p-adic cohomology: from theory to practice

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These are the notes from a lecture series at the 2007 Arizona Winter School; the first four top-level sections correspond roughly to the individual lectures, and the fifth gives some ideas for the student project associated to the lecture series.

The overall goal of the lectures is to survey some of the ways in which p-adic analytic methods are used to associate meaningful cohomology theories, and associated extra structures, to algebraic varieties over finite fields and local fields. A key theme is the explicit, computable nature of these constructions, which makes them suitable for numerical calculations. For instance, if you ask the computer algebra system `Magma` for the order of the Jacobian of a hyperelliptic curve over a field of small characteristic, this order is computed using p-adic cohomology. The same is true if you ask the system `SAGE` for the p-adic regulator of an elliptic curve over $\mathbb{Q}$, for p a good ordinary prime.

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1 Algebraic de Rham cohomology

In this section, we introduce the algebraic approach to constructing de Rham cohomology for smooth varieties, as originally introduced by Grothendieck [19] based on ideas of Atiyah and Hodge. Notice I said "varieties" rather than "algebraic varieties"; that's because the same approach works also for complex analytic varieties, and for rigid analytic varieties. In fact, we will take this point of view as foundational for our study of p-adic cohomology.

1.1 Affine varieties

Definition 1.1.1. Let $X = \operatorname{Spec} A$ be an affine variety over a field K of characteristic zero. Let $\Omega^1_{A/K}$ denote the module of Kähler differentials; that is, $\Omega^1_{A/K}$ is the A-module generated by symbols da for $a \in A$, modulo the relations da for $a \in K$, and $d(ab) - a\,db - b\,da$ for $a, b \in A$. This module is finitely generated over A; it has the universal property that for any K-linear derivation $D : A \to M$ into an A-module, there is a unique A-linear map $\psi : \Omega^1_{A/K} \to M$ such that $D = \psi \circ d$.

We assume hereafter that X/K is smooth, which forces $\Omega^1_{A/K}$ to be a locally free A-module. Put

$$\Omega^i_{A/K} = \wedge^i_A \Omega^1_{A/K};$$

that is, $\Omega^i_{A/K}$ is the free A-module generated by symbols $\omega_1 \wedge \cdots \wedge \omega_i$, modulo the relations

$$(a\omega_1 + a'\omega_1') \wedge \omega_2 \wedge \cdots \wedge \omega_i - a\omega_1 \wedge \omega_2 \cdots \wedge \omega_i - a'\omega_1' \wedge \omega_2 \cdots \wedge \omega_i$$

for $a, a' \in A$, and $\omega_1 \wedge \cdots \wedge \omega_i$ whenever two of the factors are equal. (That is, $\Omega^i_{A/K}$ is the i-th *alternating power*, or *wedge power*, of $\Omega_{A/K}$ over A. Beware that the elements of the form $\omega_1 \wedge \cdots \wedge \omega_i$, the so-called *decomposable elements*, are not all of $\Omega^i_{A/K}$; e.g., if $A = k[x_1, x_2, x_3, x_4]$ then the map $d : A \to \Omega^1_{A/K}$ sending a to da induces maps $d : \Omega^i_{A/K} \to \Omega^{i+1}_{A/K}$. Moreover, the composition $d \circ d$ is always zero.

We thus have a complex $\Omega^\cdot_{A/K}$, called the *de Rham complex* of A (or X). The cohomology of this complex is called the *(algebraic) de Rham cohomology* of A, or X, denoted $H^i_{\mathrm{dR}}(X)$. Note that this vanishes above the dimension of X, because the complex itself vanishes there.

Example 1.1.2. If $A = K[x_1, \dots, x_n]$ represents an affine space, then

$$H^0(X) = K, \qquad H^i(X) = 0 \quad (i > 0).$$

Terminology 1.1.3. An element of Ω^i is often called an *i-form*. An i-form is *closed* if it is in the kernel of d, and *exact* if it is in the image of d. In this terminology, $H^i_{\mathrm{dR}}(X)$ is the quotient of the space of closed i-forms by the subspace of exact i-forms.

1.2 Example: an elliptic curve

Example 1.2.1. Put

$$A = K[x, y]/(y^2 - x^3 - ax - b),$$

and assume that A is smooth, i.e., $P(x) = x^3 + ax + b$ has no repeated roots. That is, X is the affine part of an elliptic curve.

Then $H^0_{\mathrm{dR}}(X)$ is one-dimensional, generated by 1, and $H^i_{\mathrm{dR}}(X)$ vanishes for $i > 1$. To calculate $H^1_{\mathrm{dR}}(X)$, we use the relation

$$2y\, dy = (3x^2 + a)\, dx.$$

Since P has no repeated roots, we can choose polynomials $A, B \in K[x]$ such that $AP + BP' = 1$. Now put

$$\omega = 2B\, dy$$

so that

$$dx = y\omega, \qquad dy = \frac{P'}{2}\omega.$$

Consequently, we can write every 1-form on X uniquely as $(C + Dy)\,\omega$ for some $C, D \in K[x]$. For this form to be exact, it must be of the form $d(E + Fy)$ for some $E, F \in K[x]$. But

$$\begin{aligned} d(E + Fy) &= E'\, dx + F'y\, dx + F\, dy \\ &= \left(\frac{1}{2}P'F + E'y + F'P\right)\omega. \end{aligned}$$

Hence any $Dy\omega$ is exact. As for $C\omega$, if F has leading term cx^d, then $\frac{1}{2}P'F + F'P$ has leading term $\left(\frac{3}{2} + d\right) cx^{d+2}$. We can thus clear all terms except those of degree ≤ 1, and this implies that

$$\omega, x\omega$$

form a basis of $H^1_{\mathrm{dR}}(X)$.

Note that by writing

$$\omega = \frac{dx}{y} = \frac{2\,dy}{3x^2+a},$$

we can see that ω actually extends to a 1-form on the complete elliptic curve, whereas $x\omega$ has a double pole at infinity.

Exercise 1.2.2. Repeat this derivation for the affine part of a hyperelliptic curve $y^2 = P(x)$ with $\deg(P)$ odd. Then modify it for the case where $\deg(P)$ is even.

Exercise 1.2.3. Let $P(x) \in K[x]$ be a squarefree polynomial. Compute $H^i_{\mathrm{dR}}(X)$ for the punctured affine line $X = \operatorname{Spec} K[x,y]/(yP(x)-1)$.

1.3 Sheaf cohomology

Definition 1.3.1. Let X be a variety over a field K of characteristic zero, which is still smooth but not necessarily affine. Now $\Omega^1_{X/K}$ and the other $\Omega^i_{X/K}$ are not modules, but locally free coherent sheaves of $\mathcal{O}_X$-modules; however, they still form a complex using the exterior derivative maps d, and we would like to look at kernel mod image like we did before.

The right thing to do is to define $H^i_{\mathrm{dR}}(X)$ as the *hypercohomology* $\mathbb{H}^i(\Omega^{\cdot}_{X/K})$ of the complex $\Omega^{\cdot}_{X/K}$. At a suitably abstract level, this is completely straightforward to define: construct a quasi-isomorphism of the given complex to a complex $\mathcal{I}^{\cdot}$ of acyclic sheaves and compute the cohomology of the complex $\Gamma(X, \mathcal{I}^{\cdot})$ of global sections. (We secretly just used the fact that the category of bounded-below complexes of quasicoherent sheaves of $\mathcal{O}_X$-modules has enough injectives, but this follows from the same statement for the category of quasicoherent sheaves of $\mathcal{O}_X$-modules.)

Definition 1.3.2. To actually compute hypercohomology explicitly, we need a particular acyclic resolution, which we produce using a Čech complex. First let's recall how to do this for a single sheaf, rather than a complex. Let $\{U_i\}_{i \in I}$ be a finite cover of X by open affine subschemes. For $j = 0, 1, \dots$, let I_j be the set of $(j+1)$-element subsets of I. For $J \in I_j$, let X_J be the intersection of the U_i for $i \in J$. For a quasicoherent sheaf $\mathcal{F}$ on X, the corresponding Čech complex $\check{C}^{\cdot}(X, \mathcal{F})$ has j-th term $\prod_{J \in I_j} \Gamma(X_J, \mathcal{F})$, and

$$d((s_J)_{J \in I_j}) = \left(\sum_{i=0}^{j+1} (-1)^i s_{J - \{j_i\}} \right)_{J \in I_{j+1}},$$

where the elements of $J \in I_{j+1}$ are labeled $j_0, \dots, j_{j+1}$ in increasing order. Its cohomology computes the sheaf cohomology $H^i(X, \mathcal{F})$ [22, Theorem III.4.5], [20, Proposition 1.4.1]. (Note: what I really did was take the complex of sheaves, where the j-th term is the sum of the sheaves $(i_J)_* i_J^* \mathcal{F}$ for each $J \in I_j$, and $i_J : X_J \hookrightarrow X$ is the open immersion; this is an acyclic resolution of $\mathcal{F}$.)

Now if you do this with each term of a complex of sheaves $\mathcal{F}^{\cdot}$, what you end up with is a *double complex*:

$$\begin{tikzcd}
\check{C}^0(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \check{C}^1(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \check{C}^2(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \cdots \\
\check{C}^0(X, \Omega^1_{X/K}) \arrow[r] \arrow[d] & \check{C}^1(X, \Omega^1_{X/K}) \arrow[r] \arrow[d] & \check{C}^2(X, \Omega^1_{X/K}) \arrow[r] \arrow[d] & \cdots \\
\check{C}^0(X, \Omega^2_{X/K}) \arrow[r] \arrow[d] & \check{C}^1(X, \Omega^2_{X/K}) \arrow[r] \arrow[d] & \check{C}^2(X, \Omega^2_{X/K}) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Let d_X and $\check{d}$ denote the vertical and horizontal maps, respectively. To take cohomology here, you take cohomology of the total complex in which

$$C^i = \oplus_{j+k=i} \check{C}^j(X, \Omega^k_{X/K}),$$

with the differentials

$$d((\omega_{j,k})_{j+k=i}) = (d_X(\omega_{j,k-1}) + (-1)^j \check{d}(\omega_{j-1,k}))_{j+k=i+1}.$$

(Notice you have to twiddle the sign of the maps in one direction, in order to get $d \circ d = 0$.)

Example 1.3.3. Let X be the complete elliptic curve defined by

$$y^2 = x^3 + ax + b,$$

i.e., the Zariski closure in $\mathbb{P}^2$ of the affine curve. Let U be the affine curve, i.e., X minus the point at infinity $[0:1:0]$. Let V be X minus the three points of the form $(x, 0)$. The everywhere holomorphic differential $\omega = dx/y$ defines an element of $H^1_{\mathrm{dR}}(X)$, which might make you think this space is one-dimensional. But you would be incorrect!

Remember that elements of $H^1_{\mathrm{dR}}(X)$ are represented by triples (ω_U, ω_V, f), where ω_U is a 1-form on U, ω_V is a 1-form on V, and f is a 0-form on $U \cap V$. The differential takes this triple to the 1-form $\omega_V - \omega_U - df$ on $U \cap V$, so we want this to be zero. One way to get this is to take $\omega_U = \omega_U = \omega$ and $f = 0$, but there are others. In fact, $H^1_{\mathrm{dR}}(X)$ is two-dimensional, which matches what you get for topological de Rham cohomology. That's no accident; see Remark 1.4.4.

Exercise 1.3.4. Construct an element of $H^1_{\mathrm{dR}}(X)$ represented by a triple (ω_U, ω_V, f) with $\omega_U = x\omega$, and show that your element is linearly independent from ω in $H^1_{\mathrm{dR}}(X)$. (Hint: since $x\omega$ has a double pole at $[0:1:0]$, your f will have to have a single pole there.)

Exercise 1.3.5. (Tricky) Prove that $H^2_{\mathrm{dR}}(X) \neq 0$ by exhibiting a 1-form on $U \cap V$ which is not the differential of a triple (ω_U, ω_V, df). Hint: first define the *residue* of a 1-form at a point on a curve.

Exercise 1.3.6. Prove that the space $H^i_{\mathrm{dR}}(\mathbb{P}^n_K)$ is one-dimensional if $i = 0, 2, \ldots, 2n$ and zero otherwise. (This may be easier to do using a spectral sequence; see the appendix to this lecture.)

Remark 1.3.7. There is an excision exact sequence in algebraic de Rham cohomology. If X is a smooth K-variety, Z is a smooth subvariety of pure codimension d, and $U = X \setminus Z$, then

$$\cdots \to H^{i-2d}_{\mathrm{dR}}(Z, K) \to H^i_{\mathrm{dR}}(X, K) \to H^i_{\mathrm{dR}}(U, K) \to H^{i-2d+1}_{\mathrm{dR}}(Z, K) \to \cdots$$

Note the shift by the codimension of Z.

Exercise 1.3.8. Use the previous remark to show that, if C is a smooth projective curve over K and Z is a nonempty zero-dimensional subscheme of length d, then

$$\dim_K H^1_{\mathrm{dR}}(Z, K) = \dim_K H^1_{\mathrm{dR}}(C, K) + d - 1.$$

The quantity $\dim_K H^1_{\mathrm{dR}}(C, K)$ will turn out to be twice the genus of C.

1.4 Comparisons via GAGA

Serre's "GAGA principle" asserts that anything you want to compute about coherent sheaves on a smooth proper algebraic variety can be computed on a corresponding analytic variety. (GAGA is an acronym for "Géométrie algébrique et géométrie analytique", the title of Serre's paper [35] introducing this idea.)

Theorem 1.4.1 (Complex-analytic GAGA)**.** *Let X be a smooth proper variety over $\mathbb{C}$, and let X^{an} be the corresponding analytic variety.*

(a) Any coherent sheaf on X^{an} is the analytification of a coherent sheaf on X.

(b) For any coherent sheaves $\mathcal{E}, \mathcal{F}$ on X, any morphism $\mathcal{E}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$ is induced by a morphism $\mathcal{E} \to \mathcal{F}$.

(c) For any coherent sheaf $\mathcal{E}$ on X, with analytification $\mathcal{E}^{\mathrm{an}}$, the natural maps $H^i(X, \mathcal{E}) \to H^i(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$ are bijections.

Proof. For X projective, proceed directly to Serre's original paper [35]. For the general case (which reduces to the projective case using Chow's lemma), see [21, Exposé XII]. $\square$

Theorem 1.4.2 (Rigid-analytic GAGA)**.** *Let K be a complete nonarchimedean field. Let X be a smooth proper variety over K, and let X^{an} be the corresponding rigid analytic variety.*

(a) Any coherent sheaf on X^{an} is the analytification of a coherent sheaf on X.

(b) *For any coherent sheaves $\mathcal{E}, \mathcal{F}$ on X, any morphism $\mathcal{E}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$ is induced by a morphism $\mathcal{E} \to \mathcal{F}$.*

(c) *For any coherent sheaf $\mathcal{E}$ on X, with analytification $\mathcal{E}^{\mathrm{an}}$, the natural maps $H^i(X, \mathcal{E}) \to H^i(X^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$ are bijections.*

Proof. The proof in the complex analytic case carries over directly, given the following inputs. (Thanks to Ravi Vakil for suggesting this list.)

- The ring of analytic functions on a closed unit polydisc is noetherian.
- On an affinoid space (a closed analytic subspace of a closed unit polydisc), coherent sheaves are acyclic for finite coverings by affinoid subspaces, and are represented by finitely generated modules (i.e., the theorems of Kiehl and Tate).
- The analytic projective n-space is proper in the sense of Kiehl, so the H^1 of a coherent sheaf on that space is finite dimensional.
- The structure sheaf on analytic projective n-space is acyclic.

□

Remark 1.4.3. One important corollary of GAGA is that (in either the complex or rigid setting) any closed analytic subvariety of a smooth proper algebraic variety (e.g., projective space) is itself algebraic.

Remark 1.4.4. For our purposes, the key corollary of GAGA (due to Grothendieck) is that for X smooth over $\mathbb{C}$, the algebraic de Rham cohomology groups of X are naturally isomorphic to the holomorphic de Rham cohomology. (For X not proper, this follows from the proper case by excision; it also follows from Theorem 1.5.2 below.) For X smooth proper, by Dolbeaut's theorem, the holomorphic de Rham cohomology groups are isomorphic to topological de Rham cohomology with complex coefficients; by de Rham's theorem, those are isomorphic to Betti cohomology with complex coefficients. In particular, the dimensions of the $H^i_{\mathrm{dR}}(X)$ coincide with the topological Betti numbers.

1.5 Cohomology with logarithmic singularities

Definition 1.5.1. By a *smooth (proper) pair* over a base S, I will mean a pair (X, Z) in which X is a smooth (proper) scheme over S and Z is a relative (to S) strict normal crossings divisor. Over a field, this means each component of Z is smooth (no self-intersections allowed; that's the "strict" part), and the components of Z always meet transversely. In general, we can say that étale locally (over S) X should look like an affine space and Z should look like an intersection of coordinate hyperplanes.

Put $U = X \setminus Z$ and let $j : U \hookrightarrow X$ be the implied open immersion. The *sheaf of logarithmic differentials* on X, denoted $\Omega^1_{(X,Z)/K}$, is the subsheaf of $j_*\Omega^1_{U/K}$ generated by

$\Omega^1_{X/K}$ and by sections of the form df/f, where f is a regular function on some open subset V of X which only vanishes along components of Z.

Again, we write $\Omega^i_{(X,Z)/K}$ for the i-th exterior power of $\Omega^1_{(X,Z)/K}$ over $\mathcal{O}_X$.

Theorem 1.5.2. *The obvious map of complexes*

$$\Omega^{\cdot}_{(X,Z)/K} \to j_* \Omega^{\cdot}_{U/K}$$

is a quasi-isomorphism, i.e., it induces isomorphisms on cohomology sheaves. Hence we obtain an isomorphism $\mathbb{H}^i(\Omega^{\cdot}_{(X,Z)/K}) \cong H^i_{\mathrm{dR}}(U)$*.*

(The last "hence" is because there is a spectral sequence that goes from cohomology sheaves to hypercohomology; see appendix to this lecture.) For $Z = \emptyset$, this is the theorem of Grothendieck [19] that introduced algebraic de Rham cohomology to the world. The general case was proved by Deligne [11] using GAGA (see below); it is possible, and important for applications to p-adic cohomology, to give a completely algebraic proof, e.g., [1, Theorem 2.2.5].

Exercise 1.5.3. Prove Theorem 1.5.2 directly for the example in Exercise 1.2.3.

Remark 1.5.4. One might be tempted to deduce from Theorem 1.5.2 that for a smooth projective curve X and a point $x \in X(K)$, every class in $H^1_{\mathrm{dR}}(X)$ is represented by a 1-form on $X \setminus \{x\}$ with a logarithmic singularity at x. This is false; for instance, in Example 1.3.3, you need to allow either a double pole at one point (to pick up $x\omega$), or poles at two different points. (This is because the sum of the residues of the poles of a 1-form is always zero.)

1.6 Example: a smooth hypersurface

Example 1.6.1. Let X be a smooth hypersurface in $\mathbb{P}^n$ defined by the homogeneous polynomial $P(x_1, \dots, x_n)$. Then there are natural maps $H^i_{\mathrm{dR}}(\mathbb{P}^n_K) \to H^i_{\mathrm{dR}}(X)$ which by the Lefschetz hyperplane theorem [18, §1.2] are isomorphisms for $i < n-1$ and injective for $i = n-1$. (That's actually a fact about complex manifolds, but by GAGA it transfers to the algebraic setting.) Since the cohomology of projective space is simple (Exercise 1.3.6), the only interesting cohomology group of X is $H^{n-1}_{\mathrm{dR}}(X)$.

There is a short exact sequence

$$0 \to \Omega^{\cdot}_{\mathbb{P}^n/K} \to \Omega^{\cdot}_{(\mathbb{P}^n,X)/K} \xrightarrow{\mathrm{Res}} j_* \Omega^{\cdot+1}_{X/K} \to 0,$$

where $j : X \to \mathbb{P}^n$ is the implied closed immersion. The map Res is a residue map, which can be described as follows: locally on $\mathbb{P}^n$, a section of $\Omega^{\cdot}_{(\mathbb{P}^n,X)/K}$ can be written as $df/f \wedge \omega$, where f is a dehomogenized form of P; Res takes this section to the restriction of ω to Z. Taking cohomology, and using Deligne's theorem, gives a long exact sequence

$$\cdots \to H^i_{\mathrm{dR}}(\mathbb{P}^n) \to H^i_{\mathrm{dR}}(U) \to H^{i-1}_{\mathrm{dR}}(X) \to H^{i+1}_{\mathrm{dR}}(\mathbb{P}^n) \to \cdots,$$

where $U = \mathbb{P}^n \setminus X$.

Upshot: if n is even, then $H^n_{\mathrm{dR}}(U)$ is isomorphic to $H^{n-1}_{\mathrm{dR}}(X)$. If n is odd, then (using Poincaré duality) $H^n_{\mathrm{dR}}(U)$ is isomorphic to the quotient of $H^{n-1}_{\mathrm{dR}}(X)$ by the (one-dimensional) image of the map $H^{n-1}_{\mathrm{dR}}(\mathbb{P}^n) \to H^{n-1}_{\mathrm{dR}}(X)$, the so-called *primitive middle cohomology* of X.

The point is that U is affine, so you can compute its de Rham cohomology on global sections. For the recipe for doing this easily, see Griffiths [17, §4, 5]; see also §5.4.

1.7 Appendix: spectral sequences

The notion of a spectral sequence is a generalization of the long exact sequence

$$\cdots \to H^i(C_1^\cdot) \to H^i(C^\cdot) \to H^i(C_2^\cdot) \xrightarrow{\delta} H^{i+1}(C_1^\cdot) \to \cdots$$

associated to a short exact sequence of complexes

$$0 \to C_1^\cdot \to C^\cdot \to C_2^\cdot \to 0.$$

The relevance of spectral sequences is described by [18, §3.5] (where you should be looking for more details) better than I ever could: "[to] someone who works with cohomology, they are essential in the same way that the various integration techniques are essential to a student of calculus."

Definition 1.7.1. A *filtered complex* is a decreasing sequence of complexes

$$C^\cdot = F^0C^\cdot \supseteq F^1C^\cdot \supseteq \cdots \supseteq F^nC^\cdot \supseteq F^{n+1}C^\cdot = 0;$$

given a filtered complex, the *associated graded complex* is

$$\operatorname{Gr} C^\cdot = \oplus_{p\geq 0} \operatorname{Gr}^p C^\cdot, \qquad \operatorname{Gr}^p C^\cdot = \frac{F^pC^\cdot}{F^{p+1}C^\cdot}.$$

For instance, a short exact sequence as above gives you a filtration with $F^2C^\cdot = 0$, $\operatorname{Gr}^1 C^\cdot \cong F^1C^\cdot = C_1^\cdot$, and $\operatorname{Gr}^0 C^\cdot = C_2^\cdot$.

Example 1.7.2. For instance, if you start with a double complex $D^{p,q}$ and flatten it into a single complex $C^i = \oplus_{p+q=i} D^{p,q}$, we can filter this by taking

$$F^pC^i = \bigoplus_{p'+q=i,\, p'\geq p} D^{p',q}.$$

We can also flip p and q to get a second filtration; this will give two distinct spectral sequences computing the cohomology of $C^\cdot$.

As payback for being easy to use, spectral sequences sacrifice the computation of the entire cohomology of filtered complexes. Instead, they only computed the graded pieces of a certain filtration on the cohomology.

Definition 1.7.3. Let Z^q and B^q be the cocycles and coboundaries in a filtered complex C^q. The *filtered cohomology* is

$$F^p H^q(C^\cdot) = \frac{F^p Z^q}{F^p B^q} = \text{image}(H^q(F^p C^\cdot) \to H^q(C^\cdot));$$

note that the map $H^q(F^p C^\cdot) \to H^q(C^\cdot)$ need not be injective. The *associated graded cohomology* is

$$\operatorname{Gr} H^\cdot(C^\cdot) = \oplus_{p,q} \operatorname{Gr}^p H^q(C^\cdot), \qquad \operatorname{Gr}^p H^q(C^\cdot) = \frac{F^p H^q(C^\cdot)}{F^{p+1} H^q(C^\cdot)}.$$

Definition 1.7.4. A *spectral sequence* is a sequence $\{E_r, d_r\}_{r=r_0}^\infty$, where each E_r is a bigraded group

$$E_r = \oplus_{p,q \geq 0} E_r^{p,q}$$

and

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \qquad d_r^2 = 0$$

is a differential such that

$$H^{p,q}(E_r) = E_{r+1}^{p,q}.$$

If at some point $E_r = E_{r+1} = \cdots$, we call this stable value the *limit* of the spectral sequence, denoted E_∞. One also says that the sequence *degenerates at* E_r, and that the sequence *converges to* E_∞..

Pictures of spectral sequences speak louder than words:

$$\begin{tikzcd}
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} \\
E_0^{0,0} \arrow[u, "d_0"] & E_0^{1,0} \arrow[u, "d_0"] & E_0^{2,0} \arrow[u, "d_0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
E_1^{0,1} \arrow[r, "d_1"] & E_1^{1,1} \arrow[r, "d_1"] & E_1^{2,1} \\
E_1^{0,0} \arrow[r, "d_1"] & E_1^{1,0} \arrow[r, "d_1"] & E_1^{2,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2^{0,1} \arrow[drr, "d_2"] & E_2^{1,1} & E_2^{2,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0}
\end{tikzcd}$$

Theorem 1.7.5. *Let $F^p C^\cdot$ be a filtered complex. Then there is a spectral sequence $\{E_r\}_{r=0}^\infty$ with*

$$\begin{aligned}
E_0^{p,q} &= \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} \\
E_1^{p,q} &= H^{p+q}(\operatorname{Gr}^p C^\cdot) \\
E_\infty^{p,q} &= \operatorname{Gr}^p H^{p+q}(C^\cdot).
\end{aligned}$$

Proof. See [18, §3.5] or [5, §14]. □

Terminology 1.7.6. In the previous theorem, one writes

$$E_r \Rightarrow H^\cdot(C^\cdot)$$

and says that the spectral sequence *abuts to* $H^\cdot(C^\cdot)$.

Example 1.7.7. For instance, in the example of a short exact sequence, we have

$$E_1^{0q} = H^q(C_2^\cdot), \quad E_1^{1q} = H^{q+1}(C_1^\cdot),$$

$d_1 : H^q(C_2^\cdot) \to H^{q+1}(C_1^\cdot)$ is the connecting homomorphism δ, and $d_2 = d_3 = \cdots = 0$ because the arrows always have a zero at one or both endpoints. For a filtered complex with $F^{n+1} = 0$, we similarly have $E_\infty = E_{n+1}$.

Remark 1.7.8. A map between filtered complexes is a quasi-isomorphism if and only if the same is true at any single stage of the spectral sequence. For instance, in the example of a short exact sequence, this is an instance of the five lemma.

Example 1.7.9. Go back to the double complex used to define the hypercohomology of the de Rham complex. As noted in Example 1.7.2, we can turn it into a filtered complex in two different ways. One way, we get a spectral sequence where we first compute the sheaf cohomology of each $\Omega^\cdot_{X/K}$, i.e.,

$$E_1^{pq} = H^q(X, \Omega^p_{X/K});$$

this is the *Hodge-de Rham spectral sequence*, and the filtration it determines on $H^i_{\mathrm{dR}}(X)$ is called the *Hodge filtration*. It is a deep theorem that the Hodge-de Rham spectral sequence degenerates already at E_1; this was originally established using analytic techniques, but can also be proved algebraically [12].

The other way, we get a spectral sequence with

$$E_1^{pq} = \prod_{J \in U_p} H^q_{\mathrm{dR}}(X_J),$$

which doesn't quite reconstruct $H^i_{\mathrm{dR}}(X)$ from the $H^i_{\mathrm{dR}}(X_J)$ (the spectral sequence only abuts to $H^i_{\mathrm{dR}}(X)$, so all you get are the graded pieces of some filtration) but still tells you a lot.

2 Rigid cohomology

In this section, we define rigid (as in rigid analytic) cohomology for smooth varieties over fields of positive characteristic, and sketch the proof of the Lefschetz trace formula (due to Monsky). We then state the comparison theorem with algebraic de Rham cohomology for liftable smooth proper varieties, and explain how this can be used to compute zeta functions.

2.1 Monsky-Washnitzer's formal cohomology

Notation 2.1.1. Throughout this lecture, let k be a perfect field of characteristic $p > 0$; eventually we'll further insist that k be a finite field. Let W be the ring of Witt vectors of k; it is the unique complete discrete valuation ring with residue field k and maximal ideal (p). Let K be the fraction field of W; it will be the coefficient field for MW cohomology. When we set $k = \mathbb{F}_q$, we'll write $\mathbb{Z}_q, \mathbb{Q}_q$ instead of W, K; note that $\mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$.

We start out considering smooth affine varieties over k. The theory of "formal cohomology" in this setting was introduced by Monsky and Washnitzer [34, 32, 33], based on ideas introduced by Dwork in his proof of the rationality of zeta functions of varieties over finite fields [13]. (Historical note: Dwork's proof predated Grothendieck's proof using étale cohomology by about 10 years.) Nowadays, we usually refer to this construction as *Monsky-Washnitzer cohomology*, or just *MW cohomology*. (See [36] for a useful overview.)

Definition 2.1.2. Let $\overline{X} = \operatorname{Spec} \overline{A}$ be a smooth affine variety over k. One could compute algebraic de Rham cohomology of $\overline{X}$, but the result would be disastrous; already H^0 would be huge, consisting of all of $\overline{A}^p$. The idea of Monsky-Washnitzer is to lift the picture to W and take cohomology there, but to try to do so in a way that stays functorial in the characteristic p data.

To start with, lift $\overline{A}$ to a smooth affine scheme A over W; that is, A is smooth affine over W and $A \otimes_W k \cong \overline{A}$. The existence of such a lift is guaranteed by a theorem of Elkik [15], but it is hopelessly non-functorial; once you choose lifts A, B of $\overline{A}, \overline{B}$, you usually cannot find any lifts of a given map $\overline{A} \to \overline{B}$.

Definition 2.1.3. To fix this, replace A by its p-adic completion $\hat{A}$. Then a "henselian" argument of Grothendieck (exposed nicely by Arabia [2]) shows that if again you choose lifts A, B of two given smooth affines $\overline{A}, \overline{B}$, you can lift any map $\overline{A} \to \overline{B}$ to a map $\hat{A} \to \hat{B}$, though not canonically. For instance, if $\overline{A} = \overline{B} = k[\overline{x}]$ and $A = B = W[x]$, then $\hat{A}$ is the ring $W\langle x \rangle$ of power series whose coefficients tend p-adically to 0, and you can lift a map $\overline{x} \mapsto \overline{P}(\overline{x})$ by sending x to any lift of $\overline{P}(\overline{x})$.

What you now want to do is take algebraic de Rham cohomology of the generic fibre $\hat{A} \otimes_W K$, but this is not a good idea either; you still get very large spaces.

Example 2.1.4. If $\overline{A} = k[\overline{x}]$ and $\hat{A} = W\langle x \rangle$, the 1-form

$$\sum_{n=0}^{\infty} p^n x^{p^n - 1} \, dx$$

is exact but not closed, so the proposed H^1 of the affine line does not vanish.

The fix is to replace the completion $\hat{A}$ by a smaller *weak completion* $A^\dagger$ that is big enough to have the good lifting properties, but is small enough to give meaningful de Rham cohomology.

Definition 2.1.5. Let $\overline{x}_1, \ldots, \overline{x}_n$ be generators of $\overline{A}$ over k, i.e., choose a surjection $k[\overline{x}_1, \ldots, \overline{x}_n] \to \overline{A}$. Choose lifts $x_1, \ldots, x_n$ of $\overline{x}_1, \ldots, \overline{x}_n$ to A, so that there is a map $W[x_1, \ldots, x_n] \to A$ which may or may not be a surjection. In any case, the image is p-adically dense, so you get a surjection $W\langle x_1, \ldots, x_n \rangle \to \hat{A}$ by taking p-adic completions. Here

$$W\langle x_1, \ldots, x_n \rangle$$

is the set of formal power series $\sum_I c_I x^I$, with I running over n-tuples of nonnegative integers and $c_I \in W$ for all I, such that $c_I \to 0$ as $I \to \infty$. (That is, for any integer n, there are only finitely many indices I such that c_I is not divisible by p^n.)

We are going to replace $W\langle x_1, \ldots, x_n \rangle$ with a dense subring $W\langle x_1, \ldots, x_n \rangle^\dagger$: it will consist of series $\sum_I c_I x^I$ for which there exist constants a, b with $a > 0$, such that

$$v(c_I) \geq a(i_1 + \cdots + i_n) + b \qquad (I \in \mathbb{Z}^n_{\geq 0}).$$

(The constants a, b may vary from series to series; this will be crucial in a moment.) Let $A^\dagger$ denote the image of $W\langle x_1, \ldots, x_n \rangle^\dagger$ in $\hat{A}$; it is called the *weak completion* of A. It can be shown that it depends only on A and not on the choice of the generators $x_1, \ldots, x_n$. Also, one has an analogue of Grothendieck's lifting theorem: any map $\overline{A} \to \overline{B}$ lifts (nonuniquely) to a map $A^\dagger \to B^\dagger$.

We define a module of differentials $\Omega^1_{A^\dagger/W}$ generated by $dx_1, \ldots, dx_n$ modulo the relations you expect: if $f(x_1, \ldots, x_n)$ is a series which vanishes in A, then the formal total derivative

$$\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

vanishes in Ω^1. We "define" the Monsky-Washnitzer cohomology of X with coefficients in K, denoted $H^i_{\mathrm{MW}}(X, K)$, to be the cohomology of the resulting de Rham complex $\Omega^i_{A^\dagger/W} \otimes_W K$, i.e., *after tensoring with* K. (Without inverting p, there would be a lot of torsion whose geometric meaning is not very clear, at least not to me.)

To remove the quotation marks, one must check that given any two lifts $f_1^\dagger, f_2^\dagger : A^\dagger \to B^\dagger$ of the same map $\overline{f} : \overline{A} \to \overline{B}$, you get the same induced maps $H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{A}, K) \to H^i_{\mathrm{MW}}(\operatorname{Spec} \overline{B}, K)$. In fact something better is true: the two maps $\Omega^\cdot_{A^\dagger/W} \otimes K \to \Omega^\cdot_{B^\dagger/W} \otimes K$ are homotopic [34, Theorem 5.5]. (I don't know whether this is also true *before* tensoring with K.) This first shows you that two different lifts give you canonically isomorphic cohomology (because you can lift the identity to maps of rings both ways, and the compositions induce identity maps on cohomology), and second implies functoriality.

Remark 2.1.6. Series like this are sometimes said to be *overconvergent*, because each such series converges on a polydisc of the form $|x_i| \leq \eta$ for some η strictly bigger than 1. (Remember: in rigid geometry it makes sense to take η to be any fractional power of p, because you should allow yourself to take points over extension fields of K.) By contrast, elements of $W\langle x_1, \ldots, x_n \rangle$ are only guaranteed to converge on the polydisc $|x_i| \leq 1$.

Remark 2.1.7. It would be helpful for certain purposes to be able to construct this theory without having to worry about checking independence from choices. (This will become apparent when we pass to nonaffines.) A canonical construction of Monsky-Washnitzer cohomology, using a variant of the de Rham-Witt complex, was proposed by James Borger; we are currently working out this construction as an ongoing project with Chris Davis.

2.2 Example: an elliptic curve

Example 2.2.1. Let's see how this works for the affine part of the elliptic curve $\overline{X} = \operatorname{Spec} \overline{A}$ with

$$\overline{A} = k[\overline{x}, \overline{y}]/(\overline{y}^2 - \overline{x}^3 - \overline{a}\overline{x} - \overline{b})$$

(we must have $p \neq 2$ for this to be able to be smooth), using a lift of the form

$$A = W[x, y]/(y^2 - x^3 - ax - b).$$

Again, $H^i_{\mathrm{MW}}(\overline{X}, K)$ vanishes for $i \geq 2$, and $H^1_{\mathrm{MW}}(\overline{X}, K)$ consists of the closed 1-forms (generated by dx and dy) modulo the exact ones. Exactly the same calculation as in Example 2.2.1 shows that we can still use $\omega, x\omega$ as a basis; this will turn out to be a special case of the comparison theorem to de Rham cohomology (Theorem 2.5.2).

But this is not the end of the story, as we also want to lift morphisms, particularly the absolute Frobenius $\overline{F}$. It will almost never lift to A, but we know it will lift to $A^\dagger$. Unfortunately, writing down such a lift is a bit complicated.

Example 2.2.2. Following [25] (see also [16, §7]), we can make constructing the Frobenius lift much easier by working with a slightly smaller affine piece of the same curve, namely $\overline{X}_0 = \operatorname{Spec} \overline{A}_0$ with

$$\overline{A}_0 = k[\overline{x}, \overline{y}, \overline{z}]/(\overline{y}^2 - \overline{x}^3 - \overline{ax} - \overline{b}, \overline{yz} - 1)$$

(that is, $\overline{A}$ with $\overline{y}$ inverted), with corresponding lift

$$A_0 = W[x, y, z]/(y^2 - x^3 - ax - b, yz - 1).$$

Then we can define a lift $F^\dagger$ of $\overline{F}$ acting as the Witt vector Frobenius on W and acting on x, y, z as

$$\begin{aligned}
&x \mapsto x^p \\
&y \mapsto y^p(1 + (F(x^3 + ax + b) - (x^3 + ax + b)^p)z^{2p})^{1/2} \\
&z \mapsto z^p(1 + (F(x^3 + ax + b) - (x^3 + ax + b)^p)z^{2p})^{-1/2}.
\end{aligned}$$

The point is that in the expressions $(1 + *)^{\pm 1/2}$ I just wrote, the $*$ is divisible by p, so expanding the binomial series gives you something overconvergent. (Word to the wise: an underlying reason why this worked is that $\operatorname{Spec} \overline{A}_0$ is étale over $\mathbb{A}^1_k$, so I can extend the obvious Frobenius lift $x \mapsto x^p$ from $W\langle x \rangle^\dagger$ to $A_0^\dagger$.)

2.3 Nonaffine varieties

We are now going to define the generalization of MW cohomology to smooth but not necessarily affine varieties; this gives *rigid cohomology* as introduced by Berthelot [3]. (Actually, Berthelot also allows singular varieties, but I didn't deal with singularities even in the affine case for algebraic de Rham cohomology.) To avoid some notational unpleasantness, I'm going to only explain the case of a variety covered by two affines $\overline{U}, \overline{V}$ and leave the general case to the reader's imagination.

Definition 2.3.1. Let $\overline{X}$ be a smooth variety over k covered by two open affines $\overline{U}, \overline{V}$. Since smooth implies separated, $\overline{U} \cap \overline{V}$ is also affine.

Choose smooth lifts of $\overline{U}, \overline{V}, \overline{U} \cap \overline{V}$ and make corresponding de Rham complexes over K; out of laziness I'll call these $\Omega^\cdot_{\overline{U}}$, $\Omega^\cdot_{\overline{V}}$, $\Omega^\cdot_{\overline{U} \cap \overline{V}}$. From functoriality, I get quasi-isomorphisms $r_u : \Omega^\cdot_{\overline{U}} \to \Omega^\cdot_{\overline{U} \cap \overline{V}}$ and $r_{\overline{V}} : \Omega^\cdot_{\overline{V}} \to \Omega^\cdot_{\overline{U} \cap \overline{V}}$ (where r stands for "restriction").

We now imitate the Čech construction of hypercohomology: form the complex

$$C^i = \Omega^i_{\overline{U}} \oplus \Omega^i_{\overline{V}} \oplus \Omega^{i-1}_{\overline{U} \cap \overline{V}},$$

with differentials

$$d(\omega_{\overline{U}}, \omega_{\overline{V}}, \omega_{\overline{U} \cap \overline{V}}) = (d(\omega_{\overline{U}}), d(\omega_{\overline{V}}), d(\omega_{\overline{U} \cap \overline{V}}) + r_{\overline{V}}(\omega_{\overline{V}}) - r_{\overline{U}}(\omega_{\overline{U}}))$$

and call its cohomology $H^i_{\mathrm{rig}}(\overline{X})$.

Checking that this is independent of the choices implicit in making the $\Omega^\cdot$ terms and the quasi-isomorphisms is left as an exercise, which may make you appreciate Remark 2.1.7. (With more than two affines in the cover, it becomes essential to use the spectral sequence from Example 1.7.9 to argue, once you have constructed a map between two different versions of $H^i_{\mathrm{rig}}(\overline{X})$, that the map is forced to be an isomorphism.)

Exercise 2.3.2. Check that in the situation considered in Definition 2.3.1, the definition of $H^i_{\mathrm{rig}}(\overline{X})$ does not depend on choices. That is, if you make a second set of data $\tilde{\Omega}^\cdot_{\overline{U}}$, $\tilde{\Omega}^\cdot_{\overline{V}}$, $\tilde{\Omega}^\cdot_{\overline{U} \cap \overline{V}}$, $\tilde{r}_{\overline{U}}$, $\tilde{r}_{\overline{V}}$ and a corresponding group $\tilde{H}^i_{\mathrm{rig}}(\overline{X})$, you have a canonical isomorphism $H^i_{\mathrm{rig}}(\overline{X}) \cong \tilde{H}^i_{\mathrm{rig}}(\overline{X})$. (Hint: use functoriality of MW cohomology over and over again. It may be easier to show that the two complexes $C^i, \tilde{C}^i$ are quasi-isomorphic to a third complex than that they are quasi-isomorphic to each other.)

A somewhat difficult theorem, proved independently by Berthelot [4] and Mebkhout [31], yields the following. (This theorem has subsequently been strengthened to allow twisted coefficients [27], about which I'll only have a tiny bit to say in the next lecture.)

Theorem 2.3.3. *The spaces $H^i_{\mathrm{rig}}(\overline{X})$ are finite dimensional for any smooth k-variety $\overline{X}$.*

However, we will typically only be interested in cases where $\overline{X}$ lifts nicely to characteristic zero, in which case dealing with rigid cohomology will prove to be much easier thanks to the comparison theorem with de Rham cohomology (Theorem 2.5.2).

2.4 Excision and the trace formula

There is an excision formula in rigid cohomology parallel to the one in algebraic de Rham cohomology (Remark 1.3.7): if $\overline{X}$ is a smooth k-variety, $\overline{Z}$ is a smooth subvariety of pure codimension d, and $\overline{U} = \overline{X} \setminus \overline{Z}$, then

$$\cdots \to H^{i-2d}_{\mathrm{rig}}(\overline{Z}) \to H^i_{\mathrm{rig}}(\overline{X}) \to H^i_{\mathrm{rig}}(\overline{U}) \to H^{i-2d+1}_{\mathrm{rig}}(\overline{Z}) \to \cdots .$$

Again, note the shift by the codimension of Z.

One really nice use of excision is to prove the Lefschetz trace formula for Frobenius in rigid cohomology, due to Monsky.

Theorem 2.4.1. *Let $\overline{X}$ be a smooth variety of dimension n over $\mathbb{F}_q$, and let F denote the induced action of the q-power Frobenius of X on $H^i_{\mathrm{rig}}(\overline{X})$. Then for all positive integers n,*

$$\#\overline{X}(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{Trace}(q^n F^{-1}, H^i_{\mathrm{rig}}(\overline{X})).$$

Sketch of proof. It suffices to check for $n = 1$, otherwise we can just replace q by q^n. Using excision, we reduce to the case where $\overline{X} = \operatorname{Spec} \overline{A}$ is affine and has no $\mathbb{F}_q$-rational points. One then observes that for any $a, b \in A^\dagger$, the operator $\phi(a)bF^{-1}$ has the same trace as $bF^{-1}\phi(a) = abF^{-1}$, so $(\phi(a) - a)bF^{-1}$ has trace zero. Since $\overline{X}$ has no $\mathbb{F}_q$-rational points, the $\phi(a) - a$ generate the unit ideal, and the desired result follows. □

2.5 The liftable case: comparison to de Rham cohomology

Life is a bit easier if $\overline{X}$ lifts to a smooth scheme X over W, as then you can use affines in a cover of X to make the MW algebras. Better yet, we can do this in a logarithmic fashion.

Definition 2.5.1. Let (X, Z) be a smooth proper pair over W, and put $U = X \setminus Z$. Let $U_K = U \times_{\operatorname{Spec} W} \operatorname{Spec} K$ be the generic fibre of U, and let $\overline{U} = U \times_{\operatorname{Spec} W} \operatorname{Spec} k$ be the special fibre.

Theorem 2.5.2. *There are canonical isomorphisms $H^i_{\mathrm{dR}}(U_K, K) \to H^i_{\mathrm{rig}}(\overline{U})$.*

Proof. There is a natural map $H^i_{\mathrm{dR}}(U_K, K) \to H^i_{\mathrm{rig}}(\overline{U})$ by simply viewing algebraic forms as overconvergent forms. To show that this map is an isomorphism, we may invoke excision to reduce to the case $U = X$. But then it is clear that $H^i_{\mathrm{rig}}(\overline{X})$ is just the de Rham cohomology of the analytification of X, and by rigid GAGA this agrees with the algebraic de Rham cohomology. □

Remark 2.5.3. Similarly, if you have a map $\overline{X} \to \overline{Y}$ which lifts to a map $X \to Y$ between smooth lifts, you can use that lifted map to give the functorial morphism $H^i_{\mathrm{rig}}(\overline{Y}) \to H^i_{\mathrm{rig}}(\overline{X})$.

Remark 2.5.4. One consequence of Theorem 2.5.2 is that the de Rham cohomology of a smooth proper variety over K with good reduction admits a canonical Frobenius action, even though the Frobenius map does not itself typically lift. In fact, you also get a Frobenius action even when the reduction is not good; this will be touched upon in Lecture 4.

2.6 Computing zeta functions: an example

Let us put this all together to see how you can compute the zeta function of an elliptic curve using rigid cohomology, as in [25]. (See also Edixhoven's course notes [14].)

Example 2.6.1. One can use the Frobenius lift in Example 2.2.2 to compute the Frobenius action on the de Rham cohomology of an elliptic curve over $\mathbb{Q}_p$ with good reduction. The comparison theorem implies $H^i_{\mathrm{dR}}(\operatorname{Spec} A_0, \mathbb{Q}_p) \cong H^i_{\mathrm{rig}}(\overline{X}_0)$. This isomorphism is respected by the hyperelliptic involution $y \mapsto -y$, and the minus eigenspaces are the de Rham cohomology of the complete curve over $\mathbb{Q}_p$ on one hand, and the rigid cohomology of the complete curve over $\mathbb{F}_p$ on the other.

To compute the Frobenius action on these spaces, first apply Frobenius formally to

$$\frac{dx}{y}, \qquad \frac{x\,dx}{y},$$

where $F(dx) = x^{p-1}\,dx$. In practice you have to truncate modulo some power of p, so that you can write down the coefficients; as a side effect, you truncate the infinite series defining $F(y)$, so you end up with a finite expression

$$\sum_{i=0}^{N} \frac{R_i(x)}{y^{2i+1}}\,dx$$

which you want to rewrite in terms of the basis. You can do this by successively eliminating the terms with the worst poles at $y = 0$: rewrite the numerator as a multiple of $P(x) = x^3 + ax + b$ plus a multiple of its derivative, then use the relations $P(x) = y^2$ and

$$0 \equiv d\left(\frac{S(x)}{y^{2N-1}}\right) = \frac{S'(x)\,dx}{y^{2N-1}} - \frac{(2N-1)S(x)P'(x)\,dx}{2y^{2N+1}}$$

to clear the y^{-2N-1} term. Repeat as needed. (I deliberately ignored issues related to how much initial p-adic precision is needed to get an answer accurate to a given final precision; this is discussed further in [25].)

Remark 2.6.2. Here is an interesting arithmetic application of Example 2.6.1 due to Mazur, Stein, and Tate [30]. Following a suggestion of Katz, they give a formula for the p-adic canonical height of an elliptic curve over $\mathbb{Q}$ in terms of the Frobenius action on the de Rham cohomology over $\mathbb{Q}_p$. (This height is not the Néron local height; it is a global height with p-adic values, which computes the regulator term in Mazur-Tate-Teitelbaum's p-adic analogue of the Birch-Swinnerton-Dyer conjecture.) This computation was implemented in `SAGE` during the summer of 2006 by Jennifer Balakrishnan, Robert Bradshaw, David Harvey, and Liang Xiao.

3 Gauss-Manin connections

In this section, we introduce the notion of a Gauss-Manin connection for a smooth proper morphism (of algebraic, complex analytic, or rigid analytic varieties) and explain how in some cases, such a connection carries a "Frobenius action" that can be used to recover zeta functions.

3.1 Connections in geometry and algebra

Definition 3.1.1. Let V be a vector bundle over a complex manifold X. (This concept originated with real differential manifolds, but never mind that here.) A *connection* on V is a bundle map $\nabla : V \to V \otimes \Omega^1_X$ which satisfies the *Leibniz rule*: for any open set $U \subseteq X$, any $f \in \Gamma(U, \mathcal{O})$ and $s \in \Gamma(U, \mathcal{O})$,

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

A section s is called *horizontal* if $\nabla(s) = 0$. The idea is that the choice of ∇ tells you which direction to move between fibres of V if you want to move "horizontally", i.e., parallel to the base.

However, it turns out that this notion of horizontality is path-dependent in general. This path dependency is measured by a device called the *curvature*.

Definition 3.1.2. To construct it, let $\nabla_1 : V \otimes \Omega^1_X \to V \otimes \Omega^2_X$ be the map

$$s \otimes \omega \mapsto \nabla(s) \wedge \omega + s \otimes d\omega,$$

where $\wedge$ is being used to represent the map $(V \otimes \Omega^1_X) \otimes \Omega^1_X \to V \otimes \Omega^2_X$ given by wedging the second and third factors. The *curvature* is then given by $\nabla_1 \circ \nabla$; if it vanishes, we say ∇ is *integrable*. This means that the notion of horizontality is locally path-independent; that is, on any contractible neighborhood of a point $x \in X$, we can write down a basis of V consisting of horizontal sections $s_1, \dots, s_n$, and the connection is given in terms of this basis by

$$\nabla(f_1 s_1 + \cdots + f_n s_n) = s_1 \otimes df_1 + \cdots + s_n \otimes df_n.$$

(Differential geometers also call these *flat* connections, but algebraic geometers have already committed this word to another meaning, so we stick to the word "integrable".)

Remark 3.1.3. Another way to think about this: let $z_1, \dots, z_n$ be local coordinates for X at a point x. Then we can contract ∇ with the vector field $\frac{\partial}{\partial z_i}$ to obtain a map from V to itself satisfying the Leibniz rule with respect to $\frac{\partial}{\partial z_i}$; you should think of this as an "action of $\frac{\partial}{\partial z_i}$ on sections of V". The connection ∇ is integrable if and only if these different actions commute with each other.

3.2 Integrable connections and local systems

Definition 3.2.1. Let X be a connected complex manifold, and choose a point $x \in X$. A *local system* on X is a homomorphism $\rho : \pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C})$. One gets a local system from an integrable connection as follows: given a loop representing a class in $\pi_1(X, x)$ and an element of the fibre V_x, write down a horizontal section on a neighborhood of the path whose initial value is the one you picked. In general the final value will be something different, and in fact the map from initial to final values will be a linear endomorphism of V_x.

This function on loops is called the *holonomy* of the connection. If the connection is integrable, then it depends only on the homotopy class of the loop (and the word *monodromy* is used instead).

Conversely, one can go backwards from local systems to integrable connections, so the two categories are equivalent.

3.3 Aside: connections and differential equations

It is worth pointing out that for purposes of explicit calculations, it is common to work with differential equations instead of connections.

Definition 3.3.1. Let's for simplicity suppose you have a trivial vector bundle V of rank n over the t-line equipped with a connection, or equivalently, an action of $\frac{d}{dt}$. A *cyclic vector* is a section s of V such that

$$s, \frac{d}{dt}s, \ldots, \frac{d^{n-1}}{dt^{n-1}}s$$

form a basis of V.

Given a cyclic vector, we can describe horizontal sections as follows. Write

$$\frac{d^n}{dt^n}s = a_0 s + a_1 \frac{d}{dt}s + \cdots + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}s.$$

Let's define an unspecified section of V by taking

$$v = f_0 s + f_1 \frac{d}{dt}s + \cdots + f_{n-1}\frac{d^{n-1}}{dt^{n-1}}s$$

where $f_0, \ldots, f_{n-1}$ are functions of t. For v to be horizontal, we need

$$\begin{aligned} 0 &= f_0' + f_{n-1}a_0 \\ 0 &= f_1' + f_0 + f_{n-1}a_1 \\ &\vdots \\ 0 &= f_{n-1}' + f_{n-2} + f_{n-1}a_{n-1}. \end{aligned}$$

Eliminating $f_0, \ldots, f_{n-1}$ leaves a differential equation of the form

$$f_{n-1}^{(n)} + * f_{n-1}^{(n-1)} + \cdots + * f_{n-1}' + * f_{n-1} = 0.$$

Important class of examples: for the Gauss-Manin connections we are about to introduce, the resulting differential equations were introduced earlier and so have their own name. They are known as *Picard-Fuchs equations.*

3.4 The Gauss-Manin connection in geometry

Definition 3.4.1. Let B be a smooth complex analytic space, and let $X \to B$ be a smooth proper morphism. We can then define the *relative de Rham cohomology* as the higher direct images $\mathbb{R}^i\Omega^{\cdot}_{X/B}$ of the complex of relative differentials. These turn out to be vector bundles whose fibres can be identified with the cohomology of the fibres X_b.

Definition 3.4.2. Suppose B is contractible. Then the fibration $X \to B$ is not trivial in the category of complex analytic spaces, but it turns out to be trivial in the category of compact manifolds. In particular, you can think of the cohomologies as forming a local system; you even have this "integrally", in that you get a map $\pi_1(B, b) \to \mathrm{GL}_n(\mathbb{Z})$ by tracing around integral cohomology classes. The corresponding connection is the *Gauss-Manin connection.*

Remark 3.4.3. I referred earlier to something called a *Picard-Fuchs equation*, which was supposed to be a differential equation derived from the Gauss-Manin connection. Where it originally came from: take a homology class on one fibre, move it to nearby fibres and integrate against a fixed differential form.

3.5 The algebraic construction (after Katz-Oda)

One of the points of this lecture is that the Gauss-Manin connection also has an algebraic description, which makes it quite easy to compute explicitly. This was originally discovered by Katz and Oda [24].

Definition 3.5.1. Let B be a smooth variety over a field K of characteristic zero, and let $\pi : X \to B$ be a smooth proper morphism. Equip the de Rham complex $\Omega^{\cdot}_{X/K}$ with the decreasing filtration

$$F^i = \mathrm{image}[\Omega^{\cdot - i}_{X/K} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{B/K}) \to \Omega^{\cdot}_{X/K}],$$

then form the corresponding spectral sequence (as in Theorem 1.7.5). The E_1 term of the result has

$$E_1^{p,q} = \Omega^p_{B/K} \otimes_{\mathcal{O}_B} H^q_{\mathrm{dR}}(X/B);$$

the *algebraic Gauss-Manin connection* is the differential $d_1 : E_1^{0,q} \to E_1^{1,q}$. Katz and Oda showed that this is an integrable connection, and it agrees with the analytically defined connection.

The way this translates into practice is as follows: given a relative i-form $\omega \in \Omega^i_{X/B}$, lift it to an absolute i-form $\tilde{\omega} \in \Omega^i_{X/K}$, then differentiate, then project the result back into $\Omega^i_{X/B} \otimes \Omega^1_{B/K}$. This is quite easy to do with real examples; see Example 3.6.1.

3.6 Example: a family of elliptic curves

Example 3.6.1. As an example, let us consider the family of elliptic curves

$$y^2 = x^3 + A(t)x + B(t)$$

over the t-line. A basis for $R^1 f_* \Omega^{\cdot}$ is given by

$$\omega, x\omega \qquad \omega = \frac{dx}{y},$$

these being a basis for the relative differentials with logarithmic singularities along the infinity section and no other poles.

To compute the connection, we lift these to 1-forms on the total surface and differentiate, using the fact that

$$2y\,dy = (3x^2 + A)\,dx + (A'x + B')\,dt.$$

First,

$$\begin{aligned}\nabla(\omega) &= d\left(\frac{dx}{y}\right) \\ &= -\frac{dy \wedge dx}{y^2} \\ &= \frac{A'x + B'}{2y^2}\frac{dx}{y} \wedge dt.\end{aligned}$$

Unfortunately, this is not quite in the right form: we need to eliminate the pole along $y = 0$. This happened because dx/y is not actually a holomorphic differential on the surface; it only looks holomorphic when you ignore the dt component, which is what we did when we looked at the complex of relative differentials.

To eliminate the pole, one computes the unique C, D, E, F, G (functions of t alone) such that

$$A'x + B' = (Cx + D)(x^3 + Ax + B) + (Ex^2 + Fx + G)(3x^2 + A),$$

so that in $R^1 f_* \Omega^{\cdot}$,

$$\begin{aligned}\frac{(A'x + B')\,dx}{2y^3} &= \frac{(Cx + D)\,dx}{2y} + \frac{(Ex^2 + Fx + G)(3x^2 + A)\,dx}{2y^3} \\ &\equiv \frac{(Cx + D)\,dx}{2y} + \frac{(2Ex + F)\,dx}{y}.\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla(x\omega) &= d\left(\frac{x\,dx}{y}\right) \\ &= \frac{A'x^2 + B'x}{2y^2}\frac{dx}{y} \wedge dt\end{aligned}$$

and writing

$$A'x^2 + B'x = (Hx + I)(x^3 + Ax + B) + (Jx^2 + Kx + L)(3x^2 + A),$$

we get

$$\frac{(A'x^2 + B'x)\,dx}{2y^3} \equiv \frac{(Hx + I)\,dx}{2y} + \frac{(2Jx + K)\,dx}{y}.$$

3.7 Gauss-Manin connections and Frobenius

The reason why Gauss-Manin connections are relevant in rigid cohomology is that they can be used to compute Frobenius actions *en masse.*

Exercise 3.7.1. Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Let A be an affinoid algebra over K with good reduction; that is, if A^0 denotes the subring of A of elements of spectral norm ≤ 1, then $\overline{A} = A^0/\mathfrak{m}_K A^0$ is smooth over $\mathbb{F}_q$.) Let $\phi : A \to A$ be a q-power absolute Frobenius lift; that is, ϕ acts on $\overline{A}$ by the q-power map. Prove that for each $\overline{x} \in \overline{A}(\mathbb{F}_q)$, there is a unique $x \in A(K)$ which specializes to $\overline{x}$, such that $\phi(x) = x$.

Definition 3.7.2. With notation as in Exercise 3.7.1, we call x the *Teichmüller lift* of $\overline{x}$ with respect to ϕ. If x is a Teichmüller lift of a point not specified, we say x is a *Teichmüller point.*

Exercise 3.7.3. State and prove a generalization of Exercise 3.7.1 to the case where the residue field is perfect but not necessarily finite.

Theorem 3.7.4 (Berthelot)**.** *Let B be a smooth $\mathbb{Z}_q$-scheme with associated analytic space B^{an}, and let ϕ be a Frobenius lift on B^{an}. (More precisely, ϕ should be defined on the completion $\hat{B}$ of B along $B_{\mathbb{F}_q}$; it then acts on B^{an}.) Let $X \to B$ be a smooth morphism, and let ∇ be the Gauss-Manin connection on $\mathcal{E} = H^i_{\mathrm{dR}}(X_{\mathbb{Q}_q}/B_{\mathbb{Q}_q})$. Then there exists an isomorphism $F : \phi^*\mathcal{E} \cong \mathcal{E}$ of modules with integrable connection on B^{an}, such that for any positive integer a, and any Teichmüller point $x \in B_{\mathbb{Q}_{q^a}}$, $F^a : (\phi^a)^*\mathcal{E} \cong \mathcal{E}$ induces the Frobenius action on $H^i_{\mathrm{dR}}(X_x, \mathbb{Q}_{q^a})$.*

The key feature here is that the Frobenius action commutes with the action of the connection; this constraint can be interpreted as a differential equation on the Frobenius action. (Note: include explicit description here.)

Remark 3.7.5.

4 Hyodo-Kato cohomology

In this section, we describe some natural structure on the de Rham cohomology of a smooth proper variety over $\mathbb{Q}_q$ that need not have good reduction. Besides the Frobenius action that one sees in the good reduction case, there is a second operator present, the *monodromy operator*, that in some sense measures to what extent the variety fails to have good reduction.

5 Proposed projects

The intended computational platform for these projects is William Stein's package `SAGE`, available for free download at

`http://sage.math.washington.edu/sage/`.

It can also be accessed directly from the above site, without local installation, using the notebook interface.

5.1 Improving the Christol-Dwork bound

The Christol-Dwork theorem [7] gives an effective bound on the convergence of solutions of certain differential equations; this bound is crucially used in the deformation method. However, some experimental evidence suggest that this bound is far from optimal; the goal of this project is to collect more such evidence, then establish an improved version of the Christol-Dwork bound. (Note: the theorem I attribute to Christol-Dwork is a bit weaker than what they proved, but it's what we need in practice.)

Definition 5.1.1. Let K be a discretely valued field of mixed characteristics $(0,p)$, with ring of integers $\mathfrak{o}_K$ and maximal ideal $\mathfrak{m}_K$. (I really only need a nonarchimedean valuation, but let us assume discreteness for simplicity.) Let R be the ring of formal power series $\sum_{i=0}^{\infty} c_i t^i \in K[\![t]\!]$ such that

$$\lim_{i\to\infty} |c_i|\rho^i = 0 \qquad (\rho \in [0,1)).$$

For $\rho \in [0,1)$, define the *ρ-Gauss norm* on R by

$$\left|\sum_{i=0}^{\infty} c_i t^i\right|_\rho = \sup_i\{|c_i|\rho^i\}. \tag{5.1.1.1}$$

Put $E = \mathfrak{o}_K[\![t]\!][1/p]$, viewed as a subring of R. Note that we can use (5.1.1.1) to define a ρ-Gauss norm on E not only for $\rho \in [0,1)$ but also for $\rho = 1$.

Exercise 5.1.2. Prove the following.

(a) The ring R consists of the rigid analytic functions on the open unit disc over K.

(b) For $\rho \in [0,1)$, the ρ-Gauss norm on R equals both the supremum on the circle $|t| = \rho$ and on the disc $|t| \leq \rho$.

(c) The subring E of R consists of the bounded functions on the open unit disc over K.

(d) The 1-Gauss norm on E equals the supremum on the open unit disc.

Definition 5.1.3. By a *differential operator of rank n* over a subring S of $K[\![t]\!]$, we will mean an operator $D : S^n \to S^n$ of the form

$$\mathbf{v} \mapsto N\mathbf{v} + t\frac{d\mathbf{v}}{dt},$$

where $N = \sum_{i=0}^{\infty} N_i t^i$ is an $n \times n$ matrix over S; we extend the action of D in the obvious way to $n \times m$ matrices for any m. We refer to N_0 as the *residue* of D. An *isomorphism* between two differential systems D_1, D_2 is an invertible S-linear map $T : S^n \to S^n$ such that $D_2 = T^{-1} \circ D_1 \circ T$; if we define the change-of-basis matrix U by $T\mathbf{e}_j = \sum_i U_{ij}\mathbf{e}_i$, then the matrices N_1, N_2 defining D_1, D_2 are related by

$$N_2 = U^{-1}N_1U + U^{-1}t\frac{dU}{dt}.$$

Note that change of basis changes the residue by a true conjugation, so it is well-defined to specify, e.g., that the residue be nilpotent.

Remark 5.1.4. You may prefer (as I do) to think of the differential operator D as giving a connection on S with a logarithmic singularity at $t = 0$.

Exercise 5.1.5. Let D be a differential operator of rank n over $K[\![t]\!]$ with nilpotent residue. Prove that there exists a unique $n \times n$ matrix $U = \sum_{i=0}^{\infty} U_i t^i$ over $K[\![t]\!]$ with U_0 equal to the identity matrix, such that $DU = 0$.

Definition 5.1.6. Let D be a differential operator of rank n over $K[\![t]\!]$ with nilpotent residue. We refer to the matrix U in Exercise 5.1.5 as the *universal solution* of D.

Definition 5.1.7. Let D be a differential operator of rank n with nilpotent residue over a subring of R. We say D is *fully convergent* if the universal solution of D has entries in R. (The term *overconvergent* is also sometimes used.) There is also a definition that works even if the residue is not nilpotent, but it is a bit more complicated to describe.

Typically, we obtain fully convergent differential operators from Frobenius actions.

Definition 5.1.8. Let q be a power of p. Define a *(relative) q-power Frobenius lift* on $\mathfrak{o}_K[\![t]\!]$ to be a ring homomorphism $\phi : \mathfrak{o}_K[\![t]\!] \to \mathfrak{o}_K[\![t]\!]$ such that

$$\phi(t) = t^q u \qquad \text{for some } u \in 1 + (t, \mathfrak{m}_K)\mathfrak{o}_K[\![t]\!].$$

(It is not necessary to assume that $\phi(K) \subseteq K$.) Any such map ϕ has the form

$$\phi\left(\sum_{i=0}^{\infty} c_i t^i\right) = \sum_{i=0}^{\infty} \phi(c_i)\phi(t)^i;$$

we can use the same formula to extend ϕ to $E, R, K[\![t]\!]$.

Definition 5.1.9. Let D be a differential operator of rank n with nilpotent residues over a subring S of $\mathfrak{o}_K[\![t]\!]$. A *Frobenius action* on D over S with respect to a Frobenius lift ϕ is an isomorphism of D with its pullback by ϕ. Geometrically, this means an isomorphism of the module with connection defined by D with its Frobenius pullback. Algebraically, if D is defined by the matrix N, the pullback is defined by the matrix

$$\frac{t}{\phi(t)} \frac{d\phi(t)}{dt} \phi(N).$$

Exercise 5.1.10. Let D be a differential operator of rank n with nilpotent residues over R admitting a Frobenius action. Prove that D is convergent. (You might want to start with the case $\phi(t) = t^p$.)

Definition 5.1.11. Let $K[t]_{(t)}$ be the localization of $K[t]$ at the maximal ideal (t). Let E_0 be the closure of $K[t]_{(t)}$ in E for the 1-Gauss norm.

Theorem 5.1.12 (Christol-Dwork). *Let D be a fully convergent differential operator of rank n with nilpotent residues over E_0, with universal solution $U = \sum_{i=0}^{\infty} U_i t^i$. Then for $i \geq 1$, $v_p(U_i) \geq -\gamma_n \lfloor \log_p(i) \rfloor$ with*

$$\gamma_n = n - 1 + v_p((n-1)!) + \min\left(n-1, v_p\left(\prod_{j=1}^{n} \binom{n}{j}\right)\right).$$

Problem 5.1.13. Decide whether Theorem 5.1.12 still hold if N has matrices in E?

Problem 5.1.14. Reduce the size of the constant γ_n in Theorem 5.1.12, possibly introducing dependence on some other parameters. These may include the index of nilpotency, and the slopes of the Frobenius action modulo t (in the sense of the Dieudonné-Manin classification). The latter is suggested by a recent preprint by Chiarellotto and Tsuzuki [6], but without consideration of effective bounds.

Remark 5.1.15. The bound given by Theorem 5.1.12 is crucial for provably computing the Frobenius structure on a Gauss-Manin connection to a specified p-adic accuracy. However, Ralf Gerkmann has noticed experimentally that the bound is far from optimal in many practical examples; it would thus be of great interest to improve these bound. It may be advisable to first collect some more experimental evidence in a systematic manner.

5.2 Coleman integrals

A theory of integration for 1-forms on curves over finite extensions of $\mathbb{Q}_p$ was introduced by Coleman [8]; computing Coleman's integrals turns out to be quite closely related to computing Frobenius actions in Monsky-Washnitzer cohomology. The usual integrals arise in the Chabauty method for finding all S-integral points on a curve over a number field [9] (for more details on which see Matt Baker); the iterated ones seem to relate to a "nonabelian Chabauty method" proposed by Minhyong Kim [28].

Problem 5.2.1. Implement the computation of Teichmüller lifts, e.g., for the Frobenius lift on an elliptic curve described in Example 2.2.2. You should be able to do this achieving quadratic convergence (as when using Newton-Raphson iteration) rather than linear convergence (as in the usual proof of Hensel's lemma).

5.3 Computing Hyodo-Kato cohomology

Problem 5.3.1. Implement an explicit computation of the Frobenius and monodromy operators on an elliptic curve over $\mathbb{Q}_p$ with (split or nonsplit) multiplicative reduction. A procedure for doing this is suggested by le Stum [29], and explained further by Coleman and Iovita [10].

Problem 5.3.2. Extend the implementation in Problem 5.3.1 first to hyperelliptic curves with totally multiplicative reduction. Then treat the case of semistable reduction (i.e., with a mix of good and multiplicative reduction).

5.4 Nondegenerate hypersurfaces in toric varieties

Those familiar with toric varieties might enjoy the following.

Problem 5.4.1. Extend and implement the algorithm described in [1] for computing zeta functions of smooth hypersurfaces in projective spaces, to nondegenerate hypersurfaces in toric varieties.

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