The Sequence

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_v Br(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Let k be a number field. Finding a primitive solution over $\mathbb{Z}$ (or $\mathbb{Q}$) of a homogenous polynomial over $\mathbb{Z}$ (or $\mathbb{Q}$) relates via the norm map to the injectivity of $Br(k) \longrightarrow \bigoplus_v Br(k)$ (which is the subject of the Brauer Hasse Noether theorem).

Example: Consider the polynomial $ax^2 + by^2 + cz^2 = 0$ with $a, b, c \in \mathbb{Z}$ $a, b, c \neq 0$. If we set $z = 1$ and divide by a we get $x^2 + \frac{b}{a}y^2 = -\frac{c}{a}$. Now consider the field extension $K = \mathbb{Q}(\sqrt{-\frac{b}{a}})$ over $\mathbb{Q}$. The norm map for this extension is given by $N(\alpha + \beta\sqrt{-\frac{b}{a}}) = \alpha^2 + \frac{b}{a}\beta^2$ $\alpha, \beta \in \mathbb{Q}$. So the polynomial has a (primitive) solution with $z \neq 0$ if and only if $-\frac{c}{a}$ is a norm. From the Brauer group sequence one gets an injective map

$$\mathbb{Q}^*/NK^* \longrightarrow \prod_{l \cup \infty} \mathbb{Q}_l/N\mathbb{Q}_l^* \qquad \text{(later)}$$

.

So one can check if $-\frac{c}{a}$ is a norm in $\mathbb{Q}_l$ for every l and the injectivity allows a conclusion for $\mathbb{Q}$. Explicitely, for the solutions this means that a tripel $(\alpha, \beta, 1)$ is a solution in $\mathbb{Q}$ if and only if it is a solution in all $\mathbb{Q}_l$ (or equivalently in all $\mathbb{Z}/l^n\mathbb{Z}$).

In this example one needs a special norm map which fits to the situation. The field extension was constructed the way that it fits. But if one just takes a field extension, one gets a norm map from which one can deduce a polynomial on which again one can apply this local-global method.

Now we want to relate the norm map of a cyclic extension L/k of a number field to the Brauer group sequence. For this we need a map

$$k^*/NL^* \longrightarrow Br(k)$$

But first:

Tate cohomology

Let G be a cyclic group and A, B and C be G-modules. Consider the functor $A \rightsquigarrow A/(G-1)A = A_G$ Its left derived functor gives homology $H_i(G, A)$.

One glues its long exact sequence with the one of the usual group cohomology via a norm map $A_G \xrightarrow{N_A} A^G$. From an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we get

$$\begin{tikzcd}
 & & \widehat{H}^0(G,A) & \widehat{H}^0(G,B) & \widehat{H}^0(G,C) \arrow[dr, dashed] & & \\
 & 0 \arrow[r] & A^G \arrow[u] \arrow[r] & B^G \arrow[u] \arrow[r] & C^G \arrow[u] \arrow[r] & H^1(G,A) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_1(G,A) \arrow[r] \arrow[dr, dashed] & A_G \arrow[u, "N_A"] \arrow[r] & B_G \arrow[u, "N_B"] \arrow[r] & C_G \arrow[u, "N_C"] \arrow[r] & 0 & \\
 & & \widehat{H}_0(G,A) \arrow[u] & \widehat{H}_0(G,B) \arrow[u] & \widehat{H}_0(G,C) \arrow[u] & &
\end{tikzcd}$$

The Tate cohomology groups are then defined to be

$$\begin{array}{lll}
\widehat{H}^n(G,A) & = H^n(G,A) & n \geq 1 \\
\widehat{H}^0(G,A) & = coker(N_A) & \\
\widehat{H}^-1(G,A) & = ker(N_A) = \widehat{H}_0(G,A) & \\
\widehat{H}^{-n}(G,A) & = H_{n-1}(G,A) & n \geq 2
\end{array}$$

.

Now the Brauer group of k (still a number field) is $Br(k) = H^2(Gal(\bar{k}/k), \bar{k}^*)$. If we have a tower of Galois extensions $\bar{k} \supseteq L \supseteq k$ where L is cyclic over k , and

$$G = Gal(\bar{k}/k), \qquad I = Gal(\bar{k}/L) \quad \text{and} \quad \Gamma = G/I$$

then from the Hochschild Serre spectral sequence

$$E_2^{r,s} = H^r(\Gamma, H^s(I, \bar{k}^*)) \Rightarrow H^{r+s}(G, \bar{k}^*)$$

we obtain a sequence of lower terms

$$H^0(\Gamma, H^1(I,\bar{k}^*)) \longrightarrow H^2(\Gamma, L^*) \longrightarrow ker(H^2(G,\bar{k}^*) \longrightarrow H^2(I,\bar{k}^*)) \longrightarrow H^1(\Gamma, H^1(I,\bar{k}^*))$$

.

Hilbert 90 implies $H^2(\Gamma, H^1(I,\bar{k}^*)) = 0 = H^1(\Gamma, H^1(I,\bar{k}^*))$ so one gets an injection

$$H^2(\Gamma, L^*) \longrightarrow Br(k)$$

For an arbitrary cyclic group Γ with generator s and a Γ-module A one has a simple projective resolution of A. Namely one uses the resolution of $\mathbb{Z}$

$$\cdots \xrightarrow{\sigma-1} \mathbb{Z}[\Gamma] \xrightarrow{N} \mathbb{Z}[\Gamma] \xrightarrow{\sigma-1} \mathbb{Z} \longrightarrow 0$$

and tensors with $-\bigotimes_{\mathbb{Z}[\Gamma]} A$ for homology and takes $Hom_{\mathbb{Z}[\Gamma]}(-,A)$ for cohomology. We then can see that $\widehat{H}^n(\Gamma,A) \cong \widehat{H}^{n+2}(G,A)$ for all $n \in \mathbb{Z}$.

For $n=0$ one obtains

$$A^\Gamma/NA \cong \widehat{H}^0(\Gamma,A) \cong \widehat{H}^2(\Gamma,A) = H^2(\Gamma,A)$$

In our situation we have L^* for A and $Gal(L/k)$ as Γ. So this gives us an injection

$$k^*/NL^* \longrightarrow Br(k)$$

We want to have the same map for the local fields of k. First we note that for two places w and w' over v, $NL_w^* = NL_{w'}^*$ because they are conjugate under $Gal(L/k)(=\Gamma)$. Then since $Gal(L_w/k_v) \subseteq Gal(L/k)$ the group $Gal(L_w/k_v)$ is also cyclic so the same way as above we get

$$k_v^*/NL_w^* \longrightarrow Br(k_v)$$

And finaly putting everything together we obtain a commutative diagram:

$$\begin{tikzcd}
k^*/NL^* \arrow[r, hook] \arrow[d] & Br(k) \arrow[d] \\
\prod_v k_v^*/NL_w^* \arrow[r, hook] & \bigoplus_v Br(k_v)
\end{tikzcd}$$

And the injectivity on the right side implies the one on the left side.

Next we want to take a closer look onto the Brauer group sequence itself.

Theorem. If k is a finite extension of $\mathbb{Q}_p$ then

$$Br(k) \cong \mathbb{Q}/\mathbb{Z} \quad .$$

Proof. We use that $Br(k) \cong H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^*,)$. Now consider the tower $\bar{k} \supseteq k^{nr} \subseteq k$ with the Galois groups $G = \mathrm{Gal}(\bar{k},/k,)$, $I = \mathrm{Gal}(\bar{k},/k^{nr},)$ and $\Gamma = G/I$ $(I \lhd G)$ where k^{nr} is the maximal unramified extension of k.

To compute the Brauer group of k it is sufficient to only consider the field k^{nr}. Namely one has $Br(k) \cong H^2(\Gamma, {k^{nr}}^*)$:
We consider the Hochschild Serre spectral sequence again

$$E_2^{r,s} = H^r(\Gamma, H^s(I, \bar{k}^*)) \Rightarrow H^{r+s}(G, \bar{k}^*)$$

Using that $H^1(I, \bar{k}^*) = 0$ (Hilbert 90) and that $H^2(I, \bar{k}^*) = Br(k^{nr}) = 0$ (k^{nr} is a C_1 field) we find $E_2^{n,1} = 0$ and $E^{n,2} = 0$ for all n. So

$$0 = E_2^{0,1} \xrightarrow{d} E_2^{2,0} \xrightarrow{d} 0$$

gives

$$E_2^{2,0} \cong E_\infty^{2,0} \cong F^2/F^3 \cong F^2$$

and also

$$\begin{array}{lcl} E^{1,1} = 0 & \Rightarrow & F^1 \cong F^2 \quad \text{and} \\ E^{0,2} = 0 & \Rightarrow & F^0 \cong F^1 \end{array}$$

.
Finaly we get $E^2 = F^0 = F^1 = F^2 = E_2^{2,0}$ which means $Br(k) = H^2(G, \bar{k}^*) = H^2(\Gamma, H^0(I.\bar{k}^*)) = H^2(\Gamma, {k^{nr}}^*)$.

Now we have a valuation

$${k^{nr}}^* \twoheadrightarrow \mathbb{Z}$$

which is G equivariant. If we apply the cohomology functor we get a morphism

$$Br(k) \xrightarrow{\varphi} H^2(\Gamma, \mathbb{Z})$$

.
At last we want to see that $H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ and that φ is an isomorphism. From the corresponding short exact sequence we have

$$H^1(\Gamma, \mathbb{Q}) \longrightarrow H^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(\Gamma, \mathbb{Z}) \longrightarrow H^2(\Gamma, \mathbb{Q})$$

But $H^i(\Gamma, \mathbb{Q}) = 0 \quad \forall i > 0$ and thus using that $\Gamma \cong \widehat{\mathbb{Z}}$ we obtain $H^2(\Gamma, \mathbb{Z}) \cong H^1(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$. Since $\widehat{\mathbb{Z}} = \lim_\rightarrow(\mathbb{Z}/n\mathbb{Z})$ this reduces to $H^2(\Gamma, \mathbb{Z}) \cong \lim_\rightarrow H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \lim_\rightarrow \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$. So we have a Morphism

$$Br(k) \longrightarrow \mathbb{Q}/\mathbb{Z} \qquad .$$

Consider the exact sequence of Γ-modules

$$0 \longrightarrow A^* \longrightarrow {k^{nr}}^* \longrightarrow \mathbb{Z} \longrightarrow 0$$

where A is just the valuation ring of this valuation. This sequence splits so ${k^{nr}}^* \cong A^* \oplus \mathbb{Z}$ The cohomology functor is additiv, therefor we have an exact sequence

$$0 \longrightarrow H^2(\Gamma, A^*) \longrightarrow Br(k) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and it only remains to show, that $H^2(\Gamma, A^*) = 0$.

From Milne's Étale Cohomology I 4.5 + IV 1.6 we have an injection $Br(A) \hookrightarrow Br(A/m)$. But A/m is a finite field and the Brauer group of a finite field is zero. So $Br(A) = H^2(\Gamma, A^*)$ must also be zero. Thus we obtain an isomorphism $Br(k) \xrightarrow{\simeq} \mathbb{Q}/\mathbb{Z}$.

$\square$

Using the fact that $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ and $Br(\mathbb{C}) = 0$ the sequence rewrites as

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_l \mathbb{Q}/\mathbb{Z} \oplus \bigoplus \mathbb{Z}/2Z \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where the last map is given by

$$(a_1, \ldots ; b_1, \ldots, b_n) \longrightarrow \Sigma a_i + \Sigma b_i \qquad .$$