p-adic cohomology: from theory to practice

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These are the notes from a lecture series at the 2007 Arizona Winter School; the first four top-level sections correspond roughly to the individual lectures, and the fifth gives some ideas for the student project associated to the lecture series.

The overall goal of the lectures is to survey some of the ways in which p-adic analytic methods are used to associate meaningful cohomology theories, and associated extra structures, to algebraic varieties over finite fields and local fields. A key theme is the explicit, computable nature of these constructions, which makes them suitable for numerical calculations. For instance, if you ask the computer algebra system `Magma` for the order of the Jacobian of a hyperelliptic curve over a field of small characteristic, this order is computed using p-adic cohomology. The same is true if you ask the system `SAGE` for the p-adic regulator of an elliptic curve over $\mathbb{Q}$, for p a good ordinary prime.

These notes are deliberately somewhat more voluminous than the contents of the actual lectures, which for reasons of time have to be somewhat more sketchy. I've indicated some things that will not be discussed in detail in the lectures as "notes-only".

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1 Algebraic de Rham cohomology

In this section, we introduce the algebraic approach to constructing de Rham cohomology for smooth varieties, as originally introduced by Grothendieck [21] based on ideas of Atiyah and Hodge. Notice I said "varieties" rather than "algebraic varieties"; that's because the same approach works also for complex analytic varieties, and for rigid analytic varieties. In fact, we will take this point of view as foundational for our study of p-adic cohomology.

1.1 Affine varieties

Definition 1.1.1. Let $X = \operatorname{Spec} A$ be an affine variety over a field K of characteristic zero. Let $\Omega^1_{A/K}$ denote the module of Kähler differentials; that is, $\Omega^1_{A/K}$ is the A-module generated by symbols da for $a \in A$, modulo the relations da for $a \in K$, and $d(ab) - a\,db - b\,da$ for $a, b \in A$. This module is finitely generated over A; it has the universal property that for any K-linear derivation $D : A \to M$ into an A-module, there is a unique A-linear map $\psi : \Omega^1_{A/K} \to M$ such that $D = \psi \circ d$.

We assume hereafter that X/K is smooth, which forces $\Omega^1_{A/K}$ to be a locally free A-module. Put

$$\Omega^i_{A/K} = \wedge^i_A \Omega^1_{A/K};$$

that is, $\Omega^i_{A/K}$ is the free A-module generated by symbols $\omega_1 \wedge \cdots \wedge \omega_i$, modulo the relations

$$(a\omega_1 + a'\omega_1') \wedge \omega_2 \wedge \cdots \wedge \omega_i - a\omega_1 \wedge \omega_2 \cdots \wedge \omega_i - a'\omega_1' \wedge \omega_2 \cdots \wedge \omega_i$$

for $a, a' \in A$, and $\omega_1 \wedge \cdots \wedge \omega_i = 0$ whenever two of the factors are equal. In other words, $\Omega^i_{A/K}$ is the i-th *alternating power*, or *wedge power*, of $\Omega_{A/K}$ over A. Beware that the elements of the form $\omega_1 \wedge \cdots \wedge \omega_i$, the so-called *decomposable elements*, are not all of $\Omega^i_{A/K}$; e.g., if $A = k[x_1, x_2, x_3, x_4]$, then $dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \in \Omega^2_{A/K}$ is not decomposable.

The map $d : A \to \Omega^1_{A/K}$ sending a to da induces maps $d : \Omega^i_{A/K} \to \Omega^{i+1}_{A/K}$. Moreover, the composition $d \circ d$ is always zero. We thus have a complex $\Omega^\cdot_{A/K}$, called the *de Rham complex* of A (or X). The cohomology of this complex is called the *(algebraic) de Rham cohomology* of A, or X, denoted $H^i_{\mathrm{dR}}(X)$. Note that this vanishes above the dimension of X, because the complex itself vanishes there.

Example 1.1.2. If $A = K[x_1, \dots, x_n]$ represents an affine space, then

$$H^0(X) = K, \qquad H^i(X) = 0 \quad (i > 0).$$

Terminology 1.1.3. An element of Ω^i is often called an *i-form.* An i-form is *closed* if it is in the kernel of d, and *exact* if it is in the image of d. In this terminology, $H^i_{\mathrm{dR}}(X)$ is the quotient of the space of closed i-forms by the subspace of exact i-forms.

Remark 1.1.4. (Notes-only) There is a construction of algebraic de Rham cohomology that allows affines which are not smooth (and non-affines by patching, as in §1.3). Roughly, you embed your non-smooth affine scheme as a closed subscheme of a smooth affine, then compute with the de Rham complex on the formal neighborhood of the image. See [24] for a fuller derivation.

1.2 Example: an elliptic curve

Example 1.2.1. Put

$$A = K[x, y]/(y^2 - x^3 - ax - b),$$

and assume that A is smooth, i.e., $P(x) = x^3 + ax + b$ has no repeated roots. That is, X is the affine part of an elliptic curve.

Then $H^0_{\mathrm{dR}}(X)$ is one-dimensional, generated by 1, and $H^i_{\mathrm{dR}}(X)$ vanishes for $i > 1$. To calculate $H^1_{\mathrm{dR}}(X)$, we use the relation

$$2y\, dy = (3x^2 + a)\, dx.$$

Since P has no repeated roots, we can choose polynomials $A, B \in K[x]$ such that $AP + BP' = 1$. Now put

$$\omega = Ay\, dx + 2B\, dy$$

so that

$$dx = y\omega, \qquad dy = \frac{P'}{2}\omega.$$

Consequently, we can write every 1-form on X uniquely as $(C + Dy)\,\omega$ for some $C, D \in K[x]$. For this form to be exact, it must be of the form $d(E + Fy)$ for some $E, F \in K[x]$. But

$$\begin{aligned} d(E + Fy) &= E'\,dx + F'y\,dx + F\,dy \\ &= \left(\frac{1}{2}P'F + E'y + F'P\right)\omega. \end{aligned}$$

Hence any $Dy\omega$ is exact. As for $C\omega$, if F has leading term cx^d, then $\frac{1}{2}P'F + F'P$ has leading term $\left(\frac{3}{2} + d\right) cx^{d+2}$. We can thus clear all terms except those of degree ≤ 1, and this implies that

$$\omega, x\omega$$

form a basis of $H^1_{\mathrm{dR}}(X)$.

Note that by writing

$$\omega = \frac{dx}{y} = \frac{2\,dy}{3x^2 + a},$$

we can see that ω actually extends to a 1-form on the complete elliptic curve, whereas $x\omega$ has a double pole at infinity.

Exercise 1.2.2. Repeat this derivation for the affine part of a hyperelliptic curve $y^2 = P(x)$. Note that the net result varies slightly depending on whether $\deg(P)$ is odd or even; see Exercise 1.3.8.

Exercise 1.2.3. Let $P(x) \in K[x]$ be a squarefree polynomial. Compute $H^i_{\mathrm{dR}}(X)$ for the punctured affine line $X = \operatorname{Spec} K[x, y]/(yP(x) - 1)$.

1.3 Sheaf cohomology

Definition 1.3.1. Let X be a variety over a field K of characteristic zero, which is still smooth but not necessarily affine. Now $\Omega^1_{X/K}$ and the other $\Omega^i_{X/K}$ are not modules, but locally free coherent sheaves of $\mathcal{O}_X$-modules; however, they still form a complex using the exterior derivative maps d, and we would like to look at kernel mod image like we did before.

The right thing to do is to define $H^i_{\mathrm{dR}}(X)$ as the *hypercohomology* $\mathbb{H}^i(\Omega^{\cdot}_{X/K})$ of the complex $\Omega^{\cdot}_{X/K}$. At a suitably abstract level, this is completely straightforward to define: construct a quasi-isomorphism of the given complex to a complex $\mathcal{I}^{\cdot}$ of acyclic sheaves and compute the cohomology of the complex $\Gamma(X, \mathcal{I}^{\cdot})$ of global sections. (We secretly just used the fact that the category of bounded-below complexes of quasicoherent sheaves of $\mathcal{O}_X$-modules has enough injectives, but this follows from the same statement for the category of quasicoherent sheaves of $\mathcal{O}_X$-modules.)

Definition 1.3.2. To actually compute hypercohomology explicitly, we need a particular acyclic resolution, which we produce using a Čech complex. First let's recall how to do this for a single sheaf, rather than a complex. Let $\{U_i\}_{i\in I}$ be a finite cover of X by open affine subschemes. For $j = 0, 1, \dots,$ let I_j be the set of $(j+1)$-element subsets of I. For $J \in I_j$, let X_J be the intersection of the U_i for $i \in J$. For a quasicoherent sheaf $\mathcal{F}$ on X, the corresponding Čech complex $\check{C}^{\cdot}(X, \mathcal{F})$ has j-th term $\prod_{J\in I_j} \Gamma(X_J, \mathcal{F})$, and

$$d((s_J)_{J\in I_j}) = \left(\sum_{i=0}^{j+1}(-1)^i s_{J-\{j_i\}}\right)_{J\in I_{j+1}},$$

where the elements of $J \in I_{j+1}$ are labeled $j_0, \dots, j_{j+1}$ in increasing order. Its cohomology computes the sheaf cohomology $H^i(X, \mathcal{F})$ [25, Theorem III.4.5], [22, Proposition 1.4.1]. (Note: what I really did was take the complex of sheaves, where the j-th term is the sum of the sheaves $(i_J)_* i_J^* \mathcal{F}$ for each $J \in I_j$, and $i_J : X_J \hookrightarrow X$ is the open immersion; this is an acyclic resolution of $\mathcal{F}$.)

Now if you do this with each term of a complex of sheaves $\mathcal{F}^{\cdot}$, what you end up with is a *double complex.* For the de Rham complex, this looks like

$$\begin{tikzcd}
\check{C}^0(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \check{C}^1(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \check{C}^2(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \cdots \\
\check{C}^0(X, \Omega^1_{X/K}) \arrow[r] \arrow[d] & \check{C}^1(X, \Omega^1_{X/K}) \arrow[r] \arrow[d] & \check{C}^2(X, \Omega^1_{X/K}) \arrow[r] \arrow[d] & \cdots \\
\check{C}^0(X, \Omega^2_{X/K}) \arrow[r] \arrow[d] & \check{C}^1(X, \Omega^2_{X/K}) \arrow[r] \arrow[d] & \check{C}^2(X, \Omega^2_{X/K}) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Let d_X and $\check{d}$ denote the vertical and horizontal maps, respectively. To take cohomology here, you take cohomology of the total complex in which

$$C^i = \oplus_{j+k=i} \check{C}^j(X, \Omega^k_{X/K}),$$

with the differentials

$$d((\omega_{j,k})_{j+k=i}) = (d_X(\omega_{j,k-1}) + (-1)^j \check{d}(\omega_{j-1,k}))_{j+k=i+1}.$$

(Notice you have to twiddle the sign of the maps in one direction, in order to get $d \circ d = 0$.)

Example 1.3.3. Let X be the complete elliptic curve defined by

$$y^2 = x^3 + ax + b,$$

i.e., the Zariski closure in $\mathbb{P}^2$ of the affine curve. Let U be the affine curve, i.e., X minus the point at infinity $[0:1:0]$. Let V be X minus the three points of the form $(x, 0)$. The everywhere holomorphic differential $\omega = dx/y$ defines an element of $H^1_{\mathrm{dR}}(X)$, which might make you think this space is one-dimensional. But you would be incorrect!

Remember that elements of $H^1_{\mathrm{dR}}(X)$ are represented by triples (ω_U, ω_V, f), where ω_U is a 1-form on U, ω_V is a 1-form on V, and f is a 0-form on $U \cap V$. The differential takes this triple to the 1-form $\omega_V - \omega_U - df$ on $U \cap V$, so we want this to be zero. One way to get this is to take $\omega_U = \omega_U = \omega$ and $f = 0$, but there are others. In fact, $H^1_{\mathrm{dR}}(X)$ is two-dimensional, which matches what you get for topological de Rham cohomology. That's no accident; see Remark 1.4.4.

Exercise 1.3.4. Construct an element of $H^1_{\mathrm{dR}}(X)$ represented by a triple (ω_U, ω_V, f) with $\omega_U = x\omega$, and show that your element is linearly independent from ω in $H^1_{\mathrm{dR}}(X)$. (Hint: since $x\omega$ has a double pole at $[0:1:0]$, your f will have to have a single pole there.)

Exercise 1.3.5. (Tricky) Prove that $H^2_{\mathrm{dR}}(X) \neq 0$ by exhibiting a 1-form on $U \cap V$ which is not the differential of a triple (ω_U, ω_V, df). Hint: first define the *residue* of a 1-form at a point on a curve.

Exercise 1.3.6. Prove that the space $H^i_{\mathrm{dR}}(\mathbb{P}^n_K)$ is one-dimensional if $i = 0, 2, \ldots, 2n$ and zero otherwise. (This may be easier to do using a spectral sequence; see the appendix to this lecture.)

Remark 1.3.7. There is an excision exact sequence in algebraic de Rham cohomology. If X is a smooth K-variety, Z is a smooth subvariety of pure codimension d, and $U = X \setminus Z$, then

$$\cdots \to H^{i-2d}_{\mathrm{dR}}(Z) \to H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}}(U) \to H^{i-2d+1}_{\mathrm{dR}}(Z) \to \cdots$$

Note the shift by the codimension of Z. Something like this holds even if Z is not smooth, but you have to make sense of the term corresponding to Z (as in Remark 1.1.4), and I don't plan to here.

Exercise 1.3.8. Use the previous remark to show that, if C is a smooth projective curve over K, Z is a nonempty zero-dimensional subscheme of length d, and $U = C \setminus Z$, then

$$\dim_K H^1_{\mathrm{dR}}(U) = \dim_K H^1_{\mathrm{dR}}(C) + d - 1.$$

The quantity $\dim_K H^1_{\mathrm{dR}}(C)$ will turn out to be twice the genus of C.

1.4 Comparisons via GAGA

Note: this subsection is notes-only except for Remark 1.4.4.

Serre's "GAGA principle" asserts that anything you want to compute about coherent sheaves on a smooth proper algebraic variety can be computed on a corresponding analytic variety. (GAGA is an acronym for "Géométrie algébrique et géométrie analytique", the title of Serre's paper [39] introducing this idea.)

Theorem 1.4.1 (Complex-analytic GAGA)**.** *Let X be a smooth proper variety over $\mathbb{C}$, and let X^{an} be the corresponding analytic variety.*

(a) Any coherent sheaf on X^{an} is the analytification of a coherent sheaf on X.

(b) For any coherent sheaves $\mathcal{E}, \mathcal{F}$ on X, any morphism $\mathcal{E}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$ is induced by a morphism $\mathcal{E} \to \mathcal{F}$.

(c) For any coherent sheaf $\mathcal{E}$ on X, with analytification $\mathcal{E}^{\mathrm{an}}$, the natural maps $H^i(X, \mathcal{E}) \to H^i(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$ are bijections.

Proof. For X projective, proceed directly to Serre's original paper [39]. For the general case (which reduces to the projective case using Chow's lemma), see [23, Exposé XII]. □

Theorem 1.4.2 (Rigid-analytic GAGA)**.** *Let K be a complete nonarchimedean field. Let X be a smooth proper variety over K, and let X^{an} be the corresponding rigid analytic variety.*

(a) Any coherent sheaf on X^{an} is the analytification of a coherent sheaf on X.

(b) For any coherent sheaves $\mathcal{E}, \mathcal{F}$ on X, any morphism $\mathcal{E}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$ is induced by a morphism $\mathcal{E} \to \mathcal{F}$.

(c) For any coherent sheaf $\mathcal{E}$ on X, with analytification $\mathcal{E}^{\mathrm{an}}$, the natural maps $H^i(X, \mathcal{E}) \to H^i(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$ are bijections.

Proof. The proof in the complex analytic case carries over directly, given the following inputs. (Thanks to Ravi Vakil for suggesting this list.)

- The ring of analytic functions on a closed unit polydisc is noetherian.
- On an affinoid space (a closed analytic subspace of a closed unit polydisc), coherent sheaves are acyclic for finite coverings by affinoid subspaces, and are represented by finitely generated modules (i.e., the theorems of Kiehl and Tate).
- The analytic projective n-space is proper in the sense of Kiehl, so the H^1 of a coherent sheaf on that space is finite dimensional.
- The structure sheaf on analytic projective n-space is acyclic.

□

Remark 1.4.3. One important corollary of GAGA is that (in either the complex or rigid setting) any closed analytic subvariety of a smooth proper algebraic variety (e.g., projective space) is itself algebraic.

Remark 1.4.4. For our purposes, the key corollary of GAGA (due to Grothendieck) is that for X smooth over $\mathbb{C}$, the algebraic de Rham cohomology groups of X are naturally isomorphic to the holomorphic de Rham cohomology. (For X not proper, this follows from the proper case by excision, after applying resolution of singularities; it also follows from Theorem 1.5.4 below.) For X smooth proper, by Dolbeaut's theorem, the holomorphic de Rham cohomology groups are isomorphic to topological de Rham cohomology with complex coefficients; by de Rham's theorem, those are isomorphic to Betti cohomology with complex coefficients. In particular, the dimensions of the $H^i_{\mathrm{dR}}(X)$ coincide with the topological Betti numbers.

1.5 Cohomology with logarithmic singularities

Definition 1.5.1. By a *smooth (proper) pair* over a base S, I will mean a pair (X, Z) in which X is a smooth (proper) scheme over S and Z is a relative (to S) strict normal crossings divisor. Over a field, this means each component of Z is smooth (no self-intersections allowed; that's the "strict" part), and the components of Z always meet transversely. In general, we can say that étale locally (over S) X should look like an affine space and Z should look like an intersection of coordinate hyperplanes.

Example 1.5.2. For instance, if X is a smooth proper curve over $\mathbb{Z}_p$, then you can form a smooth proper pair (X, Z) by taking Z to be the Zariski closure in X of a set of closed points of $X_{\mathbb{Q}_p}$ which have *distinct* images in $X_{\mathbb{F}_p}$.

Definition 1.5.3. Let (X, Z) be a smooth pair over K. Put $U = X \setminus Z$ and let $j : U \hookrightarrow X$ be the implied open immersion. The *sheaf of logarithmic differentials* on X, denoted $\Omega^1_{(X,Z)/K}$, is the subsheaf of $j_*\Omega^1_{U/K}$ generated by $\Omega^1_{X/K}$ and by sections of the form df/f, where f is a regular function on some open subset V of X which only vanishes along components of Z.

Again, we write $\Omega^i_{(X,Z)/K}$ for the i-th exterior power of $\Omega^1_{(X,Z)/K}$ over $\mathcal{O}_X$.

Theorem 1.5.4. *The obvious map of complexes*

$$\Omega^{\cdot}_{(X,Z)/K} \to j_*\Omega^{\cdot}_{U/K}$$

is a quasi-isomorphism, i.e., it induces isomorphisms on cohomology sheaves. Hence we obtain an isomorphism $\mathbb{H}^i(\Omega^{\cdot}_{(X,Z)/K}) \cong H^i_{\mathrm{dR}}(U)$ from the spectral sequence that goes from cohomology sheaves to hypercohomology (see §1.7).

Remark 1.5.5. Theorem 1.5.4 was originally proved by Deligne [12] using the GAGA principle. It is possible, and important for applications to p-adic cohomology, to give a completely algebraic proof, e.g., [1, Theorem 2.2.5].

Exercise 1.5.6. Prove Theorem 1.5.4 directly for the example in Exercise 1.2.3.

Remark 1.5.7. One might be tempted to deduce from Theorem 1.5.4 that for a smooth projective curve X and a point $x \in X(K)$, for $U = X \setminus \{x\}$, every class in $H^1_{\mathrm{dR}}(X) \cong H^1_{\mathrm{dR}}(U)$ is represented by a 1-form on U with a logarithmic singularity at x. This is false; for instance, in Example 1.3.3, you need to allow either a double pole at one point (to pick up $x\omega$), or poles at two different points. (This is because the sum of the residues of the poles of a 1-form is always zero.) A related observation is that $\Omega^{\cdot}_{(X,Z)/K}$ is a sheaf on X, not on U, so that even if U is affine one must use hypercohomology to compute $\Omega^{\cdot}_{(X,Z)/K}$.

1.6 Example: a smooth hypersurface (notes-only)

Example 1.6.1. Let X be a smooth hypersurface in $\mathbb{P}^n$ defined by the homogeneous polynomial $P(x_1, \dots, x_n)$. Then there are natural maps $H^i_{\mathrm{dR}}(\mathbb{P}^n_K) \to H^i_{\mathrm{dR}}(X)$ which by the Lefschetz hyperplane theorem [19, §1.2] are isomorphisms for $i < n-1$ and injective for $i = n-1$. (That's actually a fact about complex manifolds, but by GAGA it transfers to the algebraic setting.) Since the cohomology of projective space is simple (Exercise 1.3.6), the only interesting cohomology group of X is $H^{n-1}_{\mathrm{dR}}(X)$.

There is a short exact sequence

$$0 \to \Omega^{\cdot}_{\mathbb{P}^n/K} \to \Omega^{\cdot}_{(\mathbb{P}^n,X)/K} \stackrel{\mathrm{Res}}{\to} j_*\Omega^{\cdot+1}_{X/K} \to 0,$$

where $j : X \to \mathbb{P}^n$ is the implied closed immersion. The map Res is a residue map, which can be described as follows: locally on $\mathbb{P}^n$, a section of $\Omega^{\cdot}_{(\mathbb{P}^n,X)/K}$ can be written as $df/f \wedge \omega$, where f is a dehomogenized form of P; Res takes this section to the restriction of ω to Z. Taking cohomology, and using Deligne's theorem, gives a long exact sequence

$$\cdots \to H^i_{\mathrm{dR}}(\mathbb{P}^n) \to H^i_{\mathrm{dR}}(U) \to H^{i-1}_{\mathrm{dR}}(X) \to H^{i+1}_{\mathrm{dR}}(\mathbb{P}^n) \to \cdots,$$

where $U = \mathbb{P}^n \setminus X$.

Upshot: if n is even, then $H^n_{\mathrm{dR}}(U)$ is isomorphic to $H^{n-1}_{\mathrm{dR}}(X)$. If n is odd, then (using Poincaré duality) $H^n_{\mathrm{dR}}(U)$ is isomorphic to the quotient of $H^{n-1}_{\mathrm{dR}}(X)$ by the (one-dimensional) image of the map $H^{n-1}_{\mathrm{dR}}(\mathbb{P}^n) \to H^{n-1}_{\mathrm{dR}}(X)$, the so-called *primitive middle cohomology* of X.

The point is that U is affine, so you can compute its de Rham cohomology on global sections. For the recipe for doing this easily, see Griffiths [18, §4, 5]; see also §5.4.

1.7 Appendix: spectral sequences (notes-only)

The notion of a spectral sequence is a generalization of the long exact sequence

$$\cdots \to H^i(C^{\cdot}_1) \to H^i(C^{\cdot}) \to H^i(C^{\cdot}_2) \stackrel{\delta}{\to} H^{i+1}(C^{\cdot}_1) \to \cdots$$

associated to a short exact sequence of complexes

$$0 \to C^{\cdot}_1 \to C^{\cdot} \to C^{\cdot}_2 \to 0.$$

The relevance of spectral sequences is described by [19, §3.5] (where you should be looking for more details) better than I ever could: "[to] someone who works with cohomology, they are essential in the same way that the various integration techniques are essential to a student of calculus."

Definition 1.7.1. A *filtered complex* is a decreasing sequence of complexes

$$C^{\cdot} = F^0C^{\cdot} \supseteq F^1C^{\cdot} \supseteq \cdots \supseteq F^nC^{\cdot} \supseteq F^{n+1}C^{\cdot} = 0;$$

given a filtered complex, the *associated graded complex* is

$$\operatorname{Gr} C^{\cdot} = \oplus_{p\geq 0} \operatorname{Gr}^p C^{\cdot}, \qquad \operatorname{Gr}^p C^{\cdot} = \frac{F^pC^{\cdot}}{F^{p+1}C^{\cdot}}.$$

For instance, a short exact sequence as above gives you a filtration with $F^2C^{\cdot} = 0$, $\operatorname{Gr}^1 C^{\cdot} \cong F^1C^{\cdot} = C_1^{\cdot}$, and $\operatorname{Gr}^0 C^{\cdot} = C_2^{\cdot}$.

Example 1.7.2. For instance, if you start with a double complex $D^{p,q}$ and flatten it into a single complex $C^i = \oplus_{p+q=i}D^{p,q}$, we can filter this by taking

$$F^pC^i = \bigoplus_{p'+q=i, p'\geq p} D^{p',q}.$$

We can also flip p and q to get a second filtration; this will give two distinct spectral sequences computing the cohomology of $C^{\cdot}$.

As payback for being easy to use, spectral sequences sacrifice the computation of the entire cohomology of filtered complexes. Instead, they only computed the graded pieces of a certain filtration on the cohomology.

Definition 1.7.3. Let Z^q and B^q be the cocycles and coboundaries in a filtered complex C^q. The *filtered cohomology* is

$$F^pH^q(C^{\cdot}) = \frac{F^pZ^q}{F^pB^q} = \operatorname{image}(H^q(F^pC^{\cdot}) \to H^q(C^{\cdot}));$$

note that the map $H^q(F^pC^{\cdot}) \to H^q(C^{\cdot})$ need not be injective. The *associated graded cohomology* is

$$\operatorname{Gr} H^{\cdot}(C^{\cdot}) = \oplus_{p,q} \operatorname{Gr}^p H^q(C^{\cdot}), \qquad \operatorname{Gr}^p H^q(C^{\cdot}) = \frac{F^pH^q(C^{\cdot})}{F^{p+1}H^q(C^{\cdot})}.$$

Definition 1.7.4. A *spectral sequence* is a sequence $\{E_r, d_r\}_{r=r_0}^{\infty}$, where each E_r is a bigraded group

$$E_r = \oplus_{p,q\geq 0}E_r^{p,q}$$

and

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \qquad d_r^2 = 0$$

is a differential such that

$$H^{p,q}(E_r) = E_{r+1}^{p,q}.$$

If at some point $E_r = E_{r+1} = \cdots$, we call this stable value the *limit* of the spectral sequence, denoted E_∞. One also says that the sequence *degenerates at E_r*, and that the sequence *converges to E_∞.*.

Pictures of spectral sequences speak louder than words:

$$
\begin{tikzcd}
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} \\
E_0^{0,0} \arrow[u, "d_0"] & E_0^{1,0} \arrow[u, "d_0"] & E_0^{2,0} \arrow[u, "d_0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
E_1^{0,1} \arrow[r, "d_1"] & E_1^{1,1} \arrow[r, "d_1"] & E_1^{2,1} \\
E_1^{0,0} \arrow[r, "d_1"] & E_1^{1,0} \arrow[r, "d_1"] & E_1^{2,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2^{0,1} \arrow[drr, "d_2"] & E_2^{1,1} & E_2^{2,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0}
\end{tikzcd}
$$

Theorem 1.7.5. *Let $F^pC^\cdot$ be a filtered complex. Then there is a spectral sequence $\{E_r\}_{r=0}^\infty$ with*

$$
\begin{aligned}
E_0^{p,q} &= \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}} \\
E_1^{p,q} &= H^{p+q}(\operatorname{Gr}^p C^\cdot) \\
E_\infty^{p,q} &= \operatorname{Gr}^p H^{p+q}(C^\cdot).
\end{aligned}
$$

Proof. See [19, §3.5] or [6, §14]. □

Terminology 1.7.6. In the previous theorem, one writes

$$E_r \Rightarrow H^\cdot(C^\cdot)$$

and says that the spectral sequence *abuts to* $H^\cdot(C^\cdot)$.

Example 1.7.7. For instance, in the example of a short exact sequence, we have

$$E_1^{0q} = H^q(C_2^\cdot), \quad E_1^{1q} = H^{q+1}(C_1^\cdot),$$

$d_1 : H^q(C_2^\cdot) \to H^{q+1}(C_1^\cdot)$ is the connecting homomorphism δ, and $d_2 = d_3 = \cdots = 0$ because the arrows always have a zero at one or both endpoints. For a filtered complex with $F^{n+1} = 0$, we similarly have $E_\infty = E_{n+1}$.

Remark 1.7.8. A map between filtered complexes is a quasi-isomorphism if the same is true at any single stage of the spectral sequence (and conversely if the sequence degenerates at some stage, which it must if either the complex or the filtration is finite). For instance, in the example of a short exact sequence, this is an instance of the five lemma.

Example 1.7.9. Go back to the double complex used to define the hypercohomology of the de Rham complex. As noted in Example 1.7.2, we can turn it into a filtered complex in two different ways. One way, we get a spectral sequence where we first compute the sheaf cohomology of each $\Omega^\cdot_{X/K}$, i.e.,

$$E_1^{pq} = H^q(X, \Omega^p_{X/K});$$

this is the *Hodge-de Rham spectral sequence*, and the filtration it determines on $H^i_{\mathrm{dR}}(X)$ is called the *Hodge filtration.* It is a deep theorem that the Hodge-de Rham spectral sequence degenerates already at E_1; this was originally established using analytic techniques, but can also be proved algebraically [13].

The other way, we get a spectral sequence with

$$E_1^{pq} = \prod_{J \in U_p} H^q_{\mathrm{dR}}(X_J).$$

This looks like it reconstructs $H^i_{\mathrm{dR}}(X)$ from the $H^i_{\mathrm{dR}}(X_J)$ (which we know we can do using Čech complexes), but in general spectral sequences can't do that: the sequence only abuts to $H^i_{\mathrm{dR}}(X)$, so all you get are the graded pieces of some filtration. Nonetheless, this can be quite useful.

2 Rigid cohomology

In this section, we define rigid (as in rigid analytic) cohomology for smooth varieties over fields of positive characteristic, and sketch the proof of the Lefschetz trace formula (due to Monsky). We then state the comparison theorem with algebraic de Rham cohomology for liftable smooth proper varieties, and explain how this can be used to compute zeta functions.

2.1 Monsky-Washnitzer's formal cohomology

Notation 2.1.1. Throughout this lecture, let k be a perfect field of characteristic $p > 0$. Let W be the ring of Witt vectors of k; it is the unique complete discrete valuation ring with residue field k and maximal ideal (p). Let K be the fraction field of W; it will be the coefficient field for Monsky-Washnitzer cohomology. We will later insist that $k = \mathbb{F}_q$ be finite, and write $\mathbb{Z}_q, \mathbb{Q}_q$ instead of W, K; note that $\mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$.

Remark 2.1.2. (Notes-only) The restriction to unramified p-adic fields is largely for ease of exposition; one could repeat everything here with W replaced by a totally ramified extension, except that to talk about Frobenius, one should equip W with a lift of the appropriate q-power Frobenius on k. If I were to be more precise about integral p-adic cohomology, then restricting to unramified fields would also skirt some badnesses that arise when the absolute ramification index is too large (usually meaning bigger than $p-1$).

We start out considering smooth affine varieties over k. The theory of "formal cohomology" in this setting was introduced by Monsky and Washnitzer [37, 35, 36], based on ideas introduced by Dwork in his proof of the rationality of zeta functions of varieties over finite fields [14]. (Historical note: Dwork's proof predated Grothendieck's proof using étale cohomology by about 10 years.) Nowadays, we usually refer to this construction as *Monsky-Washnitzer cohomology*, or just *MW cohomology*. (See [40] for a useful overview.)

Definition 2.1.3. Let $\overline{X} = \operatorname{Spec} \overline{A}$ be a smooth affine variety over k. One could compute algebraic de Rham cohomology of $\overline{X}$, but the result would be disastrous; already H^0 would be huge, consisting of all of $\overline{A}^p$. The idea of Monsky-Washnitzer is to lift the picture to W and take cohomology there, but to try to do so in a way that stays functorial in the characteristic p data.

To start with, lift $\overline{A}$ to a smooth affine scheme A over W; that is, A is smooth affine over W and $A \otimes_W k \cong \overline{A}$. The existence of such a lift is guaranteed by a theorem of Elkik [16], but it is hopelessly non-functorial; once you choose lifts A, B of $\overline{A}, \overline{B}$, you usually cannot find any lifts of a given map $\overline{A} \to \overline{B}$.

Definition 2.1.4. To fix this, replace A by its p-adic completion $\hat{A}$. Then a "henselian" argument of Grothendieck (exposed nicely by Arabia [2]) shows that if again you choose lifts A, B of two given smooth affines $\overline{A}, \overline{B}$, you can lift any map $\overline{A} \to \overline{B}$ to a map $\hat{A} \to \hat{B}$, though not canonically. For instance, if $\overline{A} = \overline{B} = k[\overline{x}]$ and $A = B = W[x]$, then $\hat{A}$ is the ring $W\langle x \rangle$ of power series whose coefficients tend p-adically to 0, and you can lift a map $\overline{x} \mapsto \overline{P}(\overline{x})$ by sending x to any lift of $\overline{P}(\overline{x})$.

What you now want to do is take algebraic de Rham cohomology of the generic fibre $\hat{A} \otimes_W K$, but this is not a good idea either; you still get very large spaces.

Example 2.1.5. If $\overline{A} = k[\overline{x}]$ and $\hat{A} = W\langle x \rangle$, the 1-form

$$\sum_{n=0}^{\infty} p^n x^{p^n - 1}\, dx$$

is closed but not exact, so the proposed H^1 of the affine line does not vanish.

The fix is to replace the completion $\hat{A}$ by a smaller *weak completion* $A^\dagger$ that is big enough to have the good lifting properties, but is small enough to give meaningful de Rham cohomology.

Definition 2.1.6. Let $\overline{x}_1, \dots, \overline{x}_n$ be generators of $\overline{A}$ over k, i.e., choose a surjection $k[\overline{x}_1, \dots, \overline{x}_n] \to \overline{A}$. Choose lifts $x_1, \dots, x_n$ of $\overline{x}_1, \dots, \overline{x}_n$ to A, so that there is a map $W[x_1, \dots, x_n] \to A$ which may or may not be a surjection. In any case, the image is p-adically dense, so you get a surjection $W\langle x_1, \dots, x_n \rangle \to \hat{A}$ by taking p-adic completions. Here

$$W\langle x_1, \dots, x_n \rangle$$

is the set of formal power series $\sum_I c_I x^I$, with I running over n-tuples of nonnegative integers and $c_I \in W$ for all I, such that $c_I \to 0$ as $I \to \infty$. (That is, for any integer n, there are only finitely many indices I such that c_I is not divisible by p^n.)

We are going to replace $W\langle x_1, \dots, x_n \rangle$ with a dense subring $W\langle x_1, \dots, x_n \rangle^\dagger$: it will consist of series $\sum_I c_I x^I$ for which there exist constants a, b with $a > 0$, such that

$$v(c_I) \geq a(i_1 + \cdots + i_n) + b \qquad (I \in \mathbb{Z}^n_{\geq 0}).$$

(The constants a, b may vary from series to series; this will be crucial in a moment.) Let $A^\dagger$ denote the image of $W\langle x_1, \dots, x_n \rangle^\dagger$ in $\hat{A}$; it is called the *weak completion* of A. It can be shown that it depends only on A and not on the choice of the generators $x_1, \dots, x_n$. Also, one has an analogue of Grothendieck's lifting theorem: any map $\overline{A} \to \overline{B}$ lifts (nonuniquely) to a map $A^\dagger \to B^\dagger$.

We define a module of differentials $\Omega^1_{A^\dagger/W}$ generated by $dx_1, \dots, dx_n$ modulo the relations you expect: if $f(x_1, \dots, x_n)$ is a series which vanishes in A, then the formal total derivative

$$\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

vanishes in Ω^1. We "define" the Monsky-Washnitzer cohomology of X with coefficients in K, denoted $H^i_{\mathrm{MW}}(X)$, to be the cohomology of the resulting de Rham complex $\Omega^i_{A^\dagger/W} \otimes_W K$, i.e., *after tensoring with* K. (Without inverting p, there would be a lot of torsion whose geometric meaning is not very clear, at least not to me.)

To remove the quotation marks, one must check that given any two lifts $f_1^\dagger, f_2^\dagger : A^\dagger \to B^\dagger$ of the same map $\overline{f} : \overline{A} \to \overline{B}$, you get the same induced maps $H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{A}) \to H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{B})$. In fact something better is true: the two maps $\Omega^\cdot_{A^\dagger/W} \otimes K \to \Omega^\cdot_{B^\dagger/W} \otimes K$ are homotopic [37, Theorem 5.5]. (I don't know whether this is also true *before* tensoring with K.) This first tells you that you get a well-defined map $H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{A}) \to H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{B})$, second that the passage to cohomology is functorial (it commutes with composition), and third that $H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{A})$ is canonically independent of the choice of the lift (if $\overline{A} \cong \overline{B}$, the identity map lifts canonically to an isomorphism $H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{A}) \cong H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{B})$).

Remark 2.1.7. Series in $W\langle x_1, \dots, x_n\rangle^\dagger$ are sometimes said to be *overconvergent*, because each such series converges on a polydisc of the form $|x_i| \leq \eta$ for some η strictly bigger than 1. (Remember: in rigid geometry it makes sense to take η to be any fractional power of p, because you should allow yourself to take points over extension fields of K. In Berkovich's foundations, η can be any positive real number.) By contrast, elements of $W\langle x_1, \dots, x_n\rangle$ are only guaranteed to converge on the polydisc $|x_i| \leq 1$.

Remark 2.1.8. It would be helpful for certain purposes to be able to construct this theory without having to worry about checking independence from choices. (This will become apparent when we pass to nonaffines.) A canonical construction of Monsky-Washnitzer cohomology, using a variant of the de Rham-Witt complex, was proposed by James Borger; we are currently working out this construction as an ongoing project with Chris Davis.

Remark 2.1.9. (Notes-only) There is a theory parallel to the theory of rigid analytic geometry, using the MW algebra in place of the Tate algebra. This is the theory of *dagger spaces*, developed by Grosse-Klönne; it is related to Meredith's theory of weak formal schemes in the same way that rigid spaces are related to usual formal schemes.

2.2 Example: an elliptic curve

Example 2.2.1. Let's see how this works for the affine part of the elliptic curve $\overline{X} = \operatorname{Spec} \overline{A}$ with

$$\overline{A} = k[\overline{x}, \overline{y}]/(\overline{y}^2 - \overline{x}^3 - \overline{ax} - \overline{b})$$

(we must have $p \neq 2$ for this to be able to be smooth), using a lift of the form

$$A = W[x, y]/(y^2 - x^3 - ax - b).$$

Again, $H^i_{\mathrm{MW}}(\overline{X})$ vanishes for $i \geq 2$, and $H^1_{\mathrm{MW}}(\overline{X})$ consists of the closed 1-forms (generated by dx and dy) modulo the exact ones. Exactly the same calculation as in Example 2.2.1 shows that we can still use $\omega, x\omega$ as a basis; this will turn out to be a special case of the comparison theorem to de Rham cohomology (Theorem 2.5.2).

But this is not the end of the story, as we also want to lift morphisms, particularly the absolute Frobenius $\overline{F}$. It will almost never lift to A, but we know it will lift to $A^\dagger$. Unfortunately, writing down such a lift is a bit complicated.

Example 2.2.2. Following [28] (see also [17, §7]), we can make constructing the Frobenius lift much easier by working with a slightly smaller affine piece of the same curve, namely $\overline{X}_0 = \operatorname{Spec} \overline{A}_0$ with

$$\overline{A}_0 = k[\overline{x}, \overline{y}, \overline{z}]/(\overline{y}^2 - \overline{x}^3 - \overline{ax} - \overline{b}, \overline{yz} - 1)$$

(that is, $\overline{A}$ with $\overline{y}$ inverted), with corresponding lift

$$A_0 = W[x, y, z]/(y^2 - x^3 - ax - b, yz - 1).$$

Then we can define a lift $F^\dagger$ of $\overline{F}$ acting as the Witt vector Frobenius on W and acting on x, y, z as

$$\begin{aligned} x &\mapsto x^p \\ y &\mapsto y^p(1 + (F^\dagger(x^3 + ax + b) - (x^3 + ax + b)^p)z^{2p})^{1/2} \\ z &\mapsto z^p(1 + (F^\dagger(x^3 + ax + b) - (x^3 + ax + b)^p)z^{2p})^{-1/2}. \end{aligned}$$

The point is that in the expressions $(1 + *)^{\pm 1/2}$ I just wrote, the $*$ is divisible by p, so expanding the binomial series gives you something overconvergent. (Word to the wise: an underlying reason why this worked is that $\operatorname{Spec} \overline{A}_0$ is étale over $\mathbb{A}^1_k$, so I can extend the obvious Frobenius lift $x \mapsto x^p$ from $W\langle x \rangle^\dagger$ to $A_0^\dagger$.)

2.3 Nonaffine varieties

We are now going to define the generalization of MW cohomology to smooth but not necessarily affine varieties; this gives *rigid cohomology* as introduced by Berthelot [4]. (Actually, Berthelot also allows singular varieties, but I didn't deal with singularities even in the affine case for algebraic de Rham cohomology.) To avoid some notational unpleasantness, I'm going to only explain the case of a variety covered by two affines $\overline{U}, \overline{V}$ and leave the general case to the reader's imagination.

Definition 2.3.1. Let $\overline{X}$ be a smooth variety over k covered by two open affines $\overline{U}, \overline{V}$. Since smooth implies separated, $\overline{U} \cap \overline{V}$ is also affine.

Choose smooth lifts of $\overline{U}, \overline{V}, \overline{U} \cap \overline{V}$ and make corresponding de Rham complexes over K; out of laziness I'll call these $\Omega^\cdot_{\overline{U}}$, $\Omega^\cdot_{\overline{V}}$, $\Omega^\cdot_{\overline{U} \cap \overline{V}}$. From functoriality, I get morphisms $r_u : \Omega^\cdot_{\overline{U}} \to \Omega^\cdot_{\overline{U} \cap \overline{V}}$ and $r_{\overline{V}} : \Omega^\cdot_{\overline{V}} \to \Omega^\cdot_{\overline{U} \cap \overline{V}}$ (where r stands for "restriction").

We now imitate the Čech construction of hypercohomology: form the complex

$$C^i = \Omega^i_{\overline{U}} \oplus \Omega^i_{\overline{V}} \oplus \Omega^{i-1}_{\overline{U}\cap\overline{V}},$$

with differentials

$$d(\omega_{\overline{U}}, \omega_{\overline{V}}, \omega_{\overline{U}\cap\overline{V}}) = (d(\omega_{\overline{U}}), d(\omega_{\overline{V}}), d(\omega_{\overline{U}\cap\overline{V}}) + r_{\overline{V}}(\omega_{\overline{V}}) - r_{\overline{U}}(\omega_{\overline{U}}))$$

and call its cohomology $H^i_{\mathrm{rig}}(\overline{X})$.

Checking that this is independent of the choices implicit in making the $\Omega^{\cdot}$ terms and the morphisms between them is left as an exercise, which may make you appreciate Remark 2.1.8.

Remark 2.3.2. Beware that the picture with only two affines is a bit misleading: one can glue the two spaces together because there is no cocycle condition to worry about. This becomes a problem with three or more affines, where the cocycle condition only holds up to homotopy on the de Rham complexes.

Exercise 2.3.3. Check that in the situation considered in Definition 2.3.1, the definition of $H^i_{\mathrm{rig}}(\overline{X})$ does not depend on choices. That is, if you make a second set of data $\tilde{\Omega}^{\cdot}_{\overline{U}}$, $\tilde{\Omega}^{\cdot}_{\overline{V}}$, $\tilde{\Omega}^{\cdot}_{\overline{U}\cap\overline{V}}$, $\tilde{r}_{\overline{U}}$, $\tilde{r}_{\overline{V}}$ and a corresponding group $\tilde{H}^i_{\mathrm{rig}}(\overline{X})$, you have a canonical isomorphism $H^i_{\mathrm{rig}}(\overline{X}) \cong \tilde{H}^i_{\mathrm{rig}}(\overline{X})$. (Hint: use functoriality of MW cohomology over and over again. It may be easier to show that the two complexes $C^i, \tilde{C}^i$ are quasi-isomorphic to a third complex than that they are quasi-isomorphic to each other.)

A somewhat difficult theorem, proved independently by Berthelot [5] and Mebkhout [34], yields the following. (This theorem has subsequently been strengthened to allow twisted coefficients [30], about which I'll only have a tiny bit to say in the next lecture.)

Theorem 2.3.4. *The spaces $H^i_{\mathrm{rig}}(\overline{X})$ are finite dimensional for any smooth k-variety $\overline{X}$.*

However, in these lectures, we will only be interested in cases where $\overline{X}$ lifts nicely to characteristic zero. In this case, dealing with rigid cohomology will prove to be much easier, thanks to the comparison theorem with de Rham cohomology (Theorem 2.5.2).

2.4 Excision and the trace formula

There is an excision formula in rigid cohomology parallel to the one in algebraic de Rham cohomology (Remark 1.3.7): if $\overline{X}$ is a smooth k-variety, $\overline{Z}$ is a smooth subvariety of pure codimension d, and $\overline{U} = \overline{X} \setminus \overline{Z}$, then

$$\cdots \to H^{i-2d}_{\mathrm{rig}}(\overline{Z})(d) \to H^i_{\mathrm{rig}}(\overline{X}) \to H^i_{\mathrm{rig}}(\overline{U}) \to H^{i-2d+1}_{\mathrm{rig}}(\overline{Z})(d) \to \cdots .$$

Again, note the shift by the codimension of Z. Beware that the action of Frobenius gets twisted by a factor of q^d.

One really nice use of excision is to prove the Lefschetz trace formula for Frobenius in rigid cohomology, due to Monsky.

Theorem 2.4.1. *Let $\overline{X}$ be a smooth variety of dimension d over $\mathbb{F}_q$, and let F denote the induced action of the q-power (arithmetic) Frobenius of X on $H^i_{\mathrm{rig}}(\overline{X})$. Then for all positive integers n,*

$$\#\overline{X}(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{Trace}(q^{dn}F^{-n}, H^i_{\mathrm{rig}}(\overline{X})).$$

Sketch of proof. It suffices to check for $n = 1$, otherwise we can just replace q by q^n. Using excision, we reduce to the case where $\overline{X} = \operatorname{Spec} \overline{A}$ is affine and has no $\mathbb{F}_q$-rational points. One then observes that for any $a, b \in A^\dagger$, the operator $\phi(a)bF^{-1}$ has the same trace as $bF^{-1}\phi(a) = abF^{-1}$, so $(\phi(a) - a)bF^{-1}$ has trace zero. Since $\overline{X}$ has no $\mathbb{F}_q$-rational points, the $\phi(a) - a$ generate the unit ideal, so I can write F^{-1} as a $\mathbb{Q}_q$-linear combination of operators of the form $(\phi(a) - a)bF^{-1}$. Hence F^{-1} has trace zero. □

2.5 The liftable case: comparison to de Rham cohomology

Life is a bit easier if $\overline{X}$ lifts to a smooth scheme X over W, as then you can use affines in a cover of X to make the MW algebras. Better yet, we can do this in a logarithmic fashion.

Definition 2.5.1. Let (X, Z) be a smooth proper pair over W. Put $U_K = X_K \setminus Z_K$ and $\overline{U} = \overline{X} \setminus \overline{Z}$.

Theorem 2.5.2. *There are canonical isomorphisms $H^i_{\mathrm{dR}}(U_K) \to H^i_{\mathrm{rig}}(\overline{U})$.*

Proof. We get this by giving a canonical isomorphism from the log-de Rham cohomology $H^i_{\mathrm{dR}}(X, Z)$ to $H^i_{\mathrm{rig}}(\overline{U})$. The map is simply to view algebraic forms as overconvergent forms; this gives by rigid GAGA an isomorphism to the log-de Rham cohomology of the analytification of the pair (X, Z). One can then show that the latter agrees with $H^i_{\mathrm{rig}}(\overline{U})$; this was done by Shiho [38, §2.4]. □

Remark 2.5.3. Similarly, if you have a map $\overline{X} \to \overline{Y}$ which lifts to a map $X \to Y$ between smooth lifts, you can use that lifted map to give the functorial morphism $H^i_{\mathrm{rig}}(\overline{Y}) \to H^i_{\mathrm{rig}}(\overline{X})$.

Remark 2.5.4. One consequence of Theorem 2.5.2 is that the de Rham cohomology of a smooth proper variety over K with good reduction admits a canonical Frobenius action, even though the Frobenius map does not itself typically lift. In fact, you also get a Frobenius action even when the reduction is not good; this will be touched upon in Lecture 4.

2.6 Computing zeta functions: an example

Let us put this all together to see how you can compute the zeta function of an elliptic curve using rigid cohomology, as in [28]. (See also Edixhoven's course notes [15].)

Example 2.6.1. One can use the Frobenius lift in Example 2.2.2 to compute the Frobenius action on the de Rham cohomology of an elliptic curve over $\mathbb{Q}_p$ with good reduction. The comparison theorem implies $H^i_{\mathrm{dR}}(\operatorname{Spec} A_0, \mathbb{Q}_p) \cong H^i_{\mathrm{rig}}(\overline{X}_0)$. This isomorphism is respected by the hyperelliptic involution $y \mapsto -y$, and the minus eigenspaces are the de Rham cohomology of the complete curve over $\mathbb{Q}_p$ on one hand, and the rigid cohomology of the complete curve over $\mathbb{F}_p$ on the other.

To compute the Frobenius action on these spaces, first apply Frobenius formally to

$$\frac{dx}{y}, \qquad \frac{x\,dx}{y},$$

where $F(dx) = x^{p-1}\,dx$. In practice you have to truncate modulo some power of p, so that you can write down the coefficients; as a side effect, you truncate the infinite series defining $F(y)$, so you end up with a finite expression

$$\sum_{i=0}^{N} \frac{R_i(x)}{y^{2i+1}}\,dx$$

which you want to rewrite in terms of the basis. You can do this by successively eliminating the terms with the worst poles at $y = 0$: rewrite the numerator as a multiple of $P(x) = x^3 + ax + b$ plus a multiple of its derivative, then use the relations $P(x) = y^2$ and

$$0 \equiv d\left(\frac{S(x)}{y^{2N-1}}\right) = \frac{S'(x)\,dx}{y^{2N-1}} - \frac{(2N-1)S(x)P'(x)\,dx}{2y^{2N+1}}$$

to clear the y^{-2N-1} term. Repeat as needed. (I deliberately ignored issues related to how much initial p-adic precision is needed to get an answer accurate to a given final precision; this is discussed further in [28].)

Remark 2.6.2. Here is an interesting arithmetic application of Example 2.6.1 due to Mazur, Stein, and Tate [33]. Following a suggestion of Katz, they give a formula for the p-adic canonical height of an elliptic curve over $\mathbb{Q}$ in terms of the Frobenius action on the de Rham cohomology over $\mathbb{Q}_p$. (This height is not the Néron local height; it is a global height with p-adic values, which computes the regulator term in Mazur-Tate-Teitelbaum's p-adic analogue of the Birch-Swinnerton-Dyer conjecture.) This computation was implemented in `SAGE` during the summer of 2006 by Jennifer Balakrishnan, Robert Bradshaw, David Harvey, and Liang Xiao.

3 Gauss-Manin connections

In this section, we introduce the notion of a Gauss-Manin connection for a smooth proper morphism (of algebraic, complex analytic, or rigid analytic varieties) and explain how in some cases, such a connection carries a "Frobenius action" that can be used to recover zeta functions.

3.1 Connections in geometry and algebra

Definition 3.1.1. Let V be a vector bundle (or if you prefer, a locally free coherent sheaf) over X, which could be any of: a C^∞ real manifold, a complex manifold, a smooth algebraic variety over a field of characteristic zero, or a smooth rigid analytic variety. (This concept originated with real differential manifolds, but never mind that here.) A *connection* on V is a bundle map $\nabla : V \to V \otimes \Omega^1_X$ which satisfies the *Leibniz rule*: for any open set $U \subseteq X$, any $f \in \Gamma(U, \mathcal{O})$ and $s \in \Gamma(U, V)$,

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

A section s is called *horizontal* if $\nabla(s) = 0$.

Definition 3.1.2. Let $\nabla_1 : V \otimes \Omega^1_X \to V \otimes \Omega^2_X$ be the map

$$s \otimes \omega \mapsto \nabla(s) \wedge \omega + s \otimes d\omega,$$

where $\wedge : (V \otimes \Omega^1_X) \otimes \Omega^1_X \to V \otimes \Omega^2_X$ denotes the map given by wedging the second and third factors. The *curvature* is the map $\nabla_1 \circ \nabla : V \to V \otimes \Omega^2_X$; if it vanishes, we say ∇ is *integrable*. (This is automatic if $\dim(X) = 1$.)

Remark 3.1.3. Here is another way to think about integrability of a connection ∇. Let $z_1, \dots, z_n$ be local coordinates for X at a point x. Then $dz_1, \dots, dz_n$ form a basis of the cotangent bundle Ω^1_X on some neighborhood of x; it admits a dual basis $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ of tangent vector fields. We can contract ∇ with the vector field $\frac{\partial}{\partial z_i}$ to obtain a map $V \to V$ satisfying the Leibniz rule with respect to $\frac{\partial}{\partial z_i}$; you may think of this as an action of $\frac{\partial}{\partial z_i}$ on sections of V. Then ∇ is integrable if and only if the $\frac{\partial}{\partial z_i}$ commute with each other.

Remark 3.1.4. Here is the original differential-geometric interpretation of curvature. In real geometry, you can use a connection ∇ to tell you how to move between fibres of the bundle in a "horizontal" fashion, i.e., parallel to the base. Even in a small neighborhood of a point, moving parallel to different paths on the base leading to the same endpoint can give different results. But if the curvature vanishes, then this discrepancy does not arise; this means that on any contractible neighborhood of $x \in X$, we can write down a basis of V consisting of horizontal sections $s_1, \dots, s_n$, and the connection is given in terms of this basis by

$$\nabla(f_1 s_1 + \cdots + f_n s_n) = s_1 \otimes df_1 + \cdots + s_n \otimes df_n.$$

It also means that given $x \in X$, parallel transport (the process of moving from one fibre to another via a horizontal path) gives a well-defined homomorphism $\rho : \pi_1(X, x) \to \mathrm{GL}(V_x)$, called the *monodromy*. (Differential geometers refer to integrable connections as *flat* connections, but algebraic geometers prefer to give this word another meaning!)

One can take the relationship between integrable connections and monodromy a step further.

Definition 3.1.5. Let X be a connected complex manifold, and choose a point $x \in X$. A *local system* on X is a homomorphism $\rho : \pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C})$. As noted above, there is a natural functor from vector bundles equipped with integrable connections to local systems; it turns out to be an equivalence of categories.

3.2 Gauss-Manin connections

Definition 3.2.1. Let $\pi : X \to S$ be a smooth proper morphism between objects in one of the categories we considered in Definition 3.1.1. We can then define the *relative de Rham cohomology* $H^i_{\mathrm{dR}}(X/S)$ as the higher direct images $\mathbb{R}^i\Omega^{\cdot}_{X/S}$ of the complex of relative differentials. These turn out to be vector bundles on S whose fibres can be identified with the cohomology of the fibres X_b. (Remember that $\Omega^1_{X/S}$ is the quotient of $\Omega^1_{X/K}$ by the pullback of $\Omega^1_{S/K}$.)

Remark 3.2.2. The formation of the relative de Rham cohomology throws away some information: it only uses the "vertical" part of the differential operator d. What this means is that given a relative i-form $\omega \in \Omega^i_{X/S}$, if one lifts ω to an absolute i-form $\tilde{\omega} \in \Omega^i_{X/K}$ and differentiates the result, you may get something nonzero even if ω was a relative cocycle. If one projects the result into $\Omega^i_{X/S} \otimes \Omega^1_{S/K}$, you have essentially constructed the Gauss-Manin connection. We will give a more formal construction below, but the procedure just described is how one really computes the Gauss-Manin connection in practice; see Example 3.4.1.

Definition 3.2.3. Equip the de Rham complex $\Omega^{\cdot}_{X/K}$ with the decreasing filtration

$$F^i = \mathrm{image}[\Omega^{\cdot - i}_{X/K} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{S/K}) \to \Omega^{\cdot}_{X/K}],$$

then form the corresponding spectral sequence (as in Theorem 1.7.5). The E_1 term of the result has

$$E_1^{p,q} = \Omega^p_{S/K} \otimes_{\mathcal{O}_S} H^q_{\mathrm{dR}}(X/S);$$

the *algebraic Gauss-Manin connection* is the differential $d_1 : E_1^{0,q} \to E_1^{1,q}$. This construction was introduced by Katz and Oda [27], who showed that this is an integrable connection, and also that it agrees with the more traditional analytic description for a real or complex manifold.

Definition 3.2.4. Suppose we are working with real or complex manifolds, and that S is contractible. Then the fibration $X \to S$ is trivial in the category of real manifolds, so we get a notion of horizontality for sections of $H^i_{\mathrm{dR}}(X/S)$. If S is not contractible, this gives a connection on $H^i_{\mathrm{dR}}(X/S)$; this is the usual Gauss-Manin connection, and Katz and Oda showed that it agrees with their algebraic construction.

3.3 Connections and differential equations (notes-only)

It is worth pointing out that for purposes of explicit calculations, it is common to work with differential equations instead of connections. This discussion is formal, so it works in any of the categories we allowed when defining connections.

Definition 3.3.1. Suppose, for simplicity, that we are given a trivial vector bundle V of rank n over a subspace S of the t-line, and that V comes equipped with a connection, or equivalently, with an action of $\frac{d}{dt}$. A *cyclic vector* is a section s of V such that

$$s, \frac{d}{dt}s, \dots, \frac{d^{n-1}}{dt^{n-1}}s$$

form a basis of V. Given a cyclic vector, we can describe horizontal sections as follows. We can write

$$\frac{d^n}{dt^n}s = a_0 s + a_1 \frac{d}{dt}s + \cdots + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}s$$

for certain functions $a_0, \dots, a_{n-1}$ on S. Consider an undetermined section v of V, which must be given by

$$v = f_0 s + f_1 \frac{d}{dt}s + \cdots + f_{n-1}\frac{d^{n-1}}{dt^{n-1}}s$$

for certain functions $f_0, \dots, f_{n-1}$ on t. For v to be horizontal, we need

$$\begin{aligned}
0 &= f_0' + f_{n-1}a_0 \\
0 &= f_1' + f_0 + f_{n-1}a_1 \\
&\vdots \\
0 &= f_{n-1}' + f_{n-2} + f_{n-1}a_{n-1}.
\end{aligned}$$

Eliminating $f_0, \dots, f_{n-1}$ leaves a differential equation of the form

$$f_{n-1}^{(n)} + b_1 f_{n-1}^{(n-1)} + \cdots + b_{n-2}f_{n-1}' + b_{n-1}f_{n-1} = 0.$$

Conversely, one can turn the differential equation into a first-order differential system in the usual fashion, and thus reconstruct ∇.

Remark 3.3.2. You should imagine the above construction as being a differential analogue of the passage from a matrix to its characteristic polynomial, which can be reversed (up to similarity) by forming the companion matrix of a polynomial.

Remark 3.3.3. The differential equations corresponding to Gauss-Manin connections were introduced long before anyone had defined a connection, and so they have their own name. They are known as *Picard-Fuchs equations*; they arose by taking a homology class across different fibres and integrating against a fixed differential form on the total space. A number of classical differential equations (e.g., hypergeometric equations) arose in this fashion.

3.4 Example: a family of elliptic curves

Example 3.4.1. Consider the family of smooth affine curves $\pi : X \to S$ given by the affine equation

$$y^2 = P(x,t) = x^3 + a(t)x + b(t)$$

over the subspace S of the t-line, where $\Delta(t) = 4a^2 + 27b^3$ vanishes nowhere on S. Then there exist $A, B \in \Gamma(S, \mathcal{O}_S)[x]$ such that

$$AP + BP_x = 1,$$

using subscripted x and t for the partial derivatives in x and t, respectively. Put

$$\omega = Ay\,dx + 2B\,dy,$$

so that a basis for $H^1_{\mathrm{dR}}(X/S)$ is given by

$$\omega, x\omega.$$

The bundle $H^1_{\mathrm{dR}}(X/S)$ is the same as the relative de Rham cohomology of the corresponding family of projective curves, but (no surprise) it is easier to compute the Gauss-Manin connection on the affine family.

In the relative module of differentials $\Omega^1_{X/S}$, i.e., modulo dt, we have as before the relation

$$2y\,dy = P_x\,dx,$$

and again

$$dx = y\omega, \qquad dy = \frac{1}{2}P_x\omega.$$

But in the full module $\Omega^1_{X/K}$, the relation lifts to

$$2y\,dy = P_x\,dx + P_t\,dt,$$

and it is this discrepancy that gives rise to the connection. It follows that

$$\begin{aligned} dx \wedge dt &= y\omega \wedge dt \\ dy \wedge dt &= \frac{1}{2}P_x\omega \wedge dt \\ dx \wedge dy &= \frac{1}{2}P_t\omega \wedge dt. \end{aligned}$$

To compute the connection, we are supposed to lift the basis of relative cohomology to a set of forms on the total space, then differentiate, then project onto $H^1_{\mathrm{dR}}(X/S) \otimes \Omega^1_{S/K}$. First,

$$\begin{aligned} \nabla(\omega) &= A\,dy \wedge dx + A_t y\,dt \wedge dx + 2B_x\,dx \wedge dy + 2B_t\,dt \wedge dy \\ &= (B_xP_t - \frac{1}{2}AP_t - A_tP - B_tP_x)\omega \wedge dt. \end{aligned}$$

Second,

$$\begin{aligned} \nabla(x\omega) &= Ax\,dy \wedge dx + A_txy\,dt \wedge dx + 2(xB_x + B)\,dx \wedge dy + 2xB_t\,dt \wedge dy \\ &= (xB_xP_t + BP_t - \frac{1}{2}xAP_t - xA_tP - xB_tP_x)\omega \wedge dt. \end{aligned}$$

We then rewrite the quantities being wedged with dt as exact relative differentials plus a linear combination of $\omega, x\omega$ as in Example 2.2.1.

If you prefer, here is another way of describing essentially the same calculation.

Example 3.4.2. (Notes-only) Consider the same situation as in Example 3.4.1, but now let's redefine
$$\omega = \frac{dx}{y}.$$
Again, use $\omega, x\omega$ as the basis of relative differentials. These have poles along $y = 0$, but never mind that; we can still compute

$$\begin{aligned}\nabla(\omega) &= d\left(\frac{dx}{y}\right) \\ &= -\frac{dy \wedge dx}{y^2} \\ &= \frac{P_t}{2y^2}\frac{dx}{y} \wedge dt.\end{aligned}$$

To eliminate the pole, find the unique $C, D, E, F, G \in \Gamma(S, \mathcal{O}_S)$ (so these are functions of t alone) such that
$$P_t = a_t x + b_t = (Cx + D)P + (Ex^2 + Fx + G)P_x.$$
Then in relative de Rham cohomology,

$$\begin{aligned}\frac{P_t\,dx}{2y^3} &= \frac{(Cx + D)\,dx}{2y} + \frac{(Ex^2 + Fx + G)P_x\,dx}{2y^3} \\ &\equiv \frac{(Cx + D)\,dx}{2y} + \frac{(2Ex + F)\,dx}{y}.\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla(x\omega) &= d\left(\frac{x\,dx}{y}\right) \\ &= \frac{xP_t}{2y^2}\frac{dx}{y} \wedge dt\end{aligned}$$

and writing
$$xP_t = (Hx + I)P + (Jx^2 + Kx + L)P_x,$$
we get
$$\frac{xP_t\,dx}{2y^3} \equiv \frac{(Hx + I)\,dx}{2y} + \frac{(2Jx + K)\,dx}{y}.$$

3.5 Semistable reduction and logarithmic connections

For simplicity, I'm going to restrict the discussion of logarithmic connections to one-dimensional spaces.

Definition 3.5.1. Let (X, Z) be a smooth pair with X one-dimensional. A *logarithmic connection* on X is a bundle map $\nabla : V \to V \otimes \Omega^1_{(X,Z)}$ that satisfies the Leibniz rule.

Definition 3.5.2. Let ∇ be a logarithmic connection on (X, Z). For each $z \in Z$, ∇ induces a linear map $V_z \to V_z$, called the *residue* of ∇ at z, as follows. For a section s specializing to a given point $\overline{s} \in V_z$, write $ds = f\frac{dt}{t}$ for f a section of V and t a local parameter for z. Assign $\overline{s}$ to the specialization $\overline{f}$ of f to V_z.

Definition 3.5.3. Let $\pi : X \to S$ be a proper, flat, generically smooth morphism (in any of the categories from Definition 3.1.1) with S one-dimensional. We say π is *semistable* at $z \in S$ if the fibre X_z is a reduced divisor with simple normal crossings. That is, étale locally, S looks like $\operatorname{Spec} k[t]$ and X looks like $\operatorname{Spec} k[x_1, \dots, x_n]/(x_1 \cdot x_m - t)$ for some m, n (where m varies from point to point).

Example 3.5.4. For example, the Legendre family of elliptic curves

$$y^2 = x(x-1)(x-\lambda)$$

is smooth over $\lambda \notin \{0, 1, \infty\}$, and semistable at $\lambda = 0$ and $\lambda = 1$. To check at ∞, we reparametrize $\mu = \lambda^{-1}$, $x = X/\mu$, $y = Y/\mu^2$ to get

$$Y^2 = \mu X(X-\mu)(X-1),$$

which means that the fibre at $\lambda = \infty$ is not reduced.

If we instead consider the family

$$y^2 = x(x-1)(x-\lambda^2),$$

this family is indeed semistable everywhere.

Theorem 3.5.5. *Let $\pi : X \to S$ be a semistable morphism, smooth over $U = S \setminus Z$. Then the Gauss-Manin connection on $H^i_{\mathrm{dR}}(X_U/U)$ extends to a logarithmic connection with nilpotent residue maps.*

Exercise 3.5.6. Check that the Gauss-Manin connection for the Legendre family extends to a logarithmic connection on all of $\mathbb{P}^1$, but the residue map at infinity cannot be made nilpotent. (Hint: you can cheat by looking up this calculation in [40, §7].)

3.6 Gauss-Manin connections and Frobenius

The reason why Gauss-Manin connections are relevant in rigid cohomology is that they can be used to compute Frobenius actions *en masse.*

Exercise 3.6.1. Let $\operatorname{Spec} A$ be a smooth affine $\mathbb{Z}_q$-scheme. Let $\phi : \hat{A} \to \hat{A}$ be a q-power Frobenius lift on $\hat{A}$; that is, ϕ acts on $\overline{A}$ by the q-power map. Prove that for each $\overline{x} \in (\operatorname{Spec} \overline{A})(\mathbb{F}_q)$, there is a unique $x \in (\operatorname{Spec} \hat{A})(\mathbb{Z}_q)$ which specializes to $\overline{x}$, such that $\phi(x) = x$.

Definition 3.6.2. With notation as in Exercise 3.6.1, we call x the *Teichmüller lift* of $\overline{x}$ with respect to ϕ. If x is a Teichmüller lift of a point not specified, we say x is a *Teichmüller point.*

Exercise 3.6.3. State and prove a generalization of Exercise 3.6.1 to the case where the residue field is perfect but not necessarily finite, and/or the p-adic field is not necessarily unramified.

Theorem 3.6.4 (Berthelot)**.** *Let $S = \operatorname{Spec} A$ be a smooth affine $\mathbb{Z}_q$-scheme. Let ϕ be a q-power Frobenius lift on $\hat{A}$. Let $\pi : X \to S$ be a smooth proper morphism, and put $\mathcal{E} = H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q}/S_{\mathbb{Q}_q})$ as a vector bundle equipped with the Gauss-Manin connection. Then there exists an isomorphism $F : \phi^*\mathcal{E} \cong \mathcal{E}$ of vector bundles with integrable connection on the affinoid space $\hat{S}^{\mathrm{an}}$, such that for any positive integer a, and any Teichmüller point $x \in \hat{S}^{\mathrm{an}}(\mathbb{Q}_{q^a})$, $F^a : (\phi^a)^*\mathcal{E} \cong \mathcal{E}$ induces the Frobenius action on $H^i_{\mathrm{dR}}(X_x)$.*

Remark 3.6.5. The key feature of Theorem 3.6.4 is that the Frobenius action commutes with the action of the connection; this constraint can be interpreted as a differential equation on the Frobenius action. Let's see what this looks like for S a subscheme of the affine t-line and $\mathcal{E}$ admitting a basis $s_1, \dots, s_n$. Define the matrices A, N by

$$Fs_j = \sum_i A_{ij} s_i$$
$$\frac{d}{dt} s_j = N_{ij} s_i.$$

Then the compatibility between Frobenius and the connection is equivalent to the equation

$$NA + \frac{d}{dt}A = \left(\frac{d\phi(t)}{dt}\right) A\phi(N);$$

given N, this constitutes a differential system on the entries of A. To solve this system, we need also an initial condition, e.g., the Frobenius action on $H^i_{\mathrm{dR}}(X_x)$ for any one Teichmüller point.

Remark 3.6.6. There is also an overconvergent analogue of Theorem 3.6.4, which we will not write out explicitly but which is important for applications. As for an analogue where π is semistable, see the next lecture.

Remark 3.6.7. We can now articulate Lauder's deformation method for computing zeta functions: given a smooth proper variety over $\mathbb{Q}_q$, insert it into a smooth proper family over a one-dimensional base, in which it is simpler to compute the Frobenius action on some other fibre. (E.g., given a smooth hypersurface in $\mathbb{P}^n$, interpolate linearly between the defining polynomial and a diagonal polynomial.) Then compute the Gauss-Manin connection and its Frobenius action, and specialize to the point of interest.

4 More Frobenius actions in de Rham cohomology

In this section, we describe some natural structure on the de Rham cohomology of a smooth proper variety over $\mathbb{Q}_q$ that need not have good reduction. Besides the Frobenius action that one sees in the good reduction case, there is a second operator present, the *monodromy operator*, that in some sense measures to what extent the variety fails to have good reduction.

4.1 Motivation: p-adic Hodge theory

I recommend [3] for a gentle introduction into the formalism of p-adic Hodge theory; here I'll just mention a few relevant points.

Remark 4.1.1. Again, for ease of exposition, I have restricted attention to unramified extensions of $\mathbb{Q}_p$, but in this case, this choice actually does hide at least one interesting feature of the theory; see Definition 4.2.2

Definition 4.1.2. Let X be a smooth proper variety over $\mathbb{Z}_q$. Then $H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q})$ comes equipped with two wildly disparate extra structures: the Hodge filtration, coming from the Hodge-de Rham spectral sequence, and the action of Frobenius via the comparison theorem with rigid cohomology. Fontaine conjectured that these structures could be used to relate the de Rham cohomology of X to its étale cohomology with $\mathbb{Q}_p$-coefficients. Namely, he constructed a topological $\mathbb{Q}_p$-algebra $\mathbf{B}_{\mathrm{crys}}$ equipped with its own filtration and Frobenius actions, and conjectured the existence of a natural isomorphism

$$H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q}) \otimes_{\mathbb{Q}_q} \mathbf{B}_{\mathrm{crys}} \cong H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_q}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}}$$

respecting the filtration, Frobenius action, and $\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)$-action on both sides. (Here the Galois action on de Rham cohomology, and the filtration and Frobenius actions on étale cohomology, are taken to be trivial.) This is now known by the work of Faltings, Nizioł, Tsuji, et al.

Definition 4.1.3. Now suppose X is a proper semistable variety over $\mathbb{Z}_q$. Then $H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q})$ again admits a Hodge filtration, but it is less obvious what other structures it has. Nonetheless, Fontaine conjectured that it admits two extra endomorphisms: a Frobenius map ϕ, which is semilinear for the Witt vector Frobenius on $\mathbb{Q}_q$, and a new map called the *monodromy operator*. The monodromy operator N is a $\mathbb{Q}_q$-linear operator on $H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q})$, and we have the relation

$$N \circ \phi = p\phi \circ N,$$

which forces N to be nilpotent. Fontaine constructed a second topological $\mathbb{Q}_p$-algebra $\mathbf{B}_{\mathrm{st}}$ (for "semistable") equipped with its own filtration and Frobenius and monodromy operators, and conjectured the existence of a natural isomorphism

$$H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q}) \otimes_{\mathbb{Q}_q} \mathbf{B}_{\mathrm{st}} \cong H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_q}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}}$$

respecting the filtration, Frobenius action, monodromy action, and $\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)$-action on both sides. (Here the Galois action on de Rham cohomology, and the filtration, Frobenius, and monodromy actions on étale cohomology, are taken to be trivial.) Again, this is known by the work of Faltings, Nizioł, Tsuji, et al. (Warning: not everything above is completely natural; see Definition 4.2.2.)

4.2 The Hyodo-Kato isomorphism

Just as in the good reduction case, you should imagine the Frobenius and monodromy operators on a smooth proper $\mathbb{Q}_q$-variety with semistable reduction as arising, via a comparison isomorphism, from operators on a certain cohomology space associated to the mod p reduction. In the absence of good reduction, though, an additional wrinkle arises.

Definition 4.2.1. Let $\overline{X}$ be a proper scheme over $\mathbb{F}_q$ of pure dimension n with simple normal crossings (the phrase *strict normal crossings* is also used); that is, étale locally, $\overline{X}$ looks like $\operatorname{Spec} \mathbb{F}_q[x_1, \dots, x_{n+1}]/(x_1 \cdots x_m)$. Then there is a construction due to Hyodo and Kato [26] that associates to $\overline{X}$ a collection of vector spaces $H^i_{\mathrm{HK}}(\overline{X})$ over $\mathbb{Q}_q$ equipped with a Frobenius-semilinear operator ϕ and a nilpotent linear operator N satisfying $N \circ \phi = p\phi \circ N$.

Definition 4.2.2. Let X be a proper semistable scheme over $\mathbb{Z}_q$. Then Hyodo and Kato also exhibited a comparison isomorphism

$$\rho_\pi : H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q}) \cong H^i_{\mathrm{HK}}(\overline{X}),$$

thus equipping the left side with Frobenius and monodromy operators. But there is a catch: this isomorphism is not canonical! It depends on the choice of a uniformizer π of $\mathbb{Z}_q$; changing π by multiplying by a unit $u \in \mathbb{Z}_q^*$ has the effect

$$\rho_{\pi u} = \exp(-\log(u)N) \circ \rho_\pi.$$

In particular, the transferred monodromy operator $\rho_\pi^{-1} \circ N \circ \rho_\pi$ is independent of the choice of π, but the transferred Frobenius operator $\rho_\pi^{-1} \circ \phi \circ \rho_\pi$ is not.

Remark 4.2.3. Here is where the promised lack of naturality appears in Fontaine's comparison map: there is an ambiguity in the map which affects the Frobenius action on the left side. There is a correlated ambiguity in the filtrations: the filtration on $\mathbf{B}_{\mathrm{st}}$ depends on the choice of a branch for the p-adic logarithm. (This reduces to a choice of a uniformizer π because you want $\log_p(\pi) = 0$.) Since I am only working with unramified extensions, I can simply choose the uniformizer p to stamp out the ambiguities, but this is a real issue when working with ramified extensions.

Remark 4.2.4. (Notes-only) The variation in the filtration on $\mathbf{B}_{\mathrm{st}}$ comes from the fact that $\mathbf{B}_{\mathrm{st}}$ does not embed canonically into Fontaine's larger ring $\mathbf{B}_{\mathrm{dR}}$, which is where the filtration is obtained. By contrast, $\mathbf{B}_{\mathrm{crys}}$ does embed canonically into $\mathbf{B}_{\mathrm{dR}}$. On the other hand, $\mathbf{B}_{\mathrm{dR}}$ does not admit a continuous extension of the Frobenius on $\mathbf{B}_{\mathrm{crys}}$, whereas $\mathbf{B}_{\mathrm{st}}$ admits a canonical such extension. (The lack of a Frobenius on $\mathbf{B}_{\mathrm{dR}}$, the ring from which the filtrations arise, reflects the fact that the Frobenius action and the filtration come from incompatible worlds, one from characteristic p and the other from characteristic zero.)

Remark 4.2.5. The Hyodo-Kato cohomology itself can be made explicit: a description as "logarithmic rigid cohomology" was given by Grosse-Klönne [20]. The hard part is the comparison theorem; Grosse-Klönne's description naturally gives a comparison isomorphism with the de Rham cohomology of the generic fibre of an *equisingular* lift, but we are most interested in the case where the generic fibre is smooth and the special fibre is singular. This is what prompted the speculations in §4.3.

4.3 Gauss-Manin connections and Hyodo-Kato cohomology

Definition 4.3.1. Let $S = \operatorname{Spec} A$ be a smooth affine $\mathbb{Z}_q$-scheme of relative dimension 1. For any positive integer a and any point $\overline{x} \in \overline{S}$, let $D(\overline{x})$ be the subspace of $\hat{S}^{\mathrm{an}}$ consisting of points which specialize to $\overline{x}$; if $\overline{x} \in \overline{S}(\mathbb{F}_q)$, this is an open unit disc over $\mathbb{Q}_q$.

Definition 4.3.2. Let $T = \operatorname{Spec} B$ be an open dense subscheme of S meeting $\overline{S}$. Let $\pi : X \to S$ be a proper flat morphism such that $X_T \to T$ is smooth, and $X_{\mathbb{Q}_q} \to S_{\mathbb{Q}_q}$ is semistable. Assume moreover that there is at most one nonsmooth fibre over each disc $D(\overline{x})$. Put $\mathcal{E} = H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q}/S_{\mathbb{Q}_q})$ as a vector bundle equipped with the Gauss-Manin logarithmic connection.

The following results follow from Theorem 3.6.4 by an argument due to Dwork.

Theorem 4.3.3. *Let ϕ be a q-power Frobenius lift on $\hat{A}$ for which each point of $\hat{S}^{\mathrm{an}}$ over which π is nonsmooth is a Teichmüller point.*

(a) Given $\overline{x} \in \overline{S}(\mathbb{F}_q)$, let $x \in D(\overline{x})$ be a point such that π is smooth over $D(\overline{x}) \setminus \{x\}$, and let t be a local parameter of S at x. Then $\mathcal{E}$ admits a basis of sections over $D(\overline{x})$ via which $t\frac{d}{dt}$ acts via the residue map of ∇ at x. In particular, if π is smooth over all of $D(\overline{x})$, then $\mathcal{E}$ admits a basis of horizontal sections.

(b) The Frobenius isomorphism $F : \phi^*\mathcal{E} \cong \mathcal{E}$ over $\hat{T}^{\mathrm{an}}$ given by Theorem 3.6.4 extends over all of $\hat{A}$.

Remark 4.3.4. Theorem 4.3.3(a) is an instance of p-adic parallel transport. In contrast with the complex analytic setting, this phenomenon is not automatic in rigid geometry: a (non-logarithmic) connection on a p-adic open disc need not have a basis of horizontal sections. For instance, one has a connection on a rank 1 trivial module on the entire affine t-line whose action on a generator is given by $\frac{d}{dt}s = s$, but the horizontal section $\exp(-t)s$ is only defined for $|t| < p^{1/(p-1)}$. This observation is the departure point of the theory of p-adic differential equations, as developed by Dwork, Robba, Christol, Mebkhout, et al.

Remark 4.3.5. (Notes-only) Theorem 4.3.3(b) actually depends on the overconvergent version of Theorem 4.3.3: you glue the Frobenius isomorphism over an admissible cover consisting of finitely many $D(\overline{x})$ plus one strict neighborhood of $\hat{T}^{\mathrm{an}}$.

Here's how I would expect this to relate to the Hyodo-Kato operators.

Conjecture 4.3.1. *The action of $t\frac{d}{dt}$ on the basis of sections in Theorem 4.3.3(a) induces the Hyodo-Kato monodromy operator on $H^i_{\mathrm{dR}}(X_x)$ for any $x \in D(\overline{x})$ (not necessarily the distinguished choice made in Theorem 4.3.3(a)).*

Conjecture 4.3.2. *For any $x \in D(\overline{x})$, the Hyodo-Kato Frobenius operator on $H^i_{\mathrm{dR}}(X_x)$ (for some choice of the uniformizer) is induced by the Frobenius action on the basis of sections in Theorem 4.3.3(a).*

Remark 4.3.6. One might like to say in Conjecture 4.3.2 that the global Frobenius operator induces the Frobenius action on $H^i_{\mathrm{dR}}(X_x)$. But this can only happen if x is a Teichmüller point, and we reserved those for the points of bad reduction. So instead, you are supposed to use parallel transport to travel from x to the bad fibre, where you compute the Frobenius action.

Remark 4.3.7. For X/S of relative dimension 1, I think both conjectures follow from the explicit description of the Frobenius and monodromy operators given by Coleman and Iovita [11]. Checking this is a possible project topic; see Problem 5.3.3.

5 Proposed projects

The intended computational platform for these projects is William Stein's package `SAGE`, available for free download at

`http://sage.math.washington.edu/sage/`.

It can also be accessed directly from the above site, without local installation, using the notebook interface.

5.1 Improving the Christol-Dwork bound

The Christol-Dwork theorem [8] gives an effective bound on the convergence of solutions of certain differential equations; this bound is crucially used in the deformation method. However, some experimental evidence suggest that this bound is far from optimal; the goal of this project is to collect more such evidence, then establish an improved version of the Christol-Dwork bound. (Note: the theorem I attribute to Christol-Dwork is a bit weaker than what they proved, but it's what we need in practice.)

Definition 5.1.1. Let K be a discretely valued field of mixed characteristics $(0, p)$, with ring of integers $\mathfrak{o}_K$ and maximal ideal $\mathfrak{m}_K$. (I really only need a nonarchimedean valuation, but let us assume discreteness for simplicity.) Let R be the ring of formal power series $\sum_{i=0}^{\infty} c_i t^i \in K[\![t]\!]$ such that

$$\lim_{i \to \infty} |c_i|\rho^i = 0 \qquad (\rho \in [0, 1)).$$

For $\rho \in [0, 1)$, define the *ρ-Gauss norm* on R by

$$\left| \sum_{i=0}^{\infty} c_i t^i \right|_\rho = \sup_i \{|c_i|\rho^i\}. \tag{5.1.1.1}$$

Put $E = \mathfrak{o}_K[\![t]\!][1/p]$, viewed as a subring of R. Note that we can use (5.1.1.1) to define a ρ-Gauss norm on E not only for $\rho \in [0, 1)$ but also for $\rho = 1$.

Exercise 5.1.2. Prove the following.

(a) The ring R consists of the rigid analytic functions on the open unit disc over K.

(b) For $\rho \in [0,1)$, the ρ-Gauss norm on R equals both the supremum on the circle $|t| = \rho$ and on the disc $|t| \leq \rho$.

(c) The subring E of R consists of the bounded functions on the open unit disc over K.

(d) The 1-Gauss norm on E equals the supremum on the open unit disc.

Definition 5.1.3. By a *differential operator of rank* n over a subring S of $K[\![t]\!]$, we will mean an operator $D : S^n \to S^n$ of the form

$$\mathbf{v} \mapsto N\mathbf{v} + t\frac{d\mathbf{v}}{dt},$$

where $N = \sum_{i=0}^{\infty} N_i t^i$ is an $n \times n$ matrix over S; we extend the action of D in the obvious way to $n \times m$ matrices for any m. We refer to N_0 as the *residue* of D. An *isomorphism* between two differential systems D, D' is an invertible S-linear map $T : S^n \to S^n$ such that $D' = T^{-1} \circ D \circ T$; if we define the change-of-basis matrix U by $T\mathbf{e}_j = \sum_i U_{ij}\mathbf{e}_i$, then the matrices N, N' defining D, D' are related by

$$N' = U^{-1}NU + U^{-1}t\frac{dU}{dt}. \tag{5.1.3.1}$$

Note that change of basis changes the residue by a true conjugation, so it is well-defined to specify, e.g., that the residue be nilpotent.

Remark 5.1.4. You may prefer (as I do) to think of the differential operator D as giving a connection on S with a logarithmic singularity at $t = 0$.

Exercise 5.1.5. Let D be a differential operator of rank n over $K[\![t]\!]$ with nilpotent residue. Prove that there exists a unique $n \times n$ matrix $U = \sum_{i=0}^{\infty} U_i t^i$ over $K[\![t]\!]$ with U_0 equal to the identity matrix, such that $D^n U = 0$. (Hint: choose U so that in (5.1.3.1), we have $N' = N_0$. Then when you change basis by U, D acts on the K-span of the new basis via multiplication by N_0.)

Definition 5.1.6. Let D be a differential operator of rank n over $K[\![t]\!]$ with nilpotent residue. We refer to the matrix U in Exercise 5.1.5 as the *universal solution* of D.

Definition 5.1.7. Let D be a differential operator of rank n with nilpotent residue over a subring of R. We say D is *fully convergent* if the universal solution of D has entries in R. (The term *overconvergent* is also sometimes used.) There is also a definition that works even if the residue is not nilpotent, but it is a bit more complicated to describe.

Typically, we obtain fully convergent differential operators from Frobenius actions.

Definition 5.1.8. Let q be a power of p. Define a *(relative) q-power Frobenius lift* on $\mathfrak{o}_K[\![t]\!]$ to be a ring homomorphism $\phi : \mathfrak{o}_K[\![t]\!] \to \mathfrak{o}_K[\![t]\!]$ such that

$$\phi(t) = t^q u \qquad \text{for some } u \in 1 + (t, \mathfrak{m}_K)\mathfrak{o}_K[\![t]\!].$$

(It is not necessary to assume that $\phi(K) \subseteq K$.) Any such map ϕ has the form

$$\phi\left(\sum_{i=0}^{\infty} c_i t^i\right) = \sum_{i=0}^{\infty} \phi(c_i)\phi(t)^i;$$

we can use the same formula to extend ϕ to $E, R, K[\![t]\!]$.

Definition 5.1.9. Let D be a differential operator of rank n with nilpotent residues over a subring S of $\mathfrak{o}_K[\![t]\!]$. A *Frobenius action* on D over S with respect to a Frobenius lift ϕ is an isomorphism of D with its pullback by ϕ. Geometrically, this means an isomorphism of the module with connection defined by D with its Frobenius pullback. Algebraically, if D is defined by the matrix N, the pullback is defined by the matrix

$$\frac{t}{\phi(t)}\frac{d\phi(t)}{dt}\phi(N).$$

Exercise 5.1.10. Let D be a differential operator of rank n with nilpotent residues over R admitting a Frobenius action. Prove that D is convergent. (You might want to start with the case $\phi(t) = t^p$.)

Definition 5.1.11. Let $K[t]_{(t)}$ be the localization of $K[t]$ at the maximal ideal (t). Let E_0 be the closure of $K[t]_{(t)}$ in E for the 1-Gauss norm.

Theorem 5.1.12 (Christol-Dwork). *Let D be a fully convergent differential operator of rank n with nilpotent residues over E_0, with universal solution $U = \sum_{i=0}^{\infty} U_i t^i$. Then for $i \geq 1$, $v_p(U_i) \geq -\gamma_n \lfloor \log_p(i) \rfloor$ with*

$$\gamma_n = n - 1 + v_p((n-1)!) + \min\left(n-1, v_p\left(\prod_{j=1}^{n}\binom{n}{j}\right)\right).$$

Problem 5.1.13. Decide whether Theorem 5.1.12 still hold if N has entries in E.

Problem 5.1.14. Reduce the size of the constant γ_n in Theorem 5.1.12, possibly introducing dependence on some other parameters. These may include the index of nilpotency, and the slopes of the Frobenius action modulo t (in the sense of the Dieudonné-Manin classification). The latter is suggested by a recent preprint by Chiarellotto and Tsuzuki [7], but without consideration of effective bounds.

Remark 5.1.15. The bound given by Theorem 5.1.12 is crucial for provably computing the Frobenius structure on a Gauss-Manin connection to a specified p-adic accuracy. However, Ralf Gerkmann has noticed experimentally that the bound is far from optimal in many practical examples; it would thus be of great interest to improve these bound. It may be advisable to first collect some more experimental evidence in a systematic manner.

5.2 Coleman integrals

A theory of integration for 1-forms on curves over finite extensions of $\mathbb{Q}_p$ was introduced by Coleman [9]; computing Coleman's integrals turns out to be quite closely related to computing Frobenius actions in Monsky-Washnitzer cohomology. The usual integrals arise in the Chabauty method for finding all S-integral points on a curve over a number field [10] (for more details on which see Matt Baker); the iterated ones seem to relate to a "nonabelian Chabauty method" proposed by Minhyong Kim [31].

Problem 5.2.1. Implement the computation of Teichmüller lifts, e.g., for the Frobenius lift on an elliptic curve described in Example 2.2.2. You should be able to do this achieving quadratic convergence (as when using Newton-Raphson iteration) rather than linear convergence (as in the usual proof of Hensel's lemma).

Definition 5.2.2. Let X be a smooth projective curve over $\mathbb{Z}_q$, and let Z be a relative normal crossings divisor (i.e., a union of lifts of distinct closed points on $X_{\mathbb{F}_q}$). Put $U = X \setminus Z$, put $A = \Gamma(U, \mathcal{O})$, and let $A^\dagger$ be the weak completion of A. Let $\phi : A^\dagger \to A^\dagger$ be a q-power Frobenius lift. Given a choice of branch for the p-adic logarithm, Coleman [9, §2] constructed a canonical map

$$\int : U^{\mathrm{an}} \times U^{\mathrm{an}} \times (\Omega^1_{A^\dagger} \otimes_{\mathbb{Z}_q} \mathbb{Q}_q) \to \mathbb{C}_p$$

(where $\mathbb{C}_p$ is the completed algebraic closure of $\mathbb{Q}_p$) which obeys the usual formalism of an integral, including the following.

- Additivity in the path: $\int_P^Q \omega + \int_Q^R \omega = \int_P^R \omega$.
- Linearity in the 1-form: $\int_P^Q (a\omega + b\eta) = a \int_P^Q \omega + b \int_P^Q \eta$.
- Functoriality (change of variables): if X', Z', U' is another set of data and $f : X \to X'$ is a map (not necessarily compatible with Frobenius lifts), then $\int_{f(P)}^{f(Q)} \omega = \int_P^Q f^*(\omega)$.
- The fundamental theorem of calculus: for $f \in A^\dagger$, $\int_P^Q df = f(Q) - f(P)$.

On each copy of U^{an}, the function is only locally analytic (e.g., if $A = K[t, t^{-1}]$ and the one-form is dt/t), with analyticity on each open unit disc of U^{an}.

Problem 5.2.3. Implement a computation of Coleman integrals for some curve X (e.g., a hyperelliptic curve), as follows. Let $\omega_1, \dots, \omega_n \in \Omega^1_{A^\dagger}$ be representatives for a basis of $H^1_{\mathrm{MW}}(\overline{U})$. Let P, Q be Teichmüller points in U^{an}. Compute an approximation to the matrix A over $\mathbb{Q}_q$ and the functions $f_j \in A^\dagger \otimes \mathbb{Q}_q$ such that

$$\phi^* \omega_j = df_j + \sum_{i=1}^n A_{ij} \omega_i,$$

as if one were going to compute the zeta function of $\overline{U}$. Then apply functoriality of the Coleman integral to deduce

$$\int_P^Q \omega_j = f_j(Q) - f_j(P) + \sum_{i=1}^n A_{ij} \int_P^Q \omega_i,$$

and solve the system of linear equations. Finally, for general points R, S in the same residue discs as P, Q, respectively, compute

$$\int_R^S \omega = \int_R^P \omega + \int_P^Q \omega + \int_Q^S \omega$$

by computing the outer terms by formally integrating ω.

Problem 5.2.4. Extend Problem 5.2.3 to iterated integrals; these are analogous to path integrals

$$\int_{0 \leq t_1 \leq t_2 \leq 1} \gamma_1^*(\omega)\gamma_2^*(\omega)$$

and so on. I have some notes from Coleman that may help with this.

5.3 Hyodo-Kato cohomology

Problem 5.3.1. Implement an explicit computation of the Frobenius and monodromy operators on an elliptic curve over $\mathbb{Q}_p$ with (split or nonsplit) multiplicative reduction. A procedure for doing this is suggested by le Stum [32], and explained further by Coleman and Iovita [11].

Problem 5.3.2. Extend the implementation in Problem 5.3.1 first to hyperelliptic curves with totally multiplicative reduction. Then treat the case of semistable reduction (i.e., with a mix of good and multiplicative reduction).

Problem 5.3.3. Prove that the recipe for computing the Frobenius and monodromy operators proposed in Conjectures 4.3.1 and 4.3.2 is correct for H^1, by comparing it with the Coleman-Iovita construction.

Problem 5.3.4. Try computing some examples on H^2 of smooth surfaces in $\mathbb{P}^3$. Although I do not have an alternate recipe for computing the Hyodo-Kato operators, we can at least test the results by seeing whether the conclusion of the weight-monodromy conjecture is satisfied.

5.4 Nondegenerate hypersurfaces in toric varieties

Those familiar with toric varieties might enjoy the following.

Problem 5.4.1. Extend and implement the algorithm described in [1] for computing zeta functions of smooth hypersurfaces in projective spaces, to nondegenerate hypersurfaces in toric varieties.

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