table of G.

Remark. The result of (h) is known as the **Stone–Von Neumann Theorem**, and the representation $\mathrm{Ind}_H^G(\chi_H)$ is known as the **Heisenberg representation**. See [Bum97, Exers. 4.1.4 – 4.1.8] for a more general treatment.

2. Finite groups of Lie type – characteristic 0 coefficients

In this section and the next, we introduce the following notation: we fix a prime number p, and let G denote the group $\mathrm{GL}_2(\mathbb{F}_p)$. You should remind yourself why the group G has order $(p^2-1)(p^2-p)$.

We have the following subgroups of G:

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Here, $*$ indicates an arbitrary entry of $\mathbb{F}_p$, subject to the condition that the resulting matrix is invertible. Thus, an element of B is of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{F}_p^\times, b \in \mathbb{F}_p$. We also define Z to be the center of G; thus, we have $Z = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in \mathbb{F}_p^\times\}$.

Exercise 2.1. Calculate the conjugacy classes in G.[12] In particular:

(a) Compute the number of conjugacy classes.
(b) For each conjugacy class $\mathcal{C}$, calculate its size.
(c) For each conjugacy class $\mathcal{C}$, find a representative and calculate its order in G.
(d) Verify the class equation.
(e) Compute the number of irreducible representations of G over $\mathbb{C}$.

In the next set of exercises, we will determine all irreducible representations of $G = \mathrm{GL}_2(\mathbb{F}_p)$ over $\mathbb{C}$.

Exercise 2.2. Let $\psi : \mathbb{F}_p^\times \longrightarrow \mathbb{C}^\times$ denote a character (that is, a homomorphism).

(a) Show that as ψ ranges over characters $\mathbb{F}_p^\times \longrightarrow \mathbb{C}^\times$, the homomorphisms $\psi \circ \det : G \longrightarrow \mathbb{C}^\times$ give $p-1$ inequivalent, 1-dimensional representations of G.
(b) Show that when $p > 2$, this gives all 1-dimensional representations of G.[13]
(c) What happens for $p = 2$?

Now fix two characters $\chi_1, \chi_2 : \mathbb{F}_p^\times \longrightarrow \mathbb{C}^\times$, and let $\chi := \chi_1 \boxtimes \chi_2 : B \longrightarrow \mathbb{C}^\times$ denote the character given by

$$(\chi_1 \boxtimes \chi_2)\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \chi_1(a)\chi_2(d).$$

We then consider the representation $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$. (Recall that G acts by right translation of functions.)

Exercise 2.3 (Principal series representations)**.** Let $\chi_1, \chi_2, \psi_1, \psi_2 : \mathbb{F}_p^\times \longrightarrow \mathbb{C}^\times$ denote characters.

(a) Calculate the dimension of $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$.
(b) Calculate the central character of $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$ (cf. Exercise 4.13).

[11] Hint: if V is such an irreducible representation, then $V|_H$ is a sum of characters, all of which extend χ, and all of which are G-conjugate.

[12] Hint: rational canonical form/Jordan normal form may be helpful here.

[13] Hint: use Exercise 1.4.

(c) Consider the space
$$\mathrm{Hom}_G\left(\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2),\ \mathrm{Ind}_B^G(\psi_1 \boxtimes \psi_2)\right).$$
Show that this space is nonzero if and only if $\{\chi_1, \chi_2\} = \{\psi_1, \psi_2\}$.[14]

(d) Show that the spaces
$$\mathrm{Hom}_G\left(\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2),\ \mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)\right) \qquad \text{and} \qquad \mathrm{Hom}_G\left(\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2),\ \mathrm{Ind}_B^G(\chi_2 \boxtimes \chi_1)\right)$$
both have dimension 1 if $\chi_1 \neq \chi_2$ and have dimension 2 otherwise.

(e) Deduce that when $\chi_1 \neq \chi_2$, the representation $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$ is irreducible, and we have an isomorphism $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2) \cong \mathrm{Ind}_B^G(\chi_2 \boxtimes \chi_1)$. Also deduce that when $\chi_1 = \chi_2$, then $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$ is a direct sum of two inequivalent irreducible representations. (In particular, the semisimplification does not change when the characters are interchanged.)

Thus, unordered pairs of distinct characters $\{\chi_1, \chi_2\}$ give rise to pairwise inequivalent irreducible representation $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$. These are known as **principal series representations**. We therefore obtain $(p-1)(p-2)/2 = \binom{p-1}{2}$ representations in this way.

In fact, it is also easy to describe the reducible case. Suppose $\chi_1 = \chi_2 =: \chi$, and note that we have
$$\mathrm{Ind}_B^G(\chi \boxtimes \chi) \cong \mathrm{Ind}_B^G(1_B) \otimes \chi \circ \det,$$
where 1_B denotes the trivial character of B. The constant functions in $\mathrm{Ind}_B^G(1_B)$ give a copy of the trivial G-representation 1_G, and we let
$$\mathrm{St}_B^G := \mathrm{Ind}_B^G(1_B)/1_G$$
denote the **Steinberg representation**. (Note that since we work over $\mathbb{C}$, we have $\mathrm{Ind}_B^G(1_B) \cong 1_G \oplus \mathrm{St}_B^G$.)

Exercise 2.4.

(a) Show that the representations
$$\chi \circ \det, \qquad \mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2), \qquad \mathrm{St}_B^G \otimes \chi \circ \det$$
as $\chi, \chi_1, \chi_2, (\chi_1 \neq \chi_2)$ range over characters of $\mathbb{F}_p^\times$, are pairwise non-isomorphic.

(b) Let 1_U denote the trivial character of U. Determine the structure of the B-representation $\mathrm{Ind}_U^B(1_U)$.

(c) Suppose (π, V) is an irreducible representation of G. Prove that the following are equivalent:[15]
   (i) V is isomorphic to a representation from part (a);
   (ii) $V^U \neq 0$;
   (iii) $V_U \neq 0$.

The representations of G satisfying $V_U = 0$ (or, equivalently, $V^U = 0$) are called **cuspidal representations**. By the above exercise, we are left with classifying the cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_p)$.

Exercise 2.5 (Cuspidal representations)**.** Let $\mathbb{F}_{p^2}$ denote the quadratic extension of $\mathbb{F}_p$, and identify $\mathbb{F}_{p^2}^\times$ with a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. (For example, we have $\mathbb{F}_{p^2} \cong \mathbb{F}_p^{\oplus 2}$ as $\mathbb{F}_p$-vector spaces, and multiplication by $\mathbb{F}_{p^2}^\times$ defines an $\mathbb{F}_p$-linear map.) Fix a character $\theta : \mathbb{F}_{p^2}^\times \longrightarrow \mathbb{C}^\times$ satisfying $\theta^p \neq \theta$ (these characters are called **regular**), and a character $\psi : U \longrightarrow \mathbb{C}^\times$ satisfying $\psi \neq 1$. We first consider the representations
$$\mathrm{Ind}_{ZU}^G(\theta \boxtimes \psi) \qquad \text{and} \qquad \mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^G(\theta).$$
of G.

[14] Hint: use Frobenius reciprocity (Exercise 1.2) and the Mackey formula (Exercise 1.8).

[15] Hint: note that $\mathrm{Ind}_U^G = \mathrm{Ind}_B^G \circ \mathrm{Ind}_U^B$ and use the previous part.

(a) Prove that if ψ' is another character of U satisfying $\psi' \neq 1$, then we have $\mathrm{Ind}_{ZU}^{G}(\theta \boxtimes \psi) \cong \mathrm{Ind}_{ZU}^{G}(\theta \boxtimes \psi')$.[16]

(b) Calculate

$$\langle \boldsymbol{\chi}_{\mathrm{Ind}_{ZU}^{G}(\theta\boxtimes\psi)}, \boldsymbol{\chi}_{\mathrm{Ind}_{ZU}^{G}(\theta\boxtimes\psi)} \rangle_G.$$

It may be useful to use Frobenius reciprocity and the formula for the character of an induced representation (see [Ser77, Part I, Thm. 12]). Also, recall that if $\alpha : A \longrightarrow \mathbb{C}^\times$ is a nontrivial character of an abelian group A, then $\sum_{a\in A} \alpha(a) = 0$.

(c) Calculate

$$\langle \boldsymbol{\chi}_{\mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{G}(\theta)}, \boldsymbol{\chi}_{\mathrm{Ind}_{ZU}^{G}(\theta\boxtimes\psi)} \rangle_G \qquad \text{and} \qquad \langle \boldsymbol{\chi}_{\mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{G}(\theta)}, \boldsymbol{\chi}_{\mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{G}(\theta)} \rangle_G.$$

For this, use that if $a, b \in \mathbb{F}_{p^2}^\times \subset G$ are two elements which do not lie in Z, then a and b are G-conjugate if and only if they have the same characteristic polynomial, if and only if $b \in \{a, a^p\}$.

Now consider the virtual representation $(\pi_\theta, V_\theta) := \mathrm{Ind}_{ZU}^{G}(\theta \boxtimes \psi) - \mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{G}(\theta)$. Recall that this is a formal $\mathbb{Z}$-linear combination of irreducible representations, with character $\boldsymbol{\chi}_{\pi_\theta} = \boldsymbol{\chi}_{\mathrm{Ind}_{ZU}^{G}(\theta\boxtimes\psi)} - \boldsymbol{\chi}_{\mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{G}(\theta)}$. By part (a) above, up to isomorphism V_θ is independent of the choice of ψ.

(d) Show that $\langle \boldsymbol{\chi}_{\pi_\theta}, \boldsymbol{\chi}_{\pi_\theta} \rangle_G = 1$ and $\boldsymbol{\chi}_{\pi_\theta}(1) = p-1$. Deduce from this that V_θ is a genuine irreducible representation of G.

(e) Verify the following character values of $\boldsymbol{\chi}_{\pi_\theta}$:

$$\begin{array}{rcll} \boldsymbol{\chi}_{\pi_\theta}(z) & = & (p-1)\theta(z) & \text{if } z \in Z, \\ \boldsymbol{\chi}_{\pi_\theta}(zu) & = & -\theta(z) & \text{if } z \in Z, u \in U \smallsetminus \{1\}, \\ \boldsymbol{\chi}_{\pi_\theta}(a) & = & -\theta(a) - \theta(a^p) & \text{if } a \in \mathbb{F}_{p^2}^\times \smallsetminus Z, \\ \boldsymbol{\chi}_{\pi_\theta}(g) & = & 0 & \text{if } g \text{ is not conjugate to an element of } ZU \cup \mathbb{F}_{p^2}^\times. \end{array}$$

Use the above to show that if θ, θ' are two regular characters, then $V_\theta \cong V_{\theta'}$ if and only if $\theta' \in \{\theta, \theta^p\}$.

(f) Show that $\sum_{u\in U} \boldsymbol{\chi}_{\pi_\theta}(u) = 0$, and deduce that V_θ is cuspidal. (In fact, when $p > 2$, cuspidality follows from dimension considerations.)

(g) Consider the representations

$$\chi \circ \det, \quad \mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2), \quad \mathrm{St}_B^G \otimes \chi \circ \det, \quad V_\theta,$$

where $\chi, \chi_1, \chi_2, (\chi_1 \neq \chi_2)$ range over characters of $\mathbb{F}_p^\times$, where θ is a regular character of $\mathbb{F}_{p^2}^\times$, and where we identify $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2) \cong \mathrm{Ind}_B^G(\chi_2 \boxtimes \chi_1)$ and $V_\theta \cong V_{\theta^p}$. Show that this list constitutes all (isomorphism classes of) irreducible representations of G over $\mathbb{C}$.[17]

Important Remark. Suppose $\mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$. Then the exercises above work in exactly the same way for the group $\mathrm{GL}_2(\mathbb{F}_q)$; it suffices to "replace p with q everywhere."

Exercise 2.6.

(a) Recall that the group S_3 is isomorphic to $\mathrm{GL}_2(\mathbb{F}_2)$. Match the irreducibles from Exercise 1.5(a) with the irreducibles from Exercise 2.5(g).

(b) Repeat the above using the exceptional isomorphism $S_4 \cong \mathrm{PGL}_2(\mathbb{F}_3)$ and Exercise 1.5(b).

(c) Repeat the above using the exceptional isomorphism $S_5 \cong \mathrm{PGL}_2(\mathbb{F}_5)$ and Exercise 1.5(c).

(d)* Try the same thing with other exceptional isomorphisms, e.g., for alternating groups. See Wikipedia.

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[16] Hint: first show that any non-trivial character ψ' is of the from $\psi'(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)) = \psi(\left(\begin{smallmatrix} 1 & ax \\ 0 & 1 \end{smallmatrix}\right))$ for some $a \in \mathbb{F}_p^\times$. Then show $f \longmapsto f'$ where $f'(g) = f(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) g)$ gives the required isomorphism.

[17] Hint: the number of irreducible complex representations of a finite group is equal to the number of conjugacy classes.

Exercise 2.7 (Deligne–Lusztig representations)**.** Here is another construction of cuspidal representations of G. Consider the affine variety V over $\overline{\mathbb{F}}_p$ given by the equation

$$\begin{vmatrix} X & X^p \\ Y & Y^p \end{vmatrix}^{p-1} = (XY^p - X^pY)^{p-1} = 1.$$

This is a variant of the famous **Drinfeld curve**.

(a) Show that V decomposes into irreducible components as $\mathsf{V} = \bigsqcup_{a \in \mathbb{F}_p^\times} \mathsf{V}_a$, where V_a is defined by the equation $XY^p - X^pY = a$.
(b) Show that V is smooth.
(c) Show that the group G acts on V via $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \longmapsto g\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$, where $g \in G$.
(d) Show that the group $\mathbb{F}_{p^2}^\times$ acts on V via $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \longmapsto \left(\begin{smallmatrix} \lambda X \\ \lambda Y \end{smallmatrix}\right)$, where $\lambda \in \mathbb{F}_{p^2}^\times$. Moreover, show that this action commutes with the action of G.
(e) Show that the map $\mathsf{V} \longrightarrow \mathbb{P}^1$ given by $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \longmapsto [X : Y]$ defines a G-equivariant isomorphism of varieties $\mathsf{V}/\mathbb{F}_{p^2}^\times \xrightarrow{\sim} \mathbb{P}^1 \smallsetminus \mathbb{P}^1(\mathbb{F}_p)$.
(f) Let $a \in \mathbb{F}_p^\times$. Show that the maps $\mathsf{V}_a \longrightarrow \mathbb{A}^1$ given by $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \longmapsto Y$ define an $\mathbb{F}_{p^2}^\times$-equivariant isomorphism of varieties $\mathsf{V}/U \xrightarrow{\sim} \bigsqcup_{a \in \mathbb{F}_p^\times} \mathbb{A}^1 \smallsetminus \{0\}$. The action of $\lambda \in \mathbb{F}_{p^2}^\times$ on the right-hand side sends Z in the a-component to λZ in the $a\mathrm{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\lambda)$-component.

Let ℓ denote a prime different from p. We can associate to V its compactly supported, ℓ-adic étale cohomology groups $H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell)$. (See [Bon11, App. A] or [DM91, §10] for a quick overview.) These are finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces, which, by functoriality, obtain commuting actions of G and $\mathbb{F}_{p^2}^\times$.

(g) Show that $H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell) = 0$ if $i = 0$ or $i > 2$.[18]
(h) Show that as a representation of $G \times \mathbb{F}_{p^2}^\times$, we have

$$H_c^2(\mathsf{V}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\chi : \mathbb{F}_p^\times \to \overline{\mathbb{Q}}_\ell^\times} \chi \circ \det \boxtimes \chi \circ \mathrm{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}.$$ [19]

Let us now define $H_c^*(\mathsf{V}, \overline{\mathbb{Q}}_\ell) := \sum_{i \geq 0} (-1)^i H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell)$, which is a virtual representation of $G \times \mathbb{F}_{p^2}^\times$. By part (g), we have

$$H_c^*(\mathsf{V}, \overline{\mathbb{Q}}_\ell) = -H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell) + H_c^2(\mathsf{V}, \overline{\mathbb{Q}}_\ell).$$

Since $\mathbb{F}_{p^2}^\times$ is abelian, any finite-dimensional representation is semisimple. Given a character $\theta : \mathbb{F}_{p^2}^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$, we define $V'_\theta := H_c^*(\mathsf{V}, \overline{\mathbb{Q}}_\ell)[\theta]$ to be the alternating sum of the θ-isotypic components of the $H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell)$. This gives a virtual representation of G.

(i) Suppose θ is regular. Show that $V'_\theta = -H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell)[\theta]$.
(j) Suppose that $\theta = 1_{\mathbb{F}_{p^2}^\times}$ is the trivial character. Using the fact that $H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell)^{\mathbb{F}_{p^2}^\times} = H_c^i(\mathsf{V}/\mathbb{F}_{p^2}^\times, \overline{\mathbb{Q}}_\ell)$ ([Bon11, Eq. A.2.3]), the excision long exact sequence ([Bon11, Thm. A.2.1(d)]) and part (e), show that $H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell)[1_{\mathbb{F}_{p^2}^\times}] \cong \mathrm{St}_B^G$, and conclude that $V'_{1_{\mathbb{F}_{p^2}^\times}} \cong -\mathrm{St}_B^G + 1_G$. Similarly, if θ is of the form $\chi \circ \mathrm{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}$ for some character χ of $\mathbb{F}_p^\times$, we have $V'_{\chi \circ \mathrm{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}} \cong (-\mathrm{St}_B^G + 1_G) \otimes \chi \circ \det$.
(k) For $(g, \lambda) \in G \times \mathbb{F}_{p^2}^\times$, we define the trace of (g, λ) by

$$\mathrm{Tr}_{\mathsf{V}}(g, \lambda) := \sum_{i \geq 0} (-1)^i \mathrm{Tr}((g, \lambda) | H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell)).$$

[18] Hint: see [Bon11, Thm. A.2.1(b)].
[19] Hint: see [Bon11, Thm. A.2.1(c)].

The fixed-point formula of Deligne–Lusztig [DL76, Thm. 3.2] states that if s and u are two commuting elements with order prime to p, respectively a power of p, then

$$\mathrm{Tr}_{\mathsf{V}}(su) = \mathrm{Tr}_{\mathsf{V}^s}(u).$$

Show that if $\lambda \neq 1$, then $\mathrm{Tr}_{\mathsf{V}}(1, \lambda) = 0$. Use this to deduce that as a virtual representation of $\mathbb{F}_{p^2}^\times$, $H_c^*(\mathsf{V}, \overline{\mathbb{Q}}_\ell)$ is a multiple of the regular representation $\overline{\mathbb{Q}}_\ell[\mathbb{F}_{p^2}^\times]$.

(l) Let θ denote a regular character. Use the previous part to deduce that

$$\dim_{\overline{\mathbb{Q}}_\ell}(H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell)[\theta]) = \dim_{\overline{\mathbb{Q}}_\ell}(-V'_\theta) = \dim_{\overline{\mathbb{Q}}_\ell}(-V'_{1_{\mathbb{F}_{p^2}^\times}}) = p - 1.$$

(m) Finally, we verify that $H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell)[\theta]$ is cuspidal by showing it does not contain the trivial character of U. Using the fact that $H_c^i(\mathsf{V}, \overline{\mathbb{Q}}_\ell)^U \cong H_c^i(\mathsf{V}/U, \overline{\mathbb{Q}}_\ell)$, the excision long exact sequence, and part (f), show that we have

$$H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell)^U \cong \bigoplus_{\chi : \mathbb{F}_p^\times \to \overline{\mathbb{Q}}_\ell^\times} \chi \circ \mathrm{N}_{\mathbb{F}_{p^2}/\mathbb{F}_p}$$

as $\mathbb{F}_{p^2}^\times$-representations. Conclude that $H_c^1(\mathsf{V}, \overline{\mathbb{Q}}_\ell)[\theta]^U = 0$, which gives the desired cuspidality. (Note that by using dimension counting, we also deduce irreducibility.)

Remark. It is also possible to construct the cuspidal representations of G by means of the Weil representation (since $\mathrm{GL}_2 \cong \mathrm{GSp}_2$). See [Bum97, §4.1] for an exposition.

Exercise 2.8 (Steinberg representation in characteristic ℓ)**.** Let $\mathbb{P}^1(\mathbb{F}_p)$ denote the projective line over $\mathbb{F}_p$, endowed with an action of G by Möbius transformations: if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $[X : Y] \in \mathbb{P}^1(\mathbb{F}_p)$, we define $g[X : Y] = [aX + bY : cX + dY]$. Let $V := C(\mathbb{P}^1(\mathbb{F}_p), \mathbb{C})$ denote the space of $\mathbb{C}$-valued functions, and endow V with a linear G-action via $(g \cdot f)([X : Y]) = f(g^{-1}[X : Y])$ for $g \in G, f \in V$.

(a) Show that $V \cong \mathrm{Ind}_B^G(1_B)$.

(b) Show that the subspace of constant functions gives the trivial subrepresentation $1_G \subset V$. Thus, the Steinberg representation St_B^G may be identifed as $C(\mathbb{P}^1(\mathbb{F}_p), \mathbb{C})/\{\text{constant functions}\}$.

(c) Suppose now that ℓ is a prime number dividing $p + 1$. (Thus ℓ divides $|G|$, and Maschke's theorem does not apply.) Let $V_\ell := C(\mathbb{P}^1(\mathbb{F}_p), \overline{\mathbb{F}}_\ell)$, and define a map

$$\begin{aligned} \Sigma : V_\ell &\longrightarrow \overline{\mathbb{F}}_\ell \\ f &\longmapsto \sum_{v \in \mathbb{P}^1(\mathbb{F}_p)} f(v) \end{aligned}$$

Show that the map Σ is surjective and G-equivariant, if we endow $\overline{\mathbb{F}}_\ell$ with the trivial G action.

(d) Show that the subspace of constant functions lies in the kernel of Σ, so that Σ factors as

$$\begin{tikzcd} V_\ell \arrow[rr, two heads, "\Sigma"] \arrow[dr] & & 1_G \\ & \mathrm{St}_B^G \arrow[ur, two heads] & \end{tikzcd}$$

Deduce that the Steinberg representation is not irreducible in characteristic ℓ dividing $p + 1$.

(e)* Show that $0 \subset 1_G \subset \ker(\Sigma) \subset V_\ell$ is the unique composition series of V_ℓ.[20]

Exercise 2.9 (Whittaker model)**.** Suppose C is an algebraically closed field satisfying $\mathrm{char}(C) \neq p$, and fix a non-trivial character $\psi : U \longrightarrow C^\times$. We examine the **Gelfand–Graev representation** $\mathrm{Ind}_U^G(\psi)$.

(a) Show that if ψ' is any other non-trivial character of U, then $\mathrm{Ind}_U^G(\psi') \cong \mathrm{Ind}_U^G(\psi)$.

[20] Hint: use Exercise 2.9(e).

(b) Show that $\mathrm{Ind}_U^G(\psi)$ is an injective object in $C[G]-\mathrm{Mod}$, and therefore that $V \longmapsto \mathrm{Hom}_G(V, \mathrm{Ind}_U^G(\psi))$ is an exact functor. (The dual of this functor is often called the **twisted Jacquet module**.)
(c) Suppose (π, V) is an irreducible representation of G over C.
  (i) Show that $V = \chi \circ \det$ for some character $\chi : \mathbb{F}_p^\times \longrightarrow C^\times$ if and only if $V|_U \cong 1_U^{\oplus n}$ for some $n \geq 0$.
  (ii) Show that $V|_U \cong 1_U^{\oplus n}$ if and only if $\mathrm{Hom}_G(V, \mathrm{Ind}_U^G(\psi)) = 0$.[21]
  Thus, $\mathrm{Hom}_G(V, \mathrm{Ind}_U^G(\psi)) = 0$ if and only if $V \cong \chi \circ \det$.
(d) Let $\chi_1, \chi_2 : \mathbb{F}_p^\times \longrightarrow C^\times$ be two characters. Calculate

$$\dim_C \left(\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2), \mathrm{Ind}_U^G(\psi))\right) \qquad \text{and} \qquad \dim_C \left(\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2), \mathrm{Ind}_U^G(1_U))\right).$$ [22]

(e) Using parts (b) and (d), conclude that $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$ has length at most 3 as a G-representation.

Remark. In fact, the Gelfand–Graev representation $\mathrm{Ind}_U^G(\psi)$ satisfies

$$\dim_C \left(\mathrm{Hom}_G(V, \mathrm{Ind}_U^G(\psi))\right) \leq 1$$

for all irreducible representations V. This holds more generally for connected reductive groups over finite fields; see [DL76, §10] and [DM91, Ch. 14] for the case $C = \mathbb{C}$. The case $\mathrm{char}(C) \neq p$ then follows from properties of the ***cde* triangle** (cf. [Ser77, Ch. 15]).

Exercise 2.10 (Finite Hecke algebras)**.** Let $C = \mathbb{C}$ and consider the algebra $\mathrm{End}_G(\mathrm{Ind}_B^G(1_B))$. (This appears as a finite-dimensional subalgebra of the **Iwahori–Hecke algebra**; see Exercise 6.12.)

(a) Show that we may identify $\mathrm{End}_G(\mathrm{Ind}_B^G(1_B))$ with $\mathcal{H}(G, B, 1_B) = \mathrm{Ind}_B^G(1_B)^B$, the B-bi-invariant functions on G. Show also that the composition product on $\mathrm{End}_G(\mathrm{Ind}_B^G(1_B))$ corresponds to the convolution product on $\mathrm{Ind}_B^G(1_B)^B$.
(b) Write down a basis for $\mathrm{Ind}_B^G(1_B)^B$, and compute all possible products of the basis elements. (This yields a presentation of the algebra $\mathrm{End}_G(\mathrm{Ind}_B^G(1_B))$.)[23]
(c) Suppose (π, V) is a representation of G. Derive the formula for the right action of $\mathrm{Ind}_B^G(1_B)^B$ on a vector $v \in V^B$.[24]
(d) Calculate the right action of $\mathrm{Ind}_B^G(1_B)^B$ on V^B, where (π, V) is the trivial representation 1_G, the Steinberg representation St_B^G, and the principal series $\mathrm{Ind}_B^G(1_B)$.
(e)* Repeat the above exercise with B replaced by U.

Remark. In the previous exercise, the algebra $\mathrm{End}_G(\mathrm{Ind}_B^G(1_B))$ was commutative. This is a low-dimensional coincidence; the analogously defined algebras for $\mathrm{GL}_n(\mathbb{F}_p)$ are non-commutative for $n \geq 3$. Similarly, the algebra $\mathrm{End}_G(\mathrm{Ind}_U^G(1_U))$ of part (e)* is not commutative.

3. Finite groups of Lie type – characteristic p coefficients

We maintain the notation of the previous section, so that $G = \mathrm{GL}_2(\mathbb{F}_p)$, B is the Borel subgroup of upper triangular matrices, etc. In this section, we focus on the case of representations in "natural" or "defining" characteristic, namely $\mathrm{char}(C) = p$. For concreteness, we take $C = \overline{\mathbb{F}}_p$. Keep in mind that the category of G-representations over $\overline{\mathbb{F}}_p$ is not semisimple!

Exercise 3.1. This exercise will give the classification of mod p representations of $G = \mathrm{GL}_2(\mathbb{F}_p)$. For $0 \leq r \leq p-1$ and $0 \leq s < p-1$, define

$$V_{r,s} := \mathrm{Sym}^r(\overline{\mathbb{F}}_p^{\oplus 2}) \otimes \det{}^s.$$

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[21] Hint: use part (a).
[22] Hint: use Exercise 1.8 and the fact that any representation of U over C is semisimple.
[23] Hint: use the Bruhat decomposition.
[24] Hint: translate both v and $f \in \mathrm{Ind}_B^G(1_B)^B$ into the corresponding Hom spaces using Frobenius reciprocity.

Recall that we may view $V_{r,s}$ as the space $\overline{\mathbb{F}}_p[X,Y]_r$ of homogeneous polynomials of degree r, with action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X^iY^{r-i} = (aX+cY)^i(bX+dY)^{r-i}(ad-bc)^s.$$

(a) Prove that $V_{r,s}^U = \mathrm{span}\{X^r\}$.
(b) Show that the subrepresentation generated by X^r is equal to all of $V_{r,s}$.[25]
(c) Use the previous two points to show that $V_{r,s}$ is irreducible as a G-representation.[26]
(d) Prove that $V_{r,s} \cong V_{r',s'}$ if and only if $(r,s) = (r',s')$.
(e) Prove that any irreducible G-representation is isomorphic to one and only one $V_{r,s}$.[27]

Important Remark. Suppose $f \geq 2$. Then it is **not** the case that the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_{p^f})$ are obtained in the same way as those for $\mathrm{GL}_2(\mathbb{F}_p)$ by "replacing p by p^f everywhere." To see this, notice that the subspace $\mathrm{span}\{X^p, Y^p\}$ of $\mathrm{Sym}^p(\overline{\mathbb{F}}_p^{\oplus 2})$ is stable by the action of $\mathrm{GL}_2(\mathbb{F}_{p^f})$, and therefore $\mathrm{Sym}^p(\overline{\mathbb{F}}_p^{\oplus 2})$ is not irreducible.

The classification looks as follows. Fix an inclusion $\mathbb{F}_{p^f} \subset \overline{\mathbb{F}}_p$, and for $0 \leq i \leq f-1$ and $0 \leq r$, we let $\mathrm{Sym}^r(\overline{\mathbb{F}}_p^{\oplus 2})^{(p^i)}$ denote the vector space $\mathrm{Sym}^r(\overline{\mathbb{F}}_p^{\oplus 2})$ on which the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_{p^f})$ acts by $\left(\begin{smallmatrix} a^{p^i} & b^{p^i} \\ c^{p^i} & d^{p^i} \end{smallmatrix}\right)$. Then any irreducible representation of $\mathrm{GL}_2(\mathbb{F}_{p^f})$ is of the form

$$\mathrm{Sym}^{r_0}(\overline{\mathbb{F}}_p^{\oplus 2}) \otimes \mathrm{Sym}^{r_1}(\overline{\mathbb{F}}_p^{\oplus 2})^{(p)} \otimes \ldots \otimes \mathrm{Sym}^{r_{f-1}}(\overline{\mathbb{F}}_p^{\oplus 2})^{(p^{f-1})} \otimes \det{}^s,$$

where $0 \leq r_0, r_1, \ldots, r_{f-1} \leq p-1$ and $0 \leq s < p^f - 1$. Moreover, these representations are pairwise non-isomorphic. These statements may be proved in a similar manner as in Exercise 3.1, and may be seen as a consequence of **Steinberg's tensor product theorem** [Jan03, Part II, Cor. 3.17].

Exercise 3.2 (Principal series in characteristic p)**.** In this exercise we determine the structure of principal series representations in characteristic p.

Fix $0 \leq r \leq p-1$ and $0 \leq s < p-1$, and let $V_{r,s} = \mathrm{Sym}^r(\overline{\mathbb{F}}_p^{\oplus 2}) \otimes \det^s$ be the irreducible representation of G from the previous exercise.

(a) Use Exercise 3.1(a) to calculate the action of T on $V_{r,s}^U$.
(b) Determine the space of coinvariants $(V_{r,s})_U$ and calculate the action of T on this space.

Now suppose $\chi : B \longrightarrow \overline{\mathbb{F}}_p^\times$ is a character given by $\chi\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) = a^id^j$, where we take $0 \leq i, j < p-1$, and consider the principal series representation $\mathrm{Ind}_B^G(\chi)$. Suppose first that $i \neq j$, and let $[i-j]$ denote the unique integer satisfying $0 < [i-j] < p-1$ and $[i-j] \equiv i-j \pmod{p-1}$.

(c) Show that we have an injection $V_{p-1-[i-j],i} \hookrightarrow \mathrm{Ind}_B^G(\chi)$ and a surjection $\mathrm{Ind}_B^G(\chi) \twoheadrightarrow V_{[i-j],j}$.[28]
(d) Show that we have a non-split short exact sequence

$$0 \longrightarrow V_{p-1-[i-j],i} \longrightarrow \mathrm{Ind}_B^G(\chi) \longrightarrow V_{[i-j],j} \longrightarrow 0.$$[29]

(e) Let $\chi^s : B \longrightarrow \overline{\mathbb{F}}_p^\times$ denote the character $\chi^s(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)) = a^jd^i$. Show that $\mathrm{Ind}_B^G(\chi^s)$ and $\mathrm{Ind}_B^G(\chi)$ have the same Jordan–Hölder factors but $\mathrm{Ind}_B^G(\chi^s) \not\cong \mathrm{Ind}_B^G(\chi)$.

Suppose now that $i = j$. Twisting by a character, we can assume $i = j = 0$. Thus, we consider the representation $\mathrm{Ind}_B^G(1_B)$.

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[25] Hint: use the action of $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ and a Vandermonde determinant.

[26] Hint: suppose V' is a nontrivial subrepresentation of $V_{r,s}$, and use the p-groups lemma.

[27] Hint: the number of irreducible representations of a finite group in characteristic p is equal to the number of p-regular conjugacy classes (i.e., those which consist of elements of order prime to p). Alternatively, use the results of Exercise 3.2.

[28] Hint: use the previous two parts and Frobenius reciprocity.

[29] Hint: count dimensions.

(f) Show that we have injections $1_G = V_{0,0} \hookrightarrow \mathrm{Ind}_B^G(1_B)$ and $V_{p-1,0} \hookrightarrow \mathrm{Ind}_B^G(1_B)$.
(g) Deduce that $\mathrm{Ind}_B^G(1_B) \cong 1_G \oplus V_{p-1,0}$ and that $\mathrm{St}_B^G \cong V_{p-1,0}$.

Remark. The above exercise shows that, contrary to the case of characteristic 0 coefficients, principal series in characteristic p are *always* reducible. (This also follows for dimension reasons.) We also point out that the short exact sequence of the previous exercise no longer holds for $\mathrm{GL}_2(\mathbb{F}_{p^f})$; for "generic" χ, the principal series $\mathrm{Ind}_B^G(\chi)$ has length 2^f. For a summary of these results, see [BP12, §2].

Exercise 3.3 (Injectivity of St_B^G)**.** In this exercise, we will show that the Steinberg representation St_B^G and its twists are the only irreducible representations of G which are injective in characterstic p.

(a) Suppose first that H is a finite group and P is a Sylow p-subgroup of H. Let (π, V) be a representation of H over $\overline{\mathbb{F}}_p$. Show that V is projective as an H-representation if and only if $V|_P$ is projective as a P-representation.[30]
(b) In the setting of the previous part, show that if (π, V) is a projective H-representation, then $\dim_{\overline{\mathbb{F}}_p}(V)$ is a multiple of $|P|$. More precisely, show that $\dim_{\overline{\mathbb{F}}_p}(V) = |P| \cdot \dim_{\overline{\mathbb{F}}_p}(V^P)$.[31]
(c) Now suppose (π, V) is an irreducible representation of $G = \mathrm{GL}_2(\mathbb{F}_p)$ which is not isomorphic to a twist of St_B^G. Show that V is not projective as a G-representation.[32]
(d) Let $U^- := \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ denote the subgroup of lower triangular matrices. Fix a nonzero vector $v \in (\mathrm{St}_B^G)^U$, and consider the map

$$\begin{array}{ccc} \overline{\mathbb{F}}_p[U^-] & \longrightarrow & \mathrm{St}_B^G|_{U^-} \\ \displaystyle\sum_{u \in U^-} c_u u & \longmapsto & \displaystyle\sum_{u \in U^-} c_u u \cdot v. \end{array}$$

Show that the map is an isomorphism of U^--representations, and deduce that St_B^G is projective as a G-representation.[33]
(e) Prove that St_B^G is injective as a G-representation.[34]

Remark. It is true more generally that if G' is the group of $\mathbb{F}_q$-points of a connected semisimple group, then the Steinberg representation is the only irreducible G'-representation which is injective (equivalently, projective). See [Hum06, §§9.1 – 9.3].

Exercise 3.4 (Injective envelopes of Serre weights)**.** Let $V_{r,s}$ be an irreducible representation of G, and suppose $0 < r < p - 1$. In particular, we assume $p \geq 3$.

(a) Define a character $\chi : B \longrightarrow \overline{\mathbb{F}}_p^\times$ by $\chi(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)) = a^s d^{r+s}$. Show that $\mathrm{soc}_G(\mathrm{Ind}_B^G(\chi)) = V_{r,s}$, and deduce that we have an injection $\mathrm{Ind}_B^G(\chi) \hookrightarrow \mathrm{inj}_G(V_{r,s})$.
(b) Show that $\dim_{\overline{\mathbb{F}}_p}(\mathrm{inj}_G(V_{0,s})) \geq p$ and $\dim_{\overline{\mathbb{F}}_p}(\mathrm{inj}_G(V_{r,s})) \geq 2p$.[35]
(c) Use the previous part and Exercise 1.9(c) to show that we actually have equalities $\dim_{\overline{\mathbb{F}}_p}(\mathrm{inj}_G(V_{0,s})) = p$ and $\dim_{\overline{\mathbb{F}}_p}(\mathrm{inj}_G(V_{r,s})) = 2p$.
(d) Show that $\mathrm{cosoc}_G(\mathrm{Ind}_B^G(\chi^s)) = V_{r,s}$, and deduce that we have a surjection $\mathrm{inj}_G(V_{r,s}) \twoheadrightarrow \mathrm{Ind}_B^G(\chi^s)$.[36]

[30] Hint: for the "if" direction, suppose $W \twoheadrightarrow V$ is an H-equivariant surjection, and let $\sigma_0 : V|_P \longrightarrow W|_P$ denote a P-equivariant section. Show that $\sigma(v) := \frac{1}{|H/P|} \sum_{h \in H/P} h \cdot \sigma_0(h^{-1} \cdot v)$ is an H-equivariant section.
[31] Hint: use the previous part and that $\overline{\mathbb{F}}_p[P]$ is a local ring.
[32] Hint: there are several ways to see this, using either the current exercise or Exercise 3.2.
[33] Hint: proceed as in Exercise 3.1(b) and use part (a).
[34] Hint: use Exercise 1.3(c).
[35] Hint: use Exercise 3.3(b).
[36] Hint: note that $\mathrm{inj}_G(V_{r,s})$ is also the *projective cover* of $V_{r,s}$, cf. [Ser77, Exer. 14.6].

(e) Suppose $r = p-2$. Use the previous parts to show that the submodule structure on $\mathrm{inj}_G(V_{p-2,s})$ is given by the diagram

$$V_{p-2,s} \longleftarrow V_{1,p-2+s} \longleftarrow V_{p-2,s}.$$

To proceed further, we need the following fact: supposing $0 \leq r < p-2$, there exists a representation V_{c} of dimension $p-1$ which fits into a non-split short exact sequence

$$0 \longrightarrow V_{r,s} \longrightarrow V_{\mathrm{c}} \longrightarrow V_{p-3-r,r+s+1} \longrightarrow 0.$$

(The representation V_{c} arises as the "mod p reduction" of a cuspidal representation V_θ in characteristic 0 (more precisely, a lattice therein). Determining its composition factors involves a calculation with **Brauer characters**; see [Dia07].)

(f) Use the above fact to show that when $0 < r < p-2$, the submodule structure on $\mathrm{inj}_G(V_{r,s})$ is given by the diagram

$$\begin{tikzcd} & V_{p-1-r,r+s} \arrow[dl] & \\ V_{r,s} & & V_{r,s} \arrow[ul] \arrow[dl] \\ & V_{p-3-r,r+s+1} \arrow[ul] & \end{tikzcd}$$

Show also that the submodule structure on $\mathrm{inj}_G(V_{0,s})$ is given by the diagram

$$V_{0,s} \longleftarrow V_{p-3,s+1} \longleftarrow V_{0,s}.$$

Remark. When $f \geq 2$, the precise submodule structure of injective envelopes of irreducible $\mathrm{GL}_2(\mathbb{F}_{p^f})$-representations is more involved combinatorially. See [BP12, §§2 – 4].

Exercise 3.5.

(a) Recall that the group S_3 is isomorphic to $\mathrm{GL}_2(\mathbb{F}_2)$. Match the irreducibles from Exercise 1.5(a) over $\overline{\mathbb{F}}_2$ with the irreducibles from Exercise 3.1.

(b) Repeat the above using the exceptional isomorphism $S_4 \cong \mathrm{PGL}_2(\mathbb{F}_3)$ and Exercise 1.5(b) (with irreducible representations over $\overline{\mathbb{F}}_3$).

(c) Repeat the above using the exceptional isomorphism $S_5 \cong \mathrm{PGL}_2(\mathbb{F}_5)$ and Exercise 1.5(c) (with irreducible representations over $\overline{\mathbb{F}}_5$).

(d)* Try the same thing with other exceptional isomorphisms, e.g., for alternating groups. See Wikipedia.

Exercise 3.6 (Additional reducibilities)**.** Let $G' = \mathrm{SL}_p(\mathbb{F}_p)$, and let $V = \mathfrak{sl}_p(\overline{\mathbb{F}}_p)$ denote the space of traceless $p \times p$ matrices with entries in $\overline{\mathbb{F}}_p$. We endow V with an action of G' by conjugation: if $g \in G'$ and $X \in V$, we set $g \cdot X := gXg^{-1}$. The representation V is usually called the **adjoint representation**.

(a) Show that the subspace of scalar matrices in V is stable under the action of G' and isomorphic to the trivial representation $1_{G'}$.

(b) Show that the short exact sequence of G'-representations

$$0 \longrightarrow 1_{G'} \longrightarrow V \longrightarrow V/1_{G'} \longrightarrow 0$$

does not split.

(c) Show that $V/1_{G'}$ is irreducible.

Remark. We recall that when we work with complex coefficients, the adjoint representation of $\mathrm{SL}_n(\mathbb{C})$ on $\mathfrak{sl}_n(\mathbb{C})$ is irreducible. Thus the additional reducibility of the previous exercise is a special feature of characteristic p, and is a consequence of the **Linkage Principle** (see [Jan03, Part II, Ch. 6]).

4. Representations of (locally) profinite groups

We continue to assume C an algebraically closed coefficient field of arbitrary characteristic. These exercises deal with locally profinite groups G (such as $\mathrm{GL}_n(\mathbb{Q}_p)$) and profinite groups K (such as $\mathrm{GL}_n(\mathbb{Z}_p)$). Keep these examples in mind to make things more concrete!

Exercise 4.1. Suppose G is a locally profinite group and let (π, V) denote an abstract representation of G over C (i.e., a $C[G]$-module). Recall that we say V is **smooth** if for every $v \in V$, there exists a compact open subgroup $K \subset G$ (which depends on v) such that $k \cdot v = v$ for every $k \in K$. Prove that the following are equivalent:

(a) The representation V is smooth.
(b) For every $v \in V$, the subgroup $\mathrm{Stab}_G(v) \subset G$ is open.
(c) For every $v \in V$, the map
$$\begin{array}{ccc} G & \longrightarrow & V \\ g & \longmapsto & g \cdot v \end{array}$$
is locally constant.
(d) We have
$$V = \bigcup_{\substack{K \subset G \\ \text{compact open}}} V^K,$$
where $V^K := \{v \in V : k \cdot v = v \ \forall k \in K\}$ denotes the subspace of vectors fixed by K.
(e) The action map
$$\begin{array}{ccc} G \times V & \longrightarrow & V \\ (g, v) & \longmapsto & g \cdot v \end{array}$$
is continuous, where V is given the discrete topology.

Exercise 4.2. Suppose G is a locally profinite group and let (π, V) denote an abstract representation of G over C (i.e., a $C[G]$-module). Define
$$V^\infty := \bigcup_{\substack{K \subset G \\ \text{compact open}}} V^K.$$
The space V^∞ is often called the subspace of **smooth vectors of** V, or the **smoothification of** V.

(a) Show that V^∞ is stable by the action of G.
(b) Show that V^∞ is indeed a smooth representation of G.
(c) Show that if
$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$
is an exact sequence of $C[G]$-modules, then
$$0 \longrightarrow (V')^\infty \longrightarrow V^\infty \longrightarrow (V'')^\infty$$
is an exact sequence of smooth G-representations.
(d) Consider the short exact sequence of $C[G]$-modules
$$0 \longrightarrow \ker(\mathrm{ev}) \longrightarrow C[G] \xrightarrow{\mathrm{ev}} C \longrightarrow 0,$$
where $\mathrm{ev} : C[G] \longrightarrow C$ is given by $\sum_i c_i g_i \longmapsto \sum_i c_i$. Show that
$$0 \longrightarrow \ker(\mathrm{ev})^\infty \longrightarrow C[G]^\infty \xrightarrow{\mathrm{ev}} C$$
is not right exact.[37]

[37] Hint: explicitly compute $C[G]^\infty$.

(e)* Let $C[G]-\mathrm{Mod}$ denote the category of left $C[G]$-modules, and let $\mathrm{Rep}_C(G)$ denote the category of smooth G-representations over C. Show that the smoothification functor $(-)^\infty : C[G]-\mathrm{Mod} \longrightarrow \mathrm{Rep}_C(G)$ is right adjoint to the forgetful functor $\mathrm{Rep}_C(G) \longrightarrow C[G] - \mathrm{Mod}$.

Exercise 4.3. Let G denote a locally profinite group, and suppose (π, V) is a *finite-dimensional* smooth representation of G over C. Show that the kernel of $\pi : G \longrightarrow \mathrm{GL}(V)$ is open.

Exercise 4.4. Suppose K is a profinite group. Prove the following statements:

(a) If $J \subset K$ is an open subgroup, then J is closed and of finite index in K.

(b) Suppose $J \subset K$ is a closed subgroup. Prove that J is open in K if and only if it is of finite index.

(c) If $J \subset K$ is an open subgroup, then there exists an open **normal** subgroup $J' \lhd K$ such that $J' \subset J$. Can you explicitly construct J' from J and K?

Remark. We have the following generalization of (b): if K is a profinite group which is (topologically) finitely generated, then any finite-index subgroup J is automatically open. Note that we are not requiring J to be closed in the hypotheses! This is due to Nikolov–Segal [NS07]. On the other hand, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is not finitely generated, and contains many non-open finite index subgroups; see [Mil22, §7].

Exercise 4.5. Recall that a smooth representation (π, V) of a locally profinite group G is said to be **admissible** if V^K is finite-dimensional for all compact open subgroups $K \subset G$.

(a) Fix a compact open subgroup K. Prove that V is admissible is and only if $\mathrm{Hom}_K(U, V|_K)$ is finite-dimensional for all irreducible K-representations U.

(b) Show that if $\mathrm{char}(C) = p$ and K is an open pro-p subgroup of G, then V is admissible if and only if V^K is finite-dimensional.

Exercise 4.6.

(a) Suppose K is a profinite group, and suppose (π, V) is a smooth irreducible representation of K. Show that V is finite-dimensional.[38]

(b) Suppose G is a locally profinite group for which G/Z is compact (where Z denotes the center of G), and suppose (π, V) is a smooth irreducible representation of G. Show that V is finite-dimensional.[39]

(c) Suppose G is a locally profinite group, and $K \subset G$ is a compact open subgroup for which G/K is countable. (An example is $K = \mathrm{GL}_n(\mathbb{Z}_p) \subset G = \mathrm{GL}_n(\mathbb{Q}_p)$.) Suppose (π, V) is a smooth irreducible representation of G. Show that $\dim_C(V)$ is countable.

Exercise 4.7 (Maschke's theorem)**.** Let K denote a profinite group, and suppose $\mathrm{char}(C)$ does not divide the pro-order of K. Let $\mu : K \longrightarrow \mathbb{R}_{\geq 0}$ denote the Haar measure giving measure 1 to K (which exists by the assumption on $\mathrm{char}(C)$). Let (π, V) denote a smooth representation of K, and let $W \subset V$ denote a subrepresentation. Choose a *vector space* decomposition $V = W \oplus U'$, and let $f' : V \longrightarrow W$ denote the projection (a linear map equal to the identity on W and sending U' to 0).

(a) Define $f : V \longrightarrow W$ by
$$f(v) = \int_K k \cdot f'(k^{-1} \cdot v)\ d\mu(k).$$
Prove that f is K-equivariant, and equal to the identity on W.

(b) Deduce that $V = W \oplus \ker(f)$ is a decomposition into K-representations.

Remark. The above exercise shows that under the assumptions of the exercise, the category of smooth K-representations is semisimple: any smooth K-representation is a direct sum of irreducibles. Note that when $\mathrm{char}(C)$ divides the pro-order of K, this result breaks down; see Exercise 1.7.

[38]Hint: by irreducibility, we have $V = \langle K \cdot v \rangle = \langle (K/\mathrm{Stab}_K(v)) \cdot v \rangle$.

[39]Hint: reduce to the case where G is discrete and G/Z is finite, and use that the restriction $V|_Z$ is semisimple.

Exercise 4.8 (Completed group rings)**.** Let K be a profinite group, and write $K = \varprojlim_{N \lhd K \text{ open}} K/N$. In this exercise, we will investigate the **completed group ring**

$$C[\![K]\!] := \varprojlim_{N \lhd K \text{ open}} C[K/N].$$

Start with $K = \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

(a) Prove that we have an isomorphism of rings $C[\mathbb{Z}/p^n\mathbb{Z}] \cong C[Y]/(Y^{p^n} - 1)$.

(b) Suppose $\mathrm{char}(C) = p$. Using the substitution $X = Y - 1$, prove that $C[\mathbb{Z}/p^n\mathbb{Z}] \cong C[X]/(X^{p^n})$. Deduce that $C[\![K]\!] \cong C[\![X]\!]$, a power series ring in one variable. In particular, $C[\![K]\!]$ is noetherian.

(c) Suppose $\mathrm{char}(C) \neq p$. Prove that we have an isomorphism of rings $C[\mathbb{Z}/p^n\mathbb{Z}] \cong \prod_{j=1}^{p^n} C$, and describe the transition maps $C[\mathbb{Z}/p^{n+1}\mathbb{Z}] \longrightarrow C[\mathbb{Z}/p^n\mathbb{Z}]$. Deduce that $C[\![K]\!] \cong \prod_{j=1}^{\infty} C$, and that $C[\![K]\!]$ is not noetherian.

Suppose now that $K = \prod_{k=1}^{\infty} \mathbb{Z}/p\mathbb{Z} = \varprojlim_n \prod_{k=1}^{n} \mathbb{Z}/p\mathbb{Z}$.

(d) Prove that we have a ring isomorphism $C[\prod_{k=1}^{n} \mathbb{Z}/p\mathbb{Z}] \cong C[X_1, X_2, \ldots, X_n]/(X_1^p - 1, X_2^p - 1, \ldots, X_n^p - 1)$.

(e) Deduce that $C[\![K]\!] \cong C[\![X_1, X_2, \ldots]\!]/(X_1^p - 1, X_2^p - 1, \ldots)$, and that $C[\![K]\!]$ is not noetherian.

Exercise 4.9 (Socles (from F. Herzig's lecture))**.** Let K denote a profinite group, and let (τ, W) denote a smooth representation over C. Recall that the **socle of** W is the maximal semisimple subrepresentation of W. We denote this subrepresentation by $\mathrm{soc}_K(W)$. By Exercise 4.7, if $\mathrm{char}(C)$ does not divide the pro-order of K, then $\mathrm{soc}_K(W) = W$.

(a) Show that if $W \neq 0$, then $\mathrm{soc}_K(W) \neq 0$.[40]

(b) Show that the evaluation map

$$\bigoplus_{V \in \mathrm{Irr}(K)} \mathrm{Hom}_K(V, W) \otimes_C V \longrightarrow \mathrm{soc}_K(W)$$

is an isomorphism. Deduce that the functor $W \longmapsto \mathrm{soc}_K(W)$ is left-exact.

(c) Suppose $\mathrm{char}(C) = p$, and $J \lhd K$ is an open, normal, pro-p subgroup such that $|K/J|$ is prime to p. Show that $\mathrm{soc}_K(W) = W^J$.[41]

Exercise 4.10 (Injective envelopes (from F. Herzig's lecture))**.** Let K denote a profinite group, and let (τ, W) denote a smooth representation over C. Recall that an **injective envelope of** W is an injection $j : W \hookrightarrow \mathfrak{I}$, where $\mathfrak{I}$ is a smooth representation satisfying the following "minimality" property: any subrepresentation $U \subset \mathfrak{I}$ satisfies $U \cap j(W) \neq 0$. Injective envelopes exist and are unique up to (non-unique) isomorphism. We denote a choice of injective envelope by $W \hookrightarrow \mathrm{inj}_K(W)$.

Prove the following:

(a) If $\mathrm{char}(C)$ does not divide the pro-order of K, show that $\mathrm{inj}_K(W) \cong W$.[42]

(b) If $\mathrm{char}(C) = p$ and K is pro-p, show that $1_K \hookrightarrow C^\infty(K, C)$ is an injective envelope of the trivial representation.

($*_1$) If $K' \subset K$ is open, then $\mathrm{inj}_K(W)|_{K'}$ is injective.

($*_2$) If $J \lhd K$ is a closed normal subgroup, then $\mathrm{inj}_K(W)^J \cong \mathrm{inj}_{K/J}(W^J)$.

($*_3$) We have $\mathrm{inj}_K(W) \cong \mathrm{inj}_K(\mathrm{soc}_K(W))$ and $\mathrm{soc}_K(W) \cong \mathrm{soc}_K(\mathrm{inj}_K(W))$.

($*_4$) We have $\mathrm{inj}_K(W_1 \oplus W_2) \cong \mathrm{inj}_K(W_1) \oplus \mathrm{inj}_K(W_2)$.

($*_5$) If W is finite-dimensional, $\mathrm{inj}_K(W)$ is admissible.

[40] Hint: reduce to the case of K finite and W finite-dimensional.

[41] Hint: use Exercise 4.7 and the p-groups lemma.

[42] Hint: use Exercise 4.7.

($*_6$) If W is irreducible and (π, V) is any finite-length smooth representation of K, then
$$\dim_C(\mathrm{Hom}_K(V, \mathrm{inj}_K(W))) = [V : W],$$
where the latter denotes the number of times W appears in the Jordan–Hölder series of V.

Exercise 4.11. Let K is a profinite group, and suppose we are give a short exact sequence of smooth K-representations:
$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0.$$
Taking K-invariants is always left-exact, so we obtain
$$0 \longrightarrow (V')^K \longrightarrow V^K \longrightarrow (V'')^K.$$

(a) Suppose $\mathrm{char}(C)$ does not divide the pro-order of K. Prove that the above sequence is right exact, that is, that $V^K \longrightarrow (V'')^K$ is surjective. Thus, taking K-invariants is an exact functor.[43]
(b) Suppose $\mathrm{char}(C)$ does divide the pro-order of K. Give an example to show that $V^K \longrightarrow (V'')^K$ may fail to be exact.[44]
(c) Suppose K is pro-p and $\mathrm{char}(C) \neq p$. Prove that a quotient of an admissible K-representation over C is admissible.

Exercise 4.12 (Schur's lemma)**.** Let G be a locally profinite group, and suppose (π, V) is an irreducible representation. Schur's lemma states that under some extra hypotheses, we have $\mathrm{Hom}_G(V, V) = C$ (where C denotes the scalar endomorphisms $c \cdot \mathrm{id}_V$, $c \in C$). We prove two versions of this lemma below.

(a) Suppose in addition that V is admissible. Prove that $\mathrm{Hom}_G(V, V) = C$ using the following outline.
  (i) Suppose $f \in \mathrm{Hom}_G(V, V)$ is nonzero, and choose a compact open subgroup K such that $V^K \neq 0$. Show that f induces a linear map $f : V^K \longrightarrow V^K$ between finite-dimensional vector spaces.
  (ii) Let $\lambda \in C$ be an eigenvalue of f on V^K (recall that C is algebraically closed). Show that $f - \lambda \cdot \mathrm{id}_V$ vanishes on all of V.
(b) Suppose now that the field C is uncountable, and that there exists a compact open subgroup K such that G/K is countable. Prove that $\mathrm{Hom}_G(V, V) = C$ using the following outline.
  (i) Fix a nonzero vector $v_0 \in V$. Show that the map $\mathrm{Hom}_G(V, V) \longrightarrow V$ given by $f \longmapsto f(v_0)$ is injective. Deduce that $\mathrm{Hom}_G(V, V)$ is a division algebra over C of countable dimension.[45]
  (ii) Suppose $f \in \mathrm{Hom}_G(V, V)$ is not of the form $c \cdot \mathrm{id}_V$, so that f is transcendental over C. Prove that for each $c \in C$, the map $f - c \cdot \mathrm{id}_V$ is invertible, and that the uncountable set
$$\{(f - c \cdot \mathrm{id}_V)^{-1}\}_{c \in C}$$
  is linearly independent over C. Use this to arrive at a contradiction.

Remark. Part (b) is sometimes called Dixmier's lemma. The result and proof hold more generally when the dimension of V over C is strictly smaller than the cardinality of C (see [Bou62, §3, no. 2, Thm. 1]).

Remark. Note that when G is a finite group, admissibility is automatically satisfied, and part (a) is the usual version of Schur's lemma.

Remark. Here is an instance of the failure of Schur's lemma. Let $G = \mathrm{GL}_2(F)$, where F is a non-archimedean local field whose residue field is strictly larger than $\mathbb{F}_p$. Then the article [Le24] constructs an irreducible smooth representation (π, V) of G over $\overline{\mathbb{F}}_p$ such that $\mathrm{Hom}_G(V, V) = \overline{\mathbb{F}}_p(X)$

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[43] Hint: take any $v'' \in (V'')^K \subset V''$, and choose a lift $v \in V$. Then check that $v_1 := \int_K k \cdot v \, d\mu(k)$ is a K-invariant preimage of v''.

[44] Hint: Exercise 1.7.

[45] Hint: use Exercise 4.6(c).

(see Theorem 3.2 and Remark 3.3 in *op. cit.*). Such a representation cannot be admissible (by part (a)).

Exercise 4.13 (Central characters)**.** Let G be a locally profinite group, and let $Z \subset G$ denote its center. Suppose (π, V) is a smooth irreducible representation of G, and suppose $\mathrm{Hom}_G(V, V) = C$ (for example, if we are in the situation of Exercise 4.12). Prove that there exists a character $\omega_\pi : Z \longrightarrow C^\times$ such that $z \cdot v = \omega_\pi(z)v$ for all $z \in Z, v \in V$. (The map ω_π is called the **central character of** V.)[46]

5. Representations of p-adic groups

Throughout, F denotes a nonarchimedean local field, i.e., a finite extension of $\mathbb{Q}_p$ or a local function field $\mathbb{F}_{p^f}(\!(t)\!)$. We let $\mathcal{O}_F \supset \mathfrak{p}_F \ni \varpi$ denote the ring of integers, maximal ideal, and a choice of uniformizer in F. We let $k_F := \mathcal{O}_F/\mathfrak{p}_F$ denote the residue field of $\mathcal{O}_F$.

We also let C denote an algebraically closed coefficient field of arbitrary characteristic, unless otherwise specified. In this section we focus on representations of groups of the form $G = \mathbf{G}(F)$, where $\mathbf{G}$ is a connected reductive group defined over F (by abuse of terminology, we call these **p-adic reductive groups**). Examples of such G include $\mathrm{GL}_n(F), \mathrm{SL}_2(F), \mathrm{Sp}_{2n}(F), \mathrm{E}_8(F)$, etc.

Exercise 5.1. Suppose $G = \mathrm{GL}_n(F)$, and suppose (π, V) is a smooth, irreducible, *finite-dimensional* representation of G.

(a) For $1 \leq i, j \leq n$ with $i \neq j$, let $U_{i,j}$ denote the subgroup consisting of 1s on the diagonal, an arbitrary element of F in the (i, j)-entry, and 0s elsewhere. Show that $\ker(\pi)$ contains an open subgroup of $U_{i,j}$.[47]
(b) Show that $\ker(\pi)$ contains all of $U_{i,j}$.
(c) Show that $\ker(\pi)$ contains $\mathrm{SL}_n(F)$.
(d) Deduce that π factors through the determinant map $\det : G \longrightarrow F^\times$, and therefore $V \cong \chi \circ \det$ for some smooth character χ.

Thus, the only finite-dimensional, smooth, irreducible representations of G are in fact *one-dimensional*.

(e)* Show more generally that if G is split, semisimple and simply connected, then the only finite-dimensional, smooth, irreducible representation of G is the trivial representation.

Exercise 5.2 (Extensions)**.** Suppose $G = \mathrm{GL}_n(F)$. Consider the smooth two-dimensional representation (π, V) of G given by

$$\begin{array}{rcl} \pi : G & \longrightarrow & \mathrm{GL}_2(C) \\ g & \longmapsto & \begin{pmatrix} 1 & \mathrm{val}(\det(g)) \\ 0 & 1 \end{pmatrix} \end{array}$$

where $\mathrm{val} : F^\times \longrightarrow \mathbb{Z}$ is the normalized valuation of F.

(a) Show that V defines a nonsplit extension of 1_G by 1_G.
(b) Show that any two-dimensional representation of $\mathrm{SL}_n(F)$ is isomorphic to $1_{\mathrm{SL}_n(F)}^{\oplus 2}$.

Exercise 5.3. Recall that for $F/\mathbb{Q}_p$ a finite extension and $G = \mathrm{GL}_n(F)$, any quotient of an admissible G-representation over $\overline{\mathbb{F}}_p$ will also be admissible. The purpose of this exercise is to show that this no longer holds when F is a local function field. (By contrast, when the coefficient field is of characteristic $\ell \neq p$, then this result holds regardless of $\mathrm{char}(F)$; see Exercise 4.11(c).) For simplicity, we take $F = \mathbb{F}_p(\!(t)\!)$.

[46] Hint: the map $v \longmapsto z \cdot v$ is G-equivariant.
[47] Hint: use Exercise 4.3.

(a) Let $\mathcal{I}$ denote a countable set, and consider the profinite group $H := \prod_{i\in\mathcal{I}} \mathbb{Z}_p$. Set

$$V := C^\infty\left(\prod_{i\in\mathcal{I}} \mathbb{Z}/p\mathbb{Z},\ \overline{\mathbb{F}}_p\right),$$

the space of locally constant, $\overline{\mathbb{F}}_p$-valued functions. Let H act on this space by right translation: if $h \in H, f \in V$, we have

$$(h \cdot f)(x) = f(x + \overline{h}),$$

where $\overline{h}$ denotes the image of h in $\prod_{i\in\mathcal{I}} \mathbb{Z}/p\mathbb{Z}$. Prove that V is a smooth H-representation over $\overline{\mathbb{F}}_p$.

(b) We define an action of $\mathrm{GL}_1(F) = F^\times = t^{\mathbb{Z}} \times \mathbb{F}_p^\times \times 1 + \mathfrak{p}$ on V as follows. First, prove that the map

$$\begin{aligned} \alpha : \prod_{i\geq 1,\ p\nmid i} \mathbb{Z}_p &\longrightarrow 1 + \mathfrak{p} = 1 + t\mathbb{F}_p[\![t]\!] \\ (x_i)_{i\geq 1,\ p\nmid i} &\longmapsto \prod_i (1+t^i)^{x_i} \end{aligned}$$

is an isomorphism of topological groups.[48] We then define an action of $t^n a(1+z) \in t^{\mathbb{Z}} \times \mathbb{F}_p^\times \times 1 + \mathfrak{p} = F^\times$ on $f \in V$ by

$$(t^n a(1+z)) \cdot f := \alpha^{-1}(1+z) \cdot f.$$

(c) Prove that $V^{1+\mathfrak{p}}$ is one-dimensional, spanned by the constant functions. Consequently V is an admissible $F^\times$-representation.

(d) Let $V' := V/V^{1+\mathfrak{p}}$, which is a smooth $F^\times$-representation. Prove that $(V')^{1+\mathfrak{p}}$ is infinite-dimensional. Consequently, V' is not admissible.[49]

Exercise 5.4. Suppose $G = \mathrm{GL}_n(F)$, $P \subset G$ is a parabolic subgroup, and let (τ, W) be a smooth P-representation. Show that W is admissible if and only if $\mathrm{Ind}_P^G(W)$ is admissible.[50]

Exercise 5.5 (Jacquet modules)**.** Suppose $G = \mathrm{GL}_2(F)$, and let B, T, U, etc., denote the subgroups of G defined analogously to those in Section 2. Fix two smooth characters $\chi_1, \chi_2 : F^\times \longrightarrow C^\times$, and let $\chi := \chi_1 \boxtimes \chi_2 : B \longrightarrow C^\times$ denote the associated character of B.

The goal will be to compute the Jacquet module (i.e., the U-coinvariants) of the induced representation $\mathrm{Ind}_B^G(\chi)$. We begin with some preparation.

(a) Show that the map $\varepsilon : \mathrm{Ind}_B^G(\chi) \longrightarrow \chi$ given by $f \longmapsto f(1)$ is B-equivariant.

(b) Let $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Show that if $f \in \ker(\varepsilon)$, then there exists a compact open subgroup $U_0 \subset U$ such that $\mathrm{supp}(f) \subset BsU_0$.[51]

(c) Show that the map $f \longmapsto \phi_f$, where $\phi_f(u) = f(su)$, defines an isomorphism $\ker(\varepsilon) \xrightarrow{\sim} C_c^\infty(U, C)$. Use this isomorphism to describe the B-action on $C_c^\infty(U, C)$.

We now examine the short exact sequence of smooth B-representations

$$0 \longrightarrow C_c^\infty(U, C) \longrightarrow \mathrm{Ind}_B^G(\chi) \longrightarrow \chi \longrightarrow 0.$$

(d) Suppose first that $\mathrm{char}(C) = p$. Using the fact that U does not possess a C-valued Haar measure, show that $C_c^\infty(U, C)_U = 0$. Deduce that $\mathrm{Ind}_B^G(\chi)_U \cong \chi$ as T-representations.

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[48] See [Neu99, Ch. 2, Prop. 5.7].

[49] Hint: any linear map $\ell \in \mathrm{Hom}(\prod_{i\geq 1,\ p\nmid i} \mathbb{Z}/p\mathbb{Z},\ \overline{\mathbb{F}}_p)$ which is contained in V gives an element of $(V')^{1+\mathfrak{p}}$.

[50] Hint: use the Mackey decomposition [Yam22].

[51] Hint: use smoothness of f and the identity $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -x^{-1} & 0 \\ 0 & x \end{smallmatrix}\right) s \left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right)$.

(e) Suppose now that $\mathrm{char}(C) \neq p$. In this case, the functor of U-coinvariants is exact, and we obtain a short exact sequence of T-representations

$$0 \longrightarrow C_c^\infty(U, C)_U \longrightarrow \mathrm{Ind}_B^G(\chi)_U \longrightarrow \chi \longrightarrow 0.$$

Fix a Haar measure μ on U. Show that the map

$$\begin{array}{rcl} C_c^\infty(U, C) & \longrightarrow & C \\ \phi & \longmapsto & \displaystyle\int_U \phi(u)\ d\mu(u) \end{array}$$

induces a C-linear isomorphism $C_c^\infty(U, C)_U \xrightarrow{\sim} C$. Using this, show that the action of T on $C_c^\infty(U, C)_U$ is given by the character $\chi^s\delta_B$, where $\chi^s := \chi_2 \boxtimes \chi_1$ and where $\delta_B(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)) = q^{-\mathrm{val}(ad^{-1})}$.

Remark. Suppose $\mathrm{char}(C) \neq p$ and fix a square root of p in C. We can then define the **normalized parabolic induction** $\text{n-Ind}_B^G(W) := \mathrm{Ind}_B^G(W \otimes \delta_B^{1/2})$ and the **normalized Jacquet module** $r_B^G(V) := V_U \otimes \delta_B^{-1/2}$. (When $C = \mathbb{C}$, these functors preserve unitary representations.) The above exercise then shows that we have a short exact sequence of T-representations

$$0 \longrightarrow \chi^s \longrightarrow r_B^G(\text{n-Ind}_B^G(\chi)) \longrightarrow \chi \longrightarrow 0.$$

We therefore obtain nicer combinatorics.

Remark. One may proceed in an analogous manner for the group $\mathrm{GL}_n(F)$ to compute $r_{B_n}^{\mathrm{GL}_n}(\text{n-Ind}_{B_n}^{\mathrm{GL}_n}(\chi))$ over a field of characteristic different from p. The upshot is the following: the T_n-representation $r_{B_n}^{\mathrm{GL}_n}(\text{n-Ind}_{B_n}^{\mathrm{GL}_n}(\chi))$ has a filtration whose i^{th} graded piece is equal to $\bigoplus_{\substack{w \in S_n \\ \ell(w) \geq n(n-1)/2 - i}} \chi^w$, where χ^w is the w-conjugate of χ. This decomposition is known as the **Bernstein–Zelevinsky geometrical lemma**, and holds more generally for other p-adic reductive groups G, and for computing $r_{P'}^G(\text{n-Ind}_P^G(W))$, where P, P' are two parabolic subgroups. See [BZ77, Thm. 5.2] and [Vig96, Ch. II, §2.18].

Remark. Suppose $\mathrm{char}(C) = p$. Is there a way to recover the mod p analogues of the "missing" characters $\chi^s\delta_B$? This is possible, but in order to do this one needs to work in the derived category of smooth representations, and derive the functor $V \longmapsto V_U$. For the construction see [Hey23], and for an analogue of the geometrical lemma see [Hey24].

Exercise 5.6. Let $G = \mathrm{GL}_n(F)$, fix smooth characters $\chi_1, \ldots, \chi_n : F^\times \longrightarrow C^\times$, and set $\chi := \chi_1 \boxtimes \ldots \boxtimes \chi_n : T \longrightarrow C^\times$. Suppose $\chi_i = \chi_{i+1}$ for some i. Prove that $\mathrm{Ind}_B^G(\chi)$ is *reducible* as a G-representation.[52]

Exercise 5.7. Let $G = \mathrm{GL}_2(F)$ and recall that St_B^G is defined by the short exact sequence

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G(1_T) \longrightarrow \mathrm{St}_B^G \longrightarrow 0.$$

This exercise will give another proof of the irreducibility of St_B^G when $\mathrm{char}(C) = p$. Let $I_1 := \left(\begin{smallmatrix} 1+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{smallmatrix}\right)$ denote the pro-p Iwahori subgroup of G.

(a) Prove that $\mathrm{Ind}_B^G(1_T)^{I_1}$ is spanned by two functions f_1, f_s, where $\mathrm{supp}(f_v) = BvI_1$ and $f_v(v') = \delta_{v,v'}$ for $v, v' \in \{1, s\}$. (Recall that $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.)
(b) Show that f_1 generates $\mathrm{Ind}_B^G(1_T)$ as a G-representation. Conclude that $\overline{f_1}$ generates St_B^G as a G-representation, where $\overline{f} \in \mathrm{St}_B^G$ denotes the image of $f \in \mathrm{Ind}_B^G(1_T)$.

[52] Hint: use transitivity of parabolic induction.

(c) Choose a nonzero $\overline{f} \in (\mathrm{St}_B^G)^{I_1}$ where $f \in \mathrm{Ind}_B^G(1_T)$ (such an $\overline{f}$ exists by the pro-p-groups lemma). Explain why there exists a function $a : I_1 \longrightarrow C$ such that

$$f(gi) - f(g) = a(i)$$

for all $i \in I_1, g \in G$.

(d) We now determine the possibilities for the function a. Using judicious choices of g, show that $a(i) = 0$ in each of the following cases:
- $i \in I_1 \cap U$
- $i \in I_1 \cap U^-$
- $i \in I_1 \cap T$

(e) Show that $f(gi) = f(g)$ for all $i \in I_1, g \in G$. Explain why this shows that the induced map $\mathrm{Ind}_B^G(1_T)^{I_1} \longrightarrow (\mathrm{St}_B^G)^{I_1}$ is surjective and $\dim_C((\mathrm{St}_B^G)^{I_1}) = 1$.[53]

(f) Suppose $V' \subset \mathrm{St}_B^G$ is a nonzero G-subrepresentation. Show that that $\overline{f_1} \in V'$, and therefore that V' must be all of St_B^G.[54]

Exercise 5.8. Suppose $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(\mathcal{O})$. Fix smooth characters $\chi_1, \ldots, \chi_n : F^\times \longrightarrow C^\times$.

(a) Find conditions on the χ_i so that $\mathrm{Ind}_B^G(\chi_1 \boxtimes \ldots \boxtimes \chi_n)^K \neq 0$.

(b) Prove that if $\mathrm{Ind}_B^G(\chi_1 \boxtimes \ldots \boxtimes \chi_n)^K \neq 0$, then $\dim_C(\mathrm{Ind}_B^G(\chi_1 \boxtimes \ldots \boxtimes \chi_n)^K) = 1$. (A basis for this nonzero space is often called a **spherical vector**.)

(c) Let $P = MN \subset G$ denote a standard parabolic subgroup, and let St_P^G denote the **generalized Steinberg representation**:

$$\mathrm{St}_P^G := \mathrm{Ind}_P^G(1_M) \Big/ \sum_{P \subsetneq P' \subset G} \mathrm{Ind}_{P'}^G(1_{M'}).$$

Suppose $C = \mathbb{C}$. Prove that $(\mathrm{St}_P^G)^K \neq 0$ if and only if $P = G$.

Exercise 5.9 (From J. Fintzen's lecture)**.** Suppose G is a p-adic reductive group, and let K denote a open, compact-mod-center subgroup of G containing the center Z of G. Suppose (ρ, W) is a smooth representation of K. Recall that we say $g \in G$ **intertwines** (ρ, W) if

$$\mathrm{Hom}_{K \cap gKg^{-1}}({}^g\rho|_{K \cap gKg^{-1}},\ \rho|_{K \cap gKg^{-1}}) \neq 0,$$

where ${}^g\rho(k) := \rho(g^{-1}kg)$.

(a) Show that g intertwines (ρ, W) if and only if gk intertwines (ρ, W) for all $k \in K$, if and only if kg intertwines (ρ, W) for all $k \in K$.

(b) Suppose (ρ, W) is an irreducible cuspidal representation of $\mathrm{GL}_n(k_F)$ over $\mathbb{C}$, which we inflate to a representation of $\mathrm{GL}_n(\mathcal{O}_F)$ and extend to $Z\mathrm{GL}_n(\mathcal{O}_F)$ by letting $\varpi 1_n$ act by a scalar. We continue to denote the resulting representation of $Z\mathrm{GL}_n(\mathcal{O}_F)$ by (ρ, W). Show that $g \in \mathrm{GL}_n(F)$ intertwines (ρ, W) if and only if $g \in Z\mathrm{GL}_n(\mathcal{O}_F)$.[55]

Exercise 5.10 (From J. Fintzen's lecture)**.** Suppose G is a p-adic reductive group, and let K be a compact-mod-center open subgroup of G. Suppose (τ, W) is an irreducible representation of K over $\mathbb{C}$, and suppose the compact induction $\text{c-ind}_K^G(W)$ is irreducible.

(a) Show that $\text{c-ind}_K^G(W)$ is supercuspidal by using matrix coefficients.[56]

(b) Suppose that $G = \mathrm{GL}_n(F)$, $K = Z\mathrm{GL}_n(\mathcal{O}_F)$, and (τ, W) is an irreducible cuspidal representation of $\mathrm{GL}_n(k_F)$, inflated and extended to $Z\mathrm{GL}_n(\mathcal{O}_F)$. (Recall that an irreducible representation (τ, W) of $\mathrm{GL}_n(k_F)$ over $\mathbb{C}$ is said to be cuspidal if $W^N = 0$ (equivalently, $W_N = 0$) for all

[53] Hint: use the Iwahori decomposition.

[54] Hint: use the p-groups lemma.

[55] Hint: the method of Exercise 6.7 may help.

[56] Hint: use the fact that $(\text{c-ind}_K^G(W))^\vee \cong \mathrm{Ind}_K^G(W^\vee)$ and matrix coefficients coming from W.

unipotent radicals N of parabolic subgroups of $\mathrm{GL}_n(k_F)$.) By Exercise 5.9(b) and Lemma 3.2.3 of J. Fintzen's notes, the representation $\text{c-ind}_{Z\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(W)$ is irreducible. Using Jacquet modules, show that $\text{c-ind}_{Z\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(W)$ is supercuspidal.[57]

Exercise 5.11 (Depth zero representations)**.** Suppose that $G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O}_F)$, and (τ, W) is an irreducible cuspidal representation of $\mathrm{GL}_n(k_F)$ over $\mathbb{C}$, inflated and extended to KZ. By Exercise 5.10(b), the representation $\text{c-ind}_{KZ}^G(W)$ is irreducible and supercuspidal.

(a) Compute the restriction $\text{c-ind}_{KZ}^G(W)|_K$.
(b) Let $K_1 := 1+\varpi\mathrm{M}_n(\mathcal{O}_F)$ denote the first congruence subgroup of K. Show that $\text{c-ind}_{KZ}^G(W)^{K_1} \neq 0$, and conclude that $\text{c-ind}_{KZ}^G(W)$ has depth zero.

Exercise 5.12 (Positive depth representations)**.** Suppose $p > 2$, $G = \mathrm{SL}_2(F)$, let $Z \cong \mu_2$ denote the center, and let I_1 denote the **pro-p Iwahori subgroup** consisting of matrices of the form

$$\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix},$$

where $a, b, c, d \in \mathcal{O}_F$, and where matrices are assumed to have determinant 1. Choose an nontrivial additive character $\psi : k_F \longrightarrow \mathbb{C}^\times$, and consider the representation of ZI_1 given by

$$\psi\left(\pm\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix}\right) = \psi(\bar{b} + \bar{c}).$$

(a) Show that $g \in \mathrm{SL}_2(F)$ intertwines ψ if and only if $g \in ZI_1$. Deduce that $\text{c-ind}_{ZI_1}^G(\psi)$ is irreducible and supercuspidal.
(b) Show that $\text{c-ind}_{ZI_1}^G(\psi)^{K_1} = 0$, where K_1 is defined as in Exercise 5.11(b).
(c) Let I_2 denote the subgroup of I_1 consisting of matrices of the form $\left(\begin{smallmatrix} 1+\varpi a & \varpi b \\ \varpi^2 c & 1+\varpi d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}_F$. Show that $\text{c-ind}_{ZI_1}^G(\psi)^{I_2} \neq 0$.[58]

Remark. The representation $\text{c-ind}_{ZI_1}^G(\psi)$ is an example of a **simple supercuspidal representation**, defined by Gross–Reeder [GR10, §9.3]. The representation constructed above has depth $1/2$.

Exercise 5.13 (Smooth + irreducible $\Longrightarrow$ admissible)**.** In this exercise we show that any smooth, irreducible representation of $G = \mathrm{GL}_2(F)$ over $\mathbb{C}$ is admissible. (The argument applies with minor modification to other groups.) We let $K = \mathrm{GL}_2(\mathcal{O}_F)$, and let B denote the upper-triangular Borel subgroup of G, written as $B = TU$, where T denotes the diagonal matrices and U is the unipotent radical.

Suppose (π, V) is a smooth irreducible representation of G. We assume first that the Jacquet module V_U is nonzero.

(a) Show that $V|_B$ is finitely generated as a B-representation. Conclude that the Jacquet module V_U is finitely generated as a T-representation.[59]
(b) Using Zorn's lemma, show that the T-representation V_U admits an irreducible (and hence one-dimensional) quotient χ.
(c) Show that we have an injection $V \hookrightarrow \mathrm{Ind}_B^G(\chi)$, and consequently that V is admissible.

Suppose now that $V_U = 0$.

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[57]Hint: first calculate the restriction of $\text{c-ind}_{Z\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(W)$ to a standard parabolic subgroup $P = MN$ using the Mackey decomposition. Then calculate $\mathrm{Hom}_M(\text{c-ind}_{Z\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(W)_N, X)$ for any X.

[58]Hint: calculate the restriction $\text{c-ind}_{ZI_1}^G(\psi)|_{I_1}$ first.

[59]Hint: use the Iwasawa decomposition and the fact that $\mathrm{Stab}_G(v) \cap K$ is of finite index in K.

(d) Let $v \in V, v^\vee \in V^\vee$ be a pair of nonzero vectors (where $V^\vee$ denotes the smooth dual). Show that $\langle v^\vee, \left(\begin{smallmatrix} \varpi^n & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot v\rangle = 0$ for $n \gg 0$.[60]

(e) Consider the function $f_{v^\vee,v} : G \longrightarrow \mathbb{C}$ given by $f_{v^\vee,v}(g) = \langle v^\vee, g \cdot v\rangle$. Show that $\mathrm{supp}(f_{v^\vee,v})$ is contained in a finite union of (KZ, K)-double cosets, and consequently that $f_{v^\vee,v}$ has compact support modulo Z.

(f) Suppose by contradiction that V is not admissible, and choose a compact open subgroup K' such that $V^{K'}$ is infinite-dimensional. Explain why $V^{K'}$ is of countable dimension and $(V^\vee)^{K'}$ is of uncountable dimension.[61]

(g) Let $\mathcal{C}(\pi)$ denote the vector space of functions of of the form $f_{v^\vee,v}$, where $v \in V, v^\vee \in V^\vee$. Fix a nonzero $v \in V^{K'}$, and define a map $\mathcal{F} : (V^\vee)^{K'} \longrightarrow \mathcal{C}(\pi)$ by $\mathcal{F}(v^\vee) = f_{v^\vee,v}$. Show that $\mathcal{F}$ is injective.

(h) Use part (e) to show that the dimension of $\mathrm{im}(\mathcal{F})$ is at most countable, and arrive at a contradiction.

(i)* Sketch this argument for $\mathrm{GL}_n(F)$ (or more generally for any p-adic reductive group of your choice).

Remark. The implication "smooth + irreducible $\Longrightarrow$ admissible" also holds when $\mathrm{char}(C)$ is positive and distinct from p; see [Vig96, Ch. II, §2.8]. On the other hand, when $\mathrm{char}(C) = p$, there exist examples of smooth irreducible representations of $\mathrm{GL}_2(F)$ which are not admissible; see [GLS23] and F. Herzig's notes, Section 3.5.

Remark. If the group G is not assumed to be reductive, then the implication "smooth + irreducible $\Longrightarrow$ admissible" may fail (even over $\mathbb{C}$). For an example when $G = \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ is the mirabolic subgroup of $\mathrm{GL}_2(F)$, see [BH06, §8].

Exercise 5.14. Suppose G is a p-adic reductive group, and (π, V) a smooth representation of G over $\mathbb{C}$. Recall that V is said to be **finitely generated** if there is a finite collection of vectors $v_1, \ldots, v_n$ such that $\mathrm{span}\{g \cdot v_i : 1 \le i \le n, g \in G\} = V$.

(a) Show that if V is of finite length, then it is finitely generated and admissible.[62]

(b) Suppose conversely that V is finitely generated and admissible. Show that there exists a compact open subgroup K such that V is generated (as a G-representation) by the finite-dimensional space V^K.[63]

(c) By shrinking K if necessary, we may assume it admits an Iwahori factorization. Using finite-dimensionality of V^K and the second remark following Exercise 6.2, prove that V is of finite length.

Exercise 5.15 (Supersingular representations in characteristic p)**.** Suppose $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $C = \overline{\mathbb{F}}_p$. Let Z denote the center of G, and let $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Fix $0 \le r \le p-1$, and consider the compactly induced representation $\text{c-ind}_{KZ}^G(V_{r,0}) = \text{c-ind}_{KZ}^G(\mathrm{Sym}^r(\overline{\mathbb{F}}_p^{\oplus 2}))$, where we let K act on $V_{r,0}$ by inflation from $\mathrm{GL}_2(\mathbb{F}_p)$, and where $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially.

(a) Using the method of Exercise 6.7, show that $\mathrm{End}_G(\text{c-ind}_{KZ}^G(V_{r,0})) \cong \overline{\mathbb{F}}_p[T]$ for some endomorphism T, whose associated function $\phi : G \longrightarrow \mathrm{End}_{\overline{\mathbb{F}}_p}(V_{r,0})$ is supported on $KZ\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)KZ$. (The operator T is the representation-theoretic analog of the T_p operator in the theory of modular forms.)

[60] Hint: since $V_U = 0$, there exists a compact open subgroup U_0 such that $\int_{U_0} u \cdot v \, d\mu(u) = 0$.

[61] Hint: since any smooth representation of K' is semisimple, we have $(V^\vee)^{K'} = (V_{K'})^\vee = (V^{K'})^\vee$.

[62] Hint: use Exercises 4.11 and 5.13.

[63] Hint: consider the common stabilizer of a generating set.

It is known that the endomorphism T is not surjective. We set $\pi(r) := \text{c-ind}_{KZ}^G(V_{r,0})/\text{coker}(T)$. Breuil [Bre03] proved that

$$\pi(r)^{I_1} = \overline{\mathbb{F}}_p\overline{1 \otimes X^r} \oplus \overline{\mathbb{F}}_p\overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \otimes Y^r},$$

where we identify $\text{c-ind}_{KZ}^G(-) \cong \overline{\mathbb{F}}_p[G] \otimes_{\overline{\mathbb{F}}_p[KZ]} -$, and where the bar denotes the image under the quotient map $\text{c-ind}_{KZ}^G(V_{r,0}) \twoheadrightarrow \pi(r)$. Further, I_1 denotes the "upper triangular" pro-p Iwahori subgroup, as in Exercise 5.12.

(b) Prove that the G-representation generated by $\overline{1 \otimes X^r}$ is equal to all of $\pi(r)$.
(c) Calculate the right action of $\mathbb{1}_{I_1\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)I_1}$, $\mathbb{1}_{I_1\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)I_1}$ and $\mathbb{1}_{I_1\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)I_1}$ $(a, d \in \mathbb{Z}_p^\times)$ on $\pi(r)^{I_1}$. (These elements give a generating set for $\mathcal{H}(G, I_1, 1_{I_1})$.)
(d) Use the previous parts to show that $\pi(r)$ is irreducible.
(e) Show that $\pi(r)$ is not isomorphic to a subquotient of a principal series representation.[64]

Remark. Up to twisting by a character of G, the representations $\pi(r)$ constitute *all* smooth, irreducible supersingular representations of $\text{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$. See [BL94], [Bre03] and [Ber12].

Remark. On the other hand, when considering the group $\text{GL}_2(F)$ for F a finite nontrivial extension of $\mathbb{Q}_p$, or F a local function field, the classification of irreducible, admissible, supersingular representations over $\overline{\mathbb{F}}_p$ is unknown. See [Bre03, Rmk. 4.2.6] and [BP12].

Exercise 5.16 (Projectivity of supercuspidals)**.** Suppose $G = \text{PGL}_2(F)$, and $C = \mathbb{C}$. Suppose (π, V) is a smooth, irreducible, supercuspidal representation of G. In this exercise, we will show that V is a projective (and injective) object in the category of smooth G-representations over $\mathbb{C}$.

Our main tool will be **Schur's orthogonality relation**: if $v_1, v_2 \in V$ and $v_1^\vee, v_2^\vee \in V^\vee$, then we have

$$\int_G \langle v_1^\vee, g^{-1} \cdot v_1\rangle\langle v_2^\vee, g \cdot v_2\rangle \, d\mu(g) = d_\pi\langle v_1^\vee, v_2\rangle\langle v_2^\vee, v_1\rangle, \tag{1}$$

where d_π is a positive real number depending only on (π, V) and μ. Note that supercuspidality of V implies that the matrix coefficients appearing are compactly supported, and therefore the integral makes sense.

(a) Let (σ, U) be a smooth G-representation, and suppose we are given a G-equivariant surjection $\gamma : U \twoheadrightarrow V$. Choose $u \in U$ such that $\gamma(u) \neq 0$, and define $U' := \sigma(\mathcal{H})u = \{\sigma(f)u : f \in \mathcal{H}\} \subset U$ (using the notation of the next section). Show that U' is stable by G, and the restriction of γ to U' is surjective.
(b) Show that the map

$$\begin{array}{rcl} \Gamma : \mathcal{H} & \longrightarrow & V \\ f & \longmapsto & \sigma(f)(\gamma(u)) \end{array}$$

is G-equivariant and surjective (where we equip $\mathcal{H}$ with its left translation action).
(c) Show that if Γ splits, then γ splits.
(d) Choose a vector $v_1^\vee \in V^\vee$ such that $\langle v_1^\vee, \gamma(u)\rangle = d_\pi^{-1}$. Show that for every $v \in V$, the function $\phi_v : G \longrightarrow \mathbb{C}$ given by $\phi_v(g) = \langle v_1^\vee, g^{-1} \cdot v\rangle$ lies in $\mathcal{H}$.
(e) Show that the map

$$\begin{array}{rcl} \Delta : V & \longrightarrow & \mathcal{H} \\ v & \longmapsto & \phi_v \end{array}$$

is G-equivariant.
(f) Show that $\langle v_2^\vee, \Gamma \circ \Delta(v_2)\rangle = \langle v_2^\vee, v_2\rangle$, and conclude that Δ gives a splitting for Γ. Deduce that V is a projective object in the category of smooth G-representations.[65]

[64]Hint: use Exercise 6.13.
[65]Hint: use the orthogonality relation (1).

(g) Show that V is also an injective object in the category of smooth G-representations.

Remark. The above exercise generalizes as follows: suppose G is a general p-adic reductive group, and fix a smooth character $\chi : Z \longrightarrow \mathbb{C}^\times$ of its center. If (π, V) is a supercuspidal representation with central character χ, then it is both an injective and projective object in the category of smooth G-representations on which Z acts by χ. See [Cas, Thm. 5.4.1].

Remark. The analogous statement no longer holds when $C = \overline{\mathbb{F}}_p$. Namely, when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with $p \geq 5$, Paškūnas [Pas10, Thm. 10.13] has shown that if (π, V) is a smooth, irreducible, supersingular representation, then $\mathrm{Ext}^1_{G,\omega_\pi}(V, V) \neq 0$, where $\mathrm{Ext}^1_{G,\omega_\pi}$ denotes the Ext group in the category of smooth representations with central character ω_π.

6. Hecke algebras

We now discuss several flavors of Hecke algebras.

Suppose first that $C = \mathbb{C}$ and that G is locally profinite and **unimodular** (this condition means that the left and right Haar measures agree, and will be satisfied for p-adic reductive groups). The "big" Hecke algebra $\mathcal{H}$ is defined as $C_c^\infty(G, \mathbb{C})$, equipped with a convolution product relative to a fixed choice of Haar measure μ on G: we have

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\ d\mu(h)$$

for $f_1, f_2 \in \mathcal{H}$. Note that we have $\mathcal{H} = \bigcup_{\substack{K \subset G \\ \text{compact open}}} e_K * \mathcal{H} * e_K$, where $e_K := \mu(K)^{-1}\mathbb{1}_K$. (The subalgebra $e_K * \mathcal{H} * e_K$ is isomorphic to the algebra $\mathcal{H}(G, K, 1_K)$ defined below.)

Given a smooth representation (π, V) of G, we obtain an action of $\mathcal{H}$ on V via

$$\pi(f)v = \int_G f(g) g \cdot v\ d\mu(g) \tag{2}$$

for $f \in \mathcal{H}, v \in V$. We note that this action turns V into a **smooth** $\mathcal{H}$-module, meaning that for every $v \in V$ there exists a compact open subgroup K_v such that $\pi(e_{K_v})v = v$.

Exercise 6.1. Suppose G is a locally profinite unimodular group, and define an action of $g \in G$ on $\mathcal{H}$ via the **right translation action**: $(g \cdot f)(g') = f(g'g)$.

(a) Show that this turns $\mathcal{H}$ into a smooth G-representation.
(b) Do the same for the **left translation action**: $(g \cdot f)(g') = f(g^{-1}g')$.
(c) Show that these two actions commute.

Exercise 6.2. Suppose (π, V) is a smooth, irreducible representation of a locally profinite unimodular group G. Fix a compact open subgroup $K \subset G$, and suppose $V^K \neq 0$.

(a) Prove that $\pi(e_K) : V \longrightarrow V$ is a projection operator with image V^K.
(b) Show that under the action of $\mathcal{H}$ defined in Equation (2), the space V^K becomes a module over $e_K * \mathcal{H} * e_K$, on which $\pi(e_K)$ acts by the identity.
(c) Suppose $M \subset V^K$ is a nonzero subspace stable by $e_K * \mathcal{H} * e_K$. Show that the space $V' := \mathrm{span}\{\pi(f)m : f \in \mathcal{H}, m \in M\}$ is stable by G, and therefore $V' = V$.
(d) Show that $\pi(e_K)V' = \pi(e_K * \mathcal{H} * e_K)M$, and conclude that $M = V^K$.

Therefore, we see that if (π, V) is irreducible, then for any choice of compact open subgroup $K \subset G$, the space V^K is either 0 or a simple $e_K * \mathcal{H} * e_K$-module.

Remark. With more work, one can upgrade the conclusion of the above exercise as follows: the map $V \longmapsto V^K$ induces a *bijection* between isomorphism classes of smooth irreducible representations V satisfying $V^K \neq 0$ and isomorphism classes of simple $e_K * \mathcal{H} * e_K$-modules.

Remark. In the context of p-adic reductive groups, the conclusion of the above exercise can be further strengthened. Suppose G is a p-adic reductive group, and K is a compact open subgroup admitting an Iwahori factorization. Then the map $V \longmapsto V^K$ induces an *equivalence of categories* between the category of smooth representations generated by their K-fixed vectors, and the category of $e_K * \mathcal{H} * e_K$-modules. See [Ber84, Cor. 3.9(ii)].

Suppose now that G is locally profinite (not necessarily unimodular) and that C is arbitrary. Let H denote an open subgroup, and (τ, W) a finite-dimensional smooth representation of H over C. Recall that we may identify $\mathrm{End}_G(\text{c-ind}_H^G(W))$ with the space

$$\mathcal{H}(G,H,\tau) := \left\{ \phi : G \longrightarrow \mathrm{End}_C(W) : \begin{array}{l} \diamond\ \phi(hgh') = \tau(h)\circ\phi(g)\circ\tau(h') \quad \forall h,h' \in H, g \in G \\ \diamond\ H\backslash\mathrm{supp}(\phi)/H \text{ is finite} \end{array} \right\}.$$

This space is equipped with the convolution product of functions: if $\phi_1, \phi_2 \in \mathcal{H}(G,H,\tau)$, then we set

$$(\phi_1 * \phi_2)(g) = \sum_{H\backslash G \ni \gamma} \phi_1(g\gamma^{-1}) \circ \phi_2(\gamma).$$

Under the linear isomorphism $\mathrm{End}_G(\text{c-ind}_H^G(W)) \cong \mathcal{H}(G,H,\tau)$, composition on the left-hand side corresponds to convolution on the right-hand side. (For details, see [BL94, §2.2].)

Suppose further that we are given a smooth G-representation (π, V) over C. There is a natural right action of $\mathrm{End}_G(\text{c-ind}_H^G(W))$ on $\mathrm{Hom}_G(\text{c-ind}_H^G(W), V)$ by composition. Then via Frobenius reciprocity, the Hecke algebra $\mathcal{H}(G,H,\tau)$ acts on the right on the multiplicity space $\mathrm{Hom}_H(W, V|_H)$. Explicitly, this action is given as follows: if $f \in \mathrm{Hom}_H(W, V|_H)$ and $\phi \in \mathcal{H}(G,H,\tau)$, then the right action of ϕ on f is given by

$$(f\cdot\phi)(w) = \sum_{H\backslash G \ni \gamma} \gamma^{-1}\cdot f(\phi(\gamma)w) \tag{3}$$

where $w \in W$. In particular, when (τ, W) is the trivial representation 1_H, the algebra $\mathcal{H}(G,H,1_H)$ consists of the H-bi-invariant functions which are supported on finitely many H double cosets. The algebra $\mathcal{H}(G,H,1_H)$ then acts on $v \in V^H$ on the right via

$$v\cdot\mathbb{1}_{HgH} = \sum_{H\backslash HgH \ni \gamma} \gamma^{-1}\cdot v = \sum_{h \in H/(H\cap g^{-1}Hg)} hg^{-1}\cdot v. \tag{4}$$

Exercise 6.3. Verify Equations (3) and (4).

Exercise 6.4 (Left vs. right actions)**.** Suppose G is locally profinite and unimodular, and that $C = \mathbb{C}$. Fix a compact open subgroup K.

(a) Show that the algebras $e_K * \mathcal{H} * e_K$ and $\mathcal{H}(G,K,1_K)$ are isomorphic. (The isomorphism will depend on the choice of Haar measure μ.)
(b) Normalize the Haar measure μ so that $\mu(K) = 1$. Let (π, V) denote a smooth representation of G. On the one hand, we have a *left* action of $e_K * \mathcal{H} * e_K$ on V^K coming from Equation (2). On the other hand, we have a *right* action of $\mathcal{H}(G,K,1_K)$ on V^K coming from Equation (4). Prove that

$$v\cdot\mathbb{1}_{KgK} = \pi(\mathbb{1}_{Kg^{-1}K})v,$$

where $v \in V^K, g \in G$. Therefore, the two actions "agree," up to applying the anti-automorphism $g \longmapsto g^{-1}$ (which converts left $\mathcal{H}$-modules to right $\mathcal{H}$-modules).

Exercise 6.5. Let $G = \mathrm{GL}_2(F), K = \mathrm{GL}_2(\mathcal{O}_F)$ and 1_K the trivial representation of K over C.

(a) Show that we have a decomposition of cosets
$$K\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}K = K\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix} \sqcup \bigsqcup_{a\in\mathcal{O}_F/\mathfrak{p}_F} K\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}.$$[66]

(b) Let $\chi_1, \chi_2 : F^\times \longrightarrow C^\times$ be two unramified characters (i.e., $\chi_1(\mathcal{O}_F^\times) = \chi_2(\mathcal{O}_F^\times) = 1$), and let $f \in \mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)^K$ denote a spherical vector (see Exercise 5.8(b)). Calculate the following:
- $f \cdot \mathbb{1}_{K\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)K}$
- $f \cdot \mathbb{1}_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}$

Note that these calculations suffice to compute the action of the entire Hecke algebra $\mathcal{H}(G, K, 1_K)$.[67]

Exercise 6.6 (Gelfand pairs)**.** Suppose G is a locally profinite group and K is a compact open subgroup. In this exercise we will determine a sufficient condition for the Hecke algebra $\mathrm{End}_G(\text{c-ind}_K^G(1_K))$ to be commutative. We abbreviate $\mathcal{H}_K := \mathcal{H}(G, K, 1_K)$.

We suppose that G possesses an anti-involution ι, that is, a bijection $\iota : G \longrightarrow G$ satisfying $\iota(gh) = \iota(h)\iota(g)$ and $\iota^2 = 1$. We assume moreover that $\iota(K) = K$.

(a) Given $\phi \in \mathcal{H}_K$, define $\phi^\iota \in \mathcal{H}_K$ by $\phi^\iota(g) := \phi(\iota(g))$. Prove that $(\phi_1 * \phi_2)^\iota = \phi_2^\iota * \phi_1^\iota$.
(b) Suppose in addition that $\iota(KgK) = KgK$ (that is, we have equality as sets, *not* elementwise). Prove that $\phi^\iota = \phi$ for all $\phi \in \mathcal{H}_K$.
(c) Conclude that $\mathcal{H}_K$ is commutative.
(d) Show that the above setup applies to $(G, K) = (\mathrm{GL}_2(\mathbb{F}_p), B)$ (cf. Exercise 2.10) and $(G, K) = (\mathrm{GL}_n(F), \mathrm{GL}_n(\mathcal{O}_F))$.[68]

Remark. Pairs (G, K) possessing an anti-involution ι as above are known as **Gelfand pairs**. The representation theory of G and K then has many favorable properties; see [Gro91].

Exercise 6.7. Suppose (τ, W) is an irreducible cuspidal representation of $\mathrm{GL}_2(k_F)$ over $\mathbb{C}$, which we inflate to a representation of $K = \mathrm{GL}_2(\mathcal{O}_F)$ and extend to KZ by letting $\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)$ act by a scalar. We continue to denote the resulting representation of KZ by (τ, W), and examine the representation of $G = \mathrm{GL}_2(F)$ given by $\text{c-ind}_{KZ}^G(W)$.

(a) Suppose $\phi \in \mathcal{H}(G, KZ, \tau)$. Show that ϕ is determined by its values on the elements $\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi^n\end{smallmatrix}\right)$ for $n \geq 0$.
(b) Suppose $n \geq 1$. Show that the endomorphism $\phi(\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi^n\end{smallmatrix}\right)) : W \longrightarrow W$ has image contained in $W^{U\cap K}$. Deduce that $\phi(\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi^n\end{smallmatrix}\right)) = 0$.
(c) Show that $\phi(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)) = c \cdot \mathrm{id}_W$ for some $c \in \mathbb{C}$. Conclude that $\mathrm{End}_G(\text{c-ind}_{KZ}^G(W)) = \mathbb{C}$.[69]
(d) Repeat the above exercise for $G = \mathrm{GL}_n(F)$.
(e) Repeat the above exercise for an irreducible representation (τ, W) of $\mathrm{GL}_n(k_F)$ over $\overline{\mathbb{F}}_p$. What breaks down in the argument?

Exercise 6.8 (From J. Fintzen's lecture)**.** Suppose G is a p-adic reductive group, K a compact-mod-center open subgroup of G, and (ρ, W) a smooth representation of K. Recall that the **support of** $\mathcal{H}(G, K, \rho)$ is the set of all $g \in G$ for which there exists $\phi \in \mathcal{H}(G, K, \rho)$ satisfying $\phi(g) \neq 0$.

Show that $g \in \mathrm{Supp}(\mathcal{H}(G, K, \rho))$ if and only if g intertwines (ρ, W) (cf. Exercise 5.9).

Exercise 6.9 (Satake transform I)**.** Let $G = \mathrm{GL}_2(F)$, and let $C = \mathbb{C}$. Further, we let $K = \mathrm{GL}_2(\mathcal{O}_F)$, and $B = TU$ the upper triangular Borel subgroup. The **Satake transform/map** is an $\mathbb{C}$-algebra

[66] Hint: use the Bruhat decomposition for K and the Iwahori decomposition.

[67] Hint: we know that $f \cdot \mathbb{1}_{KgK}$ lands back in the one-dimensional space $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)^K$, and is therefore a scalar multiple of f. It therefore suffices to evaluate $f \cdot \mathbb{1}_{KgK}$ at 1 to determine this scalar.

[68] Hint: for the latter, use the transpose and the Cartan decomposition.

[69] Hint: Schur's lemma.

homomorphism $\mathcal{S} : \mathcal{H}(G, K, 1_K) \longrightarrow \mathcal{H}(T, T \cap K, 1_{T\cap K})$ defined by the formula

$$\mathcal{S}(\phi)(t) = \delta_B(t)^{1/2} \int_U \phi(tu)\ d\mu(u),$$

where δ_B is as in Exercise 5.5(e). Here we normalize Haar measures as follows: the Haar measure on G (resp., T, resp., U) gives K (resp., $T \cap K$, resp., $U \cap K$) measure 1.

(a) Show that $\mathcal{S}$ is well-defined, that is, check that $\mathcal{S}(\phi)$ is invariant by $T \cap K$ on both sides, and finitely supported modulo $T \cap K$.

(b) Show that $\mathcal{S}(\phi)(t)$ may be rewritten as

$$\mathcal{S}(\phi)(t) = \delta_B(t)^{-1/2} \int_U \phi(ut)\ d\mu(u).$$

(c) Let $\phi = \mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$ and suppose $t = \left(\begin{smallmatrix} \varpi^c & 0 \\ 0 & \varpi^d \end{smallmatrix}\right)$ $(c, d \in \mathbb{Z})$. Show that $\mathcal{S}(\mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K})(\left(\begin{smallmatrix} \varpi^c & 0 \\ 0 & \varpi^d \end{smallmatrix}\right)) \neq 0$ implies

$$K \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} K \cap \begin{pmatrix} \varpi^c & 0 \\ 0 & \varpi^d \end{pmatrix} U \neq \emptyset,$$

which in turn implies $(c, d) = (1, 0)$ or $(0, 1)$.

(d) Calculate $\mathcal{S}(\mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K}) = q^{1/2}\left(\mathbb{1}_{(T\cap K)\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)} + \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)}\right)$.[70]

(e) Proceed similarly to show that $\mathcal{S}(\mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)K}) = \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)}$.

(f)* More generally, show that if $a > b$, then we have

$$\begin{aligned} \mathcal{S}(\mathbb{1}_{K\left(\begin{smallmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{smallmatrix}\right)K}) &= q^{(a-b)/2}\left(\mathbb{1}_{(T\cap K)\left(\begin{smallmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{smallmatrix}\right)} + \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix} \varpi^b & 0 \\ 0 & \varpi^a \end{smallmatrix}\right)}\right) \\ &\quad + (q^{(a-b)/2} - q^{(a-b)/2-1}) \sum_{k=1}^{a-b-1} \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix} \varpi^{a-k} & 0 \\ 0 & \varpi^{b+k} \end{smallmatrix}\right)}. \end{aligned}$$

(g) Use the previous part to show that the image of $\mathcal{S}$ lands in the S_2-invariant functions of $\mathcal{H}(T, T\cap K, 1_{T\cap K})$. (Here, S_2 acts on the torus T by permuting entries, and this induces an action on functions on T.) Conclude that $\mathcal{S}$ induces an **isomorphism** $\mathcal{S} : \mathcal{H}(G, K, 1_K) \xrightarrow{\sim} \mathcal{H}(T, T\cap K, 1_{T\cap K})^{S_2}$.[71]

(h) Use the above formulas to calculate $\mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K} * \mathbb{1}_{K\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$.[72]

Remark. The Satake isomorphism $\mathcal{S} : \mathcal{H}(G, K, 1_K) \xrightarrow{\sim} \mathcal{H}(T, T\cap K, 1_{T\cap K})^W$ holds more generally for split connected reductive groups G, where W denotes the Weyl group of G. (In fact, we can even assume G is unramified, that is, quasi-split and split over an unramified extension of F.) The proof that $\mathcal{S}$ is an algebra homomorphism works in a manner similar to the $\mathrm{GL}_2(F)$ case, while the W-invariance of $\mathrm{Im}(\mathcal{S})$ in the general case uses clever integration tricks due to Harish-Chandra. For more details and nice properties of $\mathcal{S}$, see [Car79, §IV] and [Gro98].

Exercise 6.10 (Satake transform II)**.** We keep the notation of the Exercise 6.9., so that $G = \mathrm{GL}_2(F)$, etc.. Define an algebra homomorphism $\mathcal{S}_{\mathrm{un}} : \mathcal{H}(G, K, 1_K) \longrightarrow \mathcal{H}(T, T\cap K, 1_{T\cap K})$ by

$$\mathcal{S}_{\mathrm{un}}(\phi)(t) = \int_U \phi(tu)\ d\mu(u),$$

which we refer to as the unnormalized Satake map.

(a) Show that $\mathcal{S}_{\mathrm{un}}(\phi)(t) = \sum_{u \in U/(U\cap K)} \phi(tu)$.

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[70] Hint: for $t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, use the original definition. For $t = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, use part (b).

[71] Hint: choose a "simple" basis of $\mathcal{H}(T, T\cap K, 1_{T\cap K})^{S_2}$, and show that the matrix of $\mathcal{S}$ (for an appropriate ordering of bases) is upper-triangular with non-zero entries along the diagonal.

[72] Hint: use that $\mathcal{S}$ is an algebra homomorphism.

(b) Explain why the map $\mathcal{S}_{\mathrm{un}}$ gives an algebra homomorphism over any coefficient field C (or, more generally, over any coefficient ring R).

(c) Suppose now that $\mathrm{char}(C)$ is arbitrary, and define $\mathcal{S}_{\mathrm{un}}$ using the sum from part (a). Show that $\mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)K}) = q\mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)} + \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)}$ and $\mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)K}) = \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi & 0\\ 0 & \varpi\end{smallmatrix}\right)}$.[73]

(d) More generally, show that if $a > b$, then we have

$$\begin{aligned}\mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)K}) &= q^{a-b}\mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)} + \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^b & 0\\ 0 & \varpi^a\end{smallmatrix}\right)}\\ &\quad + \sum_{k=1}^{a-b-1}(q^{a-b-k} - q^{a-b-k-1})\mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^{a-k} & 0\\ 0 & \varpi^{b+k}\end{smallmatrix}\right)}.\end{aligned}$$

(e) Suppose further that $\mathrm{char}(C) = p$. Show that we have

$$\mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi^{b+1} & 0\\ 0 & \varpi^b\end{smallmatrix}\right)K}) = \mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi^b & 0\\ 0 & \varpi^{b+1}\end{smallmatrix}\right)K}) = \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^b & 0\\ 0 & \varpi^{b+1}\end{smallmatrix}\right)}$$

and, if $a - b \geq 2$, we have

$$\mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)K}) = \mathcal{S}_{\mathrm{un}}(\mathbb{1}_{K\left(\begin{smallmatrix}\varpi^b & 0\\ 0 & \varpi^a\end{smallmatrix}\right)K}) = \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^b & 0\\ 0 & \varpi^a\end{smallmatrix}\right)} - \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^{b+1} & 0\\ 0 & \varpi^{a-1}\end{smallmatrix}\right)}.$$

(f) Use the previous part to show that when $\mathrm{char}(C) = p$, $\mathcal{S}_{\mathrm{un}}$ induces an isomorphism $\mathcal{S}_{\mathrm{un}} : \mathcal{H}(G, K, 1_K) \xrightarrow{\sim} \mathcal{H}^-(T, T\cap K, 1_{T\cap K})$, where $\mathcal{H}^-(T, T\cap K, 1_{T\cap K})$ consists of those functions supported on the monoid $T^- = \{\left(\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right) \in T : a^{-1}d \in \mathcal{O}\}$.

Exercise 6.11 (Satake transform III)**.** Maintain the notation of the previous two exercises, and suppose $C = \overline{\mathbb{F}}_p$. We now take (τ, W) to be the $K = \mathrm{GL}_2(\mathcal{O}_F)$-representation equal to the inflation of the Steinberg representation of $\mathrm{GL}_2(k_F)$, so that

$$W \cong \mathrm{Sym}^{p-1}(\overline{\mathbb{F}}_p^{\oplus 2}) \otimes (\mathrm{Sym}^{p-1})(\overline{\mathbb{F}}_p^{\oplus 2})^{(p)} \otimes \ldots \otimes (\mathrm{Sym}^{p-1})(\overline{\mathbb{F}}_p^{\oplus 2})^{(p^{f-1})}.$$

Note that $W^{U\cap K}$ and $W_{U^-\cap K}$ are both one-dimensional, with trivial action of $T\cap K$. The Satake homomorphism $\mathcal{S} : \mathcal{H}(G, K, \tau) \longrightarrow \mathcal{H}(T, T\cap K, 1_{T\cap K})$ is given by

$$(\mathcal{S}(\phi))(t)(w) = \sum_{u\in U/(U\cap K)} \phi(tu)(w)$$

for $t \in T, w \in W^{U\cap K}$.

(a) Let $(a, b) \in \mathbb{Z}^2$ with $a \leq b$. Prove that up to scaling there exists a unique nonzero function $T_{(a,b)} \in \mathcal{H}(G, K, \tau)$ with support $K\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)K$, and describe $T_{(a,b)}(\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)) \in \mathrm{End}_{\overline{\mathbb{F}}_p}(W)$.[74]

(b) Show that $K\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)K \cap \left(\begin{smallmatrix}\varpi^c & 0\\ 0 & \varpi^d\end{smallmatrix}\right)U \neq \emptyset$ implies $(c, d) = (a + k, b - k)$ for some $k \geq 0$.[75]

(c) Using the previous part, show that

$$\mathcal{S}(T_{(a,b)}) = \mathbb{1}_{(T\cap K)\left(\begin{smallmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{smallmatrix}\right)}.$$

(d) Repeat the above calculations for (τ, W) an irreducible representation of $\mathrm{GL}_2(k_F)$ over $\overline{\mathbb{F}}_p$, inflated to K, and satisfying $\dim_{\overline{\mathbb{F}}_p}(W) > 1$. In this case, the target of $\mathcal{S}$ will be $\mathcal{H}(T, T\cap K, \chi)$, where $\chi : T\cap K \longrightarrow C^\times$ denotes the action of $T\cap K$ on $W^{U\cap K}$.

Remark. The Satake transform over characteristic p coefficient fields was defined by Herzig [Her11b] and further generalized by Henniart–Vignéras [HV15]. An explicit computation of the (inverse) Satake transform is given in [AHV22, Thm. 1.1] (see also [Her11a, Thm. 5.1]).

[73] Hint: use Exercise 6.9(d) and (e).

[74] Hint: use the strategy of Exercise 6.7.

[75] Hint: this was shown in the course of Exercise 6.9. The identity $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\ x^{-1} & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ x^{-1} & 0\end{smallmatrix}\right)$ may also be useful.

Exercise 6.12 (Iwahori–Hecke algebras)**.** Suppose $G = \mathrm{GL}_2(F)$, and let $I = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F^\times \end{pmatrix}$ denote the "upper triangular" Iwahori subgroup. We let T denote the diagonal torus in G, and N its normalizer.

(a) Show that

$$G = \bigsqcup_{w \in N/(T\cap I)} IwI = \bigsqcup_{a,b\in\mathbb{Z}} I \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} I \sqcup \bigsqcup_{a,b\in\mathbb{Z}} I \begin{pmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{pmatrix} I.$$

(b) Define $s := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\omega := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$. Show that each coset as above admits a representative of the form $\omega^\ell(\omega s)^m\omega^n$, where $\ell \in \{0,1\}, m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}$.

(c) We consider the two elements of $\mathcal{H}(G, I, 1_I)$ (over C) given by $\mathbb{1}_{IsI}$ and $\mathbb{1}_{I\omega I}$. Show that, for any $g \in G$, we have[76]
- $\mathbb{1}_{IsI}^2 := \mathbb{1}_{IsI} * \mathbb{1}_{IsI} = (q-1)\mathbb{1}_{IsI} + q\mathbb{1}_I$,
- $\mathbb{1}_{IgI} * \mathbb{1}_{I\omega I} = \mathbb{1}_{Ig\omega I}$,
- $\mathbb{1}_{I\omega I} * \mathbb{1}_{IgI} = \mathbb{1}_{I\omega g I}$,
- $\mathbb{1}_{I\omega s I}^m = \mathbb{1}_{I(\omega s)^m I}$,
- $\mathbb{1}_{I\omega^2 I}$ is central in $\mathcal{H}(G, I, 1_I)$.

(d) Deduce that $\mathcal{H}(G, I, 1_I)$ is generated as an algebra by $\mathbb{1}_{IsI}$ and $\mathbb{1}_{I\omega I}$.

Remark. Suppose G is the group of F-points of a split, semisimple, simply connected algebraic group, and I is an Iwahori subgroup of G. Then the algebra $\mathcal{H}(G, I, 1_I)$ admits a description in terms of the **affine Weyl group** of G: $\mathcal{H}(G, I, 1_I)$ is generated by $\mathbb{1}_{Is_iI}$, where the s_i are simple affine reflections. These generators satisfy **braid relations** determined by the affine Dynkin diagram of G, and **quadratic relations** of the form

$$\mathbb{1}_{Is_iI}^2 = (q-1)\mathbb{1}_{Is_iI} + q\mathbb{1}_I.$$

Thus, $\mathcal{H}(G, I, 1_I)$ is a q-deformation of the group algebra of affine Weyl group of G over C, and is generally not commutative. For more details, including the case when G is not assumed to be semisimple, see [Vig16].

Exercise 6.13. Suppose $G = \mathrm{GL}_2(F)$, and I_1 is the "upper triangular" pro-p Iwahori subgroup. Analogously to Exercise 6.12, the Hecke algebra $\mathcal{H}(G, I_1, 1_{I_1})$ over C is generated by the elements

$$\mathbb{1}_{I_1\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)I_1}, \qquad \mathbb{1}_{I_1\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)I_1}, \qquad \text{and} \qquad \mathbb{1}_{I_1\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)I_1}, \tag{5}$$

where $a, d \in \mathcal{O}_F^\times$.

Suppose $\chi_1, \chi_2 : F^\times \longrightarrow C^\times$ are two smooth characters which are *tamely ramified* (i.e., $\chi_1(1 + \mathfrak{p}_F) = \chi_2(1 + \mathfrak{p}_F) = 1$). We consider the principal series representation $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$.

(a) Prove that $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)^{I_1}$ is spanned by two functions f_1, f_s, where $\mathrm{supp}(f_v) = BvI_1$ and $f_v(v') = \delta_{v,v'}$ for $v, v' \in \{1, s\}$.

(b) Calculate $f_v \cdot \phi$ for $v \in \{1, s\}$ and ϕ as in (5).[77] (Since (5) is a generating set, this gives the structure of $\mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)^{I_1}$ as a $\mathcal{H}(G, I_1, 1_{I_1})$-module.)

(c) Describe the structure of $1_G^{I_1}$ and $(\mathrm{St}_B^G)^{I_1}$ as modules over $\mathcal{H}(G, I_1, 1_{I_1})$.[78]

7. Characters

In this section we work with the coefficient field $C = \mathbb{C}$ and assume $\mathrm{char}(F) = 0$. Suppose G is a p-adic reductive group, and let (π, V) denote a smooth, admissible representation of G.

[76]Hint: for s, inflate the Bruhat decomposition from $\mathrm{GL}_2(k_F)$; for ω; use that ω normalizes I.

[77]Hint: use Equation (4).

[78]Hint: this should not require extra calculation; use the previous part.

Since V is admissible, the linear operator $\pi(f)$ has finite rank, so it makes sense to define $\mathrm{Tr}(\pi(f))$. Letting G_{rss} denote the open dense subset of regular semisimple elements, there exists a unique, locally constant, conjugation-invariant function θ_π on G_{rss} satisfying

$$\mathrm{Tr}(\pi(f)) = \int_G f(g)\theta_\pi(g)\ d\mu(g).$$

We refer to θ_π as the **character of** (π, V).

Exercise 7.1. Suppose $\chi : G \longrightarrow \mathbb{C}^\times$ is a one-dimensional smooth representation. Show that the character θ_χ is equal to the function χ.

Exercise 7.2. Let $K \subset G$ denote a compact open subgroup. Show that $\int_K \theta_\pi(k)\ d\mu(k) = \mu(K)\dim_{\mathbb{C}}(V^K)$.

We consider the following special case: suppose that G is semisimple (for example, $G = \mathrm{SL}_n(F)$), and let (π, V) be an irreducible supercuspidal representation of the form $V = \text{c-ind}_J^G(W)$, where J is a compact open subgroup of G and (τ, W) is an irreducible representation of J. Define $\dot{\boldsymbol{\chi}}_\tau$ by

$$\dot{\boldsymbol{\chi}}_\tau(g) = \begin{cases} \boldsymbol{\chi}_\tau(g) & \text{if } g \in J, \\ 0 & \text{if } g \in G \smallsetminus J. \end{cases}$$

Then, for $g \in G_{\mathrm{rss}}$ we have

$$\theta_\pi(g) = \sum_{x \in J\backslash G/J} \sum_{J\backslash JxJ \ni y} \dot{\boldsymbol{\chi}}_\tau(ygy^{-1})$$

(see [Sal88, Thm. 1.9]; here we normalize the Haar measure on G so that $\mu(J) = 1$).

Exercise 7.3. Suppose $p > 2$, $G = \mathrm{SL}_2(F)$, and let (τ, W) denote an irreducible cuspidal representation of $\mathrm{SL}_2(k_F)$, inflated $K = \mathrm{SL}_2(\mathcal{O}_F)$. We consider the supercuspidal representation $\text{c-ind}_K^G(W)$.

(a) For every $n \geq 0$, calculate a set of coset representatives for $K\backslash K \left(\begin{smallmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{smallmatrix}\right) K$.

(b) We will compute the values of θ_π on regular (semisimple) elements of the diagonal torus T. Fix $t = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T, a \neq \pm 1$. Show that if $a \notin \pm 1 + \mathfrak{p}_F$, then $\dot{\boldsymbol{\chi}}_\tau(yty^{-1}) = 0$, where $K\backslash K \left(\begin{smallmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{smallmatrix}\right) K \ni y$.[79]

(c) Since $\theta_\pi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) g) = \omega_\tau(-1)\theta_\pi(g)$, it suffices to compute $\theta_\pi(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right))$ for $a \in 1 + \mathfrak{p}_F$. Evaluate the sum $\sum_{K\backslash K\left(\begin{smallmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{smallmatrix}\right)K \ni y} \dot{\boldsymbol{\chi}}_\tau(yty^{-1})$ and conclude the calculation. (Your answer will depend on the depth of t.)

(d)* Proceed as above to calculate $\theta_\pi(t')$, there t' lies in a non-split torus.

Exercise 7.4. Suppose $p > 2$, $G = \mathrm{SL}_2(F)$, and let (π, V) denote the irreducible, supercuspidal representation $\text{c-ind}_{ZI_1}^G(\psi)$ constructed in Exercise 5.12. We will compute some values of θ_π.

(a) Calculate convenient set of coset representatives for $ZI_1\backslash G/ZI_1$, and for each coset representative w, calculate $ZI_1\backslash ZI_1 w ZI_1$.[80]

(b) As with the previous exercise, we focus on computing θ_π on regular (semisimple) elements of T. Fix $t = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T, a \neq \pm 1$. Show that if $a \notin \pm 1 + \mathfrak{p}_F$, then $\dot{\psi}(yty^{-1}) = 0$.

(c) Suppose $a \in 1 + \mathfrak{p}_F$. For each of the convenient coset representatives from part (a), evaluate $\sum_{ZI_1\backslash ZI_1 w ZI_1 \ni y} \dot{\psi}(yty^{-1})$, and finally compute $\theta_\pi(t)$.[81]

(d)* Proceed as above to calculate $\theta_\pi(t')$, where t' lies in a non-split torus.

Remark. When $G = \mathrm{SL}_2(F)$ and $p > 2$, the characters θ_π for supercuspidal representations have been completely worked out in [ADSS11], expanding on work of Sally–Shalika.

[79] Hint: the character table of $\mathrm{SL}_2(k_F)$ may be useful here.

[80] Hint: use the affine Bruhat decomposition.

[81] Hint: it may help to split the calculation into cases where (1) w is diagonal; (2) w is anti-diagonal.

Exercise 7.5 (Jacquet–Langlands correspondence)**.** Suppose $p > 2$. Let $G = \mathrm{GL}_2(F)$ and consider $D^\times$, where D is a division algebra with center F of dimension 4 over F.

(a) Suppose $d \in D^\times \smallsetminus F^\times$. Show that the minimal polynomial of d is quadratic and irreducible over F.[82]

(b) An element $g \in G$ is called **elliptic** if its characteristic polynomial is irreducible over F. Show that if g is elliptic, there exists $g_D \in D^\times$ with the same minimal polynomial as g.

(c) Let St_B^G denote the Steinberg representation of G, and let $\theta_{\mathrm{St}_B^G}$ denote its character. Likewise, let $1_{D^\times}$ denote the trivial representation of $D^\times$, and $\theta_{1_{D^\times}}$ its character. Show that if $g \in G$ is elliptic and $g_D \in D^\times$ has the same minimal polynomial as g, then we have

$$\theta_{\mathrm{St}_B^G}(g) = -\theta_{1_{D^\times}}(g_D).$$

(d)* Suppose now that E/F is an unramified quadratic extension, and $\chi : E^\times \longrightarrow \mathbb{C}^\times$ is a character which is trivial on $1 + \mathfrak{p}_E$ and which satisfies $\chi^\sigma \neq \chi$, where σ is a generator of $\mathrm{Gal}(E/F)$. We construct two representations as follows:

- Let V_{χ_0} denote the cuspidal representation of $\mathrm{GL}_2(k_F)$ associated to the character $\chi_0 : k_E^\times \longrightarrow \mathbb{C}^\times$ (obtained as the mod ϖ reduction of the restriction of χ to $\mathcal{O}_E^\times$), and let $\widetilde{V}_{\chi_0}$ denote the representation of $Z\mathrm{GL}_2(\mathcal{O}_F)$ obtained by making $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ act by $\chi(\varpi)$. We then obtain the irreducible, supercuspidal representation (π, V) by $\text{c-ind}_{Z\mathrm{GL}_2(\mathcal{O}_F)}^G(\widetilde{V}_{\chi_0})$.
- The group $E^\times$ embeds as a subgroup of $D^\times$. We obtain a one-dimensional representation of $E^\times(1+\mathfrak{p}_D) = \mathcal{O}_D^\times \varpi^{\mathbb{Z}}$ by making $1+\mathfrak{p}_D$ act trivially on χ. We then obtain the representation (π', V') by taking $\text{c-ind}_{E^\times(1+\mathfrak{p}_D)}^{D^\times}(\chi)$.

Suppose $g \in G$ is elliptic and $g_D \in D^\times$ has the same minimal polynomial as g. Show that we have

$$\theta_\pi(g) = -\theta_{\pi'}(g_D).$$

Remark. The above exercise is an instance of the **Jacquet–Langlands correspondence**. In the context above, the correspondence states that we have a unique bijection between isomorphism classes of smooth, irreducible, essentially square-integrable representations π of $\mathrm{GL}_2(F)$ and isomorphism classes of smooth, irreducible representations π' of $D^\times$ which matches θ_π and $-\theta_{\pi'}$ on elliptic elements. For the general case, see [DKV84].

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