1. (a) Bezeichnet man $x_k = \frac{k}{n}$,

so hat mat $0 = x_0 < \ldots < x_n = 1$

und die Summe ist

$$\sum_{k=1}^{n} f(x_k)\,(x_k - x_{k-1}) = \int_0^1 g_n$$

wobei $g_n(x) = f(x_k) \quad \forall x \in (x_{k-1}, x_k)$
Treppenfunktion.

$g_n \xrightarrow{glm} f$ (VL 2, Satz 27),

also $\lim_n \sum = \lim_n \int_0^1 g_n = \int_0^1 f$.

(b). $$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + kn}} = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{1 + \frac{k}{n}}} =$$

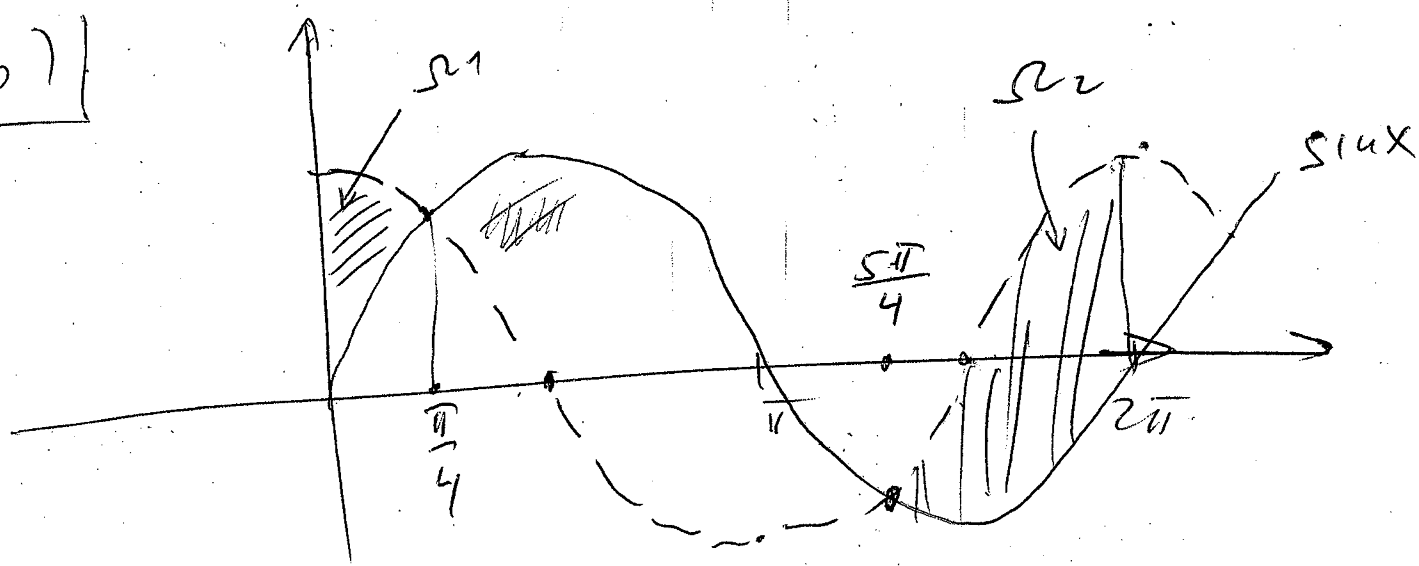
$$= \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right), \quad f(x) = \frac{1}{\sqrt{1+x}}$$

Also $$\lim = \int_0^1 \frac{dx}{\sqrt{1+x}} =$$

$$= \int_0^1 (1+x)^{-1/2}\,dx = 2(1+x)^{\frac{1}{2}} \Big|_{x=0}^{x=1}$$

$$= 2\sqrt{2} - 2.$$

4(b)



$\Omega = \Omega_1 \cup \Omega_2$

$\Omega_1 = \{0 \le x \le \frac{\pi}{4},\ \sin x \le y \le \cos x\},$

$\Omega_2 = \{\frac{5\pi}{4} \le x \le 2\pi,\ \sin x \le y \le \cos x\},$

$$|\Omega_1| = \int_0^{\pi/4} (\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_0^{\pi/4}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0 - 1$$

$$= \sqrt{2} - 1.$$

$$|\Omega_2| = \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,dx =$$

$$= (\sin x + \cos x)\Big|_{\frac{5\pi}{4}}^{2\pi} = \sin 2\pi + \cos 2\pi$$

$$- \sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) = 0 + 1 - \left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)$$

$$= 1 + \sqrt{2}.$$

$$|\Omega| = |\Omega_1| + |\Omega_2| = \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2}.$$

6) 1. Lösung: ganz direkt

~~$f(x) \longrightarrow 0$~~ Sei $\varepsilon > 0$. Aus $\lim\limits_{x \to \infty} f(x) = 0$ folgt, dass $\exists\, N > 0$: $-\varepsilon < f(x) < \varepsilon$ $\forall\, x > N$.

Also $$\int_0^x f = \underbrace{\int_0^N f}_{C_N} + \int_N^x \underbrace{f}_{|\cdot| < \varepsilon} \qquad \forall x > N$$

$$\Rightarrow C_N - \varepsilon(x-N) \leq \int_0^x f \leq C_N + \varepsilon(x-N)$$

$$\limsup_{x \to +\infty} \frac{\int_0^x f}{x} \leq \limsup_{x \to \infty} \frac{C_N + \varepsilon(x-N)}{x} = \varepsilon.$$

$$\liminf_{x \to +} \frac{\int_0^x f}{x} \geq \limsup_{x \to +\infty} \frac{C_N - \varepsilon(x-N)}{x} = -\varepsilon$$

$\varepsilon > 0$ beliebig $\Rightarrow$ $$\limsup_{x \to +\infty} \frac{\int_0^x f}{x} \leq 0$$ $$\liminf_{x \to +\infty} \frac{\int_0^x f}{x} \geq 0$$

$$\Rightarrow \lim_{x \to +\infty} \frac{\int_0^x f}{x} = 0$$

6) 2. Lösung
L'hôpitalsche Regel: mit

$$F(x) = \int_0^x f(t)\,dt$$

$$g(x) = x$$

$$\lim_{x \to +\infty} \frac{F(x)}{g(x)} = \lim_{x \to \infty} \frac{F'(x)}{g'(x)}$$

$$= \lim_{x \to \infty} \frac{f(x)}{1} = 0.$$