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Lecture 4:

Eisenstein cocycles and Stark units in case TR_p.

$k > 2$, $v = (v_1, v_2) \in \mathbb{Q}^2/\mathbb{Z}^2$, $z \in \mathcal{H}$

$$E_{k,v}(z) = \sum_{m,n \in \mathbb{Z}}' \frac{e(mv_1 + nv_2)}{(mz+n)^k} \qquad e(x) = e^{2\pi i x}$$

$$E_{k,v}\big|_\gamma(z) := (cz+d)^{-k} E_{k,v}\left(\frac{az+b}{cz+d}\right)$$

$$= E_{k,\gamma^{-1}v}(z) \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} = \Gamma$$

In particular,

$$E_{k,v}(z) \in M_k(\Gamma(N)) \text{ where } N = \text{common denominator}(v_1, v_2)$$

$\mathcal{P}_k$ = homogeneous polys of deg $k \subset \mathbb{C}[x,y] =: \mathcal{P}$

$(\mathcal{P}_{\mathbb{Q}} := \mathbb{Q}[x,y])$

$\gamma \in SL_2\mathbb{Z}$, $(\gamma P)(x,y) = P((x,y)\gamma)$

<u>Siegel's Formula</u> F real quad field

$\mathfrak{f} \subset \mathcal{O}_F$, $K = K_{\mathfrak{f}}$ ray class field of cond $\mathfrak{f}$

$R = \{\infty_1, \infty_2, \mathfrak{p} | \mathfrak{f}\}$. Fix $\mathfrak{a} \subset \mathcal{O}_F$, $(\mathfrak{a}, \mathfrak{f}) = 1$.

$\mathfrak{a}^{-1}\mathfrak{f} = \langle w_1, w_2 \rangle$

$w_1 \bar{w}_2 - \bar{w}_1 w_2 > 0$

$P(x,y) = N\mathfrak{a} \cdot \mathrm{Norm}_{F/\mathbb{Q}}(xw_1 + yw_2) \in \mathcal{P}_{\mathbb{Q},2}$

$(w_1 \;\; w_2)\,\varepsilon = (w_1 \;\; w_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$.

$\langle \varepsilon \rangle = E(\mathfrak{f})$, Define $v \in \mathbb{Q}^2$ by

$0 < \varepsilon < 1$ $\qquad 1 = v_1 w_1 + v_2 w$

Thm (Siegel) Fix $\tau \in \mathcal{H}$. For $r \geq 1$

$$\zeta_{K/F,R}(\sigma_{\Omega}, 1-r) = \frac{(2r-1)!}{(2\pi i)^{2r}} \int_{\tau}^{\gamma\tau} P(z,1) \cdot E_{2r,v}(z)\, dz$$

$$= \Psi_\tau(\gamma)(P,v).$$

$V := \mathbb{Q}^2/\mathbb{Z}^2 - \{0\}$.

Define: $\Psi_\tau(\gamma) : \mathcal{P} \times V \longrightarrow \mathbb{C}$ by

for $P \in \mathcal{P}_d$,

$$\Psi_\tau(\gamma)(P,v) = \frac{(d+1)!}{(2\pi i)^{d+2}} \int_{\tau}^{\gamma\tau} P(z,1) \cdot E_{d+2,v}(z)\, dz$$

$$M = \left\{ f : \mathcal{P} \times V \longrightarrow \mathbb{C}, \text{ linear in } \mathcal{P} \text{ and sat. dist. rel. in } V \right\}$$

$f \in M$, $\gamma \in \Gamma$,

$$(\gamma f)(P,v) = f(\gamma^t P, \gamma^{-1} v)$$

Prop $\Psi_\tau(AB) = \Psi_\tau(A) + (A\Psi_\tau)(B)$

for $A, B \in \Gamma$

i.e. $\Psi_\tau \in Z^1(\Gamma, M)$

$[\Psi_\tau] \in H^1(\Gamma, M)$ does not depend on $\tau \in \mathcal{H}$

Smoothing Fix ℓ prime.

$$E^{(\ell)}_{k,v} = \ell^{k-2}\left(E_{k,(\ell v_1, v_2)}(\ell z) - E_{k,v}(z)\right)$$

$$E^{(\ell)}_{k,v}\Big|_\gamma = E^{(\ell)}_{k,\gamma^{-1}v} \quad \text{for } \gamma \in \Gamma_0(\ell)$$

$$v \in V_\ell = \left(\mathbb{Q}^2 - \left(\tfrac{1}{\ell}\mathbb{Z} + \mathbb{Z}\right)\right) / \mathbb{Z}^2$$

$\Psi_{\tau,\ell}(\gamma)(P, V) =$ as Ψ_τ with $E^{(\ell)}_{k,v}$ instead of $E_{k,v}$

$$\Psi_{\tau,\ell} \in Z^1(\Gamma_0(\ell), M_\ell)$$

M_ℓ = same as M,

V replaced by V_ℓ.

Const term of $E^{(\ell)}_{k,v}$ is 0 at ∞ and at $\Gamma_0(\ell)\infty$

$\Longrightarrow$ can take $\tau = \infty$ in our defn, i.e. $\Psi_{\infty,\ell}$ makes sense.

"partial modular symbol"

Integrality thm

"**Thm**" (D-Darmon)

$\Psi_{\infty,\ell}(P,v) \in \mathbb{Z}[\frac{1}{\ell}]$ if $P \in \mathbb{Z}[\frac{1}{\ell}][x,y]$

<u>and</u> $P(v + \mathbb{Z}[\frac{1}{\ell}] \oplus \mathbb{Z}) \subset \mathbb{Z}[\frac{1}{\ell}]$

$\Psi_{\infty,\ell}(1,v) \in \mathbb{Z}$. $(\ell \geq 5)$

Siegel's Thm revisited

Fix ideal $\mathfrak{l} \subset \mathcal{O}_F$ s.t. $N\mathfrak{l} = \ell$

(assume exists)

$T = \{\mathfrak{l}\}$ $\qquad \mathfrak{a}^{-1}\mathfrak{f} = \langle w_1, w_2 \rangle$

$\qquad\qquad\qquad \mathfrak{a}^{-1}\mathfrak{l}^{-1}\mathfrak{f} = \langle \frac{1}{\ell} w_1, w_2 \rangle$

<u>Cor</u> $\zeta_{K/F,R,T}(\sigma_{\mathfrak{a}}, 1-r) = \Psi_{\infty,\ell}(\gamma)(P_{,v}^{r-1}) \in \mathbb{Z}[\frac{1}{\ell}]$

and $\in \mathbb{Z}$ if $r=1$, $\ell \geq 5$.

(Coates-Sinnott)

Sczech's construction of an Eisenstein cocycle.

bogus calculation:

$$\Psi_\tau(\gamma)(1,v) = \frac{1}{(2\pi i)^2} \int_\tau^{\gamma\tau} \sum_{m,n\in\mathbb{Z}}{}' \frac{e(mv_1+nv_2)}{(mz+n)^2}\, dz$$

$$\text{“}=\text{''} \frac{1}{(2\pi i)^2} \sum_{m,n\in\mathbb{Z}}{}' e(mv_1+nv_2) \underbrace{\int_\tau^{\gamma\tau} \frac{1}{(mz+n)^2}\, dz}_{= \frac{(\gamma\tau-\tau)}{(m(\gamma\tau)+n)(m\tau+n)}}$$

Formally plug in
$\tau = r/s \in \mathbb{Q}$

$$\sigma_1 = \begin{pmatrix} r \\ s \end{pmatrix} \quad \sigma_2 = \gamma \begin{pmatrix} r \\ s \end{pmatrix}, \quad \sigma = (\sigma_1, \sigma_2) \in M_2(\mathbb{Z})$$

$$\Phi(\gamma)(1, v) \text{ "="} \frac{1}{(2\pi i)^2} \sum_{z=(m,n)\in\mathbb{Z}^2}{}' \frac{\det(\sigma)\cdot e(\langle z, v\rangle)}{\langle z, \sigma_1\rangle \langle z, \sigma_2\rangle}$$

Problems: – den = 0?

– convergence?
