Silverman - Lecture 3

Lower Bounds for Canonical Heights

K number field

A/K abelian variety

D ample, symmetric divisor

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Theorem: $\hat{h}_D \geq 0$

$$\hat{h}_D(P) = 0 \iff P \in A(K)_{\text{tors}}$$

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Question: If $\hat{h}_D(P) \neq 0$,

how small can it be?

Older Analogue

"Canonical" $h : \overline{\mathbb{Q}}^* \longrightarrow [0, \infty)$

$$h(\alpha^m) = |m|\, h(\alpha)$$

Theorem: (Kronecker)

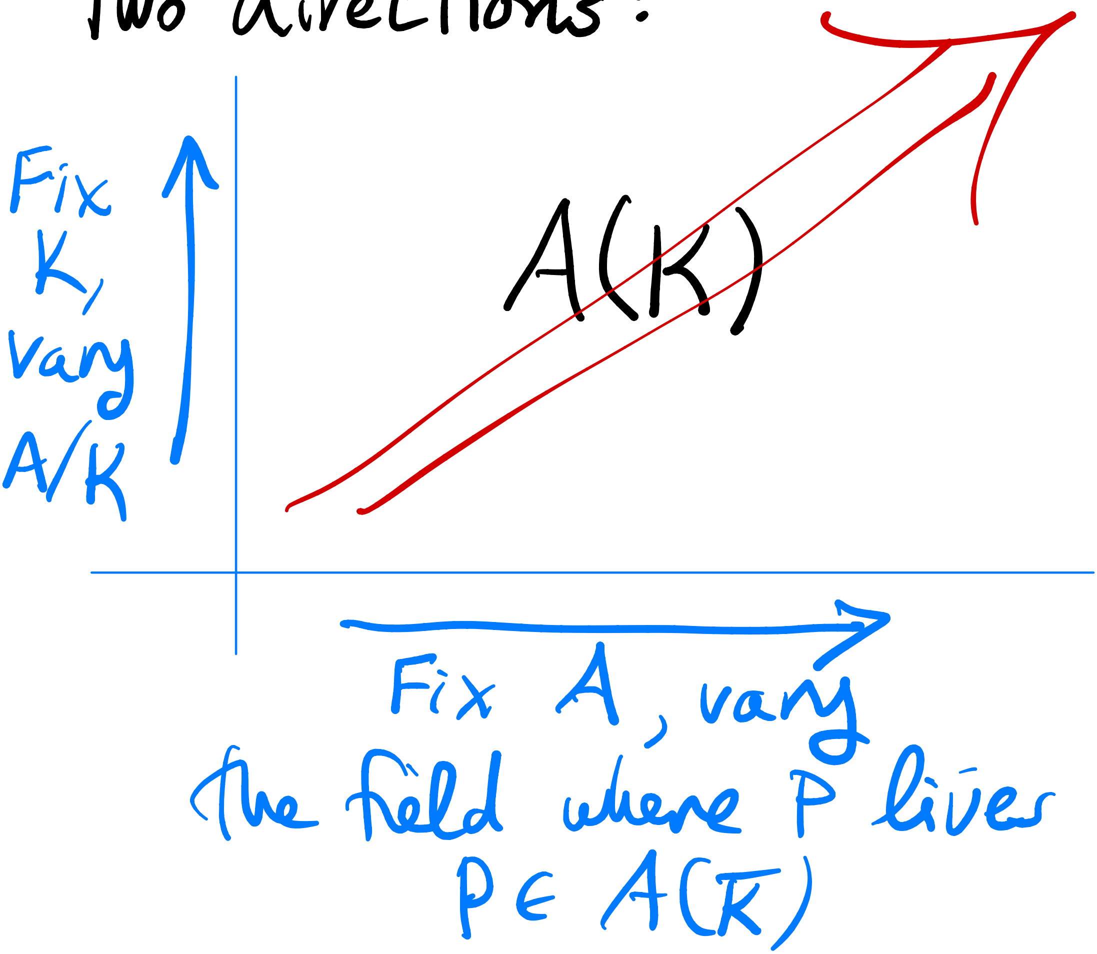
$h(\alpha) = 0 \iff \alpha$ is a root of 1

$\alpha \in \mu(\overline{\mathbb{Q}})$

Question: If $h(\alpha) \neq 0$, how small can it be?

Question: How small can $0 \neq \hat{h}_{A,D}(P)$ be?

Two directions:



$\boxed{\begin{array}{l}\text{Part I: Fix } A/K \\ \text{Vary the field of def. of } P\end{array}}$

Analogy:

If $\alpha \in \overline{\mathbb{Q}}^*$, $\alpha \notin \mu$, $h(\alpha) \geq$?

$$h(2^{1/n}) = \frac{1}{n} h(2) = \frac{\log 2}{n} \xrightarrow[n \to \infty]{} 0$$

$$(h(\alpha^m) = |m| h(\alpha))$$

$$[\mathbb{Q}(2^{1/n}) : \mathbb{Q}] = n$$

$\hat{h} = \hat{h}_{A,D}$

Ditto for A/K

Fix $P \in A(K)$ - tors

Let $Q \in A(\bar{K})$ and $mQ = P$.

$$\hat{h}(Q) = \frac{1}{m^2} \underbrace{\hat{h}(P)}_{\neq 0}$$

$$\begin{aligned} [K(Q):K] &\approx \# [m]^{-1}(P) \\ &= \# A[m] \\ &= m^{2g}, \quad g = \dim A \end{aligned}$$

$$\boxed{\hat{h}(Q) \asymp \frac{1}{[K(Q):K]^{1/g}}}$$

Lehmer's Conjecture

There is $C > 0$ s.t.
for all $\alpha \in \overline{\mathbb{Q}}^* - \mu$,

$$h(\alpha) \geq \frac{C}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$$

Thm (Dobrowolski 1979)

$$d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$$

$$h(\alpha) \geq \frac{c}{d}\left(\frac{\log\log d}{\log d}\right)^3$$

Lehmer Conj for AVs

(Masser, 1984)

$\exists\, C = C(A/K, D) > 0$ s.t

$\forall \quad P \in A(\overline{K}) - \text{tors}$

$$\hat{h}_D(P) \geq \frac{C}{[K(P):K]^{1/g}}$$

Conj

$$\boxed{\hat{h}_D(P) \geq \frac{C}{[K(P):K]^{1/g}}}$$

Some known results: Fix $\varepsilon > 0$,

$d = [K(P):K]$

$C = C(A/K, D, \varepsilon) > 0$

$P \in$ non-~~torsion~~ pt $\in A(\bar{K})$

(a) $\hat{h}(P) \geq C/d^{2g+1+\varepsilon}$

(b) If A has CM, $\hat{h}(P) \geq C/d^{1+\varepsilon}$

(c) If $g = 1$ and $j(A) \notin R_K$,

then $\hat{h}(P) \geq C/d^{2+\varepsilon}$

Really $d^{\varepsilon} \rightsquigarrow$ power of $\log(d)$.

Proof methods:

1. Transcendence Theory
2. Fourier averaging
   Harmonic analysis

$L/K \quad d = [L:K]$ $\qquad B = \frac{c_2}{d^2}$

$A(L,B) = \{P \in A(L) : \hat{h}_D(P) \leq B\}$

Goal: Show

$$\# A\left(L, \frac{c_1}{d}\right) \leq c_2 \, d^{\text{some power}}$$

Applies it to $\{mP : m \leq \text{some bd}\}$

P if $\hat{h}(P)$ is small

Goal: Bd $A(L,B) = \{P \in A(L) : \hat{h}(P) \le B\}$

$A(L,B)^{(m)} := \{P_1 + \dots + P_m : P_i \in A(L,B)\}$

Then $\# A(L,B)^{(m)} \approx \frac{\# A(L,B)^m}{m!}$ lots of pts

$A(L,B)^{(m)} \subseteq A(L, \underbrace{m^2}_{\text{not too large}} B)$

1. Constructs (small) theta fn F s.t. F vanishes $A(L,B)^{(m)}$ to high order
2. Upper bd $|\partial_\eta F(Q)|$ for $Q \in$ Cauchy theorem
3. ~~Zero estimate~~ Lower bd for non-zero
4. 2 & 3 $\Rightarrow$ ~~$\#$~~ $A(L,B)$ ~~is~~ big

(II) Fix K, vary A/K.

Intution: $\hat{h}(P) \neq 0$

$\Longrightarrow \hat{h}(P) >$ complexity of A/K

$h(A/K) =$ complexity of A/K

(1) $E : y^2 = x^3 + ax + b$

$$h(E/K) = \min_{u \in K^*} h([1, u^4 a, u^6 b])$$

(2) Fix: $\mathcal{A}_g^{pp} \xrightarrow[j]{} \mathbb{P}^N$

$h(A/K) = h(j(A/K)) + \log N_{K/\mathbb{Q}}$ (conductor of A/K)

(3) Faltings

Conj: (Dem'janenko–Lang)

$g = \dim A$

$P \in A(K)$, and $\mathbb{Z}\cdot P$ are Z-dense in A

Then

$$\hat{h}_{A,D}(P) \geq C_1(K,g)\, h(A/K) - C_2(K,g)$$

Results:

1. True if $j(A) \in \mathcal{A}_g(\mathbb{C})$ if ε-distinct from the geo simple locus in $\mathcal{A}_g(\mathbb{C})$
2. True if fix A_0, let $A = A_0^{\chi}$ twists
3. $g=1$. ABC conj $\Longrightarrow$ DL conj.

Fourier Averaging

small ht.

$\{P_1, P_2, \ldots, P_N\}$

$A = E$
$g = 1$

$$\sum_{i \neq j} \hat{h}(P_i - P_j)$$

$$= \sum_{v \in M_K} \underbrace{\sum_{i \neq j} \lambda_v(P_i - P_j)}$$

v good red ≥ 0

additive

v arch

v non-arch, mult red.

$$E(K) \xrightarrow{t} E(K)/E_0(K) \cong \tfrac{1}{M}\mathbb{Z}/\mathbb{Z}$$

good red

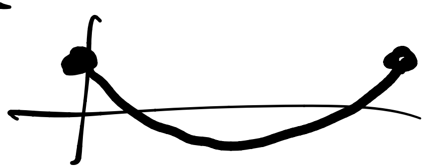
$$M = \mathrm{ord}_v(\Delta_E)$$

$$\lambda_v(P) = \tfrac{1}{2}\mathbb{B}_2(t(P)) + \text{formal gp part}$$

≥ 0

$$\mathbb{B}_2(T) = T^2 - T + \tfrac{1}{6} \quad \text{on } [0,1)$$

$\mathbb{R}/\mathbb{Z}$



$$\mathbb{B}_2(T) = \frac{1}{2\pi^2}\sum_{n\neq 0}\frac{e(nT)}{n^2}$$

$e(x) = e^{2\pi i x}$

N^2 terms

$$\sum_{i \neq j} \lambda \omega (P_i - P_j)$$

$$\geq \sum_{i \neq j} \frac{1}{2} B_2 \big(\underbrace{t(P_i - P_j)}_{\substack{t(P_i) - t(P_j) \\ t_i - t_j}} \big)$$

$$= \sum_{i \neq j} \frac{1}{2\pi^2} \cdot \frac{1}{2} \sum_{n \neq 0} \frac{e(t_i - t_j)}{n^2}$$

$$= \frac{1}{4\pi^2} \sum_{n \neq 0} \frac{1}{n^2} \underbrace{\sum_{i \neq j} e(t_i - t_j)}$$

$$\underbrace{\sum_{i,j} e(t_i) e(-t_j)} - N$$

$$\to \Big| \sum_{i=1}^{N} e(t_i) \Big|^2$$

$L(A/K, s) \sim C (s-1)^r + \ldots$

$\mathrm{Reg}(A/K) \gg_r \min \hat{h}_{A/K}(P)^r$

$E/\mathbb{Q}$ $\quad y^2 = x^3 + ax + b$

Conj

$\# E(\mathbb{Z}) \le C^{1 + \mathrm{rank}\, E(\mathbb{Q})}$

Known True

$C\left(\frac{h(E/\mathbb{Q})}{\min \hat{h}(P)}\right)$