Hyperbolische Funktionen sh, ch.

Definition Für $t \in \mathbb{R}$,

$$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}, \quad \text{Kosinus hyperbolicus}$$

$$\operatorname{sh} t = \frac{e^t - e^{-t}}{2}. \quad \text{Sinus hyperbolicus}$$

Identitäten: (einfach zu beweisen).

$$\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1,$$

$$\operatorname{ch}^2 t + \operatorname{sh}^2 t = \operatorname{ch}(2t)$$

$$2 \operatorname{sh} t \operatorname{ch} t = \operatorname{sh}(2t)$$

$$(\operatorname{sh} t)' = \operatorname{ch} t, \quad (\operatorname{ch} t)' = \operatorname{sh} t$$

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sh^{-1}, ch^{-1} ? (Tradition:

$\operatorname{Arsh} = \operatorname{sh}^{-1}$

$\operatorname{Arch} = \operatorname{ch}^{-1}$

(nicht Arcsh !!!)

$\operatorname{sh} x = y$; $\dfrac{e^x - e^{-x}}{2} = y$; $e^{2x} - 1 = 2ye^x$,

$$e^x = z \Rightarrow z^2 - 2yz - 1 = 0$$

$$z = y \pm \sqrt{y^2+1}$$

($z = e^x > 0 \Rightarrow$ nur $+$ ist möglich)

$$\Rightarrow e^x = y + \sqrt{y^2+1},$$

$$x = \ln\left(y + \sqrt{y^2+1}\right) = \operatorname{Arsh} y.$$

Ähnlich,

$\operatorname{ch} x = y \quad (y \geq 1,\ x \geq 0)$

$$\frac{e^x + e^{-x}}{2} = y; \quad e^{2x} + 1 = 2ye^x,$$

$$z = e^x, \qquad z^2 - 2yz + 1 = 0$$

$$z = y \pm \sqrt{y^2 - 1}.$$

($z = e^x \geq 1 \Rightarrow$ nur "+" möglich)

$$e^x = y + \sqrt{y^2-1},$$

$$x = \ln\left(y + \sqrt{y^2-1}\right) = \operatorname{Arch} y.$$

Beispiele:

*) $$\int \frac{dx}{\sqrt{x^2-1}} = \left\{ \begin{array}{l} x = \operatorname{ch} t \\ dx = \operatorname{sh} t \, dt \\ x^2 - 1 = \operatorname{sh}^2 t \end{array} \right\}$$

$$= \int \frac{\operatorname{sh} t}{\operatorname{sh} t} dt = \int dt = t + c$$

$$= \operatorname{Arch} x + c = \ln\left(x + \sqrt{x^2-1}\right) + c.$$

*) $$\int \sqrt{x^2-1}\, dx = \left\{ \begin{array}{l} x = \operatorname{ch} t \\ dx = \operatorname{sh} t \, dt \\ \sqrt{x^2-1} = \operatorname{sh} t \end{array} \right\} =$$

$$= \int \operatorname{sh} t \cdot \operatorname{sh} t \, dt = \int \operatorname{sh}^2 t \, dt$$

$$= \int \frac{1 + \operatorname{ch}(2t)}{2} dt =$$

$$= \frac{1}{2} t + \frac{1}{4} sh(2t) + C =$$

$$\int \cancel{\frac{1}{2} \ln(} \frac{1}{2} \operatorname{Arch} x + \frac{1}{4} sh(2 \operatorname{Arch} x) + C$$

wobei $\operatorname{Arch} x = \ln(x + \sqrt{x^2-1})$,

$$sh(2 \operatorname{Arch} x) = 2 \underbrace{sh(\operatorname{Arch} x)}_{\sqrt{x^2-1}} \cdot \underbrace{ch(\operatorname{Arch} x)}_{x}$$

$$= 2x\sqrt{x^2-1}$$

$$\int = \frac{1}{2} \ln(x + \sqrt{x^2-1}) + \frac{1}{2} x \sqrt{x^2-1} + C.$$