Arizona Winter School 2017: Adic Spaces

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February 1, 2017

1 Introduction

Warning! Incomplete version. Contains numerous embarrassing errors. Will revise in time for AWS, I promise!

1.1 What is a "space"?

Consider the different kinds of geometric "spaces" you know about. First you learned about topological spaces. Then came various sorts of manifolds, which are topological spaces which locally look like something a *model space* (an open subset of $\mathbf{R}^n$). Then you learned that manifolds could carry different structures (differentiable, smooth, complex,...). You could express these structures in terms of the transition functions between charts on your manifold. But this is a little awkward, thinking of everything in terms of charts. Later you learned a more efficient definition: a manifold with one of these structures is a *ringed space* $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, such that locally on X the pair $(X, \mathcal{O}_X)$ is isomorphic to one of the model spaces, together with its sheaf of (differentiable, smooth, complex) functions. An advantage of this point of view is that it becomes simple to define a morphism $f\colon X \to Y$ between such objects: it is a continuous map of topological spaces together with a homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (in other words, *functions pull back*).

This formulation of spaces in terms of pairs $(X, \mathcal{O}_X)$ was good preparation for learning about schemes, the modern language of algebraic geometry. This time the model spaces are affine schemes, which are spectra of rings. For a ring A, the topological space $\operatorname{Spec} A$ may have initially seemed strange–in particular it is not generally Hausdorff. But then you learn some advantages of working with schemes. For instance, an integral scheme X has a generic point η. It is enormously useful to take an object associated with X (a morphism to X, an $\mathcal{O}_X$-module, an étale sheaf on X,...) and pass to its generic fiber, which is associated with the function field of X. Usually if some property is true on the generic fiber, then it is also true "generically" on X (that is, on a dense open subset). Number theorists use this language all the time in the setting of $\operatorname{Spec} \mathbf{Z}$: if a property holds over the generic point $\operatorname{Spec} \mathbf{Q}$, then it holds at almost all special points $\operatorname{Spec} \mathbf{F}_p$.

The language of *formal schemes* is useful for studying what happens in an infinitesimal neighborhood of a closed subset of a scheme. (These often come up in the study of deformation theory.) This time the model spaces are formal spectra $\operatorname{Spf} A$, where A is a ring which is separated and complete for the I-adic topology for some ideal I. That is, $A \cong \varprojlim A/I^n$ as topological rings. Examples: $\mathbf{Z}_p$, $\mathbf{Z}[\![T]\!]$. (One gets into a little trouble without imposing some finiteness condition; *e.g.* one could assume the ideal I is finitely generated.) The notation Spf stands for "formal spectrum", and refers to the collection of open prime ideals of A. This can be given the structure of a topological space X, which is equipped with a sheaf $\mathcal{O}_X$ of topological rings.

In the theory of *complex-analytic spaces*, the model space is the vanishing locus of a collection of holomorphic functions on an open subset of $\mathbf{C}^n$. Thus it is like the theory of complex manifolds, except that some singularities are allowed. The theory of complex-analytic spaces has many nice interactions with the theory of schemes. If X is a finite-type scheme over $\operatorname{Spec} \mathbf{C}$, then there is a complex-analytic space X^{an}, the *analytification* of X, which is universal for the property of admitting a morphism of ringed spaces $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$. Conversely, if $\mathcal{X}$ is a complex-analytic space admitting a closed immerson into projective space, then $\mathcal{X}$ is the analytification of a projective complex variety X, and then X and $\mathcal{X}$ have equivalent categories of coherent sheaves, and the equivalence respects cohomology groups (Serre's *GAGA* theorem). In this situation there are *comparison isomorphisms* between the étale cohomology groups of X and $\mathcal{X}$. There are further relations known as *uniformizations*; most well-known of these is the phenomenon that if E is an elliptic curve over $\operatorname{Spec} \mathbf{C}$, then there exists a lattice $L \subset \mathbf{C}$ such that $E^{\mathrm{an}} \cong \mathbf{C}/L$ as complex-analytic spaces.

1.2 Rigid-analytic spaces

Let us turn our attention from archimedean fields ($\mathbf{R}$ and $\mathbf{C}$) to non-archimedean fields ($\mathbf{Q}_p$, $\mathbf{C}_p$, $k(\!(t)\!)$) for any field k). Both are kinds of complete metric fields, so it is natural to expect a good theory of manifolds or analytic spaces for a non-archimedean field K. Which ringed spaces $(X, \mathcal{O}_X)$ should serve as our model spaces? The naïve answer is that (to define a manifold) X should be an open subset of K^n, and $\mathcal{O}_X$ should be its sheaf of continuous K-valued functions. The problem with this approach is that X is will be totally-disconnected, which makes it too easy to glue functions together. This problem will ruin an attempt to emulate the complex theory: if $X = \mathbf{P}^1$ defined this way, then $H^0(X, \mathcal{O}_X) \neq K$ (violating GAGA) and $H^0_{\text{ét}}(X, A) \neq A$ (violating the comparison isomorphism).

Nonetheless, Tate observed that some elliptic curves over K (those with multiplicative reduction) admit an explicit uniformization by $K^\times$, which hints that there should be a good theory of analytic varieties. Tate's uniformization involved power series which converged on certain sorts of domains in $K^\times$. Tate's theory of *rigid-analytic spaces* is a language which satisfies most of the desiderata of an analytic space, including GAGA and the comparison isomorphisms.

A brief summary of the theory: we define the *Tate algebra* $K\langle T_1, \ldots, T_n\rangle$ to be the K-algebra of power series in $K[\![T_1, \ldots, T_n]\!]$ whose coefficients tend to zero. (Alternately, this is the completion of the polynomial ring $K[T_1, \ldots, T_n]$ with respect to the "Gauss norm".) The Tate algebra has various nice properties: it is Noetherian, all ideals are closed, and there is a bijection between $\operatorname{Spm} K[\![T_1, \ldots, T_n]\!]$ and the closed unit disc in $\overline{K}^n$, modulo the action of $\operatorname{Gal}(\overline{K}/K)$. An *affinoid K-algebra* is a quotient of a Tate algebra.

The model spaces in the theory of rigid-analytic spaces are $\operatorname{Spm} A$, where A is an affinoid K-algebra, and Spm means the set of maximal ideals. But the topology Tate puts on $\operatorname{Spm} A$ isn't the one coming from $\overline{K}^n$, and in fact isn't a topology at all, but rather a *Grothendieck topology*, with a collection of "admissible opens" and a notion of "admissible open covering". With this topology, $\operatorname{Spm} A$ carries a sheaf of rigid-analytic functions, whose global sections recover A. Then a rigid-analytic space over K is a pair $(X, \mathcal{O}_X)$, where X is a set carrying a Grothendieck topology and $\mathcal{O}_X$ is a sheaf of K-algebras, which is locally isomorphic to a model space $\operatorname{Spm} A$.

Despite this quirk about Grothendieck topologies, the theory of rigid-analytic spaces has had spectacular successes as a non-archimedean analogue to complex-analytic spaces: there is a rigid-analytic GAGA theorem, comparison theorems, fascinating theorems about uniformization of curves and of Shimura varieties, new moduli spaces which are local analogues of Shimura varieties (implicated in the proof of the local Langlands correspondence for GL_n over a p-adic field).

1.3 A motivation for adic spaces

Despite these successes, the theory of rigid-analytic spaces has a few shortcomings, which are addressed by the more general theory of *adic spaces*. One is the problem with topologies, illustrated in the following examples:

Example 1.3.1. Let $X = \operatorname{Spm} K\langle T\rangle$ be the rigid-analytic closed unit disk, and let Y be the disjoint union of the open unit disc U with the circle $S = \operatorname{Spm} K\langle T, T^{-1}\rangle$. There is an open immersion $Y \to X$, which is a bijection on the level of points. But it is not an isomorphism, because the two spaces have different Grothendieck topologies. (For instance, U is not an admissible open in X, because it is not a finite union of affinoid subdomains.)

Another example: let $X = \operatorname{Spm} K\langle T\rangle$, let α be an element of the completion of K which is transcendental over K, and let $Y \subset X$ be the union of all affinoid subdomains U which do not "contain" α, in the sense that α does not satisfy the collection of inequalities among power series which define U. Then the open immersion $Y \to X$ is once again a bijection on points but not an isomorphism.

In both examples there was an open immersion $Y \to X$ which is a bijection on points but which is not an isomorphism. This suggests that there are certain hidden "points" in X which Y is missing. In fact in the world of adic spaces, Y is simply the complement in X of a single point.

Another shortcoming, if we may be so greedy as to point it out, is that rigid-analytic spaces are too narrowly tailored to the class of K-affinoid algebras studied by Tate. Whereas the category of adic spaces encompasses the categories of rigid-analytic spaces, formal schemes, and even ordinary schemes. This allows to pass between these categories very easily. For instance, if X is a formal scheme over $\operatorname{Spf}\mathbf{Z}_p$ (satisfying certain finiteness assumptions), then there should be a corresponding rigid space X^{rig}, its *rigid generic fiber*. This was worked out by Raynaud and Berthelot (CITE), but is rather subtle: if $X = \operatorname{Spf}\mathbf{Z}_p[[T]]$, then X^{rig} is the rigid-analytic open unit disc, which isn't even affinoid. Whereas in the adic world, there is a formal unit disc fibered over a two-point space $\operatorname{Spa}\mathbf{Z}_p$, and its generic fiber is simply the open subset lying over the generic point $\operatorname{Spa}\mathbf{Q}_p$.

1.4 Huber rings

The model spaces in the theory of adic spaces are associated to certain topological rings A. In light of our desiderata, A should be allowed to be $\mathbf{Z}_p[[T]]$, or $\mathbf{Q}_p\langle T\rangle$, or even any ring whatsoever with its discrete topology. In the first and third case, the topology of A is generated by a finitely-generated ideal. In the second case, the topology of $\mathbf{Q}_p\langle T\rangle$ certainly isn't generated by p (since this is invertible in A), but rather there is an open subring $\mathbf{Z}_p\langle T\rangle$ whose topology *is* generated by p.

Definition 1.4.1. A *Huber ring* is a topological ring A containing an open subring A_0 carrying the linear topology induced by a finitely generated ideal $I \subset A_0$. The ring A_0 and the ideal I are called a *ring of definition* and an *ideal of definition*, respectively. (The data of A_0 and I are not packaged along with A.)

Example 1.4.2.

1. Any ring A can be given the discrete topology; then A is a Huber ring[1] with $A_0 = A$ and $I = 0$.
2. Let K be a nonarchimedean field: this means a topological field which is complete with respect to a nontrivial nonarchimedean metric $|\ |$. Since $|\ |$ is nontrivial, K contains an element $\varpi \in K^\times$ with $|\varpi| < 1$. Such an element is called a *pseudouniformizer*. Then K is a Huber ring with ring of definition $K^\circ = \{|x| \le 1\}$ and ideal of definition (ϖ), where $\varpi \in K$ is a pseudo-uniformizer.
3. Continuing with the previous example, we have the Tate K-algebra $A = K\langle T_1, \dots, T_n\rangle$; this is a Huber ring with ring of definition $A_0 = K^\circ\langle T_1, \dots, T_n\rangle$ and ideal of definition (ϖ).
4. Let R be any ring with its discrete topology; then the power series ring $A = R[[T_1, \dots, T_n]]$ is a Huber ring with ring of definition $A_0 = A$ and ideal of definition $(T_1, \dots, T_n)$.
5. Similarly, if K is a nonarchimedean field with pseudouniformizer ϖ, then $A = K^\circ[[T_1, \dots, T_n]]$ is a Huber ring with ring of definition $A = A_0$ and ideal of definition $(\varpi, T_1, \dots, T_n)$.

[1] called an f-adic ring by Huber [Hub94].

6. Let K be a nonarchimedean field which is perfect of characteristic p. The ring of Witt vectors $A = W(K^\circ)$ is a Huber ring with ring of definition $A = A_0$ and ideal of definition $(p, [\varpi])$.

7. Let $A = \mathbf{Q}_p[\![T]\!]$. It is tempting to say that A is a Huber ring with ring of definition $A_0 = \mathbf{Z}_p[\![T]\!]$ and ideal of definition (p, T). But in fact one cannot put a topology on A which makes this work. Indeed, in such a topology $T^n \to 0$, and since multiplication by p^{-1} is continuous, $p^{-1}T^n \to 0$ as well. But this sequence never enters A_0, contradicting that fact that $A_0 \subset A$ is open. (It is fine to say that $\mathbf{Q}_p[\![T]\!]$ is a Huber ring with ring of definition $\mathbf{Q}_p[\![T]\!]$ and ideal of definition (T), but then you are artificially suppressing the topology of $\mathbf{Q}_p$.) There is a similar obstruction to $\mathbf{Z}_p[\![T]\!][1/p]$ being a Huber ring.

1.5 Continuous valuations

The idea now is to associate to a Huber ring A a ringed space $\operatorname{Spa} A = (X, \mathcal{O}_X)$, which will serve as the model space for the theory of adic spaces. The points of X are quite interesting: they correspond to continuous valuations on the ring A.

Recall that an *ordered abelian group* is an abelian group Γ admitting a total order $\leq$ preserved by the group operations. These will be written multiplicatively. Examples include $\mathbf{R}_{>0}$ and any subgroup thereof. Another example is $\Gamma = \mathbf{R}_{>0} \times \mathbf{R}_{>0}$ under its *lexicographical ordering*: $(a, b) \leq (c, d)$ means that either $a < c$ or else $a = c$ and $b \leq d$. A feature of this Γ is that it contains $\mathbf{R}_{>0}$ (embedded along the first coordinate) together with, for each $a \in \mathbf{R}_{>0}$, elements (such as $(a, 1/2)$, respectively $(a, 2)$) which are between a and every real number less than (respectively, greater than) a. This concept easily generalizes to finite products $\mathbf{R}^n_{>0}$, or even infinite products of $\mathbf{R}_{>0}$ indexed by an ordinal.

Definition 1.5.1. For an ordered abelian group Γ, a subgroup $\Gamma' \subset \Gamma$ is *convex* if any element of Γ lying between two elements of Γ' must itself lie in Γ'.

It is a nice exercise to show that if $\Gamma', \Gamma'' \subset \Gamma$ are two convex subgroups then either $\Gamma' \subset \Gamma''$ or $\Gamma'' \subset \Gamma$. Therefore the set of convex subgroups forms a totally ordered set which one easily sees is well-ordered (since the intersection of any collection of convex subgroups is again a convex subgroups). The corresponding ordinal is called the *rank* of Γ. If the rank if finite, we name its rank with a natural number in such a way that the rank of $\mathbf{R}^n_{>0}$ is n.

The condition for Γ to be rank 1, which is to say that Γ is a nontrivial ordered abelian group with no convex subgroups, is equivalent to the following archimedean property: given $a, b \in \Gamma$ with $a > 1$, then there exists $n \in \mathbf{Z}$ with $b < a^n$. We remark that a rank 1 ordered abelian group can always be embedded into $\mathbf{R}_{>0}$. Also, every nontrivial ordered abelian group Γ admits a unique rank 1 quotient $\Gamma \to \widetilde{\Gamma}$.

Definition 1.5.2. Let A be a topological ring. A *continuous valuation* on A with is a map

$$|\cdot| : A \to \Gamma_{|\cdot|} \cup \{0\},$$

where Γ is a totally ordered abelian group, and $\Gamma \cup \{0\}$ is the ordered monoid with least element 0. It is required that

- $|ab| = |a|\,|b|$
- $|a+b| \leq \max(|a|, |b|)$
- $|1| = 1$
- $|0| = 0$
- (Continuity) For all $\gamma \in \Gamma_{|\cdot|}$, $\{a \in A |\, |a| < \gamma\}$ is open in A.

Two continuous valuations $|\;| : A \to \Gamma \cup \{0\}$ and $|\;|' : A \to \Gamma' \cup \{0\}$ are *equivalent* if for all $a, b \in A$ we have $|a| \geq |b|$ if and only if $|a|' \geq |b|'$. In that case, after replacing Γ by the subgroup generated by the image of A, and similarly for Γ', there exists an isomorphism $\iota \colon \Gamma \cong \Gamma'$ such that $\iota(|a|) = |a|'$ for all $a \in A$.

Note that the kernel of $|\cdot|$ is a prime ideal of A which only depends on its equivalence class.

Definition 1.5.3. Let $\mathrm{Cont}(A)$ denote the set of equivalence classes of continuous valuations of A. For an element $x \in \mathrm{Cont}(A)$, we use the notation $f \mapsto |f(x)|$ to denote a continuous valuation representing x. We give $\mathrm{Cont}(A)$ the topology generated by subsets of the form $\left\{x \;\middle|\; |f(x)| \leq |g(x)| \neq 0\right\}$, with $f, g \in A$. For $x \in \mathrm{Cont}(A)$, the rank of x is the rank of the ordered abelian group generated by the image of a continuous valuation representing x.

Some remarks on the topology of $\mathrm{Cont}(A)$: Note that sets of the form $\{|g(x)| \neq 0\}$ are open, as are sets of the form $\{|f(x)| \leq 1\}$. This blends features of the Zariski topology on schemes and topology on rigid spaces. Now suppose $x \in \mathrm{Cont}(A)$ corresponds to a continuous valuation with values in an ordered abelian group Γ. If $\Gamma \to \Gamma'$ is any homomorphism of ordered abelian groups, then we get a new point $x' \in \mathrm{Cont}(A)$. An important observation, which follows immediately from the definition of the topology on $\mathrm{Cont}(A)$, is that any open set that contains x also contains x'. Thus the closure of x' contains x, which is to say that x' is a *generalization* of x. If $\Gamma \to \Gamma'$ is the unique rank 1 quotient, then x' is the *maximal generalization* of x.

When A is a Huber ring, the set $\mathrm{Cont}(A)$ is a good candidate for the model space we want to build. For instance if A is a discrete ring, then $\mathrm{Cont}(A)$ contains one point x for each prime $\mathfrak{p} \in \operatorname{Spec} A$, namely the valuation pulled back from the trivial valuation on the residue field of $\mathfrak{p}$. The set $\mathrm{Cont}(\mathbf{Q}_p)$ is a single point, namely the equivalence class of the usual p-adic valuation on $\mathbf{Q}_p$.

Now consider $\mathrm{Cont}(\mathbf{Q}_p\langle T\rangle)$, which is our hypothetical "adic closed unit disc". For each maximal ideal $\mathfrak{m} \in \mathrm{Spm}\,\mathbf{Q}_p\langle T\rangle$, we do get a point in $\mathrm{Cont}(A)$ by pulling back the valuation on the nonarchimedean field $\mathbf{Q}_p\langle T\rangle/\mathfrak{m}$ (this is a finite extension of $\mathbf{Q}_p$). Thus there is a map $\mathrm{Spm}\,\mathbf{Q}_p\langle T\rangle \to \mathrm{Cont}\,\mathbf{Q}_p\langle T\rangle$. But the latter set contains many more points. For instance, we can let $\alpha \in \mathbf{C}_p$ be a transcendental element with $|\alpha| \leq 1$, and define a continuous valuation on $\mathbf{Q}_p\langle T\rangle$ by $f \mapsto |f(\alpha)|$. This is going to address one of the problems in classical rigid geometry brought up in Example 1.3.1.

Addressing the other problem brought up in that example, we can also define an element $x^- \in \mathrm{Cont}\,\mathbf{Q}_p\langle T\rangle$ as follows: let $\Gamma = \mathbf{R}_{\geq} \times \gamma^{\mathbf{Z}}$, where the order is determined by the relations $a < \gamma < 1$ for all real $a < 1$. (If you like, Γ can be embedded as a subgroup of $\mathbf{R}_{>0} \times \mathbf{R}_{>0}$ by $a\gamma^n \mapsto (a, 1/2^n)$. Now define x^- by

$$\sum_{n=0}^{\infty} a_n T^n \mapsto \sup_{n\geq 0} |a_n|\, \gamma^n.$$

Thus x^- "thinks" that T is infinitesimally smaller than one: we have $|T(x^-| = \gamma < 1$, but $|T(x^-)| > |a|$ for all $a \in \mathbf{Q}_p$ with $|a| < 1$. The point x^- prevents us from disconnecting $\mathrm{Cont}\,\mathbf{Q}_p\langle T\rangle$ by the disjoint open sets $\{|T(x)| < 1\}$ and $\{|T(x)| = 1\}$, because neither of these contains x^-!

However, this example suggests we have more points in $\mathrm{Cont}\,\mathbf{Q}_p\langle T\rangle$ than we bargained for. There is also a point x^+ with the same definition, except that γ is now infinitesimally greater than 1. Morally, whatever the closed adic disc is, it should not contain any points which think that T is greater than 1, and so we need to modify our model spaces a little.

1.6 Integral subrings

We need a few basic definitions.

Definition 1.6.1. A subset S of a topological ring A is *bounded* if for all open neighborhoods U of 0, there exists an open neighborhood V of 0 such that $VS \subset U$. An element $f \in A$ is *power-bounded* if $\{f^n\} \subset A$ is bounded. Let A° be the subring of power-bounded elements.

Definition 1.6.2. Let A be a Huber ring. A subring $A^+ \subset A$ is a *ring of integral elements* if it is open and integrally closed and $A^+ \subset A^\circ$. A *Huber pair*[2] is a pair (A, A^+), where A is Huber and and $A^+ \subset A$ is a ring of integral elements. Given a Huber pair, we let $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ be the subset of continuous valuations x for which $|f(x)| \leq 1$ for all $f \in A^+$.

The Huber ring A is *uniform* if A° is bounded, or equivalently if A° is a ring of definition.

Thus the closed adic disc should be $\mathrm{Spa}(A, A^+)$, where $A = \mathbf{Q}_p\langle T\rangle$ and

[2]Called an *affinoid algebra* in [Hub94].

$A^+ = A^\circ = \mathbf{Z}_p\langle T\rangle$. But one could also define an integral subring

$$A^{++} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in A^+ \;\middle|\; |a_n| < 1 \text{ for all } n \geq 1 \right\}.$$

We have $A^{++} \subset A^+$, and so $\mathrm{Spa}(A, A^+) \subset \mathrm{Spa}(A, A^{++})$. In fact the complement of $\mathrm{Spa}(A, A^+)$ in $\mathrm{Spa}(A, A^{++})$ is the single point x^+ from our discussion above. Furthermore, if we embed $\mathrm{Spa}(A, A^+)$ into an adic closed disc of larger radius, then it will be an *open* subset of the larger disc, and its closure will be $\mathrm{Spa}(A, A^{++})$.

1.7 The classification of points in the adic unit disc

Suppose C is a nonarchimedean field which is algebraically closed, and suppose that $\alpha \mapsto |\alpha|$ is an absolute value inducing the topology on C. The classification of points in $X = \mathrm{Spa}(C\langle T\rangle, C^\circ\langle T\rangle)$ is discussed in various places (for instance [Sch12]); since it is so helpful for the understanding of the theory, we review it here. The points of X are divided into five types; we warn that this division into types breaks down for other adic spaces. (The reader is invited to attempt such a classification for a Tate algebra in two variables – it gets complicated very quickly. Generally one may work with adic spaces without consciously knowing what each point looks like.)

1. Points of Type 1 correspond to elements $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$. The corresponding continuous valuation is $f \mapsto |f(\alpha)|$.

2. Points of Type 2 and 3, also called Gauss points, correspond to closed discs $D = D(\alpha, r)$. Here $\alpha \in \mathbf{C}$ has $|\alpha| \leq 1$, $0 < r \leq 1$ is a real number, and $D = \left\{ \beta \in \mathbf{C} \;\middle|\; |\alpha - \beta| \leq r \right\}$. The corresponding valuation is

$$f \mapsto \sup_{\beta \in D} |f(\beta)|.$$

   Explicitly, if we expand f as a series in $T - \alpha$, say $f(T) = \sum_{n=0}^{\infty} a_n (T - \alpha)^n$, then this works out to be $\sup_n |a_n| r^n$.

   If r belongs to $|C|$, then the point is Type 2; otherwise it is Type 3.

3. Points of Type 4 appear because of the strange phenomenon that C may not be *spherically complete*. That is, there may be a descending sequence of closed discs $D_1 \supset D_2 \supset \cdots$ with empty intersection. (For instance, this occurs when $C = \mathbf{C}_p$.) The corresponding continuous valuation is $f \mapsto \inf_i \sup_{\beta \in D_i} |f(\beta)|$.

4. Points of Type 5 have rank 2. For each $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$ each $0 < r < 1$, and each sign $\pm$ (excluding the positive sign if $r = 1$), we let $\Gamma = \mathbf{R}_{>0} \times \gamma^{\mathbf{Z}}$ be the ordered abelian group generated by $\mathbf{R}_{>0}$ and an element γ which

is infinitesimally less than or greater than r, depending on the sign. The corresponding continuous valuation is

$$\sum_{n=0}^{\infty} a_n(T-\alpha)^n \mapsto \sup_n |a_n| \gamma^n.$$

If C has value group $\mathbf{R}_{>0}$, then there are no points of Type 3. If C is spherically complete, then there are no points of Type 4 either: every descending sequence of closed discs has an intersection which is either itself a closed disc or a single point.

The only non-closed points in X are the Type 2 points, which correspond to discs D: the closure of such a point contains all Type 5 labeled with a triple $(\alpha, r, \pm)$, where $D = D(\alpha, r)$.

1.8 The structure presheaf, and the definition of an adic space

In the contruction of affine schemes, one starts with a ring A, defines the topological space $X = \operatorname{Spec} A$, and then defines the structure sheaf $\mathcal{O}_X$ this way: there is a basis of open sets of the form $U_f = \{x | f(x) \neq 0\}$ for $f \in A$, and one puts $\mathcal{O}_X(U_f) = A[1/f]$; it is easy enough to check that there is a unique sheaf of rings $\mathcal{O}_X$ with this property. (Here we use the notational convention that if x corresponds to a prime ideal $\mathfrak{p} \subset A$, then $f(x)$ is the image of x in the residue field of $\mathfrak{p}$.) The idea behind this definition is that U_f should be an affine scheme in its own right, namely $\operatorname{Spec} A[1/f]$. The key observation here is that $\operatorname{Spec} A[1/f] \to \operatorname{Spec} A$ is an open immersion with image U_f, and is universal for this property in the sense that for any A-algebra B, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ factors through U_f if and only if $A \to B$ factors through $A[1/f]$.

It is somewhat more subtle to define $\mathcal{O}_X$ for $X = \operatorname{Spa}(A, A^+)$, where (A, A^+) is a Huber pair. We single out a class of open sets called rational subsets.

Definition 1.8.1. Let $s_1, \ldots, s_n \in A$ and let $T_1, \ldots, T_n \subset A$ be finite subsets such that $T_i A \subset A$ is open for all i. We define a subset

$$U\left(\left\{\frac{T_i}{s_i}\right\}\right) = U\left(\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right) = \{x \in X |\, |t_i(x)| \leq |s_i(x)| \neq 0, \text{ for all } t_i \in T_i\}$$

This is open because it is an intersection of a finite collection of the sort of opens which generate the topology on X. Subsets of this form are called *rational subsets.*

Note that a finite intersection of rational subsets is again rational, just by concatenating the data that define the individual rational subsets.

The following theorem shows that rational subsets are themselves adic spectra.

Theorem 1.8.2. *Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ factors over U, and is universal for such maps. Moreover this map is a homeomorphism onto U. In particular, U is quasi-compact.*

Proof. (Sketch.) Choose s_i and T_i such that $U = U(\{T_i/s_i\})$. Choose $A_0 \subset A$ a ring of definition, $I \subset A_0$ a finitely generated ideal of definition. Take $(A, A^+) \to (B, B^+)$ such that $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors over U. Then

1. The s_i are invertible in B, so that we get a map $A[\{1/s_i\}] \to B$.
2. All t_i/s_i are of $|\cdot| \leq 1$ everywhere on $\mathrm{Spa}(B, B^+)$, so that $t_i/s_i \in B^+ \subset B^\circ$.
3. Since B° is the inductive limit of the rings of definition B_0, we can choose a B_0 which contains all t_i/s_i. We get a map

$$A_0[t_i/s_i | i = 1, \ldots, n, t_i \in T_i] \to B_0.$$

   Endow $A_0[\{t_i/s_i\}]$ with the $IA_0[\{t_i/s_i\}]$-adic topology.

Lemma 1.8.3. *This defines a ring topology on $A[\{1/s_i\}]$ making $A_0[\{t_i/s_i\}]$ an open subring.*

The crucial point is to show that there exists n such that $\frac{1}{s_i} I^n \subset A_0[\{t_i/s_i\}]$, so that multiplication by $1/s_i$ can be continuous. It is enough to show that $I^n \subset T_i A_0$.

Lemma 1.8.4. *If $T \subset A$ is a subset such that $TA \subset A$ is open, then TA_0 is open.*

Proof. After replacing I with some power we may assume that $I \subset TA$. Write $I = (f_1, \ldots, f_k)$. There exists a finite set R such that $f_1, \ldots, f_k \in TR$.

Since I is topologically nilpotent, there exists n such that $RI^n \subset A_0$. Then for all $i = 1, \ldots, k$, $f_i I^n \subset TRI^n \subset TA_0$. Sum this over all i and conlude that $I^{n+1} \subset TA_0$. □

Back to the proof of the proposition. We have $A[\{1/s_i\}]$, a (non-complete) Huber ring. Let $A[\{1/s_i\}]^+$ be the integral closure of the image of $A^+[\{t/s_i\}]$ in $A[\{1/s_i\}]$.

Let $(A\langle\{T_i/s_i\}\rangle, A\langle\{T_i/s_i\}\rangle^+)$ be its completion, a Huber pair. This has the desired universal property.

For the claim that Spa of this pair is homeomorphic to U: Use that Spa doesn't change under completion. (Also that the operation of taking the integral closure doesn't change much, either.) □

Definition 1.8.5. Define a presheaf $\mathcal{O}_X$ of topological rings on $\mathrm{Spa}(A, A^+)$: If $U \subset X$ is rational, $\mathcal{O}_X(U)$ is as in the theorem. On a general open $W \subset X$, we put

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U).$$

One defines $\mathcal{O}_X^+$ similarly.

Proposition 1.8.6. *For all $U \subset \mathrm{Spa}(A, A^+)$,*

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \,|\, |f(x)| \le 1, \text{ all } x \in U\}.$$

In particular $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is.

Theorem 1.8.7 ([Hub94])**.** *$\mathcal{O}_X$ is a sheaf in the following situations.*

1. *(Schemes) A is discrete.*
2. *(Formal schemes) A is finitely generated (as an algebra) over a noetherian ring of definition. This includes the case when X comes from a noetherian formal scheme.*
3. *(Rigid spaces) A is Tate*[3] *and* strongly noetherian*: the rings*

$$A\langle X_1, \dots, X_n\rangle = \left\{ \sum_{\underline{i}=(i_1,\dots,i_n)\ge 0} a_{\underline{i}} T^{\underline{i}} \;\middle|\; a_{\underline{i}} \in A,\ a_{\underline{i}} \to 0 \right\}$$

are noetherian for all $n \ge 0$.

We can now define the category of adic spaces.

Definition 1.8.8. An *adic space* consists of a topological space X, a sheaf of topological rings $\mathcal{O}_X$, and the data of a continuous valuation on $\mathcal{O}_{X,x}$ for each $x \in X$. We require that X be covered by open subsets of the form $\mathrm{Spa}(A, A^+)$, where each (A, A^+) is a sheafy Huber pair.

Example 1.8.9 (The adic open disc)**.** Let $A = \mathbf{Z}_p[\![T]\!]$. Since A is its own ring of definition and is noetherian, (A, A) is sheaf and $\mathrm{Spa}(A, A)$ is an adic space. We have a morphism $\mathrm{Spa}(A, A) \to \mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$. The latter is a two-point space, with generic point $\eta = \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. The generic fiber of $\mathrm{Spa}(A, A)$ is $\mathrm{Spa}(A, A)_\eta$, the preimage of η. It is worthwhile to study this space in detail.

Let $x \in \mathrm{Spa}(A, A)_\eta$. We have $|p(x)| \ne 0$. We also know that since p and T are topologically nilpotent in A, $|T(x)|^n \to 0$ as $n \to \infty$. Therefore there exists $n \ge 0$ with $|T^n(x)| \le |p(x)|$. Therefore x lies in the rational subset $U(T^n/p)$. From this we see that the increasing sequence of rational subsets $U(T^n/p)$ covers $\mathrm{Spa}(A, A)_\eta$. Since this covering has no finite subcovering, we can conclude that $\mathrm{Spa}(A, A)_\eta$ is not quasi-compact.

2 Perfectoid fields

We are now going to take a sudden change of direction to talk about perfectoid fields. The idea is that perfectoid fields are the one-point perfectoid spaces, so

[3] This means that A contains an invertible topologically nilpotent element.

they are rather a prerequisite to study perfectoid spaces in general. Besides, perfectoid fields have an interesting history, even if the name and formal definition did not appear until [Sch12] and [KL].

A class of perfectoid fields plays a crucial role in Tate's study of p-divisible groups [Tat67]. Let K be the fraction field of a DVR with perfect residue field of characteristic p (*e.g.*, a finite extension of $\mathbf{Q}_p$). Tate considered a tower of Galois extensions K_n/K satisfying the conditions (a) $\mathrm{Gal}(K_n/K) \cong (\mathbf{Z}/p^n\mathbf{Z})^h$ for some $h \geq 1$ and (b) K_n/K is totally ramified. (For Tate, such a tower came by adjoining the torsion in a p-divisible group.) Let $K_\infty = \cup_n K_n$ and let $\widehat{K}_\infty$ be its completion.

Tate proved some basic facts about the cohomology of C as a $\mathrm{Gal}(\overline{K}/K)$-module, using K_∞ as an intermediary. (The ultimate goal was to prove a p-adic Hodge decomposition for p-divisible groups and abelian varieties.) Along the way he proved a curious fact: if L/K_∞ is a finite extension, then the ideal of K_∞° generated by traces of elements of L° contains the maximal ideal $\mathfrak{m}_{K_\infty}$ of K_∞°. (Thus it is either $\mathfrak{m}_{K_\infty}$ or else it is all of K_∞°.) Now, if L were instead a finite extension of K, then this ideal of traces is related to the different ideal of L/K, and measures its ramification: the bigger the ideal, the less ramified L/K is. Tate's result is that any finite extension of K_∞ is *almost unramified*, or put another way, the corresponding extension of K_∞° is *almost étale*.

The next work along these lines comes from Fontaine and Wintenberger [FW79]. They considered a more general infinite algebraic extension K_∞/K where (a) $\mathrm{Gal}(K_\infty/K)$ is a p-adic Lie group and (b) K_∞/K is totally ramified. Such extensions are called *strictly arithmetically profinite* (strictly APF). To such an extension, Fontaine and Wintenberger attached a nonarchimedean field X, the *field of norms*, whose multiplicative monoid is the inverse limit $\varprojlim K_n$, where the transition maps in the limit are norms. The field X has characteristic p; in fact it is a Laurent series field over the residue field of K. Rather surprisingly, we have an isomorphism of Galois groups $\mathrm{Gal}(\overline{X}/X) \cong \mathrm{Gal}(\overline{K}/K_\infty)$. This isomorphism is fundamental to the classification of p-adic Galois representations via (ϕ, Γ)-modules (see [Ked15] for a discussion of these) and the proof of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ [Col10].

The themes of almost étale extensions and passage to characteristic p are the hallmarks of perfectoid fields, which we now define.

Definition 2.0.10. Let K be a nonarchimedean field of residue characteristic p. K is a *perfectoid field* if (a) its value group is nondiscrete, and if (b) the pth power Frobenius map on K°/p is surjective.

Example 2.0.11.

1. The basic examples of perfectoid fields are the completions of $\mathbf{Q}_p(\mu_{p^\infty})$ and $\mathbf{Q}_p(p^{1/p^\infty})$. The completion of any APF extension is perfectoid.

2. One source of APF extensions (and therefore perfectoid fields) comes from *p-divisible formal group laws*. Let F be a local field with residue characteristic p and uniformizer π. Recall that a 1-dimensional formal group law over $\mathcal{O}_F$ is a power series $\mathcal{F}(X,Y) = X + Y + O(\deg 2)$ which satisfies the

axioms of an abelian group. Iterating $\mathcal{F}$ p times produces a power series $[p]_{\mathcal{F}}(T)$. If $[p]_{\mathcal{F}}(T)$ modulo π is nonzero, then $\mathcal{F}$ is p-divisible; in that case $[p]_{\mathcal{F}}(T) \mod \pi = g(T^{p^h})$ for some power series g and some maximal h, called the height of $\mathcal{F}$. The set of roots $\mathcal{F}[p^n]$ of $[p^n]_{\mathcal{F}}$ is isomorphic to $(\mathbf{Z}/p^n\mathbf{Z})^h$, and the extension $F(\mathcal{F}[p^n])/F$ has this Galois group. The composite $F_\infty = F(\mathcal{F}[p^\infty])$ is algebraic over F with $\mathrm{Gal}(F_\infty/F) \cong T_p\mathcal{F} \cong \mathbf{Z}_p^h$; this extension is APF, and therefore the completion of F_∞ is perfectoid.

3. If a nonarchimedean field has characteristic p, then it is perfectoid if and only if it is perfect. A basic example is $k(\!(t^{1/p^\infty})\!)$, where $k/\mathbf{F}_p$ is a perfect field: this is defined to be the completion of the perfection of $k(\!(t)\!)$. This example is rather fundamental: if K is a perfectoid field of characteristic p and residue field k, then K contains $k(\!(t^{1/p^\infty})\!)$, where t is any element of K with $0 < |t| < 1$.

2.1 Tilting

Let K be a perfectoid field with absolute value $|\ |$. We let $K^\circ = \{|x| \leq 1\}$ be its ring of integers.

We define

$$K^\flat = \varprojlim K,$$

where the transition map is $x \mapsto x^p$. Thus elements of $K^\flat$ are sequences $(a_0, a_1, \dots)$ of elements of K with $a_n^p = a_{n-1}$ for all $n \geq 1$. (If K has characteristic p, then trivially $K^\flat \cong K$; this operation is only interesting in characteristic 0.) *A priori* $K^\flat$ is a topological multiplicative monoid. We define an addition law on $K^\flat$ by the rule $(a_n) + (b_n) = (c_n)$, where

$$c_n = \lim_{m\to\infty} (a_{m+n} + b_{m+n})^{p^n}. \tag{2.1.1}$$

It isn't hard to check that the limit exists (here we use the fact that K is complete). It can be verified directly that $K^\flat$ is a field, but the easiest route is to pass to the quotient K°/p. The reduction map $K^\circ \to K^\circ/p$ induces a map of topological multiplicative monoids

$$\varprojlim_{x\mapsto x^p} K^\circ \to \varprojlim_{x\mapsto x^p} K^\circ/p.$$

Now one observes that this map is an isomorphism; the inverse sends a sequence $(a_n \mod p)$ to (b_n), where

$$b_n = \lim_{m\to\infty} a_{m+n}^{p^n}.$$

Therefore $\varprojlim K^\circ$ inherits the structure of a ring, with addition law as in (2.1.1); its fraction field is $K^\flat$. Let $f \mapsto f^\sharp$ denote the projection map $K^\flat \to K$ which sends (a_n) to a_0. We define an absolute value on $K^\flat$ by $|f| = \left|f^\sharp\right|$. One checks that this is a nontrivial nonarchmedean absolute value inducing the topology on $K^\flat$, and that $K^\flat$ is complete with respect to it. Finally, the very definition of $K^\flat$ shows that it is perfect of characteristic p. Therefore $K^\flat$ is a perfectoid field of characteristic p; it is called the *tilt* of K.

The perfectoid field $K^\flat$ contains a pseudo-uniformizer ϖ with $|\varpi| = |p|$. An important observation is that $K^{\flat\circ} \cong \varprojlim_{x \mapsto x^p} K^\circ/p$, and that

$$K^{\flat\circ}/\varpi \cong K^\flat/p.$$

Example 2.1.1. 1. Let $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$. Then $K^\flat$ contains the element $t = (p, p^{1/p}, \dots)$ with $|t| = |p|$. Thus t is a pseudo-uniformizer of $K^\flat$, and since $K^\flat$ is perfectoid, $K^\flat$ contains $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$ (as remarked in Example 2.0.11). In fact $K^\flat = \mathbf{F}_p(\!(t^{1/p^\infty})\!)$. To see this, observe that $K^\circ/p = \mathbf{Z}_p[p^{1/p^\infty}]/p \cong \mathbf{F}_p[t^{1/p^\infty}]/t$, and apply $\varprojlim$ along $x \mapsto x^p$ to both sides.

2. If $K = \mathbf{Q}_p(\mu_{p^\infty})^\wedge$, then $K^\flat$ (considered as the fraction field of $\varprojlim K^\circ/p$) contains the element $t = (1-\zeta_p, 1-\zeta_{p^2}, \dots))$, and then once again $K^\flat = \mathbf{F}_p(\!(t^{1/p^\infty})\!)$. In fact if K is the completion of any APF extension of a p-adic field (see Example 2.0.11), then $K^\flat \cong k(\!(t^{1/p^\infty})\!)$, where k is the residue field of K.

2.2 The tilting equivalence for perfectoid fields

For a perfectoid field K, the structures of K and $K^\flat$ seem quite different: of course their characteristics are different, and even though there is a multiplicative map $K^\flat \to K$ ($f \mapsto f^\sharp$), this is far from being surjective in general. Nonetheless we will encounter a family of theorems known as *tilting equivalences* which relate the arithmetic of a perfectoid object and its tilt. The most basic tilting equivalence concerns the Galois groups of perfectoid fields.

Theorem 2.2.1. *Let K be a perfectoid field. Then for any finite extension L/K (necessarily separable), L is also a perfectoid field, and $L^\flat/K^\flat$ is a finite extension of the same degree as L/K. The categories of finite extensions of K and $K^\flat$ are equivalent, via $L \mapsto L^\flat$. Consequently there is an isomorphism $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K}^\flat/K^\flat)$.*

Example 2.2.2. Theorem 2.2.1 allows us to describe the tilt of the perfectoid field $\mathbf{C}_p = \overline{\mathbf{Q}}_p^\wedge$. Since $\mathbf{C}_p$ is the completion of the algebraic closure of the perfectoid field $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, $\mathbf{C}_p^\flat$ is the completion of the algebraic closure of $K^\flat \cong \mathbf{F}_p(\!(t^{1/p^\infty})\!)$.

There is an explicit inverse to $L \mapsto L^\flat$ which merits discussion. Since we want to move from characteristic p to characteristic 0, it is not surprising that Witt vectors appear. Recall that for a perfect ring R of characteristic p, we have the ring of Witt vectors $W(R)$. This is a ring which is separated and complete for the p-adic topology; there is a surjective morphism $W(R) \to R$ which admits a multiplicative (not additive) section $R \to W(R)$, written $x \mapsto [x]$. $W(R)$ has the following universal property: For a p-adically complete ring S and a map of multiplicative monoids $R \mapsto S$ for which the composition $R \to S \to S/p$ is a ring homomorphism, there exists a unique continuous ring homomorphism $W(R) \to S$ such that the diagram

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ W(R) \arrow[r] & S \end{tikzcd}$$

commutes. Elements of $W(R)$ may be written uniquely as formal power series $[x_0] + [x_1]p + [x_n]p^2 + \dots$.

In the context of Theorem 2.2.1, we have the perfect ring $K^{\flat\circ}$, the p-adically complete ring K°; the ring homomorphism $K^{\flat\circ} \to K^\circ/p$ factors through a map of multiplicative monoids $K^{\flat\circ} \to K^\circ$, namely $f \mapsto f^\sharp$. Therefore by the universal property of Witt vectors, there exists a unique continuous ring homomorphism $\theta\colon W(K^{\flat\circ}) \to K^\circ$ satisfying $\theta([f]) = f^\sharp$. Since p is invertible in K°, θ extends to a homomorphism of $\mathbf{Q}_p$-algebras $W(K^{\flat\circ})[1/p] \to K$, which we continue to call θ.

Lemma 2.2.3. *The homomorphism $\theta\colon W(K^{\flat\circ})[1/p] \to K$ is surjective. Its kernel is a principal ideal, generated by an element of the form $p + [\varpi]\alpha$, where $\varpi \in K^\flat$ is a pseudo-uniformizer and $\alpha \in W(K^{\flat\circ})$ is a unit.*

We can now describe the inverse to the tilting functor $L \mapsto L^\flat$ in Theorem 2.2.1. Suppose $M/K^\flat$ is a finite extension. Then M° is perfect, and $W(M^\circ)$ is an algebra over $W(K^{\flat\circ})$. We put

$$M^\sharp = W(M^\circ) \otimes_{W(K^{\flat\circ}),\theta} K.$$

Then $M^\sharp$ is a perfectoid field, and there is a multiplicative map $M \to M^\sharp$ given by $f \mapsto f^\sharp = [f] \otimes 1$. There is an isomorphism $M \cong M^{\sharp\flat}$ given by $f \mapsto (f^\sharp, (f^{1/p})^\sharp, \dots)$.

2.3 Untilts of a perfectoid field of characteristic p

Let K be a perfectoid field of characteristic p. Does there always exists a characteristic 0 perfectoid field whose tilt is K, and if so, can one describe the set of such "untilts"? Certainly an untilt is not unique in general: In Example 2.1.1 we saw that at least two distinct perfectoid fields have tilt isomorphic to $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$.

Definition 2.3.1. An *untilt* of K is a pair $(K^\sharp, \iota)$, where $K^\sharp$ is a perfectoid field and $\iota\colon K \xrightarrow{\sim} K^{\sharp\flat}$ is an isomorphism.

We remark that our definition includes K as an untilt of itself, since after all $K^\flat = K$.

Given an untilt $(K^\sharp, \iota)$, the multiplicative map $K^\circ \xrightarrow{\iota} K^{\sharp\flat\circ} \xrightarrow{\sharp} K^{\sharp\circ}$ induces a surjective ring homomorphism

$$\begin{aligned} \theta_{K^\sharp}\colon W(K^\circ) &\to K^{\sharp\circ} \\ \sum_{n=0}^{\infty}[f_n]p^n &\mapsto \sum_{n=0}^{\infty} f_n^\sharp p^n. \end{aligned}$$

Then $\ker\theta_{K^\sharp}$ is an ideal which is *primitive of degree 1*: this means that I is generated by an element of the form $p + [\varpi]\alpha$, where $\varpi \in K$ is a pseudo-uniformizer and $\alpha \in W(K^\circ)$ is a unit.

Theorem 2.3.2. *The map $I \mapsto (W(K^\circ)/I)[1/p]$ is a bijection between the set of primitive ideals of $W(K^\circ)$ of degree 1, and the set of isomorphism classes of untilts of K.*

Note that $I = (p)$ is the unique ideal which produces the trivial untilt K.

Theorem 2.3.2 suggests that untilts of K are parametrized by some kind of geometric object which is related to $W(K^\circ)$. An approximation to this object might be $\mathrm{MaxSpec}\, W(K^\circ)[1/p[\varpi]]$, where ϖ is a pseudo-uniformizer of K. After all, every untilt $K^\sharp$ of K induces a surjective ring homomorphism $\theta_{K^\sharp}\colon W(K^\circ)[1/p] \to K^\sharp$ for which $\theta_{K^\sharp}([\varpi]) = \varpi^\sharp$ is a pseudo-uniformizer of $K^\sharp$ (and is therefore nonzero); thus $\ker\theta_{K^\sharp}$ determines a maximal ideal of $W(K^\circ)[1/p[\varpi]]$. However, $\mathrm{MaxSpec}\, W(K^\circ)[1/p[\varpi]]$ isn't a rigid-analytic space, as $W(K^\circ)[1/p[\varpi]]$ isn't an affinoid algebra.

The approach of Fargues and Fontaine requires looking at $W(K^\circ)$ as a ring equipped with its $([\varpi], p)$-adic topology. (This is called the *weak topology* in [FF11].) This makes $W(K^\circ)$ into a Huber ring (with itself as ring of definition), and so we may make the following definition.

Definition 2.3.3 (The adic Fargues-Fontaine curves $\mathcal{Y}$ and $\mathcal{X}$)**.** Let

$$\mathcal{Y} = \mathrm{Spa}\, W(K^\circ) \backslash \{|p[\varpi] = 0|\},$$

where $\varpi \in K^\circ$ is a pseudo-uniformizer. The Frobenius automorphism on K° induces a properly discontinuous automorphism $\phi\colon \mathcal{Y} \to \mathcal{Y}$; we let $\mathcal{X} = \mathcal{Y}/\phi^{\mathbf{Z}}$.

Theorem 2.3.4 ([Ked])**.** *$\mathcal{Y}$ (and thus $\mathcal{X}$) are adic spaces.*

Theorem 2.3.5 ([FF11], Corollary 2.5.4)**.** *Suppose K is algebraically closed. Let $B = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. There is a bijection between the set of closed maximal ideals of B and the set of characteristic 0 untilts of K, given by $I \mapsto B/I$.*

This means that there is an embedding of the set of untilts of K into the set of closed points of $\mathcal{Y}$ (although this is far from being surjective).

2.4 Explicit parametrization of untilts by a formal $\mathbf{Q}_p$-vector space

Theorems 2.3.2 and 2.3.5 don't give particularly explicit paramatrizations for the set of untilts of a perfectoid field K. The problem is that, even though it is easy to exhibit elements of $W(K^\circ)$ which generate primitive ideals of degree 1, it is not easy to decide whether two such elements generate the same ideal.

We offer now a different perspective. For ease of exposition, we assume that $K = C$ is an algebraically closed perfectoid field of characteristic p. Suppose $(C^\sharp, \iota\colon C \to C^{\sharp\flat})$ is an untilt of C in characteristic 0. By Theorem 2.2.1, $C^\sharp$ is also algebraically closed. Therefore it contains a compatible system of primitive pth power roots of unity: $1, \zeta_p, \zeta_{p^2}, \dots$. Let $\varepsilon = \iota^{-1}(1, \zeta_p, \zeta_{p^2}, \dots) \in C$. The idea is that the element $\varepsilon \in C$ is an invariant of the untilt $C^\sharp$. Now, this element isn't quite well-defined, because there is an ambiguity in the choice of system of roots of unity.

Before resolving this ambiguity, let's introduce some notation. Let $H = \widehat{\mathbf{G}}_m$ be the formal multiplicative group over $\mathbf{Z}_p$: this is the completion of $\mathbf{G}_m$ along the origin. It is perhaps easiest to think of H as a functor from adic $\mathbf{Z}_p$-algebras to $\mathbf{Z}_p$-modules, which sends R to the abelian group $1+R^{\circ\circ}$ under multiplication. This group gets its $\mathbf{Z}_p$-module structure this way: for $a \in \mathbf{Z}_p$, the action of a sends x to x^a (defined using power series). The underlying formal scheme of H is isomorphic to $\operatorname{Spf} \mathbf{Z}_p[\![T]\!]$. We also define the *universal cover*

$$\widetilde{H} = \varprojlim_{x \mapsto x^p} H,$$

so that for an adic $\mathbf{Z}_p$-algebra R, $\widetilde{H}(R)$ is the $\mathbf{Q}_p$*-vector space* $\varprojlim_{x\mapsto x^p}(1+R^{\circ\circ})$. There is a reduction map

$$\widetilde{H}(R) \to \widetilde{H}(R/p), \tag{2.4.1}$$

which one checks is an isomorphism, rather along the lines of proof that $K^{\flat\circ} \cong \varprojlim_{x\mapsto x^p} K^\circ/p$ for a perfectoid field K. Consequently

$$\widetilde{H}(R) \cong \widetilde{H}(R/p) \cong \varprojlim_{x\mapsto x^p} R^{\circ\circ}/p \cong \varprojlim_{x\mapsto x^p} R^{\circ\circ},$$

so that $\widetilde{H}$ is representable by the formal scheme $\operatorname{Spf} \mathbf{Z}_p[\![T^{1/p^\infty}]\!]$. Thus $\widetilde{H}$ is a $\mathbf{Q}_p$-vector space object in the category of formal schemes, which is to say, a *formal* $\mathbf{Q}_p$*-vector space*. Whenever K is a perfectoid field, $\widetilde{H}(K^\circ) \cong \widetilde{H}(K^{\flat\circ}) \cong H(K^{\flat\circ})$ (the last isomorphism holds because $K^\flat$ is perfect).

Given a characteristic 0 untilt $C^\sharp$ of C, we obtain a nonzero element $\varepsilon \in \widetilde{H}(C^\circ)$ defined as the image of $(1, \zeta_p, \zeta_{p^2}, \dots)$ under $\widetilde{H}(C^{\sharp\circ}) \cong \widetilde{H}(C^\circ)$. This element is well-defined up to translation by an element of $\mathbf{Z}_p^\times$. Note that $\theta_{C^\sharp}([\varepsilon^{1/p^n}]) = \zeta_{p^n}$ for all $n \geq 0$; therefore the element

$$\xi = \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = [1] + [\varepsilon] + \cdots + [\varepsilon^{(p-1)/p}]$$

lies in the kernel of $\theta_{C^\sharp}$. One checks that the ideal (ξ) is primitive of degree 1, and therefore $C^\sharp$ corresponds to the ideal (ξ) under the bijection in Theorem 2.3.2.

On the other hand, we could start with a nonzero element $\varepsilon \in \widetilde{H}(C^\circ)$, form ξ as above, and from this construct the untilt $C^\sharp = W(C^\circ)[1/p]/(\xi)$. Therefore:

Theorem 2.4.1. *The map $C^\sharp \mapsto \varepsilon$ gives a bijection between equivalence classes (respectively, Frobenius equivalence classes) of characteristic 0 untilts of C and the quotient $(\widetilde{H}(C^\circ)\setminus\{0\})/\mathbf{Z}_p^\times$ (respectively, $(\widetilde{H}(C^\circ)\setminus\{0\})/\mathbf{Q}_p^\times$).*

2.5 The schematic Fargues-Fontaine curve

In this construction, we started with an element $[\varepsilon]$ with $\theta_{C^\sharp}([\varepsilon]) = 1$, and used it to concoct an element ξ satisfying $\theta_{C^\sharp}(\xi) = 0$. There are of course many way we could have done this, but informally speaking, the canonical way to turn 1 into 0 is with a logarithm. Define

$$t = \log[\varepsilon] = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n} \in B = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}).$$

One has to check here that the sum converges in the Fréchet topology on B, but this is just a matter of checking that $\left|[\varepsilon]-1\right|_r < 1$ for all $0 < r < \infty$. Then formally we have

$$\phi(t) = \log\phi([\varepsilon]) = \log[\varepsilon^p] = p\log[\varepsilon] = pt,$$

and so t lies in the $\mathbf{Q}_p$-vector space $B^{\phi=p}$ consisting of elements that exhibit this behavior.

In general we can take any element $\varepsilon \in \widetilde{H}(C^\circ)$ and produce $\log[\varepsilon] \in B^{\phi=p}$. Then for any untilt $C^\sharp$ of C, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{H}(C^\circ) \arrow[r, "{\log[\cdot]}"] \arrow[d, "\cong"'] & B^{\phi=p} \arrow[d, "\theta_{C^\sharp}"] \\
\widetilde{H}(C^{\sharp\circ}) \arrow[r, "(x_n)\mapsto\log x_0"'] & C^\sharp.
\end{tikzcd}$$

Theorem 2.5.1 ([FF11])**.** *The map $\varepsilon \mapsto \log[\varepsilon]$ defines an isomorphism of $\mathbf{Q}_p$-vector spaces $\widetilde{H}(C^\circ) \cong B^{\phi=p}$. Furthermore, for each $t \in B^{\phi=p}\backslash 0$, there is a unique Frobenius-equivalence class of untilts $C^\sharp$ such that $\theta_{C^\sharp}(t) = 0$. Therefore there is a bijection between the set of Frobenius-equivalence classes of untilts of $C^\sharp$ and the set $(B^{\phi=p}\setminus\{0\})/\mathbf{Q}_p^\times$.*

Recall that $\mathcal{Y}$ is the adic space which is (informally) supposed to parametrize equivalence classes of untilts of C, and $\mathcal{X} = \mathcal{Y}/\phi^{\mathbf{Z}}$ parametrizes Frobenius-equivalents classes of untilts. A key insight of [FF11] is $\mathcal{X}$ resembles a proper smooth analytic curve, and so should be the analytification of an algebraic curve, just as the Tate curve $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$ is the analytification of an elliptic curve. In this context, the usual thing to do is to find a very ample line bundle $\mathcal{L}$ on $\mathcal{X}$, and then define

$$X = \operatorname{Proj}\bigoplus_{n\geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes n}).$$

For the Fargues-Fontaine curve, the requisite line bundle $\mathcal{L}$ on $\mathcal{X}$ should pull back to a line bundle on $\mathcal{Y}$ which is ϕ-equivariant. And so we define a free line bundle $\mathcal{O}_{\mathcal{Y}}e$, with the ϕ-equivariance defined by $\phi(e) = p^{-1}e$. This $\mathcal{O}_{\mathcal{Y}}e$ descends to a line bundle on $\mathcal{X}$, which we call $\mathcal{O}_{\mathcal{X}}(1)$. For $n \in \mathbf{Z}$ we define $\mathcal{O}_{\mathcal{X}}(n) = \mathcal{O}_{\mathcal{X}}^{\otimes n}$ (with the usual convention regarding negative n).

The algebraic Fargues-Fontaine curve is defined by declaring $\mathcal{O}_{\mathcal{X}}(1)$ to be very ample. Note that

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)) \cong H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}e^{\otimes n})^{\phi=1} \cong B^{\phi=p^n}$$

Definition 2.5.2 (The schematic Fargues-Fontaine curve)**.** Define $X = \operatorname{Proj} P$, where

$$P = \bigoplus_{d \geq 0} P_d, \text{ where } P_d = B^{\phi=p^d}.$$

Theorem 2.5.3. *1. $H^0(X, \mathcal{O}_X) = P_0 = B^{\phi=1}$ is exactly $\mathbf{Q}_p$.*

2. P is a graded factorial ring: the irreducible homogenous elements are exactly the nonzero elements of P_1, and for every $d \geq 1$, a nonzero element of P_d admits a factorization into irreducibles in P_1, unique up to units.

3. As a result, X is an integral Noetherian scheme of dimension 1, which admits a cover by spectra of Dedekind rings (in fact PIDs).

In these respects X resembles nothing so much as the projective line $\mathbf{P}^1 = \operatorname{Proj} \mathbf{Q}_p[X, Y]$, where $\mathbf{Q}_p[X, Y]$ is graded by degree. But unlike $\mathbf{P}^1$, X isn't finitely generated over any field.

Since X is an integral Noetherian scheme of dimension 1, it is the union of its generic point together with its set of closed points $|X|$. In light of Theorem 2.5.3, it is easy to describe the closed points: they correspond to nonzero homogenous prime ideals of P (other than the irrelevant ideal); since every homogenous element of P factors as a product of elements of P_1, every such ideal is generated by a nonzero element of P_1. Since $P^\times = \mathbf{Q}_p$, we find that $|X|$ is in bijection with $(P_1 \setminus \{0\})/\mathbf{Q}_p^\times$. Summing up our investigations of untilts of $C^\sharp$ gives the following theorem.

Theorem 2.5.4. *Let C be an algebraically closed perfectoid field of characteristic p. The following sets are in bijection:*

- *Frobenius-equivalence classes of untilts of C,*
- *Primitive ideals of $W(C^\circ)$ of degree 1, modulo the action of Frobenius,*
- $(\widetilde{H}(C^\circ) \setminus \{0\})/\mathbf{Q}_p^\times$,
- *Closed points of X.*

2.6 Universal covers of other p-divisible groups

What are the $\mathbf{Q}_p$-vector spaces $P_d = B^{\phi=p^d}$ for $d \geq 2$? It isn't clear! It is easy enough to exhibit elements of P_d; for $x \in C^{\circ\circ}$ the element

$$\sum_{n\in\mathbf{Z}} \frac{[x^{p^n}]}{p^{dn}}$$

belongs to P_d. However, it is probably not the case that all elements of P_d admit such a presentation, nor is it clear that such a presentation is unique.

The situation is better for the $\mathbf{Q}_p$-vector space $B^{\phi^h=p}$, where $h \geq 1$. As in the case $h=1$, this is isomorphic to the universal cover of a p-divisible formal group. Let $H_{1/h}/\breve{\mathbf{Z}}_p$ be the 1-dimensional formal group whose logarithm is

$$\log_{H_{1/h}}(T) = \sum_{n=1}^{\infty} T^{p^{hn}}/p^n.$$

This means that the underlying formal scheme of $H_{1/h}$ is $\operatorname{Spf} \breve{\mathbf{Z}}_p[\![T]\!]$, and is addition law $+_{H_{1/h}}$ is determined by the relation

$$\log_{H_{1/h}}(X +_{H_{1/h}} Y) = \log_{H_{1/h}}(X) + \log_{H_{1/h}}(Y)$$

as power series in $\breve{\mathbf{Q}}_p[\![X,Y]\!]$. Then $H_{1/h}\otimes_{\breve{\mathbf{Z}}_p}\overline{\mathbf{F}}_p$ has height h; in fact $[p]_{H_{1/h}}(T) \equiv T^{p^h} \pmod p$ (See [Haz12] for proofs of these assertions. $H_{1/h}$ is an example of a *p-typical* formal group.) Note that $H_{1/h}$ admits endomorphisms by the ring of integers $\mathbf{Z}_{p^h}$ in $\mathbf{Q}_{p^h}$, the subfield of $\breve{\mathbf{Q}}_p$ generated by roots of $X^{p^h} - X$. In fact $H_{1/h}$ is a Lubin-Tate formal $\mathbf{Z}_{p^h}$-module in the sense of [LT65].

Let

$$\widetilde{H}_{1/h} = \varprojlim_{x\mapsto [p]_{H_{1/h}}(x)} H_{1/h},$$

a priori as a functor from adic $\breve{\mathbf{Z}}_p$-algebras to $\mathbf{Q}_p$-vector spaces. Just as with H_1, one uses the congruence between $[p]_{H_{1/h}}$ and a power of Frobenius to show that for any adic $\breve{\mathbf{Z}}_p$-algebra R, we have isomorphisms

$$\widetilde{H}(R) \cong \widetilde{H}(R/p) \cong \varprojlim_{x\mapsto x^p} R^{\circ\circ}$$

is an isomorphism.

Applied to $R = W(C^\circ)$, the first isomorphism has inverse

$$\begin{array}{rcl} \widetilde{H}_{1/h}(C^\circ) & \to & \widetilde{H}_{1/h}(W(C^\circ)) \\ (x_n) & \mapsto & (y_n), \end{array}$$

where

$$y_n = \lim_{m\to\infty} p^m [x_{m+n}].$$

This isomorphism respects the action of Frobenius ϕ on either side, and therefore the identity $\phi^h = p$ holds in $\operatorname{End} \widetilde{H}_{1/h}(W(C^\circ))$, since it holds in $\operatorname{End} \widetilde{H}_{1/h}(C^\circ)$. Given an element $(x_n) \in \widetilde{H}_{1/h}(W(C^\circ))$, its logarithm $\log_{\widetilde{H}_{1/h}}(x_n)$ lies in $B^{\phi^h=p}$.

Theorem 2.6.1. *The map* $(x_n) \mapsto \log_{\widetilde{H}_{1/n}}([x_n])$ *gives an isomorphism* $\widetilde{H}(C^\circ) \to B^{\phi^h=p}$.

We can be quite explicit about this isomorphism. There is a commutative diagram

$$\begin{tikzcd}
\widetilde{H}_{1/h}(C^\circ) \arrow[r, "\sim"] \arrow[d, "\cong"'] & \widetilde{H}_{1/h}(W(C^\circ)) \arrow[r, "(x_n)\mapsto \log_{H_{1/h}}(x_0)"] & B^{\phi^h=p} \\
C^{\circ\circ} \arrow[urr] & &
\end{tikzcd}$$

in which all maps are isomorphisms; the diagonal map is

$$x \mapsto \lim_{m\to\infty} p^m \log_{H_{1/h}}[x^{1/p^m}] = \sum_{n\in\mathbf{Z}} \frac{[x^{p^{hn}}]}{p^n}.$$

Note that the latter expression visibly lies in $B^{\phi^h=p}$.

Theorem 2.6.1 generalizes to p-divisible groups of arbitrary height $h \geq 1$ and dimension $d \geq 0$, whenever $0 \leq d/h \leq 1$. The universal cover over such a formal group parametrizes $B^{\phi^h=p^d}$.

2.7 Interpretation in terms of vector bundles on X

A major theorem in [FF11] is the classification of vector bundles on the Fargues-Fontaine curve X. This classification is in terms of isocrystals.

Definition 2.7.1. Let k be a perfect field of characteristic p, and let $K = W(k)[1/p]$. Let $\phi\colon K \to K$ be the Frobenius automorphism. An *isocrystal* over k is a finite-dimensional K-vector space N together with an isomorphism $\phi_N\colon \phi^*N \to N$.

These form an abelian tensor category. When k is algebraically closed, the category of isocrystals k is well-understood. It is a semisimple category, with one irreducible object $N_{d/h}$ for each pair (d,h), where $d \in \mathbf{Z}$ and $h \geq 1$ are relatively prime. $N = N_{d/h}$ has basis $e, \phi_N(e), \dots, \phi_N^{h-1}(e)$, and $\phi_N^h(e) = p^d e$. Morphisms between the simple objects go as follows: There are no nonzero morphisms between distinct $N_{d/h}$s, and the endomorphism algebra of $N_{d/h}$ is a central division algebra over K of rank h^2, with invariant $d/h \in \mathbf{Q}/\mathbf{Z}$.

Given an isocrystal N over $\overline{\mathbf{F}}_p$, we can define the graded P-module

$$\widetilde{N} = \bigoplus_{d\geq 0} (B \otimes_{\breve{\mathbf{Q}}_p} N)^{\phi=p^d}.$$

Let $\mathcal{E}(N)$ be the corresponding $\mathcal{O}_X$-module. Then $\mathcal{E}(N)$ is a vector bundle of rank $\dim N$. For a relatively prime pair (d,h) with $d \geq 0$ and $h \geq 1$, we let $\mathcal{E}_{d/h} = \mathcal{E}(N_{-d/h})$. Then $H^0(X, \mathcal{E}_{d/h}) \cong B^{\phi^h = p^d}$.

Theorem 2.7.2 ([FF11])**.** *Every vector bundle on X is isomorphic to $\mathcal{E}(N)$ for an isocrystal N.*

It must be emphasized that the functor $N \mapsto \mathcal{E}(N)$ is far from being an equivalence of categories, as it is not full. Each nonzero element of $B^{\phi=p}$ gives a morphism $\mathcal{O}_X \to \mathcal{O}_X(1)$ which does not arise from a map of isocrystals. However it is the case that the map $\operatorname{End} N_{d/h} \to \operatorname{End} \mathcal{E}_{-d/h}$ is an isomorphism.

In the last section we saw that if $0 \leq d/h \leq 1$, there is a p-divisible group $\overline{H} = \overline{H}_{d/h}/\overline{\mathbf{F}}_p$ and a natural isomorphism $\widetilde{\overline{H}}(C^\circ) \cong H^0(X, \mathcal{O}_X(d/h))$. Let $C^\sharp$ be an untilt of C, and let H be a lift of $\overline{H}$ to C°; then there is an exact sequence of $\mathbf{Z}_p$-modules

$$0 \to H[p^\infty](C^{\sharp\circ}) \to H(C^{\sharp\circ}) \overset{\log_H}{\to} \operatorname{Lie} H \otimes C^\sharp \to 0,$$

Taking an inverse limit along multiplication by p (this is right-exact in this context) gives an exact sequence of $\mathbf{Q}_p$-vector spaces

$$0 \to VH(C^{\sharp\circ}) \to \widetilde{H}(C^{\sharp\circ}) \overset{\log_H}{\to} \operatorname{Lie} H \otimes C^\sharp \to 0. \qquad (2.7.1)$$

Note that the middle term, which is naturally isomorphic to $\overline{H}(C^\circ)$, does not depend on the lift H. Also note that this exact sequence presents a very large $\mathbf{Q}_p$-vector space as an extension of a finite-dimensional $C^\sharp$-vector space by a finite-dimensional $\mathbf{Q}_p$-vector space; this is an instance of the theory of *Banach-Colmez spaces*, which we will investigate systematically in the last lecture.

Let $x \in |X|$ be the closed point corresponding to the Frobenius equivalence class of $C^\sharp$ under Theorem 2.5.4. The exact sequence in (2.7.1) can be reinterpreted as the global sections of the following exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X \otimes_{\mathbf{Q}_p} VH \to \mathcal{E}(d/h) \to i_* \operatorname{Lie} C^\sharp \to 0,$$

where i is the inclusion of $x = \operatorname{Spec} C^\sharp$ into X.

3 Perfectoid spaces and diamonds

3.1 Definitions

Definition 3.1.1. Let A be a Huber ring. A is a *perfectoid ring* if the following conditions hold:

1. A is Tate, meaning it contains a pseudo-uniformizer (a topologically nilpotent unit),

2. A is uniform, meaning that $A^\circ \subset A$ is bounded,

3. A contains a pseudo-uniformizer ϖ such that $\varpi^p | p$ in A°, and such that the pth power map $A/\varpi \to A/\varpi^p$ is an isomorphism.

Remark 3.1.2. In the definition above it is always possible to choose a pseudo-uniformizer ϖ which contains arbitrary pth power roots.

Theorem 3.1.3. *Let (A, A^+) be a Huber pair, with A perfectoid. Then (A, A^+) is sheafy, so that $X = \mathrm{Spa}(A, A^+)$ is an adic space. Furthermore, $\mathcal{O}_X(U)$ is a perfectoid ring for every rational subset $U \subset X$.*

Theorem 3.1.3 shows that adic spaces $\mathrm{Spa}(R, R^+)$ with R perfectoid can serve as model spaces for the category of perfectoid spaces:

Definition 3.1.4. A *perfectoid space* is an adic space which is covered by affinoids of the form $\mathrm{Spa}(A, A^+)$, where A is perfectoid.

Example 3.1.5.
- If K is a perfectoid field and $K^+ \subset K$ is a ring of integral elements, then $\mathrm{Spa}(K, K^+)$ is a perfectoid space.
- (The perfectoid closed disc.) Let K be a perfectoid field. Let $A = K\langle T^{1/p^\infty}\rangle$; this is the completion of the polynomial algebra $K[T^{1/p^\infty}]$. Then A is a perfectoid ring, and $\mathrm{Spa}(A, A^\circ)$ is a perfectoid space.
- (The perfectoid open disc.) This time let $A = K^\circ[\![T^{1/p^\infty}]\!]$, the completion of $K[T^{1/p^\infty}]$ with respect to the (ϖ, T)-adic topology. Then A is not a perfectoid ring, because it is not Tate. But the generic fiber of $\mathrm{Spa}\, A$ over $\mathrm{Spa}\, K^\circ$ is perfectoid: it is covered by the affinoids $\mathrm{Spa}(A_n, A_n^\circ)$, where $A_n = K\langle (T/\varpi^{1/p^n})^{1/p^\infty}\rangle$.
- Let k be a perfect field of characteristic p with its discrete topology. Let $A = k[\![T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]\!]$. Then A is not a perfectoid ring (it is not Tate), but the analytic locus in $\mathrm{Spa}\, A$ is perfectoid. This is the locus of $x \in \mathrm{Spa}\, A$ for which $|T_i(x)| \neq 0$ for some i. Note that this perfectoid space does not live over any particular perfectoid field.
- (Some totally disconnected perfectoid spaces.) Let K be a perfectoid field and let S be a profinite set. Let $A = \mathrm{Cont}(S, K)$ be the ring of continuous maps $S \to K$. Then $\mathrm{Spa}(A, A^\circ)$ is a perfectoid space whose underlying topological space is S. The construction globalizes to the case that S is only locally profinite. If K is understood, we write $\underline{S}$ for the resulting perfectoid space.

The tilting operation we discussed in 2.1 extends to perfectoid spaces. For a perfectoid ring A with pseudo-uniformizer ϖ as in Remark 3.1.2, we define its tilt by

$$A^\flat = \left(\varprojlim_{x \mapsto x^p} A/\varpi \right) [1/\varpi^\flat],$$

where $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots)$. Then $A^\flat$ is a perfectoid ring of characteristic p.

Theorem 3.1.6. *Let A be a perfectoid ring.*

1. *There is a bijection $A^+ \mapsto A^{\flat +} = \varprojlim_{x\mapsto x^p} A^+/\varpi$ between rings of integral elements of A and $A^\flat$.*

2. *Given a ring of integral elements $A^+ \subset A$, there is a homeomorphism*

$$\begin{array}{rcl} \mathrm{Spa}(A, A^+) & \tilde{\to} & \mathrm{Spa}(A^\flat, A^{\flat +}) \\ x & \mapsto & x^\flat \end{array}$$

which sends x to $x^\flat$, defined by $\left|f(x^\flat)\right| = \left|f^\sharp(x)\right|$ for $f \in A^\flat$. This homeomorphism identifies rational subsets on either side.

3. *Let B be an étale A-algebra. Then B is also perfectoid. The categories of étale algebras over A and $A^\flat$ are equivalent, via $B \mapsto B^\flat$.*

Another source of perfectoid spaces comes from universal covers of p-divisible groups, which we discussed in (2.6). Let k be a perfect field of characteristic p, and let H be a p-divisible group over k. We have the universal cover $\widetilde{H} = \varprojlim_p H$, which we may consider as a functor from k-algebras to $\mathbf{Q}_p$-vector spaces. For now let us assume that H is connected, so that H is representable by $\mathrm{Spf}\, k[\![T_1, \dots, T_d]\!]$, where $d = \dim H$; then $\widetilde{H}$ is representable by a formal scheme $\mathrm{Spf}\, k[\![T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty}]\!]$. (This follows from the fact that multplication by p in H factors through Frobenius.)

If $\widetilde{H}^{\mathrm{ad}}$ be the corresponding adic space. Then $\widetilde{H}^{\mathrm{ad}}$ isn't quite a perfectoid space (it isn't analytic), but the punctured version $\widetilde{H}^{\mathrm{ad}}\backslash\{0\}$ is a perfectoid space, as in Example 3.1.5. If we want to create a perfectoid space version of $\widetilde{H}$ without puncturing it, we can introduce a separate perfectoid field K/k, and define $\widetilde{H}_K$ as the adic generic fiber of $\widetilde{H} \times_{\mathrm{Spec}\, k} \mathrm{Spf}\, K^\circ$. Then $\widetilde{H}_K$ is a $\mathbf{Q}_p$-vector space object in the category of perfectoid spaces over K.

A similar object exists in characteristic 0. Suppose now that K is a perfectoid field of characteristic 0, whose residue field contains k. We may simply define $\widetilde{H}_K$ as the perfectoid space over K whose tilt is $\widetilde{H}_{K^\flat}$. Then if G is any lift of $H \otimes_k K^\circ/p$ to K°, then we have the following functorial interpretation of $\widetilde{H}_K$: it is the sheafification of the functor $R \mapsto \widetilde{G}(R^\circ)$ on perfectoid K-algebras R. Note that this does not depend on the choice of lift G.

In fact, the requirement that H be formal is just a red herring; there is a functor $H \mapsto \widetilde{H}_K$ from the whole category of p-divisible groups over k to the category of perfectoid spaces with $\mathbf{Q}_p$-vector space structure. For instance if $H = \mathbf{Q}_p/\mathbf{Z}_p$ is the constant p-divisible group, then $\widetilde{H} = \underline{\mathbf{Q}}_p$ is the constant $\mathbf{Q}_p$-vector space.

Finally, if we allow K to be any nonarchimedean field with residue field containing k, then $\widetilde{H}_K$ will be a *pre-perfectoid space*, meaning that it becomes perfectoid after extending scalars from K to any perfectoid field.

3.2 Rapoport-Zink spaces

3.3 Untilts of perfectoid spaces in characteristic p, and a motivation for diamonds

Let X be a perfectoid space lying over $\mathrm{Spa}\,\mathbf{F}_p$. As we did with perfectoid fields, we can investigate the set of equivalence classes of untilts of X. What we would like is a *moduli space* M lying over $\mathrm{Spa}\,\mathbf{F}_p$, for which there is a natural bijection between the following sets:

- Morphisms $X \to M$, and
- Equivalence classes of untilts $X^\sharp \to \mathrm{Spa}\,\mathbf{Q}_p$.

This object M will ultimately be called $\mathrm{Spd}\,\mathbf{Q}_p$, where the d stands for *diamond*; it lives in a category of diamonds, which contains the category of perfectoid spaces as a full subcategory.

In the special case $X = \mathrm{Spa}\,C$ for a perfectoid field C of characteristic p, Theorem 2.5.4 gave the following parametrizations:

1. Equivalence classes of untilts correspond to primitive ideals $I \subset W(C^\circ)$ of degree 1, via $C^\sharp \mapsto \ker\theta_{C^\sharp}$.
2. Frobenius-equivalence classes of untilts correspond to closed points on the Fargues-Fontaine curve X constructed from C as in (2.5); the inverse map sends a point to its residue field.
3. Equivalence classes of untilts correspond to elements of $(\widetilde{H}(C^\circ)\backslash\{0\})/\mathbf{Q}_p^\times$.

The parametrization described in (1) relativizes quite easily. Suppose R is a perfectoid algebra in characteristic p, with pseudo-uniformizer ϖ. Then we have the Witt ring $W(R^\circ)$, equipped with its $(p,[\varpi])$-adic topology. A *primitive ideal of degree 1* in $W(R^\circ)$ is a principal ideal generated by an element of the form $\xi = \sum_{n=0}^\infty [x_n]p^n$, where $x_1 \in R^\circ$ is a unit and $x_0 \in R^\circ$ is a pseudo-uniformizer.

Theorem 3.3.1 ([Fon13]). *Ideals $I \subset W(R^\circ)$ which are primitive of degree 1 are in bijection with untilts of R, via $I \mapsto (W(R^\circ)/I)[1/p]$.*

As in the case with perfectoid fields, however, this does not give us much in the way of defining the object $\mathrm{Spd}\,\mathbf{Q}_p$; it is not easy to tell whether two such ideals are the same, given their generators.

Let's turn now to (2). It is easy to define a relative Fargues-Fontaine curve: given a perfectoid ring $R/\mathbf{F}_p$, first define the relative adic curve

$$\mathcal{Y}_R = \mathrm{Spa}\,W(R^\circ)\backslash\{|p[\varpi]| = 0\}$$

and the ring $B_R = H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R})$. Then B_R has an action of Frobenius ϕ, and we define the *relative schematic Fargues-Fontaine curve* as

$$X_R = \mathrm{Proj} \bigoplus_{d\geq 0} B_R^{\phi=p^d}.$$

However, when R is not a field, we cannot expect X_R to have any nice properties (*e.g.* it may not be Noetherian). Nor should we expect that closed points of X_R parametrize Frobenius-equivalence classes of untilts; after all, the residue field of such a point is a field, whereas an untilt $R^\sharp$ very well may not be.

Perhaps (3) has more promise. In the case that $R = C$ is an algebraically closed field of characteristic p, Theorem 2.5.4 says that untilts of C are in bijection with $(\widetilde{H}(C^\circ)\backslash\{0\})/\mathbf{Z}_p^\times$, where H is the formal multiplicative group over $\mathbf{F}_p$. Recall the construction: if $C^\sharp$ is an untilt, we choose a compatible system $(1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive pth power roots of 1 in $C^\sharp$, which determines a nonzero element of $\widetilde{H}(C^{\sharp\circ}) \cong \widetilde{H}(C^\circ)$, well-defined up to multiplication by an element of $\mathbf{Z}_p^\times$.

Let $\mathbf{Q}_p^{\mathrm{cycl}}$ be the completion of $\mathbf{Q}_p(\mu_{p^\infty})$. Then $\mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \cong \mathbf{Z}_p^\times$ acts continuously on $\mathbf{Q}_p^{\mathrm{cycl}}$.

Lemma 3.3.2. *There is an isomorphism* $\widetilde{H}^{\mathrm{ad}}\backslash\{0\} \cong \mathbf{Q}_p^{\mathrm{cycl},\flat}$ *which is* $\mathbf{Z}_p^\times$*-equivariant.*

This is not hard to check, as $\widetilde{H}^{\mathrm{ad}} \cong \mathrm{Spa}\,\mathbf{F}_p[\![t^{1/p^\infty}]\!]$, so $\widetilde{H}^{\mathrm{ad}}\backslash\{0\} \cong \mathrm{Spa}\,\mathbf{F}_p(\!(t^{1/p^\infty})\!)$; we have already identified the latter with $\mathbf{Q}_p^{\mathrm{cycl},\flat}$ in Example 2.1.1, so one only needs to check that the $\mathbf{Z}_p^\times$-action is preserved. There is a generalization of the lemma to Lubin-Tate extensions of any local field [Wei16, Proposition 3.5.3].

Therefore we can restate our parametrization of untilts of C as follows:

$$\{\text{Untilts of } C\} \cong \mathrm{Hom}(\mathbf{Q}_p^{\mathrm{cycl},\flat}, C)/\mathbf{Z}_p^\times = \mathrm{Hom}(\mathrm{Spa}\,C, \mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times. \tag{3.3.1}$$

This suggests that $\mathrm{Spd}\,\mathbf{Q}_p$ should be the quotient "$(\mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times$", which rather makes sense, since $(\mathbf{Q}_p^{\mathrm{cycl}})^{\mathbf{Z}_p^\times}$ is just $\mathbf{Q}_p$. But the quotient doesn't exist in the category of adic spaces; the subfield of $\mathbf{Q}_p^{\mathrm{cycl},\flat}$ fixed by $\mathbf{Z}_p^\times$ is just $\mathbf{F}_p$.

We would like to formulate a generalization of (3.3.1) for general perfectoid rings $R/\mathbf{F}_p$. Let's begin with the case that $R = K$ is a perfectoid field which isn't algebraically closed. Let $K^\sharp/\mathbf{Q}_p$ be an untilt. Then $K^\sharp$ might not contain all pth power roots of unity. For each $n \geq 1$, $K_n^\sharp := K^\sharp(\mu_{p^n})$ is a perfectoid field, whose tilt K_n is a finite Galois extension of K. Let $K_\infty^\sharp$ be the completion of $\cup_{n\geq 1} K_n^\sharp$; then $K_\infty^\sharp$ is perfectoid with tilt K_∞. Let $G = \mathrm{Gal}(K^\sharp(\mu_{p^\infty})/K^\sharp)$; then G acts continuously on K_∞. If we choose a compatible sequence of pth power roots of 1 in $K^{\sharp,\mathrm{cycl}}$, we obtain a nonzero element $\varepsilon\colon \widetilde{H}(K_\infty^{\sharp\circ}) \cong \widetilde{H}(K_\infty^{\sharp\flat\circ}) = \widetilde{H}(K_\infty^\circ)$. Since G acts on ε through the cyclotomic character, the class of ε in $\widetilde{H}(K_\infty^\circ)/\mathbf{Z}_p^\times$ is G-invariant.

Thus, given an untilt $K^\sharp$, there exists a perfectoid field K_∞/K, equal to the completion of a Galois extension with group G, together with a class $\varepsilon \in \mathrm{Hom}(\mathrm{Spa}\,K_\infty, \mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cycl}})/\mathbf{Z}_p^\times$ which is G-invariant. Conversely, if we are given such data, the class ε determines an untilt $K_\infty^\sharp$ of K_∞ together with a continuous action of G; then $K^\sharp := (K_\infty^\sharp)^G$ is an untilt of K.

It may happen that two data of the form (K_∞, ε) give rise to the same untilt. The proper way to sort this out is in the language of sheaves on the *pro-étale site*, in which $\mathrm{Spa}\,K_\infty \to \mathrm{Spa}\,K$ is considered a covering.

3.4 The pro-étale topology

The extension of fields K_∞/K appearing in the previous section was the completion of the union of a tower of finite separable (that is, étale) extensions of K. Such an extension K_∞/K is said to be *pro-étale*. The definition works in families as follows.

Definition 3.4.1. A morphism $f\colon X \to Y$ of perfectoid spaces is *pro-étale* if locally on X it is of the form $\operatorname{Spa}(A_\infty, A_\infty^+) \to \operatorname{Spa}(A, A^+)$, where A and A_∞ are perfectoid rings, and

$$(A_\infty, A_\infty^+) = \left[\varinjlim (A_i, A_i^+)\right]^\wedge$$

is a filtered colimit of pairs (A_i, A_i^+), such that $\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A, A^+)$ is étale.

(The notion of an étale morphism between affinoid adic spaces appears in [Sch12, Definition 7.1].)

Example 3.4.2. Let K be a perfectoid field, and let S be a profinite set; we have the perfectoid space $\underline{S}$ as in Example 3.1.5. Then $\underline{S} \to \operatorname{Spa} K$ is pro-étale. If $K = C$ is algebraically closed and $X \to \operatorname{Spa} C$ is pro-étale, then $X = \underline{S}$ for a locally profinite set S.

Example 3.4.3. Somewhat counterintuitively, the inclusion of a Zariski-closed subset is pro-étale. For instance, let K be a perfectoid field with pseudo-uniformizer ϖ, and let $Y = \operatorname{Spa} K\langle T^{1/p^\infty}\rangle$. For $n = 1, 2, \dots$, let $Y_n = \operatorname{Spa}(A_n, A_n^+) \subset Y$ be the rational subset $\left\{|T| \leq |\varpi|^{1/n}\right\}$. Then "evaluation at 0" induces an isomorphism $\left[\varprojlim A_n\right]^\wedge \to K$, so that the inclusion-at-0 map $\operatorname{Spa} K \to Y$ is pro-étale.

Definition 3.4.4. Consider the category Perf of perfectoid spaces of characteristic p. We endow this with the structure of a site by declaring that a collection of morphisms $\{f_i\colon X_i \to X\}$ is a covering (a *pro-étale cover*) if the f_i are pro-étale, and if for all quasi-compact open $U \subset X$, there exists a finite subset $I_U \subset I$, and a quasi-compact open $U_i \subset X_i$ for $i \in I_U$, such that $U = \cup_{i \in I_U} f_i(U_i)$.

Remark 3.4.5. The finiteness condition in Definition 3.4.4 excludes certain "pointwise" morphisms from giving pro-étale. For instance if Y is the perfectoid unit disc, we can consider the inclusion $f_x\colon \operatorname{Spa}(K_x, K_x^+) \to Y$ for each point $x \in |Y|$; this is pro-étale by similar reasoning as in Example 3.4.3, but we don't want $\{f_x\}_{x \in |Y|}$ to be a pro-étale covering.

Remark 3.4.6. The notions of a pro-étale morphism of schemes and of a pro-étale site appear in [BS], where they were used to define a pro-étale fundamental group of a scheme, and also to give the "morally correct" definition of the ℓ-adic cohomology group $H^i(X, \mathbf{Q}_\ell)$ for a scheme X.

It now makes sense to talk about a sheaf on Perf: this is a contravariant set-valued functor on Perf (a presheaf) which satisfies the sheaf axioms with respect to the pro-étale topology. If X is a perfectoid space of characteristic p, we have the representable presheaf h_X defined by $h_X(Y) = \mathrm{Hom}(Y, X)$.

Proposition 3.4.7 ([SW, Proposition 8.2.7])**.** *The presheaf h_X is a sheaf.*

If $\mathcal{F}$ is a sheaf on Perf, and if X is an object of Perf, then a morphism $h_X \to \mathcal{F}$ is the same thing as a section of $\mathcal{F}(X)$. Note that the functor $X \to h_X$ exhibits Perf as a full subcategory of the category of sheaves on Perf.

Definition 3.4.8.

1. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves on Perf is *pro-étale* if for all perfectoid spaces X and maps $h_X \to \mathcal{G}$, the pullback $h_X \times_{\mathcal{G}} \mathcal{F}$ is representable by a perfectoid space Y, and the morphism $Y \to X$ (corresponding to $h_Y = h_X \times_{\mathcal{G}} \mathcal{F} \to h_X$) is pro-étale.
2. A *diamond* is a sheaf $\mathcal{F}$ on Perf such that there exists a collection of pro-étale morphisms $h_{X_i} \to \mathcal{F}$ such that $\coprod_i h_{X_i} \to \mathcal{F}$ is surjective.
3. For a perfectoid space X of any characteristic, let $X^\diamondsuit = h_{X^\flat}$.

3.5 The diamond $\operatorname{Spd} \mathbf{Q}_p$

For a Huber ring R, let us abbreviate $\operatorname{Spa} R = \operatorname{Spa}(R, R^\circ)$. If R is perfectoid (of whatever characteristic), let us write $\operatorname{Spd} R$ for the diamond $h_{\operatorname{Spa}(R^\flat)}$. Thus we have the diamond $\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$: it is the sheaf on (Perf) whose (R, R^+)-valued sections are the set of continuous homorphisms $\mathbf{Q}_p^{\mathrm{cycl},\flat} \to R$. Since $\mathbf{Q}_p^{\mathrm{cycl},\flat} \cong \mathbf{F}_p(\!(t^{1/p^\infty})\!)$, this is nothing more than the set of topologically nilpotent elements of R (these automatically lie in R^+).

We now give an *ad hoc* definition of the diamond $\operatorname{Spd} \mathbf{Q}_p$.

Definition 3.5.1. $\operatorname{Spd} \mathbf{Q}_p = \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl},\flat} / \underline{\mathbf{Z}}_p^\times$. That is, $\operatorname{Spd} \mathbf{Q}_p$ is the coequalizer of

$$\underline{\mathbf{Z}}_p^\times \times h_{\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}} \rightrightarrows h_{\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}},$$

where one map is the projection and the other is the action.

We would like to know that $\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spa} \mathbf{Q}_p$ is a $\underline{\mathbf{Z}}_p^\times$-torsor. This is something like showing that the $\mathbf{Z}_p^\times$ action on $\mathbf{Q}_p^{\mathrm{cycl},\flat} = \mathbf{F}_p(\!(t^{1/p^\infty})\!)$ is sufficiently nontrivial.

Lemma 3.5.2. *Let $g \colon \underline{\mathbf{Z}}_p^\times \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \times_{\operatorname{Spd} \mathbf{Q}_p} \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$ be the product of the projection onto the second factor by the group action. Then g is an isomorphism.*

Proof. The crucial point is that $\mathbf{Z}_p^\times \to \operatorname{Aut} \mathbf{F}_p(\!(t^{1/p^\infty})\!)$ is injective. This implies that $\mathbf{Z}_p^\times$ acts freely on geometric points of $\operatorname{Spa} \mathbf{F}_p(\!(t^{1/p^\infty})\!)$. Let us construct the inverse of g. We need a map

$$\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl}} \times_{\operatorname{Spa} \mathbf{Q}_p} \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \to \underline{\mathbf{Z}}_p^\times$$

(whereas the map to $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$ is just the projection onto the first factor). A section of the fibre product over (R,R^+) is a pair of maps $f_1,f_2\colon \mathbf{Q}_p^{\mathrm{cycl},\flat}\to R$, such that there exists an affinoid pro-étale cover $\operatorname{Spa}(\widetilde{R},\widetilde{R}^+)\to\operatorname{Spa}(R,R^+)$, and a continuous map $\widetilde{\gamma}\colon \operatorname{Spa}(\widetilde{R},\widetilde{R}^+)\to\mathbf{Z}_p^\times$ such that

$$f_1(t)=(1+f_2(t))^{\widetilde{\gamma}}-1\in\widetilde{R}.$$

We want to show that $\widetilde{\gamma}$ factors through a continuous map $\widetilde{\gamma}\colon \operatorname{Spa}(R,R^+)\to\mathbf{Z}_p^\times$. It is enough to show that $\widetilde{\gamma}$ is constant on fibres of $\operatorname{Spa}(\widetilde{R},\widetilde{R}^+)\to\operatorname{Spa}(R,R^+)$, so without loss of generality $R=C$ is an algebraically closed nonarchimedean field. If $\widetilde{\gamma}$ is not constant, we have $\gamma_0\neq\gamma_1\in\mathbf{Z}_p^\times$ such that

$$f_1(t)=(1+f_2(t))^{\gamma_i}-1\in C$$

for $i=0,1$. That is, the two composites $f_2\circ\widetilde{\gamma}_1, f_1\circ\widetilde{\gamma}_2\colon \mathbf{F}_p(\!(t^{1/p^\infty})\!)\to C$ are the same. But this can't be, since the action of $\mathbf{Z}_p^\times$ on geometric points of $\operatorname{Spa}\mathbf{F}_p(\!(t^{1/p^\infty})\!)$ is free. $\square$

Corollary 3.5.3. $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}\to\operatorname{Spd}\mathbf{Q}_p$ *is a* $\underline{\mathbf{Z}}_p^\times$*-torsor.*

Corollary 3.5.4. $(\operatorname{Spd}\mathbf{Q}_p)(R,R^+)$ *is the set of isomorphism classes of data of the form:*

1. $R\to\widetilde{R}$, *a* $\mathbf{Z}_p^\times$*-torsor,*
2. *A* $\mathbf{Z}_p^\times$*-equivariant map* $\mathbf{Q}_p^{\mathrm{cycl},\flat}\to\widetilde{R}$.

Theorem 3.5.5. *There is an equivalence of categories between perfectoid spaces over* $\mathbf{Q}_p$*, and the category of perfectoid spaces* X *of characteristic* p *together with a "structure morphism"* $X^\diamondsuit\to\operatorname{Spd}\mathbf{Q}_p$.

Proof. We will show below that for any affinoid perfectoid (R,R^+), specifying a morphism $\operatorname{Spa}(R,R^+)\to\operatorname{Spd}(\mathbf{Q}_p)$ determines a *characteristic 0 untilt* of (R,R^+), by which we mean a pair $(R^\sharp,\iota)$, where $R^\sharp$ is a perfectoid $\mathbf{Q}_p$-algebra $R^\sharp$ and an isomorphism $\iota\colon R^{\sharp\flat}\to R$ of perfectoid algebras. This pair is *uniquely functorial* in the map to $\operatorname{Spd}(\mathbf{Q}_p)$. This gives a precise sense in which specifying an untilt of (R,R^+) is functorially the same as specifying a morphism $\operatorname{Spa}(R,R^+)\to\operatorname{Spd}(\mathbf{Q}_p)$, so we may then globalize to obtain the assertion in the theorem.

Suppose $(R^\sharp,\iota)$ is an untilt of (R,R^+). Let

$$\widetilde{R}^\sharp=R^\sharp\widehat{\otimes}_{\mathbf{Q}_p}\mathbf{Q}_p^{\mathrm{cycl}}=\left(\varinjlim R^\sharp\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(\mu_{p^n})\right)^\wedge,$$

so that $\widetilde{R}^\sharp$ is a $\mathbf{Z}_p^\times$-torsor over $R^\sharp$. Tilting this gives $\widetilde{R}$, a $\underline{\mathbf{Z}}_p^\times$-torsor over R, together with a $\mathbf{Z}_p^\times$-equivariant map $\mathbf{Q}_p^{\mathrm{cycl},\flat}\to\widetilde{R}$.

In the other direction, let $R\to\widetilde{R}$ be a $\underline{\mathbf{Z}}_p^\times$-torsor, and let $\mathbf{Q}_p^{\mathrm{cycl},\flat}\to\widetilde{R}$ be a $\mathbf{Z}_p^\times$-equivariant map. Recall the tilting equivalence between perfectoid algebras over $\mathbf{Q}_p^{\mathrm{cycl},\flat}$ and $\mathbf{Q}_p^{\mathrm{cycl}}$. Thus we get a canonical untilt $\mathbf{Q}_p^{\mathrm{cycl}} \to \widetilde{R}^\sharp$ which is $\mathbf{Z}_p^\times$-equivariant. We let

$$R^\sharp = \left(\widetilde{R}^\sharp\right)^{\mathbf{Z}_p^\times},$$

and similarly for $R^{\sharp+}$. We would like to know that $(R^\sharp, R^{\sharp+})$ is "big enough" in the sense that $R^{\sharp\flat} \cong R$.

First we work modulo a pseudo-uniformizer. By an argument with cocycles, we can build a pseudo-uniformizer $\varpi \in \widetilde{R}$ such that $\varpi^\sharp \in R^\sharp$ (*i.e.*, it is $\mathbf{Z}_p^\times$-invariant), and such that $(\varpi^\sharp)^p | p$ in $\widetilde{R}^{\sharp,+}$.

Consider the short exact sequence

$$0 \longrightarrow \widetilde{R}^{\sharp+} \xrightarrow{\varpi^\sharp} \widetilde{R}^{\sharp+} \longrightarrow \widetilde{R}^{\sharp+}/\varpi^\sharp = \widetilde{R}^+/\varpi \longrightarrow 0$$

Taking $\mathbf{Z}_p^\times$ invariants, we see there is an inclusion $R^{\sharp+}/\varpi^\sharp \to H^0(\mathbf{Z}_p^\times, \widetilde{R}^+/\varpi)$. The latter H^0 is just R^+/ϖ, because $R^+ \to \widetilde{R}^+$ is a $\mathbf{Z}_p^\times$-torsor. The obstruction to $R^{\sharp+}/\varpi^\sharp \to R^+/\varpi$ being surjective lies in $H^1_{\mathrm{cont}}(\mathbf{Z}_p^\times, \widetilde{R}^{\sharp+})$. By successive approximation, this H^1 vanishes provided that $H^1_{\mathrm{cont}}(\mathbf{Z}_p^\times, \widetilde{R}/\varpi^\sharp)$ vanishes.

Lemma 3.5.6. *We have an almost isomorphism:*

$$H^i_{cont}(\mathbf{Z}_p^\times, \widetilde{R}^{\sharp+}/\varpi^\sharp) \cong \begin{cases} R^+/\varpi, & i = 0 \\ 0, & i > 0. \end{cases}$$

Proof. Use pro-étale descent for $(\mathcal{O}^+/\varpi)^a$ along $R \to \widetilde{R}$. $\square$

Thus $R^{\sharp+}/\varpi^\sharp \cong R^+/\varpi$. We can also apply the lemma using ϖ^p instead of ϖ (the lemma only required that $\varpi^\sharp | p$) to obtain an isomorphism $R^{\sharp+}/(\varpi^\sharp)^p \cong R^+/\varpi^p$. Since R is perfectoid, $\Phi\colon R^+/\varpi \to R^+/\varpi^p$ is an isomorphism, and so is $\Phi\colon R^{\sharp+}/\varpi^\sharp \to R^{\sharp+}/(\varpi^\sharp)^p$. Therefore $R^\sharp$ is perfectoid as well.

The $\mathbf{Z}_p^\times$-invariant inclusion $R^\sharp \to \widetilde{R}^\sharp$ is a map between perfectoid rings, so the functoriality of tilting gives a $\mathbf{Z}_p^\times$-invariant map $R^{\sharp\flat} \to \widetilde{R}^{\sharp\flat} = \widetilde{R}$, which factors through the subalgebra R of $\mathbf{Z}_p^\times$-invariants. As a result we have a map $R^{\sharp\flat} \to R$ which induces a map $R^{\sharp\flat+} \to R^+$. We have already seen that the reduction of the latter map modulo ϖ is an isomorphism, and so by successive approximation, $R^{\sharp\flat+} \to R^+$ is surjective. But likewise we see from being an isomorphism modulo ϖ that it is an isomorphism modulo ϖ^n for $n = 1, 2, \ldots$. Taking the inverse limit, we deduce an isomorphism $R^{\sharp\flat+} \to R^+$, which becomes an isomorphism $R^{\sharp\flat} \to R$ upon inverting ϖ. $\square$

3.6 The functor $X \mapsto X^\diamondsuit$

4 Some vector space diamonds

4.1 A diamond version of the Fargues-Fontaine curve

Let C be an algebraically closed perfectoid field, and let H/C° be the multiplicative formal group. We have seen that the universal cover $\widetilde{H}$ is a formal $\mathbf{Q}_p$-vector space, whose adic generic fiber $\widetilde{H}_C$ is a $\mathbf{Q}_p$-vector space object in the category of perfectoid spaces. The underlying perfectoid space of $\widetilde{H}_C$ is the perfectoid open unit disc. Let $\widetilde{H}_C^* = \widetilde{H}_C \backslash \{0\}$. Then $\widetilde{H}_C^*$ admits an action of $\mathbf{Q}_p^\times$.

Proposition 4.1.1 ([Wei16])**.** *Let $Z = \widetilde{H}_C^{\diamondsuit *}/\mathbf{Q}_p^*$, where the quotient is taken in the category of sheaves on Perf. Then Z is a proper diamond lying over* $\operatorname{Spd} C$.

1. *We have an isomorphism $Z \cong \mathcal{X}^\diamondsuit$, where $\mathcal{X}$ is the adic Fargues-Fontaine curve (constructed from $C^\flat$).*
2. *Accordingly, for a perfectoid ring R, $Z(R)$ is the set of Frobenius-equivalence classes of untilts of R.*
3. *The category of finite étale covers of Z is equivalent to the category of finite étale $\mathbf{Q}_p$-algebras. Thus the étale fundamental group of Z is isomorphic to* $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

There is a generalization of the above proposition which concerns a finite extension $E/\mathbf{Q}_p$. One has to replace H with the Lubin-Tate formal $\mathcal{O}_E$-module H_E. Then the diamond $Z_E = \widetilde{H}_{E,C}^{\diamondsuit *}/E^*$ classifies untilts of a perfectoid ring to a perfectoid E-algebra.

4.2 Perfectoid spaces arising from vector bundles on the Fargues-Fontaine curve

Once again we let C be an algebraically closed perfectoid field. Let k be the residue field of C. We saw in (2.7) that an isocrystal N over k gives rise to a vector bundle $\mathcal{E}(N)$ on the absolute Fargues-Fontaine curve X constructed from C. Furthermore, if N is the isocrystal attached to a p-divisible group H (this happens exactly when the slopes of N lie in the range $[0, 1]$), then we have an isomorphism

$$\widetilde{H}(C^{\flat\circ}) \xrightarrow{\sim} H^0(X_{C^\flat}, \mathcal{E}(N)).$$

In fact the construction can be made slightly more general. Let R be a perfectoid algebra containing k, and let X_R be the corresponding schematic Fargues-Fontaine curve, as in (3.3). The isocrystal N can be used to construct a vector bundle on X_R, which we still call $\mathcal{E}(N)$.

Proposition 4.2.1. *There is an isomorphism $\widetilde{H}(R^\circ) \cong H^0(X_R, \mathcal{E}(N))$.*

Sketch of proof: The right-hand side is $(B_R \otimes_{W(k)} N)^{\phi=1}$. R° is a perfect ring, so [SW13, Theorem 4.1.4] (suitably interpreted) applies to give an isomorphism $\widetilde{H}(R^\circ) \cong (B_{\text{cris}}(R) \otimes_{W(k)} N)^{\phi=1}$, where $B_{\text{cris}}(R)$ is the crystalline period ring. A hint as to why $(B_R \otimes_{W(k)} N)^{\phi=1} \cong (B_{\text{cris}}(R) \otimes_{W(k)} N)^{\phi=1}$ is [FF11, Corollaire 1.10.13], although strictly speaking that result only applies to the case that R is a field.

We may consider the formal scheme $\widetilde{H}$ as a functor on Perf_k, via $R \mapsto \widetilde{H}(R^\circ)$. This functor is not quite representable, because $\widetilde{H}$ isn't analytic. However, if K/k is any perfectoid field, then the pullback $\widetilde{H}_K$ to the category of perfectoid K-algebras is representable; it is the same as the generic fiber $\widetilde{H}_K$ we have already investigated. One refers to $\widetilde{H}$ as an *absolute perfectoid space*: it is a functor on $Perf_k$ which becomes representable when pulled back through $\mathrm{Perf}_K \to \mathrm{Perf}_k$, for any perfectoid field K/k.

Let $K^\sharp$ be an untilt of Perf_K, and let $\widetilde{H}_{K^\sharp}$ be the pullback of $\widetilde{H}$ through the tilt map $\mathrm{Perf}_{K^\sharp} \to \mathrm{Perf}_K$. Then $\widetilde{H}_{K^\sharp}$ is a perfectoid space over $\mathrm{Spa}\, K^\sharp$. Let $G/K^{\sharp\circ}$ be a lift of $H \otimes_k K^{\sharp\circ}/p$, so that $\widetilde{G}_{K^\sharp} = \widetilde{H}_{K^\sharp}$. We have the following complex of $\mathbf{Q}_p$-vector space diamonds over $\mathrm{Spd}\, K^\sharp$:

$$0 \to \underline{VG} \to \widetilde{G} \to \mathrm{Lie}\, G \otimes \mathbf{G}_{\mathrm{a}} \to 0.$$

In fact the complex is exact: to check exactness on the right is to check that for any perfectoid $K^\sharp$-algebra R, and any element $v \in \mathrm{Lie}\, G \otimes_{K^\sharp \circ} R$, there exists a pro-étale R'/R and an element $\widetilde{G}((R')^\circ)$ whose logarithm is v.

5 Projects

5.1 Basic examples of adic spaces

1. Classify points in $\mathrm{Spa}\, \mathbf{Q}_p\langle T\rangle$; describe the set-theoretic fibers of $\mathrm{Spa}\, \mathbf{C}_p\langle T\rangle \to \mathrm{Spa}\, \mathbf{Q}_p\langle T\rangle$.

2. Classify points in $\mathrm{Spa}\, W(C^\circ)$, where C is an algebraically closed perfectoid field of characteristic p.

5.2 Perfectoid fields

1. Let $K = \mathbf{Q}_2(2^{1/2^\infty})^\wedge$. Identify $K^\flat$ with $\mathbf{F}_2(\!(t^{1/2^\infty})\!)$, where t corresponds to the sequence $(2, 2^{1/2}, \dots)$. Let $L = K(\sqrt{-1})$, so that L/K has degree 2. Thus L is perfectoid. Identify $L^\flat$ as a separable extension of $\mathbf{F}_2(\!(t^{1/p^\infty})\!)$. Repeat for all other quadratic field extensions of K.

2. Let K be a perfectoid field with residue field k. Show that $K^\flat \cong k(\!(t^{1/p^\infty})\!)$ if and only if the following criterion holds: K admits no proper perfectoid subfields with the same residue field and value group.

5.3 Some commutative algebra

1. Let K be a perfectoid field. Describe the group of units in $K\langle T^{1/p^\infty}\rangle$.

2. Let C be an algebraically closed perfectoid field, and let $f \in C\langle T^{1/p^\infty}\rangle$ be a non-unit. Let $D = \{|x| \leq 1\} \subset C$. Does there always exist $\alpha \in D$ with $f(\alpha) = 0$? Is the set of zeros of f finite? Profinite? Which subsets of D are zero sets of such f?

3. Continuing this theme, let C be an algebraically closed perfectoid field of characteristic p, and let $f_1, \ldots, f_m \in A = C\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$ be elements which do not generate the unit ideal. Does there exist a common zero of the f_i in D^n? (This is something like a perfectoid Nullstellensatz statement.)

5.4 Closed subsets of adic spaces

For a scheme X, a closed subset $T \subset X$ is (rather by definition) Zariski closed: it is the zero locus of an ideal sheaf in $\mathcal{O}_X$. There is a scheme, the *reduced induced subscheme* Z, and a closed immersion $Z \to X$ whose set-theoretic image is T. This property is universal: for a reduced scheme Y, a morphism $f\colon Y \to X$ has $f(Y) \subset T$ (set-theoretically) if and only if f factors as $Y \to Z \to X$.

It is quite different with adic spaces. One difference is that closed subsets are not necessarily Zariski-closed.

1. Consider $\mathbf{Q}_p$ as a closed subset of the underlying topological space of $\mathbf{A}^1$, considered as an adic space over $\mathbf{Q}_p$. Show that $\mathbf{Q}_p$ is not Zariski closed.

2. Nonetheless, show that there exists a reduced adic space Z and a morphism $Z \to \mathbf{A}^1$, which is monic and has image $\mathbf{Q}_p$, and which satisfies a universal property.

Let $H/\overline{\mathbf{F}}_p$ be a formal p-divisible group of height 2 and dimension 1. Its universal cover $\widetilde{H}$ lifts to a formal $\mathbf{Q}_p$-vector space over $\breve{\mathbf{Z}}_p = W(\overline{\mathbf{F}}_p)$; let $\widetilde{H}_{\breve{\mathbf{Q}}_p}$ be its generic fiber. Then $\widetilde{H}_{\breve{\mathbf{Q}}_p}$ is a preperfectoid space. Let $M(H)$ be the Dieudonné module of H; this is a free $\breve{\mathbf{Z}}_p$-module of rank 2. There is a quasi-logarithm map of adic spaces

$$\mathrm{qlog}_H\colon \widetilde{H}_{\breve{\mathbf{Q}}_p} \to M(H) \otimes_{\breve{\mathbf{Z}}_p} \mathbf{G}_{\mathrm{a}} \cong \mathbf{G}_{\mathrm{a}}^2$$

which respects the $\mathbf{Q}_p$-vector space structure on either side. We describe it as a natural transformation between functors from perfectoid algebras containing $\breve{\mathbf{Q}}_p$ to $\mathbf{Q}_p$-vector spaces. Let R be a perfectoid algebra. We have an isomorphism $\widetilde{H}_{\breve{\mathbf{Q}}_p}(R) = \widetilde{H}(R^\circ) \cong (B(R^\flat) \otimes_{\breve{\mathbf{Z}}_p} M(H))^{\phi=1}$. Then $\mathrm{qlog}_H(R)$ is the composition of this map with $\theta_R \otimes 1\colon B(R^\flat) \otimes_{\breve{\mathbf{Z}}_p} M(H) \to R \otimes_{\breve{\mathbf{Z}}_p} M(H)$.

3. Prove that qlog is monic.

4. Let Z be the image of qlog_H, considered as a subset of the underlying topological space of $\mathbf{G}_{\mathrm{a}}^2$. Show that Z is closed and generalizing.

5. Show that the residue fields of points of Z are never finite extensions of $\breve{\mathbf{Q}}_p$. That is, the image of qlog_H "contains no classical points".

6. Show that if Y is a perfectoid space over $\mathrm{Spa}\,\breve{\mathbf{Q}}_p$ and $f\colon Y \to \mathbf{G}_{\mathrm{a}}^2$ has set-theoretic image contained in Z, then f factors through qlog_H.

Thus we have a closed subset of the adic space $\mathbf{G}_{\mathrm{a}}^2$ which (considered as a subfunctor on the category of perfectoid spaces) is representable by a preperfectoid space. In fact, it is a theorem of Scholze that any closed generalizing subset of a diamond, when considered as a subfunctor on the category of perfectoid spaces, is itself a diamond.

5.5 Computations with Banach-Colmez spaces

Recall our discussion of Banach-Colmez (BC) spaces, which are sheaves on the category of perfectoid spaces in characteristic p, which take values in the category of $\mathbf{Q}_p$-algebras. There are two projects here. The first has to do with Banach-Colmez spaces which are *ineffective*; *i.e.* not the space of global sections of a vector bundle on the Fargues-Fontaine curve.

1. We begin with the BC space $H^1(X, \mathcal{O}_X(-1))$, which inputs a perfectoid algebra $R/\overline{\mathbf{F}}_p$ and outputs the $\mathbf{Q}_p$-vector space $H^1(X_R, \mathcal{O}_{X_R}(-1))$. Show that there is an isomorphism of sheaves of $\mathbf{Q}_p$-vector spaces on $\mathrm{Perf}_{\breve{\mathbf{Q}}_p}$:
$$H^1(X, \mathcal{O}_X(-1)) \times_{\mathrm{Spd}\,\bar{\mathbf{F}}_p} \mathrm{Spd}\,\breve{\mathbf{Q}}_p \cong \mathbf{A}^1_{\breve{\mathbf{Q}}_p}/\mathbf{Q}_p.$$

2. $H^1(X, \mathcal{O}_X(-1))$ parametrizes extension classes $0 \to \mathcal{O}_X(-1) \to \mathcal{E} \to \mathcal{O}_X \to 0$, or (after twisting by $\mathcal{O}_X(1)$ extension classes $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X(1)$. Show that if this extension is nonsplit, then there exists an isomorphism $\mathcal{E} \cong \mathcal{O}_X(1/2)$. Recall that global sections of $\mathcal{O}_X(1/2)$ are representable by a formal scheme $\widetilde{H}$, where $H/\bar{\mathbf{F}}_p$ is a formal p-divisible group of dimension 1 and height 2. Let us abbreviate $\widetilde{H}^* = \widetilde{H}\backslash\{0\}$. Show that there is an isomorphism $H^1(X, \mathcal{O}_X(-1))\backslash\{0\} \cong (\widetilde{H}^*_{\mathbf{Q}_p} \times \mathbf{Q}_p(1)^*)/D^\times$, where $D = \mathrm{Aut}\,H \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the nonsplit quaternion algebra over $\mathbf{Q}_p$.

3. Let $\Omega = \mathbf{A}^1_{\breve{\mathbf{Q}}_p}\backslash\mathbf{Q}_p$. Combining the previous two exercises gives an isomorphism $\Omega/\mathbf{Q}_p \cong (\widetilde{H}^*_{\breve{\mathbf{Q}}_p} \times \mathbf{Q}_p(1)^*)/D^\times$. This isomorphism means there is a diamond M carrying an action of $\mathbf{Q}_p \times D^1$, whose quotient by D^1 is Ω, and whose quotient by $\mathbf{Q}_p$ is $\widetilde{H}^*_{\breve{\mathbf{Q}}_p} \times \mathbf{Q}_p(1)$. Show that M (with this action) is isomorphic to the Lubin-Tate tower for $\mathrm{GL}_2(\mathbf{Q}_p)$.

4. Is there a similar story for $H^1(X, \mathcal{O}_X(\lambda))$ for other negative values of $\lambda \in \mathbf{Q}$?

The other project is due to David Hansen. Let $M \to \operatorname{Spd} \breve{\mathbf{Q}}_p$ be the infinite-level Lubin-Tate tower for $\mathrm{GL}_2(\mathbf{Q}_p)$. Then M can be interpreted as the space of "mixed-characteristic shtukas" of a certain type. To wit, M is the sheafiffication of the presheaf which assigns to a perfectoid $\mathbf{Q}_p$-algebra R, the set of exact sequences of the form

$$0 \to \mathcal{O}^2_{X_{R^\flat}} \to \mathcal{O}_{X_{R^\flat}}(1/2) \to i_* W \to 0,$$

where $i\colon \operatorname{Spec} B_{\mathrm{dR}}(R) \to X_{R^\flat}$ is the closed immersion corresponding to the untilt R of $R^\flat$, and W is a projective module of rank 1 over $R = B_{\mathrm{dR}}(R^\flat)/\xi$. Then M admits an action of the product group $\mathrm{GL}_2(\mathbf{Q}_p) \times D^\times$, where $D = \operatorname{Aut} \mathcal{O}_X(1/2)$ is the nonsplit quaternion algebra over $\mathbf{Q}_p$.

Here is a different space of shtukas, which we'll call N: it is the sheafification of the presheaf which assigns to a perfectoid $\mathbf{Q}_p$-algebra R the set of exact sequences of the form

$$0 \to \mathcal{O}^2_{X_{R^\flat}} \to \mathcal{O}_{X_{R^\flat}}(1)^2 \to i_* V \to 0,$$

where this time V is a projective $B_{\mathrm{dR}}(R^\flat)/\xi^n$-module of rank 1. Then N admits an action of $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p)$.

5. Show that N is a perfectoid space.

6. Show that, in the category of diamonds, N is isomorphic to the quotient $(M \times M)/D^\times$, where the action of $D^\times$ is the diagonal one.

7. Are there other isomorphisms of these type, for different spaces of shtukas?

References

[BS] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, To appear in Proceedings of the conference in honour of Gérard Laumon, 2013.

[Col10] Pierre Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (ϕ, Γ)*-modules*, Astérisque (2010), no. 330, 281–509. MR 2642409

[FF11] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, preprint, 2011.

[Fon13] Jean-Marc Fontaine, *Perfectoïdes, presque pureté et monodromie-poids (d'après Peter Scholze)*, Astérisque (2013), no. 352, Exp. No. 1057, x, 509–534, Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058. MR 3087355

[FW79] Jean-Marc Fontaine and Jean-Pierre Wintenberger, *Extensions algébrique et corps des normes des extensions APF des corps locaux*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 8, A441–A444. MR 527692 (80h:12014)

[Haz12] Michiel Hazewinkel, *Formal groups and applications*, AMS Chelsea Publishing, Providence, RI, 2012, Corrected reprint of the 1978 original. MR 2987372

[Hub94] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551.

[Ked] Kiran S. Kedlaya, *Noetherian properties of Fargues-Fontaine curves*, arXiv:1410.5160 (version 1).

[Ked15] ———, *New methods for* (Γ, φ)*-modules*, Res. Math. Sci. **2** (2015), Art. 20, 31. MR 3412585

[KL] Kiran S. Kedlaya and Ruochuan Liu, *Relative p-adic Hodge theory, I: Foundations*, arXiv:1301.0792 (version 3).

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387. MR 0172878

[Sch12] Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR 3090258

[SW] Peter Scholze and Jared Weinstein, *p-adic geometry: notes from peter scholze's 2014 course at uc berkeley.*

[SW13] ———, *Moduli of p-divisible groups*, Cambridge Journal of Mathematics **1** (2013), no. 2, 145–237.

[Tat67] J. T. Tate, *$p-$divisible groups.*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183. MR 0231827 (38 #155)

[Wei16] Jared Weinstein, $\mathrm{Gal}(\overline{\mathbf{Q}}_{\mathrm{p}}/\mathbf{Q}_{\mathrm{p}})$ *as a geometric fundamental group*, Int. Math. Res. Not. (2016).