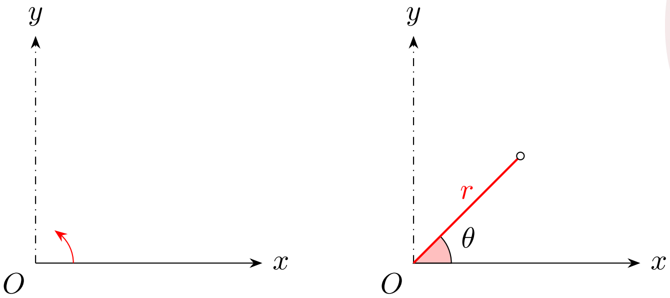
§2.7 极坐标系 · 径向速度与横向速度

一、平面极坐标系

- 选择一个点为坐标原点
- 选择一个方向为极轴 (如下图中的 x 轴)
- 规定一个方向为辐角 (极角) 增大的方向 (通常选择逆时针方向)
- 从坐标原点到某个位置之间的距离称为矢径 (极径)，用 r 表示
- 极径与极轴之间的夹角称为辐角 (极角)，用 θ 表示
- (r, θ) 称为质点位置的极坐标



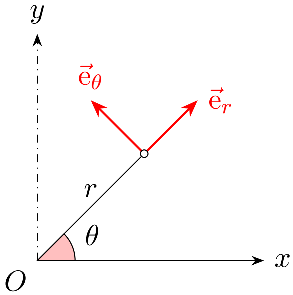
因此，极坐标系中质点的运动学方程为

$$r = r(t), \theta = \theta(t)$$

消去 t，即得轨道方程

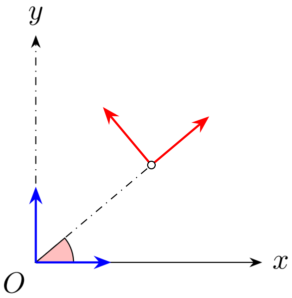
$$r = r(\theta)\text{，或 } \theta = \theta(r)\text{，或 } f(r, \theta) = 0$$

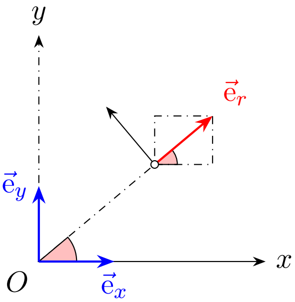
沿着极径方向的单位矢量称为径向单位矢量，记为 $\vec{\mathrm{e}}_r$；与径向单位矢量垂直且指向极角增大的方向的单位矢量称为横向单位矢量，记为 $\vec{\mathrm{e}}_\theta$

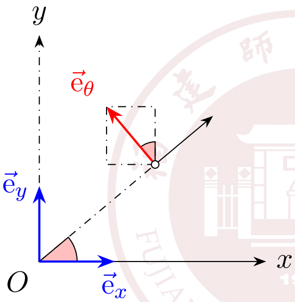


$\vec{\mathrm{e}}_r$、$\vec{\mathrm{e}}_\theta$ 与 $\vec{\mathrm{e}}_x$、$\vec{\mathrm{e}}_y$ 之间的关系为

$$\vec{\mathrm{e}}_r = \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y \qquad \vec{\mathrm{e}}_\theta = -\sin\theta\,\vec{\mathrm{e}}_x + \cos\theta\,\vec{\mathrm{e}}_y$$

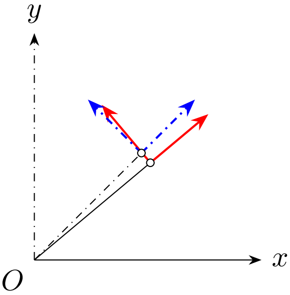


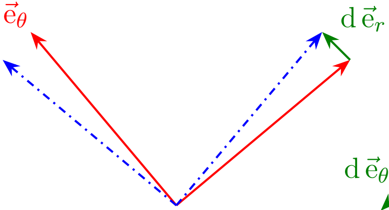




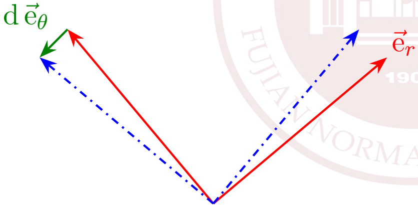
显然在不同位置 (θ 不同时)，$\vec{e}_r$ 和 $\vec{e}_\theta$ 是变化的。当 θ 发生了微小的变化，从 θ 变化到 $\theta + d\theta$ 时，两个单位矢量的改变量分别为

$$d\,\vec{e}_r = d\theta\,\vec{e}_\theta$$





$$d\,\vec{e}_\theta = d\theta(-\vec{e}_r)$$



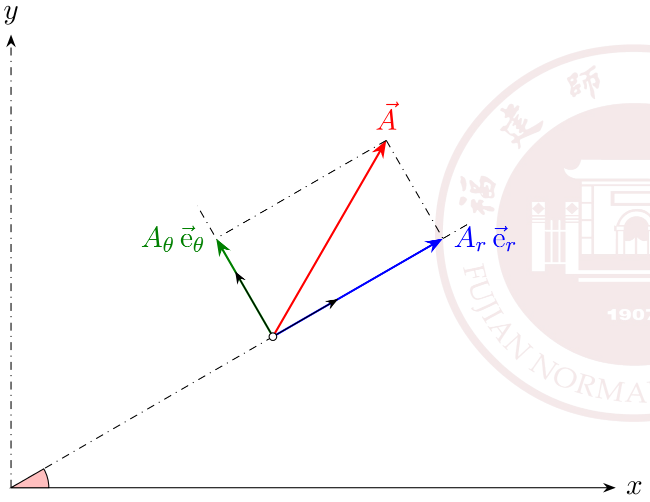
$$
\begin{aligned}
\vec{\mathrm{e}}_r &= \cos\theta\,\vec{\mathrm{e}}_x + \sin\theta\,\vec{\mathrm{e}}_y \\
\vec{\mathrm{e}}_\theta &= -\sin\theta\,\vec{\mathrm{e}}_x + \cos\theta\,\vec{\mathrm{e}}_y \\
\mathrm{d}\,\vec{\mathrm{e}}_r &= -\sin\theta\,\mathrm{d}\theta\,\vec{\mathrm{e}}_x + \cos\theta\,\mathrm{d}\theta\,\vec{\mathrm{e}}_y \\
&= \mathrm{d}\theta(-\sin\theta\,\vec{\mathrm{e}}_x + \cos\theta\,\vec{\mathrm{e}}_y) \\
&= \mathrm{d}\theta\,\vec{\mathrm{e}}_\theta \\
\mathrm{d}\,\vec{\mathrm{e}}_\theta &= -\cos\theta\,\mathrm{d}\theta\,\vec{\mathrm{e}}_x - \sin\theta\,\mathrm{d}\theta\,\vec{\mathrm{e}}_y \\
&= \mathrm{d}\theta(-\cos\theta\,\vec{\mathrm{e}}_x - \sin\theta\,\vec{\mathrm{e}}_y) \\
&= \mathrm{d}\theta(-\,\vec{\mathrm{e}}_r)
\end{aligned}
$$

矢量在平面极坐标系中的表示为

$$\vec{A} = A_r\,\vec{e}_r + A_\theta\,\vec{e}_\theta$$

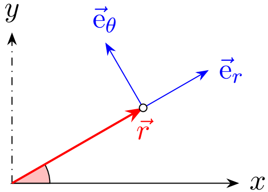
因此

$$A = |\vec{A}| = \sqrt{A_r^2 + A_\theta^2}$$



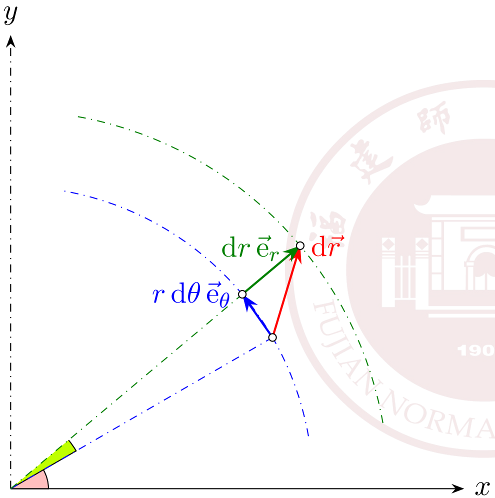
在平面极坐标系中，位置矢量表示成

$$\vec{r} = r\,\vec{\mathrm{e}}_r$$



有限大小的位移 $\Delta\vec{r}$ 无法表示，但在一个从 (r,θ) 变化到 $(r+\mathrm{d}r, \theta+\mathrm{d}\theta)$ 的无限小的元过程中，元位移

$$\begin{aligned} \mathrm{d}\vec{r} &= \mathrm{d}(r\,\vec{\mathrm{e}}_r) \\ &= \mathrm{d}r\,\vec{\mathrm{e}}_r + r\,\mathrm{d}\,\vec{\mathrm{e}}_r \\ &= \mathrm{d}r\,\vec{\mathrm{e}}_r + r\,\mathrm{d}\theta\,\vec{\mathrm{e}}_\theta \end{aligned}$$



二、速度

平面极坐标系中，速度可以一般地表示成

$$\vec{v} = v_r \, \vec{\mathrm{e}}_r + v_\theta \, \vec{\mathrm{e}}_\theta$$

而

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

$$\mathrm{d}\vec{r} = \mathrm{d}r \, \vec{\mathrm{e}}_r + r \, \mathrm{d}\theta \, \vec{\mathrm{e}}_\theta$$

所以

$$\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t} \vec{\mathrm{e}}_r + r \frac{\mathrm{d}\theta}{\mathrm{d}t} \vec{\mathrm{e}}_\theta$$

$$v_r = \frac{\mathrm{d}r}{\mathrm{d}t}, v_\theta = r \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$v = |\vec{v}| = \sqrt{v_r^2 + v_\theta^2}$$

对于圆周运动，如果取圆心为坐标原点，

$$r = R = \mathrm{Const}$$

$$v_r = \frac{\mathrm{d}r}{\mathrm{d}t} = 0$$

$$v_\theta = r \frac{\mathrm{d}\theta}{\mathrm{d}t} = R\omega$$

其中

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

三、加速度

平面极坐标系中，加速度可以一般地表示成

$$\vec{a} = a_r\,\vec{e}_r + a_\theta\,\vec{e}_\theta$$

而

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v} = v_r\,\vec{e}_r + v_\theta\,\vec{e}_\theta$$

$$v_r = \frac{dr}{dt}$$

$$v_\theta = r\frac{d\theta}{dt}$$

$$d\,\vec{e}_r = d\theta\,\vec{e}_\theta$$

$$d\,\vec{e}_\theta = d\theta(-\,\vec{e}_r)$$

$$\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}(v_r\,\vec{e}_r + v_\theta\,\vec{e}_\theta) \\
&= \frac{dv_r}{dt}\,\vec{e}_r + v_r\frac{d\,\vec{e}_r}{dt} + \frac{dv_\theta}{dt}\,\vec{e}_\theta + v_\theta\frac{d\,\vec{e}_\theta}{dt} \\
&= \frac{dv_r}{dt}\,\vec{e}_r + v_r\frac{d\theta}{dt}\,\vec{e}_\theta + \frac{dv_\theta}{dt}\,\vec{e}_\theta - v_\theta\frac{d\theta}{dt}\,\vec{e}_r \\
&= \left(\frac{dv_r}{dt} - v_\theta\frac{d\theta}{dt}\right)\vec{e}_r + \left(v_r\frac{d\theta}{dt} + \frac{dv_\theta}{dt}\right)\vec{e}_\theta
\end{aligned}$$

$$\begin{aligned}
a_r &= \frac{dv_r}{dt} - v_\theta\frac{d\theta}{dt} \\
&= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2
\end{aligned}$$

$$\begin{aligned}
a_\theta &= v_r\frac{d\theta}{dt} + \frac{dv_\theta}{dt} \\
&= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) \\
&= \frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \\
&= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}
\end{aligned}$$

对于圆周运动，如果取圆心为坐标原点，

$$r = R = \mathrm{Const}$$

$$v_r = \frac{\mathrm{d}r}{\mathrm{d}t} = 0$$

$$v_\theta = R\omega$$

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$a_r = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -R\omega^2$$

$$a_\theta = 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = R\beta$$

$$\beta = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t}$$