AWS 2020 COURSE OUTLINE: QUADRATIC CHABAUTY

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1. Course Outline

By a theorem of Faltings, there are only finitely many $\mathbf{Q}$-rational points on a smooth projective curve $X/\mathbf{Q}$ of genus $g > 1$. However, no general algorithm is known to compute $X(\mathbf{Q})$. When the rank r of the Mordell-Weil group $\mathrm{Jac}(X)(\mathbf{Q})$ is less than g, one can often use the method of Chabauty and Coleman [MP12], based on properties of $\mathrm{Jac}(X)$ and Coleman's theory of p-adic line integrals [Col85, Bes02, Bes12], to construct an explicit p-adic function with finitely many zeroes that vanishes on $X(\mathbf{Q})$. The aim of Kim's non-abelian Chabauty program [Kim05, Kim09] is to generalize this method by studying rational points [1] through Selmer varieties and iterated Coleman integrals. In this course we will discuss the simplest non-abelian instance of this program, developed in [BD18, BD17, BDM$^+$19], building on earlier work for integral points on (hyper)elliptic curves [BB15, BBM17]. The idea is that under restrictive conditions on X, one can find a particularly simple, yet non-abelian quotient of the $\mathbf{Q}_p$-étale unipotent fundamental group of X with the following properties:

(i) a simple Galois cohomology computation shows that one obtains a *finite* subset of $X(\mathbf{Q}_p)$ containing $X(\mathbf{Q})$ via the corresponding Selmer variety;
(ii) using p-adic heights, one can *construct* a function vanishing on $X(\mathbf{Q})$ with finitely many zeroes.

The function in (ii) is defined by twisting elements of the Selmer variety to obtain a mixed extension of Galois representations. Via p-adic heights [Nek93, CG89, Bes04], this leads to bilinear relations satisfied by global points. The global p-adic height decomposes into a sum of local heights, one for each prime number, and the local heights away from p can be controlled [KT08, BD]. The local height at p is defined in terms of a mixed extension of filtered ϕ-modules. For extensions coming from $X(\mathbf{Q}_p)$, this filtered ϕ-module is a fiber of a unipotent filtered connection with Frobenius structure satisfying certain universal properties [Kim09, Had11], and the local height at p is given in terms of its filtration and Frobenius structure.

In the course we will focus on theoretical aspects of quadratic Chabauty, referring to Kim's lectures for additional foundational material. For (ii) we will restrict to the case $r = g$ and $\rho > 1$, where ρ is the Picard number of $\mathrm{Jac}(X)$, as in [BDM$^+$19]. These conditions are often satisfied for modular curves. Finally, we will sketch alternative approaches. The course will be closely connected to Jennifer Balakrishnan's course, which will focus on explicit and computational methods.

2. Tentative ideas for projects

(1) *Quadratic Chabauty for $r = g$ and CM.* So far, quadratic Chabauty has only been applied for bielliptic genus 2 and real multiplication examples. In this project we will look at complex multiplication. Possible examples include Fermat curves and the genus 2 curves in [vW99]. Familiarity with CM theory will be helpful.

[1] more generally, *integral* points on hyperbolic curves

(2) *Quadratic Chabauty for $r > g$.* While the explicit methods of [BD18] and [BDM+19] assume that $r = g$, [BD18] also contains a general recipe for quadratic Chabauty when $r < g - 1 + \rho$. More generally, the Bloch-Kato conjecture predicts that quadratic Chabauty should be applicable when $r < g^2$, without any assumptions on ρ (see [BD17]). We will try to make some of this explicit in simple situations.

(3) *A geometric interpretation of* [BD18, BDM+19]. Recent work of Edixhoven-Lido gives a geometric approach to quadratic Chabauty via Poincaré torsors. The methods of [BD18, BDM+19] can be interpreted in a similar way, resulting in a quadratic Chabauty function factoring through $L^*(\mathbf{Q}_p)$, where L is a line bundle on J and L^* is the $\mathbb{G}_m$-torsor $\mathrm{Isom}_J(\mathcal{O}, L)$. The goal of this project is to make various aspects of this explicit using work of Betts [Bet17]. One potential application is to develop a Mordell-Weil sieve [BS10] type method on L^*. For this project students should have some knowledge of local heights on abelian varieties.

For all projects, students should be familiar with basic algebraic geometry and rigid analytic geometry [Sch98]. In addition, some knowledge of the background required for Kim's non-abelian Chabauty, such as non-abelian cohomology [Ser02], p-adic Hodge theory [BC09], Coleman integrals and the theory of isocrystals [Bes12] would be helpful. For the computational parts of the projects we will use `Magma`.

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