Chan Lecture 1: Deligne-Lusztig theory

irreps of $\underline{G}(\mathbb{C})$ $\longleftrightarrow$ W-orbits of regular characters of $\underline{T}(\mathbb{C})$

Cartan, Weyl ~1910s

irreps of $\underline{G}(\mathbb{F}_q)$ $\overset{*}{\longleftrightarrow}$ W-orbits of reg chars of $\underline{T}(\mathbb{F}_q)$

Deligne-Lusztig 1976

$\updownarrow$

irreps of $\underline{G}(F)$ $\overset{*}{\longleftrightarrow}$ W-orbits of reg. chars of $\underline{T}(F)$

elliptic: Kaletha 2019

* well, not quite

TODAY!

f. simple gps:

- cyclic of prime order
- alternating gps
- f. gps of Lie type

$\Bigg]$ 100%

Setup:

$\mathbb{G}$ conn red gp over $\overline{\mathbb{F}}_q$

$\sigma : \mathbb{G} \to \mathbb{G}$ Frobenius (root)

(power of σ is Frob.)

$\overline{G} := \mathbb{G}(\overline{\mathbb{F}}_q)^{\sigma}$

$\mathbb{T} \hookrightarrow \mathbb{G}$ max'l torus, σ-stable

$\overline{T} := \mathbb{T}(\overline{\mathbb{F}}_q)^{\sigma}$

<u>Example.</u> $\mathbb{G} = GL_2$.

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^q & b^q \\ c^q & d^q \end{pmatrix}$$

- $\mathbb{T} = \mathbb{G}_m \times \mathbb{G}_m \hookrightarrow GL_2 \circlearrowleft \sigma$
  $(a, b) \longmapsto \begin{pmatrix} a & \\ & b \end{pmatrix}$

$$\sigma'\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^q & c^q \\ b^q & a^q \end{pmatrix}$$

- $\mathbb{T}' = \mathbb{G}_m \times \mathbb{G}_m \hookrightarrow GL_2 \circlearrowleft \sigma'$
  $(a, b) \longmapsto \begin{pmatrix} a & \\ & b \end{pmatrix}$

$\bar{G} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \in \mathbb{F}_q, \det \neq 0 \right\}$.

$\bar{T} = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} : \in \mathbb{F}_q^\times \right\} \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$

$\bar{G} = \left\{ \begin{pmatrix} a & b \\ b^q & a^q \end{pmatrix} : \in \mathbb{F}_{q^2}, \det \neq 0 \right\}$

$\bar{T}' = \left\{ \begin{pmatrix} a & \\ & a^q \end{pmatrix} : a \in \mathbb{F}_{q^2}^\times \right\} \cong \mathbb{F}_{q^2}^\times$

Ex: classify max'l tori in GL_n

- $\bar{T} \subset \bar{B} \supset \bar{U} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $\quad \bar{B} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

For any char $\theta : \bar{T} \to \mathbb{C}^\times$,

$$\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\theta) := \{ f : \bar{G} \to \mathbb{C} \mid f(bg) = \theta(pr(b)) f(g) \ \forall b \in \bar{B}, g \in \bar{G} \}$$

$$= \{ f : \bar{U} \backslash \bar{G} \to \mathbb{C} \mid f(tg) = \theta(t) f(g) \ \forall t \in \bar{T}, g \in \bar{U} \backslash \bar{G} \}$$

$W_{\bar{G}}(\mathbb{T})$

$= \{ g \in \bar{G} : g \mathbb{T} g^{-1} = \mathbb{T} \} / \mathbb{T}$

$\cancel{* \text{If } \theta \neq \theta^m}$

$= S_2 = \{1, w\}$

~~Ex.~~ $\cancel{W_{\bar{G}}}$

* If $\theta \neq \theta^w$, then $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\theta)$ is irred.
  it has dim $q+1$.

* In $\theta = \theta^w$, then $\theta = \theta_0 \circ \det$ and

$$\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\theta) = (\theta_0 \circ \det) \otimes \mathrm{Ind}_{\bar{B}}^{\bar{G}}(1)$$

Jessica's Talk #1: $\| $

$$1 + St_G.$$

So: what about $\bar{T}'$?

$$X := \mathbb{V}\left((x^{q+1} - y^{q+1})^{q-1} = 1 \right)$$

$\circlearrowright$ $g \in \bar{G}$ $\begin{pmatrix} a & b \\ b^q & a^q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\circlearrowleft$ $\bar{T}' \cong \mathbb{F}_{q^2}^{\times}$ scaling action.

$$\rightsquigarrow H^i_c(X, \bar{\mathbb{Q}}_\ell) \circlearrowleft \bar{G} \times \bar{T}'$$

$$\cup$$

$$H^i_c(X, \bar{\mathbb{Q}}_\ell)_\theta \circlearrowleft \bar{T}' \qquad \theta : \bar{T}' \to \mathbb{C}^\times$$

$$\rightsquigarrow \sum_{i \geq 0} (-1)^i H^i_c(X)_\theta =: H^*_c(X)_\theta$$

a virtual $\bar{G}$-repn (over $\mathbb{C}$) $\cong$ s.t.

* If $\theta \neq \theta^w$ for $w \in W_{\bar{G}}(\bar{T}')$, then irred
* else, get $(\theta_0 \circ \det) \otimes [-St_G + 1]$.

<u>Def.</u> (GL_2). The DL functors are

- $R_{\mathbb{T}}^{\mathbb{G}}(\theta) = \mathrm{Ind}_{\bar{B}}^{\bar{G}}(\theta)$
- $R_{\mathbb{T}'}^{\mathbb{G}}(\theta) = H_c^*(X)_\theta$

<u>Def.</u> (general) $\mathbb{T} \hookrightarrow \mathbb{G}$ σ-stable max'l tr.
$\mathbb{B}$ Borel contains $\mathbb{T}$, $\mathbb{U}$ unip rad.

The <u>DL var</u> is:

$$X_{\mathbb{T} \subset \mathbb{G}} := \{ g \in \mathbb{G} : g^{-1}\sigma(g) \in \mathbb{U} \}.$$

$$\bar{G} \circlearrowright X_{\mathbb{T} \subset \mathbb{G}} \circlearrowleft \bar{T}$$

The <u>DL Ind</u> is: $R_{\mathbb{T}}^{\mathbb{G}} : \mathbb{Z}[\mathrm{Irr}(\bar{T})] \to \mathbb{Z}[\mathrm{Irr}(\bar{G}$

$$R_{\mathbb{T}}^{\mathbb{G}}(\theta) := H_c^*(X_{\mathbb{T} \subset \mathbb{G}})_\theta$$

<u>**Thm**</u> (scalar product form) $\mathbb{G} \circlearrowleft \sigma$ general

$\mathbb{T}_1, \mathbb{T}_2$ any σ-stable max'l tori in $\mathbb{G}$.

θ_1, θ_2 chars of $\overline{T}_1, \overline{T}_2$.

$$\left\langle R^{\mathbb{G}}_{\mathbb{T}_1}(\theta_1), R^{\mathbb{G}}_{\mathbb{T}_2}(\theta_2) \right\rangle_{\overline{G}}$$

$$= \sum_{w \in W_{\overline{G}}(\mathbb{T}_1, \mathbb{T}_2)} \left\langle \theta_1, {}^w\theta_2 \right\rangle_{\overline{T}_1}$$

<u>**Def.**</u> · We say a character θ is <u>regular</u> if

$\mathrm{stab}_W(\theta) = \{1\}$. $\Downarrow$ $\Uparrow$ $Z(\mathbb{G})$ conn.

· We say a char. θ is <u>nonsingular</u> if

$\theta \circ \mathrm{Nm}|_{\alpha^\vee(\mathbb{G}_m)} \neq 1 \ \forall$ roots $\alpha^\vee$

$\mathrm{Nm}: \mathbb{T}(\mathbb{F}_q) \to \overline{T}$.

<u>Cor.</u> If θ is ~~the gen~~ regular, then $|R_{\mathbb{T}}^{G}(\theta)|$ is irreducible.

<u>Main results.</u>

<u>Thm.</u> (DL fixed pt formula) $\overset{g \text{ f. aut.}}{\underset{}{\circlearrowright}}$ X sep., f type.

$$\mathrm{Tr}\left(g; H_c^*(X)\right) = \mathrm{Tr}\left(u; H_c^*(X^s)\right)$$

where $s =$ prime-to-p order

$u =$ p-power order

$g = su = us$

$s = g^{\cdots}$ $\quad u = g^{\cdots}$ $\quad$ s.t. $su = g^{\bigcirc + 1}$

Thm. (DL char. formula).

s = semisimple
u = unipotent

$$\Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(su) = \frac{1}{|\bar{Z}^{\circ}_{\mathbb{G}}(s)|} \sum_{g \in \bar{G}} \theta^{g}(s) \cdot \underbrace{\Theta_{R_{\mathbb{T}^{g}}^{Z^{\circ}_{\mathbb{G}}(s)}(1)}(u)}_{\text{Green fn.}}$$

Def. A repn π of $\bar{G}$ is cuspidal if

$$\langle \pi, \mathrm{Ind}_{\bar{P}}^{\bar{G}}(\rho) \rangle = 0 \quad \forall\ \sigma\text{-stable}$$

proper parabolic $\mathbb{P} \subset \mathbb{G}$,
$\rho \in \mathrm{Irr}(\bar{M})$.

<u>Thm.</u> $\mathbb{T}$ σ-stable max'l tor, <u>not</u> contained in any σ-stable proper parabolic.

For $\theta : \bar{T} \to \mathbb{C}^\times$ nonsingular, then $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is cuspidal.

<u>Ex</u> (rel to reps of p-adic grps):

Then $c\mathrm{Ind}_{SL_2(\mathcal{O}_F)}^{SL_2(F)}\left(\mathrm{Inf}_{SL_2\mathbb{F}_q}^{SL_2\mathcal{O}_F}(\pi)\right)$ is irred supercuspidal if:

- $\pi = R_{\mathbb{T}'}^{\mathbb{G}}(\theta)$ for θ regular
- $\pi \subset R_{\mathbb{T}'}^{\mathbb{G}}(\theta)$ for θ nonsigular but not reg.