AN INTRODUCTION TO BERKOVICH ANALYTIC SPACES AND NON-ARCHIMEDEAN POTENTIAL THEORY ON CURVES (DRAFT 2/19/07)

MATTHEW H. BAKER

Abstract. This is an expository set of lecture notes meant to accompany the author's lectures at the 2007 Arizona Winter School on p-adic geometry. It is partially adapted from the author's monograph with R. Rumely [BR06], and also draws on ideas from V. Berkovich's monograph [Ber90], A. Thuillier's thesis [Thu05], and Rumely's book [Rum89]. We have purposely chosen to emphasize examples, pictures, discussion, and the intuition behind various constructions rather than emphasizing formal proofs and rigorous arguments. (Indeed, there are very few proofs given in these notes!) Once the reader has acquired a basic familiarity with the ideas in the present survey, it should be easier to understand the material in the sources cited above.

These notes are still incomplete and in draft form. Comments, typos, and suggestions for clarifying the exposition are more than welcome!

Contents

I was supported by the Arizona Winter School and by NSF Research Grant DMS-0600027 during the writing of these lecture notes. I would like to thank Xander Faber and David Krumm for sending numerous comments and corrections, and Robert Rumely for allowing me to incorporate portions of [BR06] into these notes. I would also like to thank Clay Petsche for his help as my Arizona Winter School project assistant.

Notation

We set the following notation, which will be used throughout these notes.

K	an algebraically closed field which is complete with respect to a non-archimedean absolute value.
$\lvert\ \rvert$	the absolute value on K, which we assume to be nontrivial.
$\lvert K^*\rvert$	the value group of K, i.e., $\{\lvert\alpha\rvert \ : \ \alpha \in K^*\}$.
$\tilde{K}$	the residue field of K.
q_v	a fixed real number greater than 1, chosen so that $-\log_{q_v} : K^* \to \mathbb{R}$ is a suitably normalized valuation v on K.
$\log_v$	shorthand for $\log_{q_v}$

$B(a,r)$	the closed disk $\{z \in K : \|z-a\| \le r\}$ of radius r about a in K. Here r is any positive real number, and sometimes we allow the degenerate case $r=0$ as well. If $r \in \|K^*\|$ we call the disk *rational*, and if $r \notin \|K^*\|$ we call it *irrational*.
$B(a,r)^-$	the open disk $\{z \in K : \|z-a\| < r\}$ of radius r about a in K.
$\mathbb{A}^1_{\mathrm{Berk}}$	the Berkovich affine line over K.
$\mathbb{P}^1_{\mathrm{Berk}}$	the Berkovich projective line over K.
$\mathbf{H}_{\mathrm{Berk}}$	the "Berkovich hyperbolic space" $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$.
$\mathbb{Q}_p$	the field of p-adic numbers.
$\mathbb{C}_p$	the completion of a fixed algebraic closure of $\mathbb{Q}_p$ for some prime number p.

1. **Lecture I**: The Berkovich projective line

In this lecture, we will introduce in a concrete way the Berkovich affine and projective lines, and we will explore in detail their topological properties. We will also define a subset of the Berkovich projective line called "Berkovich hyperbolic space", which is equipped with a canonical metric. Finally, we define and describe various special functions (e.g. the "canonical distance") on the Berkovich projective line.

1.1. **Motivation.**

Let K be an *algebraically closed field* which is *complete* with respect to a nontrivial nonarchimedean absolute value. (These conventions will hold unless otherwise explicitly stated throughout these notes.) The topology on K induced by the given absolute value is Hausdorff, but it is also totally disconnected and not locally compact. This makes it very difficult to define in a direct way a good local notion of an analytic function on K. Tate dealt with this problem by developing the subject which is now known as *rigid analysis*, in which one works with a certain Grothendieck topology on K. This gives a satisfactory theory of analytic functions on K, but the underlying topological space is unchanged, so problems remain for certain other applications. For example, the topology on K makes it difficult to define a Laplacian operator on K analogous to the classical Laplacian on $\mathbb{C}$, or to formulate a natural notion of harmonic and subharmonic functions on K. Rigid analysis does not provide much apparent help with these difficulties.

However, these difficulties, and many more, can be resolved in an extremely satisfactory way using Berkovich's theory. The Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}$ over K is a locally compact, Hausdorff, and path-connected topological space which contains K (with the topology induced by the given absolute value) as a dense subspace. One obtains the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}$ by adjoining to $\mathbb{A}^1_{\mathrm{Berk}}$ in a suitable manner a point at infinity; the resulting space $\mathbb{P}^1_{\text{Berk}}$ is a compact, Hausdorff, and path-connected topological space which contains $\mathbb{P}^1(K)$ (with its natural topology) as a dense subspace. In fact, $\mathbb{A}^1_{\text{Berk}}$ and $\mathbb{P}^1_{\text{Berk}}$ are better than just path-connected: they are *uniquely* path-connected, in the sense that any two distinct points can be joined by a unique arc. The unique path-connectedness is closely related to the fact that $\mathbb{A}^1_{\text{Berk}}$ and $\mathbb{P}^1_{\text{Berk}}$ are endowed with a natural *profinite* $\mathbb{R}$*-tree* structure. The profinite $\mathbb{R}$-tree structure on $\mathbb{A}^1_{\text{Berk}}$ (resp. $\mathbb{P}^1_{\text{Berk}}$) can be used to define a *Laplacian operator* in terms of the classical Laplacian on a finite graph. This in turn leads to a good theory of harmonic and subharmonic functions which closely parallels the classical theory over $\mathbb{C}$.

1.2. **Multiplicative seminorms.** The definition of $\mathbb{A}^1_{\text{Berk}}$ is quite simple, and makes sense with K replaced by an arbitrary field k endowed with a (possibly archimedean or even trivial) absolute value. A *multiplicative seminorm* on a ring A is a function $| \ |_x : A \to \mathbb{R}_{\geq 0}$ satisfying:

- $|0|_x = 0$ and $|1|_x = 1$.
- $|fg|_x = |f|_x \cdot |g|_x$ for all $f, g \in A$.
- $|f + g|_x \leq |f|_x + |g|_x$ for all $f, g \in A$.

As a set, $\mathbb{A}^1_{\text{Berk},k}$ consists of all multiplicative seminorms on the polynomial ring $k[T]$ which extend the usual absolute value on k. By an aesthetically desirable abuse of notation, we will identify seminorms $| \ |_x$ with points $x \in \mathbb{A}^1_{\text{Berk},k}$, and we will usually omit explicit reference to the field k, writing $\mathbb{A}^1_{\text{Berk}}$ and assuming that we are working over a complete and algebraically closed non-archimedean field K. The topology on $\mathbb{A}^1_{\text{Berk},k}$ is defined to be the weakest one for which $x \mapsto |f|_x$ is continuous for every $f \in A$.

To motivate the definition of $\mathbb{A}^1_{\text{Berk}}$, we can observe that in the classical setting, every multiplicative seminorm on $\mathbb{C}[T]$ which extends the usual absolute value on $\mathbb{C}$ is of the form $f \mapsto |f(z)|$ for some $z \in \mathbb{C}$. (This is a consequence of the well-known Gelfand-Mazur theorem from functional analysis.) It is then easy to see that $\mathbb{A}^1_{\text{Berk},\mathbb{C}}$ is homeomorphic to $\mathbb{C}$ itself, and also to the Gelfand space of all maximal ideals in $\mathbb{C}[T]$.

1.3. **Berkovich's classification theorem.** In the non-archimedean world, K can once again be identified with the Gelfand space of maximal ideals in $K[T]$, but now there are many more multiplicative seminorms on $K[T]$ than just the ones given by evaluation at a point of K. The prototypical example is if we fix a closed disk $B(a, r) = \{z \in K \ :$

$|z-a| \le r\}$ in K, and define $|\ |_{B(a,r)}$ by

$$|f|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)| \ .$$

It is an elementary consequence of the well-known "Gauss lemma" that $|\ |_{B(a,r)}$ is *multiplicative*, and the other axioms for a seminorm are trivially satisfied. Thus each disk $B(a,r)$ gives rise to a point of $\mathbb{A}^1_{\mathrm{Berk}}$. Note that this includes disks for which $r \notin |K^*|$, i.e., *irrational disks* for which the set $\{z \in K \ : \ |z-a| = r\}$ is empty. Also, we may consider the point a itself as a "degenerate" disk of radius zero, in which case we set $|f|_{B(a,0)} = |f(a)|$.

It is not hard to see that distinct disks $B(a,r)$ with $r \ge 0$ give rise to distinct multiplicative seminorms on $K[T]$, and therefore the set of all such disks embeds naturally into $\mathbb{A}^1_{\mathrm{Berk}}$. In particular, K embeds naturally into $\mathbb{A}^1_{\mathrm{Berk}}$ as the set of disks of radius zero, and it is not hard to show that K is in fact *dense* in $\mathbb{A}^1_{\mathrm{Berk}}$ in the Berkovich topology.

Suppose $x, x' \in \mathbb{A}^1_{\mathrm{Berk}}$ are distinct points corresponding to the (possibly degenerate) disks $B(a,r), B(a',r')$, respectively. The unique path in $\mathbb{A}^1_{\mathrm{Berk}}$ between x and x' has the following very intuitive description. If $B(a,r) \subset B(a',r')$, this path consists of all points of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to disks containing $B(a,r)$ and contained in $B(a',r')$. The collection of such "intermediate disks" is totally ordered by containment, and if $a = a'$ it is just $\{B(a,t) \ : \ r \le t \le r'\}$, which is homeomorphic to the closed interval $[r,r']$ in $\mathbb{R}$. If $B(a,r)$ and $B(a',r')$ are disjoint, the unique path between x and x' consists of all points of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to disks of the form $B(a,t)$ with $r \le t \le |a-a'|$ or $B(a',t')$ with $r' \le t' \le |a-a'|$. The disk $B(a,|a-a'|)$ is the smallest one containing both $B(a,r)$ and $B(a',r')$, and if $x \vee x'$ denotes the corresponding point of $\mathbb{A}^1_{\mathrm{Berk}}$, then the unique path from x to x' is just the path from x to $x \vee x'$ followed by the path from $x \vee x'$ to x'.

In particular, if a, a' are distinct points of K, then one can visualize the unique path in $\mathbb{A}^1_{\mathrm{Berk}}$ from a to a' as follows: Start increasing the "radius" of the degenerate disk $B(a,0)$ until you have a disk $B(a,r)$ which also contains a'. This disk can also be written as $B(a',s)$ with $r = s = |a-a'|$. Now decrease s until the radius reaches zero and we have the degenerate disk $B(a',0)$. In this way we have "connected up" the totally disconnected space K by adding points corresponding to closed disks in K!

In order to obtain a *compact* space from this construction, it is usually necessary to add even more points. This is because K may not be *spherically complete* (e.g., when $K = \mathbb{C}_p$): there may be decreasing sequences of closed disks having empty intersection. Intuitively, we need to add points corresponding to such sequences in order to obtain a space which has a chance of being compact. More precisely, if we return to the definition of $\mathbb{A}^1_{\mathrm{Berk}}$ in terms of multiplicative seminorms, it is easy to see that if $\{B(a_n, r_n)\}$ is any decreasing nested sequence of closed disks, then the map

$$f \mapsto \lim_{n\to\infty} |f|_{B(a_n,r_n)}$$

defines a multiplicative seminorm on $K[T]$ extending the usual absolute value on K. Two such sequences of disks with empty intersection define the same seminorm if and only if the sequences are *cofinal* (or *interlacing*). This yields a large number of additional points of $\mathbb{A}^1_{\mathrm{Berk}}$ which we are forced to throw into the mix. According to *Berkovich's classification theorem*, we have now described all points of $\mathbb{A}^1_{\mathrm{Berk}}$. More precisely:

Theorem 1.1 (Berkovich's Classification Theorem)**.** *Every point $x \in \mathbb{A}^1_{\mathrm{Berk}}$ corresponds to a nested sequence $B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq B(a_3, r_3) \supseteq \cdots$ of closed disks, in the sense that*

$$|f|_x = \lim_{n\to\infty} |f|_{B(a_n,r_n)}.$$

Two such nested sequences define the same point of $\mathbb{A}^1_{\mathrm{Berk}}$ if and only if

(a) each has a nonempty intersection, and their intersections are the same; or

(b) both have empty intersection, and the sequences are cofinal.

Consequently, we can categorize the points of $\mathbb{A}^1_{\mathrm{Berk}}$ into four types according to the nature of $B = \bigcap B(a_n, r_n)$:

Type I: B is a point of K.

Type II: B is a closed disk with radius belonging to $|K^*|$.

Type III: B is a closed disk with radius not belonging to $|K^*|$.

Type IV: $B = \emptyset$.

The set of points of Type III can be either infinite or empty, and similarly for the Type IV points. The set of points of Type I is always infinite, as is the set of points of Type II.

The description of points of $\mathbb{A}^1_{\mathrm{Berk}}$ in terms of closed disks is very useful, because it allows us to visualize quite concretely the abstract space of multiplicative seminorms which we started out with.

As a set[1], the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}$ is obtained from $\mathbb{A}^1_{\mathrm{Berk}}$ by adding a type I point at infinity, denoted ∞. The topology on $\mathbb{P}^1_{\mathrm{Berk}}$ is that of the one-point compactification.

Remark 1.2. Both $\mathbb{P}^1(K)$ and $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$ are dense in the Berkovich topology on $\mathbb{P}^1_{\mathrm{Berk}}$.

We will denote by $\zeta_{a,r}$ the point of $\mathbb{A}^1_{\mathrm{Berk}}$ of type II or III corresponding to the closed or irrational disk $B(a,r)$. Allowing degenerate disks (i.e., $r = 0$), we can extend this notation to points of type I: we let $\zeta_{a,0}$ (or simply a, depending on the context) denote the point of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to $a \in K$.

Following the terminology introduced by Chambert-Loir in [CL06], the distinguished point $\zeta_{0,1}$ in $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to the Gauss norm on $K[T]$ will be called the *Gauss point*. We will usually write ζ_{Gauss} instead of $\zeta_{0,1}$.

1.4. **Visualizing $\mathbb{P}^1_{\mathrm{Berk}}$.**

1.4.1. *A partial order on $\mathbb{P}^1_{\mathrm{Berk}}$.* The space $\mathbb{A}^1_{\mathrm{Berk}}$ is endowed with a natural partial order, defined by saying that $x \leq y$ if and only if $|f|_x \leq |f|_y$ for all $f \in K[T]$. In terms of (possibly degenerate) disks, if $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$ are points of type I, II, or III, we have $x \leq y$ if and only if the disk corresponding to x is contained in the disk corresponding to y. (We leave it to the reader to extend this description of the partial order to points of type IV.) For each pair of points $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$, there is a unique least upper bound $x \vee y \in \mathbb{A}^1_{\mathrm{Berk}}$ with respect to this partial order. Concretely, if $x = \zeta_{a,r}$ and $y = \zeta_{b,s}$ are points of type I, II or III, then $x \vee y$ is the point of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to the smallest disk containing both $B(a,r)$ and $B(b,s)$.

We can extend the partial order to $\mathbb{P}^1_{\mathrm{Berk}}$ by declaring that $x \leq \infty$ for all $x \in \mathbb{A}^1_{\mathrm{Berk}}$. Writing

$$[x, x'] = \{z \in \mathbb{P}^1_{\mathrm{Berk}} \ : \ x \leq z \leq x'\} \cup \{z \in \mathbb{P}^1_{\mathrm{Berk}} \ : \ x' \leq z \leq x\} \ ,$$

it is easy to see that the unique path between $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$ is just

$$\ell_{x,y} := [x, x \vee y] \cup [x \vee y, y] \ .$$

[1] One should also define a sheaf of regular functions on $\mathbb{A}^1_{\mathrm{Berk}}$ and $\mathbb{P}^1_{\mathrm{Berk}}$ and view them as locally ringed spaces, but we will not emphasize this point of view at the moment.

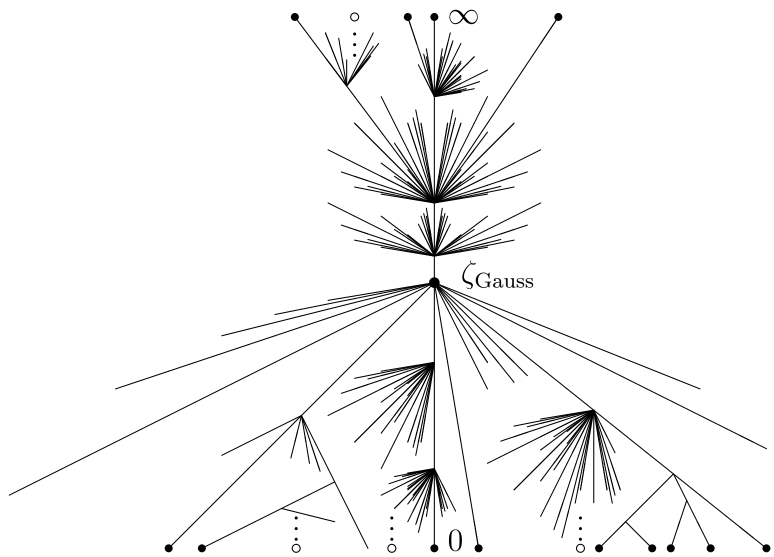

FIGURE 1. The Berkovich projective line (adapted from an illustration of Joe Silverman)

1.4.2. *Navigating* $\mathbb{P}^1_{\mathrm{Berk}}$. You can visualize "navigating" the Berkovich projective line in the following way (c.f. Figure 1). Starting from the Gauss point $\zeta_{0,1}$, there are infinitely many branches in which you can travel, one for each element of the residue field $\tilde{K}$ plus a branch leading up towards infinity. Having chosen a direction in which to move, at each point of type II along the chosen branch there are infinitely many new branches to choose from, and each subsequent branch behaves in the same way. This dizzying collection of densely splitting branches forms a configuration which Robert Rumely has christened a "witch's broom". However, the witch's broom has some structure:

- There is branching *only* at the points of type II, not those of type III.
- The branches emanating from a type II point $\zeta_{a,r}$ are in one-to-one correspondence with elements of $\mathbb{P}^1(\tilde{K})$: there is one branch going "up" to the Gauss point, with the other branches corresponding to open disks $B(a',r)^-$ of radius r contained in $B(a,r)$.
- Some of the branches extend all the way to the bottom (terminating in points of type I), while others are "cauterized off" earlier and terminate at points of type IV. In any case, every branch terminates either at a point of type I or type IV.

1.4.3. *Tangent spaces and directional derivatives.* Let $x \in \mathbb{P}^1_{\mathrm{Berk}}$. We define the space T_x of *tangent directions at* x to be the set of equivalence classes of paths $\ell_{x,y}$ emanating from x, where y is any point of $\mathbb{P}^1_{\mathrm{Berk}}$ not equal to x and two paths $\ell_{x,y_1}, \ell_{x,y_2}$ are *equivalent* if they share a common initial segment. There is a natural bijection between elements $\vec{v} \in T_x$ and connected components of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$. We denote by $U(x;\vec{v})$ the connected component[2] of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ corresponding to $\vec{v} \in T_x$. It is not hard to show that the open sets $U(x;\vec{v})$ for $x \in \mathbb{P}^1_{\mathrm{Berk}}$ and $\vec{v} \in T_x$ generate (i.e., form a sub-base for) the topology on $\mathbb{P}^1_{\mathrm{Berk}}$.

For example, consider the Gauss point ζ_{Gauss}. The different tangent directions $\vec{v} \in T_{\zeta_{\mathrm{Gauss}}}$ correspond bijectively to elements of $\mathbb{P}^1(\tilde{K})$, the projective line over the residue field of K. Equivalently, elements of $T_{\zeta_{\mathrm{Gauss}}}$ correspond to the open disks of radius 1 contained in the closed unit disk $B(0,1)$, together with the open disk

$$B(\infty,1)^- := \mathbb{P}^1(K) \backslash B(0,1).$$

The correspondence between elements of $T_{\zeta_{\mathrm{Gauss}}}$ and open disks is given explicitly by $\vec{v} \mapsto U(\zeta_{\mathrm{Gauss}};\vec{v})$.

More generally, for each point $x = \zeta_{a,r}$ of type II, the set T_x of tangent directions at x is (non-canonically) isomorphic to $\mathbb{P}^1(\tilde{K})$: there is one tangent direction going "up" to infinity, and the other tangent directions correspond to open disks $B(a',r)^-$ of radius r contained in $B(a,r)$, which (after choosing a Möbius transformation sending $B(a,r)$ to $B(0,1)$) correspond bijectively to elements of $\tilde{K}$.

For points $x = \zeta_{a,r}$ of type III, there are only two possible tangent directions: one leading "up" towards infinity, and one going "down" towards a. Similarly, since points of type I or IV are "endpoints" of $\mathbb{P}^1_{\mathrm{Berk}}$, the set T_x of tangent directions at a point $x \in \mathbb{P}^1_{\mathrm{Berk}}$ of type I or IV consists of just one element.

In particular, for $x \in \mathbb{P}^1_{\mathrm{Berk}}$, we have:

$$|T_x| = \begin{cases} |\mathbb{P}^1(\tilde{K})| & x \text{ of type II} \\ 2 & x \text{ of type III} \\ 1 & x \text{ of type I or type IV.} \end{cases}$$

Finally, we explain how to interpret the sets $U(x;\vec{v})$ as "open Berkovich disks".

[2]It is not difficult to show that a subset of $\mathbb{P}^1_{\mathrm{Berk}}$ is connected if and only if it is path-connected, and in particular that the path-connected components of $\mathbb{P}^1_{\mathrm{Berk}}$ coincide with the connected components.

For $a \in K$ and $r > 0$, write

$$\begin{aligned} \mathcal{B}(a,r)^- &= \{x \in \mathbb{A}^1_{\mathrm{Berk}} : |T-a|_x < r\} \ , \\ \mathcal{B}(a,r) &= \{x \in \mathbb{A}^1_{\mathrm{Berk}} : |T-a|_x \leq r\} \ . \end{aligned}$$

We call a set of the form $\mathcal{B}(a,r)^-$ an *open Berkovich disk* in $\mathbb{A}^1_{\mathrm{Berk}}$, and a set of the form $\mathcal{B}(a,r)$ a *closed Berkovich disk* in $\mathbb{A}^1_{\mathrm{Berk}}$.

Similarly, we can define open and closed Berkovich disks in $\mathbb{P}^1_{\mathrm{Berk}}$: an open (resp. closed) Berkovich disk in $\mathbb{P}^1_{\mathrm{Berk}}$ is either an open (resp. closed) Berkovich disk in $\mathbb{A}^1_{\mathrm{Berk}}$ or the complement of a closed (resp. open) Berkovich disk in $\mathbb{A}^1_{\mathrm{Berk}}$.

It follows from the definitions that the intersection of a Berkovich open (resp. closed) disk in $\mathbb{P}^1_{\mathrm{Berk}}$ with $\mathbb{P}^1(K)$ is an open (resp. closed) disk in $\mathbb{P}^1(K)$.

We have:

Lemma 1.3. *Every open set $U(x;\vec{v})$ with x of type II or III and $\vec{v} \in T_x$ is a Berkovich open disk in $\mathbb{P}^1_{\mathrm{Berk}}$, and conversely.*

Remark 1.4. A fundamental system of open neighborhoods for the topology on $\mathbb{P}^1_{\mathrm{Berk}}$ is given by the finite intersections of Berkovich open disks in $\mathbb{P}^1_{\mathrm{Berk}}$ (c.f. Lemma 2.7 below).

1.5. **The Berkovich hyperbolic space $\mathbf{H}_{\mathrm{Berk}}$ and its canonical metric.** Following notation introduced by Juan Rivera-Letelier, we write $\mathbf{H}_{\mathrm{Berk}}$ for the subset of $\mathbb{P}^1_{\mathrm{Berk}}$ consisting of all points of type II, III, or IV, and call $\mathbf{H}_{\mathrm{Berk}}$ "Berkovich hyperbolic space". We also write $\mathbf{H}^{\mathbb{Q}}_{\mathrm{Berk}}$ for the set of type II points, and $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ for the set of points of type II or III.

The subset $\mathbf{H}^{\mathbb{Q}}_{\mathrm{Berk}}$ is dense in $\mathbb{P}^1_{\mathrm{Berk}}$ (and therefore $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ and $\mathbf{H}_{\mathrm{Berk}}$ are also dense).

There is a canonical metric ρ on $\mathbf{H}_{\mathrm{Berk}}$, which we refer to as the *path metric*, that is of great importance for potential theory. To define this metric, we first define the *diameter function* $\mathrm{diam} : \mathbb{A}^1_{\mathrm{Berk}} \to \mathbb{R}_{\geq 0}$ by setting $\mathrm{diam}(x) = \lim r_i$ if x corresponds to the nested sequence $\{B(a_i, r_i)\}$. This is well-defined independent of the choice of nested sequence. If $x \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$, then $\mathrm{diam}(x)$ is just the diameter (= radius) of the corresponding closed disk. In terms of multiplicative seminorms, we have

$$\mathrm{diam}(x) = \inf_{a \in K} |T-a|_x.$$

Because K is complete, it is not hard to see that if x is of type IV, then necessarily $\mathrm{diam}(x) > 0$. Thus $\mathrm{diam}(x) = 0$ for $x \in \mathbb{A}^1_{\mathrm{Berk}}$ of type I, and $\mathrm{diam}(x) > 0$ for $x \in \mathbf{H}_{\mathrm{Berk}}$.

If $x, y \in \mathbf{H}_{\mathrm{Berk}}$ with $x \leq y$, we define

$$\rho(x,y) = \log_v \frac{\mathrm{diam}(y)}{\mathrm{diam}(x)} \ ,$$

where $\log_v$ denotes the logarithm to the base q_v, with $q_v > 1$ a fixed real number chosen so that $-\log_v$ is a suitably normalized valuation on K. For example, if $K = \mathbb{C}_p$, endowed with the standard absolute value $| \ |_p$ for which $|p|_p = 1/p$, then we set $q_v = p$ in order to have

$$\{-\log_v |x|_p \ : \ x \in \mathbb{C}_p^*\} = \mathbb{Q}.$$

More generally, for $x, y \in \mathbf{H}_{\mathrm{Berk}}$ arbitrary, we define

$$\rho(x,y) = \rho(x, x \vee y) + \rho(y, x \vee y) \ .$$

It is not hard to verify that ρ defines a metric on $\mathbf{H}_{\mathrm{Berk}}$ satisfying the ultrametric inequality. One can extend ρ to a singular metric on $\mathbb{P}^1_{\mathrm{Berk}}$ by declaring that if $x \in \mathbb{P}^1(K)$ and $y \in \mathbb{P}^1_{\mathrm{Berk}}$, we have $\rho(x,y) = +\infty$ if $x \neq y$ and 0 if $x = y$. However, we will usually only consider ρ as being defined on $\mathbf{H}_{\mathrm{Berk}}$.

Intuitively, $\rho(x,y)$ is just the "length" of the unique path $\ell_{x,y}$ between x and y, which for closed disks $B(a,r) \subseteq B(a,R)$ is just $\log_v R - \log_v r$.

Remark 1.5. It is important to note that the topology on $\mathbf{H}_{\mathrm{Berk}}$ defined by the metric ρ is *not* the subspace topology induced from the Berkovich topology on $\mathbb{P}^1_{\mathrm{Berk}}$. However, the inclusion map $i : \mathbf{H}_{\mathrm{Berk}} \hookrightarrow \mathbb{P}^1_{\mathrm{Berk}}$ is continuous with respect to these topologies.

The group $\mathrm{PGL}(2,K)$ of *Möbius transformations* acts continuously on $\mathbb{P}^1_{\mathrm{Berk}}$ in a natural way compatible with the usual action on $\mathbb{P}^1(K)$, and this action preserves $\mathbf{H}_{\mathrm{Berk}}, \mathbf{H}^{\mathbb{Q}}_{\mathrm{Berk}}$, and $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$. (The action of $\mathrm{PGL}(2,K)$ on $\mathbb{P}^1_{\mathrm{Berk}}$ can be described quite concretely in terms of Berkovich's classification theorem, using the fact that each $M \in \mathrm{PGL}(2,K)$ takes closed disks to closed disks, but it can also be described more intrinsically in terms of multiplicative seminorms; see §2.1 for further details.)

An important observation is that $\mathrm{PGL}(2,K)$ acts *via isometries* on $\mathbf{H}_{\mathrm{Berk}}$, i.e.,

$$\rho(M(x), M(y)) = \rho(x,y)$$

for all $x, y \in \mathbf{H}_{\mathrm{Berk}}$ and all $M \in \mathrm{PGL}(2,K)$. This shows that the metric ρ really is canonical and coordinate-free.

1.6. **The canonical distance.**

1.6.1. *The canonical distance relative to infinity.* The diameter function diam introduced in §1.5 can be used to extend the usual distance function $|x-y|$ on K to $\mathbb{A}^1_{\rm Berk}$ in a natural way. We call this extension the *canonical distance* (relative to infinity), and denote it by $[x,y]_\infty$.

Formally, for $x,y \in \mathbb{A}^1_{\rm Berk}$ we have

$$[x,y]_\infty = \mathrm{diam}(x \vee y) \ . \tag{1.6}$$

It is easy to see that if $x,y \in K$ then $[x,y]_\infty = |x-y|$. More generally, one has the formula

$$[x,y]_\infty = \limsup_{(x_0,y_0)\to(x,y)} |x_0 - y_0| \ ,$$

where $(x_0,y_0) \in K \times K$ and the lim sup is taken with respect to the product topology on $\mathbb{P}^1_{\rm Berk} \times \mathbb{P}^1_{\rm Berk}$. The canonical distance $[x,y]_\infty$ satisfies all of the axioms for an ultrametric except for the fact that $[x,x]_\infty > 0$ for $x \in \mathbf{H}_{\rm Berk}$.

Remark 1.7. In [BR06], $[x,y]_\infty$ is written as $\delta(x,y)_\infty$, and is called the *Hsia kernel*.

1.6.2. *The canonical distance relative to an arbitrary point.* In this section, we describe a function $[x,y]_z$ which extends the *canonical distance* on $\mathbb{P}^1(K)$, as introduced by Rumely in [Rum89], to the Berkovich projective line. When $z = \infty$, it coincides with the canonical distance relative to infinity as defined in the previous section.

Let x,y,z be points of $\mathbb{P}^1_{\rm Berk}$, not all equal. Following the terminology introduced by Favre and Rivera-Letelier in [FRL04b, FRL04a], we define the *Gromov product* $(x|y)_z$ by

$$(x|y)_z = \rho(w,z),$$

where w is the first point where the unique paths from x to z and y to z intersect. By convention, we set $(x|y)_z = +\infty$ if $x = y$ and x is a point of type I, and we set $(x|y)_z = 0$ if $x = z$ or $y = z$.

Remark 1.8. If $x,y,z \in \mathbf{H}_{\rm Berk}$, then one checks easily that

$$(x|y)_z = \frac{1}{2}(\rho(x,z) + \rho(y,z) - \rho(x,y)).$$

This is the usual definition of the Gromov product in Gromov's theory of δ-hyperbolic spaces, with $\mathbf{H}_{\rm Berk}$ being an example of a 0-hyperbolic space.

Remark 1.9. In [BR06], the function $(x|y)_z$ is written $j_z(x,y)$.

Next, define the *fundamental potential kernel relative to* z, written $\kappa_z(x,y)$, and the *canonical distance relative to* z, written $[x,y]_z$, by setting

$$\kappa_z(x,y) = -\log_v [x,y]_z = (x|y)_\zeta - (x|z)_\zeta - (y|z)_\zeta, \tag{1.10}$$

where $\zeta = \zeta_{\text{Gauss}}$ is the Gauss point of $\mathbb{P}^1_{\text{Berk}}$. One can define $\kappa_z(x,y)_z$ as an extended-real number for all $x,y,z \in \mathbb{P}^1_{\text{Berk}}$ by setting $\kappa_z(z,y) = \kappa_z(x,z) = -\infty$ if z is a point of type I.

Remark 1.11. 1. In [BR06, Chapter 3], the notation $\delta(x,y)_z$ is used instead of $[x,y]_z$, and $\delta(x,y)_z$ is referred to as the *generalized Hsia kernel.* For $x,y,z \in \mathbb{P}^1(K)$, our definition of $[x,y]_z$ agrees with Rumely's definition of the canonical distance in [Rum89].

2. If ζ, ζ' are arbitrary points of $\mathbf{H}_{\text{Berk}}$, one can show that

$$(x|y)_\zeta - (x|z)_\zeta - (y|z)_\zeta = (x|y)_{\zeta'} - (x|z)_{\zeta'} - (y|z)_{\zeta'} + C_{\zeta,\zeta'}$$

for some constant $C_{\zeta,\zeta'}$ independent of x,y,z. Thus a different choice of ζ in (1.10) would only change the definition of $\kappa_z(x,y)$ by an additive constant. Our choice $\zeta = \zeta_{\text{Gauss}}$ is just a convenient normalization.

3. After defining the Laplacian operator Δ on $\mathbb{P}^1_{\text{Berk}}$, we will see in Example 4.25 below that for y,z fixed, the function $f(x) = \kappa_z(x,y)$ satisfies the *Laplace equation* $\Delta(f) = \delta_y - \delta_z$, and up to an additive constant f is in fact the *unique* such function.

Since the definition of $[x,y]_z$ takes some getting used to, we will attempt to orient the reader with the following illustrative examples:

Example 1.12. If $z = \infty$, it is straightforward (but not completely trivial) to verify that the definitions of $[x,y]_\infty$ given in (1.10) and (1.6) coincide. Thus our notation is consistent, and $[x,y]_\infty$ extends the distance function $|x-y|$ on $K \times K$.

Example 1.13. If x,y are written in homogeneous coordinates as $x = (x_1 : x_2)$ and $y = (y_1 : y_2)$, the *spherical metric* on $\mathbb{P}^1(K)$ is given by

$$\|x,y\| = \frac{|x_1 y_2 - x_2 y_1|}{\max(|x_1|,|x_2|)\cdot\max(|y_1|,|y_2|)}.$$

If $z = \zeta_{\text{Gauss}}$, then $-\log_v [x,y]_{\zeta_{\text{Gauss}}}$ coincides with the Gromov product $(x|y)_{\zeta_{\text{Gauss}}}$, and the restriction of $[x,y]_{\zeta_{\text{Gauss}}}$ to $x,y \in \mathbb{P}^1(K)$ coincides with the spherical metric $\|x,y\|$ on $\mathbb{P}^1(K)$.

We will sometimes write $\|x,y\|$ for the extended function $[x,y]_{\zeta_{\text{Gauss}}}$ on $\mathbb{P}^1_{\text{Berk}} \times \mathbb{P}^1_{\text{Berk}}$.

Remark 1.14. 1. Note that unlike $[x,y]_\infty$, which is singular at infinity, the function $\|x,y\| = [x,y]_{\zeta_{\text{Gauss}}}$ is bounded and real-valued on all of $\mathbb{P}^1_{\text{Berk}} \times \mathbb{P}^1_{\text{Berk}}$

2. By (1.10), we have the identity

$$[x,y]_z = \frac{\|x,y\|}{\|x,z\|\,\|y,z\|}.$$

The following result describes some of the main properties possessed by the canonical distance $[x,y]_z$ on $\mathbb{P}^1_{\mathrm{Berk}}$. Recall that if X is a topological space, a real-valued function $f : X \to [-\infty,\infty)$ is called *upper semicontinuous* if for each $x_0 \in X$,

$$\limsup_{x\to x_0} f(x) \ \leq \ f(x_0)\ .$$

This is equivalent to requiring that $f^{-1}([-\infty,b))$ be open for each $b \in \mathbb{R}$.

Proposition 1.15. *1. For each $z \in \mathbb{P}^1_{\mathrm{Berk}}$, the canonical distance $[x,y]_z$ is nonnegative, symmetric, and continuous in each variable separately. If $z \in \mathbf{H}_{\mathrm{Berk}}$, then $[x,y]_z$ is bounded. For $z \in \mathbb{P}^1(K)$ it is unbounded, and extends the canonical distance $[x,y]_z$ from* [Rum89].

2. As a function of x and y, the canonical distance $[x,y]_z$ is upper semicontinuous. It is continuous off the diagonal, and is continuous at (x_0,x_0) for each point $x_0 \in \mathbb{P}^1(K)$ of type I, but is discontinuous at (x_0,x_0) for each point $x_0 \in \mathbf{H}_{\mathrm{Berk}}$.

3. For each $x,y \in \mathbb{P}^1_{\mathrm{Berk}}$,

$$[x,y]_z \ = \ \limsup_{\substack{(a,b)\to(x,y)\\ a,b\in\mathbb{P}^1(K)}} [a,b]_z\ .$$

4. For all $x,y,w \in \mathbb{P}^1_{\mathrm{Berk}}$, the ultrametric inequality

$$[x,y]_z \ \leq \ \max([x,w]_z,[y,w]_z)$$

holds, with equality if $[x,w]_z \neq [y,w]_z$.

5. If f is a nonzero meromorphic function on $\mathbb{P}^1$ with divisor $\mathrm{Div}(f) = \sum m_i(a_i)$, then for any $z \in X_{\mathrm{Berk}}$, there is a constant C (depending on z and f) such that

$$|f(x)| = C \cdot \prod [x,a_i]_z^{m_i}$$

for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$.

2. **Lecture II**: Introduction to Berkovich curves

In this lecture, we will explore further properties and an alternative definition of the Berkovich projective line, and then we discuss some more general Berkovich spaces. For example, after defining the Berkovich analytic space $\mathcal{M}(\mathcal{A})$ associated to an arbitrary normed ring $\mathcal{A}$, we will describe in detail the topological structure of $\mathcal{M}(\mathbb{Z})$. We will then give a brief overview of the topological structure of Berkovich analytic curves. (A more detailed description will be given in §5.)

2.1. **The Berkovich "Proj" construction.** As a topological space, we have defined the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},K}$ to be the one-point compactification of the locally compact Hausdorff space $\mathbb{A}^1_{\mathrm{Berk},K}$. However, this description depends on a choice of coordinates, and is often awkward to use. For example, it is not immediately clear from this definition how a rational function $\varphi \in K(T)$ induces a natural map from $\mathbb{P}^1_{\mathrm{Berk}}$ to itself. We therefore introduce the following alternate construction of $\mathbb{P}^1_{\mathrm{Berk},K}$, analogous to the "Proj" construction in algebraic geometry.

Let S denote the set of multiplicative seminorms $[\ ]$ on the two-variable polynomial ring $K[X,Y]$ which extend the absolute value on K, and which are not identically zero on the maximal ideal (X,Y) of $K[X,Y]$. It is easy to see that $[\ ]$ is automatically non-archimedean, and therefore the condition that $[\ ]$ is not identically zero on (X,Y) is equivalent to saying that $[X]$ and $[Y]$ are not both zero.

We put an equivalence relation on S by declaring that $[\ ]_1 \sim [\ ]_2$ if and only if there exists a constant $C > 0$ such that $[G]_1 = C^d[G]_2$ for all homogeneous polynomials $G \in K[X,Y]$ of degree d.

As a set, define $\mathbb{P}^1_{\mathrm{Berk}}$ to be the equivalence classes of elements of S.

Define the point ∞ in $\mathbb{P}^1_{\mathrm{Berk}}$ to be the equivalence class of the seminorm $[\ ]_\infty$ defined by $[G]_\infty = |G(1,0)|$. More generally, if $P \in \mathbb{P}^1(K)$ has homogeneous coordinates $(a:b)$, the equivalence class of the evaluation seminorm $[G]_P = |G(a,b)|$ is independent of the choice of homogeneous coordinates, and therefore $[\ ]_P$ is a well-defined point of $\mathbb{P}^1_{\mathrm{Berk}}$. This furnishes an embedding of $\mathbb{P}^1(K)$ into $\mathbb{P}^1_{\mathrm{Berk}}$.

We say that a seminorm $[\ ]$ in S is *normalized* if $\max\{[X],[Y]\} = 1$. One can show using the definition of the equivalence relation on S that there is a *unique* normalized seminorm within each equivalence class. We denote by $[\ ]_z^*$ the normalized seminorm corresponding to a point $z \in \mathbb{P}^1_{\mathrm{Berk}}$. Explicitly, if $[\ ]_z$ is any representative of the equivalence class of z, then

$$[G]_z^* = [G]_z / \max\{[X]_z, [Y]_z\}^d$$

for all homogeneous polynomials $G \in K[X,Y]$ of degree d.

The topology on $\mathbb{P}^1_{\mathrm{Berk}}$ is defined to be the weakest one such that $z \mapsto [G]_z^*$ is continuous for all $G \in K[X,Y]$.

Proposition 2.1. *This definition of $\mathbb{P}^1_{\mathrm{Berk}}$ as a topological space agrees with the previous one.*

Let $\varphi \in K(T)$ be a rational function of degree $d \geq 1$. To conclude this section, we explain how to extend the usual action of φ on $\mathbb{P}^1(K)$ to a continuous map $\varphi : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{P}^1_{\mathrm{Berk}}$.

Choose a homogeneous lifting $F = (F_1, F_2)$ of φ, where $F_i \in K[X, Y]$ are homogeneous of degree d and have no common zeros in K. (Recall that the field K is assumed to be algebraically closed.) The condition that F_1 and F_2 have no common zeros is equivalent to requiring that the homogeneous resultant $\mathrm{Res}(F) = \mathrm{Res}(F_1, F_2)$ is nonzero.

We define the action of φ on $\mathbb{P}^1_{\mathrm{Berk}}$ as follows: Let $G \in K[X, Y]$, and define

$$[G]_{\varphi(z)} := [G(F_1(X, Y), F_2(X, Y))]_z. \tag{2.2}$$

It is readily verified that the right-hand side of (2.2) is independent of the lifting F of φ, up to equivalence of seminorms. As it is clear that the right-hand side of (2.2) gives a continuous multiplicative seminorm on $K[X, Y]$, to see that (2.2) induces a map from $\mathbb{P}^1_{\mathrm{Berk}}$ to itself, it suffices to note that $[X]_{\varphi(z)} = [F_1(X, Y)]_z$ and $[Y]_{\varphi(z)} = [F_2(X, Y)]_z$ cannot both be zero; this can be proved using standard properties of resultants.

In particular, we see that the group $\mathrm{PGL}(2, K)$ acts naturally on $\mathbb{P}^1_{\mathrm{Berk}}$ via automorphisms, as mentioned in §1.5.

Remark 2.3. One can show that $\varphi : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{P}^1_{\mathrm{Berk}}$ is an open surjective mapping, and that every point $z \in \mathbb{P}^1_{\mathrm{Berk}}$ has at most d preimages under φ.

Remark 2.4. Note that if $z \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$, then $\varphi(z) \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$ as well because the seminorm $[G]_{\varphi(z)}$ has trivial kernel (i.e., is a norm), whereas for each $a \in \mathbb{P}^1(K)$, the corresponding seminorm has nonzero kernel.

More generally, one can verify that φ takes type I points to type I points, type II points to type II points, type III points to type III points, and type IV points to type IV points.

2.2. **$\mathbb{P}^1_{\mathrm{Berk}}$ as an inverse limit of $\mathbb{R}$-trees.** We now come to an important description of $\mathbb{P}^1_{\mathrm{Berk}}$ as a *profinite* $\mathbb{R}$-*tree*.

We will need the following definitions. An $\mathbb{R}$-*tree* is a metric space (T, d) such that for each distinct pair of points $x, y \in T$, there is a unique path in T from x to y. A topological space homeomorphic to an $\mathbb{R}$-tree (but which is not necessarily endowed with a distinguished metric) will be called a *topological tree*. A *branch point* of a topological tree is a point $x \in T$ for which $T \backslash \{x\}$ has either fewer than or more than two connected components. A *finite* $\mathbb{R}$*-tree* (resp. topological tree) is an $\mathbb{R}$-tree (resp. topological tree) with only finitely many branch points. Intuitively, a finite $\mathbb{R}$-tree is just a tree in the usual graph-theoretic sense, but where the edges are thought of as line segments having definite lengths, and a finite topological tree is the same thing as a connected one-dimensional CW-complex with finitely many cells. Finally, a *profinite* $\mathbb{R}$*-tree* is an inverse limit of finite $\mathbb{R}$-trees.

Here's how these definitions play out in the case of $\mathbb{P}^1_{\text{Berk}}$. If $S \subset \mathbb{P}^1_{\text{Berk}}$, define the *convex hull* of S to be the smallest path-connected subset of $\mathbb{P}^1_{\text{Berk}}$ containing S. (This is the same as the union of all paths between points of S.) By a *finite subgraph* of $\mathbb{P}^1_{\text{Berk}}$, we will mean the convex hull of a finite subset $S \subset \mathbf{H}^{\mathbb{R}}_{\text{Berk}}$. Every finite subgraph Γ can be thought of as a finite $\mathbb{R}$-tree, with the metric induced by the path-distance ρ on $\mathbf{H}_{\text{Berk}}$. By construction, a finite subgraph of $\mathbb{P}^1_{\text{Berk}}$ is both finitely branched and of finite total length with respect to ρ.[3]

The collection of all finite subgraphs of $\mathbb{P}^1_{\text{Berk}}$ is a directed set under inclusion. Moreover, if $\Gamma \leq \Gamma'$, then by a basic property of $\mathbb{R}$-trees, there is a continuous *retraction map* $r_{\Gamma',\Gamma} : \Gamma' \twoheadrightarrow \Gamma$. The following result can be thought of as a topological reformulation of Berkovich's classification theorem:

Theorem 2.5. *$\mathbb{P}^1_{\text{Berk}}$ is homeomorphic to the inverse limit $\varprojlim \Gamma$ over all finite subgraphs $\Gamma \subset \mathbb{P}^1_{\text{Berk}}$.*

This description of $\mathbb{P}^1_{\text{Berk}}$ as a profinite $\mathbb{R}$-tree provides a convenient way to visualize the topology on $\mathbb{P}^1_{\text{Berk}}$: two points are "close" if they retract to the same point of a "large" finite subgraph.

The following useful fact is a simple consequence of our definitions.

Lemma 2.6. *The direct limit of all finite subgraphs Γ of $\mathbb{P}^1_{\text{Berk}}$ with respect to inclusion is homeomorphic to the space $\mathbf{H}^{\mathbb{R}}_{\text{Berk}}$, endowed with the metric topology coming from ρ.*

Let r_Γ be the natural map from $\mathbb{P}^1_{\text{Berk}}$ to Γ coming from the universal property of the inverse limit. A fundamental system of open neighborhoods for the topology on $\mathbb{P}^1_{\text{Berk}}$ is given by the *connected open affinoids*, or *simple domains*, which are subsets of the form $r_\Gamma^{-1}(V)$ for Γ a finite subgraph of $\mathbb{P}^1_{\text{Berk}}$ and V a connected open subset of Γ.

Lemma 2.7. *For a subset $U \subseteq \mathbb{P}^1_{\text{Berk}}$, the following are equivalent:*

[3] We have chosen to require in addition that $\partial\Gamma \subset \mathbf{H}^{\mathbb{R}}_{\text{Berk}}$, but this could be relaxed by allowing a finite subgraph to have boundary points of type IV without creating any major differences in the resulting theory.

1. U is a simple domain.
2. U is a finite intersection of Berkovich open disks.
3. U is a connected open set whose boundary is a finite subset of $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$.

2.3. **The Berkovich spectrum of a normed ring.** In this section, we explain a general construction which associates a Berkovich analytic space to an arbitrary normed ring.

2.3.1. *Seminorms and norms.* A *seminorm* on a ring $\mathcal{A}$ is a function $|\ | : \mathcal{A} \to \mathbb{R}_{\geq 0}$ with values in the set of nonnegative reals such that for every $f, g \in \mathcal{A}$, we have

(S1) $|0| = 0, |1| = 1$.
(S2) $|f + g| \leq |f| + |g|$.
(S3) $|f \cdot g| \leq |f| \cdot |g|$.

A seminorm $|\ |$ defines a topology on $\mathcal{A}$ in the usual way, and this topology is Hausdorff if and only if $|\ |$ is a *norm*, meaning that $|f| = 0$ if and only if $f = 0$.

A *normed ring* is a pair $(\mathcal{A}, \|\ \|)$ consisting of a ring $\mathcal{A}$ and a norm $\|\ \|$. It is called a *Banach ring* if $\mathcal{A}$ is complete with respect to this norm. Any ring may be regarded as a Banach ring with respect to the *trivial norm*, for which $\|0\| = 0$ and $\|f\| = 1$ for $f \neq 0$.

A seminorm $|\ |$ on a ring $\mathcal{A}$ is called *multiplicative* if for all $f, g \in \mathcal{A}$, we have

$(S3)'$ $|f \cdot g| = |f| \cdot |g|$,

and it is called *non-archimedean* if

$(S2)'$ $|f + g| \leq \max\{|f|, |g|\}$.

A multiplicative norm on a ring $\mathcal{A}$ is also called an *absolute value* on $\mathcal{A}$.

A seminorm $|\ |$ on a normed ring $(\mathcal{A}, \|\ \|)$ is called *bounded* if

(S4) there exists a constant $C > 0$ such that $|f| \leq C\|f\|$ for all $f \in \mathcal{A}$.

Lemma 2.8. *If $|\ |$ is a multiplicative seminorm, then condition (S4) is equivalent to:*

$(S4)'$ $|f| \leq \|f\|$ *for all* $f \in \mathcal{A}$.

Proof. Since $|f^n| \leq C\|f^n\| \leq C\|f\|^n$, we have $|f| \leq \sqrt[n]{C}\|f\|$ for all $n \geq 1$. Passing to the limit as n tends to infinity yields the desired result. $\square$

2.3.2. *The Berkovich spectrum of a normed ring.* Let $(\mathcal{A}, \| \ \|)$ be a normed ring. We define a topological space $\mathcal{M}(\mathcal{A})$, called the *Berkovich spectrum* of $\mathcal{A}$, as follows. As a set, $\mathcal{M}(\mathcal{A})$ consists of all bounded multiplicative seminorms on $\mathcal{A}$. The topology on $\mathcal{M}(\mathcal{A})$ (which we will call the *Berkovich topology*[4]) is defined to be the weakest one for which all functions of the form $| \ | \mapsto |f|$ for $f \in \mathcal{A}$ are continuous.

It is useful from a notational standpoint to denote points of $X = \mathcal{M}(\mathcal{A})$ by a letter such as x, and the corresponding bounded multiplicative seminorm by $| \ |_x$. With this notation, a fundamental system of open neighborhoods for the topology on X is given by the collection

$$U(f, \alpha, \beta) \ = \ \{x \in X : \alpha < |f|_x < \beta\}$$

for all $f \in \mathcal{A}$ and all $\alpha < \beta$ in $\mathbb{R}$.

Equivalently, one may define the topology on $\mathcal{M}(\mathcal{A})$ as the *topology of pointwise convergence*: a net[5] $\langle x_\alpha \rangle$ in $\mathcal{M}(\mathcal{A})$ converges to $x \in \mathcal{M}(\mathcal{A})$ if and only if $|f|_{x_\alpha}$ converges to $|f|_x$ in $\mathbb{R}$ for all $f \in \mathcal{A}$.

Theorem 2.9. *If $\mathcal{A}$ is a Banach ring, then the spectrum $\mathcal{M}(\mathcal{A})$ is a non-empty compact Hausdorff space.*

Proof. This is proved in Theorem 1.2.1 of [Ber90]. The fact that $\mathcal{M}(\mathcal{A})$ is Hausdorff is an easy exercise. The proof that $\mathcal{M}(\mathcal{A})$ is always non-empty is rather subtle, though. (In many cases of interest, however, such as when the norm on $\mathcal{A}$ is multiplicative, the fact that $\mathcal{M}(\mathcal{A})$ is non-empty is obvious.)

Here is a quick proof, different from the one in [Ber90], of the compactness of $\mathcal{M}(\mathcal{A})$. It suffices by general topology to prove that every net in $X = \mathcal{M}(\mathcal{A})$ has a convergent subnet. Let T be the space $\prod_{f \in \mathcal{A}} [0, \|f\|]$ endowed with the product topology. By Tychonoff's theorem, T is compact. By Lemma 2.8, there is a natural map $\iota : X \to T$ sending $x \in X$ to $(|f|_x)_{f \in \mathcal{A}}$, and ι is clearly injective and continuous.

Let $\langle x_\alpha \rangle$ be a net in X. Since T is compact, $\langle \iota(x_\alpha) \rangle$ has a subnet $\langle \iota(y_\beta) \rangle$ converging to an element $(\alpha_f)_{f \in \mathcal{A}} \in T$. Define a function $| \cdot |_y : \mathcal{A} \to \mathbb{R}_{\geq 0}$ by $|f|_y = \alpha_f$. It is easily verified that $| \cdot |_y$ is a bounded multiplicative seminorm on $\mathcal{A}$, and thus defines a point $y \in X$. By

[4] This topology is also referred to as the *Gelfand topology.*

[5] Recall that a *net* in a topological space X is a mapping from a directed set I to X, with a sequence being the special case where $I = \mathbb{N}$. For non-metrizable topological spaces, nets are much better than sequences for describing the interplay between concepts like convergence and continuity. The space $\mathbb{P}^1_{\mathrm{Berk},K}$ is metrizable if and only if the residue field $\tilde{K}$ of K is countable.

construction, we have $\iota(y) = \lim_\beta \iota(y_\beta)$. This implies that $\lim_\beta |f|_{y_\beta} = |f|_y$ for all $f \in \mathcal{A}$, i.e., $y_\beta \to y$. Thus $\langle x_\alpha \rangle$ has a convergent subnet as desired. $\square$

2.4. **The analytification of an algebraic variety.** As discussed in [Ber90, §3.4.1] and [Ber93, §2.6] (see also [Duc06, §1.4]), one can associated in a functorial way to every algebraic variety X/K a locally ringed topological space X_{Berk} called the *Berkovich K-analytic space associated to X*. We will refer to the functor from algebraic varieties over K to Berkovich K-analytic spaces as the *Berkovich analytification functor*. When $X = \mathrm{Spec}(A)$ is affine, the underlying topological space of X_{Berk} is the set of multiplicative seminorms on A which extend the given absolute value on K, equipped with the weakest topology for which all functions of the form $|\ | \mapsto |f|$ for $f \in A$ are continuous. When $X = \mathbb{A}^1$ (resp. $\mathbb{P}^1$), we recover the definition of $\mathbb{A}^1_{\mathrm{Berk}}$ (resp. $\mathbb{P}^1_{\mathrm{Berk}}$) given above. The space X_{Berk} is locally compact and Hausdorff, and if X is proper then X_{Berk} is compact. Moreover, if X is connected in the Zariski topology then X_{Berk} is path-connected. Finally, there is a canonical embedding of $X(K)$ (endowed with its totally disconnected analytic topology) as a dense subspace of X_{Berk}.

As a concrete example, let X be a smooth, proper, and geometrically integral algebraic curve over K. We briefly describe the topological structure of X_{Berk}; further details will be given in §5.1.

A *finite topological graph* is just a finite connected graph whose edges are thought of as line segments; this is essentially the same thing as a connected one-dimensional CW-complex with finitely many cells. If the genus of X is at least one, there is a canonically defined subset $\Sigma \subset X_{\mathrm{Berk}}$, called the *skeleton* of X_{Berk}, which is homeomorphic to a finite topological graph. Moreover, the entire space X_{Berk} admits a deformation retraction r onto Σ. (In the case $X = \mathbb{P}^1$, there is no canonical skeleton, but after choosing coordinates, we can if we like define the skeleton of $\mathbb{P}^1_{\mathrm{Berk}}$ to be the Gauss point ζ_{Gauss}.) The skeleton of X_{Berk} is equipped with a canonical metric.

For each $x \in \Sigma$, the fiber $r^{-1}(x)$ is homeomorphic to a compact, connected subset of $\mathbb{P}^1_{\mathrm{Berk}}$, and in particular is a topological tree. Using this, one can define a notion of a "finite subgraph" of X_{Berk} in such a way that X_{Berk} is homeomorphic to the inverse limit of its finite subgraphs (see §5.1 below).

More generally, Berkovich proves in [Ber99] and [Ber04] that every smooth K-analytic space (for example, the analytification of a smooth projective variety over K) admits a deformation retraction onto a finite polyhedral complex which can be equipped in a canonical way with a piecewise-linear affine structure (generalizing the metric on the skeleton of a curve). This is a very difficult result which relies, among other things, on de Jong's theory of *alterations*, and we will not discuss the higher-dimensional case any further in these notes. See [Duc06, §2] for a nice overview of this and many other aspects of Berkovich's theory.

2.5. **The Berkovich space $\mathcal{M}(\mathbb{Z})$.** We now consider a simple but interesting example of the construction from (2.3.2): the Berkovich analytic space $\mathcal{M}(\mathbb{Z})$ associated to the normed ring $(\mathbb{Z}, |\ |_\infty)$, where $|\ |_\infty$ denotes the usual archimedean absolute value on $\mathbb{Z}$.

A famous result of Ostrowski asserts that every non-trivial absolute value on $\mathbb{Q}$ is equivalent to either $|\ |_\infty$, or to the standard p-adic absolute value $|\ |_p$ for some prime number p. (We normalize $|\ |_p$ in the usual way so that $|p|_p = \frac{1}{p}$.)

Thus, if we let $M_\mathbb{Q}$ denote the set of *places* (equivalence classes of non-trivial absolute values) of $\mathbb{Q}$, then there is a bijection

$$M_\mathbb{Q} \leftrightarrow \{\text{prime numbers } p\} \cup \{\infty\}.$$

With this notation, the *product formula* states that if $\alpha \in \mathbb{Q}^*$ is a non-zero rational number, then

$$\prod_{v \in M_\mathbb{Q}} |\alpha|_v = 1.$$

Following Berkovich[6], one can classify all *multiplicative seminorms* on $\mathbb{Z}$ as follows:

(Z1) The *p-trivial seminorms* $|\ |_{p,\infty}$ defined by

$$|n|_{p,\infty} = \begin{cases} 0 & p \mid n \\ 1 & p \nmid n. \end{cases}$$

(Z2) The *trivial seminorm* $|\ |_0$ defined by

$$|n|_0 = \begin{cases} 0 & n = 0 \\ 1 & n \neq 0. \end{cases}$$

(Z3) The *p-adic absolute values* $|\ |_{p,\epsilon}$ for $0 < \epsilon < \infty$ defined by

$$|n|_{p,\epsilon} = |n|_p^\epsilon.$$

(Z4) The *archimedean absolute values* $|\ |_{\infty,\epsilon}$ for $0 < \epsilon \leq 1$ defined by

$$|n|_{\infty,\epsilon} = |n|_\infty^\epsilon.$$

All of the seminorms in (Z1)-(Z4) are clearly bounded. Moreover:

[6]See [Ber90, Example 1.4.1], although our notation is slightly different

Lemma 2.10. *For all $n \in \mathbb{Z}$, we have:*
(1) $\lim_{\epsilon \to 0} |n|_{\infty,\epsilon} = |n|_0$.
(2) $\lim_{\epsilon \to \infty} |n|_{p,\epsilon} = |n|_{p,\infty}$.
(3) $\lim_{\epsilon \to 0} |n|_{p,\epsilon} = |n|_0$.

We will therefore write $|\ |_{\infty,0}$ or $|\ |_{p,0}$ instead of $|\ |_0$ when convenient.

This leads to the following visual representation of the Berkovich analytic space $\mathcal{M}(\mathbb{Z})$ associated to $\mathbb{Z}$:

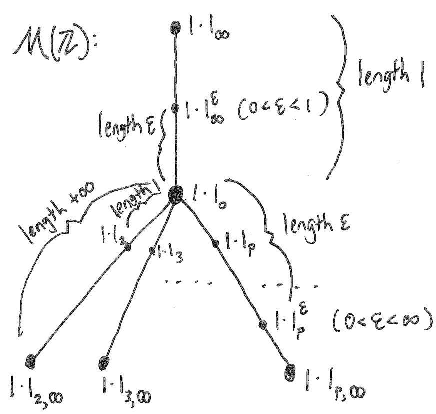

FIGURE 2. The space $\mathcal{M}(\mathbb{Z})$.

Note that the different "tangent directions" emanating from the trivial seminorm $|\ |_0$ are in one-to-one correspondence with the places of $\mathbb{Q}$. We will return to this observation later when we discuss harmonic functions and Laplacians.

Recall that the Berkovich topology on $\mathcal{M}(\mathbb{Z})$ is defined to be the weakest one for which the function $x \mapsto |n|_x$ is continuous for all $n \in \mathbb{Z}$. This can be described concretely as follows: each of the subsets

$$\ell_\infty = \{|\ |_0\} \cup \{|\ |_{\infty,\epsilon}\}_{0<\epsilon\leq 1} = \{|\ |_{\infty,\epsilon}\}_{0\leq\epsilon\leq 1}$$

and

$$\ell_p = \{|\ |_{p,\infty}\} \cup \{|\ |_{p,\epsilon}\}_{0<\epsilon<\infty} \cup \{|\ |_0\} = \{|\ |_{p,\epsilon}\}_{0\leq\epsilon\leq\infty}$$

is homeomorphic to a real interval, and the open neighborhoods of the trivial seminorm $|\ |_0$ are the subsets U of $\mathcal{M}(\mathbb{Z})$ containing $|\ |_0$ for which:
(1) $U \cap \ell_v$ is open in ℓ_v for all $v \in M_{\mathbb{Q}}$.
(2) $U \cap \ell_v = \ell_v$ for all but finitely many $v \in M_{\mathbb{Q}}$.

It is an instructive and simple exercise to verify directly using this description of the topology that $\mathcal{M}(\mathbb{Z})$ is path-connected, compact, and Hausdorff.

If we identify the segment ℓ_∞ with the real interval $[0,1]$ via the association

$$| \ |_{\infty,\epsilon} \mapsto \epsilon$$

and the segment ℓ_p with the extended-real interval $[0,\infty]$ via

$$| \ |_{p,\epsilon} \mapsto \epsilon,$$

then the complement $\mathbf{H}_\mathbb{Z}$ in $\mathcal{M}(\mathbb{Z})$ of all points of type (Z1) becomes a metric space. We let ρ denote the corresponding metric.

Remark 2.11. 1. The points of $\mathcal{M}(\mathbb{Z})$ having distance 1 from the trivial seminorm $| \ |_0$ are precisely the points corresponding to the standard absolute values $| \ |_p = | \ |_{p,1}$ and $| \ |_\infty = | \ |_{\infty,1}$.

2. If we extend ρ to a degenerate metric on all of $\mathcal{M}(\mathbb{Z})$, then a point x of type (Z1) is infinitely far away from every point $y \in \mathcal{M}(\mathbb{Z})$ distinct from x.

Remark 2.12. Like $\mathbb{P}^1_{\mathrm{Berk}}$, the space $\mathcal{M}(\mathbb{Z})$ can be viewed as an inverse limit of finite graphs. Indeed, define a *finite subgraph* of $\mathcal{M}(\mathbb{Z})$ to be the "convex hull" (in the obvious sense) of finitely many points of $\mathbf{H}_\mathbb{Z}$, endowed with the usual Euclidean topology on a finite union of real segments. The collection $\mathcal{S}$ of all such finite subgraphs $\Gamma \subseteq \mathcal{M}(\mathbb{Z})$ forms an inverse system with respect to the natural retraction maps $r_{\Gamma',\Gamma} : \Gamma' \to \Gamma$ (defined whenever $\Gamma \subseteq \Gamma'$), and one can show that $\mathcal{M}(\mathbb{Z})$ is homeomorphic to the inverse limit $\varprojlim_{\Gamma\in\mathcal{S}} \Gamma$.

Equipping each finite subgraph $\Gamma \in \mathcal{S}$ with the metric induced by ρ, the space $\mathcal{M}(\mathbb{Z}) = \varprojlim_{\Gamma\in\mathcal{S}}$ becomes a profinite $\mathbb{R}$-tree, with

$$\mathbf{H}_\mathbb{Z} = \varinjlim_{\Gamma\in\mathcal{S}} \Gamma.$$

3. **Lecture III**: Harmonic functions

In this lecture, we explore the notion of a *harmonic function* in the context of the spaces $\mathcal{M}(\mathbb{Z})$ and $\mathbb{P}^1_{\mathrm{Berk}}$.

By a *measure* on a space X, we will always mean a bounded, signed Radon measure on X. All rings will be commutative rings with an identity element 1.

3.1. **Harmonic functions on $\mathcal{M}(\mathbb{Z})$.** It is possible to give a natural definition of a "harmonic function" on $\mathcal{M}(\mathbb{Z})$, using the metric ρ introduced in §2.5.

We introduce the following convenient notation for points of $\mathcal{M}(\mathbb{Z})$:

- $\zeta_{p,\infty}$: the point of $\mathcal{M}(\mathbb{Z})$ corresponding to $|\ |_{p,\infty}$.
- ζ_0: the point of $\mathcal{M}(\mathbb{Z})$ corresponding to $|\ |_0$.
- $\zeta_{p,\epsilon}$: the point of $\mathcal{M}(\mathbb{Z})$ corresponding to $|\ |_{p,\epsilon}$.
- $\zeta_{\infty,\epsilon}$: the point of $\mathcal{M}(\mathbb{Z})$ corresponding to $|\ |_{\infty,\epsilon}$.
- ζ_v: the point $\zeta_{p,1}$ if $v \in M_{\mathbb{Q}}$ is a non-archimedean place corresponding to the prime p, or the point $\zeta_\infty = \zeta_{\infty,1}$if $v \in M_{\mathbb{Q}}$ is the archimedean place.

As in §1.4.3, for x in $\mathcal{M}(\mathbb{Z})$, we define the set T_x of *tangent directions* at x to be the connected components of $\mathcal{M}(\mathbb{Z})\backslash\{x\}$. When $x = \zeta_0$ is the point corresponding to the trivial seminorm $|\ |_0$ on $\mathbb{Z}$, there is a canonical bijection between T_x and the set $M_{\mathbb{Q}}$ of places of $\mathbb{Q}$; at all other points of $\mathcal{M}(\mathbb{Z})$, the space T_x has cardinality 1 or 2. We will refer to the segments $[\zeta_0, \zeta_v]$ for $v \in M_{\mathbb{Q}}$ as the "branches emanating from ζ_0".

Recall also from §2.5 that $\mathbf{H}_{\mathbb{Z}}$ denotes the complement of the points of type (Z1); the points of $\mathbf{H}_{\mathbb{Z}}$ are precisely the ones at finite distance from the trivial point ζ_0 with respect to the metric ρ.

Let U be a connected open subset of $\mathcal{M}(\mathbb{Z})$ (with respect to the Berkovich topology), and let $f : U \to \mathbb{R} \cup \{\pm\infty\}$ be a continuous extended-real valued function which is finite-valued on $U \cap \mathbf{H}_{\mathbb{Z}}$. For expositional simplicity, we assume that $\zeta_0 \in U$ (which is the main case of interest, since the connected components of $\mathcal{M}(\mathbb{Z})\backslash\{\zeta_0\}$ are homeomorphic to segments in $\mathbb{R}$).

We say that f is *continuous piecewise affine* on U, and write $f \in \mathrm{CPA}(U)$, if f is (i) continuous, (ii) piecewise-affine along each branch of $\mathcal{M}(\mathbb{Z})$ emanating from ζ_0, and (iii) constant on all but finitely many branches emanating from ζ_0. These conditions guarantee that if $f \in \mathrm{CPA}(U)$ and $x \in U \cap \mathbf{H}_{\mathbb{Z}}$, then the directional derivative $d_{\vec{v}} f(x)$ is well-defined for all $\vec{v} \in T_x$, and $d_{\vec{v}} f(x) = 0$ for all but finitely many $\vec{v} \in T_x$. Thus for all $x \in U \cap \mathbf{H}_{\mathbb{Z}}$ the quantity

$$\Delta_x(f) := -\sum_{v \in T_x} d_{\vec{v}} f(x)$$

is well-defined.

Let $x \in U$, and let $h \in \mathrm{CPA}(U)$.

Definition. 1. If $x \in \mathbf{H}_{\mathbb{Z}}$, we say that h is *harmonic at x* if $\Delta_x(h) = 0$.

2. If x is of type (Z1), we say that h is *harmonic at* x if h is constant on an open neighborhood of x.

Example 3.1. Let $n \in \mathbb{Z}$ be a nonzero integer, let $S_0 = \{\zeta_{p,\infty} : p \mid n\}$, and let $S = S_0 \cup \{\zeta_\infty\}$.

Define

$$F_n(x) = \begin{cases} +\infty & x \in S_0 \\ -\log|n|_x & x \in \mathcal{M}(\mathbb{Z})\backslash S_0. \end{cases}$$

Claim: $F_n(x)$ is harmonic outside S.

To see this, first note that if Λ denotes the smallest connected subset of $\mathcal{M}(\mathbb{Z})$ containing all the points of S, then Λ is finitely branched and there is a natural retraction map $r_\Lambda : \mathcal{M}(\mathbb{Z}) \twoheadrightarrow \Lambda$. Along the branch Λ_v of $\mathcal{M}(\mathbb{Z})$ emanating from ζ_0 in the tangent direction corresponding to $v \in M_\mathbb{Q}$, the function $F_n(x)$ is linear with slope equal to $-\log|n|_v$. In particular, $F_n(x)$ is locally constant off Λ: for all $x \in \mathcal{M}(\mathbb{Z})$, we have $F_n(x) = F_n(r_\Lambda(x))$. It follows from this that $F_n(x)$ is harmonic at all points $x \notin S \cup \{\zeta_0\}$. Finally, the fact that $F_n(x)$ is harmonic at ζ_0 is equivalent to the *product formula* for $\mathbb{Q}$:

$$\Delta_{\zeta_0}(F_n) = -\sum_{\vec{v} \in T_{\zeta_0}} d_{\vec{v}} F_n(\zeta_0) = \sum_{v \in M_\mathbb{Q}} \log|n|_v = 0.$$

If we think of $n \neq \pm 1$ as an analytic function on $\mathcal{M}(\mathbb{Z})$, of S_0 as the set of "zeros" of n, and of ζ_∞ as the unique "pole" [7] of n, then this example can be rephrased, in analogy with the classical situation over $\mathbb{C}$, as saying that the function $-\log|n|$ on $\mathcal{M}(\mathbb{Z})$ is harmonic outside the zeros and poles of n.

3.2. **Harmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$.** In this section, we define what it means for a real-valued function on $\mathbb{P}^1_{\mathrm{Berk}}$ to be harmonic. This is somewhat more complicated than the corresponding notion for $\mathcal{M}(\mathbb{Z})$ discussed in §3.1, since the branching behavior of $\mathbb{P}^1_{\mathrm{Berk}}$ is much more complicated than that of $\mathcal{M}(\mathbb{Z})$.

We recall from §1.4.3 that if $x \in \mathbb{P}^1_{\mathrm{Berk}}$, there is a well-defined set T_x of *tangent directions* at x, and the tangent directions at x are in one-to-one correspondence with the connected components of $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{x\}$.

Let U be a connected open subset of $\mathbb{P}^1_{\mathrm{Berk}}$, and let $f : U \to \mathbb{R} \cup \{\pm\infty\}$ be a continuous extended-real valued function which is finite-valued on $\mathbb{P}^1_{\mathrm{Berk}}\backslash\mathbb{P}^1(K)$.

[7] Somewhat peculiarly, it seems that the point ζ_∞ should be thought of as a pole of n, despite the fact that both $|n|_\infty$ and $-\log|n|_\infty$ are finite-valued, because $-\log|n|$ is not locally constant near ζ_∞; see Example 4.29 below for another explanation.

We say that f is *continuous piecewise affine* on U, and write $f \in \mathrm{CPA}(U)$ if:

(CPA1) The restriction of f to $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ is piecewise-affine with respect to the path metric ρ; concretely, this means that for each $x \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ and each sufficiently small path $\Lambda = \ell_{x,y}$ emanating from x, the restriction of f to Λ is affine.

(CPA2) If $f \in \mathrm{CPA}(U)$ and $x \in U \cap \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$, then for each $\vec{v} \in T_x$ the *directional derivative* $d_{\vec{v}} f(x)$ is well-defined. Concretely, this means that for each $\vec{v} \in T_x$, there exists a constant $m_{\vec{v}} = d_{\vec{v}} f(x)$ such that for every $y \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ representing the tangent direction $\vec{v}$, there exists a point $y' \in (x, y]$ such that for every $z \in (x, y']$ we have

$$f(z) = f(x) + m_{\vec{v}} \rho(x, z).$$

(CPA3) For each $x \in U \cap \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$, we have $d_{\vec{v}} f(x) = 0$ for all but finitely many $\vec{v} \in T_x$. In particular, the quantity

$$\Delta_x(f) := -\sum_{v \in T_x} d_{\vec{v}} f(x) \tag{3.2}$$

is well-defined for each $x \in U \cap \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$.

Definition. Let $x \in U$, and let $h \in \mathrm{CPA}(U)$.

1. If x is of type II or III, we say that h is *harmonic at* x if $\Delta_x(h) = 0$. In other words, a function $h \in \mathrm{CPA}(U)$ is harmonic at a point $x \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ if the sum of the slopes of h in all tangent directions emanating from x is zero.

2. If x is of type I or type IV, we say that h is *harmonic at* x if h is constant on an open neighborhood of x.

Example 3.3. Consider the function $G : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G(x) = \begin{cases} +\infty & x = \infty \\ \log_v \max(|T|_x, 1) & x \in \mathbb{A}^1_{\mathrm{Berk}} \end{cases}$$

whose restriction to K is the function $\log_v^+ |x| = \log_v \max(|x|, 1)$. Let $\Lambda = \ell_{\zeta_{\mathrm{Gauss}},\infty}$ be the closed path from ζ_{Gauss} to ∞ in $\mathbb{P}^1_{\mathrm{Berk}}$, and let $r_\Lambda : \mathbb{P}^1_{\mathrm{Berk}} \twoheadrightarrow \Lambda$ be the natural retraction map from $\mathbb{P}^1_{\mathrm{Berk}}$ onto Λ. Recall that if $x = \zeta_{a,r} \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$, then

$$|T|_x = \sup_{z \in B(a,r)} |z|.$$

From this, one deduces easily:

- $G(x)$ is linear with slope 1 along Λ, i.e., $G(x) = \rho(\zeta_{\mathrm{Gauss}}, x)$.

- $G(x)$ is locally constant off Λ, i.e., for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, we have $G(x) = G(r_\Lambda(x))$.

It follows that $G \in \mathrm{CPA}(\mathbb{P}^1_{\mathrm{Berk}})$ and that G is harmonic on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta_{\mathrm{Gauss}}, \infty\}$, but is not harmonic at ζ_{Gauss} or ∞. For example, the sum of the slopes of G in all directions emanating from ζ_{Gauss} is 1: in the direction heading up to infinity the slope is 1, and in all other directions the slope is 0.

As an immediate consequence of the definition of harmonic functions, we have:

Lemma 3.4. *If h_1, h_2 are harmonic on U and $c_1, c_2 \in \mathbb{R}$, then $c_1 h_1 + c_2 h_2$ is harmonic on U.*

As an application of Lemma 3.4, we discuss the following example.

Example 3.5. Let $f(T) = \prod_{i=1}^n (x - a_i) \in K[T]$ be a nonconstant polynomial, and let

$$F(x) = \begin{cases} -\infty & x = \infty \\ +\infty & x \in \{a_1, \ldots, a_n\} \\ -\log_v |f|_x & x \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\infty, a_1, \ldots, a_n\} \end{cases}$$

be the unique continuous function on $\mathbb{P}^1_{\mathrm{Berk}}$ extending the function $-\log_v |f(x)|$ on K.

Claim: $F(x)$ is harmonic outside $\{\infty, a_1, \ldots, a_n\}$.

Indeed, as far as type I points go, it follows from the ultrametric inequality that if $x \in K \backslash \{a_1, \ldots, a_n\}$, then $|f(x)|$ is constant on every disk around a not containing $a_1, \ldots, a_n$. Since F is continuous and K is dense in $\mathbb{A}^1_{\mathrm{Berk}}$, it follows that F is constant on a Berkovich open disk $\mathbb{B}(a, r)^-$ containing a.

It remains to see why F is harmonic on $\mathbf{H}_{\mathrm{Berk}}$. First, we consider the special case in which $f(T) = T - a$. In this case, if $\Lambda_a = \ell_{a,\infty}$ denotes the unique path in $\mathbb{P}^1_{\mathrm{Berk}}$ from a to ∞ and $F_a = -\log_v |T - a|_x$, then we have:

- $F_a(x)$ is linear with slope -1 along (a, ∞).
- $F_a(x)$ is locally constant off Λ_a, i.e., for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, we have $F_a(x) = F_a(r_{\Lambda_a}(x))$.

It follows in this special case that $F_a \in \mathrm{CPA}(\mathbb{P}^1_{\mathrm{Berk}})$, and that F_a is harmonic on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\infty, a\}$.

In the general case, we have $F(x) = \sum_{i=1}^n F_a(x)$, and it follows from Lemma 3.4 that F is harmonic outside $\{\infty, a_1, \ldots, a_n\}$, as claimed.

Example 3.6. Let $V = \mathcal{M}(\mathcal{A}_V)$ be an affinoid subdomain of $\mathbb{P}^1_{\mathrm{Berk}}$, and let $U \subset V$ be a connected open subset. If $f \in \mathcal{A}_V$ is a nowhere zero analytic function on V, then the function $F(x) = -\log_v |f|_x$ is harmonic on U. This can be deduced from the previous example (the special case where f is a polynomial) using some results from p-adic analysis.

Note that this example generalizes the well-known classical fact that if f is a nowhere zero analytic function on an open subset U of the complex plane, then $\log |f|$ is harmonic on U.

3.3. **Properties of harmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$.** By a *domain* in $\mathbb{P}^1_{\mathrm{Berk}}$, we will mean a connected open subset of $\mathbb{P}^1_{\mathrm{Berk}}$. In this section, we present a selection of results from [BR06, Chapter 5] concerning harmonic functions on domains in $\mathbb{P}^1_{\mathrm{Berk}}$

3.3.1. *The maximum principle.* The following result is the Berkovoch space analogue of the classical maximum principle for harmonic functions on domains in $\mathbb{C}$:

Proposition 3.7 (Maximum Principle)**.** *1. If h is a nonconstant harmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$, then h does not achieve a maximum or a minimum value on U.*

2. If h is a harmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ which extends continuously to the closure $\bar{U}$ of U, then h achieves both its minimum and maximum values on the boundary ∂U of U.

Recall from Lemma 2.7 that a *simple domain* in $\mathbb{P}^1_{\mathrm{Berk}}$ is a connected open set $U \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ whose boundary is a finite subset of $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$. One can show (see §3.3.2 below for further details) that every harmonic function on a simple domain U extends continuously to $\bar{U}$. If $U = \mathbb{P}^1_{\mathrm{Berk}}$ (resp. U is a Berkovich open disk), then ∂U is empty (resp. consists of a single point). By part 2 of the maximum principle, we therefore conclude:

Corollary 3.8. *If $U = \mathbb{P}^1_{\mathrm{Berk}}$ or U is an open Berkovich disk, then every harmonic function on U is constant.*

The conclusion of 3.8 can be better understood and generalized through the observation that the behavior of a harmonic function on a domain U in $\mathbb{P}^1_{\mathrm{Berk}}$ is controlled by its behavior on a certain special subset.

Definition. If U is a domain in $\mathbb{P}^1_{\mathrm{Berk}}$, the *main dendrite* $D(U) \subset U$ is set of all $x \in U$ belonging to paths between boundary points $y, z \in \partial U$.

The main dendrite of a domain U is empty if and only if U has at most one boundary point, which happens precisely in the following three cases:

- $U = \mathbb{P}^1_{\mathrm{Berk}}$.
- $U \cong \mathbb{P}^1_{\mathrm{Berk}} \backslash \{a\}$ for some point a of type I or IV.
- U is an open Berkovich disk.

Example 3.9. If $U = \mathcal{B}(a,R)^- \backslash \mathcal{B}(a,r)$ is a Berkovich open annulus, then $D(U)$ is the open segment joining the two boundary points $\zeta_{a,r}$ and $\zeta_{a,R}$ of U.

Example 3.10. If $K = \mathbb{C}_p$ and $U = \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(\mathbb{Q}_p)$, then the main dendrite $D(U)$ is a locally finite real tree in which the set of branch points is discrete, and every branch point has degree $p+1$. In fact, $D(U)$ can be identified with the (geometric realization of the) *Bruhat-Tits tree* $\mathrm{BT}(\mathbb{Q}_p)$ (see [FvdP04, Definition 4.9.3]).

When $D(U)$ is non-empty, it is not hard to see that there is a natural retraction map $r_{U,D(U)} : U \twoheadrightarrow D(U)$. The following result is proved in [BR06, Chapter 5]:

Proposition 3.11. *Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$.*

1. If the main dendrite $D(U)$ of a domain U is nonempty, then it is finitely branched at each point of $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$.

2. Let h be harmonic in a domain U. If the main dendrite is empty, then h is constant; otherwise, h is constant on branches off the main dendrite, i.e., $h = h \circ r_{U,D(U)}$.

3.3.2. *The Poisson Formula.* In the classical theory of harmonic functions in the complex plane, if f is harmonic on an open disk V then it has a continuous extension to the closure of V, and the Poisson Formula expresses the values of f on V in terms of its values on the boundary of V.

Specifically, if $V \subseteq \mathbb{C}$ is an open disk of radius r centered at z_0, and if f is harmonic in V, then f extends continuously to $\bar{V}$ and $f(z_0) = \int_{\partial V} f \, d\mu_V$, where μ_V is the uniform probability measure $d\theta/2\pi$ on the boundary circle ∂V. More generally, for any $z \in V$ there is a measure $\mu_{z,V}$ depending only on z and V, called the *Jensen-Poisson measure*, for which

$$f(z) = \int_{\partial V} f \, d\mu_{z,V}$$

for every harmonic function f on $\bar{V}$. We seek to generalize this type of formula to the Berkovich projective line.

In $\mathbb{P}^1_{\mathrm{Berk}}$, the basic open neigbhorhoods are the *simple domains* (or *open affinoids*), which are just the finite intersections of open Berkovich disks (c.f. Lemma 2.7). A simple domain has only a finite number of boundary points, and its main dendrite is the interior of a finite subgraph Γ of $\mathbb{P}^1_{\mathrm{Berk}}$. As we will see, every harmonic function f on a simple domain V has a continuous extension to its closure, and there is an analogue of the Jensen-Poisson measure which yields an explicit formula for f in terms of its values on the boundary. In other words, one can explicitly solve the Berkovich space analogue of the Dirichlet problem on any simple domain (using, as we will see, only simple linear algebra).

Recall from §1.6.2 that $\kappa_z(x,y) = -\log_v[x,y]_z$ denotes the fundamental potential kernel on $\mathbb{P}^1_{\mathrm{Berk}}$ relative to the point z.

Let V be a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with boundary points $x_1,\ldots,x_m \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$. For $z \in V$, let $C(z)$ be the $m \times m$ matrix whose ij^{th} entry is $\kappa_z(x_i,x_j)$. Define a *probability vector* on $\mathbb{R}^m$ to be a vector $[p_1,\ldots,p_m]^T \in \mathbb{R}^m$ such that $p_i \geq 0$ for $1 \leq i \leq m$ and $p_1+\cdots+p_m = 1$.

Proposition 3.12. *For each $z \in V$, there is a unique probability vector $\vec{p}(z) = [p_1(z), p_2(z),\ldots,p_m(z)]^T \in \mathbb{R}^m$ such that $C(z)\cdot\vec{p}(z)$ is a scalar multiple of* $[1,1,\ldots,1]^T$.

For each $1 \leq i \leq m$, define the function $h_i : V \to \mathbb{R}$, called the i^{th} *harmonic measure* with respect to V, by setting $h_i(z) = p_i(z)$. By construction, we have $0 \leq h_i(z) \leq 1$ for all $z \in V$ and $h_1+\cdots+h_m \equiv 1$ on V.

Explicitly, let

$$M(z) \;=\; \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \kappa_z(x_1,x_1) & \cdots & \kappa_z(x_1,x_m) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \kappa_z(x_m,x_1) & \cdots & \kappa_z(x_m,x_m) \end{pmatrix}$$

and for each $i = 0,1,\ldots,m$, let $M_i(z)$ be the matrix obtained by replacing i^{th} column of $M(z)$ by $[1,0,\ldots,0]^T$. If $C(z)\cdot\vec{p}(z) = [-\nu,\ldots,-\nu]^T$, then

$$M(z)\begin{bmatrix} \nu \\ p_1(z) \\ \vdots \\ p_m(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and so by Cramer's rule, we have

$$h_i(z) \;=\; \det(M_i(z))/\det(M(z)) \;.$$

Lemma 3.13. *For each $1 \leq i \leq m$, the function $h_i(z)$ is harmonic in V and extends continuously to $\bar{V}$ by setting $h_i(x_j) = \delta_{ij}$.*

Proposition 3.14 (Poisson Formula). *Let V be a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with boundary points $x_1, \ldots, x_m$. Then each harmonic function f on V has a continuous extension to $\bar{V}$, and there is a unique such function with a prescribed set of boundary values $A_1, \ldots, A_m$. Moreover, f can be computed from its boundary values using the formula*

$$f(z) = \sum_{i=1}^{m} f(x_i) \cdot h_i(z),$$

valid for all $z \in \bar{V}$, where $h_i(z)$ is the i^{th} harmonic measure with respect to V.

A useful reformulation of Proposition 3.14 is as follows (compare with [Kan89, §4.2]). For $z \in V$, define the *Jensen-Poisson measure* $\mu_{z,V}$ on V relative to the point z by

$$\mu_{z,V} = \sum_{i=1}^{m} h_i(z)\delta_{x_i}.$$

Then by Proposition 3.14, we have:

Corollary 3.15. *If V is a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$, then a continuous function $f : \bar{V} \to \mathbb{R} \cup \{\pm\infty\}$ is harmonic in V if and only if*

$$f(z) = \int_{\partial V} f \, d\mu_{z,V}$$

for all $z \in V$.

Since the closures of simple domains form a fundamental system of compact neighborhoods for the topology on $\mathbb{P}^1_{\mathrm{Berk}}$, it follows from the definitions that a function f is harmonic on an open set U if and only if its restriction to every *simple subdomain* $V \subseteq U$ is harmonic, where a simple subdomain of U denotes a simple domain whose closure is contained in U. With this terminology, we have:

Corollary 3.16. *If U is a domain in $\mathbb{P}^1_{\mathrm{Berk}}$ and $f : U \to \mathbb{R} \cup \{\pm\infty\}$ is a continuous function, then f is harmonic in U if and only if for every simple subdomain V of U we have*

$$f(z) = \int_{\partial V} f \, d\mu_{z,V}$$

for all $z \in V$.

Project #1: Let $B = B_1 \cup \cdots \cup B_m$ be a finite disjoint union of closed disks in $\mathbb{C}_p$ having radii in $|\mathbb{C}_p^*| = p^{\mathbb{Q}}$. Prove that there is a polynomial $f \in \mathbb{C}_p[T]$ such that $B = \{z \in \mathbb{C}_p \ : \ |f(z)| \leq 1\}$, and find an explicit formula for $f(z)$ in terms of the Poisson-Jensen measure associated to the simple domain $V = \mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m)$, where $\mathcal{B}_i$ is the closed Berkovich disk in $\mathbb{P}^1_{\mathrm{Berk}}$ associated to B_i.

3.3.3. *Uniform Convergence.* The Poisson formula implies that the limit of a sequence of harmonic functions is harmonic, under a much weaker condition than is required classically.

Proposition 3.17. *Let U be an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$. Suppose $f_1, f_2, \ldots$ are harmonic in U and converge pointwise to a function $f : U \to \mathbb{R}$. Then $f(z)$ is harmonic in U, and the $f_i(z)$ converge uniformly to $f(z)$ on compact subsets of U.*

Using the previous result, one can characterize harmonic functions as local uniform limits of logarithms of norms of rational functions (for a proof, see [BR06, Chapter 5]).

Proposition 3.18. *If $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ is a domain and h is harmonic in U, there are rational functions $g_1(T), g_2(T), \ldots \in K(T)$ and rational numbers $R_1, R_2, \ldots \in \mathbb{Q}$ such that*

$$h(x) \;=\; \lim_{i \to \infty} R_i \cdot \log_v(|g_i|_x)$$

uniformly on compact subsets of U.

A Berkovich space analogue of Harnack's principle holds as well:

Proposition 3.19 (Harnack's Principle)**.** *Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$, and suppose $f_1, f_2, \ldots$ are harmonic in U, with $0 \le f_1 \le f_2 \le \cdots$. Then either*

A) $\lim_{i\to\infty} f_i(z) = \infty$ for each $z \in U$, or

B) $f(z) = \lim_{i\to\infty} f_i(z)$ is finite for all z, the $f_i(z)$ converge uniformly to $f(z)$ on compact subsets of U, and $f(z)$ is harmonic in U.

3.4. **Subharmonic functions.** We give a brief introduction to the notion of a subharmonic function on $\mathbb{P}^1_{\mathrm{Berk}}$; see [BR06, Chapter 6] and [Thu05] for further details.

Definition. Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain.
A function $f : U \to [-\infty, \infty)$ with $f(x) \not\equiv -\infty$ is called *subharmonic* on U if

(SH1) f is upper semicontinuous.

(SH2) For each simple subdomain $V \subset U$ we have

$$f(z) \le \int_{\partial V} f \, d\mu_{z,V}$$

for all $z \in V$.

f is called *superharmonic* on U if $-f$ is subharmonic on U.

Remark 3.20. By Corollary 3.16, f is harmonic on U if and only if it is both subharmonic and superharmonic on U.

Corollary 3.16 also shows that condition (SH2) can be replaced by the condition that for each simple subdomain $V \subset U$ and each harmonic function h on V, if $f(x) \leq h(x)$ on ∂V then $f(x) \leq h(x)$ on V.

Example 3.21. For fixed $y, z \in \mathbb{P}^1_{\mathrm{Berk}}$ with $y \neq z$, the function $f(x) = \kappa_z(x, y)$ is superharmonic in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{z\}$, and is subharmonic in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{y\}$.

Example 3.22. If ν is a probability measure on $\mathbb{P}^1_{\mathrm{Berk}}$ and $z \notin \mathrm{Supp}(\nu)$, then the *potential function*

$$p_{\nu,z}(x) = \int_{\mathbb{P}^1_{\mathrm{Berk}}} \kappa_z(x, y)\, d\nu(y)$$

is superharmonic in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{z\}$ and is subharmonic in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathrm{Supp}(\nu)$.

Subharmonic functions obey the following maximum principle:

Proposition 3.23. *1. If f is a nonconstant subharmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$, then f does not achieve a global maximum on U.*

2. If f is a subharmonic function on a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ which extends continuously to $\bar{U}$, then f achieves its maximum value on ∂U.

Finally, we mention the following analogue of Proposition 3.11:

Proposition 3.24. *Let f be subharmonic on a domain U. Then f is non-increasing on paths leading away from the main dendrite of U. If U is a disk, then $f(x)$ is non-increasing on paths leading away from the unique boundary point of U.*

Since the main dendrite of a domain is finitely branched, Proposition 3.24 implies that at any given point, there are only finitely many tangent directions in which a subharmonic function can be increasing.

4. **Lecture IV**: Laplacians

In this lecture, we will define a Laplacian operator on the Berkovich projective line which is analogous in many ways to the classical Lapacian operator

$$\Delta(f) = -(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}) dx \wedge dy$$

on $\mathbb{C}$.

Actually, a slight abstraction of the construction from [BR06] of the Laplacian on $\mathbb{P}^1_{\mathrm{Berk}}$ yields a Laplacian operator on more general one-dimensional Berkovich spaces such as $\mathcal{M}(\mathbb{Z})$ or the analytic space X_{Berk}

associated to a complete nonsingular curve (see §5). The Laplacian on $\mathbb{P}^1_{\mathrm{Berk}}$ will be construced via a limiting process from the Laplacian on a finite $\mathbb{R}$-tree. For curves of higher genus, the associated Berkovich analytic space is no longer simply connected, so in order to construct a Laplacian in this generality, one needs to replace finite $\mathbb{R}$-trees by *metrized graphs*.

We will define a Laplacian operator in the rather abstract general setting of an *arboretum*, which is a special kind of inverse limit of metrized graphs, and then gradually specialize to the particular cases of interest to us. This involves setting up some cumbersome notation, but it has the advantage of making the entire construction more conceptually clear.

4.1. **Metrized graphs.**

4.1.1. *Definition of a metrized graph.* Intuitively, a metrized graph is a finite graph whose edges are thought of as line segments having a well-defined length. In particular, Γ is a one-dimensional manifold except at finitely many "branch points", where it looks locally like an n-pointed star. The path-length function along each edge extends to a metric on all of Γ, making it a compact metric space. One thinks of a metrized graph as an analytic object, not just a combinatorial one.

Formally, define a *star-shaped set of valence* $n_p \geq 1$ to be a set of the form

$$S(n_p, r_p) \;=\; \{z \in \mathbb{C} : z = te^{k \cdot 2\pi i/n_p} \text{ for some } 0 \leq t < r_p \text{ and some } k \in \mathbb{Z}\}.$$

Then a *metrized graph* is a compact, connected metric space Γ such that each $p \in \Gamma$ has a neighborhood U_p isometric to a star-shaped set of valence $n_p \geq 1$, endowed with the path metric.

A metrized graph with no cycles is the same thing as a compact, finite $\mathbb{R}$-tree, as defined in §2.2.

By a *vertex set* for Γ, we mean a finite set of points S such that $\Gamma \backslash S$ is a union of open intervals whose closures have distinct endpoints. (A vertex set necessarily contains all endpoints and branch points of Γ, and if Γ has loops, a vertex set also contains at least one interior point from each loop.)

There is a close connection between metrized graphs and finite weighted graphs. Given a metrized graph Γ, any choice of a vertex set S for Γ gives rise to a weighted graph G which one may call a *model* for Γ; a different choice of vertex set leads to an equivalent weighted graph. Conversely, every weighted graph G determines a metrized graph in the obvious way, so there is a one-to-one correspondence between metrized graphs and equivalence classes of finite weighted graphs (see [BR] or [BF06] for further details).

Often, when given a metrized graph Γ, one chooses without explicit comment a vertex set S, together with distinguished parametrizations of the edges of the corresponding model G. The definition of the Laplacian given below in §4 is independent of these implicit choices.

By a *path* in Γ, we will mean an injective length-preserving continuous map from the real interval $[0, L]$ into Γ. We will say that a path $\gamma : [0, L] \to \Gamma$ *emanates from* p, and *terminates at* q, if $\gamma(0) = p$ and $\gamma(L) = q$. We call two paths emanating from p *equivalent* if they share a common initial segment. For each $p \in \Gamma$, we let $T_p(\Gamma)$ (the set of *tangent directions* at p) denote the set of equivalence classes of paths emanating from p. It is easy to see that $|T_p(\Gamma)| = n_p$, i.e., there is a bijection between elements of T_p and the "edges" of Γ emanating from p.

It is useful to associate to each element of $T_p(\Gamma)$ a formal "unit tangent vector" $\vec{v}$, and to write $p+t\vec{v}$ instead of $\gamma(t)$, where $\gamma : [0, L] \to \Gamma$ is a representative path. If $f : \Gamma \to \mathbb{R}$ is a function, and $\vec{v}$ is a formal unit tangent vector at p, we define the derivative of f in the direction $\vec{v}$ to be

$$d_{\vec{v}}f(p) = \lim_{t\to 0^+} \frac{f(p + t\vec{v}) - f(p)}{t} = \lim_{t\to 0^+} \frac{f(\gamma(t)) - f(p)}{t} ,$$

provided the limit exists.

4.2. **The Laplacian on a metrized graph.**

4.2.1. *The space* CPA(Γ). Let CPA(Γ) be the space of continuous, piecewise-affine, real-valued functions on Γ, i.e., functions which have the form $t \mapsto at + b$ on each edge of Γ (with respect to some vertex set).

If $f \in$ CPA(Γ), then clearly the directional derivatives $d_{\vec{v}}f(p)$ are defined for all $p \in \Gamma$ and all $\vec{v} \in T_p(\Gamma)$.

Chinburg and Rumely ([CR93]) introduced a Laplacian operator on CPA(Γ). Their Laplacian is a map from CPA(Γ) to the space of discrete signed measures on Γ. We will take the Laplacian to be the negative of theirs, and put

$$\Delta(f) \;=\; \sum_{p\in\Gamma} \Delta_p(f)\ \delta_p(x),$$

where $\Delta_p(f) = -\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p)$ and $\delta_p(x)$ is the Dirac measure at p. The operator Δ coincides, in a natural sense, with the usual combinatorial Laplacian on a finite weighted graph – see [BF06] for details. Here are some easily verified properties of Δ:

Proposition 4.1. *Let $f, g \in \mathrm{CPA}(\Gamma)$. Then*

1. *$\Delta(f) \equiv 0$ if and only if f is constant on Γ.*
2. *$\Delta(f) = \Delta(g)$ if and only if $f = g + C$ for some constant C.*
3. *If f is nonconstant, then $f(x)$ achieves its maximum at a point p where $\Delta(f)(p) > 0$, and its minimum at a point q where $\Delta(f)(q) < 0$.*
4. *$\int_\Gamma f\,\Delta(g) = \int_\Gamma g\,\Delta(f)$.*
5. *The total mass $\Delta(f)(\Gamma)$ is 0.*

4.2.2. *The space* $\mathrm{BDV}(\Gamma)$. One can define a measure-valued Laplacian operator on a much larger class of functions than just $\mathrm{CPA}(\Gamma)$. The construction is motivated by the following "Mass Formula". If $S \subseteq \Gamma$, we let ∂S be the set of points belonging to the closures of both S and $\Gamma \backslash S$. (Note that under this definition, if $\Gamma = [0,1]$ and $S = [0, \frac{1}{2}]$, for example, then the left endpoint 0 is not a boundary point of S.)

For each $p \in \partial S$, let $\mathrm{In}(p, S)$ be the set of "inward-pointing unit vectors at p", i.e., the set of all $\vec{v} \in T_p(\Gamma)$ for which $p + t\vec{v}$ belongs to S for all sufficiently small $t > 0$. Similarly, let $\mathrm{Out}(p, S) = T_p(\Gamma) \backslash \mathrm{In}(p, S)$ be the collection of "outward-directed unit vectors at p". For example, if p is an isolated point of S, then $\mathrm{In}(p, S) = \emptyset$ and $\mathrm{Out}(p, S) = T_p(\Gamma)$.

Proposition 4.2 (Mass formula)**.** *Let $f \in \mathrm{CPA}(\Gamma)$. Then the measure $\mu = \Delta(f)$ satisfies the following properties:*

(a) If $E \subseteq \Gamma$ is a finite union of connected closed sets, then

$$\mu(E) \;=\; -\sum_{p \in \partial E} \sum_{\vec{v} \in \mathrm{Out}(p,E)} d_{\vec{v}} f(p). \tag{4.3}$$

(b) If $V \subseteq \Gamma$ is a finite union of connected open sets, then

$$\mu(V) \;=\; \sum_{p \in \partial V} \sum_{\vec{v} \in \mathrm{In}(p,V)} d_{\vec{v}} f(p). \tag{4.4}$$

Let $\mathcal{D}(\Gamma)$ be the class of all functions on Γ whose one-sided derivatives exist everywhere, i.e.,

$$\mathcal{D}(\Gamma) = \{f : \Gamma \to \mathbb{R} : \quad d_{\vec{v}} f(p) \text{ exists for each } p \in \Gamma \text{ and } \vec{v} \in T_p(\Gamma)\}\ .$$

It is easy to see that each $f \in \mathcal{D}(\Gamma)$ is continuous.

Definition. We will say that a continuous function $f \in \mathcal{D}(\Gamma)$ is of *bounded differential variation*, and write $f \in \mathrm{BDV}(\Gamma)$, if there is a (bounded signed Borel) measure μ on Γ satisfying properties (a) and (b) of Proposition 4.2. If this is the case, we define the Laplacian $\Delta(f)$ of f to be this measure. Since the connected open sets generate the topology of Γ, it is not hard to see that the measure $\Delta(f)$, if it exists, is uniquely determined by properties (a) and (b).

By Proposition 4.2, we have $\mathrm{CPA}(\Gamma) \subseteq \mathrm{BDV}(\Gamma)$. We now describe a larger class of functions for which $\Delta(f)$ exists and can be explicitly described.

Proposition 4.5. *Let* $\mathrm{Zh}(\Gamma)$ *be the space of continuous, piecewise* $\mathcal{C}^2$ *functions* f *whose one-sided directional derivatives* $d_{\vec{v}}f(p)$ *exist for all* $p \in \Gamma$*, and for which* f'' *is bounded along each edge. Then* $\mathrm{Zh}(\Gamma) \subset \mathrm{BDV}(\Gamma)$*, and for* $f \in \mathrm{Zh}(\Gamma)$ *we have*

$$\Delta(f) \ = \ -f''(x)\,dx + \sum_{p\in\Gamma} \Delta_p(f)\,\delta_p, \tag{4.6}$$

where f'' *is computed on each segment in the complement of an appropriate vertex set* X_f *for* Γ*.*

For $f \in \mathrm{Zh}(\Gamma)$, our Laplacian operator therefore coincides with the one defined by S. Zhang in [Zha93]. The Laplacian in (4.6) is a hybrid of the usual Laplacian $-f''(x)dx$ on $\mathbb{R}$ and the combinatorial Laplacian on a weighted graph. It is easy to check using integration by parts that $\int_\Gamma f\Delta(g) = \int_\Gamma g\Delta(f)$ for all $f, g \in \mathrm{Zh}(\Gamma)$. From this, it follows (taking g to be the constant function 1) that $\Delta(f)(\Gamma) = 0$.

It is clear from the definitions that the Laplacian on $\mathrm{BDV}(\Gamma)$ is a linear operator, i.e., that

$$\Delta(\alpha f + \beta g) = \alpha\Delta(f) + \beta\Delta(g)$$

for all $f, g \in \mathrm{BDV}(\Gamma)$ and all $\alpha, \beta \in \mathbb{R}$.

We note the following additional properties of the Laplacian on $\mathrm{BDV}(\Gamma)$, which extend those proved in Proposition 4.1 for functions in $\mathrm{CPA}(\Gamma)$. (We let μ^+, μ^- denote the positive and negative parts, respectively, of the Jordan decomposition of a measure μ.)

Proposition 4.7. *If* $f, g \in \mathrm{BDV}(\Gamma)$*, then*

1. $\Delta(f) \equiv 0$ *if and only if* f *is constant on* Γ*.*
2. $\Delta(f) = \Delta(g)$ *if and only if* $f = g + C$ *for some constant* C*.*
3. *If* f *is nonconstant, then* $f(x)$ *achieves its maximum at a point* p *in the support of* $\Delta(f)^+$*, and its minimum at a point* q *in the support of* $\Delta(f)^-$*.*
4. $\int_\Gamma f\,\Delta(g) = \int_\Gamma g\,\Delta(f)$*.*
5. $\Delta(f)(\Gamma) = 0$*.*

We also have the following useful, but somewhat harder to prove, result:

Theorem 4.8. *If* ν *is a measure of total mass zero on a metrized graph* Γ*, then there exists a function* $h \in \mathrm{BDV}(\Gamma)$*, unique up to an additive constant, such that* $\Delta h = \nu$*.*

It follows that there is a natural bijection between measures of total mass zero on Γ and functions in $\mathrm{BDV}(\Gamma)$ modulo constant functions.

Finally, we note that the Laplacian operator on a metrized graph satisfies the following compatibility property:

Lemma 4.9. *If Γ, Γ' are metrized graphs with $\Gamma \subseteq \Gamma'$ and there is a deformation retraction $r_{\Gamma',\Gamma}$ from Γ' onto Γ, and if $f \in \mathrm{BDV}(\Gamma')$, then*

$$\Delta_\Gamma(f|_\Gamma) = (r_{\Gamma',\Gamma})_* \Delta_{\Gamma'}(f) \ . \tag{4.10}$$

Lemma 4.9 will be used in §4.4 to define the Laplacian on an arboretum.

4.3. **Arboreta.** Recall that a *directed set* is a set I together with a reflexive and transitive relation $\leq$ satisfying:

Given $\alpha, \beta \in I$, there exists $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition. An *arboreal system of metrized graphs* is a directed set $(I, \leq)$ together with the following data:

- For each $\alpha \in I$, a metrized graph Γ_α.
- For each $\alpha \leq \beta$ in I, an isometric inclusion map $i_{\alpha,\beta} : \Gamma_\alpha \hookrightarrow \Gamma_\beta$.
- For each $\alpha \leq \beta$ in I, a deformation retraction $r_{\beta,\alpha} : \Gamma_\beta \twoheadrightarrow \Gamma_\alpha$.

Moreover, these data should be *compatible*, in the sense that if $\alpha \leq \beta \leq \gamma$, then:

- $r_{\beta,\alpha} \circ i_{\alpha,\beta}$ is the identity map on Γ_α.
- $r_{\beta,\alpha} \circ r_{\gamma,\beta} = r_{\gamma,\alpha}$.
- $i_{\beta,\gamma} \circ i_{\alpha,\beta} = i_{\alpha,\gamma}$.

Remark 4.11. Concretely, the condition that $r_{\beta,\alpha} : \Gamma_\beta \to \Gamma_\alpha$ is a deformation retraction is equivalent to saying that the graph Γ_β is obtained from Γ_α by attaching finitely many finite $\mathbb{R}$-trees, and $r_{\beta,\alpha}$ is the map obtained by contracting each of these $\mathbb{R}$-trees to a point.

We will usually write $\{\Gamma_\alpha\}$ instead of $(\Gamma_\alpha, i_{\alpha,\beta}, r_{\beta,\alpha})$ to denote an arboreal system of metrized graphs.

Definition. The *arboretum* attached to an arboreal system of metrized graphs $\{\Gamma_\alpha\}$ is the topological space $X = \varprojlim \Gamma_\alpha$, where the inverse limit is taken with respect to the maps $r_{\beta,\alpha}$.

Recall that a point of $\varprojlim \Gamma_\alpha$ is a *compatible system* $(x_\alpha) \in \prod_{\alpha \in I} \Gamma_\alpha$, where "compatible" means that if $\alpha \leq \beta$ then $r_{\beta,\alpha}(x_\beta) = x_\alpha$. The inverse limit X comes equipped with a compatible system of continuous maps $r_\alpha : X \to \Gamma_\alpha$ satisfying the following universal property:

> If Y is a topological space equipped with a compatible system of continuous maps $s_\alpha : Y \to \Gamma_\alpha$, then there is a unique continuous map $\phi : Y \to X$ such that $s_\alpha = r_\alpha \circ \phi$ for all $\alpha \in I$.

The topology on $X = \varprojlim \Gamma_\alpha$ is the weakest one for which all of the maps r_α are continuous. A fundamental system of open neighborhoods for the topology on X is given by the open sets $r_\alpha^{-1}(U_\alpha)$ for $\alpha \in I$ and $U_\alpha \subseteq \Gamma_\alpha$ a connected open set.

The following topological properties of an arboretum are easily verified:

Lemma 4.12. *If X is an arboretum, then:*

1. X is a compact, Hausdorff, and path-connected topological space.

2. For each $\alpha \in I$ and each $x_0 \in \Gamma_\alpha$, the map $r_\alpha : X \to \Gamma_\alpha$ is a retraction which induces an isomorpishm of fundamental groups $\pi_1(X, x_0) \cong \pi_1(\Gamma_\alpha, x_0)$.

Remark 4.13. If X is an arboretum, then for each $\alpha \in I$ and each $x \in \Gamma_\alpha$, the fiber $r_\alpha^{-1}(x)$ can be endowed in a natural way with the structure of a compact topological tree. Thus one can think of an arboretum as a family of (compact topological) trees fibered over a (finite) metrized graph (hence the name *arboretum*).

Next, we define the analogue for a general arboretum of the collection $\mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ of all points of type II or III in $\mathbb{P}^1_{\mathrm{Berk}}$.

Definition. The *hyperbolic space* $\mathbf{H}^{\mathbb{R}}(X)$ associated to an arboretum X is the metric space $\varinjlim \Gamma_\alpha$, where the direct limit is taken with respect to the inclusion maps $i_{\alpha,\beta}$.

By the definition of a direct limit, there are natural continuous inclusion maps $i_\alpha : \Gamma_\alpha \hookrightarrow \mathbf{H}^{\mathbb{R}}(X)$ for each $\alpha \in I$, and $\mathbf{H}^{\mathbb{R}}(X)$ satisfies a universal property relative to these maps.

It is not hard to show that there is a natural injective map of sets $i : \mathbf{H}^{\mathbb{R}}(X) \hookrightarrow X$, and the image of $\mathbf{H}^{\mathbb{R}}(X)$ under this map is a dense, path-connected subspace of X.

Remark 4.14. The metric topology on $\mathbf{H}^{\mathbb{R}}(X)$ is *not* in general the same as the subspace topology induced by the inclusion map $i : \mathbf{H}^{\mathbb{R}}(X) \hookrightarrow X$. However, i is always continous as a map from $\mathbf{H}^{\mathbb{R}}(X)$ with its metric topology to X with its inverse limit topology.

Definition. A *simple function* on X is a function of the form $f \circ r_\alpha$ with $\alpha \in I$ and $f \in \mathrm{CPA}(\Gamma_\alpha)$.

We will make use of the following consequence of the Stone-Weierstrass theorem in §4.

Proposition 4.15. *The simple functions are dense in the ring $\mathcal{C}(X)$ of continuous real-valued functions on X (endowed with the uniform topology).*

4.4. **The Laplacian on an arboretum.** Let X be an arboretum corresponding to the arboreal system $\mathcal{S} = (\Gamma_\alpha)_{\alpha \in I}$ of metrized graphs.

Definition. A system of measures $\{\mu_\Gamma\}_{\Gamma \in \mathcal{S}}$ is called *coherent* if:

- For each $\alpha \leq \beta$, we have $(r_{\beta,\alpha})_*(\mu_\beta) = \mu_\alpha$.
- There is a constant B such that $|\mu_\Gamma|(\Gamma) \leq B$ for all $\Gamma \in \mathcal{S}$.

If μ is a measure on X and we set $\mu_\Gamma = (r_\Gamma)_*(\mu)$ for each $\Gamma \in \mathcal{S}$, then $\{\mu_\Gamma\}$ is easily seen to be a coherent system of measures. Conversely, every coherent system arises in this way:

Theorem 4.16. *If $\{\mu_\Gamma\}$ is a coherent system of measures on X, there is a unique measure μ on X such that $(r_\Gamma)_*(\mu) = \mu_\Gamma$ for each $\Gamma \in \mathcal{S}$.*

The proof is based on the Riesz representation theorem, together with Proposition 4.15 (see [BR06, Chapter V] for a proof in the special case $X = \mathbb{P}^1_{\mathrm{Berk}}$, which can be easily generalized to the present setting.)

We define $\mathrm{BDV}(X)$ to be the collection of all functions $f : X \to \mathbb{R} \cup \{\pm\infty\}$ such that:

(BDV1) $f \circ i_\Gamma \in \mathrm{BDV}(\Gamma)$ for each finite subgraph $\Gamma \in \mathcal{S}$, where $i_\Gamma : \Gamma \hookrightarrow X$ is the natural inclusion.

(BDV2) The measures $|\Delta_\Gamma(f \circ i_\Gamma)|$ for $\Gamma \in \mathcal{S}$ have uniformly bounded total mass.

Note that belonging to $\mathrm{BDV}(\Gamma)$ imposes no condition on the values of f at points of $X \backslash \mathbf{H}^{\mathbb{R}}(X)$.

Using the compatibility property (4.10), one shows that if $f \in \mathrm{BDV}(X)$, then $\{\Delta_\Gamma(f \circ i_\Gamma)\}$ is a coherent system of measures on X. By Theorem 4.16, there is a unique measure $\Delta_X(f)$ on the inverse limit space X for which

$$(r_\Gamma)_* \Delta_X(f) = \Delta_\Gamma(f \circ i_\Gamma) \tag{4.17}$$

for all $\Gamma \in \mathcal{S}$.

Definition. If $f \in \mathrm{BDV}(X)$, the unique measure $\Delta_X(f)$ on X satisfying (4.17) is called the *Laplacian* of f on X.

Remark 4.18. It follows from the definitions that $\Delta_X(f)$ is the weak limit over all $\Gamma \in \mathcal{S}$ of the measures $(i_\Gamma)_* (\Delta_\Gamma(f \circ i_\Gamma))$; this is in fact an alternate way to define $\Delta_X(f)$.

As a consequence of Theorem 4.16, we have:

Corollary 4.19. *Let $F = f \circ r_\Gamma$ be a simple function on X (c.f. Proposition 4.15). Then $F \in \mathrm{BDV}(X)$ and $\Delta_X(F) = (i_\Gamma)_* \Delta_\Gamma(f)$.*

The Laplacian on an arboretum satisfies the following properties, which can be deduced from the corresponding properties of the Laplacian on a metrized graph.

Proposition 4.20. *Let X be an arboretum, and let $f, g \in \mathrm{BDV}(X)$. Then:*

1. *$\Delta_X(f) \equiv 0$ if and only if f is constant on $\mathbf{H}^{\mathbb{R}}(X)$.*
2. *$\int_X f\, \Delta_X(g) = \int_X g\, \Delta_X(f)$.*
3. *$\Delta_X(f)$ has total mass zero.*

We also have the following result, which is a generalization of Theorem 4.8.

Theorem 4.21. *If ν is a measure of total mass zero on an arboretum X, then there exists a function $h \in \mathrm{BDV}(X)$, unique up to an additive constant, such that $\Delta_X(h) = \nu$.*

It follows that there is a natural bijection between measures of total mass zero on X and functions in $\mathrm{BDV}(X)$ modulo constant functions.

4.5. **Examples.** We now compute the Laplacian in some specific examples, using the fact that we have endowed the Berkovich spaces $\mathbb{P}^1_{\mathrm{Berk}}$ and $\mathcal{M}(\mathbb{Z})$ with a canonical arboretum structure (see Theorem 2.5 and Remark 2.12).

Example 4.22. As a concrete example, fix $y \in \mathbb{A}^1_{\mathrm{Berk}}$ and let $f : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$f(x) = \kappa_\infty(x, y) = -\log_v [x, y]_\infty$$

(c.f. §1.6.2). Then $f \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$, and

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \delta_y - \delta_\infty \tag{4.23}$$

is a discrete measure on $\mathbb{P}^1_{\mathrm{Berk}}$ supported on $\{y, \infty\}$. The explanation for the formula (4.23) is as follows. The function f is locally constant off the path $\Lambda = \ell_{y,\infty}$ from y to ∞; more precisely, we have $f(x) = f(r_\Lambda(x))$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$. Moreover, the restriction of f to the open segment (y, ∞) is linear (with respect to the metric ρ) with slope -1. It follows that for every finite subgraph Γ of $\mathbb{P}^1_{\mathrm{Berk}}$, we have

$$\Delta_\Gamma(f) = \delta_{r_\Gamma(y)} - \delta_{r_\Gamma(\infty)}. \tag{4.24}$$

The weak limit over all $\Gamma \in \mathcal{S}$ of the right-hand side of (4.24) is $\delta_y - \delta_\infty$, which establishes (4.23).

Equation (4.23) shows that $\kappa_\infty(x,y) = -\log_v [x,y]_\infty$, like its classical counterpart $-\log|x-y|$ over $\mathbb{C}$, is a "fundamental solution of the Laplace equation".

Example 4.25. Generalizing the previous example, fix $y, z \in \mathbb{P}^1_{\mathrm{Berk}}$ and let $f : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$f(x) = \kappa_z(x,y).$$

Then one can show that $f(x)$ is the unique solution in $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ to the differential equation

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \delta_y - \delta_z$$

and satisfying the initial condition

$$f(\zeta_{\mathrm{Gauss}}) = \log_v \|y,z\|$$

(c.f. §1.6.2).

Example 4.26. Consider the function $G : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\}$ from Example 3.3, defined as

$$G(x) = \begin{cases} +\infty & x = \infty \\ \log_v \max(|T|_x, 1) & x \in \mathbb{A}^1_{\mathrm{Berk}}. \end{cases}$$

If Λ denotes the closed path from ζ_{Gauss} to ∞ in $\mathbb{P}^1_{\mathrm{Berk}}$, and $r_\Lambda : \mathbb{P}^1_{\mathrm{Berk}} \to \Lambda$ is the natural retraction map from $\mathbb{P}^1_{\mathrm{Berk}}$ onto Λ, then we saw in Example 3.3 that $G = g \circ r_\Lambda$, where $g : \Lambda \to \mathbb{R} \cup \{+\infty\}$ is the linear map of slope 1 along Λ for which $g(\zeta_{\mathrm{Gauss}}) = 0$.

It follows as in Example 4.22 that

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(G) = \delta_\infty - \delta_{\zeta_{\mathrm{Gauss}}}.$$

Note that $G(x) = -\log_v \|x, \infty\|$, so that this example is in fact a special case of Example 4.25.

Example 4.27. Let $f(T) = \prod_{i=1}^k (x - a_i)^{m_i} \in K[T]$ be a nonconstant polynomial of degree n, with $a_1, \ldots, a_k$ distinct, and let $F(x)$ be the unique continuous function on $\mathbb{P}^1_{\mathrm{Berk}}$ extending the function $-\log_v |f(x)|$ on K.

From Example 3.5 and the linearity of the Laplacian, we deduce easily that

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(F) = \left(\sum_{i=1}^k m_i \delta_{a_i} \right) - n\delta_\infty.$$

More generally, let $\varphi \in K(T)$ be a nonzero rational function with zeros and poles given by the divisor $\mathrm{Div}(\varphi)$ on $\mathbb{P}^1(K)$. As discussed in §2.1, the usual action of φ on $\mathbb{P}^1(K)$ extends naturally to an action of φ on $\mathbb{P}^1_{\mathrm{Berk}}$, and there is a unique continous function $-\log_v |\varphi| : \mathbb{P}^1_{\mathrm{Berk}} \to$

$\mathbb{R} \cup \{\pm\infty\}$ in $\mathrm{CPA}(\mathbb{P}^1_{\mathrm{Berk}})$ extending the usual map $x \mapsto -\log_v |\varphi(x)|$ on $\mathbb{P}^1(K)$. One can derive from (4.23) the following analogue for $\mathbb{P}^1_{\mathrm{Berk}}$ of the classical *Poincaré-Lelong formula*:

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(-\log_v |\varphi|) = \delta_{\mathrm{Div}(\varphi)} \ . \tag{4.28}$$

Example 4.29. Following Example 3.1, let $n \in \mathbb{Z}$ be a nonzero integer. As we saw previously, along the branch Λ_v of $\mathcal{M}(\mathbb{Z})$ emanating from ζ_0 in the tangent direction corresponding to $v \in M_{\mathbb{Q}}$, the function $F_n(x) = -\log|n|_x$ is linear with slope equal to $-\log|n|_v$. Taking a limit over all finite subgraphs of $\mathcal{M}(\mathbb{Z})$ yields the formula

$$\Delta_{\mathcal{M}(\mathbb{Z})}(-\log|n|_x) = \sum_{v \in M_{\mathbb{Q}} \text{ finite}} -\log(|n|_v) \cdot \delta_{\zeta_{v,\infty}} - \log(|n|_\infty) \cdot \delta_{\zeta_\infty}.$$

For example, if $n = 12$ then

$$\Delta_{\mathcal{M}(\mathbb{Z})}(-\log|12|) = \log(4) \cdot \delta_{\zeta_{2,\infty}} + \log(3) \cdot \delta_{\zeta_{3,\infty}} - \log(12) \cdot \delta_{\zeta_\infty}.$$

Thus we see that the Laplacian operator on $\mathcal{M}(\mathbb{Z})$ is basically *factoring* the integer n.

Example 4.30. More generally, it is not hard to see that if $\alpha = m/n$ is a nonzero rational number written in lowest terms, then $\Delta_{\mathcal{M}(\mathbb{Z})}(-\log|\alpha|)$ has total mass zero, and in the Jordan decomposition

$$\Delta_{\mathcal{M}(\mathbb{Z})}(-\log|\alpha|) = \mu^+ - \mu^-,$$

each of μ^+ and μ^- has total mass equal to the standard logarithmic Weil height $h(\alpha) = \log\max\{|m|, |n|\}$.

For example, if $\alpha = 4/3$ then

$$\Delta_{\mathcal{M}(\mathbb{Z})}(-\log|4/3|) = \log(4) \cdot \delta_{\zeta_{2,\infty}} - \log(3) \cdot \delta_{\zeta_{3,\infty}} - \log(4/3) \cdot \delta_{\zeta_\infty}$$

and

$$\Delta_{\mathcal{M}(\mathbb{Z})}(-\log|4/3|)^+(\mathcal{M}(\mathbb{Z})) = \Delta_{\mathcal{M}(\mathbb{Z})}(-\log|4/3|)^-(\mathcal{M}(\mathbb{Z})) = \log(4) = h(4/3).$$

Thus we see that the Laplacian on $\mathcal{M}(\mathbb{Z})$ provides an intriguing way of interpreting the Weil height on $\mathbb{Q}^*$.

Project # 2: Extend the results of Examples 4.29 and 4.30 to (a) the Berkovich analytic space associated to the ring of integers in an arbitrary number field, and (b) the ring $\overline{\mathbb{Z}}$ of all algebraic integers.

4.6. **Harmonic functions and the Laplacian.** Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$. In this section, we will define the Laplacian $\Delta_{\bar{U}}(f)$ of a function f on U as a measure supported on $\bar{U}$, and explain the connection between harmonic functions on U and the Laplacian.

The key point is that $\bar{U}$ is itself an arboretum. Indeed, define $\mathcal{S}_U$ to be the collection of all finite subgraphs Γ of $\mathbb{P}^1_{\mathrm{Berk}}$ which are contained in U. With respect to the natural inclusion and retraction maps, we have:

Lemma 4.31. *There are homeomorphisms*

$$U \cap \mathbf{H}_{\mathrm{Berk}} \cong \varinjlim_{\Gamma \in \mathcal{S}_U} \Gamma$$

(with respect to the metric topology on $\mathbf{H}_{\mathrm{Berk}}$) and

$$\bar{U} \cong \varprojlim_{\Gamma \in \mathcal{S}_U} \Gamma$$

(with respect to the Berkovich subspace topology on $\bar{U}$).

It follows that $\bar{U}$ is naturally endowed with the structure of an arboretum, and therefore we have defined a Laplacian operator on $\bar{U}$. Since the Laplacian of a function f on $\bar{U}$ depends only on the restriction of f to $\mathbf{H}(U) = U \cap \mathbf{H}_{\mathrm{Berk}}$, we see that it makes sense to speak of the Laplacian of a (suitably nice) function f on U as a measure on $\bar{U}$.

Remark 4.32. The definition of $f \in \mathrm{BDV}(\bar{U})$ only concerns the restrictions $f|_\Gamma$ to finite subgraphs $\Gamma \subset U$. By definition, such subgraphs only contain points of type II or III; in fact, as the subgraphs vary they exhaust $U \cap \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$. In particular, requiring that $f \in \mathrm{BDV}(\bar{U})$ imposes no conditions on the behavior of f on $\mathbb{P}^1(K)$. In practice, that behavior must be deduced from auxiliary hypotheses, such as continuity or upper-semicontinuity.

Using Lemma 4.9, we have:

Lemma 4.33. *If $V \subseteq U$ are domains in $\mathbb{P}^1_{\mathrm{Berk}}$, and if $r = r_{\bar{U},\bar{V}} : \bar{U} \to \bar{V}$ denotes the natural retraction map, then for all $f \in \mathrm{BDV}(\bar{U})$, we have $f|_V \in \mathrm{BDV}(\bar{V})$ and*

$$r_*(\Delta_{\bar{U}}(f)) = \Delta_{\bar{V}}(f|_V).$$

The relation between harmonic functions and the Laplacian is the following:

Theorem 4.34. *Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$, and let $f \in \mathrm{BDV}(\bar{U})$. Then f is harmonic on U if and only if $\Delta_{\bar{V}}(f)$ is supported on ∂V for every simple subdomain V of U.*

4.7. **Valuation polygons.** Let $\varphi \in K(T)$ be a nonzero rational function. We saw in §2.1 how φ induces in a natural way a map

$$\varphi : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{P}^1_{\mathrm{Berk}}$$

which, when restricted to $\mathbb{P}^1(K)$, coincides with the usual induced map.

There is a unique simple function $f = -\log_v |\varphi| : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ whose restriction to $\mathbb{P}^1(K)$ is the map $x \mapsto -\log_v |\varphi(x)|$ on $\mathbb{P}^1(K)$. As we have seen in (4.28), one has the following Berkovich space analogue of the *Poincaré-Lelong formula*:

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \delta_{\mathrm{Div}(\varphi)} \ .$$

As a consequence, we obtain:

Corollary 4.35. *For any Borel subset A of $\mathbb{P}^1_{\mathrm{Berk}}$, let $N_0(\varphi, A)$ be the number of zeros of φ in A, and let $N_\infty(\varphi, A)$ be the number of poles of φ in A (counting multiplicities). Then:*

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(-\log_v |\varphi|)(A) = N_0(\varphi, A) - N_\infty(\varphi, A). \tag{4.36}$$

We wish to interpret the formula (4.36) in terms of the classical theory of *valuation polygons* (see [Rob00, Chapter 6, §1,3]) using the *Mass Formula* (Proposition 4.2).

If D is a Berkovich open disk in $\mathbb{P}^1_{\mathrm{Berk}}$, then by Lemma 1.3, we have $D = U(x; \vec{v})$ for some $x \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ and $\vec{v} \in T_x$ (and conversely, each pair $(x, \vec{v})$ with $x \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}}$ and $\vec{v} \in T_x$ determines a unique Berkovich open disk). Following Rivera-Letelier [RL03a, RL03b] we call x the *boundary* of D and $\vec{v}$ the corresponding *end* (French: "bout").

More generally, let V be a simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$. By Lemma 2.7, we can write $V = r_\Gamma^{-1}(U)$ for some finite subgraph Γ of $\mathbb{P}^1_{\mathrm{Berk}}$ and some connected open subset U of Γ. Let $\partial U = \{x_1, \ldots, x_n\} \subseteq \Gamma$ be the boundary of U in Γ (which is the same as the boundary of U in $\mathbb{P}^1_{\mathrm{Berk}}$). For each $1 \le i \le n$, let $\vec{v}_i \in T_{x_i}(\Gamma)$ denote the unique inward-pointing tangent vector at x_i (with respect to U). We can identify $\vec{v}_i$ with an element of T_{x_i} (the set of tangent directions in $\mathbb{P}^1_{\mathrm{Berk}}$ at x_i) in a natural way, since there is a canonical inclusion $T_{x_i}(\Gamma) \subset T_{x_i}$. We call $x_1, \ldots, x_n$ the *boundary points* of V and $\vec{v}_1, \ldots, \vec{v}_n$ the *ends* of V. It is easy to see that the set of ends of V is *canonical*, and does not depend on our choice of the pair (Γ, U).

Remark 4.37. If we write $D_i = U(x_i; \vec{v}_i)$, then each D_i is a Berkovich open disk and

$$V = \bigcap_{i=1}^{n} D_i.$$

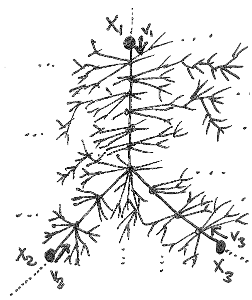

FIGURE 3. A simple domain with three boundary points and three ends.

Using the *Mass Formula* (Proposition 4.2) and Corollary 4.19, we find:

Proposition 4.38. *Let V be a simple domain with boundary $\{x_1, \ldots, x_n\}$ and ends $\{\vec{v}_1, \ldots, \vec{v}_n\}$, and let f be a simple function on $\mathbb{P}^1_{\text{Berk}}$. Then:*

$$\Delta_{\mathbb{P}^1_{\text{Berk}}}(f)(V) = \sum_{i=1}^{n} d_{\vec{v}_i} f(x_i).$$

Combining Proposition 4.38 and Corollary 4.35, we obtain the following result:

Corollary 4.39. *Let V be a simple domain with boundary $\{x_1, \ldots, x_n\}$ and ends $\{\vec{v}_1, \ldots, \vec{v}_n\}$, let $\varphi \in K(T)$ be a nonzero rational function, and let $f = -\log_v |\varphi|$. Then:*

$$N_0(\varphi, V) - N_\infty(\varphi, V) = \sum_{i=1}^{n} d_{\vec{v}_i} f(x_i).$$

In other words, one can compute the difference between the number of zeros and the number of poles of a rational function in a simple domain V in terms of the inward-pointing derivatives of $-\log_v |\varphi|$ at each boundary point of V. This is the essential content of the theory of valuation polygons.

Remark 4.40. For simplicity, we have stated Corollary 4.39 just for rational functions, but using the Weierstrass preparation theorem, one can extend Corollary 4.39 more generally to meromorphic functions.

Example 4.41. Let $\varphi(T) = T^2 - T$, and let $f(x) = -\log_v |T^2 - T|_x$ for $x \in \mathbb{A}^1_{\text{Berk}}$. Let $D = \mathcal{B}(0,1)^-$ be the open unit disk in $\mathbb{A}^1_{\text{Berk}}$, and let $\vec{v} \in T_{\zeta_{\text{Gauss}}}$ be the corresponding end (i.e., the unique tangent vector at ζ_{Gauss} pointing in the direction of 0). Since $f(x)$ is linear with slope 1 along the path from ζ_{Gauss} to 0 (with respect to the path-distance ρ), we have $d_{\vec{v}}f(\zeta_{\text{Gauss}}) = 1$. This agrees with the prediction of Corollary 4.39, since $T^2 - T$ has precisely one zero (namely 0) in the open unit disk $B(0,1)^-$. In other words:

$$\#\text{ of zeros of } T^2 - T \text{ on } B(0,1)^- \;= d_{\vec{v}}f(\zeta_{\text{Gauss}}) = 1.$$

Example 4.42. There is a variant of Corollary 4.39 for intersections of *closed* Berkovich disks (which are precisely the connected affinoid subdomains of $\mathbb{P}^1_{\text{Berk}}$). For example, to compute the number of zeros of $T^2 - T$ in the *closed* unit disk $B(0,1)$, we look at the tangent vector $\vec{w} \in T_{\zeta_{\text{Gauss}}}$ pointing in the direction of ∞ (which is the unique end of the open Berkovich disk $\mathbb{P}^1_{\text{Berk}} \backslash \mathcal{B}(0,1)$), and we find that the number of zeros of $T^2 - T$ in $B(0,1)$ is

$$\#\text{ of zeros of } T^2 - T \text{ on } B(0,1) \;= -d_{\vec{w}}f(\zeta_{\text{Gauss}}) = 2.$$

We leave the details to the reader.

We conclude this section by explaining more precisely the relationship between Corollary 4.39 and the theory of valuation polygons. For this, we shift our attention to analytic functions on a closed disk.

Let $B(a,R)$ be a closed disk in K, and let $B = \mathcal{B}(a,R)$ denote the corresponding Berkovich closed disk in $\mathbb{A}^1_{\text{Berk}}$. We let $x = \zeta_{a,R} \in \mathbf{H}^{\mathbb{R}}_{\text{Berk}}$ denote the unique boundary point of B, and let $\vec{w}$ be the corresponding "outward-pointing" tangent direction at x. More precisely, we define $\vec{w}$ to be the end of the open disk $\mathbb{P}^1_{\text{Berk}} \backslash B$, and call $\vec{w}$ the *end* of B. Let F be an analytic function on a strictly larger disk $B(a,R')$ containing $B(a,R)$, i.e., F is an element of the affinoid algebra

$$\{\sum_{n=0}^{\infty} a_n (T-a)^n \;:\; a_n \in K,\, \lim_{n\to\infty} |a_n|(R')^n = 0\}$$

for some $R' > R$.

We define the *growth modulus* of F to be the function $M_r(F) : [0,R'] \to \mathbb{R}$ defined by

$$M_r(F) = \max_{n\geq 0} |a_n| r^n = \sup_{z \in B(a,r)} |F(z)| = |F|_{\mathcal{B}(a,r)},$$

so that $M_r(F)$ is a positive, increasing real-valued function on $[0,R')$.

Replacing open disks by closed disks in the Mass Formula, and using the Weierstrass Preparation Theorem to pass from polynomials to analytic functions, one obtains the following variant of Proposition 4.38:

Proposition 4.43. *Let B be a closed Berkovich disk with boundary point x and end $\vec{w}$, and let F be analytic on an open neighborhood U*

of B in $\mathbb{P}^1_{\mathrm{Berk}}$. Then

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\log_v|F|)(B) = d_{\vec{w}}(\log_v|F|)(x).$$

It follows from the definitions that if $B = \mathcal{B}(a,R)$ is a closed Berkovich disk with unique boundary point $x = \zeta_{a,r}$ and end $\vec{w}$, then

$$d_{\vec{w}}\log_v|F|(\zeta_{a,R}) = \frac{d}{dr}\log_v M_r(F)|_{r=R}.$$

We therefore obtain a new proof using basic properties of the Laplacian on $\mathbb{P}^1_{\mathrm{Berk}}$ of the following classical result, which is the essence of the theory of valuation polygons:

Corollary 4.44. *Let $B(a,R)$ be a closed disk in K, and let F be analytic on $B(a,R')$ for some $R' > R$. Then*

$$\#\text{ of zeros of } F \text{ on } B(a,R) \ = \frac{d}{dr}\log_v M_r(F)\,|_{r=R}.$$

Remark 4.45. The valuation polygon of F is dual to the Newton polygon of F (in a certain precise sense), so Proposition 4.43 may be thought of as a geometric "explanation" for the theory of valuation and Newton polygons (see [Rob00, Chapter 6, §1.6] for further details concerning valuation and Newton polygons).

4.8. **The modulus of an open annulus.** In this section, we explore some more "classical" results from p-adic analysis from the point of view of Berkovich's theory. Our treatment of this material draws inspiration from [RL03a].

Definition. An *open annulus* is a subset of K of the form $A = B(a,R)^-\backslash B(a,r)$ with $0 < r < R$, i.e., an open disk in K with a closed disk removed from it.

The *standard open annulus A_r of height* r for $0 < r < 1$, is the subset of K given by $A_r = B(0,1)^-\backslash B(0,r)$.

The *modulus* of an open annulus A is defined to be

$$\mathrm{Mod}(A) = \log_v(R/r).$$

Remark 4.46. If $\mathcal{A} = \mathcal{B}(a,R)^-\backslash\mathcal{B}(a,r)$ is the Berkovich open annulus in $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to the open annulus A in K, then by definition, the modulus $\mathrm{Mod}(A)$ coincides with the distance $\rho(\zeta_{a,r},\zeta_{a,R})$ between the two boundary points of $\mathcal{A}$ in $\mathbf{H}_{\mathrm{Berk}}$. The invariance of ρ under Möbius transformations is therefore equivalent to the fact that the modulus of an annulus is invariant under Möbius transformations.

Definition. An *analytic function* on an open annulus $A = B(a,R)^- \backslash B(a,r)$ is a map $\varphi : A \to K$ given by a power series $\sum_{n=-\infty}^{\infty} a_n(T-a)^n$ with $a_n \in K$ which converges at every point of A.

An *analytic isomorphism* between open annuli is an analytic function $\varphi : A \to A'$ with an analytic inverse.

The following proposition shows that, just as in the complex case, the only "conformal invariant" of an open annulus is its modulus.

Proposition 4.47. *Let A, A' be open annuli in K. Then there exists an analytic isomorphism between A and A' if and only if* $\mathrm{Mod}(A) = \mathrm{Mod}(A')$.

Sketch. By applying suitable Möbius transformations, one is reduced to showing that the standard open annuli A_r and A_s are analytically isomorphic if and only if $r = s$. Suppose that $\phi : A_r \to A_s$ is an analytic isomorphism. Since ϕ is a nonzero analytic function on A_r, one can show that ϕ extends to a function $\mathcal{A}_r \to \mathcal{A}_s$, where $\mathcal{A}_r = \mathcal{B}(0,1)^- \backslash \mathcal{B}(0,r)$ denotes the corresponding simple domain in $\mathbb{P}^1_{\mathrm{Berk}}$, and that the function $f = -\log_v |\phi| : \mathcal{A}_r \to \mathbb{R}$ is harmonic and extends continuously to $\partial \mathcal{A}_r = \mathcal{A}_r \cup \{\zeta_{0,1}, \zeta_{0,r}\}$. Consequently, $\Delta_{\overline{\mathcal{A}}_r}(f) = d\delta_{\zeta_{0,1}} - d\delta_{\zeta_{0,r}}$ for some integer $d \in \mathbb{Z}$. As $g(x) = -d \log_v |T|$ has the same property, it follows that $f(x) = g(x) + C'$ for some constant C', and therefore that $\varphi(x) = C \cdot |x|^d$ for all $x \in A_r$. Letting $|x| \to 1$ shows that $C = 1$, and then letting $|x| \to r$ shows that $r = s$. $\square$

Proposition 4.48. *Let $B = B(a,R)^-$ be an open disk in K, and let $\phi : B \to K$ be an analytic function on B defined by a power series $\sum_{n=0}^{\infty} a_n(T-a)^n$ which converges for all $x \in B$. Then $B' = \phi(B)$ is also an open disk, and there exists an integer $d \geq 1$ such that $\phi : B \to B'$ has degree d (i.e., such that every point of B' has exactly d preimages in B, counting multiplicities).*

Sketch. By applying a suitable translation, we may assume that $\phi(0) = 0$. Let $\vec{v}$ be the end corresponding to the unique boundary point $\zeta_{a,R}$ of the open Berkovich disk $\mathcal{B} = \mathcal{B}(a,R)^-$. Then

$$R' := |\phi|_{\zeta_{a,R}} = \sup_{z \in B(a,R)} |\phi(z)|.$$

We claim that $\phi(B) = B(0,R')^-$. Indeed, $\phi(B) \subseteq B(0,R')^-$ by the maximum modulus principle, since $\log_v |\phi|$ is harmonic on $\mathcal{B}$ outside the zeros of ϕ (where it takes the value $-\infty$). And if $w \in B(0,R')^-$ is any point, then $|\phi|_{B(0,r)} = |\phi - w|_{B(0,r)}$ for all r with $|w| < r < R'$ by the ultrametric inequality. By considerations analogous to those in §4.7, the number of zeros of $\phi - w$ in B coincides with the directional derivative $d_{\vec{v}}(-\log_v |\phi|)(\zeta_{a,R})$, which is equal to d for some integer $d \geq 1$ that is independent of w. $\square$

Proposition 4.49. *Let A be an open annulus in K, and let $\phi : A \to K$ be an analytic function on A. Then $\phi(A)$ is also an open annulus, $\mathrm{Mod}(\phi(A)) = d\,\mathrm{Mod}(A)$ for some integer $d \geq 1$, and the map $\phi : A \to \phi(A)$ has degree d.*

Sketch. By applying suitable Möbius transformations, we may assume that $0 \notin \phi(A)$, so that ϕ has no zeros or poles on A, and that $A = A_r$ is a standard open annulus. Write $\mathcal{A} = \mathcal{B}(0,1)^- \backslash \mathcal{B}(0,r)$, so that the two boundary points of $\mathcal{A}$ are $\zeta = \zeta_{0,1}$ and $\zeta' = \zeta_{0,r}$, with corresponding ends $\vec{v}$ and $\vec{v}'$. Let $f = -\log_v |\phi| : \mathcal{A} \to \mathbb{R}$. As in the proof of Proposition 4.47, we find that there is a nonzero integer d such that $|\phi(z)| = \alpha |z|^d$ for all $z \in A$. Applying suitable Möbius transformations, we may assume that $d \geq 1$ and that $\alpha = 1$. Thus if $A' = B(0,1)^- \backslash B(0, r^d)$, then $\phi(A) \subseteq A'$.

We claim that $\phi(A) = A'$, and more generally that every point of A' has exactly d preimages in A, counting multiplicities. To see this, choose a point $w \in A'$. Then the number $m_w(\phi)$ of zeros of $\phi - w$ in A is given by

$$m_w(\phi) = d_{\vec{v}}(-\log_v |\phi - w|)(\zeta) + d_{\vec{v}'}(-\log_v |\phi - w|)(\zeta').$$

Since $|\phi(z) - w| = |z|^d$ for $|w| < |z|^d < 1$ and $|\phi(z) - w| = |w|$ for $r < |z|^d < |w|$, we find that

$$m_w(\phi) = d + 0 = d$$

as desired. $\square$

Using Remark 4.46 and Proposition 4.49, one deduces the following result, due originally to Rivera-Letelier [RL03a].

Theorem 4.50. *Let $\varphi \in K(T)$ be a nonzero rational function of degree $d \geq 1$. Suppose $x \in \mathbf{H}_{\mathrm{Berk}}$ and let $\vec{v} \in T_x$. For $y \in \mathbf{H}^{\mathbb{R}}_{\mathrm{Berk}} \cap U(x; \vec{v})$, let $\mathcal{A}_{x,y}$ denote the unique open annulus with boundary points x, y. Then there is a positive integer $\deg_{\vec{v}}(\varphi) \leq d$, depending only on φ, x, and $\vec{v}$, with the following property:*

> *If y is close enough to x so that φ has no zeros or poles in $\mathcal{A}_{x,y}$, then*
> $$\rho(\varphi(x), \varphi(y)) = \deg_{\vec{v}}(\varphi) \cdot \rho(x, y).$$

This result shows that, locally in the direction of a tangent vector $\vec{v}$, a rational function φ stretches distances in $\mathbf{H}_{\mathrm{Berk}}$ by a factor equal to the derivative of φ in the direction $\vec{v}$.

As a consequence of Theorem 4.50, one deduces from a simple compactness argument the following result:

Corollary 4.51. *Let $\varphi \in K(T)$ be a nonzero rational function of degree $d \geq 1$. Then for all $x, y \in \mathbf{H}_{\text{Berk}}$, we have*

$$\rho(\varphi(x), \varphi(y)) \leq d \cdot \rho(x, y).$$

5. **Lecture V**: Introduction to Berkovich curves

Following Berkovich [Ber90] and Thuillier [Thu05], we describe in this lecture how to generalize some of our constructions and "visualization techniques" from $\mathbb{P}^1_{\text{Berk}}$ to more general Berkovich curves. The basic idea, explained using different terminology in [Ber90], is that the Berkovich analytic space X_{Berk} attached to a smooth, proper, geometrically integral curve X/K admits a deformation retraction onto a finite topological graph called the *skeleton* of X_{Berk}. A very special case of the results of [Ber99] and [Ber04], which also apply in higher-dimensions, is that the skeleton of X_{Berk} can be naturally endowed with the structure of a *metrized graph*. More generally, the entire space X_{Berk} can be viewed in a canonical way as an *arboretum*, in the sense of §4.3. This observation will allow us to define a Laplacian operator on X_{Berk} which generalizes the one we have already defined on the Berkovich projective line.

Out of necessity, this lecture assumes significantly more background in rigid analysis and Berkovich's global theory of K-analytic spaces than the previous lectures did. Nevertheless, we have tried to keep the exposition as accessible as possible.

5.1. **Visualizing Berkovich curves via the semistable reduction theorem.** As before, K will denote a complete, algebraically closed non-archimedean field. Let X be a smooth, proper, and geometrically integral algebraic curve over K, and let X^{rig} denote the corresponding rigid-analytic space in the sense of Tate. In this section, we will explain how to visualize the associated Berkovich analytic space X_{Berk} (see §2.4 above) in a manner similar to the way we visualized $\mathbb{P}^1_{\text{Berk}}$ and $\mathcal{M}(\mathbb{Z})$ in §1.3 and §2.5, respectively. To do this, we will use a deep result from rigid analysis, namely the existence of semistable coverings. We will also utilize the rigid-analytic description (due to Bosch and Lutkebohmert) of the formal fibers of a semistable model.

Let $R = \{x \in K \ : \ |x| \leq 1\}$ denote the valuation ring of K, and let $\tilde{K}$ be its (algebraically closed) residue field. A *semistable covering* of X (or, more precisely, of X^{rig}) is a formal covering $\mathcal{U}$ of X^{rig} whose special fiber $\tilde{X}_{\mathcal{U}}$ is a semistable curve over $\tilde{K}$. (See [FvdP04, §4.8] or [BL85] for a discussion of formal coverings and their reductions.) Recall that a projective curve Z over k is called *semistable* if it is reduced, and if the only singularities of Z are ordinary double points. The semistable reduction theorem implies that every curve X as above has a semistable covering.

From now on, unless otherwise specified, we choose without comment a particular semistable covering $(X, \mathcal{U})$, and let Z denote its special fiber. We denote by π the natural reduction map $\pi : X(K) \to Z(\tilde{K})$. If $z \in Z(\tilde{K})$, we call $X(z) = \pi^{-1}(z)$ the *formal fiber* of X over z. It carries the structure of a rigid analytic space in a natural way. Let $B(0,r)$ denote the closed disk of radius r in K (i.e., the affinoid space corresponding to $K\langle r^{-1}T\rangle$), and let $B(0,r)^-$ be the corresponding open disk, which is the formal fiber over 0 in $B(0,r)$. Also, define

$$A(\alpha) = \operatorname{Max} K\langle T, \alpha T^{-1}\rangle = \{z \in B(0,1) \ : \ \alpha \le |z| \le 1\}$$

to be the *standard closed annulus of height* α in $B(0,1)$. The canonical reduction of $A(\alpha)$ is the scheme $\operatorname{Spec} \tilde{K}[S,T]/(ST)$ consisting of two affine lines intersecting at an ordinary double point. We also let

$$A(\alpha)^- = \operatorname{Max} K\langle T, \alpha T^{-1}\rangle = \{z \in B(0,1) \ : \ \alpha < |z| < 1\}$$

be the corresponding *standard open annulus of height* α, which is the formal fiber over the singular point of the canonical reduction of $A(\alpha)$.

We recall the following result due to Bosch and Lutkebohmert:

Proposition 5.1. *Let $z \in Z(\tilde{K})$. Then:*

(i) *If z is a nonsingular point of Z, then the formal fiber $\pi^{-1}(z)$ is analytically isomorphic to the open unit disk $B(0,1)^-$.*
(ii) *If z is a singular point of Z (which by hypothesis is an ordinary double point), then the formal fiber $\pi^{-1}(z)$ is analytically isomorphic to a standard open annulus $A(\alpha)^-$ for some unique $\alpha \in |K^*|$ with $0 < \alpha < 1$.*

Remark 5.2. In fact, the proof of (i) shows that there exists an affinoid neighborhood V of $X(z)$ and an analytic map $V \to B(0,1)$ which restricts to an isomorphism $\pi^{-1}(z) \xrightarrow{\sim} B(0,1)^-$. Similarly, the proof of (ii) shows that if z lies on two different components of Z, then there exists an affinoid neighborhood V of $X(z)$ and an analytic map $V \to A(\alpha)$ which restricts to an isomorphism $\pi^{-1}(z) \xrightarrow{\sim} A(\alpha)^-$.

For our purposes, it is important to note that, just as over the complex numbers, the height α is a "conformal invariant" of an annulus.

More precisely, define a *closed annulus* to be an affinoid of the form

$$A(\alpha, \beta) = \mathrm{Max}\, K\langle \beta^{-1}T, \alpha T^{-1}\rangle = \{z \in \mathbb{A}^1 \;:\; \alpha \le |z| \le \beta\},$$

and define an *open annulus* to be a rigid space of the form $A(\alpha,\beta)^-$ (i.e., the formal fiber over the singular point of the canonical reduction of a closed annulus). Define the *modulus* of a closed (resp. open) annulus $A = A(\alpha,\beta)$ (resp. $A = A(\alpha,\beta)^-$) to be $m(A) = \log_v |\beta| - \log_v |\alpha|$. It is easy to see (by applying a suitable homothety) that two annuli with the same modulus are analytically isomorphic (i.e., isomorphic as rigid analytic spaces). The converse is true as well (c.f. Proposition4.47):

Lemma 5.3. (i) *Any two closed (resp. open) disks are isomorphic as rigid spaces.*

(ii) *Two closed (resp. open) annuli A, A' are isomorphic as rigid spaces if and only if they have the same modulus.*

Let X/K be an algebraic curve, as above, let $\mathcal{U}$ be a formal covering of X having semistable reduction Z, and let $\pi : X^{\mathrm{rig}} \to Z(\tilde{K})$ be the corresponding analytic reduction map. We can write $Z = \bigcup_{i=1}^{t} Z_i$, where the Z_i are geometrically irreducible curves, each containing at least one double point of Z. Let Z_i^* be the nonsingular affine curve obtained from Z_i by removing all double points of Z lying on Z_i. The rigid analytic space $X_i^* = \pi^{-1}(Z_i^*)$ is a nonsingular affinoid with canonical reduction Z_i^*. For each ordinary double point $p \in Z(\tilde{K})$, lying on the components Z_i and Z_j, we also have an affinoid $X_p = \pi^{-1}(Z_i^* \cup Z_j^* \cup \{p\})$ whose canonical reduction is $(Z_i^* \cup Z_j^* \cup \{p\})$. As a rigid space, X is obtained by gluing the affinoids X_p along the affinoid subsets X_i^*.

Berkovich's construction of global K-analytic spaces in [Ber93] (c.f. §2.4) yields a locally ringed space X_{Berk} obtained by gluing the affinoid spaces $M(X_p)$ along the affinoid subsets $M(X_i^*)$. It can be thought of as the "Berkovich analytification" of the rigid analytic space associated to X. The reduction map $\pi : X(K) \to Z(\tilde{K})$ extends naturally to a map $\mathrm{red} : X_{\mathrm{Berk}} \to Z$, where the target space is the set of *scheme-theoretic* points of Z. According to [Ber90, Proposition 2.4.4], for each irreducible component Z_i of Z, there is a *unique* point of X_{Berk} reducing to the generic point of Z_i. We now describe, following [FvdP04, §7.2], how the underlying topological space of X_{Berk} can be understood in terms of the *dual graph* of Z.

Recall that the spaces $M(X_i^*)$ correspond to the irreducible components of Z, and the spaces $M(X_p)$ to the intersections between components. This can be conveniently encoded via the dual graph $\Sigma_{\mathcal{U}}$ of Z, which is the topological graph having a vertex for each component of Z, and an edge for each point of intersection between two components. The gluing data for the space X_{Berk} corresponds in a natural way to $\Sigma_{\mathcal{U}}$. According to [Ber90, Chapter 4], since each X_i^* is a 1-dimensional affinoid with nonsingular canonical reduction, the spaces $M(X_i^*)$ are all contractible. More concretely, if ζ_i denotes the unique point of $M(X_i^*)$ reducing to the generic point of Z_i^*, then there is a deformation retraction $r_i : M(X_i^*) \twoheadrightarrow \{\zeta_i\}$.

One can visualize the retraction r_i as follows. By Proposition 5.1, the rigid-analytic formal fiber $\pi^{-1}(z)$ over any closed point $z \in Z_i^*(\tilde{K})$ is isomorphic to the open disk $B(0,1)^-$. Applying the Berkovich analytification functor, one sees that the inverse image $\mathrm{red}^{-1}(z)$ is isomorphic to the Berkovich open disk $\mathcal{B}(0,1)^-$. The closure of $\mathrm{red}^{-1}(z)$ in $M(X_i^*)$ (or equivalently, in X_{Berk}) is precisely $\mathrm{red}^{-1}(z) \cup \{\zeta_i\}$, which is isomorphic to $\mathcal{B}(0,1)^- \cup \{\zeta_0\}$, the closure of $\mathcal{B}(0,1)^-$ in $\mathcal{B}(0,1)$. Using the tree structure of $\mathcal{B}(0,1)^-$, one therefore sees that each Berkovich formal fiber $\mathrm{red}^{-1}(z)$ admits a deformation retraction onto $\{\zeta_i\}$, and the deformation retraction $r : M(X_i^*) \twoheadrightarrow \{\zeta_i\}$ is obtained by simultaneously retracting all of these formal fibers to this point.

Similarly, for each singular point p of Z, corresponding to an intersection of Z_i and Z_j, the space $M(X_p)$ deformation retracts onto a line segment e_p with endpoints ζ_i and ζ_j. One can again visualize this retraction using Proposition 5.1, as there is an isomorphism $\psi : \pi^{-1}(p) \to A(\alpha_p)^-$ from the rigid-analytic formal fiber $\pi^{-1}(p)$ to an open annulus $A(\alpha_p)^-$ which extends to an isomorphism on Berkovich spaces. By considering the reduction map $\mathrm{red} : X_p \to Z_i^* \cup Z_j^* \cup \{p\}$, one sees that $M(X_p) = \mathrm{red}^{-1}(p) \cup M(X_i) \cup M(X_j)$ as sets. The closure of $\mathrm{red}^{-1}(p)$ in $M(X_p)$ is $\mathrm{red}^{-1}(p) \cup \{\zeta_i, \zeta_j\}$, and the open annulus $\mathbb{A}(\alpha_p)^- \subset \mathbb{B}(0,1)$ deformation retracts onto the open segment connecting its two boundary points $\psi(\zeta_i)$ and $\psi(\zeta_j)$ in $\mathbb{A}^1_{\mathrm{Berk}}$, which are the type II points corresponding to the disks $B(0,1)$ and $B(0,\alpha_p)$. (Of course, $\mathbb{A}(\alpha_p)^-$ is in fact contractible, but to get a retraction which is compatible with the gluing maps, we need to keep the points $\psi(\zeta_i)$ and $\psi(\zeta_j)$ fixed.) The segment e_p is the unique path in $M(X_p)$ from ζ_i to ζ_j, and can be thought of as a "line of embedded disks" $\{\psi^{-1}(B(0,r)) \ : \ \alpha_p \le r \le 1\}$ linking the Gauss point of X_i^* to the Gauss point of X_j^*.

Globalizing, we now see that the entire Berkovich space X_{Berk} admits a deformation retraction onto the topological space obtained by gluing the segments e_p along the points ζ_i, which is precisely the topological dual graph $\Sigma_{\mathcal{U}}$. Let $r : X_{\mathrm{Berk}} \to \Sigma_{\mathcal{U}}$ be this retraction map. For

$x \in \Sigma_{\mathcal{U}}$, the fiber $r^{-1}(x)$ is isomorphic to a compact, connected subset of $\mathcal{B}(0,1)$, and in particular is a topological tree.

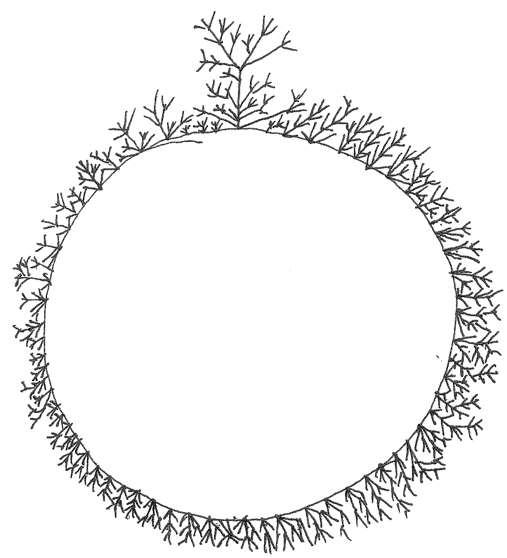

FIGURE 4. The Berkovich analytic space associated to an elliptic curve with multiplicative reduction.

We now define a path metric $\rho(x,y)$ on $\mathbf{H}(X) = X_{\mathrm{Berk}} \backslash X(K)$. It is enough to define ρ locally, so we may assume that either $x, y \in r^{-1}(z)$ for some $z \in \Sigma_{\mathcal{U}}$ or that $x, y \in e_p$ for some singular point $p \in Z$. Since both $r^{-1}(z)$ and e_p are isomorphic to subsets of $\mathcal{B}(0,1)$, we can use the path metric ρ on $\mathcal{B}(0,1)$ to define distances locally on $\mathbf{H}(X)$. Lemma 5.3 shows that this is well-defined. It is not hard to show that the metric on $\mathbf{H}(X)$ is canonical, i.e., does not depend on our choice of a particular semistable formal covering $\mathcal{U}$ of X.

Remark 5.4. The set $S(X_{\mathrm{Berk}})$ of points of X_{Berk} corresponding to generic points of irreducible components of Z having arithmetic genus at least 1 is independent of the formal covering $\mathcal{U}$. We will refer to $S(X_{\mathrm{Berk}})$ as the set of *marked points* of X_{Berk}. The set of marked points of X_{Berk} is empty if and only if X is a *Mumford curve* in the sense of [GvdP80].

Remark 5.5. If the genus of X is at least one, there is a maximal subgraph Σ of $\Sigma_{\mathcal{U}}$ containing the set $S(X_{\mathrm{Berk}})$ of marked points and having no vertices of degree 1. Following Berkovich, the metrized graph Σ is called the *skeleton* of X_{Berk}. The skeleton is canonical, and in particular does not depend on the choice of any formal covering $\mathcal{U}$. We refer to the metrized graph $\Sigma_{\mathcal{U}}$, which admits a deformation retraction onto Σ, as the skeleton of X_{Berk} with respect to the formal covering $\mathcal{U}$.

The existence of the skeleton of a curve of genus at least one, and the non-existence of such a canonical skeleton for curves of genus zero, is very closely related to the theory of minimal models for arithmetic surfaces (see, e.g., [Liu02]). For example, if X has genus at least one and is defined over a discretely valued subfield K' of K with valuation ring R', then the skeleton of the formal covering associated to the minimal regular model of X over R' coincides with the skeleton of X_{Berk}.

5.2. **Tate elliptic curves.** The semistable reduction theorem is not the only way to study Berkovich curves. For example, when studying curves with totally degenerate reduction, one can also profit considerably from the point of view offered by p-adic uniformization theory. To illustrate the utility of this point of view, in this section we give a detailed topological description of the Berkovich analytic space associated to a Tate elliptic curve (i.e., an elliptic curve with multiplicative reduction) from the point of view of Tate's non-archimedean uniformization theory.

As before, let K be a complete and algebraically closed non-archimedean field. Let E/K be an elliptic curve with multiplicative reduction, and let E_{Berk} be the corresponding Berkovich analytic space. According to Tate's theory, $E_{\mathrm{Berk}} \cong (\mathbf{G}_m^{\mathrm{rig}})/q^{\mathbb{Z}}$, where $\mathbf{G}_m^{\mathrm{rig}}$ is the rigid analytic space associate to the multiplicative group over K and $q \in K^*$ satisfies $|q| < 1$. In particular, $E(K) \cong K^*/q^{\mathbb{Z}}$. By choosing coordinates on $\mathbb{P}^1$, we can identify $\mathbf{G}_m^{\mathrm{rig}}$ with $(\mathbb{P}^1)^{\mathrm{rig}} \backslash \{0, \infty\}$. We can then view the Berkovich analytic space E_{Berk} somewhat informally as the space obtained from the affinoid annulus $V = \mathcal{B}(0,1) \backslash \mathcal{B}(0,|q|)^-$ by identifying the affinoid subspaces $V_1 = \mathcal{B}(0,1) \backslash \mathcal{B}(0,1)^-$ and $V_2 = \mathcal{B}(0,|q|) \backslash \mathcal{B}(0,|q|)^-$ in the "obvious" way. Let $\pi : V \to E_{\mathrm{Berk}}$ denote the corresponding quotient map.

We may visualize the resulting graph structure on E_{Berk}, and the path distance ρ on $\mathbf{H}(E_{\mathrm{Berk}}) := E_{\mathrm{Berk}} \backslash E(K)$, as follows. Let ζ, ζ' be the points of $\mathbb{P}^1_{\mathrm{Berk}}$ corresponding to $B(0,1)$ and $B(0,|q|)$, respectively, and let I denote the unique path in $\mathbb{P}^1_{\mathrm{Berk}}$ from ζ to ζ', so that I is isometric to a segment of length $\ell = -\log_v |q|$ with $0 < \ell < \infty$. Then the image $\pi(I)$ of I in E_{Berk} is isometric to a circle Σ of length ℓ, which we call the *skeleton* of E_{Berk}. There is also a retraction map $r : E_{\mathrm{Berk}} \to \Sigma$ induced by the retraction (which we also denote by r) of V onto I.

Note that if $x \in I$ corresponds to the disk $B(0,R)$ with $R \in |K^*|$ (i.e., x is a point of type II), then the space $\{z \in V \ : \ r(z) = x\}$ is naturally identified with $\mathcal{B}(0,R)\backslash\mathcal{B}(0,R)^-$, which itself is isomorphic (and isometric) to $\mathcal{B}(0,1)\backslash\mathcal{B}(0,1)^-$. If $R \not\in |K^*|$ (i.e., x is a point of type III), then $r^{-1}(x) = \{x\}$. In this way, we can view $E_{\rm Berk}$ as fibered over the circle Σ, and each fiber $r^{-1}(x)$ for $x \in \Sigma$ is isomorphic either to a point or to the closed annulus $W = \mathcal{B}(0,1)\backslash\mathcal{B}(0,1)^-$, which itself has the structure of an infinite metrized tree. In particular, $E_{\rm Berk}$ can be thought of as a family of infinite metrized trees fibered over the rational points of the circle Σ. For any two points $x, y \in E_{\rm Berk}$, we can define the path-distance $\rho(x,y)$ as the length of the shortest path from x to y, where path lengths are given on each annulus W by the usual path distance on $\mathbb{P}^1_{\rm Berk}$, and on Σ by identifying it with a circle of length ℓ. If $x \neq y$, we have $\rho(x,y) = \infty$ if and only if one of x and y is in $E(K)$. In particular, $\rho(x,y) < \infty$ for all $x, y \in \mathbf{H}(E_{\rm Berk})$.

The Berkovich topology on $E_{\rm Berk}$ is the quotient topology induced by the map π; it is not difficult to see that a fundamental system of open neighborhoods for the topology on $E_{\rm Berk}$ is given by $\{U_{\Gamma,V}\}$, where Γ is a finitely branched connected subgraph of $\mathbf{H}(E_{\rm Berk})$ containing Σ, $V \subseteq \Gamma$ is open in the metric topology on Γ, and $U_{\Gamma,V} = r^{-1}(V)$. With this topology, $E_{\rm Berk}$ is a path-connected compact Hausdoff space.

Remark 5.6. The space $\mathbb{P}^1_{\rm Berk}\backslash\{0,\infty\}$ is simply connected, and is in fact the universal covering space of $E_{\rm Berk}$ via the natural map $\mathbb{P}^1_{\rm Berk}\backslash\{0,\infty\}$ to $E_{\rm Berk}$. The fundamental group of $E_{\rm Berk}$ is $\mathbb{Z}$, since it admits a deformation retraction onto the circle Σ. This is yet another illustration of the power and utility of Berkovich's theory: it allows one to study non-archimedean analytic spaces using standard tools of algebraic topology!

5.3. **Harmonic functions on Berkovich curves.** In this section, we define what it means for a real-valued function on a Berkovich curve $X_{\rm Berk}$ to be harmonic. This is analogous to the discussion in §3.2.

5.3.1. *Definition of a harmonic function.* As in §1.4.3, if $x \in X_{\rm Berk}$ then there is a well-defined set T_x of *tangent directions* at x, defined as the set of equivalence classes of paths $\ell_{x,y}$ emanating from x, where y is any point of $X_{\rm Berk}$ not equal to x and two paths $\ell_{x,y_1}, \ell_{x,y_2}$ are *equivalent* if they share a common initial segment.

Define $\mathbf{H}^{\mathbb{R}}(X_{\rm Berk})$ to be the set of all points $x \in X_{\rm Berk}$ for which T_x has more than one element. (This coincides with the definition of $\mathbf{H}^{\mathbb{R}}_{\rm Berk}$ when $X = \mathbb{P}^1_{\rm Berk}$.) As with $\mathbf{H}_{\rm Berk}$, we view $\mathbf{H}^{\mathbb{R}}_{\rm Berk}$ as equipped with the metric ρ defined in §5.1.

Let U be a connected open subset of X_{Berk}, and let $f : U \to \mathbb{R} \cup \{\pm\infty\}$ be a continuous extended-real valued function which is finite-valued on $\mathbf{H}(X_{\mathrm{Berk}})$.

As in §3.2, we say that f is *continuous piecewise affine* on U, and write $f \in \mathrm{CPA}(U)$ if:

(CPA1) The restriction of f to $\mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$ is piecewise-affine with respect to the path metric ρ.

(CPA2) If $f \in \mathrm{CPA}(U)$ and $x \in U \cap \mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$, then for each $\vec{v} \in T_x$ the *directional derivative* $d_{\vec{v}} f(x)$ is well-defined.

(CPA3) For each $x \in U \cap \mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$, we have $d_{\vec{v}} f(x) = 0$ for all but finitely many $\vec{v} \in T_x$. In particular, the quantity

$$\Delta_x(f) := -\sum_{v \in T_x} d_{\vec{v}} f(x)$$

is well-defined for each $x \in U \cap \mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$.

Let $x \in U$, and let $h \in \mathrm{CPA}(U)$. As in §3.2, we make the following definition.

Definition. 1. If $x \in \mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$, we say that h is *harmonic at x* if $\Delta_x(h) = 0$.

2. If $x \notin \mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$, we say that h is *harmonic at x* if h is constant on an open neighborhood of x.

5.3.2. *Example: Logarithms of absolute values of meromorphic functions.* As before, we let X_{Berk} be a Berkovich curve, and we let f be a nonzero meromorphic function on X_{Berk}. We also let $\mathcal{U}$ be a formal covering of the underlying rigid-analytic space X^{rig}, and we let $r : X_{\mathrm{Berk}} \twoheadrightarrow \Sigma_{\mathcal{U}}$ be the corresponding retraction onto the skeleton of X_{Berk} with respect to $\mathcal{U}$.

Proposition 5.7. *Let φ be the unique continuous function from X_{Berk} to $\mathbb{R} \cup \{\pm\infty\}$ for which $\varphi(x) = -\log_v |f(x)|$ outside the zeros and poles of f. Then $\varphi \in \mathrm{CPA}(X_{\mathrm{Berk}})$.*

Sketch. Since the property of belonging to $\mathrm{CPA}(X_{\mathrm{Berk}})$ is local, and since the zeros and poles of a meromorphic function are finite in number, it suffices to check the result for an affinoid V on which either f or $1/f$ is analytic. In this case, φ is continuous by definition. Moreover, the linearity of φ can be checked locally on residue classes U isomorphic to open disks or open annuli in $\mathbb{P}^1_{\mathrm{Berk}}$. So we can reduce the assertion that φ satisfies (CPA1) to the corresponding fact on $\mathbb{P}^1_{\mathrm{Berk}}$, which we already know to be true. Furthermore, note that $X_{\mathrm{Berk}} \backslash \Sigma_{\mathcal{U}}$ is a disjoint union of branches emanating from $\Sigma_{\mathcal{U}}$, each of which is isomorphic to an open disk U in $\mathbb{P}^1_{\mathrm{Berk}}$. By the corresponding fact for open subsets of $\mathbb{P}^1_{\mathrm{Berk}}$, we obtain that φ satisfies (CPA2). Finally, if f has no zeros or poles on such a branch, then the corresponding function $\tilde{\phi} = -\log_v |\tilde{f}| : U \to \mathbb{R}$ is harmonic on an open disk, so by Corollary 3.8 it is constant. It follows that φ is constant in all but finitely many tangent directions emanating from $\Sigma_{\mathcal{U}}$, proving (CPA3). $\square$

We will now show that φ is harmonic outside the zeros and poles of f. In order to do this, we describe the translation into the language of Berkovich spaces of some results of Bosch and Lutkebohmert from rigid analysis (see [BL85, Section 3]). As before, let $Z_1, \ldots, Z_t$ be the irreducible components of Z, and for each $i = 1, \ldots, t$, choose an open affine subset $W_i \subset Z_i$. Then the preimage $V_i = \pi^{-1}(W_i)$ is an affinoid subdomain of X^{rig}.

Let $\zeta_i \in X_{\mathrm{Berk}}$ be the generic point of an irreducible component Z_i of Z. The tangent space T_{ζ_i}, which is the set of equivalence class of paths emanating from ζ_i, is naturally in bijection with the set $\tilde{Z}_i(\tilde{K})$ of closed points of $\tilde{Z}_i$, where $\tilde{Z}_i$ is the normalization of Z_i. We make this bijection explicit by writing $T_{\zeta_i} = \{\vec{v}_{\tilde{z}}\}_{\tilde{z} \in Z_i(\tilde{K})}$, where $\vec{v}_{\tilde{z}}$ is a formal unit vector emanating from ζ_i in the direction corresponding to $\tilde{z} \in \tilde{Z}_i(\tilde{K})$.

Remark 5.8. Our explicit "visualization" of X_{Berk} using the semistable reduction theorem shows that every point $x \in X_{\mathrm{Berk}} \backslash \{\zeta_1, \ldots, \zeta_t\}$ has an open neighborhood isomorphic to an open disk or open annulus in $\mathbb{P}^1_{\mathrm{Berk}}$, and by the corresponding result on $\mathbb{P}^1_{\mathrm{Berk}}$, in order to show that $\varphi(x)$ is harmonic outside the zeros and poles of f, it suffices to check that $\Delta_{\zeta_i}(\varphi) = 0$ at the finitely points $\zeta_i \in X_{\mathrm{Berk}}$ corresponding to the generic points of the irreducible components of Z.

For each component Z_i, there exists a nonzero scalar $c_i = c_i(f) \in K^*$, depending on Z_i and on f, such that $c_i^{-1} f$ reduces to a nonzero rational function on Z_i. The absolute value $|c_i|$ of c_i is uniquely determined, and is in fact equal to $|f|_{\zeta_i}$. The divisor of $\overline{c_i^{-1} f}$ on the normalization $\tilde{Z}_i$ of Z_i depends only on f, not on the choice of a particular constant c_i. We can thus define the *order of f at a point $\tilde{z} \in \tilde{Z}_i$* to be

$$\mathrm{ord}_{\tilde{z}}(f) := \mathrm{ord}_{\tilde{z}}(\overline{c_i^{-1} f}).$$

Part 2 of the following proposition is Proposition 3.1 of [BL85], and part 1 is established as the key step in the proof of this proposition.

Proposition 5.9. *Let f be a nonzero meromorphic function on X_{Berk}.*

1. *If $\tilde{z} \in \tilde{Z}_i$, then the directional derivative $d_{\vec{v}_{\tilde{z}}}(-\log_v |f|)(\zeta_i)$ equals $-\mathrm{ord}_{\tilde{z}}(f)$.*

2. *Let $z \in Z(\tilde{K})$, and denote by $\tilde{z}_1, \ldots, \tilde{z}_r$ the points in the normalization of Z lying over z (so that r equals 1 or 2, depending on whether z is non-singular or singular). Then the divisor of f has degree*

$$\deg(\mathrm{Div}(f)|_{\pi^{-1}(z)}) = \sum_{i=1}^{r} \mathrm{ord}_{\tilde{z}_i}(f).$$

on the formal fiber $\pi^{-1}(z)$.

As an immediate consequence of part 1 of the proposition, we obtain:

Corollary 5.10. *Let f be a nonzero meromorphic function on X_{Berk}, let Z_i be an irreducible component of Z, and let $\zeta_i \in X_{\mathrm{Berk}}$ be the corresponding "generic point". Then*

$$\Delta_{\zeta_i}(-\log_v |f|) = -\sum_{\tilde{z} \in \tilde{Z}_i} d_{\vec{v}_{\tilde{z}}}(-\log_v |f|)(\zeta_i) = 0.$$

Proof. The sum in question is equal to

$$\sum_{\tilde{z} \in \tilde{Z}_i} \mathrm{ord}_{\tilde{z}}(\overline{c_i^{-1} f}) = 0,$$

since a rational function on a complete nonsingular curve has the same number of zeros as poles (counting multiplicities). □

Combining Corollary 5.10 and Remark 5.8, we find:

Corollary 5.11. *Let f be a nonzero meromorphic function on X_{Berk}, and let $\varphi = -\log_v |f|$. Then φ is harmonic outside the zeros and poles of f.*

5.4. **The Laplacian on a Berkovich curve.**

5.4.1. *A higher genus analogue of Berkovich's classification theorem.* Let X_{Berk} be a Berkovich curve over K.

Define a *finite subgraph* of X_{Berk} to be a (finite and connected) metrized graph contained in $\mathbf{H}^{\mathbb{R}}(X_{\mathrm{Berk}})$. It follows from our description in §5.1 that the collection $\mathcal{S}$ of all finite subgraphs Γ_α of X_{Berk} containing $\Sigma_{\mathcal{U}}$ forms an arboreal system of metrized graphs.

Moreover, we have the following generalization of Theorem 2.5, which can be thought of as an extension of Berkovich's classification theorem from $\mathbb{P}^1_{\mathrm{Berk}}$ to arbitrary Berkovich curves.

Theorem 5.12. *X_{Berk} is homeomorphic to $\varprojlim_{\Gamma \in \mathcal{S}} \Gamma$.*

There is a useful and more natural version of Theorem 5.12 which can be formulated in terms of *formal coverings*. If $\mathcal{U}$ is a semistable formal covering of X, the dual graph $\Gamma_{\mathcal{U}}$ of the reduction of $\mathcal{U}$ can be naturally endowed with the structure of a metrized graph. Moreover, any two such formal coverings have a common refinement. Consequently, one finds that the collection

$$\{\Gamma_{\mathcal{U}} \ : \ \mathcal{U} \text{ is a formal covering of } X\}$$

forms an arboreal system of metrized graphs. The reformulation of Theorem 5.12 in terms of formal coverings is that X_{Berk} is homeomorphic to $\varprojlim_{\mathcal{U}} \Gamma_{\mathcal{U}}$.

Closely related to the formal covering version of Theorem 5.12 is a description of X_{Berk} in terms of *formal models*. If $\mathfrak{X}$ is a semistable formal model of X, the dual graph $\Gamma_{\mathfrak{X}}$ of the special fiber of $\mathfrak{X}$ can be naturally endowed with the structure of a metrized graph. Moreover, given any two such formal models $\mathfrak{X}_1$ and $\mathfrak{X}_2$, there is a third formal model $\mathfrak{X}_3$ of X which dominates both $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Consequently, one finds that the collection

$$\{\Gamma_{\mathfrak{X}} \ : \ \mathfrak{X} \text{ is a formal model of } X\}$$

forms an arboreal system of metrized graphs. The reformulation of Theorem 5.12 in terms of formal models is that X_{Berk} is homeomorphic to $\varprojlim_{\mathfrak{X}} \Gamma_{\mathfrak{X}}$.

5.4.2. *The Laplacian on* X_{Berk}. The discussion in the previous section shows that X_{Berk} is naturally endowed with the structure of an arboretum, and in particular there is a well-defined Laplacian operator on X_{Berk}.

Here is an example which generalizes both Example 4.27 and Corollary 5.11.

Example 5.13. Let φ be a nonzero meromorphic function on the algebraic curve X/K. Then there is a continous function $-\log_v |\varphi| : X_{\text{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ in $\text{CPA}(X_{\text{Berk}})$ extending the usual map $x \mapsto -\log_v |\varphi(x)|$ on $X(K)$. One can deduce from the discussion in §5.3.2 the following *Poincaré-Lelong formula for Berkovich curves*:

$$\Delta_{X_{\text{Berk}}} (-\log_v |\varphi|) = \delta_{\text{Div}(\varphi)} \ . \tag{5.14}$$

(See Thuillier [Thu05] for a generalization of this result.)

Project # 3: The left-hand side of formula (5.14) depends on the metric structure on $\mathbf{H}(X_{\text{Berk}})$. Show that ρ is the *unique* metric on $\mathbf{H}(X_{\text{Berk}})$ for which the formula (5.14) holds.

5.4.3. *Examples.* We give some examples of Laplacians of functions on Berkovich curves.

Example 5.15. If X_{Berk} is a Berkovich curve and $y, z \in X_{\mathrm{Berk}}$, there is, up to an additive constant, a unique function $f(x) = \kappa_z(x, y) : X_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ in $\mathrm{BDV}(X_{\mathrm{Berk}})$ for which $\Delta_{X_{\mathrm{Berk}}}(f) = \delta_y - \delta_z$. We call any such function a *generalized Hsia kernel*, and the corresponding function $[x, y]_z = q_v^{-\kappa_z(x,y)}$ a *canonical distance* on X_{Berk} relative to z; c.f. (4.25).

The restriction of the canonical distance $[x, y]_z$ to $x, y, z \in X(K)$ agrees with the canonical distance constructed by Rumely in [Rum89] (which is also defined only up to a constant).

As in Proposition 1.15(5), the canonical distance can be used to factorize absolute values of meromorphic functions on X: if f is a nonzero meromorphic function with divisor $\mathrm{Div}(f) = \sum m_i(a_i)$, then for any $z \in X_{\mathrm{Berk}}$ there is a constant C (depending on z and f) such that

$$|f(x)| = C \cdot \prod [x, a_i]_z^{m_i}$$

for all $x \in X_{\mathrm{Berk}}$.

Example 5.16. Let E_{Berk} be the Berkovich analytic space associated to a Tate elliptic curve E/K, and let Σ be the skeleton of E_{Berk}, which is isometric to a circle of length $\ell > 0$. Let μ be the normalized Haar measure supported on $\Sigma \subset E_{\mathrm{Berk}}$. Let $\lambda : E(K)\backslash\{O\} \to \mathbb{R}$ be the Néron canonical local height relative to the origin, as defined in [Sil94, §VI.1]. Then λ extends in a canonical way to a function $\lambda : E_{\mathrm{Berk}} \to \mathbb{R} \cup \{+\infty\}$ which is singular only at the origin O. Moreover, $\lambda \in \mathrm{BDV}(E_{\mathrm{Berk}})$ and

$$\Delta_{E_{\mathrm{Berk}}}(\lambda) = \delta_O - \mu, \tag{5.17}$$

so that λ can be considered as a kind of *Green's function* on E_{Berk}. Moreover, the normalization of λ given in [Sil94, §VI.1] ensures that

$$\int_{E_{\mathrm{Berk}}} \lambda \, d\mu = 0. \tag{5.18}$$

The canonical local height λ can in fact be *completely characterized* as the unique function satisfying (5.17) and (5.18).

5.4.4. *The Laplacian on a simple subdomain of* X_{Berk}. For domains in $\mathbb{P}^1_{\mathrm{Berk}}$, we defined a Laplacian operator $\Delta_{\bar{U}}$ on the closure $\bar{U}$ of U by viewing $\bar{U}$ as the inverse limit of all finite subgraphs of $\mathbb{P}^1_{\mathrm{Berk}}$ contained in U. If $U \subseteq X_{\mathrm{Berk}}$ is an arbitrary domain, it is no longer true in general that $\bar{U}$ is the inverse limit of all finite subgraphs of X_{Berk} contained in U. For one thing, such graphs need not belong to $\mathcal{S}$, since they might not contain the skeleton $\Sigma_{\mathcal{U}}$. Another problem is illustrated by the following: if X is a Tate elliptic curve, $r : X_{\mathrm{Berk}} \to \Sigma$ is the corresponding retraction, and $V = X_{\mathrm{Berk}} \backslash \{p\}$ for some point $p \in \Sigma$, then $U = r^{-1}(V)$ has closure equal to $U \cup \{p\}$, which is not the same as the inverse limit of all finite subgraphs of X_{Berk} contained in U.

There are various possible ways to remedy the situation. The most natural is probably the development of potential theory on Berkovich curves as in [Thu05], in terms of formal and rigid geometry.

Here is another possible approach:

Definition. A *simple compact set* in X_{Berk} is a subset of the form $r_{\Gamma}^{-1}(W)$ with $\Gamma \in \mathcal{S}$ and W a connected union of closed intervals in Γ.

If V is a simple compact set in X_{Berk}, we define

$$\mathcal{S}_V = \{\Gamma \cap V \ : \ \Gamma \in \mathcal{S} \text{ and } \Gamma \cap V \neq \emptyset\}.$$

It is not hard to show that $\mathcal{S}_V$ is an arboreal system of metrizd graphs, and that $(V, \mathcal{S}_V)$ is an arboretum. In particular, V is homeomorphic to $\varprojlim_{\Gamma' \in \mathcal{S}_V} \Gamma'$, and there is a corresponding Laplacian operator Δ_V.

References

[Ber90] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Ber93] V. G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, 78:5–161, 1993.

[Ber99] V. G. Berkovich. Smooth p-adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.

[Ber04] V. G. Berkovich. Smooth p-adic analytic spaces are locally contractible. II. In *Geometric aspects of Dwork theory*, pages 293–370. Walter de Gruyter and Co. KG, Berlin, 2004.

[BF06] M. Baker and X. Faber. Metrized graphs, Laplacian operators, and electrical networks. *Contemp. Math.*, 415, 2006.

[BL85] S. Bosch and W. Lütkebohmert. Stable reduction and uniformization of abelian varieties. I. *Math. Ann.*, 270(3):349–379, 1985.

[BR] M. Baker and R. Rumely. Harmonic analysis on metrized graphs. *Canad. J. Math.* to appear.

[BR06] M. Baker and R. Rumely. Potential theory on the Berkovich projective line. in preparation, 270 pages, 2006.

[CL06] A. Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006.

[CR93] T. Chinburg and R. Rumely. The capacity pairing. *J. Reine angew. Math.*, 434:1–44, 1993.

[Duc06] A. Ducros. Espaces analytiques p-adiques au sense de Berkovich. preprint. Available at `http://math.unice.fr/ ducros/asterisque.pdf`, 38 pages, 2006.

[FRL04a] C. Favre and J. Rivera-Letelier. Équidistribution des points de petite hauteur. preprint. Available at `arxiv:abs/math.NT/0407471`, 34 pages, 2004.

[FRL04b] C. Favre and J. Rivera-Letelier. Théorème d'équidistribution de Brolin en dynamique p-adique. *C. R. Math. Acad. Sci. Paris*, 339(4):271–276, 2004.

[FvdP04] J. Fresnel and M. van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.

[GvdP80] L. Gerritzen and M. van der Put. *Schottky groups and Mumford curves*, volume 817 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Kan89] E. Kani. Potential theory on curves. In *Théorie des nombres (Quebec, PQ, 1987)*, pages 475–543. de Gruyter, Berlin, 1989.

[Liu02] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[RL03a] J. Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. *Astérisque*, (287):xv, 147–230, 2003. Geometric methods in dynamics. II.

[RL03b] J. Rivera-Letelier. Espace hyperbolique p-adique et dynamique des fonctions rationnelles. *Compositio Math.*, 138(2):199–231, 2003.

[Rob00] A. M. Robert. *A course in p-adic analysis*, volume 198 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Rum89] R. Rumely. *Capacity theory on algebraic curves*, volume 1378 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[Sil94] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[Thu05] A. Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov.* PhD thesis, University of Rennes, 2005. preprint. Available at `http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html`.

[Zha93] S. Zhang. Admissible pairing on a curve. *Invent. Math.*, 112(1):171–193, 1993.

E-mail address: `mbaker@math.gatech.edu`

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA