SEVERAL APPROACHES TO NON-ARCHIMEDEAN GEOMETRY

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Introduction

Let k be a *non-archimedean field*: a field that is complete with respect to a specified nontrivial non-archimedean absolute value $|\cdot|$. There is a classical theory of k-analytic manifolds (often used in the theory of algebraic groups with k a local field), and it rests upon versions of the inverse and implicit function theorems that can be proved for convergent power series over k by adapting the traditional proofs over $\mathbf{R}$ and $\mathbf{C}$. Serre's Harvard lectures [S] on Lie groups and Lie algebras develop this point of view, for example. However, these kinds of spaces have limited geometric interest because they are totally disconnected. For global geometric applications (such as uniformization questions, as first arose in Tate's study of elliptic curves with split multiplicative reduction over a non-archimedean field), it is desirable to have a much richer theory, one in which there is a meaningful way to say that the closed unit ball is "connected". More generally, we want a satisfactory theory of coherent sheaves (and hence a theory of "analytic continuation"). Such a theory was first introduced by Tate in the early 1960's, and then systematically developed (building on Tate's remarkable results) by a number of mathematicians. Though it was initially a subject of specialized interest, in recent years the importance and power of Tate's theory of *rigid-analytic spaces* (and its variants, due especially to the work of Raynaud, Berkovich, and Huber) has become ever more apparent. To name but a few striking applications, the proof of the local Langlands conjecture for GL_n by Harris–Taylor uses étale cohomology on non-archimedean analytic spaces (in the sense of Berkovich) to construct the required Galois representations over local fields, the solution by Raynaud and Harbater of Abyhankar's conjecture concerning fundamental groups of curves in positive characteristic uses the rigid-analytic GAGA theorems (whose proofs are very similar to Serre's proofs in the complex-analytic case), and recent work of Kisin on modularity of Galois representations makes creative use of rigid-analytic spaces associated to Galois deformation rings.

The aim of these lectures is to explain some basic ideas, results, and examples in Tate's theory and its refinements. In view of time and space constraints, we have omitted most proofs in favor of examples to illustrate the main ideas. To become a serious user of the theory it is best to closely study a more systematic development. In particular, we recommend [BGR] for the "classical" theory due to Tate, [BL1] and [BL2] for Raynaud's approach based on formal schemes, and [Ber1] and [Ber2] for Berkovich's theory of k-analytic spaces. Some recent lecture notes by Bosch [B] explain both rigid geometry and Raynaud's theory with complete proofs, and a recent Bourbaki survey by Ducros [D] treats Berkovich's theory in greater depth. There are other points of view as well (most notably the work of Huber [H]),

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but we will pass over these in silence (except for a few comments on how Huber's adic spaces relate to Berkovich spaces).

Before we begin, it is perhaps best to tell a story that illustrates how truly amazing it is that there can be a theory of the sort that Tate created. In 1959, Tate showed Grothendieck some *ad hoc* calculations that he had worked out with p-adic theta functions in order to uniformize certain p-adic elliptic curves by a multiplicative group, similarly to the complex-analytic case. Tate wondered if his computations could have deeper meaning within a theory of global p-adic analytic spaces, but Grothendieck was doubtful. In fact, in an August 18, 1959 letter to Serre, Grothendieck expressed serious pessimism that such a global theory could possibly exist: "*Tate has written to me about his elliptic curves stuff, and has asked me if I had any ideas for a global definition of analytic varieties over complete valuation fields. I must admit that I have absolutely not understood why his results might suggest the existence of such a definition, and I remain skeptical. Nor do I have the impression of having understood his theorem at all; it does nothing more than exhibit, via brute formulas, a certain isomorphism of analytic groups; one could conceive that other equally explicit formulas might give another one which would be no worse than his (until proof to the contrary!)*"

1. Affinoid algebras

1.1. **Tate algebras.** In this first lecture, we discuss the commutative algebra that forms the foundation for the local theory of rigid-analytic spaces, much as the theory of polynomial rings over a field is the basis for classical algebraic geometry. (The primary reference for this lecture and the next one is [BGR].) The replacement for polynomial rings over a field will be Tate algebras.

Unless we say to the contrary, throughout this lecture and all subsequent ones we shall fix a non-archimedean field k, and we write R to denote its *valuation ring* and $\widetilde{k}$ its *residue field*:

$$R = \{t \in k \,|\, |t| \leq 1\}, \ \ \widetilde{k} = R/\mathfrak{m}$$

where $\mathfrak{m} = \{t \in R \,|\, |t| < 1\}$ is the unique maximal ideal of R.

Exercise 1.1.1. The *value group* of k is $|k^\times| \subseteq \mathbf{R}^\times_{>0}$. Prove that R is noetherian if and only if $|k^\times|$ is a discrete subgroup of $\mathbf{R}^\times_{>0}$, in which case R is a discrete valuation ring.

It is a basic fact that every finite extension k'/k admits a unique absolute value $|\cdot|'$ (necessarily non-archimedean) extending the given one on k, and that k' is complete with respect to this absolute value. Explicitly, if $x' \in k'$ then $|x'|' = |\mathrm{N}_{k'/k}(x')|^{1/[k':k]}$, but it is not obvious that this latter definition satisfies the non-archimedean triangle inequality (though it is clearly multiplicative). The absolute value on k therefore extends uniquely to any algebraic extension of k (using that it extends to every finite subextension, necessarily compatibly on overlaps due to uniqueness), though if k'/k is not finite then k' may not be complete. In view of the uniqueness, this extended absolute value is invariant under all k-automorphisms of k'. It is an important fact that if $\overline{k}$ is an algebraic closure of k then its completion $\overline{k}^\wedge$ is still algebraically closed and its residue field is an algebraic closure of $\widetilde{k}$. In the special case $k = \mathbf{Q}_p$, this completed algebraic closure is usually denoted $\mathbf{C}_p$.

Definition 1.1.2. For $n \geq 1$, the n-variable *Tate algebra* over k is

$$T_n = T_n(k) = \left\{ \sum a_J X^J \,|\, |a_J| \to 0 \text{ as } \|J\| \to \infty \right\},$$

where for a multi-index $J = \{j_1, \ldots, j_n\}$ we write X^J to denote $\prod X_i^{j_i}$ and $\|J\|$ to denote $\sum_i j_i$. In other words, $T_n(k)$ is the subring of formal power series in $k[\![X_1, \ldots, X_n]\!]$ that converge on R^n. This k-algebra is also denoted $k\langle X_1, \ldots, X_n \rangle$.

The *Gauss norm* (or *sup norm*, for reasons to become clear shortly) on T_n is

$$\| \sum a_J X^J \| = \max_J |a_J| \geq 0.$$

Obviously $\|f\| = 0$ if and only if $f = 0$.

Exercise 1.1.3. This exercise develops properties of the Gauss norm on T_n. This gives T_n a topological structure that goes beyond its mere algebraic structure.

(1) Prove that the Gauss norm is a k-Banach algebra norm on T_n. That is, $\|f_1 + f_2\| \leq \max(\|f_1\|, \|f_2\|)$ for all $f_1, f_2 \in T_n$, $\|cf\| = |c| \cdot \|f\|$ for all $c \in k$ and $f \in T_n$, $\|f_1 f_2\| \leq \|f_1\| \|f_2\|$ for all $f_1, f_2 \in T_n$, and T_n is complete for the metric $\|f_1 - f_2\|$.
(2) By using $k^\times$-scaling to reduce to the case of unit vectors, show that $\|f_1 f_2\| = \|f_1\| \cdot \|f_2\|$ for all $f_1, f_2 \in T_n$. That is, the Gauss norm is multiplicative.
(3) Let $\overline{k}$ be an algebraic closure of k, endowed with the unique absolute value (again denoted $|\cdot|$) extending the given one on k. Using $k^\times$-scaling to reduce to the case of unit vectors, prove that the Gauss norm computes a supremum of magnitudes over the closed unit n-ball over $\overline{k}$:

$$\|f\| = \sup_x |f(x)| = \max_x |f(x)|,$$

where $x = (x_1, \ldots, x_n)$ varies with $x_j \in \overline{k}$ and $|x_j| \leq 1$. In particular, this supremum and maximum are finite.
(4) Show that the use of $\overline{k}$ in the previous part is essential: give an example of $f \in \mathbf{Q}_p\langle X \rangle$ such that $\|f\| > \sup_{x \in \mathbf{Z}_p} |f(x)|$.

We want $T_n(k)$ to be the "coordinate ring" of the closed unit n-ball over k, but as with algebraic geometry over a field that may not be algebraically closed, we have to expect to work with points whose coordinates are not all in k. That is, the underlying space for the closed unit n-ball over k should admit points with coordinates in finite extensions of k. Let's now see that T_n admits many k-algebra maps to finite extensions of k.

Exercise 1.1.4. Let k'/k be a finite extension, and choose $c'_1, \ldots, c'_n \in k'$ with $|c'_j| \leq 1$. Prove that there exists a unique continuous k-algebra map $T_n \to k'$ (using the Gauss norm on T_n) such that $X_j \mapsto c'_j$ for all j. Conversely, prove that every continuous k-algebra map $T_n \to k'$ arises in this way. (Hint for converse: $c \in k'$ satisfies $|c| \leq 1$ if and only if the sequence $\{c^m\}_{m \geq 1}$ in k' is bounded.)

The basic properties of T_n are summarized in the following result that is analogous to properties of polynomial rings over a field. The proofs of these properties are inspired by the local study of complex-analytic spaces (via Weierstrass Preparation techniques to carry out induction on n).

Theorem 1.1.5. *The Tate algebra $T_n = T_n(k)$ satisfies the following properties:*

(1) *The domain T_n is noetherian, regular, and a unique factorization domain. For every maximal ideal $\mathfrak{m}$ of T_n the local ring $(T_n)_{\mathfrak{m}}$ has dimension n and residue class field $T_n/\mathfrak{m}$ that has finite degree over k.*
(2) *The ring T_n is Jacobson: every prime ideal $\mathfrak{p}$ of T_n is the intersection of the maximal ideals containing it. In particular, if I is an ideal of T_n then an element of T_n/I is nilpotent if and only if it lies in every maximal ideal of T_n/I.*
(3) *Every ideal in T_n is closed with respect to the Gauss norm.*

As a consequence of this theorem, we can reinterpret Exercise 1.1.3(3) and Exercise 1.1.4 in a more geometric manner, as follows. Consider the set $\mathrm{MaxSpec}(T_n)$ of maximal ideals of T_n. A point in this set will usually be denoted as x, though if we want to emphasize its nature as a maximal ideal we may denote it as $\mathfrak{m}_x$. To each such point there is associated the residue class field $k(x) = T_n/\mathfrak{m}_x$ of finite degree over k, and this field is equipped with the unique absolute value (which we also denote as $|\cdot|$) that extends the given one on k. For any $f \in T_n$ we write $f(x)$ to denote the image of f in $k(x)$. We can combine Exercise 1.1.3(3) and Exercise 1.1.4 to say that for all $f \in T_n$,

$$\|f\| = \sup_x |f(x)| = \max_x |f(x)|,$$

where now x varies through $\mathrm{MaxSpec}(T_n)$; there is no intervention of the auxiliary $\overline{k}$ here. In particular, the function $x \mapsto |f(x)|$ on $\mathrm{MaxSpec}(T_n)$ is bounded and attains a maximal value. (It is "as if" $\mathrm{MaxSpec}(T_n)$ were a compact topological space, an idea that becomes a reality within the framework of Berkovich spaces, as we shall see later.) One curious consequence of this formula for the Gauss norm in terms of $\mathrm{MaxSpec}(T_n)$ and the intrinsic k-algebra structure of T_n is that the Gauss norm is intrinsic to the k-algebra T_n and does not depend on its "coordinates" $X_j \in T_n$; in particular, it is invariant under all k-algebra automorphisms of T_n (which is not obvious from the initial definition of the Gauss norm).

1.2. **Affinoid algebras.**

Much as affine algebraic schemes over a field can be obtained from quotients of polynomial rings, and these in turn are the local model spaces from which more general algebraic schemes are constructed via gluing, the building blocks for rigid-analytic spaces will be obtained from quotients of Tate algebras. This distinguished class of k-algebras is given a special name, as follows.

Definition 1.2.1. A *k-affinoid algebra* is a k-algebra A admitting an isomorphism $A \simeq T_n/I$ as k-algebras, for some ideal $I \subseteq T_n$. The set $\mathrm{MaxSpec}(A)$ of maximal ideals of A is denoted $M(A)$.

Example 1.2.2. We have $R^n \subseteq M(T_n)$ in an evident manner, but if k is not algebraically closed (e.g., $k = \mathbf{Q}_p$) then $M(T_n)$ has many more points than just those coming from R^n. This underlies the enormous difference between rigid-analytic spaces over k and the more classical notion of a k-analytic manifold.

By Theorem 1.1.5, every k-affinoid algebra A is noetherian and Jacobson with finite Krull dimension, and $A/\mathfrak{m}$ is a finite extension of k for every $\mathfrak{m} \in M(A)$. For a point $x \in M(A)$ we write $k(x)$ to denote this associated finite extension of k and we write $a(x) \in k(x)$ to denote the image of $a \in A$ in $k(x)$. By the Jacobson property of A, $a \in A$ is nilpotent if and only if $a(x) = 0$ for all $x \in M(A)$. Obviously $a \in A^\times$ if and only if $a(x) \neq 0$ for all $x \in M(A)$. In this respect, we can view elements of A as "functions" on $M(A)$ (valued in varying fields $k(x)$) much like we do for coordinate rings of affine algebraic schemes over a field, and the function $x \mapsto a(x)$ determines a up to nilpotents.

Exercise 1.2.3. Recall that any domain of finite dimension over a field is itself a field. Using this, prove that $M(A)$ is functorial via pullback. That is, if $f : A' \to A$ is a map of k-affinoid algebras then the prime ideal $f^{-1}(\mathfrak{m}) \subseteq A'$ is a maximal ideal of A' for every maximal ideal $\mathfrak{m}$ of A.

Geometrically, if we choose an isomorphism $A \simeq T_n/I$ and we let $\{f_1, \dots, f_m\}$ be generators of I then functoriality provides an injection $M(A) \hookrightarrow M(T_n)$ onto the subset of points

$$\{x \in M(T_n) \mid f_1(x) = \cdots = f_m(x) = 0\} = \{x \in M(T_n) \mid f(x) = 0 \text{ for all } f \in I\}.$$

In this sense, we want to think of $M(A)$ as being the underlying set of the "space" of points in the closed unit n-ball over k where the f_j's simultaneously vanish. (Keep in mind that, just as for $M(T_n)$, if k is not algebraically closed then there are generally many points $x \in M(A)$ with $k(x) \neq k$, which is to say that $M(A)$ usually has many points that are not k-rational. This abundance of non-rational points over the base field is a fundamental distinction between rigid-analytic spaces and the more classical concept of a k-analytic manifold. In Berkovich's theory there will nearly always be even more points than these, and in particular lots of non-rational points even if k is algebraically closed! This is analogous to the fact that an algebraic scheme over an algebraically closed field nearly always has many non-rational points.) In order to give geometric substance to the sets $M(A)$, we need to endow them with a good function theory, and this in turn requires an understanding of the topological structure of A. Thus, we now turn to this aspect of k-affinoid algebras.

A *k-Banach space* is a k-vector space V equipped with a function $\|\cdot\| : V \to \mathbf{R}_{\geq 0}$ such that $\|v\| = 0$ if and only if $v = 0$,

$$\|v + v'\| \leq \max(\|v\|, \|v'\|), \quad \|cv\| = |c| \cdot \|v\|$$

(for all $v, v' \in V$ and $c \in k$), and V is complete for the metric $\|v - v'\|$. Likewise, a *k-Banach algebra* (always understood to be commutative) is a k-algebra $\mathscr{A}$ equipped with a k-Banach space structure $\|\cdot\|$ that is submultiplicative with respect to the multiplication law on $\mathscr{A}$: $\|a_1 a_2\| \leq \|a_1\| \cdot \|a_2\|$ for all $a_1, a_2 \in \mathscr{A}$. For example, we have seen that T_n with the Gauss norm is a k-Banach algebra, and in fact any k-affinoid algebra A admits a k-Banach algebra structure. To see this, we choose an isomorphism $A \simeq T_n/I$ as k-algebras, and since I is closed in T_n we may use the residue norm from T_n to define a k-Banach structure on the quotient T_n/I (and hence on A), as is explained in the next exercise.

Exercise 1.2.4. Let $(V, \|\cdot\|)$ be a k-Banach space and W a closed subspace. For $\overline{v} \in V/W$ define the *residue norm* on $\overline{v}$ to be

$$\|\overline{v}\|' = \inf_{v \bmod W = \overline{v}} \|v\|,$$

the infimum of the norms of all representatives of $\overline{v}$ in V. Using that W is closed, prove that this is a k-Banach space structure on V/W; what goes wrong if W is not closed in V? In the special case that $V = \mathscr{A}$ is a k-Banach algebra and $W = I$ is a closed ideal, prove that the residue norm is a k-Banach algebra structure on $\mathscr{A}/I$.

If we choose two different presentations $T_n/I \simeq A$ and $T_m/J \simeq A$ of a k-affinoid algebra A as a quotient of a Tate algebra, then the resulting residue norms on A are generally not the same. In this sense, A usually has no canonical k-Banach structure (in contrast with T_n). However, it turns out that any two k-Banach algebra structures on A (even those perhaps not arising from a presentation of A as a quotient of a Tate algebra) are bounded by positive multiples of each other, and hence the resulting k-Banach *topology* and concepts such as "boundedness" are in fact intrinsic to A. In particular, for this intrinsic k-Banach topology all ideals of A are closed (since the "residue norm" construction via an isomorphism $A \simeq T_n/I$ reduces this to the known case of Tate algebras). These and further remarkable features of the k-Banach algebra structures on k-affinoid algebras are summarized in the next result.

Theorem 1.2.5. *Let A be a k-affinoid algebra.*

(1) *If $\|\cdot\|$ and $\|\cdot\|'$ are k-Banach algebra norms on A then there exist $C \geq c > 0$ such that*
$$c\|\cdot\| \leq \|\cdot\|' \leq C\|\cdot\|,$$
so both norms define the same topology and the same concept of boundedness. In particular, for $a \in A$ the property that the sequence $\{a^n\}_{n\geq 1}$ is bounded (i.e., a is power-bounded) is independent of the choice of k-Banach algebra structure.

(2) *Any k-algebra map $A' \to A$ between k-affinoid algebras is automatically continuous for the intrinsic k-Banach topologies, or equivalently is a bounded linear map with respect to any choices of k-Banach algebra norms.*

(3) *Any A-algebra A' with module-finite structure map $A \to A'$ is necessarily a k-affinoid algebra.*

(4) (Noether normalization theorem) *If $d = \dim A \geq 0$ then there is a module-finite k-algebra injection $T_d(k) \hookrightarrow A$. In particular, if A is a domain then all of its maximal ideals have height d.*

(5) (Maximum Modulus Principle) *For any $f \in A$ we have the equality*
$$\|f\|_{\sup} := \sup_{x\in M(A)} |f(x)| = \max_{x\in M(A)} |f(x)| < \infty.$$
In particular, the function $x \mapsto |f(x)|$ on $M(A)$ is bounded and attains a maximal value. If A is reduced (i.e., has no nonzero nilpotent elements) then this is a k-Banach algebra structure on A.

The final part of this theorem provides a canonical k-Banach algebra structure on any reduced k-affinoid algebra, recovering the Gauss norm in the special case of Tate algebras. This k-Banach algebra structure may not be multiplicative, but it is clearly power-multiplicative: $\|a^n\|_{\sup} = \|a\|_{\sup}^n$ for all $a \in A$ and $n \geq 1$. In particular, for a reduced k-affinoid algebra A we deduce the important consequence that $a \in A$ is power-bounded if and only if $\|a\|_{\sup}^n$ is bounded for $n \geq 1$, which is to say $\|a\|_{\sup} \leq 1$, or in other words that $|a(x)| \leq 1$ for all $x \in M(A)$. In fact, it can be shown that this characterization of power-boundedness in k-affinoid algebras is valid without assuming reducedness. That is, if A is any k-affinoid algebra then $a \in A$ is power-bounded if and only if $|a(x)| \leq 1$ for all $x \in M(A)$.

Remark 1.2.6. For units in k-affinoid algebras A (i.e., $u \in A$ such that $u(x) \neq 0$ for all $x \in M(A)$!) there is a "minimum modulus principle": for $u \in A^\times$, $\inf_{x \in M(A)} |u(x)| = \min_{x \in M(A)} |u(x)| > 0$. Indeed, this is a reformulation of the Maximum Modulus Principle for $1/u$.

We conclude this lecture with an exercise that provides a universal mapping property for Tate algebras within the category of k-affinoid algebras (and even k-Banach algebras), reminiscent of the universal mapping property of polynomial rings.

Exercise 1.2.7. Let $\mathscr{A}$ be a k-Banach algebra, and let $\mathscr{A}^0$ be the subset of power-bounded elements: $a \in \mathscr{A}$ such that the sequence $\{a^n\}_{n \geq 1}$ is bounded with respect to the k-Banach norm on $\mathscr{A}$.

1. Prove that $\mathscr{A}^0$ is a subring of $\mathscr{A}$, and in fact is a subalgebra over the valuation ring R of k.
2. Prove that $\mathscr{A}^0$ is functorial in $\mathscr{A}$ within the category of k-Banach algebras (using continuous maps). In particular, any k-Banach algebra map $T_n = T_n(k) \to \mathscr{A}$ carries each X_j to an element $a_j \in \mathscr{A}^0$.
3. Show that the map of sets $\mathrm{Hom}(T_n, \mathscr{A}) \to (\mathscr{A}^0)^n$ defined by $\phi \mapsto (\phi(X_1), \ldots, \phi(X_n))$ is bijective. This is the universal mapping property of T_n within the category of k-Banach algebras, and in particular within the full subcategory of k-affinoid algebras.
4. As an application of the universal property, we can "recenter the polydisc" at any k-rational point. That is, for $c_1, \ldots, c_n \in k$ with $|c_j| \leq 1$, prove that there is a unique automorphism of T_n satisfying $X_j \mapsto X_j - c_j$.

2. Global rigid-analytic spaces

2.1. **Topological preparations.** In the first lecture we studied some basic algebraic and topological properties of k-affinoid algebras, and in particular for any such algebra A we introduced the set $M(A)$ of maximal ideals of A. We wish to impose a suitable topology (really a mild Grothendieck topology) on A with respect to which notions such as connectedness will have a good meaning. But before doing that we want to explain how $M(A)$ has a Hausdorff "canonical topology" that is closer in spirit to the totally disconnected topology that arises in the classical theory of k-analytic manifolds. This canonical topology is not especially useful, but it is psychologically satisfying to know that it exists; the subtle issue is that $M(A)$ usually has many points that are not k-rational and it is also not a set of $\overline{k}$-points either (unless $k = \overline{k}$). For this reason, it requires some thought to define the canonical topology. The motivation for the definition comes from the following concrete description of $M(A)$.

Exercise 2.1.1. Let A be a k-affinoid algebra, and $\overline{k}$ an algebraic closure of k. For each $x \in M(A)$ if we choose a k-embedding $i : k(x) \hookrightarrow \overline{k}$ then we get a k-algebra map $A \to \overline{k}$ whose image lies in a subextension of finite degree over k. Let $A(\overline{k})$ denote the set of k-algebra maps $A \to \overline{k}$ with image contained in a subfield of finite degree over k; this set has contravariant functorial dependence on A. Observe that $\mathrm{Aut}(\overline{k}/k)$ acts on this set via composition.

(1) Show that if we change the choice of i then the resulting map in $A(\overline{k})$ changes by the action of $\mathrm{Aut}(\overline{k}/k)$. Hence, we get a well-defined map of sets $M(A) \to A(\overline{k})/\mathrm{Aut}(\overline{k}/k)$ into the space of orbits of $\mathrm{Aut}(\overline{k}/k)$ on $A(\overline{k})$.
(2) Prove that the map $M(A) \to A(\overline{k})/\mathrm{Aut}(\overline{k}/k)$ is functorial in A, and that it is a bijection.
(3) For any $x \in A(\overline{k})$ and $f \in A$ we get a well-defined element $f(x) \in \overline{k}$ and hence a number $|f(x)|$. Show that the loci

$$\{x \in A(\overline{k}) \,|\, |f_1(x)| \geq \varepsilon_1, \ldots |f_n(x)| \geq \varepsilon_n, |g_1(x)| \leq \eta_1, \ldots, |g_m(x)| \leq \eta_m\}$$

for $f_1, \ldots, g_m \in A$ and $\varepsilon_1, \ldots, \eta_m > 0$ are a basis of open sets for a topology on $A(\overline{k})$. Give $M(A)$ the resulting quotient topology. Prove that this topology on $M(A)$ is Hausdorff and totally disconnected, and that it is functorial in A (in the sense that the pullback map $M(A) \to M(A')$ induced by a k-algebra map $A' \to A$ of k-affinoid algebras is continuous). This is the *canonical* topology on $M(A)$.
(4) Show that $M(T_n)$ is the disjoint union of two open sets, $\{|x_1| = \cdots = |x_n| = 1\}$ and its complement.

Having introduced the canonical topology, we now prepare to build up the Tate topology that will replace it. The basic idea is to artfully restrict both the open sets and the coverings of one open set by others that we permit ourselves to consider. In this way, disconnectedness will be eliminated where it is not desired. For example, the decomposition in Exercise 2.1.1(4) will be eliminated in Tate's theory, and in fact $M(T_n)$ will (in an appropriate sense) wind up becoming connected. To construct Tate's theory, we need to introduce several important classes of open subsets of $M(A)$: *Weierstrass domains*, *Laurent domains*, and *rational domains*. These are analogues of basic affine opens as used in algebraic geometry, but the main difference is that we can consider loci defined by (non-strict!) inequalities of the type $|f_1| \leq |f_2|$ on absolute values, whereas in algebraic geometry (with the Zariski topology) we can only use conditions of the type $f_1 \neq f_2$. It should be noted that later in the theory we will permit strict inequalities of the type $|f_1| < |f_2|$, but at the beginning it is non-strict inequalities that are more convenient to use as the building blocks. (Roughly what is happening is that non-strict inequalities define loci that will behave "as if" they are compact, which is in fact what happens within the framework of Berkovich's theory, whereas strict inequalities define loci that lack a kind of compactness property.)

In order to define interesting open domains within $M(A)$, it will be useful to first introduce a relative version of Tate algebras, much as we do with polynomial rings over a general commutative ring.

Definition 2.1.2. Let $(\mathscr{A}, \|\cdot\|)$ be a k-Banach algebra. The *Tate algebra over* $\mathscr{A}$ (in n variables) is

$$\mathscr{A}\langle Y_1, \ldots, Y_n\rangle = \left\{\sum a_J Y^J \in \mathscr{A}[\![Y_1, \ldots, Y_n]\!] \,|\, a_J \to 0\right\};$$

this is also denoted $\mathscr{A}\langle \underline{Y}\rangle$ if n is understood from context. We define a norm on this ring as follows:

$$\|\sum a_J Y^J\| = \max_J \|a_J\|.$$

Exercise 2.1.3. Let $\mathscr{A}$ be a k-Banach algebra.

(1) Check that in the preceding definition, the norm is a k-Banach algebra structure on $\mathscr{A}\langle \underline{Y}\rangle$, and that if the k-Banach algebra structure on $\mathscr{A}$ is replaced with an equivalent such norm (i.e., one bounded above and below by a positive constant multiple of the given one) then the resulting norm on the Tate algebra over $\mathscr{A}$ is also replaced with an equivalent one. In particular, if $\mathscr{A}$ is k-affinoid then all of its k-Banach algebra structures define equivalent norms on the Tate algebras over $\mathscr{A}$.

(2) If $\mathscr{A} = A$ is k-affinoid, and say $T_m/I \simeq A$ is an isomorphism, show that the resulting natural map $T_{n+m} \to A\langle Y_1, \dots, Y_n\rangle$ is surjective, so the Tate algebras over A are also k-affinoid.

(3) In the category of k-Banach algebras over $\mathscr{A}$ (i.e., the category of k-Banach algebras equipped with a continuous map from $\mathscr{A}$, and morphisms are as $\mathscr{A}$-algebras), state and prove a universal mapping property similar to that for $T_n(k)$ in the category of k-Banach algebras. Using this, construct a "transitivity" isomorphism of the type $(\mathscr{A}\langle \underline{X}\rangle)\langle \underline{Y}\rangle \simeq \mathscr{A}\langle \underline{X}, \underline{Y}\rangle$.

Let A be a k-affinoid algebra. The following class of rings will wind up being the coordinate rings of subsets of $M(A)$ to be called *Laurent domains*. Let A be a k-affinoid algebra. For $\underline{a} = (a_1, \dots, a_n) \in A^n$ and $\underline{a}' = (a'_1, \dots, a'_m) \in A^m$, define

$$A\langle \underline{a}, \underline{a}'^{-1}\rangle = A\langle X_1, \dots, X_n, Y_1, \dots, Y_m\rangle/(X_1 - a_1, \dots, X_n - a_n, a'_1 Y_1 - 1, \dots, a'_m Y_m - 1).$$

Remark 2.1.4. Beware that relative Tate algebras cannot be treated as easily as polynomial rings. For example, consider $A\langle a\rangle = A\langle X\rangle/(X - a)$. This is generally *not* the same as A; geometrically what happens is that we are "forcing" a to become power-bounded, which may not be the case in A at the outset. We will see that the natural map $M(A\langle a\rangle) \to M(A)$ is an injection onto the set of $x \in M(A)$ such that $|a(x)| \le 1$.

In view of the universal property of relative Tate algebras, the A-algebra $A\langle \underline{a}, \underline{a}'^{-1}\rangle$ has the following universal property: for any map of k-Banach algebras $\phi : A \to B$, we can fill in a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\phi"'] & A\langle \underline{a}, \underline{a}'^{-1}\rangle \arrow[d, "?"] \\ & B \end{tikzcd}$$

in at most one way, and such a diagram exists if and only if $\phi(a_i) \in B^0$ for all i (i.e., all $\phi(a_i)$ are power-bounded) and $\phi(a'_j) \in B^\times$ with $\phi(a'_j)^{-1} \in B^0$ for all j. In more geometric language, if B is k-affinoid then we can say that the structure map $\phi : A \to B$ factors through $A\langle \underline{a}, \underline{a}'^{-1}\rangle$ if and only if the map of sets $M(\phi) : M(B) \to M(A)$ factors through the subset

$$\{x \in M(A) \mid |a_1(x)| \le 1, \dots, |a_n(x)| \le 1, |a'_1(x)| \ge 1, \dots, |a'_m(x)| \ge 1\} \subseteq M(A).$$

Exercise 2.1.5. By taking B to vary through finite extensions of k, use the above universal property to deduce that the map of k-affinoid algebras $A \to A\langle \underline{a}, \underline{a'}^{-1}\rangle$ induces a bijection

$$M(A\langle \underline{a}, \underline{a'}^{-1}\rangle) \to \{x \in M(A) \,|\, |a_1(x)| \leq 1, \dots, |a_n(x)| \leq 1, |a'_1(x)| \geq 1, \dots, |a'_m(x)| \geq 1\}.$$

The significance of the conclusion of exercise is that the purely algebraic condition that a map of k-affinoid algebras $\phi : A \to B$ factors through the canonical map $A \to A\langle \underline{a}, \underline{a'}^{-1}\rangle$ is equivalent to the set-theoretic condition that $M(\phi) : M(B) \to M(A)$ factors through the locus of points of $M(A)$ defined by the pointwise conditions $|a_i| \leq 1$ and $|a'_j| \geq 1$. Subsets of $M(A)$ defined by such conditions are called *Laurent domains*, and if there are no a'_j's then we call the subset a *Weierstrass domain*. In particular, by Yoneda's Lemma, a Laurent domain in $M(A)$ functorially determines the k-affinoid A-algebra $A\langle \underline{a}, \underline{a'}^{-1}\rangle$ that gives rise to it, so this latter algebra is intrinsic to the image of its MaxSpec in $M(A)$, and hence it enjoys some independence of the choice of the a_i's and a'_j's. (An analogue in algebraic geometry is that the localizations $A[1/a]$ and $A[1/a']$ are isomorphic as A-algebras if and only if there is the set-theoretic equality of the non-vanishing loci of a and a' in $\operatorname{Spec} A$, in which case such an isomorphism is unique.)

This characterization of an algebra by means of a set-theoretic condition is reminiscent of the situation for affine open subschemes of an affine scheme in algebraic geometry: if $\operatorname{Spec}(A')$ is an open subscheme of $\operatorname{Spec}(A)$, then a map of schemes $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ factors through $\operatorname{Spec}(A')$ as schemes if and only if it does so on underlying sets. Note that closed subschemes rarely have such a set-theoretic characterization (unless they are also open), since we can replace the defining ideal by its square without changing the underlying set but this nearly always changes the closed subscheme. This set-theoretic mapping property suggests that we ought to consider a Laurent domain as an "open subset" of $M(A)$ with associated coordinate ring given by its canonically associated A-algebra as above.

Exercise 2.1.6. Let us work out an example of a Laurent domain (explaining the reason for the name "Laurent domain"). Pick $c \in k$ with $0 < |c| \leq 1$, and consider the Laurent domain in $M(T_1)$ defined by the conditions $|c| \leq |t| \leq 1$, where $T_1 = k\langle t\rangle$; this is an "annulus". The associated coordinate ring is

$$k\langle t, X, Y\rangle/(X - t, c^{-1}tY - 1) = k\langle t, X, Y\rangle(X - t, tY - c).$$

Prove that the natural map $k\langle t\rangle \to k\langle t, X\rangle/(X - t)$ is an isomorphism by considering universal mapping properties, and deduce that the annulus has associated k-affinoid algebra $k\langle t, Y\rangle/(tY - c)$.

Prove that this is a domain. (Hint: show that every element of $k\langle t, Y\rangle/(tY - c)$ can be represented by a unique series of the form $c_0 + \sum_{j\geq 1} c_j t^j + \sum_{j\geq 1} c_{-j} Y^j$ with $c_j \to 0$ as $j \to \infty$ and $c_{-j} \to 0$ as $j \to \infty$. Show that this defines an injection into a k-algebra of doubly infinite Laurent series in t satisfying certain convergence properties.)

Exercise 2.1.7. If you are familiar with étale maps of schemes, then to put in perspective the role of open subschemes in set-theoretic mapping properties, consider the following problem. Let $i : U \to X$ be a locally finitely presented map of schemes with the property that a map of schemes $X' \to X$ factors through i if and only if its image is contained in $i(U)$, in which case such a factorization is unique. Prove that i is an open immersion. (Hint: Show that i is étale via the functorial criterion, and that it is set-theoretically injective and induces purely inseparable residue field extensions. Thus, it is an open immersion by [EGA, IV_4, 17.9.1].)

Now we introduce another important class of subsets of $M(A)$, the *rational domains*. For these subsets, the relevant input is a collection of elements $a_1, \dots, a_n, a' \in A$ with *no common zero*. Given such data, we define

$$A\langle \frac{a_1}{a'}, \dots, \frac{a_n}{a'} \rangle = A\langle X_1, \dots, X_n \rangle / (a'X_1 - a_1, \dots, a'X_n - a_n).$$

What is happening in this A-algebra is that we are forcing $|a_j| \leq |a'|$ at all points (easier to remember as the imprecise condition $|a_j/a'| \leq 1$). To make this precise, we state and prove a universal mapping property.

Lemma 2.1.8. *Let A be a k-affinoid algebra, and $a_1, \dots, a_n, a' \in A$ with no common zero. For any map of k-affinoid algebras $\phi : A \to B$, there is at most one way to fill in the commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\phi"'] & A\langle \frac{a_1}{a'}, \dots, \frac{a_n}{a'} \rangle \arrow[d, "?"] \\ & B \end{tikzcd}$$

and such a diagram exists if and only if $M(\phi) : M(B) \to M(A)$ factors through the subset of $x \in M(A)$ such that $|a_j(x)| \leq |a'(x)|$ for all j, or in other words $|\phi(a_j)(y)| \leq |\phi(a')(y)|$ for all $y \in M(B)$.

Proof. By the universal property of relative Tate algebras, to give such a diagram is to give power-bounded elements $b_1, \dots, b_n \in B$ such that $\phi(a')b_j = \phi(a_j)$ for all j. This implies that $\phi(a')$ must be a unit in B because at any $y \in M(B)$ where it vanishes we get that all $\phi(a_j)$ also vanish, so the point $M(\phi)(y) \in M(A)$ is a common zero of the a_j's and a', contrary to hypothesis. Hence, the b_j's are uniquely determined if they exist, so we get the uniqueness of the diagram if it exists. Moreover, power-boundedness of such b_j's forces $|\phi(a_j)(y)|/|\phi(a')(y)| = |b_j(y)| \leq 1$ for all $y \in M(B)$, which is to say that $M(\phi)$ factors through the desired subset of $M(A)$. Conversely, if this set-theoretic condition holds then $|\phi(a_j)| \leq |\phi(a')|$ pointwise on $M(B)$, so $\phi(a') \in B$ has to be a unit because if it is not a unit then there would exist some $y \in M(B)$ at which $\phi(a')$ vanishes, and hence all $\phi(a_j)$ vanish, yielding the point $M(\phi)(y) \in M(A)$ as a common zero of the a_j's and a' (contrary to hypothesis). But with $\phi(a')$ a unit in B (even if a' is not a unit in A) it makes sense to consider $b_j = \phi(a_j)/\phi(a') \in B$. To construct the desired commutative diagram, the problem is to prove that b_j is power-bounded in B, or equivalently that $|b_j(y)| \leq 1$ for all $y \in M(B)$. This is exactly the assumed system of inequalities $|\phi(a_j)(y)| \leq |\phi(a')(y)|$ for all $y \in M(B)$ since $|\phi(a')(y)| \neq 0$ for all such y. ■

We call a subset in $M(A)$ of the form

$$\{x \in M(A) \,|\, |a_j(x)| \leq |a'(x)|\}$$

for $a_1, \dots, a_n, a' \in A$ with no common zero a *rational domain*. The universal property in the preceding lemma shows that such a subset canonically determines the A-algebra $A\langle a_1/a', \dots, a_n/a' \rangle$.

Example 2.1.9. We write $\sqrt{|k^\times|} \subseteq \mathbf{R}_{>0}^\times$ to denote the divisible subgroup that is generated by $|k^\times|$, which is to say the set of positive real numbers α such that $\alpha^N \in |k^\times|$ for some integer $N > 0$. Note that this is a dense subgroup of $\mathbf{R}_{>0}^\times$. If $\alpha \in \sqrt{|k^\times|}$ and $\alpha^N = |c|$ with $c \in k^\times$, then for k-affinoid A and $f \in A$ the inequality $|f(x)| \leq \alpha$ for $x \in M(A)$ is equivalent to the inequality $|c^{-1}f^N(x)| \leq 1$ Thus, in the definitions of Weierstrass, Laurent, and rational domains it is no more general to permit real scaling factors from $\sqrt{|k^\times|}$ in the inequalities.

For example, in the closed unit disc over $k = \mathbf{Q}_p$, the locus $|t| \leq 1/\sqrt{p}$ is a Weierstrass domain: it is the same as the condition $|pt^2| \leq 1$, and so has associated "coordinate ring" $k\langle t, X\rangle/(X - pt^2)$. In contrast, it is true and unsurprising (but perhaps not obvious how to prove) that for $r \notin \sqrt{|\mathbf{Q}_p^\times|} = p^{\mathbf{Q}}$ the locus $\{|t| \leq r\}$ in the closed unit disc $M(\mathbf{Q}_p\langle t\rangle)$ over $\mathbf{Q}_p$ is *not* a Weierstrass domain.

2.2. **Affinoid subdomains and admissible opens.** Weierstrass, Laurent, and rational domains are the most important examples of the following general concept:

Definition 2.2.1. Let A be a k-affinoid algebra. A subset $U \subseteq M(A)$ is an *affinoid subdomain* if there exists a map $i : A \to A'$ of k-affinoids such that $M(i) : M(A') \to M(A)$ lands in U and is universal for this condition in the following sense: for any map of k-affinoid algebras $\phi : A \to B$, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\phi"'] & A' \arrow[d, "?"] \\ & B \end{tikzcd}$$

if and only if $M(\phi)$ carries $M(B)$ into U, in which case such a diagram is unique.

By Yoneda's Lemma, if $U \subseteq M(A)$ is an affinoid subdomain then the k-affinoid A-algebra A' as in the preceding definition is unique up to unique A-algebra isomorphism. It is therefore legitimate to denote this A-algebra as A_U: it is functorially determined by U. We call A_U the *coordinate ring* of U (with respect to A). For example, the universal property of the domains of Weierstrass, Laurent, and rational types shows that each is an affinoid subdomain and provides an explicit description of A_U in such cases. By chasing points valued in finite extensions of k, it is not hard to show that the natural map $M(A_U) \to M(A)$ is an injection onto $U \subseteq M(A)$. By the universal property we likewise see that if $V \subseteq U$ is an inclusion of affinoid subdomains of $M(A)$ then there is a unique A-algebra map of coordinate rings $\rho_V^U : A_U \to A_V$, and by uniqueness this is transitive with respect to another inclusion $W \subseteq V$ of affinoid subdomains in $M(A)$ (in the sense that $\rho_W^V \circ \rho_V^U = \rho_W^U$). This is to be considered as analogous to restriction maps for the structure sheaf of a scheme, so for $f \in A_U$ we usually write $f|_V$ to denote $\rho_V^U(f) \in A_V$. Akin to the case of schemes, it can be shown (by a method entirely different from the case of schemes) that A_U is A-flat for any affinoid subdomain $U \subseteq M(A)$.

Exercise 2.2.2. Prove that Weierstrass, Laurent, and rational domains in $M(A)$ are all open for the canonical topology. If the condition "no common zero" is dropped from the definition of a rational domain then it still makes sense to consider the underlying set in $M(A)$ defined by the simultaneous conditions $|a_j(x)| \leq |a'(x)|$. Show by example that this locus can fail to be open if there is a common zero.

Exercise 2.2.3. This exercise develops two kinds of completed tensor product operations that arise in rigid-analytic geometry. The theory of the completed tensor product can be developed in greater generality than we shall do, but for our limited purposes we adopt a more utilitarian approach.

(1) Let A and A' be k-affinoid algebras. We wish to construct a k-affinoid "completed tensor product" $A\widehat{\otimes}_k A'$. To do this, first choose presentations $A \simeq T_n/I$ and $A' \simeq T_{n'}/I'$. Using the natural maps $T_n \to T_{n+n'}$ and $T_{n'} \to T_{n+n'}$ onto the first n and last n' variables, it makes sense to let $J, J' \subseteq T_{n+n'}$ be the ideals generated by I and I' respectively. Consider the k-affinoid algebra $T_{n+n'}/(J+J')$. There are evident k-algebra maps $\iota : A \to T_{n+n'}/(J+J')$ and $\iota' : A' \to T_{n+n'}/(J+J')$. Prove that this pair of maps is universal in the following sense: for any k-Banach algebra B and any k-Banach algebra maps $\phi : A \to B$ and $\phi' : A' \to B$, there is a unique k-Banach algebra map $h : T_{n+n'}/J \to B$ so that $h\circ\iota = \phi$ and $h\circ\iota' = \phi'$. In view of this universal property, the triple $(T_{n+n'}/(J+J'), \iota, \iota')$ is unique up to unique isomorphism, so we may denote $T_{n+n'}/(J+J')$ as $A\widehat{\otimes}_k A'$. The product $\iota(a)\iota'(a')$ is usually denoted $a\widehat{\otimes}a'$.

(2) Let $j : A'' \to A$ and $j' : A'' \to A'$ be a pair of maps of k-affinoid algebras. Define the k-affinoid algebra
$$A\widehat{\otimes}_{A''}A' := (A\widehat{\otimes}_k A')/(j(a'')\widehat{\otimes}1 - 1\widehat{\otimes}j'(a'') \,|\, a'' \in A'').$$
Formulate and prove a universal property for this in the category of k-Banach A''-algebras (analogous to the universal property of tensor products of rings).

(3) Let A be a k-affinoid algebra, and let K/k be an analytic extension field (i.e., a non-archimedean field K equipped with a structure of extension of k respecting the absolute values). Beginning with the case of Tate algebras, define a K-affinoid algebra $K\widehat{\otimes}_k A$ as a solution to a universal mapping problem for continuous maps (over $k \to K$) from A to K-Banach algebras.

Exercise 2.2.4. Let A be a k-affinoid algebra.

(1) If $U, U' \subseteq M(A)$ are affinoid subdomains, then prove that $U \cap U'$ is one as well: the k-affinoid A-algebra $A_{U\cap U'}$ is $A_U\widehat{\otimes}_A A_{U'}$. Check this via universal mapping properties.

(2) Let $\phi : A \to B$ be a map of k-affinoid algebras, and let $U \subseteq M(A)$ be an affinoid subdomain. Show that the pullback $M(\phi)^{-1}(U) \subseteq M(B)$ is also an affinoid subdomain: its coordinate ring is $A_U\widehat{\otimes}_A B$. Check this via universal mapping properties. Is there a similar result for Weierstrass, Laurent, and rational domains?

(3) Let $U \subseteq M(A)$ be an affinoid subdomain, with corresponding coordinate ring A_U. Using the natural bijection $U = M(A_U)$, prove that a subset $U' \subseteq U = M(A_U)$ is an affinoid subdomain of $M(A_U)$ if and only if it is an affinoid subdomain of $M(A)$.

The introduction of the concept of affinoid subdomains was a genuine advance beyond Tate's original work, in which he got by with just Weierstrass, Laurent, and rational subdomains. In order to make affinoid subdomains easy to handle (e.g., are they open?), the crucial result required is the *Gerritzen–Grauert theorem* that describes them in terms of rational domains:

Theorem 2.2.5. *Let A be a k-affinoid algebra. Every affinoid subdomain $U \subseteq M(A)$ is a finite union of rational domains. In particular, affinoid subdomains are open with respect to the canonical topology.*

It is very hard to determine when a given finite union of rational domains (let alone affinoid subdomains) is an affinoid subdomain. This is analogous to the difficulty of detecting when a finite union of affine open subschemes of a scheme is again affine. Since Laurent domains are a basis for the canonical topology, in order to get a good theory of non-archimedean analytic spaces we cannot permit ourselves to work with arbitrary unions of affinoid subdomains (or else we will encounter the total disconnectedness problem). Tate's idea is to restrict attention to a class of open subsets (for the canonical topology) and a *restricted* collection of coverings of these opens by such opens so as to "force" affinoid subdomains to appear to be compact. The key definition in the theory is as follows.

Definition 2.2.6. Let A be a k-affinoid algebra. A subset $U \subseteq M(A)$ is an *admissible open* subset if it has a set-theoretic covering $\{U_i\}$ by affinoid subdomains $U_i \subseteq M(A)$ with the following finiteness property under affinoid pullback: for any map of k-affinoid algebras $\phi : A \to B$ such that $M(\phi) : M(B) \to M(A)$ has image contained in U, there are finitely many U_i's that cover this image; equivalently, the open covering $\{M(\phi)^{-1}(U_i)\}$ of $M(B)$ by affinoid subdomains has a finite subcovering.

A collection $\{V_j\}$ of admissible open subsets of $M(A)$ is an *admissible cover* of its union V if, for any k-affinoid algebra map $\phi : A \to B$ with $M(\phi)(M(B)) \subseteq V$, the covering $\{M(\phi)^{-1}(V_j)\}$ of $M(B)$ has a refinement by a covering consisting of finitely many affinoid subdomains. (This forces V to be admissible open, by using the affinoid subdomain covering $\{V_{jk}\}_{j,k\in K_j}$ where $\{V_{jk}\}_{k\in K_j}$ is a covering of each V_j by affinoid subdomains as in the definition of admissibility of each V_j.)

Note that the set-theoretic covering $\{U_i\}$ of U in the definition of an admissible open subset of $M(A)$ is necessarily an admissible cover.

Example 2.2.7. Let $U_1, \ldots, U_n$ be affinoid subdomains of $M(A)$ for a k-affinoid algebra A. Then $U = \cup U_j$ is an admissible open subset, with $\{U_j\}$ an admissible covering of U. (For example, for $a \in A$ the Laurent domains $M(A\langle a\rangle) = \{|a| \leq 1\}$ and $M(A\langle 1/a\rangle) = \{|a| \geq 1\}$ constitute an admissible covering of $M(A)$.) The content here is that the pullback of each U_j under $M(\phi)$ for a k-affinoid algebra map $\phi : A \to B$ is an affinoid subdomain of $M(B)$. It is difficult to determine if such a U is an affinoid subdomain.

Now we come to a key example that shows the significance of the finiteness requirement in the definition of admissibility.

Example 2.2.8. Let $T_1 = k\langle t\rangle$ be the Tate algebra in one variable. Within the closed unit ball $M(T_1)$, the locus $V = \{|t| = 1\}$ is a Laurent domain. The subset $U = \{|t| < 1\}$ is open for the canonical topology, and more importantly it is an admissible open. Indeed, it is covered by the Weierstrass domains $U_n = \{|t| \leq |c|^{1/n}\}$ for a fixed $c \in k$ with $0 < |c| < 1$ and $n \geq 1$, and these satisfy the admissibility condition due to the Maximum Modulus Principle: if $\phi : T_1 \to B$ is a map to a k-affinoid algebra such that $M(B) \to M(T_1)$ lands in U, then the function $\phi(t) \in B$ has absolute value < 1 pointwise on $M(B)$, and so the Maximum

Modulus Principle on $M(B)$ provides $0 < \alpha < 1$ such that $|\phi(t)(y)| \le \alpha$ for all $y \in M(B)$. Hence, for n_0 so large that $\alpha < |c|^{1/n_0} < 1$ we have $M(\phi)(M(B)) \subseteq U_{n_0}$. Thus, the required finite subcover property is satisfied.

The pair of admissible opens $V = \{|t| = 1\}$ and $U = \{|t| < 1\}$ covers $M(T_1)$ set-theoretically, and these are disjoint. However, this is not an *admissible* covering. Indeed, by the definition of admissibility of a covering (applied to the identity map of $M(T_1)$) it would follow that $\{U, V\}$ has as a refinement a finite covering of $M(T_1)$ by affinoid subdomains. But by the Maximum Modulus Principle, any affinoid subdomain of $M(T_1)$ contained in U is contained in some $U_n = \{|t| \le |c|^{1/n}\}$, and hence if there were a refinement of $\{U, V\}$ by a finite collection of affinoid subdomains then we would get that $M(T_1)$ is covered by V and by U_{n_0} for some large n_0. By using a suitable finite extension of k we can certainly find a point in $M(T_1)$ lying in the locus $\{|t| = |c|^{1/(n_0+1)}\}$ that is disjoint from $U_{n_0} \cup V$. This gives a contradiction, so $\{U, V\}$ is not an admissible covering of $M(T_1)$. (Note that here it is essential that our spaces have points that are not necessarily k-rational.)

Exercise 2.2.9. Choose $0 < r < 1$ with $r \notin \sqrt{|k^\times|}$. Prove that $\{|t| \le r\} = \{|t| < r\}$ is an admissible open subset of $M(k\langle t\rangle)$, and give an admissible covering by Weierstrass domains. Prove that this locus does not have an admissible covering by finitely many affinoid subdomains. (Hint: use the Maximum Modulus Principle.)

Exercise 2.2.10. Generalize the method of Example 2.2.8 to show that for any k-affinoid algebra A and $a, a' \in A$, the set $U = \{x \in M(A) \,|\, |a(x)| < |a'(x)|\}$ is an admissible open subset; give an admissible open affinoid covering of U (and be careful about points in $M(A)$ where a' vanishes).

Exercise 2.2.11. Let A be a k-affinoid algebra. Prove that if an admissible open $U \subseteq M(A)$ is covered set-theoretically by some admissible opens U_i then $\{U_i\}$ is an admissible covering of U if and only if it admits an admissible refinement. Also show that admissibility for subsets is a "local" property in the following sense: if $U \subseteq M(A)$ is an admissible open and $\{U_i\}$ is an admissible covering of U by admissible opens then a subset $V \subseteq U$ is admissible open in $M(A)$ if and only if $V \cap U_i$ is admissible open in $M(A)$ for all i.

2.3. **The Tate topology.** The admissible opens and their admissible coverings within $M(A)$ lead to the definition of a very mild Grothendieck topology (in the sense that it only involves subsets of the ambient spaces, which is not a requirement in the general theory of Grothendieck topologies, such as the étale topology on a scheme):

Definition 2.3.1. The *Tate topology* (or *G-topology*) on $M(A)$ has as objects the admissible open subsets and as coverings the admissible open coverings.

Exercise 2.3.2. Let A be a k-affinoid algebra. For $a \in A$ let $V(a)$ be the locus of $x \in M(A)$ for which $a(x) \neq 0$. Prove that this is an admissible open (give an admissible Laurent covering), and show that the $V(a)$'s are a base of opens for a topology on $M(A)$; this is called the *analytic Zariski topology*. Show that the closed sets for this topology are the subsets $M(A/I)$ for ideals I of A (these are called *analytic sets* in $M(A)$), and that all Zariski-opens and Zariski-open covers of Zariski-opens are admissible. (Hint for admissibility: if B is k-affinoid with $\|\cdot\|$ a k-Banach algebra norm on B and $b_1, \dots, b_n \in B$ some elements that generate 1, say $\sum \beta_j b_j = 1$ with $\beta_j \in B$, then show that $M := \max(\|\beta_1\|, \dots, \|\beta_n\|) > 0$ and the Laurent domains $\{|b_j| \geq 1/M\}$ cover $M(B)$.)

The Tate topology is generally not a topology on $M(A)$ in the usual sense, but is instead a Grothendieck topology. It is not crucial (for our purposes) to delve into the general formalism of Grothendieck topologies. The main point that matters for working with the Tate topology is to keep in mind that we do not consider general unions of admissible opens, and when doing sheaf theory we only consider admissible coverings of admissible opens. It generally does not make sense to evaluate a sheaf for the Tate topology on a general open set for the canonical topology, and even for evaluation on admissible opens we cannot expect a sheaf for the Tate topology to satisfy the sheaf axioms for a non-admissible covering of an admissible open by admissible opens (e.g., the pair $\{U, V\}$ in the closed unit disc in Example 2.2.8). When we define disconnectedness later, it will be expressed in terms of an *admissible* covering by a pair of disjoint non-empty admissible opens. Example 2.2.8 shows that this rules out many classical sources of disconnectedness of the canonical topology. The fundamental result that gets the theory off the ground is the existence of a "structure sheaf" with respect to the Tate topology. This is Tate's *Acylicity Theorem*:

Theorem 2.3.3. *Let A be a k-affinoid algebra. The assignment $U \mapsto A_U$ of the coordinate ring to every affinoid subdomain of $M(A)$ uniquely extends to a sheaf $\mathscr{O}_A$ with respect to the Tate topology on $M(A)$. In particular, if $\{U_i\}$ is a finite collection of affinoid subdomains with $U = \cup U_i$ also an affinoid subdomain of $M(A)$ then the evident sequence*

$$0 \to A_U \to \prod A_{U_i} \to \prod A_{U_i \cap U_j}$$

is exact.

Tate proved this theorem by heavy use of Čech-theoretic methods to reduce to the special case of a Laurent covering of $M(A)$ by the pair $\{M(A\langle a\rangle), M(A\langle 1/a\rangle)\}$ for $a \in A$. In this special case he could carry out a direct calculation. The next exercise gives the simplest instance of this calculation.

Exercise 2.3.4. Choose $c \in k$ such that $0 < |c| < 1$. In $M(T_1)$, let $U = \{|t| \leq |c|\}$ and $V = \{|t| \geq |c|\}$, so $U \cap V = \{|t| = |c|\}$. By calculating with convergent Laurent series in t (using Exercise 2.1.6 to describe $A_V \simeq k\langle t, Y\rangle/(tY - c)$ as a $k\langle r\rangle$-algebra of certain Laurent series $\sum_{n \in \mathbf{Z}} c_n t^n$), show that if $f \in A_U$ and $g \in A_V$ satisfy $f|_{U \cap V} = g|_{U \cap V}$ in $A_{U \cap V}$ then there is a unique $h \in T_1$ such that $h|_U = f$ and $h|_V = g$.

Definition 2.3.5. Let A be a k-affinoid algebra. The *affinoid space* $\mathrm{Sp}(A)$ is the pair $(M(A), \mathscr{O}_A)$ consisting of the set $M(A)$ endowed with its Tate topology and sheaf of k-algebras $\mathscr{O}_A$ with respect to the Tate topology. If $A = T_n = T_n(k)$ then this is denoted $\mathbf{B}^n = \mathbf{B}^n_k$. Usually we write $\mathscr{O}_X$ rather than $\mathscr{O}_A$ (with $X = \mathrm{Sp}(A)$).

Exercise 2.3.6. Prove that for $x \in X = \mathrm{Sp}(A)$, the stalk

$$\mathscr{O}_{X,x} = \varinjlim_{x \in U} \mathscr{O}_X(U)$$

(limit over admissible opens, or equivalently affinoid subdomains, containing x) is a local ring. Describe $\mathscr{O}_{\mathbf{B}^1,0}$ as an intermediate ring strictly between the algebraic local ring $k[t]_{(t)}$ and the completion $k[\![t]\!]$.

In general, $\mathscr{O}_{X,x}$ is a noetherian ring that is faithfully flat over the algebraic local ring $A_{\mathfrak{m}_x}$, and in fact it has the same completion, but this requires more work to prove. This is the key to developing a good *dimension theory* for rigid-analytic spaces.

In order to make global definitions, we have to first define the category in which we will be working. A *G-topologized space* is a set X equipped with a set $\mathfrak{U}$ of subsets $U \subseteq X$ (the "open subsets" for the G-topology) and a set of set-theoretic coverings $\mathrm{Cov}(U)$ of each $U \in \mathfrak{U}$ by collections of members of $\mathfrak{U}$ such that certain natural properties from ordinary topology are satisfied; we will not give the list here (see [BGR, Ch. 9] for a precise discussion) except to note that it includes basic locality and transitivity conditions (e.g., $\mathfrak{U}$ is stable under finite intersections, if $\{U_i\} \in \mathrm{Cov}(U)$ and $V \subseteq U$ then $V \in \mathfrak{U}$ if and only if $V \cap U_i \in \mathfrak{U}$ for all i, $\{U\} \in \mathrm{Cov}(U)$, $\emptyset \in \mathfrak{U}$, etc.) and we *omit* the requirement that $\mathfrak{U}$ is stable under arbitrary (or even finite) unions.

One very important construction for G-topologized subspaces is the analogue of the "open subspace" topology. To be precise, if X is a G-topologized space with associated collection $\mathfrak{U}$ of open subsets, then for any $U \in \mathfrak{U}$ we endow U with a structure of G-topologized space by using $\mathfrak{U}_U = \{V \in \mathfrak{U} \,|\, V \subseteq U\}$ as the collection of "open subsets" of U and using the same collection of coverings; that if, for each $V \in \mathfrak{U}_U$ its associated collection $\mathrm{Cov}_U(V)$ of set-theoretic coverings is $\mathrm{Cov}(V)$. This construction does satisfy all of the axioms to be a G-topologized space, and it is called the *open subspace structure* on U.

A *sheaf* (of sets, groups, etc.) on a G-topologized space X is a contravariant assignment $U \mapsto \mathscr{F}(U)$ of a set (or group, etc.) to each $U \in \mathfrak{U}$ such that the usual sheaf axioms are satisfied for coverings in $\mathrm{Cov}(U)$ for all $U \in \mathfrak{U}$. In this respect, we are restricting both the concept of openness and the concept of open covering from ordinary topology. Note that if $U \in \mathfrak{U}$ and $\mathscr{F}$ is a sheaf on X then the functor $\mathscr{F}|_U : V \mapsto \mathscr{F}(V)$ on $\mathfrak{U}_U$ is easily seen to be a sheaf on U with respect to its open subspace structure. We leave it to the reader's imagination (or see [BGR, Ch. 9]) to formulate how one glues G-topologized spaces or sheafifies presheaves (this latter issue requires some care).

Example 2.3.7. Consider a pair $(X, \mathscr{O}_X)$ consisting of a G-topologized space X and a sheaf of k-algebras $\mathscr{O}_X$ on this space. If $U \in \mathfrak{U}$ then with the open subspace structure on U we get another such pair $(U, \mathscr{O}_U)$ where $\mathscr{O}_U$ is the sheaf of k-algebras $\mathscr{O}_X|_U$. This is called an *open subspace* of $(X, \mathscr{O}_X)$.

For a second such pair $(X', \mathscr{O}_{X'})$, a *morphism* $(X', \mathscr{O}_{X'}) \to (X, \mathscr{O}_X)$ is a pair $f : X' \to X$ and $f^\sharp : \mathscr{O}_X \to f_*(\mathscr{O}_{X'})$ where f is continuous (in the sense that pullback under f respects the class of "opens" and their "coverings") and $f^\sharp$ is a map of sheaves of k-algebras (with f_* defined in the usual way; f_* carries sheaves to sheaves because f is continuous). Composition of morphisms is defined exactly as in the theory of ringed spaces. In case the stalks of the structure sheaves are local rings, we can also define the more restrictive notion of a morphism of locally ringed G-topologized spaces.

As a fundamental example, for any k-affinoid algebra A we have constructed such a space $\mathrm{Sp}(A)$, and if $U \subseteq \mathrm{Sp}(A)$ is an affinoid subdomain then the corresponding open subspace

$(U, \mathscr{O}_A|_U)$ is naturally identified with $\mathrm{Sp}(A_U)$ due to Exercise 2.2.4(3). More importantly, if $\phi : A \to B$ is a map of k-affinoid algebras then we get a morphism of locally ringed G-topologized spaces $\mathrm{Sp}(\phi) : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ as follows: on underlying spaces we use the map $f = M(\phi) : M(B) \to M(A)$, and the map $\mathscr{O}_A \to f_*(\mathscr{O}_B)$ that is defined on affinoid subdomains $U \subseteq \mathrm{Sp}(A)$ via the k-algebra map $A_U \to A_U \widehat{\otimes}_A B = B_{f^{-1}(U)}$ and is uniquely extended to general admissible opens via the sheaf axioms. One checks readily that this assignment $A \rightsquigarrow \mathrm{Sp}(A)$ is thereby a contravariant functor, and one shows by copying the proof in the case of affine schemes that this is a *fully faithful* functor. In this way, the (opposite category of the) category of k-affinoid algebras is identified with a full subcategory of the category of locally ringed G-topologized spaces with k-algebra structure sheaf (and maps that respect this k-structure).

An unfortunate fact of life is that the concept of stalk (at a point of X) is not as useful as in ordinary topology. For example, even for the G-topologized spaces $X = M(A)$ it can and does happen that there exist abelian sheaves $\mathscr{F}$ on X and nonzero $s \in \mathscr{F}(X)$ such that $s_x \in \mathscr{F}_x$ vanishes for all $x \in X$. The reason that this does not violate the sheaf axiom is that the vanishing in the stalk merely provides $U_x \in \mathfrak{U}$ containing $x \in X$ so that $s|_{U_x} \in \mathscr{F}(U_x)$ vanishes, but perhaps $\{U_x\}_{x \in X}$ is not in $\mathrm{Cov}(X)$! Hence, one cannot conclude $s = 0$. For the purposes of coherent sheaf theory in rigid-analytic geometry, this pathology will not intervene. However, it is a very serious issue when working with more general abelian sheaves, and so it arises in any attempt to set up a good theory of étale cohomology on such spaces. The work of Berkovich and Huber enlarges the underlying sets of affinoid spaces (and their global counterparts) to have "enough points" so as to permit stalk arguments to work as in classical sheaf theory. This is one of the technical merits of these other approaches to non-archimedean geometry.

2.4. **Globalization.** Having introduced the notion of G-topologized spaces and sheaves on them, we can now make the key global definition.

Definition 2.4.1. A *rigid-analytic space* over k is a pair $(X, \mathscr{O}_X)$ consisting of a locally ringed G-topologized space whose structure sheaf is a sheaf of k-algebras such that there is a covering $\{U_i\} \in \mathrm{Cov}(X)$ with each open subspace $(U_i, \mathscr{O}_X|_{U_i})$ isomorphic to an affinoid space $\mathrm{Sp}(A_i)$ for a k-affinoid algebra A_i. Morphisms are taken in the sense of locally ringed G-topologized spaces with k-algebra structure sheaf (and maps respecting this k-structure), as in Example 2.3.7.

Example 2.4.2. Let $(X, \mathscr{O}_X)$ be a rigid-analytic space and let $U \subseteq X$ be an admissible open subset. The pair $(U, \mathscr{O}_U)$ is a rigid-analytic space. Indeed, let $\{U_i\}$ be an admissible open covering of X such that each $(U_i, \mathscr{O}_{U_i})$ is an affinoid space. By the axioms for a G-topologized space, $\{U_i \cap U\}$ is an admissible open covering of U. Thus, if we can find an admissible open covering of each $U_i \cap U$ by affinoid spaces V_{ij} ($j \in J_i$) then the entire collection $\{V_{ij}\}_{i,j}$ is an admissible covering of U (by the G-topology axioms and the definition of the open subspace structure), so $(U, \mathscr{O}_U)$ thereby has an admissible covering by affinoid spaces, as required. Hence, we can rename U_i as X to reduce to the case when $X = \mathrm{Sp}(A)$ is an affinoid space. But then by definition of the G-topology on $M(A)$, any admissible open $U \subseteq \mathrm{Sp}(A)$ has an admissible covering by affinoid domains.

For any rigid-analytic space X and $x \in X$, the stalk $\mathscr{O}_{X,x}$ can be computed using any affinoid open around x, so its residue field $k(x)$ is a finite extension of k. For any $f \in \mathscr{O}_X(X)$ we therefore get a value $f(x) \in k(x)$ and hence a number $|f(x)| \geq 0$. We can imitate the definitions of Weierstrass, Laurent, and rational domains by imposing non-strict inequalities on the $|f(x)|$'s, and these are admissible opens in X (since they meet each admissible affinoid open in an admissible open subset). One can also construct rigid-analytic spaces by gluing procedures that are similar to the case of gluing ringed spaces. Rather than delve into general details, we illustrate with some examples.

Example 2.4.3. We first construct rigid-analytic affine n-space over k. This rigid space, $\mathbf{A}_k^{n,\mathrm{an}}$, is defined by choosing $c \in k$ with $0 < |c| < 1$ and gluing a rising chain of closed balls centered at the origin with polyradius $|c|^{-j}$ for $j \geq 1$. More precisely, let $D_j = \mathbf{B}_k^n$ be the closed unit n-ball over k, with coordinates $\xi_{1,j}, \dots, \xi_{n,j}$. We define the map $D_j \to D_{j+1}$ to correspond to the k-affinoid algebra map $\xi_{i,j+1} \mapsto c\xi_{i,j}$ (which makes sense since $|c| \leq 1$). This identifies D_j with an affinoid subdomain $\{|\xi_{i,j}| \leq |c|\}$ in D_{j+1}, and in particular for each $1 \leq i \leq n$ the analytic functions $\xi_{i,j}/c^j$ on D_j are compatible with change in j. Thus, on the gluing of the D_j's we get unique global sections ξ_i of the structure sheaf such that $\xi_i|_{D_j} = \xi_{i,j}/c^j$ for all j. This gluing is denoted $\mathbf{A}_k^{n,\mathrm{an}}$, and the locus $\{|\xi_1| \leq |c|^{-j}, \dots, |\xi_n| \leq |c|^{-j}\}$ is the open subspace D_j (with $c^j\xi_i|_{D_j} = \xi_{i,j}$).

It is instructive to see that this deserves to be called an affine n-space by showing that it has the right universal property. Namely, for any rigid space X and any morphism $f : X \to \mathbf{A}_k^{n,\mathrm{an}}$ we get pullback functions $f^\sharp(\xi_i) \in \mathscr{O}_X(X)$, and hence a natural map of sets

$$\mathrm{Hom}(X, \mathbf{A}_k^{n,\mathrm{an}}) \to \mathscr{O}_X(X)^n$$

given by $f \mapsto (f^\sharp(\xi_1), \dots, f^\sharp(\xi_n))$. We claim that this is bijective, so it provides a universal property: affine n-space is the universal rigid space equipped with an ordered n-tuple of global functions. This gives a viewpoint that is independent of the auxiliary choice of c. To prove the bijectivity, by naturality and gluing for morphisms it suffices to treat the case when X is affinoid. But then by the Maximum Modulus Principle, the k-affinoid ring A of global functions on X is the rising union of its subsets A_j of elements with sup-norm at most $|c|^{-j}$ for $j \geq 1$ (since $0 < |c| < 1$). Since a map $f : X \to \mathbf{A}_k^{n,\mathrm{an}}$ lands in D_j if and only if each $f^\sharp(\xi_i) \in A$ has sup-norm at most $|c|^{-j}$ (why?), we reduce ourselves to a problem for the D_j's separately. Upon working with $\xi_{i,j} = c^j\xi_i|_{D_j}$, the problem for each D_j viewed as a closed unit n-ball becomes exactly the universal property of the n-variable Tate algebra!

Example 2.4.4. By imitating the gluing procedures used to make projective spaces as a union of affine spaces, we can construct rigid-analytic projective spaces. (These can also be constructed by gluing closed unit polydiscs.) These satisfy the usual universal property in terms of line bundles, by the same method of proof as in algebraic geometry, once the theory of coherent sheaves (to be discussed later) is fully developed.

Example 2.4.5. In Exercise 2.2.3 we saw how to define the completed tensor product $A\widehat{\otimes}_{A''}A'$ for a pair of maps of k-affinoid algebras $A'' \to A$ and $A'' \to A'$. Via its universal property, one readily checks (much like in the case of affine schemes with tensor products) that $\mathrm{Sp}(A\widehat{\otimes}_{A''}A')$ equipped with its evident morphisms to $\mathrm{Sp}(A)$ and $\mathrm{Sp}(A')$ (agreeing upon composition to $\mathrm{Sp}(A'')$) is a fiber product $\mathrm{Sp}(A) \times_{\mathrm{Sp}(A'')} \mathrm{Sp}(A')$ in the category of affinoid spaces, and then (by gluing maps) that it is such a fiber product in the category of rigid-analytic spaces. One can then copy the same gluing method as for schemes to globalize this construction to obtain the existence of fibers products $X \times_{X''} X'$ for any pair of maps $X \to X''$ and $X \to X''$ of rigid-analytic spaces.

Example 2.4.6. In complex-analytic geometry, a very useful tool is the procedure of analytification for both algebraic $\mathbf{C}$-schemes and coherent sheaves on them. The resulting functors $\mathscr{X} \rightsquigarrow \mathscr{X}^{\mathrm{an}}$ and $\mathscr{F} \rightsquigarrow \mathscr{F}^{\mathrm{an}}$ from algebraic $\mathbf{C}$-schemes to complex-analytic spaces (and on their categories of sheaves of modules) satisfy a number of nice properties that we will not list here. The one aspect we note is that there is a natural map $i_{\mathscr{X}} : \mathscr{X}^{\mathrm{an}} \to \mathscr{X}$ of locally ringed spaces with $\mathbf{C}$-algebra structure sheaves such that $i_{\mathscr{X}}$ carries $\mathscr{X}^{\mathrm{an}}$ bijectively onto $\mathscr{X}(\mathbf{C})$, it induces isomorphisms on completed local rings, and it is final among maps from complex-analytic spaces to $\mathscr{X}$. For any $\mathscr{O}_X$-module $\mathscr{F}$ on X, one defines $\mathscr{F}^{\mathrm{an}} = i_{\mathscr{X}}^*(\mathscr{F})$. The GAGA theorems of Serre (as extended by Grothendieck from the projective to the proper case) concern three aspects: the equivalence of categories of coherent sheaves on $\mathscr{X}$ and $\mathscr{X}^{\mathrm{an}}$ when $\mathscr{X}$ is proper, the comparison isomorphisms of cohomology for $\mathscr{F}$ and $\mathscr{F}^{\mathrm{an}}$ when $\mathscr{X}$ is proper and $\mathscr{F}$ is coherent, and the full faithfulness of the functor $\mathscr{X} \rightsquigarrow \mathscr{X}^{\mathrm{an}}$ when $\mathscr{X}$ is proper.

A similar procedure works in the rigid-analytic setting as follows. First of all, algebraic affine n-space $\mathbf{A}_k^n$ equipped with its standard ordered n-tuple of global functions can be shown to be a final object in the category of locally ringed G-topologized spaces equipped with k-algebra structure sheaf. In particular, we get a unique morphism

$$\mathbf{A}_k^{n,\mathrm{an}} \to \mathbf{A}_k^n$$

compatible with the standard ordered n-tuple of global functions on each space. By the universal property of the source and target, this is final among all rigid-analytic spaces over k equipped with a morphism to $\mathbf{A}_k^n$. In general we define an *analytification* of a locally finite type k-scheme $\mathscr{X}$ to be a map $i_{\mathscr{X}} : \mathscr{X}^{\mathrm{an}} \to \mathscr{X}$ that is final among all maps from rigid spaces over k to $\mathscr{X}$ (as locally ringed G-topologized spaces with k-algebra structure sheaf, and maps that respect this k-structure). The preceding shows that for $\mathscr{X} = \mathbf{A}_k^n$ an analytification exists: rigid-analytic affine n-space. Then one uses arguments with coherent ideal sheaves (discussed in the next lecture!) to pass from this case to all affine algebraic k-schemes, and finally to the general case by gluing arguments. We omit the details, except to remark that analytification is naturally compatible with the formation of fiber products and that $i_{\mathscr{X}}$ carries $\mathscr{X}^{\mathrm{an}}$ bijectively onto the set of closed points in $\mathscr{X}$ (inducing an isomorphism on completed local rings).

Exercise 2.4.7. Let X be a rigid-analytic space and $\{U_i\}$ a collection of admissible opens that cover X set-theoretically. Show that this is an admissible covering if and only if for every morphism $f : \mathrm{Sp}(A) \to X$ from an affinoid space, the set-theoretic covering $\{f^{-1}(U_i)\}$ by admissible opens has a finite affinoid open refinement. (This is an easy exercise in unwinding definitions, with "morphism" defined as in Example 2.3.7. Keep in mind that by definition X has an admissible covering by open affinoid subspaces, and that the functor $A \rightsquigarrow \mathrm{Sp}(A)$ is a fully faithful functor in the sense discussed near the end of Example 2.3.7.)

Exercise 2.4.8. A rigid-analytic space X is said to be *quasi-compact* if it has an *admissible* covering consisting of finitely many affinoid opens. A morphism $f : X' \to X$ of rigid-analytic spaces is *quasi-compact* if there is an admissible covering of X by affinoid opens U_i such that each admissible open $f^{-1}(U_i)$ in X' (endowed with its open subspace structure) is quasi-compact for all i.

(1) Prove that if $X = \mathrm{Sp}(A)$ is affinoid and $f : X' \to X$ is quasi-compact, then for every affinoid subdomain $V = \mathrm{Sp}(B) \subseteq \mathrm{Sp}(A)$ the preimage $f^{-1}(V)$ is quasi-compact. In particular, X' is quasi-compact. (Hint: Use the Gerritzen-Grauert Theorem (Theorem 2.2.5) to reduce to the case when V is a rational domain.)
(2) Prove that if $f : X' \to X$ is quasi-compact in the sense of the above definition then for every quasi-compact admissible open $U \subseteq X$ the preimage admissible open $f^{-1}(U) \subseteq X'$ is quasi-compact. (Hint: Show that it suffices to treat the case when U is affinoid.)
(3) Assume that $f : X' \to X$ is a local isomorphism in the sense that there is an admissible open covering $\{U_i'\}$ of X' such that f maps U_i' isomorphically onto an admissible open $U_i \subseteq X$. If f is bijective and quasi-compact then prove that it is an isomorphism. Also give an example of such an f that is bijective but not an isomorphism.

We conclude with an exercise that demonstrates the power of Tate's theory by rescuing connectedness.

Exercise 2.4.9. A rigid-analytic space X is *disconnected* if there exists an admissible open covering $\{U, V\}$ of X with $U, V \neq \emptyset$ and $U \cap V = \emptyset$. Otherwise we say that X is *connected*. (Under this definition, $X = \emptyset$ is connected.)

(1) Using that the k-algebra of global functions on $\mathrm{Sp}(A)$ is A, prove that $\mathrm{Sp}(A)$ is connected if and only if A has no nontrivial idempotents. (This includes $A = 0$ as an uninteresting special case.) Equivalently, $\mathrm{Sp}(A)$ is connected if and only if $\mathrm{Spec}(A)$ is connected. Also show that if $A = A_1 \times A_2$ is a product of nonzero k-affinoid algebras then $\mathrm{Sp}(A_1)$ and $\mathrm{Sp}(A_2)$ are affinoid subdomains of $\mathrm{Sp}(A)$ (hint: impose inequalities on pointwise absolute values of idempotents) with $\{\mathrm{Sp}(A_1), \mathrm{Sp}(A_2)\}$ an admissible covering of $\mathrm{Sp}(A)$.
(2) Let X be a rigid-analytic space. For any $x \in X$, let U_x be the set of points $x' \in X$ that can be linked to x by a connected chain of finitely many connected admissible affinoid opens. That is, there exist connected admissible affinoid opens $U_1, \ldots, U_n$ in X such that $x \in U_1$, $x' \in U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all $1 \leq i < n$. Prove that the U_x's are admissible open in X and that for any $x_1, x_2 \in X$ either $U_{x_1} = U_{x_2}$ or $U_{x_1} \cap U_{x_2} = \emptyset$. Prove that the collection of U_x's (without repetition) is an admissible cover of X.
(3) Building on the previous part, prove that X is connected if and only if $\mathscr{O}_X(X)$ has no nontrivial idempotents (just like for locally ringed spaces). Moreover, in the context of the previous part, show that the U_x's are connected and that any connected admissible open in X is contained in some U_x. For this reason, we call the U_x's the *connected components* of X.

3. Coherent sheaves and Raynaud's theory

3.1. **Coherent sheaves.** To go further in the theory (e.g., to define closed immersions, separatedness, etc.), we need to discuss coherent sheaves. Kiehl extended Tate's methods to prove the following basic result.

Theorem 3.1.1. *Let $X = \mathrm{Sp}(A)$ be an affinoid space over k, and let M be a finite A-module. The assignment $U \mapsto A_U \otimes_A M$ for affinoid subdomains $U \subseteq X$ uniquely extends to an $\mathscr{O}_X$-module $\widetilde{M}$. In particular, $M \simeq \widetilde{M}(X)$ and the natural map*

$$\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}) \to \mathrm{Hom}_A(M, \mathscr{F}(X))$$

induced by the global sections functor is bijective for any $\mathscr{O}_X$-module $\mathscr{F}$.

Kiehl also proved the following globalization.

Theorem 3.1.2. *Let X be a rigid space, and $\{U_i\}$ an admissible affinoid cover. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. The following properties are equivalent:*

1. *For every admissible affinoid open $V \subseteq X$, $\mathscr{F}|_V \simeq \widetilde{M_V}$ for a finite $\mathscr{O}_X(V)$-module M_V.*
2. *For every i, $\mathscr{F}|_{U_i} \simeq \widetilde{M_i}$ for a finite $\mathscr{O}_X(U_i)$-module M_i.*

An $\mathscr{O}_X$-module $\mathscr{F}$ that satisfies these equivalent conditions is called *coherent*. As for locally noetherian schemes, coherence is inherited by kernels, cokernels, tensor products, and extensions. There is a naive approach to trying to define quasi-coherence, but it is not satisfactory, as we now explain. Motivated by the case of locally noetherian schemes, one may consider to define a quasi-coherent sheaf on X to be an $\mathscr{O}_X$-module that, locally on the space (i.e., on the constituents of an admissible covering) can be expressed as a direct limit of coherent sheaves. (This is the definition suggested in [FvP, Exer. 4.6.7].) It can be shown that this property is preserved under the formation of kernels, cokernels, extensions, tensor products, and direct limits, and that it suffices to work with coherent *subsheaves* in the local direct limit process used in the definition. However, it is generally not true that on an arbitrary admissible affinoid open in the space such a sheaf is a direct limit of coherent sheaves (thereby answering in the negative the "open problem" mentioned in [FvP, Exer. 4.6.7]). More specifically, Gabber has given an example of a sheaf of modules $\mathscr{F}$ on the closed unit disk $\mathbf{B}^1$ such that $\mathscr{F}$ is locally a direct limit of coherent sheaves but with nonzero degree-1 sheaf cohomology, so $\mathscr{F}$ cannot be expressed as a direct limit of coherent sheaves over the entire affinoid space (because the formation of sheaf cohomology commutes with the formation of direct limits on an affinoid rigid space; this is left as an exercise for readers who are familiar with the Čech to derived functor cohomology spectral sequence, which has to be carried over to the rigid-analytic case). There may be a better definition of quasi-coherence that enjoys the stability properties under basic operations as in algebraic geometry and is equivalent over an affinoid space to some module-theoretic data (perhaps with topological structure) over the coordinate ring, but I do not know what such a definition should be.

Exercise 3.1.3. If $X = \mathrm{Sp}(A)$ is affinoid and M is a finite A-module, prove that $\widetilde{M}_x \simeq M \otimes_A \mathscr{O}_{X,x}$ for all $x \in X$. Using that $\mathscr{O}_{X,x}$ is a local noetherian ring with the same completion as $A_{\mathfrak{m}_x}$, deduce that if $m \in M$ vanishes in $\widetilde{M_x}$ for all $x \in X$ then $m = 0$. More globally, deduce that a global section of a coherent sheaf on any rigid-analytic space vanishes if and only if it vanishes in all stalks on the space. (This property is special to coherent sheaves; it fails for general abelian sheaves.)

Example 3.1.4. Let $X = \mathrm{Sp}(B) \to \mathrm{Sp}(A) = Y$ be a map of k-affinoid spaces and let I be the kernel of the natural map $B\widehat{\otimes}_A B \to B$ induced by multiplication. Then I/I^2 is a finite module over $(B\widehat{\otimes}_A B)/I = B$. This is generally not the same as the module of relative algebraic Kähler differentials (which is typically huge, since the usual tensor product $B \otimes_A B$ may not be noetherian). This finite B-module gives rise to a coherent sheaf $\Omega^1_{X/Y}$ on X. There is an $\mathscr{O}_Y$-linear derivation $\mathscr{O}_X \to \Omega^1_{X/Y}$ that can be globalized in accordance with local formulas similar to the situation in algebraic geometry.

Example 3.1.5. A map of rigid spaces $f : X' \to X$ is a *closed immersion* if there exists an admissible affinoid covering $\{U_i\}$ of X such that $U'_i = f^{-1}(U_i)$ is affinoid and the map of affinoids $U'_i \to U_i$ corresponds to a surjection on coordinate rings. In that case it can be proved that for every admissible affinoid open $U \subseteq X$ the preimage $f^{-1}(U)$ is affinoid and $f^{-1}(U) \to U$ corresponds to a surjection on coordinate rings, and moreover that $\mathscr{O}_{X'} \to f_*(\mathscr{O}_X)$ is surjective with coherent kernel $\mathscr{I}$. In fact, this coherent ideal sheaf $\mathscr{I}$ determines the map $f : X' \hookrightarrow X$ up to unique X-isomorphism, and conversely every coherent ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$ arises in this way (by gluing $\mathrm{Sp}(A/I)$'s for admissible affinoid open $\mathrm{Sp}(A) \subseteq X$ with $I = \mathscr{I}(\mathrm{Sp}(A)) \subseteq \mathscr{O}_X(\mathrm{Sp}(A)) = A$).

Using closed immersions and quasi-compactness, we can carry over some notions from algebraic geometry involving diagonal maps:

Definition 3.1.6. A map $f : X \to Y$ of rigid spaces is *separated* if the diagonal map $\Delta_f : X \to X \times_Y X$ is a closed immersion. In case $Y = \mathrm{Sp}(k)$, we say that X is *separated*. If Δ_f is merely quasi-compact (equivalently, the overlap of any two affinoid opens in X over a common affinoid open in Y is quasi-compact) then f is *quasi-separated*.

A map of rigid spaces $f : X' \to X$ is *finite* if there exists an admissible affinoid covering $\{U_i\}$ of X such that each $U'_i = f^{-1}(U_i)$ is affinoid and the map of coordinate rings $\mathscr{O}_X(U_i) \to \mathscr{O}_{X'}(U'_i)$ is module-finite. (In this case it can then be shown that $f^{-1}(U)$ is affinoid with coordinate ring finite over that of U for any affinoid open $U \subseteq X$.)

Remark 3.1.7. There is no rigid-analytic notion of "affinoid morphism" akin to the concept of affine morphism in algebraic geometry. The problem is that there is no good analogue of Serre's cohomological criterion for affineness; see [Liu] for counterexamples (i.e., quasi-compact and separated non-affinoid spaces whose coherent sheaves all have vanishing higher cohomology). Interestingly, [Liu] also gives an example of a quasi-compact and separated non-affinoid space admitting a finite surjection from an affinoid space, in contrast with a theorem of Chevalley in the case of schemes (a separated scheme admitting a finite surjection from an affine scheme is necessarily affine).

Exercise 3.1.8. Copy the proof from the case of schemes to show that if $f : X \to Y$ is a map of rigid spaces and Y is separated then for any admissible affinoid opens $U \subseteq Y$ and $V \subseteq X$, the overlap $V \cap f^{-1}(U)$ is affinoid. (Hint: Show that the graph map $(1, f) : X \to X \times Y$ is a closed immersion and consider $U \times V \subseteq X \times Y$.) Taking f to be the identity map, deduce that an overlap of finitely many admissible affinoid opens in a separated rigid space is again affinoid.

Exercise 3.1.9. Let X be a rigid space over k. Prove that X is quasi-separated if and only if it has an admissible covering by affinoid opens U_i such that each overlap $U_i \cap U_j$ is quasi-compact, in which case $U \cap V$ is quasi-compact for any quasi-compact admissible opens U and V in X. Prove that if X is a quasi-separated rigid space then for any finite collection $\{U_i\}$ of quasi-compact admissible opens in X, the union $U = \cup U_i$ is an admissible open in X for which the U_i's are an admissible covering.

Give an example of a quasi-compact rigid space that is not quasi-separated, and one such for which some finite union of admissible affinoid opens is not an admissible open subset.

Remark 3.1.10. By a gluing procedure, on the category of quasi-separated rigid spaces over k one can define change of base field functors $X \rightsquigarrow X_K$ from rigid spaces over k to ones over K for any analytic extension field K/k, compatibly with fiber products. This is very useful, such as with $k = \mathbf{Q}_p$ and $K = \mathbf{C}_p$, or more generally $K = \overline{k}^{\wedge}$ for any k. (It is a general fact that if $[\overline{k} : k]$ is infinite then $\overline{k}$ is not complete, so $\overline{k}^{\wedge}$ contains elements that are transcendental over k in such cases; a notable such example is $k = \mathbf{Q}_p$.) The idea underlying the definition of this functor is to first define it in the affinoid case via the operation $A \rightsquigarrow K\widehat{\otimes}_k A$ from k-affinoid algebras to K-affinoid algebras compatibly with completed tensor products, and to then globalize by gluing in the separated case (since an overlap of affinoids is affinoid). The quasi-separated case is obtained by another repetition of this process, using that an overlap of admissible affinoid opens in a quasi-separated space may fail to be affinoid but is at least quasi-compact and separated.

We omit the details, except to remark that this is merely a "construction" and it is not really a fiber product or characterized by an abstract universal property as in the case of schemes if K/k has infinite degree because in such cases rigid spaces over K cannot be mapped to rigid spaces over k in any reasonable way: MaxSpec is not functorial (with respect to pullback of prime ideals) between k-affinoid algebras and K-affinoid algebras! This lack of functoriality is a real nuisance, but in the approaches of Berkovich and Huber there are many more points in the underlying spaces and one can view change of base field functors as actual fiber products (and more specifically one can consider analytic spaces over k and K as part of a common category). For purposes of analogy, consider the classical concepts of a variety over $\overline{\mathbf{Q}}$ and over $\mathbf{C}$ (using only "closed points" from the scheme perspective; no "universal domain") and try to formulate the idea of the map $V_{\mathbf{C}} \to V$ for a $\overline{\mathbf{Q}}$-variety V: the trascendental points in $V_{\mathbf{C}}$ have nowhere to go in V since the variety V does not have generic points. This is due to the use of MaxSpec in classical algebraic geometry, and exactly the same problem arises in rigid geometry (i.e., rigid spaces lack "enough points").

Exercise 3.1.11. Since elements of $k[X_1, \dots, X_n]$ have only finitely many nonzero coefficients whereas elements of $k\langle X_1, \dots, X_n\rangle$ can have infinitely many nonzero coefficients, change of the base field in rigid geometry exhibits some features that may be surprising from the viewpoint of algebraic geometry. For example, it can happen that an affinoid space $X = \mathrm{Sp}(A)$ over k remains reduced after any finite extension on k but not after some infinite-degree analytic extension on k. (This never happens for algebraic schemes over a field.) Indeed, consider the following example. Let k be a non-archimedean field of characteristic $p > 0$ such that $[k : k^p]$ is infinite.

(1) Show that an example of such a k is $F(\!(y)\!)$ with the y-adic absolute value, where F is a field of characteristic p such that $[F : F^p]$ is infinite. Find such an F.
(2) Show that there exists an infinite sequence $\{a_n\}$ in k tending to 0 such that $|a_n| \leq 1$ for all n and the $a_n^{1/p}$ generate an infinite-degree extension of k.
(3) Choose a sequence $\{a_n\}$ as just constructed, and let $f = \sum a_n X^{np} \in T_1$ and $A = T_1[Y]/(Y^p - f)$ (which is T_1-finite, hence affinoid). Prove that $A \simeq T_1\langle Y\rangle/(Y^p - f)$ and that $A \otimes_k k'$ is a domain for any k'/k of finite degree, but $A \widehat{\otimes}_k k^{p^{-1}}$ is not reduced. (Also show that $k^{p^{-1}}$ is complete!)
(4) Assume $p \neq 2$ and let $B = k\langle X, Y, t\rangle/(t^2 - (Y^p - f))$. Prove that $B \otimes_k k'$ is a normal domain for any finite-degree extension k'/k but that $B \widehat{\otimes}_k k^{p^{-1}}$ is reduced and not normal. It can be shown that $B \widehat{\otimes}_k K$ is reduced for any analytic extension field K/k.

Exercise 3.1.12. This exercise addresses some subtle features of the general concept of an admissible open subset, even within an affinoid space. Let X be a quasi-separated rigid space over k and let K/k be an analytic extension field. Let $i : U \to X$ be the natural inclusion from an admissible open $U \subseteq X$. We get an induced map of rigid spaces $i_K : U_K \to X_K$ over K. Is this an isomorphism onto an admissible open? To appreciate where the difficulties lie, consider some special cases as follows.

(1) Assume that $X = \mathrm{Sp}(A)$ is affinoid and that U is an affinoid subdomain. Prove that $U_K \to X_K$ is an isomorphism onto an affinoid subdomain. Do this by first showing that U_K has admissible open image in X_K via the Gerritzen–Grauert theorem (Theorem 2.2.5), and then work with the scalar extension of the coordinate ring A_U of U. (Exercise 2.4.8(2) will be useful here.) The reason that this special case requires serious input (Theorem 2.2.5) is that the universal property for $U \subseteq X$, even when formulated in purely algebraic terms via k-affinoid algebras, only involves maps with k-affinoid spaces, and not also K-affinoid spaces (if $[K : k]$ is infinite).
(2) If X is affinoid and U is a quasi-compact admissible open, use the previous part to show that i_K is an isomorphism onto a quasi-compact admissible open in X_K.
(3) Prove an affirmative answer if i is a quasi-compact map. The key difficulty in the general case appears to be to determine if i_K has admissible open image in X_K.

3.2. **Cohomology, properness, and flatness.** No discussion of coherent sheaves would be complete without addressing their cohomology, especially in the proper case. We first make some general observations concerning how to define sheaf cohomology on rigid spaces. Despite the problematic nature of stalks at points on rigid spaces when working with general abelian sheaves, one can adapt some methods of Grothendieck to prove that the category of abelian sheaves and the category of $\mathscr{O}_X$-modules on a rigid space X each have enough injectives, so we may (and do) define sheaf cohomology via derived functors in both cases. The concept of flasque sheaf as traditionally used on ringed spaces is a bit problematic in the general rigid-analytic case, but nonetheless the following result can be proved (and it is left as an exercise just for those whose taste is inclined toward such questions!):

Exercise 3.2.1. Let X be a quasi-separated rigid space. Prove that for an $\mathscr{O}_X$-module $\mathscr{F}$, the natural map from its sheaf cohomology in the sense of $\mathscr{O}_X$-modules to its sheaf cohomology in the sense of abelian sheaves is an isomorphism. More specifically, prove that an injective $\mathscr{O}_X$-module has vanishing higher sheaf cohomology in the sense of abelian sheaves, and that restriction to an admissible open preserves the property of being an injective sheaf of modules (resp. an injective abelian sheaf).

Thus, on quasi-separated spaces the theory of sheaf cohomology via derived functors presents no ambiguities. Kiehl showed that there is a good cohomology theory for coherent sheaves on rigid spaces. His work, coupled with some auxiliary arguments, gives the following result.

Theorem 3.2.2. *Let X be a rigid space and $\mathscr{F}$ a coherent sheaf on X.*

(1) (Acyclicity theorem for coherent sheaves). *If X is affinoid and $\mathfrak{U}$ is a finite covering of X by admissible affinoid opens then the Čech cohomology $\mathrm{H}^i(\mathfrak{U}, \mathscr{F})$ vanishes for all $i > 0$. Moreover, the sheaf cohomology* (*defined via derived functors*) $\mathrm{H}^i(X, \mathscr{F})$ *vanishes for all $i > 0$.*

(2) *If X is a quasi-compact and separated rigid space and $\mathfrak{U}$ is a finite admissible affinoid open covering of X then the natural map $\mathrm{H}^i(\mathfrak{U}, \mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F})$ is an isomorphism for all i.*

We next turn to the definition of properness. In view of the nature of the Tate topology, the condition of universal closedness that is used for schemes is not the right one to use in rigid geometry. Instead, we adapt a formulation similar to one that works in the complex-analytic case.

Definition 3.2.3. A map $f : X \to Y$ of rigid spaces is *proper* if it is separated and quasi-compact and there exists an admissible affinoid open covering $\{U_i\}$ of Y and a pair of finite (necessarily admissible) affinoid open coverings $\{V_{ij}\}_{j \in J_i}$ and $\{V'_{ij}\}_{j \in J_i}$ (same index set J_i of j's!) of $f^{-1}(U_i)$ such that two conditions hold: $V_{ij} \subseteq V'_{ij}$ for all j, and for all $j \in J_i$ there is an $n \geq 1$ and a closed immersion $V'_{ij} \hookrightarrow U_i \times \mathbf{B}^n$ over U_i such that $V_{ij} \subseteq U_i \times \{|t_1|, \dots, |t_{n_i}| \leq r\}$ for some $0 < r < 1$ with $r \in \sqrt{|k^\times|}$. (Equivalently, by the Maximum Modulus Principle, we can replace "$\leq r$" with "< 1".)

The condition on the inclusion $V_{ij} \subseteq V'_{ij}$ over U_i in Definition 3.2.3 is called *relative compactness* of V_{ij} in V'_{ij} over U_i. It is a replacement for saying that V_{ij} has U_i-proper closure in V'_{ij} in ordinary topology. Algebraically, if the coordinate rings of U_i, V_{ij}, and V'_{ij} are A_i, B_{ij}, and B'_{ij} then the condition is that the k-affinoid A_i-algebra B'_{ij} can be expressed as a quotient of a relative n-variable Tate algebra over A_i such that the images $b'_1, \dots, b'_n$ in B'_{ij} of the standard variables $X_1, \dots, X_n$ from the Tate algebra have images $b_1, \dots, b_n \in B_{ij}$ with sup-norms less than 1. (That is, all $b'_r|_{\mathrm{Sp}(B_{ij})}$ have sup-norm less than 1 for all $j \in J_i$.)

Exercise 3.2.4. In the case of a submersion of complex manifolds, use the implicit function theorem to describe the condition of properness of the underlying map of topological spaces in terms similar to the definition of properness in the rigid-analytic case.

Example 3.2.5. Let $X = \mathbf{P}_k^{n,\mathrm{an}}$ be rigid-analytic projective n-space. This can be constructed by gluing $n+1$ affine n-spaces $U_0, \dots, U_n$ exactly as in algebraic geometry so as to establish a universal property much like in algebraic geometry. It is therefore also identified with the analytification of the algebraic scheme $\mathbf{P}^n_{\mathrm{Spec}(k)}$. But this rigid-analytic space can likewise be constructed by analogous gluing using closed unit polydiscs $\mathbf{B}^n_k$! More precisely, let V_j be the closed unit polydisc centered at the origin in the jth standard affine n-space U_j and let V'_j be the closed polydisc centered at the origin with radius $1/|c|$ for some $c \in k^\times$ with $|c| < 1$. We have that V_j is relatively compact in V'_j for all j (over $\mathrm{Sp}(k)$), since identifying V'_j with a closed unit polydisc via scaling its coordinates by $|c|$ carries the affinoid subdomain $V_j \subseteq V'_j$ isomorphically onto the closed subdisc of radius $|c| < 1$ centered at the origin. But $\{V_j\}$ and $\{V'_j\}$ are each admissible covers of X, so X is proper.

The definition of properness is tricky to work with because the requirement on the U_i's does not obviously apply to all open affinoids in Y. For example, if Y is affinoid then it is not clear if one can find a pair of finite affinoid open coverings $\{V_j\}$ and $\{V'_j\}$ of X such that V_j is relatively compact in V'_j over Y for all j. To appreciate the difficulties of the situation, consider the following exercise.

Exercise 3.2.6. Prove that properness is preserved by any base change, as well as any change of the base field functor (assuming in the latter case that Y, and hence X, is quasi-separated, so the change of base field functors can be constructed by gluing across affinoids). Also show that if $Z \to X$ is a closed immersion and Y is proper over a rigid-analytic space S then Z is also proper over Y. (In particular, any projective rigid-analytic space is proper, by Example 3.2.5.) But try to prove that properness is preserved under composition. It is not easy (and was not proved in general until quite recently, by Temkin via Berkovich spaces)!

In algebraic geometry, proper maps enjoy some important cohomological properties, such as the theorem on coherence of higher direct images. Likewise, one has Grothendieck's theory of cohomology and base change, as well as his theorem on formal functions, that relates fibral cohomology to the structure of higher direct images. Results along these lines in the rigid-analytic case were proved by Kiehl, adapting both analytic techniques used to prove such results in the complex-analytic case as well as algebraic techniques used by Grothendieck. We just record the first of these results:

Theorem 3.2.7. *If $f : X \to Y$ is a proper map of rigid spaces and $\mathscr{F}$ is a coherent sheaf on X then the higher direct image sheaves $\mathrm{R}^i(f_*)(\mathscr{F})$ on Y are coherent. In particular, if X is proper over $\mathrm{Sp}(k)$ then $\mathrm{H}^i(X, \mathscr{F})$ is finite-dimensional over k for all coherent sheaves $\mathscr{F}$ on X and all i.*

Remark 3.2.8. As for ringed spaces, higher direct images may be computed via sheafified cohomology. However, it is worth noting that one does not see a priori (as one does for schemes) that for affinoid Y and a proper map $f : X \to Y$ the natural map $\delta_{i,\mathscr{F}} : \mathrm{H}^i(Y, \mathscr{F}) \to \Gamma(X, \mathrm{R}^i(f_*)(\mathscr{F}))$ is an isomorphism for coherent $\mathscr{F}$ on X. In fact, Kiehl's proof does not directly show this; the proof establishes such a result only over affinoids in an admissible covering of Y as in the definition of properness for f. But a posteriori such an isomorphism claim is true over the entire affinoid base Y. The point is that once Kiehl's coherence theorem for higher direct images is known, then for affinoid Y it follows from Kiehl's acyclicity theorem for coherent sheaves on Y that the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(Y, \mathrm{R}^q(f_*)(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

degenerates (with $E_2^{p,q} = 0$ for $p > 0$). All that survives are the maps $\delta_{p,\mathscr{F}}$ as edge maps, so these are isomorphisms. In particular, if $Y = \mathrm{Sp}(A)$ and X is Y-proper then $\mathrm{H}^p(X, \mathscr{F})$ is a finite A-module for every coherent sheaf $\mathscr{F}$ on X and every $p \geq 0$.

Example 3.2.9. The theory of proper maps provides the framework for Serre's GAGA theorems. We now explain this just in the case of proper objects over a field, though there are variants for proper morphisms to any rigid-analytic space. Let X be a proper k-scheme. It is a non-obvious fact (proved for the first time in general only recently, by Temkin, without restrictions on k) that the associated rigid-analytic space X^{an} is proper over $\mathrm{Sp}(k)$ in the sense defined above; in the special case that X is projective this is trivial because analytification carries closed immersions to closed immersions, and we have seen that projective rigid-analytic spaces are proper.

The GAGA theorems assert that the functor $X \rightsquigarrow X^{\mathrm{an}}$ from proper k-schemes to proper rigid-analytic spaces is fully faithful, and that for a fixed such X the analytification functor $\mathscr{F} \rightsquigarrow \mathscr{F}^{\mathrm{an}}$ on $\mathscr{O}_X$-modules is an equivalence of categories from the category of coherent sheaves on X to the category of coherent sheaves on X^{an} with the natural comparison morphism $\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$ an isomorphism for all i and all coherent $\mathscr{F}$ on X. The prove for projective k-schemes goes almost exactly as in Serre's original arguments over $\mathbf{C}$, except that one has to be careful about the presence of non-rational points (if k is not algebraically closed) and about admissibility issues (since Serre uses some pointwise compactness arguments). Also, Grothendieck's generalization to the proper case via Chow's Lemma carries over to the rigid-analytic case essentially verbatim as well.

Let us now pose some exercises that illustrate shortcomings of Tate's theory, and whose only known solutions are via later approaches developed by Raynaud, Berkovich, and others.

Exercise 3.2.10. Let $f : X' \to X$ be a map of rigid spaces over k. Say that f is *flat* if the local map $\mathscr{O}_{X,f(x')} \to \mathscr{O}_{X',x'}$ is flat for every $x' \in X'$. If in addition f is surjective then say it is *faithfully flat*. Prove that if $X' = \mathrm{Sp}(A')$ and $X = \mathrm{Sp}(A)$ are affinoid then f is flat (resp. faithfully flat) if and only if the map of coordinate rings $A \to A'$ is flat (resp. faithfully flat).

In algebraic geometry, it is a basic fact that a flat map between algebraic k-schemes is an open map. So it is natural to ask if a flat map $f : X' \to X$ has admissible open image. This is too much to expect to be true in general (due to the subtle nature of admissible opens), but if X and X' are quasi-compact and quasi-separated then it is a more reasonable question (since at least finite unions of admissible open affinoids in such spaces are necessarily admissible open). This problem has an affirmative answer, but it seems hopeless to attack this by the methods of rigid geometry. Raynaud's theory of formal scheme models provides the right techniques (to ultimately reduce this openness problem to the known analogue in algebraic geometry for schemes of finite type over the residue field $\widetilde{k}$!).

Exercise 3.2.11. If $f : X' \to X$ is a flat map between separated rigid spaces over k and if K/k is an analytic extension field, then is $f_K : X'_K \to X_K$ flat? Reduce this to the problem of showing that if $A \to A'$ is a flat map of k-affinoid algebras then $K \widehat{\otimes}_k A \to K \widehat{\otimes}_k A'$ is flat. This affinoid special case appears to be beyond the reach of the methods of rigid geometry. Once again, Raynaud's theory of formal scheme models (which in this case requires using *non-affine* formal schemes, even in the case of affinoid rigid spaces) provides the methods needed to give an affirmative answer (by reducing the problem to the known case of preservation of flatness under base change for schemes over suitable quotients of the valuation ring R of k).

Exercise 3.2.12. If you are familiar with descent theory for schemes, scheme maps, and quasi-coherent sheaves, try to formulate an analogue of faithfully flat and quasi-compact descent theory for rigid spaces, their maps, and coherent sheaves on them. How much can you prove in this direction? Without Raynaud's theory, you will probably find yourself not able to prove very much!

Exercise 3.2.13. If you are familiar with étale cohomology for schemes, how would you define the étale site on a rigid space? (An *étale map* $f : X' \to X$ of rigid spaces can be defined in several equivalent ways, just like in algebraic geometry, one definition being flatness and the requirement that for all $x \in X$ the fiber $f^{-1}(x)$ over $k(x)$ is a disjoint (admissible) union of $\mathrm{Sp}(k')$ for various finite separable extensions $k'/k(x)$.) Admissibility will make things more subtle than one may have expected. The works of Huber and Berkovich were inspired in part by the aim of developing a good theory of étale cohomology in non-archimedean geometry. Their theories are not the same, but enjoy certain compatibilities and each has its own merits for various purposes.

3.3. **Raynaud's formal models.** The remainder of this lecture will be concerned with explaining some basic aspects of Raynaud's theory of formal scheme models for rigid spaces. In the final two lectures we will discuss Berkovich's theory. Roughly speaking, Raynaud's theory is very useful for "algebraic" questions (flatness, fiber dimension, behavior of properties with respect to base change, etc.) whereas Berkovich's theory is useful for deeper cohomological questions and for carrying out "pointwise" intuition in a manner that sometimes cannot be done within the framework of usual rigid geometry. (We will see some striking examples of this in later lectures.)

Recall that we let R denote the valuation ring of k. (This is noetherian if and only if k is discretely-valued.) A *topologically finitely presented* (tfp) *R-algebra* is an R-algebra of the form

$$\mathscr{A} = R\{X_1, \ldots, X_n\}/\mathscr{I}$$

where $\mathscr{I}$ is a finitely generated ideal in the ring $R\{X_1, \ldots, X_n\}$ of restricted powers series in n variables over R: power series $\sum a_J X^J$ with coefficients $a_J \in R$ such that $a_J \to 0$ as $\|J\| \to \infty$. If it is R-flat as well then it is called an *admissible* R-algebra.

Exercise 3.3.1. Choose $\pi \in R$ with $0 < |\pi| < 1$. Prove that $R\{X_1, \ldots, X_n\}$ is the π-adic completion of $R[X_1, \ldots, X_n]$. If k is discretely-valued, so R is a discrete valuation ring, this is the same as the $\mathfrak{m}$-adic completion, with $\mathfrak{m}$ the maximal ideal of R. But show that if k is algebraically closed (e.g., $k = \mathbf{C}_p$) then $\mathfrak{m}^2 = \mathfrak{m}$, and deduce that in such cases the $\mathfrak{m}$-adic completion is $\widetilde{k}[X_1, \ldots, X_n]$!

In general (any k), show that $k \otimes_R R\{X_1, \ldots, X_n\} = R\{X_1, \ldots, X_n\}[1/\pi] \simeq T_n(k)$ is the n-variable Tate algebra over k. Deduce that if $\mathscr{A}$ is a tfp R-algebra then $A = k \otimes_R \mathscr{A}$ is a k-affinoid algebra, and if $\mathscr{M}$ is a finitely generated $\mathscr{A}$-module then $k \otimes_R \mathscr{M}$ is a finite A-module.

Using some special properties of the valuation rings R (even in the non-noetherian case), one can prove the following useful lemma.

Lemma 3.3.2. *Let $\mathscr{A}$ be a tfp R-algebra. It is π-adically separated and complete, and the ideal $\mathscr{A}[\pi^\infty]$ of π-power torsion elements is finitely generated. Also, if I is any ideal in T_n then its intersection with $R\{X_1, \dots, X_n\}$ is finitely generated.*

Exercise 3.3.3. Prove that an R-module is flat if and only if it is torsion-free (even in the non-noetherian case). Deduce from Lemma 3.3.2 that every k-affinoid algebra A has the form $k \otimes_R \mathscr{A}$ with $\mathscr{A}$ an admissible R-algebra. (In particular, the natural map $\mathscr{A} \to A$ is injective.)

The preceding exercise shows that every k-affinoid algebra A admits a flat integral model in the sense of admissible R-algebras (i.e., $A \simeq k \otimes_R \mathscr{A}$ with $\mathscr{A}$ admissible over R.) Rather less evident is how this can globalize to non-affinoid rigid spaces over k. In fact, even in the affinoid setting there are further non-obvious questions concerning integral models: if M is a finite A-module, can we choose the admissible R-algebra $\mathscr{A}$ so that M arises from a finitely presented $\mathscr{A}$-module $\mathscr{M}$? And if M is A-flat then can we arrange that $\mathscr{M}$ is $\mathscr{A}$-flat? The key to these latter affinoid questions is to attack them within a more global theory of formal scheme models for rigid spaces. This is a remarkable achievement of Raynaud, as we shall now see. First we develop the required notions from the theory of formal schemes.

Exercise 3.3.4. Let $\mathscr{A}$ be a tfp R-algebra. Consider $\overline{X} = \mathrm{Spec}(\mathscr{A}/\mathfrak{m}\mathscr{A})$ as a topological space. For any $\pi \in R$ with $0 < |\pi| < 1$ and any $n \geq 1$, explain how to naturally identify $\overline{X}$ with $X_n = \mathrm{Spec}(\mathscr{A}/(\pi^n))$ as topological spaces. For any $f \in \mathscr{A}$, the non-vanishing locus $\overline{X}_f$ of f in X_n has coordinate ring $\mathscr{A}[1/f]/(\pi^n)$. Show that this ring only depends on n and f mod $\mathfrak{m}$; the inverse limit of these coordinate rings is the π-adic completion $\mathscr{A}_{\{f\}}$ of $\mathscr{A}[1/f]$. Prove that this completion is isomorphic to $\mathscr{A}\{X\}/(1 - fX)$, and so is a tfp R-algebra.

Explain why this completion only depends on the non-vanishing locus $\overline{X}_f$ of f in $\overline{X}$ (in particular, it is naturally independent of π and of the choice of f giving rise to it). Thus, it is well-defined to assign $\mathscr{A}_{\{f\}}$ to this open subset of $\overline{X}$. Prove that this assignment satisfies the sheaf axioms for the covering of one such open subset by others of this type. (Hint: use the existence of structure sheaves on the schemes X_n.) Deduce that this uniquely extends to a sheaf of R-algebras $\mathscr{O}_{\mathscr{A}}$ on $\overline{X}$, and that the stalks of this sheaf are local rings.

Definition 3.3.5. With notation as in the preceding exercise, the *tfp affine formal scheme* $\mathrm{Spf}(\mathscr{A})$ over R is the locally ringed space $(\overline{X}, \mathscr{O}_{\mathscr{A}})$. (Note that its R-algebra of global functions is $\mathscr{A}$.) A *tfp formal scheme* over R is a quasi-compact locally ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ with R-algebra structure sheaf such that it is locally isomorphic (respecting the R-structure) to a tfp affine formal scheme over R. *Morphisms* of tfp formal schemes are morphisms of locally ringed spaces that respect the R-structure.

An *admissible formal R-scheme* is a tfp formal R-scheme $\mathfrak{X}$ whose local rings are R-flat. (It is equivalent, but not obviously so in the non-noetherian case, to require that the coordinate rings of some open affine covering to be R-flat, or for the coordinate ring of every open affine to be R-flat.)

Observe that if $\mathfrak{X}$ is a tfp formal R-scheme then its underlying topological space is noetherian (even if R is not noetherian!), since it is identified with the underlying space of a scheme of finite type over the residue field $\widetilde{k}$.

Exercise 3.3.6. Prove that the assignment $\mathscr{A} \rightsquigarrow \mathrm{Spf}(\mathscr{A})$ is a fully faithful contravariant functor from the category of tfp R-algebras to the category of tfp formal schemes over R. For any $f \in \mathscr{A}$, show that the set $\{f \neq 0\}$ in $\mathrm{Spf}(\mathscr{A})$ is an open subset whose induced open formal subscheme structure is naturally isomorphic to $\mathrm{Spf}(\mathscr{A}_{\{f\}})$; these are called *basic open affines* in $\mathrm{Spf}(\mathscr{A})$.

If $\mathfrak{X}$ is a tfp formal R-scheme then a *formal open affine* in $\mathfrak{X}$ is an open subspace $(\mathfrak{U}, \mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}})$ that is a tfp formal affine scheme. Prove that any such $\mathfrak{X}$ has a base of open subsets that are formal open affines (when endowed with their open subspace structure as locally ringed spaces).

By using finitely presented modules over formal open affines, one can create a good theory of coherent sheaves on such formal schemes (requiring some work in the non-noetherian case). Globally, a coherent sheaf on such an $\mathfrak{X}$ is simply an $\mathscr{O}_{\mathfrak{X}}$-module that is locally finitely presented. (The hard part in the non-noetherian case is to justify that this latter definition is equivalent to a concrete one using modules in the affine case.)

One may be annoyed by the non-noetherian case, but in fact it is a great feature of the theory: if we want to study base change from $\mathbf{Q}_p$ to $\mathbf{C}_p$ for rigid spaces and prove results about preservation of flatness and other kinds of properties (fiber dimension, etc.), it is very useful to have the theory of these formal schemes available over both $\mathbf{Z}_p$ and the (non-noetherian) valuation ring of $\mathbf{C}_p$.

Exercise 3.3.7. Let $f : X' \to X$ be a map between quasi-compact and quasi-separated rigid spaces over k. Let K/k be an analytic extension field. If all fibers of f have dimension $\leq d$, prove the same is true for $f_K : X'_K \to X_K$. (This exercise is meant to make you appreciate how hard the problem is from a classical point of view when $[K : k]$ is not finite; without using further deep results in Raynaud's theory, it is probably too hard to solve.)

The interest in tfp formal R-schemes is revealed in the next exercise.

Exercise 3.3.8. Explain how the assignment $\mathrm{Spf}(\mathscr{A}) \rightsquigarrow \mathrm{Sp}(k \otimes_R \mathscr{A})$ from tfp formal affine schemes over R to affinoid rigid spaces over k is functorial, and prove that it carries Zariski-open immersions to quasi-compact admissible opens. (Hint: First show that basic open affines go over to Laurent domains!)

Once you have handled the affine case, prove that this construction uniquely extends to a functor $\mathfrak{X} \rightsquigarrow \mathfrak{X}_k$ from tfp formal R-schemes to rigid spaces over k such that open immersions go over to admissible opens and fiber products are preserved (in particular, there is compatibility with overlaps of opens). This is Raynaud's *generic fiber* functor.

Example 3.3.9. Let X be an R-scheme of finite presentation. A rigid-analytic space can be associated to X in *two ways*. First, we can form the analytification X_k^{an} of the generic fiber $X_k = X \otimes_R k$. Second, if we let $\mathfrak{X}$ be its π-adic completion (a tfp formal R-scheme, admissible if X is R-flat), then we can form the Raynaud generic fiber $\mathfrak{X}_k$. For example, if $X = \mathbf{A}_R^n$ then $X_k^{\mathrm{an}} = \mathbf{A}_k^{n,\mathrm{an}}$ is rigid-analytic affine n-space whereas $\mathfrak{X} = \mathrm{Spf}(R\{t_1, \ldots, t_n\})$ is formal affine n-space over $\mathrm{Spf}(R)$, so $\mathfrak{X}_k = \mathbf{B}_k^n$, which is much smaller than X_k^{an}. The natural quasi-compact open immersion $\mathbf{B}_k^n \hookrightarrow \mathbf{A}_k^{n,\mathrm{an}}$ can be uniquely generalized for any such X to a functorial morphism of rigid spaces $i_X : \mathfrak{X}_k \to X_k^{\mathrm{an}}$ that is compatible with fiber products and recovers the canonical map in case $X = \mathbf{A}_R^1$. Moreover, as long as X is R-separated, it follows from the valuative criterion for separatedness (and auxiliary arguments) that this map is a quasi-compact open immersion, and (via the valuative criterion for properness) that it is an isomorphism when X is R-proper. For example, if $X = \mathbf{P}^n_R$ then X_k^{an} is rigid-analytic projective n-space from the viewpoint of gluing $n+1$ affine n-spaces whereas $\mathfrak{X}_k$ is projective n-space from the viewpoint of gluing $n+1$ closed unit polydiscs of dimension n (and i_X is the canonical isomorphism between these two gluing constructions).

Exercise 3.3.10. Show that if $\mathfrak{X}$ is quasi-compact then the associated generic fiber rigid space $\mathfrak{X}_k$ is quasi-compact and quasi-separated. Also show that on the category of admissible formal $\mathfrak{X}$'s (i.e., the tfp formal R-schemes whose open affines have R-flat coordinate rings), the generic-fiber functor is faithful (i.e., if two maps between such formal schemes agree on the associated rigid spaces then they must be the same map). For this faithfulness aspect you may wish to just treat the case of R-separated admissible formal schemes over R (with R-separatedness defined in an evident manner); the general case requires some notions (rig-points) that we have not developed.

The preceding exercise naturally raises the question of whether every quasi-compact and quasi-separated rigid space X over k admits a *formal model*: does there exist an admissible formal R-scheme $\mathfrak{X}$ whose generic fiber is X? If so, what relations are there between the various formal models of X? More specifically, given a map $f : X' \to X$ between such rigid spaces, can we choose the formal models $\mathfrak{X}'$ and $\mathfrak{X}$ so that f is induced by a map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ between such formal schemes over R? We also want to know if $\mathfrak{f}$ can be arranged to satisfy analogues of properties of f (such as separatedness, flatness, having fiber dimension at most d, having geometrically reduced fibers, etc., when such notions are appropriately defined for morphisms between tfp formal schemes over R).

Raynaud gave definitive (and largely affirmative) answers to such questions. Before we explain part of his answer, it is instructive to see an example of how a formal model can be changed without affecting the associated rigid space. This operation intuitively corresponds to the idea of making a blow-up in the "closed fiber" (over $\widetilde{k}$), and so it should not affect the "generic fiber" (over k).

Example 3.3.11. Let $\mathfrak{X} = \mathrm{Spf}(\mathscr{A})$ be an affine admissible formal R-scheme (so $\mathscr{A}$ if tfp and flat over R). Let $\mathscr{I}$ be a finitely generated ideal in $\mathscr{A}$, say with generators $f_1, \dots, f_n$. Pick $\pi \in R$ with $0 < |\pi| < 1$. Assume that some power of π lies in $\mathscr{I}$. (Intuitively, this means that this ideal cuts out a closed subscheme "supported in the closed fiber" over R.) Imitating the charts in a blow-up, consider the R-algebra A_i that is the quotient of $\mathscr{A}[T_{i1}, \dots, T_{in}]/(f_i T_{ij} - f_j)$ modulo its f_i-power torsion. Finally, let $\mathscr{A}_i$ denote the π-adic completion of A_i. This is an admissible R-algebra (in particular, it is R-flat). We can glue the $\mathrm{Spf}(\mathscr{A}_i)$'s much as we glue the charts in a blow-up. The resulting glued admissible formal scheme over R is called the *admisible formal blow-up* of $\mathfrak{X}$ along $\mathscr{I}$. It can be characterized by a universal property so as to show it is independent of the choice of f_j's (and of π). We shall denote it $\mathrm{Bl}_{\mathscr{I}}(\mathfrak{X})$.

Consider the natural map $\mathrm{Bl}_{\mathscr{I}}(\mathfrak{X}) \to \mathfrak{X}$. This is generally far from an isomorphism (e.g., the target is affine but the source almost never is). However, the associated generic fiber is an isomorphism! This follows from the next exercise.

Exercise 3.3.12. With notation as at the end of the previous example, let $A = k \otimes_R \mathscr{A}$. Show that the f_i's as elements of A have no common zero, so the rational domains

$$A\langle f_1/f_i, \ldots, f_n/f_i\rangle$$

make sense for $1 \leq i \leq n$. Show that the generic fibers of the charts of the formal admissible blow-up correspond to these rational domains, and that these rational domains cover $\mathfrak{X}_k = \mathrm{Sp}(A)$. Deduce that the map $\mathrm{Bl}_{\mathscr{I}}(\mathfrak{X}) \to \mathfrak{X}$ from the formal admissible blow-up induces an isomorphism on generic-fiber rigid spaces.

We conclude this lecture by briefly stating some of Raynaud's main results. He proved that every quasi-compact and quasi-separated rigid space over k does admit a formal model, that any two formal models are dominated by a common admissible formal blow-up, and that every map between quasi-compact and quasi-separated rigid spaces arises from a map between suitable formal models. More precisely, he showed that the category of quasi-compact and quasi-separated rigid spaces over k is equivalent to the localization of the category of quasi-compact admissible formal R-schemes with respect to the admissible formal blow-up morphisms. In addition, he proved that any quasi-compact admissible open subset arises from a Zariski-open subset in a suitable formal model (and something similar on the level of finite covers by quasi-compact admissible opens) and that every finite collection of coherent sheaves on such a rigid space arises from a collection of formal coherent sheaf models on a suitable formal model. Raynaud also proved that it is often possible to transfer properties of rigid-analytic maps (such as flatness and fiber dimension) to suitable choices of formal models. This theory is powerful, because it settles in the affirmative many vexing questions of "algebraic" nature that appear to be beyond the reach of the methods of rigid geometry. The basic principle is that using suitable formal models can reduce hard questions in rigid geometry over k to known standard results in algebraic geometry over $\widetilde{k}$ or over quotient rings $R/(\pi^n)$ (by killing powers of π or all of $\mathfrak{m}$ in the structure sheaves of formal models). We refer to [BL1] and [BL2] for a systematic development of the first steps in this extremely useful theory.

4. Berkovich spaces I

4.1. **A topological construction.** Let X be a quasi-compact and quasi-separated rigid space. In the previous lecture we saw that Raynaud's theory provides formal models $\mathfrak{X}$ of X, where $\mathfrak{X}$ is an admissible formal R-scheme. It was also seen within Raynaud's theory that the collection of all such models for a fixed X naturally forms an inverse system (with formal admissible blow-ups providing a cofinal system of transition maps). In particular, although the underlying topological space $|\mathfrak{X}|$ of any particular formal model is not intrinsic to X, the inverse limit topological space

$$X^{\mathrm{ad}} = \varprojlim |\mathfrak{X}|$$

formed over the collection of all formal models of X is intrinsic to X. Hochster's work on spectral spaces shows that this is a quasi-compact topological space (not obvious, since quasi-compactness need not be preserved under the formation of inverse limits in the non-Hausdorff case), and it is also quasi-separated (i.e., the overlap of any two quasi-compact opens is quasi-compact). More interesting is the following:

Exercise 4.1.1. Using Raynaud's result that maps of rigid spaces extend to maps between a suitable pair of formal models (and that such an extension is unique if it exists, when the pair of formal models is fixed), prove that X^{ad} is naturally functorial in X.

By a procedure that we will not explain here (the theory of rig-points), one can construct a natural (and functorial) injective map $X \to X^{\mathrm{ad}}$ whose image is dense. The topological space X^{ad} is the underlying space of the *adic space* associated to X in the sense of Huber [H]. A technical merit of the adic space associated to the quasi-compact and quasi-separated rigid space X is that its G-topology is set-theoretically quasi-compact (in the sense that any set-theoretic covering of X^{ad} by opens for the G-topology has a finite subcovering) and has associated category of sheaves of sets is equivalent to the category of sheaves of sets on X with respect to the Tate topology. There is no "admissibility" condition required on coverings as there is on X. Thus, for cohomological questions on X we can work on X^{ad}, where it is possible to use pointwise arguments. (In fact, the set of points of X^{ad} is the set of points of the topos defined by the Tate topology on X.) Unfortunately, in contrast with X, X^{ad} exhibits very un-Hausdorff features.

An intermediate space between X and X^{ad} is the topological space X^{an} underlying the *Berkovich space* associated to X. This is naturally a compact Hausdorff space with remarkable topological properties (e.g., its connected components are path-connected!), and its coherent sheaf theory is equivalent to that of X. Moreover, it has so many points that traditional stalk arguments can be carried out on X^{an} (in contrast with X). The final two lectures are devoted to explaining some basic aspects of Berkovich's theory; the main references are [Ber1, Ch. 1-2] and early parts of [Ber2]. In this lecture we will focus on definitions and examples in the affinoid setting, and in the final lecture we discuss the global theory and some simple striking applications (by no means the most impressive ones).

4.2. **Modified Tate algebras and affinoid algebras.**

An amazing feature of Berkovich's theory is that it permits working with discs whose radius may be any positive real number at all, not just an element of $\sqrt{|k^\times|}$. Another curious feature (which does have some applications) is that it is permissible for the base field to have a *trivial* absolute value. For these reasons, at the beginning of the theory it is convenient to introduce the following innocuous-looking generalization of the concept of a Tate algebra over a non-archimedean field.

Definition 4.2.1. Let k be a field complete with respect to a non-archimedean absolute value $|\cdot|$ (perhaps the trivial one!). For $r_1, \dots, r_n > 0$, define the k-algebra

$$T_{\underline{r}}(k) = T_{r_1,\dots,r_n}(k) = k\langle r_1^{-1}X_1, \dots, r_n^{-1}X_n\rangle = \left\{\sum a_J X^J \in k[\![X_1,\dots,X_n]\!] \,|\, |a_J| r^J \to 0\right\}.$$

This algebra is also denoted $k\langle \underline{r}^{-1}\underline{X}\rangle$ if n and the r_i's are understood from context.

Example 4.2.2. If k is endowed with the trivial absolute value and $r_i < 1$ for all i then $T_{\underline{r}}(k) = k[\![X_1,\dots,X_n]\!]$. If $r_i \geq 1$ for all i then $T_{\underline{r}}(k)$ is the polynomial ring in n variables over k. For other possibilities with the r_i's, it is some mixture of the two (formal power series in some variables, polynomial in others).

Exercise 4.2.3. Prove that $k\langle r_1^{-1}X_1, \dots, r_n^{-1}X_n\rangle$ is a k-Banach algebra via the norm

$$\|\sum a_J X^J\|_{\underline{r}} = \max_J |a_J| r^J,$$

and prove that this is a multiplicative norm. (Hint: Induct on n, so it may be convenient to prove a more general result for relative generalized Tate algebras over k-Banach algebras with a multiplicative norm.) In the special case that k has the trivial absolute value and $r_i < 1$ for all i, prove that when this k-algebra is identified with $k[\![X_1, \dots, X_n]\!]$ in the evident manner then this power series ring acquires (from the norm) the maximal-adic topology.

Example 4.2.4. Choose $r_1, \dots, r_n > 0$ and let K/k be an analytic extension field such that $r_1, \dots, r_n \in |K^\times|$. (We will see soon that such a K always exists.) Using the appropriate notion of completed tensor product over k for k-Banach spaces (and k-Banach algebras), there is a continuous K-algebra isomorphism $K\widehat{\otimes}_k T_{\underline{r}}(k) \simeq T_n(K)$ via $1\widehat{\otimes}X_i \mapsto c_i X_i$ with $c_i \in K$ satisfying $|c_i| = r_i$. In this sense we think of $T_{\underline{r}}(k)$ as the coordinate ring of a closed polydisc over k with polyradius $\underline{r}$. If $r_1, \dots, r_n \in \sqrt{|k^\times|}$ with all $r_i \leq 1$ and $|k^\times| \neq \{1\}$ then $T_{\underline{r}}(k)$ has a bounded k-algebra isomorphism to the coordinate ring of the Weierstrass domain $\{|t_1| \leq r_1, \dots, |t_n| \leq r_n\}$ in $\mathbf{B}_k^n$.

Example 4.2.5. Let k be a field equipped with a non-archimedean (perhaps trivial) absolute value $|\cdot|_0$. Let $|\cdot|$ be a multiplicative norm on k satisfying $|\cdot| \leq C|\cdot|_0$ for some $C > 0$. We now show that the only such norm is $|\cdot|_0$. For $a \in k$ we have $|a^n| \leq C|a^n|_0$ for all $n \geq 1$, so extracting nth roots yields $|a| \leq C^{1/n}|a|_0$. Taking $n \to \infty$ gives $|a| \leq |a|_0$. If $a \neq 0$ then we apply this to $1/a$ to get $|1/a| \leq |1/a|_0$, or equivalently $|a| \geq |a|_0$. Thus, $|\cdot| = |\cdot|_0$.

Exercise 4.2.6. Choose $r > 0$ with $r \notin \sqrt{|k^\times|}$, and consider

$$k\langle r^{-1}X, rX^{-1}\rangle := k\langle r^{-1}X, rY\rangle/(XY - 1).$$

(1) Show that every element has a unique expression as a convergent sum $\sum_{n\in\mathbf{Z}} a_n X^n$ with $|a_n| r^n \to 0$ as $n \to \pm\infty$. Explain why $|a_n| r^n \neq |a_m| r^m$ whenever $n \neq m$ and $a_n, a_m \in k^\times$.
(2) Prove that the residue norm $\|\cdot\|$ induced by the multiplicative norm $\|\cdot\|_{r,1/r}$ from Exercise 4.2.3 assigns $\sum_{n\in\mathbf{Z}} a_n X^n$ the norm $\max_n |a_n| r^n$, and that this is a multiplicative norm. Conclude that $k\langle r^{-1}X, rX^{-1}\rangle$ is a domain.
(3) Let $|\cdot| : k\langle r^{-1}X, rX^{-1}\rangle \to \mathbf{R}$ be a seminorm (norm except with perhaps nonzero kernel) that is multiplicative and also bounded in the sense that $|\cdot| \leq C\|\cdot\|$ for some $C > 0$. Applying this to nth-powers and extracting nth roots, deduce that $|\cdot| \leq \|\cdot\|$. Conclude that $|X| = r$ and that $|\cdot|$ on k is the given absolute value, and then that $|\cdot| = \|\cdot\|$. In particular, $k\langle r^{-1}X, rX^{-1}\rangle$ has exactly one bounded multiplicative seminorm.

The point of the preceding exercise is that (by combining it with Theorem 4.3.4(1) and the closedness of ideals) it implies that the ring $k\langle r^{-1}X, rX^{-1}\rangle$ cannot have any nonzero ideals, so it is a *field*. Hence, via $\|\cdot\|$ it is an analytic extension of k such that r is in its value group (in fact, r is the norm of the element X). By repeating this process, for any finite set of positive real numbers there is an analytic extension of k whose value group contains the finitely many such given real numbers.

At the possible risk of confusion, we now change some terminology from classical rigid geometry:

Definition 4.2.7. A *k-affinoid algebra* is a k-Banach algebra $\mathscr{A}$ for which there is a continuous surjection

$$k\langle r_1^{-1}X_1, \dots, r_n^{-1}X_n\rangle \to \mathscr{A}$$

for some $r_1, \dots, r_n > 0$ such that the residue norm induced on $\mathscr{A}$ is equivalent to the given norm on $\mathscr{A}$. If we can take all $r_i = 1$ then $\mathscr{A}$ is called a *strict k-affinoid algebra* (these are the classical k-affinoid algebras of rigid geometry when k has a nontrivial absolute value). A *morphism* between k-affinoid algebras is a k-algebra map that is continuous.

Exercise 4.2.8. Let $\mathscr{A}$ be a k-Banach algebra. Define relative generalized Tate algebras

$$\mathscr{A}\langle r_1^{-1}X_1, \dots, r_n^{-1}X_n\rangle$$

and formulate and prove a universal property (similar to the case of relative Tate algebras in rigid geometry).

The theory of completed tensor products can be extended to all k-Banach algebras. In particular, if $\mathscr{A}$ is k-affinoid in the above sense and K/k is an analytic extension field then $K\widehat{\otimes}_k\mathscr{A}$ is K-affinoid in the above sense and $\mathscr{A} \to K\widehat{\otimes}_k\mathscr{A}$ is a faithfully flat map that is an isometry onto its (closed) image. By using K with sufficiently large value group, this enables one to prove basic properties of k-affinoid algebras in the new sense (such as being noetherian, all ideals being closed, etc.) by reducing to the known strict case. For example, this technique enables one to show that if $\mathscr{C} \rightrightarrows \mathscr{A}, \mathscr{B}$ are morphisms of k-affinoid algebras then $\mathscr{A}\widehat{\otimes}_{\mathscr{C}}\mathscr{B}$ is again k-affinoid. In particular, if $\mathscr{A}$ is k-affinoid then any relative generalized Tate algebra over $\mathscr{A}$ is also k-affinoid. Also, if the absolute value on k is nontrivial then the norm-equivalence condition in the preceding definition of a k-affinoid algebra is automatically satisfied, due to the non-archimedean version of the Banach Open Mapping Theorem (which is proved over any such k via a simple variant on the proof in the classical case). In general, the absence of the Banach Open Mapping Theorem in the case of trivial absolute value is the reason why the case of the trivial absolute value on the base field k often requires separate consideration.

One basic way in which these new kinds of k-affinoid algebras are a bit harder to work with than the strict (i.e., classical) type is seen in dimension theory. For example, in Exercise 4.2.6 we gave an example of a k-affinoid algebra that is a field but should correspond to the coordinate ring of the circle of radius r: after a sufficiently large ground field extension (to put r into the value group of the ground field) it becomes 1-dimensional, but the given affinoid algebra over k is 0-dimensional. Thus, to define dimension theory in the k-analytic setting, one has to first show (e.g., using Noether normalization) that dimension theory for strict affinoid algebras is well-behaved with respect to extension of the base field, and then make definitions in the general case by using an auxiliary field extension to pass to the strict case (where Krull dimension provides an adequate theory). It is a nontrivial matter to set up a geometrically satisfactory dimension theory for k-affinoid algebras that enjoys properties one expects out of analogy with the strict case and algebraic geometry, but such a theory has been developed (in some respects, quite recently, by Ducros). The main point is that ring-theoretic dimension theory alone is not an adequate viewpoint over the base field in Berkovich's theory.

4.3. **Spectrum of Banach algebras.** We have introduced the wider class of k-Banach algebras that will be used in the local theory of Berkovich spaces. It is now time to introduce the underlying topological spaces to be used, replacing the role of $M(A)$ in rigid geometry. To motivate what we are about to do, let us return to the setting of rigid geometry and consider a strict $\mathbf{Q}_p$-affinoid algebra A as well as its associated strict $\mathbf{C}_p$-affinoid fiber $A_{\mathbf{C}_p} = \mathbf{C}_p \widehat{\otimes}_{\mathbf{Q}_p} A$. There is a continuous map $A \to A_{\mathbf{C}_p}$ via $a \mapsto 1 \widehat{\otimes} a$, but there is no corresponding natural map of sets $M(A_{\mathbf{C}_p}) \to M(A)$ because the "transcendental points" with respect to $\mathbf{Q}_p$ in $A_{\mathbf{C}_p}$ have nowhere to go. We now reformulate the problem in a way that leads us to the way out of this conundrum.

For each point $x \in M(A_{\mathbf{C}_p})$, the corresponding maximal ideal $\mathfrak{m}_x \subseteq A_{\mathbf{C}_p}$ contracts to the prime ideal $\mathfrak{p}_x \subseteq A$ that is the kernel of the composite map

$$| \cdot |_x : A \to A_{\mathbf{C}_p} \xrightarrow{x} \mathbf{C}_p \xrightarrow{|\cdot|} \mathbf{R}.$$

This composite map is a seminorm on A (i.e., it satisfies all of the requirements of a norm except for possibly having nonzero kernel), it is bounded (in the sense that $| \cdot |_x$ is bounded by a constant multiple of a fixed $\mathbf{Q}_p$-Banach algebra norm on A), and it is *multiplicative*. More specifically, this corresponds to an absolute value on $\mathrm{Frac}(A/\mathfrak{p}_x)$ whose restriction to $A/\mathfrak{p}_x$ is bounded above by a constant multiple of the residue norm on $A/\mathfrak{p}_x$. Points of $M(A)$ gives rise to such structures as a special case, with the additional condition that $\mathrm{Frac}(A/\mathfrak{p}_x)$ is finite over $\mathbf{Q}_p$ (and equipped with an embedding into $\mathbf{C}_p$).

Exercise 4.3.1. Let A be a k-Banach algebra, and let $| \cdot | : A \to \mathbf{R}$ be a bounded multiplicative seminorm. Prove that its kernel $\mathfrak{p}$ is a prime ideal of A and that $| \cdot |$ arises from an absolute value on $\mathrm{Frac}(A/\mathfrak{p})$ that restricts to the given absolute value on k. If we let K denote the completion of $\mathrm{Frac}(A/\mathfrak{p})$ with respect to this absolute value, then show that $| \cdot |$ arises by pullback of the absolute value on K via a bounded k-algebra map from A to the analytic extension field K/k in which A generates a dense subfield. (By "bounded" k-algebra map $A \to K$ we mean boundedness in the sense of k-Banach space maps.)

Show that this procedure sets up a bijection between the set of bounded multiplicative seminorms on A and the set of bounded k-algebra maps $A \to K$ to analytic extension fields K/k in which A generates a dense subfield (and these maps are taken up to composition with isometric isomorphism in K).

It is very interesting that the consideration of bounded multiplicative seminorms makes sense even on abstract rings like $\mathbf{Z}$ (with the trivial norm) which contain no field. More specifically, let $(\mathscr{A}, \| \cdot \|)$ be a commutative Banach algebra (a commutative ring endowed with a submultiplicative norm with respect to which it is complete as a metric space); this could be an abstract ring endowed with the trivial norm. It makes sense to consider multiplicative seminorms $| \cdot | : \mathscr{A} \to \mathbf{R}$ that are bounded in the sense that $|f| \leq C\|f\|$ for all $f \in \mathscr{A}$ for some $C > 0$. Applying this to f^n and using multiplicativity, taking $n \to \infty$ after extracting nth roots gives the consequence $|f| \leq \|f\|$. As was seen in Exercise 4.3.1, if $\mathscr{A}$ is a k-Banach algebra then all such $|\cdot|$'s are automatically compatible with the absolute value on k.

Definition 4.3.2. The *spectrum* of a Banach algebra $(\mathscr{A}, \|\cdot\|)$ is the set $\mathscr{M}(\mathscr{A})$ of bounded multiplicative seminorms on $\mathscr{A}$. This set is given the topology for which a base of opens around a point $|\cdot|_0 \in \mathscr{M}(\mathscr{A})$ is provided by the sets

$$U_{f_1,\dots,f_n,\varepsilon_1,\dots,\varepsilon_n} = \{|\cdot| \in \mathscr{M}(\mathscr{A}) \,|\, ||f_j|_0 - |f_j|| < \varepsilon_j \text{ for all } j\}.$$

Equivalently, this is the weakest topology with respect to which the functions $\mathscr{M}(\mathscr{A}) \to \mathbf{R}$ defined by $|\cdot| \mapsto |f|$ for each $f \in \mathscr{A}$ are all continuous.

Remark 4.3.3. Observe that the definition of the spectrum as a topological space is unaffected by replacing the Banach norm on $\mathscr{A}$ with an equivalent one (i.e., one that is bounded above and below by a constant positive multiple of the initial one). Hence, if k has nontrivial absolute value and $\mathscr{A}$ is a strict k-affinoid algebra then we can use any k-Banach algebra norm on $\mathscr{A}$ to unambiguously define the topological space $\mathscr{M}(\mathscr{A})$.

The preceding exercise for k-Banach algebras carries over to (commutative) Banach algebras in general, so as a set we can identify $\mathscr{M}(\mathscr{A})$ with the set of ring maps $\mathscr{A} \to K$ (up to isometric isomorphism in K) that are bounded as maps of complete normed additive groups and have target K that is a complete valued field in which $\mathscr{A}$ generates a dense subfield. This is reminiscent of how the points of an affine scheme can be interpreted as the set of "all" field-valued points of the coordinate ring; we now work with non-archimedean fields (with possibly trivial absolute value)!

The spectrum of a (commutative) Banach algebra enjoys some fundamental topological properties that we now summarize.

Theorem 4.3.4. *Let $\mathscr{A}$ be a commutative Banach algebra.*

1. $\mathscr{M}(\mathscr{A}) = \emptyset$ *if and only if* $\mathscr{A} = 0$.
2. $\mathscr{M}(\mathscr{A})$ *is compact and Hausdorff.*
3. *If $\phi : \mathscr{A} \to \mathscr{A}'$ is a bounded map between commutative Banach algebras then the pullback map $\mathscr{M}(\mathscr{A}') \to \mathscr{M}(\mathscr{A})$ defined by $|\cdot|' \mapsto |\cdot|' \circ \phi$ is continuous.*

The proof of the first two parts is an application of Zorn's Lemma, and the third part is a matter of unwinding definitions. As an interesting application of the theorem, let k be a non-archimedean field (nontrivial absolute value) and let K/k and K'/k be two analytic extension fields with at least one of them of countable type over k (in the sense of having a dense k-linear subspace of countable dimension). The Banach algebra $K\widehat{\otimes}_k K'$ is nonzero by general results on countable-type k-Banach spaces [BGR]. Thus, its spectrum is nonempty by Theorem 4.3.4, so we obtain a third non-archimedean extension of k into which K and K' isometrically embed. This underlies the fact that if $\mathscr{A}$ is a (perhaps non-strict) k-affinoid algebra (with k having possible trivial valuation) then for any non-archimedean extension field K/k such that $K\widehat{\otimes}_k\mathscr{A}$ is strict, the dimension of this strict K-affinoid algebra is independent of K/k.

Example 4.3.5. A very interesting example of functoriality of the spectrum (as a topological space) is to let $\mathscr{A}$ be a k-affinoid algebra and K/k a non-archimedean extension field (with compatible absolute value). The natural map $\mathscr{A} \to \mathscr{A}_K = K\widehat{\otimes}_k\mathscr{A}$ from a k-affinoid algebra to a K-affinoid algebra induces a continuous (in fact, surjective) map $\mathscr{M}(\mathscr{A}_K) \to \mathscr{M}(\mathscr{A})$. So already we can do something that is not generally possible in rigid geometry, which is to have maps from a geometric object "over K" to one "over k". This will be improved to a morphism of analytic spaces once we endow these spectra with more structure beyond their topologies.

For each $x \in \mathscr{M}(\mathscr{A})$, let the prime ideal $\mathfrak{p}_x \subseteq \mathscr{A}$ be the kernel of this multiplicative seminorm, and let $\mathscr{H}(x)$ denote the completion of the field $\mathrm{Frac}(\mathscr{A}/\mathfrak{p}_x)$ with respect to the absolute value induced by x. For each $f \in \mathscr{A}$ we write $f(x)$ to denote the image of f under the natural map $\mathscr{A} \to \mathscr{H}(x)$ and $|f(x)|$ to denote the absolute value of $f(x)$ (using the canonical absolute value on $\mathscr{H}(x)$).

Remark 4.3.6. There are nearly always points $x \in \mathscr{M}(\mathscr{A})$ with $\mathscr{H}(x)$ of infinite degree over k, even if k is algebraically closed. For example, for any k the space $\mathscr{M}(k\langle T\rangle)$ contains the point ξ corresponding to the Gauss norm, and $\mathscr{H}(\xi)$ is the completion of $\mathrm{Frac}(k\langle T\rangle)$ with respect to the absolute value induced by the Gauss norm on $k\langle T\rangle$. (We can replace $k\langle T\rangle$ with $k[T]$ in this latter description, for denseness reasons.) This abundance of points with large residue field (in contrast with the case of complex-analytic spaces) can be considered to be a consequence of the lack of a non-archimedean Gelfand–Mazur theorem. Even if k is discretely valued, the space $\mathscr{M}(k\langle T\rangle)$ has many points whose completed residue field is not discretely valued (see Exercise 4.3.8 below), and so it will follow that to pass between problems in rigid geometry and Berkovich's theory even over a discretely-valued field we will inexorably be led to consider rigid spaces over extension fields whose valuation ring is not a discrete valuation ring!

Exercise 4.3.7. Let $\mathscr{A}$ be a commutative Banach algebra. For $r > 0$ and $f, g \in \mathscr{A}$, prove that the subset $\{x \in \mathscr{M}(\mathscr{A}) \,|\, |f(x)| \leq r|g(x)|\}$ is closed in $\mathscr{M}(\mathscr{A})$. Also prove that $f \in \mathscr{A}$ is a unit if and only if $f(x) \neq 0$ in $\mathscr{H}(x)$ for all $x \in \mathscr{M}(\mathscr{A})$, and that f is nilpotent in $\mathscr{A}$ if and only if $f(x) = 0$ in $\mathscr{H}(x)$ for all $x \in \mathscr{M}(\mathscr{A})$. (Hint: it is a consequence of Zorn's Lemma that the natural map $\mathscr{M}(\mathscr{A}) \to \mathrm{Spec}(\mathscr{A})$ assigning to each bounded multiplicative seminorm its prime ideal kernel is a surjective map.)

Exercise 4.3.8. Let k be a non-archimedean field (nontrivial absolute value), and let $\mathscr{A} = k\langle T\rangle$.

(1) Fix $0 \leq r \leq 1$ and define $\|\cdot\|_r$ by $\|\sum a_n T^n\|_r = \max_n |a_n| r^n$. Prove that this is a bounded multiplicative seminorm on $\mathscr{A}$. Also show that it deserves to be called the sup-norm on the closed disc of radius r (with center at 0) in the sense that $\|f\|_r$ is the supremum of the numbers $|f|_x$ as $|\cdot|_x$ ranges through all bounded k-algebra maps $\mathscr{A} \to K$ to non-archimedean fields such that the image of T in K has absolute value at most r. It is useful to think of $\|\cdot\|_r$ as the "generic point" of the closed disc of radius r centered at the origin, though it is not a generic point in the usual topological sense used for schemes.

(2) Prove that the function $[0, 1] \to \mathscr{M}(\mathscr{A})$ defined by $r \mapsto \|\cdot\|_r$ is *continuous*, and so provides a path connecting the origin to the Gauss norm!

(3) Prove that the map $M(\mathscr{A}) \to \mathscr{M}(\mathscr{A})$ defined by assigning to each $x \in M(\mathscr{A})$ the seminorm induced by $\mathscr{A} \to k(x)$ is injective with dense image, and induces on $M(\mathscr{A})$ the canonical topology as the subspace topology. By using the trick of "recentering the disk", prove that any two k-rational points in $M(\mathscr{A})$ are linked by a path in $\mathscr{M}(\mathscr{A})$! (In fact, $\mathscr{M}(\mathscr{A})$ is a path-connected space, but this requires some more work to generalize the trick of linking via the Gauss norm.)

The final part of the preceding exercise holds rather more generally (but requires more effort to prove):

Theorem 4.3.9. *Assume that k has nontrivial absolute value and let A be a strict k-affinoid algebra. The natural map of sets $M(A) \to \mathscr{M}(A)$ is injective with dense image and induces the canonical topology as the subspace topology. Moreover, if A has no nontrivial idempotents then $\mathscr{M}(A)$ is path-connected.*

4.4. **Affinoid subdomains.** In order to work with the spectrum of a k-affinoid algebra $\mathscr{A}$, we need to define the analogues of affinoid subdomains (including Weierstrass, Laurent, and rational domains). As in the case of rigid geometry, we will use a set-theoretic mapping property. One enormous difference is that affinoid subdomains in $\mathscr{M}(\mathscr{A})$ will no longer be open sets in general: they will be closed subsets.

Definition 4.4.1. Let $\mathscr{A}$ be a k-affinoid algebra. A subset $U \subseteq \mathscr{M}(\mathscr{A})$ is an *affinoid subdomain* if there exists a bounded map $i : \mathscr{A} \to \mathscr{A}'$ of k-affinoid algebras such that two conditions hold: $\mathscr{M}(i)$ has image contained in U, and for any bounded map $\phi : \mathscr{A} \to \mathscr{B}$ of k-affinoid algebras with $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ having image contained in U, there is a unique factorization of ϕ through i via a bounded map $\mathscr{A}' \to \mathscr{B}$ of $\mathscr{A}$-algebras.

As in rigid geometry, this definition uniquely determines $\mathscr{A}'$ up to a unique bounded $\mathscr{A}$-algebra isomorphism, so we can write $\mathscr{A}_U$ to denote this $\mathscr{A}$-algebra (with norm determined up to equivalence); it is called the *coordinate ring* of U. By chasing appropriate field-valued points and using the stronger universal property given in Example 4.4.3, one shows that $\mathscr{M}(\mathscr{A}_U) \to U$ is a homeomorphism and that for any analytic extension field K/k the affinoid K-algebra $K\widehat{\otimes}_k\mathscr{A}_U$ has the analogous universal property for the preimage U_K of U under the surjective map $\mathscr{M}(K\widehat{\otimes}_k\mathscr{A}) \to \mathscr{M}(\mathscr{A})$. In particular, the concept of affinoid domain is well-behaved with respect to extension of the base field. Somewhat less evident (but unsurprising) is that $\mathscr{A}_U$ is $\mathscr{A}$-flat. By using completed tensor products, it may be proved exactly as in rigid geometry that an overlap of finitely many affinoid subdomains is again an affinoid subdomain.

Exercise 4.4.2. We now work out the basic examples of Weiestrass, Laurent, and rational domains in the setting of Berkovich spaces. One new feature is that we can use arbitrary positive real scaling factors on the non-strict inequalities used to define such domains. (In the rigid-analytic case we saw that it only made sense to impose scaling factors in $\sqrt{|k^\times|}$, and that the use of such factors provided no extra generality than not using them.)

(1) Choose $f_1, \dots, f_n, g_1, \dots, g_m$ in a k-affinoid algebra $\mathscr{A}$, and $r_1, \dots, r_n, s_1, \dots, s_m > 0$. Prove that the subset

$$\{x \in \mathscr{M}(\mathscr{A}) \,|\, |f_1(x)| \le r_1, \dots, |f_n(x)| \le r_n, |g_1(x)| \ge s_1, \dots, |g_m(x)| \ge s_m\}$$

is an affinoid subdomain with coordinate ring

$$\mathscr{A}\{r_1^{-1}X_1, \dots, r_n^{-1}X_n, s_1Y_1, \dots, s_mY_m\}/(X_1 - f_1, \dots, X_n - f_n, g_1Y_1 - 1, \dots, g_mY_m - 1)$$

(usually denoted $\mathscr{A}\langle \underline{r}^{-1}\underline{f}, \underline{s}\underline{g}^{-1}\rangle$).

(2) Let $f_1, \dots, f_n, g \in \mathscr{A}$ be elements which generate the unit ideal (equivalently, they have no common zero on $\mathscr{M}(\mathscr{A})$). Choose $r_1, \dots, r_n > 0$. Prove that the subset

$$\{x \in \mathscr{M}(\mathscr{A}) \mid |f_1(x)| \le r_1|g(x)|, \dots, |f_n(x)| \le r_n|g(x)|\}$$

is an affinoid subdomain with coordinate ring

$$\mathscr{A}\{r_1^{-1}X_1, \dots, r_n^{-1}X_n\}/(gX_1 - f_1, \dots, gX_n - f_n)$$

(usually denoted $\mathscr{A}\langle \underline{r}^{-1}\underline{f}/g\rangle$).

These affinoid subdomains are respectively called *Laurent domains* and *rational domains*. A Laurent domain is called *Weierstrass* if there are no g_j's. Prove that every point of $\mathscr{M}(\mathscr{A})$ has a base of closed neighborhoods consisting of Laurent domains.

Recall that in rigid geometry (i.e., when working with strict k-affinoid algebras, with k having nontrivial absolute value), the natural map $A \to A\langle a_1, \dots, a_n\rangle$ to the coordinate ring of a Weierstrass domain has dense image. The same obviously holds in our new setting (for $\mathscr{A} \to \mathscr{A}\langle \underline{r}^{-1}\underline{a}\rangle$), but Berkovich proved more: if the natural map $\mathscr{A} \to \mathscr{A}_U$ from a k-affinoid algebra to the coordinate ring of an affinoid subdomain has dense image then U is *necessarily* a Weierstrass domain!

Example 4.4.3. Let $\mathscr{A}$ be a k-affinoid algebra and $U \subseteq \mathscr{M}(\mathscr{A})$ be an affinoid subdomain with associated coordinate ring $\mathscr{A}_U$. Choose $x \in U = \mathscr{M}(\mathscr{A}_U) \subseteq \mathscr{M}(\mathscr{A})$. Let us show that the completed residue fields at x for $\mathscr{A}$ and $\mathscr{A}_U$ coincide (via the natural map). In other words, if $\mathscr{H}(x)$ denotes the completion of the fraction field of $\mathscr{A}/\ker(|\cdot|_x)$ then we claim that the map $\mathscr{A} \to \mathscr{H}(x)$ factors (uniquely) through $\mathscr{A} \to \mathscr{A}_U$ (and so we then get the claim, due to denseness and absolute-value compatibility reasons). To prove this, it is enough to show that affinoid subdomains satisfy their universal mapping property with respect to maps from $\mathscr{A}$ to arbitrary k-Banach algebras, not merely to k-affinoid algebras. For Laurent and rational domains this is clear. In the general case, the important theorem of Gerritzen–Grauert from rigid geometry (Theorem 2.2.5) can be proved to remain true in the setting of the spaces $\mathscr{M}(\mathscr{A})$, and from this it may be deduced that affinoid subdomains do indeed satisfy the desired stronger universal mapping property.

The theorems of Tate and Kiehl from rigid geometry carry over to our new setting. The proofs go via using appropriate extension of the base field to reduce to the strict cases that were proved by Tate and Kiehl. Here is the result (in a slightly weaker form):

Theorem 4.4.4. *Let $\mathscr{A}$ be a k-affinoid algebra and M a finite $\mathscr{A}$-module. Let $\mathfrak{U}$ be a finite collection of affinoid subdomains of $X = \mathscr{M}(\mathscr{A})$ that cover X. Define $M_U := \mathscr{A}_U \otimes_{\mathscr{A}} M$ for any affinoid subdomain U of X. The Čech complex $\mathrm{C}^\bullet(\mathfrak{U}, M)$ built on the modules $M_{U_{i_1}\cap\cdots\cap U_{i_n}}$ for $U_{i_1}, \dots, U_{i_n} \in \mathfrak{U}$ is an exact sequence.*

4.5. **Relative interior and boundary.** We conclude our discussion of the preliminary parts of the affinoid Berkovich theory (prior to the introduction of sheaves, to be discussed in the final lecture) by explaining some new concepts with no good analogue in rigid geometry. It may be appealing to think that in $\mathscr{M}(k\langle T\rangle)$, the points x for which $|T|_x = 1$ should be the "boundary points" whereas the points x for which $|T|_x < 1$ should be the "interior points". Such a distinction is not meaningful because the automorphism group of a Tate algebra acts transitively on its set of k-rational points by recentering the polydisc. Nonetheless, Berkovich's theory does have an intrinsic theory of relative boundary and relative interior for morphisms which roughly exhibits properties of boundaries and interiors in topology. These notions require some experience to become familiar, and before giving the relevant definitions and examples in the affinoid case we consider a simple example.

Exercise 4.5.1. Let $\xi \in \mathscr{M}(k\langle T\rangle)$ be the point corresponding to the Gauss norm, and let D be the subset of points $x \in \mathscr{M}(k\langle T\rangle)$ for which $|T|_x = 1$.

1. Prove that the interior of D in $\mathscr{M}(k\langle T\rangle)$ is $D - \{\xi\}$. In particular, the compact subset $D \subseteq \mathscr{M}(k\langle T\rangle)$ becomes an open subset upon deletion of the single point $\xi \in D$. (Hint: For each $|\cdot|_x$ on $k\langle T\rangle$, its restriction to $R[T]$ has an ideal of topological nilpotents that contains $\mathfrak{m}[T]$ and so corresponds to an ideal in $R[T]/\mathfrak{m}[T] = \widetilde{k}[T]$. Show that this ideal is prime and that the resulting map $\mathscr{M}(k\langle T\rangle) \to \mathrm{Spec}(\widetilde{k}[T]) = \mathbf{A}^1_{\widetilde{k}}$ has ξ as the only point over the generic point of the affine line over $\widetilde{k}$.)
2. Prove that $\mathscr{M}(k\langle T\rangle) - \{\xi\}$ is topologically a disjoint union of open non-empty preimages of the closed points of $\mathbf{A}^1_{\widetilde{k}}$. (Thus, the connectedness properties of $\mathscr{M}(k\langle T\rangle)$ depend crucially on the presence of the point ξ!)

Definition 4.5.2. Let $\phi : \mathscr{A} \to \mathscr{A}'$ be a bounded map of k-affinoid algebras. The *relative interior* of $\mathscr{M}(\mathscr{A}')$ over $\mathscr{M}(\mathscr{A})$ is

$$\mathrm{Int}(\mathscr{M}(\mathscr{A}')/\mathscr{M}(\mathscr{A})) = \{x' \in \mathscr{M}(\mathscr{A}') \,|\, \mathscr{A}' \to \mathscr{H}(x') \text{ is inner with respect to } \mathscr{A}\},$$

where "inner" means that there is a bounded surjection of $\mathscr{A}$-algebras $\mathscr{A}\{\underline{r}^{-1}\underline{X}\} \twoheadrightarrow \mathscr{A}'$ (inducing a residue norm on $\mathscr{A}'$ that is equivalent to the given one) such that $|X_j|_{x'} < r_j$ for all j. (Geometrically, this says that a point $x' \in \mathscr{M}(\mathscr{A}')$ lies in the relative interior precisely when there is a closed immersion of $\mathscr{M}(\mathscr{A}')$ into the product of $\mathscr{M}(\mathscr{A})$ and a closed polydisc of polyradius $\underline{r}$ such that x' avoids the boundary of the polydisc.)

The complement of $\mathrm{Int}(\mathscr{M}(\mathscr{A}')/\mathscr{M}(\mathscr{A}))$ in $\mathscr{M}(\mathscr{A}')$ is the *relative boundary* of $\mathscr{M}(\mathscr{A}')$ over $\mathscr{M}(\mathscr{A})$; it is denoted $\partial(\mathscr{M}(\mathscr{A}')/\mathscr{M}(\mathscr{A}))$.

Exercise 4.5.3. Prove that $\mathrm{Int}(\mathscr{M}(\mathscr{A}')/\mathscr{M}(\mathscr{A}))$ is open in $\mathscr{M}(\mathscr{A}')$.

Example 4.5.4. We give several (non-obvious) examples to illustrate the interest in the concepts of relative interior and relative boundary.

1. If $U \subseteq \mathscr{M}(\mathscr{A})$ is an affinoid subdomain with associated coordinate ring $\mathscr{A}_U$ then $\mathrm{Int}(\mathscr{M}(\mathscr{A}_U)/\mathscr{M}(\mathscr{A}))$ is the topological interior of U with respect to $\mathscr{M}(\mathscr{A})$.
2. The "naive" version of the Maximum Modulus Principle is trivially true on the compact Hausdorff space $\mathscr{M}(\mathscr{A})$: for any $f \in \mathscr{A}$ the continuous $\mathbf{R}$-valued function $x \mapsto |f(x)|$ on $\mathscr{M}(\mathscr{A})$ attains a maximal value. Less evident is that the maximum

must be attained on $\partial(\mathscr{M}(\mathscr{A})/\mathscr{M}(k))$ (and in fact it is attained on a certain distinguished finite subset of this relative boundary that is independent of f).

(3) If $\mathscr{A} \to \mathscr{A}'$ is a morphism of k-affinoid algebras then $\partial(\mathscr{M}(\mathscr{A}')/\mathscr{M}(\mathscr{A})) = \emptyset$ if and only if $\mathscr{A} \to \mathscr{A}'$ is finite and (as is automatic if k has nontrivial absolute value) has closed image whose subspace and residue norms are equivalent.

(4) Let $\mathscr{B}$ be a k-affinoid algebra, and $\mathscr{B} \to \mathscr{A}$ and $\mathscr{B} \to \mathscr{A}'$ be two morphisms of k-affinoid algebras. Let $\mathscr{A} \to \mathscr{A}'$ be a bounded $\mathscr{B}$-algebra map. We say that this latter map is *inner with respect to* $\mathscr{B}$ if there is a bounded surjection of $\mathscr{A}$-algebras

$$\mathscr{B}\{\underline{r}^{-1}\underline{X}\} \twoheadrightarrow \mathscr{A}$$

inducing the quotient topology and such that the map

$$\mathscr{M}(\mathscr{A}') \to \mathscr{M}(\mathscr{A}) \subseteq \mathscr{M}(\mathscr{B}\{\underline{r}^{-1}\underline{X}\})$$

lands inside of the locus of points $|\cdot|$ satisfying $|X_j| < r_j$ for all j. One can show that this property (which motivates the terminology "inner") is equivalent to the condition that $\mathscr{M}(\mathscr{A}') \to \mathscr{M}(\mathscr{A})$ lands inside of $\mathrm{Int}(\mathscr{M}(\mathscr{A})/\mathscr{M}(\mathscr{B}))$.

The theory of the relative boundary (especially when it is empty) will underlie the theory of properness in the final lecture, where properness will be defined essentially as the condition "topological properness and the absence of a relative boundary".

5. Berkovich spaces II

In this final lecture we discuss sheaf theory on $\mathscr{M}(\mathscr{A})$ and then globalize to define k-*analytic spaces* (also called *Berkovich spaces*). We also discuss the relationship between k-analytic spaces and rigid-analytic spaces (when k has nontrivial absolute value), and we briefly address the theory of étale maps. We also mention some elegant concrete applications of Berkovich's theory.

5.1. **Globalization revisited.** To make $\mathscr{M}(\mathscr{A})$ into an object that globalizes, we have to define its "structure sheaf" in a manner that makes sense to evaluate on affinoid subdomains, which are generally closed sets that are not open (with respect to the compact Hausdorff topology on $\mathscr{M}(\mathscr{A})$).

Consider a quasi-compact and separated rigid space X. This can be described by gluing together finitely many affinoids $\mathrm{Sp}(A_i)$ along affinoid open overlaps $\mathrm{Sp}(A_{ij})$. If we try to carry out the same gluing topologically with the spaces $\mathscr{M}(A_i)$ and their affinoid subdomains $\mathscr{M}(A_{ij})$, the resulting topological space X will be compact and Hausdorff, but there may be points $x \in X$ around which no $\mathscr{M}(A_i)$ is a neighborhood in X. However, if $V_1 = \mathscr{M}(A_{i_1}), \ldots, V_n = \mathscr{M}(A_{i_n})$ are the $\mathscr{M}(A_i)$'s that contain x then the V_j's are compact Hausdorff subsets of X such that $x \in \cap V_j$ and $\cup V_j$ is a neighborhood of x in X. In general, a Berkovich space will be a kind of "structured" locally Hausdorff space X such that each $x \in X$ has a neighborhood of the form $V_1 \cup \cdots \cup V_n$ with $x \in \cap V_j$, each V_j compact and Hausdorff, and the V_j's endowed with "compatible k-affinoid structure" (with respect to the overlaps $V_j \cap V_{j'}$). To axiomatize the properties of such collections of V_j's, we shall use the following notion.

Definition 5.1.1. A *quasi-net* on a locally Hausdorff topological space X is a collection τ of compact Hausdorff subsets $V \subseteq X$ such that each $x \in X$ has a neighborhood of the form $\cup V_j$ for finitely many $V_j \in \tau$ with $x \in \cap V_j$.

Example 5.1.2. The (local) finiteness aspect of quasi-nets is the key reason why they can provide a workable substitute for usual coverings by open sets. A suggestive topological example to keep in mind (though not logically relevant to k-analytic geometry) is the closed unit disk X in $\mathbf{C}$ decomposed into a finite union of non-overlapping closed sectors V_j. The most interesting points x from the viewpoint of being a quasi-net are the points on edges where two sectors meet, and especially the center of the disk.

To define the gluing process for imposing a k-analytic structure on a topological gluing of $\mathscr{M}(A)$'s along affinoid (or more general kinds of) overlaps, we cannot work within the framework of ringed spaces because the building blocks of the theory will be compact subsets rather than open subsets. The following definition introduces the replacement for the structure of a ringed space.

Definition 5.1.3. Let X be a locally Hausdorff topological space. A *k-affinoid atlas* on X is the data consisting of a quasi-net τ on X such that

1. for all $U, U' \in \tau$, $\{V \in \tau \mid V \subseteq U \cap U'\}$ is a quasi-net on $U \cap U'$ (think of the collection of all affinoid subdomains in a rigid space as a motivating analogy),
2. to each $V \in \tau$ there is assigned a k-affinoid algebra $\mathscr{A}_V$ and a homeomorphism $V \simeq \mathscr{M}(\mathscr{A}_V)$ such that if $V' \in \tau$ and $V' \subseteq V$ then V' is an affinoid subdomain of $\mathscr{M}(\mathscr{A}_V)$ with coordinate ring $\mathscr{A}_{V'}$ (as a Banach $\mathscr{A}_V$-algebra).

The triple $(X, \mathscr{A}, \tau)$ is a *k-analytic space*. If all $\mathscr{A}_V$ are strictly k-analytic then this triple is called a *strictly k-analytic space*.

Example 5.1.4. Let $\mathscr{A}$ be a k-affinoid algebra. Letting $X = \mathscr{M}(\mathscr{A})$ and letting τ be the collection of affinoid subdomains (with $V \mapsto \mathscr{A}_V$ the usual assignment of coordinate rings to such subsets), we get a k-analytic space. If we instead take $\tau = \{X\}$ with $\mathscr{A}_X = \mathscr{A}$ then we get "another" k-analytic space. Clearly we want to consider these two k-analytic spaces to be naturally isomorphic (and to call these kinds of k-analytic spaces the *k-affinoid spaces*). If $\mathscr{A}$ is a strict k-affinoid algebra then we can take τ to be the collection of affinoid subdomains with strict coordinate ring, and this also should give rise to a k-analytic space that is naturally isomorphic to the other two. Making this idea precise requires defining the concept of *morphism* of k-analytic spaces.

Because we are not working with sheaves of rings on the underlying topological space X, we cannot easily pass to the sheaf language so as to hide the atlas as is possible in differential geometry. Thus, the definition of morphism (and in particular, isomorphism) between k-analytic spaces is rather subtle (in comparison with the definition of a morphism of ringed spaces). Rather than give the definition of a morphism of k-analytic spaces, we shall describe how one constructs morphisms in practice. Suppose that $(X, \mathscr{A}, \tau)$ and $(X', \mathscr{A}', \tau')$ are two k-analytic spaces and $\phi : X' \to X$ is a continuous map such that for all $V' \in \tau'$ and $V \in \tau$ with $V' \subseteq \phi^{-1}(V)$ there is given a compatible k-Banach algebra map $\mathscr{A}_V \to \mathscr{A}_{V'}$ that is transitive in the pair (V, V'). This data defines what is called a *strong morphism*

$(X', \mathscr{A}', \tau') \to (X, \mathscr{A}, \tau)$, and more general kinds of morphisms can be defined by a gluing process at the level of affinoids in the quasi-net and their affinoid subdomains. The key point is that to construct the most general morphisms that one wants to have, it is necessary to define a calculus of fractions to formally invert certain kinds of strong morphisms, much as one does (with quasi-isomorphisms of complexes) in the theory of derived categories. We omit the details, and instead present the key example of the kind of strong morphism that has to be "inverted", corresponding to the formation of a maximal atlas in differential geometry:

Example 5.1.5. Let $(X, \mathscr{A}, \tau)$ be a k-analytic space. Let $\widehat{\tau}$ be the collection of compact Hausdorff subsets $W \subseteq X$ such that W is covered by a (possibly infinite) collection of $W_i \in \tau$ satisfying

(1) $W_i \cap W_j \in \tau$ for all i and j,
(2) the natural k-Banach algebra map
$$\mathscr{A}_{W_i} \widehat{\otimes}_k \mathscr{A}_{W_j} \twoheadrightarrow \mathscr{A}_{W_i \cap W_j}$$
is a surjection inducing a residue norm equivalent to the given norm on the target (this norm condition is automatically satisfied if k has nontrivial absolute value),
(3) the k-Banach subalgebra $\mathscr{A}_{\{W_i\}} \subseteq \prod \mathscr{A}_{W_i}$ consisting of elements with the same image under both (continuous) maps $\prod_i \mathscr{A}_{W_i} \rightrightarrows \prod_{i,j} \mathscr{A}_{W_i \cap W_j}$ is k-affinoid.

It can be shown that $\mathscr{A}_{\{W_i\}}$ only depends on W (and so is independent of the choice of $\{W_i\}$) up to norm equivalence; in particular, if $W \in \tau$ then this k-Banach algebra is naturally isomorphic to the originally given $\mathscr{A}_W$. Thus, we may and do write $\mathscr{A}_W$ to denote this k-affinoid algebra for general W as above. The assignment $\widehat{\mathscr{A}} : W \mapsto \mathscr{A}_W$ for $W \in \widehat{\tau}$ is a k-affinoid atlas, and so $(X, \widehat{\mathscr{A}}, \widehat{\tau})$ is a k-analytic space. The canonical strong morphism $(X, \mathscr{A}, \tau) \to (X, \widehat{\mathscr{A}}, \widehat{\tau})$ is an *isomorphism* of k-analytic spaces (with respect to the correct definition of morphism, which essentially creates inverses for maps of this type). The sense in which this procedure is like the formation of a maximal atlas in differential geometry is that $\widehat{\widehat{\tau}} = \widehat{\tau}$ and $\widehat{\widehat{\mathscr{A}}} = \widehat{\mathscr{A}}$. For this reason, $(\widehat{\tau}, \widehat{\mathscr{A}})$ is called the *maximal k-affinoid atlas* on the given k-analytic space.

In the definition of k-analytic spaces as a category (including morphisms), one can consider the subcategory obtained by using only strictly k-analytic affinoids in the data of τ (and likewise for $\widehat{\tau}$ as above). This is the subcategory of *strictly k-analytic spaces*, and it is a very natural category of interest when considering how to promote a rigid-analytic space to a k-analytic space in a functorial manner. Likewise, if K/k is an extension of complete fields endowed with compatible absolute values then we can define the notion of a *morphism* from a K-analytic space to a k-analytic space by using maps of k-Banach algebras $\mathscr{A} \to \mathscr{B}$ from a k-affinoid algebra to a K-affinoid algebra (over $k \to K$) and the associated maps of topological spaces $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$. In this way, we can consider as a *single category* all K-analytic spaces for all K/k with a fixed k, and morphisms induce continuous maps on underlying topological spaces. There is nothing like this in rigid-analytic geometry for infinite-degree extensions K/k.

An important yet subtle issue is that it is not at all obvious that a pair of strictly k-analytic spaces cannot admit a morphism as k-analytic spaces that is not already a morphism in the subcategory of strictly k-analytic spaces. The point is that going outside of the strictly k-analytic category permits the possibility of gluing morphisms between non-strict affinoids that "cover" the spaces. It is a recent (not at all easy) theorem of Temkin that the category of strictly k-analytic spaces is in fact a full subcategory of the category of k-analytic spaces (i.e., one gets no new morphisms between strictly k-analytic spaces by viewing them as merely k-analytic spaces). Thus, we do not need to distinguish these two kinds of morphisms when working with strictly k-analytic spaces.

An interesting class of k-analytic spaces is those for which every point admits an affinoid neighborhood. This will include the k-analytic spaces associated to arbitrary algebraic k-schemes, but it does not include the k-analytic spaces that one obtains from arbitrary quasi-compact and (quasi-)separated rigid spaces (by a process to be described later). Thus, this class of k-analytic spaces (which were the only ones considered in [Ber1]) are rather special and not sufficient for the purposes of using Berkovich's theory to study questions in rigid-analytic geometry. Nonetheless, these special spaces do play an important role in the theory (e.g., in the definition of properness, to be given later), and so they deserve a name:

Definition 5.1.6. A k-analytic space $(X, \mathscr{A}, \tau)$ is *good* if every $x \in X$ has a neighborhood V with $V \in \widehat{\tau}$ (i.e., V is an affinoid neighborhood of x in the sense of the "maximal" atlas).

For a quasi-compact and quasi-separated rigid space X, it is a difficult matter to describe in rigid-analytic terms when the associated k-analytic space X^{an} (in the sense of Example 5.2.3) is a good space; see Exercise 5.2.5. In proofs of "local" theorems for k-analytic spaces, it is typical to first settle the affinoid case, then the good case, and finally (by appropriate gluing arguments with quasi-nets) the general case. In [Ber2] the more general (i.e., possibly not good) kind of k-analytic spaces as defined above was introduced.

In order to explain how to associate a k-analytic space to an algebraic k-scheme or to a (reasonable) rigid-analytic space, and more generally to work with k-analytic spaces, we need to define an analogue of the admissible open subsets within a rigid space. These distinguished subsets of a k-analytic space are rarely open, and so we need to use a terminology other than "admissible open":

Definition 5.1.7. Let $X = (X, \mathscr{A}, \tau)$ be a k-analytic space. A *k-analytic domain* in X is a subset $Y \subseteq X$ such that for all $y \in Y$ there exists $V_1, \dots, V_n \in \widehat{\tau}$ with $y \in \cap V_j$ and $\cup V_j$ a neighborhood of y in Y (in particular, $V_j \subseteq Y$ for all j).

Example 5.1.8. If Y is a k-analytic domain in $X = (X, \mathscr{A}, \tau)$ then the assignment $Y' \mapsto \widehat{\mathscr{A}_{Y'}}$ for Y' in the quasi-net $\widehat{\tau} \cap Y$ on Y naturally gives Y a structure of k-analytic space with respect to which a morphism $X' \to X$ of k-analytic spaces factors through $Y \subseteq X$ set-theoretically if and only if it does so as k-analytic spaces (in which case such a factorization is unique). This is the analogue of endowing an admissible open subset of a rigid space with a natural structure of rigid space. Any open subset of X is a k-analytic domain, but many interesting k-analytic domains (such as the affinoid ones) are generally not open.

Example 5.1.9. Although k-analytic domains are to be thought of as analogues of admissible opens, in fact they provide an abundance of subsets with no analogue in rigid geometry. For example, if k has nontrivial absolute value and A is a strict k-affinoid algebra then for $x \in \mathscr{M}(A)$ with $\mathscr{H}(x)$ of infinite degree over k, the open subset $\mathscr{M}(A) - \{x\}$ is a k-analytic domain in the k-affinoid space $\mathscr{M}(A)$ and it has no natural counterpart within $\mathrm{Sp}(A)$. Perhaps the most basic example of this is to remove the Gauss norm from $\mathscr{M}(k\langle T\rangle)$.

5.2. **Fiber products, analytification, and the G-topology.** By using k-analytic domains, one can glue k-analytic spaces and morphisms between them in a manner similar to what is done with ringed spaces. Here are some basic and important examples of this gluing process (which we will not formally define, because it is a bit lengthy to explain due to the role of quasi-nets for spaces that are not good).

Example 5.2.1. Let us explain fiber products and extension of the base field. First we consider the affinoid case. For a pair of morphisms $\mathscr{A} \rightrightarrows \mathscr{A}', \mathscr{A}''$ of k-affinoid algebras we get a k-affinoid algebra $\mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{A}''$, and for an analytic extension field K/k (i.e., one with compatible absolute value with respect to which it is complete) we get a K-affinoid algebra $\mathscr{A}_K = K\widehat{\otimes}_k\mathscr{A}$. The natural morphisms of k-analytic spaces $\mathscr{M}(\mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{A}'') \rightrightarrows \mathscr{M}(\mathscr{A}'), \mathscr{M}(\mathscr{A}'')$ have the same composition to $\mathscr{M}(\mathscr{A})$, and as such this makes $\mathscr{M}(\mathscr{A}'\widehat{\otimes}_{\mathscr{A}}\mathscr{A}'')$ into a fiber product $\mathscr{M}(\mathscr{A}') \times_{\mathscr{M}(\mathscr{A})} \mathscr{M}(\mathscr{A}'')$ in the category of k-analytic spaces. (The verification of this rests on the ability to uniquely determine and define morphisms from a k-analytic space by specification of the morphism on a suitable covering by affinoid k-analytic domains.) Likewise, the natural morphism $\mathscr{M}(\mathscr{A}_K) \to \mathscr{M}(\mathscr{A})$ from a K-affinoid space to a k-affinoid space serves as a fiber product $\mathscr{M}(\mathscr{A}) \times_{\mathscr{M}(k)} \mathscr{M}(K)$ in the category of analytic spaces over arbitrary analytic extension fields K'/K. This latter extension of the ground field functor is naturally transitive with respect to further extension of the base field.

These constructions in the affinoid case can be uniquely globalized to define fiber products of arbitrary k-analytic spaces and extension of the ground field functors for any K/k. Moreover, the formation of fiber products of k-analytic spaces is naturally compatible with any extension of the ground field. We omit any discussion of the details, except to make several remarks. First, the gluing process in these globalizations is a bit more involved than the analogous kind of gluing used in the case of schemes (though it follows a similar strategy), and fiber products are well-behaved with respect to k-analytic domains: for a pair of maps $X', X'' \rightrightarrows X$ of k-analytic spaces and a compatible collection of k-analytic domains $Y \subseteq X$, $Y' \subseteq X'$, and $Y'' \subseteq X''$, the natural map $Y' \times_Y Y'' \to X' \times_X X''$ is a k-analytic domain. The other aspect we wish to emphasize is that the topological space $|X' \times_X X''|$ has more subtle behavior than that of fiber products of complex-analytic spaces. As for schemes, the natural map $|X' \times_X X''| \to |X'| \times_{|X|} |X''|$ is generally *not* bijective (even if k is algebraically closed), essentially because of the nature of completed tensor products of analytic extension fields of k, but this map is a proper surjection of topological spaces (ultimately because of the k-affinoid case); recall that a map of topological spaces is *proper* if it is separated (i.e., has diagonal that is a closed embedding) and universally closed. Moreover, a point in $X' \times_X X''$ can fail to have a base of neighborhoods of the type $Y' \times_Y Y''$ for k-analytic domains $Y \subseteq X$, $Y' \subseteq X'$, $Y'' \subseteq X''$ (even if $X = Y = \mathscr{M}(k)$).

The notion of a *closed immersion* between k-analytic spaces can be defined in a manner that is similar to the rigid-analytic case (though the definition is a little more involved for spaces that are not good). In particular, closed immersions of k-analytic spaces are closed embeddings on topological spaces. Applying this concept of closed immersion to diagonal maps, we can define the notion of *separatedness* for a morphism $f : X' \to X$ of k-analytic spaces. Beware that, unlike for schemes, a diagonal map with closed image need not be a closed immersion. (For example, there are compact Hausdorff k-analytic spaces X that are not separated over $\mathscr{M}(k)$. Any quasi-compact and quasi-separated rigid space that is not separated gives rise to such an X, by the procedure in Example 5.2.3 below. The distinction between the Hausdorff condition and separatedness is due to the distinction between topological fiber products and k-analytic fiber products, as well as to the fact that diagonal maps for k-analytic spaces can lack a kind of "immersion" property that they always have in the categories or schemes and complex-analytic spaces.)

Exercise 5.2.2. Let $f : X' \to X$ be a map of k-analytic spaces, and let $\Delta_f : X' \to X' \times_X X'$ be the diagonal. Prove that the induced map $|f| : |X'| \to |X|$ on underlying topological spaces is separated (in the sense that its relative diagonal is a closed embedding) if and only if the preimage under $|f|$ of a Hausdorff subset of $|X|$ is Hausdorff. (Hint: k-analytic spaces are locally Hausdorff, and the natural map $|X' \times_X X'| \to |X'| \times_{|X|} |X'|$ is a proper surjection of topological spaces.) Also prove that if f is separated then $|f| : |X'| \to |X|$ is separated. In particular, if X is separated over $\mathscr{M}(k)$ then $|X|$ is Hausdorff. The converse is *false*.

Example 5.2.3. The relationship between k-analytic spaces, algebraic k-schemes, and rigid-analytic spaces over k goes as follows. First assume that k has a nontrivial absolute value. For any quasi-compact and quasi-separated rigid space X over k, we can describe X in terms of gluing of finitely many k-affinoids along quasi-compact admissible opens. By choosing a finite (necessarily admissible) covering of each of these quasi-compact overlaps by affinoid opens, we have a gluing description of X entirely in terms of finitely many k-affinoid spaces and affinoid subdomains of these spaces. This can be carried over to the k-analytic category, and it yields a compact Hausdorff strictly k-analytic space $X^{\rm an}$ that is called the *analytification* of X (in the sense of k-analytic spaces). The underlying set of X is the set of points $x \in X^{\rm an}$ such that $[\mathscr{H}(x) : k] < \infty$ (ultimately because of the affinoid case). This construction is independent of the choice of the covering, and it can be made into a functor from the category of quasi-compact and quasi-separated rigid spaces to the category of compact Hausdorff strictly k-analytic spaces. Spaces of this latter type may fail to be good! This is why the good spaces (the only ones considered in [Ber1]) are inadequate for a satisfactory general theory.

The functor $X \rightsquigarrow X^{\rm an}$ from quasi-compact and quasi-separated rigid spaces to compact Hausdorff strictly k-analytic spaces is an equivalence of categories, and it is compatible with the formation of fiber products and extension of the ground field. A map $f : X' \to X$ between quasi-compact and quasi-separated rigid spaces is a closed immersion if and only if $f^{\rm an}$ is a closed immersion, and so applying this to diagonal maps gives that f is separated if and only if $f^{\rm an}$ is separated. Thus, if X is a quasi-compact and quasi-separated rigid space that is not separated (such as a gluing of a closed unit disk to itself along a proper subdisk) then $X^{\rm an}$ is a strictly k-analytic space that is compact and Hausdorff but not separated (over $\mathscr{M}(k)$).

In fact, this procedure can be carried out more generally without quasi-compactness of the spaces, provided that we impose a local finiteness condition on both sides. The main result is that one gets an equivalence of categories between the category of quasi-separated rigid spaces admitting a locally finite (for the Tate topology) admissible covering by affinoid opens and the category of paracompact Hausdorff strictly k-analytic spaces. In a related manner, there is a natural analytification functor from algebraic k-schemes to strictly k-analytic spaces (compatible with fiber products and extension of the base field), and this is compatible with the analytification functors from algebraic k-schemes to rigid spaces over k and from (suitable) rigid spaces over k to strictly k-analytic spaces. We omit a precise statement.

Example 5.2.4. Assume that k has nontrivial absolute value and let $X = \mathscr{M}(k\langle T\rangle)$ and $U = X - \{\xi\}$, where ξ is the Gauss norm. These are both paracompact Hausdorff strictly k-analytic spaces, and under the fully faithful functor from such k-analytic spaces into the category of rigid-analytic spaces over k we get that the associated rigid spaces X_0 and U_0 are as follows: $X_0 = \mathbf{B}^1_k$ and U_0 is a disjoint union of "twisted" open unit discs labeled by the closed points of $\mathbf{A}^1_{\widetilde{k}}$. Thus, although $U \to X$ is an open immersion, the corresponding map $U_0 \to X_0$ is a bijective local isomorphism that is not an isomorphism: this rigid-analytic map identifies U_0 with the disjoint union of the "open residue discs" for $\mathbf{B}^1_k$ (which are a non-admissible cover of $\mathbf{B}^1_k$). This gives a rigid-analytic interpretation for the operation of removing the non-classical point ξ from the space $\mathscr{M}(k\langle T\rangle)$.

Exercise 5.2.5. Let X be a quasi-compact and quasi-separated rigid space. What rigid-analytic conditions on X are equivalent to the property that the associated k-analytic space X^{an} is good (i.e., each point of X^{an} has an affinoid neigborhood)? This is a difficult exercise.

Remark 5.2.6. A remarkable feature of k-analytic spaces is that they have very nice topological properties. For example, a k-analytic space is locally path-connected (so it is connected if and only it is path-connected!), and the topological dimension (in any of the usual senses of topological dimension theory) is bounded above by the analytic dimension (defined in terms of coordinate rings of affinoid subdomains). This relation with topological dimension theory is extremely useful in the study of étale cohomology for analytic spaces, where there is an interesting spectral sequence that relates étale cohomology of an étale sheaf to topological cohomology of the underlying topological sheaf and Galois cohomology for the stalks at the the residue fields of points of the space. Coupled with bounds on the cohomological dimension of such fields, one gets vanishing results in étale cohomology via vanishing results in topological sheaf theory (using topological dimension theory).

On k-analytic spaces $X = (X, \mathscr{A}, \tau)$ we can use k-analytic subdomains to define a G-topology that is analogous to the Tate topology on rigid-analytic spaces. The objects of the *G-topology* on X are the k-analytic subdomains $Y \subseteq X$, and a *covering* $\{Y_i\}$ of such a Y by k-analytic subdomains of X is a set-theoretic covering with the "quasi-net" property that each $y \in Y$ has a neighborhood in Y of the form $Y_{i_1} \cup \cdots \cup Y_{i_n}$ with $y \in \cap_j Y_{i_j}$. This encodes the local finiteness property of admissible coverings in rigid-analytic geometry. We write X_G to denote X endowed with the G-topology. There is a *unique* way to define a sheaf $\mathscr{O}_{X_G} : Y \mapsto \mathscr{A}_Y$ on the G-topology of X such that for all $V \in \widehat{\tau}$ it recovers the coordinate ring $\mathscr{A}_V$ of the k-affinoid subdomain V. This is what underlies the procedure by which every k-analytic subdomain $Y \subseteq X$ naturally acquires a structure of k-analytic space such that it satisfies the set-theoretic mapping property that a morphism $X' \to X$ factoring through Y

set-theoretically uniquely does so as k-analytic spaces. When we restrict the sheaf $\mathscr{O}_{X_G}$ on the G-topology to the collection of open subsets of X then we get a sheaf of k-algebras $\mathscr{O}_X$ on the topological space X. The pair $(X, \mathscr{O}_X)$ is a locally ringed space that is functorial in the k-analytic space X.

If X is not a good space then it is difficult to work with the stalks $\mathscr{O}_{X,x}$ when x lacks an affinoid neighborhood. This is the reason why flatness is a difficult concept to work with for general morphisms between k-analytic spaces. Often one studies "local" problems by first treating the good case (where $\mathscr{O}_{X,x}$ and its residue field can be computed using an affinoid neighborhood and so have some nice properties), and then bootstrapping to the general case by gluing arguments. For example, building up from the affinoid case (with Kiehl's theory of coherent sheaves in the strict case) permits one to construct a satisfactory theory of *coherent sheaves* on X_G and on the locally ringed space $(X, \mathscr{O}_X)$. An important example of a coherent sheaf is $\Omega^1_{X_G/Y_G}$ on X_G for any map of k-analytic spaces $f : X \to Y$ (and $\Omega^1_{X/Y}$ on $(X, \mathscr{O}_X)$ if X and Y are good). This sheaf of relative 1-forms is defined in terms of the coherent ideal sheaf of the relative diagonal in the separated (e.g., affinoid) case, much like in the case of schemes, and it is defined in general by a gluing procedure.

If X is a good k-analytic space then the categories of coherent sheaves on X_G and $(X, \mathscr{O}_X)$ are naturally equivalent. In general, the theory of coherence underlies the theory of *finite morphisms* (using coherent sheaves of algebras), and the coherent sheaf $\Omega^1_{X_G/Y_G}$ underlies the theories of smooth and étale morphisms of k-analytic spaces. The definition of a *finite morphism* in general (for spaces that are not necessarily good) illustrates how to work pointwise and to bootstrap from the affinoid case via quasi-nets, so let us give this definition. (Compare this with the analogous definition for maps of rigid spaces.)

Definition 5.2.7. A map $f : X' \to X$ of k-analytic spaces is *finite* if, for all $x \in X$, there exist k-affinoid subdomains $V_1, \dots, V_n \subseteq X$ such that $\cup V_j$ is a neighborhood of x in X, $x \in \cap V_j$, and each $V'_j = f^{-1}(V_j) \subseteq X'$ is a k-affinoid subdomain in X' with $\mathscr{A}_{V_j} \to \mathscr{A}'_{V'_j}$ a finite map of Banach $\mathscr{A}_{V_j}$-algebras such that $\mathscr{A}'_{V'_j}$ has its norm equivalent to the natural one induced by the finite $\mathscr{A}_{V_j}$-module structure (this final condition being automatically satisfied when k has nontrivial absolute value).

Remark 5.2.8. The requirement on the pairs (V_j, V'_j) in the preceding definition can be shown to be satisfied by the pair $(V, f^{-1}(V))$ for any k-affinoid subdomain $V \subseteq X$ when $f : X' \to X$ is a finite morphism.

Let us now illustrate the power of k-analytic spaces to give an intuitive proof of a fact in rigid geometry that is difficult to prove within the framework of rigid geometry. Assume that k has a nontrivial absolute value, and let A be a strict k-affinoid algebra. Let $X = \mathrm{Sp}(A)$ and $Z = \mathrm{Sp}(A/I)$ for an ideal $I \subseteq A$. Let $U \subseteq X$ be an admissible open subset such that $Z \subseteq U$. Finally, let $\{f_1, \dots, f_n\}$ be a set of generators of I. I claim that for some $\varepsilon > 0$ (with $\varepsilon \in \sqrt{|k^\times|}$), the "tube" $\{|f_1| \le \varepsilon, \dots, |f_n| \le \varepsilon\}$ around Z is contained in U. This can be proved by the methods of rigid geometry, but the only proof along such lines which I know is long and complicated. A short proof was given by Kisin via Raynaud's formal models, but let us now give an appealing short proof via Berkovich spaces. Without loss of generality we can assume that the open subset U is quasi-compact, so it is a finite union of rational domains in

X by the Gerritzen–Grauert theorem. Thus, U^{an} is a k-analytic domain in X^{an}. The space X^{an} is compact Hausdorff, so the decreasing collection of neighborhoods $N_\varepsilon = \{|f_j| \leq \varepsilon\}$ of the closed set $Z^{\mathrm{an}} = \{f_j = 0\} = \cap_\varepsilon N_\varepsilon$ must be cofinal. Thus, it suffices to show that U^{an} is a neighborhood of Z^{an} in X^{an}. Since U^{an} is a (possibly non-affinoid) k-analytic domain in X^{an}, by the global theory of the relative interior (which we have only discussed for morphisms between affinoid spaces, but can be developed for morphisms between any pair of k-analytic spaces) it follows that the topological interior $\mathrm{int}_{X^{\mathrm{an}}}(U^{\mathrm{an}})$ of U^{an} in X^{an} is equal to the relative interior $\mathrm{Int}(U^{\mathrm{an}}/X^{\mathrm{an}})$. But the closed immersion $Z^{\mathrm{an}} \to X^{\mathrm{an}}$ is a finite map, so $Z^{\mathrm{an}} = \mathrm{Int}(Z^{\mathrm{an}}/X^{\mathrm{an}})$. There is a general transitivity property of relative interior with respect to a composition of morphisms, and applying this in the case of the composition $Z^{\mathrm{an}} \to U^{\mathrm{an}} \to X^{\mathrm{an}}$ yields

$$Z^{\mathrm{an}} = \mathrm{Int}(Z^{\mathrm{an}}/X^{\mathrm{an}}) = \mathrm{Int}(Z^{\mathrm{an}}/U^{\mathrm{an}}) \cap \mathrm{Int}(U^{\mathrm{an}}/X^{\mathrm{an}}) \subseteq \mathrm{Int}(U^{\mathrm{an}}/X^{\mathrm{an}}) = \mathrm{int}_{X^{\mathrm{an}}}(U^{\mathrm{an}}),$$

as desired.

5.3. **Proper and étale morphisms.** Another interesting application of k-analytic spaces within rigid geometry is in the theory of properness. Recall that although proper morphisms between rigid spaces do enjoy properties as in algebraic and complex-analytic geometry (such as coherence of higher direct images, and the theory of cohomology and base change), such maps are not as easy to work with; for example, it is not at all obvious that properness is preserved under composition. However, this latter problem was elegantly solved by Temkin by comparison with the notion of properness in Berkovich's theory. The key definition is as follows:

Definition 5.3.1. A map $f : X' \to X$ of k-analytic spaces is *proper* if the map $|f| : |X'| \to |X|$ on underlying topological spaces is proper and for all morphisms $Y = \mathscr{M}(\mathscr{A}) \to X$ from k-affinoid spaces the pullback $Y' = X' \times_X Y$ is a good space and the relative boundary $\partial(Y'/Y)$ is empty. (Since we have not defined the relative boundary in a non-affinoid setting, let us at least give a concrete translation of the condition that $\partial(Y'/Y)$ is empty: for all $y' \in Y'$ and k-affinoid domains $U' \subseteq Y'$ that are neighborhoods of y' in the good space Y', we have that y' lies in the relative interior $\mathrm{Int}(U'/Y)$ as defined in our discussion of the affinoid theory.)

Roughly speaking, a proper map is one that is topologically proper and boundaryless in a relative sense; this is akin to the idea of a compact manifold. In the setting of proper k-schemes and coherent sheaves on them, Serre's methods can be carried over to prove versions of the GAGA theorems for analytification in the sense of k-analytic spaces. Also, a morphism of k-analytic spaces is finite if and only if it is proper and has finite fibers (exactly as for rigid spaces, schemes, and complex-analytic spaces).

Elementary properties of the relative interior in the affinoid case ensure that properness in the k-analytic category is stable under composition (and under base change and change of the ground field). Thus, the following theorem of Temkin elegantly disposes of the general problem of stability of properness under composition in rigid-analytic geometry (which was settled earlier by Lütkebohmert in the discretely-valued case by means of sophisticated arguments with formal models).

Theorem 5.3.2. *Assume that k has a nontrivial absolute value. Let $f : X' \to X$ be a map between quasi-compact and separated rigid spaces. Then f is proper in the sense of rigid-analytic spaces if and only if f^{an} is proper in the sense of k-analytic spaces.*

We conclude by briefly discussing some aspects of the theory of étale maps in the k-analytic category; this is exhaustively developed in [Ber2].

Definition 5.3.3. A map $f : X' \to X$ of k-analytic spaces is (locally) *quasi-finite* if, for all $x' \in X'$, there exist open subsets $U' \subseteq X'$ around x' and $U \subseteq X$ around $f(x')$ such that $U' \subseteq f^{-1}(U)$ and $f : U' \to U$ is finite. If moreover for every affinoid domain $V \subseteq U$ the affinoid pullback $V' = V \times_U U'$ has coordinate ring that is flat over the coordinate ring of V then f is called *flat quasi-finite*.

This definition of (local) quasi-finiteness is inspired by the local structure theorem for maps with discrete fibers between complex-analytic spaces, and it is very different from the definition of quasi-finiteness in algebraic geometry; the Zariski topology is too coarse to permit using a definition as above in the algebraic theory. As a partial "justification" for the above definition of quasi-finiteness in the k-analytic setting, we remark that if $\phi : \mathscr{X}' \to \mathscr{X}$ is a map between k-schemes of finite type then it is quasi-finite (in the sense of algebraic geometry) if and only if $\phi^{\mathrm{an}} : \mathscr{X}'^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is quasi-finite in the sense of the preceding definition. Likewise, one has that ϕ is flat quasi-finite if and only if ϕ^{an} is flat quasi-finite.

Flatness for a morphism of k-analytic spaces is generally a subtle matter, but for quasi-finite maps it is not too hard to develop the theory of flatness via the definition introduced above because such maps locally (on the source and target) are finite maps, and so one can work algebraically with finite maps between coordinate rings. For a general map of k-analytic spaces the condition of flatness in the sense of ringed spaces is not a good notion. If $f : X' \to X$ is a flat quasi-finite map then for each $x \in X$ the fiber $f^{-1}(x)$ as an $\mathscr{H}(x)$-analytic space has the form $f^{-1}(x) = \coprod_i \mathscr{M}(\mathscr{B}_i)$ for $\mathscr{H}(x)$-finite algebras $\mathscr{B}_i$.

Definition 5.3.4. A map $f : X' \to X$ between k-analytic spaces is *étale* if it is flat quasi-finite and for each $x \in X$ the fiber $f^{-1}(x)$ as a $\mathscr{H}(x)$-analytic space is a disjoint union of analytic spaces of the form $\mathscr{M}(\mathscr{K}_i)$ where $\mathscr{K}_i$ is a finite separable extension field of $\mathscr{H}(x)$.

Étale maps are always open on underlying topological spaces, and an *étale cover* is defined as in algebraic geometry: a collection of étale maps whose (open) images cover the target space. Also, the property of being étale is preserved under base change and change of the base field, and any X-map between étale k-analytic spaces over X is necessarily étale. By using étale covers one can define an *étale topology* on the category of étale objects over any k-analytic space. It can be shown that all representable functors on the category of k-analytic spaces are sheaves for the étale topology (proved in [Ber2] under a mild restriction).

The theory of étale cohomology can be developed in a style that parallels the development for schemes, though the proofs require many new ideas and exhibit an interesting interplay between algebraic and analytic points of view. A notable feature of the theory, ultimately resting on the nice topological structure of k-analytic spaces (especially the role of para-compactness in the theory), is that étale cohomology with proper supports in the k-analytic theory can be defined exactly as in ordinary topological sheaf theory (as opposed to the case of schemes, where one has to bring in compactifications, which may not exist in analytic settings): it is the derived functor of the functor of sections with proper support! There are also comparison isomorphisms with étale cohomology in algebraic geometry via the analytification functor, analogous to Artin's comparison results for étale and topological cohomology over **C** but resting on proofs that lead to some simplifications even in the classical case over **C**.

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