

Cases where HP and WA hold.

• Quadratics.

• X/k smooth proj. g.c. $\exists G/k$ linear alg gp
| $G \times X \rightarrow X$ $g_*(x) = (g_1 g_2)_* \circ x + X(\bar{k})$
conch
transit $\forall x_0 \in X(\bar{k}) \quad G(\bar{k})x_0 = X(\bar{k})$

• HF and WA for principal homogeneous

| spaces of semisimpli simply connected

linear alg gps

Eichler D/R

$$\alpha \in k^\times \cap Nrd D^\times \stackrel{H^4(k, SL(2))}{\longrightarrow}$$
$$\Leftrightarrow \alpha_v^\times \in Nrd_v D_v^\times \quad \forall v \in \mathbb{Z}$$

there are many counterexamples to HP and WA or

- Curves of genus > 0

- $\sum_{K/Q} (\gamma_1 w_1 + \dots + \gamma_n w_n) = C$

e.g. K/Q Galois

$$\text{Gal}(K/Q) = 2/2 \times 2/2$$

- PHS of G nonempty (not really needed) (Serre)
in which HF fails

- $y^2 + z^2 = \beta - x^2(x^2 - 2)/\mathbb{Q}$
Iskorskikh

X/k smooth proj d.c.

$$X(k)^{\text{top}} \subset X(A_k)^{B_{\text{rk}} X} \subset X(A_k) \quad 1570$$

Naire Jyoti.

Is $X(k)^{\text{top}} = X(A_k)^{B_{\text{rk}} X}$? $\overline{X}(k)$

No (Shorabogta)

CONJ [If X/k \overline{X} is RATIONALLY CONNECTED
then $X(k)^{\text{top}} = X(A_k)^{B_{\text{rk}} X}$]

Kollar - Miyaoka - Mori:

$$k \subset \bar{k} \subset \mathbb{C}$$

\overline{X} is rat connected

any two pts $A, B \in X(\mathbb{C})$
are connected by a $P^1_{\mathbb{C}}$

$$\begin{array}{ccc} P^1_{\mathbb{C}} & \longrightarrow & X_{\mathbb{C}} \\ 0 & \longmapsto & t \\ \infty & \longmapsto & g \end{array}$$

Known case of $X(k)^{\text{top}} \cong X(A_k)^{\text{Br}}$

Sakska, Biruroi'

$\otimes X \supset U$

smooth proj.

U is a homogeneous space

of a connected linear group G

with $\forall x \in U(\mathbb{F})$

$G_x = \text{Stab}(x)$ is connected

Thm $X(B)^{\text{top}} \cong X(D_B)^{\text{Br}}$

Châtellet surfaces

$$y^2 - az^2 = P(z) \quad \text{separable} \quad \text{X} \quad \text{xx}$$
$$d^0 P = \mathfrak{Z} \cup \mathfrak{L}$$

X smooth conic.

Thm $\left[\begin{array}{l} \text{X}(-, \text{Saito}, \text{Swinnerton-Dyer } 1984-87) \\ X(\mathbb{C})^{+,\dagger} = X(\mathbb{D}_\mathbb{C})^{\text{Br}X} \end{array} \right]$

FACT $\overline{[}$ this X is not stably b-bitational
to one of the previous ones

For X^* $\text{Br}X_{\mathbb{F}_r}/\text{Br}k_r = 0$ for almost all r.

For xx not true

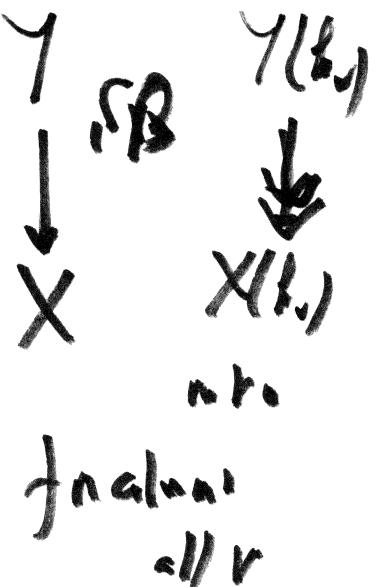
Pf using | discat
+
fibration

X/k # fact
smooth proj $A \in \text{Br}X$

For all $v \in \Sigma_A$ except finitely many

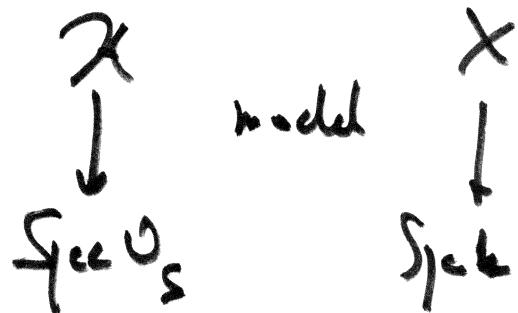
$$X(k_v) \xrightarrow{\text{er}_A} \text{Br}_{k_v}$$

has $\text{wige} = 0$



What if X is not projective?

Thm: X/k smooth g.c.



$$\left\{ \begin{array}{l} U \subset X \quad \alpha \in Br U \setminus Br X \\ \text{then} \end{array} \right.$$

$$\alpha \in Br U \setminus Br X.$$

$$q_u$$

thus \exists nitely many $v \in \mathcal{D}_k$ such that

$$\exists M_v \in \mathcal{T}(k_v) \cap \mathcal{X}(O_v) \text{ and } \alpha(M_v) \neq 0$$

Example. $0 = \boxed{y^2 - a_2 t^2 = P(t)Q(t) \neq 0}$ irreducible
 $a \notin k_v^{x^2}$ of odd degree

Assume $\prod U(k_v) \neq \emptyset$

$S = \text{obvious bad at v of prms}$

$$\alpha = (a, P(t)) \quad \text{If } v \notin S' \quad a \notin k_v^{x^2} \quad (\text{Tchebotarev})$$

$$\text{if I take } t_v = \frac{1}{\pi_v} \quad \sim (P(t_v), Q(t_v)) \text{ even}$$

$$\Rightarrow \text{solution } (y_v, z_v, t_v) \in M_v \subset U(k_v)$$

$$\text{If } v \notin S' \quad \exists M_v \subset U(k_v) \quad \alpha(M'_v) = 0$$

$$\Rightarrow \exists (M_v)_{v \in S'} \text{ such that } \sum_{v \in S'} \alpha(M_v) = 0$$

~~THEOREM~~

$U(1D_\ell)$

$$\begin{array}{c} \cup -k^*/NK^* \rightarrow \bigoplus k_v^*/Nk_v^* \rightarrow \mathbb{Z}/2 \rightarrow \\ K = k(\sqrt{a}) \cup -Brk \longrightarrow \bigoplus Br_{k_v} \longrightarrow \mathbb{Q}/\mathbb{Z} \end{array}$$

CFT

$$c \in k^* \longrightarrow (P(t_r)) \longrightarrow 0$$

*

$$\begin{array}{l} \rightarrow \left. \begin{array}{l} y_1^2 - a z_1^2 = c P(t) \neq 0 \quad | \text{ has solns} \\ y_2^2 - a z_2^2 = c^{-1} Q(t) \neq 0 \end{array} \right| \text{ in all } k_v \\ \text{Solvability implies } h \in P \end{array}$$

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$$y^2 - a z^2 = P(t) Q(t)$$

If (*) has pts in all k_v , then $\exists c \in k^*$

(*)_c has pts in all k_v

Harari's Tamal Lemma

(1994)

X/ℓ smooth
irreducile

$U \subset X$
open

$B \subset \text{Br } U$
take subgrp

$(P_v) \in U(R_v)$ Assume:

Assume: $\forall \alpha \in B \cap (\text{Br } X) \quad \sum_{v \in D_\alpha} \alpha(P_v) = 0$

fix S' finite.

Then $\exists (P'_v) \in U(R_v)$ with ~~$P'_v = P_v$~~
 $P'_v = P_v \quad v \in S'$

such that

$\forall \alpha \in B \quad \sum_{v \in S'} \alpha(P'_v) = 0$

$$\Leftrightarrow y^2 - c_2^2 = \prod_{i=1}^n P_i(t) \neq 0 \quad P_i: \text{irreducible}$$

$$X > 0$$

X smooth compact. If $X(A_\lambda) \stackrel{B_r X}{\neq} \emptyset$

and if Schauder is true, then $X(h) \neq \emptyset$.

$$(g, P_i(t)) \quad \left\{ \begin{array}{l} P_i(t) = c_i(y^2 - c_2^2) \\ (i=1-h) \end{array} \right.$$

2010 Green, Tao, Ziegler.

$$l_i(u, v) = a_i u + b_i v \quad a_i, b_i \in \mathbb{Z}$$

$$i=1, \dots, n$$

Assume some obvious restrictions —

then \exists ~~only~~ many pairs of integers

~~(c, d)~~ such that

$$\left. \begin{aligned} l_2(c, d) &= \text{prime} \\ l_n(c, d) &= \text{prime} \end{aligned} \right\}$$

Browning - Matthiessen - Skorobogatov

Harper - Skorobogatov - Wittmann

Thm. $\int_{\mathbb{Q}/\mathbb{Q}} y^2 - az^2 = b \prod_{i=1}^{2n} (t - e_i)$ $e_i \neq e_j$

$a \in \mathbb{Q}$

$e_i, e_j \in \mathbb{Q}$

D/\mathbb{Q} DCX mod \mathbb{K} copies

Thm $\left| X(\mathbb{Q}) \right|^{\text{tors}} = X(B_{\mathbb{Q}})^{\text{tors}}$

$$\underline{\text{Pf.}} \quad \nabla \cdot y^2 - a^2 z^2 = b \prod_{i=1}^{2^n} (u - e_i v) \neq 0 \quad (k^2, u, v)$$

$$y = \frac{u}{v^n} \quad z = \frac{z}{v^n} \quad t = \frac{u}{v} \quad \nabla \cong U \times G_n$$

(then) look at $B \subset B_r \nabla$

generated by the quat. elgs. $(a, u - e_i v)$

Apply Zarank's fatal lemma.

$$\hookrightarrow c_i \in k^\times$$

has ord.
with hv

$$\left[\overline{y_i^2 - a^2 z_i^2} = c_i \cdot (u - e_i v) \right]_{i=1-2^n}$$

$$b = \prod c_i \quad y^2 - a^2 z^2 = b \prod_{i=1}^{2^n} (u - e_i v)$$

u_0, v_0

$$\left| \gamma_i^2 - a z_i^2 = c, \left[u_0 - e_i v_0 \right] \stackrel{\in Q}{\neq} 0 \right. \\ \left. (i=1 \dots l_n) \right.$$

hence ord. is odd $\mathbb{Q}P_1 - \mathbb{R}$

$$\text{and } (u_0 - e_i v_0) \in p_i \pi G_{\mathfrak{S}(e_i)}$$