Z 16

Vor: $K = \mathbb{Z}/3\mathbb{Z}$, $F: K^{2\times 2} \to K[t]_{\leq 2}$,

$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := (a+b) + (b+c)t + (c+d)t^2$

$B := (v_1, v_2, v_3, v_4) := \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$ Basis von $K^{2\times 2}$,

$C = (w_1, w_2, w_3) = (1, t, t^2)$ Basis von $K[t]_{\leq 2}$

a) Beh: $M_C^B(F) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Bew: $M_C^B(F) \overset{K.5.48}{=} \left(I_C(F(v_1)) \; I_C(F(v_2)) \; I_C(F(v_3)) \; I_C(F(v_4)) \right)$,

wobei $I_C: \overset{K[t]_{\leq 2} \to K^3}{\lambda_0 + \lambda_1 t + \lambda_2 t^2 \mapsto} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}$

$F(v_1) = F\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1 + 0t + 0t^2$

$\vdots$

$\Rightarrow M_C^B(F) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ q.e.d.

b) Weitere Vor: $S := (p_1, p_2, p_3) = (1+t, t+t^2, t^2)$

Beh: S Basis von $K[t]_{\leq 2}$

Bew: i) Zeige S lin. unabh.:

$\lambda_1(1+t) + \lambda_2(t+t^2) + \lambda_3 t^2 = 0$

$\Leftrightarrow \lambda_1 + (\lambda_1 + \lambda_2)t + (\lambda_2 + \lambda_3)t^2 = 0 + 0t + 0t^2$

$\Leftrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = 0 \\ \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_3 = 0 \end{cases}$ ✓

ii a) Da $\dim(K[t]_{\leq 2}) = 3$ ist S maximale lin. unabh. Menge und damit Basis nach Beob. 3.2.21. q.e.d.

ii b) Zeige $\operatorname{span}(S) = K[t]_{\leq 2}$:

Sei $v = a_0 + a_1 t + a_2 t^2 \in K[t]_{\leq 2}$ beliebig.

$$\lambda_1(1+t) + \lambda_2(t+t^2) + \lambda_3 t^2 = a_0 + a_1 t + a_2 t^2$$

$$\Leftrightarrow \begin{cases} \lambda_1 = a_0 \\ \lambda_1 + \lambda_2 = a_1 \Rightarrow \lambda_2 = a_1 - a_0 \\ \lambda_2 + \lambda_3 = a_2 \Rightarrow \lambda_3 = a_2 - a_1 + a_0 \end{cases}$$

q.e.d.

c) weitere Vor.: $G: K[t]_{\leq 2} \to K[t]_{\leq 2}$,

$$G(a_0 + a_1 t + a_2 t^2) := a_0 + (a_0 + 2a_1)t + (2a_1 + a_2)t^2$$

Beh.: $M_S^C(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Bew.: $M_S^C(G) = (I_S(G(w_1)) \quad I_S(G(w_2)) \quad I_S(G(w_3)))$

$G(w_1) = G(1) = 1 + t = 1p_1 + 0p_2 + 0p_3$

$G(w_2) = G(t) = 2t + 2t^2 = 0p_1 + 2p_2 + 0p_3$

$G(w_3) = G(t^2) = t^2 = 0p_1 + 0p_2 + 1p_3$

$\Rightarrow M_S^C(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ q.e.d.

d) Beh.: $M_S^B(G \circ F) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Bew.: $M_S^B(G \circ F) \overset{S.\,5.4.13}{=} M_S^C(G) \cdot M_C^B(F) =$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ q.e.d.

Z.17

Vor.: $a, b, c, d \in \mathbb{R}$

$$T := \begin{pmatrix} 0 & a & b & c & d \\ a & 0 & b & c & d \\ b & a & 0 & c & d \\ c & a & b & 0 & d \\ d & a & b & c & 0 \end{pmatrix} \in \mathbb{R}^{5\times 5}$$

Beh.: $\det(T) = (a+b+c+d)\cdot a\cdot b\cdot c\cdot d$

Bew.: Wissen: $\det(S_k(\lambda)) = \lambda$ (Korollar 6.1.6)

$$\det(Q_{kl}(\mu)) = 1$$

$$\det(P_{kl}) = -1$$

Vereinfache T durch elem. Zeilenop.:

$$T \xrightarrow{\substack{Q_{21}(-1)\\ Q_{31}(-1)\\ Q_{41}(-1)\\ Q_{51}(-1)}} \begin{pmatrix} 0 & a & b & c & d \\ a & -a & 0 & 0 & 0 \\ b & 0 & -b & 0 & 0 \\ c & 0 & 0 & -c & 0 \\ d & 0 & 0 & 0 & -d \end{pmatrix} \longrightarrow$$

$$\xrightarrow{\substack{Q_{12}(1)\\ Q_{13}(1)\\ Q_{14}(1)\\ Q_{15}(1)}} \begin{pmatrix} a+b+c+d & 0 & 0 & 0 & 0 \\ a & -a & 0 & 0 & 0 \\ b & 0 & -b & 0 & 0 \\ c & 0 & 0 & -c & 0 \\ d & 0 & 0 & 0 & -d \end{pmatrix} =: T'$$

$$\det(T) \underset{\det(Q_{kl}(\mu)) = 1}{=} \det(T') = (a+b+c+d)(-a)(-b)(-c)(-d) =$$

$$= (a+b+c+d)\,abcd \qquad \text{q.e.d.}$$

Z.18

Vor.: $A := \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$

a) Beh.: $B := (v_1, v_2, v_3) := \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} \right)$ ist Basis von $\mathbb{C}^3$ aus EV von A.

Bew.: Eigenwerte berechnen:

$$0 = \det(A - \lambda E_3) = \det \begin{pmatrix} -\lambda & 0 & -1 \\ 1 & -1-\lambda & 1 \\ 1 & 0 & -\lambda \end{pmatrix} \overset{\text{LPE nach 2. Spalte}}{=}$$

$$= (-1-\lambda)(\lambda^2 + 1) = -(\lambda+1)(\lambda \mp i)(\lambda + i)$$

Genau

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = i,\ \lambda_3 = -i$ sind EW von A

mit $\mu(\lambda_{1,2,3}, A) = 1$

Eigenräume berechnen:

$\operatorname{Eig}(A, -1) = \operatorname{Lös}(A - (-1)E_3 \mid 0)$:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[Q_{21}(-1)]{Q_{32}(-1)} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[Q_{12}(1)]{S_2(\frac{1}{2})}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \operatorname{Eig}(A, -1) = \operatorname{span}(\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{= v_1})$

$\mathrm{Eig}(A,i) = \mathrm{Lös}(A - iE_3|0)$:

$$\left(\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 1 & -1-i & 1 & 0 \\ 1 & 0 & -i & 0 \end{array}\right) \xrightarrow[Q_{31}(-i)]{Q_{21}(-i)} \left(\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 0 & -1-i & 1+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[S_2\left(\frac{i-1}{2}\right)]{S_1(i)}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \mathrm{Eig}(A,i) = \mathrm{span}(\underbrace{\begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}}_{=v_2})$$

$$\mathrm{Eig}(A,-i) = \mathrm{Lös}(A + iE_3|0) = \mathrm{Lös}\left(\left(\begin{array}{ccc|c} i & 0 & -1 & 0 \\ 1 & -1+i & 1 & 0 \\ 1 & 0 & i & 0 \end{array}\right)\right) \overset{Q_{21}(i)}{\underset{Q_{31}(i)}{=}}$$

$$= \mathrm{Lös}\left(\left(\begin{array}{ccc|c} i & 0 & -1 & 0 \\ 0 & -1+i & 1-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\right) \overset{S_1(-i)}{\underset{S_2\left(\frac{-(i+1)}{2}\right)}{=}} \mathrm{Lös}\left(\left(\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\right) =$$

$$= \mathrm{span}(\underbrace{\begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}}_{=v_3})$$

~~q.e.d.~~

Nach Satz 7.2.4 ist $B = (v_1, v_2, v_3)$ lin. unabh. und damit Basis von $\mathbb{C}^3$.

q.e.d.

Z. 18 b)

weitere Vor.: $Q := (v_1\ v_2\ v_3) \in Gl(3, \mathbb{C})$

Beh.: $D := Q^{-1} A Q$ diagonal

Bew.: $A Q = A(v_1\ v_2\ v_3) = (Av_1\ Av_2\ Av_3) =$

$= (\lambda_1 v_1\ \lambda_2 v_2\ \lambda_3 v_3) = \cancel{\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}} (v_1\ v_2\ v_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

$Q^{-1}()$

$\Longleftrightarrow \underbrace{Q^{-1} A Q}_{D} = Q^{-1} Q \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\cancel{\Rightarrow D}$

q. e. d.

c) Beh.: $A^{2021} = A$

Bew.: $D = Q^{-1} A Q \quad | \ Q(\ )Q^{-1}$

$\Longleftrightarrow Q D Q^{-1} = A$

$\Rightarrow A^{2021} = \underbrace{QD\cancel{Q^{-1}}\,\cancel{Q}D\cancel{Q^{-1}} \cdots \cancel{Q}DQ^{-1}}_{2021 \text{ mal}} = Q D^{2021} Q^{-1} =$

$= Q \begin{pmatrix} (-1)^{2021} & 0 & 0 \\ 0 & (i)^{2021} & 0 \\ 0 & 0 & (-i)^{2021} \end{pmatrix} Q^{-1} = Q \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}}_{=D} Q^{-1} = A$

q. e. d.

NR: $i^4 = 1$, ~~2024 : 4 =~~ $[2021]_4 = [1]_4$

$(-i)^4 = 1$

2.19

Vor.: Skalarprodukt $\langle , \rangle : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$,

$\langle x, y \rangle := x^T A y$ mit

$$A := \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{9}{4} \end{pmatrix},$$

also $\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \right\rangle = \frac{1}{4} x_1 y_1 + 9 x_2 y_2 + 1 x_3 y_3 + \frac{9}{4} x_4 y_4.$

$U \subset \mathbb{R}^4$ Unter-VR. mit Basis $B := (v_1, v_2, v_3) := \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \\ -2 \end{pmatrix} \right)$

Beh.: $C := (w_1, w_2, w_3) := \left(\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/5 \\ 4/5 \\ 0 \end{pmatrix} \right)$ ist ONB

von U mit $\langle , \rangle$.

Bew.: Gram-Schmidt-Orthonormalisierung:

$\|v_1\| = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{\frac{1}{4} \cdot 1 \cdot 1} = \frac{1}{2}$

$\underline{w_1 = \frac{v_1}{\|v_1\|}} = 2v_1 = \underline{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$

$\tilde{w}_2 = v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle}_{= \frac{1}{4} \cdot 1 \cdot 2 = \frac{1}{2}} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

$\|\tilde{w}_2\| = \sqrt{\frac{9}{4} \cdot (-1)(-1)} = \frac{3}{2}$

$\underline{w_2 =} \frac{\tilde{w}_2}{\|\tilde{w}_2\|} = \underline{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -2/3 \end{pmatrix}}$

$$\widehat{w_3} = v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2 =$$

$$= \begin{pmatrix} -4 \\ 1 \\ 4 \\ -2 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} -4 \\ 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle}_{= \frac{1}{4}(-4)\cdot 2 = -2} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} -4 \\ 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{2}{3} \end{pmatrix} \right\rangle}_{= \frac{9}{4}(-2)(-\frac{2}{3}) = 3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{2}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} -4 \\ 1 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

$$\|\widehat{w_3}\| = \sqrt{9 \cdot 1 \cdot 1 + 1 \cdot 4 \cdot 4} = \sqrt{25} = 5$$

$$\underline{w_3} = \frac{\widehat{w_3}}{\|\widehat{w_3}\|} = \begin{pmatrix} 0 \\ 1/5 \\ 4/5 \\ 0 \end{pmatrix} \qquad \text{q.e.d.}$$

Weitere Vor.: $v = (6 \;\; 1 \;\; -4 \;\; 12)^T \in U$

b) Beh.: $v = 3 w_1 - 18 w_2 - 5 w_3$

Bew.: Da (w_1, w_2, w_3) ONB gilt

$$v = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 + \langle v, w_3 \rangle w_3$$

$$\langle v, w_1 \rangle = \frac{1}{4} \, 6 \, 2 = 3$$

$$\langle v, w_2 \rangle = \frac{9}{4} \cdot 12 \cdot \left(-\frac{2}{3}\right) = -18$$

$$\langle v, w_3 \rangle = 9(-1)\frac{1}{5} + 1(-4)\frac{4}{5} = -\frac{9}{5} - \frac{16}{5} = -\frac{25}{5} = -5$$

q.e.d.