ARITHMETIC OF K3 SURFACES
ARIZONA WINTER SCHOOL 2015

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1. Course Outline

The qualitative features of the arithmetic of curves is strongly governed by geometry. Elliptic curves form a fascinating class of varieties to study because they are varieties "of intermediate type", i.e., they are neither (geometrically) birational to $\mathbb{P}^1$, nor are they varieties of general type. K3 surfaces occupy a similar place in the theory of surfaces. This class of surfaces includes double covers of $\mathbb{P}^2$ ramified over a sextic plane curve, quartic surfaces in $\mathbb{P}^3$, and complete intersections of three quadrics in $\mathbb{P}^5$. The last fifteen years have seen a surge of activity on the arithmetic of K3 surfaces. The goal of this course is to survey some of these developments, with an emphasis on explicit methods and examples.

Geometry of K3 surfaces. We will start with a crash course (light on proofs) on the geometry of K3 surfaces: topological properties, including the lattice structure of $H^2(X, \mathbb{Z})$ and simple connectivity; the period point of K3 surface, the Torelli theorem and surjectivity of the period map. Good references for this material include [BHPVdV04, Ch. VIII] and [LP80].

Potential Density. A variety X over a number field k is said to satisfy potential density if there is a finite extension L/k such that $X(L)$ is Zariski dense in X. After a quick survey of some known results for several classes of varieties, we will explain work of Bogomolov and Tschinkel that shows that K3 surfaces X endowed with an elliptic fibration or with an infinite automorphism group satisfy potential density [BT98, BT99, BT00, Has03].

Picard groups. It is known that over a number field k, the (geometric) Picard group $\operatorname{Pic}(\overline{X})$ of a projective K3 surface X is a free $\mathbb{Z}$-module of rank $1 \leq \rho(\overline{X}) \leq 20$. Determining $\rho(\overline{X})$ for a given K3 surface is a difficult task; we will explain how work of van Luijk, Kloosterman, Elsenhans-Jahnel and Charles [vL07, Klo07, EJ11, Cha14] solves this problem.

Brauer Groups. The Galois module structure of $\operatorname{Pic}(\overline{X})$ allows one to compute an important piece of the Brauer group $\operatorname{Br}(X) = \mathrm{H}^2(X_{\mathrm{et}}, \mathbb{G}_m)$ of a locally solvable K3 surface X, consisting of the classes of $\operatorname{Br}(X)$ that are killed by passage to an algebraic closure (modulo Brauer classes coming from the ground field). These classes can be used to construct counter-examples to the Hasse principle on K3 surfaces via Brauer-Manin obstructions, a topic which will dovetail with Viray's course.

For surfaces of negative Kodaira dimension (e.g., cubic surfaces), we have $\operatorname{Br}_1(X) = \operatorname{Br}(X)$, so the algebraic Brauer group already gives all the information needed to determine

2000 *Mathematics Subject Classification.* Primary 11 G35; Secondary 14 G05.

Brauer-Manin obstructions to the Hasse principle and weak approximation. In contrast, for a K3 surface X, we know that $\operatorname{Br}(X(\mathbb{C})) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho}$. However, a remarkable theorem of Skorobogatov and Zarhin [SZ08] says that over a number field the quotient $\operatorname{Br}(X)/\operatorname{Br}(k)$ is finite! The remainder of the course will be devoted to ongoing work by several authors on the computation of the non-algebraic Brauer classes on K3 surfaces, and their impact on the arithmetic of such surfaces [HVAV11, HVA13, MSTVA14].

2. Project Description

2.1. **Diagonal K3 surfaces of degree 2.** The goal of this project is to understand the geometric Picard group, as a Galois module, of certain double covers of $\mathbb{P}^2$ ramified along a sextic. More concretely, over a number field k, we want to study the hypersurface in the weighted projective space $\mathbb{P}(1,1,1,3) = \operatorname{Proj} k[x,y,z,w]$ given by

$$X_{A,B,C}/k: \qquad w^2 = Ax^6 + By^6 + Cz^6$$

for some A, B and $C \in k^\times$.

(1) What is the rank of $\operatorname{Pic}(\overline{X}_{A,B,C})$? Note that to compute this number we may assume that $A = B = C = 1$. What upper bounds are suggested by reduction modulo 3 and point counting?

(2) The double cover map $\pi\colon X_{A,B,C} \to \mathbb{P}^2_k = \operatorname{Proj} k[x,y,z]$ gives us a large supply of divisors on $X_{A,B,C}$, namely, the components of the pullback of a line in $\mathbb{P}^2_k$ tritangent to the branch curve $Ax^6 + By^6 + Cz^6 = 0$. What is the rank of the sublattice of $\operatorname{Pic}(\overline{X}_{A,B,C})$ generated by these divisors? Does it equal $\rho(\overline{X}_{A,B,C})$? If so, is the sublattice saturated, i.e., is it all of the Picard group? If not, what are the missing divisor classes?

(3) What is the Galois module structure of $\operatorname{Pic}(\overline{X}_{A,B,C})$? The answer should depend on A, B and C. What is the group $\mathrm{H}^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic}(\overline{X}_{A,B,C}))$?

(4) The Hochschild-Serre spectral sequence gives rise to an isomorphism

$$\operatorname{Br}_1(X_{A,B,C})/\operatorname{Br}_0(X_{A,B,C}) \xrightarrow{\sim} \mathrm{H}^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic}(\overline{X}_{A,B,C})),$$

where $\operatorname{Br}_1(X_{A,B,C}) = \ker\left(\operatorname{Br}(X_{A,B,C}) \to \operatorname{Br}(\overline{X}_{A,B,C})\right)$ is the algebraic Brauer group, and $\operatorname{Br}_0(X_{A,B,C}) = \operatorname{im}(\operatorname{Br}(k) \to \operatorname{Br}(X_{A,B,C}))$ is the subgroup of constant algebras. Can you invert this map and produce central simple algebras over the function field $k(X_{A,B,C})$ that represent nonconstant algebraic classes in $\operatorname{Br}(X_{A,B,C})$? Can you use these classes to give examples of Brauer-Manin obstructions to weak approximation or the Hasse principle? The paper [VA08, §3] could be of help here.

(5) Specialize to $k = \mathbb{Q}$. Look at the "box"

$$\mathcal{B} := \{(A,B,C) \in \mathbb{Z}^3 : |A|, |B|, |C| \leq 100\}.$$

For which $(A,B,C) \in \mathcal{B}$ is there an algebraic obstruction to the Hasse principle on $X_{A,B,C}$? If there is no obstruction, can you find a rational point on $X_{A,B,C}$?

(6) Can you construct a cubic fourfold containing a plane having X as its associated K3 surface? See [HVAV11] for details on this construction. If so, can you construct a transcendental element of $\operatorname{Br}(X)[2]$ as a quaternion algebra over the function field

$k(X)$? How about transcendental elements in $\mathrm{Br}(X)[2]$ arising from K3 surfaces of degree 8? See [MSTVA14] for the geometry involved here.

2.2. **Twisted derived equivalence and rational points.** The goal of this project is to explore a recent question coming out of work of Hassett and Tschinkel. FYI: *You don't have to know much about twisted derived categories to work on this project!* However, a good understanding of the paper [HVA13] would be most helpful.

Question 2.1. *Let X and Y be locally solvable K3 surfaces over a number field, and suppose there is an equivalence of twisted derived categories $\mathrm{D}^b(X,\alpha) \cong \mathrm{D}^b(Y,\beta)$ for some $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$. Assume that α obstructs the Hasse principle on X. Is $Y(k) = \emptyset$?*

Here is a concrete instance where we can explore this problem: Let W be a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a type $(2,2)$ divisor. The two projections $\pi_i \colon Y \to \mathbb{P}^2$ $(i = 1, 2)$ give quadric bundle fibrations, and the degeneracy locus of this fibration is a plane sextic in $\mathbb{P}^2$. Taking the double cover of $\mathbb{P}^2$ ramified along the branch locus of π_i gives a K3 surface. We thus obtain two K3 surfaces X and Y out of W. In [HVA13] we explain how to use W to construct elements $\alpha \in \mathrm{Br}(X)[2]$ and $\beta \in \mathrm{Br}(Y)[2]$. It turns out that $\mathrm{D}^b(X,\alpha) \cong \mathrm{D}^b(Y,\beta)$. This way we get a good supply of surfaces on which to test Question 2.1. Our goal is then to

(1) Produce a supply of (X,α) and (Y,β) as above over $\mathbb{Q}$, in such a way that $X(\mathbb{Q}) = \emptyset$ on account of the class α. The delicate point here is to do this in a way that the defining equations of W have small coefficients (this will require an implementation of invariant calculations on 2-adic points of X). In order to do this, it'd be nice to guarantee that $\rho(\overline{X}) = 1$ (this will ensure that $\rho(\overline{Y}) = 1$, and thus there is no "interference" from algebraic Brauer classes).

(2) For the surfaces in our catalogue, does β obstruct rational points on Y? If not, can we develop an efficient algorithm to search for points on K3 surfaces of degree 2?

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