RATIONAL POINTS AND OBSTRUCTIONS TO THEIR EXISTENCE
2015 ARIZONA WINTER SCHOOL PROBLEM SET
EXTENDED VERSION

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The primary goal of the problems below is to build up familiarity with some useful lemmas and examples that are related to the theme of the Winter School.

Notation. For a field k, we denote by $\overline{k}$ a choice of its algebraic closure, and by $k^s \subset \overline{k}$ the resulting separable closure. If k is a number field and v is its place, we write k_v for the corresponding completion. If $k = \mathbb{Q}$, we write $p \leqslant \infty$ to emphasize that p is allowed be the infinite place; for this particular p, we write $\mathbb{Q}_p$ to mean $\mathbb{R}$. For a base scheme S and S-schemes X and Y, we write $X(Y)$ for the set of S-morphisms $Y \to X$. When dealing with affine schemes we sometimes omit Spec for brevity: for instance, we write $\mathrm{Br}\, R$ in place of $\mathrm{Br}(\mathrm{Spec}\, R)$. A 'torsor' always means a 'right torsor.'

Acknowledgements. I thank the organizers of the Arizona Winter School 2015 for the opportunity to design this problem set. I thank Alena Pirutka for helpful comments.

1. Rational points

In this section, k is a field and X is a k-scheme. A *rational point* of X is an element $x \in X(k)$, i.e., a section $x\colon \mathrm{Spec}\, k \to X$ of the structure map $X \to \mathrm{Spec}\, k$.

1.1. Suppose that $X = \mathrm{Spec}\, \frac{k[T_1, \dots, T_n]}{(f_1, \dots, f_m)}$. Find a natural bijection

$$X(k) \longleftrightarrow \{(x_1, \dots, x_n) \in k^n \quad \text{such that} \quad f_i(x_1, \dots, x_n) = 0 \quad \text{for every } i = 1, \dots, m\}.$$

Hint. Inspect the images of the T_j under the k-algebra homomorphism $\frac{k[T_1, \dots, T_n]}{(f_1, \dots, f_m)} \to k$ corresponding to a rational point $x\colon \mathrm{Spec}\, k \to X$.

Side remark. In conclusion, finding rational points is tantamount to finding k-rational solutions of systems of polynomial equations with coefficients in k.

1.2. (a) Prove that the image of a rational point $x\colon \mathrm{Spec}\, k \to X$ is necessarily a closed point of the underlying topological space of X; in fact, prove that x is a closed immersion.

Hint. Work Zariski locally on X: for instance, a subset of Z of a scheme X is closed if and only if $Z \cap U$ is closed in U for every affine open $U \subset X$.

Side remark. For an arbitrary scheme morphism $f\colon X \to S$, a section $s\colon S \to X$ of f is always an immersion; if f is separated, then s is even a closed immersion (see [SP, 01KT] for the proofs of these facts).

Date: March 18, 2015.

(b) Deduce a strengthening of the first part of (a): for any finite extension L/k, the image of any k-morphism $\operatorname{Spec} L \to X$ is a closed point of the underlying topological space of X. In particular, a point $x \in X$ whose residue field $k(x)$ is a finite extension of k is a closed point.

Hint. The morphism $\operatorname{Spec} L \to \operatorname{Spec} k$ is finite, and hence universally closed (due to the going up theorem).

Side remark. The claim remains true with 'finite extension' replaced by 'algebraic extension'—the same proof works because integral morphisms are still universally closed (going up holds for integral morphisms; see [SP, 01WM] for a detailed argument).

(c) Prove a partial converse: if X is locally of finite type over k and $x \in X$ is a closed point, then the residue field $k(x)$ is a finite extension of k.

Hint. Apply Noether's normalization lemma to $\operatorname{Spec} k(x)$: an affine k-scheme of finite type admits a finite surjection to some affine space $\mathbb{A}^n_k$. Then use the fact that if a field L is module-finite over its subalgebra $R \subset L$, then R must also be a field. See [AK12, 15.4] or [AM69, 7.9] for a complete proof.

1.3. If k is finite and X is of finite type, prove that $X(k)$ is finite.

Hint. Work locally on X.

Side remark. Combine **1.2.** (a) and **1.3.** to see that rational points are *never* Zariski dense in a positive-dimensional variety over a finite field.

1.4. Use **1.2.** (a) to prove that every k-group scheme G is necessarily separated.

Hint. Show that the following diagram is Cartesian:

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times_k G \arrow[d, "g_1 g_2^{-1}"] \\ \operatorname{Spec} k \arrow[r, "e"] & G. \end{tikzcd}$$

Side remark. Separatedness of group schemes fails over non-field bases. To see this, choose a discrete valuation ring R and glue the two generic points of the constant group $\underline{\mathbb{Z}/2\mathbb{Z}}_R$ together to obtain a nonseparated R-group scheme. See [SP, 06E7] for further examples.

1.5. Suppose that X is of finite type over k and connected. If X has a rational point, prove that X is *geometrically connected*, i.e., that the base change $X_{\overline{k}}$ is still connected.

Hint. Reduce to showing connectedness of $X_{k'}$ for every finite extension k'/k. Then exploit the fact that $\operatorname{Spec} k' \to \operatorname{Spec} k$ is finite flat, and hence both universally closed and universally open, to conclude that the image of the base change $x_{k'}$ of a rational point $x\colon \operatorname{Spec} k \to X$ must meet every connected component of $X_{k'}$.

Side remark. See [EGA IV$_2$, 4.5.13] for a generalization: if $Y \to X$ is a morphism of arbitrary k-schemes with X connected and Y geometrically connected (and hence nonempty!), then X must also be geometrically connected. Note that this implies in particular that every connected group scheme over a field is geometrically connected.

1.6. Suppose that X is smooth over k and nonempty. Prove that there is a closed point $x \in X$ with $k(x)/k$ finite separable; in fact, prove that such x are Zariski dense.

Hint. Exploit the local structure of a smooth morphism: if $f\colon X \to S$ is smooth of relative dimension n, then there exists an open $U \subset X$ for which $f|_U$ factors as $U \xrightarrow{g} \mathbb{A}^n_S \to S$ with g being étale (see [SP, 054L] for this structure theorem).

Side remark. See [EGA IV$_4$, 17.16.3] for generalizations to arbitrary bases.

1.7. Let $\mathfrak{o}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$, and let $\mathbb{F}$ be the residue field of $\mathfrak{o}$. Prove Hensel's lemma: for a smooth $\mathfrak{o}$-scheme $\mathcal{X}$, the pullback map $\mathcal{X}(\mathfrak{o}) \to \mathcal{X}(\mathbb{F})$ is surjective, i.e., every $\mathbb{F}$-point of $\mathcal{X}$ may be lifted to an $\mathfrak{o}$-point.

Hint. As in **1.6.**, exploit the local structure of a smooth morphism to reduce to the case when $\mathcal{X}$ is étale over $\mathfrak{o}$. In the étale case argue that every local ring of $\mathcal{X}$ with residue field $\mathbb{F}$ must be $\mathfrak{o}$.

Side remark. See [EGA IV$_4$, 18.5.17] for a generalization: $\mathfrak{o}$ could be any Henselian local ring.

1.8. Suppose that k is a finite extension of $\mathbb{Q}_p$ and that X is of finite type, irreducible, and extends to a smooth $\mathfrak{o}$-scheme $\mathcal{X}$ of finite type whose special fiber is nonempty. Prove that the points of X valued in unramified extensions of k are Zariski dense.

Hint. If $Z \subset X$ is a closed subscheme with $\dim Z < \dim X$, then its schematic closure in $\mathcal{X}$ cannot cover the special fiber of $\mathcal{X}$ because "the relative dimension of a flat family is locally constant on the source" (see [BLR90, 2.4/4] for a precise statement). With this at hand, recall from [EGA IV$_2$, 2.8.5] that $\mathfrak{o}$-flat closed subschemes of $\mathcal{X}$ correspond to the schematic images of closed subschemes of X and use **1.6.** and **1.7.**

Side remark. This problem is a special case of [EGA IV$_3$, 14.3.11].

1.9. Suppose that X is of finite type over k and regular. Prove that every rational point $x\colon \operatorname{Spec} k \to X$ factors through a k-smooth open subscheme $U \subset X$.

Hint. Choose a regular system of parameters at the image of x to obtain a morphism $g\colon U \to \mathbb{A}^n_k$ for some affine open $U \subset X$ containing x. To check that g is étale at x, note that g is unramified at x by construction and flat at x because it induces an isomorphism $\widehat{\mathscr{O}}_{\mathbb{A}^n_k, g(x)} \cong \widehat{\mathscr{O}}_{U,x}$ (and because $R \to \widehat{R}$ is faithfully flat for every Noetherian local ring R).

Side remark. The claim still holds if k is an arbitrary regular local ring (e.g., a discrete valuation ring); for the proof, combine [BLR90, 3.1/2] and [EGA IV$_4$, 17.5.3].

2. Rational points on torsors

In this section, k is a field and G is a k-group scheme of finite type.

- A *right action* of G on a k-scheme X is a morphism $X \times_k G \to X$ that induces a right $G(S)$-action on $X(S)$ for every k-scheme S.
- A *trivial torsor* under G is a k-scheme X equipped with the right action of G such that X is isomorphic to G equipped with its right translation action (the isomorphism is required to respect the actions of G). A choice of such an isomorphism is a *trivialization* of X.
- A *torsor* under G (or a *G-torsor*) is a k-scheme X equipped with a right action of G such that for some finite extension k'/k the base change $X_{k'}$ is a trivial torsor under $G_{k'}$.

If G is commutative and smooth (smoothness is automatic if $\operatorname{char} k = 0$), then there is a bijection

$$\{\text{isomorphism classes of } G\text{-torsors } X\} \longleftrightarrow H^1(k, G(k^s)). \qquad (\bigstar)$$

2.1. For a G-torsor X, find a natural bijection

$$\{\text{trivializations of } X\} \longleftrightarrow \{\text{rational points } x \in X(k)\}.$$

Hint. Inspect the image of the identity section $e \in G(k)$ under a trivialization.

Side remark. This is the basic link between the study of torsors and that of rational points.

2.2. If G fits into a short exact sequence $1 \to G \to H \to Q \to 1$ of k-group schemes of finite type and $x \in Q(k)$, prove that the fiber $H_x := H \times_{Q,x} \operatorname{Spec} k$ is a G-torsor. When is it trivial?

Hint. The sequence induced on S-points is left exact (but typically not right exact). To find a criterion for triviality, inspect the map $H(k) \to Q(k)$.

Side remark. The assignment $x \mapsto [H_x]$ defines the connecting map of nonabelian cohomology: $Q(k) \to H^1(k, G)$, where $H^1(k, G)$ is *defined* to be the pointed set of isomorphism classes of G-torsors.

2.3. For $a, b \in k^\times$, prove that $G := \operatorname{Spec}\left(\frac{k[x,y]}{(x^2 - ay^2 - 1)}\right)$ has a structure of a k-group scheme and $X := \operatorname{Spec}\left(\frac{k[x,y]}{(x^2 - ay^2 - b)}\right)$ has a structure of its torsor.

Hint. Imitate multiplication in $\frac{k[t]}{(t^2 - a)}$ to define a group structure on

$$G(S) = \{(u, v) \in \Gamma(S, \mathcal{O}_S) \quad \text{such that} \quad u^2 - av^2 = 1\}$$

and its action on

$$X(S) = \{(u, v) \in \Gamma(S, \mathcal{O}_S) \quad \text{such that} \quad u^2 - av^2 = b\}.$$

Side remark. The name of G is the "norm-1 torus" of the quadratic k-algebra $\frac{k[t]}{(t^2-a)}$; to convince yourself that the name is apt, compute the norm of an element $x + yt \in \frac{k[t]}{(t^2-a)}$. This computation, interpreted in suitable generality, leads to a solution of the problem.

2.4. Prove that every $\mathbb{G}_m$-torsor over k is trivial. Prove the same for $\mathbb{G}_a$-torsors. Deduce that every G-torsor is trivial if G admits a filtration whose subquotients are either $\mathbb{G}_m$ or $\mathbb{G}_a$.

Hint. Use ($\bigstar$) and recall Hilbert's Theorem 90. For the last sentence, use the long exact nonabelian cohomology sequence.

Side remark. Some important cases when the required filtration exists are when G is a Borel subgroup of a split reductive group or when k is perfect and G is a smooth connected unipotent group (see [SGA 3_{II}, Exp. XVII, Cor. 4.1.3] for the latter case).

2.5. If G is smooth, prove that every G-torsor trivializes over a finite *separable* extension k'/k.

Hint. Torsors under smooth G inherit smoothness, so **1.6.** applies.

Side remark. More generally, if S is a base scheme and G is a *smooth* S-group scheme, then every G-torsor X trivializes over an étale cover $S' \to S$. The proof is the same modulo the technicality that in this generality a G-torsor fppf sheaf X may not be a scheme but is always an algebraic space (cf. [SP, 04SK]); namely, [SP, 0429] guarantees smoothness of X, so [EGA IV_4, 17.16.3 (ii)] applied to an étale cover of X by a scheme provides a desired S'.

2.6. Suppose that the field k is finite.

(a) If G is an abelian variety, prove that every G-torsor is trivial.

Hint. Use the sequences $0 \to G[n] \to G \xrightarrow{n} G \to 0$, which, due to perfectness of k, stay exact on k^s-points even when $\operatorname{char} k \mid n$. Also use the vanishing of the Herbrand quotient of a finite module.

Side remark. If k is finite, then any torsor under a *connected* k-group scheme of finite type is trivial: this is a theorem of Lang, who in [Lan56, Thm. 2] treated the key case when G is in addition smooth.

(b) If X is a proper smooth geometrically connected k-curve of genus 1, prove that $X(k) \neq \varnothing$.

Hint. Recall that the Jacobian $\mathrm{Pic}^0_{X/k}$ of X is an elliptic curve over k. Pass to $\overline{k}$ to prove that the canonical k-morphism $X \to \mathrm{Pic}^1_{X/k}$ is an isomorphism. Conclude that X is a $\mathrm{Pic}^0_{X/k}$-torsor and apply (a).

2.7. Suppose that k is a finite extension of $\mathbb{Q}_p$ and that A is a nonzero abelian variety over k.

(a) Prove that up to isomorphism there are only finitely many A-torsors X for which the associated class in $H^1(k, A)$ is killed by an integer that is prime to p.

Hint. Prove that even $\#H^1(k, A[n])$ is uniformly bounded as long as n is prime to p. For this, use the local Euler characteristic formula, local duality, and the fact that torsion subgroups of abelian varieties over k are finite.

Side remark. One could also apply Tate local duality for abelian varieties: if $A^\vee$ denotes the dual abelian variety, then $H^1(k, A)$ and $A^\vee(k)$ are Pontryagin dual, see [Mil06, I.3.4]. The claim then follows from $A^\vee(k)$ being pro-p up to finite index.

(b) Prove that up to isomorphism there are infinitely many A-torsors.

Hint. Use the same technique to obtain the growth rate of $\#H^1(k, A[p^n])$ as $n \to \infty$ and prove that $A(k)/p^n A(k)$ grows slower. For the latter, use the fact that $A(k)$ has a finite index subgroup isomorphic to $\mathbb{Z}_p^{[k:\mathbb{Q}_p]\cdot \dim A}$ (this fact is proved in [Mat55]).

2.8. Suppose that k is a finite extension of $\mathbb{Q}_p$. Let E be an elliptic curve over k, and let X be a torsor under E. The *period* of X is the order n of the corresponding class in $H^1(k, E)$. The *index* of X is the greatest common divisor of the degrees of closed points on X. Lichtenbaum has proved in [Lic68, Thm. 3] that period equals index under our assumptions. Assuming Lichtenbaum's result, prove that X even has a closed point of degree n.

Hint. Use Lichtenbaum's result to get a zero-cycle z of degree n on X. Apply Riemann–Roch to the line bundle $\mathscr{O}(z)$ to replace z by an *effective* zero cycle of degree n on X.

Side remark. The analogue of Lichtenbaum's "period = index" conclusion fails for torsors under higher dimensional abelian varieties over k.

3. Brauer groups

In this section, X is a scheme.

- An *Azumaya algebra* over X is a coherent $\mathscr{O}_X$-algebra $\mathscr{A}$ such that for some étale cover $\{f_i : X_i \to X\}$ there are $\mathscr{O}_{X_i}$-algebra isomorphisms $f_i^*\mathscr{A} \cong \mathrm{Mat}_{n_i \times n_i}(\mathscr{O}_{X_i})$ for some $n_i \in \mathbb{Z}_{>0}$.

- Azumaya algebras $\mathscr{A}$ and $\mathscr{A}'$ over X are *similar* if there is an $\mathscr{O}_X$-algebra isomorphism
$$\mathscr{A} \otimes_{\mathscr{O}_X} \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E}) \cong \mathscr{A}' \otimes_{\mathscr{O}_X} \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E}')$$
for some locally free coherent $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{E}'$ that are stalkwise nonzero.
- The set of similarity classes of Azumaya algebras over X forms an abelian group with $-\otimes_{\mathscr{O}_X}-$ as the group operation. This *Azumaya Brauer group* of X is denoted by $\mathrm{Br}_{\mathrm{Az}}\, X$.
- The *Brauer group* of X is $\mathrm{Br}\, X := H^2_{\text{ét}}(X, \mathbb{G}_m)$.
- If X is regular, Noetherian, and has an ample invertible sheaf (in the sense of [EGA II, 4.5.3]), then $\mathrm{Br}_{\mathrm{Az}}\, X = \mathrm{Br}\, X$ and both of these groups are torsion, see [Gro68a, Prop. 1.4] and [dJ, Thm. 1.1]. For example, this holds if X is a smooth quasi-projective scheme over a field.

Caution. Some authors use different definitions! For example, instead of meaning $\mathrm{Br}\, X$ the term *the Brauer group of X* may mean either $\mathrm{Br}_{\mathrm{Az}}\, X$ or $(\mathrm{Br}\, X)_{\mathrm{tors}}$.

3.1. Prove that similarity of Azumaya algebras over X is an equivalence relation.

Hint. Construct an $\mathscr{O}_X$-algebra isomorphism $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E}) \otimes_{\mathscr{O}_X} \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E}') \cong \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}')$.

3.2. Prove that an Azumaya algebra $\mathscr{A}$ over X is in particular a locally free $\mathscr{O}_X$-module whose rank at every point $x \in X$ is a square. Deduce that for every fixed $n \in \mathbb{Z}_{>0}$, the locus where the rank of $\mathscr{A}$ is n^2 is an open and closed subscheme of X.

Hint. Use étale descent: a quasi-coherent $\mathscr{O}_X$-module is locally free of finite rank if and only if it is so after pullback to an étale cover, see [SGA 1_{new}, Exp. VIII, Prop. 1.10].

3.3. Suppose that $X = \mathrm{Spec}\, k$ with k a field.

(a) Find a natural bijection
$$\{\mathrm{PGL}_n\text{-torsors over } X\}/\simeq \quad \longleftrightarrow \quad \{\text{Azumaya algebras over } X \text{ of rank } n^2\}/\simeq .$$
(On both sides, "$/\simeq$" means "up to isomorphism.")

Hint. For any field K, by the Skolem–Noether theorem, $\mathrm{PGL}_n(K)$ is the group of automorphisms of the matrix algebra $\mathrm{Mat}_{n\times n}(K)$. To conclude, exploit **2.5.**, effectivity of Galois descent for K-algebras, and the formalism of nonabelian Galois cohomology.

Side remark. The bijection continues to hold for any scheme X.

(b) Find the following maps:
$$H^1(k, \mathrm{PGL}_n) \hookrightarrow H^2(k, \mu_n) \xrightarrow{\sim} (\mathrm{Br}\, k)[n].$$
Combine them with (a) to prove that every Azumaya k-algebra gives an element of $\mathrm{Br}\, k$.

Hint. For the first map, use the long exact nonabelian cohomology sequence of the central extension $1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$ and prove that $H^1(k, \mathrm{SL}_n) = 1$. For the second map, use **2.4.**

Side remark. In fact, the first map is also bijective, see [Ser79, Ch. X, §5]. Loc. cit. also proves that $\mathrm{Br}_{\mathrm{Az}}\, k = \mathrm{Br}\, k$.

3.4. Prove that if a field k is a filtered union of its subfields k_i, then
$$\mathrm{Br}\, k = \varinjlim \mathrm{Br}\, k_i.$$
Hint. An Azumaya k-algebra has a description involving only finitely many elements of k.

Side remark. More generally, if $(X_i)_{i \in I}$ is a filtered inverse system of quasi-compact and quasi-separated schemes with affine transition morphisms, then $\mathrm{Br}(\varprojlim X_i) = \varinjlim \mathrm{Br}\, X_i$ (and $\varprojlim X_i$ is representable by a scheme), see [SGA 4_{II}, Exp. VII, Cor. 5.9]. For example, the assumptions are met if each X_i is affine: $X_i = \mathrm{Spec}\, R_i$; in this case, the conclusion is the equality $\mathrm{Br}(\varinjlim R_i) = \varinjlim \mathrm{Br}\, R_i$.

3.5. Prove that the Brauer group of a finite field is trivial.

Hint. A finite field has cohomological dimension 1 (but does not have strict cohomological dimension 1, so a small argument is still needed!).

3.6. If k is a finite extension of $\mathbb{Q}_p$ and $\mathfrak{o}$ is its ring of integers, prove that $\mathrm{Br}\, \mathfrak{o} = 0$.

Hint. To prove that every Azumaya $\mathfrak{o}$-algebra is a matrix algebra, use the side remark of **3.3.** (a) to reduce to proving triviality of PGL_n-torsors over $\mathfrak{o}$. Deduce this triviality from Hensel's lemma (see **1.7.**) and Lang's theorem (see the side remark of **2.6.** (a)).

Side remark. More generally, $\mathrm{Br}\, \mathfrak{o} = 0$ whenever $\mathfrak{o}$ is a Henselian local ring whose residue field has a trivial Brauer group, see [Gro68b, 11.7 2°)].

3.7. If K is a number field and K^{ab} is its maximal abelian extension, prove that $\mathrm{Br}\, K^{\mathrm{ab}} = 0$.

Hint. Combine **3.4.** with the reciprocity sequence

$$0 \to \mathrm{Br}\, K' \to \textstyle\bigoplus_v \mathrm{Br}\, K'_v \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

to reduce to proving that for every completion K'_v of a finite abelian extension K'/K and every integer $n \geqslant 1$ there are a finite abelian extension L/K containing K'/K and a place w of L lying above v such that $[L_w : K'_v]$ is divisible by n. To find a required L, adjoin roots of unity to K' or, alternatively, reduce further to the case when n is a power of a prime p, recall that no finite place of K' splits completely in the cyclotomic $\mathbb{Z}_p$-extension of K', and treat real v separately.

Side remark. The claim also holds for global fields of positive characteristic.

3.8. Let R be a discrete valuation ring and K its field of fractions.

(a) Prove that $H^1(R, \mathrm{PGL}_n) \to H^1(K, \mathrm{PGL}_n)$ has trivial kernel, i.e., that there is no nontrivial $(\mathrm{PGL}_n)_R$-torsor $\mathcal{T}$ whose base change to K is trivial.

Hint. Suppose that the class of $\mathcal{T}$ belongs to the kernel, let $B \subset (\mathrm{PGL}_n)_R$ be a Borel subgroup scheme, and consider the fppf quotient $\mathcal{Q} := \mathcal{T}/B$ (which is a scheme[1]). Use the valuative criterion and the properness of $\mathcal{Q}$ inherited from the projective R-scheme $(\mathrm{PGL}_n)_R/B$ to show that $\mathcal{Q}(R) \neq \varnothing$. Then choose a $q \in \mathcal{Q}(R)$, note that the q-fiber of $\mathcal{T} \to \mathcal{Q}$ is a B-torsor, and apply the analogue of **2.4.** over R to conclude that $\mathcal{T}(R) \neq \varnothing$.

Side remark. The same proof shows that $H^1(R, G) \to H^1(K, G)$ has trivial kernel for any split reductive R-group scheme G. It is an (open in general!) conjecture of Grothendieck and Serre that $H^1(R, G) \to H^1(K, G)$ is *injective* whenever R is a regular local ring with fraction field K and G is a reductive R-group scheme.

(b) Prove that $\mathrm{Br}\, R \to \mathrm{Br}\, K$ is injective.

[1] Representability of $\mathcal{Q}$ by a scheme follows, for instance, from the combination of [Ana73, 4.D] and [SGA $3_{\mathrm{III\ new}}$, Exp. XXII, Cor. 5.8.5]. However, for the argument at hand the representability of $\mathcal{Q}$ by an algebraic space would suffice; such representability follows already from [SP, 06PH].

Hint. Combine (a) with the side remark of **3.3.** (a) and use the nonabelian cohomology sequence of the central extensions

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

Side remark. In fact, $\mathrm{Br}\, X \to \mathrm{Br}\, k(X)$ is injective whenever X is an integral Noetherian regular scheme with function field $k(X)$, see [Gro68a, 1.8].

3.9. (a) Prove that $\mathrm{Br}\, \mathbb{Z} = 0$.

Hint. Combine the injectivity of $\mathrm{Br}\, \mathbb{Z} \to \mathrm{Br}\, \mathbb{Q}$ mentioned in the side remark of **3.8.** (b) with the exact sequence

$$0 \to \mathrm{Br}\, \mathbb{Q} \to \textstyle\bigoplus_{p \leqslant \infty} \mathrm{Br}\, \mathbb{Q}_p \xrightarrow{\sum_p \mathrm{inv}_p} \mathbb{Q}/\mathbb{Z} \to 0$$

and with the vanishing of $\mathrm{Br}\, \mathbb{Z}_p$ for $p < \infty$ supplied by **3.6.**

Side remark. If $\mathcal{O}_K$ is the ring of integers of a number field K, then $\mathrm{Br}\, \mathcal{O}_K \cong (\mathbb{Z}/2\mathbb{Z})^{r-1}$, where $r = \max(1, \#\{\text{real places of } K\})$, see [Gro68b, Prop. 2.4].

(b) For a proper smooth curve X over a finite field, prove that $\mathrm{Br}\, X = 0$.

Hint. Reduce to the case when X is connected and argue as in (a).

3.10. Recall Tsen's theorem: if $\overline{k}(X)$ is the function field of an integral curve X over an algebraically closed field $\overline{k}$, then $\mathrm{Br}\, \overline{k}(X) = 0$.

Prove that if k is a perfect field, then $\mathrm{Br}\, k \to \mathrm{Br}\, \mathbb{P}^1_k$ is an isomorphism.

Hint. Use the spectral sequence $H^i(\mathrm{Gal}(k^s/k), H^j(\mathbb{P}^1_{k^s}, \mathbb{G}_m)) \Rightarrow H^{i+j}(\mathbb{P}^1_k, \mathbb{G}_m)$ and write out its exact sequence of low degree terms:

$$0 \to \mathrm{Pic}\, \mathbb{P}^1_k \to (\mathrm{Pic}\, \mathbb{P}^1_{k^s})^{\mathrm{Gal}(k^s/k)} \to \mathrm{Br}\, k \to \mathrm{Ker}\left(\mathrm{Br}\, \mathbb{P}^1_k \to \mathrm{Br}\, \mathbb{P}^1_{k^s}\right) \to H^1(\mathrm{Gal}(k^s/k), \mathrm{Pic}\, \mathbb{P}^1_{k^s}).$$

Then combine Tsen's theorem with the side remark of **3.8.** (b) to get $\mathrm{Br}\, \mathbb{P}^1_{k^s} = 0$. Conclude by noting that Galois acts trivially on $\mathrm{Pic}\, \mathbb{P}^1_{k^s} \cong \mathbb{Z}$ because the degree of a divisor is preserved under any automorphism of $\mathbb{P}^1_{k^s}$.

Side remark. By [Gro68b, 5.8], $\mathrm{Br}\, X = 0$ for every proper curve X over a separably closed field, so the same proof gives the bijectivity $\mathrm{Br}\, k \to \mathrm{Br}\, \mathbb{P}^1_k$ even when k is imperfect.

3.11. Suppose that X is equipped with a structure map $f\colon X \to \mathrm{Spec}\, k$ for some field k.

(a) If $X(k) \neq \varnothing$, prove that $\mathrm{Br}\, k \xrightarrow{\mathrm{Br}(f)} \mathrm{Br}\, X$ is injective.

Hint. Inspect the Brauer group morphisms induced by f and by its section $\mathrm{Spec}\, k \to X$.

Side remark. The question has little to do with Brauer groups and is a special case of the following generality: if $f\colon X \to Y$ is a morphism of schemes, s is its section, and $F\colon \underline{\text{Schemes}}^{\mathrm{op}} \to \underline{\text{Sets}}$ is a functor, then $F(f)$ is injective because $F(s) \circ F(f) = \mathrm{id}_{F(Y)}$.

(b) If k is a number field and $\prod_v X(k_v) \neq \varnothing$, prove that $\mathrm{Br}\, k \xrightarrow{\mathrm{Br}(f)} \mathrm{Br}\, X$ is injective even when $X(k) = \varnothing$.

Hint. If $(p_v)_v \in \prod_v X(k_v)$, then the map $(\mathrm{Br}(p_v) \circ \mathrm{Br}(f))_v \colon \mathrm{Br}\, k \to \prod_v \mathrm{Br}\, k_v$ is nothing else than the usual injection $\mathrm{Br}\, k \to \bigoplus_v \mathrm{Br}\, k_v$.

4. The Hasse principle

In this section, K is a number field and X is a K-scheme of finite type. A rational point $x \in X(K)$ gives rise to local points $x_v \in X(K_v)$, one for each place v of K. In particular,

$$X(K) \neq \varnothing \qquad \Longrightarrow \qquad \textstyle\prod_v X(K_v) \neq \varnothing.$$

One may wonder whether the existence of local points forces the existence of a global point:

$$\textstyle\prod_v X(K_v) \neq \varnothing \qquad \overset{?}{\Longrightarrow} \qquad X(K) \neq \varnothing.$$

If it does, then X *satisfies the Hasse principle*. If it does not, then X *violates the Hasse principle*.

4.1. (a) Prove that $\mathrm{Proj}\left(\frac{\mathbb{Q}[x,y]}{(x^2-ay^2)}\right)$ satisfies the Hasse principle for every $a \in \mathbb{Q}^\times$.

Hint. If a is positive and has an even p-adic valuation for every finite prime p, then a is a square in $\mathbb{Q}^\times$.

Side remark. The question is a simple special case of the theorem of Hasse–Minkowski: if $f \in \mathbb{Q}[x_1, \ldots, x_n]$ is a homogeneous quadratic polynomial, then $\mathrm{Proj}\left(\frac{\mathbb{Q}[x_1,\ldots,x_n]}{(f)}\right)$ satisfies the Hasse principle, see [Ser73, Ch. IV, §3.2, Thm. 8] for a proof.

(b) Prove that $\mathrm{Proj}\left(\frac{K[x,y]}{(x^2-ay^2)}\right)$ satisfies the Hasse principle for every $a \in K^\times$.

Hint. Reduce to proving the injectivity of $H^1(K, \mu_2) \to \prod_v H^1(K_v, \mu_2)$, and then to that of $H^1(K, \mathbb{Z}/2\mathbb{Z}) \to \prod_v H^1(K_v, \mathbb{Z}/2\mathbb{Z})$. To conclude, note that no quadratic extension of K is split everywhere. If passage to $\mathbb{Z}/2\mathbb{Z}$ is undesirable, then use "Kummer theory" to carry out the same argument.

Side remark. In fact, the full Hasse–Minkowski theorem also holds over an arbitrary number field.

(c) Prove that $\mathrm{Proj}\left(\frac{\mathbb{Q}(\sqrt{7})[x,y]}{(x^8-16y^8)}\right)$ violates the Hasse principle (over $\mathbb{Q}(\sqrt{7})$).

Hint. Use the factorization $T^8 - 2^4 = (T^2-2)(T^2+2)((T+1)^2+1)((T-1)^2+1)$ to prove the absence of rational points over $\mathbb{Q}(\sqrt{7})$. Combine the factorization with Hensel's lemma to prove that 16 is an 8^{th} power already in $\mathbb{Q}_p$ if p is odd. To also treat the place above 2, note that 2 ramifies in $\mathbb{Q}(\sqrt{7})$ and that the logarithm isomorphism $(1 + 2^n\mathbb{Z}_2, \times) \cong (2^n\mathbb{Z}_2, +)$ for $n \geqslant 2$ shows that $\mathbb{Q}_2(\sqrt{7}) = \mathbb{Q}_2(\sqrt{-1})$. To handle the infinite places, note that 16 is an 8^{th} power in $\mathbb{R}$.

Side remark. For the necessary and sufficient conditions on K and $n \in \mathbb{Z}_{>0}$ so that $\mathrm{Proj}\left(\frac{K[x,y]}{(x^n-ay^n)}\right)$ satisfies the Hasse principle for every $a \in K^\times$, see [NSW08, 9.1.11 (i)].

4.2. Prove that $\mathbb{Z}/n\mathbb{Z}$-torsors over K satisfy the Hasse principle.

Hint. Use the correspondence between $\mathbb{Z}/n\mathbb{Z}$-torsors over K and elements of $H^1(K, \mathbb{Z}/n\mathbb{Z})$ to reduce to proving the injectivity of $H^1(K, \mathbb{Z}/n\mathbb{Z}) \to \prod_v H^1(K_v, \mathbb{Z}/n\mathbb{Z})$. Then exploit the identification $H^1(K, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(K^s/K), \mathbb{Z}/n\mathbb{Z})$ and note that no nontrivial extension of number fields is completely decomposed at every place.

4.3. A *Severi–Brauer variety* over K is a K-scheme S for which there is a K^s-isomorphism $S_{K^s} \cong \mathbb{P}^n_{K^s}$ for some $n \geqslant 0$.

(a) Find a natural bijection

$$\{(\mathrm{PGL}_{n+1})_K\text{-torsors}\}/\simeq \quad \longleftrightarrow \quad \{n\text{-dimensional Severi–Brauer varieties over } K\}/\simeq .$$

(On both sides, "$/\simeq$" means "up to isomorphism.")

Hint. Recall from [MFK94, Ch. 0, §5, b) on p. 20] that the automorphism functor of $\mathbb{P}^n$ is representable by PGL_{n+1}. Then combine the general principle of "twists" together with the effectivity of Galois descent for quasi-projective varieties to deduce the bijection

$$H^1(K^s/K, \mathrm{PGL}_{n+1}(K^s)) \longleftrightarrow \{n\text{-dimensional Severi–Brauer varieties over } K\}/\simeq .$$

Apply **2.5.** to conclude.

Side remark. The assumption that the field K be a number field is irrelevant for (a).

(b) Prove that an n-dimensional Severi–Brauer variety S is isomorphic to $\mathbb{P}^n_K$ if and only if $S(K) \neq \varnothing$.

Hint. The 'only if' direction is trivial. For the 'if' direction, fix an $s \in S(K)$ and note that (S, s) is a twist of $(\mathbb{P}^n_K, (1:0:\ldots:0))$. To conclude, either find a short exact sequence

$$1 \to \mathbb{G}_a^n \to \mathrm{Aut}(\mathbb{P}^n, (1:0:\ldots:0)) \to \mathrm{GL}_n \to 1$$

and use the resulting nonabelian cohomology sequence over K, or note that the passage to the dual projective space identifies twists of $(\mathbb{P}^n_K, (1:0:\ldots:0))$ with those of $(\mathbb{P}^n_K, \text{hyperplane } \{x_0 = 0\})$ and that the latter are all trivial because the morphism determined by the very ample line bundle corresponding to the twist of the divisor $\{x_0 = 0\}$ is an isomorphism, as may be checked after passage to the separable closure.

Side remark. The assumption that the field K be a number field is irrelevant for (b).

(c) Prove that Severi–Brauer varieties satisfy the Hasse principle.

Hint. Combine (a) and (b) to reduce to proving the injectivity of

$$H^1(K, \mathrm{PGL}_{n+1}) \to \textstyle\prod_v H^1(K_v, \mathrm{PGL}_{n+1})$$

for every $n \geqslant 0$. Then use **3.3.** (b) and the injectivity of

$$\mathrm{Br}\, K \to \textstyle\prod_v \mathrm{Br}\, K_v.$$

4.4. The goal of this question is to work out an example of Lind [Lin40] and Reichardt [Rei42]:

$$X := \mathrm{Proj}\left(\tfrac{\mathbb{Q}[x,y,z]}{(x^4-17y^4-2z^2)}\right), \quad \text{where the grading has } x \text{ and } y \text{ in degree 1 and } z \text{ in degree 2,}$$

violates the Hasse principle; in other words, $x^4 - 17y^4 = 2z^2$ has a nonzero solution in $\mathbb{Q}_p$ for every $p \leqslant \infty$, but does not have any nonzero solution in $\mathbb{Q}$.

(a) Prove that X is a smooth geometrically connected curve of genus 1.

Hint. Compute on affine coordinate patches. The genus of a proper smooth plane curve of degree d is $\frac{(d-1)(d-2)}{2}$.

(b) Prove that $x^4 - 17y^4 = 2z^2$ has a nonzero solution in $\mathbb{F}_p$ for every prime $p \notin \{2, 17\}$.

Hint. Apply **2.6.** (b).

(c) Prove that $X(\mathbb{Q}_p) \neq \varnothing$ for every prime $p \notin \{2, 17\}$.

Hint. Apply **1.7.** and (b) to a suitable a smooth model of X over $\mathrm{Spec}\, \mathbb{Z}[\frac{1}{2\cdot 17}]$.

(d) Use the 2-adic logarithm to prove that $X(\mathbb{Q}_2) \neq \varnothing$. Prove that $X(\mathbb{Q}_{17}) \neq \varnothing$ by exploiting the fact that 17 splits in $\mathbb{Q}(\sqrt{2})$. Observe that $X(\mathbb{R}) \neq \varnothing$.

Hint. The logarithm induces an isomorphism $(1 + 2^n\mathbb{Z}_2, \times) \cong (2^n\mathbb{Z}_2, +)$ for every $n \geqslant 2$.

(e) Prove that $X(\mathbb{Q}) = \varnothing$.

Hint. Start with integers (a, b, c) satisfying $a^4 - 17b^4 = 2c^2$ and prove that every prime dividing c must be a square in $\mathbb{F}_{17}$. Conclude that c must be a square in $\mathbb{F}_{17}$. Check that 2 is not a fourth power in $\mathbb{F}_{17}$.

5. The Brauer–Manin obstruction

In this section, K is a number field, $\mathbb{A}_K$ is its ring of adeles, and X is a separated K-scheme of finite type, so that $X(\mathbb{A}_K) \subset \prod_v X(K_v)$ (see **5.1.** (d)).

- The *Brauer–Manin set* of X is
$$X(\mathbb{A}_K)^{\mathrm{Br}} := \{(x_v)_v \in X(\mathbb{A}_K) \quad \text{for which} \quad \textstyle\sum_v \mathrm{inv}_v(x_v^*(B)) = 0 \quad \text{for every} \quad B \in \mathrm{Br}\, X\}.$$
The Brauer–Manin set fits into inclusions
$$X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K) \subset \textstyle\prod_v X(K_v).$$
- If $\prod_v X(K_v) \neq \varnothing$ but $X(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$, so that necessarily $X(K) = \varnothing$, then X has a *Brauer–Manin obstruction* to the local-global principle. In this case, the absence of rational points of X is explained by the emptiness of the Brauer–Manin set.

The aim of the first few questions is to solidify the understanding of these ideas.

5.1. (a) Prove that there is a nonempty open $U \subset \mathrm{Spec}\, \mathcal{O}_K$ and a separated U-scheme $\mathcal{X}$ of finite type for which one may fix an isomorphism $\mathcal{X}_K \cong X$. Prove uniqueness of $\mathcal{X}$ up to shrinking U: if $\mathcal{X} \to U$ and $\mathcal{X}' \to U'$ both extend X, then the composite isomorphism $\mathcal{X}_K \cong X \cong \mathcal{X}'_K$ extends to an isomorphism $\mathcal{X}_{U''} \cong \mathcal{X}'_{U''}$ for some nonempty open $U'' \subset U \cap U'$.

Hint. Describe X as a separated K-scheme of finite type by finitely many equations and use these equations over a small enough U to "reglue" to a desired $\mathcal{X}$. To extend an (iso)morphism, work locally on the source and use the functor of points definition of being locally of finite presentation.

Side remark. The question is an instance of general "limit formalism." See [EGA IV$_3$, §8, esp. 8.10.5] for many useful results of this sort.

(b) With $\mathcal{X}$ as in (a), prove that the restricted product $\prod'_v (X(K_v), \mathcal{X}(\mathcal{O}_v))$ is an independent of $\mathcal{X}$ subset of $\prod_v X(K_v)$.

Hint. Use the valuative criterion of separatedness for the subset claim. Use the uniqueness of $\mathcal{X}$ up to shrinking U for the independence claim.

(c) For an $\mathcal{X}$ as in (a) and each finite set of places Σ containing the places that do not correspond to a closed point of U, prove that pullback maps induce an isomorphism
$$\mathcal{X}(\textstyle\prod_{v \in \Sigma} K_v \times \prod_{v \notin \Sigma} \mathcal{O}_v) \xrightarrow{\sim} \prod_{v \in \Sigma} \mathcal{X}(K_v) \times \prod_{v \notin \Sigma} \mathcal{X}(\mathcal{O}_v).$$

Hint. To treat the case of an affine $\mathcal{X}$, translate into a statement about rings. To then prove the surjectivity, choose an element $(p_v)_v$ of the target, cover $\mathcal{X}$ by finitely many affines $\mathcal{X}_i$, and subdivide the v's into finitely many parts $\mathcal{P}_j$ in such a way that all the p_v's with $v \in \mathcal{P}_j$ factor through the same $\mathcal{X}_i$. To prove the injectivity, use that the diagonal of $\mathcal{X}$ is a closed immersion and that the only ideal of $\prod_{v\in\Sigma} K_v \times \prod_{v\notin\Sigma} \mathcal{O}_v$ with vanishing projections is the zero ideal.

Side remark. More generally, as proved in [Bha14, Thm. 1.3], $X(\prod_i R_i) \xrightarrow{\sim} \prod_i X(R_i)$ for every set of rings $\{R_i\}_i$ and every quasi-compact and quasi-separated scheme X.

(d) Using (b) to interpret the restricted product, prove that $X(\mathbb{A}_K) = \prod'_v X(K_v)$.

Hint. Combine the functor of points definition of being locally of finite presentation with (c).

(e) If X is proper, prove that $X(\mathbb{A}_K) = \prod_v X(K_v)$.

Hint. To ensure that $\mathcal{X} \to U$ in (a) is proper, apply Chow's lemma [SP, 0200] to $X \to \operatorname{Spec} K$ and shrink U (alternatively, apply [EGA IV$_4$, 8.10.5 (xii)]). Then conclude by combining the valuative criterion of properness with (d).

5.2. For an $(x_v) \in X(\mathbb{A}_K) \subset \prod_v X(K_v)$ and a $B \in \operatorname{Br} X$, prove that $\operatorname{inv}_v(x_v^*(B)) = 0$ for all but finitely many v.

Hint. Use the side remark of **3.4.** to find an $\mathcal{X}$ as in **5.1.** (a) for which B extends to a $\mathcal{B} \in \operatorname{Br} \mathcal{X}$. Then apply **3.6.**

Side remark. In particular, the infinite sum in the definition of $X(\mathbb{A}_K)^{\mathrm{Br}}$ makes sense.

5.3. Prove that $X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}}$.

Hint. Use the reciprocity sequence

$$0 \to \operatorname{Br} K \to \bigoplus_v \operatorname{Br} K_v \xrightarrow{\sum_v \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

to show that for every $B \in \operatorname{Br} X$ and every $x \in X(K)$ with pullbacks $x_v \in X(K_v)$ one has $\sum_v \operatorname{inv}_v(x_v^*(B)) = 0$.

5.4. Suppose that $f\colon X \to X'$ is a morphism of separated K-schemes of finite type.

(a) Prove that $f(\mathbb{A}_K)\colon X(\mathbb{A}_K) \to X'(\mathbb{A}_K)$ maps $X(\mathbb{A}_K)^{\mathrm{Br}}$ into $X'(\mathbb{A}_K)^{\mathrm{Br}}$.

Hint. If $(x_v)_v \in X(\mathbb{A}_K)^{\mathrm{Br}}$ and $B' \in \operatorname{Br} X'$, then $f^*B' \in \operatorname{Br} X$ so

$$\textstyle\sum_v \operatorname{inv}_v((f \circ x_v)^*B') = \sum_v \operatorname{inv}_v(x_v^*(f^*B')) = 0.$$

(b) Assume that $\prod_v X(K_v) \neq \varnothing$, so that necessarily $\prod_v X'(K_v) \neq \varnothing$. If X' has a Brauer–Manin obstruction to the local–global principle, prove that so does X.

Hint. If $X'(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$, then the conclusion of (a) forces $X(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$.

5.5. Recall that each $X(K_v)$ has a "v-adic topology" inherited from K_v: if X has a closed immersion into some $\mathbb{A}^n$, then the v-adic topology on $X(K_v)$ is just the subspace topology of the v-adic topology on $\mathbb{A}^n(K_v) = K_v^n$; in general, the v-adic topology on $X(K_v)$ is described by also requiring that $U(K_v) \subset X(K_v)$ be open for each affine open $U \subset X$. The identification $X(\mathbb{A}_K) = \prod' X(K_v)$ of **5.1.** (d) then endows $X(\mathbb{A}_K)$ with the restricted product topology.

For (a), (b), and (c) below, suppose that the separated finite type K-scheme X is *regular*.

(a) For a Brauer class $B \in \operatorname{Br} X$ and a place v, prove that the map

$$X(K_v) \to \mathbb{Q}/\mathbb{Z}, \qquad\qquad x_v \mapsto \operatorname{inv}_v(x_v^* B)$$

is locally constant for the v-adic topology on $X(K_v)$.

Hint. Reduce to the case of an affine connected X. Then use the equality $\operatorname{Br} X = \operatorname{Br}_{\mathrm{Az}} X$ and the side remark of **3.3.** (a) to represent B by a $(\mathrm{PGL}_n)_X$-torsor $T \to X$, and hence reduce to proving local constancy on $X(K_v)$ of the fibral isomorphism class of T. For this local constancy, fix a $(\mathrm{PGL}_n)_{K_v}$-torsor $T_0 \to \operatorname{Spec} K_v$ and consider the isomorphism functor $\operatorname{Isom}_{\mathrm{PGL}_n}(T, (T_0)_X)$ defined by

$$\underline{X\text{-schemes}} \quad \ni \quad S \mapsto \{(\mathrm{PGL}_n)_S\text{-torsor isomorphisms } T_S \xrightarrow{\sim} (T_0)_S\} \quad \in \quad \underline{\text{Sets}}$$

and prove that it is an fppf (or even étale) torsor under the inner form $\operatorname{Aut}_{\mathrm{PGL}_n}(T)$ of $(\mathrm{PGL}_n)_X$ (to make this step easier, first solve the question **6.1.**). Then deduce that $\operatorname{Isom}_{\mathrm{PGL}_n}(T, (T_0)_X)$ is representable by a smooth X-scheme, that the locus where the fiber of T is isomorphic to T_0 is the image of $\operatorname{Isom}_{\mathrm{PGL}_n}(T, (T_0)_X)(K_v) \to X(K_v)$, and that this image is open due to the smoothness of $\operatorname{Isom}_{\mathrm{PGL}_n}(T, (T_0)_X) \to X$.

Side remark. Neither regularity nor separatedness of the finite type K-scheme X is needed for the claim to hold: without these assumptions, the local constancy on $X(K_v)$ of the fibral isomorphism class of B may be viewed as a special case of [Česk14, 3.4 (b)].

(b) Prove that the evaluation of a Brauer class $B \in \operatorname{Br} X$ defines a continuous map

$$X(\mathbb{A}_K) \to \mathbb{Q}/\mathbb{Z}, \qquad\qquad (x_v) \mapsto \textstyle\sum_v \operatorname{inv}_v(x_v^* B),$$

where $\mathbb{Q}/\mathbb{Z}$ is endowed with the discrete topology.

Hint. Combine (a) with the techniques used in the hint of **5.2.**

(c) Prove that $X(\mathbb{A}_K)^{\mathrm{Br}}$ is closed in $X(\mathbb{A}_K)$.

Hint. Use (b) and the fact that an arbitrary intersection of closed sets is closed.

Side remark. Due to the side remark of (a), the regularity of the separated finite type K-scheme X is not needed for the claims of (b) and (c) to hold.

5.6. The goal of this question is to work out an example of Birch and Swinnerton-Dyer [BSD75]:

$$X := \operatorname{Proj}\left(\frac{\mathbb{Q}[u, v, x, y, z]}{(uv - x^2 + 5y^2, (u+v)(u+2v) - x^2 + 5z^2)}\right)$$

has a Brauer–Manin obstruction to the local-global principle. In other words, $X(\mathbb{Q}_p) \neq \varnothing$ for every $p \leqslant \infty$ but $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \varnothing$, so that $X(\mathbb{Q}) = \varnothing$, too.

(a) Prove that X is a smooth, projective, geometrically connected surface over $\mathbb{Q}$.

Hint. To prove that X is smooth and of dimension 2 at every point, use the Jacobi criterion from [BLR90, 2.2/7] to reduce to checking that the differentials of the two defining equations are pointwise linearly independent in $\Omega^1_{\mathbb{P}^4_{\mathbb{Q}}/\mathbb{Q}}$. The geometric connectedness then follows because any two proper surfaces in $\mathbb{P}^4_{\mathbb{Q}}$ intersect, cf. [Har77, Thm. I.7.2].

Side remark. Another way to argue geometric connectedness is to appeal to the general fact that positive dimensional complete intersections in projective space are always geometrically connected, see [Liu02, Exercise 5.3.3 (c)].

(b) A smooth, projective, geometrically connected surface Y over a field k is a *del Pezzo surface* if the line bundle $\omega_{Y/k}^{-1}$ is ample, where $\omega_{Y/k} = \bigwedge^2 \Omega^1_{Y/k}$. Prove that X is a del Pezzo surface over $\mathbb{Q}$.

Hint. Prove that $\omega_{X/\mathbb{Q}} \cong \mathscr{O}_X(-1)$ by using the formula for the canonical sheaf of a complete intersection, cf. [Liu02, Exercise 6.4.11 (b)]. Conclude that $\omega_{X/\mathbb{Q}}^{-1}$ is ample.

(c) The *degree* of a del Pezzo surface $Y \to \operatorname{Spec} k$ is the self-intersection number of the canonical line bundle $\omega_{Y/k}$. Prove that the degree of X is 4.

Hint. Using that the self-intersection number of $\omega_{X/\mathbb{Q}}$ equals that of $\omega_{X/\mathbb{Q}}^{-1} \cong \mathscr{O}_X(1)$, prove that it also equals the degree of X as a surface in $\mathbb{P}^4_{\mathbb{Q}}$, cf. [Har77, Exercise V.1.2]. To compute this degree, recall some intersection theory in the form of [Har77, Thm. I.7.7] and note that X is a smooth geometrically connected intersection in $\mathbb{P}^4_{\mathbb{Q}}$ of two hypersurfaces of degree 2.

(d) Verify that the points

$$(0:0:\sqrt{5}:1:1), \qquad (1:1:1:0:\sqrt{-1}),$$
$$(1:0:0:0:\tfrac{1}{\sqrt{-5}}), \qquad (-5:1:0:1:\sqrt{\tfrac{-12}{5}})$$

lie on X. Use these points to prove that $X(\mathbb{Q}_p) \neq \varnothing$ for every $p \leqslant \infty$.

Hint. Prove that every $p \neq 2$ splits in at least one of the quadratic number fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-5})$. Use the hint of **4.4.** (d) to prove that 2 splits $\mathbb{Q}(\sqrt{-15})$.

(e) Let F be the function field of X. Use the cup product

$$(\ ,\ ) \colon H^1(F, \mathbb{Z}/2\mathbb{Z}) \times H^1(F, \mu_2) \to H^2(F, \mu_2) = (\operatorname{Br} F)[2]$$

to make sense of the following 2-torsion classes in $\operatorname{Br} F$:

$$(5, \tfrac{u+v}{u}), \qquad (5, \tfrac{u+v}{v}), \qquad (5, \tfrac{u+2v}{u}), \qquad (5, \tfrac{u+2v}{v}).$$

Hint. Interpret the first entry 5 as the element of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ corresponding to the quadratic extension $F(\sqrt{5})/F$. Interpret the second entry as the image in $F^\times/F^{\times 2} = H^1(F, \mu_2)$ of the indicated nonzero rational function on X.

(f) For a suitable finite extension F'/F, use the projection formula

$$(\,\cdot\,, \operatorname{Norm}_{F'/F}(-)) = \operatorname{Norm}_{F'/F}((\operatorname{Res}_{F'/F}(\cdot), -))$$

to prove that $(5, \frac{u+v}{u}) = (5, \frac{u+v}{v}) = (5, \frac{u+2v}{u}) = (5, \frac{u+2v}{v})$ in $\operatorname{Br} F$.

Hint. Use the bilinearity of $(\ ,\ )$, take $F' = F(\sqrt{5})$, and note that

$$\frac{(u+v)^2}{uv} = \operatorname{Norm}_{F(\sqrt{5})/F}\left(\frac{u+v}{x+\sqrt{5}y}\right), \qquad \frac{(u+v)(u+2v)}{u^2} = \operatorname{Norm}_{F(\sqrt{5})/F}\left(\frac{x+\sqrt{5}z}{u}\right),$$
$$\frac{(u+v)(u+2v)}{uv} = \operatorname{Norm}_{F(\sqrt{5})/F}\left(\frac{x+\sqrt{5}z}{x+\sqrt{5}y}\right), \qquad \frac{(u+v)(u+2v)}{v^2} = \operatorname{Norm}_{F(\sqrt{5})/F}\left(\frac{x+\sqrt{5}z}{v}\right),$$
$$\frac{(u+2v)^2}{uv} = \operatorname{Norm}_{F(\sqrt{5})/F}\left(\frac{u+2v}{x+\sqrt{5}y}\right).$$

(g) Admit the existence and exactness of the residue sequences

$$0 \to \operatorname{Br} U \to \operatorname{Br} F \to \textstyle\bigoplus_{u \in U^{(1)}} H^1(k(u), \mathbb{Q}/\mathbb{Z}),$$

in which $U \subset X$ is a nonempty open, the direct sum is indexed by height 1 points $u \in U$, the residue field of u is denoted by $k(u)$, and the maps $\operatorname{Br} F \to H^1(k(u), \mathbb{Q}/\mathbb{Z})$ do not depend on the choice of an open U containing u. Use these sequences to prove that the element $b \in \operatorname{Br} F$ exhibited in (f) extends to a $B \in \operatorname{Br} X$.

Hint. Combine the residue sequences with the different presentations of b given in (f) to prove that b extends to a $\widetilde{b} \in \operatorname{Br}(X \backslash Z)$, where $Z \subset X$ is the locus where $u(u+v)$, $v(u+v)$, $u(u+2v)$, and $v(u+2v)$ all vanish. Conclude by proving that $X \backslash Z$ covers the height 1 points of X, i.e., that Z is a finite set of points.

(h) If $p \leqslant \infty$ is a prime different from 5 and $x_p \in X(\mathbb{Q}_p)$, prove that $\operatorname{inv}_p(x_p^* B) = 0$.

Hint. Start by proving that $X(\mathbb{Q}_p)$ endowed with its p-adic topology has no isolated points: combine the local structure of smooth morphisms with the fact that étale morphisms of finite type $\mathbb{Q}_p$-schemes induce open maps on $\mathbb{Q}_p$-points. Then use **5.5.** (a) to reduce to considering those x_p which, in the notation of the hint of (g), factor through $X \backslash Z$. To then settle the cases when $5 \in \mathbb{Q}_p^{\times 2}$, locally on $X \backslash Z$ represent B by one of the cup products of (e). To treat the remaining p, write $x_p = (\widetilde{u} : \widetilde{v} : \widetilde{x} : \widetilde{y} : \widetilde{z})$ with all entries in $\mathbb{Z}_p$ but not all in $p\mathbb{Z}_p$. Then exploit the unramifiedness of $\mathbb{Q}_p(\sqrt{5})/\mathbb{Q}_p$ to reduce to proving that the valuation of $\frac{\widetilde{u}+\widetilde{v}}{\widetilde{u}}$ is even if $\widetilde{u}(\widetilde{u}+\widetilde{v}) \neq 0$, or that the valuation of $\frac{\widetilde{u}+2\widetilde{v}}{\widetilde{u}}$ is even if $\widetilde{u}(\widetilde{u}+2\widetilde{v}) \neq 0$, etc. Proceed to prove that $\widetilde{x}^2 \equiv 5\widetilde{y}^2 \bmod p$ would force $\widetilde{x}, \widetilde{y} \in p\mathbb{Z}_p$, and conclude that either $p \nmid \widetilde{u}$ or $p \nmid \widetilde{v}$, and that also either $p \nmid \widetilde{u}+\widetilde{v}$ or $p \nmid \widetilde{u}+2\widetilde{v}$.

(i) If $x_5 \in X(\mathbb{Q}_5)$, prove that $\operatorname{inv}_5(x_5^* B) = \frac{1}{2}$.

Hint. As in the hint of (h), reduce to considering those x_5 that factor through $X \backslash Z$ and write $x_5 = (\widetilde{u} : \widetilde{v} : \widetilde{x} : \widetilde{y} : \widetilde{z})$ with the same normalization of the entries. Observe that if $5 \mid \widetilde{u}$, or $5 \mid \widetilde{v}$, or $5 \mid \widetilde{u}+\widetilde{v}$, or $5 \mid \widetilde{u}+2\widetilde{v}$, then $5 \mid \widetilde{x}$, so that $5 \mid \widetilde{u}\widetilde{v}$ and $5 \mid (\widetilde{u}+\widetilde{v})(\widetilde{u}+2\widetilde{v})$, to the effect that $5 \mid \widetilde{u}$ *and* $5 \mid \widetilde{v}$, which contradicts the normalization. Deduce that $\frac{\widetilde{u}}{\widetilde{v}} \equiv 1 \text{ or } 2 \mod 5$, and hence that $1 + \frac{\widetilde{u}}{\widetilde{v}} \neq a^2 - 5b^2$ with $a, b \in \mathbb{Z}_5$. Conclude that $\frac{\widetilde{u}+\widetilde{v}}{\widetilde{v}}$ is not a norm of an element of $\mathbb{Q}_5(\sqrt{5})$, and hence that $\operatorname{inv}_5(x_5^* B) = \frac{1}{2}$.

(j) Prove that $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \varnothing$.

Hint. Combine (h) and (i) to prove that the element $B \in \operatorname{Br} X$ constructed in (g) satisfies $\sum_{p \leqslant \infty} \operatorname{inv}_p(x_p^* B) = \frac{1}{2}$ for every $(x_p)_p \in X(\mathbb{A}_{\mathbb{Q}})$ and hence forces $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \varnothing$.

6. The étale Brauer–Manin obstruction

As in §5, we assume that K is a number field and X is a separated K-scheme of finite type.

- For an X-group scheme $\mathcal{G}$ and $\mathcal{G}$-torsors $Y \to X$ and $Y' \to X$, the *isomorphism functor* $\operatorname{Isom}_{\mathcal{G}}(Y, Y')$ is the fppf sheaf

  $$S \mapsto \{\mathcal{G}_S\text{-torsor isomorphisms } Y_S \xrightarrow{\sim} Y'_S\}, \qquad \text{where } S \text{ is a variable } X\text{-scheme.}$$

  If $\mathcal{G} \to X$ is affine, then $\operatorname{Isom}_{\mathcal{G}}(Y, Y')$ is representable[2] by an X-scheme that is X-affine.

- The *étale Brauer–Manin set* of X is

  $$X(\mathbb{A}_K)^{\text{ét,Br}} := \bigcap_{G \text{ and } Y \to X} \; \bigcup_{[\mathcal{T}] \in H^1(K,G)} \operatorname{Im}\left((\operatorname{Isom}_{G_X}(Y, \mathcal{T}_X)(\mathbb{A}_K))^{\mathrm{Br}} \to X(\mathbb{A}_K) \right),$$

[2] If $\mathcal{G} \to X$ is not assumed to be affine, then $\operatorname{Isom}_{\mathcal{G}}(Y, Y')$ is only representable by an algebraic space. This is "good enough" for most practical purposes.

where the intersection is taken over the isomorphism classes of finite étale K-group schemes G and over the isomorphism classes of G_X-torsors $Y \to X$, and the union is taken over the isomorphism classes of G-torsors $\mathcal{T}$. The étale Brauer–Manin set fits into inclusions

$$X(K) \subset X(\mathbb{A}_K)^{\text{ét,Br}} \subset X(\mathbb{A}_K)^{\text{Br}} \subset X(\mathbb{A}_K) \subset \textstyle\prod_v X(K_v).$$

- If $\prod_v X(K_v) \neq \varnothing$ but $X(\mathbb{A}_K)^{\text{ét,Br}} = \varnothing$, so that necessarily $X(K) = \varnothing$, then X has an *étale Brauer–Manin obstruction* to the local-global principle.

The questions below are intended to help internalize the above notions.

6.1. Let $\mathcal{G}$ be an X-group scheme, and let $Y \to X$ and $Y' \to X$ be $\mathcal{G}$-torsors.

(a) Consider the *automorphism functor* $\mathrm{Aut}_{\mathcal{G}}(Y)$:

$$S \mapsto \{\mathcal{G}_S\text{-torsor isomorphisms } Y_S \xrightarrow{\sim} Y_S\}, \qquad \text{where } S \text{ is a variable } X\text{-scheme.}$$

Prove that $\mathrm{Aut}_{\mathcal{G}}(Y)$ is a sheaf on the fppf site of X.

Hint. Recall (say, from [SP, 02W0]) that for an fppf $S' \to S$ the base change functor

$$\{S\text{-schemes}\} \to \{S'\text{-schemes equipped with a descent datum with respect to } S'/S\}$$

is fully faithful. Use this to descend a $\mathcal{G}_{S'}$-torsor isomorphism $Y_{S'} \xrightarrow{\sim} Y_{S'}$ with equal $(S' \times_S S')$-pullbacks to a unique S-isomorphism $Y_S \xrightarrow{\sim} Y_S$. Conclude by noting that the latter is automatically $\mathcal{G}_S$-equivariant.

(b) If $\mathcal{G} \to X$ is affine, prove that $\mathrm{Aut}_{\mathcal{G}}(Y)$ is representable by an X-scheme that is X-affine.

Hint. If $X' \to X$ is an fppf cover trivializing Y, prove that $(\mathrm{Aut}_{\mathcal{G}}(Y))_{X'} \cong \mathcal{G}_{X'}$. To then prove the representability of $\mathrm{Aut}_{\mathcal{G}}(Y)$, use the effectivity of fppf descent for relatively affine schemes (which essentially is just fppf descent for quasi-coherent sheaves, cf. [BLR90, p. 135, paragraph after the proof of 6.1/4]). This descent result simultaneously proves the X-affineness of $\mathrm{Aut}_{\mathcal{G}}(Y)$, which also follows from [SP, 02L5] or from checking the isomorphism property of $\mathrm{Aut}_{\mathcal{G}}(Y) \to \underline{\mathrm{Spec}}(f_*\mathcal{O}_{\mathrm{Aut}_{\mathcal{G}}(Y)})$ fppf locally on X (where f denotes the structure map $\mathrm{Aut}_{\mathcal{G}}(Y) \to X$).

(c) Prove that $\mathrm{Isom}_{\mathcal{G}}(Y, Y')$ is an $\mathrm{Aut}_{\mathcal{G}}(Y)$-torsor fppf sheaf.

Hint. Use the methods of the hint of (a) to prove that $\mathrm{Isom}_{\mathcal{G}}(Y, Y')$ is an fppf sheaf. Define the $\mathrm{Aut}_{\mathcal{G}}(Y)$-action by precomposition. Then prove the torsor property

$$\mathrm{Aut}_{\mathcal{G}}(Y) \times_X \mathrm{Isom}_{\mathcal{G}}(Y, Y') \xrightarrow{\sim} \mathrm{Isom}_{\mathcal{G}}(Y, Y') \times_X \mathrm{Isom}_{\mathcal{G}}(Y, Y'), \qquad (a, i) \mapsto (ia, i)$$

by working fppf locally on X.

(d) If $\mathcal{G} \to X$ is affine, prove that $\mathrm{Isom}_{\mathcal{G}}(Y, Y')$ is representable by an X-scheme that is X-affine.

Hint. The argument is very similar to the one sketched in the hint of (b).

6.2. (a) If G is a finite étale K-group scheme, $\mathcal{T} \to \operatorname{Spec} K$ is a G-torsor, and $Y \to X$ is a G_X-torsor, prove that $\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X) \to X$ is finite étale. Conclude that $\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X)$ is a separated K-scheme of finite type, so that $(\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X)(\mathbb{A}_K))^{\mathrm{Br}}$ makes sense.

Hint. In checking that $\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X) \to X$ is finite étale, work étale locally on X to reduce to the case when the G_X-torsors Y and $\mathcal{T}_X$ are trivial. In this case, prove that $\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X) \cong G_X$.

(b) Prove that $X(\mathbb{A}_K)^{\text{ét,Br}} \subset X(\mathbb{A}_K)^{\text{Br}}$.

Hint. Combine (a) with **5.4.** (a).

6.3. Prove that $X(K) \subset X(\mathbb{A}_K)^{\text{ét,Br}}$.

Hint. For a finite étale K-group scheme G, a G_X-torsor $Y \to X$, and a $p \in X(K)$, let $Y_p := Y \times_X p$ be the p-fiber of $Y \to X$. Prove that $p \in \operatorname{Im}(\operatorname{Isom}(Y, (Y_p)_X)(K) \to X(K))$ and use **5.3.** to conclude that $p \in \operatorname{Im}\left((\operatorname{Isom}(Y, (Y_p)_X)(\mathbb{A}_K))^{\text{Br}} \to X(\mathbb{A}_K)\right)$.

6.4. Suppose that $f\colon X \to X'$ is a morphism of separated K-schemes of finite type.

(a) Prove that $f(\mathbb{A}_K)\colon X(\mathbb{A}_K) \to X'(\mathbb{A}_K)$ maps $X(\mathbb{A}_K)^{\text{ét,Br}}$ into $X'(\mathbb{A}_K)^{\text{ét,Br}}$.

Hint. If $(x_v)_v \in X(\mathbb{A}_K)^{\text{ét,Br}}$ and $Y' \to X'$ is a $G_{X'}$-torsor for some finite étale K-group scheme G, then $Y \to X$ with $Y := Y' \times_{X'} X$ is a G_X-torsor and

$$\operatorname{Isom}_{G_X}(Y, \mathcal{T}_X) = \left(\operatorname{Isom}_{G_{X'}}(Y', \mathcal{T}_{X'})\right) \times_{X'} X \qquad \text{for every } G\text{-torsor } \mathcal{T}.$$

Thus, if $\mathcal{T}$ is such that $(x_v)_v \in \operatorname{Im}\left((\operatorname{Isom}_{G_X}(Y, \mathcal{T}_X)(\mathbb{A}_K))^{\text{Br}} \to X(\mathbb{A}_K)\right)$, then the commutativity of

$$\begin{tikzcd} \operatorname{Isom}_{G_X}(Y, \mathcal{T}_X) \arrow[r] \arrow[d] & \operatorname{Isom}_{G_{X'}}(Y', \mathcal{T}_{X'}) \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

and **5.4.** (a) force $(f \circ x_v)_v \in \operatorname{Im}\left(\left(\operatorname{Isom}_{G_{X'}}(Y', \mathcal{T}_{X'})(\mathbb{A}_K)\right)^{\text{Br}} \to X'(\mathbb{A}_K)\right)$.

(b) Assume that $\prod_v X(K_v) \neq \varnothing$, so that necessarily $\prod_v X'(K_v) \neq \varnothing$. If X' has an étale Brauer–Manin obstruction to the local–global principle, prove that so does X.

Hint. If $X'(\mathbb{A}_K)^{\text{ét,Br}} = \varnothing$, then the conclusion of (a) forces $X(\mathbb{A}_K)^{\text{ét,Br}} = \varnothing$.

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