ARITHMETIC DYNAMICS AND INTERSECTION PROBLEMS

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LAURA DEMARCO

The goal of this Arizona Winter School lecture series is to present some complex-analytic and dynamical techniques that have been useful for studying algebraic and arithmetic intersection problems. I do not plan to focus on specific Unlikely Intersection problems – though I will mention several in passing and give explicit examples in the final lecture – but I want to describe tools that might help us solve more of these problems.

In choosing a theme for the lectures, I was motivated by recent developments in arithmetic intersection theory, especially as presented in the manuscript of Yuan and Zhang [YZ], in the use of height bounds and equidistribution theorems for the study of abelian varieties and, more generally, families of algebraic dynamical systems parameterized by a quasiprojective variety defined over $\overline{\mathbb{Q}}$. For example, in my recent preprint with Myrto Mavraki [DM], we build on the work of Gauthier-Vigny [GV] and her earlier work with Schmidt [MS], in combination with an equidistribution theorem of [YZ], to study the intersections of preperiodic points for families of maps on $\mathbb{P}^1$. Our proof methods are closely related to – and very much inspired by – the recent works of Kühne [Kü1, Kü2], Dimitrov-Gao-Habegger [DGH1, DGH2], and Gauthier [Ga]. Some of this theory will be discussed in the final lecture; especially, I want to emphasize how purely complex-analytic input can force intersections and lead to "positivity" of an arithmetic nature.

1. Lecture 1. The Lattès family

There is an important class of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$, the *Lattès examples*, that has inspired many of the developments in arithmetic dynamical systems, building on parallels between the study of elliptic curves and dynamics in dimension 1. Such a map f arises as the quotient of an endomorphism of an elliptic curve $\varphi : E \to E$. We begin by introducing these examples and presenting some fundamental concepts from 1-dimensional complex dynamics. Helpful references include [Mi2, Si1, Mi1, FS].

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1.1. **Lattès maps.** Take any elliptic curve E defined over $\mathbb{C}$. The identification of a point P with its additive inverse $-P$ defines a degree 2 projection $\pi : E \to \mathbb{P}^1$. Not that, if E is presented as $\mathbb{C}/L$ for a lattice L and if we choose coordinates on $\mathbb{P}^1$, the associated Weierstrass $\wp$-function can be viewed as the composition of the quotient $\mathbb{C} \to \mathbb{C}/L$ with the projection π.

Now let φ be an endomorphism of E. For example, let's take

$$\varphi(P) = P + P = 2P.$$

Since $\varphi(-P) = -\varphi(P)$, the map φ descends by π to define an endomorphism f_φ on $\mathbb{P}^1$, making the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f_\varphi"] & \mathbb{P}^1 \end{tikzcd} \tag{1.1}$$

The degree of f_φ, as a rational function of one variable, coincides with the degree of φ, which is 4 for my example with $\varphi(P) = 2P$.

Note that a point P is torsion on E *if and only if* it has finite orbit under iteration of φ. That is, the sequence of points

$$P,\ 2P,\ 4P,\ 8P, \ldots$$

must be finite. And it follows that the projection $\pi(P)$ is preperiodic for f_φ *if and only if* P is torsion on E. This allows us to use dynamics – and the full power of machinery developed to study the iteration of holomorphic maps on $\mathbb{P}^1(\mathbb{C})$ – to study properties (arithmetic or geometric) of torsion points on elliptic curves.

These rational functions $f : \mathbb{P}^1 \to \mathbb{P}^1$ that are quotients $f = f_\varphi$ of an endomorphism $\varphi : E \to E$ are called *Lattès maps*. More generally, we call an endomorphism f *Lattès* if it arises as the quotient of any morphism $\varphi : E \to E$ (not necessarily a homomorphism) via a finite-degree quotient π (not necessarily of degree 2) as in the diagram (1.1). A classification and summary of the dynamical features of Lattès maps is given in [Mi2].

Here's a concrete example. Consider the elliptic curves in Legendre form,

$$E_t = \{y^2 = x(x-1)(x-t)\} \subset \mathbb{P}^2(\mathbb{C})$$

for $t \in \mathbb{C} \setminus \{0,1\}$, with the projection $\pi : E \to \mathbb{P}^1$ given by $\pi(x,y) = x$. Take endomorphism $\varphi(P) = 2P$ on E. The action of φ on the x-coordinate induces a rational function f_t that depends on the parameter t with formula

$$f_t(x) = \frac{(x^2 - t)^2}{4x(x-1)(x-t)}. \tag{1.2}$$

A derivation of this formula can be found in, for example, [Si2, Chapter III].

1.2. **Higher-dimensional Lattès maps.** The same basic quotient construction does not work for general endomorphisms of abelian varieties in higher dimensions, at least not if we hope to induce a morphism on $\mathbb{P}^N$. A classification of Lattès maps in dimension $N = 2$ arising as in (1.1) – replacing E with an abelian surface A defined over $\mathbb{C}$ and replacing $\mathbb{P}^1$ with $\mathbb{P}^2$ – is presented in Section 5 of [Dup], and see the references given there. Given an abelian surface A, it is rare for a quotient A/G, for a finite group of automorphisms G of A, to be isomorphic to $\mathbb{P}^2$. You will observe that each A in the table appearing in [Dup, §5.1] is actually the square $E \times E$ of an elliptic curve! The maps are then built from product endomorphisms. Dupont observes (in his remark 5.1) that examples exist in every degree $d > 1$ and every dimension N. I do not know if there is a known classification in dimensions > 2.

On the other hand, though the quotient construction does not always work, we can often *extend* an endomorphism $\varphi : A \to A$ to a large projective space for any choice of A. If X is a projective variety over $\mathbb{C}$, then a morphism $f : X \to X$ is said to be *polarizable* if there is an ample line bundle L on X for which $f^*L \simeq L^d$ for some integer $d > 1$. If polarizable, there exists an embedding $X \hookrightarrow \mathbb{P}^N$ so that f extends to a morphism on all of $\mathbb{P}^N$ [Fa, Corollary 2.2]. In particular, beginning with the multiplication-by-2 endomorphism $\varphi : A \to A$ on an arbitrary abelian variety A, Fakhruddin describes an extension of φ in the proof of [Fa, Corollary 2.4]. Note that the torsion points on A are precisely the points in A with finite forward orbit under φ.

As Fakhruddin points out in [Fa], various questions and conjectures about abelian varieties can thus be reformulated in dynamical terms. For example, the uniform boundedness question about torsion points on abelian varieties becomes a special case of the Morton-Silverman Uniform Boundedness Conjecture for endomorphisms of $\mathbb{P}^N$ [MS]; see also [Si1].

1.3. **Julia sets and canonical measures.** Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $d > 1$. Its *filled Julia set* is the compact subset of $\mathbb{C}$ defined by

$$K(f) = \{z \in \mathbb{C} : \sup_n |f^n(z)| < \infty\}.$$

The boundary of $K(f)$ is called the *Julia set* $J(f)$ of f and turns out to be equal to the closure of the set of repelling periodic points of f. That is, the points $z_0 \in \mathbb{C}$ for which $f^n(z_0) = z_0$ for some $n > 0$ and so that $|(f^n)'(z_0)| > 1$ are all contained in $J(f)$ and form a dense subset of $J(f)$.

In general, the *Julia set* $J(f)$ of a (possibly non-polynomial) map $f : \mathbb{P}^1 \to \mathbb{P}^1$ can be defined as the closure of the set of all repelling periodic points.

For **Lattès maps**, it isn't hard to see that the Julia set must be all of $\mathbb{P}^1(\mathbb{C})$. The torsion points on the elliptic curve E that are periodic for the endomorphism are dense in $E(\mathbb{C})$. All of the periodic points are repelling because the original endomorphism φ is everywhere expanding.

For polynomial maps f, the Julia set might be a complicated fractal subset of $\mathbb{C}$, but there is an easy way to visualize these sets with an escape-time algorithm. That is, we iterate all the points z in some fine grid and color z according to how many iterates are required until $|f^n(z)|$ is large (where "large" depends on the coefficients of f). We might color a pixel black if $|f^n(z)|$ remains "small" for all iterates tested. See Figure 1.2.

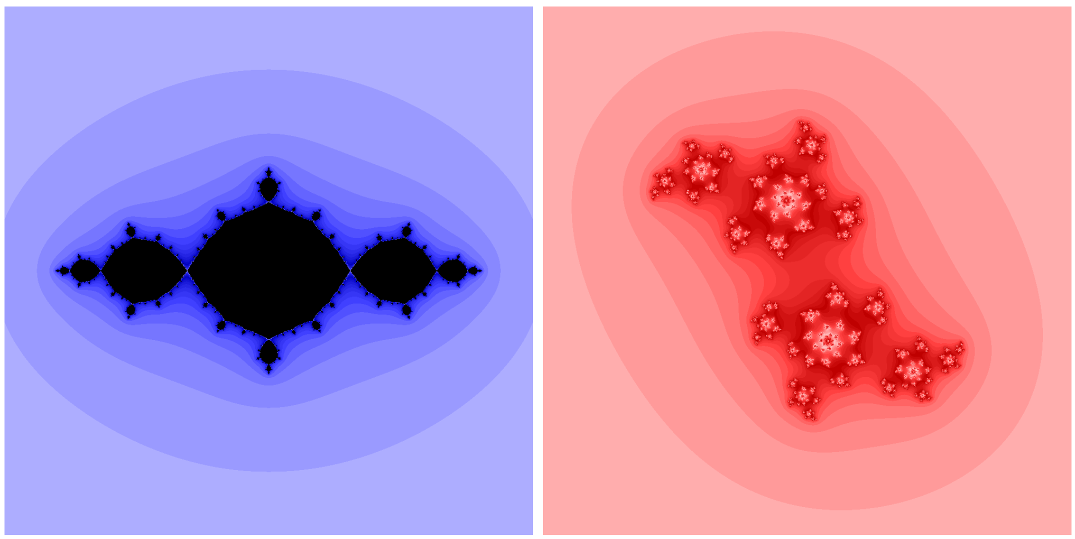

FIGURE 1.1. The filled Julia sets for $f(z) = z^2 - 1$ (at left) and for $f(z) = z^2 + 0.1 + 0.7i$ (at right), both of degree $d = 2$.

The images of Figure 1.2 illustrate more. The color transitions approximate level curves of the escape-rate function, defined by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

where $\log^+ = \max\{\log, 0\}$. These level curves are called "equipotential" curves, because G_f turns out to be the Green's function (with pole at ∞) for the domain $\mathbb{C} \setminus K(f)$. In particular, its Laplacian – computed in the sense of distributions – is the *equilibrium measure*

$$\mu_f = \frac{1}{2\pi} \Delta G_f$$

for the compact set $K(f)$. That is, it gives the "optimal distribution" of an electric charge (if $K(f)$ were a conducting material in some ideal world), in the sense of Newtonian potential theory; see, for example, [Ra1]. The measure μ_f turns out to also be the *unique measure of maximal entropy* for f, and so it plays a very important role in our study of these types of dynamical systems [Br, Ly1, FLM].

In fact, there is also a potential-theoretic interpretation for the measure of maximal entropy for any map $f : \mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{C}$, and even for maps $f : \mathbb{P}^N \to \mathbb{P}^N$ with $N > 1$. Working in homogeneous coordinates, choose a presentation

$$f = (f_0, \ldots, f_N),$$

where $f_0, \ldots, f_N$ are homogeneous polynomials in $N + 1$ variables of degree $d > 1$, having no common zeroes except at the origin in $\mathbb{C}^{N+1}$. As for polynomials in dimension 1, we define an escape rate in $\mathbb{C}^{N+1}$ by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|f^n(z)\|. \tag{1.3}$$

There is a canonical dynamical "Green current" T_f on $\mathbb{P}^N(\mathbb{C})$ defined by

$$dd^c G_f = \pi^* T_f \tag{1.4}$$

in the sense of distributions, where $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ is the tautological projection; currents and the operations of dd^c and π^* will be discussed in the second lecture. The canonical measure $\mu_f := (T_f)^{\wedge N}$ on $\mathbb{P}^N(\mathbb{C})$ turns out to be the unique measure of maximal entropy for f. As far as I'm aware, this theory was first developed by Fornaess and Sibony [FS, FS1, FS2].

In the case of a **Lattès map** arising as the quotient of a morphism on elliptic curve E as in (1.1), the measure μ_f on $\mathbb{P}^1(\mathbb{C})$ is equal to the projection $\pi_*\omega$ of the Haar measure ω on E.

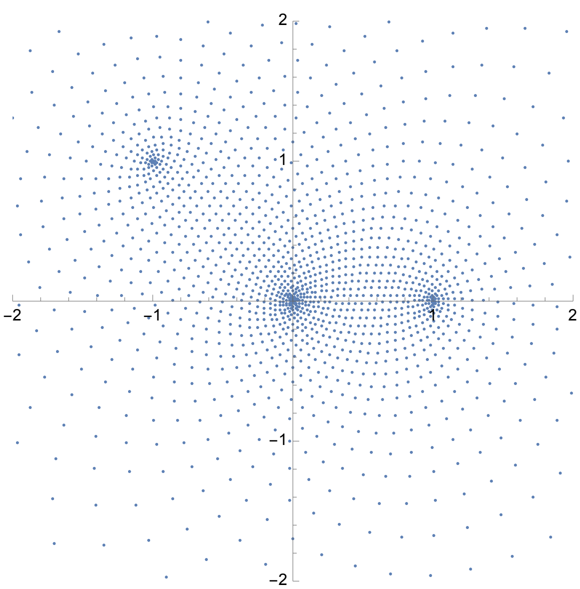

FIGURE 1.2. Illustrating the distribution μ_f for the Lattès map of (1.2) with $t = -1+i$. The preimages $f^{-n}(a)$ of any point $a \in \mathbb{P}^1(\mathbb{C})$ are uniformly distributed with respect to μ_f as $n \to \infty$; here, $n = 6$ and $a = 1$.

1.4. **Canonical heights.** Let's finish this section with the important construction of Call and Silverman that defined canonical heights for endomorphisms of $\mathbb{P}^N$ [CS]. For simplicity, we return to dimension $N = 1$. Assume that $f : \mathbb{P}^1 \to \mathbb{P}^1$ has degree $d > 1$ and is defined over a number field. Let h denote the naive logarithmic Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. The canonical height function associated to f is defined by

$$\hat{h}_f(\alpha) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(\alpha))$$

for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$. It is the unique height function for which

$$\frac{1}{d}\,\hat{h}_f \circ f = \hat{h}_f$$

and for which there exists a constant $C = C(f)$ so that

$$|\hat{h}_f - h| \leq C$$

on $\mathbb{P}^1(\overline{\mathbb{Q}})$ [CS, Theorem 1.1]. Note, in particular, that $\hat{h}_f(\alpha) = 0$ if and only if α is preperiodic for f: one implication is clear, and the other is a consequence of Northcott's theorem that for any $B, D > 0$, we have

$$\#\{\alpha \in \overline{\mathbb{Q}} : h(\alpha) \leq B \text{ and } \deg \alpha \leq D\} < \infty.$$

A local height decomposition of $\hat{h}_f$ can be given in terms of the functions G_f of (1.3), replacing $\|\cdot\|$ with appropriately-defined p-adic norms on the affine space $\mathbb{A}^2(\overline{\mathbb{Q}})$; details can be found in [BR, Si1].

Note that, in the case of **Lattès maps** f, where the projection π of (1.1) has degree 2, we have

$$\hat{h}_f(\pi(P)) = 2\,\hat{h}_E(P)$$

for all $P \in E(\overline{\mathbb{Q}})$, where $\hat{h}_E$ is the Néron-Tate canonical height on the elliptic curve E. See [Si1] for details.

2. Lecture 2. Pluripotential theory

In this lecture, we introduce key tools in the study of complex analysis and dynamics in dimensions > 1, namely the theory of currents and plurisubharmonic functions. Unlikely Intersection problems in arithmetic dynamics quickly lead to "intersections" of unwieldy fractal objects, and we need to build on the intersection theory of currents. Helpful references: [DS, Dem, Kl, FS], and [Ra1, BR] for 1-dimensional potential theory (including non-archimedean!).

2.1. **What is a current?** A summary of the basics can be found in [DS] (without proofs) or in Chapter I of [Dem]. Let Ω be an open subset of $\mathbb{R}^N$. For integer p with $0 \leq p \leq N$, a p-current (or a *current of degree* p) on Ω is a continuous linear functional on the space $\mathcal{D}^{N-p}(\Omega)$ of smooth $(N-p)$-forms on Ω with compact support. The continuity is with respect to the C^∞ topology on this space of forms. If the current defines a bounded linear functional with respect to the C^0-topology, then it extends to the space of compactly-supported forms with continuous coefficients, and we say that the current has *order 0*. Thus, an N-current is the same thing as a *distribution*, and an N-current with order 0 is the same as a *measure* μ on Ω.

Example 2.1. A smooth p-form ω on Ω defines a p-current by

$$\langle \omega, \alpha \rangle := \int_\Omega \omega \wedge \alpha$$

for all $\alpha \in \mathcal{D}^{N-p}(\Omega)$.

Example 2.2. An oriented, closed, C^∞ submanifold Y in Ω of codimension p defines a p-current by

$$\langle [Y], \alpha \rangle := \int_Y \alpha$$

for all $\alpha \in \mathcal{D}^{N-p}(\Omega)$. It is called the *current of integration* on Y.

The exterior derivative d acts on p-currents T by duality with its action on forms:

$$\langle dT, \alpha \rangle := (-1)^{p+1} \langle T, d\alpha \rangle$$

for all $\alpha \in \mathcal{D}^{N-p-1}(\Omega)$, so that dT is a $(p+1)$-current. As with forms, we say the current is *closed* if $dT = 0$. A p-current can naturally be pushed forward by a smooth map $F : \Omega \to V$ which is *proper*, meaning that the preimage $F^{-1}(K)$ of any compact set in V is compact in Ω. We simply set

$$\langle F_* T, \alpha \rangle := \langle T, F^* \alpha \rangle$$

for all $\alpha \in \mathcal{D}^{\dim V - (N-p)}(V)$, so $F_* T$ is a $(p - N + \dim V)$-current on V. A pullback operation on currents is more delicate, but if the map $F : \Omega \to V$ is a submersion, then there is a sensible way to push forward a form (by integrating it over the fibers of F). If T is a p-current on V, then we can define $F^* T$ as a p-current on Ω by

$$\langle F^* T, \alpha \rangle := \langle T, F_* \alpha \rangle.$$

This was the meaning of π^* mentioned in (1.4) in Lecture 1.

In complex manifolds or complex algebraic varieties, recall that a smooth form α has bidegree (p, q) if it can be expressed in local coordinates as

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} \, dz^I \wedge d\bar{z}^J$$

where $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, with $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$. Note that $i\,dz \wedge d\bar{z} = 2\,dx \wedge dy$ in $\mathbb{R}^2 = \mathbb{C}$.

A (p,q)-*current* on a complex manifold X of dimension N is a continuous linear functional on the space $\mathcal{D}^{(N-p,N-q)}(X)$ of smooth $(N-p,N-q)$-forms with compact support. Recall that the d operator can be decomposed as

$$d = \partial + \bar{\partial} = \sum_j \frac{\partial}{\partial z_j} dz_j + \sum_k \frac{\partial}{\partial \bar{z}_j} d\bar{z}_k$$

in local coordinates. We define

$$d^c = \frac{1}{2\pi i}(\partial - \bar{\partial}),$$

so that

$$dd^c = \frac{i}{\pi}\partial\bar{\partial}.$$

2.2. **Positive (1,1)-currents.** Positivity is introduced in Chapter III of [Dem]. A (p,p)-current T on a complex manifold X of dimension N is said to be *positive* if $\langle T, \alpha \rangle \geq 0$ for all positive test forms

$$\alpha = (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (i\alpha_{N-p} \wedge \bar{\alpha}_{N-p})$$

with $\alpha_i \in \mathcal{D}^{(1,0)}(X)$. Positivity implies the current has order 0 [Dem, Proposition 1.14].

An important class of examples of bidegree (1,1) comes from plurisubharmonic functions. Suppose Ω is a domain in $\mathbb{C}^N$. An upper-semi-continuous (*usc* for short) function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* (or *psh* for short) if $u|_{L\cap\Omega}$ is subharmonic on every complex line L in $\mathbb{C}^N$. Recall that, in one complex dimension, a function u is subharmonic if it is usc and

$$u(x_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\,e^{i\theta})\,d\theta$$

for all closed balls $\overline{B(x_0,r)}$ in the domain of u. (It follows that $u \in L^1_{loc}$ if u is not the contant $-\infty$ function.) Equivalently, if you assume that $u \in L^1_{loc}$ and usc on a domain $\Omega \subset \mathbb{C}$, then u is subharmonic if the Laplacian

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

defined in the sense of distributions, is non-negative. Note that, in dimension 1, we have

$$\frac{1}{2\pi}\Delta u\,dx \wedge dy = dd^c u.$$

In particular, for any plurisubharmonic function u on an open set $\Omega \subset \mathbb{C}^N$,

$$T = dd^c u$$

is a closed and positive $(1,1)$-current. A Poincaré-type lemma for dd^c implies the converse: A closed, positive $(1,1)$-current T on a complex manifold X can be expressed, locally, as $T = dd^c u$ for a plurisubharmonic function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$, where Ω is an open neighborhood in X, identified with a domain in $\mathbb{C}^N$. See, for example, [Dem, Chapter III §1.18]. The function u is called a *local potential function* for T.

As a special case, consider $u(z) = \log|z|$ in $\mathbb{C}$. Then

$$dd^c u = \delta_0$$

in the sense of distributions. In other words,

$$\frac{1}{2\pi}\int_{\mathbb{C}} u\,\Delta\varphi\,dx \wedge dy = \varphi(0)$$

for all smooth functions $\varphi : \mathbb{C} \to \mathbb{R}$ with compact support. In higher dimensions, if $f : \Omega \to \mathbb{C}$ is holomorphic and not $\equiv 0$, then $u(z) = \log|f(z)|$ is plurisubharmonic, and $dd^c u$ is the current of integration along the analytic hypersurface $\{f(z) = 0\}$.

2.3. **Intersection of currents and Monge-Ampère.** We would like to have a good theory of intersecting currents that extends the notions in the context of Examples 2.1 and 2.2: given two smooth forms ω_1 and ω_2, we have a smooth form $\omega_1 \wedge \omega_2$, and given two smooth subvarieties Z_1 and Z_2 in a complex manifold X, we may consider their intersection $Z_1 \cap Z_2$ (in some appropriate sense).

In the case of positive $(1,1)$-currents with locally-bounded potentials, this can be done. Let Ω be an open subset of $\mathbb{C}^N$. If T is a closed and positive (p,p)-current, and if $u : \Omega \to \mathbb{R}$ is a bounded plurisubharmonic function, then we can set

$$dd^c u \wedge T := dd^c(uT)$$

where the right-hand-side is defined in the sense of distributions. By a continuity argument, it was proved in [BT] that this is the "right" definition, as it extends the notion for smooth forms. The wedge product is again a closed and positive current of bidegree $(p+1, p+1)$.

Working inductively, we can define the (complex) *Monge-Ampère* measure

$$(dd^c u)^N = dd^c u \wedge \cdots \wedge dd^c u$$

of a locally-bounded plurisubharmonic function u on a complex manifold X of dimension N. As pointed out in [Dem, Chapter III, §3], if u is smooth, then this is simply

$$(dd^c u)^N = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\,\frac{N!}{\pi^N}\,(i\,dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (i\,dz_N \wedge d\bar{z}_N).$$

Plurisubharmonic functions u satisfying $(dd^c u)^N = 0$ are called *maximally plurisubharmonic*; see [Kl] for an extensive treatment of these functions.

2.4. **Dynamics.** Returning to the setting of morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ over $\mathbb{C}$, we now have the language to talk about the concepts introduced in §1.3. The function G_f is plurisubharmonic and continuous on $\mathbb{C}^{N+1}$ (as a locally uniform limit of plurisubharmonic functions away from 0). Note that $G_f(\beta z) = G_f(z) + \log|\beta|$ for all $\beta \in \mathbb{C}^*$; we say that G_f is log-homogeneous.

Positive (1,1)-currents on $\mathbb{P}^N$ (of mass 1) are in a natural 1-to-1 correspondence with log-homogeneous psh functions on $\mathbb{C}^{N+1}$, up to the addition of a constant. That is, given any log-homogeneous psh u on $\mathbb{C}^{N+1}$, we can define a current on a local chart $U \subset \mathbb{P}^N$ by $T_u = dd^c(u \circ s)$ for any holomorphic section s of the projection $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ over U. These definitions patch together by the log-homogeneity of u, as $\log|\psi|$ is harmonic for any non-vanishing holomorphic function ψ. Then one can check that $dd^c u = \pi^* T_u$. For the converse, see [FS, Theorem 5.9].

Thus, the dynamical Green current T_f is well-defined by setting $\pi^* T_f = dd^c G_f$. Moreover, it is the unique positive (1,1)-current on $\mathbb{P}^N(\mathbb{C})$ (of total mass 1, with bounded potentials) such that

$$\frac{1}{d} f^* T_f = T_f.$$

Caution: f is not a submersion, so this pullback is first defined for the covering map $f : \mathbb{P}^N \setminus f^{-1}(f(C)) \to \mathbb{P}^N \setminus f(C)$ where C is the critical locus of f and then extended to all of $\mathbb{P}^N$; see page 159 of [FS]. Moreover, the measure

$$\mu_f := T_f \wedge \cdots \wedge T_f$$

plays an important role: it is the unique measure of maximal entropy and – as in dimension $N = 1$ – is the limiting distribution of the repelling periodic points of f or of iterated preimages of (typical) points in $\mathbb{P}^N(\mathbb{C})$ [BD1, BD2].

3. Lecture 3. Dynamical stability

We introduce the dynamical concept of structural stability for families of maps. We illustrate this concept in the setting of the Lattès family and other important examples on $\mathbb{P}^1$, and we relate stability to the values of certain (geometric) height functions. We conclude the section with a few words on the stability theory for holomorphic families of maps on $\mathbb{P}^N$ with $N > 1$. Helpful references include [De3] [Mc2, Chapter 4] [BB].

3.1. **Structural stability and J-stability.** Suppose we have a continuously varying family of continuous maps $f_t : X_t \to X_t$ on compact metric spaces, for t in a parameter space S (with, for example, topology of uniform convergence on the family). We say the family is *structurally stable* at $t_0 \in S$ if there exists a continuous family $\varphi_t : X_{t_0} \to X_t$ of homeomorphisms so that $f_t = \varphi_t \circ f_{t_0} \circ \varphi_t^{-1}$ for all t near t_0. That is, all the maps – up to a continuous change of coordinates – define the "same" dynamical system.

A *holomorphic family* of maps

$$f_t : \mathbb{P}^1 \to \mathbb{P}^1$$

for t in a complex manifold S, is a holomorphic map $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ that preserves the fibers of the projection to S. (This definition implies that the family f_t is continuous in the topology of uniform convergence, so in particular they all have the same degree, and the coefficients of f_t are holomorphic functions of t.) Assume the degree of f_t is > 1. We will say that the family f_t is *periodic-point stable* at $t_0 \in S$ if every periodic point of f_t can be holomorphically parameterized in a neighborhood of t_0 without collisions. That is, the graphs of these infinitely-many periodic points fit together like leaves of a foliation in $S \times \mathbb{P}^1$. Note: by the Implicit Function Theorem, for each individual periodic point z_0 of period n, to solve the equation $f_t^n(z) = z$ for $z = z(t)$ with $z(t_0) = z_0$, we require that $(f^n)'(z_0) \neq 1$. The stability condition requires that this can be done uniformly in a neighborhood for all periodic points and without collisions.

Theorem 3.1. [MSS, Ly2] *A holomorphic family of maps f_t with degree > 1 is periodic-point stable at $t_0 \in S$ if and only if it is structurally stable on the Julia set $J(f_t)$ in a neighborhood of t_0. Moreover, these conditions hold on an open and dense subset of S.*

If the family f_t satisfies these stability conditions at t_0, we shall say that f_{t_0} is *J-stable*. See [Mc2, Chapter 4] for an exposition of the proof of this theorem, along with additional characterizations of J-stability.

In practice, neither condition – periodic-point stability nor structural stability on $J(f)$ – is easily checkable. We often work with a third equivalent notion of stability, namely *critical point stability*. In a neighborhood of a point t_0, we can pass to a (finite, branched) cover on which we can holomorphically parameterize the critical points of f_t near t_0, as $c_1(t), \ldots, c_{2d-2}(t)$. Then critical point stability means that the sequence of functions $\{t \mapsto f_t^n(c_i(t))\}$ form normal families in a neighborhood of t_0, for each i. That is, every sequence of iterates has a subsequence that converges uniformly on compact subsets of the neighborhood of t_0.

As a simple application of Montel's Theorem on normal families, the family $z^2 + t$ for $t \in S = \mathbb{C}$, is J-stable for all $t_0 \notin \partial\mathcal{M}$, where

$$\mathcal{M} = \{t \in \mathbb{C} : \sup_n |f_t^n(0)| < \infty\}$$

is the famous Mandelbrot set, and $\partial\mathcal{M}$ is its topological boundary. See Figure 3.1. Note that J-stability does not imply that f_t is structurally stable on all of $\mathbb{P}^1$. A simple example is given by the family $f_t(z) = z^2 + t$ at $t_0 = 0$. The map $f_0(z) = z^2$ is

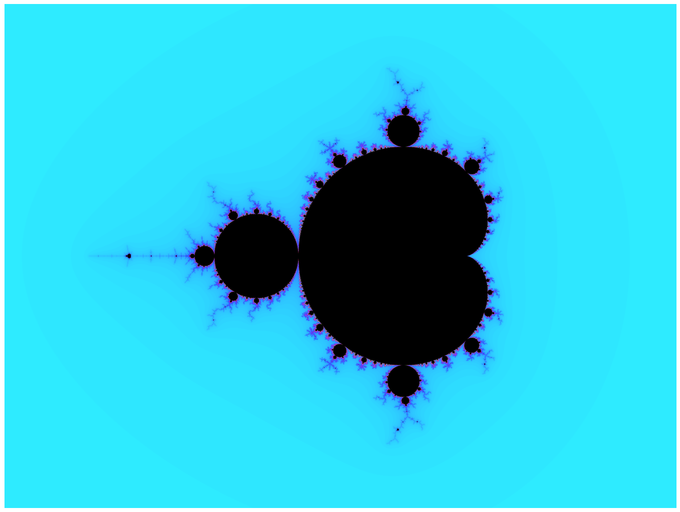

FIGURE 3.1. The Mandelbrot set.

periodic-point stable, but it is not structurally stable on all of $\mathbb{P}^1$: the critical point $c = 0$ is a fixed point at $t_0 = 0$, while it is not fixed for any $t \neq 0$. The orbit structure of a critical point must be preserved under topological conjugacy. On the other hand, it turns out that this critical orbit requirement is the only obstruction to extending the conjugacies to all of $\mathbb{P}^1$ [McS].

3.2. **Stability in the Lattès family.** Now suppose that f_t is a family of **Lattès maps**, such as the ones given by (1.2) for $t \in S = \mathbb{C} \setminus \{0, 1\}$. As we have already observed, all of the periodic points of f_t are repelling. They can be followed holomorphically with t over the entire parameter space (though with some nontrivial monodromy as you move around the three punctures of S). Viewing the graphs of these points in $S \times \mathbb{P}^1$, they form a countable, dense subset of the leaves of a holomorphic foliation of $S \times \mathbb{P}^1$. This foliation coincides with the quotient of the *Betti foliation* of the elliptic surface E. See, for example, [ACZ, CDMZ, UU1, UU2] for information about the Betti foliation. In a family of elliptic curves – or of abelian varieties – one can identify (diffeomorphically) each element of the family with a given real torus of $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ of appropriate dimension. The leaves of the Betti foliation are, by definition, the fibers of this "horizontal" projection to $\mathbb{T}^m$, and they are holomorphic.

3.3. **Rigidity and intersections.** It turns out that the Lattès maps are the only families that can be everywhere stable, at least when working with algebraic families.

For simplicity, let us assume throughout this subsection that S is a smooth, irreducible quasi-projective curve over $\mathbb{C}$. We will say $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ is an *algebraic family* of maps on $\mathbb{P}^1$ if the coefficients define *meromorphic* functions on a compactification $\overline{S}$ of S. Equivalently, f is defined by a rational function over the function field $\mathbb{C}(\overline{S})$, and we assume that the induced map on $S \times \mathbb{P}^1$ is regular.

Theorem 3.2. [Mc1] *Suppose that $f : S\times\mathbb{P}^1 \to S\times\mathbb{P}^1$ is an algebraic family of maps of degree $d > 1$. Then f is J-stable on all of S if and only if it is either isotrivial or a Lattès family.*

The map f is *isotrivial* if all elements of the family f_t are conjugate by a Möbius transformation.

McMullen's theorem is proved in two steps. The stability is analyzed by studying the orbit behavior of the critical points of f, and he deduces that stability on S implies each critical orbit must be finite, persistently, for all f_t in the family. Then the conclusion that a non-isotrivial such f is Lattès follows from the rigidity theorem of Thurston [DH].

The critical-orbit part of McMullen's theorem was extended to treat individual critical points, and later arbitrary points. A holomorphic map $a : S \to \mathbb{P}^1$ defines a *marked point over* S. A pair (f, a) is *stable* at $t_0 \in S$ if the sequence of functions $\{t \mapsto f_t^n(a(t))\}$ forms a normal family in a neighborhood of t_0.

Theorem 3.3. [De2, DF] *Suppose that $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ is an algebraic family of maps of degree $d > 1$. Suppose that $a \in \mathbb{P}^1(\mathbb{C}(\overline{S}))$ defines a marked point over S. The pair (f, a) is stable on all of S if and only if it is either isotrivial or persistently preperiodic.*

A pair (f, a) is *isotrivial* if all elements of the family f_t are conjugate by a Möbius transformation to a single map f_0 and the point a in this new coordinate system is constant.

We can immediately deduce from Theorem 3.3 that intersections must take place between algebraic curves in $S \times \mathbb{P}^1$ and the preperiodic curves for f in $S \times \mathbb{P}^1$, as follows. A curve V in $S \times \mathbb{P}^1$ is preperiodic if there exists $n > m \geq 0$ so that $f^n(V) = f^m(V)$. There are infinitely many preperiodic curves in $S \times \mathbb{P}^1$ because there are infinitely many preperiodic points for each f_t with $t \in S$. Let $\mathcal{V}(f)$ denote the union of all preperiodic curves in $S \times \mathbb{P}^1$.

Corollary 3.4. [De2, Theorem 1.6] *Let S be a smooth and irreducible quasi-projective curve over $\mathbb{C}$. Suppose that $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ is a non-isotrivial algebraic family of maps of degree $d > 1$, and suppose that C is any algebraic curve in $S \times \mathbb{P}^1$. Then the set of all preperiodic points in C, namely*

$$\bigcup_{V\cap\mathcal{V}(f)} C \cap V,$$

is an infinite subset of C.

Proof. We may assume that C is irreducible. If C is vertical, meaning a fiber of the projection $S \times \mathbb{P}^1 \to S$, then the conclusion is clear, because each f_t has infinitely many preperiodic points. Otherwise, we apply Theorem 3.3 to the pair (f, a) where

$a \in \mathbb{P}^1(k)$ for a finite extension k of $\mathbb{C}(\overline{S})$, where C becomes the graph of a over a finite branched cover of S. The non-isotriviality of f implies that the pair (f, a) will be either persistently preperiodic (in which case C is itself a preperiodic curve) or unstable. In the latter case, we apply Montel's Theorem on normal families to deduce that the orbits of C must intersect the elements of $\mathcal{V}(f)$; see, for example, [De2, Proposition 5.1]. □

3.4. **Intersections in Lattès families.** We apply Corollary 3.4 in a well-known setting: Suppose that f_t is a Lattès family, such as given in (1.2), parameterized by $S = \mathbb{C} \setminus \{0, 1\}$. Let C be an algebraic curve in $S \times \mathbb{P}^1$. Then C is either a preperiodic curve itself or it must intersect infinitely many of the preperiodic curves. In particular, lifting this C to the corresponding elliptic surface defined by the family E_t over S, this shows that *the only closed (i.e., algebraic) leaves of the Betti foliation are the torsion points.* This is well known and has several different proofs.

3.5. **Higher-dimensional stability theory.** For holomorphic families of maps on $\mathbb{P}^N$, there is also a theory of J-stability, though there are still lots of interesting questions about how many of the equivalences described above for $\mathbb{P}^1$ can carry over to higher dimensions. See [BBD] and the survey [BB] for definitions and comparisons to the dimension 1 case.

4. Lecture 4. Geometric heights, bifurcation measures, and arithmetic equidistribution

Working over number fields, we broaden the notion of dynamical stability into the general framework of the theory of adelic line bundles on quasiprojective varieties. We present results from the recent work of Yuan-Zhang [YZ, §6] and Gauthier-Vigny [GV].

4.1. **Geometric heights and polarized endomorphisms.** I just wrote that we would work over number fields, but let me first say a few things about the function field setting. Specifically, we can relate concepts from the previous Lecture to statements about canonical height values.

More precisely, let S be a smooth, quasiprojective algebraic curve over $\mathbb{C}$, and recall that a pair (f, a), consisting of an algebraic family $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ of degree $d > 1$ and a marked point $a : S \to \mathbb{P}^1$ is *stable* if the sequence of functions $\{t \mapsto f_t^n(a(t))\}$ is normal on all of S. It turns out this holds if and only the (geometric) canonical height

$$\hat{h}_f(a) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(a))$$

is equal to 0 [De2, Theorem 1.1]. Here, we view $f(z) \in k(z)$ as one rational function defined over the function field $k = \mathbb{C}(\overline{S})$, and h is the naive logarithmic Weil height on

$\mathbb{P}^1(k)$. In other words, $h(b) = \deg(b : S \to \mathbb{P}^1)$ for each $b \in \mathbb{P}^1(k)$. This equivalence was used in [De2] to give an alternative proof of Baker's theorem that – assuming the family f is not isotrivial – a point $a \in \mathbb{P}^1(k)$ has canonical height 0 if and only if it has finite orbit [Ba]. (Recall from §1.4 that for $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$, the canonical height of a point in $\mathbb{P}^1(\overline{\mathbb{Q}})$ vanishes if and only if the point is preperiodic; this follows easily from the Northcott property of the height on $\overline{\mathbb{Q}}$ but that argument fails for the naive geometric height.)

Gauthier and Vigny have recently given a new proof of Baker's theorem and of Theorem 3.3 above, and they extended the results to a much more general context. (See also [CH1] for a model-theoretic approach.)

Theorem 4.1. [GV, Theorem A] *Suppose that S is a smooth and irreducible quasiprojective variety, $\pi : \mathcal{X} \to S$ a family of projective varieties X_t for $t \in S(\mathbb{C})$, and*

$$f : \mathcal{X} \to \mathcal{X}$$

an algebraic family of polarized endomorphisms. Then a section $a : S \to \mathcal{X}$ of π is stable if and only if $\hat{h}_f(a) = 0$ if and only if a is either preperiodic or lies in an "isotrivial part" of f in $\mathcal{X}$.

There is a lot to define here. Recall that polarizable endomorphisms $f : X \to X$ of a projective variety over $\mathbb{C}$ were introduced in §1.2; this means that there is an ample line bundle L so that $f^*L \simeq L^d$ for some $d > 1$. Here we work with an endomorphism $f : X \to X$ defined over $k = \mathbb{C}(S)$, with $\pi : \mathcal{X} \to S$ a model over $\mathbb{C}$, and we assume there is a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ providing a polarization on $X_t = \pi^{-1}(t)$ for each $t \in S(\mathbb{C})$. Stability of the pair (f, a) may be defined in terms of normal families, as for maps on $\mathbb{P}^1$. The *isotrivial part* is defined as you might expect, though I will avoid technicalities: there is an f-invariant subvariety over S along which which the restricted family is isotrivial. The canonical height $\hat{h}_f$ is defined on $X(k)$, starting with a choice of Weil height h on $k = \mathbb{C}(S)$.

Note that Theorem 4.1 includes the case of polarized endomorphisms on a family $\mathcal{A}$ of abelian varieties, for example taking multiplication by 2 on each fiber A_t, with $\hat{h}_A$ the Néron-Tate canonical height. Thus, the theorem extends known results in the setting of abelian varieties [LN] to this more general setting of polarized endomorphisms.

4.2. **Bifurcation currents.** The proof of Theorem 4.1 (and the proof of Theorem 3.3) involves a study of certain positive closed currents on the parameter space S.

Suppose that $Z \subset X$ is a subvariety of dimension ℓ defined over $k = \mathbb{C}(S)$, and suppose that $\mathcal{Z}$ is a (flat) family of subvarieties in the model $\mathcal{X}$ over S. Following [GV], we may define

$$\hat{T}_{f,Z} := \pi_* \left((\hat{T}_f)^{\wedge(\ell+1)} \wedge [\mathcal{Z}] \right) \tag{4.1}$$

where $\pi : \mathcal{X} \to S$ is the projection; it is a positive (1,1)-current on S with continuous potentials. The current $\hat{T}_f$ is a positive (1,1)-current on $\mathcal{X}$ defined analogously to the dynamical Green current in §2.4. Namely, we choose a smooth (1,1)-form ω on $\mathcal{X}$ which represents the class $c_1(L_t)$ for the polarization L_t on each fiber X_t. We have

$$\hat{T}_f = \lim_{n\to\infty} \frac{1}{d^n}(f^n)^*\omega$$

where d is the polarization degree. When $Z = a$ is a single point, it is not hard to see that $\hat{T}_{f,a} = 0$ if and only if the pair (f, a) is stable on S. More generally, we can say that the pair (f, Z) is *stable over* S if $\hat{T}_{f,Z} = 0$. (Note that the stability definition makes sense over any complex manifold S, while the canonical height in Theorem 4.1 is a "global" notion.)

The current $\hat{T}_{f,Z}$ extends the notion of bifurcation current introduced in [De1] to study J-stability as defined in Lecture 3. More precisely, we consider a holomorphic family $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ with critical locus $\mathrm{Crit}(f) \subset S \times \mathbb{P}^1$, and we set

$$\hat{T}_{f,\mathrm{bif}} := \pi_*\left(\hat{T}_f \wedge [\mathrm{Crit}(f)]\right) \tag{4.2}$$

where $\pi : S \times \mathbb{P}^1 \to S$ is the projection. Then the family f_t for $t \in S$ is J-stable at t_0 if and only if the current $\hat{T}_{f,\mathrm{bif}}$ vanishes in a neighborhood of t_0 [De1, Theorem 1.1].

Towards proving Theorem 4.1, Gauther and Vigny proved that the geometric canonical height of $Z \subset X$ (of dimension ℓ, defined over the function field $k = \mathbb{C}(S)$) is given by

$$\hat{h}_f(Z) = \int_{\mathcal{X}} (\hat{T}_f)^{\wedge(\ell+1)} \wedge [\mathcal{Z}] \wedge (\omega_S)^{\dim S - 1} \tag{4.3}$$

where $\mathcal{Z}$ is the corresponding variety in $\mathcal{X}$ over S, and ω_S is the pull-back to $\mathcal{X}$ of a certain Kähler form on S. This integral formula was known in the case where $\mathcal{X}$ is an elliptic surface [CDMZ] and a version appears also in [CGHX] for families of abelian varieties.

Gauthier and Vigny also describe conditions on the dynamics of the map f that guarantee positivity of the current $\hat{T}_{f,Z}$ (so also of $\hat{h}_f(Z)$) and its higher wedge powers [GV, Lemma 4.8]. Their instability criterion has its origins in proofs that powers of the current $\hat{T}_{f,\mathrm{bif}}$ of (4.2) are positive [BB, BE] and the general theory of stability for families of higher-dimensional maps [BBD].

4.3. **Arithmetic equidistribution.** Building on a series of works studying the geometry of points of small height on projective varieties, starting with the work of Szpiro-Ullmo-Zhang for abelian varieties [SUZ] and generalizing the recent equidistribution results of Kühne [Kü1] and Gauthier [Ga], Yuan and Zhang recently proved:

Theorem 4.2. [YZ, Theorem 5.4.3] *Suppose that X is a quasi-projective variety over a number field K. Let $\overline{L}$ be a nef adelic line bundle on X for which $\deg_{\tilde{L}}(X/K)$ is* *positive. Suppose that* $\{x_m\} \subset X(\overline{K})$ *is a generic sequence with*

$$h_{\overline{L}}(x_m) \to h_{\overline{L}}(X).$$

Then for each place v *of* K*, the Galois orbits* $\mathrm{Gal}(\overline{K}/K) \cdot x_m$ *are equidistributed in the Berkovich analytification* X_v^{an} *with respect to the measure* $\mu_{\overline{L},v}$*, as* $m \to \infty$*.*

Without going into all the details and definitions, it is important to note that

$$\deg_{\tilde{L}}(X/K) = \int_{X_v^{an}} c_1(\overline{L})_v^{\wedge n} \tag{4.4}$$

at any place v, and $\mu_{\overline{L}_v}$ is the probability measure $\frac{1}{\deg_{\tilde{L}}(X/K)} c_1(\overline{L})_v^{\wedge n}$ [YZ, Lemma 5.4.4]. Here, $n = \dim X$ and $c_1(\overline{L})_v$ is the curvature form (i.e., a positive (1,1)-current) at the place v. In particular, the positivity of $\deg_{\tilde{L}}(X/K)$ can be formulated complex-analytically, by working at an archimedean place, where we might understand the current $c_1(\overline{L})$ best.

When working with particular examples of adelically-metrized line bundles $\overline{L}$ on a quasiprojective variety X, it is not always clear when this positivity of $\deg_{\tilde{L}}(X/K)$ holds. But for dynamical examples, we can now use the positivity of bifurcation currents and their wedge powers, as described in §4.2, to show that the hypotheses of Theorem 4.2 are satisfied.

4.4. **Example application: Post-critically finite maps on $\mathbb{P}^1$.** Yuan-Zhang present an important dynamical example in §6 of [YZ], which was also proved by Gauthier in [Ga] (and certain cases were known earlier). Namely, we work in the *moduli space* M_d of all maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$. This is an affine algebraic variety, defined over $\mathbb{Q}$, which parameterizes the $\mathrm{PGL}_2\mathbb{C}$-conjugacy classes of maps on $\mathbb{P}^1$. See [Si3] for background. Within M_d, we are interested in the geometry and distribution of the *post-critically finite* (or PCF) maps f; namely, the maps for which each critical point has a finite forward orbit. The PCF maps are known to form a Zariski dense subset of M_d; see, for example, [De3, Theorem A]. Note that all **Lattès maps** are PCF, though these constitute only 1-parameter families in M_d (for square degrees d) and finite sets in M_d (coming from elliptic curves with complex multiplication). Outside of those "flexible" Lattès maps, all PCF maps can be defined over $\overline{\mathbb{Q}}$, as a consequence of Thurston's Rigidity Theorem [DH].

Theorem 4.2 implies that the PCF maps are uniformly distributed with respect to the bifurcation measure

$$\mu_{\mathrm{bif}} := (\hat{T}_{f,\mathrm{bif}})^{\wedge(2d-2)}, \tag{4.5}$$

where f is the universal family of all maps of degree d and $\dim \mathrm{M}_d = 2d-2$. Indeed, Silverman introduced a *critical height function* on M_d given by

$$\hat{h}_{\mathrm{crit}}(f) = \sum_{c_i} \hat{h}_f(c_i),$$

where the c_i are the critical points of f. This height is associated to a nef adelically metrized line bundle $\overline{L}$. To apply Theorem 4.2, we need positivity of $\deg_{\tilde{L}}(\mathrm{M}_d)$, but (4.4) tells us that it suffices to know the positivity of the measure μ_{bif}. This positivity was first proved in [BB] by observing that the (continuous) potential function for $\hat{T}_{f,\mathrm{bif}}$ has an isolated minimum at each *rigid* Lattès maps (i.e., the quotient of a rigid endomorphism on an elliptic curve with complex multiplication).

4.5. **Example application: Pairs of elliptic curves.** Questions were posed and studied in [BT, BFT] about the geometry of torsion points in pairs of elliptic curves. Given elliptic curves E_1, E_2 defined over $\mathbb{C}$, and degree-two projections $\pi_i : E_i \to \mathbb{P}^1$ satisfying $\pi_i(P) = \pi_i(-P)$ for every $P \in E_i$, an application of the Manin-Mumford theorem of Raynaud [Ra2] implies that either

$$\pi_1(E_1[\infty]) = \pi_2(E_2[\infty]) \quad \text{or} \quad \#\, \pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty]) < \infty.$$

Here $E_i[\infty]$ denotes the set of all torsion points in $E_i(\mathbb{C})$. The first case holds if and only if there exists an isomorphism $\varphi : E_1 \to E_2$ so that $\pi_2 \circ \varphi = \pi_1$. Otherwise finiteness comes from considering the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ and the torsion points lying on its preimage $(\pi_1 \times \pi_2)^{-1}(\Delta)$ in the abelian surface $E_1 \times E_2$. Bogomolov-Fu-Tschinkel asked: *Is there a uniform bound on the size of the intersection* $\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$*, assuming the sets do not coincide?*

The existence of a uniform bound has recently been established by Poineau [Po], and it can also be deduced from the recent results of Kühne [Kü2] and Gao-Ge-Kühne [GGK]; see also [DKY] treating a certain 2-parameter family of pairs. Note that a (moduli) space of all pairs $((E_1, \pi_1), (E_2, \pi_2))$ has dimension 5.

In [DM], Mavraki and I presented yet another proof of the uniform bound on the size of $\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$, related to the ideas of this lecture series, that I will outline here. We rely on Theorem 4.2, and we followed the general proof outline appearing in Mavraki's earlier work with Schmidt [MS] (where they treated 1-parameter families of pairs of maps (f, g) acting on $\mathbb{P}^1 \times \mathbb{P}^1$). We work with pairs (f, g) of **Lattès maps** acting on $\mathbb{P}^1 \times \mathbb{P}^1$, parameterized by a 5-dimensional space S of pairs $((E_1, \pi_1), (E_2, \pi_2))$.

It is first worth observing that the intersections $\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$ are not generally empty. In fact, by repeated application of Theorem 3.3 and Corollary 3.4 (similar to what is done in §3.4), it is possible to prove that

$$\#\, \pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty]) \geq 5$$

for a Zariski-dense set of pairs $((E_1, \pi_1), (E_2, \pi_2))$ in S. In fact, the method also shows that the 5-tuples of points from these intersections form a Zariski-dense subset of $S \times (\mathbb{P}^1)^5$. See [DM, Theorem 1.5].

For the uniform upper bound on $\#\, \pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$, we study the bifurcation current $\hat{T}_\Delta$ of (4.1), associated to the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ over S and the family

(f, g) of Lattès pairs. We prove that the top wedge power $\mu_\Delta := (\hat{T}_\Delta)^{\wedge 5}$ is nonzero on S, using Theorem 3.3 and a dynamical criterion for instability that was also used in [GV] (and, as mentioned above in §4.2, this criterion was originally used to study positivity of traditional bifurcation currents and measures). This turns out to imply positivity of an associated measure μ_{Δ^5} on the product space $S \times \Delta^5$ and, consequently, positive degree of a certain adelically-metrized line bundle $\overline{L}$ on $S \times \Delta^5$; the line bundle $\overline{L}$ is defined so that the zeroes of the associated height function $h_{\overline{L}}$ in $(S \times \Delta^5)(\overline{\mathbb{Q}})$ are precisely the 5-tuples of points in $\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$ over a parameter $s \in S(\overline{\mathbb{Q}})$. As observed above, these zeroes form a generic sequence (being Zariski dense) in $S \times \Delta^5 \simeq S \times (\mathbb{P}^1)^5$. Now we are in a setting where we can apply Theorem 4.2. (A subtle point: we passed to Δ^5 so that we could get positivity of this degree; it fails to be positive on $S \times \Delta^m$ for $m < 5$.)

Now suppose there is no uniform bound on the cardinality of $\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$ for a generic sequence of points in S. This implies that, for every positive integer m – and not just the $m \leq 5$ case we already know – the m-tuples common preperiodic points for the Lattès maps (f, g) form a *generic* subset of the space $S \times \Delta^m$. We take $m = 6$ and construct two metrized line bundles on this space with height functions

$$h_{\overline{L},f}(t, x, y) := h_{\overline{L}}(t, x) + \hat{h}_{f_t}(y) \text{ and } h_{\overline{L},g}(t, x, y) := h_{\overline{L}}(t, x) + \hat{h}_{g_t}(y)$$

for coordinates $(t, x, y) \in S \times \Delta^5 \times \mathbb{P}^1$. Applying Theorem 4.2 to these line bundles, we obtain equidistribution of the 6-tuples of common torsion projections with respect to two measures $\mu_{\Delta^5} \otimes \hat{T}_f$ and $\mu_{\Delta^5} \otimes \hat{T}_g$ on $S \times \Delta^6 \simeq S \times \Delta^5 \times \mathbb{P}^1$. Consequently, these two measures must now be equal. By slicing this measures, we would find that the canonical measures on $\mathbb{P}^1$ of §1.3 satisfy $\mu_{f_t} = \mu_{g_t}$ for a positive μ_Δ-measure set of parameters t in S.

But recall from Lecture 1 that the measure μ_f for a Lattès map f is simply the image of the Haar measure on E from the projection $\pi_* : E \to \mathbb{P}^1$ of (1.1). In particular, the measure μ_f knows the branch points of π and so the isomorphism class of the elliptic curve E. In particular, we deduce that $\mu_{f_t} = \mu_{g_t}$ if and only if $\pi_1(E_1[\infty]) = \pi_2(E_2[\infty])$ if and only if the elliptic curves pairs are the same (up to isomorphism) with the same projection. As this holds for a positive-measure set in S, and since the measure μ_Δ is built from a current with *bounded* potentials, we know that the support of μ_Δ is itself Zariski dense in S, from which we can deduce that E_1 is isomorphic to E_2 for all pairs in S. This is nonsense. So we conclude that there is a uniform bound on $\pi_1(E_1[\infty]) \cap \pi_2(E_2[\infty])$ over a Zariski open subset of S. Working inductively on the dimension of S, we see that the uniform bound can only fail when $\pi_1(E_1[\infty]) = \pi_2(E_2[\infty])$.

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E-mail address: `demarco@math.harvard.edu`