

Reminder : The Grothendieck
Ring R

Additive generators : $[X]$ varieties/
iso

Additive relations $[X] = [U] + [Z]$

Multiplication $[X \cdot Y] = [X] \cdot [Y]$

The motivic zeta function

$$Z_X(t) = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n \in R[[t]]$$

$$\begin{aligned} Z: R &\longrightarrow 1 + tR[[t]] \subset R[[t]] \\ + \curvearrowright & x \end{aligned}$$

example: $k = \mathbb{F}_q$ ζ Weil zeta function.

$$\# : Z_X(t) \xrightarrow{\text{Ex.}} \zeta_X(t) \in \mathbb{Z}[[t]]$$

Question 2000
Kapranov

$Z_X(t)$ rational?

evidence: curves.

/c

Hodge structures.
TRUE!

Cheah '94

FFg pt counting ✓

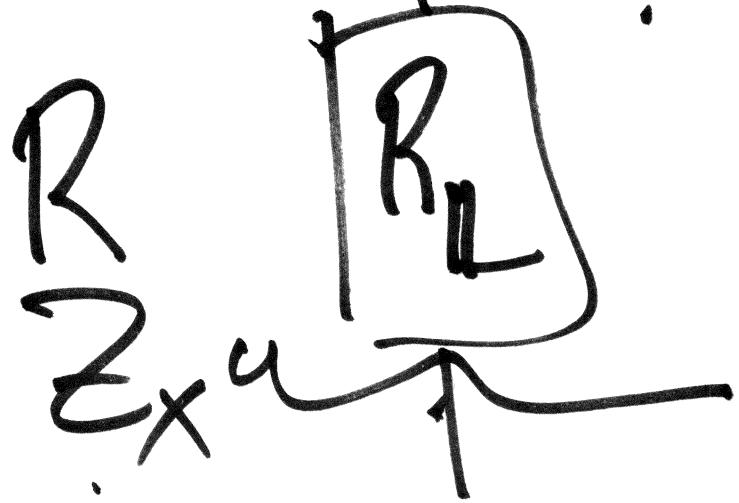
$f \in C([t])$

$= \sum_h g_h h \in C(t).$ $f h = g.$

Theorem (Larsen-Lunts)

No!

$b = f g$



$k = \mathbb{C}$

Thm ~~"~~

$$\frac{(1-t)^{h_1} (1-t)^{h_3} \dots}{(1-t)^{h_0} (1+t)^{h_2} \dots}.$$

Thm (Cheah)

Thm X complex variety

$$\sum_{n \geq 0} h^{p,q} \cdot (H_c^r(\text{Sym}^n X)) t^n x^p y^q (-z)^r$$

$$= \prod_{p,q,r} \left(\frac{1}{1 - x^p y^q z^r t} \right) (-1)^r h^{p,q} (H_c^r (X))$$

Theorem ...

$H^i : \text{Var}_K \xrightarrow[\text{functor.}]{\text{contr}} (\text{VS}^+ \text{ more.})$

↳ field char 0.

$\oplus \otimes \wedge$.

$$X(a) := \sum (-1)^i H^i(x) \in K(\text{VS}^+)$$

Axiom : $H^i(a) = 0$ for $i \gg 0$

Theorem

$$\sum_{i=0}^{\infty} X(\text{Sym}^i X) t^i = \frac{1}{(1-t)^{X(X)}}$$
$$\in K[[t]]$$

$$(1-t)^V = 1 - (V)t + (V^2)t^2 - (V^3)t^3 \dots \stackrel{def}{=} \sum (V^k)t^k$$

$$(1-t)^V (1-t)^W = (1-t)^{V+W}$$

Ex

$$\frac{1}{(1-t)^n} = \sum_{k=0}^{\infty} S_{nk} t^n$$

Axiom (Künneth) $H^*(X \times Y) = \bigoplus_{i=0}^n H^i(X) \otimes H^{n-i}(Y)$

Axiom $G \leq T$
finite gp.

$\phi: T \rightarrow T/G$.

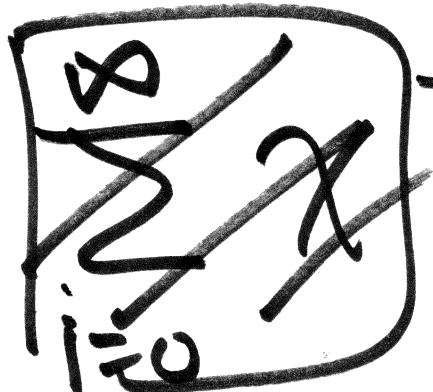
$\phi^*: H^i(T/G) \rightarrow H^i(T)$

$H^i(T/G)$ with $(H^i(T))^G$.

identifies (Groth. To.)

$$H^*(\text{Sym}^n X) \xrightarrow{\sim} H^*(X)^{S_n}$$

\approx in terms of $H^*(X)$



Cor. Hodge structures
 $\equiv \Rightarrow$ Cheah.

Cor $k = \mathbb{R} F_g$ #. $F \in H_{\text{et}}^r(X, \mathbb{Q}_\ell)$

$$\sum_i (-1)^i \text{tr}(F|_{H^r(X, \mathbb{Q}_\ell)})$$

Larsen-Lunt:

$$\mu(X) = 1 + h^0(\mathcal{I}^1) t + h^0(\mathcal{I}^2) t^2 + \dots + h^0(\mathcal{I}^{dix}) t^{dix}$$

smooth
projective

$$\in 1 + t \mathbb{Z}[[t]]$$

$$\begin{aligned}\mu(L) &= 0 \\ &= \mu(P') - \mu(pt)\end{aligned}$$

Motivation

$$S_x(t) = \frac{P_1(t) \cdots P_{2d-1}(t)}{q P_0(t) \cdots P_{2d}(t)} = \sum q_i t^i$$

$$q_i \sim (g^d)^i \quad (1 - g^d t^{d+1})$$

~ Hasse-Weil bounds

Conj (Motivic Stabilization of Powers). TMSSP

$$\frac{\text{Sym}^n X}{\text{L}^n \text{dim } X} = \mathbb{E}_{\substack{X \\ \in R_L}} \hat{}$$

converges.

-
- Evidence.
- [Stably rational varieties] pt. counting ✓
 - Hodge structures ✓
 - curves ✓

Topology.

$p \in X$

$$\text{Sym}^n X \longrightarrow \text{Sym}^{n+1} X$$

effective Arnav
Tripathy.

$$\text{Sym}^\alpha X = \overline{\pi \cdot K(\tilde{H}_i(X), n)}.$$

Thm (Tripathy).
“Sym” étale realization.
commute.

$$\begin{array}{ccc} X \times X \times A^n & \xrightarrow{\text{“homotopiz”}} & \\ \text{Sym}^n X & \xrightarrow{\quad \quad \quad} & \text{Sym}^n(X \times A^n) \end{array}$$

Thm (Totaro)
In R ,

$$\begin{aligned} & \left[\text{Sym}^n (x = A^m) \right] \\ &= \left[(\text{Sym}^n x) \underset{\text{sym}}{\sim} \right] A^{mn}. \end{aligned}$$

$$\left[\text{Sym}^n A^m \right] = \boxed{\text{?}} \left[A^{mn} \right]$$