MOD-p REPRESENTATIONS OF p-ADIC GROUPS
ARIZONA WINTER SCHOOL 2025 LECTURE NOTES

FLORIAN HERZIG

Contents

1. Introduction 3
1.1. p-adic groups 3
1.2. Smooth and admissible representations of G 3
1.3. There is no C-valued Haar measure 5
1.4. Induced representations 5
1.5. Duality 8
2. Hecke algebras and classification 11
2.1. Serre weights 11
2.2. Hecke algebras 12
2.3. mod-p Satake transforms 14
2.4. Supersingular representations 17
2.5. Generalized Steinberg representations 18
2.6. Classification of irreducibles 18
2.7. On the proofs 19
2.8. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ and the mod p local Langlands correspondence 21
3. $\mathrm{GL}_2(F)$ 24
3.1. Diagrams 24
3.2. Injective envelopes of Serre weights and the extension graph 28
3.3. An aside on reduction modulo p 33
3.4. Breuil–Paškūnas diagrams 34
3.5. Exotic diagrams 39
4. Global aspects 43
4.1. Modular forms on quaternion algebras 43
4.2. Comparison with automorphic forms over $\mathbb{C}$ 44
4.3. Fixed central character 45
4.4. An injectivity property 46
4.5. Galois representations 46
4.6. Weights in Serre's conjecture 47
4.7. Local-global compatibility 49
4.8. Existence of supersingular representations 49
4.9. Patching 51
4.10. Multiplicity one 53
4.11. Some recent progress 54
Appendix A. Appendix 57
A.1. A few words about topological groups 57
A.2. Representations of finite groups 57

A.3. Multiplicity-free representations 58
A.4. Representations of reductive algebraic groups 60
Appendix B. Project outlines 62
B.1. Explicit Satake transforms 62
B.2. Supersingular representations with prescribed properties 62
B.3. Universal supersingular modules 62
References 63

In these lecture notes we use the following notation.

$F/\mathbb{Q}_p$ is a finite extension with ring of integers $\mathcal{O}_F$, uniformizer ϖ, residue field k_F of degree f over $\mathbb{F}_p$, $q = |k_F| = p^f$. Let $\mathrm{ord}_p : F^\times \twoheadrightarrow \mathbb{Z}$ denote the p-adic valuation of F. The Artin map $F^\times \to \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$ of local class field theory is normalized so that it sends uniformizers to geometric Frobenius elements.

C denotes a field of characteristic p (coefficient field). It will be algebraically closed in sections 2 and 3.

Acknowledgments: I would like to thank the organizers for inviting me to give this lecture series at the Arizona Winter School 2025. I thank Hymn Chan, Mathilde Gerbelli-Gauthier, Yongquan Hu, Karol Kozioł, Heejong Lee, Stefano Morra, Chol Park, and Yitong Wang for helpful comments.

1. Introduction

1.1. **p-adic groups.** Suppose that $G = \mathrm{GL}_n(F)$. Then G is naturally a topological group, by giving it the subspace topology inside $\mathrm{M}_n(F) \cong F^{n^2}$. In particular, this topology is Hausdorff and totally disconnected. More precisely, a fundamental system of neighbourhoods of the identity is given by the open subgroups $K_m = 1 + \varpi^m \mathrm{M}_n(\mathcal{O}_F)$ for $m \geq 1$.

Lemma 1.1. *The groups K_m are pro-p.*

Proof. First note that $K_m \cong \mathrm{M}_n(\mathcal{O}_F) \cong \mathcal{O}_F^{n^2}$ (as topological spaces, not groups) is compact. As K_m is compact, Hausdorff, and totally disconnected, it is profinite (cf. §A.1). As the sequence $(K_s)_{s \geq m}$ forms a fundamental system of neighbourhoods of the identity, it suffices to show that the index $(K_m : K_s)$ is a p-power for any $s \geq m$. By multiplicativity of indices this reduces to showing that $(K_s : K_{s+1})$ is a power of p for $s \geq m$. But as groups, $K_s/K_{s+1} \cong M_n(k_F)$ via the map sending $1+p^s A \in K_s$ to the reduction of A modulo ϖ, and therefore $(K_s : K_{s+1}) = q^{n^2}$. $\square$

We give three more examples of compact open (hence profinite) subgroups of $G = \mathrm{GL}_n(F)$:

(i) The subgroup $K := \mathrm{GL}_n(\mathcal{O}_F)$ is a maximal compact subgroup of G, and it is the unique such up to conjugation [Her12b, Ex. 1].
(ii) The *Iwahori subgroup* $I := \{g \in K : g \text{ is upper-triangular mod } \varpi\}$.
(iii) The *pro-p Iwahori subgroup*

$$I_1 := \{g \in K : g \text{ is unipotent upper-triangular mod } \varpi\}.$$

Note that I_1 is pro-p by Lemma 1.1, since I_1/K_1 is of order $q^{n(n-1)/2}$. In fact, I_1/K_1 is a Sylow subgroup of $K/K_1 \cong \mathrm{GL}_n(k_F)$, so I_1 is a pro-p Sylow subgroup of K.

More generally we are interested in $G = \underline{G}(F)$, where $\underline{G}$ is a connected reductive algebraic group over F (e.g. a classical group, such as orthogonal, symplectic or unitary). For the purpose of smooth representation theory we can even consider any topological group G that is *locally profinite* (i.e. has an open subgroup that is profinite), but in practice we mainly care about closed subgroups of $\mathrm{GL}_n(F)$.

1.2. **Smooth and admissible representations of G.** We want to study representations of a locally profinite group G over a field C of characteristic p. Since G is a topological group we do not want to allow all representations.

Definition 1.2. A representation V of G is *smooth* if any of the following equivalent conditions hold:

- the action map $G \times V \to V$ is continuous, where V is given the discrete topology;
- the stabilizer of any vector of V in G is open;
- $V = \bigcup_U V^U$, where U runs through all (compact) open subgroups U of G.

Here, $V^U := \{v \in V : uv = v \ \forall u \in U\}$ denotes the subspace of U-invariants.

Exercise 1.3. Check that the above conditions are equivalent.

Importantly, observe that the vector space V underlying a smooth representation is discrete (or, if you prefer, carries no topology), so the notions of sub/quotient/irreducible representation are the same as for abstract representations.

Note that for any representation V of G we can functorially associate a smooth representation $V^\infty := \bigcup_U V^U$, which may be zero even if V is nonzero. (The functor $V \mapsto V^\infty$ is the right adjoint of the forgetful functor.)

The following finiteness condition is very useful:

Definition 1.4. A smooth representation V of G is *admissible* if $\dim_C V^U < \infty$ for all (compact) open subgroups U.

Example 1.5. Here are basic examples of smooth representations.

(i) Suppose that H is finite (with the discrete topology). Then any H-representation is smooth, and it is admissible if and only if it is finite-dimensional.

(ii) Suppose that H is any profinite group. Then the space $\mathcal{C}^\infty(H, C)$ of continuous, equivalently locally constant, functions $H \to C$ is an admissible smooth representation of H, where H acts by right translation. (Smoothness follows from compactness and admissibility since H/U is finite for all open subgroups U, again by compactness.)

It is clear that any subrepresentation of an admissible representation is admissible. This is also true for quotient representations, at least if G is a closed subgroup of $\mathrm{GL}_n(F)$, but it is far less obvious (see Cor. 1.32 below).

So far, these definitions work over any field, and are the same as in the other courses. Here are some peculiarities that arise over a field of characteristic p.

Lemma 1.6 ("p-group lemma")**.** *Suppose that V is a nonzero smooth representation of a pro-p group H over a field C of characteristic p. Then H fixes a nonzero vector of V, i.e. $V^H \neq 0$.*

Proof. There are several possible proofs. Here is one: by considering V as $\mathbb{F}_p$-vector space we may assume that $C = \mathbb{F}_p$ (the conclusion does not change!). Fix any nonzero vector $x \in V$. By smoothness, x is fixed by an open normal subgroup $N \lhd H$. Then the subspace $W := \mathbb{F}_p[H]x = \mathbb{F}_p[H/N]x$ is a finite-dimensional subrepresentation, so we may replace V by W and H by H/N and assume that $\dim_{\mathbb{F}_p} V < \infty$ and that H is a finite p-group. Finally we can use a counting argument: by the orbit–stabilizer theorem, the cardinality of any H-orbit on V is a power of p; since $|V|$ is a power of p, we deduce that $|V^H| \equiv 0 \bmod p$, hence $V^H \neq 0$. $\square$

Remark 1.7. In particular, if V is a semisimple representation of a pro-p group H, then H acts trivially on V! This reflects the fact that we are dealing with modular representation theory, where Maschke's theorem breaks down completely.

Exercise 1.8. Continue Example 1.5, taking $H = \mathbb{Z}_p$ and $V = \mathcal{C}^\infty(\mathbb{Z}_p, C)$. For $n \geq 0$ let V_n be the subspace spanned by all functions $\overline{\binom{x}{k}} : \mathbb{Z}_p \to \mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p \hookrightarrow C$, $0 \leq k < n$. Show that V_n is the unique subrepresentation of V that is of dimension n, and show that any proper subrepresentation of V is of the form V_n for some $n \geq 0$. (Perhaps the easiest proof uses the duality in §1.27.)

Later we will need the notion of a socle of a representation.

Definition 1.9. If π is any smooth representation of a profinite group H, the H-*socle* $\mathrm{soc}_H\, \pi$ is the maximal semisimple subrepresentation of π, or equivalently the sum of all irreducible subrepresentations of π.

Exercise 1.10. If $\pi \neq 0$, then $\operatorname{soc}_H \pi \neq 0$. Also, if $f : \pi \to \pi'$ is a homomorphism of smooth H-representations, then $f|_{\operatorname{soc}_H \pi}$ injective implies that f is injective.

On the other hand, π need not have any irreducible quotients, see Exercise 1.8.

Exercise 1.11. Suppose that H is a profinite group with open normal subgroup N such that N is pro-p and H/N is (finite) of order prime to p. Show that $\operatorname{soc}_H \pi = \pi^N$.

1.3. **There is no C-valued Haar measure.** In the complex smooth representation theory of G, the Haar measure of G plays a fundamental role (see for example Tasho's lectures). Unfortunately there is no good analog of Haar measure μ with values in a field of characteristic p.

Suppose in this section that G is locally pro-p and does not carry the discrete topology. The basic observation is that if H is a non-discrete pro-p group and μ a C-valued translation-invariant measure on H, then H has an open subgroup H' with index $(H : H') = p^m$ for some $m > 0$, so $\mu(H) = \sum_{h \in H/H'} \mu(hH') = (H : H')\mu(H')$, by translation invariance, which equals zero as $p = 0$ in C. But perhaps it is easiest to give a rigorous statement in the spirit of integration. Let $\mathcal{C}_c^\infty(G, C)$ denote the vector space of compactly supported locally constant functions $G \to C$, on which G acts on the left by right translation on the domain (G).

Lemma 1.12. *Any left G-invariant linear functional $\int : \mathcal{C}_c^\infty(G, C) \to C$ is zero.*

Proof. As above we deduce that $\int(1_H) = 0$ for any open pro-p subgroup H (as G, hence H, is non-discrete by assumption). Suppose now that $f : G \to C$ is any locally constant function of compact support. By compactness we see that there exists a compact open subgroup H such that $f(gh) = f(g)$ for all $g \in G$ and all $h \in H$. Hence $f = \sum_{i=1}^N \lambda_i 1_{g_i H}$ for some $\lambda_i \in C$ and $g_i \in G$, and hence $\int(f) = 0$ by left translation invariance. $\square$

1.4. **Induced representations.** This is exactly the same as over $\mathbb{C}$, so we will keep the discussion short.

Suppose that $H \leq G$ is any closed subgroup (note that this implies that $H\backslash G$ is Hausdorff with the quotient topology) and let σ be a smooth H-representation.

We can form several kinds of induced representations.

(i) The abstractly induced representation is ${}^\circ\mathrm{Ind}_H^G \sigma := \{f : G \to \sigma : f(hg) = hf(g)\}$, with G acting by right translation: $(gf)(\gamma) = f(\gamma g)$ (just like in the representation theory of finite groups).

(ii) The *smoothly induced representation* $\mathrm{Ind}_H^G \sigma := ({}^\circ\mathrm{Ind}_H^G \sigma)^\infty$. Concretely, $\mathrm{Ind}_H^G \sigma$ consists of all $f \in {}^\circ\mathrm{Ind}_H^G \sigma$ such that there is an open subgroup $U \leq G$ such that $f(gu) = f(g)$ for all $u \in U$ and $g \in G$.

(iii) The *compactly induced representation* $\text{c-}\mathrm{Ind}_H^G \sigma$ is the G-subrepresentation of $\mathrm{Ind}_H^G \sigma$ consisting of all f such that the image of $\operatorname{supp}(f)$ in $H\backslash G$ is compact. (This is a subrepresentation of $\mathrm{Ind}_H^G \sigma$ since the action of G translates the support of functions.)

Here, for $f \in \mathrm{Ind}_H^G \sigma$ we define the support of f as $\operatorname{supp}(f) = \{g \in G \mid f(g) \neq 0\}$; it is a union of right cosets of H. By the smoothness condition in the definition of $\mathrm{Ind}_H^G \sigma$, f is locally constant and hence $\operatorname{supp}(f)$ is open and closed, which implies that the image of $\operatorname{supp}(f)$ in $H\backslash G$ is also open and closed.

Thus we have subrepresentations
$$\text{c-Ind}_H^G \sigma \subseteq \operatorname{Ind}_H^G \sigma \subseteq {}^\circ\operatorname{Ind}_H^G \sigma.$$

Remark 1.13.

(i) If $\sigma = 1_H$, the trivial 1-dimensional representation, then $\operatorname{Ind}_H^G 1_H$ is the space of uniformly locally constant functions $f : H\backslash G \to C$.
(ii) If $H\backslash G$ is compact, then $\operatorname{Ind}_H^G \sigma = \text{c-Ind}_H^G \sigma$.
(iii) The functors Ind_H^G and c-Ind_H^G are left exact.

In contrast to complex representation theory, Ind_H^G is not always exact. However, we will see important examples when Ind_H^G and c-Ind_H^G are exact in the remainder of this section.

Proposition 1.14 (Frobenius Reciprocity)**.** *Suppose that π is a smooth G-representation and σ a smooth H-representation. We have natural isomorphisms:*

(i) $\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) \cong \operatorname{Hom}_H(\pi|_H, \sigma)$.
(ii) $\operatorname{Hom}_G(\text{c-Ind}_H^G \sigma, \pi) \cong \operatorname{Hom}_H(\sigma, \pi|_H)$ *if H is* open *in G. Moreover, the functor* c-Ind_H^G *is exact in this case.*

Useful notation: if $g \in G$ and $x \in \sigma$ we let $[g, x] \in \text{c-Ind}_H^G \sigma$ denote the function supported on Hg^{-1}, sending g^{-1} to x. Note that $[gh, x] = [g, hx]$ for $h \in H$ and $\gamma[g, x] = [\gamma g, x]$ for $\gamma \in G$.

Remark 1.15. Note that the isomorphism in (i) is realized by composition with the map $\operatorname{Ind}_H^G \sigma \to \sigma$, which is evaluation at 1. The isomorphism in (ii) is realized by composition with the map $\sigma \to \text{c-Ind}_H^G \sigma$, which sends $x \in \sigma$ to the function $[1, x]$. (It's a good exercise to write down explicit formulas for the inverse isomorphism in each case.)

Remark 1.16. One way to prove (ii) is to note that $\text{c-Ind}_H^G \sigma \cong C[G] \otimes_{C[H]} \sigma$, the isomorphism being given by $g \otimes x \mapsto [g, x]$. The exactness follows from the fact that $C[G]$ is free over $C[H]$ (generated as a left module by a set of representatives of $H\backslash G$).

Remark 1.17. There is a useful Mackey formula for $(\text{c-Ind}_H^G \sigma)|_{H'}$ if at least one of the closed subgroups H, H' is open and for $(\operatorname{Ind}_H^G \sigma)|_{H'}$ if H' is open, cf. [Yam].

An important special case of induction is *parabolic induction*. In this case, $G = \mathrm{GL}_n(F)$ (or another reductive group) and H (the F-points of) a parabolic subgroup. Importantly, the quotient $H\backslash G$ is compact in the quotient topology if H is a parabolic subgroup.

Concretely, if $G = \mathrm{GL}_n(F)$, then a *standard parabolic subgroup* $P = P_{n_1,\dots,n_r}$ consists of all upper block triangular matrices[1] in G with square blocks of sizes n_1, $\dots$, n_r (for some fixed integers $n_i \geq 1$ such that $n = \sum_i n_i$). Let $\overline{P} := {}^tP$ denote its transpose and let $M = M_{n_1,\dots,n_r} := P \cap \overline{P}$ (a *standard Levi subgroup*[2]). Note that we have a surjective homomorphism $P \twoheadrightarrow M$ (resp. $\overline{P} \twoheadrightarrow M$) that restricts to the identity on M and sends other matrix entries to zero, and we denote its kernel by $N = N_{n_1,\dots,n_r}$ (resp. $\overline{N} = {}^tN$), the *unipotent radical* of P (resp. $\overline{P}$). Then we have semidirect product decompositions $P = M \ltimes N$, $\overline{P} = M \ltimes \overline{N}$. We write

[1] `https://en.wikipedia.org/wiki/Triangular_matrix#Upper_block_triangular`.

[2] Unfortunately this is non-standard terminology...

$B := P_{1,\dots,1}$ (a *Borel subgroup*) and $T := M_{1,\dots,1}$ (a *maximal torus*), $U := N_{1,\dots,1}$. Here is a picture for $n = 3$ and $n_1 = 2, n_2 = 1$:

(1.1)
$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},\ M = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix},\ N = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},\ \overline{P} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix},$$
$$B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},\ T = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix},\ U = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, a *parabolic subgroup* is a subgroup conjugate to a (in fact, unique) standard parabolic subgroup.

Remark 1.18. The subset $\overline{N}P = \overline{N}MN \subseteq G$ is open.

Proposition 1.19.

(i) *Assume that $H\backslash G$ is compact and that the projection map $\pi : G \to H\backslash G$ admits a continuous section.*[3] *Then the functor Ind_H^G is exact.*

(ii) *If $G = \mathrm{GL}_n(F)$ and P is a parabolic subgroup, then the functor Ind_P^G is exact.*

Proof. (i) Compactness of $H\backslash G$ ensures that $\operatorname{Ind}_H^G \sigma$ consists of all locally constant functions in ${}^\circ\operatorname{Ind}_H^G \sigma$. Let $s : H\backslash G \to G$ denote a continuous section of π. Define a map $\operatorname{Ind}_H^G \sigma \to \mathcal{C}^\infty(H\backslash G, \sigma)$ by $f \mapsto (\overline{g} \mapsto f(s(\overline{g})))$ and a map the other way by
$$\varphi \mapsto \overline{\varphi} = (g = hs(\overline{g}) \mapsto h\varphi(\overline{g})) \quad \forall h \in H,\ \overline{g} \in H\backslash G.$$
The first map is well defined as f is locally constant. To check that the second map is well defined, write $h\varphi(\overline{g}) = gs(\pi(g))^{-1}\varphi(\pi(g))$ and note this is locally constant in g, as $gs(\pi(g))^{-1}$ is continuous, σ is smooth and φ is locally constant. Then check that these maps are inverses to one another and define a natural isomorphism of functors $\operatorname{Ind}_H^G \cong \mathcal{C}^\infty(H\backslash G, -)$ (the spaces $\mathcal{C}^\infty(H\backslash G, \sigma)$ have a left action of G coming from the right action on $H\backslash G$). As Ind_H^G is left exact, it suffices to show that it preserves surjections, i.e. if $\sigma \twoheadrightarrow \tau$ is any surjection of smooth G-representations, then $\mathcal{C}^\infty(H\backslash G, \sigma) \twoheadrightarrow \mathcal{C}^\infty(H\backslash G, \tau)$. Forget G-actions and choose any vector space section $\alpha : \tau \to \sigma$, which is automatically continuous in the discrete topology. Then $\overline{g} \mapsto \alpha(\varphi(\overline{g}))$ is a preimage of φ.

(ii) Write $P = MN$, which we may suppose to be standard. By (i) we need to construct a continuous section $s : P\backslash G \to G$. By Remark 1.18 the image of $\overline{N} \hookrightarrow P\backslash G$ is open, hence the image Ω of the compact open subgroup $\overline{N} \cap K$ is open and closed, so we can define a continuous section on Ω. By translation we may define a continuous section on Ωg for any $g \in G$; these sets form an open cover of $P\backslash G$. Since $P\backslash G$ is compact we can take a finite subcover $\{\Omega g_i\}_{1 \le i \le r}$. Now chop these up into 2^r disjoint open and closed subsets $\left(\bigcap_{i \in I} \Omega g_i\right) \cap \left(\bigcap_{i \in I^c} (\Omega g_i)^c\right)$ (for $I \subseteq \{1, \dots, r\}$) on which a continuous section exists, and then glue to get a continuous section on all of $P\backslash G$. $\square$

Remark 1.20. In fact, the functor $\text{c-}\operatorname{Ind}_H^G$ is always exact, since $\text{c-}\operatorname{Ind}_H^G \cong \mathcal{C}_c^\infty(H\backslash G, -)$ by [AHV19, Lemma 2.4].

Here is another useful property, cf. [Her12b, Ex. 15]:

[3] In fact the second assumption is automatic, see [AHV19, Lemma 2.3].

Proposition 1.21 (Transitivity of parabolic induction)**.** *Suppose that $Q = LN' \subseteq P = MN$ are standard parabolic subgroups, so $L \subseteq M$. For σ a smooth L-representation we have a natural isomorphism*

$$\operatorname{Ind}_P^G(\operatorname{Ind}_{Q\cap M}^M \sigma) \cong \operatorname{Ind}_Q^G \sigma.$$

(Note that $Q \cap M$ is a parabolic subgroup of M with Levi L and unipotent radical $N' \cap M$.)

We now give a basic application of induced representation to admissibility. This result is very far from true over $\mathbb{C}$!

Lemma 1.22. *Suppose that H is any open pro-p subgroup of G. Then V is admissible if and only if $\dim_C V^H < \infty$.*

Proof. For the "if" direction, it suffices to show that $\dim_C V^{H'} < \infty$ for all (compact) open subgroups $H' \leq H$. By Frobenius reciprocity we have

$$V^{H'} = \operatorname{Hom}_{H'}(1, V) = \operatorname{Hom}_H(\text{c-Ind}_{H'}^H 1, V). \tag{1.2}$$

We prove by induction that for any finite-dimensional smooth representation W of H we have

$$\dim_C \operatorname{Hom}_H(W, V) \leq (\dim_C W)(\dim_C V^H). \tag{1.3}$$

(For the induction step: if $W \neq 0$, then $0 \to 1 \to W \to W' \to 0$ by Lemma 1.6 and so $0 \to \operatorname{Hom}_H(W', V) \to \operatorname{Hom}_H(W, V) \to \operatorname{Hom}_H(1, V) = V^H$ is exact.) So we conclude by (1.2) and (1.3) that

$$\dim_C V^{H'} \leq (H : H')(\dim_C V^H) < \infty. \qquad \square$$

1.5. **Duality.** References: [Vig12], [Hen09], [Eme10], [Koh17].

There are at least two dualities one can consider on a smooth G-representation V:

(i) smooth duality $\operatorname{Hom}_C(V, C)^\infty$ (defined in §1.2),
(ii) the naive duality $V^\vee := \operatorname{Hom}_C(V, C)$,

where G acts by $(g \cdot f)(x) = f(g^{-1}x)$ for $f \in \operatorname{Hom}_C(V, C)$, $g \in G$, $x \in V$.

For now we focus on the second kind, and we forget about G-actions for the moment.

1.5.1. *Step 1. Vector spaces.* We cannot hope to recover V from the abstract vector space $V^\vee$ (for example, $V^{\vee\vee}$ is much bigger than V if V is of infinite dimension), but we can give $V^\vee$ a natural topology and then recover V as the continuous dual of $V^\vee$, as we now explain.

Since V is the union (or direct limit) of all its finite-dimensional subspaces we obtain that $V^\vee = \operatorname{Hom}_C(V, C) \xrightarrow{\sim} \varprojlim_W \operatorname{Hom}_C(W, C)$, where the index runs through all finite-dimensional subspaces $W \subseteq V$. We then give $V^\vee$ the inverse limit topology, where each $\operatorname{Hom}_C(W, C)$ carries the discrete topology. Concretely, the basic open neighbourhoods of $0 \in V^\vee$ consist of the annihilator subspaces $W^\perp$ for $W \subseteq V$ finite-dimensional. Thus $V^\vee$ is *pseudocompact*, which is defined to mean a topological vector space that is isomorphic to an inverse limit of finite-dimensional vector spaces (with inverse limit topology).

Exercise 1.23. Show that the double duality map $V \to \operatorname{Hom}_C(V^\vee, C)$, $x \mapsto (f \mapsto f(x))$ induces a vector space isomorphism $V \xrightarrow{\sim} \operatorname{Hom}_C^{\text{cts}}(V^\vee, C)$, where the right-hand side denotes continuous linear maps.

Exercise 1.24. If $V = \bigoplus_{i \in I} C$, show that $V^\vee \cong \prod_{i \in I} C$ (topological isomorphism).

Proposition 1.25. *The functor $V \mapsto V^\vee$ induces an equivalence of categories between C-vector spaces and pseudocompact C-vector spaces. A quasi-inverse is given by $M \mapsto \mathrm{Hom}_C^{\mathrm{cts}}(M, C)$.*

(This is not too hard, based on the exercises above.)

Remark 1.26. If $C = \mathbb{F}_p$, then $V \mapsto V^\vee$ is the same as classical Pontryagin duality. Let $S^1 := \{z \in \mathbb{C}^\times : |z| = 1\}$ and choose a group isomorphism $\mathbb{F}_p \cong \mu_p(\mathbb{C})$ (the group of p-th roots of unity). Then note that $\mathrm{Hom}_C(V, C) = \mathrm{Hom}(V, S^1)$, where the right-hand side denotes group homomorphisms. Pontryagin duality is a perfect duality $A \mapsto \mathrm{Hom}^{\mathrm{cts}}(A, S^1)$ on locally compact abelian groups, giving $\mathrm{Hom}^{\mathrm{cts}}(A, S^1)$ the so-called compact open topology, which coincides with the topology of $V^\vee$ defined above if $A = V$ (discrete). In this case, $V^\vee$ is literally compact, even profinite.

1.5.2. *Step 2. Representations of compact groups.* If V is a smooth representation of a profinite group H, then it is not hard to check that the induced action of H on $V^\vee$ is continuous, i.e. the action map $H \times V^\vee \to V^\vee$ is continuous.

We can do better: since H is profinite, it has lots of normal compact open subgroups and we can define the *completed group algebra*

$$C[\![H]\!] := \varprojlim_{N \lhd H} C[H/N],$$

where the limit runs over all normal compact open subgroups N. This is a topological ring with the inverse limit topology, where all $C[H/N]$ are discrete (so $C[\![H]\!]$ is pseudocompact as topological vector space).

As $V = \varinjlim_N V^N$ is smooth, it becomes a $C[\![H]\!]$-module, by letting $C[\![H]\!]$ act on V^N via the projection $C[\![H]\!] \twoheadrightarrow C[H/N]$. Moreover the action $C[\![H]\!] \times V \to V$ is continuous. (It extends the action of the usual group ring $C[H]$ on V.) As a consequence, $V^\vee$ is a module over $C[\![H]\!]$, extending the above action of $C[H]$ on $V^\vee$.

Exercise 1.27. The action map $C[\![H]\!] \times V^\vee \to V^\vee$ is continuous.

From above one then deduces with a bit of work:

Proposition 1.28. *The functor $V \mapsto V^\vee$ induces an equivalence of categories between smooth representations of H and pseudocompact C-vector spaces M together with a continuous action $C[\![H]\!] \times M \to M$. A quasi-inverse is given by $M \mapsto \mathrm{Hom}_C^{\mathrm{cts}}(M, C)$.*

Exercise 1.29. Going back to Example 1.5, show that $\mathcal{C}^\infty(H, C)^\vee \cong C[\![H]\!]$. (It might be easier to compute the quasi-inverse.)

1.5.3. *Application to admissibility.* Suppose that G is a closed subgroup of $\mathrm{GL}_n(F)$.

Proposition 1.30. *Suppose that V is a smooth G-representation and suppose H is a compact open subgroup of G. Then V is admissible if and only if $V^\vee$ is a finitely generated $C[\![H]\!]$-module.*

Sketch. First one reduces to the case where H is pro-p (by passing to an open subgroup H' of H, so $C[\![H]\!]$ is a finitely generated $C[\![H']\!]$-module). Then $C[\![H]\!]$ is a local ring, as mentioned below, and note that its maximal ideal $\mathfrak{m}_H$ is also the augmentation ideal: $\mathfrak{m}_H = \ker(C[\![H]\!] \to C)$. Then

$$\begin{aligned} V \text{ is admissible } &\iff \dim_C V^H < \infty \iff \dim_C (V^\vee)_H < \infty \\ &\iff \dim_C V^\vee/\mathfrak{m}_H V^\vee < \infty \iff V^\vee \text{ is f.g. as } C[\![H]\!]\text{-module.} \end{aligned}$$

Here, $(-)_H$ denotes the H-coinvariants. The first equivalence is Lemma 1.22, the second and third hold because $(V^H)^\vee \cong (V^\vee)_H \cong V^\vee/\mathfrak{m}_H V^\vee$, the fourth because of a topological version of Nakayama's lemma, noting that $V^\vee$ is complete in the $\mathfrak{m}_H$-adic topology (cf. [Mat89, Thm. 8.4] in the commutative setting). □

Fact 1.31. *The ring $C[\![H]\!]$ is noetherian.*

(This was first proved by Lazard, but see [DdSMS99, §7] if $C = \mathbb{F}_p$. The main idea is to reduce to an open subgroup that is "uniform" pro-p and then show that the associated graded ring with respect to the unique maximal ideal of $C[\![H]\!]$ is noetherian; more precisely, it is a polynomial ring in finitely many variables over C. Note that $C[\![H]\!]$ is local, as H is pro-p and C of characteristic p. This is related to Lemma 1.6!)

Corollary 1.32. *The quotient of an admissible G-representation V is admissible.*

Proof. Fix any pro-p subgroup H of G. If $V \twoheadrightarrow W$ is a quotient representation, then $W^\vee \hookrightarrow V^\vee$ is a (closed) $C[\![H]\!]$-submodule. The result follows from Prop. 1.30 together with Fact 1.31. □

Remark 1.33. This result fails if F is of characteristic p, even if $G = F^\times$, cf. [AHV19, Ex. 4.4]. (Here is why this should not be so surprising: by above the result is equivalent to $C[\![H]\!]$ being noetherian for one/any compact open subgroup H. But $\mathcal{O}_F^\times$ is no longer topologically finitely generated when $\operatorname{char} F = p$.)

Remark 1.34. In fact, from Fact 1.31 we deduce by duality that the category of admissible smooth G-representation is artinian (any descending chain stabilizes). For example, any nonzero admissible G-representation contains an irreducible subrepresentation.

2. Hecke algebras and classification

Useful references: [Her12a] (together with exercises [Her12b]), [Her15], and [Bre07] (for §2.8).

In this section we will explain how irreducible admissible representations can be classified in terms of supersingular representations of smaller groups. This was first done for GL_2 by Barthel–Livné. To define the notion of a supersingular (or supercuspidal) representation, we will need to talk about Serre weights, Hecke algebras, and a mod-p analogue of the Satake isomorphism. Surprisingly, supersingular representations have only been classified for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ and some closely related groups (apart from the trivial case where the group is a torus).

We will suppose in this section (at least starting in §2.4) that C is algebraically closed. It will be convenient to fix an embedding $\kappa_0 : k_F \to C$, and let $\kappa_i := \kappa_0 \circ (x \mapsto x^{p^i})$ for $i \in \mathbb{Z}/f\mathbb{Z}$.

In this section we focus on $G = \mathrm{GL}_n(F)$, though we also need to consider Levi subgroups, which are products of general linear groups. Recall that $K = \mathrm{GL}_n(\mathcal{O}_F)$.

2.1. **Serre weights.** The first observation is that any nonzero smooth representation contains an irreducible representation of K (by smoothness, any K-representation generated by one element is finite-dimensional). In stark contrast to complex representations, there are only finitely many irreducible K-representations up to isomorphism:

Lemma 2.1. *If V is any irreducible representation of K, then K_1 acts trivially, i.e. V is an irreducible representation of the finite group $K/K_1 \cong \mathrm{GL}_n(k_F)$.*

Proof. Note that V^{K_1} is a subrepresentation of V, as $K_1 \lhd K$. But $V^{K_1} \neq 0$ by Lemma 1.6, so $V^{K_1} = V$. □

An irreducible representation of K, or equivalently of $\mathrm{GL}_n(k_F)$ is called a *Serre weight* (of G). (The terminology comes from §4.6.)

Example 2.2. If $n = 1$, then a Serre weight is nothing but a 1-dimensional representation $k_F^\times \to C^\times$, and there are $q-1$ of them.

If $n = 2$, then the Serre weights can be described as follows (cf. [Her12b, Ex. 4]): for any $\underline{a} = (a_i)_i, \underline{b} = (b_i)_i \in \mathbb{Z}^f$ such that $0 \le a_i - b_i \le p-1$ for all i, define

$$F(\underline{a}, \underline{b}) := \bigotimes_{i=0}^{f-1} \Big[(\mathrm{Sym}^{a_i - b_i}(k_F^2) \otimes \det{}^{b_i}) \otimes_{k_F, \kappa_i} C \Big],$$

where $\mathrm{GL}_2(k_F)$ acts by left multiplication on k_F^2 (the standard representation). Thus $\dim_C F(\underline{a}, \underline{b}) = \prod_{i=0}^{f-1}(a_i - b_i + 1)$. In §3 we will use the short-hand notation $(a_0, \dots, a_{f-1}) := F(\underline{a}, \underline{0})$.

To make this more concrete, let us use κ_0 to identify k_F with a subfield of C. We can think of $\mathrm{Sym}^n(k_F^2)$ as homogeneous polynomials f in two variables X, Y, with $((\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}) f)(X, Y) = f(\alpha X + \gamma Y, \beta X + \delta Y)$. Let $d_i := a_i - b_i \in \{0, 1, \dots, p-1\}$, $d := \sum_{i=0}^{f-1} d_i p^i$, and $b := \sum_{i=0}^{f-1} b_i p^i$. Then the natural map

$$\begin{aligned} F(\underline{a}, \underline{b}) &\to \mathrm{Sym}^d(C^2) \otimes \det{}^b \\ \otimes_i (f_i \otimes \lambda_i) &\mapsto \prod_i (f_i^{p^i} \lambda_i) \end{aligned} \tag{2.1}$$

is an injection with image spanned by the monomials $X^c Y^{d-c}$, $c = \sum_{i=0}^{f-1} c_i p^i$, $0 \le c_i \le d_i$ for all i. Here, $f_i \in k_F[X,Y]$ is homogeneous of degree d_i and $\lambda_i \in C$. The map is $\mathrm{GL}_2(k_F)$-equivariant via the inclusion $\kappa_0 : \mathrm{GL}_2(k_F) \hookrightarrow \mathrm{GL}_2(C)$. Thus we can think of $F(\underline{a}, \underline{b})$ as homogeneous polynomials of degree d. Alternatively, in the language of algebraic representations (§A.4), $F(\underline{a}, \underline{b}) \cong (L(d,0) \otimes \det^b)|_\Gamma$ and the isomorphism (2.1) follows from Fact A.16.

If $n > 2$, there are unfortunately no simple formulas for all the Serre weights, but at least they can be described in terms of the representation theory of the algebraic group GL_n in characteristic p, which is often sufficient in practice; see §A. There are $q^{n-1}(q-1)$ Serre weights of G.

Suppose that $P = MN$ is a standard parabolic subgroup. Then $P \cap K = (M \cap K)(N \cap K)$ and $\overline{P} \cap K = (M \cap K)(\overline{N} \cap K)$ (again semidirect products).

Fact 2.3. *If V is a Serre weight of G, then the invariants $V^{\overline{N} \cap K}$ and the co-invariants $V_{N \cap K}$ are Serre weights of M, i.e. irreducible representations of $M \cap K$. Moreover, the composition $V^{\overline{N} \cap K} \hookrightarrow V \twoheadrightarrow V_{N \cap K}$ is an isomorphism.*

Exercise 2.4. Suppose that $n = 2$. Show that $V^{\overline{U} \cap K}$ is 1-dimensional, spanned by the homogeneous polynomial Y^d in the above parametrization. Similarly show that $V \to V_{U \cap K}$ can be identified with the map $V \to C \cdot Y^d$, $f(X,Y) \mapsto f(0,Y)$ in the above parametrization. Deduce that Fact 2.3 holds when $n = 2$.

In particular, $V^{\overline{U} \cap K}$ is an irreducible representation of the abelian group $T \cap K$, hence 1-dimensional.

Fact 2.5. *Given a Serre weight V, let $\psi_V : T \cap K \to C^\times$ denote the action of $T \cap K$ on $V^{\overline{U} \cap K}$. The map $V \mapsto \psi_V$ is surjective and almost a bijection. More precisely, let $P_V = M_V N_V$ denote the largest standard parabolic such that $V^{\overline{N}_V \cap K}$ is 1-dimensional. Then the map $V \mapsto (\psi_V, M_V)$ is a bijection onto the set of pairs (ψ, M), where $\psi : T \cap K \to C^\times$ and M is a standard Levi such that ψ extends to a character of M.*

Note that P_V is well defined since $V^{\overline{U} \cap K}$ is always 1-dimensional, and that $M_V = G$ if and only if V is 1-dimensional.

Exercise 2.6. Verify Fact 2.3 for $n = 2$.

2.2. **Hecke algebras.** The second observation is the following: if π is any smooth G-representation and V any irreducible subrepresentation of $\pi|_K$, then we say that V *occurs in* π. In this case the "multiplicity space"

$$\mathrm{Hom}_K(V, \pi|_K) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \pi) \tag{2.2}$$

is naturally a right module over the C-algebra $\mathcal{H}_G(V) := \mathrm{End}_G(\text{c-Ind}_K^G V)$ (by pre-composition, hence a right module). We will see in §2.3 that the *Hecke algebra* $\mathcal{H}_G(V)$ of V is commutative, so we do not need to distinguish between left and right $\mathcal{H}_G(V)$-modules.

Remark 2.7.

(i) This is exactly analogous to the Hecke algebras associated to types in the complex representation theory of G.

(ii) The same formalism works for $\mathrm{End}_G(\text{c-Ind}_H^G V)$ if H is a compact open subgroup and V a finite-dimensional smooth H-representation over *any* field. Moreover, Proposition 2.9 generalizes to this context (cf. Jessica's lectures) though $\mathrm{End}_G(\text{c-Ind}_H^G V)$ is usually noncommutative.

Exercise 2.8. Show that π is admissible if and only if $\mathrm{Hom}_K(V, \pi|_K)$ is finite-dimensional for each Serre weight V if and only if the K-socle $\mathrm{soc}_K \pi$ is finite-dimensional. (Hint: for the first "if", use the same ideas as in the proof of Lemma 1.22.)

Proposition 2.9. *The Hecke algebra $\mathcal{H}_G(V)$ is isomorphic to the algebra of functions $\varphi : G \to \mathrm{End}_C(V)$ such that $\mathrm{supp}(\varphi)$ is compact and $\varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2$ for all $k_1, k_2 \in K$, $g \in G$. The product on this algebra is convolution $\varphi_1 \star \varphi_2$, where*

$$(\varphi_1 \star \varphi_2)(g) := \sum_{\gamma \in K\backslash G} \varphi_1(g\gamma^{-1}) \circ \varphi_2(\gamma). \tag{2.3}$$

Note that the sum in (2.3) is finite, since φ_1 (or φ_2) has compact support. Also the sum is well defined since $\varphi_1(g\gamma^{-1}k^{-1})\varphi_2(k\gamma) = \varphi_1(g\gamma^{-1})\varphi_2(\gamma)$ for $k \in K$. (Here and in the future, when we sum over cosets we mean that the index runs through a set of coset representatives and that the sum is independent of this choice.)

Proof. Write $\mathrm{Map}(X, Y)$ for the set of functions $X \to Y$, which is a vector space if Y is. As vector spaces,

$$\begin{aligned}\mathcal{H}_G(V) = \mathrm{End}_G(\text{c-Ind}_K^G V) &\cong \mathrm{Hom}_K(V, \text{c-Ind}_K^G V)\\ &\subseteq \mathrm{Map}(V, \mathrm{Map}(G, V)) = \mathrm{Map}(G, \mathrm{Map}(V, V)),\end{aligned}$$

by Frobenius reciprocity. To see that the image of $\mathrm{Hom}_K(V, \text{c-Ind}_K^G V)$ inside $\mathrm{Map}(G, \mathrm{Map}(V, V))$ is as claimed, take $\psi \in \mathrm{Hom}_K(V, \text{c-Ind}_K^G V)$, so its image is given by $\varphi(g)(v) = \psi(v)(g)$. It is a nice exercise to check that the conditions on ψ (one compact support and two K-equivariance conditions) to lie in $\mathrm{Hom}_K(V, \text{c-Ind}_K^G V) \subseteq \mathrm{Map}(V, \mathrm{Map}(G, V))$ correspond precisely to the conditions on φ in $\mathrm{Map}(G, \mathrm{Map}(V, V))$ stated in the first sentence of the statement of this proposition.

We need to check that under the above vector space isomorphism, the product in $\mathcal{H}_G(V)$ (composition) corresponds to the convolution product (2.3). Take φ corresponding to $\psi' \in \mathcal{H}_G(V)$ and to $\psi \in \mathrm{Hom}_K(V, \text{c-Ind}_K^G V)$. Then

$$\psi'([1, x])(\gamma) = \psi(x)(\gamma) = \varphi(\gamma)(x)$$

and hence by K-equivariance

$$\psi'([1, x]) = \sum_{\gamma \in K\backslash G} [\gamma^{-1}, \varphi(\gamma)(x)] = \sum_{\gamma \in K\backslash G} \gamma^{-1}[1, \varphi(\gamma)(x)].$$

If φ_1, φ_2 correspond to $\psi'_1, \psi'_2 \in \mathcal{H}_G(V)$, it is then an exercise to check that $\varphi_1 \star \varphi_2$ corresponds to $\psi'_1 \circ \psi'_2$. $\square$

We will often think of $\mathcal{H}_G(V)$ as the algebra of functions $G \to \mathrm{End}_C(V)$ as in Proposition 2.9.

Example 2.10. Let V be the trivial representation, then $\mathcal{H}_G(V) = \mathcal{C}_c(K\backslash G/K, C)$, the algebra of bi-$K$-equivariant functions on G with compact support, under convolution. Let 1_{KgK} denote the characteristic function of the double coset KgK. Then, for a smooth G-representation π it acts on $\pi^K = \mathrm{Hom}_K(V, \pi)$ on the right[4]: if $KgK = \coprod_{i=1}^r Kg_i$, then for $x \in \pi^K$, $1_{KgK}(x) = \sum_{i=1}^r g_i^{-1} x$.

Exercise 2.11. Make the right action of $\mathcal{H}_G(V)$ on $\mathrm{Hom}_K(V, \pi|_K)$ explicit. Suppose that $f \in \mathrm{Hom}_K(V, \pi|_K)$ as in (2.2) and $\varphi : G \to \mathrm{End}_C(V)$ in $\mathcal{H}_G(V)$. Then $f\varphi \in \mathrm{Hom}_K(V, \pi|_K)$ is given by $x \mapsto \sum_{g \in K\backslash G} g^{-1} f(\varphi(g)x)$.

2.3. **mod-p Satake transforms.** The classical Satake isomorphism over $\mathbb{C}$ is of the form $\mathcal{H}_G^{\mathbb{C}}(1) \xrightarrow{\sim} \mathcal{H}_T^{\mathbb{C}}(1)^W$. Here, T is the diagonal maximal torus (1.1), $\mathcal{H}_G^{\mathbb{C}}(1) := \mathrm{End}_G(\text{c-Ind}_K^G 1)$, $\mathcal{H}_T^{\mathbb{C}}(1) := \mathrm{End}_T(\text{c-Ind}_{T\cap K}^T 1)$ both over the field $\mathbb{C}$, and $W \cong S_n$ is the Weyl group. Note that $\mathcal{H}_T^{\mathbb{C}}(1)$ is isomorphic to the group algebra $\mathbb{C}[T/(T \cap K)]$, as T is commutative (every double coset is a single coset), and W naturally acts on it. In particular, $\mathcal{H}_G^{\mathbb{C}}(1)$ is commutative. Explicitly, the classical Satake isomorphism is given by the formula

$$\begin{aligned} \mathcal{H}_G^{\mathbb{C}}(1) &\to \mathcal{H}_T^{\mathbb{C}}(1) \\ \varphi &\mapsto \left(t \mapsto \delta(t)^{-1/2} \sum_{u \in (U\cap K)\backslash U} \varphi(ut) \right), \end{aligned}$$

where $\delta : B \to p^{\mathbb{Z}} \subseteq \mathbb{R}^\times$ is the modulus character of B. The normalizing factor $\delta(t)^{-1/2}$ is what makes the image W-invariant.

In order to have an analog of the Satake isomorphism in characteristic p, we are forced to drop the normalizing factor. We would also like to allow non-trivial Serre weights as coefficients. Recall that if V is a Serre weight of G, then $V_{U\cap K}$ is a (1-dimensional) Serre weight of T and we let $p_U : V \twoheadrightarrow V_{U\cap K}$ denote the natural projection onto the coinvariants. Let $T^+ \subseteq T$ denote submonoid

$$T^+ := \{\mathrm{diag}(t_1, \dots, t_n) \in T : \mathrm{ord}_p(t_1) \geq \cdots \geq \mathrm{ord}_p(t_n)\}$$

and let $\mathcal{H}_T^+(V_{U\cap K}) \subseteq \mathcal{H}_T(V_{U\cap K})$ denote the subalgebra

$$\mathcal{H}_T^+(V_{U\cap K}) := \{\varphi \in \mathcal{H}_T(V_{U\cap K}) : \mathrm{supp}(\varphi) \subseteq T^+\}.$$

Theorem 2.12 (H.)**.** *The map*

$$\begin{aligned} S^G : \mathcal{H}_G(V) &\to \mathcal{H}_T(V_{U\cap K}) \\ \varphi &\mapsto \left(t \mapsto \sum_{u \in (U\cap K)\backslash U} p_U \circ \varphi(ut) \right) \end{aligned}$$

is an injective algebra homomorphism with image $\mathcal{H}_T^+(V_{U\cap K})$. In particular, $\mathcal{H}_G(V)$ is commutative.

Here we think of $\mathcal{H}_G(V)$ as functions $G \to \mathrm{End}_C(V)$ and likewise for $\mathcal{H}_T(V_{U\cap K})$. Implicit in the above theorem is that the sum of all $p_U \circ \varphi(ut) : V \to V_{U\cap K}$ factors through a map $V_{U\cap K} \to V_{U\cap K}$.

[4]It is more common in the literature to see the left action of double cosets where we decompose KgK into left cosets.

Exercise 2.13. Show how the mod-p Satake isomorphism follows from the the classical one if $V = 1$, using the explicit formula $\delta(t) = |t_1|^{n-1}|t_2|^{n-3} \cdots |t_n|^{-(n-1)}$ for $t \in T$, where $|\cdot|$ is the normalized absolute value of F. (You may assume Step 2 of the sketch proof of Theorem 2.12 below.)

Exercise 2.14. Check that $\mathcal{H}_T^{(+)}(V_{U\cap K}) \cong C[T^{(+)}/(T\cap K)]$, where the right-hand side denotes the algebra on the monoid $T^{(+)}/(T \cap K)$ (even a group if we omit the $+$).

The following calculation is instructive.

Exercise 2.15. Suppose that $n = 2$ and $\varphi \in \mathcal{H}_G(V)$ is supported on the double coset $K\begin{pmatrix} \varpi & \\ & 1\end{pmatrix}K$. Then $k_1 \circ \varphi(\begin{pmatrix} \varpi & \\ & 1\end{pmatrix}) = \varphi(\begin{pmatrix} \varpi & \\ & 1\end{pmatrix}) \circ k_2$ whenever $k_1\begin{pmatrix} \varpi & \\ & 1\end{pmatrix} = \begin{pmatrix} \varpi & \\ & 1\end{pmatrix}k_2$ for some $k_1, k_2 \in K$. Deduce that $\varphi(\begin{pmatrix} \varpi & \\ & 1\end{pmatrix}) \in \mathrm{End}_C(V)$ induces a $T \cap K$-linear map $V_{U\cap K} \to V^{\overline{U}\cap K}$, and conversely that any such $T\cap K$-linear map arises from a unique φ supported on $K\begin{pmatrix} \varpi & \\ & 1\end{pmatrix}K$.

Exercise 2.16. Continue Exercise 2.15, showing that the vector space of $\varphi \in \mathcal{H}_G(V)$ supported on $K\begin{pmatrix} \varpi & \\ & 1\end{pmatrix}K$ is 1-dimensional. Show there is a unique nonzero such φ such that $\varphi(\begin{pmatrix} \varpi & \\ & 1\end{pmatrix})$ is a linear projection (namely the composition $V \twoheadrightarrow V_{U\cap K} \xleftarrow{\sim} V^{\overline{U}\cap K} \hookrightarrow V$).

Proof of Theorem 2.12 (sketch). Let $\Lambda := \mathbb{Z}^n$ and $\Lambda^+ := \{\lambda \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}$. For $\lambda \in \Lambda$, let $t_\lambda := \mathrm{diag}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_n})$. The Cartan decomposition says that $G = \coprod_{\lambda\in\Lambda^+} Kt_\lambda K$.

Step 0: S^G is an algebra homomorphism. This is a calculation, using the Iwasawa decomposition: $G = KB$ (so instead of summing over $K\backslash G$ in Proposition 2.9 we can sum over $(B \cap K)\backslash B$).

Step 1: as C-vector spaces, $\mathcal{H}_G(V)$ has basis $(T_\lambda)_{\lambda\in\Lambda^+}$ and $\mathcal{H}_T(V_{U\cap K})$ has basis $(\tau_\lambda)_{\lambda\in\Lambda}$, where $T_\lambda \in \mathcal{H}_G(V)$ is determined uniquely by $\mathrm{supp}(T_\lambda) = Kt_\lambda K$ and $T_\lambda(t_\lambda) \in \mathrm{End}_C(V)$ is a linear projection (resp. $\mathrm{supp}(\tau_\lambda) = t_\lambda(T \cap K)$ and $\tau_\lambda(t_\lambda) = 1$).

This is easy in case of τ_λ. For T_λ, this is a generalization of Exercise 2.16, showing that for $\varphi \in \mathcal{H}_G(V)$, $\varphi(t_\lambda)$ induces a $M_\lambda \cap K$-linear map $V_{N_\lambda\cap K} \to V^{\overline{N}_\lambda\cap K}$, where $P_\lambda = M_\lambda N_\lambda$ is the standard parabolic such that M_λ is the centralizer of t_λ in G. Then $T_\lambda(t_\lambda)$ is the composition $V \twoheadrightarrow V^{\overline{N}_\lambda\cap K} \xleftarrow{\sim} V_{N_\lambda\cap K} \hookrightarrow V$ (see Fact 2.3).

Step 2: show that S^G has triangular form with respect to the bases of Step 1. More precisely, for $\lambda \in \Lambda^+$,

$$S^G(T_\lambda) = \sum_{\mu\in\Lambda,\mu\leq\lambda} a_{\lambda\mu}\tau_\mu,$$

where $a_{\lambda\mu} \in C$ and $a_{\lambda\lambda} = 1$. Here, $\mu \leq \lambda$ is the partial order defined by $\sum_{i=1}^r \mu_i \leq \sum_{i=1}^r \lambda_i$ for all $1 \leq r \leq n$, with equality when $r = n$.

Step 3: show that $S^G(\mathcal{H}_G(V)) \subseteq \mathcal{H}_T^+(V_{U\cap K})$. This step is where the proof really differs from the classical proof. It is not hard when $V = 1$, see Exercise 2.13, but a bit tricky in general.

Step 4: The triangular form in Step 2 now becomes

$$S^G(T_\lambda) = \sum_{\mu\in\Lambda^+,\mu\leq\lambda} a_{\lambda\mu}\tau_\mu, \tag{2.4}$$

for $\lambda \in \Lambda^+$, with $a_{\lambda\lambda} = 1$. It now follows inductively that $S^G : \mathcal{H}_G(V) \to \mathcal{H}_T^+(V_{U\cap K})$ is an isomorphism, using that for any $\lambda \in \Lambda^+$ there are only finitely many $\mu \in \Lambda^+, \mu \le \lambda$. $\square$

Remark 2.17. For $1 \le i \le n$ let $T_i := T_{\lambda_i}$ and $\tau_i := \tau_{\lambda_i}$, where $\lambda_i = (1, \dots, 1, 0, \dots, 0)$ with i 1's followed by $n-i$ 0's. By (2.4) it follows that $S^G(T_i) = \tau_i$. This is however rather special to the group $\mathrm{GL}_n(F)$.

Corollary 2.18. *We have $\mathcal{H}_T^+(V_{U\cap K}) \cong C[x_1, \dots, x_n, x_n^{-1}]$ inside $\mathcal{H}_T(V_{U\cap K}) \cong C[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where x_i corresponds to $\tau_i \in \mathcal{H}_T^+(V_{U\cap K})$. There exists $\varphi \in \mathcal{H}_G(V)$ such that S^G induces an algebra isomorphism $\mathcal{H}_G(V)[\varphi^{-1}] \xrightarrow{\sim} \mathcal{H}_T(V_{U\cap K})$.*

Proof. Note that $\tau_{\lambda+\mu} = \tau_\lambda\tau_\mu$. Hence we have an isomorphism $C[\Lambda^{(+)}] \cong \mathcal{H}_T^{(+)}(V_{U\cap K})$ with $\lambda \in \Lambda$ corresponding to $\tau_\lambda \in \mathcal{H}_T(V_{U\cap K})$. Now it suffices to observe that $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}\lambda_i$ and $\Lambda^+ = \bigoplus_{i=1}^{n-1} \mathbb{N}\lambda_i \oplus \mathbb{Z}\lambda_n$ as monoids, where $\lambda_i \in \Lambda^+$ is defined as in Remark 2.17. For the final part, Λ is obtained from Λ^+ by inverting $\sum_{i=1}^{n-1} \lambda_i$, so $\mathcal{H}_T(V_{U\cap K}) = \mathcal{H}_T^+(V_{U\cap K})[(\tau_1 \cdots \tau_{n-1})^{-1}]$, hence we can take φ such that $S^G(\varphi) = \tau_1 \cdots \tau_{n-1}$. (There is a unique such φ by Theorem 2.12.) $\square$

Remark 2.19. Barthel–Livné (for $n = 2$) do not work with the Hecke algebra $\mathcal{H}_G(V)$, but instead with $\mathrm{End}_G(\text{c-Ind}_{KZ}^G V)$, where $\varpi \in Z$ acts trivially on V. The comparison of these Hecke algebras is discussed in [Her15, §10].

More generally, for any standard parabolic $P = MN$ we also need a Satake transform $S_M^G : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N\cap K})$. First note that the above results generalize to Levi subgroups (in fact all connected reductive groups), so we have a Satake map $S^M : \mathcal{H}_M(W) \hookrightarrow \mathcal{H}_T(W_{U\cap M\cap K})$. Let $p_N : V \to V_{N\cap K}$ denote the natural projection. It is not hard to deduce the following result from Theorem 2.12.

Proposition 2.20. *The map*

$$\begin{aligned} S_M^G &: \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N\cap K}) \\ &\varphi \mapsto \Bigg(m \mapsto \sum_{n\in(N\cap K)\backslash N} p_N \circ \varphi(nm)\Bigg) \end{aligned} \tag{2.5}$$

is an injective algebra homomorphism. We have $S^M \circ S_M^G = S^G$. Moreover the map S_M^G is a localization at one element, i.e. there exists $\varphi \in \mathcal{H}_G(V)$ such that S_M^G induces an algebra isomorphism $\mathcal{H}_G(V)[\varphi^{-1}] \xrightarrow{\sim} \mathcal{H}_M(V_{N\cap K})$.

Remark 2.21. The group $U \cap K$ is a semidirect product of $N \cap K$ (normal) and $U \cap M \cap K$, so $(V_{N\cap K})_{U\cap M\cap K} \cong V_{U\cap K}$.

Remark 2.22. Similar to the argument in Corollary 2.18 we can see that the following φ works in Proposition 2.20: $S^G(\varphi) = \prod_{i=1}^{r-1} \tau_{n_1+\cdots+n_i}$ if $M = M_{n_1,\dots,n_r}$.

Remark 2.23. There is a variant of the Satake map S^G that is of the form ${}'S^G : \mathcal{H}_G(V) \to \mathcal{H}_T(V^{U\cap K})$, defined by

$$\varphi \mapsto \Bigg(t \mapsto \sum_{u\in U/(U\cap K)} \varphi(tu)|_{V^{U\cap K}}\Bigg).$$

Both S^G and ${}'S^G$ fit into a family of Satake maps defined by Heyer, see the projects.

2.4. **Supersingular representations.** Suppose that π is an admissible G-representation. Then for any Serre weight V, $\mathrm{Hom}_K(V, \pi)$ is finite-dimensional and $\mathcal{H}_G(V)$ acts on it. Since $\mathcal{H}_G(V)$ is commutative and $C = \overline{C}$, if $\mathrm{Hom}_K(V, \pi) \neq 0$, it contains an $\mathcal{H}_G(V)$-eigenvector. In other words, there exists $0 \neq f \in \mathrm{Hom}_K(V, \pi)$ such that $f\varphi = \chi(\varphi)f$ for all $\varphi \in \mathcal{H}_G(V)$ for some $\chi : \mathcal{H}_G(V) \to C$ (necessarily an algebra homomorphism). In this case we say that χ is a *Hecke eigenvalue of* π (in weight V).

Proposition 2.24. *Suppose $P = MN$ is a standard parabolic and σ a smooth representation of M. For any Serre weight V we have an isomorphism of vector spaces*

$$\mathrm{Hom}_K(V, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_{M\cap K}(V_{N\cap K}, \sigma), \tag{2.6}$$

which is compatible with the action of $\mathcal{H}_G(V)$, acting via $S_M^G : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N\cap K})$ on the right-hand side.

Exercise 2.25. Using the Iwasawa decomposition $G = PK$ check that $(\mathrm{Ind}_P^G \sigma)|_K = \mathrm{Ind}_{P\cap K}^K \sigma$ and deduce (2.6). Then complete the proof of Proposition 2.24.

Exercise 2.26. Suppose $P = B$ and $\theta : T \to C^\times$ is a smooth character, so $\theta = \theta_1 \otimes \cdots \otimes \theta_n$, where $\theta_i : F^\times \to C^\times$ is a smooth character. If $n = 2$, prove that $\mathrm{soc}_K(\mathrm{Ind}_B^G \theta)$ has length 2 if and only if $\theta_1|_{\mathcal{O}_F^\times} = \theta_2|_{\mathcal{O}_F^\times}$, and length 1 otherwise. In general, show that $\mathrm{soc}_K(\mathrm{Ind}_B^G \theta)$ has length at most 2^{n-1}.

Exercise 2.27. Continue Exercise 2.26 for $n = 2$. Show that for any Hecke eigenvalue χ of $\mathrm{Ind}_B^G \theta$ we have $\chi(\tau_1) = \theta_1(p)^{-1}$ and $\chi(\tau_2) = (\theta_1(p)\theta_2(p))^{-1}$.

Remark 2.28. As observed by Henniart–Vignéras, we can use the naturality of the isomorphism (2.6) in σ to *define* a unique algebra homomorphism $S_M^G : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N\cap K})$ such that Proposition 2.24 holds, and then compute that it is given by formula (2.5). See [Her12a, §12].

Corollary 2.29. *Keep the notation of Proposition 2.24. If σ is admissible, then $\mathrm{Ind}_P^G \sigma$ is admissible.*

Proof. By taking dimensions in (2.6) we deduce that $\mathrm{Hom}_K(V, \mathrm{Ind}_P^G \sigma)$ is finite-dimensional for each V, as σ is admissible, so the result follows from Exercise 2.8. □

As a result of Proposition 2.24, any Hecke eigenvalue of $\mathrm{Ind}_P^G \sigma$ necessarily factors through S_M^G (note that S_M^G is a localization). From the point of view of Hecke eigenvalues, the most complicated representations then are the supersingular representations, which were first singled out for $n = 2$ by Barthel–Livné:

Definition 2.30. We say that an irreducible admissible G-representation π is *supersingular* if the following equivalent conditions hold:

(i) if any Hecke eigenvalue χ of π factors through S_M^G, then $M = G$;
(ii) for any Hecke eigenvalue χ of π, we have $\chi(\tau_1) = \cdots = \chi(\tau_{n-1}) = 0$.

Here, we identified χ with a homomorphism $\mathcal{H}_T^+(V_{U\cap K}) \to C$ via S^G.[5] The two conditions are equivalent by Remark 2.22.

[5] We could have written $\chi(T_1) = \cdots = \chi(T_{n-1}) = 0$, but this is not ideal in light of Remark 2.17.

Remark 2.31. We remark that Vignéras gave another definition of supersingularity, using the action of (the center of) the *pro-p Iwahori-Hecke algebra* $\mathrm{End}_G(\text{c-Ind}_{I_1}^G 1)$ on π^{I_1}. It was shown to be equivalent to the above definition by Ollivier–Vignéras.

2.5. **Generalized Steinberg representations.** At the other extreme of supersingular representations are the irreducible factors of $\mathrm{Ind}_B^G(1)$. Note that any standard parabolic subgroup P defines a subrepresentation $\mathrm{Ind}_P^G(1) \subseteq \mathrm{Ind}_B^G(1)$, and we have $\mathrm{Ind}_Q^G(1) \subseteq \mathrm{Ind}_P^G(1)$ whenever $Q \supseteq P$. Therefore it makes sense to define

$$\mathrm{Sp}_P := \frac{\mathrm{Ind}_P^G(1)}{\sum_{Q \supsetneq P} \mathrm{Ind}_Q^G(1)},$$

where the sum runs through all standard parabolic subgroups Q that strictly contain P. In particular, $\mathrm{Sp}_G = 1_G$, the trivial representation of G, and $\mathrm{St} := \mathrm{Sp}_B$ is known as the *Steinberg representation*.

By work of Große-Klönne we know:

Theorem 2.32. *Let P be any standard parabolic subgroup. Then the admissible representation Sp_P is irreducible, $\mathrm{soc}_K(\mathrm{Sp}_P)$ is irreducible, and its unique Hecke eigenvalue is given by $\chi(\tau_i) = 1$ for all $1 \le i \le n$.*

More precisely, the Serre weight $\mathrm{soc}_K(\mathrm{Sp}_P)$ is parametrized by the pair $(1, M)$ (as in Fact 2.5), where $P = MN$. (See [Her12b] for two proofs that St is irreducible when $n = 2$.)

It is not so hard to deduce:

Corollary 2.33. $\mathrm{Ind}_B^G(1)$ *is of length 2^{n-1}, with irreducible constituents the Sp_P, each occurring with multiplicity 1.*

2.6. **Classification of irreducibles.**

Theorem 2.34 (H.). *Let $P = P_{n_1,\dots,n_r}$ be a standard parabolic. Suppose that σ_i is an irreducible admissible representation of $\mathrm{GL}_{n_i}(F)$ such that for each i either*

(a) *σ_i is supersingular and $n_i > 1$ or*
(b) *$\sigma_i = \mathrm{Sp}_{P_i} \otimes (\eta_i \circ \det)$ for some standard parabolic $P_i \subseteq \mathrm{GL}_{n_i}(F)$ and smooth character $\eta_i : F^\times \to C^\times$.*

Suppose moreover that $\eta_i \ne \eta_{i+1}$ if σ_i and σ_{i+1} are of type (b). Then $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible and admissible.

Conversely, every irreducible admissible G-representation arises uniquely in this way, meaning P is unique and the tuple $(\sigma_1, \dots, \sigma_r)$ is unique up to isomorphism.

Exercise 2.35. Use Theorem 2.34 and Corollary 2.33 to show that a principal series $\mathrm{Ind}_B^G(\theta_1 \otimes \cdots \otimes \theta_n)$ has length 2^r, where $r := |\{1 \le i < n : \theta_i = \theta_{i+1}\}|$. (Hint: first use transitivity of parabolic induction, inducing to the largest Levi to which the character $\theta_1 \otimes \cdots \otimes \theta_n$ extends.) In particular, it is irreducible if and only if $\theta_i \ne \theta_{i+1}$ for all i, as was first proved by Ollivier.

This theorem allows one to compare the notion of supersingularity with supercuspidality, where we say that an irreducible admissible π is *supercuspidal* if π does not occur as subquotient of $\mathrm{Ind}_P^G \sigma$ for any proper standard parabolic $P \subsetneq G$ and any irreducible admissible representation σ of the Levi of P. (This is very similar but not exactly analogous to the usual definition over $\mathbb{C}$.)

Corollary 2.36. *Suppose that π is an irreducible admissible G-representation.*

(i) *The representation π is supersingular if and only if π is supercuspidal.*
(ii) *The Hecke eigenvalues of π are constant in the sense that there exist $\lambda_i \in C$ such that for each Hecke eigenvalue χ of π and each $1 \le i \le n$ we have $\chi(\tau_i) = \lambda_i$.*
(iii) *If σ is an irreducible admissible M-representation, then $\mathrm{Ind}_P^G \sigma$ is of finite length and multiplicity free.*

By part (ii), to check whether an irreducible admissible representation is supersingular, it is enough to check Definition 2.30 for *one* Hecke eigenvalue.

As a special case of Theorem 2.34 we recover for $n = 2$:

Theorem 2.37 (Barthel–Livné)**.** *The irreducible admissible representations of $\mathrm{GL}_2(F)$ are given by*

(i) *principal series $\mathrm{Ind}_B^G(\theta_1 \otimes \theta_2)$ with $\theta_1 \neq \theta_2$,*
(ii) *one-dimensional representations $\theta \circ \det$,*
(iii) $\mathrm{St} \otimes (\theta \circ \det)$*,*
(iv) *supersingular representations,*

where $\theta, \theta_1, \theta_2$ denote smooth characters $F^\times \to C^\times$, and there are no non-trivial isomorphisms between these representations.

Here, in the notation of Theorem 2.34, case (i) arises from $P = B$ and $\eta_1 = \theta_1 \neq \theta_2 = \eta_2$ (type (b)); cases (ii) and (iii) from $P = G$ and $\eta_1 = \theta$ (type (b)); case (iv) from $P = G$ (type (a)).

Corollary 2.38. *If π is an irreducible admissible representation of $\mathrm{GL}_2(F)$ such that $\mathrm{soc}_K(\pi)$ is of length > 2, then π is supersingular.*

This follows from Theorem 2.37 by recalling Exercise 2.26 and Theorem 2.32.

2.7. **On the proofs.** Let us briefly explain how one can prove that $\mathrm{Ind}_P^G \sigma$ is irreducible for some $P = MN$ and some irreducible admissible M-representations σ. It suffices to show that any nonzero G-subrepresentation $\pi \subseteq \mathrm{Ind}_P^G \sigma$ we have $\pi = \mathrm{Ind}_P^G \sigma$. As $\pi \neq 0$ we can pick a Serre weight $V \subseteq \pi|_K$ and Hecke eigenvalues χ that occur in $\mathrm{Hom}_K(V, \pi)$. By Frobenius reciprocity we have a G-equivariant map $\text{c-Ind}_K^G V \twoheadrightarrow \pi$, which is surjective as π is irreducible. By definition, this map has to factor through a G-equivariant map $(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C \twoheadrightarrow \pi$. It is thus important to understand the left-hand side. Note that $\chi : \mathcal{H}_G(V) \to C$ has to factor as $\chi_M \circ S_M^G$ for a unique algebra homomorphism $\chi_M : \mathcal{H}_M(V_{N\cap K}) \to C$ as a result of Proposition 2.24 (since $\pi \subseteq \mathrm{Ind}_P^G \sigma$).

We say that V is *M-regular* if $M_V \subseteq M$, where M_V is as in Fact 2.5.

Proposition 2.39. *Suppose that V is M-regular and that $\chi_M : \mathcal{H}_M(V_{N\cap K}) \to C$ is an algebra homomorphism. Then we have an isomorphism of G-representations*

$$(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C \cong \mathrm{Ind}_P^G \Big((\text{c-Ind}_{M\cap K}^M V_{N\cap K}) \otimes_{\mathcal{H}_M(V_{N\cap K}),\chi_M} C \Big), \tag{2.7}$$

where $\chi := \chi_M \circ S_M^G$.

The inducing representation on the right looks complicated, but it's of exactly the same form as the left-hand side, for M instead of G. Note that the isomorphism (2.7) is induced by the G- and $\mathcal{H}_G(V)$-equivariant map $\text{c-Ind}_K^G V \to \mathrm{Ind}_P^G(\text{c-Ind}_{M\cap K}^M V_{N\cap K})$ (with $\mathcal{H}_G(V)$ acting via S_M^G on the right) coming from Proposition 2.24 with $\sigma = \text{c-Ind}_{M\cap K}^M V_{N\cap K}$.

Let us now show if π contains an M-regular Serre weight V, then $\pi = \mathrm{Ind}_P^G \sigma$. Pick an $\mathcal{H}_G(V)$-eigenvector $V \hookrightarrow \pi|_K \hookrightarrow (\mathrm{Ind}_P^G \sigma)|_K$ with eigenvalues χ. By Proposition 2.24 it corresponds to an $\mathcal{H}_M(V_{N\cap K})$-eigenvector $V_{N\cap K} \hookrightarrow \sigma$ with eigenvalues χ_M, where $\chi = \chi_M \circ S_M^G$. The induced map $(\text{c-Ind}_{M\cap K}^M V_{N\cap K}) \otimes_{\mathcal{H}_M(V_{N\cap K}),\chi_M} C \to \sigma$ is surjective, as σ is irreducible. We obtain a commutative diagram, showing that $\pi = \mathrm{Ind}_P^G \sigma$ (the commutativity comes by construction):

$$\begin{tikzcd}
(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C \arrow[r] \arrow[d, "\text{Prop. 2.39}"', "\cong"] & \pi \arrow[r, hook] & \mathrm{Ind}_P^G \sigma \\
\mathrm{Ind}_P^G \Big((\text{c-Ind}_{M\cap K}^M V_{N\cap K}) \otimes_{\mathcal{H}_M(V_{N\cap K}),\chi_M} C \Big) \arrow[urr, two heads] & &
\end{tikzcd}$$

Exercise 2.40. Suppose that $n = 2$. Note that a Serre weight is T-regular if and only if it isn't 1-dimensional. Show that all Serre weights that occur in a principal series $\mathrm{Ind}_B^G(\theta_1 \otimes \theta_2)$ are T-regular, unless $\theta_1|_{\mathcal{O}_F^\times} = \theta_2|_{\mathcal{O}_F^\times}$. Thus Proposition 2.39 shows that $\mathrm{Ind}_B^G(\theta_1 \otimes \theta_2)$ is irreducible if $\theta_1|_{\mathcal{O}_F^\times} \neq \theta_2|_{\mathcal{O}_F^\times}$.

Exercise 2.41. Suppose that $n = 2$ and $\theta := \theta_1 \otimes \theta_2$. Here is a *direct* argument showing that $\mathrm{Ind}_B^G \theta$ is irreducible if $\theta_1|_{\mathcal{O}_F^\times} \neq \theta_2|_{\mathcal{O}_F^\times}$. Suppose that $V \hookrightarrow (\mathrm{Ind}_B^G \theta)|_K$ is any Serre weight, corresponding to a $B \cap K$-linear map $\eta : V \twoheadrightarrow \theta$ by Frobenius reciprocity. Let $0 \neq v \in V^{\overline{U}\cap K}$ and let f denote its image in $\mathrm{Ind}_B^G \theta$, so $f(bk) = \theta(b)\eta(kv)$ for $b \in B, k \in K$.

(i) Show that $f(1) = \eta(v)$ $(\neq 0)$ and $f(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)) = 0$, using Exercise 2.4. (For the second part note that $\dim_C V > 1$ by Exercise 2.40!)

(ii) Deduce that the support of f is $B(\overline{B} \cap K) = B(\overline{U} \cap K)$. (The Bruhat decomposition says that $K = (B \cap K)(\overline{B} \cap K) \sqcup (B \cap K)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)(\overline{B} \cap K)$.)

(iii) Show that the restriction map gives an isomorphism of $\overline{B}$-representations

$$\begin{aligned} \{h \in \mathrm{Ind}_B^G \theta : \mathrm{supp}(h) \subseteq B\overline{U}\} &\to \mathcal{C}_c^\infty(F, C) \\ h &\mapsto h' := \Big(a \mapsto h(\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right))\Big), \end{aligned} \tag{2.8}$$

where $(\bar{b}h')(a) = \theta(\bar{b})h'(\alpha(\bar{b})a)$ $\forall \bar{b} \in \overline{B}, a \in F$ and $\alpha(\left(\begin{smallmatrix} x & 0 \\ z & w \end{smallmatrix}\right)) := xw^{-1}$.

(iv) Suppose $0 \neq \pi \subseteq \mathrm{Ind}_B^G \theta$ is a G-subrepresentation. Using Exercise 2.26, show that $f \in \pi$, which corresponds to $1_{\mathcal{O}_F} \in \mathcal{C}_c^\infty(F, C)$. Use the $\overline{B}$-action (translating/scaling!) to show that π contains the left-hand side of (2.8).

(v) Use the G-action to show that $\pi = \mathrm{Ind}_B^G \theta$.

A second ingredient is needed to deal with non-M-regular weights. For simplicity let us specialize to $n = 2$. By Exercise 2.40, if V is a Serre weight that isn't T-regular, then $\dim_C V = 1$, and there exists a (unique) T-regular Serre weight V' such that $V^{\overline{U}\cap K} \cong (V')^{\overline{U}\cap K}$. (In terms of Fact 2.5, V is parametrized by (ψ_V, G) and V' by (ψ_V, T).) Via the Satake isomorphism S^G we obtain a natural identification $\mathcal{H}_G(V) \cong \mathcal{H}_G(V')$. The following result can be obtained by constructing explicit maps $\text{c-Ind}_K^G V \leftrightarrows \text{c-Ind}_K^G V'$ using the Satake formalism:

Proposition 2.42. *With the above notation, for any $\chi : \mathcal{H}_G(V) \to C$ we have*

$$(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C \cong (\text{c-Ind}_K^G V') \otimes_{\mathcal{H}_G(V'),\chi} C \tag{2.9}$$

whenever $\chi(\tau_1)^2 \neq \chi(\tau_2)$.

We continue the argument before Exercise 2.40 (for $n = 2$, $P = B$). Suppose π doesn't contain a T-regular weight. Then π contains some V that isn't T-regular, with some Hecke eigenvalue χ. If χ satisfies the condition in Proposition 2.42, then we can use isomorphism (2.9) to show that π contains the T-regular weight V' (with Hecke eigenvalue χ), contradiction!

Exercise 2.43. Continue Exercise 2.40 to show $\operatorname{Ind}_B^G(\theta_1 \otimes \theta_2)$ is irreducible if $\theta_1 \neq \theta_2$. (Note that if $\theta_1 = \theta_2 =: \theta$, then $\operatorname{Ind}_B^G(\theta_1 \otimes \theta_2)$ is irreducible of length 2, having irreducible subrepresentation $\theta \circ \det$ and irreducible quotient $\mathrm{St} \otimes (\theta \circ \det)$.)

Exercise 2.44. Here is a *direct* argument showing that $\operatorname{Ind}_B^G \theta$ is irreducible if $\theta_1 \neq \theta_2$. By Exercise 2.41 we may suppose that $\theta_1|_{\mathcal{O}_F^\times} = \theta_2|_{\mathcal{O}_F^\times}$. By Exercise 2.26, $\operatorname{Ind}_B^G \theta$ contains precisely two Serre weights V_1 and V_2 and precisely one of them is 1-dimensional, say V_1. If $0 \neq \pi \subseteq \operatorname{Ind}_B^G \theta$ is a G-subrepresentation, then $\pi|_K$ contains V_1 or V_2, and if it contains V_2 we get $\pi = \operatorname{Ind}_B^G \theta$ by Exercise 2.41. Suppose $V_1 \subseteq \pi|_K$. Note that the action of K on V_1 factors through the determinant.

(i) Show that following Exercise 2.41(i) we now get $f(1) = f(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)) = \eta(v)$, and that $f \in \pi$.
(ii) Using our assumptions on θ deduce that $h := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right) f - \theta_1(-\varpi) f$ is such that $\operatorname{supp}(h) \subseteq B\overline{U}$ and that h' in (2.8) is a nonzero multiple of $1_{\mathcal{O}_F}$.
(iii) Deduce that $\pi = \operatorname{Ind}_B^G \theta$ as in Exercise 2.41.

Conversely, why is every irreducible admissible G-representation π of the form listed in Theorem 2.34? Again pick a Hecke eigenvalue χ occurring in $\operatorname{Hom}_K(V, \pi)$ for some Serre weight V. Then π is a quotient of $(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C$. If we can find a standard parabolic $P = MN$, $P \neq G$ such that Proposition 2.39 applies, then we can write π as quotient of a parabolically induced representation $\operatorname{Ind}_P^G \sigma$ for some (perhaps huge) smooth M-representation σ. Using Emerton's theory of ordinary parts (a right adjoint of parabolic induction) we can deduce that π is a quotient of $\operatorname{Ind}_P^G \sigma'$ for some irreducible admissible M-representation σ'. Then we are in good shape, since we can inductively suppose that σ' is as in Theorem 2.34 and decompose $\operatorname{Ind}_P^G \sigma'$ using Theorem 2.34 and Corollary 2.33.

If such a parabolic P does not exist, one can try to use Proposition (2.9) to reduce to the previous case. If this approach gets stuck, one can show there are only two possibilities: π is supersingular (and we are done) or π contains the trivial Serre weight 1 with the same Hecke eigenvalue as the one occurring in $\operatorname{Ind}_B^G 1$ (up to twist). In the final case, a separate approach is needed to show that π is 1-dimensional.

2.8. **The group $\mathrm{GL}_2(\mathbb{Q}_p)$ and the mod p local Langlands correspondence.** Suppose that $G := \mathrm{GL}_2(\mathbb{Q}_p)$. Breuil completed the classification of irreducible admissible G-representations:

Theorem 2.45. *Any irreducible admissible supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is of the form $(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C$ for some Serre weight V and some algebra homomorphism $\chi : \mathcal{H}_G(V) \to C$ such that $\chi(\tau_1) = 0$. Precisely the following non-trivial isomorphisms exist in this list:*

$$(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C \cong (\text{c-Ind}_K^G V') \otimes_{\mathcal{H}_G(V'),\chi'} C,$$

where $V \cong \mathrm{Sym}^r(C^2) \otimes \det^s$, $V' \cong \mathrm{Sym}^{p-1-r}(C^2) \otimes \det^{r+s}$, $\chi(\tau_1) = \chi'(\tau_1) = 0$, $\chi(\tau_2) = \chi'(\tau_2)$ for some $0 \leq r \leq p-1$, $s \in \mathbb{Z}$.

Let $\pi := (\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C$ as in Theorem 2.45. The second part of the theorem follows from showing that $\operatorname{soc}_K \pi \cong V \oplus V'$, where V' is as in the statement. There are several known proofs, and I refer to [Her15, §10] for a relatively short proof of Paškūnas.

Remark 2.46. Unfortunately such a simple classification is very far from true in general! We will see this for $\mathrm{GL}_2(F)$, where $F/\mathbb{Q}_p$ is a nontrivial unramified extension at the end of §3.2 (as well as in §3.4). In general it is still true that any irreducible admissible supersingular representation is a quotient of a "universal supersingular quotient" $(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C$ for some V and χ such that $\chi(\tau_1) = \cdots = \chi(\tau_{n-1}) = 0$, but this quotient is nonadmissible if $n = 2$ and $F/\mathbb{Q}_p$ is a nontrivial unramified extension (Breuil). One of the projects will study this quotient in case $n = 3$ and $F = \mathbb{Q}_p$.

Remark 2.47. We can write more concretely,

$$(\text{c-Ind}_K^G V) \otimes_{\mathcal{H}_G(V),\chi} C \cong \frac{\text{c-Ind}_K^G V}{(T_1, T_2 - \chi(\tau_2))(\text{c-Ind}_K^G V)}$$

if $\chi(\tau_1) = 0$. As $T_2 = \begin{pmatrix} p & \\ & p \end{pmatrix}^{-1}$ on $\text{c-Ind}_K^G V$, it follows that this representation has central character with $p \in Z$ acting as $\chi(\tau_2)^{-1}$ and $a \in \mathbb{Z}_p^\times$ acting via the central character of V.

Breuil observed that the classification in Theorem 2.45 uses the same data as the classification of irreducible continuous 2-dimensional Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(C)$. Recall first that any continuous character $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to C^\times$ is uniquely of the form $\omega^s \operatorname{nr}(\alpha)$, where ω is the mod-p cyclotomic character, $s \in \mathbb{Z}/(p-1)\mathbb{Z}$, and $\operatorname{nr}(\alpha)$ is the unramified character sending an arithmetic Frobenius element to $\alpha \in C^\times$. Then if $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(C)$, either $\overline{\rho}$ is reducible and $\overline{\rho} \cong \begin{pmatrix} \theta_1 & * \\ & \theta_2 \end{pmatrix}$, or $\overline{\rho}$ is irreducible and $\overline{\rho}|_{I_p} \cong \omega_2^i \oplus \omega_2^{pi}$ for some $i \in \mathbb{Z}/(p^2-1)\mathbb{Z}$ such that $p+1 \nmid i$. Here, I_p is the inertia subgroup and $\omega_2 : I_p \to C^\times$ is Serre's tame level 2 character, cf. §4.6, satisfying $\omega_2^{p+1} = \omega|_{I_p}$. For $p+1 \nmid i$ let $\overline{\rho}(i)$ denote the unique irreducible $\overline{\rho}$ such that $\overline{\rho}|_{I_p} \cong \omega_2^i \oplus \omega_2^{pi}$ and $\det \overline{\rho} = \omega^i$.

This led Breuil to propose a mod-p Langlands correspondence between 2-dimensional Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(C)$ and certain admissible representations of G:

representations of G		Galois representations
$\dfrac{\text{c-Ind}_K^G(\mathrm{Sym}^r(C^2) \otimes \det^s)}{T_1, T_2 - \alpha}$	$\longleftrightarrow$	$\overline{\rho}(r+1) \otimes \omega^s \operatorname{nr}(\alpha)$
$\mathrm{Ind}_B^G(\theta_1 \otimes \theta_2\omega^{-1}) \oplus \mathrm{Ind}_B^G(\theta_1 \otimes \theta_2\omega^{-1})$	$\longleftrightarrow$[6]	$\begin{pmatrix} \theta_1 & \\ & \theta_2 \end{pmatrix}$

where $0 \le r \le p-1$ and the characters $\theta_i : \mathbb{Q}_p^\times \to C^\times$ are considered as characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ via local class field theory (and vice versa for ω).

Remark 2.48. Note that $\overline{\rho}(r+1) \cong \overline{\rho}(p-r) \otimes \omega^r$, consistently with Theorem 2.45. Likewise note that in the second line on the right we cannot distinguish the order of θ_1, θ_2, so it is not possible to match the Galois representation $\begin{pmatrix} \theta_1 & \\ & \theta_2 \end{pmatrix}$ with a single principal series (by the uniqueness part of Theorem 2.34 or Theorem 2.37).

[6] At least if the two principal series on the left are irreducible, i.e. if $\theta_1\theta_2^{-1} \neq \omega^{\pm 1}$.

This was at the beginning of much work by Breuil, Colmez, Kisin, Emerton, Paškūnas, … culminating in the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. See [Bre10] and [Ber11] for an introduction as well as further references.

Unfortunately there are at least two major problems for $\mathrm{GL}_2(F)$, $F \neq \mathbb{Q}_p$: (i) there are far more irreducible admissible supersingular representations of $\mathrm{GL}_2(F)$ than irreducible 2-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ (see §3) and (ii) the irreducible admissible supersingular representations of $\mathrm{GL}_2(F)$ are not of finite presentation, as shown by Schraen and Wu. Therefore, there seems to be no simple way to construct a local mod-p Langlands correspondence for $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$. Local-global compatibility may provide one path forward (see §4.7 and §4.11). Another way to generalize the mod-p Langlands correspondence is the proposed categorical p-adic Langlands program of Emerton–Gee–Hellmann [EGH].

3. $\mathrm{GL}_2(F)$

In this section we will discuss the construction of many supersingular representations of $G := \mathrm{GL}_2(F)$. First, Paškūnas found a general construction using diagrams that is explained in §3.1 and illustrated in Example 3.33. Then, Breuil–Paškūnas defined a special class of diagrams that is consistent with global expectations (see §3.4 and also §4). In §3.5 we explain a recent construction of nonadmissible irreducible representation, first discovered by Daniel Le.

Throughout, we will assume that $p > 2$. Recall that $I = \left(\begin{smallmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \varpi\mathcal{O}_F & \mathcal{O}_F^\times \end{smallmatrix}\right)$ and $I_1 = \left(\begin{smallmatrix} 1+\varpi\mathcal{O}_F & \mathcal{O}_F \\ \varpi\mathcal{O}_F & 1+\varpi\mathcal{O}_F \end{smallmatrix}\right)$. Note that $I/I_1 \cong k_F^\times \times k_F^\times$ is finite abelian of order prime to p. Let $Z := \{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) : x \in F^\times\}$ denote the center of G. If H is a group, V a finite length representation of H and τ an irreducible representation of H, we write $[V : \tau]$ for the number of times τ occurs as Jordan–Hölder factor of V.

3.1. **Diagrams.** The construction is based on the following fact, going back to work of Ihara and Serre. Note that G acts on the *Bruhat–Tits tree* $\mathcal{T}$, which is an unoriented graph whose vertices consist of all homothety classes $[L]$ of $\mathcal{O}_F$-lattices $L \subseteq F^2$ (homothety is the equivalence relation by scaling by $F^\times$).[7] The edges consist of all pairs $([L], [L'])$ such that $L' \subsetneq L$ is a maximal inclusion, i.e. $L/L' \cong k_F$. Note that $g \in G$ acts on F^2 (column vectors) and hence on $\mathcal{T}$, sending $[L]$ to $[g(L)]$ and likewise for edges. It is not so hard to see that $\mathcal{T}$ is indeed a tree [Bre07, Lemma 3.1], and that each vertex has degree $|\mathbb{P}^1(k_F)| = q + 1$. Note also that the center Z acts trivially on $\mathcal{T}$.

Let $x_0 := [\mathcal{O}_F \oplus \mathcal{O}_F]$, $x_1 := [\mathcal{O}_F \oplus \varpi\mathcal{O}_F]$, which are adjacent vertices, and let $e := (x_0, x_1)$, an edge of $\mathcal{T}$. Then $\mathrm{Stab}_G(x_0) = KZ$ and $\mathrm{Stab}_G(x_1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right) KZ \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)^{-1}$. We see that G acts transitively on vertices (as each lattice is finite free of rank 2) and on edges (as $\mathrm{Stab}_G(x_0)$ is transitive on edges with vertex x_0). Let $\mathcal{N} := \mathrm{Stab}_G(e)$ and note that $IZ = \mathrm{Stab}_G(x_0) \cap \mathrm{Stab}_G(x_1)$. Note that $(\mathcal{N} : IZ) = 2$, since $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ exchanges x_0 and x_1.[8]

Proposition 3.1. *We have a group isomorphism $G \cong KZ *_{IZ} \mathcal{N}$. Here, $KZ *_{IZ} \mathcal{N}$ denotes an amalgam (i.e. pushout) in the category of groups, satisfying the universal property for all pairs of morphisms $KZ \to \Gamma$, $\mathcal{N} \to \Gamma$ that agree on $IZ = KZ \cap \mathcal{N}$).*

Proof. Let $\mathcal{T}'$ denote the tree obtained by barycentric subdivision of $\mathcal{T}$, i.e. vertices are all vertices and all midpoints of edges of $\mathcal{T}$. Each edge of $\mathcal{T}$ is thus subdivided into two pieces. By above, G acts transitively on edges of $\mathcal{T}'$ and there are two orbits of vertices: vertices of $\mathcal{T}$ and midpoints. If m denotes the midpoint of e and e' the edge (x_0, m), then $\mathrm{Stab}_G(x_0) = KZ$, $\mathrm{Stab}_G(m) = \mathrm{Stab}_G(e) = \mathcal{N}$, and $\mathrm{Stab}_G(e') = \mathrm{Stab}_G(x_0) \cap \mathrm{Stab}_G(x_1) = IZ$. By the universal property we get a homomorphism $\mathrm{Stab}_G(x_0) *_{\mathrm{Stab}_G(e')} \mathrm{Stab}_G(m) \to G$, which is an isomorphism by the same argument as in [Bre07, Thm. 3.3].[9] $\square$

[7] Seen as a simplicial complex, this is the reduced Bruhat–Tits building of G; cf. Jessica's lectures.

[8] In fact, $\mathcal{N}$ is the normalizer of I (and of IZ), but we won't need this.

[9] The general statement is: if a group G acts on a tree X such that the action is transitive on edges, but not on vertices, then $G \cong \mathrm{Stab}_G(x) *_{\mathrm{Stab}_G(e)} \mathrm{Stab}_G(y)$, where $e = (x, y)$ is any choice of edge, and it is proved in the same way.

To construct a representation of G is thus equivalent to constructing actions of KZ and $\mathcal{N}$ on the same vector space that moreover agree on IZ. The main tool to carry this out will be:

Definition 3.2. A *diagram* is a triple (D_0, D_1, r) such that

- D_0 is a smooth representation of KZ;
- D_1 is a smooth representation of $\mathcal{N}$;
- $r : D_1 \to D_0$ is an injective IZ-equivariant map;
- D_0 is admissible and $\varpi \in Z$ acts trivially on D_0.[10]

A map of diagrams $(D_0, D_1, r) \to (D'_0, D'_1, r')$ consists of a KZ-equivariant map $\varphi_0 : D_0 \to D'_0$ and an $\mathcal{N}$-equivariant map $\varphi_1 : D_1 \to D'_1$ such that $r' \circ \varphi_1 = \varphi_0 \circ r$.

Example 3.3. The easiest examples of a diagram are $(\pi|_{KZ}, \pi|_{\mathcal{N}}, \mathrm{incl})$ and its subdiagram $(\pi^{K_1}, \pi^{I_1}, \mathrm{incl})$, where π is an admissible smooth G-representation with ϖ acting trivially and incl denotes the inclusion.

Theorem 3.4. *For any diagram (D_0, D_1, r) there exists an admissible smooth G-representation π together with an injection of diagrams $(D_0, D_1, r) \hookrightarrow (\pi|_{KZ}, \pi|_{\mathcal{N}}, \mathrm{incl})$ such that moreover (i) π is generated by D_0 as G-representation and (ii)* $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0$.

To prove this, we first need injective envelopes. If τ and J are smooth representations of a profinite group H we say that an injection $j : \tau \hookrightarrow J$ is an *injective envelope* if J is injective (i.e. the functor $\mathrm{Hom}_H(-, J)$ is exact) and j is essential in the sense that any subrepresentation $V \neq 0$ of J satisfies $V \cap \tau \neq 0$.

Injective envelopes always exist (the notes [Bre07, §5.2] prove this in important special cases), and they are unique up to isomorphism (good exercise!) but not unique up to unique isomorphism! We write $\tau \hookrightarrow \mathrm{inj}_H \tau$ for the choice of an injective envelope.

Example 3.5. When H is finite of order prime to p, the category of smooth H-representations is semisimple (Maschke). Hence any τ is injective and any essential injection is an isomorphism, so τ is its own injective envelope. On the other hand, if H is a pro-p group, then $1_H \hookrightarrow \mathcal{C}^\infty(H, C) = \mathrm{Ind}_1^H C$ is an injective envelope (where 1_H is the subspace of constant functions), because any subrepresentation V contains a nonzero H-invariant element by Lemma 1.6 and $\mathcal{C}^\infty(H, C)^H = 1_H$.

Exercise 3.6. The following are nice exercises and will be used in the arguments below.

($*_1$) $(\mathrm{inj}_H \tau)|_{H'}$ is injective if $H' \leq H$ open;
($*_2$) $(\mathrm{inj}_H \tau)^N \cong \mathrm{inj}_{H/N}(\tau^N)$ for $N \lhd H$ a closed normal subgroup;[11]
($*_3$) $\mathrm{inj}_H \tau \cong \mathrm{inj}_H(\mathrm{soc}_H \tau)$ and $\mathrm{soc}_H(\tau) = \mathrm{soc}_H(\mathrm{inj}_H \tau)$;
($*_4$) $\mathrm{inj}_H(\tau_1 \oplus \tau_2) \cong \mathrm{inj}_H \tau_1 \oplus \mathrm{inj}_H \tau_2$;[12]
($*_5$) $\mathrm{inj}_H \tau$ is admissible if τ is finite-dimensional;[13]

[10] This last condition and the injectivity of r are not usually part of the definition, but they are convenient for us.

[11] Note that $\mathrm{Hom}_{H/N}(-, (\mathrm{inj}_H \tau)^N) \cong \mathrm{Hom}_H(-, \mathrm{inj}_H \tau)$ on H/N-representations. Incidentally, we implicitly assume that the isomorphism ($*_2$) is compatible with the maps from τ^N to these representations. Similarly in ($*_3$) and ($*_4$).

[12] To show $\tau_1 \oplus \tau_2 \hookrightarrow \mathrm{inj}_H \tau_1 \oplus \mathrm{inj}_H \tau_2$ is essential, it suffices to show that $X \oplus Y \hookrightarrow X \oplus Z$ is essential whenever $Y \hookrightarrow Z$ is essential, and that a composition of essential maps is essential.

[13] Use ($*_2$) to reduce to the case where H is finite (discrete), then notice that $C[H] \cong \mathrm{Ind}_1^H(C)$ is injective.

($*_6$) if τ is irreducible and V any finite-dimensional smooth representation of H, then $\dim_C \operatorname{Hom}_H(V, \operatorname{inj}_H \tau) = [V : \tau]$.

Lemma 3.7. *Suppose that H is a profinite group with open normal subgroup M_1 such that M_1 is pro-p and H/M_1 is prime-to-p. Then for any subgroup $M_1 \le M \le H$ we have $(\operatorname{inj}_H \tau)|_M \cong \operatorname{inj}_M \tau$.*

Proof. As $(\operatorname{inj}_H \tau)|_M$ is injective by ($*_1$), it suffices to check that $\tau \hookrightarrow (\operatorname{inj}_H \tau)|_M$ is essential. Suppose that $V \subseteq (\operatorname{inj}_H \tau)|_M$ is such that $\tau \cap V = 0$. Then $\tau \oplus V \hookrightarrow (\operatorname{inj}_H \tau)|_M$, so $\tau^{M_1} \oplus V^{M_1} \hookrightarrow (\operatorname{inj}_H \tau)^{M_1} \cong \operatorname{inj}_{H/M_1}(\tau^{M_1})$ by ($*_2$). As H/M_1 is prime-to-p, we have $\operatorname{inj}_{H/M_1}(\tau^{M_1}) = \tau^{M_1}$. It follows that $V^{M_1} = 0$, so $V = 0$ by Lemma 1.6. □

Let $\overline{G} := G/\varpi^{\mathbb{Z}}$ and for any subgroup $H \le G$ let $\overline{H} := H/(H \cap \varpi^{\mathbb{Z}}) \le \overline{G}$ denote its image in $\overline{G}$.[14] Note that $\overline{H} = H$ if H is compact, so e.g. $\overline{I} = I$.

An important observation is that $\overline{\mathcal{N}} = I \rtimes \langle \Pi \rangle$ (as $\overline{IZ} = I$ and Π has order 2 in $\overline{G}$). For a smooth character $\chi : I \to C^\times$ we let $\chi^s := \chi(\Pi(\cdot)\Pi^{-1})$ denote its conjugate by Π, so $\chi^{ss} = \chi$.

Lemma 3.8. *Let V be a finite-dimensional smooth representation of I/I_1. Then $V \cong \widetilde{V}|_I$ for some smooth $\overline{\mathcal{N}}$-representation $\widetilde{V}$ if and only if $\dim_C \operatorname{Hom}_I(\chi, V) = \dim_C \operatorname{Hom}_I(\chi^s, V)$ for all smooth characters $\chi : I \to C^\times$.*

Proof. It is easy to check "$\Rightarrow$", so let us check "$\Leftarrow$". Note that V is semisimple, so it suffices to check that $\chi \oplus \chi^s$ can be extended to $\overline{\mathcal{N}}$, and that χ itself can be extended to $\overline{\mathcal{N}}$ if $\chi = \chi^s$. In the first case note that $(\operatorname{Ind}_I^{\overline{\mathcal{N}}} \chi)|_I \cong \chi \oplus \chi^s$. In the second case, as $\overline{\mathcal{N}}$ is a semidirect product and $\chi = \chi^s$ we check by hand that there is a unique homomorphism $\widetilde{\chi} : \overline{\mathcal{N}} \to C^\times$ that extends χ and that sends Π to 1 (or -1) in $C^\times$. □

Lemma 3.9. *Suppose that V is any finite-dimensional smooth representation of K. Then for any smooth character $\chi : I \to C^\times$ we have $\dim_C \operatorname{Hom}_I(\chi, (\operatorname{inj}_K V)^{I_1}) = \dim_C \operatorname{Hom}_I(\chi^s, (\operatorname{inj}_K V)^{I_1})$.*

Proof. By ($*_3$) we may replace V by $\operatorname{soc}_K V$, so assume that V is semisimple. By ($*_4$) we may furthermore assume that V is irreducible. By Frobenius reciprocity and ($*_6$), $\dim_C \operatorname{Hom}_I(\chi, (\operatorname{inj}_K V)^{I_1}) = \dim_C \operatorname{Hom}_I(\chi, \operatorname{inj}_K V) = \dim_C \operatorname{Hom}_K(\operatorname{Ind}_I^K \chi, \operatorname{inj}_K V) = [\operatorname{Ind}_I^K \chi : V]$. We conclude as $\operatorname{Ind}_I^K \chi$ and $\operatorname{Ind}_I^K \chi^s$ have the same semisimplification.[15] □

Proposition 3.10. *Suppose (D_0, D_1, r) is a diagram. Then $\operatorname{inj}_K D_0$ can be equipped with an action of $\overline{\mathcal{N}}$ extending the given action of I such that moreover the composition $D_1 \hookrightarrow D_0 \hookrightarrow \operatorname{inj}_K D_0$ is $\overline{\mathcal{N}}$-equivariant.*

Proof. The K-representation $\operatorname{inj}_K D_0 = \operatorname{inj}_K(\operatorname{soc}_K D_0)$ (by ($*_3$)) is admissible by ($*_5$) and Exercise 2.8, as D_0 is admissible. In particular, $(\operatorname{inj}_K D_0)^{I_1}$ is finite-dimensional. On the other hand, $D_1 \hookrightarrow D_0 \hookrightarrow \operatorname{inj}_K D_0$, so $D_1^{I_1} \hookrightarrow (\operatorname{inj}_K D_0)^{I_1}$. As

[14] If the group G/Z seems friendlier, you can put $\overline{G} = G/Z$ and restrict attention to representations on which Z acts trivially.

[15] This follows from the corresponding claim for a character χ over a field of characteristic zero (cf. the scalar product formula in Charlotte's lectures or [Pra07, Thm. 2.8]) followed by reduction mod p (cf. §3.3).

I/I_1 is prime-to-p, the I/I_1-representation $(\mathrm{inj}_K D_0)^{I_1}$ is semisimple (Maschke!), so we can write

$$(\mathrm{inj}_K D_0)^{I_1} = D_1^{I_1} \oplus E \tag{3.1}$$

for some I/I_1-subrepresentation E. Note that $D_1 \cap E = 0$.

By assumption, the action of I on D_1 extends to a smooth representation of $\overline{\mathcal{N}}$, hence the same is true for $D_1^{I_1}$, as I_1 is normal in $\overline{\mathcal{N}}$. We now check that the same is true for E. By Lemma 3.9 and Lemma 3.8 we know for all $\chi : I \to C^\times$:

$$\dim_C \mathrm{Hom}_I(\chi, (\mathrm{inj}_K D_0)^{I_1}) = \dim_C \mathrm{Hom}_I(\chi^s, (\mathrm{inj}_K D_0)^{I_1}),$$
$$\dim_C \mathrm{Hom}_I(\chi, D_1^{I_1}) = \dim_C \mathrm{Hom}_I(\chi^s, D_1^{I_1}).$$

By subtraction,

$$\dim_C \mathrm{Hom}_I(\chi, E) = \dim_C \mathrm{Hom}_I(\chi^s, E),$$

so E extends to an $\overline{\mathcal{N}}$-representation by Lemma 3.8.

Observe that $(\mathrm{inj}_K D_0)|_I$ is an injective envelope of $D_1 \oplus E$: $(\mathrm{inj}_K D_0)|_I$ is injective by $(*_1)$ and $D_1 \oplus E \hookrightarrow (\mathrm{inj}_K D_0)|_I$ is essential, since $D_1 \oplus E$ contains $D_1^{I_1} \oplus E = (\mathrm{inj}_K D_0)^{I_1}$ (cf. Lemma 1.6). Thus, applying inj_I to (3.1) we deduce

$$(\mathrm{inj}_K D_0)|_I \cong \mathrm{inj}_I(D_1 \oplus E) \cong (\mathrm{inj}_{\overline{\mathcal{N}}}(D_1 \oplus E))|_I,$$

where the second isomorphism comes from Lemma 3.7 (using $M_1 = I_1$, $M = I$, $H = \overline{\mathcal{N}}$ and that $p > 2!$). Thus the $\overline{\mathcal{N}}$-action on $\mathrm{inj}_{\overline{\mathcal{N}}}(D_1 \oplus E)$ provides the desired $\overline{\mathcal{N}}$-action on $\mathrm{inj}_K D_0$, which agrees with the I-action on $\mathrm{inj}_K D_0$ and the $\overline{\mathcal{N}}$-action on D_1 by construction. □

Proof of Theorem 3.4. We let $\pi' := \mathrm{inj}_K D_0$, equipped with compatible actions of K and $\overline{\mathcal{N}}$ by Proposition 3.10, i.e. compatible actions of $KZ = K\varpi^{\mathbb{Z}}$ and $\mathcal{N}$ such that ϖ acts trivially. By Proposition 3.1, π' is a G-representation, and by construction, $D_0 \hookrightarrow \mathrm{inj}_K D_0 = \pi'|_K$ is K-equivariant and the composition $D_1 \hookrightarrow D_0 \hookrightarrow \pi'$ is $\mathcal{N}$-equivariant. The action of G on π' is automatically smooth, since the K-action on π' is smooth and $K \le G$ is open.

Let $\pi := \langle G \cdot D_0 \rangle \subseteq \pi'$ be the G-subrepresentation generated by $D_0 \subseteq \pi'$. Then $\mathrm{soc}_K D_0 \subseteq \mathrm{soc}_K \pi \subseteq \mathrm{soc}_K \pi' = \mathrm{soc}_K D_0$ (see $(*_3)$). Hence $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0$, and as D_0 is admissible, the admissibility of π follows from Exercise 2.8. □

Corollary 3.11. *Suppose that (D_0, D_1, r) is an irreducible diagram, i.e. does not contain any nonzero proper subdiagram. Then any admissible smooth G-representation π satisfying the properties in Theorem 3.4 is irreducible.*

Proof. Suppose that $\sigma \subseteq \pi$ is a nonzero G-subrepresentation. Then $0 \neq \mathrm{soc}_K \sigma \subseteq \mathrm{soc}_K \pi = \mathrm{soc}_K D_0$, hence $\sigma \cap D_0 \neq 0$, so $(\sigma \cap D_0, \sigma \cap D_1, r)$ is a nonzero subdiagram of (D_0, D_1, r). By the irreducibility assumption, $D_0 \subseteq \sigma$, so $\pi = \langle G \cdot D_0 \rangle \subseteq \sigma$ and hence $\sigma = \pi$. □

Here is a converse result. We say that a smooth G-representation π *arises from the diagram* (D_0, D_1, r) if π is one of the representations produced in the proof of Theorem 3.4, i.e. there exists an action of $\overline{\mathcal{N}}$ on $\mathrm{inj}_K D_0$ extending the given action of I such that the composition $D_1 \hookrightarrow D_0 \hookrightarrow \mathrm{inj}_K D_0$ is $\overline{\mathcal{N}}$-equivariant and $\pi = \langle G \cdot D_0 \rangle \subseteq \mathrm{inj}_K D_0$ and $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0$.

Corollary 3.12. *Suppose that (D_0, D_1, r) is a diagram and π is any (admissible) smooth G-representation such that $(D_0, D_1, r) \hookrightarrow (\pi|_{KZ}, \pi|_{\mathcal{N}}, \mathrm{incl})$, $\pi = \langle G \cdot D_0 \rangle$, and $\mathrm{soc}_K\, \pi = \mathrm{soc}_K\, D_0$. Then π arises from the diagram (D_0, D_1, r).*

Proof. Let $(\widetilde{D}_0, \widetilde{D}_1, \mathrm{incl})$ denote the diagram $(\pi|_{KZ}, \pi|_{\mathcal{N}}, \mathrm{incl})$. Applying the proof of Theorem 3.4 to the diagram $(\widetilde{D}_0, \widetilde{D}_1, \mathrm{incl})$, we obtain an action of $\overline{\mathcal{N}}$ on $\mathrm{inj}_K\, \widetilde{D}_0$ extending the given action of I such that the composition $\widetilde{D}_1 \hookrightarrow \widetilde{D}_0 \hookrightarrow \mathrm{inj}_K\, \widetilde{D}_0$ is $\overline{\mathcal{N}}$-equivariant. We have inclusions of diagrams $(D_0, D_1, r) \hookrightarrow (\widetilde{D}_0, \widetilde{D}_1, \mathrm{incl}) \hookrightarrow (\mathrm{inj}_K\, \widetilde{D}_0, \mathrm{inj}_K\, \widetilde{D}_0|_{\mathcal{N}}, \mathrm{incl})$. It remains to observe that $\mathrm{inj}_K\, \widetilde{D}_0 = \mathrm{inj}_K\, D_0$ by $(*_3)$, as $\mathrm{soc}_K\, \widetilde{D}_0 = \mathrm{soc}_K\, D_0$, and that $\langle G \cdot D_0 \rangle = \widetilde{D}_0 \subseteq \mathrm{inj}_K\, \widetilde{D}_0$. □

3.2. **Injective envelopes of Serre weights and the extension graph.** Let $\Gamma := K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_q)$.

For the work of Breuil–Paškūnas, it is important to understand the finite-dimensional representation $\mathrm{inj}_\Gamma\, \sigma$ for any Serre weight σ. In fact, only the following subrepresentation is needed:

Lemma 3.13. *There is a largest subrepresentation V_σ of $\mathrm{inj}_\Gamma\, \sigma$ such that $[V_\sigma : \sigma] = 1$.*

Proof. Note that any nonzero subrepresentation contains $\mathrm{soc}_\Gamma(\mathrm{inj}_\Gamma\, \sigma) = \sigma$. It suffices to show that if V_1, V_2 are nonzero subrepresentations with $[V_i : \sigma] = 1$, then $[V_1 + V_2 : \sigma] = 1$. This follows by counting multiplicities of σ in the short exact sequence $0 \to V_1 \cap V_2 \to V_1 \oplus V_2 \to V_1 + V_2 \to 0$. □

Exercise 3.14. Show that $\mathrm{JH}(V_\sigma) = \mathrm{JH}(\mathrm{inj}_\Gamma\, \sigma)$. (Hint: show by induction on $[V : \sigma]$ that any subrepresentation V of $\mathrm{inj}_\Gamma\, \sigma$ admits a filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$ such that each graded piece V_i/V_{i-1} injects into V_σ. This argument works for any finite group Γ.)

It is possible to explicitly describe V_σ as follows (this is much harder!). Recall from Example 2.2 that $(r_0, \ldots, r_{f-1})$ denotes the Serre weight $F(\underline{r}, \underline{0})$. Recall from §A.4 that $V(\lambda)$ denotes a Weyl module and $[i]$ the i-th Frobenius twist.

Proposition 3.15 ([BP12, §3])**.** *Suppose $\sigma = (r_0, \ldots, r_{f-1}) \notin \{(0, \ldots, 0), (p-1, \ldots, p-1)\}$. Then*

$$V_\sigma \cong \Big(V(2p-2-r_0, 0) \otimes \cdots \otimes V(2p-2-r_{f-1}, 0)^{[f-1]}\Big)|_\Gamma \otimes \det^{\sum_j r_j p^j}.$$

Remark 3.16.

(i) If $\sigma = (p-1, \ldots, p-1)$, then in fact $\sigma = V_\sigma = \mathrm{inj}_\Gamma\, \sigma$ (of dimension q). If $\sigma = (0, \ldots, 0)$, then V_σ is a direct summand of the above representation.

(ii) As we can see, V_σ is the restriction of an algebraic representation of GL_2. In fact, the same is true for $\mathrm{inj}_\Gamma\, \sigma$ under the conditions of the proposition.

It is not hard to then work out the irreducible constituents of V_σ in case $1 \le r_j \le p-3$ for all j. For this, use the following facts about algebraic group representations (all of which can be proven by hand; see §A.4 for the notation):

(i) $0 \to L(p-1, p-1-r) \to V(2p-2-r, 0) \to L(2p-2-r, 0) \to 0$ if $0 \le r \le p-2$;

(ii) $L(2p-2-r, 0) \cong L(p-2-r, 0) \otimes L(1, 0)^{[1]}$ if $0 \le r \le p-2$, cf. Fact A.16;

(iii) $L(a, 0) \otimes L(1, 0) \cong L(a+1, 0) \oplus L(a, 1)$ if $1 \le a \le p-2$.

In particular, one deduces:

Proposition 3.17 ([BP12, §4]). *Suppose $\sigma = (r_0, \dots, r_{f-1})$ with $1 \le r_j \le p-3$ for all j. Then V_σ is multiplicity free and of length 3^f.*

Example 3.18. If $f = 1$, then V_σ has 3 constituents $\sigma = (r_0)$, $(p-1-r_0) \otimes \det^{r_0}$, $(p-3-r_0) \otimes \det^{r_0+1}$.

If $f = 2$, then V_σ has 3^2 constituents $\sigma = (r_0, r_1)$, $(p-2-r_0, r_1 \pm 1)$, $(r_1 \pm 1, p-2-r_0)$, $(p-2-r_0 \pm 1, p-2-r_1 \pm 1)$, where we have systematically omitted determinant twists.[16]

The general formulas are combinatorially complicated [BP12, §4], but the structure of V_σ is easy to visually using the "extension graph", as we now explain. First one uses Proposition 3.15 (and a bit more if $\underline{r} = \underline{0}$ or $\underline{p-1}$) to determine $\mathrm{Ext}^1_\Gamma(\sigma, \tau)$ for all Serre weights σ, τ. (Note that if a nonsplit Γ-extension $0 \to \sigma \to E \to \tau \to 0$ exists with $\sigma \not\cong \tau$, then E embeds into $\mathrm{inj}_\Gamma \sigma$ and hence into V_σ. Some extra work is needed to rule out the case $\sigma \cong \tau$.)

Proposition 3.19 ([BP12, Cor. 5.6]). *Suppose $\sigma = (r_0, \dots, r_{f-1})$ and τ are Serre weights. Then $\mathrm{Ext}^1_\Gamma(\tau, \sigma) \ne 0$ if and only if there exists $0 \le i \le f-1$ such that $\tau \cong (r_0, \dots, r_{i-2}, p-2-r_{i-1}, r_i \pm 1, \dots, r_{f-1}) \otimes \det^{p^{i-1}(r_{i-1}+1) - p^i(\frac{1 \pm 1}{2})}$.[17] In this case, $\dim_C \mathrm{Ext}^1_\Gamma(\tau, \sigma) = 1$. We say that the nonsplit extension is of* type $(i, \pm)$.

Remark 3.20. The statement $\dim_C \mathrm{Ext}^1_\Gamma(\tau, \sigma) = 1$ is equivalent to saying that there exists a unique Γ-representation E up to isomorphism that is a nonsplit extension of the form $0 \to \sigma \to E \to \tau \to 0$.

Remark 3.21. Note that a nonsplit extension $0 \to \sigma \to E \to \tau \to 0$ is of type $(i, +)$ iff the dual nonsplit extension $0 \to \tau^\vee \to E^\vee \to \sigma^\vee \to 0$ is of type $(i, -)$.

Exercise 3.22. If $0 \to \sigma \to E \to \tau \to 0$ is a nonsplit extension, show that $\omega_\sigma = \omega_\tau$ (equality of central characters). On the other hand, show that $\sigma^\vee \cong \sigma \otimes (\omega_\sigma^{-1} \circ \det)$. Deduce that the we get a nonsplit extension $0 \to \tau \to E^\vee \otimes (\omega_\sigma \circ \det) \to \sigma \to 0$, which explains why there is symmetry between σ and τ in Proposition 3.19.

Remark 3.23. From Proposition 3.19 it follows that if $\sigma = (p-1, \dots, p-1)$ up to twist, then σ is injective and projective as Γ-representation.

Following [LMS22, §2], we define the *extension graph* to be the graph with vertices all Serre weights and edges all pairs (σ, τ) such that $\mathrm{Ext}^1_\Gamma(\tau, \sigma) (\cong \mathrm{Ext}^1_\Gamma(\sigma, \tau))$ is nonzero. By Proposition 3.19 the extension graph near a generic σ (say $0 < r_j < p-1$ for all j) locally looks like $\mathbb{Z}^f$.[18] Let us draw a picture for $f = 2$, omitting det twists:

[16]Here is how to recover the twists: all these constituents necessarily have the same central character as σ, and each Serre weight has precisely two twists that have a given central character, as $p > 2$. We take the twist that does not contain any $\frac{p-1}{2}$. For example, the Serre weight $(p-3-r_0, p-1-r_1)$ on this list should be $(p-3-r_0, p-1-r_1) \otimes \det^{(r_0+1)+pr_1}$, noting that σ has central character $(-)^{r_0+pr_1}$.

[17]The indices are taken mod f, so e.g. $-1 = f-1$. If $f = 1$, the formula simplifies to $(p-2-r_0 \pm 1) \otimes \det^{r_0 + \frac{1 \mp 1}{2}}$.

[18]One can give an explicit formula as follows, ignoring determinant twists: fix any Serre weight $(r_0, \dots, r_{f-1})$ to be the origin and then send $(a_0, \dots, a_{f-1}) \in \mathbb{Z}^f$ to the Serre weight $(s_0, \dots, s_{f-1})$, where $s_j = r_j + a_j$ if a_{j+1} is even, $s_j = p-2-(r_j + a_j)$ otherwise. Of course it is only defined on the subset of $\mathbb{Z}^f$ such that the image lands in the range $[0, p-1]^f$.

$$\begin{tikzcd}[column sep=small]
 & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \\
\dots \arrow[r, dash] & (p-1-r_0, p-3-r_1) \arrow[r, dash] \arrow[d, dash] & (p-2-r_0, r_1+1) \arrow[r, dash] \arrow[d, dash] & (p-3-r_0, p-3-r_1) \arrow[r, dash] \arrow[d, dash] & (p-4-r_0, r_1+1) \arrow[r, dash] \arrow[d, dash] & \dots \\
\dots \arrow[r, dash] & (r_0-1, p-2-r_1) \arrow[r, dash] \arrow[d, dash] & (r_0, r_1) \arrow[r, dash] \arrow[d, dash] & (r_0+1, p-2-r_1) \arrow[r, dash] \arrow[d, dash] & (r_0+2, r_1) \arrow[r, dash] \arrow[d, dash] & \dots \\
\dots \arrow[r, dash] & (p-1-r_0, p-1-r_1) \arrow[r, dash] \arrow[d, dash] & (p-2-r_0, r_1-1) \arrow[r, dash] \arrow[d, dash] & (p-3-r_0, p-1-r_1) \arrow[r, dash] \arrow[d, dash] & (p-4-r_0, r_1-1) \arrow[r, dash] \arrow[d, dash] & \dots \\
\dots \arrow[r, dash] & (r_0-1, p-r_1) \arrow[r, dash] \arrow[d, dash] & (r_0, r_1-2) \arrow[r, dash] \arrow[d, dash] & (r_0+1, p-r_1) \arrow[r, dash] \arrow[d, dash] & (r_0+2, r_1-2) \arrow[r, dash] \arrow[d, dash] & \dots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Exercise 3.24. Suppose $f = 2$. Draw all connected components of the extension graph for $p = 3$ and $p = 5$. (Up to twist you should only find 3 components: the components of the Serre weights $(0,0)$, $(0,1)$, and $(p-1,p-1)$. In general, the components are indexed by central characters, at least once the trivial components are removed.) You may need to put in the determinant twists, as $\sigma \mapsto \sigma \otimes \det^{(p^f-1)/2}$ induces a fixed-point free automorphism of order 2 of the extension graph.

Note that an edge i is parallel to the i-th coordinate axis iff it corresponds to an extension of type $(i, \pm)$ in Proposition 3.19.

Using Proposition 3.15, it is not hard to see that the constituents of V_σ (if $0 < r_j < p-1$ for all j) are then given by all vertices in the extension graph that differ from σ by at most 1 in each coordinate direction, indicated in red above.

More precisely, as V_σ is multiplicity free, with some more work it is possible to determine its submodule structure. Recall from §A.3 that the submodule structure of a multiplicity-free representation is given by a partial order on the set $\mathrm{JH}(V_\sigma)$ of its irreducible constituents.

Proposition 3.25 ([BP12, Thm. 4.7]). *The partial order on* $\mathrm{JH}(V_\sigma)$ *is given as follows:* $\tau \leq \tau'$ *in* $\mathrm{JH}(V_\sigma)$ *if and only if there is a shortest path from* τ' *to* σ *in the extension graph that passes through* τ*.*

Recall that we visualize the partial order by arrows pointing in a descending direction. For $f = 2$, the partial order of V_σ is given as follows:

$$\begin{tikzcd}
(p-1-r_0, p-3-r_1) \arrow[r] \arrow[d] & (p-2-r_0, r_1+1) \arrow[d] & (p-3-r_0, p-3-r_1) \arrow[l] \arrow[d] \\
(r_0-1, p-2-r_1) \arrow[r] & (r_0, r_1) & (r_0+1, p-2-r_1) \arrow[l] \\
(p-1-r_0, p-1-r_1) \arrow[r] \arrow[u] & (p-2-r_0, r_1-1) \arrow[u] & (p-3-r_0, p-1-r_1) \arrow[l] \arrow[u]
\end{tikzcd} \tag{3.2}$$

Let us now show that Proposition 3.25 is quite powerful.

Example 3.26. From (3.2) it follows for example that the unique subrepresentation of V_σ with cosocle $(p-1-r_0, p-3-r_1)$ is given by:

$$\begin{tikzcd}
& (p-1-r_0,p-3-r_1) \arrow[dl] \arrow[dr] & \\
(r_0-1,p-2-r_1) \arrow[dr] & & (p-2-r_0,r_1+1) \arrow[dl] \\
& (r_0,r_1) &
\end{tikzcd}$$

Equivalently, by Proposition 3.13, it is the unique Γ-representation V with socle σ, cosocle $(p-1-r_0, p-3-r_1)$ and $[V:\sigma]=1$. (First embed V into $\mathrm{inj}_\Gamma \sigma$, using injectivity, then notice it is contained in V_σ as $[V:\sigma]=1$.)

Example 3.27. We determine the submodule structure of a principal series $P := \mathrm{Ind}_I^K \chi^s$, which is a Γ-representation. Let's say $\chi = \sigma^{I_1}$, where $\sigma = (r_0, r_1)$ and assume that $1 \le r_j \le p-2$ for all j. Then by Frobenius reciprocity we see that $\mathrm{soc}_\Gamma P \cong \sigma$ and $\mathrm{cosoc}_\Gamma P \cong \sigma^{[s]}$, where $\sigma^{[s]} := (p-1-r_0, p-1-r_1)$. In particular, P injects into $\mathrm{inj}_\Gamma \sigma$. If $[P:\sigma]=1$, then P injects into V_σ and hence by (3.2) above, P has submodule structure given by:

$$\begin{tikzcd}
& (p-1-r_0,p-1-r_1) \arrow[dl] \arrow[dr] & \\
(r_0-1,p-2-r_1) \arrow[dr] & & (p-2-r_0,r_1-1) \arrow[dl] \\
& (r_0,r_1) &
\end{tikzcd}$$

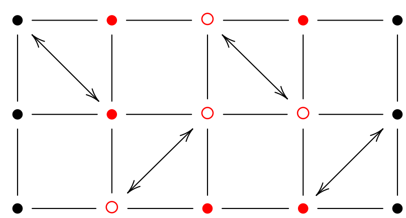
Now let's show that in fact $[P:\sigma]=1$! If not, then P/σ has a quotient with socle σ (take the image of any nonzero map $P/\sigma \to \mathrm{inj}_\Gamma \sigma$, using $(*_6)$), so by induction P has a quotient P' with socle σ and $[P':\sigma]=1$. As $P \twoheadrightarrow P'$ we also have $\mathrm{cosoc}_\Gamma P \twoheadrightarrow \mathrm{cosoc}_\Gamma P'$, so $\mathrm{cosoc}_\Gamma P' \cong \sigma^{[s]}$. By the previous argument we know P' and in particular can compute $\dim_C P' = (p-r_0)(p-r_1) + r_0(p-1-r_1) + (p-1-r_0)r_1 + (r_0+1)(r_1+1) = p^2+1$, which equals $\dim_C P = (K:I)$, so $P'=P$!

Remark 3.28. To make this more memorable, starting from the Γ-socle $\sigma = (r_0, r_1)$ we know that the cosocle is $\sigma^{[s]} = (p-1-r_0, p-1-r_1)$. Now the two intermediate constituents are the only Serre weights that admit nonsplit extensions with σ and $\sigma^{[s]}$, by Proposition 3.19. Alternatively, up to forgetting about directions of the arrows, it is the unique square in the extension graph that has σ and $\sigma^{[s]}$ as vertices.

Remark 3.29. Note also that σ^{I_1} and $(\sigma^{[s]})^{I_1}$ are interchanged by $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$: as in Exercise 2.4, $\sigma^{I_1} = \sigma^{U \cap K}$ is spanned by $X^{r_0+pr_1}$, so given by the character $(-)^{r_0+pr_1} \otimes 1$. Similarly, $(\sigma^{[s]})^{I_1} = 1 \otimes (-)^{r_0+pr_1}$, remembering that more precisely, $\sigma^{[s]} = (p-1-r_0, p-1-r_1) \otimes \det^{r_0+pr_1}$. In the extension graph picture we indicate the action of $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ on I_1-invariants of Serre weights by diagonal arrows (see the next few figures).

Exercise 3.30. Describe the submodule structure of a principal series $\mathrm{Ind}_I^K \chi^s$ in the remaining cases for $f=2$. In particular, show that any principal series always multiplicity free of length 2 or 3 or 4. (The multiplicity freeness is in fact true for all f.)

Example 3.31. Let us determine the I-representation $V_\sigma^{I_1}$ when $f=2$ and $1 \le r_j \le p-2$ for all j. This is an I/I_1-representation, an abelian group of order prime to p, so a direct sum of 1-dimensional representations. As $(-)^{I_1}$ is a left exact functor and $\chi_\tau = \tau^{I_1}$ is 1-dimensional for each Serre weight τ we see that $V_\sigma^{I_1}$ has dimension at most 9, and it suffices to determine for each constituent τ of V_σ whether χ_τ contributes to $V_\sigma^{I_1}$. Now $\chi_\tau \hookrightarrow V_\sigma^{I_1}$ if and only if there exists a nonzero K-equivariant map $\operatorname{Ind}_I^K \chi_\tau \to V_\sigma$ by Frobenius reciprocity. Keeping in mind Remark 3.28 we just need to locate the square bounded by τ and $\tau^{[s]}$ in the extension graph; if this square does not contain σ as a vertex, then χ_τ does not contribute to $V_\sigma^{I_1}$ (if the image of $\operatorname{Ind}_I^K \chi_\tau \to V_\sigma$ is nonzero, it has socle σ). In fact, otherwise it will: in this case, σ occurs in $\operatorname{Ind}_I^K \chi_\tau$, so $\operatorname{Ind}_I^K \chi_\tau$ has a unique quotient with socle σ (and cosocle τ), which is multiplicity free so embeds into V_σ, hence we get a nonzero map $\operatorname{Ind}_I^K \chi_\tau \to V_\sigma$. Concretely, the extension graph picture is as follows, where diagonal arrows denote the action of $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ on I_1-invariants, red vertices denote the constituents of V_σ, and the 4 red circles denote those τ such that χ_τ contributes to $V_\sigma^{I_1}$:

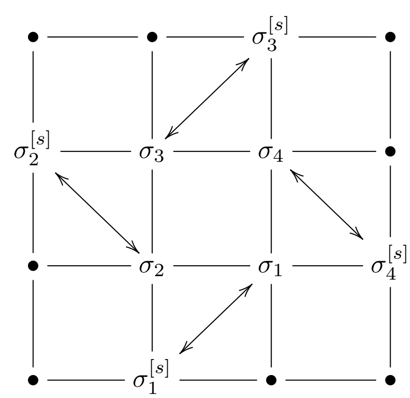


Exercise 3.32. Deduce from Example 3.31 that a nonsplit extension $0 \to \sigma \to E \to \tau \to 0$ of Γ-representations satisfies

$$E^{I_1} \cong \begin{cases} \chi_\sigma \oplus \chi_\tau & \text{if } E \text{ is of type } (j,+) \text{ for some } j; \\ \chi_\sigma & \text{if } E \text{ is of type } (j,-) \text{ for some } j. \end{cases}$$

We can now construct some interesting families of irreducible supersingular representations!

Example 3.33 ([Bre07, Ex. 5.8]). Suppose that $f = 2$. Let $\sigma_1 = (r_0, r_1)$, $\sigma_2 = (r_0 - 1, p - 2 - r_1)$, $\sigma_3 = (p - 1 - r_0, p - 3 - r_1)$, $\sigma_4 = (p - 2 - r_0, r_1 + 1)$ (ignoring twists), where $1 \le r_0 \le p - 2$, $0 \le r_1 \le p - 3$. In the extension graph the picture is as follows:



Let $D_0 := E_{14} \oplus E_{21} \oplus E_{32} \oplus E_{43}$, where $E_{i,i-1}$ denotes the unique Γ-representation (up to isomorphism) that is a nonsplit extension $0 \to \sigma_i \to E_{i,i-1} \to \sigma_{i-1}^{[s]} \to 0$

(with indices taken modulo 4). We make D_0 into a KZ-representation by letting $p \in Z$ act trivially. By Exercise 3.32, $D_1 := D_0^{I_1}$ is isomorphic to $\bigoplus_{i=1}^4 (\chi_i \oplus \chi_i^s)$, where $\chi_i := \chi_{\sigma_i}$ for short. Let v_i (resp. v_i^s) be a C-basis of the χ_i- (resp. χ_i^s-) eigenspace in D_1. We extend the IZ-action on D_1 into an $\mathcal{N}$-action by declaring that $\Pi(v_i) = v_i^s$ for $1 \le i \le 3$ and $\Pi v_4 = \lambda v_4^s$ for some $\lambda \in C^\times$ (which determines the Π-action as $\Pi^2 = 1$ on D_1), and denote this $\mathcal{N}$-representation by D_1^λ.

Exercise 3.34. Show that the diagram $D(\lambda) := (D_0, D_1^\lambda, \mathrm{incl})$ in Example 3.33 is irreducible for any $\lambda \in C^\times$. Show that $D(\lambda) \cong D(\mu)$ are isomorphic diagrams if and only if $\lambda = \mu$. (Hint: without loss of generality, the isomorphism takes v_1 to v_1, hence v_1^s to v_1^s. Use that $\langle \Gamma \cdot v_1^s \rangle = E_{21}$ contains v_2 to show that it takes v_2 to v_2, ...)

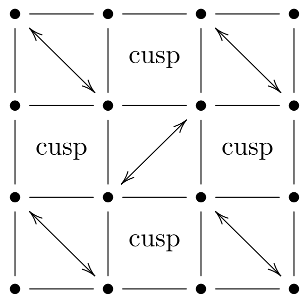
Continuing Example 3.33 we can now construct an infinite family of irreducible admissible supersingular G-representations that have the same K-socle. Let $\pi(\lambda)$ be any irreducible admissible G-representation obtained from Corollary 3.11 applied to the irreducible diagram $D(\lambda)$ ($\lambda \in C^\times$). In particular, $\mathrm{soc}_K \pi(\lambda) = \mathrm{soc}_K D_0$ has length 4, so $\pi(\lambda)$ is supersingular by Corollary 2.38. Suppose that $\theta : \pi(\lambda) \xrightarrow{\sim} \pi(\mu)$ is any G-isomorphism. Then θ takes $\mathrm{soc}_K D_0 = \mathrm{soc}_K \pi(\lambda)$ to $\mathrm{soc}_K D_0 = \mathrm{soc}_K \pi(\mu)$. As $D_0 = \langle K \cdot \Pi(\mathrm{soc}_K D_0)^{I_1} \rangle$ we deduce $\theta(D_0) = D_0$ and hence $\theta(D_1^\lambda) = D_1^\mu$. In other words, there is a diagram isomorphism $D(\lambda) \cong D(\mu)$, hence $\lambda = \mu$ by Exercise 3.34.

3.3. **An aside on reduction modulo p.** The extension graph is also very useful for visualizing the reduction modulo p of irreducible Γ-representations over a (large enough) field L of characteristic 0. Recall from Charlotte's lectures that the irreducible Γ-representations over L come in four kinds, arising from two families of Deligne–Lusztig representations, where $B_\Gamma := \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \le \Gamma$:

(i) irreducible principal series representations $\mathrm{Ind}_{B_\Gamma}^\Gamma(\chi_1 \otimes \chi_2) \cong \mathrm{Ind}_{B_\Gamma}^\Gamma(\chi_2 \otimes \chi_1)$ (of dimension $q+1$), where $\chi_1 \ne \chi_2$;
(ii) cuspidal representations (of dimension $q-1$);
(iii) 1-dimensional representations $\chi \circ \det$;
(iv) Steinberg representations (of dimension q).

Let L denote the fraction field of a discrete valuation ring $\mathcal{O}_L$ with residue field C, for example $\mathcal{O}_L = W(C)$ (Witt vectors of C). If V is any irreducible Γ-representation over L, its reduction modulo p (cf. §A.2) is a Γ-representation over C, well-defined up to semisimplification.

In case (i), the reduction is $\mathrm{Ind}_{B_\Gamma}^\Gamma(\overline{\chi}_1 \otimes \overline{\chi}_2)$, up to semisimplification, where $\overline{\chi}_i : \Gamma \to C^\times$ is the reduction of $\chi_i : \Gamma \to \mathcal{O}_L^\times$. We already know that in this case the Jordan–Hölder factors are given by those squares in the extension graph that are decorated with a diagonal arrow (Example 3.30 and the following remarks there). The new fact is that the reduction of the cuspidal representations is described by the remaining squares (and the reduction is again multiplicity free):



Remark 3.35. In case (iv) it turns out that the reduction is irreducible (giving a trivial component in the extension graph). On can see (iii)+(iv) as a degenerate case of (i) in the picture.

3.4. **Breuil–Paškūnas diagrams.** By mod-p Langlands philosophy §2.8 we would hope to associate a (family of?) mod-p representations $\pi(\overline{\rho})$ of admissible G-representations to a given mod-p Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(C)$. *For the remainder of section 3 we assume that $F/\mathbb{Q}_p$ is unramified with uniformizer $\varpi := p$.*

Using diagrams we can construct admissible G-representations having a given K-socle. Now the K-socle of $\pi(\overline{\rho})$ is predicted by the global theory, namely one expects that

$$\mathrm{soc}_K \, \pi(\overline{\rho}) = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma,$$

where $W(\overline{\rho})$ is a certain finite set arising in generalizations of Serre's modularity conjecture (§4.6). For now it will suffice to say (assuming $\overline{\rho}$ is suitably generic) that $W(\overline{\rho})$ is always a hypercube of dimension $\in [0, f]$ in the extension graph, and more precisely:

(i) if $\overline{\rho}$ is irreducible, then $W(\overline{\rho})$ consists of a hypercube that arises as the reduction of a principal series (resp. cuspidal representation) if f is odd (resp. even);[19]
(ii) if $\overline{\rho}$ is split reducible, then $W(\overline{\rho})$ consists of a hypercube that arises as the reduction of a principal series (resp. cuspidal representation) if f is even (resp. odd);
(iii) if $\overline{\rho}$ is nonsplit reducible, then $W(\overline{\rho})$ consists of a proper sub-hypercube of $W(\overline{\rho}^{\mathrm{ss}})$ in (ii).

In particular, $|W(\overline{\rho})| = 2^f$ if $\overline{\rho}$ is semisimple and $|W(\overline{\rho})| = 2^d$ for some $0 \le d < f$ otherwise. (All d can arise, and d depends in a subtle way on the extension class in $\overline{\rho}$.)

An immediate problem is then how to extend $\bigoplus_{\sigma \in W(\overline{\rho})} \sigma$ to a KZ-representation D_0 such that the IZ-action on $D_0^{I_1}$ extends to $\mathcal{N}$. Already in the rather simple Example 3.33 (which corresponds to a case where $\overline{\rho}$ is irreducible), we see that the four Serre weights $\sigma_i^{[s]}$ needed to be "added" to $\mathrm{soc}_K \, \pi = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma$ in order to produce a diagram.

[19]See §3.3.

Breuil–Paškūnas solve this problem as follows: given $\overline{\rho}$ define $D_0(\overline{\rho})$ to be the largest subrepresentation of $\operatorname{inj}_\Gamma(\bigoplus_{\sigma\in W(\overline{\rho})}\sigma)$ containing $\bigoplus_{\sigma\in W(\overline{\rho})}\sigma$ such that $[D_0(\overline{\rho}):\sigma]=1$ for all $\sigma\in W(\overline{\rho})$. This exists by the same reasoning as in Lemma 3.13.

Lemma 3.36. *We have the following properties:*

(i) *We have $D_0(\overline{\rho})\cong\bigoplus_{\sigma\in W(\overline{\rho})}D_{0,\sigma}(\overline{\rho})$, where $D_{0,\sigma}(\overline{\rho})$ is the largest subrepresentation of $\operatorname{inj}_\Gamma\sigma$ such that $[D_{0,\sigma}(\overline{\rho}):\tau]=\delta_{\sigma\tau}$ for $\tau\in W(\overline{\rho})$.*[20]

(ii) *The Γ-representation $D_0(\overline{\rho})$ is multiplicity free and*

$$\mathrm{JH}(D_0(\overline{\rho}))=\mathrm{JH}(\bigoplus_{\sigma\in W(\overline{\rho})}V_\sigma).$$

(iii) *The I-representation $D_0(\overline{\rho})^{I_1}$ is multiplicity free and if $\chi:I\to C^\times$ occurs in it, then so does χ^s.*

Proof. (i) Again $D_{0,\sigma}(\overline{\rho})$ exists by the argument in Lemma 3.13. Then by definition, $\bigoplus_{\sigma\in W(\overline{\rho})}\sigma\subseteq\bigoplus_{\sigma\in W(\overline{\rho})}D_{0,\sigma}(\overline{\rho})\subseteq D_0(\overline{\rho})$. By $(*_4)$ we have $D_0(\overline{\rho})\subseteq\bigoplus_{\sigma\in W(\overline{\rho})}\operatorname{inj}_\Gamma\sigma$, so

$$D_0(\overline{\rho})/\bigoplus_{\sigma\in W(\overline{\rho})}D_{0,\sigma}(\overline{\rho})\hookrightarrow\bigoplus_{\sigma\in W(\overline{\rho})}(\operatorname{inj}_\Gamma\sigma)/D_{0,\sigma}(\overline{\rho}).$$

If the left-hand side is nonzero, then it contains an irreducible subrepresentation τ. Hence τ injects into $(\operatorname{inj}_\Gamma\sigma)/D_{0,\sigma}(\overline{\rho})$ for some $\sigma\in W(\overline{\rho})$, so $\tau\in W(\overline{\rho})$ by maximality of $D_{0,\sigma}(\overline{\rho})$, but this contradicts that $[D_0(\overline{\rho}):\tau]=1$!

(ii) First note that by definition $D_{0,\sigma}(\overline{\rho})$ embeds into V_σ, so $D_{0,\sigma}(\overline{\rho})$ is multiplicity free by Proposition 3.17. Now suppose that τ is any Serre weight occurring in $\bigoplus_{\sigma\in W(\overline{\rho})}V_\sigma$. As $W(\overline{\rho})$ is a hypercube (of perhaps lower dimension), there is a unique $\sigma\in W(\overline{\rho})$ that is closest to τ.[21] For the same reason, for any $\sigma'\in W(\overline{\rho})$ there exists a geodesic between σ' and τ that passes through σ, and no geodesic between σ and τ passes through any other element of $W(\overline{\rho})$. Hence τ occurs in $D_{0,\sigma'}(\overline{\rho})$ if and only if $\sigma'=\sigma$ by Proposition 3.25.

(iii) The multiplicity freeness of $D_0(\overline{\rho})^{I_1}$ follows from (ii). If the character χ injects into the semisimple I-representation $D_0(\overline{\rho})^{I_1}$, we get by Frobenius reciprocity a nonzero map $\operatorname{Ind}_I^K\chi\to D_0(\overline{\rho})\subseteq\operatorname{inj}_\Gamma(\bigoplus_{\sigma\in W(\overline{\rho})}\sigma)$ and hence a nonzero map $\operatorname{Ind}_I^K\chi^s\to\operatorname{inj}_\Gamma(\bigoplus_{\sigma\in W(\overline{\rho})}\sigma)$ (see the proof of Lemma 3.9). In particular, $\mathrm{JH}(\operatorname{Ind}_I^K\chi^s)\cap W(\overline{\rho})\neq\varnothing$ and hence $\operatorname{Ind}_I^K\chi^s$ has to admit a quotient Q such that precisely one of its irreducible constituents, say σ', is in $W(\overline{\rho})$. (Keep quotienting out irreducible subrepresentations until that is the case.) Then the unique up to scalar nonzero map $\operatorname{Ind}_I^K\chi^s\to\operatorname{inj}_\Gamma\sigma'$ has to have image contained in $D_{0,\sigma'}(\overline{\rho})$, i.e. χ^s contributes to $D_{0,\sigma'}(\overline{\rho})^{I_1}$. □

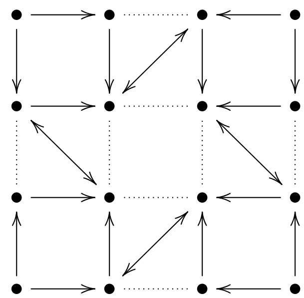
Let now $p\in Z$ act trivially[22] on $D_0(\overline{\rho})$, and by (iii) we can (non-uniquely) extend the action of IZ on $D_1(\overline{\rho}):=D_0(\overline{\rho})^{I_1}$ to an action of $\mathcal{N}$. A *Breuil–Paškūnas diagram* is any diagram $(D_0(\overline{\rho}),D_1(\overline{\rho}),\mathrm{incl})$ obtained in this way.

[20]Here, $\delta_{xy}=1$ if $x=y$, $\delta_{xy}=0$ otherwise.

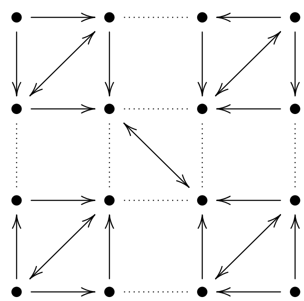
[21]We take as distance between vertices the length of a shortest path connecting them.

[22]It's best to suppose that $\det(\overline{\rho})(p)=1$, since the central character of $\pi(\overline{\rho})$ should match $\det(\overline{\rho})\omega^{-1}$ under local class field theory, just as in §2.8.

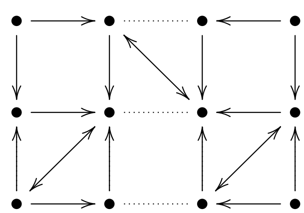
Using Proposition 3.25 we can then determine all possible Breuil–Paškūnas diagrams for $f = 2$. Horizontal/vertical arrows indicate the submodule structure and dotted edges do not occur in $D_0(\overline{\rho})$:



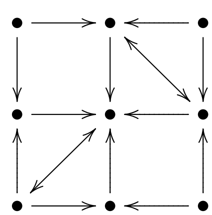
$\overline{\rho}$ irreducible,
$|W(\overline{\rho})| = 4$,
$|\operatorname{JH}(D_{0,\sigma}(\overline{\rho}))| = 4 \ \ \forall \sigma \in W(\overline{\rho})$,
$\dim_C D_0(\overline{\rho})^{I_1} = 8$.



$\overline{\rho}$ split reducible,
$|W(\overline{\rho})| = 4$,
$|\operatorname{JH}(D_{0,\sigma}(\overline{\rho}))| = 4 \ \ \forall \sigma \in W(\overline{\rho})$,
$\dim_C D_0(\overline{\rho})^{I_1} = 10$.



$\overline{\rho}$ nonsplit reducible,
$|W(\overline{\rho})| = 2$,
$|\operatorname{JH}(D_{0,\sigma}(\overline{\rho}))| = 6 \ \ \forall \sigma \in W(\overline{\rho})$,
$\dim_C D_0(\overline{\rho})^{I_1} = 6$.



$\overline{\rho}$ nonsplit reducible,
$|W(\overline{\rho})| = 1$,
$|\operatorname{JH}(D_{0,\sigma}(\overline{\rho}))| = 9 \ \ \forall \sigma \in W(\overline{\rho})$,
$\dim_C D_0(\overline{\rho})^{I_1} = 4$.

Note that none of these diagrams is irreducible (when $f > 1$), since for example $\langle K \cdot D_1(\overline{\rho})\rangle \subsetneq D_0(\overline{\rho})$, as is clear from the pictures when $f = 2$. (If $\overline{\rho}$ is irreducible, we recover Example 3.33 in this way!) Still we have the following surprising result:

Theorem 3.37. *If $\overline{\rho}$ is irreducible, then any smooth representation π satisfying the properties in Theorem 3.4 (applied to the diagram $(D_0(\overline{\rho}), D_1(\overline{\rho}), \mathrm{incl})$) is irreducible admissible and supersingular.*

Proof. We give a proof when $f = 2$, which illuminates the general idea. Label σ_i again as in Example 3.33 and let $\chi_i := \sigma_i^{I_1}$.

Suppose $0 \neq \pi' \subseteq \pi$ is any G-subrepresentation. Then $0 \neq \operatorname{soc}_K \pi' \subseteq \operatorname{soc}_K \pi = \operatorname{soc}_K D_0 = \bigoplus_{i=1}^4 \sigma_i$. Hence $\operatorname{Hom}_K(\sigma_i, \pi') \neq 0$ – and hence is 1-dimensional – for some $1 \leq i \leq 4$. It will suffice to show that this implies π' contains $D_{0,\sigma_{i+1}}(\overline{\rho})$

(in 4 steps we obtain that π' contains $D_0(\overline{\rho})$ and hence π). Let's take $i = 1$ for concreteness.

Step 1 ("Weight cycling"). The I-equivariant map $\chi_1 = \sigma_1^{I_1} \hookrightarrow \pi' \cap D_1$ gives rise to an I-equivariant map $\chi_1^s \hookrightarrow \pi' \cap D_1$ by composing by $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) : \pi' \to \pi'$. By Frobenius reciprocity we get a nonzero K-equivariant map $\theta_\varnothing : \operatorname{Ind}_I^K \chi_1^s \to \pi' \cap D_0$. In the extension graph:

$$\begin{tikzcd}
\sigma_2^{[s]} \arrow[r, dash, dotted] \arrow[d, dash, dotted] \arrow[dr, leftrightarrow] & \sigma_3 \arrow[r, dash, dotted] \arrow[d, dash, dotted] & \sigma_4 \arrow[r, dash, dotted] \arrow[d, dash, dotted] \arrow[dr, leftrightarrow] & \bullet \arrow[d, dash, dotted] \\
\bullet \arrow[r, dash, dotted] \arrow[d, dash, dotted] & \sigma_2 \arrow[r] & \sigma_1 \arrow[r, dash, dotted] & \sigma_4^{[s]} \arrow[d, dash, dotted] \\
\bullet \arrow[r, dash, dotted] & \sigma_1^{[s]} \arrow[u] \arrow[r] \arrow[ur, leftrightarrow, "P_\varnothing"] & \bullet \arrow[u] \arrow[r, dash, dotted] & \bullet
\end{tikzcd}$$

We use arrows to indicate the submodule structure of $P_\varnothing := \operatorname{Ind}_I^K \chi_1^s$ (Example 3.27). The image of $\theta_\varnothing$ is a quotient of $P_\varnothing$ whose socle is contained in $\operatorname{soc}_K D_0$, so $\operatorname{im}(\theta_\varnothing)$ is the unique quotient of $P_\varnothing$ whose socle is σ_2 (the element of $W(\overline{\rho})$ closest to $\sigma_1^{[s]}$): the only other option, $P_\varnothing$ itself, is not possible because $\operatorname{im}(\theta_\varnothing) \subseteq D_0$ and $D_0/\operatorname{soc}_K D_0$ does not contain any element of $W(\overline{\rho})$ as constituent, by definition! We indicate the image of $\theta_\varnothing$ in red color above, which is thus contained in π'.

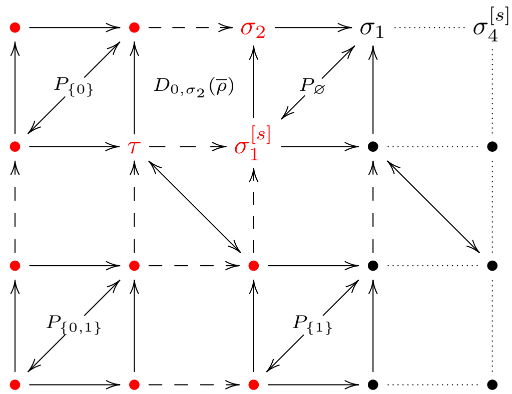
Step 2. Let's push further. The subrepresentation $\chi_1 \subseteq \sigma_1|_I$ of Step 1 is contained in a larger I-subrepresentation $V \subseteq \sigma_1|_I$ that has submodule structure

$$\begin{tikzcd}
& \chi_1\alpha_0^{-1}\alpha_1^{-1} \arrow[dl] \arrow[dr] & \\
\chi_1\alpha_0^{-1} \arrow[dr] & & \chi_1\alpha_1^{-1} \arrow[dl] \\
& \chi_1 &
\end{tikzcd}$$

where $\alpha_j(\left(\begin{smallmatrix} x & * \\ & y \end{smallmatrix}\right)) = \kappa_j(xy^{-1})$ on $I/K_1 \leq \Gamma$. (Let V be the subspace of $\sigma_1 = (r_0, r_1)$ spanned by $X^{r_0'+pr_1'}$ with $r_j - 1 \leq r_j' \leq r_j$ for $j = 0, 1$.) As $V \subseteq \sigma_1 \subseteq \pi'$, by using the action of $\Pi \in G$ we see that $\Pi(V) \subseteq \pi'$ and hence we get a K-equivariant map $\theta : \operatorname{Ind}_I^K \Pi(V) \to \pi'$, which extends $\theta_\varnothing$ from Step 1.[23] We want to understand the submodule structure of $\operatorname{Ind}_I^K \Pi(V)$, which is an extension of the four principal series $P_J := \operatorname{Ind}_I^K \Pi(\chi_1 \prod_{j \in J} \alpha_j^{-1}) = \operatorname{Ind}_I^K(\chi_1^s \prod_{j \in J} \alpha_j)$. By Frobenius reciprocity, P_J has cosocle $(p-1-r_0+2\delta_0, p-1-r_1+2\delta_1)$ up to twist, where $\delta_j = 1$ if $j \in J$, $\delta_j = 0$ if $j \notin J$. We can locate them in the extension graph, ignoring colors for

[23] $\operatorname{Ind}_I^K \Pi(V)$ is not K_1-invariant! (V is, but $\Pi(V)$ isn't.)

now:[24]



Note that in particular $\mathrm{Ind}_I^K \Pi(V)$ is multiplicity free. We use the arrows to indicate the submodule structure: the solid arrows come from the principal series (Example 3.27), the dashed arrows may or may not occur a priori, but if they occur they have to have the indicated direction (e.g. $P_\varnothing$ is a subrepresentation). In fact, all dashed arrows do occur in $\mathrm{Ind}_I^K \Pi(V)$, by Exercise 3.39 below and we will assume this in the following.

Step 3. We explain how the submodule structure of $\mathrm{Ind}_I^K \Pi(V)$ in Step 2 implies that $D_{0,\sigma_2}(\overline{\rho}) \subseteq \pi'$. By Step 1, $\mathrm{im}(\theta_\varnothing)$ has socle σ_2, so the same is true for $\mathrm{im}(\theta)$ because the other principal series do not contain elements of $W(\overline{\rho})$ as constituents. Thus $\mathrm{im}(\theta) \subseteq \pi'$ is the unique quotient of $\mathrm{Ind}_I^K \Pi(V)$ that has socle σ_2, and its constituents are indicated in red above.

Let D denote the unique K-subrepresentation of $\mathrm{im}(\theta)$ that has cosocle τ (indicated in the figure above). It is K_1-invariant, hence a Γ-representation, by a very convenient criterion of Breuil–Paškūnas:

Lemma 3.38 ([BP12, Cor. 5.7])**.** *Suppose that $p > 3$ or $f > 1$. If W is a multiplicity-free finite-dimensional smooth K-representation that does not contain as constituents any pair of Serre weights (σ, τ) of the shape*

$$\sigma \;\text{———}\; \bullet \;\text{———}\; \tau$$

in the extension graph (a straight edge of length 2 in any coordinate direction), then $W = W^{K_1}$.

Hence $D \subseteq V_{\sigma_2}$ and we deduce that $D \cong D_{0,\sigma_2}(\overline{\rho})$. Finally we have two injective maps $\iota_1, \iota_2 : D_{0,\sigma_2}(\overline{\rho}) \to \pi$, namely $\iota_1 = \mathrm{incl}$ and $\iota_2 = \theta|_D$. As $\mathrm{JH}(D_{0,\sigma_2}(\overline{\rho})/\sigma_2) \cap W(\overline{\rho}) = \varnothing$ we deduce that the restriction map $\mathrm{Hom}_K(D_{0,\sigma_2}(\overline{\rho}), \pi) \to \mathrm{Hom}_K(\sigma_2, \pi)$ is injective, so we deduce that ι_1, ι_2 are scalar multiples of each other. Therefore, $D_{0,\sigma_2}(\overline{\rho}) = \mathrm{im}(\iota_1) = \mathrm{im}(\iota_2) \subseteq \pi'$, as desired.

We have shown that π is irreducible. The proof that π is admissible and supersingular are as in the discussion of Example 3.33. $\square$

Here is the exercise filling the gap in Step 2 (it needs to be applied four times).

[24]Strictly speaking, we use that $\mathrm{Ext}^1_{K/Z_1}(\sigma, \tau) = \mathrm{Ext}^1_\Gamma(\sigma, \tau)$ for (almost all) Serre weights σ, τ, where $Z_1 := Z \cap K_1$. A proof can be found in [BHH$^+$, Lemma 4.3.4].

Exercise 3.39 ($f = 2$). Suppose χ is a smooth character of I and $0 \le j \le 1$. Let E denote any smooth I-representation that is a nonsplit extension $0 \to \chi \to E \to \chi\alpha_j \to 0$. As above, $\operatorname{Ind}_I^K E$ looks as follows in the extension graph:

$$\begin{tikzcd}
\bullet \arrow[r] & \tau \arrow[r, dashed] & \sigma \arrow[r] & \bullet \\
\bullet \arrow[u] \arrow[r] \arrow[ur, <->, "P_{\{j\}}" description] & \tau' \arrow[u] \arrow[r, dashed] & \sigma' \arrow[u] \arrow[r] \arrow[ur, <->, "P_{\varnothing}" description] & \bullet \arrow[u]
\end{tikzcd}$$

where again $P_{\{j\}} := \operatorname{Ind}_I^K \chi\alpha_j$ and $P_\varnothing := \operatorname{Ind}_I^K \chi$. Use Frobenius reciprocity to show that $\operatorname{soc}_K(\operatorname{Ind}_I^K E) = \operatorname{soc}_K(P_\varnothing)$ is irreducible and likewise $\operatorname{cosoc}_K(\operatorname{Ind}_I^K E) = \operatorname{cosoc}_K(P_{\{j\}})$ is irreducible. (The point is that E is not K_1-invariant: if it were, then $0 \ne \operatorname{Ext}^1_{I/K_1}(\chi\alpha_j, \chi) = H^1(I/K_1, \alpha_j^{-1})$ and use inflation-restriction to obtain a contradiction.) Deduce that the top (resp. bottom) dashed edge needs to occur in $\operatorname{Ind}_I^K E$, otherwise τ (resp. σ') would contribute to the cosocle (resp. socle) of $\operatorname{Ind}_I^K E$.

Remark 3.40. For general f the picture consists of two adjacent f-dimensional hypercubes in the extension graph, connected by 2^{f-1} dashed arrows. Exercise 3.39 is still valid, i.e. all dashed edges actually occur in $\operatorname{Ind}_I^K E$. Here is a short argument: as in the figure just above, let τ denote the socle of $P_{\{j\}}$, σ' the cosocle of $P_\varnothing$, σ the unique constituent of $P_\varnothing$ connected by a dashed arrow to τ, and τ' the unique constituent of $P_{\{j\}}$ connected by a dashed arrow to σ'. By using the cosocle as in Exercise 3.39, the dashed edge $\tau' \to \sigma'$ occurs in $\operatorname{Ind}_I^K E$. In particular, $\sigma \le \sigma' \le \tau'$ in the submodule structure of $\operatorname{Ind}_I^K E$, so $\operatorname{Ind}_I^K E$ admits a unique subquotient W with socle σ and cosocle τ'. Its constituents consist precisely of the "middle hypercube" in the figure. By Lemma 3.38, W is K_1-invariant, so W is the unique subrepresentation of V_σ with cosocle τ'. By the submodule structure of V_σ, we conclude that all 2^{f-1} dashed arrows occur.

If $\overline{\rho}$ is split reducible, then the Breuil–Paškūnas diagrams decompose into a direct sum of $f+1$ subdiagrams: $D_0(\overline{\rho}) = \bigoplus_{i=0}^f D_0(\overline{\rho})_i$. In general, there are precisely two $\sigma \in W(\overline{\rho})$ such that $D_{0,\sigma}(\overline{\rho})$ is a principal series and we fix one of them. We let $D_0(\overline{\rho})_i$ be the direct sum of all $D_{0,\tau}(\overline{\rho})$ that differ from $D_{0,\sigma}(\overline{\rho})$ in precisely i coordinate directions in the extension graph. One then checks that the IZ-action on $D_0(\overline{\rho})_i^{I_1}$ extends to $\mathcal{N}$. (When $f = 2$, all this should be evident from the picture above!) Then Theorem 3.37 generalizes to each $D_0(\overline{\rho})_i$, with a similar proof, the cases $i = 0$ and $i = f$ giving principal series representations and the others giving supersingular representations.

This led to the following speculation: if $\overline{\rho}$ is irreducible, then $\pi(\overline{\rho})$ is irreducible supersingular, whereas if $\overline{\rho}$ is split reducible, then $\pi(\overline{\rho})$ is a direct sum of two principal series and $f-1$ irreducible supersingular representations. This of course generalizes the $f = 1$ case, §2.8.

3.5. **Exotic diagrams.** The following idea of "spliced" diagrams appears in [GLS23]. Suppose that $f = 2$ for simplicity, and $p > 3$.

We first construct funny irreducible supersingular representations that have a socle consisting of 7 distinct Serre weights (such representations do not occur globally, see §4). As $p > 3$ we may consider an extension of the configuration in the extension graph from Example 3.33 (spliced out of two isomorphic copies there):

$$\begin{tikzcd}
 & & \sigma_6^{[s]} \arrow[d, dash] \arrow[dr, leftrightarrow] & & \\
 & & \sigma_3^{[s]} \arrow[r, dash] \arrow[d, dash] \arrow[dl, leftrightarrow] & \sigma_6 \arrow[r, dash] \arrow[d, dash] & \sigma_5^{[s]} \arrow[dl, leftrightarrow] \\
\sigma_2^{[s]} \arrow[r, dash] \arrow[dr, leftrightarrow] & \sigma_3 \arrow[r, dash] \arrow[d, dash] & \sigma_4 \arrow[r, dash] \arrow[d, dash] \arrow[dr, leftrightarrow] & \sigma_5 \arrow[d, dash] & \\
 & \sigma_2 \arrow[r, dash] \arrow[d, dash] & \sigma_1 \arrow[r, dash] & \sigma_4^{[s]} & \\
 & \sigma_1^{[s]} \arrow[ur, leftrightarrow] & & &
\end{tikzcd}$$

Let D_0 be the following Γ-representation, which is a direct sum of 6 indecomposable pieces as follows:

$$\begin{tikzcd}
\sigma_2^{[s]} \arrow[d] & \sigma_1^{[s]} \arrow[d] & \sigma_3^{[s]} \arrow[dr] & \sigma_4^{[s]} \arrow[dl] \arrow[dr] & \sigma_7^{[s]} \arrow[dl] & \sigma_5^{[s]} \arrow[d] & \sigma_6^{[s]} \arrow[d] \\
\sigma_3 & \sigma_2 & \sigma_1 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7
\end{tikzcd}$$

Here, $\sigma_7 := \sigma_3^{[s]}$ and the arrows indicate once again the submodule structure (each indecomposable piece being multiplicity free). Note that the two length 3 indecomposable pieces can be obtained as subrepresentation of V_{σ_4}, respectively as quotient of the principal series with cosocle $\sigma_4^{[s]}$. Let E_i $(1 \leq i \leq 7)$ denote the direct summand of D_0 such that $\sigma_i^{[s]}$ occurs in the cosocle of E_i, so $E_3 = E_7$ and $D_0 = \bigoplus_{i \neq 7} E_i$.

As in Example 3.33 we see that $D_1 := D_0^{I_1}$ is of dimension 14 and stable under $\chi \mapsto \chi^s$. (In the case of the 3-dimensional summand with cosocle $\sigma_4^{[s]}$, this is true because it's a quotient of a principal series.) Therefore the action of IZ can be extended to an action of $\mathcal{N}$.

Exercise 3.41. Use the same method as in Exercise 3.34 to show that any representation π associated to the diagram $(D_0, D_1, \mathrm{incl})$ is irreducible.

Now we modify this construction to construct nonadmissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$! (again following [GLS23]).

Theorem 3.42. *There exist nonadmissible irreducible representations of* $\mathrm{GL}_2(\mathbb{Q}_{p^2})$.

Note that over $\mathbb{C}$ it is known that any irreducible representation of G (any p-adic reductive group) is admissible.

Proof. For $1 \leq i \leq 7$ choose a C-basis v_i of the χ_i-eigenspace in $(\mathrm{soc}_\Gamma D_0)^{I_1}$, where $\chi_i := \sigma_i^{I_1}$. Also choose a C-basis v_i^s of the χ_i^s-eigenspace in $E_i^{I_1}$. By construction there exist $T_i \in C[\Gamma]$ such that $T_i(v_i^s) = v_{i+1}$, where $v_8 := v_4$; this is simply because $\langle \Gamma \cdot v_i^s \rangle$ contains σ_{i+1} (if $\sigma_8 := \sigma_4$). Likewise there exists $T_8 \in C[\Gamma]$ such that $T_8(v_4^s) = v_1$.

Now let $\widetilde{D}_0 := \bigoplus_{n\in\mathbb{Z}} D_0$ and let $v_{i,n}^{(s)} \in \widetilde{D}_0$ denote the element that equals $v_i^{(s)}$ in the n-th coordinate and is zero elsewhere. Extend the action of K to KZ by letting $p \in Z$ act trivially. Extend the action of IZ on $\widetilde{D}_1 := \widetilde{D}_0^{I_1}$ to $\mathcal{N}$ as follows:

$$\Pi(v_{i,n}) := \begin{cases} v_{i,n}^s & \text{if } i \notin \{1,7\}, \\ v_{i,n-1}^s & \text{if } i = 1, \\ \lambda_{n+1} v_{i,n+1}^s & \text{if } i = 7, \end{cases} \tag{3.3}$$

where $\lambda_n \in C^\times$ are any *pairwise distinct* scalars.

Unfortunately, $(\widetilde{D}_0, \widetilde{D}_1, \mathrm{incl})$ is not a diagram in our sense, since $\widetilde{D}_0$ is not admissible, so we cannot apply Theorem 3.4. But the trick is to extend $\Omega := \bigoplus_{n\in\mathbb{Z}} \mathrm{inj}_K D_0$ into a G-representation rather than $\mathrm{inj}_K \widetilde{D}_0$. Referring back to the proof of Proposition 3.10, write $(\mathrm{inj}_K D_0)^{I_1} = D_1 \oplus E$ as in (3.1) (noting that I_1 acts trivially on D_1 here). Then, as in that proof,

$$\begin{aligned} \Omega|_I &\cong \bigoplus_{n\in\mathbb{Z}} (\mathrm{inj}_I D_1 \oplus \mathrm{inj}_I E) \cong \bigoplus_{\substack{n\in\mathbb{Z}\\ 1\le i\le 7}} (\mathrm{inj}_I Cv_{i,n}) \oplus \bigoplus_{\substack{n\in\mathbb{Z}\\ 1\le i\le 7}} (\mathrm{inj}_I Cv_{i,n}^s) \oplus \bigoplus_{n\in\mathbb{Z}} (\mathrm{inj}_I E) \\ &\cong \bigoplus_{\substack{n\in\mathbb{Z}\\ i\notin\{1,7\}}} \mathrm{inj}_I(Cv_{i,n} \oplus Cv_{i,n}^s) \oplus \bigoplus_{n\in\mathbb{Z}} \mathrm{inj}_I(Cv_{1,n} \oplus Cv_{1,n-1}^s) \oplus \cdots \oplus \bigoplus_{n\in\mathbb{Z}} (\mathrm{inj}_I E). \end{aligned}$$

In the last row, each argument of inj_I extends to an $\overline{\mathcal{N}}$-representation, using (3.3) and the old extension on each copy of E, hence we can again extend $\Omega|_I$ to an $\overline{\mathcal{N}}$-representation by replacing each inj_I by $\mathrm{inj}_{\overline{\mathcal{N}}}$. By construction, the injection $\widetilde{D}_1 \hookrightarrow \widetilde{D}_0 \hookrightarrow \Omega$ becomes $\overline{\mathcal{N}}$-equivariant. Thus Ω is a G-representation by Proposition 3.1 and we let $\pi := \langle G \cdot \widetilde{D}_0 \rangle \subseteq \Omega$. Then $\mathrm{soc}_K \pi = \mathrm{soc}_K \widetilde{D}_0 = \bigoplus_{n\in\mathbb{Z}} \mathrm{soc}_K D_0$, so π is nonadmissible as π^{K_1} has infinite dimension.

We show that π is irreducible. If $0 \ne \pi' \subseteq \pi$ is any nonzero subrepresentation, then $\pi'|_K$ has to contain an irreducible subrepresentation, so $\mathrm{Hom}_K(\sigma_i, \pi') \ne 0$ for some $1 \le i \le 7$ (by our knowledge of $\mathrm{soc}_K \pi$). Suppose first that $\mathrm{Hom}_K(\sigma_1, \pi') \ne 0$. Then π' contains an element of the form $v := \sum_{n\in\mathbb{Z}} c_n v_{1,n}$ for some $c_n \in C$, not all zero. We will show that $v_{1,n} \in \pi'$ for some $n \in \mathbb{Z}$.

Suppose that at least two c_n are nonzero, and say in particular $c_m \ne 0$. By the definitions, $\Pi(v) = \sum_{n\in\mathbb{Z}} c_n v_{1,n-1}^s$ and so $T_1\Pi(v) = \sum_{n\in\mathbb{Z}} c_n v_{2,n-1} \in \pi'$. Keep cycling:

$$T_6\Pi \cdots T_3\Pi T_2\Pi T_1\Pi(v) = \sum_{n\in\mathbb{Z}} c_n v_{7,n-1} \in \pi',$$

$$v' := T_8\Pi \cdots T_2\Pi T_1\Pi(v) = \sum_{n\in\mathbb{Z}} c_n \lambda_n v_{1,n} \in \pi'.$$

As the λ_n are pairwise distinct, we can replace v by $v' - \lambda_m v$ and have reduced the number of nonzero coefficients by 1. After finitely many steps we arrive at just one nonzero coefficient, i.e. $v_{1,n} \in \pi'$ for some $n \in \mathbb{Z}$. It is then a fun exercise to show that we obtain $v_{i,n} \in \pi'$ for all $1 \le i \le 7$ and $n \in \mathbb{Z}$, and hence $\mathrm{soc}_K \widetilde{D}_0 \subseteq \pi'$, which generates π, so $\pi' = \pi$.

Finally if instead $\mathrm{Hom}_K(\sigma_i, \pi') \ne 0$ for some $i \ne 1$ we use a similar cycling procedure as above to reduce to the case $i = 1$ already covered. $\square$

Remark 3.43. We remark that Le even proved the existence of irreducible G-representations that do not have a central character. (Note that irreducible admissible representations have a central character; exercise!)

4. Global aspects

Useful reference in this section: [Gee22] (esp. sections 4 and 5.3).

Suppose that $E/\mathbb{Q}$ is a totally real field. For any place v of E, let E_v denote the completion at v and $\mathcal{O}_v$ its ring of integers if $v \nmid \infty$. Let D be a quaternion algebra over E such that $D \otimes_E E_v$ is definite (i.e. nonsplit) at all real places $v \mid \infty$. We assume that D *splits at* p, and hence we can fix an isomorphism $i_p : D \otimes_E E_p \xrightarrow{\sim} M_2(E_p)$: we do this since we want to obtain representations of the p-adic group $\mathrm{GL}_2(E_p)$. Moreover, for simplicity we assume that p is *inert* in E, so that E_p is a finite unramified extension of $\mathbb{Q}_p$. Fix any $\mathcal{O}_E$-order $\mathcal{O}_D$ of D such that $i_p(\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_p) = M_2(\mathcal{O}_p)$. Note that $(D \otimes_E \mathbb{A}_E^\infty)^\times \cong \prod'_{v\nmid\infty}(D \otimes_E E_v)^\times$, where $\mathbb{A}_E^\infty$ is the ring of finite adeles of E and the restricted direct product is formed with respect to the subgroups $(\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_v)^\times$ of $(D \otimes_E E_v)^\times$. Let U denote any compact open subgroup of $(D \otimes_E \mathbb{A}_E^\infty)^\times$ of the form $U = \prod_{v\nmid\infty} U_v$, where U_v is a compact open subgroup of $(D \otimes_E E_v)^\times$ such that $U_v = (\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_v)^\times$ for almost all v.

4.1. **Modular forms on quaternion algebras.** Let k be a topological field (or even topological ring). Consider the space of "modular forms"

$$S(U,k) := \{f : D^\times\backslash(D \otimes_E \mathbb{A}_E^\infty)^\times/U \to k\},$$

which is finite-dimensional, as $D^\times\backslash(D \otimes_E \mathbb{A}_E^\infty)^\times/U$ is finite (see [PR94, Thm. 5.1]). More generally, suppose that σ is a finite-dimensional continuous[25] representation of U over k, we define

$$S(U,\sigma) := \{f : D^\times\backslash(D \otimes_E \mathbb{A}_E^\infty)^\times \to \sigma : \\ f(du) = u^{-1}f(d) \quad \forall d \in (D \otimes_E \mathbb{A}_E^\infty)^\times, u \in U\}.$$

To get an action of the p-adic group $\mathrm{GL}_2(E_p)$, we form a direct limit (i.e. union) over all levels at p and obtain a space

$$S(U^p,k) := \varinjlim_{U_p} S(U^pU_p,k) = \{f : D^\times\backslash(D\otimes_E\mathbb{A}_E^\infty)^\times/U^p \to k \text{ locally constant}\},$$

where U_p runs through all compact open subgroups of $(D \otimes_E E_p)^\times$. The group $(D \otimes_E E_p)^\times \cong \mathrm{GL}_2(E_p)$ acts on $S(U^p,k)$ *smoothly* by right translations:

$$(g_pf)(d) = f(dg_p) \quad \forall g_p \in (D \otimes_E E_p)^\times, d \in (D \otimes_E \mathbb{A}_E^\infty)^\times, f \in S(U^p,k).$$

Concretely, $g_p \in (D\otimes_E E_p)^\times$ sends $S(U_pU^p,k)$ to $S((g_pU_pg_p^{-1})U^p,k)$, and $S(U^p,k)^{U_p} = S(U_pU^p,k)$. Moreover, $S(U^p,k)$ is an *admissible* representation of $\mathrm{GL}_2(E_p)$, since $S(U,k)$ is finite-dimensional for all U.

The spaces $S(U,k)$ admit actions of Hecke operators, similarly to the theory of classical modular forms. Let Σ denote a finite set of places of E, including p and all infinite places, such that for all $v \notin \Sigma$ the algebra D splits at v and $(\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_v)^\times$ is a maximal compact subgroup of $(D \otimes_E E_v)^\times$. For each $v \notin \Sigma$ we have an action of the double coset algebra $\mathcal{H}_v := k[(\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_v)^\times\backslash(D \otimes_E E_v)^\times/(\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_v)^\times]$ on $S(U,k)$ and $S(U^p,k)$, with the double coset $[U_vg_vU_v]$ sending f to $\sum_{i=1}^r x_if = \sum_{i=1}^r f((-)x_i)$, where $U_vg_vU_v = \coprod_{i=1}^r x_iU_v$. By construction, the actions of $\mathcal{H}_v$ and $\mathcal{H}_{v'}$ commute whenever $v \neq v'$.

Fix an isomorphism $(D \otimes_E E_v)^\times \cong \mathrm{GL}_2(E_v)$ that identifies $(\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_v)^\times$ with $\mathrm{GL}_2(\mathcal{O}_v)$ and a uniformizer ϖ_v of E_v. In particular, we have the Hecke operators

[25] Equivalently smooth if k is discrete.

$T_v = [U_v\left(\begin{smallmatrix}\varpi_v & \\ & 1\end{smallmatrix}\right)U_v]$ and $S_v = [U_v\left(\begin{smallmatrix}\varpi_v & \\ & \varpi_v\end{smallmatrix}\right)U_v]$ in $\mathcal{H}_v$, and S_v is central. Hence the polynomial algebra $\mathbb{T}_k := k[T_v, S_v : v \notin \Sigma]$ acts on $S(U,k)$ and $S(U^p,k)$. We say that $f \in S(U,k)$ is an *eigenform* if it is a $\mathbb{T}_k$-eigenvector. Note that the actions of $\mathbb{T}_k$ and $\mathrm{GL}_2(E_p)$ on $S(U^p,k)$ commute!

4.2. **Comparison with automorphic forms over** $\mathbb{C}$**.** We quickly explain the comparison with automorphic forms when $k = \overline{\mathbb{Q}}_p$, following Gross. It will be convenient to denote $G_D(A) := (D \otimes_{\mathbb{Q}} A)^\times$ for any (commutative) $\mathbb{Q}$-algebra A.

For the comparison we need to fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Suppose that σ is a finite-dimensional representation of $G_D(\overline{\mathbb{Q}}_p)$ over $\overline{\mathbb{Q}}_p$, so that we can consider $\sigma \otimes_\iota \mathbb{C}$ as representation of $G_D(\mathbb{C})$ via ι.[26] We consider σ as representation of U via the composition $U \to U_p \le (D \otimes_E E_p)^\times = G_D(\mathbb{Q}_p) \to G_D(\overline{\mathbb{Q}}_p)$ and likewise $\sigma \otimes_\iota \mathbb{C}$ is a representation of $G_D(\mathbb{R})$ via $G_D(\mathbb{R}) \subseteq G_D(\mathbb{C})$.

Then

$$S(U,\sigma) = \{f : G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q}})/G_D(\mathbb{R}) \to \sigma : f(du) = u^{-1}f(d)\}$$

and

$$S(U,\sigma) \otimes_\iota \mathbb{C} \cong \{f' : G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q}})/U \to \sigma \otimes_\iota \mathbb{C} : f'(dh_\infty) = h_\infty^{-1} f'(d)\}, \tag{4.1}$$

where $u \in U$, $h_\infty \in G_D(\mathbb{R})$ and $f'(d) := d_\infty^{-1}\iota(d_p f(d))$ (so the trick is to "interchange p and ∞", just as when one associates a Galois character to an algebraic Hecke character). By construction, this isomorphism is compatible with Hecke operators at places $v \nmid p\infty$.

The right-hand side of (4.1) is a space of automorphic forms of the group $G_D(\mathbb{A}_{\mathbb{Q}})$ over $\mathbb{C}$, analogous to a space of classical modular forms of fixed weight and level, in the adelic description. Automorphic forms on $G_D(\mathbb{A}_{\mathbb{Q}})$ can be compared with Hilbert modular forms over E via the Jacquet–Langlands correspondence. If σ is the trivial representation, then $S(U,\sigma) = S(U,\overline{\mathbb{Q}}_p)$ corresponds to Hilbert modular forms of parallel weight 2. If $\sigma = \bigotimes_{\tau : E \to \overline{\mathbb{Q}}_p} (\mathrm{Sym}^{k_\tau - 2}(E^2) \otimes \det^{\ell_\tau}) \otimes_{E,\tau} \overline{\mathbb{Q}}_p$ over $k = \overline{\mathbb{Q}}_p$ for some $k_\tau \in \mathbb{Z}_{\ge 2}$ and $\ell_\tau \in \mathbb{Z}$, then $S(U,\sigma)$ corresponds to Hilbert modular forms of weights $(k_{\iota^{-1}\tau'}, \ell_{\iota^{-1}\tau'})_{\tau' : E \to \mathbb{R}}$ (often ℓ_τ is suppressed in the notation).[27]

Exercise 4.1. Justify the isomorphism (4.1).

Remark 4.2. A more trivial kind of comparison is possible when σ is a finite-dimensional *smooth* U-representation, in which case $S(U,\sigma) \otimes_\iota \mathbb{C}$ is already a space of automorphic forms (taking $f'(d) = \iota(f(d))$). These are particular automorphic forms of trivial weight and level $U' := \ker(\sigma)$:

$$S(U,\sigma) \otimes_\iota \mathbb{C} \cong \mathrm{Hom}_{U/U'}(\sigma^\vee, \mathcal{C}^\infty(G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q}})/G_D(\mathbb{R})U', \mathbb{C})).$$

One says that they have *type* σ.

---

[26] It would be slightly more natural to recognize that G_D is the functor of points of an algebraic group and to take for σ an algebraic representation of $G_D \times_{\mathbb{Q}} \overline{\mathbb{Q}}_p$.

[27] Note that $G_D(\overline{\mathbb{Q}}_p) \cong \prod_{\tau : E \to \overline{\mathbb{Q}}_p} \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ as $E \otimes \overline{\mathbb{Q}}_p \cong \prod_{\tau : E \to \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p$, and we let the τ-th copy of $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ act on the τ-th factor $\mathrm{Sym}^{k_\tau - 2}(\overline{\mathbb{Q}}_p^2) \otimes \det^{\ell_\tau}$ in the tensor product.

4.3. **Fixed central character.** In general the theory of automorphic forms is more complicated when the action of the center is not fixed, as the center does not act semisimply. Therefore, we now fix $\psi : E^\times\backslash(\mathbb{A}_E^\infty)^\times \to k^\times$, a continuous character. If ψ is trivial on $U \cap (\mathbb{A}_E^\infty)^\times$, then ψ extends to a continuous character of $U(\mathbb{A}_E^\infty)^\times = U \cdot (\mathbb{A}_E^\infty)^\times$ that is trivial on U and we can define

$$S_\psi(U,k) := \{f : D^\times\backslash(D \otimes_E \mathbb{A}_E^\infty)^\times \to k : \\ f(duz) = \psi(z)f(d) \quad \forall d \in (D \otimes_E \mathbb{A}_E^\infty)^\times, uz \in U(\mathbb{A}_E^\infty)^\times\}.$$

as the subspace of $S(U,k)$ on which the center $(\mathbb{A}_E^\infty)^\times$ acts via ψ. More generally, suppose that σ is a finite-dimensional continuous representation of U such that $U \cap (\mathbb{A}_E^\infty)^\times$ acts on σ via ψ^{-1}. Then σ extends to a representation of $U(\mathbb{A}_E^\infty)^\times$ such that $(\mathbb{A}_E^\infty)^\times$ acts via ψ^{-1}, and we define

$$S_\psi(U,\sigma) := \{f : D^\times\backslash(D \otimes_E \mathbb{A}_E^\infty)^\times \to \sigma : \\ f(duz) = (uz)^{-1}f(d) \quad \forall d \in (D \otimes_E \mathbb{A}_E^\infty)^\times, uz \in U(\mathbb{A}_E^\infty)^\times\}$$

as the subspace of $S(U,\sigma)$ on which the center $(\mathbb{A}_E^\infty)^\times$ acts via ψ. The Hecke action preserves the subspaces $S_\psi(U,k) \subseteq S(U,k)$ and $S_\psi(U,\sigma) \subseteq S(U,\sigma)$.

Letting

$$S_\psi(U^p,k) := \varinjlim_{U_p} S_\psi(U_pU^p,k),$$

where U_p runs through all compact open subgroups of $(D \otimes_E E_p)^\times$, the group $(D \otimes_E E_p)^\times \cong \mathrm{GL}_2(E_p)$ acts on $S_\psi(U^p,k)$ by right translations and the action is admissible smooth as before.

Exercise 4.3. If $f \in S(U,\sigma)$ is an eigenform, show that $f \in S_\psi(U,\sigma)$ for a unique character ψ. (Hint: first show that $E^\times \prod_{v\notin\Sigma} E_v^\times \subseteq (\mathbb{A}_E^\infty)^\times$ acts by a character, then use class field theory to show that the inclusion $E^\times \prod_{v\notin\Sigma} E_v^\times \subseteq (\mathbb{A}_E^\infty)^\times$ is dense.)

Exercise 4.4. Let's return to $\sigma = \bigotimes_{\tau:E\to\overline{\mathbb{Q}}_p}(\mathrm{Sym}^{k_\tau-2}(E^2) \otimes \det^{\ell_\tau}) \otimes_{E,\tau} \overline{\mathbb{Q}}_p$ over $k = \overline{\mathbb{Q}}_p$ for some $k_\tau \in \mathbb{Z}_{\geq 2}$ and $\ell_\tau \in \mathbb{Z}$, considered as U-representation via $U \to U_p$. Show that if $U \cap (\mathbb{A}_E^\infty)^\times$ acts via ψ^{-1}, then the integer $w_\tau := k_\tau + 2\ell_\tau$ is independent of τ. This is known as the "parity condition" in the context of Hilbert modular forms. Here is a series of steps:

(i) Show that $\psi(x) = \prod_{\tau:E\to\overline{\mathbb{Q}}_p} \tau(x_p)^{-w_\tau}$ for all $x \in (\mathbb{A}_E^\infty)^\times$ near the identity. (Use the compatibility between ψ and σ!)

(ii) Fix any isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and define $\chi : \mathbb{A}_E^\times \to \mathbb{C}^\times$ by $\chi(x) := \iota\big(\psi(x)\prod_{\tau:E\to\overline{\mathbb{Q}}_p} \tau(x_p)^{w_\tau}\big)\prod_{\tau':E\to\mathbb{C}} \tau'(x_\infty)^{-w_{\iota^{-1}\tau'}}$, where $\tau'(x_\infty)$ denotes $\tau'(x_v)$ for the unique infinite place $v \mid \infty$ induced by the embedding $\tau' : E \to \mathbb{C}$. Check that χ is trivial on $E^\times$ and by (i) that χ is continuous.

(iii) By (ii) deduce that $|\chi|$ is trivial on $(E_\infty^\times)^{|\cdot|=1}$, using that $(E^\times\backslash\mathbb{A}_E^\times)^{|\cdot|=1}$ is compact. Conclude that w_τ is independent of τ.

Conversely, show that for any such σ and any U, there exists a (finite index) open subgroup U' of U such that $U' \cap \mathcal{O}_E^\times$ acts trivially and hence that we can choose ψ such that $U' \cap (\mathbb{A}_E^\infty)^\times$ acts via ψ^{-1}.

Exercise 4.5. Continuing Exercise 4.4, show that if the parity condition fails, then $S(U,\sigma) = 0$. (Hint: if $S(U,\sigma) \neq 0$, then it contains an eigenform.)

Exercise 4.6. Suppose that σ is a smooth representation of U_p over $\overline{\mathbb{F}}_p$ (for example, a Serre weight if $U_p = \mathrm{GL}_2(\mathcal{O}_p)$). Show that there exists a (finite index) open subgroup U' of U such that $U' \cap \mathcal{O}_E^\times$ acts trivially on σ via $U \to U_p$. Hence there is no parity issue in characteristic p.

4.4. **An injectivity property.** Suppose that $k = C$ has characteristic $p > 3$. We show, following Breuil, that $S(U^p, k)$ is injective in the category of smooth U_p-representation, for any compact open subgroup $U_p \le (D \otimes_E E_p)^\times$.

Exercise 4.7. Suppose that $U = U_p U^p$. Show that for any finite-dimensional smooth U_p-representation σ we have

$$\begin{aligned} S(U,\sigma) &\xrightarrow{\sim} \mathrm{Hom}_{U_p}(\sigma^\vee, S(U^p,k)) \\ f &\mapsto (\lambda \mapsto \lambda \circ f), \end{aligned} \tag{4.2}$$

where on the left we consider σ as U-representation via the projection $U \to U_p$.

Let d_i $(1 \le i \le r)$ be a set of coset representatives for $D^\times \backslash (D \otimes_E \mathbb{A}_E^\infty)^\times / U$. Then

$$\begin{aligned} S(U,\sigma) &\xrightarrow{\sim} \bigoplus_{i=1}^r \sigma^{U \cap d_i^{-1} D^\times d_i} \\ f &\mapsto (f(d_i))_i. \end{aligned} \tag{4.3}$$

The groups $U \cap d_i^{-1} D^\times d_i$ are always finite since $D^\times \subseteq (D \otimes_E \mathbb{A}_E^\infty)^\times$ is discrete, and they are of order prime to p [Gee22, §5.3]. Therefore the functor $\sigma \mapsto S(U,\sigma)$ is exact by (4.3), and hence we get the desired injectivity property by Exercise 4.7.

Remark 4.8. We used implicitly that if π is a smooth representation of a profinite group H, then $\mathrm{Hom}_H(-,\pi)$ is exact on smooth representations if it is exact on finite-dimensional smooth representations. This can be done by a standard application of Zorn's lemma.

4.5. **Galois representations.** By the work of many people one knows:

Theorem 4.9. *Suppose that* $\sigma = \bigotimes_{\tau : E \to \overline{\mathbb{Q}}_p} (\mathrm{Sym}^{k_\tau - 2}(E^2) \otimes \det^{\ell_\tau}) \otimes_{E,\tau} \overline{\mathbb{Q}}_p$ *for some* $k_\tau \in \mathbb{Z}_{\ge 2}$ *and* $\ell_\tau \in \mathbb{Z}$. *If* $f \in S(U,\sigma)$ *is an eigenform with* $T_v f = \lambda_v f$, $S_v f = \mu_v f$ *for all* $v \notin \Sigma$, *then there exists a unique semisimple Galois representation* $r : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *such that*

- (i) *for each* $v \notin \Sigma$, *the representation* r *is unramified at* v *and the characteristic polynomial of* $r(\mathrm{Frob}_v)$ *equals* $X^2 - \lambda_v X + N(v)\mu_v$;
- (ii) r *is de Rham (equivalently, potentially semistable) at* p, *with* τ*-Hodge–Tate weights* $k_\tau + \ell_\tau - 1$, ℓ_τ *for any* $\tau : E \to \overline{\mathbb{Q}}_p$;
- (iii) r *is odd, i.e.* $\det(r(c)) = -1$ *for any complex conjugation* $c \in \mathrm{Gal}(\overline{E}/E)$.

Moreover, r *is crystalline at* p *if* $U_p = \mathrm{GL}_2(\mathcal{O}_p)$ *and the crystalline Frobenius on* $\mathrm{D}_{\mathrm{cris}}(r|_{\mathrm{Gal}(\overline{E}_p/E_p)})$ *has characteristic polynomial* $X^2 - \lambda_p X + N(p)\mu_p$ *(assuming in addition that* $T_p f = \lambda_p f$, $S_p f = \mu_p f$*).*[28]

Here, Frob_v denotes an arithmetic Frobenius element at v, $N(v)$ denotes the cardinality of the residue field at v, and Hodge–Tate weights are normalized such that the cyclotomic character of $\mathrm{Gal}(\overline{E}/E)$ has τ-Hodge–Tate weights 1 for all τ.

From Theorem 4.9 it is not so hard to deduce over $\overline{\mathbb{F}}_p$:

[28]The action of T_p and S_p is defined in §4.8.)

Theorem 4.10. *Suppose that $U_p = \mathrm{GL}_2(\mathcal{O}_p)$ and that σ is a Serre weight over $\overline{\mathbb{F}}_p$. If $f \in S(U,\sigma)$ is an eigenform with $T_v f = \lambda_v f$, $S_v f = \mu_v f$ for all $v \notin \Sigma$, then there exists a unique odd semisimple Galois representation $\overline{r} : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ such that for each $v \notin \Sigma$, the representation r is unramified at v and the characteristic polynomial of $\overline{r}(\mathrm{Frob}_v)$ equals $X^2 - \lambda_v X + N(v)\mu_v$.*

Serre's conjecture is about the converse question if any irreducible odd $\overline{r}$ arises in this way. Serre formulated this in the context of classical modular forms (and classical weights and levels), and again this has only been proved in that context, by Khare–Wintenberger.

Given $\overline{r}$ that is unramified at all $v \notin \Sigma$ we let $\mathfrak{m}_{\overline{r}}$ denote the maximal ideal $(T_v - \lambda_v \mu_v^{-1}, S_v - N(v)\mu_v^{-1} : v \notin \Sigma)$ of $\mathbb{T}_{\overline{\mathbb{F}}_p}$, where $X^2 - \lambda_v X + N(v)\mu_v$ is the characteristic polynomial of $\overline{r}(\mathrm{Frob}_v)$.[29] We say that $\overline{r} : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *modular* if $S(U,\sigma)[\mathfrak{m}_{\overline{r}}] \neq 0$ for some U such that $U_p = \mathrm{GL}_2(\mathcal{O}_p)$ and some Serre weight σ.

Exercise 4.11. Show that if $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}] \neq 0$ (or even just $S(U^p, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}} \neq 0$), then $\overline{r}$ is modular in the above sense.

4.6. **Weights in Serre's conjecture.** Suppose that $U_p = \mathrm{GL}_2(\mathcal{O}_p)$. Given a modular $\overline{r} : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, the weight part of Serre's conjecture is the question of determining the finite set

$$W(\overline{r}) := \{\text{Serre weights } \sigma : S(U,\sigma^\vee)[\mathfrak{m}_{\overline{r}}] \neq 0\}. \tag{4.4}$$

By definition, $W(\overline{r}) \neq \varnothing$.[30]

The following result was conjectured by Buzzard–Diamond–Jarvis and established by Gee–Liu–Savitt and Gee–Kisin (under mild hypotheses). Let $f := [E_p : \mathbb{Q}_p] = [k_p : \mathbb{F}_p]$. Let $I_p \lhd D_p \leq \mathrm{Gal}(\overline{E}/E)$ denote a choice of inertia and decomposition subgroups at p. For any $n \geq 1$ let $\omega_{nf} : I_p \to \overline{\mathbb{F}}_p^\times$ denote Serre's tame level nf character: it is of order $p^{nf} - 1$ and under local class field theory corresponds to $\mathcal{O}_{p,n}^\times \twoheadrightarrow k_{p,n}^\times \xrightarrow{\kappa_{0,n}} \overline{\mathbb{F}}_p^\times$, where $E_{p,n}/E_p$ is the unramified extension of degree n (so $\mathrm{Gal}(\overline{E}_p/E_{p,n}) \leq \mathrm{Gal}(\overline{E}_p/E_p)$ have the same inertia subgroup), with ring of integers $\mathcal{O}_{p,n}$, residue field $k_{p,n}$, and $\kappa_{0,n} : k_{p,n} \to \overline{\mathbb{F}}_p$ is any field embedding that extends our chosen $\kappa_0 : k_p \to \overline{\mathbb{F}}_p$.

Theorem 4.12. *The set $W(\overline{r})$ only depends on $\overline{r}|_{D_p}$, and even only on $\overline{r}|_{I_p}$.*

(i) *If $\overline{r}|_{D_p}$ is a direct sum of two characters, then $F(\underline{a},\underline{b}) \in W(\overline{r})$ if and only if there exists $J \subseteq \{0,1,\dots,f-1\}$ such that*

$$\overline{r}|_{I_p} \cong \begin{pmatrix} \omega_f^{\sum_{j\in J}(a_j - b_j + 1)p^j} & 0 \\ 0 & \omega_f^{\sum_{j\notin J}(a_j - b_j + 1)p^j} \end{pmatrix} \otimes \omega_f^{\sum_{j=0}^{f-1} b_j p^j}.$$

---

[29] The maximal ideal $\mathfrak{m}'_{\overline{r}} = (T_v - \lambda_v, S_v - \mu_v : v \notin \Sigma)$ is more standard, but we chose these conventions to fit with $\sigma^\vee$ in formula (4.4). Note that $\mathfrak{m}_{\overline{r}} = \mathfrak{m}'_{\overline{r}^\vee \otimes \omega}$, where ω is the mod-p cyclotomic character..

[30] We ignore the level part of Serre's conjecture, keeping U^p fixed throughout.

(ii) *If $\overline{r}|_{D_p}$ is irreducible, then $F(\underline{a}, \underline{b}) \in W(\overline{r})$ if and only if there exists $J \subseteq \{0, 1, \dots, 2f-1\}$ such that $j \in J \iff j + f \notin J$ (inside $\mathbb{Z}/2f\mathbb{Z}$) and*

$$\overline{r}|_{I_p} \cong \begin{pmatrix} \omega_{2f}^{\sum_{j \in J}(a_j - b_j + 1)p^j} & 0 \\ 0 & \omega_{2f}^{\sum_{j \notin J}(a_j - b_j + 1)p^j} \end{pmatrix} \otimes \omega_f^{\sum_{j=0}^{f-1} b_j p^j}.$$

If $\overline{r}|_{D_p}$ is not semisimple, then the recipe is also known but considerably more complicated. By abuse of notation we will often write $W(\overline{r}|_{D_p})$ for $W(\overline{r})$.

Exercise 4.13. Suppose that $f = 2$.

(i) If $\overline{r}|_{I_p} \cong \omega_2^{(r_0+1)+(r_1+1)p} \oplus 1$ with $0 \le r_j \le p-3$ and $(r_0, r_1) \notin \{(0,0), (p-3, p-3)\}$, determine $W(\overline{r})$ and show that it forms a square in the extension graph.

(ii) If $\overline{r}|_{I_p} \cong \omega_4^{(r_0+1)+(r_1+1)p} \oplus \omega_4^{(r_0+1)p^2+(r_1+1)p^3}$ with $1 \le r_0 \le p-2$ and $0 \le r_1 \le p-3$, determine $W(\overline{r})$ and show that it forms a square in the extension graph.

Theorem 4.12 tells us the $\mathrm{GL}_2(\mathcal{O}_p)$-socle of the smooth $\mathrm{GL}_2(E_p)$-representation $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ as follows. By the isomorphism (4.2), for any Serre weight σ over $\overline{\mathbb{F}}_p$,
(4.5)
$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_p)}(\sigma, S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]) = \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_p)}(\sigma, S(U^p, \overline{\mathbb{F}}_p))[\mathfrak{m}_{\overline{r}}] \cong S(U, \sigma^\vee)[\mathfrak{m}_{\overline{r}}],$$

is nonzero if and only if $\sigma \in W(\overline{\rho})$, where $\overline{\rho} := \overline{r}|_{D_p}$. Note that if we shrink U^p, then the space $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ grows. It is possible to impose reasonable conditions such that the vector spaces (4.5) are of dimension ≤ 1 (cf. §4.10), so that

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_p)}(S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]) \cong \bigoplus_{\sigma \in W(\overline{\rho})} \sigma. \tag{4.6}$$

We will assume this from now on and refer to it as our *multiplicity one hypothesis.*

We can now explain why $D_0(\overline{\rho}) \subseteq S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ is a $\mathrm{GL}_2(\mathcal{O}_p)$-subrepresentation. First, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_p)}(D_0(\overline{\rho})) = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_p)}(S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}])$ by (4.6). On the other hand, $S(U^p, \overline{\mathbb{F}}_p)$, and hence $S(U^p, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}}$, is an injective $\mathrm{GL}_2(\mathcal{O}_p)$-representation by §4.4 (noting that localization is an exact functor), so we obtain a map $D_0(\overline{\rho}) \to S(U^p, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}}$ that is injective, since it is an isomorphism on socles.

Lemma 4.14. *The image of the map $D_0(\overline{\rho}) \hookrightarrow S(U^p, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}}$ of $\mathrm{GL}_2(\mathcal{O}_p)$-representations is contained in $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}] \subseteq S(U^p, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}}$.*

Proof. As $D_0(\overline{\rho})$ is finite-dimensional, its image lands in $S(V, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}}$ for $V = V_p U^p$, where $V_p \le U_p$ is some open subgroup. Let A denote the image of $\mathbb{T}_{\overline{\mathbb{F}}_p}$ in $\mathrm{End}_{\overline{\mathbb{F}}_p}(S(V, \overline{\mathbb{F}}_p))$, which is finite-dimensional as $S(V, \overline{\mathbb{F}}_p)$ is. Therefore A is an artinian ring, so $A = \prod_{\mathfrak{n}} A_{\mathfrak{n}}$ and $S(V, \overline{\mathbb{F}}_p) = \bigoplus_{\mathfrak{n}} S(V, \overline{\mathbb{F}}_p)_{\mathfrak{n}}$, where the index runs over the finitely many maximal ideals $\mathfrak{n}$ of A. It follows that $S(V, \overline{\mathbb{F}}_p)_{\mathfrak{n}} = S(V, \overline{\mathbb{F}}_p)[\mathfrak{n}^i]$ for $i \gg 0$. Applying this to the image $\mathfrak{n}$ of $\mathfrak{m}_{\overline{r}}$ in A, the image of $D_0(\overline{\rho})$ in $S(V, \overline{\mathbb{F}}_p)_{\mathfrak{m}_{\overline{r}}}$ is contained in $S(V, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}^i] \subseteq S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}^i]$.

Choose now $i \ge 1$ minimal, in the sense that the image is not contained in $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}^{i-1}]$. Consider the maps

$$D_0(\overline{\rho}) \to S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}^i]/S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}^{i-1}] \xrightarrow{a} S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}],$$

where $a \in \mathfrak{m}_{\overline{r}}^{i-1}$ is chosen such that the image I is nonzero. If $i > 1$, then the map kills $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_p)}(D_0(\overline{\rho}))$, so any irreducible subrepresentation of I is a constituent of $D_0(\overline{\rho})/\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_p)}(D_0(\overline{\rho}))$ but also of $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_p)}(S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]) = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma$, contradiction (by the construction of $D_0(\overline{\rho})$). □

4.7. **Local-global compatibility.** We continue to assume that $\overline{r}$ is modular in level $U = U_p U^p$ ($U_p = \mathrm{GL}_2(\mathcal{O}_p)$) and that our multiplicity one hypothesis (4.6) is satisfied. The isomorphism (4.6) together with Theorem 4.12 gives us the first hint that the representation $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ might be purely local, i.e. only depends on $\overline{\rho} = \overline{r}|_{D_p}$.

More generally we can consider

$$S[\mathfrak{m}_{\overline{r}}] := \varinjlim_{U} S(U, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}],$$

where the limit runs over *all* compact open subgroups U of $(D \otimes_E \mathbb{A}_E^\infty)^\times$. (To even make sense of this limit, one first needs to check that $S(U, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ does not depend on the choice of Σ above, see [BDJ10, Lemma 4.6].) As for $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$, the space $S[\mathfrak{m}_{\overline{r}}]$ is naturally a smooth representation of $(D \otimes_E \mathbb{A}_E^\infty)^\times$. Note that $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}] = S[\mathfrak{m}_{\overline{r}}]^{U^p}$.

Conjecture 4.15 ([BDJ10, Conj. 4.7])**.** *The $(D \otimes_E \mathbb{A}_E^\infty)^\times$-representation $S[\mathfrak{m}_{\overline{r}}]$ factors as restricted tensor product*

$$S[\mathfrak{m}_{\overline{r}}] \cong \bigotimes_{v \nmid \infty}{}' \pi_v,$$

where π_v is a smooth representation of $(D \otimes_E E_v)^\times$, which is explicit and only depends on $\overline{r}|_{D_v}$ if $v \nmid p\infty$.

The hope is that π_p also depends only on $\overline{r}|_{D_p}$.

In the case where $E = \mathbb{Q}$ the analog of Conjecture 4.15 and the hope about π_p was *proved* by Emerton, in the context of classical modular forms (instead of modular forms on definite quaternion algebras).

4.8. **Existence of supersingular representations.** Our aim is to sketch a global argument that irreducible supersingular representations of $G := \mathrm{GL}_2(E_p)$ exist, without using diagrams §3. This method generalizes to $\mathrm{GL}_n(F)$ for any p-adic field F, cf. [HKV20, §6].

Let $\tau_i : E_p \to \overline{\mathbb{Q}}_p$ denote the unique embedding that induces $\kappa_i : k_p \to \overline{\mathbb{F}}_p$ ($0 \le i \le f-1$) on residue fields.

Suppose that $\underline{a} = (a_i)_i, \underline{b} = (b_i)_i \in \mathbb{Z}^f$ such that $0 \le a_i - b_i \le p-1$. We suppose moreover that $a_i + 2b_i$ is independent of i, to avoid parity issues. Let

$$\sigma^0 := \bigotimes_{i=0}^{f-1} \Big[(\mathrm{Sym}^{a_i - b_i}(\mathcal{O}_p^2) \otimes \det{}^{b_i}) \otimes_{\mathcal{O}_p, \tau_i} \overline{\mathbb{Z}}_p \Big],$$

which we consider as representation of $U_p = \mathrm{GL}_2(\mathcal{O}_p)$, as usual. Then $\overline{\sigma} := \sigma^0 \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong F(\underline{a}, \underline{b})$ (Serre weight), and we let $\sigma := \sigma^0 \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$. Suppose for simplicity that U is sufficiently small in the sense that $U \cap d^{-1} D^\times d = 1$ for all $d \in (D \otimes_E \mathbb{A}_E^\infty)^\times$, so that $S(U, \sigma^0)$ is a $\overline{\mathbb{Z}}_p$-lattice in $S(U, \sigma)$ and $S(U, \sigma^0) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \cong S(U, \overline{\sigma})$, cf. (4.2).

We first compare classical and mod-p Hecke actions at p. To define classical Hecke operators on $S(U, \sigma)$ at p, we let $g_p f := g_p f((-)g_p)$ for $g_p \in \mathrm{GL}_2(E_p)$, so that $\mathrm{GL}_2(\mathcal{O}_p)$ acts trivially and hence $\mathcal{H}_p = \overline{\mathbb{Z}}_p[\mathrm{GL}_2(\mathcal{O}_p)\backslash \mathrm{GL}_2(E_p)/\mathrm{GL}_2(\mathcal{O}_p)]$ acts by the same formula $f \mapsto \sum_{i=1}^r x_i f$ as in §4.1. (This really corresponds to the usual Hecke action at p under the comparison isomorphism (4.1).) On the other hand,

$$S(U,\overline{\sigma}) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_p)}(\overline{\sigma}^\vee, S(U^p,\overline{\mathbb{F}}_p)) \tag{4.7}$$

by (4.2), so $\mathcal{H}_G(\overline{\sigma}^\vee)$ acts on it, see §2.2. Let $T := T_{(0,-1)} \in \mathcal{H}_G(\overline{\sigma}^\vee)$ in the notation of §2.3.

Lemma 4.16.

(i) *We have* $\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)\sigma^0 \subseteq p^{\sum_i b_i}\sigma^0$ *and the induced map* $p^{-\sum_i b_i}\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right) : \overline{\sigma} \to \overline{\sigma}$ *is the projection* $\overline{\sigma} \twoheadrightarrow \overline{\sigma}_{\left(\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right)} \xleftarrow{\sim} \overline{\sigma}^{\left(\begin{smallmatrix} 1 & 0 \\ * & 1\end{smallmatrix}\right)} \hookrightarrow \overline{\sigma}$.

(ii) *We have a commutative diagram:*

$$\begin{tikzcd} S(U,\sigma^0) \arrow[r, "p^{-\sum_i b_i} T_p"] \arrow[d] & S(U,\sigma^0) \arrow[d] \\ S(U,\overline{\sigma}) \arrow[r, "T"] & S(U,\overline{\sigma}) \end{tikzcd}$$

Proof. (i) It suffices to note that the action of the torus on the algebraic representation $\mathrm{Sym}^k(\mathrm{std}) \otimes \det^\ell$ of GL_2 is the direct sum of characters $(-)^{\ell+i} \otimes (-)^{\ell+k-i}$, $0 \le i \le k$, and p acts by the lowest valuation only on the lowest weight vector $(-)^\ell \otimes (-)^{\ell+k}$.

(ii) An easy check shows that we have an automorphism of (commutative) Hecke algebras $\mathcal{H}_G(\overline{\sigma}) \cong \mathcal{H}_G(\overline{\sigma}^\vee)$, $\varphi \mapsto \varphi'$ with $\varphi'(g) = \varphi(g^{-1})^\vee$. The algebra $\mathcal{H}_G(\overline{\sigma})$ acts on $S(U,\overline{\sigma})$ by the formula $f \mapsto (d \mapsto \sum_{g \in \mathrm{GL}_2(E_p)/\mathrm{GL}_2(\mathcal{O}_p)} \varphi(g) f(dg))$ (for $\varphi \in \mathcal{H}_G(\overline{\sigma})$, $f \in S(U,\overline{\sigma})$). A slightly tedious check show that with these identifications this action of $\varphi \in \mathcal{H}_G(\overline{\sigma})$ on the left is identified with the action of $\varphi' \in \mathcal{H}_G(\overline{\sigma}^\vee)$ on the right of (4.7).

Now take $\varphi = T_1 = T_{(1,0)} \in \mathcal{H}_G(\overline{\sigma})$, so $\varphi' = T$. We check $p^{-\sum_i b_i} g : \sigma^0 \to \sigma^0$ induces $T_1(g) : \overline{\sigma} \to \overline{\sigma}$ for all $g \in \mathrm{GL}_2(\mathcal{O}_p)\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)\mathrm{GL}_2(\mathcal{O}_p)$: it is true for $g = \left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)$ by (i) and Exercise 2.16, and hence follows in general.

Finally,

$$(p^{-\sum_i b_i} T_p f)(d) = \sum_g p^{-\sum_i b_i} g f(dg),$$

where the index g runs through $\mathrm{GL}_2(\mathcal{O}_p)\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)\mathrm{GL}_2(\mathcal{O}_p)/\mathrm{GL}_2(\mathcal{O}_p)$ which reduces mod p to

$$\sum_g T_1(g) f(dg) = (T_1 f)(d)$$

by the preceding paragraph. $\square$

Suppose now that $S(U,\overline{\sigma})[\mathfrak{m}_{\overline{r}}] \neq 0$ for some Galois representation $\overline{r}$ such that $\overline{r}|_{D_p}$ is irreducible.[31]

Claim: Any Hecke eigenvalue of $\mathcal{H}_G(\overline{\sigma}^\vee)$ on $S(U,\overline{\sigma})[\mathfrak{m}_{\overline{r}}]$ is supersingular.

If this is true, we pick any $f : \overline{\sigma}^\vee \to S(U^p,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ that is a T-eigenvector, so $Tf = 0$. Let π' be the $\mathrm{GL}_2(E_p)$-subrepresentation of $S(U^p,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ generated by the image of f. As π' is by definition finitely generated as $\mathrm{GL}_2(E_p)$-representation, it

[31] This is possible, for example by using automorphic induction from a suitable Hecke character, though the details are a bit involved.

admits a maximal subrepresentation M by a standard application of Zorn's lemma. Let $\pi := \pi'/M$ be the corresponding irreducible quotient of π'. Then π is admissible, as it is a subquotient of the admissible representation $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$. Moreover, $f : \overline{\sigma}^\vee \to \pi' \twoheadrightarrow \pi$ is nonzero, giving a supersingular Hecke eigenvalue, so π is supersingular by Corollary 2.36(ii), as desired.

To justify the claim, suppose by contradiction that there is an eigenform $f \in S(U, \overline{\sigma})[\mathfrak{m}_{\overline{r}}]$ such that $Tf = cf$ with $c \in \overline{\mathbb{F}}_p^\times$. By an abstract algebra argument (using only that we have a commuting set of linear operators), the Hecke eigenvalues lift, i.e. there exists an eigenform f' in $S(U, \sigma^0)$ whose Hecke eigenvalues reduce to those of f. In particular, (i) by Lemma 4.16, $T_p f' = \lambda_p f'$ with $\lambda_p \in p^{\sum_i b_i} \overline{\mathbb{Z}}_p^\times$ and (ii) the Galois representation r associated to f' by Theorem 4.9 lifts $\overline{r}$ (checking on Hecke eigenvalues at $v \notin \Sigma$).

We now show using a little bit of p-adic Hodge theory that $r|_{D_p}$ is reducible, which gives the desired contradiction, since $\overline{r}|_{D_p}$ is irreducible. To keep the argument as simple as possible, let's suppose that the residue degree $f = 1$ (in general, see [EGH13, §4.5]). The last sentence of Theorem 4.9 says that the crystalline Frobenius of $\mathrm{D}_{\mathrm{cris}}(r|_{D_p})$ has characteristic polynomial $X^2 - \lambda_p X + p\mu_p$, where $\lambda_p \in p^{b_0} \overline{\mathbb{Z}}_p^\times$. On the other hand, the Hodge–Tate weights of $\mathrm{D}_{\mathrm{cris}}(r|_{D_p})$ are $a_0 + b_0 + 1$ and b_0. This means that both the Newton and the Hodge polygon of $\mathrm{D}_{\mathrm{cris}}(r|_{D_p})$ have slopes b_0 and $a_0 + b_0 + 1$[32], giving that $r|_{D_p}$ is reducible.

Remark 4.17. One of the projects concerns the existence of supersingular representations.

4.9. **Patching.** A key to proving Theorem 4.12, as well as to various other results on the global representations $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ is Taylor–Wiles–Kisin patching (here, [Gee22] provides a great introduction).

Since the patching construction is ultimately based on a pigeonhole principle, it is important that we work with a finite residue field, so we fix $L/\mathbb{Q}_p$ a *finite* extension and take an absolutely irreducible, modular Galois representation $\overline{r} : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(k_L)$. We assume L is large, in particular $[k_L : \mathbb{F}_p] \geq f$ so that all Serre weights can be defined over k_L. We also assume $L/\mathbb{Q}_p$ unramified, which will be useful in §4.10.

To keep the notation as simple as possible and to focus on the essential ideas, we now assume that D is split at all finite places, $U_v = \mathrm{GL}_2(\mathcal{O}_v)$ for all $v \nmid p\infty$, and that $\overline{r}$ is unramified at all finite places.[33] We also cheat slightly by not fixing central characters on the Galois side (this is important to get correct statements, but in the parallel context of a definite unitary group of rank 2 over E this is not needed).

Let R denote the universal framed deformation ring of $\overline{r}|_{D_p}$, which is the ring representing the functor

$$\begin{aligned} &\mathcal{C} \to \mathrm{Set} \\ &A \mapsto \{\text{homomorphisms } \rho : D_p \to \mathrm{GL}_2(A) \text{ reducing to } \overline{r}|_{D_p}\}, \end{aligned}$$

[32] For the Newton polygon we only calculated one slope; however, the endpoints of the two polygons always coincide.

[33] These assumptions are too strong, since e.g. the assumption on D forces that $[E : \mathbb{Q}]$ is even. Instead, one allows finitely many bad places and R_∞ will be a power series ring over a completed tensor product of local Galois deformation rings at all bad places.

where $\mathcal{C}$ is the category of complete noetherian local $\mathcal{O}_L$-algebras together with an isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} k_L$.

The patching method gives (non-canonically) the following: a power series ring $R_\infty = R[\![x_1, \dots, x_h]\!]$ for some $h \geq 1$ and an exact functor

$$M_\infty : \{\text{f.g. } \mathcal{O}_L\text{-modules with continuous } \mathrm{GL}_2(\mathcal{O}_p)\text{-action}\} \to \{\text{f.g. } R_\infty\text{-modules}\}$$

such that

(4.8) the action of R_∞ on $M_\infty(\sigma^0(\lambda,\tau))$ factors through $R_\infty^{\lambda;\tau}$ and $M_\infty(\sigma^0(\lambda,\tau))$ is a maximal Cohen–Macaulay $R_\infty^{\lambda;\tau}$-module;

and

$$M_\infty(W)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_p)}(W, S(U^p, k_L)[\mathfrak{m}_{\overline{r}}])^\vee \quad \forall W, \tag{4.9}$$

where $\mathfrak{m}_\infty$ denotes the maximal ideal of R_∞.

Remark 4.18.

(i) The original Taylor–Wiles–Kisin patching construction works with a fixed pair (λ,τ) and a fixed lattice $\sigma^0(\lambda,\tau)$, producing the maximal Cohen–Macaulay module $M_\infty(\sigma^0(\lambda,\tau))$. Emerton–Gee–Savitt, building on earlier ideas of Kisin, realized that the patching argument can be done simultaneously for all coefficients. The exactness of M_∞ ultimately comes from the exactness of $S(U,-)$ in §4.4.

(ii) You do no not need to know the definition of maximal Cohen–Macaulay (CM) modules. The important thing is that it implies that the support of the module $M_\infty(\sigma^0(\lambda,\tau))$ is a union of (maximal dimensional) irreducible components of $\mathrm{Spec}\, R_\infty^{\lambda;\tau}$.

In particular, by (4.9) and (4.6) and Nakayama's lemma we see that

$$W(\overline{r}) = \{\overline{\sigma} : M_\infty(\overline{\sigma}) \neq 0\}. \tag{4.10}$$

To give one illustration of the power of patching functors, let us show how it can be used to prove Theorem 4.12, following Emerton–Gee–Savitt.[34] Let $W^?(\overline{r})$ denote the explicit set of Serre weights described in Theorem 4.12, and recall that $W^?(\overline{r})$ is a hypercube in the extension graph. To show that $W(\overline{r}) = W^?(\overline{r})$ we need two (non-trivial!) inputs that come from p-adic Hodge theory:

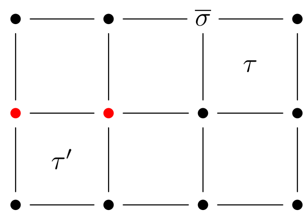
(i) for any tame τ, R^τ is either 0 or an integral domain such that R^τ/p has precisely $|\,\mathrm{JH}(\overline{\sigma}(\tau)) \cap W^?(\overline{r})|$ irreducible components;

(ii) for any low weight λ (i.e. $0 \leq \lambda_\tau \leq p-1$ for all τ), R^λ is either 0 or a power series ring over $\mathcal{O}_L$.

We first show that $W(\overline{r}) \subseteq W^?(\overline{r})$. If $\overline{\sigma} \in W(\overline{r}) \setminus W^?(\overline{r})$, then by looking at the extension graph we can choose a tame τ such that $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))$ and $\mathrm{JH}(\overline{\sigma}(\tau)) \cap W^?(\overline{r}) = \varnothing$ (cf. §3.3; we are using that the $\sigma(\tau)$ are the Deligne–Lusztig

[34]Unfortunately the argument does not quite work like this in the context of quaternion algebras, due to parity constraints, but it does work for definite unitary groups. Also, strictly speaking we need to impose a slight genericity condition on $\overline{r}|_{D_p}$, to ensure that $W^?(\overline{\rho})$ is a hypercube and that (i) and (ii) are valid.

representations appearing there), where the red dots denote $W^?(\overline{r})$:

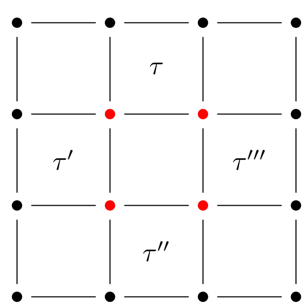


Then $M_\infty(\overline{\sigma}) \neq 0$ by (4.10). Using the exactness of M_∞, we have $M_\infty(\overline{\sigma}(\tau)) \neq 0$ and hence $M_\infty(\sigma^0(\tau)) \neq 0$, as $M_\infty(\sigma^0(\tau))/p \cong M_\infty(\overline{\sigma}(\tau)) \neq 0$, again by exactness. As $M_\infty(\sigma^0(\tau))$ is an R_∞^τ-module by (4.8), it follows that $R^\tau \neq 0$, contradicting (i)!

Now we show that $W^?(\overline{r}) \subseteq W(\overline{r})$. Again by the extension graph, see picture above, there exists a tame τ' such that $\mathrm{JH}(\overline{\sigma}(\tau')) \supseteq W^?(\overline{r})$. By modularity of $\overline{r}$, there exists a Serre weight $\overline{\sigma}' \in W(\overline{r})$, so $\overline{\sigma}' \in W^?(\overline{r}) \subseteq \mathrm{JH}(\overline{\sigma}(\tau'))$. As in the preceding paragraph we deduce that $M_\infty(\sigma^0(\tau')) \neq 0$. By (4.8) and (i), $M_\infty(\sigma^0(\tau'))$ is supported on all of $R_\infty^{\tau'}$, and hence $M_\infty(\sigma^0(\tau'))/p \cong M_\infty(\overline{\sigma}(\tau'))$ is supported on all of $R_\infty^{\tau'}/p$. Using the exactness of M_∞ again, the support of $M_\infty(\overline{\sigma}(\tau'))$ is the union of the supports of all $M_\infty(\overline{\sigma})$ for $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau'))$, or equivalently for $\overline{\sigma} \in W(\overline{r}) \subseteq W^?(\overline{r})$ by (4.10) and our choice of τ'. As $R_\infty^{\tau'}/p$ has precisely $|W^?(\overline{r})| (\geq |W(\overline{r})|)$ irreducible components by (i) and our choice of τ', it suffices to show that each $M_\infty(\overline{\sigma})$ is supported on just one irreducible component. For this, choose a low weight λ such that $\overline{\sigma}(\lambda) \cong \sigma$. By (4.8) and (ii), we deduce (similarly to the case of τ') that $M_\infty(\overline{\sigma})$ is supported precisely on R_∞^λ/p, which is irreducible.

4.10. **Multiplicity one.** We finally justify our multiplicity one hypothesis! First note that any maximal CM module over a *regular* local ring is necessarily free, by [Mat89, Thm. 19.1, Thm. 19.2] (first exploited by Diamond in this context). In particular, $M_\infty(\sigma^0(\lambda))$ is free over R_∞^λ for any low weight λ (by (ii) above). So necessarily, $R_\infty^\lambda = R_\infty / \mathrm{Ann}_{R_\infty}(M_\infty(\sigma^0(\lambda))$ and we just say that $M_\infty(\sigma^0(\lambda))$ is free over its support. By reduction, $M_\infty(\overline{\sigma})$ is free over its support for any $\overline{\sigma} \in W(\overline{r})$.

Choose any tame type τ such that $|\mathrm{JH}(\overline{\sigma}(\tau)) \cap W(\overline{r})| = 2$ (cf. §3.3):



Then, by exactness of M_∞ and (4.10) we have

$$0 \to M_\infty(\overline{\sigma}_1) \to M_\infty(\overline{\sigma}(\tau)) \to M_\infty(\overline{\sigma}_2) \to 0, \tag{4.11}$$

where $\mathrm{JH}(\overline{\sigma}(\tau)) \cap W(\overline{r}) = \{\overline{\sigma}_1, \overline{\sigma}_2\}$. Another input from p-adic Hodge theory is that R^τ is regular if $|\mathrm{JH}(\overline{\sigma}(\tau)) \cap W(\overline{r})| \leq 2$ (here we use that $L/\mathbb{Q}_p$ is unramified), so that $M_\infty(\sigma(\tau))$, hence also $M_\infty(\overline{\sigma}(\tau))$, is free over its support. By localizing the sequence (4.11) at the two minimal primes of R^τ/p, we deduce that all three modules have the same rank (over their respective supports).

By the extension graph, any two elements of $W(\overline{r})$ can be connected by a sequence of tame types as above, so by the preceding paragraph all $M_\infty(\overline{\sigma})$ ($\overline{\sigma} \in W(\overline{r})$) have the same rank, so all $\operatorname{Hom}_K(\overline{\sigma}^\vee, \pi(\overline{r}))$ have the same dimension by (4.9). With a bit more effort one can achieve rank (resp. dimension) equal to 1.

4.11. **Some recent progress.** We would like to understand the $\mathrm{GL}_2(E_p)$-representation $\pi(\overline{r}) := S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ further, under our multiplicity one hypothesis. Recall that we hope that $\pi(\overline{r})$ only depends on $\overline{\rho} := \overline{r}|_{D_p}$. The following results give some, albeit weak, evidence for that.

As in §4.9 we work over the coefficient field k_L, where $L/\mathbb{Q}_p$ is a large finite extension. Let $K_1 \subseteq I_1 \subseteq I \subseteq K$ be defined as in sections 1–3 with $F := E_p$.

Theorem 4.19 (Emerton–Gee–Savitt). *We have* $\pi(\overline{r})^{I_1} \cong D_0(\overline{\rho})^{I_1}$ *as* I*-representations.*

For any smooth character $\overline{\chi} : I \to k_L^\times$ we have by Frobenius reciprocity:

$$\operatorname{Hom}_I(\overline{\chi}, \pi(\overline{r})^{I_1}) = \operatorname{Hom}_I(\overline{\chi}, \pi(\overline{r})) = \operatorname{Hom}_K(\operatorname{Ind}_I^K \overline{\chi}, \pi(\overline{r})),$$

which is dual to $M_\infty(\operatorname{Ind}_I^K \overline{\chi})/\mathfrak{m}_\infty$.

The key result is:

Proposition 4.20 ([EGS15]). *The patched module* $M_\infty(\sigma^0(\tau))$ *is cyclic for any tame type* τ *and all lattices* $\sigma^0(\tau)$ *such that* $\overline{\sigma}(\tau)$ *has an irreducible cosocle.*

(We already saw the case where $|\operatorname{JH}(\overline{\sigma}(\tau)) \cap W(\overline{r})| \le 2$ above; in that case we didn't even need an irreducible cosocle!) Applied to $\sigma^0(\tau) = \operatorname{Ind}_I^K \chi$, where $\chi : I \to \mathcal{O}_L^\times$ is the Teichmüller lift of $\overline{\chi}$, Proposition 4.20 gives that $\operatorname{Hom}_I(\overline{\chi}, \pi(\overline{r})^{I_1})$ is at most 1-dimensional for any $\overline{\chi}$. As I/I_1 acts semisimply, it remains to check:

Lemma 4.21. *We have* $\operatorname{Hom}_I(\overline{\chi}, \pi(\overline{r})^{I_1}) \neq 0$ *if and only if* $\operatorname{Hom}_I(\overline{\chi}, D_0(\overline{\rho})^{I_1}) \neq 0$.

Proof. Recall that

$$\bigoplus_{\overline{\sigma} \in W(\overline{\rho})} \overline{\sigma} \subseteq D_0(\overline{\rho}) \subseteq \pi(\overline{r})^{K_1} \subseteq \operatorname{inj}_\Gamma\Big(\bigoplus_{\overline{\sigma} \in W(\overline{\rho})} \overline{\sigma}\Big) \tag{4.12}$$

by §4.4, respectively (4.6), so $D_0(\overline{\rho})^{I_1} \subseteq \pi(\overline{r})^{I_1}$ and "$\Leftarrow$" holds. Conversely, suppose that $0 \neq \operatorname{Hom}_I(\overline{\chi}, \pi(\overline{r})^{I_1}) = \operatorname{Hom}_K(\operatorname{Ind}_I^K \overline{\chi}, \pi(\overline{r}))$. The image V of such a nonzero K-homomorphism is a Γ-representation such that $\operatorname{JH}(V) \cap W(\overline{\rho}) \neq \varnothing$ by (4.6). Let V' be a minimal quotient of V such that $\operatorname{JH}(V') \cap W(\overline{\rho}) \neq \varnothing$. Then $\operatorname{soc}_K(V') = \overline{\sigma} \in W(\overline{\rho})$ and $\operatorname{JH}(V'/\overline{\sigma}) \cap W(\overline{\rho}) = \varnothing$, so V' injects into $D_{0,\overline{\sigma}}(\overline{\rho}) \subseteq D_0(\overline{\rho})$ by Lemma 3.36(i), so we get $\operatorname{Ind}_I^K \overline{\chi} \twoheadrightarrow V \twoheadrightarrow V' \hookrightarrow D_0(\overline{\rho})$, as desired. □

Remark 4.22. Assume $\det \overline{\rho}(p) = 1$. Theorem 4.19 implies that there exists an $\mathcal{N}$-action on $D_1(\overline{\rho}) = \pi(\overline{r})^{I_1}$ extending the given I-action such that we have an inclusion of diagrams $(D_0(\overline{\rho}), D_1(\overline{\rho}), \mathrm{incl}) \hookrightarrow (\pi(\overline{r})|_{KZ}, \pi(\overline{r})|_{\mathcal{N}}, \mathrm{incl})$. By Corollary 3.12, $\pi(\overline{r})$ contains a Breuil–Paškūnas representation, meaning a representation arising from the diagram $(D_0(\overline{\rho}), D_1(\overline{\rho}), \mathrm{incl})$.

Theorem 4.23 (Le–Morra–Schraen, Hu–Wang, Le). *We have* $\pi(\overline{r})^{K_1} \cong D_0(\overline{\rho})$ *as* Γ*-representations.*

Exercise 4.24. Show that this statement is equivalent to $M_\infty(\operatorname{proj}_\Gamma \overline{\sigma})$ being cyclic for any $\overline{\sigma} \in W(\overline{\rho})$. (Use (4.9) and the definition of $D_0(\overline{\rho})$.)

If $\overline{\rho}$ is semisimple, Theorem 4.23 follows by proving the following representation-theoretic criterion: if $D_0(\overline{\rho}) \subseteq W \subseteq \mathrm{inj}_\Gamma(\bigoplus_{\overline{\sigma}\in W(\overline{\rho})} \overline{\sigma})$ is any Γ-representation such that

$$\dim_{k_L} \mathrm{Hom}_\Gamma(\overline{\sigma}(\tau), W) \leq 1$$

for all tame types τ and all lattices $\sigma^0(\tau)$ such that $\overline{\sigma}(\tau)$ has an irreducible cosocle, then $W = D_0(\overline{\rho})$. Applying this to $W = \pi(\overline{r})^{K_1}$ using Proposition 4.20 and (4.9) gives the theorem. If $\overline{\rho}$ is non-semisimple, the proof is more subtle and requires more inputs from p-adic Hodge theory.

Remark 4.25. Hu [Hu10] showed that there exist Breuil–Paškūnas representations π such that $D_0(\overline{\rho}) \subsetneq \pi^{K_1}$, so Theorem 4.23 shows that some Breuil–Paškūnas representations do not occur globally!

Another application of patched modules:

Theorem 4.26 (Dotto–Le)**.** *The diagram* $(\pi(\overline{r})^{K_1}, \pi(\overline{r})^{I_1}, \mathrm{incl})$ *is purely local, i.e. only depends on* $\overline{\rho}$*.*

Again this rules out many Breuil–Paškūnas representations, since it constrains the action of $\mathcal{N}$ on $D_1(\overline{\rho})$.

The next important step was to understand the "size" of the representation $\pi(\overline{r})$, which can be measured by its so-called Gelfand–Kirillov dimension. It is known that $\pi(\overline{r})$ has a central character, so that it is a smooth representation of G/Z_1, where $Z_1 := Z \cap K_1$. Therefore the linear dual $\pi(\overline{r})^\vee$ is a finitely generated over the noetherian ring $k_L[\![H]\!]$ (see §1.5.3), where H is any fixed compact open subgroup of G/Z_1. The G-representation $\pi(\overline{r})$ is said to have *Gelfand–Kirillov dimension* $d \in \mathbb{Z}_{\geq 0}$ if $j := 3f - d$ is minimal such that $\mathrm{Ext}^j_{k_L[\![H]\!]}(\pi(\overline{r})^\vee, k_L[\![H]\!]) \neq 0$, which is a noncommutative generalization of the "codimension of the support" of the module $\pi(\overline{r})^\vee$ and is independent of the choice of H. Concretely, this is true if and only if there exist real constants $0 < a \leq b$ such that $ap^{md} \leq \dim_{k_L}(\pi(\overline{r})^{K_m}) \leq bp^{md}$ for all $m \geq 1$.

Theorem 4.27 (Breuil–H.–Hu–Morra–Schraen; Hu–Wang)**.** *Assume some Taylor–Wiles hypotheses and a genericity condition on* $\overline{\rho}$*. Then* $\pi(\overline{r})$ *has Gelfand–Kirillov dimension* f*.*

The crucial input comes again from patching. Let $\mathfrak{m}$ denote the unique maximal of the local ring $\Lambda := k_L[\![I_1/Z_1]\!]$. Patching allowed us to determine $\pi(\overline{r})[\mathfrak{m}^3]$, which was sufficient to show that the associated graded module $\mathrm{gr}_\mathfrak{m}(\pi(\overline{r})^\vee) := \bigoplus_{i\geq 0} \mathfrak{m}^i\pi(\overline{r})^\vee/\mathfrak{m}^{i+1}\pi(\overline{r})^\vee$ over $\mathrm{gr}_\mathfrak{m}\Lambda$ is killed by an explicit ideal J such that $(\mathrm{gr}_\mathfrak{m}\Lambda)/J$ is commutative of dimension f. This gave the desired upper bound on the Gelfand–Kirillov dimension (the lower part is easier and was known earlier).

We briefly list several results that use Theorem 4.27 as key input:

Theorem 4.28 (Breuil–H.–Hu–Morra–Schraen)**.** *Assume some Taylor–Wiles hypotheses and a genericity condition on* $\overline{\rho}$*. Then the* G*-representation* $\pi(\overline{r})$ *is generated by* $D_0(\overline{\rho})$*.*

In particular, under these hypotheses, $\pi(\overline{r})$ is itself a Breuil–Paškūnas representation. If $\overline{\rho}$ is reducible, we could recently give evidence for the speculation mentioned at the end of §3.4.

Theorem 4.29 (Breuil–H.–Hu–Morra–Schraen)**.** *Assume some Taylor–Wiles hypotheses and a genericity condition on $\overline{\rho}$. Then the G-representation $\pi(\overline{r})$ is of finite length. More precisely, it is of length at least 3 (if $f \geq 2$) and at most $f+1$.*

Appendix A. Appendix

A.1. **A few words about topological groups.** Let H be any topological group, i.e. a group that carries a topology such that multiplication and inverse are continuous.

Remark A.1.

(i) Left and right translations $H \to H$ (i.e. $h \mapsto h'h$, $h \mapsto hh'$ for a fixed $h' \in H$) are homeomorphisms.
(ii) Any open subgroup of H is also closed (the complement is a union of cosets, hence open).
(iii) Any closed subgroup of finite index of H is open (the complement is a finite union of cosets, hence closed).
(iv) If H is compact then any open subgroup of H has finite index (the cosets form a disjoint open cover).

A *profinite group* is a compact Hausdorff topological group H such that H has a fundamental system of neighbourhoods of 1 consisting of normal subgroups. (One can show that the last condition can be replaced with H being totally disconnected.) Equivalently, H is an inverse limit of finite groups, carrying the coarsest topology such that all projection maps are continuous. We say that H is *pro-p* if the index $(H : N)$ is a power of p for each open normal subgroup N. The Sylow theorems generalize to profinite groups; in particular, any profinite group contains a maximal (closed) pro-p subgroup, which is unique up to conjugation.

Any closed subgroup or quotient group of a profinite (resp. pro-p) group is profinite (resp. pro-p). An arbitrary product of profinite (resp. pro-p) groups is profinite (resp. pro-p).

Basic examples of profinite groups are finite groups with discrete topology, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, and $\mathbb{Z}_p^\times = \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times$, where the limit is over integers $n \geq 1$. The second example is pro-p, whereas the third has pro-p Sylow subgroup $1 + p\mathbb{Z}_p$ of index $p - 1$.

A.2. **Representations of finite groups.** Suppose Γ is a finite group and k a field. Then the category of Γ-representations over k (equivalently $k[\Gamma]$-modules) is semisimple by Maschke if $|\Gamma|$ is invertible in k, and if $k = \overline{k}$ the number of irreducible representations equals the number of conjugacy classes in Γ.

Now suppose that $\operatorname{char} k = p > 0$ with $p \mid |\Gamma|$. Then the category of Γ-representations is no longer semisimple, and if $k = \overline{k}$ the number of irreducible representations equals the number of p-regular conjugacy classes in Γ, where $\gamma \in \Gamma$ is p-regular if its order is prime to p. (See [Her12b, Ex. 3].)

If V is a finite-dimensional Γ-representation, then $\mathrm{JH}(V)$ denotes the set of its Jordan–Hölder factors (up to isomorphism) and the *length* of V denotes the length of any Jordan–Hölder filtration (so $|\mathrm{JH}(V)|$ is at most the length of V). We define the *semisimplification* V^{ss} to be the direct sum of all Jordan–Hölder factors of V (with multiplicities).

Note that we may consider Γ-representations as smooth by giving Γ the discrete topology. In particular, we have the notion of socle, soc_Γ (Definition 1.9), which is functorial and left exact. If V is a finite-dimensional representation, it also has a maximal semisimple quotient, its *cosocle* $\mathrm{cosoc}_\Gamma V$, which is functorial and right exact (note $\mathrm{cosoc}_\Gamma V \cong (\mathrm{soc}_\Gamma V^\vee)^\vee$).

We also have an induced representation $\mathrm{Ind}_\Delta^\Gamma W(= \text{c-}\mathrm{Ind}_\Delta^\Gamma W)$ for any representation W of a subgroup $\Delta \le \Gamma$, and both forms of Frobenius reciprocity hold (Prop. 1.14). It follows that $k[\Gamma] \cong \mathrm{Ind}_1^\Gamma 1$ is both projective and injective.

By $(*_3)$ in §3.1 we then have $k[\Gamma] = \mathrm{inj}_\Gamma(k[\Gamma]) \cong \mathrm{inj}_\Gamma(\mathrm{soc}_\Gamma k[\Gamma])$. By Frobenius reciprocity we compute $\mathrm{soc}_\Gamma k[\Gamma] \cong \bigoplus_\sigma \sigma^{\oplus \dim_k \sigma}$, where the sum runs over all irreducible representations of Γ. It follows using $(*_4)$ that

$$k[\Gamma] \cong \bigoplus_\sigma (\mathrm{inj}_\Gamma \sigma)^{\oplus \dim_k \sigma}.$$

A further consequence is that each $\mathrm{inj}_\Gamma \sigma$ (for σ irreducible) is projective, as $k[\Gamma]$ is. In fact $\mathrm{cosoc}_\Gamma(\mathrm{inj}_\Gamma \sigma) \cong \sigma$ (see Exercise A.2 for a special case of the proof), which implies that $\mathrm{inj}_\Gamma \sigma \cong \mathrm{proj}_\Gamma \sigma$. Here, $\mathrm{proj}_\Gamma \sigma$ is the *projective cover* of σ, which satisfies the dual definition of an injective envelope (it is a projective object surjecting onto σ such that no proper subobject surjects onto σ).

Exercise A.2. Prove directly that $\mathrm{inj}_\Gamma 1$ surjects onto 1, by showing that $\sum_{\gamma\in\Gamma} \gamma$ induces an isomorphism $P_\Gamma \xrightarrow{\sim} P^\Gamma$ for any finite-dimensional projective representation P. (Hint: reduce to $P = k[\Gamma]$ by using that P is a direct summand of a finite free $k[\Gamma]$-module.)

Suppose now that $\mathcal{O}_L$ is a discrete valuation ring with residue field k, for example $\mathcal{O}_L = W(k)$ (Witt vectors of k). Let L denote the fraction field and ϖ a uniformizer. If V is any finite-dimensional Γ-representation over L, then there exists a Γ-stable lattice $V^0 \subseteq V$, where lattice means a finite free $\mathcal{O}_L$ module such that $V^0[1/\varpi] = V$. (For example, let $M \subseteq V$ denote any lattice and take $V^0 := \sum_{\gamma\in\Gamma} \gamma M$.) By the Brauer–Nesbitt theorem, the semisimplified *reduction* $(V^0/\varpi V^0)^{\mathrm{ss}}$ of V is well defined, i.e. independent of the choice of V^0.

Exercise A.3. Suppose that p^n is the largest power of p dividing Γ. Show that p^n divides $\dim_k(\mathrm{inj}_\Gamma \sigma)$ for any irreducible representation σ. (Hint: use $(*_1)$ to restrict to a Sylow subgroup and recall Example 3.5.)

A.3. **Multiplicity-free representations.** The following is very useful and less well-known than it should be (see [Alp80]).

Suppose that Γ is any group and W a finite-dimensional Γ-representation over a field k. Suppose that W is multiplicity free, i.e. each irreducible constituent of W occurs only once in W. Then the "submodule structure" is determined completely by a partial order on the set $\mathrm{JH}(W)$ of Jordan–Hölder factors, as follows.

For any $\sigma \in \mathrm{JH}(W)$ there exists a unique submodule W_σ such that $\mathrm{cosoc}_\Gamma W_\sigma \cong \sigma$. The easiest way to see this is that there is a unique nonzero homomorphism $\mathrm{proj}_\Gamma \sigma \to W$ up to scalar, dually to $(*_6)$, and we let W_σ be its image.

We define a partial order $\le$ (or more precisely $\le_W$) on $\mathrm{JH}(W)$: for $\sigma, \tau \in \mathrm{JH}(W)$ we say $\sigma \le \tau$ if $W_\sigma \subseteq W_\tau$. A *lower set* in the poset $\mathrm{JH}(W)$ is any subset X such that ($\sigma \le \tau$ and $\tau \in X$) implies $\sigma \in X$. The following is a fun exercise!

Lemma A.4. *We have an isomorphism of lattices*[35]*:*

$$\begin{array}{ccc} \{\textit{subrepresentations of } W\} & \xrightarrow{\sim} & \{\textit{lower sets in } (\mathrm{JH}(W), \le)\} \\ W' & \longmapsto & \mathrm{JH}(W'). \end{array}$$

[35] I.e. preserving min and max, or equivalently preserving inclusions (both sides are lattices with respect to $\subseteq$).

with inverse $X \mapsto \sum_{\sigma \in X} W_\sigma$.

Dually, for each $\sigma \in \mathrm{JH}(W)$, W admits a unique quotient W^σ with socle σ, and quotient representations W' correspond to upper sets in $(\mathrm{JH}(W), \leq)$ under $W' \mapsto \mathrm{JH}(W')$.

Note however that $(\mathrm{JH}(W), \leq)$ is not enough to recover W up to isomorphism, which can already be seen in case $|\mathrm{JH}(W)| = 2$.

Exercise A.5. Any subquotient representation W' of W is also multiplicity free, hence has its own partial order $\leq_{W'}$. Show that $\leq_{W'}$ is the restriction of $\leq_W$ to $\mathrm{JH}(W')$. (First consider when W' is a subrepresentation, resp. quotient.)

Exercise A.6. Show that $\mathrm{soc}_\Gamma W \cong \bigoplus_{\sigma \in \mathrm{JH}(W)_{\min}} \sigma$ and $\mathrm{cosoc}_\Gamma W \cong \bigoplus_{\sigma \in \mathrm{JH}(W)_{\max}} \sigma$, where $\mathrm{JH}(W)_{\min}$ (resp. $\mathrm{JH}(W)_{\max}$) denotes the set of all minimal (resp. maximal) elements of $(\mathrm{JH}(W), \leq)$.

To visualize the partial order $\leq$, we draw an arrow $\sigma \leftarrow \tau$ precisely if $\sigma < \tau$ and there is no $\rho \in \mathrm{JH}(W)$ such that $\sigma < \rho < \tau$. In particular, we then see the Γ-socle of W as the collection of terminal vertices.

Example A.7. If the Γ-representation E is a nonsplit extension $0 \to \sigma \to E \to \tau \to 0$ with $\sigma \not\cong \tau$ both irreducible, then the associated partial order is given by $\sigma \leq \tau$, which is visualized by the single arrow $\sigma \leftarrow \tau$.

Example A.8. Suppose $(\mathrm{JH}(W), \leq)$ is given by:

$$\begin{tikzcd} & d \arrow[dl] \arrow[dr] & & e \arrow[dl] \\ b \arrow[dr] & & c \arrow[dl] & \\ & a & & \end{tikzcd}$$

Then $\mathrm{soc}_\Gamma W \cong a$, $\mathrm{soc}_\Gamma(W/\mathrm{soc}_\Gamma W) \cong b \oplus c$, and $\mathrm{cosoc}_\Gamma W \cong d \oplus e$. Moreover, the quotient W^c with socle c is an extension of $d \oplus e$ by c.

Exercise A.9. If $\sigma, \tau \in \mathrm{JH}(W)$ show that there exists a subquotient W' of W with $\mathrm{soc}_\Gamma W' \cong \sigma$, $\mathrm{cosoc}_\Gamma W' \cong \tau$ if and only if $\sigma \leq \tau$.

Exercise A.10. Show that W can be uniquely written as $W = \bigoplus_i W_i$ with W_i indecomposable and that the $\mathrm{JH}(W_i)$ are the equivalence classes under the equivalence relation generated by $\leq$. Moreover, show that any subrepresentation (resp. quotient) W' of W is of the form $\bigoplus_i W_i'$, where W_i' is a subrepresentation (resp. quotient) of W_i.

Remark A.11. Why is the multiplicity freeness important? If σ is irreducible, then $\sigma \oplus \sigma$ has many subrepresentations: $\mathrm{Hom}_C(\sigma, \sigma \oplus \sigma) \cong \mathrm{End}_C(\sigma)^2$ ($= C^2$ if σ is absolutely irreducible). However, Lemma A.4 is still valid for representations that satisfy the weaker condition that they do not admit any subquotient of the form $\sigma \oplus \sigma$ for some irreducible σ. The only difference is that $\mathrm{JH}(W)$ is now a multi-set. (I do not know a reference, but it is not hard to show.) For example, if $\Gamma = \mathbb{Z}/p\mathbb{Z}$ and C is of characteristic p, then the only irreducible representation is 1 (Lemma 1.6). The Γ-representation $W = C[\Gamma]$ is uniserial, so verifies the condition. You can then verify that its partial order is visualized by $1 \leftarrow 1 \leftarrow \cdots \leftarrow 1$ (p copies of 1).

A.4. **Representations of reductive algebraic groups.** General reference, which contains much more than we need: [Jan03, Part II]. For a quick introduction for GL_n as well as the relation to representations of $\mathrm{GL}_n(\mathbb{F}_q)$, see [Her09, §3].

We briefly discuss representations of reductive algebraic groups over an algebraically closed field k. For simplicity and concreteness we restrict to GL_2. Notation: $G := \mathrm{GL}_2(k)$ with subgroups $B := \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, $T := \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$, $U := \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.

We can either think of G as an affine algebraic variety in the classical sense (cut out by polynomial equations in affine space) or as a functor of points $A \mapsto \mathrm{GL}_2(A)$ for any commutative k-algebra A.

An *algebraic representation* of G is then a morphism $G \to \mathrm{GL}_N(k)$ of algebraic groups, or better in a coordinate-independent way: $G \to \mathrm{GL}(V)$, where V is a finite-dimensional k-vector space. In the functor-of-points language, this is a functorial group homomorphism $\mathrm{GL}_2(A) \to \mathrm{GL}(V \otimes_k A)$.

Any algebraic representation of T is a direct sum of 1-dimensional representations, which in turn are described by the character group $X^*(T) = \{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) \mapsto x^{i_1}y^{i_2} : i_1, i_2 \in \mathbb{Z}\} \cong \mathbb{Z}^2$. Hence if V is an algebraic representation of G, we have

$$V = \bigoplus_{\mu \in X^*(T)} V_\mu,$$

where $V_\mu \subseteq V$ is the subspace on which T acts via μ. There is a partial order $\leq$ on $X^*(T)$, where $\lambda \leq \mu$ if and only if $\lambda_1 \leq \mu_1$ and $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$ (cf. the proof of Theorem 2.12). If $V_\mu \neq 0$, we say μ is a *weight of* V and $\lambda \in X^*(T)$ is a *highest weight* of V if λ a weight and any other weight is $\leq \lambda$. Any irreducible algebraic representation of G has a (unique) highest weight. By considering the action of the normalizer of T, one sees that the weights of V are closed under $\mu \mapsto w_0\mu := (\mu_2, \mu_1)$, so a highest weight λ has to be *dominant*, i.e. $\lambda_1 \geq \lambda_2$. Let $X^*(T)_+ \subseteq X^*(T)$ denote the subset of dominant characters.

Fact A.12. *The irreducible algebraic representations of G (up to isomorphism) are in bijection with $X^*(T)_+$, by sending V to the highest weight of V.*

For $\lambda \in X^*(T)_+$ let $L(\lambda)$ denote the irreducible algebraic representations of highest weight λ.

If $\operatorname{char} k = 0$, then everything is very neat: any algebraic representation of G is semisimple, and the irreducibles are given explicitly by $L(\lambda) = \mathrm{Sym}^{\lambda_1-\lambda_2}(k^2) \otimes \det^{\lambda_2}$, which we can think of as homogeneous polynomials of degree $\lambda_1 - \lambda_2$ in X, Y, cf. §2.1. Note that the weights of $L(\lambda)$ are all μ such that $w_0\lambda \leq \mu \leq \lambda$, each appearing with multiplicity 1.

If $\operatorname{char} k = p$, as we suppose from now on, then things are a lot more complicated. For $\lambda \in X^*(T)_+$ we can still define $H^0(\lambda) := \mathrm{Sym}^{\lambda_1-\lambda_2}(k^2) \otimes \det^{\lambda_2}$ (it can also be thought of as an algebraic induction of λ from B to G, or equivalently as global sections of a line bundle on the flag variety), and $H^0(\lambda)$ contains $L(\lambda)$ as unique irreducible subrepresentation. If $\lambda_1 - \lambda_2 \leq p - 1$, then $H^0(\lambda) = L(\lambda)$, but we have a nonsplit exact sequence

$$0 \to L(p, 0) \to H^0(p, 0) \to L(p-1, 1) \to 0. \tag{A.1}$$

Exercise A.13. Verify (A.1) by showing that $L(p, 0) = C \cdot X^p + C \cdot Y^p$ is G-stable. For nonsplitness, show that $\dim_k H^0(p, 0)^U = 1$.

Exercise A.14. More generally show that we have a nonsplit exact sequence

$$0 \to L(ap+b,0) \to H^0(ap+b,0) \to L(ap-1,b+1) \to 0.$$

for any $1 \le a \le p-1$, $0 \le b \le p-2$.

It is also useful to define a *Weyl module* $V(\lambda)$, which has $L(\lambda)$ as unique irreducible quotient. Concretely, $V(\lambda) := H^0(-w_0\lambda)^\vee$ and note that $L(-w_0\lambda)^\vee \cong L(\lambda)$ by comparing highest weights. We remark that $V(\lambda)$ and $H^0(\lambda)$ have isomorphic semisimplifications, and that $V(\lambda)$ is a universal highest weight module of weight λ (for any algebraic representation V, V is generated by a highest weight vector of weight λ if and only if V is a nonzero quotient of $V(\lambda)$).

Remark A.15. We have $L(\lambda + (1,1)) \cong L(\lambda) \otimes \det$ and likewise for $H^0(\lambda)$, $V(\lambda)$.

For V an algebraic representation of G let $V^{[i]}$ ($i \ge 0$) denote the composition $G \xrightarrow{\varphi^i} G \to \mathrm{GL}(V)$, where φ denotes the Frobenius morphism of G, raising coordinate entries to their p-th powers.

Fact A.16 (Steinberg). *If $\lambda^{(i)} \in X^*(T)_+$ such that $0 \le \lambda_1^{(i)} - \lambda_2^{(i)} \le p-1$ for all $0 \le i \le n$, then*

$$L\Big(\sum_{i=0}^n p^i \lambda^{(i)}\Big) \cong \bigotimes_{i=0}^n L(\lambda^{(i)})^{[i]}.$$

In particular, we can compute $\dim_k L(\lambda)$ for any $\lambda \in X^*(T)_+$. However, this, as well as the explicit descriptions of $L(\lambda)$, is very particular to the group GL_2.

Exercise A.17. Deduce that $L(\lambda) = H^0(\lambda)$ if and only if $\lambda_1 - \lambda_2 = rp^n - 1$ for some $1 \le r \le p-1$ and $n \ge 0$. (Hint: reduce to $\lambda = (a,0)$ with $a = \sum_{i=0}^n a_i p^i$ with $0 \le a_i \le p-1$. Compare dimensions, using induction on n.)

Appendix B. Project outlines

Project Assistant: Heejong Lee

Let $G := \mathrm{GL}_n(F)$ (for simplicity) and keep the notation of §1 and §2.

B.1. **Explicit Satake transforms.** Recently, using derived techniques, Heyer [Hey23, §4.3] defined Satake transforms which are algebra homomorphisms

$$S^i : \mathrm{End}_G(\text{c-Ind}_K^G V) \to \mathrm{End}_T(\text{c-Ind}_{T\cap K}^T H^i(U \cap K, V))$$

for $0 \le i \le \dim U$ and a Serre weight V, where dim here denotes the dimension as a p-adic Lie group. (In fact, V can be any smooth K-representation.) The two extreme cases appear in §2.3 as S^G and ${}'S^G$ (note that $H^{\dim U}(U \cap K, V) \cong V_{U\cap K}$).

The purpose of this project is to study the new cases $0 < i < \dim U$, starting with $n = 2$ and $F/\mathbb{Q}_p$ quadratic unramified (so $\dim U = 2$ and $i = 1$). You will work on finding explicit formulas for S^i in terms of standard bases of the two Hecke algebras (in the extreme cases, there are nice formulas for the "inverse" of S^i). It would even be interesting to find a "hands-on" formula for S^i, as in the extreme cases.

The recent paper [KP] may be helpful. Though derived categories initially appear (for which [Yek] provides a helpful introduction), I do not think that much is needed in that direction. It will be essential to know some group cohomology.

B.2. **Supersingular representations with prescribed properties.** So far, extremely little is known about the supersingular irreducible admissible representations of G for $n > 2$. In fact, to the best of my knowledge, only their existence is known, cf. [HKV20]! The goal of this project is to show that such representations exist with some prescribed properties, e.g. containing a given Serre weight (at least for Serre weights in the lowest alcove). This will most likely require global arguments, following the approach in §4.8 (based on [HKV20]) but requires unitary groups instead of quaternion algebras. The global setup discussed in [EGH13] may be helpful.

B.3. **Universal supersingular modules.** Let V be a Serre weight, which we consider as KZ-representation by letting $\varpi \in Z$ act by a fixed scalar. The universal supersingular module is the quotient $\mathcal{U}(V) := (\text{c-Ind}_{KZ}^G V)/(T_1, \dots, T_{n-1})$, where $T_i \in \mathrm{End}_G(\text{c-Ind}_{KZ}^G V)$ is (as in §2.3) the Hecke operator supported on the double coset KZt_iKZ, sending t_i to a linear projection, where $t_i := \mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$ (i copies of ϖ followed by $n - i$ copies of 1). It is a smooth representation of G.

For $n = 2$ it is known that $\mathcal{U}(V)$ is admissible iff $F = \mathbb{Q}_p$ (see e.g. [Bre03], [Sch14], [Hen19]). What about $n > 2$? The representation $\mathcal{U}(V)$ is then expected to be non-admissible for all F.

It should be possible to show that $\mathcal{U}(V)$ is at least of infinite length when $n = 3$ and $F = \mathbb{Q}_p$ (and V in the lowest alcove), using global methods from [HLM17]. The idea is to use $V = F(a-1, b, c+1)$ with (a, b, c) as in [HLM17, §1] and to use a global Galois representation $\overline{r}$ as in [HLM17, Thm. D(ii)]. The G-representation generated by V inside $S(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ will also contain a second Serre weight $F(a, c, b - p + 1)$, with non-supersingular (ordinary) Hecke eigenvalue. Infinitely many such Hecke eigenvalues are possible. The project consists of first verifying this in detail, and then trying to go further.

References

[AHV19] N. Abe, G. Henniart, and M.-F. Vignéras, *Modulo p representations of reductive p-adic groups: functorial properties*, Trans. Amer. Math. Soc. **371** (2019), no. 12, 8297–8337. MR 3955548

[Alp80] J. L. Alperin, *Diagrams for modules*, J. Pure Appl. Algebra **16** (1980), no. 2, 111–119. MR 556154

[BDJ10] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, *On Serre's conjecture for mod ℓ Galois representations over totally real fields*, Duke Math. J. **155** (2010), no. 1, 105–161. MR 2730374

[Ber11] Laurent Berger, *La correspondance de Langlands locale p-adique pour* $\mathrm{GL}_2(\mathbf{Q}_p)$, no. 339, 2011, Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026, pp. Exp. No. 1017, viii, 157–180. MR 2906353

[BHH$^+$] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, *Finite length for unramified* GL_2, `https://arxiv.org/abs/2501.03644`.

[BP12] Christophe Breuil and Vytautas Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc. **216** (2012), no. 1016, vi+114. MR 2931521

[Bre03] Christophe Breuil, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$. *I*, Compositio Math. **138** (2003), no. 2, 165–188. MR 2018825

[Bre07] ———, *Representations of Galois and of* GL_2 *in characteristic p*, `https://www.imo.universite-paris-saclay.fr/~christophe.breuil/PUBLICATIONS/New-York.pdf`, 2007.

[Bre10] ———, *The emerging p-adic Langlands programme*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 203–230. MR 2827792

[DdSMS99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, second ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999. MR 1720368

[EGH] Matthew Emerton, Toby Gee, and Eugen Hellmann, *An introduction to the categorical p-adic Langlands program*, `https://arxiv.org/abs/2210.01404`.

[EGH13] Matthew Emerton, Toby Gee, and Florian Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649–1722. MR 3079258

[EGS15] Matthew Emerton, Toby Gee, and David Savitt, *Lattices in the cohomology of Shimura curves*, Invent. Math. **200** (2015), no. 1, 1–96. MR 3323575

[Eme10] Matthew Emerton, *Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties*, Astérisque (2010), no. 331, 355–402. MR 2667882

[Gee22] Toby Gee, *Modularity lifting theorems*, Essent. Number Theory **1** (2022), no. 1, 73–126. MR 4573253

[GLS23] Eknath Ghate, Daniel Le, and Mihir Sheth, *Non-admissible irreducible representations of p-adic* GL_n *in characteristic p*, Represent. Theory **27** (2023), 1088–1101. MR 4664337

[Hen09] Guy Henniart, *Sur les représentations modulo p de groupes réductifs p-adiques*, Automorphic forms and L-functions II. Local aspects, Contemp. Math., vol. 489, Amer. Math. Soc., Providence, RI, 2009, pp. 41–55. MR 2533002

[Hen19] Yotam I. Hendel, *On the universal* $\bmod p$ *supersingular quotients for* $\mathrm{GL}_2(F)$ *over* $\overline{\mathbb{F}}_p$ *for a general* $F/\mathbb{Q}_p$, J. Algebra **519** (2019), 1–38. MR 3873949

[Her09] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116. MR 2541127

[Her12a] ———, *The mod p representation theory of p-adic groups*, notes of a graduate course at Fields Institute, available at `https://www.math.utoronto.ca/~herzig/modpreptheory.pdf`, 2012.

[Her12b] ———, *The mod p representation theory of p-adic groups (exercises)*, available at `https://www.math.utoronto.ca/~herzig/MAT1104-exercises.pdf`, 2012.

[Her15] ———, *p-modular representations of p-adic groups*, Modular representation theory of finite and p-adic groups, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 30, World Sci. Publ., Hackensack, NJ, 2015, pp. 73–108. MR 3495745

[Hey23] Claudius Heyer, *The left adjoint of derived parabolic induction*, Math. Z. **305** (2023), no. 3, Paper No. 46, 60. MR 4658617

[HKV20] Florian Herzig, Karol Kozioł, and Marie-France Vignéras, *On the existence of admissible supersingular representations of p-adic reductive groups*, Forum Math. Sigma **8** (2020), Paper No. e2, 73. MR 4061972

[HLM17] Florian Herzig, Daniel Le, and Stefano Morra, *On mod p local-global compatibility for* GL_3 *in the ordinary case*, Compositio Math. **153** (2017), no. 11, 2215–2286.

[Hu10] Yongquan Hu, *Sur quelques représentations supersingulières de* $\mathrm{GL}_2(\mathbb{Q}_{p^f})$, J. Algebra **324** (2010), no. 7, 1577–1615. MR 2673752

[Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057

[Koh17] Jan Kohlhaase, *Smooth duality in natural characteristic*, Adv. Math. **317** (2017), 1–49. MR 3682662

[KP] Karol Kozioł and Cédric Pépin, *Derived Satake morphisms for p-small weights in characteristic p*, `https://arxiv.org/abs/2407.11269`.

[LMS22] Daniel Le, Stefano Morra, and Benjamin Schraen, *Multiplicity one at full congruence level*, J. Inst. Math. Jussieu **21** (2022), no. 2, 637–658. MR 4386824

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263

[Pra07] Amritanshu Prasad, *Representations of* $\mathrm{GL}_2(\mathbb{F}_q)$ *and* $\mathrm{SL}_2(\mathbb{F}_q)$*, and some remarks about* $\mathrm{GL}_n(\mathbb{F}_q)$, `https://arxiv.org/abs/0712.4051`, 2007.

[Sch14] Michael M. Schein, *On the universal supersingular* $\bmod p$ *representations of* $\mathrm{GL}_2(F)$, J. Number Theory **141** (2014), 242–277. MR 3195399

[Vig12] Marie-France Vignéras, *Représentations p-adiques de torsion admissibles*, Number theory, analysis and geometry, Springer, New York, 2012, pp. 639–646. MR 2867935

[Yam] Yuki Yamamoto, *On Mackey decomposition for locally profinite groups*, `https://arxiv.org/abs/2203.14262`.

[Yek] Amnon Yekutieli, *A course on derived categories*, `https://arxiv.org/abs/1206.6632`.

(F. Herzig) Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, ON M5S 2E4, Canada.