ABELIAN VARIETIES OVER FINITE FIELDS: PROBLEM SET 5

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Instructions: The goal of this problem set is to venture into the world of p-divisible groups and Dieudonné modules. Problems marked ($\star$), ($\star\star$), and ($\star\star\star$) denote beginner, intermediate, and advanced problems, respectively.

Notation: As customary, p will be a prime, and q will be a power of p.

In the first two problems, we explore the Newton polygon of a polynomial and use it to define the q-Newton polygon of an abelian variety. These problems are inspired by problems from [Poo06], which serves as a good complementary reference.

Problem 1 ($\star$)

Let K be a field with a non-archimedean valuation $v\colon K^\times \to \mathbb{R}$. The Newton polygon of a polynomial $P(T) = a_0T^n + a_1T^{n-1} + \cdots + a_{n-1}T + a_n$ is the lower convex hull of the finite set $\{(j, v(a_j)) \in \mathbb{R}^2 : 0 \leq j \leq n \text{ and } a_j \neq 0\}$. We will denote it by $\mathcal{N}(P) = \mathcal{N}(P, v)$. We define the width of a line segment from (a, b) to (c, d) (with $a < c$) to be $c - a$.

Theorem A. *Suppose that (K, v) above is complete, so that there is a unique extension v_L of v to any algebraic field extension $L \supset K$. Let $\bar{K}$ be an algebraic closure of K, and let $\bar{v}$ denote the extension of v to $\bar{K}$. Then,*

$$\#\{\alpha \in \bar{K} : P(\alpha) = 0 \text{ and } \bar{v}(\alpha) = s\} = \text{ width of the segment of slope } s \text{ in } \mathcal{N}(P).$$

(1) Prove Theorem A.[a]
(2) Let m be a positive integer. How does $\mathcal{N}(P)$ compare to $\mathcal{N}(P^m)$?
(3) How does the Newton polygon of a product of polynomials relate to the Newton polygons of the factors?

[a] Hint: By changing $P(T)$ to $P(\lambda T)$ for some suitable $\lambda \in \bar{K}$, reduce to the case of slope $s = 0$. Start with $P(T)$ in factored form, and in terms of the number of zeros with positive and negative valuation, determine the location of the slope-zero part of the Newton polygon.

In the context of abelian varieties over finite fields, we focus on the case where $K = \mathbb{Q}_p$, and p is the characteristic of our base field $\mathbb{F}_q$.

Problem 2 ($\star$)

Let the q-valuation $\bar{v}\colon \overline{\mathbb{Q}}_p^\times \to \mathbb{R}$ to be the p-adic valuation renormalized so that $\bar{v}(q) = 1$. We can define the q-Newton polygon of an abelian variety $A/\mathbb{F}_q$ to be the Newton polygon of the characteristic polynomial of Frobenius $P_A(T)$ with respect to the q-valuation $\bar{v}$. We write $\mathcal{N}(A) := \mathcal{N}(P_A(T), \bar{v})$. Newton polygons of g-dimensional abelian varieties over $\mathbb{F}_q$ satisfy the following properties:[a]

a. The left endpoint is $(0, 0)$ and the right endpoint is $(2g, g)$.
b. The vertices are all integer points with nonnegative second coordinate.
c. The vertices are symmetric: (i, j) is a vertex if and only if $(2g - i, g - j)$ is a vertex. Equivalently, (i, j) lies above the polygon if and only if $(2g - i, g - j)$ does so.

We say a Newton polygon is admissible if it satisfies properties a, b, c.

(1) Describe the admissible Newton polygons for $g \leq 3$.
(2) Are all admissible Newton polygons realized by some abelian variety of dimension $g \leq 3$? Find explicit examples in the `LMFDB` for each one.
(3) How does the Newton polygon of an abelian variety relate to the Newton polygons of its simple factors in the isogeny category?
(4) How does the q-Newton polygon of A compare to the q^r-Newton polygon of $A_{\mathbb{F}_{q^r}}$?
(5) Calculate the Newton polygon of the varieties described in PSET 4 Problem 11.

[a]See how many of these you can prove!

The following problem establishes the basics of the ring of Witt vectors attached to a commutative ring. It is taken from [Neu13, Chapter II. Exercise 2-5].

Problem 3 ($\star\star$)
Let $X_0, X_1, \dots$ be an infinite sequence of variables, and p a prime number. For each $n \in \mathbb{Z}_{\geq 1}$, let $W_n(X_0, \dots, X_n) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$.

(1) Show that there exists polynomials $S_0, S_1, \dots ; P_0, P_1, \dots \in \mathbb{Z}[X_0, X_1, \dots ; Y_0, Y_1, \dots]$ such that

$$W_n(S_0, S_1, \dots, S_n) = W_n(X_0, X_1, \dots, X_n) + W_n(Y_0, Y_1, \dots, Y_n)$$
$$W_n(P_0, P_1, \dots, P_n) = W_n(X_0, X_1, \dots, X_n) \cdot W_n(Y_0, Y_1, \dots, Y_n)$$

Now, let A be a commutative ring such that $pA = 0$. Let $\underline{a} := (a_0, a_1, \dots)$ be an infinite tuple with $a_i \in A$. We make the set of such tuples into a commutative ring $W(A)$ as follows. For two such tuples $\underline{a} = (a_0, a_1, \dots), \underline{b} = (b_0, b_1, \dots)$, define addition and multiplication

$$\underline{a} + \underline{b} := (S_0(a, b), S_1(a, b), \dots) \text{ and } \underline{a} \cdot \underline{b} := (P_0(a, b), P_1(a, b), \dots).$$

$W(A)$ is **the ring of (p-typical) Witt vectors** attached to A.

(2) Check that $1 := (1, 0, \dots)$ is the multiplicative identity of $W(A)$, and that $p := 1 + 1 + \cdots + 1$ is the element $(0, 1, 0, \dots)$ in $W(A)$.
(3) Show that $\underline{a} = \underline{b}$ in $W(A)$ if and only if $W_n(\underline{a}) = W_n(\underline{b})$.
(4) For every Witt vector $\underline{a} = (a_0, a_1, \dots) \in W(A)$, we define the **ghost components** $a^{(n)}$ as

$$\underline{a}^{(n)} := W_n(\underline{a}) = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n.$$

Consider mappings $V, F : W(A) \to W(A)$ defined by

$$V(\underline{a}) := (0, a_0, a_1, \dots) \text{ and } F(\underline{a}) := (a_0^p, a_1^p, \dots).$$

Show that

$$V(\underline{a})^{(n)} = p\underline{a}^{(n-1)} \text{ and } \underline{a}^{(n)} = (F(\underline{a}))^{(n)} + p^n a_n.$$

(5) Now let K be a field of characteristic p. Show that V is a homomorphism of $W(K)$ as an additive group, F is a homomorphism of $W(K)$ as a ring, and

$$V \circ F(\underline{a}) = F \circ V(\underline{a}) = p \cdot \underline{a} = (0, a_0^p, a_1^p, \dots)^a$$

(6) ($\star\star\star$) If K is a perfect field of characteristic p, then $W(K)$ is a complete discrete valuation ring with residue field K and maximal ideal $pW(K)$.
(7) ($\star\star\star$) Show that $W(\mathbb{F}_{p^n}) \cong \mathbb{Z}_{p^n}$, which is the valuation ring of $\mathbb{Q}_{p^n}$, the unique degree n unramified extension of $\mathbb{Q}_p$.

[a]To show that f, g are the same map from $W(A) \to W(A)$, it suffices to show that $W_n \circ f = W_n \circ g$ from $W(A)$ to A.

The next problem is Exercise 7.4.5 in [BC09], which gives a different way to understand the Witt vectors.

Problem 4 ($\star\star$)
Let k be an arbitrary field of characteristic $p > 0$.

(1) Use the addition law on the truncated Witt ring W_n defined in Problem 3 (applied to all k-algebras), to explain how this gives $\mathbb{A}_k^n$ the structure of a smooth group variety W_n.
(2) Describe the group variety structure explicitly for $n = 2$ and any k.
(3) Recall from PSET 1, Problem 8 the idea of a ring variety. Write down the axioms to define a "commutative ring scheme" and exhibit W_n as such an example.

The following is Lemma/Exercise after Definition 4.28 in [CO09]. It introduces the notion of the Dieudonné ring and the local Cartier ring.

Problem 5 ($\star$)
Let K be a perfect field of characteristic p. Let $W(K)$ be the ring of Witt vectors and let $\sigma\colon W(K) \to W(K)$ be the homomorphism $(a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots)$. The Dieudonné ring D_K is defined to the polynomial ring $W(K)[F, V]$ satisfying $FV = VF = p$, $F\underline{a} = \underline{a}^\sigma F$, $V\underline{a}^\sigma = \underline{a}V$.

(1) Show that the Dieudonné ring D_K can be naturally identified with the $\mathbb{Z}$-graded ring $\bigoplus_{i\in\mathbb{Z}} c_i V^i W(K)$ with the relation $\underline{a}V^n = V^n \underline{a}^{\sigma^n}$, where $c_i = p^{-i}$ if $i < 0$ and $c_i = 1$ otherwise. This means $W(k)[F, V]$ is the ring consisting of finite sums $\sum_i a_i V^i$ where $a_i \in W(K)$, $v_K(a_i) \geq \max\{0, -i\}$.
(2) Let $W(K)[[V, F\rangle\rangle$ be the ring consisting of formal Laurent series $\sum_i a_i V^i$ where $a_i \in W(K)$, $v_K(a_i) \geq \max\{0, -i\}$, and $v_p(a_i) + i \to \infty$ as $|i| \to \infty$. Again the relation $\underline{a}V^n = V^n \underline{a}^{\sigma^n}$ is given. Let $v\colon W(K)[[V, F\rangle\rangle \to \mathbb{Z}$ be defined by $v(\sum_i a_i V^i) = \min_i\{v_K(a_i) + i\}$. Show that v is a discrete valuation on $W(K)[[V, F\rangle\rangle$. [a]
(3) Show that the inclusion $W(K)[F, V] \hookrightarrow W(K)[[V, F\rangle\rangle$ is a ring homomorphism whose image is dense. [b]

[a] This ring can be naturally identified with the local Cartier ring $\mathrm{Cart}_p(K)$.
[b] This indicates that the Dieudonné ring can be naturally identified as a dense subring of the local Cartier ring.

We compute the Cartier duals of some finite flat group schemes.

Problem 6 ($\star\star$)
Let k be a field. Compute the Cartier duals of the following commutative k-groups.

(1) $\underline{\mathbb{Z}/n\mathbb{Z}}$. Recall that as a k-scheme, this is given by $\operatorname{Spec} A$ where $A := \prod_{i\in\mathbb{Z}/n\mathbb{Z}} e_i k$. The multiplication on A is defined by $e_i \cdot e_j = \delta_{ij} e_i$, and the co-multiplication is given by $\Delta(e_r) = \sum_{i+j=r} e_i \otimes e_j$.
(2) When k has characteristic p, the group $\alpha_p := \operatorname{Spec} k[x]/(x^p)$, considered as a subgroup of $\mathbb{A}_k^1$.

In problem 7 and 8, we use Dieudonné modules to classify the commutative finite flat group schemes of order p defined over an algebraically closed field k of characterstic p, and apply this to study the p-torsion group scheme of a supersingular elliptic curve over k. If you get stuck, the solutions can be found here.

Problem 7 ($\star$)
Let k be an algebraically closed field of characterstic p. Let $D_k = W(k)[F, V]$ be the Dieudonné ring.

(1) Using [BC09, Theorem 7.2.4], there is an equivalence of categories between commutative order p finite flat group schemes over k and D_k-modules M whose underlying $W(k)$-module is of length 1. Use (6) from Problem 3 to show that such an M must be isomorphic to $W(k)/(p)$ as a $W(k)$-module.
(2) To specify the D_k-module structure on M, it suffices to write down the action of F and V. Let e be a basis element of M as a 1-dimensional k-vector space. Let $\alpha, \beta \in k$ be such that
$$Fe = \alpha e, \quad Ve = \beta e.$$
Show that at least one of α, β is zero.
(3) Conversely, show that upon fixing a basis element e, any choice of (α, β) with at least one of α and β being 0 uniquely determines a Dieudonné module over $W(\bar{k})$ of length 1.
(4) Use a change of basis $e' := \lambda e$ for some $\lambda \in k^\times$ to show that if one of α, β is nonzero, then it equals 1.

(5) Now we have reduced to the cases (α, β) being $(0,0)$, $(1,0)$, or $(0,1)$. There are three well-known finite flat group schemes of order p over a characteristic p field: μ_p, $\underline{\mathbb{Z}/p\mathbb{Z}}$, and α_p. For each group scheme, find out whether it is connected, étale, or neither.
(6) Show that the relative Frobenius kills a connected order p group scheme over k, and is trivial on an étale group scheme.[a] Deduce that the $(1,0)$ Dieudonné module must correspond to $\underline{\mathbb{Z}/p\mathbb{Z}}$.
(7) Use the definition of the Verschiebung morphism on a group scheme together with Problem 6 to decide which of μ_p, α_p correspond to $(0,1)$, and which to $(0,0)$.

[a]Hint: See these notes by Andrew Snowden.

Problem 8 ($\star\star$)
Let $E/\overline{\mathbb{F}}_p$ be a supersingular elliptic curve. We will show there is a unique group scheme G over $\overline{\mathbb{F}}_p$ of order p^2 such that $E[p] \cong G$.
(1) Using [BC09, Theorem 7.2.4] again, a group scheme G over k of order p^2 corresponds to a Dieudonné module $M(G)$ of length 2 as a $W(\overline{\mathbb{F}}_p)$-module. Show that if G is p-torsion, then so is $M(G)$. In particular, $M(G)$ must be isomorphic to $W(\overline{\mathbb{F}}_p)/(p) \oplus W(\overline{\mathbb{F}}_p)/(p)$ as a $W(\overline{\mathbb{F}}_p)$-module.
(2) ($\star\star$) Use the connected-étale sequence and the fact that $\#E[p](\overline{\mathbb{F}}_p) = 1$ to show that $E[p]$ is connected.
(3) ($\star\star$) As an extension of Part (6) of Problem 7, one can show the relative Frobenius ϕ_G is a finite flat morphism of degree p, and is nilpotent on any connected finite flat group scheme G over a field. Use this to show that the kernel of ϕ_G is an order p flat group scheme, and so the Dieudonné module of $\ker(\phi_G)$ must be isomorphic to $\overline{\mathbb{F}}_p$ as a $W(\overline{\mathbb{F}}_p)$-module.
(4) The induced action of Frobenius on the Dieudonné module $M(E[p])$ is also nilpotent by functoriality, so we can choose an $\overline{\mathbb{F}}_p$-basis e_1, e_2 of $M(E[p])$ so that
$$Fe_1 = e_2, \quad Fe_2 = 0.$$
Show that $Ve_2 = 0$, and $Ve_1 = \alpha e_2$ for some $\alpha \in \overline{\mathbb{F}}_p$. Show that $\alpha \neq 0$.
(5) By scaling e_1 and using that $\overline{\mathbb{F}}_p$ is algebraically closed, show that we can let $\alpha = 1$. In particular, there is a unique Dieudonné module corresponding to the group scheme $E[p]$ for a supersingular elliptic curve.

The case of E ordinary is more straightforward. Use the fact that $\#E[p](\overline{\mathbb{F}}_p) = p$ and the fact that the connected-étale exact sequence splits for group schemes over a perfect field to show that $E[p] \cong \mu_p \times \underline{\mathbb{Z}/p\mathbb{Z}}$.

The following problem is adapted from [CO09, Exercise 4.6]. Here we investigate the endomorphism algebra of simple Dieudonné modules over an algebraically closed base field.

Problem 9 ($\star\star$)
Let k be an algebraically closed field containing $\mathbb{F}_p$. Let D_k be the Dieudonné ring as in Problem 5, and $D_k[\frac{1}{p}]$ be the rational Dieudonné ring. Now, let (m,n) be a pair of non-negative integer such that $\gcd(m,n) = 1$. Let $N_{m,n} := D_k[\frac{1}{p}]/D_k[\frac{1}{p}](F^m - V^n)$. $N_{m,n}$ is a simple object in the isogeny category of Dieudonné module over k. We want to compute $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$.
(1) Show that $N_{m,n} \cong D_k[\frac{1}{p}]/D_k[\frac{1}{p}](F^{m+n} - p^n)$.
(2) Let $\varphi \in \mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$. Suppose $\varphi(1) = \sum_{i=0}^{m+n-1} a_i F^i$ with $a_i \in W(k)[\frac{1}{p}]$. Use the fact that $(F^{m+n} - p^n)\varphi(1) \in D_k[\frac{1}{p}](F^{m+n} - p^n)$ to show that all the a_i's lie in $W(\mathbb{F}_{p^{m+n}})[\frac{1}{p}] = \mathbb{Q}_{p^{m+n}}$[a]
(3) Show that the center of $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$ is $\mathbb{Q}_p$.
(4) Use the fact that $N_{m,n}$ is a simple left $D_k[\frac{1}{p}]$-module, show that $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$ is a central division algebra over $\mathbb{Q}_p$.

(5) Recall the definition and notation of $D_{p,h,n}$ from PSET 2, Problem 4. It can be written as $\mathbb{Q}_{p^h}[F]/(F^h - p^n)$, where $F\alpha = \alpha^\sigma F$ for $\alpha \in \mathbb{Q}_{p^h}$. Show that $\varphi \mapsto \varphi(1)$ gives an isomorphism $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n}) \cong \mathbb{Q}_{p^{m+n}}[F]/(F^{m+n} - p^n)$.

(6) Conclude that $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$ is a central simple algebra over $\mathbb{Q}_p$ with Hasse-invariant $\frac{n}{m+n}$.

[a] That is, show that $a_i^{\sigma^{m+n}} = a^i$ for all a_i.

The next problem is Exercise 7.4.8 in [BC09]. It displays the role p-divisible groups play compared to ℓ-adic Tate-modules: they are more suitable for encoding information at p!

Problem 10 ($\star\star\star$)
Let A and B be abelian varieties over a perfect field k of characteristic $p > 0$. Recall that there is an additive antiequivalence of categories $G \mapsto \mathbb{D}(G)$ between the category of p-divisible groups over k and the category of left $W(k)[F,V]$-modules which are also finite as $W(k)$-modules.

(1) Show that the natural map
$$\mathrm{Hom}_k(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Hom}_{W(k)[F,V]}(\mathbb{D}(B[p^\infty]), \mathbb{D}(A[p^\infty]))$$
is injective.

(2) Show however, the natural map
$$\mathrm{Hom}_k(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Hom}_{\mathbb{Z}_p}(T_pA, T_pB)$$
is never injective.

(3) If $f \in \mathrm{End}_k(A)$ is a nonzero endomorphism of A then the common characteristic polynomial $P_f \in \mathbb{Z}[T]$ of all $T_\ell(f) \in \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell A)$ with $\ell \neq \mathrm{char} k$ is also the characteristic polynomial of $\mathbb{D}(f) \in \mathrm{End}_{W(k)}(\mathbb{D}(A[p^\infty]))$.

In Problem 7, we have considered examples of finite flat group schemes of order p. The following problem expands on these examples to give examples of p-divisible groups of height 1.

Problem 11 ($\star\star$)
Let k be an algebraically closed field of characteristic p.

(1) Let $\mathbb{G}_m/k$ be the multiplicative group scheme defined over k.
   (a) Show that the multiplication $[p^i]$ is given by $x \mapsto x^{p^i}$ on the coordinate ring. Determine the Hopf algebra of the group scheme $\mathbb{G}_m[p^i]$, i.e. the kernel of $[p^i]$.
   (b) Define $G_i := \mathbb{G}_m[p^i]$. Show that $\mathbb{G}_m[p^\infty] := \{G_i\}_{i\geq 1}$, together with the inclusion $j_i : G_i \to G_{i+1}$, is a p-divisible group of height 1.
   (c) Show that the relative Frobenius $\mathrm{F}_{G_i/k} : G_i \to G_i^{(p)} \cong G_i$, agrees with $[p] : G_i \to G_i$. Conclude that $V_{G_i/k} : G_i \to G_i$ is the identity.
   (d) Let $G_{m,n}$ be the p-divisible group whose Dieudonné module is $M_{m,n} := D_k/D_k(F^m - V^n)$. By comparing the action of Frobenius and Verschiebung and using the Dieudonné-Mannin classfication[a], show that $\mathbb{G}_m[p^\infty]$ is isogenous to $G_{0,1}$. That is, $\mathbb{D}(\mathbb{G}_m[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong M_{0,1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(2) Let $H_i = \underline{p^{-i}\mathbb{Z}/\mathbb{Z}}_k$ be the constant group scheme over k attached to the finite group $p^{-i}\mathbb{Z}/\mathbb{Z}$.
   (a) Show that $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_k := \{H_i\}_{i\geq 1}$, together with the inclusion $j_i : H_i \to H_{i+1}$, is a p-divisible group of height 1.
   (b) Show that $\mathrm{F}_{H_i/k} : H_i \to H_i^{(p)} \cong H_i$ is the identity. Conclude that $V_{H_i/k}$ is $[p]$.
   (c) Show that $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_k$ is isogenous to $G_{1,0}$.[b]

[a] Use the statement of [BC09, Theorem 8.1.4]. $M_{m,n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is $D_{m,m+n}$ in the notation of Theorem 8.1.4, and is $N_{m,n}$ in the notation of Problem 9.
[b] Use the fact that $\mathbb{D}(G^t) = \mathbb{D}(G)^\vee$ and $M_{m,n}^\vee = M_{n,m}$, we see that $\mathbb{G}_m[p^\infty]$ is the Serre dual (see below) of $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_k$.

The following problem gives the construction of Dieudonné module associated to the Serre dual of a p-divisible group.

Problem 12 $(\star\star)$
Let k be an algebraically closed field of characteristic p. Let M be a Dieudonné module[a], i.e. a finite free $W(k)$-module with left D_k action. We construct its dual $M^\vee$ as follows. As a $W(k)$ module, $M^\vee = \mathrm{Hom}_{W(k)}(M, W(k))$, with the action of V and F given as

$$(V \cdot h)(m) = (h(F(m)))^{\sigma^{-1}}, \ \ (F \cdot h)(m) = (h(V(m)))^{\sigma}$$

for all $h \in M^\vee$ and $m \in M$.

(1) For a pair of non-negative integer (m, n) such that $\gcd(m, n) = 1$, let $M_{m,n}$ be as in Problem 11. Show that $M_{m,n}^\vee \cong M_{n,m}$.

(2) We have the following facts:
- Let X/k an abelian variety. Let $X[p^\infty]$ denote its p-divisible group, and $X[p^\infty]^{\mathrm{t}}$ the Serre dual of $X[p^\infty]$, then
$$X[p^\infty]^{\mathrm{t}} \cong X^\vee[p^\infty],$$
where $X^\vee$ is the dual abelian variety.
- If G is a p-divisible group over k, and $D(G)$ is its Dieudonné module, then
$$D(G^t) \cong D(G)^\vee$$

Use the above facts, show that the Newton polygon of an abelian variety is symmetric. That is, $X[p^\infty]$ is isogenous to $\bigoplus_i (G_{m_i,n_i} \oplus G_{n_i,m_i})^{r_i}$ for some (m_i, n_i) non-negative and $\gcd(m_i, n_i) = 1$.

[a] There is an unfortunate clash of terminology with the Dieudonné module of a finite flat group scheme, which isn't necessarily torsionfree. We hope that the meanings are clear from the context.

The following problem explores examples of p-divisible groups attached to an abelian variety and compute their Newton polygons.

Problem 13 $(\star\star)$

(1) Recall that if $f : X \to Y$ is an isogeny between abelian varieties over a field k, then $\deg(f) = \mathrm{rank}(\ker(f))$, i.e. the rank of the finite group scheme $\ker(f)$ over k. Show that the p-divisible group of a g-dimensional abelian variety over k is of height $2g$.

(2) Now let $E/\mathbb{F}_q$ be an elliptic curve.
   (a) Suppose $E/\mathbb{F}_q$ is supersingular. Recall in PSET 3, problem 7, we have shown that $\mathrm{End}^0(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D_{p,2,1}$, the central division algebra over $\mathbb{Q}_p$ with Hasse-invariant $\frac{1}{2}$. Combine Problem 10 and Problem 9 Part (5) to conclude that $E_{\overline{\mathbb{F}}_q}[p^\infty]$ is isogenous to $G_{1,1}$.
   (b) Suppose $E/\mathbb{F}_q$ is ordinary. Recall in PSET 3, problem 9, we have shown that $L = \mathrm{End}^0(E)$ is an imaginary quadratic extension over $\mathbb{Q}$ generated by ϕ_q. Furthermore, the characteristic polynomial of ϕ_q is $T^2 - aT + q$, where $v_p(a) = 0$. Show that $L \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$. Use the injection
$$\mathrm{End}^0(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}^0(E_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}(E_{\overline{\mathbb{F}}_p}[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
to conclude that $E_{\overline{\mathbb{F}}_q}[p^\infty]$ is isogenous to $G_{1,0} \oplus G_{0,1}$.

(3) Recall that in PSET 4, Problem 11, for a pair of non-negative integers (m, n) with $n < m$ and $\gcd(m, n) = 1$, we have a simple abelian variety $A/\mathbb{F}_q$ of dimension $g = m + n$, and the Frobenius ϕ_q on A has minimal polynomial $h_A(T) = T^2 - p^n T + p^g$. Moreover, $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D_{p,g,m} \oplus D_{p,g,n}$. Use these to show that $A_{\overline{\mathbb{F}}_q}[p^\infty]$ is isogenous to $G_{n,m} \oplus G_{m,n}$.

As an important notion to study p-divisible groups, we introduce the Tate module of a p-divisible group.

Problem 14 ($\star\star$)
Let G be a p-divisible group over a perfect scheme S of characteristic p. Consider the inverse limit

$$TG := \varprojlim_{\times p} G[p^n].$$

Show that this limit exists in the category of schemes and TG is an affine scheme, flat over S. This is called the (schematic) Tate module of the p-divisible group G.[a]

(1) Show that the functor of points of TG identifies with the following functor

$$(T \to S) \mapsto \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G_T),$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ is the constant p-divisible group over T, and G_T denotes the base change.

(2) Show that over a noetherian scheme of characteristic p, the Tate module of μ_{p^∞} is trivial.

[a] Depending on conventions, sometimes the Tate module of G refers to the sections of TG, which is a finite free $\mathbb{Z}_p$-module.

References

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