EXERCISES ON THE MODULI SPACES AND THE TORELLI MAP

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CONTENTS

1. INTRODUCTION

These exercises were written for the 2024 Arizona Winter School on Abelian varieties. They are intended as an introduction to moduli spaces, with a focus on the moduli space of curves, the moduli space of abelian varieties, and the Torelli map relating them.

1.1. **How to use these exercises.** These exercises are not written in a linear order. In particular, many of the exercises at the beginning may be more challenging and assume more background than what you are interested in learning. I suggest you that skip around and find a part of the exercises you are interested in, and start working there.

1.2. **Assumed background.** I have also tried to indicate in the title of an exercise when it involves additional background beyond thorough familiarity with material in an introductory algebraic geometry book, such as [Har77] or [Vak]. We also assume basic familiarity with the theory of Abelian varieties. Certainly thorough knowledge of Mumford's book [Mum08] will be more than sufficient, but we will only require knowledge of a small portion of that, and try to point out when we do use things from there.

We first have some exercises on the moduli space of curves in §2, we then have exercises on the moduli space of abelian varieties, and then exercises on the torelli map relating these two in §3. Finally, we include some exercises on modui spaces in general in §5, with a particular focus on moduli spaces of divisors and finite covers. These are examples of some moduli spaces which are simpler than the moduli space of curves and abelian varieties.

2. The moduli space of curves

2.1. Defining the moduli stack of curves.

Definition 2.1.1. The *moduli stack of curves of genus g $\mathscr{M}_g$* is defined as the algebraic stack whose S points are the groupoid of smooth proper $f : C \to S$ of relative dimension 1 with geometrically connected fibers of genus g. The groupoid structure is given by letting isomorphisms from $[C, f] \to [C', f']$ be commutative triangles

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & S & \end{tikzcd} \tag{2.1}$$

where ϕ is an isomorphism.

More generally, $\mathscr{M}_{g,n}$ has S points given by smooth proper $f : C \to S$ of relative dimension 1 with geometrically connected fibers of genus g, together with n disjoint sections $\sigma_1, \ldots, \sigma_n : S \to C$. An isomorphisms of S points $[C, f, \sigma_1, \ldots, \sigma_n]$ to $[C', f', \sigma'_1, \ldots, \sigma'_n]$ be given by commutative triangles above so that $\sigma'_i = \phi \circ \sigma_i$ for all i.

Note that above S may be any scheme, so we may equivalently consider S as a scheme over $\operatorname{Spec} \mathbb{Z}$.

Notation 2.1.2. As a convention, unless otherwise stated, throughout these exercises, we will simply use curve to mean a smooth proper curve with geometrically connected fibers.

Exercise 2.1.3. Show $\mathscr{M}_g$ is nonempty by exhibiting a curve of genus g for every $g \geq 0$.

Exercise 2.1.4.
(1) Show $\mathscr{M}_g$ is not a scheme for any $g \geq 0$. *Hint:* If $\mathscr{M}_g$ were a scheme there would be an injection $\mathscr{M}_g(K) \to \mathscr{M}_g(\overline{K})$ for any field K. Take $K = \mathbb{Q}$ and consider curves of the form $y^2 = f(x)$ and $2y^2 = f(x)$.
(2) Show that $\mathscr{M}_g \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C}$ is not a scheme. *Hint:* One can adapt the trick suggested in the hint to Exercise 2.1.4(1) to answer this question.

2.2. **Examples of $\mathscr{M}_g$ for small g.**

Exercise 2.2.1. Show that $\mathscr{M}_{0,3} \simeq \operatorname{Spec} \mathbb{Z}$ and $\mathscr{M}_{0,4} \simeq \mathbb{P}^1_{\mathbb{Z}} - \{0, 1, \infty\}$.

Exercise 2.2.2 (Assumes background with stable genus 0 curves)**.** There is a notion of the moduli stack of stable genus g curves with (ordered) marked points, $\overline{\mathscr{M}}_{g,n}$. Assuming familiarity with this, show $\overline{\mathscr{M}}_{0,4} \simeq \mathbb{P}^1$.

Exercise 2.2.3 (genus 2 curves)**.**
(1) Show that every genus 2 curve C over an algebraically closed field k is hyperelliptic, meaning that there is a degree 2 map $C \to \mathbb{P}^1$.
(2) Show that a genus 2 curve over any field has a degree two map to $\mathbb{P}^1$

Exercise 2.2.4. Show that every genus 3 curve over an algebraically closed field is either a plane curve or hyperelliptic. Can you give a similar description of every genus 3 curve over an arbitrary field?

Exercise 2.2.5. Show that every genus 4 curve over an algebraically closed field is an intersection of a cubic and quadric in $\mathbb{P}^3$ or is hyperelliptic.

Exercise 2.2.6. Show that any genus 5 curve over an algebraically closed field is either a complete intersection of three quadrics, is trigonal (has a degree 3 map to $\mathbb{P}^1$) or is hyperelliptic.

2.3. **Automorphism groups of generic curves.** In the following sequence of exercises, we do a dimension count to show that a generic curve of genus $g \geq 3$ has trivial automorphism group.

Exercise 2.3.1. Suppose C is a genus g curve with automorphism group G over $\mathbb{C}$. Show that the set of genus g curves with automorphism group G can be partitioned into a disjoint union of classes described by surjections $\phi : \pi_1(\Sigma_{h,n}) \to G$ where $\Sigma_{h,n}$ is an n-times punctured genus h topological surface, where C/G has genus h and $C \to C/G$ has n branch points in C/G.

Call M_ϕ the space of genus g curves with automorphism group of type ϕ, as in the previous exercise.

Exercise 2.3.2. With notation as above and as in the previous exercise, show that the dimension of M_ϕ is $3h - 3 + n$.

Exercise 2.3.3 (exercise assuming background with stacks)**.** Show that M_ϕ is in fact a gerbe for the center of G over a finite étale cover of $\mathscr{M}_{g,n}/S_n$, where S_n acts by permuting the marked points.

Exercise 2.3.4. Show that if $g \geq 3$, the dimension of every such M_ϕ associated to nontrivial groups G has dimension strictly less than $3g - 3$. What goes wrong with this argument when $g = 2$?

2.4. **Deformation theory.** Readers less comfortable with algebraic geometry may be advised to skip this subsection, and jump to §2.5, as it assumes a fair amount of mathematical maturity. You may assume the facts from deformation theory that for X a scheme:

(1) $H^0(X, T_X)$ parameterizes infinitesimal automorphisms of X,
(2) $H^1(X, T_X)$ parameterizes deformations of X,
(3) and $H^2(X, T_X)$ parameterizes obstructions to deformations of X.

Exercise 2.4.1. Assume X is a curve. Show $H^2(X, T_X) = 0$. Deformation theory then implies the moduli stack of such X, $\mathscr{M}_g$ is smooth and its dimension is equal to its tangent space, (assuming $\mathscr{M}_g$ is Deligne-Mumford, see Exercise 2.4.3,) which can be identified with $H^1(X, T_X)$. Compute this dimension.

Definition 2.4.2. An algebraic stack is *Deligne-Mumford* if its automorphism group schemes at any point are étale.

Exercise 2.4.3. Show that for any $[X] \in \mathscr{M}_g$, $H^0(X, T_X) = 0$ if $g \geq 2$ and conclude that $\mathscr{M}_g$ is Deligne-Mumford if $g \geq 2$.

2.5. **Hyperelliptic curves.** By definition a hyperelliptic curve is a genus g curve with a degree 2 finite locally free map to a genus 0 curve.

Exercise 2.5.1 (Assumes more background with stacks)**.** In this exercise, we construct the moduli stack of hyperelliptic curves.

(1) For $g \geq 2$, construct a moduli stack $\mathscr{H}_g$ of hyperelliptic curves of genus g over $\mathbb{Z}[1/2]$ as a global quotient stack of an open in affine space by a group action. *Hint: Every genus g hyperelliptic curve in characteristic not 2 can be written as $y^2 = f(x, z)$ for some polynomial f of degree $2g + 2$. This expression is unique up to automorphisms of the base $\mathbb{P}^1$ and scaling the polynomial f. Be careful with the fact that when one scales*

the x and z coordinates by a factor of c, the y coordinate must be scaled by a factor of c^{g+1} to preserve the equation. From the description you give, conclude this stack is smooth. Show that it indeed parameterizes hyperelliptic curves.

(2) Show the moduli space of hyperelliptic curves has relative dimension $2g-1$ over $\operatorname{Spec}\mathbb{Z}[1/2]$.
(3) Construct a moduli stack of hyperelliptic curves over $\mathbb{Z}$.

Exercise 2.5.2. In this exercise, we generalization Exercise 2.2.3 to hyperelliptic curves.

(1) Show that a genus 0 curve over k with a k point is isomorphic to $\mathbb{P}^1$.
(2) More generally, show that a genus 0 curve with an odd degree divisor is isomorphic to $\mathbb{P}^1$.
(3) Show that a hyperelliptic curve over an algebraically closed field of genus g has a map to $\mathbb{P}^1$.
(4) Show that a hyperelliptic curve of even genus g has a map to $\mathbb{P}^1$.
(5) For any odd genus g, show there exists a field k and a hyperelliptic curve over k which has no degree 2 map to $\mathbb{P}^1$.
(6) Specifically can you do the above exercise in the case $k=\mathbb{R}$? What about the case $k=\mathbb{F}_q$? Can you characterize which fields have this property?
(7) (trickier) Show the genus 0 curve must be $\mathbb{P}^1$ in the case k is separably closed (but not necessarily algebraically closed). *Possible hint:* Show the genus 0 curve is $\mathbb{P}^1$ if and only if a certain class in $H^1(k, \mathrm{PGL}_2)$ or $H^2(k, \mu_2)$ vanishes, and use that this is invariant under inseparable base extension.

Thanks to Bjorn Poonen for suggesting the following exercise.

Exercise 2.5.3 (trickier exercise, assumes some background with stacks)**.** In this exercise, we investigate the relation between hyperelliptic curves of genus g and genus 0 curves with $2g+2$ unordered marked points.

(1) To understand better what is going on in Exercise 2.2.3, construct a map from the moduli space of hyperelliptic curves to the moduli space of genus 0 curves with $2g+2$ unordered marked points. Show that this map is a so-called $\mathbb{Z}/2\mathbb{Z}$ gerbe, which loosely means it is a bijection on geometric points, but each point of the source has twice as many automorphisms as each point of the target. That is, each geometric fiber is $B(\mathbb{Z}/2\mathbb{Z})$.
(2) Construct a PGL_2 bundle over the space of genus 0 curves with $2g+2$ unordered marked points and show that the genus 0 curve is $\mathbb{P}^1$ if and only if this bundle is trivial.

(3) Use this and Exercise 2.5.2 to show that the μ_2 gerbe mentioned in the first part of this exercise is trivial whenever the genus is even and nontrivial whenever the genus is odd. *Possible hint:* It may be helpful to use that μ_2 gerbes over B are classified by $H^2(B, \mu_2)$ and the exact sequence

$$0 \longrightarrow \mu_2 \longrightarrow \mathrm{SL}_2 \longrightarrow \mathrm{PGL}_2 \longrightarrow 0 \tag{2.2}$$

yields a map $H^1(B, \mathrm{PGL}_2) \to H^2(B, \mu_2)$ sending the PGL_2 torsor from the previous part to the corresponding μ_2 gerbe.

3. The moduli space of polarized abelian varieties

Given an abelian scheme A over a base S, recall that the dual abelian scheme $A^\vee := \mathrm{Pic}^0_{A/S}$ is the identity component of the scheme parameterizing line bundles on A. There is a universal Poincaré line bundle $\mathscr{P}_A$ on $A \times A^\vee$ whose fiber over a point of $A^\vee$ is the corresponding line bundle on A. A *polarization* is a map $\phi : A \to A^\vee$ is a symmetric isogeny so that $(\mathrm{id}, \phi)^* \mathscr{P}_A$ is an ample line bundle on A; here, symmetric means the the composition $A \to (A^\vee)^\vee \xrightarrow{\widehat{\phi}} A^\vee$ agrees with ϕ, for $A \to (A^\vee)^\vee$ the canonical map.

Definition 3.0.1. Fix an integer ν. The moduli stack of polarized abelian abelian schemes over $\mathbb{Z}$ with polarization of degree ν assigns to a scheme S the groupoid parameterizing abelian schemes $f : A \to S$ (meaning a smooth proper group scheme) together with a polarization $\phi : A \to A^\vee$ of degree ν. An isomorphism morphism in this category between $[A, f, \phi]$ and $[A', f', \phi']$ is a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"] & & A' \arrow[dl, "f'"] \\ & S & \end{tikzcd} \tag{3.1}$$

where h is an isomorphism and so that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\phi"] & A' \arrow[d, "\phi'"] \\ A^\vee \arrow[r, "h^\vee"] & (A')^\vee. \end{tikzcd} \tag{3.2}$$

commutes.

In the case $\nu = 1$, we call the polarization a *principal polarization*.

3.1. **Elliptic curves.** Abelian varieties of dimension 1 are called elliptic curves.

Exercise 3.1.1. Show that every elliptic curve has a principal polarization.

Exercise 3.1.2. Show that there is a bijection between elliptic curves over an algebraically closed field k and elements of $\mathbb{A}^1_k(k) = k$. (It turns out $\mathbb{A}^1_k$ is a coarse moduli space for the moduli stack of elliptic curves.)

We now provide a sequence of exercises relating to elliptic curves. We start with the complex perspective.

Definition 3.1.3. If V is a complex vector space, a lattice $\Lambda \subset V$ is a discrete subgroup so that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq V$, i.e., Λ generates V as a $\mathbb{R}$ vector space (viewing $\mathbb{C}$ as a 2-dimensional $\mathbb{R}$ vector space).

Concretely, if $V = \mathbb{C}$, a lattice can be written in the form $\{av + bw : a, b \in \mathbb{Z}\}$ so long as neither of v and w are $\mathbb{R}$ multiples of the other. We can now relate lattices to elliptic curves.

Exercise 3.1.4. Show that for any lattice $\Lambda \subset \mathbb{C}$, one can make sense of the quotient $\mathbb{C}/\Lambda$ as a Riemann surface. Show this quotient is a genus 1 Riemann surface with a marked point.

Fact 3.1.5. By Riemann existence, every compact Riemann surface is algebraic, which means that this genus 1 surface is actually the complex points of an algebraic variety. As a trickier exercise, you can try to prove this directly in the case of elliptic curves. As a hint, look up the Weierstrass $\wp$ function, and show that one can express the square of the derivative of $\wp$ in terms of a cubic polynomial in $\wp$, and then show that the complex points of the elliptic curve precisely satisfy this differential equation.

Exercise 3.1.6. Let $\mathcal{L}$ denote the set of pairs $(\omega_1, \omega_2) \in (\mathbb{C}^\times)^2$ so that ω_1, ω_2 are not $\mathbb{R}$ multiples of each other. There is an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{R}$ by treating (ω_1, ω_2) as a length 2 vector and letting matrices act on these vectors. Show there is a bijection between $\mathcal{L}/\mathrm{SL}_2(\mathbb{Z})$ and the set of lattices $\Lambda \subset \mathbb{C}$. *Hint:* Send $(\omega_1, \omega_2) \mapsto \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

Definition 3.1.7. Let the *upper half plane* be $\mathbb{H} := \{z \in \mathbb{C} : z > 0\}$.

We consider two lattices Λ_1, Λ_2 in $\mathbb{C}$ isomorphic if there is some $k \in \mathbb{C}^\times$ so that $k\Lambda_1 = \Lambda_2$.

Exercise 3.1.8. Show there is a bijection between $\mathbb{H}$ and the set of lattices up to isomorphism. Specifically, show $\mathbb{H}$ is the quotient of $\mathcal{L}/\mathrm{SL}_2(\mathbb{Z})$ by the scaling action of $\mathbb{C}^\times$ where $k \in \mathbb{C}^\times$ sends $(\omega_1, \omega_2) \mapsto (k\omega_1, k\omega_2)$.

Exercise 3.1.9. In Exercise 3.1.6, show that the bijection there respects stabilizer groups in the following sense: Show that the stabilizer of the $\mathbb{C}^\times$ action on $\mathcal{L}/\mathrm{SL}_2(\mathbb{Z})$ via scaling (whose quotient is $\mathbb{H}$) is identified with the set of automorphisms of a lattice. More specifically, show the following:

(1) show that the hexagonal lattice (corresponding to $\omega_1 = 1$ and ω_2 a primitive 6th root of unity) has automorphism group $\mathbb{Z}/6\mathbb{Z}$.
(2) Show the square lattice (corresponding to $\omega_1 = 1$ and ω_2 a primitive 4th root of unity has automorphism group $\mathbb{Z}/4\mathbb{Z}$).
(3) Show that all other lattices have automorphism group $\mathbb{Z}/2\mathbb{Z}$.
(4) Conclude using Fact 3.1.5 that the above describes all automorphism groups of ellpitic curves.

Exercise 3.1.10 (Explicit description of elliptic curves)**.** (1) Show that any elliptic curve over a field k can be described as a plane cubic.
(2) Moreover, by choosing the point to lie at $[0,1,0] \in \mathbb{P}^2$ and specifying the tangent space to this point show the equation may be taken to be of the form $zy^2 + f(x,z)y + g(x,z) = 0$ where f is quadratic and g is cubic.
(3) When k has characteristic which is not 2, show that by completing the square, one can arrange $f = 0$.
(4) When k has characteristic neither 2 nor 3, show that one can write the equation in the form $zy^2 = x^3 + Axz^2 + Bz^3$ for $A, B \in k$, and that this is uniquely determined up to replacing (A, B) by (c^4A, c^6B) for $c \in k^\times$.

Exercise 3.1.11 (Fun exercise)**.** Show that every genus 1 curve over a field k with a degree 2 line bundle can be described as a double cover of $\mathbb{P}^1_k$, and can be described by an equation of the form $y^2 = f(x)$ for $f(x)$ a quartic equation. (In degenerate cases where the cover is branched over infinity, it can be that $f(x)$ is a cubic; for simplicity we will assume this is not the case, but you can also try to figure out how to generalize this to that case.)

When one learns about solving the quartic equation $x^4 + ax^3 + bx^2 + cx + d$, the method of solving this typically reduces to solving the *resolvent cubic*. In the case the quartic is of the form $(x-A)(x-B)(x-C)(x-D)$, the resolvent cubic is the cubic polynomial $(x-(AB+CD))(x-(AC+BD))(x-(AD+BC))$. In general, the resolvent cubic is the unique polynomial over k whose base change to $\overline{k}$ has the above property.

(1) Show that the resolvent cubic exists.
(2) Show that if C is the genus 1 curve $y^2 = f(x)$, then $\mathrm{Pic}^0_{C/k}$ defines an elliptic curve whose 2-torsion is the disjoint union of the identity point

and a degree 3 scheme. Show this degree 3 scheme is the resolvent cubic of $f(x)$.

(3) Suppose the quartic equation has Galois group S_4 and interpret the resolvent cubic as sitting inside the Galois closure of the S_4 extension.
(4) Fixing a specified cubic polynomial h with S_3 Galois group, show there is a bijection between separable degree 2 extensions of the S_3 extension of k corresponding to h and degree 4 extensions whose resolvent is h. This is a special case of the *Recillas correspondence*.

3.2. **Higher dimensional abelian varieties.**

Exercise 3.2.1 (Abel-Jacobi map)**.** Given a curve C and a point $p \in C$, define a map $C \to \mathrm{Pic}^0_C$ sending $q \mapsto \mathscr{O}(q-p)$. Show this map is injective. (It turns out this map is also injective on tangent vectors, and hence is a closed immersion; this is trickier to prove and a proof is outlined in Exercise 4.1.4.)

We next aim to describe all principally polarized abelian varieties in dimensions 2 and 3, in the next few exercises.

Definition 3.2.2. If C is a genus g curve with a point $p \in C$, the *Theta divisor* is the subset $\Theta_p \subset \mathrm{Pic}^0_{C/k}$ of divisors on C of the form $D-(g-1)p$ where D is an effective Cartier divisor of degree $g-1$ divisor.

Later in Definition 4.2.1 will define the Theta divisor in families.

Fact 3.2.3. Recall that a stable curve refers to a with at worst nodal singularities so that every genus 0 component has at least 3 singular points. A curve is compact type if the dual graph of its irreducible components has no loops. The Torelli map is the map sending a curve to its principally polarized Jacobian, where the principal polarization is induced by the Theta divisor. We give a detailed construction of this in §4.1. In particular, the Jacobian of any compact type curve is an abelian variety. For the next few exercises, you may take for granted that the Torelli map in $g \geq 2$ extends to a *proper* map from the space of compact type stable curves. The theta divisor associated to a compact type curve still corresponds to degree $g-1$ effective Cartier divisors on the curve.

Exercise 3.2.4. For this exercise, it may be helpful to see Fact 3.2.3. Describe explicitly the theta divisor on the following nodal curves:

(1) Two genus 1 curves meeting a node
(2) Three genus 1 curves meeting $\mathbb{P}^1$ at the points $0, 1, \infty$.
(3) A genus 2 curve meeting a genus 1 curve at a node.
(4) A genus 1 curve meeting two other genus 1 curves at two distinct points.

(5) In general, compute the number of components of the Theta divisor associated to a curve in terms of discrete data associated to that curve. (What discrete data you will need is left to you to figure out.)

Exercise 3.2.5. In Exercise 4.4.2, we will show that a general g dimensional ppav is the Jacobian of a curve when $g = 2$ and $g = 3$. Assuming this, show that the Torelli map is dominant when $g = 2$ and $g = 3$. Conclude using Fact 3.2.3 that the extended Torelli map to compact type curves is surjective when $g = 2$ and $g = 3$.

Exercise 3.2.6. Describe principally polarized abelian varieties (ppav's) of dimension 2 via the following classification: Show that every ppav of dimension 2 over an algebraically closed field is either the Jacobian of a genus 2 curve or a product of elliptic curves. *Hint:* Use Fact 3.2.3 and Exercise 3.2.5.

Classify what type of (possibly singular) genus 2 curve the abelian variety is the Jacobian of in terms of whether the Theta divisor is reducible.

Exercise 3.2.7. Generalizing the above exercise, describe all points of $\mathscr{A}_3$ by describing all compact type genus 3 curves, and describing ppav's as the Jacobian of a compact type genus 3 curve.

Exercise 3.2.8. Construct a g-dimensional abelian variety for $g > 1$ which does not have a principal polarization as follows:

(1) Show that if A is a g-dimensional ppav over $\mathbb{C}$ with endomorphism group $\mathbb{Z}$ and $G \subset A$ is a finite subgroup of order n, with n not a gth power of an integer, then $B = A/G$ has no principal polarization. *Hint:* If B did have a principal polarization, consider the degree of the composite map $A \to B \to B^\vee \to A^\vee$.
(2) You may assume that the Jacobian of a general hyperelliptic curve over $\mathbb{C}$ has endomorphism group $\mathbb{Z}$. Using this, for every $g > 1$, show there exist examples of abelian varieties of dimension g with no principal polarization.

4. The Torelli map

4.1. Defining the Torelli map.

Definition 4.1.1. Let $f : C \to S$ be a smooth proper curve of genus at least 1 with geometrically connected fibers. The Abel-Jacobi map is defined functorially by

$$\begin{aligned} i : C &\to \mathrm{Pic}^1_{C/S} \\ x &\mapsto \mathscr{O}_C(x). \end{aligned}$$

In the next few exercises, we will show that the Abel-Jacobi map is closed immersion.

Exercise 4.1.2. Let C be a curve over a field k. Show that the Abel-Jacobi map $C \to \mathrm{Pic}^1_{C/k}$ induces an isomorphism $H^1(\mathrm{Pic}^1_{C/k}, \mathscr{O}_{\mathrm{Pic}^1_{C/k}}) \simeq H^1(C, \mathscr{O}_C)$ in the following steps.

(1) Show that both vector spaces are g dimensional, so reduce to showing the map is injective.
(2) Show the following diagram is commutative.

$$\begin{tikzcd} C \arrow[r, "x \mapsto \mathscr{O}(x)"] \arrow[d, "x \mapsto (x,p,\dots,p)"] & \mathrm{Pic}^1_{C/k} \arrow[d, "\mathscr{L} \mapsto \mathscr{L} \otimes \mathscr{O}((g-1)p)"] \\ C^g \arrow[r, "(x_1,\dots,x_g) \mapsto \mathscr{O}(x_1+\cdots+x_g)"'] & \mathrm{Pic}^g_{C/k} \end{tikzcd} \tag{4.1}$$

(3) Reduce to showing that the composite of the left and lower maps in the above diagram is injective.
(4) Show that the lower map factors through $\mathrm{Sym}^g_{C/k}$ and hence reduce to showing injectivity of the composite

$$H^0(C, \Omega_C) \leftarrow H^0(C^g, \Omega_{C^g}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}}) \leftarrow H^0(\mathrm{Pic}^g_{C/k}, \Omega_{\mathrm{Pic}^g_{C/k}}).$$

(5) Show that $\mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$ is birational.
(6) Conclude that $H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}}) \leftarrow H^0(\mathrm{Pic}^g_{C/k}, \Omega_{\mathrm{Pic}^g_{C/k}})$ is an isomorphism. Reduce to showing $H^0(C, \Omega_C) \leftarrow H^0(C^g, \Omega_{C^g}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}})$ is injective.
(7) Let $h : C^g \to \mathrm{Sym}^g_{C/k}$ denote the quotient. Show this is generically separable to conclude that $h^*\Omega_{\mathrm{Sym}^g_{C/k}} \to \Omega^1_{C^g}$ is an injection.
(8) Show the composite $\Omega_{\mathrm{Sym}^g_{C/k}} \to h_* h^* \Omega_{\mathrm{Sym}^g_{C/k}} \to h_* \Omega^1_{C^g}$ is an injection of sheaves.
(9) Deduce $H^0(C^g, \Omega_{C^g}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}})$ is an injection.
(10) Show that the image of $H^0(C^g, \Omega_{C^g}) \leftarrow H^0(\mathrm{Sym}^g_{C/k}, \Omega^1_{\mathrm{Sym}^g_{C/k}})$ is invariant under the action of S_g, so the image factors through the S_g invariants $H^0(C^g, \Omega_{C^g})^{S_g}$.
(11) Identify $H^0(C^g, \Omega_{C^g}) \simeq \oplus_{i=1}^g H^0(C, \Omega_C)$ and conclude $H^0(C^g, \Omega_{C^g})^{S_g}$ corresponds to elements of $\oplus_{i=1}^g H^0(C, \Omega_C)$ in the image of the diagonal.
(12) Show that under the above identification $H^0(C^g, \Omega_{C^g})^{S_g}$ is sent injectively to $H^0(C, \Omega_C)$ via the map $C \to C^g$ induced by $x \mapsto (x, p, \dots, p)$. *Hint:* Show the induced map on differentials $\oplus_{i=1}^g H^0(C, \Omega_C) \to$

$H^0(C, \Omega_C)$ is given by sending $(\omega_1, \ldots, \omega_g) \mapsto \omega_1$, using that the $C \to C^g, x \mapsto (x, p, \ldots, p)$ is the identity in the first coordinate and constant on all other coordinates.

Exercise 4.1.3. Let C be a curve over a field k. Show that the Abel-Jacobi map $C \to \mathrm{Pic}^1_{C/k}$ induces an isomorphism $H^1(\mathrm{Pic}^1_{C/k}, \mathscr{O}_{\mathrm{Pic}^1_{C/k}}) \simeq H^1(C, \mathscr{O}_C)$. *Hint:* Use Exercise 4.1.2.

We next wish to show that the Abel-Jacobi map is a closed immersion.

Exercise 4.1.4.
(1) Show the Abel Jacobi map is proper.
(2) Reduce to showing the Abel-Jacobi map is injective on points and injective on tangent vectors *Hint:* A proper map which is injective on points and injective on tangent vectors is a closed immersion.
(3) Show it is enough the verify the previous part in the case $S = \operatorname{Spec} k$ for k a field.
(4) Reduce further to the case that k is algebraically closed.
(5) Verify the abel Jacobi map is injective on points, using that the genus of C is at least 1. (What goes wrong if the genus is 0?)
(6) It remains to show the Abel-Jacobi map is injective on tangent vectors when $S = \operatorname{Spec} \overline{k}$. This is the trickiest part. Rephrase this as showing that $i^*\Omega_{Pic^1_{C/k}} \to \Omega_C$ is surjective.
(7) Show $i^*\Omega_{Pic^1_{C/k}} \simeq H^0(\mathrm{Pic}^1_{C/k}, \Omega_{\mathrm{Pic}^1_{C/k}}) \otimes_k \mathscr{O}_C$.
(8) Use the adjunction associated to $f : C \to S$ to produce the commutative square

$$\begin{array}{ccc} H^0(\mathrm{Pic}^1_{C/k}, \Omega_{C/k}) \otimes \mathscr{O}_C & \longrightarrow & H^0(C, \Omega_{C/k}) \otimes \mathscr{O}_C \\ \downarrow & & \downarrow \\ H^0(\mathrm{Pic}^1_{C/k}, \Omega^1_{\mathrm{Pic}^1/k}) \otimes \mathscr{O}_C \simeq i^*\Omega_{\mathrm{Pic}^1_{C/k}} & \longrightarrow & \Omega_C \end{array} \tag{4.2}$$

(9) Show the bottom map in the previous part is surjective by showing the top and right map are surjective. *Hint:* For the right map, show $\Omega_{C/k}$ is globally generated using Serre duality and that C has genus at least 1. Show the top map is an isomorphism by Exercise 4.1.2.
(10) Deduce that the Abel-Jacobi map is an immersion.

4.2. **Constructing the principal polarization over an algebraically closed field.** Having constructed the Abel-Jacobi map, our next goal is to construct the Torelli map.

Definition 4.2.1. Suppose we have a curve $C \to S$ with a section $p : S \to C$. Let $C^{g-1} := C \times_S \cdots \times_S C$ denote the $g-1$ fold fiber product of C over S.

Define the *Theta divisor relative to* p, Θ_p,, as the image

$$\begin{aligned} C^{g-1} &\to \mathrm{Pic}^0_{C/S} \\ (p_1, \ldots, p_{g-1}) &\mapsto \mathscr{O}_C\left(\left(\sum_{i=1}^{g-1} p_i\right) - (g-1)\cdot p\right). \end{aligned}$$

We next wish to show that Θ_p defines a principal polarization. The Torelli map will then be the map which sends C to $\mathrm{Pic}^0_{C/S}$ along with this principal polarization. We will first outline in exercises how to show Θ_p defines a principal polarization over an algebraically closed field, and then show that Θ_p is independent of the choice of p, and can in fact be defined over a general base S when $C \to S$ has no sections.

The following is a standard result in the theory of abelian varieties, which are a bit tricky to prove directly, but are contained, for example, in Mumford's book on abelian varieties. It will allow us to verify that Θ_p is a principal polarization.

Proposition 4.2.2. *Suppose A is an abelian variety over an algebraically closed field k, and $\mathscr{L} = \mathscr{O}_A(D)$ for an effective Cartier divisor D. Let $t_x : A \to A$ denote translation by $x \in A(k)$. If $t_x^*\mathscr{L} \not\simeq \mathscr{L}$ for every $x \neq e$ in $A(k)$ and $h^0(A, \mathscr{L}) = 1$ then $\phi_{\mathscr{L}}$ is a principal polarization.*

Exercise 4.2.3. Suppose $\mathrm{Pic}^0_{C/k}$ is an abelian variety over a field and $\mathscr{L}$ is the theta divisor Θ_p. Supposing we know $h^0(\mathrm{Pic}^0_{C/k}, \Theta_p) = 1$, conclude $t_x^*\mathscr{L} \not\simeq \mathscr{L}$ for any $x \in \mathrm{Pic}^0_{C/k}$.

Exercise 4.2.4 (Easy exercise, given the above)**.** In order to show Θ_p is a principal polarization over a field, show that it suffices to prove $h^0(A, \Theta_p) = 1$.

We next aim to prove $h^0(A, \Theta_p) = 1$.

Exercise 4.2.5 (Easy exercise)**.** Show $h^0(A, \Theta_p) \geq 1$.

Define the map

$$\begin{aligned} i_D : C &\to \mathrm{Pic}^0_{C/k} \\ x &\mapsto K_C - D + x. \end{aligned}$$

Exercise 4.2.6. Show i_D is an immersion. *Hint:* Use that the Abel-Jacobi map is an immersion.

Exercise 4.2.7. Show that a general divisor D on C of degree g satisfies $h^0(C, D) = 1$.

Exercise 4.2.8. If $h^0(\mathrm{Pic}^0_{C/k}, \Theta_p) = r + 1$ and D has degree d with $h^0(C, D) = 1$, show $i_D^{-1}(\Theta_p) = D$ as a divisor on C.

Exercise 4.2.9. With notation as in the previous exercise, show that the subspace of sections of $H^0(\mathrm{Pic}^0_{C/k}, \Theta_p)$ which vanish on $i_D(C)$ has dimension r. *Hint:* Consider the map on H^0 induced by the map i_D.

Exercise 4.2.10. Consider the incidence correspondence
(4.3)

$$\begin{tikzcd}
& \Phi := \left\{ (E, \Psi) : E \in \mathrm{Sym}^g_{C/k}, \Psi \in \mathbb{P}H^0(\mathrm{Pic}^0_{C/k}, \Theta_p, i_E(C) \subset \Psi \right\} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\
\mathrm{Pic}^g_{C/k} & \mathbb{P}H^0(\mathrm{Pic}^0_{C/k}, \Theta_p).
\end{tikzcd}$$

(1) Assuming $r > 0$ as in Exercise 4.2.8, show $\dim \Phi = r + g - 1$ and conclude the generic fiber of π_2 has dimension $g - 1$.
(2) We want to show $r = 0$, so it remains to deduce a contradiction. Show it suffices to prove the fiber of π_2 over the point corresponding to Θ_p has dimension at most $g - 2$.
(3) Reduce to showing that the set of $[E] \in \mathrm{Sym}^g_{C/k}$ with $h^0(C, E) > 1$ are constrained in a subscheme of $\mathrm{Sym}^g_{C/k}$ of codimension at least 2.
(4) Show the set of $[E] \in \mathrm{Sym}^g_{C/k}$ with $h^0(C, E) > 1$ does have codimension 2 by identifying this set with the subset of $\mathrm{Sym}^g_{C/k}$ where the fiber dimension of $\mathrm{Sym}^g_{C/k} \to \mathrm{Pic}^g_{C/k}$ is positive. Use that this is a birational map of irreducible varieties to conclude this subset has codimension at least 2.
(5) Conclude $h^0(\mathrm{Pic}^0_{C/k}, \Theta_p) = 1$. Conclude that Θ_p induces a principal polarization $\phi_{\mathscr{O}(\Theta_p)}$.

4.3. **Constructing Θ over a general base.**

Exercise 4.3.1. Show that for any $p : S \to C$, Θ_p is an effective relative Cartier divisor. *Possible hint:* Being a relative effective Cartier divisor is equivalent to being a Cartier divisor and each fiber being a Cartier divisor; check this by pulling back from the moduli space of curves, which is regular, so Cartier and Weil divisors coincide.

An important construction in the theory of abelian varieties is the map $\phi_{\mathscr{L}}$ defined as follows. Let $A \to S$ be an abelian scheme and $\mathscr{L}$ an invertible sheaf on A. Define

$$\begin{aligned}\phi_{\mathscr{L}} A &\to A^\vee \\ x &\mapsto t_x^*\mathscr{L} \otimes \mathscr{L}^\vee.\end{aligned}$$

Exercise 4.3.2. Show that if A and B are two abelian schemes over S, $\mathrm{Hom}_S(A, B)$ is unramified by showing it is formally unramified. *Hint:* Use the rigidity lemma.

Exercise 4.3.3. Show that if $p, q : S \to C$ are two sections, the induced maps $\phi_{\mathscr{O}(\Theta_p)}$ and $\phi_{\mathscr{O}(\Theta_q)}$ from $(\mathrm{Pic}^0_{C/S})^\vee \to ((\mathrm{Pic}^0_{C/S})^\vee)^\vee \simeq \mathrm{Pic}^0_{C/S}$ coincide. *Hint:* Use the previous exercise and the rigidity lemma to reduce to the case that $S = \operatorname{Spec} k$ is an algebraically closed field. In this case, use that a map from a curve to an unramified scheme over a field is constant.

Exercise 4.3.4 (Assumes knowledge of descent)**.** Show that for any smooth proper relative curve $C \to S$ with geometrically connected fibers, there is a unique map $\phi_\theta : \left(\mathrm{Pic}^0_{C/S}\right)^\vee \to \mathrm{Pic}^0_{C/S}$ such that for any flat $T \to S$ with section $p : T \to C_T$, the base change of ϕ_Θ to T agrees with $\phi_{\mathscr{O}(\Theta_p)}$. *Hint:* This is an application of fppf descent.

Exercise 4.3.5. For $C \to S$ a smooth proper curve with geometrically connected fibers, we have defined a map $\phi_\Theta : \left(\mathrm{Pic}^0_{C/S}\right)^\vee \to \mathrm{Pic}^0_{C/S}$ in Exercise 4.3.4. Show that ϕ_Θ is a principal polarization. *Hint:* Reduce to checking it defines an isomorphism on fibers over S and use Exercise 4.2.10.

Finally, we can define the Torelli map!

Definition 4.3.6. For $g \geq 0$, the Torelli map is the map $\mathscr{M}_g \to \mathscr{A}_g$ given by sending $C \to S$ to the principally polarized abelian scheme $(\mathrm{Pic}^0_{C/S}, \phi_\Theta)$.

4.4. **The infinitesimal Torelli theorem.** For $g \geq 2$, we have a map $\tau_g : \mathscr{M}_g \to \mathscr{A}_g$ sending a curve to its principally polarized Jacobian. Recall the usual Torelli theorem for curves says this is injective on geometric points.

Exercise 4.4.1. Show that for $g \geq 3$, the Torelli map is generically $2:1$, so it is *not* an generically immersion on stacks. *Hint:* Generic curves have no automorphisms, but abelian varieties all have at least 2 automorphisms.

Even ignoring this $2:1$ issue, it turns out the Torelli map is ramified along the hyperelliptic locus. We next present a series of exercises to explain this. You may take for granted the following deformation theory facts:

(1) The tangent space to $\mathscr{M}_g$ at $[C]$ is identified with $H^1(C, T_C)$ for C a genus g curve.

(2) Let $\mathrm{Sym}_2 V$ denote the kernel of the map $V^{\otimes 2} \to \wedge^2 V$. The tangent space to $\mathscr{A}_g$ at $[A, \phi]$ is identified with $\mathrm{Sym}_2 H^1(A, \mathscr{O}_A) \subset H^1(A, \mathscr{O}_A)^{\otimes 2} \simeq H^1(A, \mathscr{O}_A) \otimes H^0(A, \Omega_A)^\vee \simeq H^1(A, \Omega_A^\vee) \simeq H^1(A, T_A)$ where $H^1(A, T_A)$ parameterizes deformations of the abstract abelian variety A and $\mathrm{Sym}_2 H^1(A, \mathscr{O}_A)$ parameterizes deformations of the polarized abelian variety. See [OS79, Theorem 2.6] for a proof.

Exercise 4.4.2. Compute the dimension of the tangent space to $\mathscr{A}_g$ at a point. Assuming that $\mathscr{A}_g$ is smooth, use this and the Torelli theorem to give another proof that the Torelli maps is dominant in genus 2 and 3.

Exercise 4.4.3. (1) Show that a general 4-dimensional ppav is not the Jacobian of a curve. *Hint:* Do a dimension count. You may want to do Exercise 4.4.2 first.

(2) Given a genus 5 curve C, and a connected finite étale double cover $D \to C$ show that $\mathrm{Pic}^0_D / \mathrm{Pic}^0_C$ is a 4-dimensional ppav. This is called a Prym abelian variety.
(3) Make a dimension count which makes it plausible that a general 4-dimensional ppav is obtained from such a Prym construction. (This is in fact true, prove it as a bonus.)
(4) Make a dimension count which makes it plausible that a general 5-dimensional ppav is also obtained from such a Prym construction. (This is in fact true, prove it as a bonus.)
(5) Show that a general 6-dimensional ppav is not obtained from such a Prym construction.

Exercise 4.4.4. Let C be a curve and J be its principally polarized Jacobian.

(1) Identify the map on tangent spaces $T_{[C]}\tau_g : T_{[C]}\mathscr{M}_g \to T_{[J]}\mathscr{A}_g$ with a map $H^1(C, T_C) \to \mathrm{Sym}^2 H^1(J, \mathscr{O}_J)$.
(2) Using the identification $H^1(J, \mathscr{O}_J) \simeq H^1(C, \mathscr{O}_C)$ of Exercise 4.1.3 and Serre duality, show the above map $H^1(C, T_C) \to H^1(J, T_J)$ can be re-expressed as a map $\mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$. This turns out to agree with the multiplication map sending two sections to their tensor product, see [Voi07, Lemma 10.19, Theorem 10.21, and Lemma 10.22] and also [Lan19] for a more algebraic proof.
(3) Show the Torelli map is injective on tangent vectors if and only if $\mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ is surjective.

Exercise 4.4.5. For this exercise, we work over a field k of characteristic not 2.

(1) Show that if C is given by an equation of the form $y^2 = f$, then a basis for $H^0(C, \omega_C)$ is given by $x^i\, dx/y$ for $0 \le i \le g - 1$.

(2) Show that if $g \geq 2$, the image of $\operatorname{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ has dimension $2g - 1$.
(3) Conclude that if C is a hyperelliptic curve of genus $g > 2$, the map $\mu : \operatorname{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ is not surjective.
(4) Deduce that the Torelli map is not injective on tangent vectors along the hyperelliptic locus.
(5) Compute an explicit basis for $H^0(C, \omega_C^{\otimes 2})$ of dimension $3g - 3$.
(6) Compute a basis for the kernel of $\mu : \operatorname{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$.

We can also extend the above exercise to work in characteristic 2. It is possible to prove the map μ fails to be surjective in characteristic 2, although here we content ourself with giving an explicit description of the image of μ, while not computing the dimension of the image.

Exercise 4.4.6 (Tricky exercise)**.** Suppose k has characteristic 2.

(1) Show that any hyperelliptic curve can be written in the form $sy^2 + hy + f = 0$ for h a polynomial of degree $g + 1$ and f a polynomial of degree $2g + 2$ and $s \in k^\times$.
(2) Show that a general such curve can be written in the form $y^2 - y = f$ for $f \in k(x)$ of the form $f = \alpha_0 x + \sum_{i=1}^{g} \frac{\alpha_i}{x - a_i}$ for $\alpha_i, a_i \in k$. *Hint:* Reduce to the case $s = 1$ and replace y by by, and divide the result by b^2 to obtain an equation of the form $y^2 - y = c/b^2$. Then, apply transformations of the form $y \mapsto \frac{y+1}{(x - a_i)^{r_i}}$ where $b = \prod (x - a_i)^{r_i}$.
(3) For C of the above form, show that the hyperelliptic map $C \to \mathbb{P}^1$ is ramified to order 2 at each of the points $a_1, \ldots, a_g, \infty$.
(4) In the above setting, show that $\frac{dx}{x - a_i}$ for $1 \leq i \leq g$ form a basis for $H^0(C, \omega_C)$.
(5) Show that the image of μ is the span of the elements $\frac{(dx)^2}{(x - a_i)(x - a_j)}$. (It turns out these span a $2g - 1$ dimensional subspace, but we will not prove this.)

Exercise 4.4.7. Recall that an abelian variety A in characteristic p over an algebraically closed field k is *ordinary* if $\# A[p] = p^g$. Show that a hyperelliptic curve which can be written in the form $y^2 - y = f$ for f of the form $\alpha_0 x + \sum_{i=1}^{g} \frac{\alpha_i}{x - a_i}$ for $\alpha_i, a_i \in k$ is ordinary. *Hint:* Construct 2^g many 2-torsion line bundles in terms of the preimages of $x - a_i$ on the hyperelliptic curve.

Remark 4.4.8. More generally, the Deuring-Shafarevich formula shows a hyperelliptic curve in the above form is ordinary if and only if all the poles of f have order 1. See [Sub75, Theorem 4.2].

Exercise 4.4.9. Show that a general hyperelliptic curve of genus g in characteristic 2 can be written via an equation as in the previous exercise. Conclude that a general hyperelliptic curve of genus g in characteristic 2 is ordinary.

A classic result of Max Noether says that if C is a curve of genus $g \geq 3$, then the multiplication map $\operatorname{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ is surjective if C is not hyperelliptic.

Exercise 4.4.10 (Easy exercise)**.** Using Max Noether's theorem mentioned above, show that τ_g is injective on tangent vectors at any non-hyperelliptic curve.

Exercise 4.4.11. Show that the one can put the structure of a ring on the vector space $\oplus_{i=0}^{\infty} H^0(C, \omega_C^{\otimes i})$. This is called the *canonical ring* of C. Show that the canonical ring is generated in degree 1 if and only if C is hyperelliptic.

The canonical ring as defined in the previous exercise is the subject of Petri's theorem.

Theorem 4.4.12 (Petri's theorem)**.** *Suppose C is a nonhyperelliptic curve over an algebraically closed field of genus $g \geq 4$ and C is neither trigonal (has a degree 3 map to $\mathbb{P}^1$) nor a plane quintic. Then, the canonical ring is generated in degree 1 with relations in degree 2.*

The following exercise is an infinitesimal version of Torelli's theorem for curves. While Torelli's theorem says one can recover C from its Jacobian, the infinitesimal Torelli theorem says one can recover C from the derivative of the Torelli map.

Exercise 4.4.13 (Fun exercise)**.** Using Petri's theorem, show that if $g \geq 4$ and C is a curve with is not hyperelliptic, a plane quintic, or trigonal then one can recover C from the linear map $T_{[C]}\tau_g : \operatorname{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$.

We now include a few more exercises relating to canonical rings.

Exercise 4.4.14. Show that if C is trigonal the canonical ring has a relation in degree 3.

Exercise 4.4.15. Show that no curves of genus 3 have canonical ring generated in degree 1 with relations in degree 2.

Exercise 4.4.16. If C is a hyperelliptic curve of the form $y^2 = f$, work out a presentation of the canonical ring of C. *Hint:* One may write a basis of differentials for C in the form $dx/y, \ldots, x^{g-1}dx/y$.

5. Introduction to moduli spaces

The general idea of a moduli space is that its points should parameterize isomorphism classes of a certain class of object of interest. For example, $\mathscr{M}_g$, the moduli space of curves, is a stack whose complex points parameterize curves over $\mathbb{C}$. The moduli space of principally polarized Abelian varieties, $\mathscr{A}_g$, is a stack whose complex points are in bijection with principally polarized Abelian varieties. More generally, we would like moduli spaces so that, not only do their $\mathbb{C}$ points parameterize objects of interest, but for any scheme S, their S points parameterize relative objects over S.

We start with a few examples of this to warm up before getting to the more involved moduli spaces of curves and moduli space of principally polarized Abelian varieties.

5.1. Moduli spaces of 0-cycles.

We start with a few exercises relating to moduli spaces of 0 dimensional subschemes of a given scheme.

Exercise 5.1.1 (Easy exercise; every scheme is a moduli space of its points)**.** Let B be a scheme and let $f : X \to B$ be a map of schemes.

(1) Explain why maps $g : B \to X$ so that $f \circ g = \mathrm{id}$ are in bijection with B-points of X (over B).
(2) More generally, for $h : S \to B$ a scheme, explain why maps $g : S \to X$ with $h = f \circ g$ are in bijection with S-points of X over B.
(3) Interpret the above as saying that X is a moduli space of points on X.
(4) Also interpret this as saying that X is a moduli space of degree 1 effective 0-cycles on X.
(5) Describe a universal family for this moduli space. *Hint:* The fiber over $x \in X$ should be the same point $x \in X$. What subscheme of $X \times X$ does this correspond to?

Exercise 5.1.2. In the setting of Exercise 5.1.1, show that X^n is a moduli space for ordered tuples of n-points on X. Here $X^n := \underbrace{X \times_B \cdots \times_B X}_{n \text{ times}}$. (Part of the exercise is interpreting what this means in terms of S-points.) Construct a finite locally free universal family $Z \to X^n$, with $Z \subset X \times_B X^n$, so that the fiber of Z over a point $t = (x_1, \ldots, x_n)$ is the subscheme $\{x_1, \ldots, x_n\} \subset X$.

Exercise 5.1.3. In the setting of Exercise 5.1.1, show that $\mathrm{Sym}^n_{X/B} := X^n / S_n$ parameterizes effective 0-cycles of degree n on X. Here, this denotes the quotient space of X^n by the S^n action; you may assume X is quasi-projective so that this exists as a scheme, but in general it exists as an algebraic space. (Part of the exercise is interpreting what this means in terms of S-points). Construct a finite locally free universal family $Z \to \mathrm{Sym}^n_{X/B}$ with $Z \subset$

$X \times_B \mathrm{Sym}^n_{X/B}$ whose fiber over a 0 cycle $z = \{x_1, \ldots, x_n\}$ is the subscheme $\{x_1, \ldots, x_n\} \subset X$. *Hint:* Use the universal family from the previous exercise and take the quotient by an S_n action.

Exercise 5.1.4. In the setting of Exercise 5.1.3, construct an open subscheme $\mathrm{Conf}^n_{X/B} \subset \mathrm{Sym}^n_{X/B}$ parameterizing degree n effective 0-cycles corresponding to n distinct points. Describe the S-points of this space and construct a universal family over $\mathrm{Conf}^n_{X/B}$.

The space $\mathrm{Conf}^n_{X/B}$ from the above exercise is called the (unordered) configuration space of n points of X over B.

Exercise 5.1.5. Suppose that in the setting of Exercise 5.1.3, $X \to B$ is a smooth proper curve with geometrically connected fibers. Show that $\mathrm{Sym}^n_{X/B}$ is also a moduli space for degree n effective divisors. Show also in this case that $\mathrm{Sym}^n_{X/B}$ also parameterizes closed subschemes of degree n.

Exercise 5.1.6. For k a field, show that $\mathrm{Sym}^n_{\mathbb{A}^1/k} \simeq \mathbb{A}^n$.

Exercise 5.1.7. For k a field, Describe $\mathrm{Conf}^n_{\mathbb{A}^1} \subset \mathrm{Sym}^n_{\mathbb{A}^1/k}$ as an S_n quotient of a complement of a union of hyperplanes in $(\mathbb{A}^1)^n$.

Exercise 5.1.8. For k a field, show that $\mathrm{Sym}^n_{\mathbb{P}^1/k} \simeq \mathbb{P}^n$.

Exercise 5.1.9. Describe explicitly $\mathrm{Conf}^2_{\mathbb{P}^1/k}$ as an open subscheme of $\mathbb{P}^2 \simeq \mathrm{Sym}^2_{\mathbb{P}^1/k}$. As a bonus, make sure your description also works in characteristic 2.

5.2. **Moduli spaces of finite locally free algebras.**

Exercise 5.2.1. Fix a field k and construct a moduli space of finite locally free degree d covers over k as follows.

(1) First, construct a moduli space of based finite locally free degree d covers, meaning a scheme whose $\mathrm{Spec}\, R$ points correspond to algebras with underlying R-module structure of the form $R^{\oplus n}$. *Hint:* Express this moduli space as a closed subscheme of $\mathbb{A}^{n^3}$, where the $\mathbb{A}^{n^3}$ has coordinates given by $a_{i,j,k}$ in the equations $e_i \cdot e_j = \sum_k a_{i,j,k} e_k$, for e_i the basis elements for $R^{\oplus n}$. The closed subscheme we will take corresponds to imposing conditions that this be an algebra. For example, we need to impose the conditions that it is commutative and associative. Commutativity amounts to the constraints that $a_{i,j,k} = a_{j,i,k}$.
(2) Construct a moduli space of covers by quotienting the space of based algebras by the GL_n action on the basis vectors.

Exercise 5.2.2. Explicitly work out the moduli stack constructed in Exercise 5.2.1 in the cases $d = 1$ and $d = 2$. When $d = 2$ and 2 is invertible, show this moduli stack is isomorphic to the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by the weight 2 action given by $t \cdot x = t^2 x$ for $t \in \mathbb{G}_m(S)$ and $x \in \mathbb{A}^1(S)$. In general, describe this as a quotient stack of $\mathbb{A}^2$ by a certain action of $\mathbb{G}_m \ltimes \mathbb{G}_a$. *Possible hint:* It may be helpful to fix the first basis vector to be the element 1.

Exercise 5.2.3 (Assumes background with stacks)**.** Show that there is an open substack of the moduli stack of degree d covers which parameterizes finite étale covers of degree d. Show that this substack has isotropy group S_d.

Exercise 5.2.4. Fix a finite group G. Construct a moduli space of G covers via a similar method to that of Exercise 5.2.1, where now the moduli space has S points given by finite locally free covers $X \to S$ with a G-action on X so that the quotient X/G is B. *Hint:* Use a similar strategy in terms of writing out generators for the algebra as in Exercise 5.2.1, but now impose additional conditions so that the cover respects the G action.

Exercise 5.2.5. Generalize the above exercises, starting with Exercise 5.2.1 to work over a general base B instead of just over a field k.

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