ARIZONA WINTER SCHOOL PROJECT SUGGESTIONS

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1. $\mathbf{A}^1$-ENUMERATIVE GEOMETRY PROJECTS

There are many beautiful results in enumerative geometry, and it is not so far-fetched to suggest turning any such result into a project, where the classical count over $\mathbf{C}$ becomes an arithmetic count over a field. Some known difficulties with this perspective include the existence of orientations (some vector bundles which occur naturally in enumerative geometry are not relatively oriented), and obtaining enumerative descriptions of local contributions to Euler classes. The following is an outline of projects which (hopefully) are of a nature to be educational in the Arizona Winter School. Since these projects are intended to be tackled in a period of six days with some success, this list is not intended to be representative of the open problems in the field.

1.1. **Configurations over** k**.** Let $\mathcal{S} \to \mathrm{GW}(1, n)$ denote the tautological bundle on the Grassmannian of lines in $\mathbf{P}^n$. In [Wen18] and [SW18] the Euler class of $\wedge^2\mathcal{S}^* \to \mathrm{Gr}(1,3)$, and more generally, $\oplus_{i=1}^{2n-2}\mathcal{S} \to \mathrm{Gr}(1,n)$ is computed. This corresponds to an arithmetic count of the number of lines meeting $2n-2$ codimension 2 hyperplanes. However, these hyperplanes are all defined over k. One could ask more generally for a count of lines meeting a configuration of $2n-2$ codimension 2 hyperplanes, where the configuration is defined over k, but where the individual codimension 2 hyperplanes are not necessarily defined over k. In other words, we have $2n-2$ hyperplanes defined over $\overline{k}$, which are permuted by the $\mathrm{Gal}(\overline{k}/k)$-action, but where the $\mathrm{Gal}(\overline{k}/k)$-action is potentially non-trivial. An arithmetic count of these lines would correspond to the computation of an Euler class of the vector bundle whose fiber over $[W]$ in $\mathrm{GW}(1,n)$ is $(W^* \wedge W^*)^{2n-2}/S_{2n-2}$, where W denotes a linear subspace of dimension 2 of an $n+1$ dimensional vector space, S_{2n-2} denotes the symmetric group on $2n-2$ objects, and the action of S_{2n-2} on $(W^* \wedge W^*)^{2n-2}$ is by permutation.

Question 1. (1) *Is this vector bundle relatively orientable for certain* n*?*
(2) *If so, what is the Euler class?*
(3) *Can you give an enumerative interpretation of the local indices?*
(4) *If so, what is the resulting theorem?*

Question 2. *Can this be generalized to other counts of subspaces meeting a configuration over* k*? For example, can the count of balanced subspaces of* [Wen18, Section 9.2] *be generalized to where the subspaces are not defined over the ground field? Similarly, if we only require that the two quadrics of* [Wen18, Example 9.4] *are permuted by* $\mathrm{Gal}(\overline{k}/k)$ *instead of being individually defined over* k*, could we obtain a more general result?*

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1.2. **Bitangents to a smooth plane quartic.** A bitangent to a curve in projective space is a line which is tangent to the curve at 2 points, counted with multiplicity. It is a classical theorem that there are 28 bitangents to a smooth degree 4 curve in $\mathbf{P}^2$ over $\mathbf{C}$. This is closely related to the 27 lines on a complex smooth cubic surface. Namely, let $S \subset \mathbf{P}^3_{\mathbf{C}}$ be a smooth cubic surface, and let p be a point of S not on any line. Projection π_p from p defines a rational map $\pi_p : \mathbf{P}^3_k \dashrightarrow \mathbf{P}^2_k$, and a map $\pi_p^{\mathrm{Bl}} : \mathrm{Bl}_p S \to \mathbf{P}^2$, which expresses S as a degree 2 cover of $\mathbf{P}^2$ branched over a quartic curve C. The images of the 27 lines on S and the exceptional divisor of the blow up give the 28 bitangents of C.

Question 3. *Is there an arithmetic count of the bitangents to a smooth plane quartic?*

One idea along these lines is as follows: perhaps a good local contribution for a bitangent line L is related to the two points of contact with the curve. If these points are defined over $k(L)[\sqrt{D}]$, then $\langle D \rangle$ in $\mathrm{GW}(K(L))$ may be useful. Consider the involution I which appears in the arithmetic count of the lines on a cubic surface described in [KW17]. Do the images of these fixed points have an independent interpretation as points on the bitangent, for example as points of contact with C?

1.3. **Eisenbud–Khimshaishvili–Levine form is $\mathbf{A}^1$-local degree over extensions of k.** The main result of [KW16] identifies the $\mathbf{A}^1$-local degree $\deg_0 f$ at 0 of a function $f : \mathbf{A}^n_k \to \mathbf{A}^n_k$ with an isolated zero at the origin with the bilinear form ω^{EKL} appearing in the Eisenbud–Levine–Khimshiashvili signature formula

$$\deg_0 f = \omega^{\mathrm{EKL}}.$$

Question 4. *Can this be generalized to equate the local degree* $\deg_p f$ *with* ω^{EKL} *when the residue field of* p *is not* k*?*

References

[KW16] Jesse Kass and Kirsten Wickelgren, *The class of Eisenbud–Khimshiashvili–Levine is the local $\mathbb{A}^1$-brouwer degree*, 2016. Accepted for publication in Duke Mathematical Journal, available at `https://arxiv.org/abs/1608.05669`. ↑1.3

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[SW18] Padmavathi Srinivasan and Kirsten Wickelgren, *An arithmetic count of the lines meeting four lines in $P\hat{3}$*, arXiv:1810.03503 [math] (October 2018), available at `1810.03503`. ↑1.1

[Wen18] Matthias Wendt, *Oriented Schubert calculus in Chow–Witt rings of Grassmannians*, 2018. *Preprint* available at `https://arxiv.org/abs/1808.07296`. ↑1.1, 2

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