Asymptotics For Number Fields And Class Groups

Project Group: Expected expectations

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Distributions of Class Groups

Quadratic fields

- conditions at infinity (Real vs. Imag.) has a big effect on the class group.
- MB-heuristics imply non-arch local cond. have no effect.

Goal: Investigate this in further cases (higher degree) (more conditions)

Cubic fields (K cubic field)

How to count $\#Sur(Cl(K), \mathbb{Z}/2\mathbb{Z})$?

Do local conditions affect this?

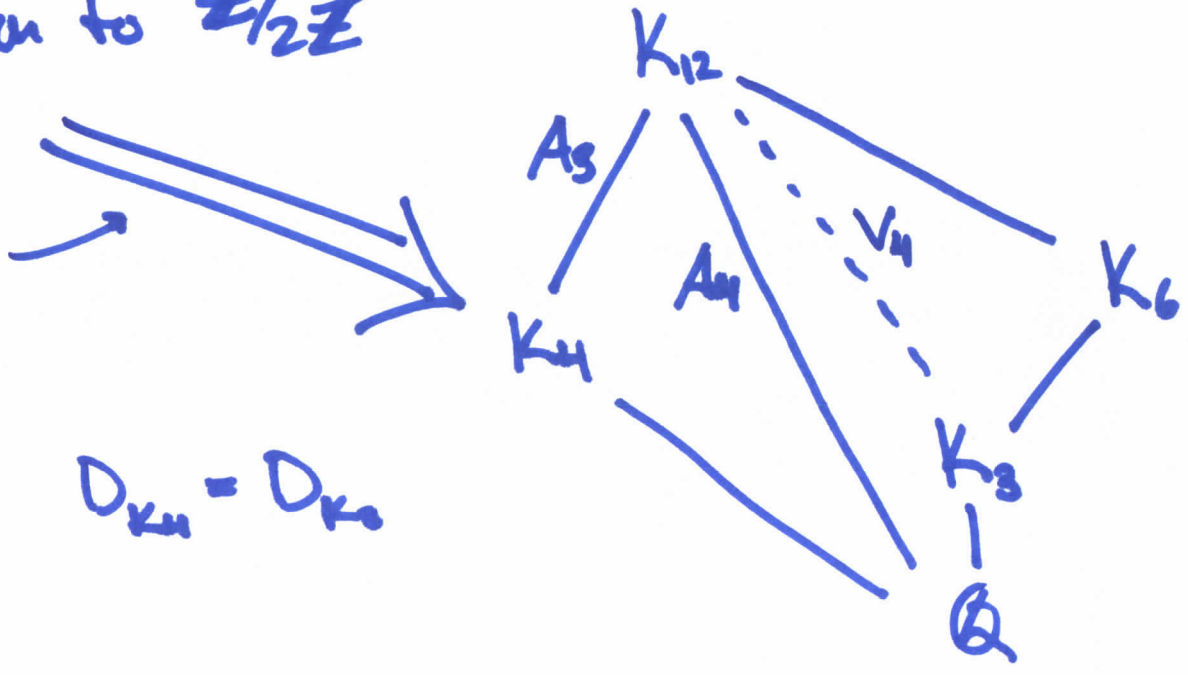
Condition at ∞: K_3 cyclic cubic (totally real)

+

Surjection to $\mathbb{Z}/2\mathbb{Z}$

Heilbronn

s.t. $D_{K_{12}} = D_{K_6}$



Decomposition Group	Inertia Group	Splitting Type in K_3	Splitting Type in K_4
1	1	$(1,1,1)$	$(1,1,1,1)$
$\langle(123)\rangle$	1	(3)	$(1,3)$
* $\langle(123)\rangle$	(123)	(1^3)	$(1,1^3)$
$\langle(12)(34)\rangle$	1	$(1,1,1)$	$(2,2)$
* $\langle(12)(34)\rangle$	$(12)(34)$	$(1,1,1)$	$(1^2,1^2)$
$W_4 = V_4$	$(12)(34)$	$(1,1,1)$	(2^2)

$$\#\left\{\mathrm{Gal}(\mathrm{Cl}(K_3), \mathbb{Z}/2\mathbb{Z}) : K_3 \text{ cyclic cubic}, \text{ totally split at } 7 \right\}$$
$$|D_{K_3}| < X$$

??

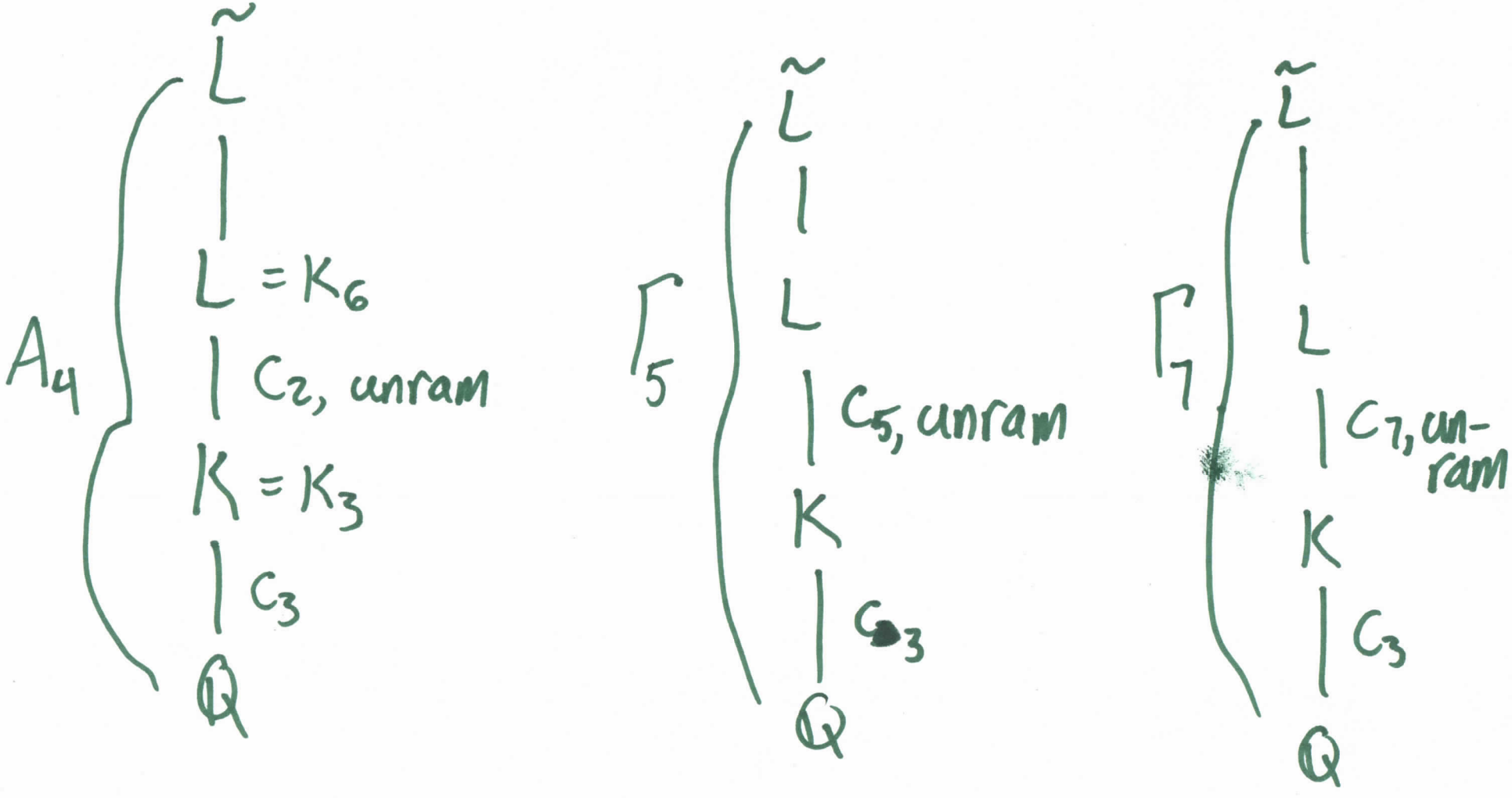
$$\#\left\{ |D_{K_4}| < X : K_4 \ A_4\text{-quartic}, \text{ totally split at } 7 \text{ or } (7) = P_1 P_2 \right\}$$
$$+(\text{Extra conditions})$$

Bhargava has results on counting S_4-quartics

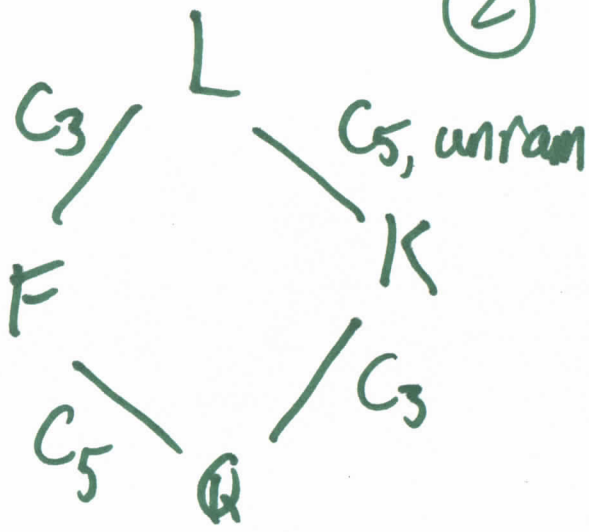
. . . but we dont know how to count A_4-quartics.

We can still compute local factors and make predictions using MB-heuristics, in this and more difficult cases.

Possible Galois Groups



$\Gamma_5 \cong (C_5)^K \rtimes C_3$; $\underline{K \neq 1}$:



$\underline{K \neq 3}$: $C_3 = \langle \alpha \rangle$. α acting on $(C_5)^3$ must have 1 as an eigenvalue.

$\Rightarrow \Gamma_5 \cong *(C_5 \times C_5) \rtimes (C_5 \times C_3)$

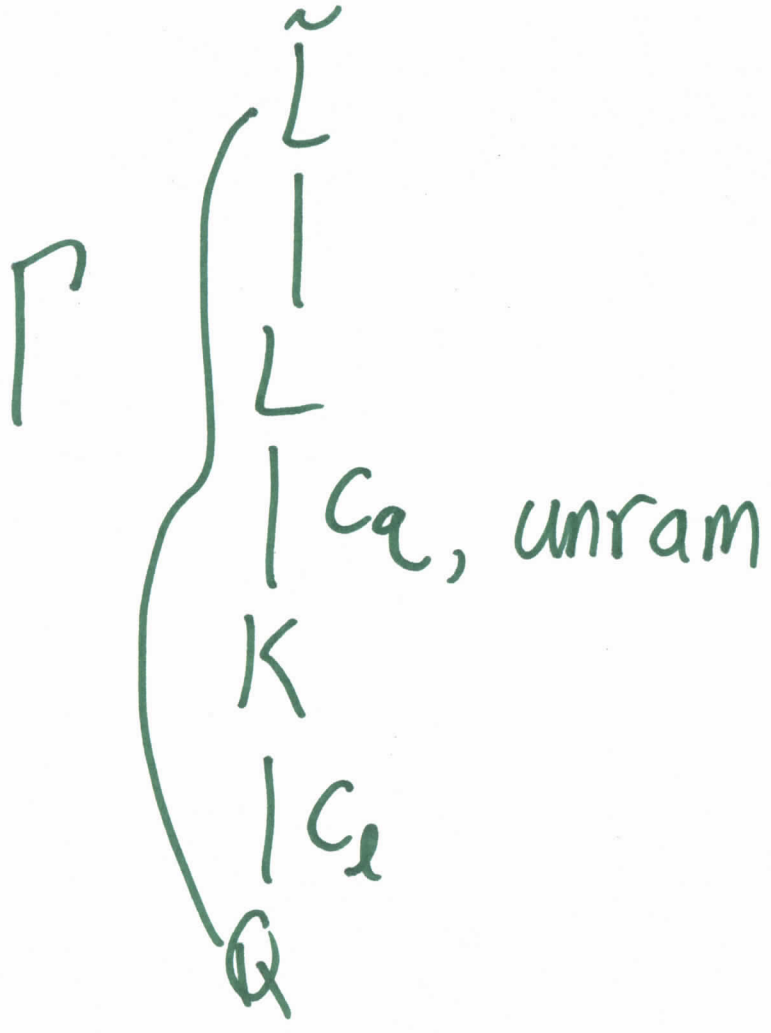
$\Rightarrow$ Unram extension of $\mathbb{Q}$. #

$\Gamma_7 \cong (C_7)^k \rtimes C_3$; $k = 1, 2, \text{ or } 3$.

Although: • α acting on $(C_7)^k$ cannot have 1 as an eigenvalue.

• If $k \geq 2$, all eigenvalues of α cannot be the same.

More generally: $\Gamma \cong (C_q)^K \rtimes C_\ell$



Notation:

- $C_\ell = \langle \alpha \rangle$, α acts on $(C_q)^K$ by φ_α.
- $\forall \alpha \in \Gamma$, $v \in (C_q)^K$

Proposition

1) φ_a cannot have any eigenvalues $= 1$

2) If $\kappa \geq 2$, all eigenvalues of φ_a cannot be the same.

3) $q^\kappa \equiv 1 \pmod{\ell}$

And, 1) implies:

4) There are $\ell - 1$ conjugacy classes of elements of order ℓ.

proof of 4)

a) Total # elts order ℓ is $q^k(\ell-1)$:

$$(V_\alpha)^\ell = (v + \varphi_\alpha(v) + \cdots + \varphi_\alpha^{\ell-1}(v)) \cdot \alpha^\ell$$

$$= (I + \varphi_\alpha + \cdots + \varphi_\alpha^{\ell-1})(v)$$

$= 0$ b/c 1 not an eigenvalue of φ_α.

b) # elts in each C. class of order ℓ element is $(q^k \cdot \ell)/\ell = q^k$.

c) Therefore, # conjugacy classes of elts of order ℓ is

$$\frac{q^k(\ell-1)}{q^k} = \ell - 1. \quad //$$

Local Factors

$\tilde{L}$
$|$
L
$\text{nr} \,|\, C_q$
K
$|\, C_\ell$
$\mathbb{Q}$

$$\Gamma = \mathrm{Gal}(\tilde{L}/\mathbb{Q}) = \underset{\substack{\uparrow \\ A}}{C_q^k} \rtimes C_\ell$$

Goal: Determine the local factors in Malle-Bhargava for counting these.

$$\mathbb{E}(\#\mathrm{Surj}(\mathrm{Cl}(K) \to C_q)) = ?$$

Local factor at p: $(p \neq \ell)$

$$\sum_{\substack{[y] \in \Gamma \\ \text{with } [y] = [y^p]}} (\operatorname{disc} y)^{-s}$$

Also restrict to $y \notin A$ (else L/K ram.)

$\iff$ $y = v\alpha$ for $v \in A$, $\alpha \in C_\ell$, $\alpha \neq id$

$\iff$ $\operatorname{ord}(y) = \ell$

(or $y = 1$)

Facts:

- Γ has $(\ell-1)$ conjugacy classes of elements of order ℓ.
- If y has order ℓ, its centralizer is $\langle y \rangle$.
- $[y] = \{v'\alpha : v' \in A\}$ $(y = v\alpha)$
- $y^p = v'\alpha^p$ for some $v' \in A$.

Thus $[y]=[y^p] \iff \alpha = \alpha^p$

$\iff p \equiv 1 \mod \ell$

Local factor is:

- 1 if $p \not\equiv 1 \mod \ell$
- $1 + (\ell-1)p^{-(\ell-1)s}$ if $p \equiv 1 \mod \ell$
- Hard if $p = \ell$.

Same as local factor for counting C_ℓ fields!

Local restrictions?

Ex: Require p to split completely in K.

Requires $K/\mathbb{Q}$ unram. at p,
and $\mathrm{Frob}_p \in A$. (For C_ℓ $\mathrm{Frob}_p = 1$.)

$$\text{Local factor} = \frac{1}{|\Gamma|} \sum_{\rho : G_{\mathbb{Q}_p} \to \Gamma} (\text{disc}\, \rho)^{-s}$$

$$= \frac{1}{|\Gamma|} (\#\, \text{Frob})$$

For $\Gamma = A \times C_\ell$, $\quad \frac{1}{q^k \cdot \ell} \cdot q^k = \frac{1}{\ell}$

For C_ℓ, $\quad \frac{1}{\ell} \cdot 1 = \frac{1}{\ell}$

Even if we have to sum up over several possible Γ, this $\frac{1}{2}$ is independent of Γ, so it doesn't matter.

Ramified local conditions?

Ex: Let $\mathcal{K}/\mathbb{Q}_p$ $(p \equiv 1 \bmod \ell)$ be a totally ramified C_ℓ extn.

Require $K_p = \mathcal{K}$.

$\rho : G_{\mathbb{Q}_p} \to \Gamma$ is given by

x, y with $xyx^{-1} = y^p = y$.

$\Longrightarrow x \in \langle y \rangle$

$\Longrightarrow \operatorname{im} \rho \simeq C_\ell$

$\Longrightarrow \rho : G_{\mathbb{Q}_p} \xrightarrow{\rho_1} C_\ell \xrightarrow{\rho_2} \Gamma$

ρ_1 determines K_p.

ρ_1, ρ_2 are independent,

so all K occur equally often.

Jones-Roberts: There are ℓ of them.

$$\text{Local factor} = \frac{\ell - 1}{\ell} p^{-(\ell-1)s}$$

Same for C_ℓ.

Moral:

Consider $\mathbb{E}\left(\#\mathrm{Surj}\left(Cl(K) \to C_\ell\right)\right)$

as K ranges through

C_ℓ-extensions of $\mathbb{Q}$.

Tame, non-arch

The Malle-Bhargava $\Rightarrow$ <u>any</u> local condition on K does not affect this.

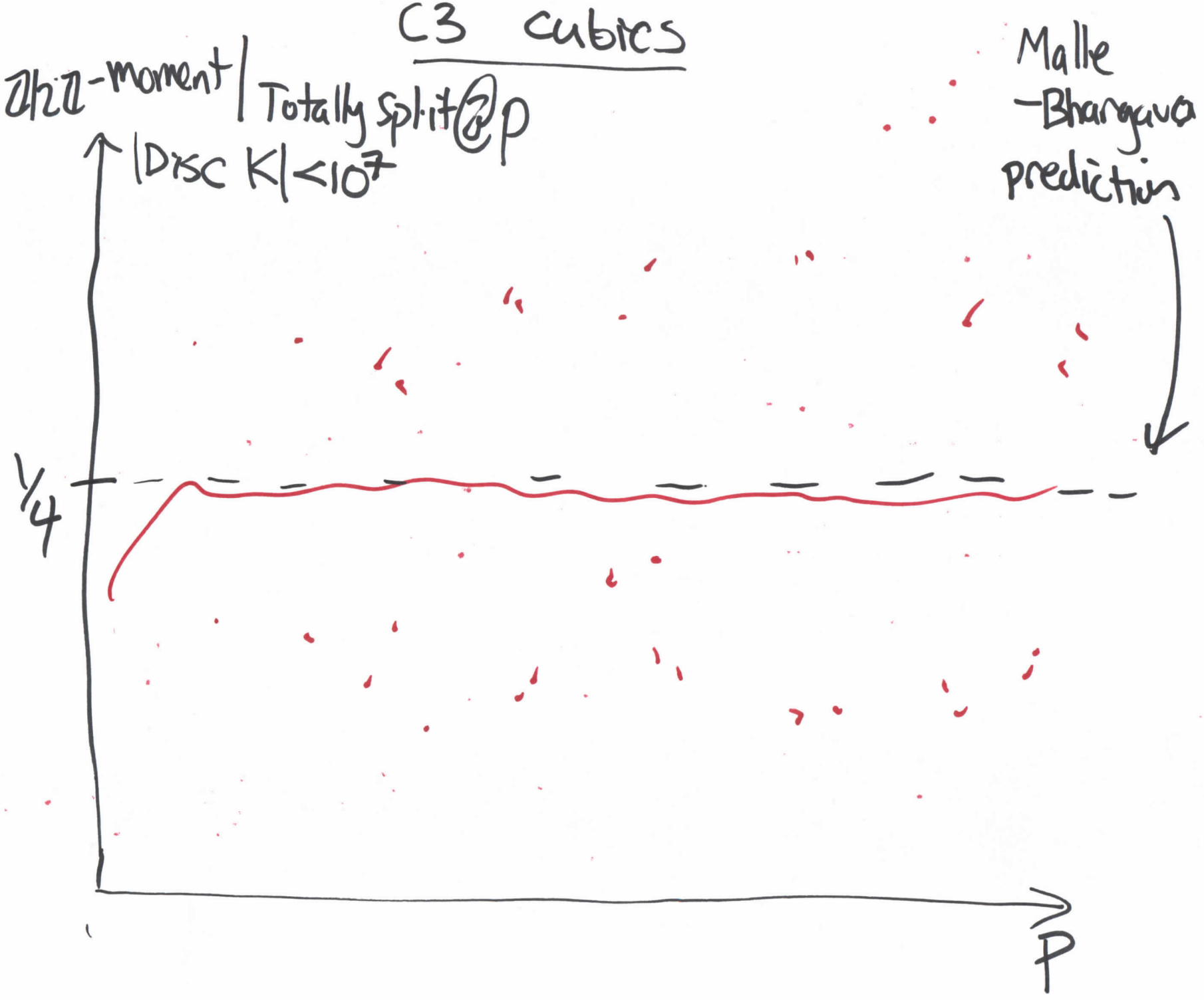
Let $D(G,X) = \left\{ \begin{array}{c} \text{cubic fields} \\ K \end{array} \;\middle|\; \begin{array}{l} \mathrm{Gal}(K) \simeq G \\ |\mathrm{Disc}\, K| < X \end{array} \right\}$

Using data for cubics with $|\mathrm{Disc}\, K| < 10^7$, we computed

$$\frac{1}{\#D(C_3, 10^7)} \sum_{K \in D(C_3, 10^7)} \frac{\#\mathrm{Sur}(\mathrm{Cl}(\mathcal{O}_K), \mathbb{Z}/2\mathbb{Z})}{\#\mathrm{Cl}(\mathcal{O}_K)[2] - 1} \approx 0.26$$

and for primes $p < 1000$

$$\frac{1}{\#D(C_3, 10^7; p \text{ totally split})} \sum_{\substack{K \in D(C_3, 10^7), \\ p \text{ totally split}}} \#\mathrm{Sur}(\mathrm{Cl}(\mathcal{O}_K), \mathbb{Z}/2\mathbb{Z})$$

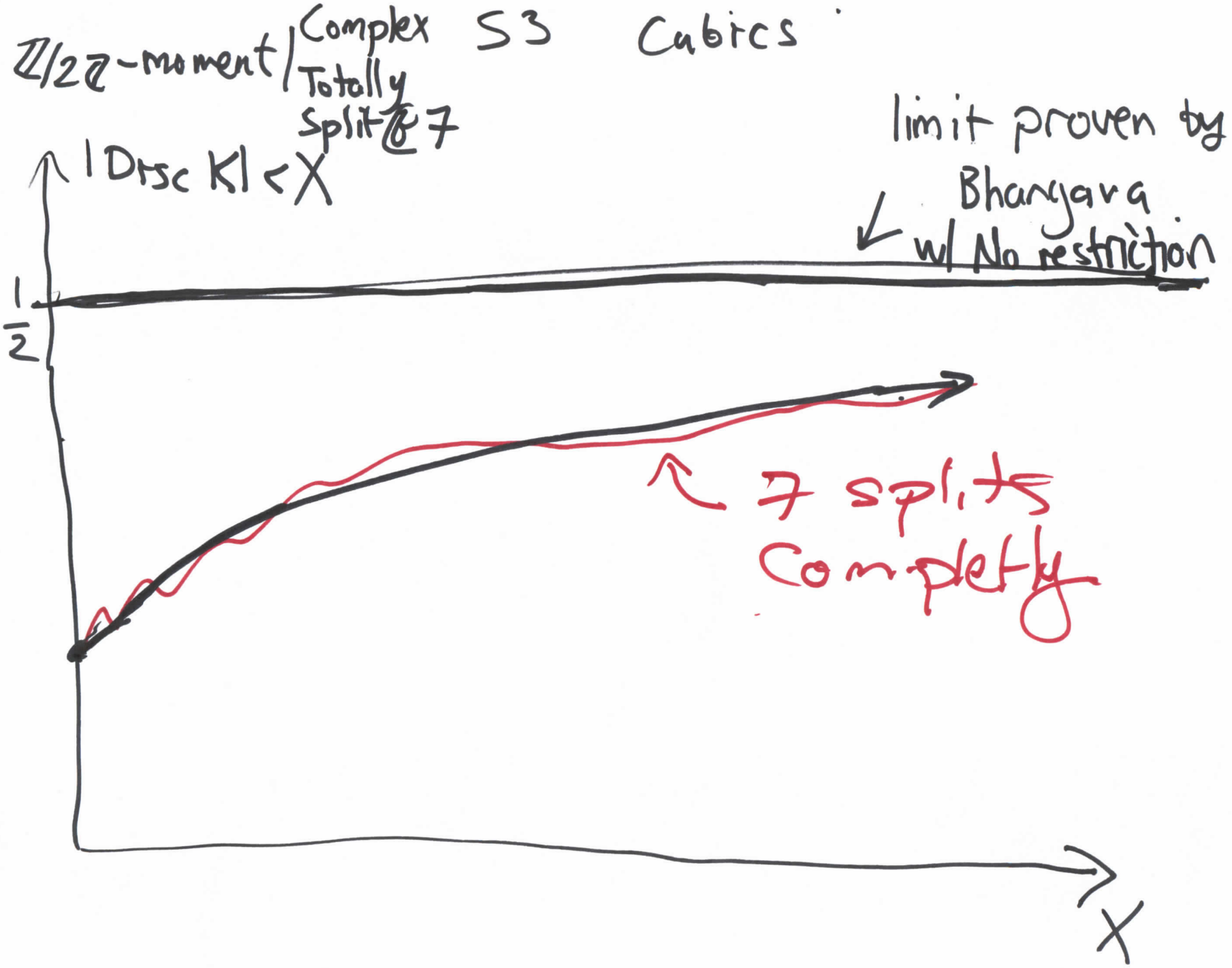

C3 cubics
2nd-moment | Totally split@p
|Disc K|<10^7
Malle
-Bhargava
prediction
1/4
p

For various $X < 10^7$ we computed and compared.

$$\frac{\sum_{\text{complex } K/\mathbb{Q}\ S_3\text{-cubic } |\mathrm{Disc}\, K| < X} \#\mathrm{Sur}(\mathrm{Cl}(\mathcal{O}_K), \mathbb{Z}/2\mathbb{Z})}{\sum_{K/\mathbb{Q}\ S_3\text{-cubic, complex } |\mathrm{Disc}\, K| < X} 1} \to \frac{1}{2}$$

AND

$$\frac{\sum_{\text{complex } K/\mathbb{Q}\ S_3\text{-cubic } |\mathrm{Disc}\, K| < X,\ 7 \text{ totally split in } \mathcal{O}_K} \#\mathrm{Sur}(\mathrm{Cl}(\mathcal{O}_K), \mathbb{Z}/2\mathbb{Z})}{\sum_{\text{same}} 1}$$


1/2-moment
Complex
Totally
Split@7
S3
Cubics
|Disc K| < X
limit proven by
Bhargava
w/ No restriction
1/2
7 splits
Completly
X

Non-Cyclic Cubic Fields:

(1) Establish the relationship between unramified quadratic extensions K_6 of cubic fields K_3 and quartic fields K_4

(2) When K_3 is split completely at some rational prime l into $l_1 l_2 l_3$, determine what the condition on K_4 is for Frob_{l_1} to be trivial in the Galois group $\mathrm{Gal}(K_6/K_3)$ of the unramified extn.

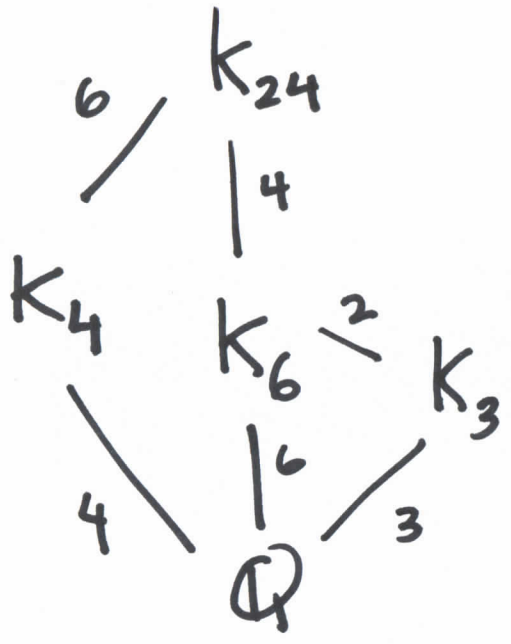
(3) Combine (1) & (2) in order to translate the following global question:

Q: How frequently are l_1, l_2, l_3 all trivial in order two quotients of the class group of K_3? (The other option is l_1 trivial and l_2, l_3 nontrivial.)

→ case 1

→ case 2

(4) Apply a theorem of Bhargava to answer the question above, namely count quartic fields with certain local conditions.



$\text{Gal}(K_{24}/\mathbb{Q}) \cong S_4$

$\text{Gal}(K_{24}/K_6) \cong V_4$

$\text{Gal}(K_{24}/K_3) \cong D_4$

$\text{Gal}(K_{24}/K_4) \cong S_3$

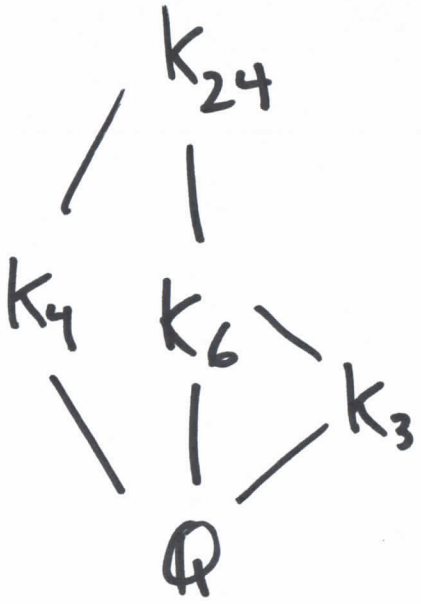
$\text{Gal}(K_6/K_3) \cong \mathbb{Z}/2\mathbb{Z}$

→ What do we need to know about K_4 in order to say that a prime ℓ is split completely in K_3?

→ How can we detect when K_6/K_3 is unramified in terms of K_4?

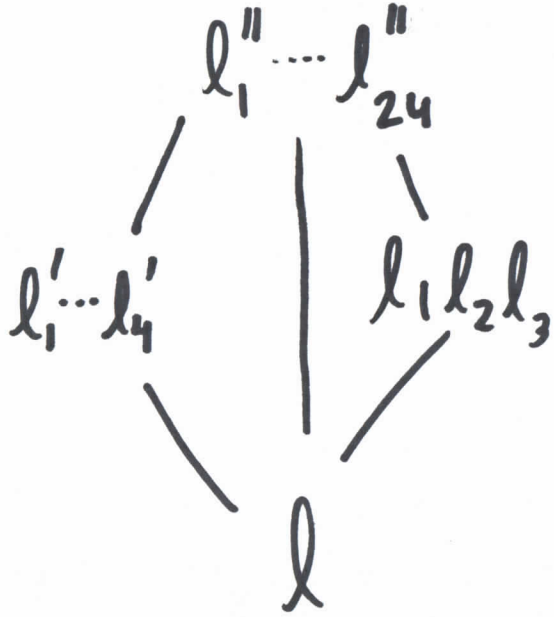
Decomposition Group	Inertia Group	Splitting Type in K_3	Splitting Type in K_4
(*) 1	1	(1, 1, 1)	(1, 1, 1, 1)
(12)	1	(1, 2)	(1, 1, 2)
(12)	(12)	$(1^2, 1)$	$(1, 1, 1^2)$
(*) (12)(34)	1	(1, 1, 1)	(2, 2)
(12)(34)	(12)(34)	(1, 1, 1)	$(1^2, 1^2)$
(123)	1	(3)	(1, 3)
(123)	(123)	(1^3)	$(1^3, 1)$
⋮	⋮	⋮	⋮

Now, we are ready to translate our global condition on K_3:



Case (i):

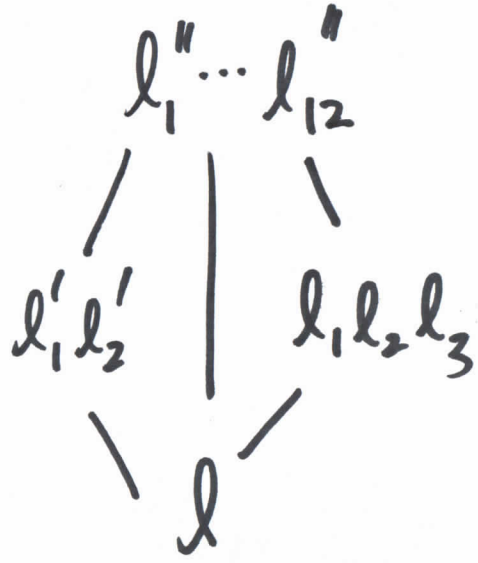
$\text{Frob}_\ell = 1$



$$\text{Frob}_{\ell_i} = \left(\frac{\ell_i}{K_6/K_3}\right) = 1, \forall i.$$

Case (ii):

$\text{Frob}_\ell = (12)(34)$



$$\text{Frob}_{\ell_1} = \left(\frac{\ell_1}{K_6/K_3}\right) = 1$$

$$\text{Frob}_{\ell_j} = \left(\frac{\ell_j}{K_6/K_3}\right) = -1, \; j = 2, 3.$$

By a theorem of Bhargava (2004) about counting quartic number fields, we obtain

THM: For a fixed rational prime ℓ, as we vary over S_3-cubic number fields in which ℓ splits completely into $\ell_1\ell_2\ell_3$, the $\mathbb{Z}/2\mathbb{Z}$-moment of case (i) is $\frac{1}{3}$ the $\mathbb{Z}/2\mathbb{Z}$-moment of case (ii).

Why might this happen?

Option 1:
(no ordering)

(0,0,0)
(1,1,0)

Option 2:
(ordering)

(0,0,0)
(0,1,1)
(1,0,1)
(1,1,0)

Our theorem supports the conjecture that if we restrict to S_3-cubic number fields K in which ℓ splits completely as $\ell_1\ell_2\ell_3$, then ℓ_1, ℓ_2, ℓ_3 are distributed as random ordered elements of $Cl(K)$.