Blatt 1. (Analysis IIa) Lösungen zu einigen Präsenzaufgaben

B. 2 $\displaystyle\int \frac{dx}{x^2+4x+5}$.

Das ist eine rationale Funktion, und es ist wichtig, die Nullstellen des Nenners zu finden. Hier hat man

$$x^2+4x+5 = (x+2)^2+1,$$

also keine reelle Nullstellen, und

$\displaystyle\frac{1}{x^2+4x^2+5}$ ist schon in der Form der Partialbruchzerlegung.

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} =$$

$$= \left\{ \text{mit Hilfe von } \int f(x+a)\,dx = \Phi(x+a) \right\}$$

$$= \arctan(x+2) + C.$$

B.4 $$\int \underbrace{x^2}_{g} \underbrace{e^{-3x}}_{f'} dx \overset{PI}{=} \underbrace{x^2}_{g} \cdot \underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{f}$$

$$- \int \underbrace{2x}_{g'} \cdot \underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{f} dx =$$

$$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int \underbrace{x}_{g}\underbrace{e^{-3x}}_{f'}\,dx \overset{PI}{=} -\frac{1}{3}x^2e^{-3x}$$ ②

$$+\frac{2}{3}\left(\underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{f}\cdot\underbrace{x}_{g} - \int \underbrace{1}_{g'}\cdot\left(-\frac{1}{3}e^{-3x}\right)dx\right)$$

$$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\underbrace{\int e^{-3x}\,dx}_{-\frac{1}{3}e^{-3x}+\text{const}} =$$

$$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C.$$

B.7 $$\int\sqrt{1-x^2}\,dx = \int \underbrace{1}_{f'}\cdot\underbrace{\sqrt{1-x^2}}_{g}\,dx \overset{PI}{=}$$

$$= \underbrace{x}_{f}\underbrace{\sqrt{1-x^2}}_{g} - \int \underbrace{x}_{f}\cdot\underbrace{\left(-\frac{x}{\sqrt{1-x^2}}\right)}_{g'}dx$$

$$= x\sqrt{1-x^2} + \int\frac{x^2}{\sqrt{1-x^2}}\,dx = \left\{\begin{aligned} x^2 &= x^2-1+1 \\ &= -(1-x^2)+1 \end{aligned}\right\}$$

$$= x\sqrt{1-x^2} + \int\left(-\frac{1-x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right)dx =$$

$$= x\sqrt{1-x^2} - \int\sqrt{1-x^2}\,dx + \underbrace{\int\frac{1}{\sqrt{1-x^2}}\,dx}_{\arcsin x+\text{Const}}$$

Also für $I = \int\sqrt{1-x^2}dx$ gilt

$I = x\sqrt{1-x^2} - I + \arcsin x + C$, und

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

B.10 $\int e^{\sqrt{x}}\,dx = \left\{ \begin{array}{l} \text{Substitution} \\ \sqrt{x} = t, \text{ also } t^2 = x, \\ \text{und } dx = 2t\,dt \end{array} \right\}$ 

$= \int e^{t} \cdot 2t\,dt = 2\int \underbrace{t}_{g}\underbrace{e^{t}}_{f'}\,dt \quad \underline{\underline{PI}}$

$= 2\left( \underbrace{t}_{g}\underbrace{e^{t}}_{f} - \underbrace{\int \underbrace{1}_{g'} \cdot \underbrace{e^{t}}_{f}\,dt}_{e^t} \right) =$

$= 2te^{t} - 2e^{t} + C = \{ t = \sqrt{x} \} = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

B.13 $\int \frac{1}{x^3-1}\,dx$ : Rationale Funktion, man sucht nach der Partialbruchzerlegung.

Der Nenner $Q(x) = x^3 - 1$ hat 1 als Nullstelle, $Q(1)$, also $Q(x) = (x-1)(\underbrace{???}_{})$

$ax^2 + bx + c$.

$(x-1)(ax^2+bx+c) =$

$= ax^3 - ax^2 + bx^2 - bx + cx - c$

$= x^3 - 1$

$$\Rightarrow \begin{cases} a = 1 \\ -a + b = 0 \\ -b + c = 0 \\ -c = -1 \end{cases} \quad \left| \begin{array}{l} \text{koef. bei} \\ x^3 \\ x^2 \\ x \\ 1. \end{array} \right.$$

$\Rightarrow a = b = c = 1$

$$\Rightarrow \quad x^3 - 1 = (x-1)(x^2+x+1).$$

Das Polynom $x^2+x+1$ hat keine relle Nullstellen, also das ist die gesuchte Zerlegung des Nenners in elementare Faktoren. Jetzt muss man die Darstellung

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x^2+1}$$

finden. Es muss gelten:

$$A(x^2+x+1) + (Bx+C)(x-1) = 1,$$

$$Ax^2 + Ax + A + Bx^2 + Cx - Bx - C = 1$$

$$\begin{cases} A + B = 0 & \Rightarrow B = -A \\ A - B + C = 0 & A-(-A)+A-1=0 \Rightarrow A = \frac{1}{3} \\ A - C = 1. & \Rightarrow C = A - 1 \\ A = \frac{1}{3}, \quad B = -\frac{1}{3}, \quad C = -\frac{2}{3} & \end{cases}$$

$$\Rightarrow \quad \frac{1}{x^3-1} = \frac{1}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{x+2}{x^2+x+1}$$

und $$\int \frac{dx}{x^3-1} = \frac{1}{3}\underbrace{\int \frac{dx}{x-1}}_{\ln|x-1|} - \frac{1}{3}\underbrace{\int \frac{x+2}{x^2+x+1}\,dx}_{I}$$

$$I = \int \frac{x+\frac{1}{2} + \frac{3}{2}}{x^2+x+1}\,dx = \frac{1}{2}\underbrace{\int \frac{2x+1}{x^2+x+1}\,dx}_{J_1}$$

$$+ \frac{3}{2}\int \frac{dx}{x^2+x+1}$$

$$J_1 = \int \frac{2x+1}{x^2+x+1}\,dx = \left\{ \begin{array}{l} \text{Substitution} \\ \varphi(x) = x^2+x+1 \\ \varphi'(x) = 2x+1 \\ (2x+1)\,dx = d\varphi \end{array} \right\}$$

$$= \int \frac{d\varphi}{\varphi} = \ln|\varphi| + C$$

$$= \ln(x^2+x^2+1) + C.$$

(da $\varphi > 0$, $|\varphi| = \varphi$)

$$J_2 = \int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \left\{ \begin{array}{l} x = \frac{\sqrt{3}}{2}t - \frac{1}{2} \iff t = \frac{2x+1}{\sqrt{3}} \\ dx = \frac{\sqrt{3}}{2} t \end{array} \right\} =$$

$$= \int \frac{\frac{\sqrt{3}}{2}\,dt}{\left(\frac{\sqrt{3}}{2}\right)^2 t^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \int \frac{dt}{t^2+1} =$$

$$= \frac{2}{\sqrt{3}} \arctan t + C = \frac{2}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C.$$

Also,

$$I = \frac{1}{2} J_1 + \frac{3}{2} J_2 = \frac{1}{2} \ln(x^2+x+1) + C$$

$$+ \sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} + C, \quad \text{und}$$

$$\int \frac{dx}{x^3-1} = \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln(x^2+x+1)$$

$$- \frac{1}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C.$$

B.16 $\int \sin^2 x \, dx = \left\{ \begin{array}{l} \text{Trigonometrie:} \\ \sin^2\alpha = \dfrac{1-\cos(2\alpha)}{2} \end{array} \right\}$ 6

$$= \int \frac{1-\cos(2x)}{2} \, dx = \frac{1}{2} \int 1 \cdot dx$$

$$- \frac{1}{2} \int \cos(2x) \, dx = \frac{1}{2} x - \frac{1}{2} \cdot \frac{1}{2} \sin(2x) + C$$

$$= \frac{1}{2} x - \frac{1}{4} \sin(2x) + C .$$