$\mathbb{A}^1$-enumerative geometry : k field

Goal to record arithmetic information about geometric objects whose # is fixed over $\bar{k}$, (but not over k)

tool : $\mathbb{A}^1$-htpy thy

L1: User's guide to $\mathbb{A}^1$ htpy thy

$$S^n = \{(x_0, \dots, x_n) \mid \sum x_i^2 = 1\} = \mathbb{P}^n(\mathbb{R}) / \mathbb{P}^{n-1}(\mathbb{R})$$

$$\deg : \underbrace{[S^n, S^n]}_{\text{pointed homotopy classes}} \longrightarrow \mathbb{Z}$$

Given $f: S^n \longrightarrow S^n$ and $p \in S^n$ s.t. $f^{-1}(y) = \{q_1, \dots, q_N\}$

$$\deg f = \sum_{i=1}^{N} \deg_{q_i} f$$

where $\deg_{q_i} f$ is the local degree :

Let $V \ni p$ be a small ball

Let $f^{-1}(V) \supset U \ni q_i$ be a small ball s.t. $f^{-1}(p) \cap U = q_i$

$$\begin{array}{ccc} U/\partial U \simeq S^n & & V/\partial V \simeq S^n \\ \cong & & \cong \\ U/(U - q_i) & \xrightarrow{\bar{f}} & V/(V-p) \end{array}$$

$\deg_{q_i} f := \deg \bar{f}$

Formula from differential topology

Let $(x_1, \ldots, x_n)$ be oriented coordinates near q_i

Let $(y_1, \ldots, y_n)$ — " — p

Then $f = (f_1, \ldots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Let $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$

$$\deg_{q_i} f = \begin{cases} +1 & J(q_i) > 0 \\ -1 & J(q_i) < 0 \\ ? & J(q_i) = 0 \end{cases}$$

Eisenbud - H. Levine / Khimshiashvili signature formula

Lannes / Morel : k field

degree for $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

valued in $GW(k)$

$GW(k) =$ Grothendieck–Witt group of k

$:=$ group completion under $\oplus$ of semi-ring $\otimes \oplus$ of non-degenerate, symmetric, bilinear forms generators: $\langle a \rangle \qquad a \in k^*/(k^*)^2$

$\langle a \rangle$ element associated to bilinear form

$$\langle a \rangle : k \times k \longrightarrow k$$
$$(x, y) \longmapsto axy$$

relations: (1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

(2) $\langle u \rangle + \langle v \rangle = \langle uv(u+v) \rangle + \langle u+v \rangle \qquad u+v \neq 0$

(3) $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$ ← hyperbolic form

$W(k) = \dfrac{GW(k)}{\mathbb{Z}\mathbb{H}}$

Exercise: (2) $\Rightarrow$ (3) $\qquad$ char $k \neq 2$

<u>Ex</u>: $GW(\mathbb{C}) \overset{\text{rank}}{\cong} \mathbb{Z}$

$$(B : V \times V \to \mathbb{C}) \mapsto \dim V$$
$$\langle a \rangle \mapsto 1$$

<u>Ex</u>: $GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$

Silvester's theorem

$B : V \times V \to \mathbb{C}$, there is a basis $\{v_1, \dots, v_r\}$ of V s.t.

Gram matrix $B(v_i, v_j)$ is
$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{bmatrix}$$

Signature $B = \#1\text{'s} - \#(-1)\text{'s}$

$$\begin{array}{ccc} GW(\mathbb{R}) & \xrightarrow{\text{rank}\times\text{sign}} & \mathbb{Z}\times\mathbb{Z} \\ & & \cup \\ & \searrow^{\cong} & \{(r,s)\in\mathbb{Z}\times\mathbb{Z} \mid r+s\equiv 0 \bmod 2\} \cong \mathbb{Z}\times\mathbb{Z} \end{array}$$

Ex: $GW(\mathbb{F}_q) \xrightarrow[\cong]{\text{rank}\times\text{disc}} \mathbb{Z}\times \mathbb{F}_q^{*}/(\mathbb{F}_q^{*})^2$

Ex: (Springer's theorem) K complete discrete valued field
k residue field

e.g. $K=\mathbb{Q}_p$ $K=\mathbb{F}_p((t))$, $k=\mathbb{F}_p$

Assume char $k\neq 2$

$$GW(K) \cong \frac{GW(k)\oplus GW(k)}{\mathbb{Z}(\mathbb{H},-\mathbb{H})}$$

Back to Lannes's formula:

$f: \mathbb{P}^1_k \to \mathbb{P}^1_k$ $p\in\mathbb{P}^1(k)$ $f^{-1}(p)=\{q_1,\dots,q_N\}$

s'pose $J(q_i)=f'(q_i)\neq 0$ $\forall i$

$$\deg f = \sum_{i=1}^{N} \langle J(q_i)\rangle$$

This doesn't depend on p!

<u>Exercise</u>: 1) $\deg\left(\mathbb{P}^1 \xrightarrow[z \mapsto az]{[x:y] \mapsto [ax:y]} \mathbb{P}^1\right) = \langle a \rangle$

2) $\deg\left(\mathbb{P}^1 \xrightarrow[z \mapsto z^2]{} \mathbb{P}^1\right) = \langle 1 \rangle + \langle -1 \rangle$

Cazanavé: studied naïve homotopy classes of maps $\mathbb{P}^1 \to \mathbb{P}^1$ (Karoubi-Villamayor)

$\mathbb{A}^1 = \operatorname{Spec} k[t]$ replaces $[0,1]$

<u>Def</u>: A naïve homotopy h between to maps of schemes

$f, g : X \to Y$ is a map

$h : X \times \mathbb{A}^1 \to Y$ s.t. $h|_{X \times \{0\}} = f$ and $h|_{X \times \{1\}} = g$

maps $\mathbb{P}^1 \to \mathbb{P}^1$ are rational functions $\frac{f}{g}$

$\deg\left(\frac{f}{g}\right) = \max(\deg f, \deg g)$ relatively prime

Bézout associated a bilinear form $\text{Béz}\left(\frac{f}{g}\right)$ to f/g constructed as follows:

$$\frac{f(X)g(Y) - f(Y)g(X)}{X - Y} = \sum_{1 \le i,j \le \underset{n}{\overset{\shortparallel}{\deg\left(\frac{f}{g}\right)}}} B_{ij} X^{i-1} Y^{j-1}$$

$\text{Béz}(f/g) : k^n \times k^n \to k$

Gram matrix $(\text{Béz}(f/g)) = [B_{ij}]$

Exercise: (1) $\text{Béz}(\mathbb{P}^1 \xrightarrow{z \mapsto z^n} \mathbb{P}^1) = \begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix} \simeq \begin{cases} \frac{n}{2}\mathbb{H} & n \text{ even} \\ \frac{n-1}{2}\mathbb{H} + \langle 1 \rangle & n \text{ odd} \end{cases}$

(2) $\text{Béz}(\mathbb{P}^1 \xrightarrow{z \mapsto az^n} \mathbb{P}^1)$ $a \in k^*$

$$\boxed{\text{Béz}\left(\frac{f}{g}\right) = \deg\left(\frac{f}{g}\right)}$$

Cazanavé gave formula for addition of naive homotopy classes of maps and showed $[\mathbb{P}^1, \mathbb{P}^1]_N \to [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$ is a group completion

Morel: $\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1} \longrightarrow GW(k)$

To make sense of this we need: $\mathbb{P}^n/\mathbb{P}^{n-1}$

$[-,-]_{\mathbb{A}^1}$

$\mathbb{P}^n/\mathbb{P}^{n-1}$ is $\operatorname{colim}\begin{pmatrix} \mathbb{P}^{n-1} & \longrightarrow & \mathbb{P}^n \\ \downarrow & & \\ * & & \end{pmatrix} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$

i.e. maps out of $\mathbb{P}^n/\mathbb{P}^{n-1}$ are maps out of $\mathbb{P}^n$ & $*$ which agree on $\mathbb{P}^{n-1}$

ex: $\begin{array}{ccc} U \cap V & \to & U \\ \downarrow & & \downarrow \\ V & \to & U \cup V \end{array} \simeq$ colim or pushout

But colimits don't always exist in schemes

We want colimits and we want colimits of $\simeq$ schemes to produce $\simeq$ colimits. So colimit $\rightsquigarrow$ hocolim

Sm_k = smooth schemes$/k$

simplicial sets or think top. spaces

$Pre(Sm_k)$ = Presheaves on Sm_k = Functors$(Sm_k^{op}, sSet)$

$Sm_k \xrightarrow{\text{Yoneda}} Pre(Sm_k)$ ← has colimits

$X \longmapsto Map(-, X)$

homotopy theory: Simplicial model category
or
∞-cat (= quasicategory)

notion of weak equivalence or homotopy equivalence

$Sm_k \xrightarrow{\text{Yoneda}} Pre(Sm_k)$

freely adding colimits.

<u>Problem</u>: We want colimits from "open covers" of schemes to be colimits.

Notion of an "open cover" of a scheme is a <u>Grothendieck topology</u>

Let τ be a Grothendieck topology

Fix: Bousfield localization imposes additional w.e.

$$\begin{tikzcd} Sm_k \arrow[r] & Pre(Sm_k) \arrow[r, "L_\tau"'] & Sh_k \arrow[l, bend right, "\top"'] \arrow[r, "L_{\mathbb{A}^1}"'] & Spc_k \arrow[l, bend right, "\top"'] \end{tikzcd}$$

$\operatorname{Cosk}^0 \coprod_\alpha U_\alpha \to X$ is a weak equivalence

$X \times \mathbb{A}^1 \to X$ is a weak equivalence

Topologies: Zariski, Nisnevich, étale

$\mathbb{A}^1$-homotopy thy

$\longrightarrow$ more open sets

Def: $f: X \to Y$ is étale at x if $(f^* \Omega_{Y/k})_x \xrightarrow{\sim} \Omega_{X/k,x}$

cotangent space

$\iff$ $\operatorname{Jac} f \neq 0$ in $k(x)$

Def: $\coprod_{\alpha \in A} U_\alpha \to X$ is an étale cover if it is étale and surjective

Def: $\coprod_{\alpha \in A} U_\alpha \to X$ is a Nisnevich cover if it is étale, surjective, and for every $x \in X$ $\exists$ $\alpha \in A$ and $y \in U_\alpha$ s.t. $y \mapsto x$ and $k(x) \xrightarrow{\cong} k(y)$

Facts: • Any $Z \hookrightarrow X$ in Sm_k is Nisnevich locally equivalent to $\mathbb{A}^d_k \hookrightarrow \mathbb{A}^{d+c}_k$ (like étale top)

- Nisnevich coh can be computed using Čech cochains (like étale top)
- Alg K-thy satisfies Nisnevich descent (like Zariski top)
- Nis coh dim = Krull dim (like Zariski top)

Notation: $X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$

<u>ex</u>: In top, $S^n \wedge S^m \simeq S^{n+m}$

L2: Local degree and $\mathbb{A}^1$-Milnor numbers

<u>Def</u> Given pointed spaces X and Y, the <u>smash product</u> is $X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$

<u>Ex</u>: In classical alg top, $S^n \wedge S^m \simeq S^{n+m}$

<u>Spheres</u>:

$$\mathbb{G}_m = \operatorname{Spec} k[\tfrac{1}{z}, z] = \mathbb{A}^1 - \{0\}$$
$$S^{p+q\alpha} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q} \simeq S^{p+q,q}$$

<u>Ex</u>:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "z \mapsto z"] \arrow[d, "z \mapsto 1/z"'] & \mathbb{A}^1 \simeq * \arrow[d, hook] \\ \mathbb{A}^1 \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd}$$
$$\mathbb{A}^1 \simeq *$$

pushout $\Rightarrow$ $\Sigma \mathbb{G}_m \simeq \mathbb{P}^1$, $\Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$

<u>Ex</u>: $\mathbb{A}^n - \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$

<u>pf</u>: Induction and

$$\begin{tikzcd}
(\mathbb{A}^1-\{0\})\times(\mathbb{A}^{n-1}-\{0\}) \arrow[r] \arrow[d] & \mathbb{A}^1\times(\mathbb{A}^{n-1}-\{0\}) \simeq \mathbb{A}^{n-1}-\{0\} \arrow[d] \\
(\mathbb{A}^1-\{0\})\times\mathbb{A}^n \arrow[r] & \mathbb{A}^n-\{0\}
\end{tikzcd}$$

pushout

$$\begin{matrix} \wr\| \\ \mathbb{A}^1-\{0\} \end{matrix}$$

$$\begin{tikzcd}
X\times Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & \Sigma X\wedge Y = S^1\wedge X\wedge Y
\end{tikzcd}$$

pushout

$$\Rightarrow \quad \mathbb{A}^n-\{0\} \simeq \Sigma(\mathbb{A}^n-\{0\})\wedge(\mathbb{A}^1-\{0\})$$

<u>Ex</u>: $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq (S^1)^{\wedge n}\wedge(\mathbb{G}_m)^{\wedge n}$

<u>pf</u>: $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/\mathbb{P}^n-\{0\} \simeq \mathbb{A}^n/\mathbb{A}^n-\{0\}$

$$\simeq */\mathbb{A}^n-\{0\} \simeq \operatorname{colim} \begin{tikzcd} \mathbb{A}^n-\{0\} \arrow[r] \arrow[d] & * \\ * \end{tikzcd}$$

$$\simeq \Sigma(\mathbb{A}^n-\{0\})$$

Thom spaces: $V \longrightarrow X$ vector bundle

$X \hookrightarrow V$ zero section

trivial bundle

$$\mathrm{Thom}(V,X) = \mathrm{Th}(V) = X^V = V/_{V-X} \cong_{\mathbb{A}^1} \frac{\mathbb{P}(V\oplus\mathcal{O})}{\mathbb{P}(V)}$$

Purity: (Morel-Voevodsky) Let $Z \hookrightarrow X$ be a closed immersion in Sm_k.

$$X/_{X-Z} \cong_{\mathbb{A}^1} \mathrm{Th}(N_Z X)$$

where $N_Z X \longrightarrow Z$ is the normal bundle

Morel: $\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \longrightarrow GW(k)$

$$\begin{tikzcd}
{[S^n, S^n]} \arrow[d, "\deg"] & {[\mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}]} \arrow[l, "\mathbb{R}\text{-pts}"'] \arrow[r, "\mathbb{C}\text{-pts}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] \\
\mathbb{Z} & GW(\mathbb{R}) \arrow[l, "\text{signature}"'] \arrow[r, "\text{rank}"] & \mathbb{Z}
\end{tikzcd}$$

Local $\mathbb{A}^1$-degree: following joint work with Jesse Kass

Thanks to M. Hoyois and F. Morel!

Suppose $f: \mathbb{A}^n \longrightarrow \mathbb{A}^n$, $x \in \mathbb{A}^n(k)$,

x is isolated in $f^{-1}(f(x))$, meaning $\exists$ $U \subset \mathbb{A}^n$ Zariski open, $x \in U$ s.t. $f^{-1}(f(x)) \cap U = x$

The <u>local $\mathbb{A}^1$-degree</u> $\deg_x f$ of f at x is defined to be the degree of

$$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \underbrace{Th\, T_x\mathbb{A}^n \simeq}_{\text{Purity}} U/U-\{x\} \xrightarrow{f} \mathbb{A}^n/\mathbb{A}^n - \{f(x)\} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$$

Now, drop the assumption that $k(x) = k$, but assume $k(f(x)) = k$

<u>Def:</u> $\deg_x f$ is the degree of the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/\mathbb{P}^n_k - \{x\} \simeq U/U-\{x\} \xrightarrow{f} \mathbb{A}^n/\mathbb{A}^n - f(x) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$$

<u>Prop:</u> These two definitions agree

<u>Prop:</u> (global degree is sum of local degrees) Let $f: \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$ be a finite map s.t. $f^{-1}(\mathbb{A}^n_k) = \mathbb{A}^n_k$

Let $\bar{f} : \mathbb{P}^n/\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n/\mathbb{P}^{n-1}$ denote the induced map. Then for any $y \in \mathbb{A}^n(k)$

$$\deg \bar{f} = \sum_{x \in f^{-1}(y)} \deg_x f$$

Computing $\deg_x f$:

In top, $f : \cancel{S^n \to} S^n$
$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$\deg_x f = \begin{cases} +1 & \operatorname{Jac} f(x) > 0 \\ -1 & \operatorname{Jac} f(x) < 0 \\ ? & \operatorname{Jac} f(x) = 0 \end{cases}$$

<u>Eisenbud-Levine/Khimshiashvili Signature formula</u> :

$$\deg_0 f = \operatorname{Signature} w^{EKL}$$

where w^{EKL} is the isomorphism class of the following bilinear form:

$f = (f_1, \dots, f_n)$ $\qquad Q := \dfrac{\mathbb{R}[x_1, \dots, x_n]_0}{\langle f_1, \dots, f_n \rangle}$ finite dimensional complete intersection

$\Rightarrow Q$ Gorenstein, $\operatorname{Hom}_k(Q, k) \cong Q$ Better: distinguished iso, coming from distinguished socle element $\rightsquigarrow$ bilinear form

Explicitly: $\operatorname{Jac} f = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$

Choose $\eta : Q \longrightarrow \mathbb{R}$ s.t. $\eta(\operatorname{Jac} f) = \dim_k Q$

$$\omega^{EKL} : Q \times Q \longrightarrow k \qquad \omega^{EKL}(a,b) = \eta(ab)$$

isomorphism class of ω^{EKL} does not depend on choice of η.

Ex: $f : \mathbb{A}^1 \to \mathbb{A}^1 \qquad f(z) = z^2$

Eisenbud Q: ω^{EKL} defined over a field k, and acts as "degree." Does ω^{EKL} have top interpretation?

Thm: (Kass-W) $\quad \deg_x f = \omega^{EKL}$ in $GW(k)$

reference for proof: "The local $\mathbb{A}^1$-Brouwer degree is the quadratic form of Eisenbud-Khimshiashvili-Levine"

Exercises: 1) $f : \mathbb{A}^2_k \to \mathbb{A}^2_k \quad f(x,y) = (4x^3, 2y)$

char $k \neq 2$

Compute $\deg_0 f$

2) f étale at $0 \Rightarrow \deg_0 f = \langle \operatorname{Jac} f \rangle$

with descent data

f étale at $x \Rightarrow \deg_x f = \operatorname{Tr}_{k(x)/k} \langle \operatorname{Jac} f \rangle$

$\mathbb{A}^1$-Milnor #'s

char $k \neq 2$

hypersurface singularity $p \in \{f=0\}$ bifurcates into nodes $\cong x_1^2 + \dots + x_n^2 = 0$ over k^s

in family $f(x_1, \dots, x_n) + a_1 x_1 + \dots + a_n x_n = t$ parameterized by t, fibers are smooth or have nodes

$k = \mathbb{C}$

For any $(a_1, \dots, a_n)$ sufficiently close to 0

$$\#\text{ nodes in family} = \deg_p^{top} \operatorname{grad} f \quad (\text{Milnor})$$

$$\| \quad \mu_p \quad \text{"Milnor \#"}$$

Over k? char $k \neq 2$

nodes at k-rat'l points: $\operatorname{type}(x_1^2 + a x_2^2 = 0) := \langle a \rangle$

$\operatorname{type}(\sum a_i x_i^2 = 0) := \langle 2^n \prod a_i \rangle$

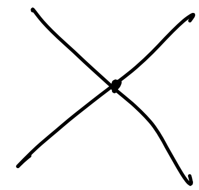
node at p with $k(p) = L$ $\quad \operatorname{type}(\sum a_i x_i^2 = 0) := \operatorname{Tr}_{L/k} \langle 2^n \prod a_i \rangle$

Fact: $k \subseteq L$ is separable

Pictures $k=\mathbb{R}$ $\quad$ $\mathbb{R}$-rat'l nodes

$x_1^2+x_2^2=0$

non-split =
non-rat'l
tgt directions



$x_1^2-x_2^2=0$

split =
rat'l tgt directions

<u>Thm</u> (Kass-W)

For generic $(a_1,\dots,a_n)$

$$\sum_{\substack{\text{nodes } p \\ \text{in family}}} \text{type}(p) = \deg_p^{\mathbb{A}^1} \operatorname{grad} f$$

$$\deg_p^{\mathbb{A}^1} \operatorname{grad} f =: \mu_p^{\mathbb{A}^1} \quad \text{"}\mathbb{A}^1\text{-Milnor \#"}$$

<u>Ex</u> $\quad f(x,y)=x^3-y^2$ $\qquad$ char $\neq 2,3$

$p=(0,0)\in\{f=0\}$

$\operatorname{grad} f=(3x^2,-2y)$

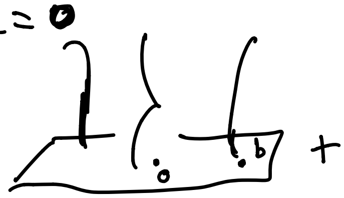
$$\deg \operatorname{grad} f=\deg(3x^2)\deg(-2y)$$

$$=\begin{array}{c|cc} & 1 & x \\ \hline 1 & 0 & \tfrac{1}{3} \\ x & \tfrac{1}{3} & 0 \end{array} \langle -2\rangle = h \qquad \text{rank}=2$$

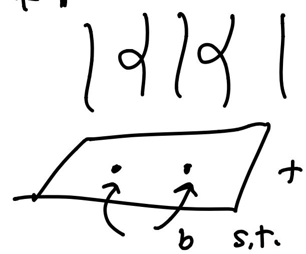
$$=\langle 1\rangle+\langle -1\rangle \text{ so } \mu=2$$

family: $y^2 = x^3 + ax + t$

$a = 0$



$a \neq 0$



b s.t. Disc $x^3 + ax + t = 0$

$$\Leftrightarrow \quad -4a^3 - 27b^2 = 0$$

bifurcates into 2 nodes

over $\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle \Rightarrow$ can't bifurcate into a split & non-split node

over $\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle \Rightarrow$ can't bifurcate into 2 split or 2 non-split nodes

<u>Exercise</u> : compute $\mu^{\mathbb{A}^1}$ for "ADE singularities"

singularity	equation
A_n	$x_1^2 + x_2^{n+1}$
D_n	$x_2(x_1^2 + x_2^{n-2})$
E_6	$x_1^3 + x_2^4$
E_7	$x_1(x_1^2 + x_2^3)$
E_8	$x_1^3 + x_2^5$

which ones are <u>not</u> multiples of $\mathbb{H}$?

L3 Euler class:

Following joint work with Kass

Alg top

$V \to X$ rank r vector bundle on dim r mfld

Def: V is oriented if $\Lambda^r V \cong L^{\otimes 2}$ where L is a line bundle

chosen iso

Def: V is relatively oriented if $\mathrm{Hom}(\Lambda^r TX, \Lambda^r V)$ is oriented.

Euler number $e(V)$: choose a section $s: X \to V$ with only isolated zeros

$$e(V) = \sum_{\substack{p \in X \\ s(p)=0}} \deg_p s$$

where $\deg_p s$ is a local degree or index defined as follows:

Choose
- local coords near p
- local trivialization V near p

$\Big\}$ compatible with relative orientation

Then $\quad s : (\mathbb{R}^r, p) \longrightarrow (\mathbb{R}^r, 0)$

$$\deg_p s = \deg\left(\partial B(p,\varepsilon) \xrightarrow{s/|s|} \partial B(0,1) \right)$$

$$\deg\left(\frac{B(p,\varepsilon)}{B(p,\varepsilon)-p} \xrightarrow{\bar{s}} \mathbb{R}^r \big/ \mathbb{R}^r - 0 \right)$$

<u>$\mathbb{A}^1$-alg top</u> : $\quad V \longrightarrow X$ alg vector bundle
rank r
relatively oriented

X sm, proper over k

s section with only isolated zeros

s'pose $p \in X$ s.t. $s(p) = 0$

choose: • étale $\varphi : \underset{\substack{\psi \\ p}}{U} \longrightarrow \mathbb{A}^r$ s.t. $k(\varphi(p)) \overset{\cong}{\subseteq} k(p)$

called <u>Nisnevich local coord</u>

- trivialisation V near φ

These must be compatible with relative orientation, meaning distinguished elt

$$\mathrm{Hom}(\underbrace{\Lambda^r TX}_{\text{from local coord}}, \underbrace{\Lambda^r V}_{\text{from local triv}})(U)$$

is $\otimes$-square section in $L(U)$

Now, $s = (s_1, \ldots, s_r) \in \mathcal{O}_{X,p}^r$

Choose $(g_1, \ldots, g_r) \in \mathfrak{p}^m \quad m \gg 0$

s.t. $(s_1 + g_1, \ldots, s_r + g_r) \in \mathrm{im}(\varphi^*: \mathcal{O}^r_{\mathbb{A}^r, \varphi(p)} \to \mathcal{O}^r_{X,p})$

(Need finite determinacy local $\mathbb{A}^1$-deg)

Then $\deg_p s := \deg_{\varphi(p)}(s_1+g_1, \ldots, s_r + g_r)$

<u>Prop</u>: well-defined

$$\text{Euler number} = \sum_{\substack{p \in X \\ \text{s.t.} \\ s(p)=0}} \deg_p s$$

<u>Prop</u>

$$\sum_{\substack{p \in X \\ \text{s.t.} \\ s(p)=0}} \deg_p s = \sum_{\substack{p \in X \\ \text{s.t.} \\ s(p)=0}} \deg_p s'$$

when s and s' are sections with only isolated zeros which lie in a family over $\mathbb{A}^1$ of sections with only isolated zeros.

to complete: W^{EKL} in families (Srinivasan? Scheja-Storch)

Harder's thm

other approaches to Euler class

L4 will be on the results of

1) An Arithmetic Count of the Lines on a Smooth Cubic Surface, with Jesse Leo Kass, https://arxiv.org/abs/1708.01175

2) An Arithmetic Count of the Lines Meeting Four Lines in P^3, with Padma Srinivasan, https://arxiv.org/abs/1810.03503

Notes for a talk on the first paper are available on MSRI's website in the handouts for the Hot Topics: Galois Theory of Periods and Applications workshop

https://docs.google.com/viewer?url=http://www.msri.org/workshops/826/schedules/22057/documents/3036/assets/28375