# NON-ARCHIMEDEAN DYNAMICS IN DIMENSION ONE: PROJECT DESCRIPTIONS

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**1. Repelling periodic points:** This project is related to the following open question: Is the set of repelling periodic points of $\phi$ in $\mathbb{P}^1(\mathbb{C}_v)$ dense in the $\mathbb{P}^1(\mathbb{C}_v)$-Julia set of $\phi$? (The answer to the question is "yes" if we change the two appearances of $\mathbb{P}^1(\mathbb{C}_v)$ to $\mathbb{P}^1_{\mathrm{Ber}}$.) If so, we say that repelling density holds for $\phi$.

(a) Prove that the closure of the set of *all* periodic points in $\mathbb{P}^1(\mathbb{C}_v)$ contains the $\mathbb{P}^1(\mathbb{C}_v)$-Julia set. (This is the main result in Hsia's paper "Closure of periodic points over a nonarchimedean field." *J. London Math. Soc.* (2) **62** (2000), 685–700.)

(b) For any polynomial $\phi$, if $\mathbb{C}_v$ has residue characteristic zero, or if $\deg \phi$ is smaller than the residue characteristic, prove that repelling density holds for $\phi$. What about rational functions?

(c) Suppose $\mathbb{C}_v$ has positive residue characteristic $p$. For any $a \in \mathbb{C}_v$ with $0 < |a| < 1$, let $\phi_a(z) = (a^p - a)z^{p+1} + z^p$, which has a repelling fixed point at $z = 1/a$. Show that there is an infinite sequence of attracting periodic points (not all in the same orbit, of course) accumulating at $z = 1/a$.

(d) Is there also a sequence of indifferent periodic points accumulating at $z = 1/a$ in the previous example?

(e) If $\phi(z) \in \mathbb{C}_v(z)$ is a rational function with at least one repelling periodic point in $\mathbb{P}^1(\mathbb{C}_v)$, prove that repelling density holds for $\phi$. (This is the main result in Bézivin's paper "Sur les points périodiques des applications rationnelles en analyse ultramétrique," *Acta Arith.* **100** (2001), 63–74.)

(f) Suppose that $\mathbb{C}_v$ has characteristic zero but residue characteristic $p > 0$. For any $a \in \mathbb{C}_v$, define $\phi_a(z) = z^{p+1}(z-a)^{2p}$. For any $n \geq 0$, prove that there is a choice of $a \in \mathbb{C}_v$ with $|a| > 1$ so that $\phi_a$ has no repelling periodic points of period $n$ or smaller, but $\phi_a$ *does* have a repelling periodic point of period $n+1$.

(g) If $\operatorname{char} \mathbb{C}_v = p > 0$, is there some fixed degree $d \geq 2$ such that for any (or at least, for infinitely many) $n \geq 0$, there is a degree-$d$ polynomial (or rational function) over $\mathbb{C}_v$ with no repelling periodic points up to period $n$ but a repelling periodic point of period $n+1$?

(h) If $\mathbb{C}_v$ has residue characteristic zero, and $\phi$ is a polynomial or rational function over $\mathbb{C}_v$ with at least one repelling periodic point, what restrictions (if any) are there on the length of the shortest repelling periodic cycle?

(i) Relate the existence of a map with nonempty $\mathbb{P}^1(\mathbb{C}_v)$-Julia set but no repelling periodic points in $\mathbb{P}^1(\mathbb{C}_v)$ to the question of Misiurewicz bifurcations near active critical points in families $\Phi_a(z)$ of dynamical systems. (See Project 4). Can such a relationship shed any light on the existence or nonexistence of such maps?

**2. Lattès maps and related functions:** A Lattès map is a morphism of $\mathbb{P}^1$ induced by an endomorphism of an elliptic curve. They exhibit relatively simple dynamics, compared with what else is out there, but they are good examples to know. This project mostly concerns Lattès maps but also includes a few more complicated maps.

(a) A map $\phi \in \mathbb{C}_v(z)$ is a *Lattès* map if there is an elliptic curve $E/\mathbb{C}_v$, a finite morphism $\psi : E \to E$, and a finite separable morphism $h : E \to \mathbb{P}^1$ such that $h \circ \psi = \phi \circ h$. Show that if $E$ has Weierstrass equation $y^2 = x^3 + ax + b$, then $\phi(z) = (z^4 - 2az^2 - 8bz + a^2)/(4z^3 + 4az + 4b)$ is a Lattès map, by choosing $\psi$ to be the multiplication-by-2 map $[2] : E \to E$, and $h$ to be the $x$-coordinate map $h(x, y) = x$.

(b) For an arbitrary elliptic curve $E$ in Weierstrass form, and with $h = x$ and $\psi = [2]$ as in the previous part, find the associated Lattès map $\phi$.

(c) If $E$ has potentially good reduction, show that any associated Lattès map $\phi$ has potentially good reduction.

(d) If $E$ does not have potentially good reduction, and if $\psi : E \to E$ is not an automorphism, prove that the Julia set of the Lattès map $\phi$ is a line segment consisting of points of types II and III.

(e) In the previous part, if $\psi = [m] : E \to E$, prove that $\phi$ maps the line segment $\mathcal{J}_{\mathrm{Ber}}$ that is the Julia set onto itself by folding into $m$ segments of equal length (in the big model metric), with each point mapping with multiplicity $m$.

(f) Suppose $\mathbb{C}_v$ has a locally compact subfield $K$, and let $\pi$ be a uniformizer for $K$. For each integer $n \geq 1$, let $E_n$ be the elliptic curve $y^2 + xy = x^3 + \pi^n$. (Note: $E_n$ has Néron model reduction type $\mathrm{I}_n$, if that means anything to you.) Let $\phi_n$ be the Lattès map associated to $x$ and $[2] : E_n \to E_n$. Define $M_n$ to be the number of periodic cycles of Fatou components of $\phi_n$ containing $K$-rational points. Find a (probably messy) formula for $M_n$, and prove that $\lim_{n\to\infty} M_n = \infty$. (Thus, there is no non-archimedean upper bound on periodic Fatou cycles, even over locally compact subfields.)

(g) Inspired by the previous part, find a (non-Lattès) family of morphisms $\phi_n$ defined over $K$ so that: the degree of $\phi_n$ is bounded independently of $n$, the number of periodic cycles of $K$-rational Fatou components is unbounded, but now $\phi_n$ has nonempty type I Julia set— that is, $\mathcal{J}(\phi_n) \cap \mathbb{P}^1(\mathbb{C}_v) \neq \varnothing$.

(h) In "Théorie ergodique des fractions rationnelles sur un corps ultramétrique" (preprint available on arxiv), Favre and Rivera-Letelier consider functions of the following type. Pick $k \geq 2$, integers $d_1, \ldots, d_k > 1$, and $a_2, \ldots, a_k \in \mathbb{C}_v$ with $|a_2| > \cdots > |a_k| > 0$, and define $\phi(z) = z^{d_1} \prod_{j=1}^{k} \Big(1 + (a_2 z)^{d_{j-1}+d_j}\Big)^{(-1)^{j-1}}$. Denoting the line segment in $\mathbb{P}^1_{\mathrm{Ber}}$ connecting 0 to $\infty$ by $L$, give a precise description of how $\phi$ maps $L$ to itself. If $\sum_{j=1}^{k} d_j^{-1} \leq 1$, show that the Julia set is contained in a bounded subinterval of $L$, and describe this Julia set in some special cases.

(i) The centerpiece of Favre and Rivera-Letelier's construction (and of the Lattès map) was a closed interval $I$ inside $\mathbb{P}^1_{\mathrm{Ber}}$ whose preimage was a subset of itself. Try to construct a similar sort of map if we replace $I$ with a more complicated tree $S$ (for example, Y-shaped or X-shaped).

Note that it's too much to ask that $\phi^{-1}(S) \subseteq S$, because each successive preimage will have more branch points— so you'll need to come up with a weaker condition that is still enough to keep everything under control.

**3. The Classification Theorem:** In his thesis, Rivera-Letelier proved his classification theorem for Fatou components and described the structure of periodic components, but only over $\mathbb{C}_p$. This project concerns the same results over arbitrary fields $\mathbb{C}_v$.

Given $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$, define the *indifference domain* $\mathcal{I}$ of $\phi$ to be the set of all points $P \in \mathbb{P}^1_{\mathrm{Ber}}$ for which there is a neighborhood $U \ni P$ and a positive integer $n \geq 1$ such that $\phi^n$ maps $U$ bijectively onto itself.

(a) Prove the first part of the classification theorem: every point in the (Berkovich) Fatou set either is in the basin of attraction of an attracting periodic point, or is in a wandering domain, or lands in $\mathcal{I}$ after a finite number of iterations.

(b) Prove the second part of the classification theorem: no component $W$ of the Fatou set can intersect the attracting basin of more than one periodic cycle, or any two of: an attracting basin, a wandering domain, and the indifference domain $\mathcal{I}$.

(c) Prove the structure theorem for attracting components: Let $W$ be a connected component of the Fatou set containing an attracting fixed point $Q$. Prove that $Q$ is the only attracting periodic point in $W$, that every point of $W$ approaches $Q$ under iteration, and that $W$ is either an open disk or a "Cantor type" domain.

(A domain of Cantor type is an increasing union of open affinoids $W_n \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ such that each component of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus W_n$ contains at least two boundary points of $W_{n+1}$.)

(d) Prove that any connected component $W$ of $\mathcal{I}$ is a connected open affinoid, and that if $n \geq 1$ is the smallest positive integer such that $\phi^n(W) \subseteq W$, then $\phi^n(W) = W$, and $\phi^n$ permutes the (finitely many) boundary points of $W$.

(e) Let $W$ be an indifferent component (i.e., a connected component of $\mathcal{I}$). If the residue field of $\mathbb{C}_v$ is algebraic over a finite field, prove that $W$ is *quasiperiodic*— i.e., for all $x \in W$, the sequence $\{\phi^n(x)\}_{n\geq 1}$ accumulates at $x$.

(f) If the residue field of $\mathbb{C}_v$ is *not* algebraic over a finite field, it is possible to have non-quasiperiodic indifferent components. Give examples of such components, in both the cases of residue characteristic zero and nonzero.

(g) Give examples of all the component types over arbitrary fields $\mathbb{C}_v$: both kinds of attracting components, and indifferent components that are affinoids with $n$ holes, for arbitrary $n$.

(h) If the residue characteristic is zero, prove that fixed attracting components must contain critical points. Over other fields, give examples of attracting components with no critical points.

(i) What can you say about the possible dynamics on a quasiperiodicity component and on a non-quasiperiodicity indifferent component?

**4. One-parameter families:** Explicit families of maps have been used to construct certain rational functions with pathological dynamics. This project considers such families.

Fix an integer $d \geq 2$ and a set $A \subseteq \mathbb{P}^1(\mathbb{C}_v)$ with an associated ring $\mathbb{C}_v[A]$ of (rigid) analytic functions. (For ease, consider $A$ to be a rational closed disk.) Let $\Phi_a(z) = \Phi(a, z) \in \mathbb{C}_v[A](z)$ be a rational function in $z$ with coefficients in $\mathbb{C}_v[A]$, and assume that for each $a \in A$, the specific rational function $\Phi_a(z) \in \mathbb{C}_v(z)$ always has the same degree $d \geq 2$.

(a) Fix a parameter $a_0$, and suppose that the map $\Phi_{a_0}$ has a periodic point $P_{a_0} \in \mathbb{P}^1(\mathbb{C}_v)$ of period $n \geq 1$ and multiplier $\lambda_{a_0}$. Show that for any $\varepsilon > 0$ and neighborhood $P_{a_0} \in V \subseteq \mathbb{P}^1(\mathbb{C}_v)$, there is a neighborhood $a_0 \in U \subseteq A$ such that for any $a \in U$, $\Phi_a$ has a periodic point $P_a \in V$ of period $n$ and multiplier $\lambda_a$ satisfying $|\lambda_a - \lambda_{a_0}| < \varepsilon$. If $\lambda_{a_0} \neq 1$, show that $P_a$ and $\lambda_a$ can be written as elements of $\mathbb{C}_v[A]$.

(Note, in particular, that the type of the periodic point (attracting, repelling, indifferent) does not change under small enough perturbations.)

(b) In part (a), suppose instead that $P \in \mathbb{P}^1_{\text{Ber}}$ is a point of type II, III, or IV, periodic of period $n$ and multiplier $\lambda$ (which must be a positive integer) for $\Phi_{a_0}$. Show that there is a neighborhood $a_0 \in U \subseteq A$ such that for any $a \in U$, the point $P$ is still a periodic point of $\Phi_a$ of period $n$ and muliplier $\lambda$.

(c) If $W \subseteq \mathbb{P}^1_{\text{Ber}}$ is a connected affinoid such that $\Phi_{a_0}(W) \subseteq W$, prove that there is a neighborhood $a_0 \in U \subseteq A$ such that $\Phi_a(W) \subseteq W$ for all $a \in U$.

(In particular, any indifferent or disk-shaped attracting periodic Fatou component is preserved under small perturbations. Cantor attracting components can move only near the extremeties; in fact, only at type I extremeties.)

(d) Assume $\mathbb{C}_v$ has residue characteristic $p > 0$, and $\Phi_a$ has both a repelling periodic point $P(a)$ and a periodic type II point $\nu$ with a repelling direction expanding the big model metric by a factor divisible by $p$. Prove that there is some $a \in A$ for which $\Phi_a$ has a wandering domain accumulating at both $P(a)$ and $\nu$. Assume some extra technical hypotheses if you need to. (This generalizes my paper, "Examples of wandering domains in p-adic polynomial dynamics" *C.R. Math. Acad. Sci. Paris* **335** (2002), 615–620.)

(e) A pair $(\Phi_a, c(a))$ of a family $\Phi_a$ as above and $c \in \mathbb{C}_v[A]$ such that $c(a)$ is a critical point of $\Phi_a$ for every $a \in A$ is said to have a *Misiurewicz bifurcation* at $a = a_0$ if:
  (i) $\Phi^{\ell}_{a_0}(c(a_0))$ is a repelling $m$-periodic point, for minimal integers $\ell, m \geq 1$.
  (ii) There is some $a \in A$ such that $\Phi^{\ell}_a(c(a))$ is *not* $m$-periodic under $\Phi_a$.
  (iii) For all $a \in A$, the multiplicity of $\Phi_a$ at $c(a)$ is the same integer $d \geq 2$.
  (iv) For every $n \geq 0$, there is some $y \in \Phi^{-n}_{a_0}(c(a_0))$ such that $\Phi^n_{a_0}$ is unramified at $y$.

Prove that if $(\Phi(a, x), c(a))$ has a Misiurewicz bifurcation at $a_0$, then for any neighborhood $a_0 \in V \subseteq A$, the same family also has a Misiurewicz bifurcation (with the same $m$ but different $\ell$) at some $a_1 \in V \setminus \{a_0\}$. (See Rivera-Letelier's, "Wild Recurrent Julia Critical Points," *J. London Math. Soc.* **72** (2005), 305–326.)

(f) The marked critical point $c(a)$ of the family $(\Phi_a, c(a))$ is *active* at $a = a_0$ if the family $\{\Phi^n_a(c(a))\}_{n \geq 0}$ (of analytic functions of the parameter $a$) fails to be equicontinuous on any neighborhood $a_0 \in U \subseteq A$. Prove that if the family has a Misiurewicz bifurcation at $a = a_0$, then it is active there. Next, consider a question of Rivera-Letelier: if the family is active at $a = a_0$, is $a_0$ accumulated by Misiurewicz bifurcations? (This question may be related to Project 1.)

(g) Another question of Rivera-Letelier: show that every Misiurewicz bifurcation is accumulated by maps with wandering domains accumulating at the repelling point.