ARIZONA WINTER SCHOOL 2016: ANALYTIC THEORY OF TRACE FUNCTIONS

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1. Functions on congruence rings

As was emphasized by C.F. Gauss in the *Disquisitiones Arithmeticae* (and probably much earlier), the concept of congruences (a.k.a arithmetic progressions of some modulus q, a.k.a the ring $\mathbf{Z}/q\mathbf{Z}$) is central in number theory. It is therefore fundamental to understand spaces of functions over the ring $\mathbf{Z}/q\mathbf{Z}$.

One of the first classes of such functions to have been studied systematically (and one of the most interesting even today) is the group of Dirichlet's characters of modulus q, $\widehat{(\mathbf{Z}/q\mathbf{Z})^{\times}}$, the group homomorphisms

$$\chi : (\mathbf{Z}/q\mathbf{Z})^{\times} \to \mathbf{C}^{\times}.$$

Indeed these functions enter crucially in Dirichlet's proof that there are infinitely many primes in any admissible arithmetic progression of modulus q [7].

The studying of integral solutions of diophantine equations is another important source of functions on congruence rings: these arise as a manifestation of some general local-global principles or more directly when using the circle method. For instance, while investigating the set of representations of an integer d by some integral, positive, definite, quadratic form in four variables $q(x, y, z, t)$, ie. the integral solutions to the equation

$$d = q(x, y, z, t),\ x, y, z, t \in \mathbf{Z}, \tag{1.1}$$

Kloosterman came up with the very important function modulo q, the so-called "Kloosterman sums",

$$m \,(\mathrm{mod}\, q) \mapsto \mathrm{Kl}_2(m; q) := \frac{1}{\sqrt{q}} \sum_{xy=1 \,(\mathrm{mod}\, q)} e_q(mx + y),$$

where $e_q(\cdot)$ denote the additive character of $(\mathbf{Z}/q\mathbf{Z}, +)$

$$e_q(x) := \exp(\frac{2\pi i x}{q}).$$

By evaluating the fourth moment of this function over $\mathbf{Z}/q\mathbf{Z}$, Kloosterman established the following non-trivial bound

$$\mathrm{Kl}_2(m; q) \ll q^{2/3-1/2+o(1)} \tag{1.2}$$

and deduced from it an asymptotic formula for the number of solutions to the equations (1.1). Kloosterman sums are also fundamental for the analytic theory of modular forms, ie. analytic functions living on arithmetic quotients of the Poincare upper-half space:

$$f : z \in \mathbf{H}/\Gamma \to \mathbf{C},\ \mathbf{H} = \{z = x + iy \in \mathbf{C},\ y > 0\},\ \Gamma \subset \mathrm{SL}_2(\mathbf{Z}).$$

One possible connection is via Petersson's formula (for holomorphic forms) and Kuznetsov's formula (for Maass forms) which compute averages of Fourier coefficients of these in terms of sums of Kloosterman sums. This together with Kloosterman's bound (1.2) implies non-trivial bounds for the Fourier coefficients of modular forms and non-trivial lower bounds for eigenvalue of the hyperbolic Laplacian. This, in turn, together with the theta correspondance implies that the set of representations (1.1) suitably normalized becomes equidistributed [7, 13].

With further developments in analytic number theory and other areas, new and more complicated functions have appeared, like the *hyper-Kloosterman sums*

$$m \mapsto \mathrm{Kl}_k(m;q) := \frac{1}{q^{\frac{k-1}{2}}} \sum_{x_1.\cdots.x_k=1 \,(\mathrm{mod}\, q)} e_q(mx_1 + \cdots + x_k)$$

and it became apparent that not only the size of these functions is important but also the way these functions vary with m and q. When trying to understand these questions, the Chinese Reminder Theorem is certainly a good start; one is then reduced to the case of q a prime power and there the hardest case is when q is itself a prime; the congruence ring $\mathbf{Z}/q\mathbf{Z}$ is then a finite field noted $\mathbf{F}_q$.

2. Trace functions

When q is a prime, the Dirichlet characters, Kloosterman sums and hyper-Kloosterman sums are special case of *trace functions*[1].

Trace function have their origin in the problem of counting the number of points on algebraic varieties defined over the finite filed $\mathbf{F}_q$. After some initial work of Hasse for elliptic curves, Weil solved the problem completely for curves of any genus and on the way established the Riemann Hypothesis for zeta functions of curves [14]. As a consequence, Weil also obtained the optimal bound for Kloosterman sums of prime modulus,

$$|\,\mathrm{Kl}_2(m;q)| \leqslant 2$$

and similar bounds for more general algebraic exponential sums in one variables. Weil also made a serie of precise conjectures to address the counting problem fo higher dimensional varieties. The Weil conjectures triggered the invention of etale (ℓ-adic) cohomology ($\ell \neq q$ a prime) by Grothendieck which was further developed by him and his school through the SGAs and which culminated in Deligne's proof of the Weil conjectures [1, 2].

A trace function

$$K : \mathbf{F}_q \to \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$$

is associated an ℓ-adic sheaf noted $\mathcal{F}$; the datum $\mathcal{F}$ is "essentially" an ℓ-adic Galois representation

$$\varrho_{\mathcal{F}} : \mathrm{Gal}(\mathbf{F}_q[X]^{sep}/\mathbf{F}_q) \mapsto \mathrm{GL}(\mathcal{F}_{\overline{\xi}})$$

where $\mathcal{F}_{\overline{\xi}}$ is a finite dimensional $\overline{\mathbf{Q}_\ell}$-vector space. Given some $\mathbf{F}_q$-point $x \in \mathbf{A}^1(\mathbf{F}_q)$ at which the representation $\varrho_{\mathcal{F}}$ is unramified (the inertia group I_x acts trivially on $\mathcal{F}_{\overline{\xi}}$) the value $K(x)$ is given as the trace of the Frobenius conjugacy class

$$K(x) = \mathrm{tr}(\mathrm{Frob}_x\,|\mathcal{F}_{\overline{\xi}});$$

more generally, K could be extended to all x by

$$K(x) = \mathrm{tr}(\mathrm{Frob}_x\,|\mathcal{F}_{\overline{\xi}}^{I_x})$$

although there are other possibilities of extension.

For applications, we will impose some extra boundedness conditions:

- The eigenvalues of the Frobenius elements view as complex numbers via the above embedding have absolute values bounded by 1 (more precisely the ℓ-adic sheaf is mixed of weight $\leqslant 0$).

[1]That Kloosterman and hyper-Kloosterman sums are are deep results of Deligne and Katz

- Whenever ones has to consider families of traces functions possibly for varying characteristics q, we assume that the dimension and the Artin conductors of all the representations involved are bounded independently of q.

These imply that the trace function is bounded independently of q in absolute value.

For our purpose, the important properties satisfied by trace functions are the Lefschetz trace formula which expresses the sum of K over $\mathbf{A}^1(\mathbf{F}_q)$ as a sum of traces of the Frobenius $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$ acting on related ℓ-adic vector spaces (the ℓ-adic cohomology groups of $\mathcal{F}$)

Theorem (Lefschetz trace formula)**.**

$$\sum_{x\in\mathbf{F}_q} K(x) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\mathrm{Frob}_q\,|H^i_c(\mathbf{A}^1_{\overline{\mathbf{F}_q}}|\mathcal{F}).$$

The dimensions of these spaces are under control (for instance $H^2_c(\mathbf{A}^1_{\overline{\mathbf{F}_q}}|\mathcal{F})=0$ if $\varrho_{\mathcal{F}}$ is geometrically irreducible (ie. when restricted to $\mathrm{Gal}(\mathbf{F}_q[X]^{sep}/\overline{\mathbf{F}_q})$, $\varrho_{\mathcal{F}}$ is irreducible). The second main result control the size of the eigenvalues

Theorem (Deligne's Theorem on Weights)**.** *The eigenvalues of* Frob_q *acting on* $H^i_c(\mathbf{A}^1_{\overline{\mathbf{F}_q}}|\mathcal{F})$ *are bounded in absolute values by* $q^{1/2}$.

In particular, if $\varrho_{\mathcal{F}}$ is geometrically irreducible (which is the case of hyper-Kloosterman sums) one has

$$\sum_{x\in\mathbf{F}_q} K(x) = O_{\mathcal{F}}(q^{1/2})$$

which beat the trivial bound $O_{\mathcal{F}}(q)$ by a factor $q^{1/2}$.

3. COURSE AND PROJECT

This course will not be a presentation ℓ-adic cohomology per se but rather a presentation of trace functions from a very utilitarian point of view, with an eye on applications to analytic number theoretic problems and by using ℓ-adic cohomology as much as possible as a black box using the representation theory language along the line initiated by Fouvry, Kowalski and myself [3–5].

A trace function while of geometric origins is naturally identified with a (q-periodic) function on $\mathbf{Z}$ which itself is contained in the field of real numbers and so can and should be compared with other functions on $\mathbf{Z}$ of more analytical origin; building on the works and books of Katz [8–10] we will discuss specific method permitting to evaluate how much a given trace function correlate with other arithmetic functions: for instance, identifying $\mathbf{F}_q$ with the interval $\mathbf{Z}\cap[0,q-1]$ it is natural to try to evaluate the partial sum

$$\sum_{0\leqslant x\leqslant X} K(x)$$

which has no clear geometric interpretation unless $X=q-1$ (in which case one case invoque the Lefschetz trace formula and Deligne's theorem.) The Polya-Vinogradov method enable to handle this question in a satisfactory way as long as X is sufficiently large (a bit larger than $q^{1/2}$) and it eventually reduces to evaluating the following complete sums for $y\in\mathbf{F}_q$

$$\mathrm{FT}(K)(y) := \frac{1}{q^{1/2}} \sum_{x\in\mathbf{F}_q} K(x)e_q(xy)$$

to which one can use the above mentioned theorems because the function

$$x \mapsto K(x)e_q(xy)$$

is most of the time a trace function[2]. This is of course a very simple example and more complicated problems yield more complicated manipulations and then more complicated trace functions to evaluate on average.

We will discuss a variety of methods coming mainly from the works of Deligne and Katz to decide when two trace functions correlate or not (ie. when the sum

$$\sum_{x\in\mathbf{F}_q} K(x)\overline{K'(x)}$$

is large or small) and apply these to specific classical problems including relevant parts of Y. Zhang proof of the existence of bounded gaps between primes and the improvement that followed within the Polymath8a project [6, 12, 15].

The project sessions will be concerned with handling new analytic number theoretic problems or with improving existing results, using the tools on trace function gathered during the course.

References

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[2] It is a deep result of Laumon that the function $y \mapsto \mathrm{FT}(K)(y)$ is most of the time a trace function [11]