AWS Project: Eisenstein cohomology classes— a comparison of constructions

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1 The Problem

In the previous "Project Preparation" documents, we defined three "Eisenstein" 1-cocycles on $\mathbf{SL}_2(\mathbf{Z})$, valued in certain related modules. Roughly speaking, these three cocycles are:

- One arising from integration of Eisenstein series; we call it the Siegel-Stevens cocycle.
- One arising from Shintani's simplicial cones construction; we call it the Solomon-Shintani cocycle.
- One arising from certain conditionally convergent sums; we call it the Sczech cocycle.

The purpose of this project is to prove a relationship between these cocycles—namely, that their images in the cohomology of a certain module where they can all be compared, are equal. Such a theorem is not present in the literature and would constitute an interesting new result. (Solomon raises this question explicitly in the introduction of [16].) Through this study, we hope to gain better insight into Eisenstein cohomology, with an eye towards applying the various methods above in higher rank.

Another feature to note is that all three cocycles can be "smoothed" using an auxiliary prime ℓ. Computations with the smoothed cocycles will be easier that with the original cocycles. In addition, the smoothing process is essential for studying the finer p-adic properties of the Eisenstein cocycles.

We begin by recalling the three Eisenstein cocycles.

2 Modular Symbols method

Define $\mathcal{V} := \mathbf{Q}^2/\mathbf{Z}^2 - \{(0,0)\}$, viewed as column vectors. The set $\mathcal{V}$ is endowed with a natural action of $\Gamma := \mathbf{SL}_2(\mathbf{Z})$. Let

$$\mathcal{P} := \mathbf{C}[x_1, x_2] = \bigoplus_{k=0}^{\infty} \mathcal{P}_k,$$

where $\mathcal{P}_k$ denotes the subspace of homogeneous polynomials of degree k. The space $\mathcal{P}$ is endowed with the Γ-action $(\gamma P)(x,y) = P((x,y)\gamma)$. Let $\mathcal{P}^\vee := \mathrm{Hom}_{\mathbf{Q}}(P, \mathbf{C})$. A $\mathcal{P}^\vee$-*valued distribution* is a function $f : \mathcal{V} \longrightarrow \mathcal{P}^\vee$ such that

$$\sum_{w \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2} f\left(\frac{v}{N} + w\right)(P) = N^{-k} \cdot f(v)(P) \tag{1}$$

for $P \in \mathcal{P}_k$ and any positive integer N. We write $f(P, v)$ for $f(v)(P)$. The $\mathbf{C}$-vector space of $\mathcal{P}^\vee$-valued distributions $\mathcal{M}$ is endowed with a Γ-action via

$$\gamma f(P, v) := f(\gamma^t P, \gamma^{-1} v).$$

Fix a $\tau \in \mathcal{H}$, the complex upper half-plane. Define an inhomogeneous 1-cocycle $\Psi_{E,\tau} \in Z^1(\Gamma, \mathcal{M})$ by

$$\Psi_{E,\tau}(\gamma)(P, v) = \frac{(k+1)!}{(2\pi i)^{k+2}} \int_\tau^{\gamma\tau} P(z, 1) E_{k+2,v}(z) dz$$

for $P \in \mathcal{P}_k$. Here the Eisenstein series $E_{k,v}$ is defined by

$$E_{k,v}(z) := \sum_{m,n}{}' \frac{e(mv_1 + nv_2)}{(mz+n)^k}, \qquad e(x) := e^{2\pi i x}. \tag{2}$$

where we discussed convergence issues for $k = 2$ in [7].

- ⋆ Prove that $\Psi_{E,\tau}$ is indeed a 1-cocycle, and that its class in $H^1(\Gamma, \mathcal{M})$ is independent of the choice of τ.

For all of our cocycles, we will be able to smooth them by choosing a fixed auxiliary prime ℓ, and considering the subgroup $\Gamma_0(\ell) \subset \Gamma$. Let

$$\pi_\ell = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Let $\gamma \in \Gamma_0(\ell)$, and let $v \in \mathcal{V}_\ell := \mathbf{Q}^2 - (\frac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z})$ modulo $\mathbf{Z}^2$. Let $\mathcal{M}_\ell$ be defined as $\mathcal{M}$, with $\mathcal{V}$ replaced by $\mathcal{V}_\ell$. (Check that $\mathcal{V}_\ell$, and hence $\mathcal{M}_\ell$ are $\Gamma_0(\ell)$-modules.) For $P \in \mathcal{P}_k$ define

$$\Psi_{E,\ell,\tau}(\gamma)(P, v) = \ell^k \left(\Psi_{\pi_\ell \tau}(\pi_\ell \gamma \pi_\ell^{-1})(\pi_\ell^{-1} P, \pi_\ell v) - \ell \Psi_\tau(\gamma)(P, v) \right). \tag{4}$$

- ⋆ Prove that $\Psi_{E,\ell,\tau} \in Z^1(\Gamma_0(\ell), \mathcal{M}_\ell)$ from the fact that $\Psi_{E,\tau}$ is a 1-cocycle on Γ together with (4) and the observation that conjugation by π_ℓ sends $\Gamma_0(\ell)$ into Γ.

- ⋆ For $v = (v_1, v_2) \in \mathcal{V}_\ell$ define

$$E^{(\ell)}_{k,v}(z) := \ell^{k-2} \cdot (E_{k,(\ell v_1, v_2)}(\ell z) - E_{k,v}(z)).$$

  Prove that for $P \in \mathcal{P}_k$, we have

$$\Psi_{E,\ell,\tau}(\gamma)(P, v) = \frac{(k+1)!}{(2\pi i)^{k+2}} \int_\tau^{\gamma\tau} P(z, 1) E^{(\ell)}_{k+2,v}(z) dz \in \mathbf{C}.$$

- $\star$ Using the q-expansion of $E_{k,v}$ from [7], prove that $E_{k,v}^{(\ell)}$ has constant term 0 at each cusp in the $\Gamma_0(\ell)$-orbit of ∞. Conclude that it makes sense to take $\tau = \infty$ as a representative for the cohomology class $[\Psi_{E,\ell,\tau}] \in H^1(\Gamma_0(\ell), \mathcal{M}_\ell)$. Given an explicit formula for $\Psi_{E,\ell,\infty}(P, v)$ in terms of Bernoulli polynomials. Note: this is not as routine as the previous exercises! Do the case $P = 1$ first. You may want to check out Section 4.4 of [3] where a similar computation is carried out.

3 The Sczech cocycle

Sczech's cocycle is described in detail in [7]. We will denote Sczech's (homogenous) 1-cocycle by $\Psi_Z \in Z^1(\Gamma, \mathcal{M})$, and the ℓ-smoothed version by $\Psi_{Z,\ell} \in Z^1(\Gamma_0(\ell), \mathcal{M}_\ell)$.

- $\star$ Read carefully through the definition of Sczech's cocycle in Section 2 of [7], at least through the proof of Theorem 2.4. Understand Theorems 2.3 and 2.4—at least the statements, if not the proofs—which give explicit formulas for the cocycles Ψ_Z and $\Psi_{Z,\ell}$.

4 The Solomon–Shintani cocycle

Solomon's algebraic *Shintani sum* $Z(L, x, V_1, V_2)$ is defined in [8]. Define $\mathcal{Q}$ to be the set of 2×2 matrices $Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that α and γ are linearly independent over $\mathbf{Q}$, as are β and δ. Endow $\mathcal{Q}$ with the natural left Γ-action. Define $\mathcal{M}_S$ to be the space of maps $f : \mathcal{Q} \times \mathbf{Z}_{\geq 0} \times \mathcal{V} \longrightarrow$ that satisfy the distribution property

$$\sum_{w \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2} f\left(Q, m, \frac{v}{N} + w\right) = N^{-2m} \cdot f(Q, m, v). \tag{5}$$

The $\mathbf{C}$-vector space $\mathcal{M}_S$ has a Γ-module structure given by $(\gamma f)(Q, m, v) := g(\gamma^t Q, m, \gamma^{-1} v)$.

- $\star$ For $A_1, A_2 \in \Gamma$, let $\rho_i = A_i \binom{1}{0}$ denote the first column of A_i, and let ρ be the matrix with columns (ρ_1, ρ_2). Define the function

  $$\Psi_S(A_1, A_2)(Q, m, v) := \Delta^{(m)} Z(\mathbf{Z}^2, v, \rho_1, \rho_2)(zQ^t) \in \mathbf{C}.$$

  Show that $\Psi_S(A_1, A_2) \in \mathcal{M}_S$.

- $\star$ Show that $\Psi_S \in Z^1(\Gamma, \mathcal{M}_S)$.

- $\star$ Give an explicit formula for Ψ_S in terms of Bernoulli polynomials.

- $\star$ Define a smoothed cocycle $\Psi_{S,\ell} \in Z^1(\Gamma_0(\ell), \mathcal{M}_{S,\ell})$ from Ψ_S as in (4). Here $\mathcal{M}_{S,\ell}$ is defined as $\mathcal{M}_S$ except with $\mathcal{V}_\ell$ replacing $\mathcal{V}$. Give an explicit formula for $\Psi_{S,\ell}$ in terms of Bernoulli polynomials.

5 Comparing the cocycles

Given $Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{Q}$, define the polynomial

$$P_Q(x, y) = \mathrm{N}((x, y)Q) = (\alpha x + \gamma y)(\beta x + \delta y).$$

There is a natural map $\iota : \mathcal{M} \longrightarrow \mathcal{M}_S$ given by

$$(\iota f)(Q, m, v) = f(P_Q^m, v).$$

- $\star$ Prove that ι is a Γ-module map.

The map ι induces a natural map on cohomology $\iota_* : H^1(\Gamma, \mathcal{M}) \longrightarrow H^1(\Gamma, \mathcal{M}_S)$. The goal of this project is:

- $\star$ Prove:

$$\iota_*[\Psi_{E,\tau}] = \iota_*[\Psi_Z] = [\Psi_S] \text{ in } H^1(\Gamma, \mathcal{M}_S). \tag{6}$$

Before doing this problem, it might be easier to prove the ℓ-smoothed version:

- $\star$ Prove:

$$\iota_*[\Psi_{E,\ell,\tau}] = \iota_*[\Psi_{Z,\ell}] = [\Psi_{S,\ell}] \text{ in } H^1(\Gamma, \mathcal{M}_{S,\ell}). \tag{7}$$

We have not done these problems! There are two reasons why we think the ℓ-smoothed version is easier—(1) As we saw in the first section, it is possible to give an explicit formula for $\Psi_{E,\ell,\tau}$ in terms of Bernoulli polynomials by taking $\tau = \infty$. It should also be possible to give a rational cocycle representing $\Psi_{E,\tau}$, but this is not so clear. (2) Also, we believe that there is actually an equality of *cocycles*

$$\iota_*\Psi_{E,\ell,\tau} = \iota_*\Psi_{Z,\ell} = \Psi_{S,\ell},$$

which implies (7). On the other hand, this is definitely not the case for the unsmoothed cocycles, so you will have to demonstrate the differences of the cocycles as explicit coboundaries in order to prove (6).

- $\star$ Is is possible to formally deduce (6) from (7)? In other words, is the sum of the ℓ-smoothing maps

$$H^1(\Gamma, \mathcal{M}_S) \longrightarrow \bigoplus_\ell H^1(\Gamma, \mathcal{M}_{S,\ell})$$

  injective?

6 Further problems

If you've completed the proofs of (6) and (7), here are some extra problems:

- ⋆ We excluded $(0,0)$ from $\mathbf{Q}^2/\mathbf{Z}^2$ in the definition of $\mathcal{V}$, and $\frac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}$ from the definition of $\mathcal{V}_\ell$ in order to simplify matters. However, it is possible to make cocycles that work on all of $\mathbf{Q}^2/\mathbf{Z}^2$, but we need to change the modules $\mathcal{M}$ and $\mathcal{M}_S$ in order to do it. For example, this is explained for Sczech's cocycle in Section 4.5 of the notes. Can you prove an analogue of (6) and (7) with an appropriately defined module replacing $\mathcal{M}_S$?

- ⋆ Can you give a *conceptual* rather than computational proof of (6) and (7)? For example, here is an idea: Define Hecke operators on the space $H^1(\Gamma, \mathcal{M}_S)$. Prove that all the classes $[\Psi]$ defined here live in the "Eisenstein subspace," i.e. the subspace on which T_p acts by $1 + p^{m-1}$. Is it true that $H^1(\Gamma, \mathcal{M}_S)$ is 1-dimensional? If so, we can prove the desired equality of cohomology classes by evaluating the given cocycles on any cycle that yields a nonzero result; is there an easy such candidate?

- ⋆ Here's a big one—generalize these results to $\mathbf{SL}_n(\mathbf{Z})$ for $n \geq 2$. Solomon's cocycle has been generalized by Richard Hill [11], and the general case for Sczech's cocycle is described in the notes. What about the analogue of the method of integration of Eisenstein series? Can we prove (6) and (7) for these higher dimensional cocycles?

If you answer all of these questions during the five days in Arizona, then perhaps you should work on:

- ⋆ Prove the rank one abelian Stark conjecture.

- ⋆ Prove the refinements described in the notes.

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