NON-ARCHIMEDEAN DYNAMICS IN DIMENSION ONE: LECTURE NOTES

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These notes for my short course at the 2010 Arizona Winter School are a bit long, but for two (hopefully good) reasons. The first is that these notes will form the core of a book that I am planning to write on the topic, and therefore I often proved even relatively minor lemmas in far greater detail than is usually appropriate for a set of lecture notes. The second, and more relevant for students in the course, is because of the nature of the subject, as I will now attempt to explain.

To study complex dynamics, naturally one ought to have completed a full course on complex analysis. For the sake of argument, though, one could probably learn quite a bit of complex dynamics with almost *no* complex analysis coursework if one were willing to accept certain facts on faith, like the fact that any function whose derivative exists on a given open disk is in fact given by a power series converging on that disk. It would of course be unwise to try to approach complex dynamics that way, but my point is that it *could* be done, if all one wanted was an overview of the subject.

The same is true of non-archimedean dynamics: to study the subject, you really ought (at least in theory) to have a solid course in non-archimedean analysis already under your belt. However, whereas complex analysis is commonly taught and even required, courses purely on non-archimedean analysis are rather rare. In practice, then, almost no one who learns non-archimedean dynamics starts by taking a non-archimedean analysis course first; I know I certainly didn't. Instead, we learn both subjects together.

But this is a short course, and we don't have time for that. Thus, in my lectures, I will ask you to accept certain facts on faith, like Proposition 3.18, which says (among other things) that rational functions have power series expansions away from their poles, or like the all-important Corollary 3.10, which says that the image of a disk under a power series (and hence under a rational function, assuming there are no poles in the disk) is again a disk, the precise center and radius of which we can easily read off of the power series itself. In my lectures, I will simply state those facts, along with others (like certain technical properties of Berkovich spaces, a key tool in modern non-archimedean analysis), without proof.

To learn the subject well, however, you should have a better idea of why such facts are true, just as you wouldn't learn complex dynamics very well if you didn't actually know the nuts and bolts of complex analysis. So these lecture notes include far more details and proofs, to complement my lectures.

If you are preparing for the Winter School by reading these notes, that's fantastic, but there is probably *too* much to read here, unless for some reason you have nothing else to do in the next few weeks. To focus you, then, I would recommend that you prepare mainly by reading Sections 2 and 3, as those are essentially only about pure non-archimedean analysis. You can also feel free to read about some elementary dynamics in Section 1, which is less technical and probably easier to manage.

If you get through all that and are looking for more, I'd recommend turning next to Sections 6.1, 6.2, and 6.3, which are about Berkovich spaces. If you somehow manage to slog through all of *that* as well, then, and only then, would I recommend that you turn to learning the actual dynamics in Section 4 (and maybe also Section 5, although that is less important) and the rest of Section 6 and beyond.

1. Basic dynamics on $\mathbb{P}^1(K)$

Given an arbitrary field K and a rational function $\phi \in K(z)$, we can of course view ϕ as a morphism $\phi : \mathbb{P}^1_K \to \mathbb{P}^1_K$. In particular, for any field L containing K (such as K itself), ϕ acts as a function from $\mathbb{P}^1(K) := K \cup \{\infty\}$ to itself; we will consider the dynamics of this action.

1.1. **Elementary discrete dynamics.** We begin with some basic terminology applying in a more general setting. A (*discrete*) *dynamical system* on a set X is a function $\phi : X \to X$. For any integer $n \geq 0$, $\phi^n : X \to X$ denotes the *n-th iterate* of ϕ,

$$\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}},$$

that is, $\phi^0 = \mathrm{id}_X$, and $\phi^{n+1} = \phi \circ \phi^n$.

The (*forward*) *orbit* of a point $x \in X$ is $\{\phi^n(x) : n \geq 0\}$, and the *backward orbit* of x is $\bigcup_{n\geq 0} \phi^{-n}(x)$. The *grand orbit* of x is the union of the backward orbits of all points in the forward orbit of x.

The point x is said to be *fixed* (under ϕ) if $\phi(x) = x$; x is *periodic* (under ϕ, of period n) if $\phi^n(x) = x$ for some positive integer $n \geq 1$; and x is *preperiodic* (under ϕ) if $\phi^n(x) = \phi^m(x)$ for distinct integers $n > m \geq 0$.

If x is a periodic point of period n, the forward orbit of x is called a *periodic cycle* of period n, or an n-periodic cycle. The smallest positive integer $n \geq 1$ such that $\phi^n(x) = x$ is called the *minimal period* or *exact period* of x under ϕ.

Several elementary observations are in order. First, grand orbits partition X, because $y \in X$ belongs to the grand orbit of $x \in X$ if and only if there are integers $m, n \geq 0$ such that $\phi^m(x) = \phi^n(y)$. Second, every point in the forward orbit of a periodic point is periodic of the same minimal period; thus, the period is really a property of the periodic cycle. Third, a point is preperiodic if and only if its forward orbit is finite. Fourth, if ϕ is onto (as will be the case for the dynamical systems we will consider), and if x has finite backward orbit, then x also has finite forward orbit; in fact, the grand orbit of x consists solely of a periodic cycle.

Example 1.1. Let $X = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, and let $\phi(z) = z^2$. The forward orbit of the point -2 is $\{-2, 4, 16, 256, \ldots\}$; more generally, the forward orbit of any $x \in \mathbb{P}^1(\mathbb{C})$ is $\{x^{2^n} : n \geq 0\}$. The points $0, 1, \infty$ are fixed by ϕ. The grand orbit of 0 is simply $\{0\}$, and that of ∞ is simply $\{\infty\}$. Every other point $x \in \mathbb{C} \setminus \{0\}$ has backward orbit $\{\zeta \cdot \sqrt[n]{x} : n \geq 0, \zeta^{2^n} = 1\}$. The preperiodic points of ϕ are 0, ∞, and all roots of unity. A primitive m-th root of unity is periodic if and only if m is odd; its minimal period is the smallest integer n such that $2^n \equiv 1 \pmod{m}$.

More generally, if we set $\phi(z) = z^d$ for some integer $d \in \mathbb{Z} \setminus \{0, 1, -1\}$, the preperiodic points are still the roots of unity. A primitive m-th root of unity is now periodic if and only if $d \nmid m$, and the minimal period is the multiplicative order of d modulo m.

The previous example is perhaps misleadingly simple; everything works out nicely because z^2 is an endomorphism of the multiplicative group $\mathbb{G}_m$. (See Chapter 6, and especially Section 6.1, of [52].) The following seeming slight variation is far more complicated, and is far more illustrative of the general situation.

Example 1.2. Let $X = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, and let $\phi(z) = z^2 - 1$. The forward orbit of the point -2 is $\{-2, 3, 8, 63, 3968, \ldots\}$; there is no general closed formula for $\phi^n(x)$. The point ∞ is fixed, and its grand orbit is just $\{\infty\}$, but no other point has finite backward orbit. The two roots of $z^2 - z - 1 = 0$ (that is, of $\phi(z) - z = 0$) are the only other fixed points of ϕ. The points 0 and -1 form a periodic cycle of period 2, and 1 is preperiodic, as $\phi(1) = 0$ is periodic. (In fact, there are no other rational preperiodic points besides $0, \pm 1, \infty$.) In general, it is difficult to tell whether a given point $x \in X$ is preperiodic or not.

1.2. **Morphisms and coordinate changes.** A *morphism* between two dynamical systems $\phi : X \to X$ and $\psi : Y \to Y$ is a function $h : X \to Y$ such that $h \circ \phi = \psi \circ h$; that is, so that the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "h"'] & X \arrow[d, "h"'] \\ Y \arrow[r, "\psi"] & Y \end{tikzcd}$$

commutes. In that case, $h \circ \phi^n = \psi^n \circ h$ for all $n \geq 0$. Thus, h preserves the structure of the dynamical system: it maps fixed points of ϕ to fixed points of ψ, periodic points to periodic points, preperiodic points to preperiodic points, forward orbits onto forward orbits, backward orbits into backward orbits, and grand orbits into grand orbits. If h is onto, then backward and grand orbits are mapped onto their counterparts. If h is one-to-one, then the minimal period of all periodic points is preserved by ϕ.

If h is one-to-one and onto, then h is an isomorphism of dynamical systems, and we have $\psi = h \circ \phi \circ h^{-1}$. If $X = Y$ in this case, then h can be considered to be a change of coordinates on X. Thus, a coordinate change on the underlying space corresponds to a conjugation of the map ϕ.

Example 1.3. Let $X = \mathbb{P}^1(\mathbb{C})$ and $\phi(z) = z^2$, as in Example 1.1. Meanwhile, let $Y = X$ and $\psi(z) = z^2 - 2$. Then the map $h : X \to Y$ given by $h(z) = z + z^{-1}$ is a morphism from ϕ to ψ, because

$$h \circ \phi(z) = z^2 + z^{-2} = (z + z^{-1})^2 - 2 = \psi \circ h(z).$$

The preperiodic points of ψ are precisely points of the form $h(x)$, where x is a preperiodic point of ϕ. That is, the preperiodic points are $\infty = h(0) = h(\infty)$ and any point of the form $\zeta + \zeta^{-1}$, where ζ is a root of unity. (The latter kinds of points can be written $2\cos(2\pi q)$, for $q \in \mathbb{Q}$ rational.) Because ϕ is onto, the periodic points of ψ are also precisely the images under h of the periodic points of ϕ; however, the minimal periods may change. For example, a primitive cube root of unity ζ_3 is periodic of minimal period 2 under ϕ, but $h(\zeta_3) = -1$ is fixed by ψ.

More generally, for any integer $d \geq 2$, the d-th *Chebyshev polynomial* is the unique polynomial T_d such that $T_d(z + z^{-1}) = z^d + z^{-d}$. Note that T_d has integer coefficients and also satisfies $T_d(2\cos\theta) = 2\cos(d\theta)$. See Section 6.2 of [52] for more on the dynamics of Chebyshev polynomials.

Examples like Example 1.3 are relatively rare. Usually, the only morphisms of dynamical systems of particular interest to us will be coordinate changes.

Example 1.4. Let $X = \mathbb{P}^1(\mathbb{C})$, and let $\psi(z) = z^2/(2z+2)$. One can check that the fixed points of ψ are exactly 0, -2, and ∞. Further investigation reveals that the only preimage of -2 is -2 itself, and similarly, the only preimage of 0 is 0. Inspired by this observation, we write down the unique linear fractional transformation $h(z)$ mapping $0 \mapsto 0$, $\infty \mapsto -2$, and $1 \mapsto \infty$, which is $h(z) = 2z/(1-z)$, and we can check that $\psi = h \circ \phi \circ h^{-1}$, where $\phi(z) = z^2$. Thus, ψ is just the squaring map under a change of coordinates. In particular, this is one of the rare dynamical systems for which we can write down a simple closed form for ψ^n, namely $\psi^n(z) = h(h^{-1}(z)^{2^n}) = 2z^{2^n}/[(z+2)^{2^n} - z^{2^n}]$.

Example 1.5. Let K be a field of characteristic different from 2 with algebraic closure $\overline{K}$, let $X = \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$, and let $\phi(z) = Az^2 + Bz + C \in K[z]$ be a quadratic polynomial. The two ramification points ∞ and $-B/(2A)$ of ϕ have dynamical significance because they are the only two points x for which $\phi^{-1}(x)$ does not consist of exactly two distinct points. (The ramification points of arbitrary rational functions will be of interest for the same reason.) We can conjugate by $h_1(z) = z + B/(2A)$ to obtain $\psi_1(z) = Az^2 + D$, where $D = C - B(B-2)/(4A^2)$. Next, to make the polynomial monic, we can conjugate by $h_2(z) = Az$ to obtain $\psi_2(z) = z^2 + c$, where $c = AD$.

Meanwhile, a little more work shows that if $h \in \mathrm{PGL}(2, K)$ is a linear fractional transformation conjugating $z^2 + c_1$ to $z^2 + c_2$, then $c_1 = c_2$. Thus, we have shown that over any field not of characteristic 2, the set of quadratic polynomial dynamical systems is parametrized by K itself, with the parameter $c \in K$ corresponding to the conjugacy class of $z^2 + c$.

Hence, the study of maps of the form $z^2 + c$ is common in complex dynamics, as they give all quadratic polynomials. However, the corresponding moduli spaces of higher degree polynomials and rational functions are far more complicated; we refer the interested reader to Chapter 4 of [52] as a starting point for learning about such spaces.

1.3. **Degrees and multipliers.** We now turn our attention more fully to the case that K is a field, L/K is a field extension (often $L = K$), $\phi \in K(z)$ is a rational function, and we consider the dynamical system $\phi : \mathbb{P}^1(L) \to \mathbb{P}^1(L)$.

Writing $\phi = f/g$, where $f, g \in K[z]$ are relatively prime polynomials, we define the *degree* of ϕ to be

$$\deg \phi := \max\{\deg f, \deg g\}.$$

This notion of degree coincides with the geometric degree of ϕ. In other words, for any point $x \in \mathbb{P}^1(\overline{K})$, x has exactly $\deg \phi$ preimages under ϕ, counting multiplicity. It is easy to check (by either the maximum or the geometric description of degree) that $\deg(\phi \circ \psi) = (\deg \phi) \cdot (\deg \psi)$ for any $\phi, \psi \in K(z)$. As a result, we also have $\deg(\phi^n) = (\deg \phi)^n$.

Unless ϕ is constant or is the identity function, it has exactly $1 + \deg \phi$ fixed points in $\mathbb{P}^1(\overline{K})$, counting multiplicity. More precisely, the fixed points are precisely the roots of the equation $\phi(z) - z = 0$, and hence of the polynomial $f(z) - zg(z) = 0$, along with, perhaps, the point ∞. Thus, when counting multiplicity of a fixed point $x \neq \infty$ earlier in this paragraph, we mean multiplicity as a root of this polynomial $f(z) - zg(z)$. If ∞ is fixed, its multiplicity is that of 0 as a root of $1/\phi(1/z) - z = 0$, or equivalently, of the polynomial $z^d g(1/z) - z^{d+1} f(1/z)$.

Alternately, to consider $\mathbb{P}^1$ holistically, we should use homogeneous coordinates $[x, y]$ and write ϕ as $\phi(x, y) = [f(x, y), g(x, y)]$, where $f, g \in K[x, y]$ are homogeneous of degree d. (Indeed, the new f can be written in terms of the old f as $y^d f(x/y)$, and similarly the new g is $y^d g(x/y)$.) The old equation $\phi(z) - z = 0$ now becomes $yf(x, y) - xg(x, y) = 0$; and as a homogeneous polynomial of degree $d + 1$, $yf(x, y) - xg(x, y)$ does indeed have $d + 1$ roots in $\mathbb{P}^1$, counting multiplicity.

Considering ϕ^n in place of ϕ, it follows that ϕ has exactly $1 + (\deg \phi)^n$ periodic points in $\mathbb{P}^1(\overline{K})$ of period n (that is, of minimal period dividing n), as long as ϕ^n is neither constant nor the identity function. (That condition is obviously true under our usual assumption that $\deg \phi \geq 2$.) If we denote by M_n the number of periodic points of minimal period n, then barring the coalescing of higher-period periodic points on smaller-period points, we would have $\sum_{m|n} M_m = d^{n+1}$. By the Möbius inversion formula, then, we have $M_n = \sum_{m|n} \mu(n/m) \cdot (d^m + 1)$. Thus, for example, a rational function ϕ of degree 2 has exactly $(2^6 + 1) - (2^3 + 1) - (2^2 + 1) + (2^1 + 1) = 54$ points of minimal period 6, assuming none of the periodic points involved coalesce on points of smaller period. (Note that 54 is divisible by 6, giving nine cycles of period 6.)

However, the coalescing of periodic points can occur. The simplest example arises for $\phi(z) = z^2 - z$. We have the expected $2 + 1 = 3$ fixed points (at 0, 2, and ∞), and the previous paragraph predicts $(2^2 + 1) - 3 = 2$ points of period 2. However, we compute $\phi^2(z) - z = z^4 - 2z^3$, giving 2-periodic points only at the same three points 0, 2, ∞ that were already fixed. The problem is that 0 is now a triple root of $\phi^2(z) - z$; the two extra roots that were supposed to be points of period 2 were instead copies of a fixed point. Thus, the two points of the expected 2-cycle have coalesced at a fixed point. More generally, this can happen any time a periodic point (in this case, the fixed point 0) has multiplier, to be defined in a moment, equal to a root of unity.

Given a periodic point $x \in \mathbb{P}^1(L)$ of minimal period $n \geq 1$, the *multiplier* of x to be $\lambda := (\phi^n)'(x)$, at least in the case that $x \neq \infty$. If $x = \infty$, then define the multiplier to be $\lambda := (\psi^n)'(0)$, where $\psi(z) = 1/(\phi(1/z))$; that is, change coordinates to move $x = \infty$ to 0, and compute the multiplier at 0 as before. It is easy to check, using the chain rule, that the multiplier is unaffected by coordinate change; in other words, given $h \in \mathrm{PGL}(2, K)$, the multiplier of a periodic point x of ϕ is the same as that of the periodic point $h(x)$ of $h \circ \phi \circ h^{-1}$. The chain rule also tells us that

$$(\phi^n)'(x) = \prod_{i=0}^{n-1} \phi'(\phi^i(x))$$

(at least, assuming none of $x, \phi(x), \ldots, \phi^n(x)$ is ∞), from which it follows that every point in the periodic cycle of x has the same multiplier. Thus, like the period, the multiplier is really a property of the periodic cycle, rather than the individual periodic point.

Note that a periodic point $x \in \mathbb{P}^1$ of minimal period n has multiplicity at least two (as an n-periodic point) if and only if its multiplier is exactly 1. This is clear if $x \neq \infty$, because such a periodic point is a multiple root of $\phi^n(z) - z$ if and only if it is also a root of the derivative $(\phi^n)'(z) - 1$ of that same polynomial. It can also be checked at $x = \infty$ either by changing coordinates or working in homogeneous coordinates. More generally, a periodic point $x \in \mathbb{P}^1$ of minimal period n has multiplicity at least two *as an mn-periodic point* if and only if its multiplier is an m-th root of unity. It is in precisely this situation that the phenomenon of coalescing of periodic points occurs: if x has minimal period n, with multiplier $\lambda \neq 1$ a primitive m-th root of unity, then some mn-periodic points will coalesce at x. See Section 4.1 of [52] and Section 6.2 of [3] for more on this phenomenon.

The following result is well known in complex dynamics (even in 1919, Julia himself called it “bien connue” in Section 9 of his Mémoire [34]) but holds over any field.

Theorem 1.6 (The Holomorphic Fixed-Point Formula). *Let K be a field, let $\phi \in K(z)$ be a rational function of degree $d \geq 0$ with no fixed points of multiplier exactly 1. Let $x_1, \ldots, x_{d+1} \in \mathbb{P}^1(\overline{K})$ be the (distinct) fixed points of ϕ, with corresponding multipliers $\lambda_1, \ldots, \lambda_{d+1} \in \overline{K}$. Then*

$$\sum_{i=1}^{d+1} \frac{1}{1 - \lambda_i} = 1.$$

Proof. After a $\overline{K}$-rational change of coordinates, assume that ∞ is not a fixed point of ϕ. (Note that if K is a finite field, all points of $\mathbb{P}^1(K)$ might be fixed; that is why we may have to pass to $\overline{K}$.) Let ω be the meromorphic differential form

$$\omega = \frac{dz}{\phi(z) - z} \in \Omega^1(\mathbb{P}^1(\overline{K})),$$

which is holomorphic except for simple poles at $\infty, x_1, \ldots, x_{d+1}$. Clearly, the residue at x_i is $\phi'(x_i) - 1 = \lambda_i$. In the coordinate $w = 1/z$, we have

$$\omega = -\frac{dw}{w^2\phi(1/w) - w} = \frac{dw}{w + O(w^2)},$$

because $\phi(1/w)$ is holomorphic at $w = 0$. Thus, the residue of ω at $w = 0$, and hence at $z = \infty$, is 1.

Recall that the Residue Theorem states that the sum of the residues of a differential form at all poles is 0. This equality holds over arbitrary fields, not just over $\mathbb{C}$— see, for example, Section II.7 of [50], where the proof for $\mathbb{P}^1$ is a simple matter of considering each term of the partial fractions decomposition of $f(t)$ for the differential form $f(t)\,dt$. In particular, we may apply the Residue Theorem to our form ω, giving us

$$1 + \sum_{i=1}^{d+1} \frac{1}{\lambda_i - 1} = 0,$$

from which the desired formula is immediate. $\square$

1.4. **Critical points and exceptional sets.** Given a nonconstant rational function $\phi(z) \in K(z)$ and a point $x \in \mathbb{P}^1(\overline{K})$, one can choose $\eta_1, \eta_2 \in \mathrm{PGL}(2, \overline{K})$ such that $\eta_1(\phi(x)) = \eta_2(x) = 0$, and define the function

$$\psi := \eta_1 \circ \phi \circ \eta_2^{-1},$$

so that $\psi(0) = 0$, and hence ψ may be written as $\psi(z) = z^e h(z)$, for some integer $e \geq 1$ and some rational function $h(z) \in \overline{K}(z)$ for which $h(0) \in \overline{K}^{\times}$. The integer e is the *multiplicity* or *ramification index* of ϕ at x, and it is independent of the choices of η_1 and η_2. If $e \geq 2$, we say that x is a *critical point*, or *ramification point* of ϕ, and that ϕ is *ramified* at x. If $e = \deg \phi \geq 2$, then we say ϕ is *totally ramified* at x.

Of course, if $x, \phi(x) \neq \infty$, then the multiplicity e is simply the order of the zero of the function $\phi(z) - \phi(x)$ at $z = x$. Thus, such a point x is critical for ϕ if and only if $\phi'(x) = 0$.

If x is a critical point of ϕ and we set $\psi = \eta_1 \circ \phi \circ \eta_2^{-1}$ as above, then the order of vanishing $\rho \geq 1$ of the derivative $\psi'(z)$ at $z = 0$ is not usually given a name, but since we will want to discuss it, we will call it the *order* of the critical point x of ϕ. (Any non-critical point has order zero.) Like the multiplicity, the order of a critical point is independent of the choices of η_1 and η_2. If $\operatorname{char} K = 0$, then the multiplicity $m \geq 2$ and the order $\rho \geq 1$ of a critical point x are related by $\rho = m - 1$. The same is true if $\operatorname{char} K = p > 0$ and $p \nmid m$. However, if $\operatorname{char} K = p > 0$ and $p | m$, then $\rho \geq m$.

In the extreme case, if $\operatorname{char} K = p > 0$ and $\phi(z) \in K(z^p)$, then $\phi'(z) = \psi'(z) = 0$; by the above definitions, every point $x \in \mathbb{P}^1$ is a critical point with multiplicity divisible by p but infinite order. Maps of this sort are said to be *inseparable*; any other map is *separable*. Any rational function ϕ over a characteristic p field K can be written uniquely as $\tilde{\phi}(z^{p^j})$, where $\tilde{\phi}$ is separable and $j \geq 0$; we say that ϕ is *totally inseparable* if $\deg \tilde{\phi} = 1$, or equivalently, in this notation, if $\deg \phi = p^j$.

The location and orbits of the critical points will turn out to be very important for understanding the dynamics of the map ϕ. Fortunately, the Riemann-Hurwitz formula gives us great control over the number of critical points.

Theorem 1.7 (The Riemann-Hurwitz Formula for Morphisms of $\mathbb{P}^1$)**.** *Let K be a field, let $\phi(z) \in K(z)$ be a separable rational function of degree $d \geq 1$, and let $R \subseteq \mathbb{P}^1(\overline{K})$ be the set of all critical points of ϕ. For each $x \in R$, define $\rho_x \geq 1$ to be the order of x as a critical point of ϕ. Then*

$$\sum_{x \in R} \rho_x = 2d - 2.$$

Proof. This is Corollary IV.2.4 of [31], in the case that the curves X and Y are both $\mathbb{P}^1$ and hence have genus zero. □

Corollary 1.8. *Let K be a field, and let $\phi(z) \in K(z)$ be a separable rational function of degree $d \geq 1$. Then ϕ has at most $2d - 2$ critical points.*

Proof. Immediate from Theorem 1.7, because $e_x \geq 1$ for all $x \in R$. □

Corollary 1.9. *Let K be a field, and let $\phi(z) \in K(z)$ be rational function of degree $d \geq 2$ that is not totally inseparable. Then ϕ has at most two totally ramified critical points. Furthermore, if ϕ is separable and has two totally ramified critical points, then it has no other critical points.*

Proof. We first consider the case that ϕ is separable. In the notation of Theorem 1.7, at a totally ramified critical point $x \in R$, the multiplicity of ϕ at x is d, and hence the ramification index satisfies $\rho_x \geq d - 1$. The conclusion of the Corollary is now immediate from Theorem 1.7.

Now consider $\phi(z) = \tilde{\phi}(z^{p^j})$, where $p = \operatorname{char} K > 0$, $j \geq 0$, and $\tilde{\phi}$ is separable and of degree $\tilde{d} \geq 2$. Then ϕ is totally ramified at a point $x \in \mathbb{P}^1(\overline{K})$ if and only if $\tilde{\phi}$ is totally ramified at x^{p^j}. However, by the previous paragraph, $\tilde{\phi}$ is totally ramified at at most two points. Finally, as a map of sets (although not as an algebraic morphism), the Frobenius map $x \mapsto x^p$ is a bijective map from the set $\mathbb{P}^1(\overline{K})$ to itself, since $\operatorname{char} K = p$. Thus, there are at most two points $x \in \mathbb{P}^1(\overline{K})$ for which $\tilde{\phi}$ is totally ramified at x^{p^j}, and we are done. $\square$

Definition 1.10. Let K be a field, and let $\phi \in K(z)$ be a rational function. A point $x \in \mathbb{P}^1(\overline{K})$ is said to be an *exceptional* point of ϕ if the grand orbit of x in $\mathbb{P}^1(\overline{K})$ is finite. The set of all exceptional points of ϕ is called the *exceptional set* of ϕ.

If K contains a finite subfield $\mathbb{F}_p$, and if $\phi(z) = z^{p^m}$ for some $m \geq 1$, then the exceptional set of ϕ is $\mathbb{P}^1(\overline{\mathbb{F}}_p)$. However, this is essentially the only way the exceptional set can have more than two elements, as the following result shows.

Theorem 1.11. *Let K be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with exceptional set $E_\phi \subseteq \mathbb{P}^1(\overline{K})$.*

a. *If $\#E_\phi > 2$, then $\#E_\phi = \infty$, K contains a finite field $\mathbb{F}_p$, and ϕ is conjugate over $\overline{K}$ to z^{p^m} for some $m \geq 1$.*
b. *If $\#E_\phi = 2$, then ϕ is conjugate over $\overline{K}$ to either z^d or z^{-d}.*
c. *If $\#E_\phi = 1$, then ϕ is conjugate over $\overline{K}$ to a polynomial.*

Proof. We begin by claiming that ϕ is totally ramified at every exceptional point. To see this, given $x \in E_\phi$, let $GO(x)$ denote its grand orbit, which is finite by definition of exceptional. Moreover, by definition of grand orbit, ϕ maps $GO(x)$ onto itself; since $\#GO(x) < \infty$, the mapping is bijective. Thus, for some $n \geq 1$, ϕ^n fixes every point of $GO(x)$. In particular, $\phi^{-n}(x) = \{x\}$. Thus, ϕ^n is totally ramified at x, and hence ϕ itself must also be totally ramified at x, as claimed.

Thus, by Corollary 1.9, if $\#E_\phi > 2$, then ϕ is totally inseparable of degree $d = p^m$ for some $m \geq 1$. Because $\phi'(z) = 0$, all the fixed points of ϕ have multiplier 0. Since $0 \neq 1$, then, there are $d + 1$ distinct fixed points; see Section 1.3. In particular, ϕ has at least three distinct fixed points, and so we may move them to 0, 1, and ∞ by some $\overline{K}$-rational change of coordinates. Thus, we have $\phi(z) = \eta(z^{p^m})$, where $m \geq 1$, $\eta \in \mathrm{PGL}(2, \overline{K})$, and ϕ fixing each of 0, 1, ∞. It follows that η also fixes each of 0, 1, ∞. Therefore, $\eta(z) = z$, and hence $\phi(z) = z^{p^m}$, proving part (a).

For part (b), if $\#E_\phi = 2$, then we can change coordinates over $\overline{K}$ so that $E_\phi = \{0, \infty\}$. On the one hand, if $\phi(0) = 0$, then $\phi^{-1}(0) = \{0\}$ and $\phi^{-1}(\infty) = \{\infty\}$; we must therefore have $\phi(z) = az^d$ for some $a \in \overline{K}^\times$. Conjugating again, we get $a^{1/(d-1)}\phi(a^{-1/(d-1)}z) = z^d$. On the other hand, if $\phi(0) \neq 0$, then we must have $\phi^{-1}(0) = \{\infty\}$ and $\phi^{-1}(\infty) = \{0\}$, and hence $\phi(z) = az^{-d}$. This time, the conjugation $a^{-1/(d+1)}\phi(a^{1/(d+1)}z)$ gives z^{-d}, as desired.

For part (c), move the unique point of E_ϕ to ∞ by a $\overline{K}$-rational change of coordinates. Then $\phi^{-1}(\infty) = \{\infty\}$, and hence ϕ is a polynomial. $\square$

1.5. **Dynamics in degree less than two.** Most of the interesting dynamics of rational functions occurs when the degree of the function is at least two. For completeness, however, we now describe what happens in degrees zero and one.

If $\phi(z) \in K(z)$ has degree zero, then ϕ is simply a constant function $\phi(z) = c$, and we have $\phi^n(z) = c$ for all $n \geq 1$. The point c is fixed, and there are no other dynamical features to speak of.

For the remainder of this section, then, suppose that $\phi(z) \in K(z)$ has degree one; that is, $\phi \in \mathrm{PGL}(2, K)$. In other words, ϕ can be represented by a 2×2 matrix, and iterating ϕ corresponds to taking powers of the matrix. Thus, ϕ^n can be described fully using elementary linear algebra. We can understand the iterates of such a function dynamically, as well. After all, ϕ has at least one fixed point, and, unless $\phi(z) = z$, at most $1 + \deg \phi = 2$. We consider two cases: that ϕ has two distinct fixed points, or that it has only one.

If ϕ has (at least) two distinct fixed points, then after a $\overline{K}$-rational change of coordinates, we can move one of those points to 0 and the other to ∞. (This change corresponds to conjugating the matrix representation of ϕ by a matrix of eigenvectors; one eigenvector is moved to $(1, 0)$, corresponding to ∞, and the other to $(0,1)$, corresponding to 0.) Thus, $\phi(z)$ is now of the form az, for some $a \in \overline{K}^{\times}$. For any $n \geq 1$, then, $\phi^n(z) = a^n z$.

Hence, if a is a root of unity, then some iterate of ϕ is the identity function. Otherwise, ϕ has no preperiodic points besides the two original fixed points. Note also that $\phi(z) = az$ is conjugate to $\psi(z) = bz$ if and only if either $a = b$ or $a = 1/b$. After all, any conjugation must map fixed points to fixed points, and hence either fix both 0 and ∞, or else exchange them; and conjugating by $z \mapsto cz$ leaves $\phi(z) = az$ unaffected.

Finally, if ϕ has only one fixed point, then that point must have multiplier 1 (see Section 1.3). By a $\overline{K}$-rational change of coordinates, we can move the fixed point to ∞, so that $\phi(z) = az + b$. However, the multiplier of ∞ for $az+b$ is $1/a$, and thus we must have $\phi(z) = z+b$; moreover, $b \neq 0$, since ϕ has only one fixed point. Conjugating by $z \mapsto z/b$, then, ϕ becomes $\phi(z) = z + 1$. (This coordinate change corresponds to the case that the associated 2×2 matrix has a repeated eigenvalue with eigenspace of dimension one. The matrix can then only be reduced to Jordan canonical form, and not diagonalized.) Clearly, then, $\phi^n(z) = z + n$ for all $n \geq 0$. Thus, if $\operatorname{char} K = p > 0$, then ϕ^p is the identity function; otherwise, if $\operatorname{char} K = 0$, then ϕ has no preperiodic points besides the fixed point at ∞.

Clearly, the dynamics of rational functions of degree smaller than two is far too easy to describe to be interesting. Thus, from now on, when studying the dynamics of a rational function $\phi(z) \in K(z)$, we will almost always assume that $\deg \phi \geq 2$

2. Some Background on Non-archimedean Fields

The study of non-archimedean analysis is huge, and we will only scrape the surface in this brief survey of those results most relevant to our goals in dynamics. For a more extensive background, see [30, 35] for a more detailed but still very friendly treatment of the p-adic field $\mathbb{C}_p$, as well as [47, 48] for more advanced analysis (including deeper discussions of arbitrary continuous and differentiable functions) that are still for p-adic fields. The treatise [17] gives a fully general treatment of rigid analysis, while [27] presents a somewhat more accessible treatment of much of the same theory, and [29] gives a very readable introduction to the fundamentals underlying the theory. Although less relevant to our goals in these lectures, the reader interested in learning more non-archimedean functional analysis should also consult [25, 26, 49]. Finally, anyone interested in modern non-archimedean analysis should certainly learn about the Berkovich theory of analytic spaces; see Berkovich's original monograph [14], as well as the more accessible (and specific to $\mathbb{P}^1$) treatment in [2].

2.1. **Absolute values.** An *absolute value* on a field K is a function $|\cdot| : K \to [0,\infty)$ that is non-degenerate (i.e, $|x| = 0$ if and only if $x = 0$), multipicative (i.e., $|xy| = |x||y|$), and satisfies the triangle inequality $|x + y| \leq |x| + |y|$. The *trivial* absolute value on K satisfies $|0| = 0$ and $|x| = 1$ for all $x \in K^{\times}$; any other absolute value is said to be *nontrivial*. We will not be interested in the trivial absolute value, and so from now on, we set the convention that *all of our absolute values will be nontrivial*.

A nontrivial absolute value $|\cdot|$ is said to be *non-archimedean* if it satisfies the stronger *ultrametric* triangle inequality

$$|x + y| \leq \max\{|x|, |y|\}. \tag{1}$$

It is easy to check that for non-archimedean absolute values, one has equality in (1) if $|x| \neq |y|$. Colloquially, then, in a non-archimedean field (i.e., in a field equipped with a non-archimedean absolute value), all triangles are isoceles.

Any non-archimedean field K has a *ring of integers*

$$\mathcal{O}_K := \{x \in K : |x| \leq 1\},$$

which is indeed a ring under the multiplication and addition inherited from K, and which has a unique maximal ideal

$$\mathcal{M}_K := \{x \in K : |x| < 1\}.$$

The quotient field

$$k := \mathcal{O}_K/\mathcal{M}_K$$

is called the *residue field* of K. If K has positive characteristic $p > 0$, then $\operatorname{char} k = p$ as well; however, if $\operatorname{char} K = 0$, then $\operatorname{char} k$ can be 0 or any positive prime. We say that K is a field of *equal characteristic* if $\operatorname{char} k = \operatorname{char} K$; otherwise, we say K is of *mixed characteristic*.

The set $|K^\times| \subseteq (0, \infty)$ of absolute values attained by nonzero elements of K is called the *value group* of K, and it is indeed a subgroup of $(0, \infty)$ under multiplication. If $|K^\times|$ is a discrete subgroup of $(0, \infty)$ (or equivalently, if it is cyclic), we say that K is *discretely valued*. In that case, recalling our convention that non-archimedean absolute values are nontrivial, there is a largest real number ε in $(0, 1) \cap K$. Any element $\pi \in K$ such that $|\pi| = \varepsilon$ is called a *uniformizer* for K; note that $|K^\times| = |\pi|^{\mathbb{Z}}$. Note that a discretely valued field cannot be algebraically closed; for example, if $\pi \in K$ is a uniformizer, there is no element $x \in K$ such that $x^2 = \pi$.

We will almost always consider non-archimedean fields K that are *complete*, that is, for which all $|\cdot|$-Cauchy sequences in K converge in K. Any non-archimedean field K has a completion K_v formed by taking the set of $|\cdot|$-Cauchy sequences modulo an appropriate equivalence. There is a natural embedding $K \hookrightarrow K_v$, and the absolute value $|\cdot|$ on K extends canonically to K_v. The value group of K_v coincides with that of K, and any uniformizer of K is still a uniformizer of K_v. In addition, the residue fields of K and K_v are also canonically isomorphic.

If K is a *complete* non-archimedean field and L/K is an *algebraic* extension, then the absolute value on K extends canonically to L. In particular, for any $\alpha \in L$ with minimal polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in K[x]$, the absolute value of α is $|\alpha| := |a_0|^{1/n}$. Clearly, the value group $|L^\times|$ contains $|K^\times|$ as a subgroup. The index $e := [|L^\times| : |K^\times|]$, called the *ramification degree of* L/K, is of course either a positive integer or ∞; if $e = 1$, we say that L/K is *unramified*. Meanwhile, there is a natural inclusion of the residue field k of K in the residue field ℓ of L; the associated *residue field extension degree* $[\ell : k]$ is usually denoted f. We have the identity

$$[L : K] = ef.$$

A non-archimedean field K is locally compact (in the topology induced by $|\cdot|$) if and only if its ring of integers $\mathcal{O}_K$ is compact. Equivalently, K is locally compact if and only if it is discretely valued and has finite residue field. If K is locally compact and if L/K is a *finite* extension, then L is also locally compact. However, if K is a complete non-archimedean field and L is an *infinite* extension, then L is not locally compact.

Given any complete non-archimdean field K, then, its algebraic closure $\overline{K}$ may not be complete, but it still has the absolute value $|\cdot|$. We can then form the completion $\mathbb{C}_K$ of $\overline{K}$ with respect to $|\cdot|$. This field $\mathbb{C}_K$ is, fortunately, both complete *and* algebraically closed; in fact, it is the smallest field extension of K with both properties. The value group $|\mathbb{C}_K^\times| = |\overline{K}^\times|$ is dense in $(0, \infty)$, and thus neither $\mathbb{C}_K$ nor $\overline{K}$ is discretely valued. Meanwhile, the residue field of $\mathbb{C}_K$, which is canonically isomorphic to that of $\overline{K}$, is an algebraic closure of the residue field of K.

Example 2.1. Given any field $\mathbb{F}$, the associated field $\mathbb{F}((t))$ of formal Laurent series over $\mathbb{F}$ can be equipped with a non-archimdean absolute value $|\cdot|$ by fixing a real number $\varepsilon \in (0, 1)$ and defining $|\sum a_n t^n| = \varepsilon^{n_0}$, where n_0 is the smallest integer $n \in \mathbb{Z}$ such that $a_n \neq 0$. The value group is $|\mathbb{F}((t))^\times| = \varepsilon^{\mathbb{Z}}$; thus, $\mathbb{F}((t))$ is discretely valued, and $t \in \mathbb{F}((t))$ is a uniformizer, with $|t| = \varepsilon$. The associated ring of integers is simply the ring of power series

$$\mathbb{F}[[t]] := \Big\{ \sum_{n \geq 0} a_n t^n : a_n \in \mathbb{F} \Big\} = \{f \in \mathbb{F}((t)) : |f| \leq 1\}.$$

The maximal ideal $t\mathbb{F}[[t]]$ consists of all power series with zero constant term, and the quotient field $\mathbb{F}[[t]]/t\mathbb{F}[[t]]$ is canonically isomorphic to $\mathbb{F}$. In particular, $\mathbb{F}((T))$ is a complete non-archimedean field of equal characteristic, because $\operatorname{char}\mathbb{F} = \operatorname{char}\mathbb{F}((t))$.

If $\overline{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$, then the field $\overline{\mathbb{F}}((t))$ is an unramified extension of $\mathbb{F}((t))$ (that is, the ramification index is $e = 1$), and in fact it is a maximal unramified extension. The direct limit

$$\overline{\mathbb{F}}\langle\langle t\rangle\rangle := \varinjlim_{m} \overline{\mathbb{F}}((t^{1/m}))$$

under the obvious inclusions is an algebraic closure of $\mathbb{F}((t))$. (If you're shaky on direct limits, think of it as $\mathbb{F}((t))$ with *every* root $t^{1/m}$ of t adjoined.) However, $\overline{\mathbb{F}}\langle\langle t\rangle\rangle$ is not complete. (For example, $\overline{\mathbb{F}}\langle\langle t\rangle\rangle$ does not contain $\sum_{m\geq 1} t^{m+1/m}$, because any given element of $\overline{\mathbb{F}}\langle\langle t\rangle\rangle$ can only involve finitely many different $t^{1/m}$'s.) Its completion $\mathbb{C}_t$, the field of formal *Puiseux series* over $\overline{\mathbb{F}}$, consists of all formal sums of the form

$$\sum_{n\geq 0} a_n t^{r_n},$$

where $\{r_n\}_{n\geq 0}$ is a sequence of rational numbers increasing to ∞, and $a_n \in \overline{\mathbb{F}}$. The value group $|\mathbb{C}_t^\times|$ is $\varepsilon^{\mathbb{Q}}$; note that although it is dense in $(0,\infty)$, it is *not* the whole interval.

The original field $\mathbb{F}((t))$ and its finite extensions are locally compact if and only if $\mathbb{F}$ is finite. None of the infinite extensions of $\mathbb{F}((t))$ are locally compact.

Example 2.2. Let $p \geq 2$ be a prime number. The p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ is defined by $|ap^r|_p = p^{-r}$, where $r \in \mathbb{Z}$, and a is a rational number with numerator and denominator not divisible by p. In particular, all integers $n \in \mathbb{Z}$ satisfy $|n| \leq 1$; positive powers of p are small; and negative powers of p are big. The p-adic field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$, and it can be thought of as the set of all Laurent series in the "variable" p with "coefficients" in $\{0, 1, \ldots, p-1\}$. That is,

$$\mathbb{Q}_p := \Big\{ \sum_{n\geq n_0} a_n p^n : n_0 \in \mathbb{Z} \text{ and } a_n \in \{0,1,\ldots,p-1\}\Big\},$$

except that the addition and multiplication operations involve carrying of digits to higher powers of p.

The absolute value $|\cdot|_p$ extends to $\mathbb{Q}_p$ by $|\sum a_n p^n| = p^{-n_0}$, where n_0 is the smallest integer $n \in \mathbb{Z}$ such that $a_n \neq 0$. The value group is $|\mathbb{Q}_p^\times| = p^{\mathbb{Z}}$; in particular, $\mathbb{Q}_p$ is discretely valued, and the element $p \in \mathbb{Q}_p$ is a uniformizer, satisfying $|p| = 1/p$. The associated ring of integers $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is the subset consisting of power series (as opposed to Laurent series) in p; it is also the completion of $\mathbb{Z}$ with respect to $|\cdot|_p$. The maximal ideal $p\mathbb{Z}_p$ consists of all power series with zero constant term, and the quotient field $\mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to $\mathbb{F}_p$, the field of p elements. In particular, $\mathbb{Q}_p$ is a complete non-archimedean field of mixed characteristic, because $\operatorname{char}\mathbb{Q}_p = 0$, but $\operatorname{char}\mathbb{F}_p = p > 0$.

The completion of an algebraic closure of $\mathbb{Q}_p$ is denoted $\mathbb{C}_p$. As was the case for $\mathbb{C}_t$, the value group $|\mathbb{C}_p^\times| = p^{\mathbb{Q}}$ is dense in $(0,\infty)$ but is not the whole interval. Although $\mathbb{Q}_p$ and its finite extensions are locally compact, $\mathbb{C}_p$ is complete but not locally compact. Examples of the many elements of $\mathbb{C}_p \smallsetminus \overline{\mathbb{Q}}_p$ include

$$\sum_{n=1}^{\infty} p^{1/n} p^n \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n p^n,$$

where $\alpha_n \in \mathbb{Q}_p$ is a primitive $(p^n - 1)$-st root of unity.

2.2. **Disks.** Given a complete non-archimedean field K with absolute value $|\cdot|$, we denote the *open disk* and *closed disk* centered at $a \in \mathbb{C}_K$ and of radius $r > 0$ by

$$D(a,r) := \{x \in \mathbb{C}_K : |x-a| < r\}, \quad \text{and} \quad \overline{D}(a,r) := \{x \in \mathbb{C}_K : |x-a| \le r\},$$

where $\mathbb{C}_K$ is the completion of an algebraic closure of K. Note that we are using the convention that *all disks have positive radius.*

Disks in non-archimedean fields behave somewhat counterintuitively, as the following Proposition demonstrates.

Proposition 2.3. *Let K be a complete non-archimedean field Then:*

a. *Given $a, b \in \mathbb{C}_K$ and $s \ge r > 0$ such that $a \in D(b,s)$, we have $D(a,r) \subseteq D(b,s)$, and $D(a,s) = D(b,s)$.*
b. *Given $a, b \in \mathbb{C}_K$ and $s \ge r > 0$ such that $a \in \overline{D}(b,s)$, we have $\overline{D}(a,r) \subseteq \overline{D}(b,s)$, and $\overline{D}(a,s) = \overline{D}(b,s)$.*
c. *If $D_1, D_2 \subseteq \mathbb{C}_K$ are two disks such that $D_1 \cap D_2 \ne \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.*
d. *All disks in $\mathbb{C}_K$ are both open and closed* topologically.
e. *K and $\mathbb{C}_K$ are totally disconnected as topological spaces; that is, their only nonempty connected subsets are singletons.*

Proof. **(a.)** Given $x \in D(a,r)$, because $|a-b| < r \le s$ and $|x-a| < s$, we have $|x-b| \le \max\{|x-a|, |a-b|\} < s$, and hence $x \in D(b,s)$. The reverse inclusion in the case $r = s$ is similar.

(b.) Similar to (a.)

(c.) Pick $c \in D_1 \cap D_2$. By parts (a) and (b), the point c is a center of each disk; that is, each D_i can be written as either $D(c, r_i)$ or $\overline{D}(c, r_i)$. After possibly exchanging D_1 and D_2, either $r_1 > r_2$, or else $r_1 = r_2$ with either D_1 closed or D_2 open. Then $D_1 \supseteq D_2$.

(d.) To show that an open disk $D(a,r)$ is closed, pick any $x \in \mathbb{C}_K \smallsetminus D(a,r)$. If the two disks $D(x,r)$ and $D(a,r)$ intersect, then one contains the other, by part (c), and hence they coincide by part (a). That contradicts our assumption that $x \notin D(a,r)$, and therefore $D(x,r) \subseteq \mathbb{C}_K \smallsetminus D(a,r)$, as desired.

To show that a closed disk $\overline{D}(a,r)$ is open, pick any $x \in \overline{D}(a,r)$. Since the disks $\overline{D}(x,r)$ and $\overline{D}(a,r)$ intersect at x, one contains the other by part (c), and hence they coincide by part (b). Thus, $D(x,a) \subseteq \overline{D}(x,r) = \overline{D}(a,r)$, as desired.

(e.) Suppose $C \subseteq \mathbb{C}_K$ is set containing two distinct points a, b. Let $r = |a-b| > 0$. Then by part (d), $C \cap D(a,r) \ni a$ and $C \smallsetminus D(a,r) \ni b$ are both nonempty open subsets of C, and hence C is disconnected. Thus, the only connected subsets of $\mathbb{C}_K$ (and hence the only connected subsets of K) are the empty set and singletons. □

Proposition 2.3.a–b shows that non-archimedean disks do not have well-defined centers; indeed, every point of a disk can be called its center. The radius, however, *is* well-defined, at least in our context, because $\mathbb{C}_K$ is *not* discretely valued.

Proposition 2.4. *Let K be a complete non-archimedean field with absolute value $|\cdot|$. Let $D \subseteq \mathbb{C}_K$ be a disk, and define $r = diam(D) := \sup\{|x-y| : x, y \in D\}$.*

a. *If $r \in |\mathbb{C}_K^\times|$ and there exist $x, y \in D$ such that $|x-y| = r$, then for any $a \in D$, we have $D = \overline{D}(a,r)$. Moreover, this is the only way to write D as a disk; it cannot be written as an open disk, nor as a disk of radius other than r.*
b. *If $r \in |\mathbb{C}_K^\times|$ and $|x-y| < r$ for all $x, y \in D$, then for any $a \in D$, we have $D = D(a,r)$. Moreover, this is the only way to write D as a disk; it cannot be written as a closed disk, nor as a disk of radius other than r.*
c. *If $r \notin |\mathbb{C}_K^\times|$, then for any $a \in D$, we have $D = \overline{D}(a,r) = D(a,r)$. Moreover, this is the only way to write D as a disk; it cannot be written as a disk of radius other than r.*

Proof. For any $a \in D$, Proposition 2.3.a–b says that D may be written as either $D(a,s)$ or $\overline{D}(a,s)$ for some $s > 0$. Before considering the cases (a–c), we claim that s can only be r.

Given any $x, y \in D \subseteq \overline{D}(a,s)$, then $|x-y| \leq \max\{|x-a|, |y-a|\} \leq s$. Thus, $r \leq s$, by the definition of r as the diameter of D. However, if $r < s$, then $D \supseteq D(a,s)$. Because $\mathbb{C}_K$ is not discretely valued, there is some $c \in \mathbb{C}_K$ such that $r < |c| < s$. Thus, $a, a+c \in D(a,s) \subseteq D$, and hence

$$r = \operatorname{diam}(D) \geq |(a+c) - a| = |c| > r,$$

a contradiction. Thus, we must have $r = s$, as claimed.

We now consider the individual cases.

(a). We only need to show that $D \neq D(a,r)$. Given $x, y \in D$ such that $|x-y| = r$, we have

$$r \geq \max\{|x-a|, |y-a|\} \geq |x-y| = r,$$

and thus $|x-a| = r$ or $|y-a| = r$. Either way, $D \neq D(a,r)$, as desired.

(b). We only need to show that $D \neq \overline{D}(a,r)$. Pick $c \in \mathbb{C}_K$ such that $|c| = r$. If $D = \overline{D}(a,r)$, then choosing $x = a$ and $y = a + c$ gives $x, y \in D$ with $|x-y| = r$, contradicting the hypotheses.

(c). We only need to show that $D(a,r) = \overline{D}(a,r)$. The forward inclusion is obvious. For the reverse, given $x \in \overline{D}(a,r)$, we have $|x-a| \leq r$ but also, because $x - a \in \mathbb{C}_K$, we have $|x-a| \in |\mathbb{C}_K^\times| \cup \{0\}$. In particular, $|x-a| \neq r$. Thus, $|x-a| < r$, giving $x \in D(a,r)$, as desired. $\square$

Closed disks with radius in $|\mathbb{C}_K^\times|$ (as in Proposition 2.4.a) are called *rational closed disks*. Similarly, open disks with radius in $|\mathbb{C}_K^\times|$ (as in Proposition 2.4.b) are called *rational open disks*. Meanwhile, disks with radius not in $|\mathbb{C}_K^\times|$ (as in Proposition 2.4.c) are called *irrational disks*.

Any rational closed disk $\overline{D}(a,r)$ can be written as a disjoint union of open disks $D(b_\alpha, r)$ of the same radius, with one such disk for each element α of the residue field k. Indeed, after translating, we may assume $a = 0$; then, scaling by $z \mapsto cz$, where $r = |c|$ for some $c \in K^\times$, we may assume that $r = 1$. The rational closed disk $\overline{D}(0,1)$ is now just the ring of integers $\mathcal{O}_K$, and the open disk $D(0,1)$ is the unique maximal ideal $\mathcal{M}_K$. By the definition of the residue field as $k := \mathcal{O}_K/\mathcal{M}_K$, we have

$$\overline{D}(0,1) = \bigcup_{\alpha \in k} b_\alpha + \mathcal{M}_K,$$

where $b_\alpha \in \mathcal{O}_K$ is a coset representative for $\alpha \in k$. However, each set $b_\alpha + \mathcal{M}_K$ is simply $D(b,1)$, as desired.

We wrap up this section by discussing a phenomenon that does not arise in the archimedean context. The following technical definition is more appropriate for Section 2.1, but it could not be stated until we had discussed disks.

Definition 2.5. A K be a field with absolute value $|\cdot|$ is said to be *spherically complete* if for every decreasing sequence of disks $D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$, the intersection $\bigcap_{n \geq 1} D_n$ is nonempty.

Any spherically complete field is complete, and any locally compact field is spherically complete. (In particular, the archimedean fields $\mathbb{R}$ and $\mathbb{C}$ are spherically complete.) In addition, if K is a complete valued field, and if the radii of the disks D_n decrease to zero, then $\bigcap_{n \geq 1} D_n$ is a (nonempty) singleton.

However, it is possible for a complete nonarchimedean field K to allow the existence of decreasing sequences of disks with radii decreasing to a *positive* limit and with nonempty intersection. For example, the complete field $\mathbb{C}_p$ is not spherically complete, because there are decreasing sequences of disks like $D_n := \overline{D}(a_n, r_n)$, where

$$a_n := \sum_{j=1}^{n} p^{1-1/j}, \quad \text{and} \quad r_n := |p|_p^{1-1/n}.$$

The intersection $\bigcap_{n\geq 1} D_n$ is empty, because $r_n \searrow |p|_p$, and thus any point in the intersection would have to have all of the infinitely many terms $p^{1-1/j}$ in its p-adic expansion. However, that is impossible, since $\lim_{j\to\infty} p^{1-1/j}$ diverges in $\mathbb{C}_p$. Similarly, the function fields $\mathbb{C}_t$ of Example 2.1 are complete but not spherically complete.

Remark 2.6. It may seem strange to be given a field K but to work with disks in the field $\mathbb{C}_K$. Some words of explanation are probably in order.

To study Fatou and Julia sets, we will often need to take inverse images, and as a result, we will want to work with points defined over an algebraically closed field like $\overline{K}$. Although working with points in $\overline{K}$ will suffice for much of what we will do (for example, all of the results of this section still hold if we replace $\mathbb{C}_K$ by $\overline{K}$), it is often more convenient to work with points in its completion $\mathbb{C}_K$, even when completion is not strictly necessary. In addition, there will be times when completeness really *is* needed.

On the other hand, there will also be times when we want our *functions* $\phi(z)$ to be defined over a locally compact field K, even if the *points* that we want ϕ to act on are defined over the larger (and not locally compact) field $\overline{K}$ or $\mathbb{C}_K$. (This state of affairs is somewhat analogous to considering polynomials with rational coefficients acting on the complex plane.) This stipulation (which is largely a matter of taste) still allows us to deal with functions defined over $\mathbb{C}_K$ itself, of course, if we simply set K to be $\mathbb{C}_K$ to begin with. However, we will need to preserve the freedom to allow $K \subsetneq \mathbb{C}_K$, and thus we keep the long-winded language of having two fields, K and the completion $\mathbb{C}_K$ of its algebraic closure.

Remark 2.7. We noted in Remark 2.6 that the results of this section hold if we work with disks in $\overline{K}$ instead of in $\mathbb{C}_K$. However, If K is discretely valued, the conclusions of Proposition 2.4 fail if we work with disks not in $\mathbb{C}_K$ or $\overline{K}$ but in K itself. (That is, define $\overline{D}_K(a,r) := \overline{D}(a,r)\cap K$ and $D_K(a,r) := D(a,r)\cap K$.) After all, in that case, if π is a uniformizer of K, then

$$\overline{D}_K(a,|\pi|^n) = \overline{D}_K(a,r) = D_K(a,r) = D_K(a,|\pi|^{n-1})$$

for any $a\in K$, any integer $n\in\mathbb{Z}$, and any radius $r\in(|\pi|^n,|\pi|^{n-1})$. However, if K is not discretely valued, then Proposition 2.4 still applies to $\overline{D}_K(a,r)$ and $D_K(a,r)$, because our proof relied only on the fact that $|\mathbb{C}_K^\times|$ is dense in $(0,\infty)$.

3. Power series on disks

3.1. **Convergence of power series.** The convergence of series in non-archimedean fields is much simpler than in $\mathbb{R}$ or $\mathbb{C}$, because of the following elementary result.

Proposition 3.1. *Let K be a complete non-archimedean field, and let $\{a_n\}_{n\geq 0}\subseteq K$ be a sequence. Then*

$$\sum_{n\geq 0} a_n \text{ convereges in } K \quad \text{if and only if} \quad \lim_{n\to\infty} a_n = 0.$$

Proof. Let $s_n = a_0+\cdots+a_n$ denote the n-th partial sum of the series.

One direction is the same as in first-year calculus, as follows. If the series converges in K, then the sequence $\{s_n\}_{n\geq 0}$ is Cauchy. Given $\varepsilon>0$, pick $N\geq 0$ so that $|s_n - s_m|<\varepsilon$ for all $m,n\geq N$. Then choosing $m=n-1$, we have $|a_n|<\varepsilon$ for all $n\geq N+1$, as desired.

Conversely, and more interestingly, suppose $\lim_{n\to\infty} a_n = 0$. It suffices to show that $\{s_n\}_{n\geq 0}$ is Cauchy. Given $\varepsilon>0$, pick $N\geq 0$ such that $|a_n|<\varepsilon$ for all $n\geq N$. Then for any $n\geq m\geq N$, we have

$$|s_n - s_m| = |a_{m+1}+a_{m+2}+\cdots+a_n| \leq \max\{|a_{m+1}|,\ldots,|a_n|\} < \varepsilon \qquad \square$$

Corollary 3.2. *Let K be a complete non-archimedean field, let $\{a_n\}_{n\geq 0}\subseteq K$ be a sequence, and let $\sigma:\mathbb{N}\to\mathbb{N}$ be any permutation of the natural numbers. If $\sum_{n\geq 0} a_n$ converges, then so does $\sum_{n\geq 0} a_{\sigma(n)}$, and the two series have the same sum.*

Proof. The fact that $\sum_{n\geq 0} a_{\sigma(n)}$ converges is an immediate corollary of Proposition 3.1. However, the fact that the two sums are the same requires a little more work, and recovers the convergence as a byproduct, as follows.

Let $A = \sum_{n\geq 0} a_n$. Given $\varepsilon > 0$, there is an integer $N_1 \geq 0$ such that $|\sum_{n=0}^m a_n - A| \leq \varepsilon$ for all $m \geq N_1$. By Proposition 3.1, there is an integer $N_2 \geq 0$ such that that $|a_n| < \varepsilon$ for all $n \geq N_2$. (In fact, one can take $N_2 = N_1$, but we do not need that here.) Set $N' := \max\{N_1, N_2\}$, and $N := \max\{\sigma^{-1}(n) : 0 \leq n \leq N'\} \geq N'$.

Then for any $m \geq N$, the set $T := \{\sigma^{-1}(0), \sigma^{-1}(1), \ldots, \sigma^{-1}(N')\}$ is contained in the set $S := \{0, 1, \ldots, m\}$, and $|a_{\sigma(n)}| < \varepsilon$ for all $n \in S \smallsetminus T$. Thus,

$$\Big|\sum_{n=0}^{m} a_{\sigma(n)} - A\Big| = \Big|\Big(\sum_{n\in T} a_{\sigma(n)} - A\Big) + \sum_{n\in S\smallsetminus T} a_{\sigma(n)}\Big| \leq \max\Big\{\Big|\sum_{n=0}^{N'} a_n - A\Big|, \max_{n\in S\smallsetminus T}\{|a_{\sigma(n)}|\}\Big\} < \varepsilon,$$

as desired. □

Corollary 3.3. *Let K be a complete non-archimedean field, let $a \in K$, and let $\{c_n\}_{n\geq 0} \subseteq K$ be a sequence. Define $f(z)$ to be the power series*

$$f(z) = \sum_{n\geq 0} c_n (z-a)^n \in K[[z-a]].$$

a. *For any $r > 0$, if $\lim_{n\to\infty} |c_n| r^n = 0$, then f converges on $\overline{D}(a, r)$.*
b. *For any $r \in |\mathbb{C}_K^\times|$, $\lim_{n\to\infty} |c_n| r^n = 0$ if and only if f converges on $\overline{D}(a, r)$.*
c. *The region of convergence $X = \{x \in \mathbb{C}_K : f(x) \text{ converges}\}$ is either all of $\mathbb{C}_K$, a disk containing a, or the singleton $\{a\}$.*

Proof. Part (a) is immediate from Proposition 3.1, and part (b) follows from the same Proposition by choosing $x \in \mathbb{C}_K$ with $|x| = r$ and considering $f(a + x)$. For part (d), let

$$R := \sup\{r \in |\mathbb{C}_K^\times| : f \text{ converges on } \overline{D}(a, r)\}.$$

Since $D(a, R) = \bigcup_{0<r<R} \overline{D}(a, r)$, we have $X \supseteq D(a, R)$. Given any $b \in K$ with $|b| > R$, then by part (b), $f(a + b)$ does not converge; hence, $X \subseteq \overline{D}(a, R)$. Thus, it remains to consider what happens to $a + b$ if $|b| = R$. However, if $f(a + b)$ converges for even one such b, then f converges on $\overline{D}(a, R)$ by part (b), and hence $X = \overline{D}(a, R)$; otherwise, $X = D(a, R)$. Either way, then, we are done. □

As a contrast to part (b) of Corollary 3.3, if $r \notin |\mathbb{C}_K^\times|$, or if we are considering the open disk $D(a, r)$, the necessary and sufficient conditions for convergence on the disk are not quite so simple. Fortunately, the convergence conditions for rational closed disks will usually suffice for our purposes.

In complex analysis, an analytic function can be written as a power series centered at a given point a in the domain, and the power series converges on the largest disk centered at a and contained in the domain. The analogous result is true for non-archimedean power series, where we recall that any point of a disk can be considered its center.

Proposition 3.4. *Let K be a complete non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk, let $a, b \in D$, and let $f(z) \in K[[z - a]]$ be a power series converging on D. Then there is a power series $g(z) \in \mathbb{C}_K[[z - b]]$ converging on D such that $g(z) = f(z)$ for all $z \in D$. Moreover, the coefficients of g lie in the completion of the field $K(b - a)$.*

Proof. Write $f(z) = \sum_{n\geq 0} c_n (z - a)^n$, and define $g(z) := \sum_{n\geq 0} c'_n (z - b)^n$, where

$$c'_n := \sum_{m=n}^{\infty} \binom{m}{n} (b-a)^{m-n}.$$

This formula is chosen by writing the expression $(z-a)^n$ from the definition of $f(z)$ as $((z-b)+(b-a))^n$, expanding the latter expression to $\sum_{k=0}^{n}\binom{n}{k}(b-a)^{n-k}(z-b)^k$, and finally switching the order of summation in the expression for f. The two series have the same region of convergence and the same sum by Corollary 3.2. □

We close this section with a technical result that underlies much of the theory of non-archimedean analytic functions. However, it is used directly only rarely; instead, other theorems (especially Theorem 3.9 and Corollary 3.10) that invoke it are more commonly quoted. Thus, the reader may wish to skip its proof until sometime after seeing how it fits into the broader theory.

Theorem 3.5 (Weierstrass Preparation Theorem for Disks)**.** *Let K be a complete non-archimedean field, and let $\mathbb{C}_K$ be the completion of an algebraic closure of K. Let $r \in |\mathbb{C}_K^\times|$, let $f(z) = \sum_{n\geq 0} c_n z^n \in K[[z]]$ be a nonzero power series converging on $\overline{D}(0,r)$, and let $j \geq 0$ be the largest integer such that $|c_j|r^j = \max_{n\geq 0}|c_n|r^n$. Then there is a monic polynomial $g(z) \in K[z]$ of degree j and a power series $h[z] \in K[[z]]$ converging on $\overline{D}(0,r)$ such that $f = gh$, and $h(x) \neq 0$ for all $x \in \overline{D}(0,r)$.*

Note that the integer j does indeed exist, because the convergence of g on $\overline{D}(0,r)$ guarantees that $\lim_{n\to\infty}|c_n|r^n = 0$.

Theorem 3.5 is slightly stronger than the usual statement of the (dimension 1) Weierstrass Preparation Theorem, which sets $r = 1$; see Sections 5.2.1–5.2.2 of [17], for example. If K is algebraically closed (in which case $|\mathbb{C}_K^\times| = |K^\times|$), then Theorem 3.5 is a corollary of the $r = 1$ version, obtained by considering $f(bz)$, where $|b| = r$. However, the version above, which still keeps g and h defined over K even if $r \notin |K^\times|$, is slightly stronger. Still, its proof is essentially the same as that of the $r = 1$ statement.

The main tool needed to prove Theorem 3.5 is the following analogue of Euclidean division. We will give a nearly complete proof, but we will skip some of the more tedious details.

Lemma 3.6 (Weierstrass Division Theorem for Disks)**.** *Let K, $\mathbb{C}_K$, r, f, and j be as in Theorem 3.5. Let $F(z) \in K[[z]]$ be a power series converging on $\overline{D}(0,r)$. Then F can be written as $F(z) = Q(z)f(z)+R(z)$, where $Q \in K[[z]]$ is a power series converging on $\overline{D}(0,r)$, and $R(z) \in K[z]$ is a polynomial of degree strictly less than j.*

Sketch of Proof. We preview Section 3.3 and define the norm of a power series $F(z) = \sum_{n\geq 0} b_n z^n \in K[[z]]$ converging on $\overline{D}(0,r)$ by $\|F\| := \max\{|b_n|r^n\}$. It is not difficult to show that $\|F\| \geq 0$ with equality if and only if $F = 0$, and that $\|FG\| = \|F\| \cdot \|G\|$ and $\|F+G\| \leq \max\{\|F\|, \|G\|\}$. (See Proposition 3.20.)

Define $f_j(z)$ to be the degree-j truncation $f_j(z) := c_0 + \cdots + c_j z^j$ of $f(z)$, and note that $\|f - f_j\| < \|f\| = \|f_j\|$. We will first prove Lemma 3.6 for the polynomial f_j in place of the power series f.

Write $F(z) = \sum_{n\geq 0} b_n z^n$. For every $n \geq 0$, Euclidean division tells us that $b_n z^n$ may be written as $q_n f_j + g_n$, where $q_n, g_n \in K[z]$ with $\deg g_n < j$. Moreover, because $|c_j|r^j = \max_{n\geq 0}\{|c_n|r^n\}$, a closer look at the equation $b_n z^n = q_n f_j + g_n$ shows that $\|q_n\|\|f\|, \|g_n| \leq |b_n|r^n$. By Proposition 3.1, since $\lim_{n\to\infty}|b_n|r^n = 0$, we can sum across all n to obtain $F = Qf_j + R$ as desired. In addition, we again get the additional fact that $\|Q\|\|f_j\|, \|R\| \leq \|F\|$.

We now turn to the original statement of the Lemma. Given $F \in K[[z]]$ converging on $\overline{D}(0,r)$, set $F_0 = F$ and invoke the previous paragraph to define $Q_0 \in K[[z]]$ and $R_0 \in K[z]$ so that $F_0 = Q_0 f_j + R_0$, with all the associated properties. Next, define $F_1 := Q_1 \cdot (f_j - f)$, and obtain Q_1 and R_1 satisfying $F_1 = Q_1 f_j + R_1$. In general, given Q_{i-1} and R_{i-1}, set $F_i := Q_{i-1} \cdot (f_j - f)$, and obtain Q_i and R_i such that $F_i = Q_i f_j + R_i$.

Thus, we have a sequence $\{Q_i\}$ of power series in $K[[z]]$ converging on $\overline{D}(0,r)$, and a sequence $\{R_i\}$ of polynomials of degree at most $j-1$. Moreover, $\|F_i\| \leq \|F\| \cdot (\|f - f_j\|/\|f_j\|)^i$, and therefore $\|Q_i\| \leq \|F\| \cdot \|f_j\|^{-1}(\|f - f_j\|/\|f_j\|)^i$ and $\|R_i\| \leq \|F\| \cdot (\|f - f_j\|/\|f_j\|)^i$. Because $\lim_{i\to\infty}(\|f -$

$f_j\|/\|f_j\|)^i = 0$, we can define $Q := \sum_{i\geq 0} Q_i$ and $R := \sum_{i\geq 0} R_i$, from which it is a simple matter to check that Q and R have the desired properties, and that $F = Qf + R$. $\square$

Proof of Theorem 3.5. Write $z^j = Q(z)f(z) + R(z)$ according to Lemma 3.6, and set $g(z) := Q(z)f(z) = z^j - R(z)$, which is a monic polynomial of degree j. Setting $h = Q$ and writing $Q(z) = \sum_{n\geq 0} b_n z^n$, it suffices to show that $|b_n|r^n < |b_0|$ for all $n > 0$, because then for any $x \in \overline{D}(0,r)$, we have $|h(x) - b_0| < |b_0|$, and hence $h(x) \neq 0$, as desired.

To prove the claim, pick the largest $M \geq 0$ maximizing $|b_M|r^M$, and consider the z^{M+j} term of $Q(z)f(z)$. Its coefficient is a sum of terms of the form $b_m c_n$ with $m + n = M + j$, and satisfying $|b_m c_n| \leq |b_M c_j|$, with strict inequality unless $m = M$ and $n = j$. Thus, the z^{M+j}-coefficient of $Q(z)f(z)$ has absolute value $|b_M c_j| > 0$. However, $Q(z)f(z) = z^j - R(z)$ is a polynomial of degree j, and therefore $M + j \leq j$; that is, $M = 0$, as claimed. $\square$

3.2. **Zeros of power series.** The zeros of non-archimedean power series are generally far easier to study than their complex counterparts. Although a complete theory can be just as easily presented for arbitrary power series, however, it will be more convenient for our purposes to narrow our view to a slightly smaller class of power series.

Definition 3.7. Let K be a complete non-archimedean field, let $a \in \mathbb{C}_K$, and let $D \subseteq \mathbb{C}_K$ be a disk containing a. A *restricted power series on D* is a power series $f = \sum_{n\geq 0} c_n(z-a)^n \in \mathbb{C}_K[[z-a]]$ that converges on D, with the extra condition that if $D = D(a,r)$ is open, then there is some $m \geq 0$ such that $|c_n|r^n \leq |c_m|r^m$ for all $n \geq 0$.

If in addition, $a \in K$ and $c_n \in K$ for all n, we say that f is a *restricted power series on D defined over K.*

Recall that if $D = \overline{D}(a,r)$ is a rational closed disk, then the convergence of f is equivalent to the condition $\lim_{n\to\infty} |c_n|r^n = 0$, by Corollary 3.3.b. Meanwhile, the condition on restricted power series for open disks, while slightly weaker than this limit condition still excludes some power series convergent on $D(a,r)$, namely those for which $\{|c_n|r^n : n \geq 0\}$ does not attain its supremum, but for which $\lim_{n\to\infty} |c_n|s^n = 0$ for all $s < r$.

In complex analysis, one can define the order of vanishing of an analytic function at a point b as either the smallest derivative that does not vanish at b, or as the smallest order term in the power series expansion of the function at b. For power series over fields of characteristic $p > 0$, however, the p-th derivative is already identically zero, and therefore the first definition should not be used. Fortunately, the second definition still works perfectly well, as follows.

Definition 3.8. Let K be a complete non-archimedean field, let $a \in K$, and let $f(z) \in K[[z-a]]$ be a power series converging on a disk $D \subseteq \mathbb{C}_K$ containing a. We say that a point $b \in D$ is a *zero* of f if $f(b) = 0$. Recentering the power series at b as $f(z) = \sum_{n\geq 0} c'_n(z-b)^n$, we say that the *multiplicity* $\mathrm{ord}_b(f)$ of the zero of f at b, also known as the *order of vanishing* of f at b, is the smallest integer $m \geq 0$ such that $c'_m \neq 0$, or $m = \infty$ if f is identically zero.

We leave it to the reader to check that ord_b is a valuation; that is, $\mathrm{ord}_b(fg) = \mathrm{ord}_b(f) + \mathrm{ord}_b(g)$, and $\mathrm{ord}_b(f+g) \geq \min\{\mathrm{ord}_b(f), \mathrm{ord}_b(g)\}$. (Note that the first of these properties would fail in positive characteristic if we used derivatives to define the order of vanishing.) We need Definition 3.8 to properly state the following result, which is essentially a repackaging of the Weierstrass Preparation Theorem. It is used constantly in non-archimedean analysis, especially when computing examples.

Theorem 3.9 (Newton Polygons). *Let K be a complete non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk, let $a \in D \cap K$, and let*

$$f(z) = \sum_{n\geq 0} c_n(z-a)^n \in K[[z-a]]$$

be a nonzero restricted power series on D defined over K.

For any $r > 0$ such that $D(a,r) \subseteq D$, let $i \geq 0$ be the smallest integer such that

$$|c_i|r^i \max_{n\geq 0}\{|c_n|r^n\}.$$

In addition, if $\overline{D}(a,r) \subseteq D$, let $j \geq i$ be the largest such integer.

Then f has exactly i zeros in $D(a,r)$ and, if $\overline{D}(a,r) \subseteq D$, exactly j in $\overline{D}(a,r)$, counting multiplicity. More precisely, there is a polynomial $g_i \in K[x]$ of degree i whose roots are exactly the roots of f in $D(a,r)$, repeated according to multiplicity; and if $\overline{D}(a,r) \subseteq D$, there is a polynomial $g_j \in K[x]$ of degree j whose roots are exactly the roots of f in $\overline{D}(a,r)$.

In particular, if $\overline{D}(a,r) \subseteq D$, then f has exactly $j-i$ zeros $x \in \mathbb{C}_K$ satisfying $|x-a| = r$, and they are roots of a polynomial in $K[z]$ of degree $j-i$.

Proof. First, the restricted hypothesis implies that f converges on D, that the maximum is attained, and that the integer i (as well as the integer j, if $\overline{D}(a,r) \subseteq D$) is defined.

Second, if $\overline{D}(a,r) \subseteq D$ is a rational closed disk, then Theorem 3.5 gives us the existence of the degree j polynomial $g_j \in K[z]$, and the roots of g_j are exactly the zeros of f in $\overline{D}(a,r)$.

Next, consider an open disk $D(a,r) \subseteq D$, and let i be the smallest integer for which $|c_i|r^i = \max_{n\geq 0}\{|c_n|r^n\}$. Given any $s \in |\mathbb{C}_K^\times|$ with $s < r$, the largest integer j for which $|c_j|s^j = \max_{n\geq 0}\{|c_n|s^n\}$ must satisfy $j \leq i$. Thus, by the previous paragraph, f has no more than i zeros in any such $\overline{D}(a,s)$, and hence no more than i zeros in $D(a,r)$. Conversely, we can choose $s \in |\mathbb{C}_K^\times|$ with $s < r$ close enough to r so that $|c_n|s^n < |c_i|s^i$ for each $n = 0, \ldots, i-1$; and because $s < r$, the same inequality also holds for $n > i$. By the previous paragraph, then, f has exactly i zeros in $\overline{D}(a,s)$, and hence exactly i zeros in $D(a,r)$, all of which are roots of a degree i polynomial $g_i \in K[z]$.

Finally, to prove the last statement (in the case of the closed disk $\overline{D}(a,r)$), simply observe that the polynomial g_i divides the polynomial g_j in $K[z]$, because all the roots of g_i are roots of g_j with the same multiplicity. Thus, g_j/g_i is a polynomial in $K[z]$ of degree $j-i$. $\square$

The term "Newton polygon" in the name of Theorem 3.9 actually refers to the following object. Fix an element $\pi \in K$ with $0 < |\pi| < 1$; if K is discretely valued, we usually choose π to be a uniformizer. Define $v : \mathbb{C}_K \to \mathbb{R} \cup \{\infty\}$ by $v(x) = \log|x|/\log|\pi|$. (Thus, $v(\pi^n) = n$, and $v(K) = \mathbb{Z} \cup \{\infty\}$; note that $\log|\pi| < 0$. In fact, v is a valuation on K, if you know what that means.) Given a nonconstant power series $f(z) = \sum_{n\geq 0} c_n(z-a)^n$, we will construct a region in the Cartesian plane $\mathbb{R}^2$ as follows. For each $n \geq 0$, let R_n be the vertical ray extending upward from the point $(n, v(c_n))$. Finally, define P_f, the Newton polygon of f at a, to be the convex hull of $\bigcup_{n\geq 0} R_n$.

Clearly, P_f is a convex region in the plane, extending upward infinitely but with piecewise linear boundary along the bottom and sides. In spite of the moniker "polygon", it is of course technically an unbounded planar region with piecewise linear boundary.

The polygon P_f gives a convenient way to visualize the content of Theorem 3.9. For the purposes of this discussion, call a radius $r > 0$ *critical* for f if there is at least one zero $x \in \mathbb{C}_K$ of f with $|x-a| = r$. The reader can check that

r is a critical radius, with f having i zeros in $D(a,r)$ and $j > i$ zeros in $\overline{D}(a,r)$,

if and only if

$(i, v(c_i))$ and $(j, v(c_j))$ are vertices of the Newton polygon P_f at opposite ends of a single line segment on the boundary of slope $-\dfrac{\log r}{\log|\pi|}$.

(Verifying this equivalence is really just a matter of playing with the equation $|c_i|r^i = |c_j|r^j = \max_{n\geq 0}\{|c_n|r^n\}$ for i minimal and j maximal, as well as with the Newton polygon. See also

Section 6.5 of [30] or Section VI.2.2 of [47].) Moreover, Theorem 3.9 also says that all the zeros of f with that particular distance from a are the zeros of a degree $j-i$ polynomial in $K[z]$; in particular, for any such zero $\alpha \in \mathbb{C}_K$, we have $[K(\alpha):K] \le j-i$. Incidentally, it is common to call this quantity $j-i$ the *length* of the boundary segment to be $j-i$, even though, strictly speaking, it is really the length of the segment's shadow on the x-axis.

Less commonly noted, but equally easy to see, is that for *any* radius $r>0$, whether critical or not, the Newton polygon can also tell us the value of $t := \max_{n\ge 0}|c_n|r^n$, as least as long as f converges on some rational closed disk containing $\overline{D}(a,r)$. Indeed, because the maximum is attained, there is a unique line in the plane of slope $-\log r/\log|\pi|$ that intersects the boundary, but not the interior, of P_f. (Moreover, by our convergence assumption, the intersection is either one of the boundary segments or else one of the boundary vertices.) Then the y-intercept of this line is exactly $\log t/\log|\pi|$.

Corollary 3.10. *Let $\mathbb{C}_K$ be a complete non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk of radius $r>0$ containing a point $a \in \mathbb{C}_K$, and let $f(z) = \sum_{n\ge 0} c_n(z-a)^n$ be a nonconstant restricted power series on D. Then $f(D)$ is a disk of the same type (rational open, rational closed, or irrational) as D, and there is an integer $d \ge 1$ such that $f : D \to f(D)$ is everywhere d-to-1, counting multiplicity.*

More precisely, the radius $t := \max_{n\ge 1}\{|c_n|r^n\}$ is defined, and

a. *if $D = \overline{D}(a,r)$ is rational closed, then d is the* largest *integer $j \ge 1$ such that $|c_j|r^j = t$, and the image is $f(D) = \overline{D}(f(a),t)$.*
b. *if $D = D(a,r)$ is rational open or irrational, then d is the* smallest *integer $i \ge 1$ such that $|c_i|r^i = t$, and the image is $f(D) = D(f(a),t)$.*

Proof. Clearly $f(a) = c_0$, t is defined because f is restricted, and $t>0$ because f is nonconstant.

We first note that $f(D)$ is contained in $\overline{D}(c_0,t)$ or $D(c_0,t)$, as appropriate. Indeed, if $D = \overline{D}(a,r)$ is rational closed, then $|c_n(x-a)^n| \le |c_n|r^n \le t$ for any $x \in D$ and any $n \ge 1$; hence $|f(x)-c_0| \le t$. Similarly, if $D = D(a,r)$ is open, then $|c_n(x-a)^n| < |c_n|r^n \le t$ for any $x \in D$ and any $n \ge 1$; therefore $|f(x)-c_0| < t$, as desired.

Finally, Theorem 3.9 tells us that every point in $\overline{D}(f(a),t)$ has exactly d preimages in D, and we are done. □

Definition 3.11. The degree $d \ge 1$ of the map $f : D \to f(D)$ in Corollary 3.10 is called the *Weierstrass degree* of the power series f on the disk D.

Remark 3.12. If the disk D in Corollary 3.10 is not rational closed, and we only assume that power series f converges on D (rather than assuming it is restricted), the situation is a little more complicated. In that case, the image $f(D)$ can be either a disk or all of $\mathbb{C}_K$. If $t := \sup_{n\ge 1}|a_n|r^n$ is finite but not attained, then $f(D) = D(a,t)$, and the Weierstrass degree (i.e., the number of preimages of every point in $f(D)$, counting multiplicity) is infinite. In this case, even though D and $f(D)$ are both open, it is possible for one to be rational and the other to be irrational. Meanwhile, if $t = \infty$, then $f(D) = \mathbb{C}_K$, and the Weierstrass degree is again infinite.

The following two results, which push Corollary 3.10 a little farther, will prove very useful in the study of non-archimedean dynamics.

Proposition 3.13. *Let K be a complete non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk of radius $r>0$, and let $f(z)$ be a nonconstant restricted power series on D. Suppose that the image $f(D)$ is a disk of radius $s>0$. Then for all $x,y \in D$,*

$$|f(x)-f(y)| \le \frac{s}{r}\cdot|x-y|. \tag{2}$$

Moreover, if the Weierstrass degree of f on D is 1, then inequality (2) becomes an equality.

Proof. Given $x, y \in D$, write f as a power series $f(z) = \sum_{n\geq 0} c_n(z-y)^n$ centered at y. Then by Corollary 3.10, the radius s of $f(D)$ is $s = \max_{n\geq 1}\{|c_n|r^n\}$. Thus,

$$\begin{aligned} |f(x)-f(y)| &= \Big|\sum_{n\geq 1} c_n(x-y)^n\Big| = \Big|\sum_{n\geq 1} c_n(x-y)^{n-1}\Big| \cdot |x-y| \\ &\leq \max_{n\geq 1}\{|c_n||x-y|^{n-1}\} \cdot |x-y| \leq \max_{n\geq 1}\{|c_n|r^{n-1}\} \cdot |x-y| = \frac{s}{r}|x-y|. \end{aligned} \tag{3}$$

Finally, if the Weierstrass degree is 1, then the first inequality in (3) is equality because the $c_1(x-y)$ term has strictly larger absolute value than any of the terms $c_n(x-y)^n$ for $n \geq 2$. Meanwhile, the second inequality is also equality in that case, because the $n = 1$ term of the second maximum attains the maximum (even if other terms also attain the maximum), and so each term of each maximum is simply $|c_1|$. ☐

Bijective power series have inverses also given by power series, and defined over the same complete field, as the following result shows.

Proposition 3.14. *Let K be a complete non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk, and let $f \in \mathcal{A}_D$ be a restricted power series on D defined over K with Weierstrass degree 1. Then f has an inverse function $g : f(D) \to D$ given by is a restricted power series $g \in \mathcal{A}_{f(D)}$ defined over K.*

Proof. We will consider the case that $D = D(a, r)$ is open; the (rational) closed case is similar. Write $f(z) = \sum_{n\geq 0} c_n(z-a)^n$, with $|c_n|r^n \leq |c_1|r$ for all $n \geq 1$. Then $f(D) = D(c_0, s)$, where $s := |c_1|r$, by Corollary 3.10. Note that $c_1 \neq 0$, because the Weierstrass degree is 1 and not 0.

We will first find a formal power series that is the inverse to f under composition of formal power series. Write an arbitrary formal power series $g \in K[[z-c_0]]$ with constant term a as $g(z) = a + \sum_{n\geq 1} b_n(z-c_0)^n$. Then $g \circ f \in K[[z-a]]$ is of the form

$$g \circ f(z) = a + b_1c_1(z-a) + \sum_{n\geq 2}\Big(b_nc_1^n + \sum_{i=1}^{n-1} b_iF_{n,i}(c_1, \ldots, c_{n-1})\Big)(z-a)^n,$$

where $F_{n,i}$ is a polynomial, each monomial of which is of the form $\prod_{j=1}^{n-1} c_j^{e_j}$ with $\sum_j e_jc_j = n$ and $\sum_j e_j = i$. In particular, since $|c_j| \leq |c_1|r^{1-j}$, it is easy to compute that

$$|F_{n,i}(c_1, \ldots, c_{n-1})| \leq |c_1|^i r^{i-n} = s^i r^{-n}. \tag{4}$$

Thus, we may set $b_1 := c_1^{-1}$, and for each $n \geq 2$,

$$b_n := -c_1^{-n}\sum_{i=1}^{n-1} b_iF_{n,i}(c_1, \ldots, c_{n-1}) \in K,$$

and we have $g \circ f(z) = z$ as formal power series. Moreover, we claim that $|b_n|s^n \leq r$ for all $n \geq 1$. The claim is obvious for $n = 1$. For $n \geq 2$, by inequality (4) and the inductive assumption of the claim for all $1 \leq i < n$, we have

$$\Big|\sum_{i=1}^{n-1} b_iF_{n,i}(c_1, \ldots, c_{n-1})\Big| \leq \max_{1\leq i\leq n-1}\{|b_i|s^i r^{-n}\} \leq r^{1-n}.$$

Therefore, $|b_n|s^n \leq |c_1|^{-n}r^{1-n}s^n = r$, proving the claim.

By the claim, together with Corollary 3.10, g is a restricted power series on $D(c_0, s)$ of Weierstrass degree 1, with $g(D(c_0, s)) = D(a, r)$, and as previously noted, we have $g \circ f(z) = z$ on $D(a, r)$. It remains to show that $f \circ g(z) = z$ on $D(c_0, s)$. Given any $w \in D(c_0, s)$, we may write $w = f(z)$ for some $z \in D(a, r)$, and we compute

$$f \circ g(w) = f(g(f(z))) = f(z) = w.$$

☐

Another key result on zeros of non-archimedean polynomials is Hensel's Lemma, which we state in two versions.

Theorem 3.15 (Hensel's Lemma, version 1)**.** *Let K be a complete non-archimedean field with residue field k, and let $f(z) \in \mathcal{O}_K[z]$ be a monic polynomial. Suppose that when we reduce the coefficients of f modulo $\mathcal{M}_v$, the resulting polynomial $\overline{f} \in k[z]$ factors as $\overline{f} = \overline{g} \cdot \overline{h}$, where $\overline{g}, \overline{h} \in k[z]$ are relatively prime polynomials. Then there are polynomials $g, h \in \mathcal{O}_K[z]$ satisfying $\deg g = \deg \overline{g}$, $\deg h = \deg \overline{h}$, $f = gh$, and reducing the coefficients of g and h modulo $\mathcal{M}_v$ gives $\overline{g}$ and $\overline{h}$, respectively.*

Proof. See [17], Section 3.3.4. □

Theorem 3.16 (Hensel's Lemma, version 2)**.** *Let K be a complete non-archimedean field, let $f(z) \in \mathcal{O}_K[z]$ be a polynomial, and let $a \in \mathcal{O}_K$ be a point such that $|f(a)| < |f'(a)|^2$. Then f has a unique zero in $\overline{D}(a, |f(a)|/|f'(a)|)$, and this zero is K-rational.*

Sketch of Proof. Apply Newton's method to produce a sequence $a_0 = a, a_1, a_2, \ldots \in K$, prove that the sequence is Cauchy (using the $|f(a)| < |f'(a)|^2$ hypothesis), and observe that the resulting limit must be a root of f. Then prove uniqueness, again using $|f(a)| < |f'(a)|^2$. For a more detailed proof over $\mathbb{Q}_p$, see Section II.1.5 of [47] or Theorem 3.4.1 of [30]. □

Both versions of Hensel's Lemma can be used in combination with the Weierstrass Preparation Theorem (Theorem 3.5) to give more precise descriptions of the zeros of a power series and its polynomial divisors. In fact, both versions (especially the second) can be easily restated and proven for power series. Thus, after the Weierstrass Preparation Theorem and the Newton polygon detect the absolute value of zeros, Hensel's Lemma can go from there to help one decide whether the $j - i$ zeros of a given absolute value lie in a degree $j - i$ extension of K, or whether the associate polynomial factors, and the zeros actually lie in smaller extensions. However, for our purposes, we will make far more use of just the Weierstrass Preparation Theorem, especially by way of Corollary 3.10.

3.3. **Rings and norms arising from disks.** Recalling Definition 3.7, we now define the set of all restricted power series on a disk.

Definition 3.17. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $D \subseteq \mathbb{C}_K$ be a disk. If $D = \overline{D}(a, r)$ is rational closed, we define

$$\mathcal{A}_D = \overline{\mathcal{A}}(a, r) := \Big\{ \sum_{n \geq 0} c_n (z - a)^n \in \mathbb{C}_K[[z - a]] : \lim_{n \to \infty} |c_n| r^n = 0 \Big\},$$

and if $D = D(a, r)$ is open, we define

$$\mathcal{A}_D = \mathcal{A}(a, r) := \Big\{ \sum_{n \geq 0} c_n (z - a)^n \in \mathbb{C}_K[[z - a]] : \sup_{n \geq 0} \{|c_n| r^n\} \text{ is attained} \Big\}$$

to be the *ring of restricted power series on D*.

Clearly, if D and E are any two disks such that $D \subseteq E$, then $\mathcal{A}_E \subseteq \mathcal{A}_D$. In addition, for any point $b \in \overline{D}(a, r)$, we know from Proposition 2.3 that $\overline{D}(a, r) = \overline{D}(b, r)$, and it is a simple matter to check, after re-centering power series, that $\overline{\mathcal{A}}(a, r) = \overline{\mathcal{A}}(b, r)$. The analogous statement is true for open disks. Thus, the ring depends only on the disk itself, and not on our choice of center when describing it.

The fact that $\mathcal{A}_D$ is indeed a ring is a consequence of the following elementary result about the arithmetic of non-archimedean power series, which is completely analogous to the arithmetic of complex power series. Although we could have presented the following Proposition earlier (and applying to any convergent power series, not just restricted power series), it is more convenient, given our notation, to state it here.

Proposition 3.18. *Let K be a complete non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk, let $a \in D \cap K$, and let $f, g \in \mathcal{A}_D$. Then $f + g$, $f - g$, and fg also belong to $\mathcal{A}_D$. If $g(x) \neq 0$ for all $x \in D$, then $f/g \in \mathcal{A}_D$ also.*

Moreover, if $E \subseteq \mathbb{C}_K$ is another disk containing the image $f(D)$, and if $h \in \mathcal{A}_E$, then $h \circ f \in \mathcal{A}_D$.

Proof. We will assume that D is a rational closed disk $\overline{D}(a, r)$; the open case is similar. Write $f(z) = \sum_{n \geq 0} c_n(z - a)^n$, $g(z) = \sum_{n \geq 0} c'_n(z - a)^n$, and $h(z) = \sum_{n \geq 0} d_n(z - b)^n$. (Here, $f(D)$ is contained in some rational closed disk $\overline{D}(b, s) \subseteq E$.)

Clearly $f + g, f - g \in \mathcal{A}_D$. Next, we turn to $fg(z) = \sum_{n \geq 0}(c_0c'_n + \cdots + c_nc'_0)(z - a)^n$. Since $\lim_{i\to\infty} |c_i|r^i = \lim_{j\to\infty} |c'_j|r^j = 0$, we also get $\lim_{n\to\infty} |c_0c'_n + \cdots + c_nc'_0|r^n = 0$, as desired.

We skip f/g for the moment and turn to $h \circ f$. This time, we can formally rearrange terms to write

$$h \circ f(z) = \sum_{n\geq 0} d_n\Big(c_0 - b + \sum_{m\geq 1} c_m(z - a)^m\Big)^n = \sum_{n\geq 0} C_n(z - a)^n,$$

where each C_n is itself an infinite sum of terms of the form $B = d_i(c_0 - b)^{i-n}c_1^{e_1} \cdots c_i^{e_i}$, with $i \geq n$ and $\sum je_j = n$. Matching $(r^j)^{e_j}$ with $c_j^{e_j}$ for each $j \geq 1$, and noting that $|c_0-b| \leq s$ and $|c_m|r^m \leq s$, we obtain $|B|r^n \leq |d_i|s^i$ for some $i \geq n$. Thus, $|C_n|r^n \leq \max_{i\geq n} |d_i|s_i$, which approaches zero as $n \to \infty$.

Finally, to prove that $f/g \in \mathcal{A}_D$, it suffices to consider $1/g$. Because $g(D)$ omits 0, we must have $c'_0 \neq 0$, and $g(D)$ must be contained in the open disk $D(c'_0, |c'_0|)$, by Corollary 3.10. Writing $g(z) = c'_0(1 - \tilde{g}(z))$, the power series $\tilde{g}(z) = 1 - (c'_0)^{-1}g(z)$ has image $\tilde{g}(D)$ contained in $D(0, 1)$. Meanwhile, $1/g(z) = h \circ \tilde{g}(z)$, where $h(z) = (c'_0)^{-1}/(1 - z) = \sum_{n\geq 0}(c'_0)^{-1}z^n$, which satisfies $h \in \mathcal{A}(0, 1)$. By the previous paragraph, then, $1/g(z)$ belongs to $\mathcal{A}_D$. □

As foreshadowed in the proof of Lemma 3.6, we also have the following definition.

Definition 3.19. Given disks $E \subseteq D \subseteq \mathbb{C}_K$, we define $\|\cdot\|_E : \mathcal{A}_D \to \mathbb{R}$ by

$$\|f\|_E := \sup\{|f(x)| : x \in E\}.$$

If $E = \overline{D}(a, r)$ or $E = D(a, r)$, we will sometimes denote $\|\cdot\|_E$ by $\|\cdot\|_{\zeta(a,r)}$, or simply by $\zeta(a, r)$. As the notation suggests, and as is clear from the definition, $\zeta(a, r)$ depends only on the disk $\overline{D}(a, r)$, and not on the particular point a of the disk chosen to denote its center.

Meanwhile, the seemingly problematic use of the same notation $\zeta(a, r)$ to refer to the norms on two different disks, $\overline{D}(a, r)$ and $D(a, r)$, is in fact perfectly consistent. Indeed, as we shall see in Corollary 3.21, the norms associated to those disks agree for any power series in $\overline{\mathcal{A}}(a, r)$.

As a consequence of the previous paragraph, note that if $D = D(a, r)$ and $E = D(b, r)$ are two open disks such that $\overline{D}(a, r) = \overline{D}(b, r)$, then the associated norms $\|\cdot\|_D$ and $\|\cdot\|_D$ coincide (at least for power series in $\overline{\mathcal{A}}(a, r)$), even though D and E are disjoint. Thus, although the set of these disk-induced norms is in one-to-one correspondence with the set of closed disks in $\mathbb{C}_K$, it is *not* in one-to-one correspondence with the set of open disks. Instead, different open disks are identified with each other when we compare norms. This observation will be important when we discuss the Berkovich projective line. In fact, the use of the letter ζ is meant to evoke a complex variable, precisely because it will be an actual point in the Berkovich projective line, which is a space containing $\mathbb{P}^1(\mathbb{C}_K)$ but that, like the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, has nicer topological properties.

For the moment, however, we will treat $\|\cdot\|_{\zeta(a,r)}$ solely as a norm, and not yet as some kind of geometric point. First, however, we must verify that it *is* indeed a norm, and that it is bounded by $\|\cdot\|_D$.

Proposition 3.20. *Given disks $E \subseteq D \subseteq \mathbb{C}_K$, the function $\|\cdot\|_E : \mathcal{A}_D \to \mathbb{R}$ is a non-archimedean multiplicative norm, meaning that for all $f, g \in \mathcal{A}_D$,*

i. $\|f\|_E \geq 0$, *with equality if and only if $f = 0$.*

ii. $\|f+g\|_E \leq \max\{\|f\|_E, \|g\|_E\}$.
iii. $\|fg\|_E = \|f\|_E \cdot \|g\|_E$.

In addition, we have the bound

iv. $\|f\|_E \leq \|f\|_D$.

Moreover, for any $f \in \mathcal{A}_D$, if E is either the open disk $D(a,r)$ or the closed disk $\overline{D}(a,r)$, and if we write $f(z) = \sum_{n\geq 0} c_n(z-a)^n \in \mathbb{C}_K[[z-a]]$, then

$$\|f\|_E = \max_{n\geq 0}\{|c_n|r^n\}. \tag{5}$$

Proof. We begin with the final statement. We assume that $D = \overline{D}(a,r)$ is rational closed; the open case is similar. Set $t = \max_{n\geq 1}\{|c_n|r^n\}$, so that $\|f\|_{\zeta(a,r)} = \max\{|c_0|, t\}$, and, by Corollary 3.10, $f(\overline{D}(a,r)) = \overline{D}(c_0, t)$. If $|c_0| \leq t$, then $\overline{D}(c_0,t) = \overline{D}(0,t)$, and therefore $\sup\{|f(x)| : x \in \overline{D}(a,r)\} = t$. On the other hand, if $|c_0| > t$, then $|f(x) - c_0| < |c_0|$ for all $x \in \overline{D}(a,r)$, and therefore $\sup\{|f(x)| : x \in \overline{D}(a,r)\} = |c_0|$. Either way, then, we have $\|f\|_{\zeta(a,r)} = \max_{n\geq 0}\{|c_n|r^n\}$.

Properties (i) and (iv) are now immediate from equation (5). Property (ii) is clear from the non-archimedean inequality $|f(x)+g(x)| \leq \max\{|f(x)|, |g(x)|\}$ for all $x \in E$; similarly, $\|fg\|_E \leq \|f\|_E \cdot \|g\|_E$. Thus, it suffices to prove that $\|fg\|_E \geq \|f\|_E \cdot \|g\|_E$, and we turn to the characterization of $\|\cdot\|_E$ in equation (5).

Write $g(z) = \sum_{n\geq 0} c'_n(z-a)^n$, and let $M, N \geq 0$ be the smallest integers such that $|c_M|r^M = \|f\|_E$ and $|c'_N|r^N = \|g\|_E$. Then the coefficient of $(z-a)^{M+N}$ in the expansion of fg is $C_{M+N} := c_0c'_{M+N} + \cdots + c_{M+N}c'_0$, and the term $c_Mc'_N$ in this sum has absolute value strictly larger than all of the other terms. Thus, $|C_{M+N}|r^{M+N} = |c_Mc'_N|r^{M+N} = \|f\|_E\|g\|_E$, verifying that $\|fg\|_E \geq \|f\|_E \cdot \|g\|_E$, as desired. □

Corollary 3.21. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $a \in \mathbb{C}_K$, and let $r > 0$. Set $D = \overline{D}(a,r)$ and $E = D(a,r)$. Then for every $f \in \mathcal{A}_D$, $\|f\|_D = \|f\|_E$.*

Proof. Immediate from the final statement of Proposition 3.20. □

In light of the alternate description of $\|\cdot\|_E$ given by equation (5) in Proposition 3.20, we may now rephrase a portion of Corollary 3.10 in the following more compact fashion.

Corollary 3.22. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk, let $a \in D$, and let $f \in \mathcal{A}_D$ be a restricted power series on D. Then*

$$f(D) = \begin{cases} \overline{D}\big(f(a), \|f-f(a)\|_D\big) & \textit{if } D \textit{ is closed, or} \\ D\big(f(a), \|f-f(a)\|_D\big) & \textit{if } D \textit{ is open.} \end{cases}$$

Proof. Immediate from Corollary 3.10 and Proposition 3.20. □

Remark 3.23 (The Valuation Polygon)**.** If we fix $a \in \mathbb{C}_K$ and $f \in K[[z-a]]$, then in light of Corollary 3.22, it is natural to ask how fast the image disk $f(\overline{D}(a,s))$ grows as s grows— or perhaps, to simplify things a bit, how fast $\|f\|_{\zeta(a,s)}$ grows. (That is, consider the norm of f, not of $f - f(a)$.) It turns out that the best way to understand this growth is by taking the logarithm of both radii.

More precisely, if $f \in \mathcal{A}(a,r)$, define the function $\mu_{f,a} : (-\infty, \log r) \to \mathbb{R}$ by

$$N_{f,a}(\log s) := \log \|f\|_{\zeta(a,s)}.$$

By the definition of the norm, we have $N_{f,a}(\log s) = \max_{n\geq 0}\{|c_n| + n\log s\}$, from which it is clear that $N_{f,a}$, as the maximum of various linear functions, is concave down, continuous, and piecewise linear. If we assume for simplicity that $f(a) = 0$, or at least that f has at least one zero in all disks $\overline{D}(a,s)$ that we consider, then $N_{f,a}$ tells us that at a particular radius $s = s_0$, the image radius $\|f\|_{\zeta(a,s)}$ is growing like a degree-d monomial (specifically, $|c_d|s^d$), where d is the number of zeros of f in $\overline{D}(a,s)$.

The graph of the function $N_{f,a}$ is called either the *valuation polygon* or the *Newton copolygon* of f at a, and as the latter name suggests, it is in some sense dual to the Newton polygon. Indeed, each vertex of the copolygon corresponds to a critical radius, and hence to a line segment of the polygon. Meanwhile, each segment of the copolygon corresponds to an interval of radii $[s_i, s_j)$ on which the number of zeros is constant, equal to some integer d, and hence the copolygon segment, which has slope equal to d, corresponds to the vertex with x-coordinate d on the polygon. For more on valuation polygons and Newton polygons, see Section 4.3 of [1] Section 3 of [46], or for a broader and more accessible exposition, Section VI.1.6 of [47].

We close this section with a result about *inverse* images of disks under *polynomials.* The proof uses the following lemma, suggested by Remark 3.23.

Lemma 3.24. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $D, E \subseteq \mathbb{C}_K$ be disks, let $a \in D$, let $f \in \mathcal{A}_D$ be a restricted power series on D, and suppose that $f(a) \in E \subseteq f(D)$. Then there is a unique disk $U \subseteq D$ with $a \in U$ such that $f(U) = E$.*

Proof. Write $f(z) = \sum_{n\geq 0} c_n(z-a)^n$. Note that f is nonconstant because $E \subseteq f(D)$; in particular, we have $c_n \neq 0$ for some $n \geq 1$. Write D as either $D = D(a, r)$ or $D = \overline{D}(a, r)$; similarly, write E as either $\overline{D}(f(a), R)$ or $D(f(a), R)$.

Define $T : [0, r] \to \mathbb{R}$ by $T(s) = \max_{n\geq 1}\{|c_n|s^n\}$, which is continuous and, because $c_n \neq 0$ for some $n \geq 1$, strictly increasing function. Clearly $T(0) = 0$, and by hypothesis, $T(r) \geq R$. Thus, by the Intermediate Value Theorem, there is some radius $s_0 \in (0, r]$ such that $T(s_0) = R$. Moreover, because T is strictly increasing, s is unique.

If D is open and $s_0 = r$, then $\phi(D) = D(f(a), R)$, by Corollary 3.10. In that case, the assumption that $E \subseteq f(D)$ forces $E = D(f(a), R)$, and we may set $U = D$. Otherwise, $\overline{D}(f(a), R) \subseteq D$, and again by Corollary 3.10, we have $f(D(a, s_0)) = D(f(a), R)$, and $f(\overline{D}(a, s_0)) = \overline{D}(f(a), R)$. Either way, then, we have the existence of U.

Finally, if V is another such disk, then it must be either $D(a, s_0)$ or $\overline{D}(a, s_0)$, by Corollary 3.10 and the uniqueness of s_0. Without loss, then, we have $U = D(a, s_0)$ and $V = \overline{D}(a, s_0)$. Then by the same Corollary again, $D(f(a), R) = f(U) = f(V) = \overline{D}(f(a), R)$, which implies that $R \notin |\mathbb{C}_K^\times|$. Yet again by the same Corollary, $s_0 \notin |\mathbb{C}_K^\times|$ as well, and hence $U = D(a, s_0) = \overline{D}(a, s_0) = V$, proving uniqueness. □

With only a little more work, one can extend Lemma 3.24 to arbitrary power series converging on D, rather that just restricted power series; see, for example, Lemma 2.6 of [10]. However, we will not need that stronger result, and so we turn to the promised application to inverse images of disks.

Proposition 3.25. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $D \subseteq \mathbb{C}_K$ be a disk, and let $\phi \in \mathbb{C}_K[z]$ be polynomial of degree $d \geq 1$. Then $\phi^{-1}(D)$ is a disjoint union $D_1 \cup \cdots \cup D_m$ of disks of the same type as D, with $1 \leq m \leq d$. Moreover, for each $i = 1, \ldots, m$, there is an integer $1 \leq d_i \leq d$ such that every point in D has exactly d_i preimages in D, and such that $d_1 + \cdots + d_m = d$.*

Proof. Note that every point in $\mathbb{C}_K$ has exactly d preimages in $\mathbb{C}_K$, counting multiplicity. We will construct the disks $D_1, \ldots, D_m$ inductively.

Given $j \geq 1$, disjoint disks $D_1, \ldots, D_{j-1}$, and integers $d_1, \ldots, d_{j-1} \geq 1$ such that each D_i maps j-to-1 onto D, if $\phi^{-1}(D) = D_1 \cup \cdots \cup D_{j-1}$, then we must have $d_1 + \ldots + d_{j-1} = d$ and $1 \leq j-1 \leq d$, and so we are done by setting $m = j - 1$.

Otherwise, choose $a \in \phi^{-1}(D) \smallsetminus (D_1 \cup \ldots \cup D_{j-1})$, and view $\phi(z)$ as a restricted power series on some very large disk (i.e., large enough so that its image contains D). By Lemma 3.24, there is a unique disk D_j containing a for which $\phi(D_j) = D$, and by Corollary 3.10, there is an integer $d_j \geq 1$ such that $\phi : D_j \to D$ is d_j-to-1. Finally, D_j cannot intersect any other D_i. Indeed, if $D_j \cap D_i \neq \varnothing$, then one disk would contain the other, and by fixing any point in the smaller disk, the uniqueness statement of Lemma 3.24 would imply $D_j = D_i$, contradicting our choice of $a \notin D_i$. □

3.4. $\mathbb{P}^1(\mathbb{C}_K)$**-disks and affinoids.** Since we will be interested in the action of rational functions on the projective line, we will need the following more general definition of a disk in $\mathbb{P}^1(\mathbb{C}_K)$, to allow for disks containing the point ∞.

Definition 3.26. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field. A $\mathbb{P}^1(\mathbb{C}_K)$-*disk*, or simply a *disk* if the meaning is clear from context, is a subset of the form either D or $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus D$, where $D \subseteq \mathbb{C}_K$ is a disk.

A $\mathbb{P}^1(\mathbb{C}_K)$-disk is *rational closed* (respectively, *rational open*, *irrational*) if it is either of the form $D \subseteq \mathbb{C}_K$, where D is rational closed (respectively, rational open, irrational), or of the form $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus D$, where $D \subseteq \mathbb{C}_K$ is rational open (respectively, rational closed, irrational).

We leave it to the reader to check that any linear fractional transformation $\eta \in \mathrm{PGL}(2, \mathbb{C}_K)$ maps $\mathbb{P}^1(\mathbb{C}_K)$-disks to $\mathbb{P}^1(\mathbb{C}_K)$-disks, preserving the type (rational closed, rational open, or irrational). Indeed, a set $X \subseteq \mathbb{P}^1(\mathbb{C}_K)$ is a $\mathbb{P}^1(\mathbb{C}_K)$-disk if and only if it is the image of a disk $D \subseteq \mathbb{C}_K$ of the same type under some linear fractional transformation. Moreover, we have the following stronger result, applying to the action of any rational function.

Proposition 3.27. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let D be a $\mathbb{P}^1(\mathbb{C}_K)$-disk, and let $\phi(z) \in \mathbb{C}_K(z)$ be a nonconstant rational function. Then the image $\phi(D)$ is either all of $\mathbb{P}^1(\mathbb{C}_K)$ or else a $\mathbb{P}^1(\mathbb{C}_K)$-disk of the same type as D. In the latter case, there is some integer $d \geq 1$ such that every point of $\phi(D)$ has exactly d preimages in D, counting multiplicity.*

Proof. If $\phi(D) = \mathbb{P}^1(\mathbb{C}_K)$, we are done. Otherwise, replacing ϕ by $\eta \circ \phi$, where $\eta \in \mathrm{PGL}(2, \mathbb{C}_K)$ is a linear fractional transformation mapping some point outside of $\phi(D)$ to ∞, we may assume that ϕ has no poles in D.

Viewing $\phi = f/g$ as a quotient of two restricted power series on D (even though both happen to be polynomials), Proposition 3.18 tells us that ϕ can be written as a restricted power series on D, because g has no zeros in D. Thus, by Corollary 3.10, $\phi(D)$ is a disk of the same type as D, and $\phi : D \to \phi(D)$ is everywhere d-to-1, for some integer $d \geq 1$. □

The following result is a simple corollary of Proposition 2.3.

Proposition 3.28. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $D, E \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be $\mathbb{P}^1(\mathbb{C}_K)$-disks. If $D \cap E \neq \varnothing$, then either $D \cup E = \mathbb{P}^1(\mathbb{C}_K)$, $D \subseteq E$, or $E \subseteq D$.*

Proof. If $D \cup E \neq \mathbb{P}^1(\mathbb{C}_K)$, pick $a \in \mathbb{P}^1(\mathbb{C}_K) \smallsetminus (D \cup E)$. By applying a linear fractional transformation, we may assume without loss that $a = \infty$. Thus, D and E are both disks in $\mathbb{C}_K$, and Proposition 2.3.c completes the proof. □

Although Proposition 3.27 tells us that the image of a $\mathbb{P}^1(\mathbb{C}_K)$-disk under a rational function is also a disk (or else all of $\mathbb{P}^1(\mathbb{C}_K)$), the inverse image is usually not even a finite union of disks, as it is for polynomials, by Proposition 3.25. Similarly, although the union of two intersecting $\mathbb{P}^1(\mathbb{C}_K)$-disks is either $\mathbb{P}^1(\mathbb{C}_K)$ or one of the two disks, their intersection is often not a disk, as it is for disks in $\mathbb{C}_K$, by Proposition 2.3.c. Those observations are among many factors motivating the following definition.

Definition 3.29. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field. A *connected affinoid* is a nonempty intersection of finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks.

If all of the $\mathbb{P}^1(\mathbb{C}_K)$-disks $D_1, \ldots, D_n$ are closed (respectively, open, rational closed, rational open, irrational), we say that the connected affinoid $D_1 \cap \cdots \cap D_n$ is also closed (respectively, open, rational closed, rational open, irrational).

Equivalently, a connected affinoid is either $\mathbb{P}^1(\mathbb{C}_K)$ or else a $\mathbb{P}^1(\mathbb{C}_K)$-disk with finitely many sub-disks removed.

Note that we allow an intersection of zero $\mathbb{P}^1(\mathbb{C}_K)$-disks in Definition 3.29, and thus we consider $\mathbb{P}^1(\mathbb{C}_K)$ itself to be a connected affinoid. This convention, as well as our convention that the disks in the intersection can be arbitrary, rather than rational closed, conflicts with the usual definition in rigid analysis; see Remark 3.32. However, the above definition is more convenient for our purposes.

Note also that in spite of their name, connected affinoids are *not* connected as topological spaces, as they are subsets of the totally disconnected space $\mathbb{P}^1(\mathbb{C}_K)$. However, as we will see in Propositions 3.30 and 3.31, they behave in many ways like connected sets; see also Remark 3.32.

Proposition 3.30. *Let $\mathbb{C}_K$ be a complete non-archimedean field, let $U, V \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be connected affinoids, and suppose that $U \cap V \neq \varnothing$. Then $U \cap V$ and $U \cup V$ are also connected affinoids. If U and V are both of the same type (i.e., rational closed, etc.), then so are $U \cap V$ and $U \cup V$.*

Proof. Write $U = D_1 \cap \cdots \cap D_m$ and $V = E_1 \cap \cdots \cap E_n$ as nonempty intersections of $\mathbb{P}^1(\mathbb{C}_K)$-disks.

Clearly, $U \cap V$ is a finite intersection of $\mathbb{P}^1(\mathbb{C}_K)$-disks, and because it is nonempty by hypothesis, it is a connected affinoid, and of the same type as U and V.

Meanwhile, $U \cup V = \bigcap_i \bigcap_j (D_i \cup E_j)$. Because each $D_i \cap E_j$ contains $U \cap V \neq \varnothing$, Proposition 3.28 says that $D_i \cap E_j$ is either $\mathbb{P}^1(\mathbb{C}_K)$ or a $\mathbb{P}^1(\mathbb{C}_K)$-disk. Thus, $U \cup V$ is a finite intersection of $\mathbb{P}^1(\mathbb{C}_K)$-disks and of copies of $\mathbb{P}^1(\mathbb{C}_K)$, and hence it can be written as simply a finite intersection of $\mathbb{P}^1(\mathbb{C}_K)$-disks, all of the same type as U and V. In addition, $U \cup V \supseteq U \neq \varnothing$; thus, $U \cup V$ is a connected affinoid. □

We also have the following analogue of Corollary 3.10 and Proposition 3.25 for rational functions and connected affinoids, instead of polynomials and disks.

Proposition 3.31. *Let $\mathbb{C}_K$ be a complete non-archimedean field, let $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a connected affinoid, and let $\phi(z) \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 1$. Then $\phi(U)$ is also a connected affinoid of the same type, if any, as V. In addition, $\phi^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_m$ of connected affinoids, each of the same type, if any, as V, with $1 \leq m \leq d$. Moreover, for each $i = 1, \ldots, m$, there is an integer $1 \leq d_i \leq d$ such that every point in U has exactly d_i preimages in D, and such that $d_1 + \cdots + d_m = d$.*

Sketch of Proof. This result requires the deeper theory of rigid analysis (see Remark 3.32) and thus we must refer most of it to other sources.

In the case that U is a rational closed connected affinoid, the fact that $\phi(U)$ is also a rational closed connected affinoid is the content of Lemma A5.6 of [22]. The fact that $\phi^{-1}(U)$ is a finite union of rational closed connected affinoids is a little easier but still requires the machinery of rigid analysis; see Proposition 7.2.4 of [17].

However, the fact that each V_i maps d_i-to-1 onto U requires the discussion in [22] following Lemma A5.6; see Lemma 2.5.4 of [4], for example.

A more general connected affinoid U can be written as an infinite increasing union of rational closed connected affinoids. Thus, $\phi(U)$ is also such an infinite increasing union of connected affinoids, and $\phi^{-1}(U)$ is a finite union of infinite increasing unions of connected affinoids. Some more work then shows that these unions are in fact still connected affinoids, and of the same type (if any) as U. □

Remark 3.32 (Rigid Analysis)**.** Connected affinoids are a key object of study in the theory of rigid analysis, which was developed by Tate [55] in the 1960s in notes that were circulated but were not published until 1971. The impetus for the theory is the following problem. In complex analysis, it is appropriate to define an analytic function on a domain D to be a function that satisfies a certain local condition at every point of D. However, any analogous definition on a non-archimedean field K would be too general, because K is totally disconnected. Thus, for example, the function that is

1 on the open unit disk $D(0,1)$ but 0 on its complement would be (locally) analytic at every point of K. With the set of locally analytic functions being so large, many of the useful consequences of complex analysis, especially analytic continuation and the fact that a function meromorphic everywhere on $\mathbb{P}^1$ must be rational, would utterly collapse in the non-archimedean setting.

Rather than attempting to develop the whole theory of analytic functions and power series from a foolhardy local condition, then, Tate cut to the chase and considered only functions given by (restricted) power series, also called *rigid* analytic functions, giving the subject its name. The "rigidity" here is the fact that the function does not merely satisfy a local condition at each point, but instead is already given by a power series known to converge on a large set. Meanwhile, Tate also stipulated that one should only attempt to do analytic continuation by glueing series expansions on *overlapping* domains. However, because two disks can overlap only if one contains the other, he needed to work with domains more general than disks, and hence he developed affinoids.

The rigorous definition of affinoid domain, and especially affinoid subdomains, is rather abstract, involving maximal ideal spectra, representable functors, and other technicalities. The idea, however, is that one starts with a ring of (restricted) power series in several variables and takes a quotient by an appropriate ideal, to obtain an *affinoid algebra*. Then the set of maximal ideals of this algebra, equipped an appropriate topology, is an affinoid domain. Intuitively, the affinoid domain is the set of points at which all the series in the ring converge; conversely, the affinoid algebra is some appropriate ring of restricted power series converging on the domain.

For example, if we let $\mathcal{A}$ be the set of two-variable power series in $\mathbb{C}_K[[x,y]]$ whose coefficients approach zero, then the corresponding affinoid is the bidisk $\overline{D}(0,1) \times \overline{D}(0,1) \subseteq \mathbb{C}_K^2$. If we opt to take the quotient of $\mathcal{A}$ by the principal ideal $I = \langle xy - a\rangle$, for some $a \in \mathcal{O}_K \smallsetminus \{0\}$, then the affinoid corresponding to $\mathcal{A}/I$ is the annulus $\overline{D}(0,1) \smallsetminus D(0,|a|) \subseteq \mathbb{C}_K$. After all, we are effectively setting $y = a/x$, and we still have the stipulation that $|x|, |y| \leq 1$, which means that we are considering all $x \in \mathbb{C}_K$ such that $|a| \leq |x| \leq 1$.

Thus, in rigid analysis, as in algebraic geometry, one considers not just the set of points in a given domain, but also an appropriate ring of functions attached to that set. An affinoid is now said to be *connected* if its associated affinoid algebra cannot be written as the direct product of two nontrivial affinoid algebras. (Compare this condition with the complex case: if $U, V \subseteq \mathbb{C}$ are disjoint open sets, then the ring of analytic functions on $U \cup V$ is the direct product of the ring for U and the ring for V.) It turns out that any connected affinoid domain of $\mathbb{P}^1(\mathbb{C}_K)$ is simply a rational closed $\mathbb{P}^1(\mathbb{C}_K)$-disk with finitely many open $\mathbb{P}^1(\mathbb{C}_K)$-disks removed, and an arbitrary affinoid domain in $\mathbb{P}^1$ is simply a finite union of such connected affinoids.

We should also mention, as described in the comments following Definition 3.29, that our definition of "connected affinoid" is more general than the rigid analysis definition in two ways. First, we consider $\mathbb{P}^1(\mathbb{C}_K)$ itself to be a connected affinoid, whereas this is not the case in rigid analysis, as the root "affine" of the word "affinoid" suggests. Second, the connected affinoids of rigid analysis are all rational closed connected affinoids, in our terminology. For example, an open disk would *not* be considered an affinoid domain in the rigid analytic sense.

Still, we opt for the broader definition because our aims are somewhat different from those of rigid analysis. The goal of rigid analysis is to do non-archimedean analysis on algebraic varieties, and especially to do analytic continuation; thus, one cares about rational closed domains, where the restricted power series are *precisely* those that converge, and hence the analysis is a little easier. By contrast, our main interest is in overcoming the totally disconnected topology on $\mathbb{P}^1(\mathbb{C}_K)$ by working with large sets that behave like connected sets. Hence, we want to broaden our library of fundamental connected sets, even if the associated rings of power series are a little messier. After all, we will ultimately study only rational functions, which only make up a thin slice of any of the various affinoid algebras; power series that converge on a given open disk but are not in the associated ring of restricted power series will not concern us.

For a more detailed development of rigid analysis, the reader can consult Tate's published notes [55] or the exposition of the same material by Goss [29]. For a longer, more detailed, but still very readable exposition with more examples, especially in dimension one, see the excellent text [27]. Various other useful results in the theory appear in Coleman's paper [22]. Finally, the serious rigid analyst must consult [17], the utterly thorough, very general, and highly intimidating bible on the subject.

4. Fundementals of non-archimedean dynamics

In Section 1, we considered dynamics of a rational function $\phi(z)$ defined over an arbitrary field K. If we further assume that K is a valued field, i.e., that K is equipped with a nontrivial absolute value, then there is much more dynamical structure to study. Our focus, of course, is on the case that K is a non-archimedean field, and we model our theory on the case that $K = \mathbb{C}$; we refer the reader to [3, 21, 38] for introductions to the subject of complex dynamics.

4.1. **Classifying periodic points.**

Definition 4.1. Let K be a field with nontrivial absolute value $|\cdot|$, let $\phi(z) \in K(z)$ be a rational function, and let $x \in \mathbb{P}^1(K)$ be a periodic point with multiplier $\lambda \in K$. We say that x is

attracting if $|\lambda| < 1$,
repelling if $|\lambda| > 1$, or
indifferent, or *neutral*, if $|\lambda| = 1$.

In addition, we say that x is

superattracting if $\lambda = 0$,
rationally indifferent if λ is a root of unity, or
irrationally indifferent if $|\lambda| = 1$ but λ is not a root of unity.

The existence of certain types of fixed points is forced by the following result, which is an immediate corollary of the Holomorphic Fixed-Point Formula.

Proposition 4.2. *Let K be a field with nontrivial absolute value $|\cdot|$, and let $\phi(z) \in K(z)$ be a rational function.*

a. *If K is archimedean and $\deg\phi \geq 2$, then ϕ has repelling or rationally indifferent fixed point.*
b. *If K is non-archimedean, then ϕ has a nonrepelling fixed point.*

Proof. In either case, if there is a fixed point of multiplier 1, then we are done. In the non-archimedean case, if $d = 0$, then ϕ is constant, and hence the constant value is a superattracting fixed point of ϕ. Excluding these cases, then, we may assume that ϕ has no fixed points of multiplier 1 and that $\deg\phi \geq 1$, and hence we may invoke Theorem 1.6, the Holomorphic Fixed-Point Formula.

If K is archimedean, note that for any $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, we have $\mathrm{Re}(1/(1-\lambda)) \geq 1/2$. Therefore, if all the fixed points were nonrepelling, taking real parts of both sides of the Holomorphic Fixed-Point Formula gives $(d+1)/2 \leq 1$, contradicting the hypothesis that $d \geq 2$. Thus, ϕ must have a repelling fixed point.

If K is non-archimedean, then for any $\lambda \in \mathbb{C}_K$ with $|\lambda| > 1$, we have $|1/(1-\lambda)| = |\lambda|^{-1} < 1$. Therefore, if all the fixed points were repelling, taking absolute values of both sides of the Holomorphic Fixed-Point Formula gives $1 > 1$. Thus, there must be at least one non-repelling fixed point. □

The next result justifies the use of the adjectives "attracting" and "repelling". Like Proposition 4.2, we state it for both archimedean and non-archimedean fields, to emphasize some of the parallels and contrasts between the two cases.

Proposition 4.3. *Let K be a complete valued field, let $\phi(z) \in K(z)$ be a rational function, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$ be a periodic point of period $m \geq 1$.*

a. *If x is attracting, then there is an open set $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ containing x such that for all $y \in U$,*

$$\lim_{n\to\infty} \phi^{mn}(y) = x.$$

b. *If x is repelling, then there is an open set $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ containing x such that for all $y \in U \smallsetminus \{x\}$, there is some $n \geq 1$ such that $\phi^{mn}(y) \notin U$.*

c. *If K is non-archimedean, and if x is nonrepelling, then there is an open $\mathbb{P}^1(\mathbb{C}_K)$-disk $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ containing x such that $\phi^m(U) \subseteq U$.*

Proof. Taking the completion of an algebraic closure, we may assume that K is complete and algebraically closed. We will devote most of the proof to the case that $K = \mathbb{C}_K$ is a complete and algebraically closed non-archimedean field.

Considering ϕ^m in place of ϕ, we may assume that $m = 1$; and after a change of coordinates, we may assume that $x = 0$. Expanding ϕ as a restricted power series in $\mathbb{C}_K[[z]]$ (which is possible by Propopsition 3.18), we write

$$\phi(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots,$$

which converges on some rational closed disk $\overline{D}(0,r)$.

If $\lambda = 0$ (i.e., if 0 is a superattracting point), then let $j \geq 2$ be the smallest integer such that $a_j \neq 0$, and choose $s \in (0,r)$ small enough so that $|a_n|s^n < |a_j|s^j$ for all $n > j$. (The choice of such s is possible because ϕ is a restricted power series on $\overline{D}(0,r)$.) Furthermore, assume s is small enough so that in addition, $|a_j|s^{j-1} < 1$; this choice is possible because $j \geq 2$. Set $\varepsilon = |a_j|s^{j-1} < 1$ and $U = D(0,s)$. Thus, for any $z \in \overline{D}(0,s)$, we have

$$|\phi(z)| = |a_j z^j| \leq \varepsilon |a_j| s^{j-1} |z| = \varepsilon |z|.$$

In particular, $\phi(z) \in U$ also, and therefore, by induction, $|\phi^n(z)| \leq \varepsilon^n |z|$ for all $n \geq 0$. Hence, $\lim_{n\to\infty} \phi^n(z) = 0$, proving the superattracting portion of part (a).

If $\lambda \neq 0$, we can choose $s \in (0,r)$ small enough so that $|a_n|s^n < |\lambda|s$ for all $n > 1$. Set $U = D(0,s)$. Then for all $z \in U$, we have

$$|\phi(z)| = |\lambda| \cdot |z|.$$

If $|\lambda| \leq 1$, then we have $\phi(z) \in U$, and hence by induction, $|\phi^n(z)| = |\lambda|^n \cdot |z|$ for all $n \geq 0$. Hence, $\phi(U) \subseteq U$, proving part (c). If $|\lambda| < 1$, we also obtain $\lim_{n\to\infty} \phi^n(z) = 0$, proving part (a).

If $|\lambda| > 1$ and $y \in U \smallsetminus \{0\}$, set n to be the smallest integer for which $|\lambda|^n |y| \geq |s|$. Then $\phi^(y) \in U$ for all $i = 0, \ldots, n-1$, but $|\phi^n(y)| = |\lambda|^n |y|$, and hence $\phi^n(y) \notin U$, proving part (b).

The only other possibility is that $K = \mathbb{C}$. In this case, the proof (of parts (a) and (b) only, of course), is well known from complex dynamics. The argument is essentially the same as for the non-archimedean case, except that the formulas like $|\phi(z)| = |\lambda| \cdot |z|$ become approximate, rather than exact equalities. For the attracting and repelling cases, the small errors are swamped, and the result goes through; see Section 6.3 of [3], Sections II.2–II.3 of [21], or Section 8 of [38]. ☐

Remark 4.4 (Indifferent Points and Local Linearization)**.** Proposition 4.3.c begs the question of what happens at indifferent periodic points over $\mathbb{C}$. This question leads to a whole series of other questions involving local linearization, Diophantine approximation, and more. We will briefly discuss these ideas in this Remark, but to simplify notation, let us assume that the periodic point x is simply $x = 0$, and let us assume that it is a fixed point, i.e., that the period is $m = 1$.

If 0 is a rationally indifferent fixed point of $\phi \in \mathbb{C}(z)$, then the local dynamics are intricate but understood. Specifically, there are finitely many repelling directions extending radially from 0, and in between those directions, points are attracted to 0. However, this attraction is not like the exponential attraction $\phi^n(y) \cong \lambda^n y$ near attracting fixed points; instead, it is like the far slower attraction of points to ∞ under the map $z \mapsto z+1$. This arrangement of attracting "petals" separated by repelling rays is often called the "Leau-Fatou flower". See Section 6.5 of [3], Section II.5 of [21], or Section 10 of [38] for more information.

If 0 is an irrationally indifferent fixed point of $\phi \in \mathbb{C}(z)$, the situation is even more complicated. Specifically, understanding the dynamics near 0 comes down to the question of whether ϕ is locally linearizable at 0. That is, is there a power series $h \in \mathbb{C}[[z]]$ that converges and is injective on a connected open neighborhood U of 0, and such that $h(\phi(z)) = \lambda \cdot h(z)$, i.e., such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & U \\ h\downarrow & & h\downarrow \\ \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \end{array}$$

commutes? If so, then the dynamics on U are easily described in terms of h and λ; the maximal such open set U is called a *Siegel disk*, and such a domain is indeed always homeomorphic to the open disk. If not, then the dynamics are far more complicated; in fact, the fixed point 0 in this case belongs to the chaotic "Julia set" that we will soon introduce.

The determination of which of the two cases applies turns out to be related to Diophantine approximation. Specifically, the multiplier λ of the fixed point may be written as $\lambda = e^{2\pi i\theta}$, and we are assuming that θ is an irrational real number. If θ is badly approximable by rationals (for a certain very precise description of "badly"), then by a result of Bryuno [19], ϕ is locally linearizable at 0, and hence 0 lies in a Siegel disk. Conversely, at least in the case that ϕ is a quadratic polynomial, if θ does not satisfy Bryuno's bad approximability condition, then by a theorem of Yoccoz [58], ϕ is not locally linearizable at 0, and hence 0 lies in the Julia set.

The question of existence of the linearization in this setting is often called the "problem of small denominators". Although the results of Bryuno, Yoccoz, and others tell us quite a bit, a full understanding of complex dynamics near an irrationally indifferent fixed point remains open in general. For more on the subject, see Sections 6.6 and 6.7 of [3], Sections II.6 and V.1 of [21], or Section 11 of [38].

Incidentally, in both the archimedean and non-archimedean settings, one can always (and easily) produce a local linearization at a periodic point that is either repelling or attracting but not superattracting. This result was first proven over $\mathbb{C}$ by Kœnigs in 1884 [36]; see Sections 6.2 and 6.3 of [3], Sections II.2 and II.3 of [21], or Section 8 of [38] for modern presentations. The same proof works essentially verbatim over a complete non-archimedean field K; in fact, the resulting linearization $h(z)$ is defined over K itself, with no need to pass to $\mathbb{C}_K$. Moreover, at repelling points, the (inverse of the) linearization can actually be extended, by a clever trick attributed to Poincaré, to a (no longer injective) meromorphic h mapping all of $\mathbb{C}$ (respectively, $\mathbb{C}_K$) onto some larger neighborhood U of 0 in such a way that $h(\lambda z) = \phi \circ h(z)$ for all $z \in \mathbb{C}$ (respectively, $z \in \mathbb{C}_K$). See Corollary 8.10 of [38] for this result over $\mathbb{C}$, or Lemme 1 of [15] for the same result over $\mathbb{C}_p$; essentially the same proof also works for arbitrary $\mathbb{C}_K$.

As for superattracting points, clearly there cannot be a local linearization, for then the nonconstant map ϕ would be locally conjugate to a constant. Instead, the best thing that could be true is that if the power series for $\phi(z)$ at 0 is of the form $c_j z^j + O(z^{j+1})$ for some $j \geq 2$, then ϕ is locally conjugate to z^j. Böttcher [18] proved the existence of this conjugation in 1904; see Section 6.10 of [3], Section II.4 of [21], or Section 9 of [38] for the proof. Once again, the same proof works over $\mathbb{C}_K$. More precisely, over any complete non-archimedean field K, one can locally conjugate ϕ to $c_j z^j$ via a power series defined over K; then, to obtain z^j, one needs to pass to the field extension $K(c_j^{1/(d-1)})$.

Meanwhile, for irrationally indifferent points over $\mathbb{C}_K$, Herman and Yoccoz [32] proved a very general result, applying to arbitrary non-archimedean fields and maps on spaces of arbitrary dimension, about the existence of local linearizations. In the case of $\mathbb{P}^1$, if $\operatorname{char} K = 0$, their result settles the question completely: for *any* irrationally indifferent point, there is a neighborhood on which a local linearization is defined. However, if $\operatorname{char} K = p > 0$, the question remains open. Lindahl [37] completely settled the question in characteristic 2 for quadratic polynomials and gave some sufficient conditions in slightly greater generality, but beyond his results, it appears that virtually nothing is known about local linearization in positive characteristic.

Lastly, unless ϕ itself is linear, there cannot be a local linearization at a rationally indifferent point over any field. Indeed, since $\lambda^e = 1$ for some $e \geq 1$, if $\phi(z)$ were locally conjugate to λz, then $\phi^e(z)$ would be locally conjugate to z, implying that $\phi^e(z) = z$, and hence ϕ itself is linear.

In light of the importance of linearization over $\mathbb{C}$, then, this last fact contrasts with Proposition 4.3.c, which featured a map remaining controlled even near a rationally indifferent point, at which there is no local linearization (When we develop the language of Fatou and Julia sets, we will be able to say that this fixed point lies in the Fatou set.) Moreover, this is no isolated case. Even for irrationally indifferent points, at which there *is* a linearization, the maximum possible domain of the linearization is usually far smaller than the controlled region U of Proposition 4.3.c. (Again, in our upcoming language, the associated component of the Fatou set properly contains the domain of linearization, whereas in the complex setting, the two regions coincide.) Indeed, as we shall see, at least over $\mathbb{C}_p$, the maximal region U will always contain *infinitely many* other periodic points, and thus the linearization cannot possibly extend to all of U.

Given the conclusion of Proposition 4.3 and the fact that a periodic point can be classified as lying in the Fatou set or not merely by checking the absolute value of its multiplier, local linearization has not seen much study in non-archimedean dynamics. The interested reader is encouraged to read both [32] and [37], which by themselves comprise essentially all of the current literature on non-archimedean linearization; after that, see what you can prove yourself!

4.2. **Good reduction.** Let K be a non-archimedean field with absolute value $|\cdot|$, ring of integers $\mathcal{O}_K$, maximal ideal $\mathcal{M}_K \subseteq \mathcal{O}_K$, and residue field $k = \mathcal{O}_K/\mathcal{M}_K$. The natural reduction map $\mathcal{O}_K \twoheadrightarrow k$, denoted $a \mapsto \overline{a}$, induces a map

$$\mathbb{P}^1(K) \to \mathbb{P}^1(k),$$

also called the reduction map, sending $[x, y] \mapsto [\overline{x}, \overline{y}]$, for $x, y\mathcal{O}_K$ chosen so that at least one of x, y has absolute value 1. (The reader can check that such x, y can always chosen, and that this definition is independent of the choices of x, y subject to those constraints.) Of course, the reduction map sends every point in $\mathcal{O}_K \subseteq \mathbb{P}^1(K)$ to the corresponding residue class in $k \subseteq \mathbb{P}^1(k)$, and it sends every point in $\mathbb{P}^1(K) \smallsetminus \mathcal{O}_K$ to the point $\overline{\infty}$ at infinity in $\mathbb{P}^1(k)$.

It is common in number theory to talk about whether a given object makes sense modulo $\mathcal{M}$. The following definition of what it means for a dynamical system to make sense modulo $\mathcal{M}$ was first proposed in the introduction of [39]. To state it, given a polynomial $f(x, y) \in \mathcal{O}_K[x, y]$, we define its reduction $\overline{f}(x, y) \in k[x, y]$ by replacing each coefficient in $\mathcal{O}_K$ by its reduction in k.

Definition 4.5. Let K be a non-archimedean field with absolute value $|\cdot|$, residue field k and algebraic closure $\overline{K}$. Let $\overline{k}$ be an algebraic closure of k. Let $\phi(z) \in K(z)$ be a rational function with homogenous presentation

$$\phi([x, y]) = [f(x, y), g(x, y)],$$

where $f, g \in \mathcal{O}_K[x, y]$ are relatively prime homogeneous polynomials of degree $d = \deg \phi$, and at least one coefficient of f or g has absolute value 1. We say that ϕ has *good reduction* at v if $\overline{f}, \overline{g} \in k[x, y]$ have no common zeros in $k \times k$ besides $(x, y) = (0, 0)$; that is, if $\overline{f}, \overline{g}$ are still relatively prime in $k[x, y]$.

If ϕ does not have good reduction, we say it has *bad reduction*.

If there is some linear fractional transformation $\eta \in \mathrm{PGL}(2, \overline{K})$ such that $\eta \circ \phi \circ \eta^{-1}$ has good reduction, we say that ϕ has *potentially good reduction*.

Equivalently, in scheme-theoretic language, ϕ has good reduction if and only if it extends from a $\operatorname{Spec} K$-morphism of the schemes $\mathbb{P}^1_K \to \mathbb{P}^1_K$ to a $\operatorname{Spec} \mathcal{O}_K$-morphism of the schemes $\mathbb{P}^1_{\mathcal{O}_K} \to \mathbb{P}^1_{\mathcal{O}_K}$. Put another way, if two points $z, w \in \mathbb{P}^1(\mathbb{C}_K)$ have the same reduction $\overline{z} = \overline{w} \in \mathbb{P}^1(\overline{k})$, then also $\overline{\phi(z)} = \overline{\phi(w)}$, and this common point is $\overline{\phi}(\overline{z})$. We leave it to the reader to check that the notion of good reduction, as well as the reduced map $\overline{\phi} := [\overline{f}, \overline{g}]$, is unaffected by the choice of $f, g \in \mathcal{O}_K[x, y]$, as long as they fit the requirements that at least one of them have a coefficient of absolute value 1.

Note that ϕ will have bad reduction if there is a residue class of $\mathbb{P}^1(\mathbb{C}_K)$ containing both a zero and a pole of ϕ. However, ϕ can also have bad reduction if all coefficients of the numerator lie in $\mathcal{M}_K$ while at least one coefficient of the denominator has absolute value 1 (and the rest lie in $\mathcal{O}_K$), or vice versa. In addition, whether or not a map has good reduction can depend on the choice of coordinate, which is why we have defined the notion of potential good reduction.

Example 4.6. Pick $\pi \in K$ such that $0 < |\pi| < 1$. Then the function $\phi_1(z) = \pi z^2 \in \mathbb{C}_K[z]$ has bad reduction, because in homogeneous coordinates it is $[\pi x^2, y^2]$, the reduction of which is $[0, y^2]$, and both 0 and y^2 are zero at the point $[1, 0]$.

However, conjugating by $\eta(z) = \pi z$ gives $\psi(z) := \eta \circ \phi_1 \circ \eta^{-1}(z) = z^2 \in \mathbb{C}_K[z]$. In homogeneous coordinates, ψ is $[x^2, y^2]$, which is degenerate only at the already degenerate point $[0, 0]$. Thus, ψ has good reduction, and therefore ϕ_1 has potentially good reduction.

Similarly, $\phi_2(z) = z^2/\pi$ has bad reduction, because in homogeneous coordinates it is $[x^2, \pi y^2]$, which reduces to the degenerate map $[x^2, 0]$. However, conjugating by $z \mapsto z/\pi$ gives the good reduction map $\psi(z) = z^2$ again; thus ϕ_2 also has potentially good reduction.

On the other hand, $\phi_3(z) = (z^2 - 1)/\pi$, whose homogeneous presentation $[x^2 - y^2, \pi y^2]$ reduces to the degenerate map $[x^2 - y^2, 0]$ (which, this time, is degenerate at both $[-1, 1]$ and $[1, 1]$). Unlike ϕ_1 and ϕ_2, if the residue characteristic of K is not 2, then it turns out that ϕ_3 does not even have potentially good reduction. Indeed, ϕ_3 has a fixed point at each of the roots of $z^2 - \pi z - 1 = 0$, and the condition on the residue characteristic guarantees that both of these fixed points are repelling. As we will soon see, maps of good reduction cannot have repelling periodic points, and hence maps of potentially good reduction cannot have such points either.

Proposition 4.7. *Let K be a non-archimedean field with absolute value $|\cdot|$, and let $\phi_1, \phi_2 \in K(z)$ be rational functions of good reduction. Then the composition $\phi_1 \circ \phi_2$ also has good reduction.*

Proof. Write $\phi_1 = [f_1, g_1]$ and $\phi_2 = [f_2, g_2]$. Then the reductions of the two coordinates of $\phi_1 \circ \phi_2$ are $\overline{F}(x, y) := \overline{f}_1(\overline{f}_2(x, y), \overline{g}_2(x, y))$ and $\overline{G}(x, y) := \overline{g}_1(\overline{f}_2(x, y), \overline{g}_2(x, y))$. If $F(x, y) = G(x, y) = 0$ for some point $(x, y) \in \overline{k} \times \overline{k}$, then because ϕ_1 has good reduction, we must have $\overline{f}_2(x, y) = \overline{g}_2(x, y) = 0$. Thus, because ϕ_2 has good reduction, we have $(x, y) = (0, 0)$, as desired. $\square$

For polynomials, it is easy to check for good reduction, at least in the given coordinate.

Proposition 4.8. *Let K be a non-archimedean field with absolute value $|\cdot|$ and ring of integers $\mathcal{O}_K$. A non-constant polynomial $\phi(z) \in K[z]$ has good reduction if and only if all of its coefficients lie in $\mathcal{O}_K$, and the lead coefficient has absolute value* 1.

Proof. Write $\phi(z) = c_d z^d + \cdots + c_0$, and let $j \geq 0$ be the largest integer maximizing $|c_j|$. If $|c_j| > 1$, then ϕ has homogeneous presentation

$$[c_j^{-1} c_d x^d + \cdots + c_j^{-1} c_1 x y^{d-1} + c_j^{-1} c_0 y^d, c_j^{-1} y^d],$$

which reduces to $[h(x, y), 0]$, where $h \in k[x, y]$ is a nonzero homogeneous polynomial of degree d. Thus, h has some nontrivial zeros in $\overline{k} \times \overline{k}$ Similarly, if $|c_j| < 1$, then ϕ has homogeneous presentation

$$[c_d x^d + \cdots + c_1 x y^{d-1} + c_0 y^d, y^d],$$

which reduces to $[0, y^d]$, which is degenerate at $[1, 0]$.

Hence, if ϕ has good reduction, then $\phi \in \mathcal{O}_K$, and at least one coefficient of K has absolute value 1. Given such a polynomial ϕ, if $|c_d| < 1$, then the reduction of the homogeneous presentation $[c_d x^d + \cdots + c_0 y^d, y^d]$ of ϕ is $[\overline{c}_j x^j y^{d-j} + \cdots + c_0 y^d, y^d]$, which is degenerate at $[1, 0]$ Thus, we have proven that good reduction implies that ϕ has the desired form.

Conversely, if $\phi \in \mathcal{O}_K[z]$ with $|c_d| = 1$, then the reduction of the homogeneous presentation of ϕ is $[\overline{c}_d x^d + \cdots + c_0 y^d, y^d]$. To evaluate to $[0, 0]$, we must have $y = 0$, and hence also $\overline{c}_d x^d = 0$, which implies $x = 0$. Thus, ϕ has good reduction, as claimed. □

Remark 4.9 (The Homogeneous Resultant)**.** For rational functions $\phi \in K(z)$, it harder to tell at a glance whether or not ϕ has good reduction. Fortunately, the reduction type (good or bad) can be determined quickly by computing the resultant of the numerator and denominator of ϕ.

Recall that if $f, g \in K[z]$ are monic polynomials of degrees d and e, respectively, then the *resultant* of f and g is the quantity

$$\operatorname{Res}(f, g) := \prod_{\alpha} \prod_{\beta} (\alpha - \beta),$$

where the products are taken over all roots $\alpha \in \overline{K}$ of f and $\beta \in \overline{K}$ of g, each repeated according to multiplicity. To extend to polynomials that are not necessarily monic, multiply the above product by $a_d^e b_e^d$, where a_d is the lead coefficient of f, and where b_e is the lead coefficient of g. This same quantity can be computed directly from the coefficients of f and g as the determinant of the *Sylvester matrix*, which is a certain $(d + e) \times (d + e)$ matrix with e rows featuring the coefficients of f, and d rows featuring the coefficients of g. The first of the f rows is

$$[a_0 \, a_1 \, \cdots \, a_d \, 0 \, 0 \, \cdots \, 0],$$

and each of the next $e - 1$ rows shifts the columns one more place to the right; the e rows for g are similar.

However, in our case, because we are considering homogeneous polynomials, it makes more sense to consider the *homogeneous resultant* of $f, g \in K[x, y]$, where f and g are homogeneous *of the same degree* $d \geq 1$. Then f and g can be factored as

$$f(x, y) = \prod_{i=1}^{d} (\beta_i x - \alpha_i y) \quad \text{and} \quad g(x, y) = \prod_{i=1}^{d} (\delta_i x - \gamma_i y),$$

with $[\alpha_i, \beta_i], [\gamma_i, \delta_i] \in \mathbb{P}^1(\overline{K})$. (The quantity $\beta_i x - \alpha_i y$ is really the wedge (or exterior) product $(x, y) \wedge (\alpha_i, \beta_i)$, although one does not need to know anything about exterior products to work with this expression.) Then the homogeneous resultant is

$$\operatorname{Res}(f, g) := \prod_{i=1}^{d} \prod_{j=1}^{d} (\alpha_i, \beta_i) \wedge (\gamma_j, \delta_j).$$

Again, the homogeneous resultant can be computed directly from the coefficients of f and g as the determinant of the homogeneous Sylvester matrix, which is a $2d \times 2d$ matrix constructed in exactly the same way as the usual Sylvester matrix.

The relationship of the resultant to good reduction is that $\operatorname{Res}(f, g)$ is zero if and only if f and g have a root in common. Similarly, the homogeneous resultant is zero if and only if $(\alpha_i, \beta_i) \wedge (\gamma_i, \delta_j) = 0$ for some i, j, which happens exactly when (α_i, β_i) and (γ_i, δ_j) lie on a common line, or, equivalently, if they describe the same point in $\mathbb{P}^1$. Thus, if we choose $f, g \in \mathcal{O}_K$ as in Definition 4.5, i.e., with at least one coefficient having absolute value 1, then their homogeneous resultant is zero modulo the maximal ideal $\mathcal{M}$ if and only if some $(\alpha_i, \beta_i) \wedge (\gamma_i, \delta_j)$ is zero modulo the maximal ideal, or equivalently, if and only if f and g have a common nontrivial root in $\overline{k} \times \overline{k}$. In other words, the homogeneous resultant is zero modulo $\mathcal{M}$ if and only if ϕ has bad reduction.

For example, the map $\phi(z) = (10z^2+4z-11)/(3z-2)$ can be written in homogeneous coordinates as $[10x^2 + 4xy - 5y^2, 3xy - 19y^2]$, and this presentation fits the conditions of Definition 4.5 for $\mathbb{Q}_p$ for each prime p of $\mathbb{Z}$. The associate Sylvester matrix is

$$\begin{bmatrix} 10 & 4 & -11 & 0 \\ 0 & 10 & 4 & -11 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix},$$

which has determinant $350 = 2 \cdot 5^2 \cdot 7$. Thus, ϕ has bad reduction over $\mathbb{Q}_p$ for $p = 2, 5, 7$. While it is fairly obvious that $\overline{\phi}$ is problematic modulo 2 or 5 (due to a problem point at $\overline{z} = \infty$, evidenced by the fact that the degree of $\overline{\phi}$ drops from 2 down to 1), the problem at $p = 7$ was less obvious. In retrospect, one can check that $\overline{\phi}$ wants to have both a zero and a pole at $\overline{z} = 3$ modulo 7. However, one of the big advantages of computing the resultant is that we can be sure that at all *other* p, the map ϕ has good reduction.

4.3. **The spherical metric.** The absolute value on K induces a metric not just on $\mathbb{C}_K$, but also on $\mathbb{P}^1(\mathbb{C}_K)$.

Definition 4.10. Let K be a non-archimedean field with absolute value $|\cdot|$. The *chordal metric*, or *spherical metric*, on K is the function $d : \mathbb{P}^1(\mathbb{C}_K) \times \mathbb{P}^1(\mathbb{C}_K) \to [0, \infty)$ given by

$$d(z, w) := \frac{|z - w|}{\max\{1, |z|\} \max\{1, |w|\}}.$$

Equivalently (except with the advantage of being more clearly defined if z or w is the point at ∞), if $z = [x, y]$ and $w = [u, v]$ in homogeneous coordinates, we have

$$d(z, w) := \frac{|xv - yu|}{\max\{|x|, |y|\} \max\{|u|, |v|\}}. \tag{6}$$

We leave it to the reader to check that the two definitions agree on $K \times K$, and that the spherical metric is indeed a metric on $\mathbb{P}^1(\mathbb{C}_K)$.

Remark 4.11. The reader should be warned that in general, $\mathbb{P}^1(\mathbb{C}_K)$-disks are *not* disks in the metric space $(\mathbb{P}^1(\mathbb{C}_K), d)$, where d is the spherical metric. As we will see in Proposition 4.13, for example, any d-disk containing both the points 0 and 1 must have radius at least 1 and therefore contain the whole of $\mathbb{P}^1(\mathbb{C}_K)$. Still, any d-disk is a $\mathbb{P}^1(\mathbb{C}_K)$-disk, and any $\mathbb{P}^1(\mathbb{C}_K)$-disk contained in a single residue class is a d-disk. In addition, the *topology* induced by the metric d coincides with the topology generated by the set of $\mathbb{P}^1(\mathbb{C}_K)$-disks.

Remark 4.12. Over the complex numbers, the chordal metric is the distance in $\mathbb{R}^3$ between two points of $\mathbb{P}^1(\mathbb{C})$ under the usual embedding of the Riemann sphere in $\mathbb{R}^3$, which is

$$d_{\mathbb{C},\text{chord}}(z, w) := \frac{|z - w|}{\sqrt{(1 + |z|^2(1 + |w|^2)}},$$

and the spherical metric is the distance over the surface of the sphere,

$$d_{\mathbb{C},\text{sph}}(z, w) := \arcsin\big(d_{\mathbb{C},\text{chord}}(z, w)\big).$$

However, there is no need to devise two different metrics (which are, incidentally, equivalent as metrics) in the non-archimedean setting.

Proposition 4.13. *Let K be a non-archimedean field with absolute value $|\cdot|$, ring of integers $\mathcal{O}_K$, and maximal ideal $\mathcal{M}_K$. Let $d(\cdot, \cdot)$ denote the spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$.*

a. *d is non-archimedean; that is, $d(a, c) \le \max\{d(a, b), d(b, c)\}$ for all $a, b, c \in \mathbb{P}^1(\mathbb{C}_K)$.*
b. *$d(z, w) = |z - w|$ for all $z, w \in \mathcal{O}_K$.*

c. $d(z,w) = 1$ *for all* $z, w \in \mathbb{P}^1(\mathbb{C}_K)$ *which have different reductions modulo* $\mathcal{M}_K$*, i.e., for which* $\overline{z} \neq \overline{w}$.

d. $d(z,w) < 1$ *for all* $z, w \in \mathbb{P}^1(\mathbb{C}_K)$ *with the same reduction* $\overline{z} = \overline{w}$.

e. $d(\eta(z), \eta(w)) = d(z,w)$ *for all* $z, w \in \mathbb{P}^1(\mathbb{C}_K)$ *and all* $\eta \in \mathrm{PGL}(2, \mathcal{O}_K)$.

Proof. Part (b) is obvious from Definition 4.10.

For part (e), recall that $\mathrm{PGL}(2, \mathcal{O}_K)$ is the group of linear fractional transformations $(az + b)/(cz + d)$ with coefficients in $\mathcal{O}_K$ and with determinant $ad - bc$ a unit in $\mathcal{O}_K$, that is, of absolute value 1. In particular, $\mathrm{PGL}(2, \mathcal{O}_K)$ is generated by maps of the form $z \mapsto z + a$ for $a \in \mathcal{O}_K$, maps of the form $z \mapsto cz$ for $c \in \mathcal{O}_K$, and $z \mapsto 1/z$. We leave to the reader the simple exercise of checking that the spherical metric is unaffected by each such change.

For part (a), after replacing K by a finite extension if necessary, we may assume that the residue field of K contains at least three elements. Thus, there is some $w \in \mathbb{P}^1(\mathbb{C}_K)$ such that $\overline{w} \neq \overline{a}, \overline{b}, \overline{c}$. By a $\mathrm{PGL}(2, \mathcal{O}_K)$-change of coordinates moving w to ∞, then, we may assume that $a, b, c \in \mathcal{O}_K$. Hence, the desired non-archimedean inequality is immediate from part (b).

For part (c), after a $\mathrm{PGL}(2, \mathcal{O}_K)$-change of coordinates, we may assume that $w = 0$, and hence $z \in \mathcal{M}_K$. Thus, by part (b), $d(z,w) = |z - 0| < 1$.

Finally, for part (d), after a $\mathrm{PGL}(2, \mathcal{O}_K)$-change of coordinates, we may assume that $w = \infty$, and hence $z \in \mathcal{O}_K$. Thus, using the homogeneous definition (6) of the spherical metric, and writing z as $[z, 1]$ and $w = \infty$ as $[1, 0]$, we have $d(z, \infty) = 1/1 = 1$. ☐

Morton and Silverman observed in Proposition 5.2 of [40] that rational functions are Lipschitz with respect to the spherical metric, and that maps of good reduction have Lipschitz constant 1, as we now show.

Proposition 4.14. *Let* K *be a non-archimedean field with absolute value* $|\cdot|$ *and spherical metric* $d(\cdot,\cdot)$*, and let* $\phi \in K(z)$*. Then there is a constant* $C \geq 1$ *such that*

$$d(\phi(z), \phi(w)) \leq C d(z,w)$$

for all $z, w \in \mathbb{P}^1(\mathbb{C}_K)$*. Moreover, if* ϕ *has good reduction, then we can choose* $C = 1$*.*

Proof. Let $r_1 \in (0,1]$ be the minimum value of $d(\alpha, \beta)$ for all $\alpha \in \phi^{-1}(0)$ and $\beta \in \phi^{-1}(\infty)$. Let r_2 be the analogous distance for $\phi^{-1}(1)$ and $\phi^{-1}(\infty)$, let r_3 be the analogous distance for $\phi^{-1}(0)$ and $\phi^{-1}(1)$, and let $r = \min\{r_1, r_2, r_3\} \in (0,1]$. Set $C = 1/r \geq 1$.

Given $z, w \in \mathbb{P}^1(\mathbb{C}_K)$, if $d(z,w) \geq r$, then because $d(\phi(z), \phi(w)) \leq 1$ (by Proposition 4.13.c–d), we get $d(\phi(z), \phi(w)) \leq C d(z,w)$.

On the other hand, if $d(z,w) < r$, then since $r \leq 1$, we have $\overline{z} = \overline{w}$ by Proposition 4.13.c. Replacing ϕ by $\phi \circ \eta$ for some appropriate $\eta \in \mathrm{PGL}(2, \mathcal{O}_K)$, which does not affect the value of r or C by Proposition 4.13.e, we may assume that $w = 0$, and hence that $z \in D(0,r)$. By our choice of r, and by Proposition 4.13.b, the image $\phi(D(0,r))$ cannot contain more than one of $0, 1, \infty$. By Proposition 3.27, then, $\phi(D(0,r))$ is a $\mathbb{P}^1(\mathbb{C}_K)$-disk containing at most one of $0, 1, \infty$. It follows that $\phi(D(0,r))$ is completely contained in the residue class $\overline{\phi(0)}$.

We will no longer need the original properties of r, but just the conclusion of the previous paragraph. Thus, replacing ϕ by $\theta \circ \phi$ for some appropriate $\theta \in \mathrm{PGL}(2, \mathcal{O}_K)$, we may assume without loss that $\phi(0) = 0$, and hence $\phi(D(0,r)) \subseteq D(0,1)$. Thus, by Proposition 3.13, we have $|\phi(z)| \leq (1/r)|z|$. Since $C = 1/r$, and because $|\phi(z)| = d(\phi(z), \phi(w))$ and $|z| = d(z,w)$ by Proposition 4.13.b, we are done.

To prove the final statement, it suffices to show that $r = 1$ if ϕ has good reduction. First, given $\alpha \in \phi^{-1}(0)$ and $\beta \in \phi^{-1}(\infty)$, then $\overline{\alpha} \neq \overline{\beta}$, because ϕ has good reduction, and $\overline{\phi(\alpha)} \neq \overline{\phi(\beta)}$. Thus, $d(\alpha, \beta) = 1$, by Proposition 4.13.c, and hence $r_1 = 1$.

Meanwhile, $\theta_2(z) := 1 - z$ and $\theta_3(z) := z/(z-1)$ have good reduction, and therefore $\theta_2 \circ \phi$ and $\theta_3 \circ \phi$ do too, by Proposition 4.7. Applying the argument of the previous paragraph to $\theta_2 \circ \phi$ and $\theta_3 \circ \phi$, we obtain $r_2 = 1$ and $r_3 = 1$, respectively. Thus, $r = 1$, as desired. ☐

4.4. **Fatou and Julia sets.** Roughly speaking, the Fatou set of a rational function $\phi \in \mathbb{C}_K(z)$ is the set of order, where points that are close together stay close together under arbitrarily many iterations, and the Julia set, as the complement of the Fatou set, is the set of chaos. To define them rigorously, however, we begin by recalling the following definition from analysis.

Definition 4.15. Let X and Y be metric spaces, and let $\mathcal{G}$ be a set of functions from X to Y. We say that $\mathcal{G}$ is *equicontinuous* on X if for every $\varepsilon > 0$, there is some $\delta > 0$ such that for every $x, y \in X$ satisfying $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \varepsilon$.

The key to Definition 4.15 is that the choice of δ does *not* depend on the function $g \in \mathcal{G}$. We will apply the definition to set of iterates of a given rational function, as follows.

Definition 4.16. Let K be a complete non-archimedean field, and let $\phi \in K(z)$ be a rational function. We define the (*classical*) *Fatou set* of ϕ, denoted $\mathcal{F}_{\phi,\mathrm{I}}$, $\mathcal{F}_\phi$, or simply $\mathcal{F}$, to be the subset of $\mathbb{P}^1(\mathbb{C}_K)$ consisting of all points $x \in \mathbb{P}^1(\mathbb{C}_K)$ having a neighborhood U on which the family of iterates $\{\phi^n : n \geq 0\}$ is equicontinuous with respect to the spherical metric.

We define the (*classical*) *Julia set* of ϕ, denoted $\mathcal{J}_{\phi,\mathrm{I}}$, $\mathcal{J}_\phi$ or simply $\mathcal{J}$, to be the complement $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \mathcal{F}_\phi$ of the Fatou set.

A few comments are in order. First, we are saying that $\{\phi^n : n \geq 0\}$ is equicontinuous as a family of maps from U to $\mathbb{P}^1(\mathbb{C}_K)$, where *both* U and $\mathbb{P}^1(\mathbb{C}_K)$ are equipped with the spherical metric. Second, note that Definition 4.16 does indeed convey the idea mentioned above: that nearby points stay nearby under iteration if they lie in the Fatou set, but they may not if they lie in the Julia set. Third, the reason for the optional use of the word "classical" and the optional subscript I in Definition 4.16 will become clear when we define versions of the Fatou and Julia sets in the Berkovich projective line. Finally, in practice, as Lemma 4.18 below shows, we will be able use use a different property, which should probably be called *uniformly equi-Lipschitz*, when discussing the equicontinuity of families of non-archimedean rational functions.

Lemma 4.17. *Let K be a complete non-archimedean field, let U be a subset of $\mathbb{P}^1(\mathbb{C}_K)$, and let $\mathcal{G}$ be a set of functions from U to $\mathbb{P}^1(\mathbb{C}_K)$. Let $\eta \in \mathrm{PGL}(2, K)$, and for each $f \in \mathcal{G}$, let $\theta_f \in \mathrm{PGL}(2, K)$, and define*

$$\mathcal{G}' := \{\theta_f \circ f \circ \eta : f \in \mathcal{G}\}.$$

Suppose there is a constant $C \geq 1$ such that

$$C^{-1} d(x, y) \leq d(\theta_f(x), \theta_f(y)) \leq C d(x, y) \tag{7}$$

for all $x, y \in \mathbb{P}^1(\mathbb{C}_K)$ and all $f \in \mathcal{G}$. Then $\mathcal{G}$ is equicontinuous on U with respect to the spherical metric if and only if $\mathcal{G}'$ is equicontinuous on $\eta^{-1}(U)$ with respect to the spherical metric.

Proof. By applying Proposition 4.14 to η and η^{-1}, we may assume that C is large enough so that inequality (7) applies not only to each θ_f, but also to η. We will prove that if $\mathcal{G}$ is equicontinuous, then so is $\mathcal{G}'$; the converse follows by the symmetry of the hypotheses.

Given $\varepsilon > 0$, pick $\delta > 0$ for the family $\mathcal{G}$ corresponding to ε/C. Then for any $x, y \in \eta^{-1}(U)$ satisfying $d(x, y) < \delta/C$ and any $g \in \mathcal{G}'$, we can write $g = \theta_f \circ f \circ \eta$ for some $f \in \mathcal{G}$. Since $\eta(x), \eta(y) \in U$ and $d(\eta(x), \eta(y)) < \delta$, we have

$$d(g(x), g(y)) \leq C d\big(f(\eta(x)), f(\eta(y))\big) < C(\varepsilon/C) = \varepsilon. \qquad \square$$

Lemma 4.18. *Let K be a complete non-archimedean field, let $\mathcal{G} \subseteq K(z)$ be a family of rational functions, and let $x_0 \in \mathbb{P}^1(\mathbb{C}_K)$ be a point. Then $\mathcal{G}$ is equicontinuous with respect to the spherical metric on some neighborhood of x_0 if and only if there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x_0 and a constant $C > 0$ such that*

$$d\big(f(x), f(y)\big) \leq C d(x, y)$$

for all $x, y \in D$ and all $f \in \mathcal{G}$.

Proof. Without loss, $K = \mathbb{C}_K$. Pick a linear fractional transformation $\eta \in \mathrm{PGL}(2, \mathcal{O}_K)$ such that $\eta(0) = x_0$. Similarly, for each $f \in \mathcal{G}$, pick a linear fractional transformation $\theta_f \in \mathrm{PGL}(2, \mathcal{O}_K)$ such that $\theta(f(x_0)) = 0$. By Proposition 4.13.e, we can use a constant of $C = 1$ in Lemma 4.17 for each θ_f, and hence we may assume without loss that $x_0 = 0$, and that $f(0) = 0$ for all $f \in \mathcal{G}$.

If C and D exist, then $\mathcal{G}$ is clearly equicontinuous on D, and we are done.

Conversely, suppose $\mathcal{G}$ is equicontinuous on some neighborhood U of 0. Pick $\delta > 0$ such that for all $x, y \in U$ with $d(x, y) < \delta$ and all $f \in \mathcal{G}$, we have $d(f(x), f(y)) < 1$. Without loss, δ is small enough that $\delta < 1$ and $D(0, \delta) \subseteq U$. Set $C := 1/\delta$ and $D := D(0, \delta)$.

Thus, for any $f \in \mathcal{G}$ and any $x \in D$, we have $d(f(x), 0) = d(f(x), f(0)) < 1$, and hence $\overline{x} = \overline{0}$ by Proposition 4.13.c. That is, $f(D) \subseteq D(0, 1)$. By Proposition 3.13, then, for all $x, y \in D$,

$$d\big(f(x), f(y)\big) = |f(x) - f(y)| \leq \frac{1}{r} \cdot |x - y| = Cd(x, y). \qquad \square$$

The following third characterization of equicontinuity for non-archimedean rational functions, this time using the reduction map $\bar{\cdot} : \mathbb{P}^1(\mathbb{C}_K) \to \mathbb{P}^1(\overline{k})$, is also useful.

Lemma 4.19. *Let K be a complete non-archimedean field, let $\mathcal{G} \subseteq K(z)$ be a family of rational functions, and let $x_0 \in \mathbb{P}^1(\mathbb{C}_K)$ be a point. Then $\mathcal{G}$ is equicontinuous with respect to the spherical metric on some neighborhood of x_0 if and only if there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x_0 such that for every $f \in \mathcal{G}$ and every $x \in D$, $\overline{f(x)} = \overline{f(x_0)}$.*

Proof. As in the proof of Lemma 4.18, we may assume without loss that $x_0 = 0$ and that $f(0) = 0$ for all $f \in \mathcal{G}$.

If $\mathcal{G}$ is equicontinuous on a neighborhood of 0, choose a disk D and a constant C for $\mathcal{G}$ and 0 as in Lemma 4.18. Assume without loss that $C \geq 1$, and then that $D \subseteq D(0, C^{-1})$. Then for all $x \in D$ and all $f \in \mathcal{G}$, we have

$$d(f(x), 0) = d(f(x), f(0)) \leq Cd(x, 0) < 1,$$

and hence, by Proposition 4.13.e, $\overline{f(x)} = \overline{f(x_0)}$.

Conversely, given D satisfying the condition of the current Lemma, shrink D if necessary so that $D \subseteq D(0, 1)$, and write $D = \overline{D}(0, r)$ with $0 < r < 1$ or $D = D(0, r)$ with $0 < r \leq 1$. Set $C := r^{-1}$. For any $f \in \mathcal{G}$, then, we have $f(D(0, r)) \subseteq D(0, 1)$. Hence, by Proposition 3.13, for any $x, y \in D(0, r)$,

$$d(f(x), f(y)) = |f(x) - f(y)| \leq r^{-1}|x - y| = Cd(x, y) \qquad \square$$

Remark 4.20 (Normality vs. Equicontinuity)**.** In complex dynamics, the Fatou and Julia sets are usually defined in terms of normality rather than equicontinuity. (A family $\mathcal{G}$ of functions is *normal* if it is precompact, meaning that its closure is compact. In other words, $\mathcal{G}$ is normal on X if every sequence of functions in $\mathcal{G}$ has a convergent subsequence; the limit function of the subsequence may or may not lie in $\mathcal{G}$ itself.) By the Arzelà-Ascoli Theorem (see, for example, Theorem VII.1.23 of [23]), normality is equivalent to equicontinuity when working over $\mathbb{C}$, and therefore complex Fatou and Julia sets can be defined using either normality or equicontinuity interchangeably.

However, the Arzelà-Ascoli Theorem fails over fields like $\mathbb{C}_K$ that are not locally compact. For example, let $\{a_n\}_{n \geq 0}$ be any sequence of points in $\mathbb{P}^1(\mathbb{C}_K)$ with no accumulation points; then the family of constant functions $\{a_n : n \geq 0\}$ is equicontinuous but not normal on any open subset $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$.

It turns out that if ϕ is defined over a locally compact subfield K of the non-archimedean field $\mathbb{C}_K$, then one can prove that the family $\{\phi^n : n \geq 0\}$ is indeed normal on a neighborhood of every point of the classical Fatou set, but the proof requires some of the more advanced dynamical theorems yet to come. In addition, normality can fail even on the classical Fatou set if ϕ is not defined over any such locally compact subfield. In light of these complications, then, the classical

Fatou set of a rational function over a non-archimedean field is usually defined in the literature via equicontinuity instead of normality.

We close this remark by pointing out that our upcoming definitions of the Berkovich Fatou and Julia sets will involve neither equicontinuity nor normality. Instead, the definitions will be inspired by the alternate description of the classical Fatou and Julia sets in Theorem 4.29 below.

The following simple properties of non-archimedean Fatou and Julia sets also hold over $\mathbb{C}$, by essentially the same proof.

Proposition 4.21. *Let K be a complete non-archimedean field, and let $\phi \in K(z)$ be a rational function with Fatou set $\mathcal{F}_\phi$ and Julia set $\mathcal{J}_\phi$.*

a. *$\mathcal{F}_\phi$ is an open subset of $\mathbb{P}^1(\mathbb{C}_K)$, and $\mathcal{J}_\phi$ is a closed subset of $\mathbb{P}^1(\mathbb{C}_K)$.*
b. *$\phi^{-1}(\mathcal{F}_\phi) = \phi(\mathcal{F}_\phi) = \mathcal{F}_\phi$, and $\phi^{-1}(\mathcal{J}_\phi) = \phi(\mathcal{J}_\phi) = \mathcal{J}_\phi$.*
c. *For every positive integer $m \geq 1$, $\mathcal{F}_{\phi^m} = \mathcal{F}_\phi$, and $\mathcal{J}_{\phi^m} = \mathcal{J}_\phi$.*
d. *For any $\eta \in \mathrm{PGL}(2,K)$, if we set $\psi = \eta \circ \phi \circ \eta^{-1}$, then $\mathcal{F}_\psi = \eta(\mathcal{F}_\phi)$, and $\mathcal{J}_\psi = \eta(\mathcal{J}_\phi)$.*

Proof. Assume without loss that $K = \mathbb{C}_K$.

Because $\mathcal{F}_\phi$ is defined by a property of neighborhoods, part (a) is immediate from Definition 4.16.

For part (b), to show $\phi^{-1}(\mathcal{F}) \subseteq \mathcal{F}$, given any $x_0 \in \phi^{-1}(\mathcal{F})$, choose a corresponding $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing $\phi(x_0)$ for the family $\{\phi^n : n \geq 0\}$ as in Lemma 4.19. Pick a $\mathbb{P}^1(\mathbb{C}_K)$-disk D' containing x_0 and contained in the residue class of x_0 so that $\phi(D') \subseteq D$. Then $\phi^n(D')$ is contained in the residue class of $\phi^n(x_0)$ for all $n \geq 0$. Therefore, by Lemma 4.19, the family $\{\phi^n : n \geq 0\}$ is equicontinuous on D'.

For the reverse inclusion, given $x_0 \in \mathcal{F}$, we need to show that $\phi(x_0) \in \mathcal{F}$ also. Pick a $\mathbb{P}^1(\mathbb{C}_K)$-disk D as in Lemma 4.19 for x_0. Then $\phi(D)$ is a disk containing $\phi(x_0)$ such that $\phi^n(\phi(D))$ is contained in the residue class of $\phi^{n+1}(x_0)$ for all $n \geq 0$. Thus, by the same Lemma, the family $\{\phi^n\}$ is equicontinuous on $\phi(D)$.

The remaining set equalities $\phi(\mathcal{F}_\phi) = \mathcal{F}_\phi$ and $\phi^{-1}(\mathcal{J}_\phi) = \phi(\mathcal{J}_\phi) = \mathcal{J}_\phi$ follow immediately from $\phi^{-1}(\mathcal{F}_\phi) = \mathcal{F}_\phi$

For part (c), the inclusion $\mathcal{F}_{\phi^m} \supseteq \mathcal{F}_\phi$ is immediate, because the family $\{(\phi^m)^n : n \geq 0\}$ is contained in the family $\{\phi^n : n \geq 0\}$. As for the reverse inclusion, let $C' \geq 1$ be the maximum of the Lipschitz constants from Proposition 4.14 for ϕ^i, for $i = 0, \ldots, m-1$. Given $x_0 \in \mathcal{F}_{\phi^n}$ and $\varepsilon > 0$, pick a $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x_0 and a constant $C > 0$ as in Lemma 4.18 for the family $\{\phi^{mn}\}$ at x_0. Then for any $x, y \in D$ and any integer $j \geq 0$, writing $j = mn + i$ for some $i = 0, \ldots, n-1$,

$$d(\phi^j(x), \phi^j(y)) \leq C' d(\phi^{mn}(x), \phi^{mn}(y)) < C'C d(x,y),$$

yielding equicontinuity by Lemma 4.18, and proving the reverse inclusion. Taking complements, we also obtain $\mathcal{J}_{\phi^n} = \mathcal{J}_\phi$.

Finally, the Fatou portion of part (d) is immediate by applying Proposition 4.14 to η and η^{-1} and then applying Lemma 4.17. The Julia portion then follows by taking complements. ☐

Proposition 4.22. *Let K be a complete non-archimedean field, and let $\phi \in K(z)$ be a rational function of good reduction. Then the classical Fatou set $\mathcal{F}_\phi$ is all of $\mathbb{P}^1(\mathbb{C}_K)$, and the classical Julia set $\mathcal{J}_\phi$ is empty.*

Proof. Given any point $x \in \mathbb{P}^1(\mathbb{C}_K)$, let $U = \mathbb{P}^1(\mathbb{C}_K)$. By Proposition 4.7, ϕ^n has good reduction for all $n \geq 0$. Therefore, by Proposition 4.14,

$$d(\phi^n(a), \phi^n(b)) \leq d(a,b)$$

for all $a, b \in U$ and all $n \geq 0$. Thus, $x \in \mathcal{F}_\phi$, as desired. ☐

Combining Propositions 4.22 and 4.21.d, it follows that any map of potentially good reduction also has empty classical Julia set. However, as we will see, there are also maps without potentially good reduction that still have empty classical Julia set. Another easy way, which is possible only over fields of positive characteristic, is illustrated by the following Proposition.

Proposition 4.23. *Let K be a complete non-archimedean field of characteristic $p > 0$, and let $\phi \in K(z^p)$ be an inseparable rational function. Then all periodic points of ϕ are superattracting, and $\mathcal{J}_\phi = \varnothing$.*

Proof. Because $\phi' = 0$, the fact that all periodic points are superattracting is immediate.

Write $\phi(z) = \psi(z^p)$ for some $\psi \in K(z)$. Let $C \geq 1$ be the constant from Proposition 4.14 for ψ. We claim that for all $x, y \in \mathbb{P}^1(\mathbb{C}_K)$, if $d(x, y) \leq C^{-1/(p-1)}$, then $d(x^p, y^p) \leq C^{-1} d(x, y)$. From the claim, it follows that $d(\phi(x), \phi(y)) \leq d(x, y)$, from which equicontinuity follows.

To prove the claim, given $x, y \in \mathbb{P}^1(\mathbb{C}_K)$ with $d(x, y) < C^{-1/(p-1)}$, observe that

$$d(x^p, y^p) = \frac{|x^p - y^p|}{\max\{1, |x|^p\} \max\{1, |y|^p\}} = \frac{|x - y|^p}{\max\{1, |x|\}^p \max\{1, |y|\}^p} = d(x, y)^p \leq C^{-1} d(x, y),$$

proving the claim and hence the Proposition. ☐

Over a non-archimedean field, the absolute value of the multiplier of a periodic point determines whether the point is in the Fatou set or Julia set, as the following result shows. It first appeared, along with Corollary 4.25.b below, as Corollaries 2.4.4 and 2.4.5 in [4], and then as Proposition 1.1 and Corollary 1.3 in [6].

Proposition 4.24. *Let K be a complete non-archimedean field, let $\phi \in K(z)$ be a rational function, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$ be a periodic point. If x is repelling, then $x \in \mathcal{J}_\phi$; and If x is nonrepelling, then $x \in \mathcal{F}_\phi$.*

Proof. Let $m \geq 1$ be the period of x. If x is repelling, let D be any $\mathbb{P}^1(\mathbb{C}_K)$-disk containing x, and let $C > 0$ be any constant; we will prove that the family $\{\phi^n\}$ fails the conditions of Lemma 4.18 for C and D. is not equicontinuous on U. Shrinking D if necessary, we may assume, by Proposition 4.3.b, that for every $y \in D \setminus \{x\}$, there is some $n \geq 0$ such that $\phi^{mn}(y) \notin D$. Further shrinking D, we may also assume that D is a d-disk (that is, a disk with respect to the spherical metric d) of d-radius $0 < r < 1$.

Pick $y \in D$ such that $0 < d(x, y) < C^{-1} r$. By the previous paragraph, there is some $n \geq 0$ such that $\phi^{mn}(y) \notin D$, and hence

$$d\big(\phi^{mn}(x), \phi^{mn}(y)\big) = d\big(x, \phi^{mn}(y)\big) \geq r > C d(x, y),$$

and hence we have indeed disproven equicontinuity. Thus, $x \in \mathcal{J}_\phi$.

On the other hand, if x is nonrepelling, then by Proposition 4.3.c, there is some open $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x such that $\phi^m(D) \subseteq D$. After a change of coordinates (and bearing in mind Proposition 4.21.c–d), we may assume that $m = 1$ and that $U \subseteq D(0, 1)$. In particular, $D = D(x, r)$ for some $0 < r \leq 1$, and we have $\phi^n(D) \subseteq D$ for all $n \geq 0$. By Proposition 3.13, we have $d(\phi^n(a), \phi^n(b)) \leq d(a, b)$ for all $a, b \in D$ and all $n \geq 0$. Thus, $x \in \mathcal{F}_\phi$ by Lemma 4.18. ☐

As noted in Remark 4.4, the analogue of Proposition 4.24 is true in complex dynamics (by essentially the same proof) only for attracting and repelling points. In particular, rationally indifferent points in complex dynamics are always in the Julia set (provided $\deg \phi \geq 2$), and irrationally indifferent points may be either in the Fatou set or Julia set, depending on more subtle factors.

Combining Proposition 4.24 with Proposition 4.2, we obtain the following result.

Corollary 4.25. *Let K be a complete valued field, and let $\phi(z) \in K(z)$ be a rational function.*

a. *If K is archimedean and $\deg \phi \geq 2$, then $\mathcal{J}_\phi \neq \varnothing$.*
b. *If K is non-archimedean, then $\mathcal{F}_\phi \neq \varnothing$.*

Proof. In the archimedean case, both repelling and rationally indifferent points are in the Julia set (see Remark 4.4), and hence the result is immediate from Proposition 4.2.a. In the non-archimedean case, all nonrepelling points are in the Fatou set by Proposition 4.24, and therefore the result follows from Proposition 4.2.b. □

As we have seen in Proposition 4.22, it is possible for the classical Julia set of a non-archimedean rational function to be empty, unlike the archimedean case described in Corollary 4.25.a. Conversely, it is also possible for the *Fatou* set of an *archimedean* rational function to be empty, unlike the non-archimedean case described in Corollary 4.25.b. The most famous examples of such archimedean maps are *Lattès* maps, which are induced by morphisms of elliptic curves; see also Project #2. However, there are other such maps— see, for example, Theorem V.1.2 of [21].

Corollary 4.26. *Let K be a complete valued field, and let $\phi(z) \in K(z)$ be a rational function of degree $d \geq 2$. Then the exceptional set of ϕ is contained in the Fatou set of ϕ.*

Proof. If there are more than two exceptional points, then by Theorem 1.11.a, ϕ is conjugate to z^{p^m}, where $m \geq 1$ and $p = \operatorname{char} K > 0$. Thus, by Proposition 4.23, we have $\mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}_K)$, which clearly contains the exceptional set.

If there are exactly two exceptional points, then by Theorem 1.11.b, ϕ is conjugate to either z^d or z^{-d}, in which case the exceptional points are 0 and ∞, both of which are superattracting periodic points, of period either 1 or 2. Thus, by Proposition 4.24, both belong to the Fatou set.

Similarly, if there is exactly one exceptional point, then by Theorem 1.11.a, ϕ is conjugate to a polynomial, and the exceptional point ∞ is a superattracting fixed point, which again is Fatou by Proposition 4.24. □

4.5. **Further properties of Fatou and Julia sets.** Montel's Theorem in complex analysis, sometimes called the Montel-Carathéodory Theorem, says that a family of meromorphic functions $\mathcal{G}$ on a domain $U \subseteq \mathbb{C}$ that omits three points of $\mathbb{P}^1(\mathbb{C})$ (i.e., for which there are three distinct points $a, b, c \in \mathbb{P}^1(\mathbb{C})$ such that $f(U) \cap \{a, b, c\} = \varnothing$ for all $f \in \mathcal{G}$) must in fact be normal on U. (See, for example, Theorem XII.4.1 of [23]) By the Arzelà-Ascoli Theorem, this three-point criterion can be viewed equivalently as a sufficient condition for equicontinuity; In Theorem 2.2 of [33], Hsia proved the following two-point non-archimedean analogue of Montel's Theorem.

Theorem 4.27 (Hsia's Non-archimedean Montel Theorem)**.** *Let K be a complete non-archimedean field, let $D \subseteq \overline{D}(0,1)$ be a disk, and let $\mathcal{G} \subseteq \mathcal{A}_D$ be a set of restricted power series on D. Let $a \in \mathbb{C}_K$, and suppose that for all $f \in \mathcal{G}$, the image $f(D)$ does not contain a. Then $\mathcal{G}$ is equicontinuous on D with respect to the spherical metric.*

Proof. First, we claim that we may assume $a = 0$. To see this, define $\mathcal{G}' := \{f - a : f \in \mathcal{G}\}$, and apply Proposition 4.14 to produce a constant $C \geq 1$ for both $z \mapsto z - a$ and $z \mapsto z + a$. Then, the claim follows from Lemma 4.17.

We will now find a constant $C > 0$ such that $d(f(x), f(y)) \leq C d(x, y)$ for all $x, y \in D$ and all $f \in \mathcal{G}$. Even though each f may only be a restricted power series, rather than a rational function, the existence of a constant C with this property clearly implies equicontinuity.

Write $D = D(b, r)$ or $D = \overline{D}(b, r)$, where $b \in \overline{D}(0,1)$ and $0 < r \leq 1$. Set $C := r^{-1}$. For any $f \in \mathcal{G}$, the image $f(D)$ is a disk not containing 0, and therefore $f(D) \subseteq D(c_f, |c_f|)$ for some $c_f \in \mathbb{C}_K^\times$. Hence, for any $x, y \in D$, we have $|f(x)| = |f(y)| = |c_f|$, and by Proposition 3.13, $|f(x) - f(y)| \leq r^{-1}|c_f| \cdot |x - y|$. It follows that

$$d\big(f(x), f(y)\big) = \frac{|f(x) - f(y)|}{\max\{|f(x)|, 1\} \max\{|f(y)|, 1\}} \leq \frac{r^{-1}|c_f| \cdot |x - y|}{\max\{|c_f|^2, 1} \leq r^{-1}|x - y| = C d(x, y),$$

as desired. □

Corollary 4.28. *Let K be a complete non-archimedean field, let $D \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a $\mathbb{P}^1(\mathbb{C}_K)$-disk, and let $\mathcal{G} \subseteq K(z)$ be a set of rational functions. Let $a, b \in \mathbb{P}^1(\mathbb{C}_K)$ be distinct, and suppose that for all $f \in \mathcal{G}$, the image $f(D)$ does not contain a or b. Then $\mathcal{G}$ is equicontinuous on D with respect to the spherical metric.*

Proof. Pick $\eta, \theta \in \mathrm{PGL}(2, K)$ such that $\eta(b) = \infty$ and $\theta(D) \subseteq \overline{D}(0, 1)$. Let $\mathcal{G}' = \{\eta \circ f \circ \theta^{-1} : f \in \mathcal{G}\}$, which is a set of rational functions on $\theta(D)$ omitting a and ∞, and hence is a set of restricted power series on $\theta(D)$ omitting a. By Theorem 4.27, $\mathcal{G}'$ is equicontinuous on $\theta(D)$. Choosing a constant $C \geq 1$ for Proposition 4.14 applying to η, θ, and their inverses, then, the result follows from Lemma 4.17. $\square$

Thus, we have the following alternate definition of the Fatou set of a non-archimedean rational function.

Theorem 4.29. *Let K be a complete non-archimedean field, and let $\phi \in K(z)$ be a rational function. Let $x \in \mathbb{P}^1(\mathbb{C}_K)$. Then $x \in \mathcal{F}_\phi$ if and only if for every $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x,*

$$\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \Big[\bigcup_{n \geq 0} \phi^n(D) \Big] \quad \textit{has more than one element.}$$

In that case, it is possible to choose the $\mathbb{P}^1(\mathbb{C}_K)$-disk D so that the set $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \Big[\bigcup_{n \geq 0} \phi^n(D) \Big]$ is uncountable.

To prove Theorem 4.29, we will also need the following lemma.

Lemma 4.30. *Let K be a complete non-archimedean field, let $\phi \in K(z)$ be a rational function, and let $D \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a $\mathbb{P}^1(\mathbb{C}_K)$-disk. Suppose that $\phi(D) \supsetneq D$. Then there is some $n \geq 0$ such that $\phi^n(D)$ is not contained in any residue class of $\mathbb{P}^1(\mathbb{C}_K)$.*

Proof. And after a change of coordinates, assume that $0 \in D$. We will prove the Lemma in the case that $D = D(0, r)$ is an open disk; the case that D is closed is similar.

If the conclusion of the Lemma is false, then $\phi^n(D) \subseteq D(0, 1)$ for all $n \geq 0$. In particular, we can write $\phi^n(D) = D(0, s_n)$ for $n \geq 1$, where $r < s_1 < s_2 < \cdots$. Note that because ϕ has no poles in $\phi^{n-1}(D)$, we may write ϕ as a restricted power series $\phi(z) = \sum_{n \geq 0} c_n z^n$ on each $D(0, s_n)$. In particular, by Corollary 3.10, we have $s_1 = |c_d| r^d$, for some $d \geq 1$.

We claim that $s_n \geq (s_1/r)^n r$ for all $n \geq 1$. Since $s_1/r > 1$, we will get $s_n > 1$ for some n, contradicting our assumption that $\phi^n(D) \subseteq D(0, 1)$.

The $n = 1$ case of the claim is trivial. For $n \geq 2$, Corollary 3.10 again tells us that $s_n \geq |c_d| s_{n-1}^d$. However, because $|c_d| = s_1 r^{-d}$ and $s_{n-1} \geq (s_1/r)^{n-1} r$, we have

$$s_n \geq s_1 s_{n-1}^d r^{-d} \geq s_1^{dn-d+1} r^{d-dn} = (s_1/r)^{dn-d+1} r \geq (s_1/r)^n r,$$

since $dn - d - n + 1 = (d-1)(n-1) \geq 0$ and $s_1/r > 1$. Thus, we have proven the claim and hence the Lemma. $\square$

Proof of Theorem 4.29. If there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x such that the complement of $\bigcup_{n \geq 0} \phi^n(D)$ contains at least two points, then $\{\phi^n\}_{n \geq 0}$ is equicontinuous on D by Corollary 4.28.

Conversely, suppose that $x \in \mathcal{F}_\phi$. Choose a $\mathbb{P}^1(\mathbb{C}_K)$-disk $D \subseteq \mathbb{P}^1(\mathbb{C}_K)$ as in Lemma 4.19, so that $\phi^n(D)$ is contained in a residue class for each $n \geq 0$. We consider three cases.

First, if there is some $m \geq 1$ for which $D \subsetneq \phi^m(D)$, then by Lemma 4.30 applied to ϕ^m, there is some $n \geq 0$ for which $\phi^{mn}(D)$ is not contained in a residue class, contradicting our choice of D.

Second, if $D \cap \phi^n(D) = \varnothing$ for all $n \geq 1$, then let $D' \subsetneq D$ be a strictly smaller $\mathbb{P}^1(\mathbb{C}_K)$-disk containing x. Then $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \bigcup_{n \geq 0} \phi^n(D')$ contains the infinite set $D \smallsetminus D'$, and we are done.

Finally, by Proposition 3.28, the only remaining case is that there is some integer $m \geq 1$ for which $\phi^m(D) \subseteq D$. Then for every $n \geq 0$, $\phi^n(D)$ is contained in one of the m $\mathbb{P}^1(\mathbb{C}_K)$-disks $\phi^i(D)$, for $i = 0, \ldots, m-1$, each of which is itself contained in a residue class, by our choice of D. Because there are infinitely many residue classes, it follows that $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \bigcup_{n\geq 0} \phi^n(D)$ contains an infinite set. □

Corollary 4.31. *Let K be a complete non-archimedean field, and let $\phi(z) \in K(z)$ be a rational function with classical Julia set $\mathcal{J}_\phi$ and exceptional set E_ϕ. Let $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be an open set intersecting $\mathcal{J}_\phi$. Then $\#E_\phi \leq 1$, and*

$$\bigcup_{n\geq 0} \phi^n(U) \supseteq \mathbb{P}^1(\mathbb{C}_K) \smallsetminus E_\phi.$$

Proof. The hypotheses assume that $\mathcal{J}_\phi \neq \varnothing$, and hence, by Theorem 1.11.a–b and Proposition 4.22, we must have $\#E_\phi \leq 1$.

Without loss, U is a $\mathbb{P}^1(\mathbb{C}_K)$-disk containing a point $x \in \mathcal{J}_\phi$. By Theorem 4.29, then, $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \bigcup_{n\geq 0} \phi^n(U)$ consists of at most one point. If this set is empty, we are done. Otherwise, the one point a in this set cannot have any preimages in $\bigcup_{n\geq 0} \phi^n(U)$, and therefore $\phi^{-1}(a) = \{a\}$. Thus, a is in the exceptional set, and we are done. □

Corollary 4.32. *Let K be a complete non-archimedean field, and let $\phi(z) \in K(z)$ be a rational function with classical Julia set $\mathcal{J}_\phi$. Then the interior of $\mathcal{J}_\phi$ is empty.*

Proof. Pick $x \in \mathcal{J}_\phi$ and an open set $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ containing x. Then by Corollary 4.31, $\bigcup_{n\geq 0} \phi^n(U)$ omits at most one point of $\mathbb{P}^1(\mathbb{C}_K)$. However, by Proposition 4.21.b, if $U \subseteq \mathcal{J}_\phi$, then $\bigcup_{n\geq 0} \phi^n(U) \subseteq \mathcal{J}_\phi$. Thus, $\mathcal{J}_\phi$ is a closed set omitting at most one point of $\mathbb{P}^1(\mathbb{C}_K)$. Then $\mathcal{J}_\phi = \mathbb{P}^1(\mathbb{C}_K)$, contradicting Corollary 4.25.b. Hence, $U \not\subseteq \mathcal{J}_\phi$ after all. □

Proposition 4.33. *Let K be a complete non-archimedean field, and let $\phi(z) \in K(z)$ be a rational function with Julia set $\mathcal{J}_\phi$. For any non-exceptional point $a \in \mathbb{P}^1(\mathbb{C}_K)$, the closure of the backward orbit of a contains $\mathcal{J}_\phi$, with equality if and only if $a \in \mathcal{J}_\phi$.*

Proof. For the first statement, we need to show that for any point $x \in \mathcal{J}_\phi$ and any open set $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ containing x, the backward orbit of a intersects U. But given such x and U, Corollary 4.31 tells us that $a \in \phi^n(U)$ for some $n \geq 0$, as desired.

For the second statement, write $\mathrm{Orb}^-(a)$ for the backward orbit of a. If $\overline{\mathrm{Orb}^-(a)} = \mathcal{J}_\phi$, then clearly $a \in \mathcal{J}_\phi$. Conversely, if $a \in \mathcal{J}_\phi$, then $\mathrm{Orb}^-(a) \subseteq \mathcal{J}_\phi$ by Proposition 4.21.b. Because $\mathcal{J}_\phi$ is closed, then, $\overline{\mathrm{Orb}^-(a)} \subseteq \mathcal{J}_\phi$. Finally, the reverse inclusion is the content of the first statement of the Proposition. □

The next result says that the Julia set is the smallest closed and completely invariant set not contained in the exceptional set.

Corollary 4.34. *Let K be a complete non-archimedean field, and let $\phi(z) \in K(z)$ be a rational function with Julia set $\mathcal{J}_\phi$. Let $S \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a closed subset of $\mathbb{P}^1(\mathbb{C}_K)$ that is not contained in the exceptional set and for which $\phi^{-1}(S) \subseteq S$. Then $\mathcal{J}_\phi \subseteq S$.*

Proof. Pick any $a \in S$ not in the exceptional set. By hypothesis, the backward orbit $\mathrm{Orb}^-(a)$ is contained in S. Because S is closed, then, the closure $\overline{\mathrm{Orb}^-(a)}$ is also contained in S, and hence, by Proposition 4.33, $\mathcal{J}_\phi \subseteq S$. □

Recall that a *perfect set* in a topological space is a closed subset with no isolated points, and that all nonempty perfect sets are uncountable.

Corollary 4.35. *Let K be a complete non-archimedean field, and let $\phi(z) \in K(z)$ be a rational function with Julia set $\mathcal{J}_\phi$. Then $\mathcal{J}_\phi$ is a perfect set. In particular, $\mathcal{J}_\phi$ is either empty or uncountable.*

Proof. We already know from Proposition 4.21.a that $\mathcal{J}_\phi$ is closed. For any $a \in \mathcal{J}_\phi$, we know that the backward orbit of ϕ is infinite, or else a would be exceptional and hence Fatou. Thus, there is some non-periodic point b in the backward orbit of a, and hence in $\mathcal{J}_\phi$. In particular, the backward orbit $\mathrm{Orb}^-(b)$ of b does not contain the a. However, by Proposition 4.33, $\mathrm{Orb}^-(b)$, which is contained in $\mathcal{J}_\phi$, has an accumulation point at a, as desired. □

We close this section by noting an alternate description of the Julia set in the case that ϕ is a polynomial.

Definition 4.36. Let K be a complete valued field, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. The *filled Julia set* of ϕ is

$$\mathcal{K} = \mathcal{K}_\phi := \big\{x \in \mathbb{C}_K : \{\phi^n(x)\}_{n\geq 0} \text{ is bounded}\big\}.$$

Because ∞ is a superattracting fixed point, $\mathcal{K}_\phi$ can alternately be described as

$$\mathcal{K}_\phi = \big\{x \in \mathbb{C}_K : \lim_{n\to\infty} \phi^n(x) \neq \infty\big\}.$$

Proposition 4.37. *Let K be a complete valued field, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$ with filled Julia set $\mathcal{K}$ and Julia set $\mathcal{J}$. Then $\mathcal{J} = \partial\mathcal{K}$, where ∂ denotes the topological boundary, as subsets of $\mathbb{P}^1(\mathbb{C}_K)$.*

Proof. Because the point at ∞ is superattracting and hence lies in the Fatou set, and because the Fatou set is open, there is some radius $R > 0$ such that $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(0, R) \subseteq \mathcal{F}_\phi$, and such that $\lim_{n\to\infty} \phi^n(a) = \infty$ for all $|a| > R$. After a change of coordinates, we may assume that $R < 1$.

Clearly $\mathcal{J} \subseteq \mathcal{K}$. Given $x \in \mathcal{J}$ and any $\mathbb{P}^1(\mathbb{C}_K)$-disk U containing x, we wish to show that U intersects both $\mathcal{K}$ and its complement. We already have $x \in \mathcal{J} \subseteq \mathcal{K}$. By Theorem 4.29, $\bigcup_{n\geq 0} \phi^n(U)$ omits at most two points of $\mathbb{P}^1(\mathbb{C}_K)$, and hence there is some $b \in U$ and $n \geq 0$ such that $|\phi^n(b)| > R$. Therefore, by our choice of R, $\lim_{n\to\infty} \phi^n(b) = \infty$, as desired.

For the reverse inclusion, given $x \in \partial\mathcal{K}$, we wish to show that $x \in \mathcal{J}$. Given any $\mathbb{P}^1(\mathbb{C}_K)$-disk U containing x, there are points $a, b \in U$ such that $a \in \mathcal{K}$ but $b \notin \mathcal{K}$. Hence, $|\phi^n(a)| \leq R < 1$ for all $n \geq 0$, there is some $m \geq 0$ such that $|\phi^m(b)| > 1$. In particular, $\phi^m(U)$ is not contained in a single residue class. By Lemma 4.19, then, $x \in \mathcal{J}$. □

Remark 4.38. Almost all of the results of this section are true in complex dynamics, with occasional slight modifications, and for essentially the same reasons. As noted, Hsia's Theorem is an analogue of the Montel's Theorem, with one fewer point involved. Similarly, the analogues of Corollary 4.28, Theorem 4.29, and Corollary 4.31 all hold over $\mathbb{C}$ but with one more point, because they are all based on Montel-Carathéodory instead of Hsia. Meanwhile, Proposition 4.33, Corollary 4.34 Corollary 4.35, and Proposition 4.37 all hold over $\mathbb{C}$, while Corollary 4.32 must be modified only slightly, to allow the possibility that the Julia set is the whole Riemann sphere. For more details, see Chapters 3 and 4 of [3], Sections III.1 and III.4 of [21], or Sections 3 and 4 of [38].

4.6. **Examples of non-archimedean Fatou and Julia sets.** Throughout this section, K will denote a complete non-archimedean field, and $\mathbb{C}_K$ will be the completion of an algebraic closure of K.

Example 4.39. Let $\phi(z) = Az^2 + Bz + C \in K[z]$ be a quadratic polynomial. As we saw in Example 1.5, if $\operatorname{char} K \neq 2$, then ϕ is conjugate over K to a polynomial $z^2 + c$, with $c \in K$. The dynamics of $z^2 + c$ over a complete non-archimedean field was first considered in [56] for $K = \mathbb{Q}_p$ and $p \neq 2$, and in Section 3.3.1 of [4] for $K = \mathbb{C}_p$ for any p. A full description of the dynamics of any quadratic polynomial over any complete non-archimedean field (even of characteristic 2) appeared in [13], and we follow that strategy here.

Instead of conjugating ϕ to $z^2 + c$, we consider a fixed point α of ϕ. Thus, α is a root of the polynomial $Az^2 + (B-1)z + C = 0$, and either $\alpha \in K$ or else the field $K(\alpha)$ is a quadratic extension of K.

Conjugating by translation by α and then by scaling by A, we obtain a map of the form $\psi(z) = z^2 + \lambda z$ for some $\lambda \in K(\alpha)$. If $|\lambda| \le 1$, then ψ has good reduction, and hence *phi* has potentially good reduction.

If $|\lambda| > 1$, however, then $z = 0$ is a repelling fixed point, and hence $\mathcal{J}_\psi$ and $\mathcal{J}_\phi$ are nonempty, implying that ϕ is not even of potentially good reduction. Let $R = |\lambda| > 1$; then for $|z| > R$, we have $|z|^2 > |\lambda z|$, so that $|\phi(z)| = |z|^2$, from which a simple inductive argument shows that $\lim_{n\to\infty} \phi^n(z) = \infty$. Thus, the filled Julia set $\mathcal{K}_\phi$ is contained in $U_0 := \overline{D}(0, R)$. It is not difficult to check that $U_1 := \psi^{-1}(U_0)$ is the disjoint union of the two disks $A := \overline{D}(0, 1)$ and $B := \overline{D}(-\lambda, 1)$, and that the mappings $\psi : A \to U_0$ and $\psi : B \to U_0$ are both bijective. Therefore, by Proposition 3.14, there are inverse maps $h_A : U_0 \to A$ and $h_B : U_0 \to B$ given by restricted power series $h_A, h_B \in \mathcal{A}_{U_0}$ defined over $K(\alpha)$.

Given the structures of the previous paragraph, it is not difficult to prove that $U_n := \phi^{-n}(U_0)$ is a dijoint union of 2^n rational closed disks, each of radius R^{1-n}, arranged so that each disk of U_{n-1} contains exactly two of the disks of U_n, and those two disks are distant R^{2-n} (the radius of the U_{n-1} disks) apart. Moreover, each disk of U_n contains a point in $K(\alpha)$ and has radius in $|K(\alpha)^\times|$. From that arrangement, a little more work shows that $\mathcal{J}_\psi = \mathcal{K}_\phi = \bigcap_{n\ge 0} U_n$, which is homeomorphic to a Cantor set. In particular, the Julia set is compact and contained in $K(\alpha)$, and every point of the Fatou set is attracted to ∞ under iteration.

The following example, first considered by Smart and Woodcock in [53] and then again by Hsia in Example 4.1 of [33], is of a higher-degree polynomial map with properties similar to the quadratic polynomials in Example 4.39 above. Although similar dynamics can be exhibited for higher degree polynomials over any non-archimedean field, we present the following example (as in [33, 53]) only over p-adic fields, because of the particular form the Julia set takes in that case.

Example 4.40. Let $K = \mathbb{Q}_p$ for any prime p, so that $\mathbb{C}_K = \mathbb{C}_p$. Let

$$\phi(z) = \frac{z^p - z}{p}.$$

Then $U_0 := \overline{D}(0, 1)$ contains the filled Julia set, and as in Example 4.39, it is easy to show that the filled Julia set $\mathcal{K}_\phi$ is just $\bigcap_{n\ge 0} U_n$, where $U_n := \phi^{-n}(U_0)$. Each successive preimage U_n consists of p^n disks of radius $|p|^n$, with p disks in each disk of U_{n-1}, and each mapping bijectively onto its image. By the same techniques as in Example 4.39, then, it is not difficult to show that both the Julia set and the filled Julia set are exactly $\mathcal{J}_\phi = \mathcal{K}_\phi = \mathbb{Z}_p$, and that the whole Fatou set $\mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{Z}_p$ is attracted to ∞ under iteration.

In complex dynamics, quadratic polynomials display a wide variety of dynamical properties, and the dynamics even of most maps of higher degree look locally like dynamics found among quadratic polynomials. However, as Example 4.39 shows, quadratic polynomials are simply not as interesting in the non-archimedean setting— either their Julia sets are Cantor sets arranged in a very understandable way, with all Fatou points attracted to the same attracting fixed point, or else they have good reduction. Indeed, the complex Mandelbrot set, which is the set of complex c-values for which the critical point $z = 0$ of $\phi_c(z) := z^2 + c$ lies in the filled Julia set of ϕ_c, has a famously complicated structure, as it distinguishes those maps with one broad class of dynamics (connected Julia set), from those with another (totally disconnected Julia set). By constract, the non-archimedean version of the Mandelbrot set does exist, but it is simply a disk. If $|\lambda| \le 1$, then $z^2 + \lambda z$ is in the Mandelbrot set; otherwse, it is not. (Or, if one works with $z^2 + c$, presumably over a field of characteristic not equal to 2, then the set of c values of potentially good reduction again forms a disk; it is the closed unit disk if the residue characteristic is also not 2, but it is $\overline{D}(0, |2|^{-2})$ if the residue characteristic *is* 2.)

However, even non-archiemdean cubic polynomials display far more intricate dynamics than their quadratic counterparts, as the following example illustrates. It was first considered by Hsia in Example 4.2 of [33].

Example 4.41. Let K be a complete non-archimedean field of residue characteristic different from 2. Pick $a, b, c \in K$ with $0 < |a| < 1$ and $|b|, |c| \leq 1$, and set

$$\phi(z) = az^3 + z^2 + bz + c.$$

By the various mapping properties of power series in Section 3, we can make the following observations about ϕ. First, ϕ has one fixed point $\alpha \in K$ with $|\alpha| = |a|^{-1} > 1$, and the multiplier λ of α satisfies $|\lambda| = |a|^{-1} > 1$, so that α is repelling. (There are also two other fixed points in $\overline{D}(0,1)$, as well as the point at ∞, of course.) On the other hand, $\phi(\overline{D}(0,1)) = \overline{D}(0,1)$. Thus, $\alpha \in \mathcal{J}_\phi$, whereas $\overline{D}(0,1) \subseteq \mathcal{F}_\phi$.

Meanwhile, the complement of the disk $U_0 := \overline{D}(0, |a|^{-1})$ is attracted to ∞ under iteration, and it is not difficult to show that the filled Julia set is $\mathcal{K}_\phi = \bigcap_{n\geq 1} \phi^{-n}(U_0)$. This time, however, $U_1 := \phi^{-1}(U_0)$ is the disjoint union of $A := \overline{D}(0, |a|^{-1/2})$ and $B := \overline{D}(\alpha, 1)$, and although B maps onto U_0 with Weierstrass degree 1, A maps onto U_0 with Weierstrass degree 2. The description of $U_n := \phi^{-n}(U_0)$ becomes far more complicated than that in Example 4.39, although such a description can be given, as this ϕ is still relatively tame compared with some other maps out there.

Moreover, the Julia set is not compact, for the following reason. The point α has a preimage α_1 with $|\alpha_1| = |a|^{-1/2}$. (In fact, there are two such preimages, but we will pick only one of them.) Similarly, α_1 has a preimage α_2 with $|\alpha_2| = |a|^{-1/4}$; and in general, α has a preimage α_n under ϕ^n with $|\alpha_n| = |a|^{-1/2^n}$. The sequence $\{\alpha_n\}_{n\geq 1}$ lies in $\mathcal{J}_\phi$ but has no accumulation points in $\mathbb{P}^1(\mathbb{C}_K)$, and thus $\mathcal{J}_\phi$ is not compact.

Still, the Julia set is (somehow) still the topological boundary of the filled Julia set. In addition, for this particular map ϕ, one can show that every element of the Fatou set either is eventually mapped into $\overline{D}(0,1)$ or is attracted to ∞ under iteration. The dynamics on $\overline{D}(0,1)$ itself are given by the dynamics of the reduced map $\overline{\phi}(z) = z^2 + \overline{b}z + \overline{c}$, in the sense that residue classees in $\overline{D}(0,1)$ are mapped to each other as specified by $\overline{\phi}(z)$. Any periodic residue class within $\overline{D}(0,1)$ containing a critical point (either in the residue class itself or in one of its forward iterates) also contains an attracting periodic point of the same period, to which every point of the residue class is attracted under iteration. Periodic residue classes not containing critical points in their forward orbits do not contain attracting periodic points, but they do contain indifferent periodic points, and in fact, often, infinitely many. Finally, if the residue field of $\mathbb{C}_K$ is not algebraic over a finite field, then there will be wandering (i.e, non-preperiodic) residue classes within $\overline{D}(0,1)$ as well.

The driving force behind Example 4.41 is that on $\overline{D}(0,1)$, ϕ acts essentially like the good reduction quadratic polynomial $z^2 + bz + c$, but once we get far enough outside the closed unit disk, the az^3 term becomes powerful enough to make other things happen. Similar phenomena will occur for any polynomial of the form $aq(z)z^d + r(z)$, where $r \in \mathcal{O}_K[z]$ is a monic polynomial of good reduction, $q(z) \in \mathcal{O}_K[z]$, $|a| < 1$ and $d \geq 1 + \deg r$. Such maps can exhibit even more complicated dynamics than the cubic polynomial in Example 4.41 did, because of critical points just outside $\overline{D}(0,1)$, for example. Even greater complications can occur if any of the disks of interest (like those making up the sets $U_n = \phi^{-n}(U_0)$) map to each other with Weierstrass degree divisible by the residue characteristic. Indeed, even the cubic map of Example 4.41 becomes far more complicated if we consider fields K of residue characteristic 2.

We now consider a rational function, which first appeared in Example 1 of [8].

Example 4.42. Let K be a complete non-archimedean field of residue characteristic different from 2, pick $a \in K$ with $0 < |a| < 1$, and set

$$\phi(z) = z^2 + \frac{az^2}{z+1} = \frac{z^3 + (a+1)z^2}{z+1}.$$

Besides the (superattracting) fixed points at $0, \infty$, ϕ also has a fixed point α in $D(1,1)$ and a fixed point β in $D(-1,1)$. Although α is indifferent, it turns out that β, because of its proximity to the pole at -1, is repelling.

Thus, if $|x| > 1$, then the iterates of x are attracted to ∞; and if $|x| < 1$, then the iterates of x are attracted to 0. Meanwhile, if $|x| = 1$, then because $\overline{\phi}(z) = z^2$, we get $\phi(x) \approx x^2$, in the sense that $\phi(x) \in D(0,1)$, *unless* $x \in D(-1,1)$. In that residue class, the presence of the pole means that the term $az^2/(z+1)$ is *not* insignificant, as evidenced by the repelling fixed point β.

Hence, if the reduction $\overline{b}$ of a point $b \in \mathcal{O}_{\mathbb{C}_K}$ does *not* ever hit -1 under repeated application of $z \mapsto z^2$ in $\mathbb{P}^1(\overline{k})$, then $\phi^n(D(b,1)) = D(b^{2^n}, 1)$. In particular, residue classes that are preperiodic under the squaring map *and avoid* -1 (or that are wandering, which can happen if the residue field k is not algebraic over a finite field) behave essentially the same way under ϕ. However, residue classes that eventually hit $D(-1,1)$ under the squaring map contain preimages of β and hence intersect the Julia set. Thus, there are many Fatou periodic and preperiodic (and maybe wandering) residue classes, but there are also many residue classes intersecting $\mathcal{J}_\phi$ jumbled in with them. In addition, within those non-Fatou residue classes, it is very difficult to see what the dynamics looks like. Are there more Fatou periodic points hiding in such classes, for example, or do all of the nonrepelling periodic points lie in the periodic Fatou residue classes? (It turns out that because of the residue characteristic and the location of the critical points, one can show that it is the former; all periodic points in the complicated residue classes are repellling.) As in Example 4.41, the Julia set is not compact (as it intersects infinitely many residue classes), but the dynamics on the Fatou set is substantially more intricate than in that example.

5. Components of the Fatou set

In complex dynamics, one partitions the Fatou set of $\phi \in \mathbb{C}(z)$ into connected components and considers the action of ϕ on the set of components. Specifically, if U is a component of the complex Fatou set, then so is $\phi(U)$, and hence we may discuss fixed, periodic, and preperiodic components. By the combined work of Fatou, Julia, Siegel, Herman, and others, it can be shown that every periodic component U of minimal period m falls into one of four types:

- *Attracting*: U contains a unique attracting periodic point a, also of period m, and every point in U is attracted to a under iteration of ϕ^m.
- *Parabolic*: the boundary of U contains a unique rationally indifferent periodic point a of period j dividing m (more precisely, so that the multiplier λ is a primitive ℓ-th root of unity, and $j\ell = m$), and every point in U is attracted to a under iteration of ϕ^m.
- *Siegel disk*: U is simply connected and contains a unique irrationally indifferent fixed point a with multiplier λ, also of period m, and ϕ^m is analytically conjugate on U to the rotation $z \mapsto \lambda z$ of the open unit disk.
- *Herman ring*: U is topologically an annulus, and ϕ^m is analytically conjugate on U to some irrational rotation of an annulus.

(Some texts count five types, by breaking the attracting case into two separate cases: attracting and superattracting.)

Moreover, each type is associated to a critical point: the periodic cycle of an attracting or parabolic component must contain a critical point, and the boundary of a Siegel disk or Herman ring also must contain a critical point. Fatou used this fact to prove in 1919 that there can be at most $6 \deg \phi - 6$ periodic cycles of components [28], although the *length* of those cycles can be arbitrarily long; it was not until 1987 that Shishikura improved the bound to the sharp value of $2\deg\phi - 2$ using a method known as quasiconformal surgery [51].

Meanwhile, Sullivan proved in 1985, also using equisconformal methods, that there are no "wandering domains," meaning that all Fatou components of a complex rational function are preperiodic [54]. Together, then, these results give a very clear description of the possible dynamics on the Fatou set over $\mathbb{C}$. For more information on this theory, see Chapters 6–9 of [3], Chapters II and IV–VI of [21], or Sections 10, 11, 15, and 16 of [38].

Much of the remaining portions of these lectures are devoted to transplanting the theory of components in complex dynamics to the non-archimedean setting. As we will see, although much has been done, some complex results are false over non-archimedean fields, some are true for some non-archimedean fields and false for others, some hold only after some modifications to the hypotheses or conclusions, and some simply remain open.

5.1. **Analytic components and D-components.** Motivated by Theorem 4.29, we make the following definition.

Definition 5.1. Let K be a complete non-archimedean field, let $\phi \in K(z)$ be a rational function, and let $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be an open set. We say that U is *dynamically stable* under ϕ if $\bigcup_{n\geq 0} \phi^n(U)$ omits at least three points of $\mathbb{P}^1(\mathbb{C}_K)$.

Note that any dynamically stable open set U must be contained in the Fatou set of ϕ. Indeed, given any $x \in U$, there is some $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x and contained in U. Clearly D is also dynamically stable, and hence, by Theorem 4.29, x belongs to the Fatou set. Meanwhile, the reason we require the omission of three points rather than two is because if U is a connected affinoid whose iterates omit at least three points of $\mathbb{P}^1(\mathbb{C}_K)$, then in fact they omit a set with nonempty interior. However, if $\phi(z) = z^d$ (with $d \geq 2$) and U is the annulus $\{z \in \mathbb{P}^1(\mathbb{C}_K) : r < |z| < s\}$ for some real numbers $0 < r < 1 < s$, then the union of the iterates of U omits only the two exceptional points 0 and ∞ themselves.

As we saw in Section 3, even though $\mathbb{P}^1(\mathbb{C}_K)$ is a totally disconnected topological space, $\mathbb{P}^1(\mathbb{C}_K)$-disks (together with $\mathbb{P}^1(\mathbb{C}_K)$ itself) and connected affinoids behave like connected sets because, for example, the image of such an object under a rational function is another such object. In more general topological spaces, we can build new connected sets by taking the union of smaller overlapping connected sets; however, finite unions of overlapping $\mathbb{P}^1(\mathbb{C}_K)$-disks or overlapping connected affinoids are again $\mathbb{P}^1(\mathbb{C}_K)$-disks or connected affinoids. With these ideas in mind, we present the following definition

Definition 5.2. Let K be a complete non-archimedean field, let $\phi \in K(z)$ be a rational function with Fatou set $\mathcal{F}_\phi$, and let $\mathcal{B}$ be the set of all

i. connected affinoids in $\mathbb{P}^1(\mathbb{C}_K)$, or
ii. $\mathbb{P}^1(\mathbb{C}_K)$-disks, or
iii. open connected affinoids in $\mathbb{P}^1(\mathbb{C}_K)$, or
iv. open $\mathbb{P}^1(\mathbb{C}_K)$-disks

that are dynamically stable under ϕ. Let $x \in \mathcal{F}_\phi$. The *$\mathcal{B}$-component* of $\mathcal{F}_\phi$ containing x is the set of all $y \in \mathcal{F}_\phi$ for which there are sets $U_1, \ldots, U_n \in \mathcal{B}$ such that $x \in U_1$, $y \in U_n$, and $U_i \cap U_{i+1} \neq \varnothing$ for all $i = 1, \ldots, n-1$.

For clarity, we will usually call the $\mathcal{B}$-components

i. analytic components, or
ii. D-components, or
iii. open analytic components, or
iv. open D-components,

if $\mathcal{B}$ is defined by i, ii, iii, iv above, respectively.

One can restate Definition 5.2 for $\mathbb{C}$ instead of K, by replacing all appearances of K and $\mathbb{C}_K$ by $\mathbb{C}$, and by setting $\mathcal{B}$ to be either the set of open disks, the set of all disks, the set of open connected sets, or the set of all connected sets in $\mathbb{P}^1(\mathbb{C})$. We leave it to the reader to check that in all four cases, the resulting complex $\mathcal{B}$-components are simply the usual connected components of the Fatou set. In fact, the dynamical stability condition is irrelevant; if we omit that from the definition and consider *any* open subset $\mathcal{F}$ of $\mathbb{P}^1(\mathbb{C})$, the resulting $\mathcal{B}$-components of $\mathcal{F}$ are exactly the connected components of $\mathcal{F}$.

The relation "y belongs to the $\mathcal{B}$-component containing x" is clearly an equivalence relation. Still, Proposition 5.4 below gives a simpler characterization of the four types of Fatou components, for which transitivity is less immediately obvious. The main reason this result holds is that unions of intersecting disks or connected affinoids are again objects of the same type. However, the dynamical stability condition causes complications. To address them, we first introduce a technical definition, needed solely to make the verbal statements of the next few results less intricate.

Definition 5.3. Let K be a complete non-archimedean field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. We say that ϕ is a *quasimonomial* if ϕ is conjugate over $\mathbb{C}_K$ to a rational function $\psi \in \mathbb{C}_K(z)$ of good reduction, and the reduction $\overline{\psi}$ is $\overline{\psi}(z) = z^d$.

Proposition 5.4. *Let K be a complete non-archimedean field, let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_\phi$, and let $\mathcal{B}$ be one of the four sets in Definition 5.2. If $\mathcal{B}$ is either the set of connected affinoids or the set of $\mathbb{P}^1(\mathbb{C}_K)$-disks (i.e., set i or ii), assume that ϕ is* not *a quasimonomial. Then for any $x \in \mathcal{F}_\phi$, the $\mathcal{B}$-component of $\mathcal{F}_\phi$ containing x is the union of all (dynamically stable) $U \in \mathcal{B}$ for which $x \in U \subseteq \mathcal{F}_\phi$. Moreover,*

a. *every open D-component of $\mathcal{F}_\phi$ is an open $\mathbb{P}^1(\mathbb{C}_K)$-disk, and*
b. *if ϕ is not a quasimonomial, then every D-component of $\mathcal{F}_\phi$ is a $\mathbb{P}^1(\mathbb{C}_K)$-disk.*

We delay the proof of Proposition 5.4 until the end of Section 6.4, because it requires some results on dynamics in Berkovich space. However, if $\mathcal{J}_\phi \neq \varnothing$, the proof is far simpler, because in that case, all subsets of the Fatou set are dynamically stable.

Proof of Proposition 5.4 when $\mathcal{J}_\phi$ is nonempty. For the first statement, it suffices to show that for each choice of $\mathcal{B}$, and for any $U_1, U_2 \in \mathcal{B}$ for which $U_1 \cap U_2 \neq \varnothing$, we have $U_1 \cup U_2 \in \mathcal{B}$. We know from Propositions 3.28 and 3.30 that the union of two intersecting (open) connected affinoids are again (open) connected affinoids, and that the union of two interesecting (open) $\mathbb{P}^1(\mathbb{C}_K)$-disks is again either an (open) disk or else all of $\mathbb{P}^1(\mathbb{C}_K)$. Thus, we only need to show that $U_1 \cup U_2$ is still dynamically stable.

However, all iterates of $U_1 \cup U_2 \subseteq \mathcal{F}_\phi$ avoid the Julia set $\mathcal{J}_\phi$. Because we assume $\mathcal{J}_\phi$ is nonempty, and hence uncountable by Corollary 4.35, it follows that $U_1 \cup U_2$ is dynamically stable, as desired.

We will prove the final statement for D-components; the case of open D-components is similar.

Because $\mathcal{J}_\phi$ has more than one point, we may assume, after changing coordinates, that $x = 0$ and $1, \infty \in \mathcal{J}_\phi$. Thus, every dynamically stable disk containing $x = 0$ is contained in $\mathcal{F}_\phi$ and hence is contained in $D(0, 1)$.

Thus, every such disk is of the form either $D(0, r)$ for $0 < r < 1$ or $\overline{D}(0, r)$ for $0 < r \leq 1$. Taking the supremum R of all such r, it follows that the Fatou D-component containing x is either $\overline{D}(0, R)$ or $D(0, R)$. □

The most commonly used components are the open analytic components, which are closely related to the connected components of the Berkovich Fatou set that we will encounter later. However, the other three component types are each sometimes useful as well. In addition, in many cases, some or all four of the types coincide.

The map ϕ itself induces a function from the set of $\mathcal{B}$-components to itself, as follows. By Corollary 3.10 and Proposition 3.31, the image $\phi(U)$ of a dynamically stable (open) disk or connected affinoid $U \in \mathcal{B}$ is again an element of $\mathcal{B}$. Thus, it is immediate from Definition 5.2 that if W is a $\mathcal{B}$-component of the Fatou set of ϕ, then $\phi(W)$ is a subset of some (probably different) $\mathcal{B}$-component. We can therefore define the action of ϕ on the $\mathcal{B}$-components by

$$\phi : W \longmapsto \phi[W],$$

where

$$\phi[W] := \text{ the } \mathcal{B}\text{-component containing } \phi(W).$$

Remark 5.5. If W is an (open) analytic component, then in fact $\phi[W] = \phi(W)$, because by Proposition 3.31, the inverse image $\phi^{-1}(V)$ of each of the connected affinoids V contained in $\phi[W]$ is itself a finite union of connected affinoids, and those affinoids clearly dynamically stable because V is.

However, if W is an (open) D-component, then it is possible that $\phi(W) \subsetneq \phi[W]$; see Example 5.7, for instance. Put another way, there may be (open) D-components V such that $\phi^{-1}(V)$ consists of more than one (and in fact, infinitely many) D-components. In that case, however, there are at most $\deg\phi - 1$ D-components V whose inverse images are not also D-components. See Theorem 3.1.2 of [4] for a proof of this fact.

Example 5.6. Let $\phi(z) \in K(z)$ be a map of good reduction. Then the reader can check that any residue class is dynamically stable, but any strictly larger open connected affinoid is not dynamically stable. Thus, the open analytic components and the open D-components are each precisely the residue classes, and ϕ maps components to each other exactly as prescribed by the way the reduced map $\overline{\phi}$ maps points of $\mathbb{P}^1(\overline{k})$ to each other.

We now turn to the (*a priori* larger) D-components and analytic components. If $\overline{\phi}$ is not conjugate over $\overline{k}$ to a polynomial, then any connected affinoid (even a closed one) strictly larger than a residue class is also not dynamically stable. Thus, the analytic components and D-components, like their open counterparts, are also precisely the residue classes, and they are therefore mapped to each other in the same way as above.

If $\overline{\phi}$ is conjugate to a polynomial but not to z^d, then assume $\overline{\phi}$ is already a polynomial. In that case, the closed disk $\overline{D}(0,1)$ maps to itself, and therefore it (and any connected affinoid subset of it) is dynamically stable. As a result, ϕ has only two analytic components, both of which are also D-components— namely, the disk $\overline{D}(0,1)$ and its complement $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(0,1)$.

The only remaining case is that ϕ is a quasimonomial, i.e., $\overline{\phi}$ is conjugate to z^d. After a change of coordinates, assume $\overline{\phi}(z) = z^d$. Then $U = \overline{D}(0,1)$ maps to itself, but so does the rational closed disk $V = \mathbb{P}^1(\mathbb{C}_K) \smallsetminus D(0,1)$, and both contain the point $z = 1$. As a result, the whole of $\mathbb{P}^1(\mathbb{C}_K)$ is a single analytic component and a single D-component. This is the one special side case in Proposition 5.4, because there is no dynamically stable connected affinoid containing both the points 0 and ∞, even though they lie in the same (non-open) component.

Example 5.7. Consider again the quadratic polynomial $\phi(z) = z^2 + \lambda z \in K[z]$ from Example 4.39, and assume that $|\lambda| > 1$. (The case $|\lambda| \leq 1$ is subsumed by Example 5.6 above.) Recall that the complement of the disk $U_0 := \overline{D}(0, |\lambda|)$ is attracted to ∞, that $U_n := \phi^{-n}(U_0)$ is a union of 2^n rational closed disks contained in U_0, and that $\mathcal{J}_\phi = \bigcap_{n\geq 0} U_n$. Thus, for each $n \geq 0$, $V_n := \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_n$ is an open connected affinoid containing the point $\infty \in \mathcal{F}_\phi$. Moreover, V_n is certainly dynamically stable, because the Julia set is nonempty (and therefore uncountable, by Corollary 4.35), and all the iterates of V_n certainly avoid $\mathcal{J}_\phi$. Thus, the entire Fatou set $\mathcal{F}_\phi = \bigcup_{n\geq 0} V_n$ is a single analytic (or open analytic) component. Note that this set is not itself a connected affinoid; Rivera-Letelier dubbed such components *attracting components of Cantor type.* (See Definition 7.6.)

Meanwhile, the only $\mathbb{P}^1(\mathbb{C}_K)$-disks containing ∞ and contained in $\mathcal{F}_\phi$ are also contained in $V_0 = \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0$, which is an open and dynamically stable $\mathbb{P}^1(\mathbb{C}_K)$-disk. Thus, V_0 is a D-component (and an open D-component) of $\mathcal{F}_\phi$. However, $\phi^{-1}(V_0) = V_1$ is *not* a disk, and hence, by Proposition 5.4, it is neither a D-component nor an open D-component. Instead, it is a disjoint union of infinitely many D-components (each of which is also an open D-component) W, where $\phi[W] = V_0$.

This is the phenomenon mentioned in Remark 5.5, of D-components mapping into but not onto each other. Indeed, each $V_n = \phi^{-n}(V_0) = \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_n$ decomposes into infinitely many disjoint disks which are both D-components and the open D-components. Each such disk is a maximal disk contained in V_n, or equivalently, in $\mathcal{F}_\phi$. The D-components within V_{n+1} map onto those in V_n, and hence all the D-components eventually land in the fixed D-component V_0, each after finitely many iterations.

Example 5.8. Consider the map

$$\phi(z) = az^3 + z^2 + bz + c$$

of Example 4.41 over a field of residue characteristic not equal to 2, where $0 < |a| < 1$, and $|b|, |c| \leq 1$. The D-component and open D-component at ∞ are both again the open $\mathbb{P}^1(\mathbb{C}_K)$-disk $V_0 := \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0$, where $U_0 := \overline{D}(0, |a|^{-1})$, and the analytic component and open analytic component are again the Cantor-type domain $\bigcup_{n\geq 0} V_n$, where $V_n := \phi^{-n}(V_0) = \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_n$, and where $U_n = \phi^{-n}(U_0)$. The dynamics of D-components mapping into each other within this fixed analytic component is similar to that of Example 5.7— all the D-components map onto D-components, with the exception of those that ϕ maps into U_0. Moreover, all of the D-components in any V_n land in the fixed D-component V_n after finitely many iterations.

This time, however, there are also Fatou components that are not attracted to ∞. In particular, the closed disk $\overline{D}(0, 1)$ maps into itself, but because of the sequence of Julia points $\{\alpha_n\}_{n\geq 1}$ with $|\alpha_n| = |a|^{-1/2^n}$, there are no connected affinoids properly containing $\overline{D}(0, 1)$ that are still contained in the Fatou set. Thus, $\overline{D}(0, 1)$ is a fixed D-component and a fixed analytic component. On the other hand, it consists of infinitely many *open* D-components and *open* analytic components, which are all the open disks $D(x, 1)$ in $\overline{D}(0, 1)$— that is, all the finite residue classes. These open disks map to one another as prescribed by the reduced map $\overline{\phi}(z) = z^2 + \overline{b}z + \overline{c}$. If the residue field $\overline{k}$ is algebraic over a finite field, then all such open components are preperiodic; but otherwise, many of them will wander.

Finally, the component(s) in $\overline{D}(0, 1)$ have preimages outside $\overline{D}(0, 1)$. Indeed, because of the stipulation that the residue characteristic is not 2, it can be shown that all Fatou points eventually map into either V_0 or into $\overline{D}(0, 1)$. Those that evenutally land in V_0 are already in the fixed analytic component $\bigcup_{n\geq 0} V_n$. Meanwhile, the D-components and analytic components of those that eventually land in $\overline{D}(0, 1)$ are rational closed disks W for which $\phi[W] = \phi(W)$, and which eventually map onto $\overline{D}(0, 1)$ after finitely many iterations. The open D-components and open analytic components of such points are rational open disks W' for which $\phi[W'] = \phi(W')$, and which eventually map onto residue classes in $\overline{D}(0, 1)$.

In particular, this example shows that the four different types of components are truly different but are closely related. The analytic components are $\overline{D}(0, 1)$ and its preimages, along with the Cantor-type domain at ∞. The D-components are $\overline{D}(0, 1)$ and its preimages, V_0, and the D-components of each V_n. The open analytic components are the finite residue classes and their preimages, along with the Cantor-type domain at ∞. Finally, the open D-components are the finite residue classes and their preimages, along with V_0 and the D-components of each V_n.

Example 5.9. Consider the map

$$\phi(z) = z^2 + \frac{az^2}{z+1} = \frac{z^3 + (a+1)z^2}{z+1}$$

where $0 < |a| < 1$, over a field of residue characteristic not equal to 2, as in Example 4.42. Recall that the Julia set intersects the residue class containing -1, as well as all preimages of that residue class under the squaring map; all other residue classes are contained in the Fatou set.

Because of the infinitely many non-Fatou residue classes, for any Fatou residue classes $W \subseteq \mathcal{F}_\phi$, W itself is the only connected affinoid containing W. Thus, the residue class W is a component of all four types, and the dynamics of these components is as determined by the squaring map acting on $\mathbb{P}^1(\overline{k})$. If the residue field $\overline{k}$ is algebraic over a finite field, then all such components are preperiodic; otherwise, some will be wandering.

Meanwhile, as noted in Example 4.42, it can be shown that all Fatou points in the bad residue classes (that is, in $D(-1, 1)$ and its preimages) eventually land in one of the good residue classes. In fact, all of the components (of any of the four types) for such Fatou points are again open disks.

6. The Berkovich projective line

In his thesis [41], Rivera-Letelier pioneered the use of the Berkovich projective line $\mathbb{P}^1_{\mathrm{Ber}}$ in the study of p-adic dynamics. In the years since then, it has become clear that Berkovich space is the appropriate setting for much of non-archimedean dynamics. Intuitively, $\mathbb{P}^1_{\mathrm{Ber}}$ is a topological space containing $\mathbb{P}^1(\mathbb{C}_K)$, where most of the extra points that have been added correspond to certain equivalence classes of disks. The space $\mathbb{P}^1_{\mathrm{Ber}}$ has many topological advantages over $\mathbb{P}^1(\mathbb{C}_K)$, such as the fact that it is both compact and path-connected. (See Chapter 3 of [14], as well as Proposition 2.6 and Lemma 2.10 of [2].) However, it is not simply an abstract compactification—the extra points that have been added have genuine significance, and the action of a rational function ϕ on the extra points in $\mathbb{P}^1_{\mathrm{Ber}}$ conveys a far clearer understanding of the action of ϕ even on the original space $\mathbb{P}^1(\mathbb{C}_K)$.

For deeper background on Berkovich spaces, we refer the reader to Berkovich's monograph [14]. Related seminorm constructions also appeared in earlier papers, such as [24], but Berkovich's more complete theory seems to be the proper language for non-archimedean dynamics. The book [2] includes a more accessible introduction to the theory, given that it focuses solely on the projective line (rather than arbitrary varieties, as in [14]), and its later chapters are devoted to certain aspects of non-archimedean dynamics. Finally, Rivera-Letelier's papers [41, 42] construct the Berkovich projective line in a different way, motivated more directly by the dynamical applications.

6.1. **Berkovich disks.** The points of Berkovich space are defined using seminorms. Recall that a *multiplicative seminorm* on a ring A is a function $\|\cdot\| : A \to [0, \infty)$ such that $\|0\| = 0$, $\|1\| = 1$, and for all $f, g \in A$, $\|fg\| = \|f\| \cdot \|g\|$ and $\|f + g\| \le \|f\| + \|g\|$; notably absent from this definition is the requirement that $\|f\| = 0$ implies $f = 0$. In addition, another seminorm $\|\cdot\|_\zeta$ is said to be *bounded* with respect to $\|\cdot\|$ if $\|f\|_\zeta \le \|f\|$ for all $f \in A$. (We leave it to the reader to check that this boundedness condition is equivalent to the *a priori* weaker condition that there is a constant $C > 0$ such that $\|f\|_\zeta \le C\|f\|$ for all $f \in A$.) Finally, recall the ring $\mathcal{A}_D$ of functions on a disk $D \subseteq \mathbb{C}_K$ and the norms $\zeta(a, r) = \|\cdot\|_{\zeta(a,r)}$ from Section 3.3.

Definition 6.1. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $a \in \mathbb{C}_K$, and let $r > 0$.

a. The *closed Berkovich disk* $\overline{D}_{\mathrm{Ber}}(a, r)$ is the set of all those multiplicative seminorms on $\overline{\mathcal{A}}(a, r)$ that are bounded with respect to $\zeta(a, r)$.
b. The *open Berkovich disk* $D_{\mathrm{Ber}}(a, r)$ is the set of all those multiplicative seminorms on $\mathcal{A}(a, r)$ *other than* $\zeta(a, r)$ *itself* that are bounded with respect to $\zeta(a, r)$.

We equip $\overline{D}_{\mathrm{Ber}}(a, r)$ (and hence also its subspace $D_{\mathrm{Ber}}(a, r)$) with the *Gel'fand topology*, which is the weakest topology such that for every $f \in \overline{\mathcal{A}}(a, r)$, the map $\zeta \mapsto \|f\|_\zeta$ from $\overline{D}_{\mathrm{Ber}}(a, r)$ to $\mathbb{R}$ is continuous.

Given any disk $\overline{D}(b,s) \subseteq \overline{D}(a,r)$, the associated norm $\zeta(b,s)$ is a point in the Berkovich disk $\overline{D}_{\mathrm{Ber}}(a,r)$, by Proposition 3.20. In addition, for any point $x \in \overline{D}(a,r)$, it is easy to check that the function $\|\cdot\|_x$ given by $\|f\|_x := |f(x)|$ is also a bounded multiplicative seminorm (although *not* a norm) on $\overline{\mathcal{A}}(a,r)$, and hence a Berkovich point. If $\mathbb{C}_K$ is not spherically complete, then given any decreasing sequence of disks $D_1 \supseteq D_2 \supseteq \cdots$ with empty intersection and with associated norms $\zeta_1, \zeta_2, \ldots$, one can define a corresponding bounded multiplicative norm $\|\cdot\|$ given by $\|f\| := \lim_{n\to\infty} \|f\|_{\zeta_n}$. (We leave it to the reader to check that this limit converges, $\|\cdot\|$ is indeed a norm, and any equivalent sequence of disks $\{E_n\}$ gives the same norm. Here, we say that two decreasing sequences $\{D_n\}$, $\{E_n\}$ of disks with empty intersection are *equivalent* if for every $N \geq 0$, there is some $M \geq N$ such that $E_M \subseteq D_N$ and $D_M \subseteq E_N$.)

Berkovich's classification theorem (see Section 1.4.4 of [14], or Theorem 1.2 of [2]) says that the previous paragraph describes all possible points in $\overline{D}_{\mathrm{Ber}}(a,r)$. More precisely, the classification theorem distinguishes four types of points:

- Type I: seminorms $\|\cdot\|_x$ corresponding to points $x \in \overline{D}(a,r)$.
- Type II: norms $\|\cdot\|_{\zeta(b,s)}$ corresponding to rational closed disks $\overline{D}(b,s)$, where $b \in \overline{D}(a,r)$ and $s \in (0,r] \cap |\mathbb{C}_K^\times|$.
- Type III: norms $\|\cdot\|_{\zeta(b,s)}$ corresponding to irrational disks $\overline{D}(b,s)$, where $b \in \overline{D}(a,r)$ and $s \in (0,r] \smallsetminus |\mathbb{C}_K^\times|$.
- Type IV: norms $\|\cdot\|_{\{\zeta_n\}}$ corresponding to decreasing sequences of disks in $\mathbb{C}_K$ with empty intersection.

Note that only the type I points fail to be norms. It is easy to check that all four types of seminorms are in fact non-archimedean: that is, $\|f+g\|_\zeta \leq \max\{\|f\|_\zeta, \|g\|_\zeta\}$ for all $f,g \in \overline{\mathcal{A}}(a,r)$.

In particular, every $\zeta \in \overline{D}_{\mathrm{Ber}}(a,r)$ falls into exactly one of the four types above. In addition, two points coincide if and only if their corresponding geometric data is equivalent. That is, two type I points $\|\cdot\|_x$ and $\|\cdot\|_y$ are equal if and only if $x = y$; two type II or III points $\zeta(b,s)$ and $\zeta(c,t)$ coincide if and only if the *closed* disks $\overline{D}(b,s)$ and $\overline{D}(c,t)$ are equal (as sets); and two type IV points coincide if and only if the underlying sequences $\{D_n\}$ and $\{E_n\}$ of disks are equivalent.

From the type II and III points, we can get an idea of how $\overline{D}_{\mathrm{Ber}}(a,r)$ is path-connected. For example, if $x \in \mathbb{C}_K$ and $r, R > 0$ are real numbers such that $0 < r < |x| < R$, consider the arrangement of the points 0 and x and the disks $\overline{D}(x,r)$ and $\overline{D}(0,R)$ in Figure 1(a). Bearing this picture in mind, we can see the path in Berkovich space from 0 to $\zeta(x,r)$ as follows. Start at the type I point 0, which we can temporarily think of as a disk of radius $s = 0$. Increase the radius s through a path of type II and III points of the form $\zeta(0,s)$ until we get to radius $s = R$. Then $\zeta(0,R) = \zeta(x,R)$ (since $\overline{D}(0,R) = \overline{D}(x,R)$), and so we may now consider disks $\overline{D}(x,s)$ centered at s. Decrease the radius s towards the new center x until we arrive at $\zeta(x,r)$, corresponding to the disk $\overline{D}(x,r)$; see Figure 1(b).

Intuitively, the space $\overline{D}_{\mathrm{Ber}}(a,r)$ looks like a tree branching out from the root point $\zeta(a,r)$ with infinitely many branches at every type II point (which are dense in the tree), and with limbs ending at the type I and type IV points. See Figure 2, which shows a few of the (type II) points of branching, with a few of the branches at each such point; all of the end points at the bottom are type I points. The infinitely many branches at a type II point $\zeta(b,s)$ correspond to the infinitely many open subdisks $D(c,s)$ of $\overline{D}(b,s)$ of the same radius, as well as (if $s < r$) one more branch corresponding to increasing the radius. The type III points, meanwhile, are interior points with no branching. See Sections 1.4 and 2.5 of [2] for a more rigorous description of the tree structure on Berkovich disks and on $\mathbb{P}^1_{\mathrm{Ber}}$.

The topology, however, is slightly weaker than the tree image might suggest at first. For example, the type I points (at the tips of the branches) are dense in the full space. The same is true of each of the other types, assuming there are any points of that type at all. (To clarify the previous sentence:

if the value group $|\mathbb{C}_K^\times|$ is all of $(0,\infty)$, then there are no type III points; and if $\mathbb{C}_K$ is spherically complete, then there are no type IV points.)

So what are the open sets? Consider the polynomial $f(z) = z - b$, for any $b \in \overline{D}(a,r)$. Then for any $s > 0$, the set U of seminorms $\zeta \in \overline{D}_{\text{Ber}}(a,r)$ for which $\|z-b\|_\zeta < s$ is an open set, as is the set V of ζ for which $\|z-b\|_\zeta > s$, by the definition of the Gel'fand topology. If we restrict to just the type I points, the corresponding sets are the open disk $D(b,s)$ and the complement of the closed disk $\overline{D}(b,s)$. Meanwhile, a given point of type II or III belongs to U if and only if the corresponding closed disk $D \subseteq \mathbb{C}_K$ is *properly* contained $D(b,s)$, and a type IV point belongs to U if and only if for some large enough N, the disk D_N in the corresponding sequence $\{D_n\}$ is properly contained in $D(b,s)$. That is, U is exactly $D_{\text{Ber}}(b,s)$, under the obvious inclusion of $D_{\text{Ber}}(b,s)$ into $\overline{D}_{\text{Ber}}(a,r)$. The analogous statements are true for concerning the membership of points of type II–IV in V.

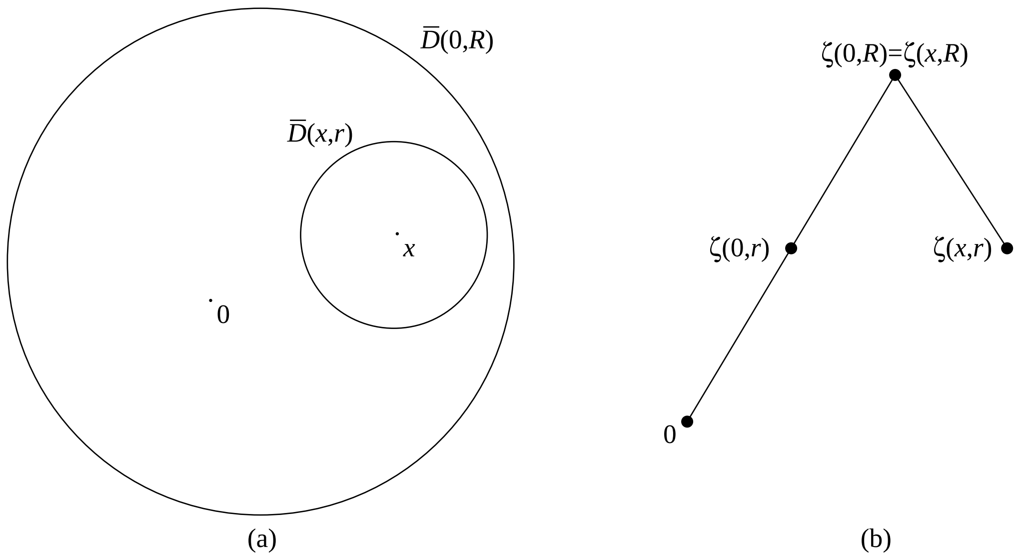

FIGURE 1. The path from 0 to $\zeta(x,r)$ in $\overline{D}_{\text{Ber}}(0,R)$

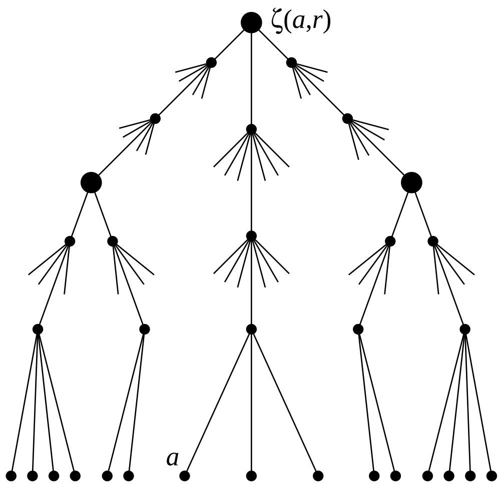

FIGURE 2. The closed Berkovich disk $\overline{D}_{\text{Ber}}(a,r)$

Summarizing the previous paragraph, then, $D_{\mathrm{Ber}}(b,s)$ is an open subset of $\overline{D}_{\mathrm{Ber}}(a,r)$, and $\overline{D}_{\mathrm{Ber}}(b,s)$ is a closed subset. Thus, any *open connected Berkovich affinoid*, defined to be a finite intersection of open Berkovich disks and complements of closed Berkovich disks, is also an open set. (See also Definition 6.3, which is for affinoids in the Berkovich projective line.) Clearly, such a Berkovich affinoid W_{Ber} corresponds to an open connected affinoid W in $\mathbb{P}^1(\mathbb{C}_K)$ by taking the intersection of analogous $\mathbb{P}^1(\mathbb{C}_K)$-disks. Both consists of the same set of type I points, and the membership of a type II–IV point in W_{Ber} is determined by whether its corresponding closed disk (or some limiting closed disk, in the case of a type IV point) is contained in W.

Proposition 6.2. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $a \in \mathbb{C}_K$, and let $r > 0$. The set of open connected Berkovich affinoids in $\overline{D}_{\mathrm{Ber}}(a,r)$ form a basis for the Gel'fand topology.*

Sketch of Proof. Using the Weierstrass Preparation Theorem and the fact that $\mathbb{C}_K$ is algebraically closed, it can be shown that any of the sets U in the subbasis used to generate the Gel'fand topology (see Definition 6.1) is a finite union of open disks. Thus, the open Berkovich disks and complements of closed Berkovich disks together form a subbasis for the same topology. Taking finite intersections, then, the open connected Berkovich affinoids form a basis. □

Thus, there is a strong parallel between the basis of open sets in a Berkovich disk and those in a $\mathbb{P}^1(\mathbb{C}_K)$-disk. However, there are substantial differences in the resulting topologies. In particular, unlike $\mathbb{P}^1(\mathbb{C}_K)$-disks, closed Berkovich disks are *not* topologically open, and open Berkovich disks are *not* topologically closed. After all, as previously mentioned, Berkovich disks are not only Hausdorff but also path-connected and (in the case of closed disks) compact.

For example, the closure of an open disk $D_{\mathrm{Ber}}(b,s)$ is $D_{\mathrm{Ber}}(b,s) \cup \{\zeta(b,s)\}$. Note also that if $s \in |\mathbb{C}_K^\times|$, that closure is *not* the full closed disk $\overline{D}_{\mathrm{Ber}}(b,s)$, which also contains the other open disks $D_{\mathrm{Ber}}(c,s)$, for $c \in \overline{D}(b,s) \smallsetminus D(b,s)$. Incidentally, the interior of a closed disk $\overline{D}_{\mathrm{Ber}}(b,s)$ is the open disk $D_{\mathrm{Ber}}(b,s)$ if $s \notin |\mathbb{C}_K^\times|$, and it is the union of the many disks $D_{\mathrm{Ber}}(c,s)$ for $c \in \overline{D}(b,s)$ if $s \in |\mathbb{C}_K^\times|$.

Most of the discussion in this Section has been stated for convenience with respect to the closed disk $\overline{D}_{\mathrm{Ber}}(a,r)$, but of course it is all true for the open disk $D_{\mathrm{Ber}}(a,r)$ as well, with only the occasional (and obvious) modification. We leave it to the reader to verify this claim.

6.2. **The Berkovich projective line.** We define the Berkovich projective line $\mathbb{P}^1_{\mathrm{Ber}}$ by glueing two copies of $\overline{D}_{\mathrm{Ber}}(0,1)$ as follows. A type I point x on one copy with $|x| = 1$ is identified with $1/x$ on the other copy. Meanwhile, a type II or III point $\zeta(b,s)$ with $|b| = 1$ and $|s| \le 1$ is identified with $\zeta(1/b,s)$, since $\overline{D}(1/b,s)$ is the image of $\overline{D}(b,s)$ under $z \mapsto 1/z$. The type IV points are glued similarly.

Thus, $\mathbb{P}^1_{\mathrm{Ber}}$ looks like $\overline{D}_{\mathrm{Ber}}(0,1)$ with an extra copy of the open tree $D_{\mathrm{Ber}}(0,1)$ attached to the top (i.e., the ∞ end) of the point $\zeta(0,1)$. The new top portion contains all points x of $\mathbb{P}^1(\mathbb{C}_K)$ with $|x| > 1$, including ∞, as well as points $\zeta(a,r)$ with $|a| > 1$ or $r > 1$. Like $\overline{D}_{\mathrm{Ber}}(0,1)$, the space $\mathbb{P}^1_{\mathrm{Ber}}$ is path-connected, Hausdorff, and compact. For a rough idea of the space, see Figure 3, which highlights the type I points $0, 1, \infty, \alpha$, for some $\alpha \in K$ with $0 < |\alpha| < 1$, as well as numerous type II points, including $\zeta(0,1)$ and $\zeta(0,|\alpha|)$. As is the case with $\overline{D}_{\mathrm{Ber}}(a,r)$, the space looks like a tree with infinite branching at a dense set of points along every edge.

Any $\mathbb{P}^1(\mathbb{C}_K)$-disk $\overline{D}(a,r)$, $D(a,r)$, or their complements is associated with the point $\zeta(a,r) \in \mathbb{P}^1_{\mathrm{Ber}}$ of type II or III. Conversely, a type III point $\zeta(a,r)$ is associated with exactly two $\mathbb{P}^1(\mathbb{C}_K)$-disks, namely $D(a,r) = \overline{D}(a,r)$ and its complement. Meanwhile, a type II point $\zeta(a,r)$ is associated with infinitely many disks: every open disk $D(b,r)$ for $b \in \overline{D}(a,r)$, the open $\mathbb{P}^1(\mathbb{C}_K)$-disk $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(a,r)$ containing ∞, and the (closed) complements of each of these open disks. Note, however, that there is a one-to-one correspondence between type II and III points of $\mathbb{P}^1_{\mathrm{Ber}}$ and closed disks in K; the point $\zeta(a,r)$ corresponds uniquely to $\overline{D}(a,r)$.

Viewed another way, a type II point $\zeta(a,r)$ induces a partition of $\mathbb{P}^1(\mathbb{C}_K)$ into the infinitely many open disks $D(b,r)$ contained in $\overline{D}(a,r)$, along with $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(a,r)$; or, in $\mathbb{P}^1_{\text{Ber}}$, a partition of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta(a,r)\}$ into infinitely many connected components, each of which is a rational open Berkovich disk. Similarly, a type III point $\zeta(a,r)$ partitions $\mathbb{P}^1(\mathbb{C}_K)$ into two disks: $\overline{D}(a,r)$ and $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(a,r)$; again, in $\mathbb{P}^1_{\text{Ber}}$, $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta(a,r)\}$ has exactly two connected components, each of which is an irrational open Berkovich disk. In each case, we call the disks of the partition the *residue classes* of $\zeta(a,r)$, or the *directions* at $\zeta(a,r)$. Meanwhile, there is only one residue class (or equivalently, direction) at a point $\zeta \in \mathbb{P}^1_{\text{Ber}}$ of type I or IV, namely its complement $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta\}$. A point $\zeta \in \mathbb{P}^1_{\text{Ber}}$ is of type II or III if and only if $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta\}$ is not connected. In that case, the intersections of the various components of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta\}$ with $\mathbb{P}^1(\mathbb{C}_K)$ (i.e, the type I points) give precisely the residue classes of ζ described above.

We leave it to the reader to check that any Berkovich disk $\overline{D}_{\text{Ber}}(a,r)$ or $D_{\text{Ber}}(a,r)$ can be viewed as a subset of $\mathbb{P}^1_{\text{Ber}}$ in the obvious way. Thus, an open (resp., closed) $\mathbb{P}^1_{\text{Ber}}$-disk is either an open (resp., closed) Berkovich disk or the complement of a closed (resp., open) one. By the obvious parallel between, say, $\overline{D}_{\text{Ber}}(a,r)$ and $\overline{D}(a,r)$, each $\mathbb{P}^1(\mathbb{C}_K)$-disk has a unique $\mathbb{P}^1_{\text{Ber}}$-disk counterpart, and each $\mathbb{P}^1_{\text{Ber}}$-disk that is not irrational closed has a unique $\mathbb{P}^1(\mathbb{C}_K)$-disk counterpart.

Definition 6.3. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field. A *connected Berkovich affinoid* is a nonempty intersection of finitely many $\mathbb{P}^1_{\text{Ber}}$-disks.

If all of the $\mathbb{P}^1_{\text{Ber}}$-disks $D_1, \ldots, D_n$ are closed (respectively, open, rational, irrational), we say that the connected Berkovich affinoid $D_1 \cap \cdots \cap D_n$ is also closed (respectively, open, rational, irrational).

The following result, whose proof we leave to the reader, will be useful for working with the topology on $\mathbb{P}^1_{\text{Ber}}$.

Proposition 6.4. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field.*

a. *The set of open connected Berkovich affinoids in $\mathbb{P}^1_{\text{Ber}}$ forms a basis for the Gel'fand topology.*

b. *Let W be a rational closed connected Berkovich affinoid, and let V be the set of type I points in W. Then W is the closure of V in $\mathbb{P}^1_{\text{Ber}}$.*

As previously noted, both $\mathbb{P}^1_{\text{Ber}}$ and $\overline{D}_{\text{Ber}}(a,r)$ have a tree-like structure. More formally, they *are* $\mathbb{R}$-trees, which are topological spaces that are inverse limits of finite trees, i.e., unions of finitely many real intervals, glued only at endpoints, forming a simply-connected topological space. See

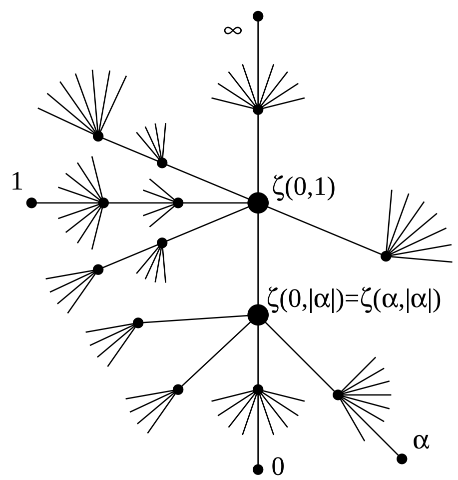

FIGURE 3. The Berkovich projective line $\mathbb{P}^1_{\text{Ber}}$

Section 1.4 of [2] for more on $\mathbb{R}$-trees in this context. We will not be especially concerned with $\mathbb{R}$-trees, but we should briefly point out one or two key properties of $\mathbb{P}^1_{\text{Ber}}$ that are related to the tree structure.

In particular, $\mathbb{P}^1_{\text{Ber}}$ is not only path-connected, but *uniquely path-connected*, meaning that for any two distinct points $\zeta_0, \zeta_1 \in \mathbb{P}^1_{\text{Ber}}$, there is a unique subset $I \subseteq \mathbb{P}^1_{\text{Ber}}$ that is homeomorphic to the interval $[0,1]$ by a homeomorphism sending 0 to ζ_0 and 1 to ζ_1. We say that a point $\xi \in \mathbb{P}^1_{\text{Ber}}$ *lies between* ζ_0 and ζ_1 if ξ belongs to this subset I. (Or, if $\zeta_0 = \zeta_1$, ξ lies between them if and only if $\xi = \zeta_0 = \zeta_1$.) Equivalently, either $\xi = \zeta_0$, $\xi = \zeta_1$, or ζ_0 and ζ_1 belong to different connected components of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\xi\}$, i.e., to different residue classes at ξ. It is a simple exercise (that we leave to the reader) to show that unique path-connectedness implies the following result.

Proposition 6.5. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{P}^1_{\text{Ber}}$. Then there is a unique point $\xi \in \mathbb{P}^1_{\text{Ber}}$ lying between ζ_0 and ζ_1, between ζ_0 and ζ_2, and between ζ_1 and ζ_2.*

Moreover, if none of ζ_0, ζ_1, or ζ_2 lies between the other two, then ξ is of type II.

We close these comments on the notion of lying in between by making the following definition.

Definition 6.6. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $S \subseteq \mathbb{P}^1_{\text{Ber}}$ be any subset of the Berkovich projective line. The *convex hull* of S is the set of all points $\xi \in \mathbb{P}^1_{\text{Ber}}$ for which there exist points $\zeta_0, \zeta_1 \in S$ such that ξ lies between ζ_0 and ζ_1.

Because of the unique path-connectedness, it is easy to show that the convex hull of S is connected. In fact, it is the smallest connected subset of $\mathbb{P}^1_{\text{Ber}}$ containing S, in the sense that it is the intersection of all connected subsets of $\mathbb{P}^1_{\text{Ber}}$ containing S.

; indeed, it is the smallest

Remark 6.7. The Berkovich projective line can also be constructed directly, using homogeneous coordinates $[X, Y]$, as a space of seminorms on $\mathbb{C}_K[X, Y]$; see Section 2.2 of [2]. The idea of that construction, translated to the language of affine coordinates, is roughly as follows.

Each point $\zeta \in \mathbb{P}^1_{\text{Ber}}$ gives a function $\|\cdot\|_\zeta : \mathbb{C}_K(z) \to [0, \infty]$ by

$$\|h\|_\zeta := \frac{\|f\|_\zeta}{\|g\|_\zeta}, \tag{8}$$

where $h = f/g$ for $f, g \in \mathbb{C}_K[z]$. Note that the expression (8) does not define an actual seminorm on $\mathbb{C}_K(z)$ because, for example, $\|h\|_\zeta$ can take on the value ∞. Indeed, this happens if and only if $\zeta = b$ is a type I point and h has a pole at b. Meanwhile, at points of types II and III, it is appropriate to think of $\|h\|_{\zeta(a,r)}$ as the generic value of $|h(x)|$ for $x \in \overline{D}(a, r)$.

Of course, given our construction of $\mathbb{P}^1_{\text{Ber}}$ as the glueing of two copies of $\overline{D}_{\text{Ber}}(0,1)$, equation 8 technically only makes sense for ζ in one of the copies U. In the other copy V, which consists of points $1/\zeta$ for $\zeta \in \overline{D}_{\text{Ber}}(0,1)$, we should actually define $\|h\|_\zeta := \frac{\|\tilde{f}\|_\zeta}{\|\tilde{g}\|_\zeta}$, where $\tilde{f}(z) = z^d f(1/z)$ and $\tilde{g}(z) = z^d g(1/z)$, for $d := \max\{\deg f, \deg g\}$.

Thus, $\mathbb{P}^1_{\text{Ber}}$ can be thought of as the set of multiplicative seminorms on $\mathbb{C}_K(z)$ that are allowed to take on the value ∞. Of course, allowing ∞ as the value of a seminorm is problematic, which is why the rigorous definition requires the homogenous coordinates discussed in [2].

Remark 6.8 (Rivera-Letelier's Construction)**.** In [41, 42], Rivera-Letelier constructed $\mathbb{P}^1_{\text{Ber}}$ by a different method, and using different terminology. In this remark, we give a rough sketch of his construction, including a dictionary between his terminology and the standard Berkovich space terminology, for the convenience of readers wishing to study those papers.

Rivera-Letelier begins by considering a strictly increasing sequence $D_1 \subsetneq D_2 \subsetneq \cdots$ of closed $\mathbb{P}^1(\mathbb{C}_K)$-disks, and he sets $B := \bigcup_{n \geq 1} D_n$ which is either the complement of a single point, a rational open $\mathbb{P}^1(\mathbb{C}_K)$-disk, an irrational $\mathbb{P}^1(\mathbb{C}_K)$-disk, or all of $\mathbb{P}^1(\mathbb{C}_K)$. Ultimately, those possibilities will, of course, correspond to points of type I, II, III, or IV, respectively. (Actually, he disallows the first case, because his goal is to construct a "hyperbolic space" $\mathbb{H}$, which in our notation is $\mathbb{P}^1_{\text{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$. Moreover, in [41] he also disallows the last case, because in that paper he does not make use of type IV points.)

He then considers the decreasing sequence $\{B \smallsetminus D_n\}_{n\geq 1}$, which he calls an *evanescent chain* ("chaine évanescente"), because the intersection $\bigcap_{n\geq 1}(U \smallsetminus D_n)$ is empty. The sets $C_n := B \smallsetminus D_n$ are either disks with a single point removed (later to correspond to type I), or annuli ("couronnes", for types II and III), or $\mathbb{P}^1(\mathbb{C}_K)$-disks (type IV). He then declares two evanescent chains $\{C_n\}$, $\{C'_n\}$ to be equivalent if for every $N \geq 1$ there is some $M \geq N$ such that $C_M \subseteq C'_N$ and $C'_M \subseteq C_N$. Next, he defines an *end* ("bout") to be an equivalence class of evanescent chains. He calls an end *singular* if its corresponding evanescent chains are chains of disks (ultimately, type IV points); *irrational* if its evanescent chains are annuli all of the form $U \smallsetminus D_n$ where U is an irrational disk (ultimately, type III points); or *rational* if its evanescent chains are annuli all of the form $U \smallsetminus D_n$ where U is a rational open disk (ultimately, type II points). Again, he does not discuss the remaining case of type I points, which be excludes because he is only constructing $\mathbb{P}^1_{\text{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$ in these papers.

Given an end $\mathcal{P}$, he declares $B_{\mathcal{P}}$ to be the set B used to define one of the chains associated with $\mathcal{P}$; recall that U is of the form either $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \{b\}$ for some point $b \in \mathbb{P}^1(\mathbb{C}_K)$, or an (irrational or rationa) open $\mathbb{P}^1(\mathbb{C}_K)$-disk, or all of $\mathbb{P}^1(\mathbb{C}_K)$. He notes that $B_{\mathcal{P}}$ is independent of the choice of chain representing $\mathcal{P}$. He then defines the points of $\mathbb{P}^1_{\text{Ber}}$ as sets of ends, as follows. The type IV, or singular, points are singleton sets $\{\mathcal{P}\}$, where $\mathcal{P}$ is a singular end. For the type II and III, or rational and irrational, points, he declares two open $\mathbb{P}^1(\mathbb{C}_K)$-disks D_1 and D_2 to be *associated* if $D_1 \cap D_2 = \varnothing$ and they are maximal with respect to that property. (That is, if $E_1 \supseteq D_1$ and $E_2 \supseteq D_2$ are open $\mathbb{P}^1(\mathbb{C}_K)$-disks for which $E_1 \cap E_2 = \varnothing$, then $E_1 = D_1$ and $E_2 = D_2$.) The type III points are then maximal sets of irrational ends whose associated sets $B_{\mathcal{P}}$ are associated, and the type II points are then maximal sets of rational ends whose associated sets $B_{\mathcal{P}}$ are associated. The type III points always consist of exactly two ends $\{\mathcal{P}, \mathcal{P}'\}$, where $\mathbb{P}^1(\mathbb{C}_K) = B_{\mathcal{P}} \cup B_{\mathcal{P}'}$, and the type II points consist of infinitely many ends, whose associated $\mathbb{P}^1(\mathbb{C}_K)$-disks $B_{\mathcal{P}}$ are the various residue classes of that point in Berkovich space. Thus, for all points ζ of types II–IV, the sets $B_{\mathcal{P}}$, for $\mathcal{P} \in \zeta$, form a partition of $\mathbb{P}^1(\mathbb{C}_K)$. Finally, the type I points, had Rivera-Letelier actually defined them, would again be singleton sets of ends $\{\mathcal{P}\}$ for which $B_{\mathcal{P}}$ is of the form $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \{b\}$; in our notation, the corresponding type I point is b itself.

6.3. **Rational functions acting on the Berkovich projective line.** A rational function $\phi \in \mathbb{C}_K(z)$ has a unique continuous extension, which we also denote ϕ, to a function from $\mathbb{P}^1_{\text{Ber}}$ to itself; see Definition 6.9 below. Of course, the restriction of $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ to the set of type I points $\mathbb{P}^1(\mathbb{C}_K)$ will be just the usual action of ϕ on $\mathbb{P}^1(\mathbb{C}_K)$. A fully rigorous description of $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ requires either defining general Berkovich spaces as locally ringed spaces with patches given by general Berkovich affinoids, or else a direct definition of $\mathbb{P}^1_{\text{Ber}}$ using homogeneous coordinates and seminorms on $\mathbb{C}_K[X, Y]$. We refer the reader to Section 3.4 of [14] or to Section 2.3 of [2] for the details. Here, in lieu of a rigorous presentation, we will describe ϕ precisely, but we will skip the proofs.

Definition 6.9. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function. The induced function $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ is defined as follows. Given $\zeta \in \mathbb{P}^1_{\text{Ber}}$, the image point $\phi(\zeta)$ is the unique point (i.e., seminorm) in $\mathbb{P}^1_{\text{Ber}}$ such that

$$\|h\|_{\phi(\zeta)} = \|\phi \circ h\|_{\zeta} \tag{9}$$

for all $h \in K(z)$.

Here, the function $\|\cdot\|_{\zeta} : K(z) \to [0, \infty]$ is the variation on a seminorm mentioned in equation (8) in Section 6.2. It is not difficult to check that $\|\cdot\|_{\phi(\zeta)} : K(z) \to [0, \infty]$ is again such a seminorm.

Unfortunately, equation (9) does not give much immediate insight into what $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ really looks like. Following Section 2.3 of [2] and Section 4 of [42], then, we present the following equivalent description. Along the way, we also define the *local degree*, or *multiplicity*, $\deg_\zeta \phi$ of ϕ at each point $\zeta \in \mathbb{P}^1_{\text{Ber}}$.

Of course, if $\phi = c$ is constant, then ϕ maps all points of $\mathbb{P}^1_{\text{Ber}}$ to the type I point c. Thus, in what follows, we assume that ϕ is not constant.

First, if $\zeta = x$ is a type I point, then

$$\phi(\zeta) := \phi(x)$$

is simply the type I point representing $\phi(x)$. In this case, the local degree of ϕ at x is defined to be

$$\deg_x \phi := \text{ the multiplicity with which } \phi \text{ maps } x \text{ to } \phi(x).$$

That is, if $\phi(x) \in \mathbb{C}_K$, then $\deg_x \phi$ is the order of vanishing of $\phi(z) - \phi(x)$ at $z = x$; or, if $\phi(x) = \infty$, then $\deg_x \phi$ is the order of vanishing of $1/\phi(z)$.

Second, if $\zeta = \zeta(b, s)$ is a point of type II or III and ϕ is a polynomial, we know from Corollary 3.10 that $\phi(\overline{D}(b, s))$ is a closed disk $\overline{D}(f(b), t)$; then

$$\phi(\zeta(b, s)) := \zeta(f(b), t),$$

and

$$\deg_{\zeta(b,s)} \phi := \text{ the Weierstrass degree of } \phi \text{ on } \overline{D}(b, s).$$

However, if ϕ is not a polynomial, then $\phi(\overline{D}(b, s))$ may be all of $\mathbb{P}^1(\mathbb{C}_K)$. Fortunately, it can be shown that (whether ϕ is a polynomial or rational function) there is a radius $s_0 < s$ such that for all s' with $s_0 < s' < s$, the image $\phi(V(s'))$ of the annulus $V(s') = \{x \in \mathbb{C}_K : s' < |z - b| < s\}$ is itself an annulus of the form

$$\{y \in \mathbb{C}_K : \sigma' < |z - \beta| < \sigma\} \quad \text{or} \quad \{y \in \mathbb{C}_K : \sigma < |z - \beta| < \sigma'\},$$

where σ' varies with s' but β and σ are independent of s'. (See, for example, Lemma 9.33 of [2], or Lemme 4.3 of [42].) Moreover, ϕ maps $V(s')$ onto its image in such a way that, counting multiplicity, every point of $\phi(V(s'))$ has exactly d preimages in $V'(s)$, for some integer $d \geq 1$, which we will call the *Weierstrass degree of ϕ on $V(s')$*. Then it turns out that

$$\phi(\zeta(b, s)) := \zeta(\beta, \sigma),$$

and

$$\deg_{\zeta(b,s)} \phi := \text{ the Weierstrass degree of } \phi \text{ on the annulus } V(s').$$

Finally, if ζ is a type IV point that is the limit of a sequence $\{\zeta_n\}$ of points of types II or III, then

$$\phi(\zeta) := \lim_{n\to\infty} \phi(\zeta_n),$$

and

$$\deg_\zeta \phi := \lim_{n\to\infty} \deg_{\zeta_n} \phi,$$

meaning that there is some N such that for all $n \geq N$, the degree $\deg_{\phi_n} \phi$ stabilizes at some constant value $d \geq 1$.

Note in particular that a nonconstant map $\phi \in \mathbb{C}_K(z)$ takes type I points to type I points, type II points to type II points, and so on. In addition, just as is the case for type I points, it can be shown that for any point $\zeta \in \mathbb{P}^1_{\text{Ber}}$,

$$\sum_{\xi \in \phi^{-1}(\zeta)} \deg_\xi \phi = \deg \phi. \tag{10}$$

See Section 9 (especially Theorem 9.1) of [2], or Section 2.3 of [41] for more on equation (10) and local degrees (i.e., multiplicities).

Remark 6.10. Any type II point can be moved to the point $\zeta(0,1)$ by a linear fractional transformation. More precisely, if $b,c \in \mathbb{C}_K$ are distinct, and $r := |b-c| \in |\mathbb{C}_K^\times|$, then $\eta(z) := (b-c)^{-1}(z-c)$ takes $b \mapsto 1$, $c \mapsto 0$, and $\infty \mapsto \infty$, and hence $\eta(\overline{D}(b,r)) = \overline{D}(0,1)$; therefore $\eta(\zeta(b,r)) = \zeta(0,1)$. The connected components of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \zeta(0,1)$ correspond naturally to the elements $\mathbb{P}^1(\overline{k})$, where $\overline{k}$ is the residue field of $\mathbb{C}_K$. Thus, because η is a homeomorphism (as η and its inverse η^{-1} are both continuous), it follows that the connected components of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \zeta(b,s)$ also correspond to the elements of $\mathbb{P}^1(\overline{k})$, although the bijection itself now depends on η and hence on the choice of b and c. Still, this correspondence is the reason for our use of the term *residue classes* for the connected components of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta\}$.

In addition, the elements of $\text{PGL}(2,\mathbb{C}_K)$ that fix $\zeta(0,1)$ are precisely the maps in $\text{PGL}(2,\mathcal{O}_{\mathbb{C}_K})$. Thus, the type II points of $\mathbb{P}^1_{\text{Ber}}$ can be viewed as corresponding to the left cosets (or equivalently, the right cosets) of $\text{PGL}(2,\mathcal{O}_{\mathbb{C}_K})$ in $\text{PGL}(2,\mathbb{C}_K)$.

As noted in Remark 6.10, the point $\zeta(0,1)$ is not really any different from any other type II point, except after we specify a particular choice of coordinates. Nevertheless (or, perhaps, *because* we sometimes *do* want to specify a certain coordinate), $\zeta(0,1)$ has been dubbed the *Gauss point* in [2], the *canonical point* in [41, 42].

Rivera-Letelier proved a result essentially like Theorem 6.11 below as Corollaire 2.2 and Proposition 2.4 of [41]. To state it, we note that if we write a rational function $\phi(z) \in \mathbb{C}_K(z)$ as $f(z)/g(z)$ with $f,g \in \mathcal{O}_{\mathbb{C}_K}[z]$ and at least one coefficient of f or g having absolute value 1, then we can define a rational function $\overline{\phi}(z) = \overline{f(z)}/\overline{g}(z)$, which will have degree smaller than $\deg\phi$ if ϕ has bad reduction. (We already made use of this idea in Examples 4.41 and 4.42 to understand the action of ϕ on residue classes in $\overline{D}(0,1)$, for instance, even though ϕ had bad reduction.)

We also set some notation: given a point $x \in \mathbb{P}^1(\mathbb{C}_K)$ with reduction $\overline{x} \in \mathbb{P}^1(\overline{k})$, we denote the corresponding residue class, i.e., the connected component of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta(0,1)\}$ containing x, by $D_{\overline{x}}$.

Theorem 6.11. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 1$, and let $\zeta \in \mathbb{P}^1_{\text{Ber}}$ be a type II point. Choose $\eta,\theta \in \text{PGL}(2,\mathbb{C}_K)$ such that $\eta(\zeta) = \theta(\phi(\zeta)) = \zeta(0,1)$, and set $\psi(z) := \theta\circ\phi\circ\eta^{-1}(z)$. Then $1 \leq \deg\overline{\psi} \leq d$, and there is a finite set $T \subseteq \mathbb{P}^1(\overline{k})$ such that*

a. *$\psi(D_{\overline{x}}) = \mathbb{P}^1_{\text{Ber}}$ for all $\overline{x} \in T$.*

b. *$\psi(D_{\overline{x}}) = D_{\overline{\psi}(\overline{x})}$ for all $\overline{x} \in \mathbb{P}^1(\overline{k}) \smallsetminus T$, and in fact ψ maps the disk $D_{\overline{x}}$ onto its image with Weierstrass degree $\deg_{\overline{x}}\overline{\phi}$, the multiplicity with which $\overline{\psi}$ maps $\overline{x}$ to $\overline{\psi}(\overline{x})$.*

Moreover, for any $\theta' \in \text{PGL}(2,\mathbb{C}_K)$, if $\psi' := \theta'\circ\phi\circ\eta^{-1}$, then its reduction $\overline{\psi'}$ is nonconstant if and only if $\theta'(\phi(\zeta)) = \zeta(0,1)$, or equivalently, if $\theta'\circ\theta^{-1} \in \text{PGL}(2,\mathcal{O}_{\mathbb{C}_K})$. In addition, if $\deg\overline{\psi} = d$, then $T = \varnothing$.

We refer the reader to Rivera-Letelier's proof in [41], which is stated for $\mathbb{C}_p$ but works for any complete, algebraically closed non-archimedean field $\mathbb{C}_K$. Still, we sketch the proof here.

Sketch of Proof of Theorem 6.11. Without loss, assume $\zeta = \phi(\zeta) = \zeta(0,1)$, so that $\eta(z) = \theta(z) = z$ and $\psi = \phi$. Write $\phi = f/g$, with $f,g \in \mathcal{O}_{\mathbb{C}_K}[z]$, with at least one coefficient of f or g having absolute value 1. If $\overline{f}/\overline{g}$ is constant, then after a $\text{PGL}(2,\mathcal{O}_{\mathbb{C}_K})$-change of coordinates we can assume that $\overline{f}/\overline{g} = 0$, from which it can be shown that $\phi(\zeta(0,1)) \in D_{\text{Ber}}(0,1)$. Thus, we have proven the bound $1 \leq \deg\overline{\phi} \leq d$, as well as one direction of the penultimate statement of the Theorem. The other direction will be immediate from the proof of part (b).

Thus, we need to prove (a) and (b), assuming that $\overline{f}/\overline{g}$ is not constant. Let $T \subseteq \mathbb{P}^1(\overline{k})$ be the set of residue classes that contain both a zero of g and a zero of f. (Here, we consider the point ∞ to be a zero of g if $\deg(g) < \deg(f)$, or a zero of f if $\deg(f) < \deg(g)$.) Note that if $\deg\overline{\psi} = d$, then there was no cancellation between $\overline{f}$ and $\overline{g}$, implying that $T = \varnothing$, proving the final statement of the Theorem.

For each $\overline{x} \in T$, we can assume by a $\mathrm{PGL}(2, \mathcal{O})$-change of coordinates that $x = 0$, and hence that f and g both have zeros in $D(0,1)$. It is then a straightforward calculation, using Newton polygons, to verify that for every $y \in \mathbb{P}^1(\mathbb{C}_K)$, the polynomial $f(z) - yg(z)$ has a root in $D(0,1)$. (Of course, we must remember our assumptions that $\overline{f}/\overline{g}$ is nonconstant, and that both f and g have roots in $D(0,1)$, to complete this calculation.) Part (a) then follows.

For any $x \in \mathbb{P}^1(\mathbb{C}_K)$ for which $\overline{x} \notin T$, pick $\eta_x, \theta_x \in \mathrm{PGL}(2, \mathcal{O})$ such that $\eta_x(x) = \theta_x(\phi(x)) = 0$. Then it can be shown that $\psi_x := \theta_x \circ \phi \circ \eta_x^{-1}$ can be written as f_x/g_x, where $f_x, g_x \in \mathcal{O}_{\mathbb{C}_K}[z]$ each have a least one coefficient, including the constant term of g, of absolute value 1, and moreover that the constant term of f has absolute value strictly less than 1. Using Newton polygons, we can then show that ψ_x maps $D(0,1)$ m-to-1 onto itself, where $m := \mathrm{ord}_0 \overline{f}_x$ is the order of the zero of the reduction $\overline{f}_x$ at 0. Undoing the coordinate changes by η_x and θ_x, we have proven (b). $\square$

6.4. **Dynamics on Berkovich space.** We begin our discussion of the dynamics of a rational function on $\mathbb{P}^1_{\mathrm{Ber}}$ with the following observation.

Proposition 6.12. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$, let $W \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a connected affinoid, and let $W_{\mathrm{Ber}} \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ be the corresponding Berkovich connected affinoid. Then $\phi(W_{\mathrm{Ber}})$ is the Berkovich connected affinoid corresponding to $\phi(W)$, and $\phi^{-1}(W_{\mathrm{Ber}})$ is the Berkovich affinoid corresponding to $\phi^{-1}(W)$.*

Proof. First, note that W_{Ber} is closed in $\mathbb{P}^1_{\mathrm{Ber}}$ and hence is compact. Since $\phi : \mathbb{P}^1_{\mathrm{Ber}} \to \mathbb{P}^1_{\mathrm{Ber}}$ is continuous, it follows that $\phi(W_{\mathrm{Ber}}$ is also compact. Therefore, since $\mathbb{P}^1_{\mathrm{Ber}}$ is Hausdorff, $\phi(W_{\mathrm{Ber}})$ is a closed subset of $\mathbb{P}^1_{\mathrm{Ber}}$ containing $\phi(W)$.

On the other hand, by Proposition 6.4.b, W_{Ber} is the closure of W in $\mathbb{P}^1_{\mathrm{Ber}}$. Thus, given any closed set $C \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ containing $\phi(W)$, its inverse image $\phi^{-1}(C)$ contains W and hence W_{Ber}. Hence, C contains $\phi(W_{\mathrm{Ber}})$, and $\phi(W_{\mathrm{Ber}})$ is the closure of $\phi(W)$. Again by Proposition 6.4.b, then, $\phi(W_{\mathrm{Ber}})$ is the Berkovich connected affinoid corresponding to $\phi(W)$.

The proof of the statement about $\phi^{-1}(W)$ is similar. Finally, the general version of the Proposition follows from the fact that an arbitrary connected affinoid (either in $\mathbb{P}^1(\mathbb{C}_K)$ or in $\mathbb{P}^1_{\mathrm{Ber}}$) can be written as a union of an increasing sequence of rational closed affinoids, because an arbitrary disk can be written as a union of an increasing sequence of rational closed disks. $\square$

Inspired by Theorem 4.29 and Definition 5.1, then, we state the following definition.

Definition 6.13. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$.

We say an open set $U \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ is *dynamically stable* under ϕ if $\bigcup_{n \geq 0} \phi^n(U)$ omits infinitely many points of $\mathbb{P}^1_{\mathrm{Ber}}$.

We define the (*Berkovich*) *Fatou set* of ϕ, denoted $\mathcal{F}_{\phi,\mathrm{Ber}}$ or $\mathcal{F}_{\mathrm{Ber}}$, or simply $\mathcal{F}$ if our meaning is clear, to be the subset of $\mathbb{P}^1_{\mathrm{Ber}}$ consisting of all points $x \in \mathbb{P}^1_{\mathrm{Ber}}$ having a dynamically stable neighborhood.

We define the (*Berkovich*) *Julia set* of ϕ, denoted $\mathcal{J}_{\phi,\mathrm{Ber}}$ or $\mathcal{J}_{\mathrm{Ber}}$, or simply $\mathcal{J}$ if our meaning is clear, to be the complement $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathcal{F}_{\phi,\mathrm{Ber}}$ of the Berkovich Fatou set.

The astute reader will note that our condition for dynamical stability in Berkovich space is *a priori* stronger than Definition 5.1 for $\mathbb{P}^1(\mathbb{C}_K)$, where the omission of three points, rather than infinitely many, was required. However, in light of Theorem 1.11 and Proposition 6.19 below, the two conditions (three points versus infinitely many) are equivalent *unless* $\mathrm{char}\, \mathbb{C}_K = p$ and ϕ is conjuate to $z \mapsto z^{p^m}$ for some $m \geq 1$. In particular, requiring infinitely many points rather than three affects only the question of whether the Gauss point $\zeta(0,1)$ over a characteristic p field is in the Julia set of $z \mapsto z^{p^m}$; see also the proof of Theorem 6.18 below.

Theorem 6.15 below will show that the classical Fatou set is precisely the set of type I points in the Berkovich Fatou set, and similarly for the Julia sets. For the moment, however, we begin by noting that the Berkovich Fatou and Julia sets have the same basic properties as were described in Proposition 4.21, as follows.

Proposition 6.14. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function with Berkovich Fatou set $\mathcal{F}_\phi$ and Berkovich Julia set $\mathcal{J}_\phi$.*

a. *$\mathcal{F}_\phi$ is an open subset of $\mathbb{P}^1(\mathbb{C}_K)$, and $\mathcal{J}_\phi$ is a closed subset of $\mathbb{P}^1(\mathbb{C}_K)$.*
b. *$\phi^{-1}(\mathcal{F}_\phi) = \phi(\mathcal{F}_\phi) = \mathcal{F}_\phi$, and $\phi^{-1}(\mathcal{J}_\phi) = \phi(\mathcal{J}_\phi) = \mathcal{J}_\phi$.*
c. *For every positive integer $m \geq 1$, $\mathcal{F}_{\phi^m} = \mathcal{F}_\phi$, and $\mathcal{J}_{\phi^m} = \mathcal{J}_\phi$.*
d. *For any $\eta \in \mathrm{PGL}(2, \mathbb{C}_K)$, if we set $\psi = \eta \circ \phi \circ \eta^{-1}$, then $\mathcal{F}_\psi = \eta(\mathcal{F}_\phi)$, and $\mathcal{J}_\psi = \eta(\mathcal{J}_\phi)$.*

Proof. Left to reader. □

The type I and Berkovich Fatou and Julia sets are related to each other in the following unsurprising way.

Theorem 6.15. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$. Then the classical Fatou set $\mathcal{F}_{\phi,\mathrm{I}}$ is the intersection of the Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$ with the set $\mathbb{P}^1(\mathbb{C}_K)$ of type I points. Similarly, the classical Julia set $\mathcal{J}_{\phi,\mathrm{I}}$ is the intersection of the Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$ with $\mathbb{P}^1(\mathbb{C}_K)$.*

Proof. Given $x \in \mathcal{F}_{\phi,\mathrm{I}}$, then by the final statement of Theorem 4.29, there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk D containing x such that the iterates $\phi^n(D)$ omit uncountably many points of $\mathbb{P}^1(\mathbb{C}_K)$. Let D_{Ber} be the corresponding Berkovich disk. By Proposition 6.12, then, the iterates $\phi^n(D_{\mathrm{Ber}})$ omit the same set of $\mathbb{P}^1(\mathbb{C}_K)$-points, and hence, by definition, $x \in \mathcal{F}_{\phi,\mathrm{Ber}}$.

Conversely, given $x \in \mathcal{F}_{\phi,\mathrm{Ber}} \cap \mathbb{P}^1(\mathbb{C}_K)$, there is an open connected Berkovich affinoid U containing x whose iterates $\phi^n(U)$ omit infinitely many points of $\mathbb{P}^1_{\mathrm{Ber}}$. Shrinking U if necessary, we may assume that U is a disk. (Note that this is possible because x is of type I; it is often not possible for points of type II and III.)

Choose two distinct points $\zeta_0, \zeta_1 \in \mathbb{P}^1_{\mathrm{Ber}}$ omitted by $\bigcup_{n\geq 0} \phi^n(U)$, and choose a type II point ξ lying between them. By a change of coordinates, we may assume that $\xi = \zeta(0,1)$.

We claim that $\xi \notin \bigcup_{n\geq 0} \phi^n(U)$. Indeed, if $\xi \in \phi^n(U)$ for some $n \geq 0$, then the Berkovich disk $\phi^n(U)$ must contain all but one residue class of ξ. Since ζ_0 and ζ_1 lie in different residue classes of ξ, it follows that at least one of ζ_0 or ζ_1 lies in $\phi^n(U)$, giving a contradiction and proving the claim.

Thus, for each $n \geq 0$, $\phi^n(U)$ lies in a residue class. If D is the $\mathbb{P}^1(\mathbb{C}_K)$-disk corresponding to U (i.e., $D := U \cap \mathbb{P}^1(\mathbb{C}_K)$), then also $\phi^n(D)$ is always in a residue class. By Lemma 4.19, then, $x \in \mathcal{F}_{\phi,\mathrm{I}}$. □

Just as all periodic points in $\mathbb{P}^1(\mathbb{C}_K)$ can easily be classified as Fatou or Julia based on the absolute value of their multipliers, one can make a similar determination for periodic points in Berkovich space.

Definition 6.16. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$, and let $\zeta \in \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$ be a point of type II, III, or IV. Suppose that ζ is a periodic point of ϕ of minimal period $n \geq 1$. Then the *multiplier* of ζ is the local degree

$$\lambda := \deg_\zeta(\phi^n)$$

of ϕ^n at ζ. We say that ζ is *repelling* if $\lambda \geq 2$; otherwise, if $\lambda = 1$, we say that ζ is *indifferent*.

A number of comments are in order regarding this definition. First, while the multiplier of a type I periodic point is an element of $\mathbb{C}_K$, the multiplier of a periodic point of types II–IV is a positive integer, which we choose *not* consider as an element of $\mathbb{C}_K$; in particular, we do not want to take the $\mathbb{C}_K$-absolute value of λ. Second, there is no such thing as an attracting point of type II–IV; that is simply because there are no positive integers strictly less than 1. Third, the name "repelling" is a bit misleading here, because repelling periodic points do not actually repel most nearby points! Instead, a repelling periodic point ζ will have repelling directions, but usually such directions will make only a tiny slice of all the residue classes at ζ, as the following very simple example illustrates.

Example 6.17. For example, if $d \geq 2$, the function $\phi(z) = z^d$ has a repelling fixed point of multipler d at the Gauss point $\zeta(0,1)$. The directions (i.e., residue classes) $\overline{0}$ and $\overline{\infty}$ are repelling directions, because, for example, for any $0 < r < 1$ $\phi(\zeta(0,r)) = \zeta(0,r^d)$, and $\phi(\zeta(0,r^{-1})) = \zeta(0,r^{-d})$. Thus, $\zeta(0,r)$ and $\zeta(0,r^{-1})$ have been pushed further away (along the tree) from $\zeta(0,1)$. (This idea can actually be formalized by placing a natural metric, dubbed the "big model metric" in [2], or simply "la distance" in [42, 44], on the set $\mathbb{H} := \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$ of points of types II–IV. We will discuss this metric a bit more in the proof of Proposition 6.19.) However, if d is not divisible by the residue characteristic of $\mathbb{C}_K$, then for any $a \in \mathbb{C}_K$ with $|a| \leq 1$, one can check that $\phi(\zeta(a,r)) = \zeta(a^d, r)$, so that $\zeta(a,r)$ is moved in some kind of orbit around $\zeta(0,1)$, but it is not actually pushed any farther away.

On the other hand, if the residue characteristic p *does* divide d (specifically, $p^e | d$ but $p^{e+1} \nmid d$, for some $e \geq 1$), then, at least for $0 < r < 1$ close enough to 1, we have $\phi(\zeta(a,r)) = \zeta(a^d, r^{p^e})$, and hence $\zeta(0,1)$ *is* repelling in all directions.

Nevertheless, the following result shows that the names "repelling" and "indifferent" are appropriate, as least as regards the Fatou and Julia sets.

Theorem 6.18. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$, and let $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}\mathbb{P}^1(\mathbb{C}_K)$ be a periodic point of type II, III, or IV.*

a. *If ζ is of type III or IV, then ζ is indifferent and in the Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$.*
b. *If ζ is of type II and indifferent, and if the reduced map $\overline{\phi}$ at ζ is of finite order under composition, then $\zeta \in \mathcal{F}_{\phi,\mathrm{Ber}}$.*
c. *If ζ is repelling, then it is in the Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$.*

Note that points of type III or IV *can* be in the Julia set, but *not* if they are periodic. Note also that if the residue field of $\mathbb{C}_K$ is algebraic over a finite field (as is the case for the p-adic field $\mathbb{C}_p$, for example), then the condition in part (b) of Theorem 6.18 that $\overline{\phi}$ has finite order is vacuous. After all, in the case of an indifferent type II point at, without loss, $\zeta = \zeta(0,1)$, in light Theorem 6.11, $\overline{\phi}$ is an element of $\mathrm{PGL}(2, \mathcal{O}_{\mathbb{C}_K})$; and our assumption on the residue field implies that every element of $\mathrm{PGL}(2, \mathcal{O}_{\mathbb{C}_K})$ has finite order. However, if the residue field is infinite, and if $\overline{\phi}$ has infinite order under composition, then ζ may or may not be in the Berkovich Julia set, depending on whether the rest of the Berkovich Julia set intersects only periodic residue classes of ζ or some of the wandering ones.

Sketch of Proof of Theorem 6.18. Without loss, assume ζ is fixed.

The first statement of part (a) is exactly Lemme 5.3 and Lemme 5.4 of [42]. The idea for the points of type III is to change coordinates so that $\zeta = \zeta(0,r)$ for some $r \notin |\mathbb{C}_K^\times|$ and write $\phi(z) = f(z)/g(z)$, where $f(z) = a_0 + \cdots a_m z^m$ and $g(z) = b_0 + \cdots b_n z^n$. Because $r \notin |\mathbb{C}_K^\times|$, there are unique integers i, j for which $|a_i| r^i$ and $|b_j| r^j$ attain their respective maxima. Thus, for all z with $|z|$ close enough to r, we must have $|a_i z^i|/|b_j z^j| = |z|$, if ζ is indeed fixed. Letting $|z| \to r$, we have $r^{i-j-1} \in |\mathbb{C}_K^\times|$, and hence $i = j+1$, and $|a_i| = |b_j|$. From this fact (and the rigorous definition of the local degree, which we have omitted), it follows that ζ is indifferent.

Meanwhile, for a type IV points, if the local degree is m, then because there is only one direction (i.e., residue class) associated with ζ (given by the nested sequence of disks $D_1 \supseteq D_2 \supseteq \cdots$), it can be shown that for n large enough, the ratio the radii of the disks $\phi(D_n)$ and $\phi(D_{n+1})$ is the m-th power of the ratio of the radii of D_n and D_{n+1}. However, because ζ is fixed, the sequences of disks $\{D_n\}_{n\geq 1}$ and $\{\phi(D_n)\}_{n\geq 1}$ must be equivalent. If $m \geq 2$, these two facts will lead to a contradiction.

The second statement of part (a) follows because for the type III points, there is a small annulus U given by $r-\varepsilon < |z| < r+\varepsilon$ such that $\phi(U) \subseteq U$; and for the type IV points, there is a disk U containing D_n for all sufficiently large n such that $\phi(U) \subseteq U$.

For part (b), assume by a change of coordinates that $\zeta = \zeta(0,1)$; thus, since ϕ maps ζ to ζ with local degree 1, the reduction $\overline{\phi}$ is a linear fractional transformation in $\mathrm{PGL}(2,\mathcal{O})$. By hypothesis, there is some $m \geq 1$ for which $\overline{\phi}^m$ is the identity; replacing ϕ by ϕ^m, we may assume that $\overline{\phi}$ itself is the identity. Let T be the set of residue classes from Theorem 6.11. Then we can remove a large enough closed subdisk from each residue class in T to obtain a connected Berkovich affinoid U such that $\phi(U) = U$, proving that $U \subseteq \mathcal{F}_{\phi,\mathrm{Ber}}$.

For part (c), assuming again that $\zeta = \zeta(0,1)$, and write denote the reduction of ϕ at ζ by $\overline{\phi}$, which is a rational function in $\overline{k}(z)$ of degree at least 2. Given an open connected Berkovich affinoid U containing ζ, note that U must contain all but finitely many residue classes at ζ. Regardless, then, of whether $\overline{\phi}$ is conjugate to $z^{\pm d}$, z^{p^m}, an arbitrary polynomial, or none of these, we can replace ϕ by an iterate so that all of the residue classes not contained in U are fixed by $\overline{\phi}$. Because they are all exceptional for $\overline{\phi}$, each such residue class $\overline{x}$ either maps onto $\mathbb{P}^1_{\mathrm{Ber}}$ (if it is in the set T of Theorem 6.11) or maps onto itself with multiplicity $\deg \overline{\phi} \geq 2$. Either way, some further work shows that the iterates of the thin annulus that is the intersection of the open Berkovich disk $\overline{x}$ with U together cover all but at most one point of U. Thus, the iterates of U together omit only finitely many points of $\mathbb{P}^1_{\mathrm{Ber}}$, for any open U containing ζ; that is, $\zeta \in \mathcal{J}_{\phi,\mathrm{Ber}}$. $\square$

As a side note, we also have the following result, which says that except for maps of potentially good reduction, the exceptional set in $\mathbb{P}^1_{\mathrm{Ber}}$ coincides with the exceptional set in $\mathbb{P}^1(\mathbb{C}_K)$.

Proposition 6.19. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$. Suppose there is a point $\zeta \in \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$ of type II, III, or IV with finite grand orbit under ϕ. Then there is only one such point, and it is fixed and of type II.*

In that case, there is a linear fractional transformation $\eta \in \mathrm{PGL}(2,\mathbb{C}_K)$ such that $\eta(\zeta) = \zeta(0,1)$, and for any such η, the function $\eta \circ \phi \circ \eta^{-1} \in \mathbb{C}_K(z)$ has good reduction.

Sketch of Proof. This is Théorème 5 (or its simplified counterpart, Théorème 4) of [44]; although the results of [44] are stated for $\mathbb{C}_p$, the proofs work just as well for general $\mathbb{C}_K$. It is also essentially equivalent to Theorem B of [7]. We sketch the strategy of [44] here.

If there are two such points ζ_1 and ζ_2, then both must be periodic. Hence, for some $n \geq 1$, $\phi^n(\zeta_i) = \zeta_i$, and in fact $\phi^{-n}(\zeta_i) = \{\zeta_i\}$, for $i = 1,2$. By equation (10), the multiplier of ζ_i as a fixed point of ϕ^n must be d^n; in particular, each ζ_i is of type II, by Theorem 6.18.

In [44], Rivera-Letelier now proves his Lemme 7.4, which says in this context that in the big model metric on $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$ mentioned in Example 6.17, the distance from $\phi^n(\zeta_1)$ to $\phi^n(\zeta_2)$ must be exactly d^n times the distance from ζ_1 to ζ_2. We will say a few more words about this lemma in a moment, but first we finish the rest of the proof.

Because the stretching factor d^n is strictly larger than 1, we must have $\zeta_1 = \zeta_2$. Hence the exceptional set of ϕ in $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$ consists of at most one point ζ, which is fixed and totally invariant under ϕ. After moving ζ to $\zeta(0,1)$, it follows by continuity that any connected component (i.e., residue class) U of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta(0,1)\}$ is mapped into a connected component. In particular, U cannot contain both a zero and a pole of ϕ. Thus, ϕ has good reduction, and we are done.

We now return to the big model metric and Lemme 7.4 of [44]. Given two distinct points $\zeta_1, \zeta_2 \in \mathbb{P}^1_{\mathrm{Ber}}$ of types II or III, let U_1 be the open Berkovich disk that is the component of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta_1\}$ containing ζ_2, and let U_2 be the open Berkovich disk that is the component of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta_2\}$ containing ζ_1. Then $U_1 \cap U_2$ is an annulus, and after applying a linear fractional transformation $\theta \in \mathrm{PGL}(2,\mathbb{C}_K)$, we can assume that it is of the form $D_{\mathrm{Ber}}(0,r_2) \smallsetminus \overline{D}_{\mathrm{Ber}}(0,r_1)$, for some radii $r_2 > r_1 > 0$. It can be shown (just as the analogous fact is true in complex analysis) that the ratio r_2/r_1 is independent of the choice of θ. We define the big model distance from ζ_1 to ζ_2 to be $\log(r_2/r_1) > 0$.

Incidentally, this distance can be extended to the type IV points by continuity; and if we allow the distance to take on the value ∞, then we can also extend it to the type I points. (However, the distance between any type I point and any other point at all is always ∞.) It can be shown that this definition gives a metric on $\mathbb{H} := \mathbb{P}^1_{\text{Ber}} \smallsetminus \mathbb{P}^1(\mathbb{C}_K)$, although the reader should be warned that the resulting metric topology, called the *strong topology*, is *strictly stronger* than the Gel'fand topology on $\mathbb{P}^1_{\text{Ber}}$.

Lemme 7.4 of [44] can then be proven in the following way. By pre- and post-composing $\psi := \phi^n$ with appropriate linear fractional transformations $\eta, \theta \in \text{PGL}(2, \mathbb{C}_K)$, we can assume that $\zeta_1 = \psi(\zeta_1) = \zeta(0,1)$, that $\zeta_2 = \zeta(0,r)$ and that $\psi(\zeta_2) = \zeta(0,s)$, for some $r, s > 1$. Because ζ_1 and ζ_2 map to their images with multiplicity equal to $\deg \psi$, the $T = \varnothing$ clause of Theorem 6.11 tells us that each residue class of ζ_i maps onto a residue class of $\psi(\zeta_i)$. In particular, the residue class at ζ_1 containing ζ_2 maps onto the residue class at $\phi(\zeta_1)$ containing $\phi(\zeta_2)$ with Weierstrass degree $\deg \psi$, and vice versa. One can then deduce that ψ maps the annulus $1 < |z| < r$ onto the annulus $1 < |z| < s$. Lemme 5.3 of [44] (which is proven by explicitly examining equations of the form $\psi(z) = a$ using tools like Newton polygons) then shows that this mapping of annuli is everywhere M-to-1, for some integer M, and that $s = r^M$. Moreover, because the Weierstrass degree on each residue class above was $\deg \psi$, it follows that $M = \deg \psi$, finishing the proof of the lemma. □

In Théorème B of [42], Rivera-Letelier proved that every rational function $\phi(z) \in \mathbb{C}_K(z)$ of degree at least two has a repelling fixed point in $\mathbb{P}^1_{\text{Ber}}$. His proof was stated for $\mathbb{C}_p$ but works over any complete and algebraically closed non-archimedean field $\mathbb{C}_K$. We now state the following weaker version, proving the existence of a repelling periodic point. Both versions require Rivera-Letelier's classification of dynamics on components of the Fatou set (See project #3.) In particular, the key result needed is that if ϕ has an indifferent periodic point in $\mathbb{P}^1(\mathbb{C}_K)$, then ϕ also has a periodic point of type II (perhaps of larger period) that lies in the Julia set. (Recall from the discussion following Theorem 6.18 that if the residue field is algebraic over a finite field, then a type II periodic point is Julia if and only if it is repelling; but over other fields, it is possible to have type II periodic points that are indifferent but still in the Julia set.)

Theorem 6.20. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2. Then ϕ either has a repelling periodic point of type I, a repelling periodic point of type II, or an indifferent but Julia periodic point of type II in $\mathbb{P}^1_{\text{Ber}}$.*

Sketch of Proof. Consider the type I fixed points of ϕ. If any of them is repelling, we are done. Similarly, by the portion of Rivera-Letelier's classification discussed in the paragraph preceding Theorem 6.20, if any of them is indifferent, then there is a type II Julia periodic point, and we are done.

Thus, we are left with the case that all periodic points of ϕ are attracting. Note that none have multiplicity greater than 1, as otherwise they would have multiplier exactly 1 and be indifferent. In particular, then, there are $d + 1 > 2$ distinct fixed points. By a change of coordinates, then, assume that both 0 and ∞ are attracting fixed points. Pick $r > 0$ small enough that $\phi(\overline{D}(0,r)) = Dbar(0,s)$ for some strictly smaller radius $s \in (0,r)$. Then the point $\xi := \zeta(0,r)$ lies between ∞ and $\phi(\xi) = \zeta(0,s)$ in Berkovich space.

In this situation, Lemme 6.1 of [41] says that there is a repelling fixed point of ϕ on the unique arc in $\mathbb{P}^1_{\text{Ber}}$ running from ∞ to $\zeta(0,s)$. For the convenience of the reader, we outline the proof of this last result now, especially since it becomes simpler given our other hypotheses.

Define $R := \sup S$, where $S \subseteq \mathbb{R}$ is the set

$$S := \{t \geq r : \phi(\zeta(0,t')) \in D_{\text{Ber}}(0,t) \text{ for all } t' \in (0,t)\}.$$

Note that $r \in S$, and hence $S \neq \varnothing$; in particular, the supremum R is defined. In addition, $R < \infty$, because for all t large enough, $\phi(\zeta(0,t)) = \zeta(0,u)$ for some $u > t$, as ∞ is attracting. Moreover, by our definition of S, we must have $\phi(\zeta(0,R)) \notin D_{\mathrm{Ber}}(0,R)$, or else we would have $R+\varepsilon \in S$ for some small enough $\varepsilon > 0$. On the other hand, we also have $\phi(\zeta(0,t')) \in D_{\mathrm{Ber}}(0,R)$ for all $t \in (0,R)$, and therefore $\phi(\zeta(0,R))$ must lie in the closure of $D_{\mathrm{Ber}}(0,R)$, although, as we have seen, not in $D_{\mathrm{Ber}}(0,R)$ itself.

However, as we noted in Section 6.1, the closure of $D_{\mathrm{Ber}}(0,R)$ is $D_{\mathrm{Ber}}(0,R) \cup \{\zeta(0,R)\}$, and therefore $\phi(\zeta(0,R)) = \zeta(0,R)$. We have found a fixed point of type II or III it remains to prove that it is repelling (and hence also, by Theorem 6.18, of type II).

If $\zeta(0,R)$ were non-repelling, then, as a point of type II or III, it would have to be indifferent, and hence have local degree 1. Thus, all nearby points, including all points $\zeta(0,t')$ for $t' \in (R-\varepsilon, R+\varepsilon)$ for some $\varepsilon > 0$, would also have local degree 1. It would follow that $\phi(\zeta(0,t')) = \zeta(0,t')$ for all such t', and hence $R+\varepsilon/2 \in S$, contradicting the definition of R as $\sup S$. Thus, R is repelling, as claimed. ☐

Note that Theorem 6.20 says nothing about whether both types I and II occur. In fact, anything can happen. Maps of good reduction have a single repelling fixed point at the (type II) Gauss point and no other repelling periodic points anywhere in $\mathbb{P}^1_{\mathrm{Ber}}$. The quadratic polynomial maps of Example 4.39 have infinitely many type I repelling periodic points and no repelling fixed points. The cubic polynomial of Example 4.41 has a repelling fixed point at the (type II) Gauss point and infinitely many type I repelling periodic points. (Note that although a map can have a positive finite number of type II repelling periodic points, but if it has even one type I repelling periodic point, then it has infinitely many, by a result of Bézivin discussed in Project #1.) There are also maps with infinitely many type II repelling periodic points and no type I repelling periodic points, and other maps with infinitely many of each; we encourage the reader to try to construct some.

For the moment, however, we have the following more obvious corollary of Theorem 6.20.

Corollary 6.21. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2. Then the Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$ of ϕ is nonempty.*

Proof. By Theorem 6.20, there is some repelling periodic point $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$, of type I or II. If ζ is type I, then it lies in the type I Julia set $\mathcal{J}_{\phi,\mathrm{I}}$ of ϕ, by Proposition 4.24; hence it lies in the Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$ by Theorem 6.15. On the other hand, if ζ is type II, then it lies in $\mathcal{J}_{\phi,\mathrm{Ber}}$ by Theorem 6.18. ☐

The reader may wonder what the Berkovich Julia set of a map of good reduction looks like, given that such maps had empty type I Julia set. As the following result indicates, although the Berkovich Julia set must be nonempty, it is still quite simple.

Proposition 6.22. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2 and of good reduction. Then the Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$ is the singleton set*

$$\mathcal{J}_{\phi,\mathrm{Ber}} = \{\zeta(0,1)\}$$

consisting solely of the Gauss point.

Proof. In light of the fact that the Julia set is nonempty by Corollary 6.21, it suffices to show that every point of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta(0,1)\}$ lies in the Berkovich Fatou set. (Still, we recommend that the reader show directly, by considering the iterates of an arbitrary open affinoid containing the Gauss point, that $\zeta(0,1)$ does indeed lie in the Julia set.)

Given $x \in \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta(0,1)\}$, let U be the connected component of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta(0,1)\}$ containing x. (That is, the open Berkovich disk U is the residue class $\overline{x}$ of x.) Because ϕ has good reduction, the image $\phi(U)$ is contained in (and in fact equal to, although we will not use that fact) the residue class $\overline{\phi(x)}$. After all, the type I points of U are mapped into the residue class $\overline{\phi(x)}$ by the comments following Definition 4.5, and therefore U itself is mapped into that same open Berkovich disk by Proposition 6.12.

Proceeding inductively, then, $\phi^m(U)$ is contained in (and in fact, is equal to) the residue class containing $\phi^m(x)$, for every $m \geq 0$. If this residue class is ever U itself for some $m \geq 1$, then $\bigcup_{n\geq 0} \phi^n(U)$ is contained in the finite union of residue classes $\bigcup_{n=0}^{m-1} \phi^n(U)$, and we are done.

Otherwise, pick $y \in \mathbb{P}^1(\mathbb{C}_K)$ such that $\phi(y) \in U$. Then the residue class $V := \overline{y}$ does not intersect any iterate $\phi^m(U)$, because if it did, then $\phi^{m+1}(U) \subseteq U$, putting us in the situation of the previous paragraph. Hence, the iterates of U all omit the whole residue class V, and we are done. □

We also have the following Berkovich analogues of several of the results from Section 4.5.

Proposition 6.23. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2 and with Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$. Let E_ϕ be the exceptional set of ϕ in $\mathbb{P}^1(\mathbb{C}_K)$. Then*

a. *Let $U \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ be an open set intersecting $\mathcal{J}_{\phi,\mathrm{Ber}}$. Then*
$$\bigcup_{n\geq 0} \phi^n(U) \supseteq \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus E_\phi.$$
b. *$\mathcal{J}_{\phi,\mathrm{Ber}}$ has empty interior.*
c. *For any point $\zeta \in \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus E_\phi$, the closure of the backward orbit of ζ contains $\mathcal{J}_{\phi,\mathrm{Ber}}$, with equality if and only if $\zeta \in \mathcal{J}_{\phi,\mathrm{Ber}}$.*
d. *Let $S \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ be a closed subset of $\mathbb{P}^1_{\mathrm{Ber}}$ that is not contained in E_ϕ and for which $\phi^{-1}(S) \subseteq S$. Then $\mathcal{J}_{\phi,\mathrm{Ber}} \subseteq S$.*
e. *If $\mathcal{J}_{\phi,\mathrm{Ber}}$ has more than one point, then it is a perfect set.*
f. *If ϕ is a polynomial and we define*
$$\mathcal{K}_{\phi,\mathrm{Ber}} := \left\{\zeta \in \mathbb{P}^1_{\mathrm{Ber}} : \lim_{n\to\infty} \phi^n(\zeta) \neq \infty\right\}$$
to be the Berkovich filled Julia set *of ϕ, then $\mathcal{K}_{\phi,\mathrm{Ber}} = \partial\mathcal{J}_{\phi,\mathrm{Ber}}$.*

Proof. (a): By definition, $W := \bigcup_{n\geq 0} \phi^n(U)$ must contain all but finitely many points of $\mathbb{P}^1_{\mathrm{Ber}}$. All preimages of each such point must also lie outside W, and thus all such points have finite grand orbit. By Proposition 6.19, there can be at most one such point that is not of type I, and after a change of coordinates, it is $\zeta(0,1)$, and ϕ has good reduction. By Proposition 6.22, then, $\mathcal{J}_{\phi,\mathrm{Ber}} = \{\zeta(0,1)\}$, and therefore $\zeta(0,1) \in U \subseteq W$. Thus, the only points that could be in $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus W$ are exceptional points of type I, as desired.

(b): Similar to the proof of Corollary 4.32.

(c): Similar to the proof of Proposition 4.33.

(d): Similar to the proof of Corollary 4.34.

(e): Similar to the proof of Corollary 4.35, except that we need to set aside as a special case the possibility that the backward orbit of the given point $a \in \mathcal{J}_{\phi,\mathrm{Ber}}$ is *finite*. In that case, Proposition 6.19 says that $\mathcal{J}_{\phi,\mathrm{Ber}}$ has only one point.

(f): Similar to the proof of Proposition 4.37, except that we cannot assume the open set U is a disk when proving that $\partial\mathcal{K} \subseteq \mathcal{J}$. A slightly different argument, which we leave to the reader, is required here. □

Corollary 6.24. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2 and with Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$. Then the following are equivalent.*

i. *$\mathcal{J}_{\phi,\mathrm{Ber}}$ has exactly one element.*
ii. *$\mathcal{J}_{\phi,\mathrm{Ber}}$ has at most countably many elements.*
iii. *$\mathcal{J}_{\phi,\mathrm{Ber}}$ has at least one isolated point.*

iv. *ϕ has potentially good reduction.*

In particular, if ϕ does not have potentially good reduction, then $\mathcal{J}_{\phi,\mathrm{Ber}}$ is a perfect set.

Proof. The implication (iv)⇒(i) is immediate from Propositions 6.14.d and 6.22. Meanwhile, (i)⇒(ii) is obvious, and (ii)⇒(iii) follows because the Julia set is closed by Proposition 6.14.a and nonempty by Corollary 6.21.

Finally, to show (iii)⇒(iv), Proposition 6.23 says that the existence of an isolated point implies that $\mathcal{J}_{\phi,\mathrm{Ber}}$ consists of only one point. Since $\mathcal{J}_{\phi,\mathrm{Ber}}$ is totally invariant, that one point cannot be of Type I, or else it would be exceptional and hence Fatou. Thus, by Proposition 6.19, ϕ has potentially good reduction. □

We close this section with the promised proof of Proposition 5.4, although we leave some details to the reader.

Sketch of a proof of Proposition 5.4. We proceed essentially as in the conditional proof we gave in Section 4.6, which works in its entirety for the general case *except* for the claim that given any overlapping $U_1, U_2 \in \mathcal{B}$, their union $U_1 \cup U_2$ is still dynamically stable.

If ϕ has potentially good reduction (and, for the cases that $\mathcal{B}$ consists of arbitrary disks or arbitrary connected affinoids, under the assumption that ϕ is *not* a quasimonomial), the proof is straightforward but involves checking a number of cases (based on which choice $\mathcal{B}$ of components we are considering and whether ϕ is of the form z^d, z^{-d}, z^{p^m}, some other polynomial, or none of these), and we leave it to the reader.

For the remainder of the proof, then, we assume that ϕ does *not* have potentially good reduction. By Corollary 6.24, $\mathcal{J}_{\phi,\mathrm{Ber}}$ must have uncountably many elements, and hence the iterates $\phi^n(W)$ of any subset W of the Berkovich Fatou set avoid an uncountable set.

Let $U_{1,\mathrm{Ber}}$ and $U_{2,\mathrm{Ber}}$ be the Berkovich versions of the disks (or connected affinoids) U_1 and U_2. By dynamical stability for $\mathbb{P}^1(\mathbb{C}_K)$, each of the unions $\bigcup_{n\geq 0} \phi^n(U_1)$ and $\bigcup_{n\geq 0} \phi^n(U_2)$ omits at least three points of $\mathbb{P}^1(\mathbb{C}_K)$, although *a priori* those three points might be different for each union. Thus, by Proposition 6.12, the unions of iterates $\bigcup_{n\geq 0} \phi^n(U_{1,\mathrm{Ber}})$ and $\bigcup_{n\geq 0} \phi^n(U_{2,\mathrm{Ber}})$ also each omit three points of $\mathbb{P}^1(\mathbb{C}_K)$. Since we have excluded the case of potential good reduction, the exceptional set in $\mathbb{P}^1(\mathbb{C}_K)$ has at most one element (see Theorem 1.11), and therefore both unions in fact omit infinitely many.

It follows that the *interiors* of $U_{1,\mathrm{Ber}}$ and $U_{2,\mathrm{Ber}}$ are each contained in the Berkovich Fatou set, and hence their iterates avoid the (uncountable) Julia set. If $\mathcal{B}$ consists of open affinoids or open disks, we are done.

To finish the proof in the case of arbitrary affinoids or disks, we will show that the *boundaries* of $U_{1,\mathrm{Ber}}$ and $U_{2,\mathrm{Ber}}$ both avoid a certain uncountable subset of $\mathcal{J}_{\phi,\mathrm{Ber}}$. To do this, simply observe that the boundary of each of $U_{1,\mathrm{Ber}}$ and $U_{2,\mathrm{Ber}}$ consists of finitely many points of type II. (That is, there is one such point for each rational closed disk in the intersection defining U_i.) Taking all forward images of all these boundary points gives only countably many points. Even if all such points are in the Julia set, that still leaves an uncountable subset of $\mathcal{J}_{\phi,\mathrm{Ber}}$ that all iterates of both the interiors and the boundaries of both $U_{1,\mathrm{Ber}}$ and $U_{2,\mathrm{Ber}}$ avoid. □

7. Dynamics of Fatou components

The Berkovich Fatou set of a rational function $\phi \in \mathbb{C}_K(z)$ can naturally be partitioned into connected components. We can then study the dynamics of ϕ on the set of such components, according to the following result.

Proposition 7.1. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$, and let U be a connected component of $\mathcal{F}_{\phi,\mathrm{Ber}}$. Then $\phi(U)$ is also a connected component of $\mathcal{F}_{\phi,\mathrm{Ber}}$, and there* *is an integer $1 \leq m \leq d$ such that every point in $\phi(U)$ has exactly m preimages in U, counting multiplicity. Moreover, the inverse image $\phi^{-1}(U)$ is a finite union of components of $\mathcal{F}_{\phi,\mathrm{Ber}}$, each of which is mapped onto U by ϕ.*

Proof. Left to reader. □

Berkovich Fatou components correspond most naturally to open analytic components of the classical Fatou set, as the next result shows.

Proposition 7.2. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with classical Fatou set $\mathcal{F}_{\phi,\mathrm{I}}$ and Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$. Then for every connected component U of $\mathcal{F}_{\phi,\mathrm{Ber}}$, the set $U \cap \mathbb{P}^1(\mathbb{C}_K)$ of type I points in U is an open analytic component of $\mathcal{F}_{\phi,\mathrm{I}}$. Conversely, for every open analytic component V of $\mathcal{F}_{\phi,\mathrm{I}}$, there is a unique connected component of $\mathcal{F}_{\phi,\mathrm{Ber}}$ such that $V = U \cap \mathbb{P}^1(\mathbb{C}_K)$.*

Proof. Left to reader. □

7.1. **Rivera-Letelier's classification.** In his thesis [41], Rivera-Letelier classified dynamics on a fixed component of the Berkovich Fatou set. He worked over the field $\mathbb{C}_p$, which turns out not to exhibit quite the full set of dynamics that is possible more generally. However, it is not difficult to adjust his classification to apply to arbitrary non-archimedean fields.

Definition 7.3. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$. The *indifference domain* $\mathcal{I} = \mathcal{I}_\phi$ is defined to be the set of points $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$ having an open neighborhood W such that ϕ^m maps W bijectively onto itself for some integer $m \geq 1$.

A connected component of the indifference domain is called an *indifferent component.*

Clearly, the indifference domain is an open subset of $\mathbb{P}^1_{\mathrm{Ber}}$, and in fact, of the Berkovich Fatou set.

Definition 7.4. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$, and let U be a connected component of $\mathcal{F}_{\phi,\mathrm{Ber}}$. Assume that U is periodic of minimal period m; that is, $\phi^m(U) = U$ but $\phi^n(U) \cap U = \varnothing$ for $1 \leq n \leq m-1$. We say that U is an *attracting component* if there is an attracting periodic point $P \in U \cap \mathbb{P}^1(\mathbb{C}_K)$, also of minimal period m, such that for all $\zeta \in U$,

$$\lim_{n\to\infty} \phi^{mn}(\zeta) = P.$$

Rivera-Letelier's classification of Fatou components involves several separate statements. The main one is the following.

Theorem 7.5 (Rivera-Letelier)**.** *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$, and let U be a periodic connected component of $\mathcal{F}_{\phi,\mathrm{Ber}}$. Then U is either an indifferent component or an attracting component. Moreover, the indifference domain does not intersect any attracting components.*

Put another way, every component of the Berkovich Fatou set is either preperiodic to an indifferent periodic component, preperiodic to an attracting periodic component, or wandering. Moreover, these three cases are mutually exclusive; Rivera-Letelier refers to this as the "trichotomy" of the classification. The analogy is to complex dynamics, where there are two kinds of domains drawn to an attracting point (attracting and parabolic), and two more that are like our indifferent components (Siegel disks and Herman rings). That leaves us with the question of wandering domains, which been settled in complex dynamics, and which we will discuss in the non-archimdean context in Section 7.2.

The proof of Theorem 7.5 for $\mathbb{C}_p$ appears in Section 4.4 of [41]. An alternate proof, still for $\mathbb{C}_p$ but using a different method, appears in Section 9 of [43]. The first two parts of Project #3 are devoted to extending (either) proof to $\mathbb{C}_K$. To avoid spoiling the fun, then, we omit the proof here.

The next two results further restrict what periodic Fatou components can look like. Before stating them, we make the following definition, introduced by Rivera-Letelier in Section 4.1 of [41].

Definition 7.6. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $U \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ be an open set. We say that U is *a domain of Cantor type* if there is an increasing sequence of open connected affinoids

$$U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \cdots$$

such that

i. for every $n \geq 1$, each connected component of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus U_n$ contains at least two points of ∂U_{n+1}, and
ii. $U = \bigcup_{n \geq 1} U_n$.

Theorem 7.7 (Rivera-Letelier)**.** *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$, and let U be an attracting periodic component of $\mathcal{F}_{\phi,\mathrm{Ber}}$. Then U is either an open Berkovich disk or a domain of Cantor type.*

If U is a disk, then it has a unique boundary point, which is a type II point in the Berkovich Julia set that is periodic of the same period as U.

If U is of Cantor type, then it has uncountable boundary, and the boundary is contained in the Berkovich Julia set.

For attracting domains of Cantor type, the boundary generally includes both preperiodic and wandering points. It can include points of various type as well. For example, the boundary of the attracting component at ∞ for the quadratic polynomial of Example 4.39 consisted solely of type I points. Meanwhile, the boundary of the attracting component at ∞ for the cubic polynomial of Example 4.41 consisted of some type I points, like the repelling fixed point α, and some type II points, like the repelling fixed point $\zeta(0,1)$. There can also be type IV points in the boundary of an attracting domain. For Example, in Exemple 6.3 of [41], Rivera-Letelier proves that the attracting component at ∞ for the p-adic map $\phi(z) = p^{-1}(z^p - z^{p^2}) \in \mathbb{C}_p(z)$ has type IV points in its boundary. It is not known whether type III points can lie in such a boundary, but it appears likely that this can occur if there are wandering domains not in the attracting basins of periodic type II points. (See Theorem 7.13 and Definition 7.14 below.)

Theorem 7.8 (Rivera-Letelier)**.** *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$, and let U be an indifferent periodic component of $\mathcal{F}_{\phi,\mathrm{Ber}}$. Then U is an open connected affinoid.*

The boundary ∂U is nonempty, finite, and contained in the Berkovich Julia set. Moreover, if m is the minimal period of U, then ϕ^m permutes the boundary points of U; in particular, each is periodic.

As before, we leave the proofs of Theorems 7.7 and 7.8 to Project #3.

A rational function can, and in fact usually does, have infinitely many periodic Berkovich Fatou components. Maps of good reduction, for example, have infinitely many such components among the residue classes. However, there is a limit on how many of them can be anything other than disks.

Proposition 7.9. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Berkovich Fatou set $\mathcal{F}_{\phi,\mathrm{Ber}}$. Then ϕ has at* *most $d-1$ cycles of attracting periodic components of Cantor type in $\mathcal{F}_{\phi,\mathrm{Ber}}$, and at most $d-1$ cycles of indifferent periodic components in $\mathcal{F}_{\phi,\mathrm{Ber}}$ that are not disks.*

Sketch of proof. Given any Cantor type attracting periodic component W of minimal period m, let $x \in W$ be the unique attracting periodic point, which is necessarily of type I. For each $i = 0,\ldots,m-1$, write $W_i := \phi^i(W)$ and $x_i := \phi^i(x)$. Because x is of type I, there is some Berkovich disk D_0 with $x_0 \in D_0 \subseteq W_0$. For each $n \geq 0$, choose the unique $i \in \{0,\ldots,m-1\}$ such that $m|(n+i)$, and let D_n be the connected component of $\phi^{-n}(D)$ containing x_i. Then $W_0 = \bigcup_{n\geq 0} D_{nm}$, and therefore, there is some $N \geq 1$ such that D_{Nm} is not a disk. In particular, there is some $\ell \geq 0$ such that D_ℓ is a disk but $D_{\ell+1}$ is not.

Thus, each cycle C of Cantor type attracting components contains a disk D_C whose preimage includes a component that is not a disk. By Lemma 2.5.3 of [4], there can be at most $d-1$ such pairwise disjoint disks. Because cycles of attracting components cannot intersect, it follows that there are at most $d-1$ such cycles C.

We leave the (more difficult) bound for indifferent components to the reader. □

Remark 7.10. In complex dynamics, one can bound the number of cycles of Fatou components by associating a critical point with each. As remarked just before Proposition 7.9, such a strategy is impossible in non-archimedean dynamics. For example, if the residue characteristic is positive, then there can be attracting components with no critical points, as for $z \mapsto z^p$. Similarly, in any residue characteristic, infinitely many indifferent domains can share the same boundary point, as for $z \mapsto z^d$ when d does not divide the residue characteristic. Still, one might wonder whether critical points could be associated with, say, Cantor type attracting domains. However, Rivera-Letelier showed that such an association cannot be made, at least if the residue characteristic is positive. Specifically, in Exemple 6.2 of [41], he showed that for any $\lambda \in \mathbb{C}_p$ with $|p|^{-(p-1)/(p+1)} < |\lambda| < 1$, the function $\phi(z) = \lambda(z^p + z/(z-1)) \in \mathbb{C}_p(z)$ has an attracting domain of Cantor type that does not contain any critical points.

Remark 7.11 (The Iterative Logarithm)**.** In Section 3.2 of [41], Rivera-Letelier does another construction that, this time, *is* specific to fields of mixed characteristic, like $\mathbb{C}_p$. He defines the *domain of quasi-periodicity* of $\phi \in \mathbb{C}_p(z)$ to be the set of type I points $z_0 \in \mathbb{P}^1(\mathbb{C}_p)$ having a neighborhood U (which we can assume to be a disk) and an integer $m \geq 1$ for which the limit

$$\phi_*(z) := \lim_{|n|_p \to 0} \frac{\phi^{nm}(z) - z}{mn}$$

converges uniformly on U, meaning that for any sequence of positive integers $n \geq 1$ whose p-adic absolute values converge to 0, the associated sequence of functions in $\mathbb{C}_p(z)$ $(\phi^{nm}(z) - z)/(mn)$ converge uniformly to some power series converging on U. (Note that this convergence implies not only that $\phi^{nm}(z) \to z$ as $|n|_p \to 0$, but in fact that the convergence of the iterates of z to z is at least as fast as the p-adic convergence of n to 0. Also note that this definition completely falls apart in characteristic p or in residue characteristic zero— that is, over any field of equal characteristic.) The resulting limit $\phi_*(z)$ above is, by definition, a power series converging on U, and Rivera-Letelier calls it the *iterative logarithm* of ϕ.

Ultimately, he proves (for $\mathbb{C}_p$) that the domain of quasi-periodicity coincides with the indifference domain. He also shows that on this domain, the iterative logarithm is zero precisely at the periodic points. (Note, incidentally, that any preperiodic point in the indifferent domain is in fact periodic, because the mapping is injective.) In particular, the indifferent type I periodic points are isolated. Even more fascinating, for any non-periodic point x_0 in the domain, he shows that on any disk D contained in U and not containing any periodic points (i.e., not containing any zeros of the power series ϕ_*), ϕ is analytically conjugate to translation. Specifically, if we define $\phi^*(z)$ to be an antiderivative of $1/\phi_*$ on D, then $\phi^*(\phi^m(z)) = \phi^*(z) + k$ for all $z \in D$.

7.2. **Wandering domains.** The question of whether or not a rational function ϕ has wandering domains is much further from being settled than the question of dynamics on periodic components. Still, a fair amount is now known, but the answer to the question seems to depend heavily on the field $\mathbb{C}_K$ itself.

Before stating our first result, we introduce a standard bit of dynamical terminology. A point $\zeta \in \mathbb{P}^1_{\text{Ber}}$ is said to be *recurrent* if there is an increasing sequence $\{n_i\}_{i\geq 1}$ of positive integers such that

$$\lim_{i\to\infty} \phi^{n_i}(\zeta) = \zeta.$$

All periodic points are clearly recurrent, but it is also possible for non-periodic points to be recurrent.

We also introduce another standard bit of terminology, this time from number theory, albeit stretched slightly to apply to our setting. If the residue characteristic of $\mathbb{C}_K$ is $p > 0$, we say that a critical point x of $\phi \in \mathbb{C}_K(z)$ is *wild* if p divides the multiplicity of ϕ at x. Thus, $z \mapsto z^{np}$ has a wild critical point at $z = 0$, for example.

Theorem 7.12 (Benedetto)**.** *Let K be a complete and locally compact non-archimedean field of residue characteristic $p > 0$, let $\mathbb{C}_K$ be the completion of an algebraic closure of K, and let $\phi \in K(z)$ be a rational function of degree at least 2. Let $\mathcal{J}_{\phi,\text{I}}$ be the classical Julia set of ϕ, and let $\mathcal{F}_{\phi,\text{Ber}}$ be the Berkovich Fatou set of ϕ. Suppose that*

a. *$\mathcal{J}_{\phi,\text{I}}$ contains no wild critical points, if* $\operatorname{char} K = p$*, i.e., if K is of equal characteristic, or*
b. *$\mathcal{J}_{\phi,\text{I}}$ contains no recurrent wild critical points, if* $\operatorname{char} K = 0$*, i.e., if K is of mixed characteristic.*

Then $\mathcal{F}_{\phi,\text{Ber}}$ has no wandering domains. (In fact, ϕ has no wandering domains for any of the various types of components.) Moreover, $\mathcal{F}_{\phi,\text{Ber}}$ also has only finitely many periodic components that contain type I points in $\mathbb{P}^1(K)$.

References for the proof. See Corollary 3.1 of [6] (which is stated for finite extensions of $\mathbb{Q}_p$ but works for any locally compact non-archimedean field) and Theorems 1.2 and 1.3 of [5].

The idea is to consider two concentric disks $U \subseteq V$, where U is contained in the Fatou set and V is not. By taking a finite extension of K, we may assume that U contains a K-rational point.

By properties of the Fatou and Julia sets, the iterates of V want to become large, but the iterates of U must remain small. Thus, there must be times when the ratio of radii of $\phi^{n+1}(V)$ to $\phi^{n+1}(U)$ is strictly larger than the corresponding ratio for $\phi^n(V)$ and $\phi^n(U)$. That, in turn, can only happen if $\phi^n(V)$ is already of a certain large size, or if there is a critical point nearby.

The various hypotheses can be used to show that the "critical point nearby" possibility can only happen finitely often, and hence it can be effectively ignored. Thus, given any disk U contained in the Fatou set and any slightly larger disk V intersecting the Julia set, there is some $n \geq 0$ such that $\phi^n(V)$ is at least a certain large size, but the ratio of the radii of $\phi^n(V)$ and $\phi^n(U)$ is still above some absolute lower bound. Thus, every such U has an iterate of at least some (slightly smaller) certain large size.

Thus, if we have many such disks U_j (which will happen if the original U was a wandering domain, or if there were many periodic components containing K-rational points), then we get many *disjoint* disks $\phi^{n_j}(U_j)$ that are all of at least a certain size, but which also contain K-rational points. Because K is locally compact, this is impossible. □

Theorem 7.12 relies heavily on the fact that the field K over which the map ϕ is defined is locally compact. Indeed, we have already seen maps with wandering domains over other fields. Consider the case that the residue field k of K, and therefore also the residue field of $\mathbb{C}_K$, is *not* algebraic over a finite field. Then k has elements that wander under the map $z \mapsto z^2$. Thus, the rational function of Example 4.42 has a wandering domains corresponding to each such wandering element of k. Similarly, the cubic polynomial of Example 4.41 would also have wandering domains (that is, wandering open analytic components) over the same field, although it would *not* have wandering analytic components, because the full disk $\overline{D}(0,1)$ would be a fixed component.

Even for the field $\mathbb{C}_p$, whose residue field is algebraic over a finite field and which has mixed characteristic, but which is not locally compact, it is possible to have wandering domains, even if we consider analytic components. This time, the counterexample comes not from exploiting the large residue field, but from the infinite ramification— that is, by using the fact that $|\mathbb{C}_K^\times|$ is not discrete. Here is the result.

Theorem 7.13 (Benedetto). *Let p be a prime number, and let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field of residue characteristic $p > 0$. Then there is a polynomial $\phi(z) \in \mathbb{C}_K[z]$ of degree $p+1$ such that ϕ has wandering domains for any of the various types of components.*

References for the proof. See [9] for $\mathbb{C}_K = \mathbb{C}_p$ and [12] for general such $\mathbb{C}_K$. See also Project #4, which features a generalization. The idea of the proof is to consider a one-parameter family like $\phi(z) = (1-a)z^{p+1} + az^p$ that has a type II periodic point (in this case, $\zeta(0,r)$ for a certain radius $0 < r < 1$) that expands the big model metric in one direction (in this case, in the residue class of $\zeta(0,r)$ containing ∞) by a factor divisible by p. Thus, points just outside $\overline{D}(0,r)$ are pushed away from the disk, but because of the divisibility by p, the derivative is small at such points, and hence on a more local scale, the dynamics are contracting. Meanwhile, because of the repelling fixed point at $z = 1$, there is another region where the dynamics are expanding. By taking the limit of an appropriate sequence of parameters a, we can engineer a small disk that splits its time between the locally contracting region and the expanding region. By balancing the split properly, we can guarantee that the disk is contained in the Fatou set, and that its Fatou component is wandering. □

Since Theorem 7.12 is false if the hypothesis about ϕ being defined over a locally compact field is removed, it is natural to ask about the other hypotheses. It is hard to violate those hypotheses— critical points cause contraction, whereas the Julia set wants to cause expansion, and this conflict makes it difficult to put a critical point in the Julia set. Indeed, a periodic critical point is superattracting and hence Fatou.

Of course, one can force a critical point to lie in the Julia set by, say, mapping it to a repelling periodic point. For example, $\phi(z) = \pi^{-1}(z^{p+1} - z^p) + 1$, where $0 < |\pi| < 1$, sends the wild critical point $z = 0$ to the repelling fixed point $z = 1$. Thus, there are plenty of maps violating hypothesis (i) of Theorem 7.12, and very little is known about wandering domains for such maps over locally compact fields of equal characteristic.

As for the case of locally compact fields of mixed characteristic, it was not known for several years after [6] appeared whether or not maps with recurrent wild Julia critical points existed at all. It was Rivera-Letelier who first constructed such maps (in fact, he constructed polynomials in $\mathbb{Q}_p[z]$ with this property, for any prime p) in [45], using a technique of *Misiurewicz* bifurcations; see Project #4. Still, nothing whatsoever is known about whether or not such maps can have wandering domains.

Meanwhile, the wandering domains of Example 4.42 are quite common if the residue field is not algebraic over a finite field, as we will see in Theorem 7.15 below. First, we make the following definition.

Definition 7.14. Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2, let $\mathcal{F}_{\phi,\mathrm{Ber}}$ be the Berkovich Fatou set of ϕ, and let $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$ be a type II periodic point of ϕ of minimal period m. We say that a wandering component U of $\mathcal{F}_{\phi,\mathrm{Ber}}$ is in the *attracting basin of* ζ if there is some integer $N \geq 0$ such that for all $n \geq 0$,

$$\phi^{N+nm}(U) \text{ is a full residue class of } \zeta.$$

To have a wandering domain as in Definition 7.14, the residue field cannot be algebraic over a finite field, and the periodic point ζ must be in the Berkovich Julia set, although it could be either indifferent or repelling. The use of the term *attracting* is justified, because for any point $\xi \in U$, it is easy to check that in the Gel'fand topology, $\lim_{n\to\infty} \phi^{N+nm}(\xi) = \zeta$.

The following theorem is essentially Theorem A of [11], although that result is stated only in the case that the periodic point ζ is repelling.

Theorem 7.15 (Benedetto). *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field whose residue field is* not *algebraic over a finite field, let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2, and let $\zeta \in \mathbb{P}^1_{\text{Ber}}$ be a type II periodic point of ϕ lying in the Berkovich Julia set of ϕ. Then ϕ has wandering domains in the attracting basin of ζ.*

Proof. Without loss, $\zeta = \zeta(0,1)$ is the Gauss point and is fixed by ϕ. By Theorem 6.18, since ζ is in the Julia set, the reduction $\overline{\phi}$ is not of finite order. Because the residue field is not algebraic over a finite field, then, $\overline{\phi}$ has a wandering point $\overline{x} \in \mathbb{P}^1(\overline{k})$. Replacing $\overline{x}$ by $\phi^m(\overline{x})$ for some large enough m if necessary, we can assume that $\phi^n(\overline{x})$ does not lie in the finite set T of Theorem 6.11 for any $n \geq 0$. Thus, by Theorem 6.11, the residue class U corresponding to $\overline{x}$ is a wandering domain in the attracting basin of ζ. ☐

Although wandering domains are therefore common in fields with large residue characteristic, recent evidence suggests that in residue characteristic zero, Julia periodic type II points are the only source of wandering domains. In particular, we have the following theorem.

Theorem 7.16 (Benedetto, Trucco). *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field of residue characteristic zero, and let $\phi \in \mathbb{C}_K(z)$. Suppose that either*

i. *ϕ is a polynomial, and $\mathbb{C}_K$ is the field of Puiseux series over $\overline{\mathbb{Q}}$, or*
ii. *ϕ is defined over a discretely valued subfield of $\mathbb{C}_K$.*

Then any wandering domains of ϕ lie in attracting basins of type II Julia periodic points.

References for the proof. See Theorem B of [11] for the case of discretely valued fields. See Corollary B of [57] for the polynomial case. Trucco's proof in [57] can probably be extended to other fields of residue characteristic zero. ☐

7.3. **Repelling density.** In complex dynamics, it is a basic (and relatively easy) theorem that the Julia set of a rational function (of degree at least 2) is equal to the closure of the set of repelling periodic points. Fatou and Julia each gave proofs of this fact, but by very different strategies.

Fatou's strategy (see Theorem III.3.1 of [21]) was as follows. Given an arbitrary open disk U intersecting the Julia set, shrink U if necessary so that it contains no poles or critical values of ϕ; then we can define $d = \deg \phi \geq 2$ local analytic inverses $f_1, \ldots, f_d : U \to \mathbb{C}$ to ϕ, with disjoint images. By writing down an appropriate family involving f_1, f_2, and the iterates ϕ^n, Montel's Theorem shows that either the family is normal on U or else U contains a periodic point of ϕ. If the family were normal, then the family $\{\phi^n\}$ would also be normal, a contradiction. Therefore U contains a periodic point.

The hardest step of the proof is the the final step, which is the observation that in complex dynamics, ϕ has only finitely many non-repelling periodic points. Fatou proved a bound of $6d-6$ such points by first noting that attracting and parabolic periodic points must attract each a critical point. Then, some more delicate work involving one-parameter analytic families of rational functions shows that there are critical points (essentially) associated with irrationally neutral periodic points as well. (Fatou's bound was sharpened to $2d-2$ by Shishikura.) See Section III.2 of [21] for a proof of the $6d-6$ bound.

The analogous argument over $\mathbb{C}_K$ yields the following result. There are a few more technicalities, but the idea is the same, although the final conclusion is weaker.

Theorem 7.17 (Hsia)**.** *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2 with classical Julia set $\mathcal{J}_{\phi,\mathrm{I}}$. Then $\mathcal{J}_{\phi,\mathrm{I}}$ is contained in the closure of the set of* all *type I periodic points of ϕ.*

References for the proof. This is Theorem 3.1 of [33]. See also Project #1. □

The reason for the weaker conclusion is that there is no analogue of Fatou's $6d-6$ bound over $\mathbb{C}_K$. Indeed, one can easily force the number of nonrepelling cycles to be infinite simply by choosing ϕ to have good reduction. Of course, in that case, one has empty classical Julia set, and therefore the goal of repelling density holds vacuously. Thus, it is *a priori* conceivable that Fatou's argument could be sharpened to prove that we can further shrink the disk U in such a way that any periodic points in U must be repelling. However, this possibility can be dismissed by the following example.

Example 7.18. Let $K = \mathbb{C}_2$, and let $\phi(z) = \dfrac{z^3+2z}{z+4}$. Then ϕ has a repelling fixed point at $z = 0$, but for any $r > 0$, there is an *attracting* periodic point of type I in $D(0,r)$. For a proof, see Example 9 of [8].

The idea, however, is similar to that of Theorem 7.13, albeit far simpler. That is, there is an annulus W, given by $|2|^{1/2} < |z| < 1$, which is mapped 2-to-1 onto its image (which is given by $|2| < |z| < 1$), and that image $\phi(W)$ properly contains W. Because the residue characteristic is also 2, the derivative of ϕ is small on W, even though $\phi(W) \supsetneq W$. Thus, it is possible to rig a point x to spend a lot of time in W (by making $|x|$ only ever so slightly smaller than 1), thus producing a *very* small derivative $(\phi^n)'(x)$, before landing near the repelling fixed point at $z = 0$ and then eventually making its way back to x.

The phenomenon of Example 7.18 is actually fairly common. In Project #1, for instance, you will show that a certain polynomial of degree $p+1$ in $\mathbb{Q}_p[z]$ also has a repelling fixed point accumulated on by attracting periodic points. The key thing that is needed is a region like W that is expanding on a large scale (in the sense that $\phi(W) \supsetneq W$) but contracting on a small scale (in the sense that $|\phi'|$ is small on W). Of course, one also needs locally expanding regions, such as near a repelling periodic point, for a point to land in after leaving W. Otherwise, the actual periodic component would be too large to be interesting. (If W were merely a subset of an attracting component, for example, no such phenomena are possible.) Such a region W can only exist, however, if the residue characteristic is positive. Project #1 therefore suggests extending Theorem 7.17 to attain repelling density over fields of residue characteristic zero.

A modification and expansion of Hsia's argument can be used to show that repelling density *is* always attained for the Berkovich Julia set. What makes this extension possible is the fact that we now have type II repelling periodic points, and not just the type I points, at our disposal.

Theorem 7.19. *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2 with Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{Ber}}$. Then $\mathcal{J}_{\phi,\mathrm{Ber}}$ is contained in the closure of the set of repelling periodic points of ϕ in $\mathbb{P}^1_{\mathrm{Ber}}$.*

Idea of proof. Start with a connected Berkovich affinoid U intersecting $\mathcal{J}_{\phi,\mathrm{Ber}}$, and follow the same strategy as in Theorem 7.17. Repelling periodic points in Berkovich space can then be constructed using the same sort of idea we invoked in the proof of Theorem 6.20. Specifically, Proposition 9.3 of [44] says that if $\phi^n(U)$ contains the closure of U (and if in addition, U is a component of $\phi^{-n}(\phi^n(U))$, a necessary technical condition), then U contains a periodic point. Moreover, the strengthened version of this statement in Theorem 10.82 of [2] shows that in fact U contains a repelling periodic point. □

Even though Theorem 7.19 settles the question of Berkovich repelling density, the question of repelling density in the *classical* Julia set remains open. We therefore take a moment to describe Julia's strategy, mentioned earlier, for proving repelling density over $\mathbb{C}$; see also Section 14 of [38].

Julia starts from Propsition 4.2.a, which says that there is a repelling or rationally indifferent fixed point— in particular, there is a fixed point x_0 in the Julia set. (Alternately, one could start with any repelling or rationally indifferent periodic point in the Julia set.)

Given an open set U intersecting the Julia set, Julia shows that there is a sequence $\{x_n\}_{n\geq 0}$ in the backward orbit of x_0 with the following properties:

- $\phi(x_{n+1}) = x_n$ for all $n \geq 0$
- $x_M \in U$ for some $M \geq 0$, and
- $\lim_{n\to\infty} x_n = x_0$.

(We say a particular backward orbit $\cdots x_2 \mapsto x_1 \mapsto x_0$ is *homoclinic* if it satisfies the third property.) He uses the fact that U intersects the Julia set to obtain the second property, and he uses the repulsion at x_0 (at least in one direction, in the case of multiplier 1) to obtain the third.

He then takes a small neighborhood V_0 of x_0, and for each $n \geq 0$, he sets V_n to be the component of $\phi^{-n}(V_0)$ containing x_n. Without loss, V_0 is small enough that $x_m \in V_m \subseteq U$. Moreover, for some M large enough, we get $x_M \in V_0$, and in fact $x_M \in V_M \subsetneq V_0$. Thus, ϕ^M maps the very small set V_M onto the much larger set V_0 containing it. After some technical arguments to deal with the possibility of critical points, we get a *repelling* periodic point $y \in V_M$, and therefore, moving up the forward orbit of y, a repelling periodic point $\phi^{M-m}(y)$ in $V_m \subseteq U$.

The whole argument goes through over non-archimedean fields (in fact, perhaps even a bit more simply), *except* for the very first step, where Julia started from the existence of one repelling point. Thus, we have the following result.

Theorem 7.20 (Bézivin). *Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree at least 2 with classical Julia set $\mathcal{J}_{\phi,\mathrm{I}}$. If ϕ has at least one repelling periodic point in $\mathbb{P}^1(\mathbb{C}_K)$, then $\mathcal{J}_{\phi,\mathrm{I}}$ is the closure in $\mathbb{P}^1(\mathbb{C}_K)$ of the set of type I repelling periodic points of ϕ*

References for the proof. See Théorème 3 of [15]. □

However, it is again not clear how one might remove the (hopefully) extraneous hypothesis of the existence of at least one repelling periodic point. Indeed, in Project #1, it will be shown that there is a family of polynomials of degree $3p+1$ in $\overline{\mathbb{Q}}_p[z]$ (and this degree can be lowered, if one does not mind some slightly messier computations) such that for any integer $n \geq 1$, there is a polynomial in the family with no repelling periodic points of any period up to and including n, but with a repelling periodic point of period $n+1$.

In light of that example and Example 7.18, then, it is conceivable that perhaps, somehow, one could have a nonempty classical Julia set, with every point being an accumulation point of attracting periodic points, and yet not one point of the Julia set itself is periodic. It is hard to imagine how such a situation could exist, but as yet, there is neither an example showing that it does exist nor a proof that it cannot.

References

[1] Yvette Amice, *Les Nombres p-adiques*, Presses Universitaires de France, 1975.

[2] Matthew H. Baker and Robert S. Rumely, *Potential Theory and Dynamics on the Berkovich Projective Line,* Amer. Math. Soc., Providence, to appear 2010.

[3] Alan F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, New York, 1991.

[4] Robert L. Benedetto, Fatou components in p-adic dynamics, Brown University Ph.D. thesis, 1998.

[5] Robert L. Benedetto, p-adic dynamics and Sullivan's No Wandering Domains Theorem, *Compositio Math.* **122** (2000), 281–298.

[6] Robert L. Benedetto, Hyperbolic maps in p-adic dynamics, *Ergodic Theory Dynam. Systems* **21** (2001), 1–11.

[7] Robert L. Benedetto, Reduction, dynamics, and Julia sets of rational functions, *J. Number Theory* **86** (2001), 175–195.

[8] Robert L. Benedetto, Components and periodic points in non-archimedean dynamics, *Proc. London Math. Soc.* (3) **84** (2002), 231–256.

[9] Robert L. Benedetto, Examples of wandering domains in p-adic polynomial dynamics, *C. R. Acad. Sci. Paris, Ser. I* **335** (2002), 615–620.
[10] Robert L. Benedetto, Non-archimedean holomorphic maps and the Ahlfors Islands Theorem, *Amer. J. Math.*, **125** (2003), 581–622.
[11] Robert L. Benedetto, Wandering domains and nontrivial reduction in non-Archimedean dynamics, *Illinois J. Math.* **49** (2005), 167–193.
[12] Robert L. Benedetto, Wandering domains in non-Archimedean polynomial dynamics. *Bull. London Math. Soc.* **38** (2006), 937–950.
[13] Robert L. Benedetto, Jean-Yves Briend, and Hervé Perdry, Dynamique des polynômes quadratiques sur les corps locaux, *J. Théor. Nombres Bordeaux* **19** (2007), 325–336.
[14] Vladimir G. Berkovich, *Spectral Theory and Analytic Geometry over Non-archimedean Fields,* Amer. Math. Soc., Providence, 1990.
[15] Jean-Paul Bézivin, Sur les points périodiques des applications rationnelles en analyse ultramétrique, *Acta Arith.* **100** (2001), 63–74.
[16] Jean-Paul Bézivin, *Dynamique des Fractions Rationnelles p-adiques,* lecture notes, 2005. Available at `http://www.math.unicaen.fr/~bezivin/`
[17] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry,* Springer-Verlag, Berlin, 1984.
[18] L.E. Böttcher, The principal laws of convergence of interates and their applications to analysis (Russian), *Izv. Kazan. Fiz.-Mat. Obshch.* **14** (1904), 155-234.
[19] Alexander D. Bryuno, Analytical form of differential equations, *Trans. Moscow Math. Soc.* **25** (1971), 131–288.
[20] Serge Cantat and Antoine Chambert-Loir, "Dynamique p-adique (d'après des exposés de Jean-Christophe Yoccoz)", preprint, 2009. Available at `arxiv:0912.5413`
[21] Lennart Carleson and Theodore W. Gamelin, *Complex Dynamics,* Springer-Verlag, New York, 1991.
[22] Robert F. Coleman, p-adic Banach spaces and families of modular forms, *Invent. Math.* **127** (1997), 417–479.
[23] John B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, New York, 1978.
[24] Alain Escassut, The ultrametric spectral theory, *Period. Math. Hungar.* **11** (1980), 7–60.
[25] Alain Escassut, *Analytic Elements in p-adic Analysis,* World Scientific, River Edge, NJ, 1995.
[26] Alain Escassut, *Ultrametric Banach Algebras,* World Scientific, River Edge, NJ, 2003.
[27] Jean Fresnel and Marius van der Put, *Rigid Analytic Geometry and its Applications,* Birkhäuser, Boston, 2004.
[28] Pierre Fatou, Sur les équations fonctionelles, *Bull. Soc. Math. France* **47** (1919), 161–271, and **48** (1920), 33–94, 208–314.
[29] David Goss, A short introduction to rigid analytic spaces, in *The Arithmetic of Function Fields*, de Gruyter, Berlin, 1992, 131–141.
[30] Fernando Q. Gouvêa, *p-adic Numbers: An Introduction,* Springer-Verlag, Berlin, 1993.
[31] Robin C. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
[32] Michael-Robert Herman and Jean-Christophe Yoccoz, Generalizations of some theorems of small divisors to non-Archimedean fields, *Geometric Dynamics (Rio de Janairo, 1981),* LNM 1007, Springer-Verlag, Berlin, 1983, 408–447.
[33] Liang-Chung Hsia, Closure of periodic points over a nonarchimedean field, *J. London Math. Soc.* (2) **62** (2000), 685–700.
[34] Gaston Julia, Mémoire sur l'itération des fonctions rationelles, *J. Math. Pures Appl.* (7) **4** (1918), 47–245.
[35] Neal Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions,* 2nd ed., Springer-Verlag, New York, 1984.
[36] Gabriel X.P. Kœnigs, Recherches sur les intégrales de certaines éequations fonctionnelles, *Ann. Sci. École Norm. Sup.* **1** (1884), Supplément 3–41.
[37] Karl-Olof Lindahl, On Siegel's linearization theorem for fields of prime characteristic, *Nonlinearity* **17** (2004), 745–763.
[38] John W. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*, Vieweg, Braunschweig, 2nd ed., 2000.
[39] Patrick Morton and Joseph H. Silverman, Rational periodic points of rational functions, *Inter. Math. Res. Notices* **2** (1994), 97–110.
[40] Patrick Morton and Joseph H. Silverman, Periodic points, multiplicities, and dynamical units, *J. Reine Angew. Math.* **461** (1995), 81–122.
[41] Juan E. Rivera-Letelier, Dynamique des fonctions rationnelles sur des corps locaux, *Astérisque* **287** (2003), 147–230.
[42] Juan E. Rivera-Letelier, Espace hyperbolique p-adique et dynamique de fonctions rationnelles, *Compositio Math.* **138** (2003), 199–231
[43] Juan E. Rivera-Letelier, Sur la structure des ensembles de Fatou p-adiques, preprint, 2004. Available at `arxiv:math.DS/0412180`

[44] Juan E. Rivera-Letelier, Points périodiques des fonctions rationnelles dans l'espace hyperbolique p-adique, *Comment. Math. Helv.* **80** (2005), 593–629.
[45] Juan E. Rivera-Letelier, Wild recurrent critical points, *J. London Math. Soc. (2)* **72** (2005), 305–326.
[46] Philippe Robba, Fonctions analytiques sur les corps valués ultramétriques complets, *Astérisque* **10** (1973), 109–218.
[47] Alain M. Robert, *A Course in p-adic Analysis,* Springer-Verlag, New York, 2000.
[48] Wilhelmus H. Schikhof, *Ultrametric Calculus: An Introduction to p-adic Analysis,* Cambridge Univ. Press, Cambridge, 1984.
[49] Peter Schneider, *Nonarchimedean Functional Analysis,* Springer-Verlag, Berlin, 2002.
[50] Jean-Pierre Serre, *Algebraic Groups and Class Fields*, Springer-Verlag, New York, 1988.
[51] Mitsuhiro Shishikura, On the quasiconformal surgery of rational functions, *Ann. Sci. cole Norm. Sup. (4)* **20** (1987), 1–29.
[52] Joseph H. Silverman, *The Arithmetic of Dynamical Systems*, Springer, New York, 2007.
[53] Nigel P. Smart and Christopher F. Woodcock, p-adic chaos and random number generation, *Experiment. Math.* **7** (1998), 333–342.
[54] Dennis P. Sullivan, Quasiconformal homeomorphisms and dynamics, I, Solution of the Fatou-Julia problem on wandering domains, *Annals of Math.* **122** (1985), 401–418.
[55] John Tate, Rigid analytic spaces, *Invent. Math.* **12** (1971), 257–289.
[56] Emmanuel Thiran, Dirk Verstegen, and Jacques Weyers, p-adic dynamics, *J. Stat. Phys.* **54** (1989), 893–913.
[57] Eugenio Trucco, Wandering Fatou components and algebraic Julia sets, preprint, 2009. Available at `arxiv:0909.4528`
[58] Jean-Christophe Yoccoz, Linéarisation des germes de difféomorphismes holomorphes de $(\mathbb{C}, 0)$, *C.R. Acad. Sci. Paris Sér I Math.* **306** (1988), 55–58.