5 Double counting

Main idea Choose a quantity that can be expressed in two ways and obtain some information by equating the two results.

Definition A *t-design* is a collection $\mathcal{B}$ of subsets (blocks) of a set X, all of size k, such that any t elements of X lie in the same number r_t of blocks. The parameters are (v, k, r_t) where $v = |X|$.

Proposition 5.1 *If $\mathcal{B}$ is a t-design with parameters (v, k, r_t), then $\mathcal{B}$ is also a $(t-1)$-design with parameters (v, k, r_{t-1}), where*

$$r_{t-1} = \frac{v-t+1}{k-t+1} r_t$$

Corollary 5.2 *A t-design is also a s-design, for $s \leq t$, with*

$$r_s = \frac{v-t+1}{k-t+1}\frac{v-t+2}{k-t+2} \cdots \frac{v-s}{k-s} r_t$$

Proposition 5.3 *Let $\mathcal{B}$ be 2-design. Denote by $r = r_1$, $\lambda = r_2$ and b the number of blocks. Then $bk = vr$ and $r(k-1) = (v-1)\lambda$.*

Examples

1. In a tournament with n teams in which every match has one winner and one loser, denote by W_i and L_i the number of wins and, respectively, losses of team i. Prove that

$$W_1 + ... + W_n = L_1 + ... + L_n$$

2. Prove that

$$d(1) + d(2) + ... + d(n) = \lfloor\frac{n}{1}\rfloor + \lfloor\frac{n}{2}\rfloor + ... + \lfloor\frac{n}{n}\rfloor$$

where $d(k)$ denotes the number of divisors of k.

3. Prove that for n a positive integer

$$\sum_{d/n} \varphi(d) = n$$

where $\varphi(d)$ is the Euler's totient function representing the number of positive integers that are coprime with n and at most equal to n.

4. Twenty-five people form several committees. Each committee has five members, and any two committees have at most one common member. Prove that there are at most 30 committees.

5. Let n and k be positive integers, and let S be a set of n points in the plane such that no three points of S are collinear, and for any point P of S there are at least k points of S equidistant from P. Prove that

$$k < \frac{1}{2} + \sqrt{2n}$$

Problems

1. Prove that

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$ n+1 students pair up

using double counting

2. Fifteen (15) students join a summer course. Every day, three (3) students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for? ~~(15,3,1) 2-design~~

$n C_2^3 = C_2^{15}$ $n = 35$

3. Prove the propositions about designs stated at the beginning of the section.

4. Let $a_1 \leq a_2 \leq ... \leq a_n = m$ be positive integers. Denote by b_k the number of those a_i for which $a_i \geq k$. Prove that $\boxed{bk = vr}$

$$a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_m$$

5. Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, ..., n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!$$

6. Prove that

$$\sigma(1) + \sigma(2) + ... + \sigma(n) = \lfloor\frac{n}{1}\rfloor + 2\lfloor\frac{n}{2}\rfloor + ... + n\lfloor\frac{n}{n}\rfloor$$

where $\sigma(k)$ denotes the sum of divisors of k

7. In a competition, there are m contestants and n judges, where $n \geq 3$ is an odd integer. Each judge rates each contestant as either pass or fail. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}$$

8. *In a rectangular array of non-negative real numbers with m rows and n columns, each row and column contains at least one positive element. Moreover, if a row and column intersect in a positive element, then their sum of the elements is the same. Prove that $m = n$.

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if $n>m$, 0

if $n\le m$

6 Inclusion-exclusion principle

Proposition 6.1 *For $A_1, A_2, \dots, A_n$ finite sets,*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{i<j}|A_i\cap A_j| + \sum_{i<j<k}|A_i\cap A_j\cap A_k| - \dots + (-1)^{n-1}|A_1\cap \dots \cap A_n|$$

$= n^m - n(n-1)^m + \binom{n}{2}(n-2)^m + (-1)^n\binom{n}{n}(n-n)^m$

Remarks 6.2 *Moreover, the following inequalities hold:*

$$\left|\bigcup_{i=1}^{n} A_i\right| \le \sum_{i=1}^{n}|A_i|$$

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$\cancel{1 - n\left(\frac{n-1}{n}\right)^m}$

$\cancel{+\binom{n}{2}\left(\frac{n-2}{n}\right)^m}$

$$\left|\bigcup_{i=1}^{n} A_i\right| \ge \sum_{i=1}^{n}|A_i| - \sum_{i<j}|A_i\cap A_j|$$

For k odd:

$$\left|\bigcup_{i=1}^{n} A_i\right| \le \sum_{i=1}^{n}|A_i| - \sum_{i<j}|A_i\cap A_j| + \dots + \sum_{i_1<i_2<\dots<i_k}|A_{i_1}\cap A_{i_2}\cap\dots\cap A_{i_k}|$$

$= \frac{1}{e}$

For k even:

$$\left|\bigcup_{i=1}^{n} A_i\right| \ge \sum_{i=1}^{n}|A_i| - \sum_{i<j}|A_i\cap A_j| + \dots - \sum_{i_1<i_2<\dots<i_k}|A_{i_1}\cap A_{i_2}\cap\dots\cap A_{i_k}|$$

Examples

1. How many positive integers less than 1000 are divisible by one of 2, 3, 5?
2. How many permutations of $(1, 2, \dots n)$ have no fixed point? (a permutation σ has a fixed point if there is k with $\sigma(k) = k$)
3. Let A, B be two sets with m and, respectively, n elements. Determine the number of surjective functions from A to B. (a function $f : A \to B$ is surjective if for any $b \in B$ there exists $a \in A$ with $f(a) = b$)
4. Let $n \in \mathbb{N}$ and consider its decomposition in prime factors: $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$. Prove that $\varphi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\dots(1-\frac{1}{p_n})$, where $\varphi(n)$ represents the number of integers less than n that are coprime with n.

n^m functions

onto function

$n^m -$ non-surjective

$= n^m - n(n-1)^m + \binom{n}{2}(n-2)^m + (-1)^n\binom{n}{m}(n-m+1)^m$

$\sum_{1\le i\le j\le n}|A_i\cap A_j|$

$j \longrightarrow$ set of function which j isn't in the image $(1\le j\le n)$

set of non surjective

$A=\{1,2,\dots,m\}$

$B=\{1,2,\dots,n\}$

$|A_1\cup A_2\cup \dots A_n| = \sum_{1\le j\le n}|A_j| -$

$= n\cdot(n-1)^m - \binom{n}{2}(n-2)^m$

4 Recursion and Fibonacci numbers

Idea Sometimes we have to count something for n an integer, but we cannot count directly, as we did in the first day. In that case, it might prove useful to denote our number by $f(n)$ and try to express it in terms of $f(0), f(1), \ldots, f(n-1)$. After that we have some options:

- We could try small cases, then guess $f(n)$ and prove by induction that it is the right answer.

- We can try to develop a method to solve more general recursions.

First order recursions(easy): Let $a \in \mathbb{R}*$. The general solution to the recursion $x_{n+1} = a \cdot x_n$ is $x_n = ca^n$, where $c = a_0$

Second order recursions: Let $a, b \in \mathbb{R}$, $b \neq 0$. Consider the recursion: $x_{n+2} = ax_{n+1} + bx_n$

- if the equation $x^2 - ax - b = 0$ has two distinct real roots λ_1 and λ_2, the general solution of the recursion is of the form

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n$$

where the two distinct roots of the equation and c_1, c_2 are to be determined in terms of the first two terms x_0, x_1.

- if the equation $x^2 - ax - b = 0$ has one double real root λ, the general solution of the recursion is of the form

$$x_n = c_1\lambda^n + c_2n\lambda^n$$

where the two distinct roots of the equation and c_1, c_2 are to be determined in terms of the first two terms x_0, x_1.

Fibonnaci numbers The Fibonnaci numbers F_n are defined by

$$F_1 = F_2 = 1$$

$$F_{n+2} = F_n + F_{n+1}$$

Binet's Formula For any positive integer n

$$F_n = \frac{1}{\sqrt{5}}(\phi_1^n + \phi_2^n)$$

where $\phi_1 = \frac{\sqrt{5}+1}{2}$ and $\phi_2 = \frac{\sqrt{5}-1}{2}$

induction: $F_3 = 2$

$F_{n+3} = F_{n+1} + F_{n+2} = 1 + \dots + F_{n} + F_{n+1}$

Examples

1. Prove that:
$$F_{n+2} = 1 + F_1 + F_2 + \dots + F_n$$

$a_n = a_{n-1} + a_{n-2}$

$a_1 = 1$

$a_2 = 2$

$a_n = F_{n+1}$

2. A frog climbs up some stairs. It can climb either one or two steps at one time. In how many ways can the frog get on the nth step?

3. Prove that any positive integer can be written as the sum of distinct Fibonacci numbers. Is the writing unique?

4. (Towers of Hanoi) Given a stack of n disks on a pole, arranged from largest on the bottom to smallest on top, together with two empty poles, what is the minimum number of moves required to move the stack of disks from one pole to another? A "move" means moving the top disk from one pole to any other pole, provided a larger disk is never placed on top of a smaller one.

Problems

1. How many sequences of As, Bs, and Cs are there, with length n, and with no two consecutive letters identical (That is, no instance of AA, BB, or CC)?

$3 \times 2^{n-1}$

2. Prove that $gcd(F_n, F_{n+1}) = 1$.

3. Prove that
$$F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$$

4. Solve the recurrence:
$$a_{n+1} = 2a_n + 3$$
$$a_1 = 5$$

$(a_{n+1} + 3) = 2(a_n + 3)$

$a_n = 2^{n+2} - 3$

5. Solve the recurrence:

$$a_{n+2} = 3a_{n+1} + a_n$$
$$a_0 = 0; a_1 = 1$$

$a_n = \frac{1}{\sqrt{13}} \times \left(\frac{3+\sqrt{13}}{2}\right)^n - \frac{1}{\sqrt{13}} \times \left(\frac{3-\sqrt{13}}{2}\right)^n$

6. Solve the recurrence:

$$a_{n+2} = a_{n+1} + a_n + 1$$
$$a_0 = 0; a_1 = 1$$

$a_n = \frac{\sqrt{5}}{5} \times \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5} \times \left(\frac{1-\sqrt{5}}{2}\right)^n - 1$

7. Solve the recurrence:

$$a_{n+2} = a_{n+1} + a_n + n \boxed{-1}$$
$$a_0 = 0; a_1 = 0$$

$a_n = \frac{\sqrt{5}}{5} \times \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5} \times \left(\frac{1-\sqrt{5}}{2}\right)^n - n$

8. How many subsets of $\{1, 2, ..., n\}$ do not contain any pair of consecutive numbers?

$a_n = \frac{\sqrt{5}}{5} \times \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{\sqrt{5}}{5} \times \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

9. Determine the number of ways to cover a $2 \times n$ rectangle with 2×1 tiles. What about with 2×1 and 2×2 tiles?

10. Determine the maximum number of regions (finite or infinite) formed by n lines in the plane.

$\frac{n^2+n}{2} + 1$

11. An alphabet contains three letters: a, b, c. Determine the number of words of length n that contain an even number of as.

12. *In how many ways can we triangulate a convex n-gon using only non-intersecting diagonals?

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2 Combinatorial identities

Proposition 2.1 *We have the following recursive identities:*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{k_1, k_2, \ldots, k_r} = \sum_{j=1}^{r} \binom{n-1}{k_1, \ldots, k_j - 1, \ldots, k_r}$$

Proposition 2.2 *Binomial and multinomial formulas:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x_1 + \cdots + x_r)^n = \sum_{k_1+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} \ldots x_r^{k_r}$$

Proposition 2.3 *For p prime and n an integer, define $e_p(n)$ to be the exponent of p in n. Then we have:*

$$e_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$$

Legendre's

Definition The nth $\boxed{\text{Catalan}}$ number is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Examples

1. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. Prove the following identity:

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

3. Prove that:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = (n+1)\, C_n$$

If $a+b \in \mathbb{Z}$ and $a \notin \mathbb{Z}$

$[a] < a$

Then $[a]+[b]+1 = [a+b]$

$\left\lfloor \frac{m+n}{p^l} \right\rfloor - \left\lfloor \frac{m}{p^l} \right\rfloor - \left\lfloor \frac{n}{p^l} \right\rfloor \geq 1$

Problems

1. Find the exponent of 3 in 30!. $=14$
2. For which $n \in \mathbb{N}$, $n!$ ends in exactly 1000 zeros?
3. Find the coefficient of x^2yz^3 in $(x+y+z)^6$.
4. Find the coefficient of x^2yz^3 in $(3x+\frac{1}{5}y+z)^6$.
5. Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$
6. Show that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$. Can you find a closed form for $\sum_{k=0}^{m}(-1)^k\binom{n}{k}$, where $m \leq n$?
7. Compute $\sum_{k=0}^{n} k^2\binom{n}{k}$
8. Show that $\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}$
9. Compute $\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}$
10. Prove combinatorially that

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}$$

11. Let m, n be positive integers with $gcd(m,n) = 1$. Prove that $\binom{n+m}{m}$ is divisible by $m+n$. (Try to do this both by using Legendre's identity and by using a clever trick)
12. Prove that

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k}$$

Let p a prime divisor of $(m+n)$

$k = V_p(m+n)$

$\boxed{\text{prove } p^k \mid \binom{m+n}{m}}$

$$V_p\left(\frac{(m+n)!}{m!\,n!}\right) = V_p((m+n)!) - V_p(m!) - V_p(n!)$$

$$= \sum_{l=1}^{\infty}\left(\left\lfloor \frac{m+n}{p^l} \right\rfloor - \left\lfloor \frac{m}{p^l} \right\rfloor - \left\lfloor \frac{n}{p^l} \right\rfloor\right)$$

$$V_p((m+n)!) = \sum_{k=0}^{\infty}\left\lfloor \frac{m+n}{p^k} \right\rfloor$$

$$V_p(m!) = \sum_{k\geq 0}^{\infty}\left\lfloor \frac{m}{p^k} \right\rfloor$$

if $p \mid m+n$, then p doesn't divide m/n

$\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$

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11 Graphs

Definition An (undirected) *graph* consists of a vertex set V and a set of edges between the vertices, which are a set of 2-elements subsets of V, denoted by E.

- A graph is *simple* if between every two vertices there is at most one edge and there are no *loops* (edges of the form (v, v)). Unless otherwise stated, we will assume our graphs are simple
- Two vertices are *adjacent* (or *neighbours*) if there is an edge between them. Two edges are *adjacent* if they share a common endpoint.
- The *degree* of a vertex $v \in V$ is the number of edges incident to v (i.e. having v as an endpoint). Denote it by $d(v)$.
- For a vertex x, denote by $N(x)$ the set of vertices adjacent to the vertex x. For a set of vertices X, denote by $N(X)$ the set of vertices adjacent to at least one vertex in set X.
- A *path* from v to w is a sequence of distinct vertices $v = v_0, v_1, \ldots, v_n = w$ such that $v_i v_{i+1} \in E$ for $i = 0, 2, ..., n-1$. n is called the *length* of the path.
- A graph is *connected* if for any vertices v and w there is a path from v to w.
- A *cycle* is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that $v_1 = v_n$ and $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n$ (indices taken cyclically). A *circuit* is a cycle in which the vertices are pairwise distinct.
- A *tree* is a connected graph with no cycles.
- An *Hamiltonian circuit* is a circuit that that contains each vertex exactly once.
- An *eulerian cycle* is a cycle that uses every edge exactly once.
- A *subgraph* of a graph G is a subset of the vertices of G together with the edges between them that were in G.
- A graph is *regular* if all vertices have the same degree.

- K_n is the complete graph on n vertices, i.e. the graph with all possible edges

Proposition 11.1 *In every graph, the number of vertices of odd degree is even.*

Proposition 11.2 *A connected graph has an eulerian cycle if and only if every vertex has an even degree.*

Examples

1. Prove that in any graph there are two vertices of the same degree.
2. Let G be a connected graph. An edge is said to be a *cut-edge* if its removal disconnects the graph. Prove that an edge is a cut edge if and only if it is not an edge of a cycle.
3. Let G be a connected graph. Prove that two paths that are both the longest path in the graph share a common vertex.

take out

4. Show that every connected graph G has a vertex v such that $G \setminus v$ is connected.
5. Consider a graph G such that every vertex has degree at least δ. Prove that there is a path of length δ.

Problems

1. Is there a graph with degrees:

   1,2,2,3,4 ✓

   0,1,2,3,4 ×

   - ~~1, 2, 3, 4, 5~~
   - 1, 1, 2, 2, 8
   - 1, 2, 3, 4, 5, 6, 7

   0,1,1,2,2,3,3,3 ×

2. How many paths of length $n-1$ are there in K_n? 5!
3. Show that every tree contains a vertex of degree ~~exactly 1~~, which is called a *leaf*.
4. Show that every connected graph contains a spanning tree. This is a subgraph which is a tree, that includes all of the original vertices.

can delete edges

5. For G a graph with n vertices, prove that the following are equivalent:
   - G is a tree
   - G is a connected graph with $n-1$ edges induction
   - for any 2 vertices u and v in G there is a unique path from u to v
   - The graph G without any edge is disconnected (every edge is a cut-edge)

6. Suppose that the graph G has all degrees at most Δ. Prove that it is possible to color the vertices of G using at most $\Delta + 1$ colors, such that no pair of adjacent vertices receives the same color.

7. What is the maximum number of edges in a disconnected graph with 2016 vertices?

8. Prove that every n-vertex graph with $n+1$ edges contains at least two (possibly overlapping) cycles. Does it always contain at least three?

9. In a graph with n vertices, every vertex has k neighbours and every two vertices have b common neighbours. Prove that

$$n\binom{k}{2} = b\binom{n}{2}$$

10. Let G a graph with n vertices, every vertex has k neighbours, every two joined vertices have a common neighbours and any two non-joined vertices have b common neighbours (such a graph is called *strongly regular*). Prove that $k(k-a-1) = b(n-k-1)$. What strongly regular graphs can you think of with $n \leq 11$?

11. In a community of more than six people, each member exchanges letters with precisely three other members of the community. Prove that the community can be divided into two nonempty groups so that each member exchanges letters with at least two members of the group he belongs to.

12. *Let G be a connected graph with an even number of vertices. Prove that we can choose some of the edges such that every vertex is adjacent to an odd number of them.

13. *Prove that for any graph G we can divide the vertices into two sets A and B such that there are at least $\frac{|E|}{2}$ edges between A and B.

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12 Bipartite graphs, Hall's Theorem and Mantel Theorem

Definition A graph is *bipartite* if the set V of vertices can be divided into sets A and B such that every edge is between a vertex in A and one in B.

- A *matching* for the set A is a set of edges $\{ab, a \in A, b \in B\}$, no two adjacent, such that every vertex $a \in A$ is the endpoint of one of them.

Proposition 12.1. *A finite graph is bipartite if and only if every cycle has even length.*

Hall's theorem Consider a finite bipartite graph with sets A, B. If for every set $X \subset A$, we have $|N(X)| \geq |X|$, then there is a matching for A.
Note The condition $|N(X)| \geq |X|$ is called Hall's condition.

Mantel's theorem If a graph G with n vertices contains no triangle, then it contains at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.
Note: Equality is obtained when G is a bipartite graph with components of equal size or difference in size 1.

The following is a generalisation of previous theorem:

Turan's theorem If a graph G with n vertices contains no K_{r+1} as a subgraph, then the number of edges is at most $\left(1 - \frac{1}{r}\right) \frac{n^2}{2}$.
Note: Maximum number of edges is obtained for a graph with r components of equal size or nearly equal size. Such a graph is called Turan graph $T_r(n)$.

Examples

1. A deck of cards is arranged, face up, in a 4×13 array. Prove that one can pick a card from each column in such a way as to get one card of each denomination.

2. A latin $n \times n$ square is a square in which all unit squares are completed with numbers in $\{1, 2, ...n\}$ such that every number appears exactly once on every line and every column. Prove that if we have completed the first k lines of an $n \times n$ square with numbers in $\{1, 2, ..., n\}$ such that every number appears at most once on every row and every column, we can extend it to a latin $n \times n$ square.

3. A graph is called r-colourable if we can colour the vertices such that no edge has two endpoints of the same colour. Prove that the Turan graph $T_r(n)$ is the r-colourable graph with the maximum number of edges.

Problems

1. Prove that a graph is bipartite if and only if it has no cycle of odd length.

2. A country has 2016 airports connected by some direct flights. For any three airports, some two are not connected by a direct flight. What is the maximum number of direct flights that can be offered?

3. Prove that the number of edges in the Turan graph $T_r(n)$ is less or equal than $\left(1-\frac{1}{r}\right)\frac{n^2}{2}$, with equality when r divides n.

4. Let $n \in \{1, 2, \dots 8\}$. Consider an 8×8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

5. Let $X = \bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n} Y_i$ be two partitions of set X with all X_i and Y_i having the same size. Prove that we can pick elements $x_1, x_2, \dots x_n \in X$ which are in different sets in both decompositions.

6. An $n \times n$ table is filled with 0 and 1 so that if we choose randomly n cells, no two of them on the same row or column, then at least one contains 1. Prove that we can find i rows and j columns so that $i + j \geq n + 1$ and their intersection contains only 1's.

7. Consider a finite bipartite graph with sets A, B.

   - If there exists $k \in \mathbb{N}$ such that for every set $X \subset A$, we have $|N(X)| \geq k|X|$, then we can find $k|A|$ edges such that every vertex in A is incident to k of them and every vertex in B to at most one of them.
   - If there exists $d \in \mathbb{N}$ such that for every set $X \subset A$, we have $|N(X)| \geq |X| - d$, then we can find $|A| - d$ non-adjacent edges.

8. *Is Hall's theorem true for $A = \{a_1, a_2, \dots\}$ (infinite) with Hall's condition holding for any finite X?

Counting Combinations

Binomial Formula Prove that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Pascal Identity Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ when n, k are positive integers.

Multinomial Formula Show that

$$(x_1 + \ldots + x_m)^n = \sum_{\substack{i_1+\ldots+i_m=n \\ i_1,\ldots,i_m \geq 0}} \binom{n}{i_1,\ldots,i_m} x_1^{i_1} \cdot \ldots \cdot x_m^{i_m}.$$

1. (E) Show that $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$. ~~$n \; n-1 \ldots n-k+1$~~ $n+k \; n+k-1 \ldots 1+k$

2. (E) Prove that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ for any positive integers n and k. $\frac{n(n-1 \ldots)!}{k!(n-k)!}$

3. (E) Compute $\sum_{k=0}^{n} (2k-1)\binom{n}{k}$.

4. (E) Compute $\sum_{k=0}^{n} k^2 \binom{n}{k}$.

5. (E) Let p be a prime number, and let k be a positive integer less than p. Prove that $\binom{p}{k}$ is divisible by p.

6. (E) Let m, n be positive integers with $\gcd(m,n) = 1$. Prove that $\binom{n+m}{m}$ is divisible by $n+m$.

7. (M) In how many ways can we write a positive integer n as an ordered sum of m positive integers? How about non-negative?

8. (E) Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

9. (E) Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

10. (M) Find a closed formula for $\sum_{k=0}^{n/2} \binom{n}{2k} 2^k$.

11. (E) Compute $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$.

12. (M) Show that $\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$

13. (M) Show that $k, n \in \mathbb{N}$ let $S_k(n) = 1^k + 2^k + \ldots + n^k$. Show that $\sum_{k=0}^{m} \binom{m}{k} S_k(n) = (n+1)^m - 1$

14. (M) Prove that $\sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} = \sum_{r=1}^{n} \frac{2^r - 1}{r}$

15. (H) Let $n \in \mathbb{N}$. For $p = 1, 2, \ldots$ we define

$$A_p(n) = \sum_{0 \le k \le \frac{n}{2}} (-1)^k \left(\binom{n}{k} - \binom{n}{k-1}\right)^p$$

Prove that whenever n is odd $A_2(n) = nA_1(n)$.

16. (H) Show that $\sum_{i=0}^{n} \frac{1}{\binom{n}{i}} = \frac{n+1}{2^{n+1}} \sum_{i=1}^{n+1} \frac{2^i}{i}$

17. (E) Show that $\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$.

18. (E) Prove that $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$.

19. (M) (Vandermonde) Show that $\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$.

20. (M) Prove that $\binom{2p}{p} - 2$ is divisible by p^2 for all primes $p > 2$.

21. (M) Show that $\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

22. (E) A license plate consists of $n \geq 1$ digits. It is called even if it contains an even number of 0's. Find the number of even license plates.

23. (H) Prove that the number of words of length n made up of 0 and 1 which have exactly m blocks of type 01 is $\binom{n+1}{2m+1}$.

24. (M) Find the number of words of length n which do not contain two consecutive zeroes.

25. (M) Prove that the number of k element subsets of $\{1, 2, \ldots, n\}$ with the property that they do not contain two consecutive elements is $\binom{n-k+1}{k}$

26. (M) Prove that $\sum_{k=0}^{n} \binom{n-k+1}{k} = F_{n+2}$, where F_n is the n-th Fibonacci number.

27. (E)Find a combinatorial proof for the following identity:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

28. (E) In how many ways can one choose a committee of k people taken from a group of m men and n women. Condition on the number of men to derive the identity

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}.$$

29. (E) Prove the following identities combinatorially:

(a) $\sum_{j=1}^{n} (j-1)(n-j) = \binom{n}{3}$

(b) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

(c) $\sum_{k \geq 0} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$

(d) $\sum_{k \geq 0} \binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}2^{m}$

(e) For $n \geq 1$, $\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

Induction Solutions

1. Prove that $1 + 2 + ... + n = \frac{n*(n+1)}{2}$

Solution: We prove the result by induction on n.
Base case: $n = 1$
$1 = \frac{1*2}{2} = 1$
Now we induct on n.
Assume for some n that $1 + 2 + ... + n = \frac{n(n+1)}{2}$
Now consider $1 + 2 + ... + n + (n + 1)$.
By our induction hypothesis, this is also $\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+2)(n+1)}{2}$
So it is true for $n + 1$, so it is true for all $n \geq 1$

2. **(Fermat's Little Theorem)** Let p be prime number. Than p divides $n^p - n$ for all positive integers n.

Solution:

The base case $n = 1$ is true as $p|1^p - 1 = 0$.
Now, if $p|(k^p - k)$, then

$$(k+1)^p - (k+1) = k^p + \binom{p}{1}k^{p-1} + \ldots + \binom{p}{p-1}k + 1 - (k+1) = k^p - k + \sum_{i=1}^{p-1}\binom{p}{i}k^{p-i}$$

As $\binom{p}{i} = \frac{p!}{(p-i)!i!}$ and p cannot divide $i!$ or $(p-i)!$, we conclude that $\binom{p}{i}$ is divisible by p for $0 < i < p$. $k^p - k$ is divisible by p from induction hypothesis so $p|(k+1)^p - (k+1)$, proving the result.

3. Consider $a_n = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}}$, a tower of n $\sqrt{2}$. Prove that a_n is increasing and bounded above by 2.

Solution: We begin with the base cases n=1. $1 \leq \sqrt{2} \leq 2$
We use the induction hypothesis that for some n, $1 \leq a_n \leq 2$. Note that $a_{n+1} = \sqrt{2}^{a_n}$ Then, since by our induction hypothesis $a_n \leq 2$, $\sqrt{2}^2 = 2$ is the largest it could be, and $\sqrt{2}^1 = 1$ is the smallest it could be. Thus, the sequence is bounded by 2.

We will also use induction to prove that it is increasing.

We use the base case of $\sqrt{2}^{\sqrt{2}} > \sqrt{2}$

Our induction hypothesis is $a_n > a_{n-1}$
Then $a_{n+1} = \sqrt{2}^{a_n} > \sqrt{2}^{a_{n-1}} = a_n$
Therefore, it is strictly increasing, and our proof that it is bounded above by 2 is complete.

4. On a circular route, there are n identical cars. Together they have enough gas for one car to make a coplete tour. Prove that there is a car that can make a complete tour by taking gas from all the cars that it encounters.

Solution: We will use induction on n to prove this.
Consider a base case of n=1. This car has enough gas to make a complete tour, so this car can make a complete tour.
Use the induction hypothesis that for some n, and for any arrangement of n cards, at least one of the n cars can make a complete tour.
Now consider a track with $n+1$ cars. There exists a car with enough gas to reach another car. If this car did not exist, the sum of the gas in all the cars would be insufficient to make a complete circuit. Thus, we can think of this 2 car pair as being just the first car, with as much gas as the two combined. But then we only have n cars, and by our induction hypothesis there is sufficient gas for n cars to make a complete tour, so our proof is complete.

5. Show that for any $n \geq 6$, a square can be partitioned into n smaller squares

Solution: Note that given any square, we can partition it into four smaller congruent squares. So we will induct from $P(n)$ to $P(n+3)$. Hence all we need is to check that the statement holds for $n = 6, 7, 8$.

For $n = 6$ can divide the square into 9 congruent squares and then join four of them into a larger one.

For $n = 7$ first divide the square into 4 equal squares and then divide one of those further into 4 equal squares.

Finally, for $n = 8$, divide the square into 16 congruent squares and then join 9 of them into a bigger one.

6. In every country, every road is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

Solution: The one city case is trivial, as well as the 2 city case.

Let us use the induction hypothesis that for some n, a country with n cities always has the property that there exists a city which can be gotten to by traversing at most 2 roads from every other city. We call a city with this property a "center", and the induction hypothesis is there is one center per country of size n. Let its center be c, the cities one road away from it be in the group R_1, and the cities two away from c be in the group R_2.
Now for induction, we are going to add a city labeled c_k.
There are 3 cases:

1) c_k has a road to the c. In this case, c is still the center.

2) c_k does not have a road to c, but is connected to a city in R_1. In this case, c_k is exactly 2 roads away from c, so c is still the center.

3) c_k has no roads to any city in R_1, and not to c either. Equivalently, c, and every city in R_1, has a road to c_k. In addition, all cities in R_2 are at most 2 away from c_k, because from each city in R_2 we can go to R_1 and then to c_k. But then all cities are at most 2 away from c_k, and c_k is our new center.

Since when we go from n cities to $n+1$ cities there is still a center, through the principle of mathematical induction all countries have a center, and our proof is complete.

7. Given a sequence of integers x_1,x_2,...,x_n whose sum is 1, prove that exactly one of the cyclic shifts $(x_1, x_2, \ldots, x_n)$; $(x_2, x_3, \ldots, x_1)$; $\cdots$; $(x_n, x_1, \ldots, x_{n-1})$ has all its partial sums positive.

Solution: We will prove this by using induction.
For our base case, set $n = 1$. Than, x_1 =1, and the only 'cyclic shift' is x_1, and its only partial sum is positive. So our base case is complete.

For our induction hypothesis, for some n and any integers x_i summing to 1, exactly one cyclic shift of (x_1,x_2,...,x_n) has all partial sums positive.
Now consider (x_1,x_2,...,x_n,x_{n+1}). This must have at least one positive value x_k. Consider x_{k+1}, where this would be x_1 if $k = n+1$. First, we show that that particular cyclic shift cannot work: if it could, then $(x_1 + \cdots + x_n) + x_{n+1} = 1$, while x_{n+1} is positive. This means that the nth partial sum is non-positive, so this cyclic shift doesn't work. Therefore the only cyclic shifts that could possibly have all positive partial sums are the ones where x_k and x_{k+1} truly are adjacent. Then we can consider them as being one value $x_j = x_k + x_{k+1}$. Formally, we would define a sequence $y_1, \ldots, y_n$ such that, if $i < k$ then $y_i = x_i$, if $i = k$ then $y_i = x_k + x_{k+1}$, and if $i > k$ then $y_i = x_{i+1}$. We can then use the induction hypothesis on the set $(y_1, y_2, \ldots, y_n)$, as these are integers that sum to 1. Exactly one of the cyclic shifts of those numbers has all positive partial sums, and that particular shift directly corresponds to a cyclic shift of the x_i that has all positive partial sums. Note that all cyclic shifts of the x_i are considered, excluding the one where x_k is in the $n+1$st position. Therefore we have proven that exactly one cyclic shift of the x_i produces all positive partial sums, which completes the inductive step of this problem.

As we have finished the inductive step, this concludes the entire proof of this statement.

8. Prove that every positive integer can be written in infinitely many ways under the

form $n = 1^2 \pm 2^2 ... \pm m^2$

Solution: To do this proof, we are going to prove first that we can find at least one sum for it, and then afterwards prove that there are an infinite number of ways to do it.

We have 4 base cases: We need to prove sums exist for n=1,2,3, and 4.
n=1: 1^2
n=2:$-1^2 - 2^2 - 3^2 + 4^2 = -(1+4+9) + 16 = -14 + 16 = 2$
n=3:$-1^2 + 2^2 = -1 + 4 = 3$
n=4:$1^2 - 2^2 - 3^2 + 4^2 = 1 - (4+9) + 16 - 17 - 13 = 4$

Now we will assume that a sum exists for n, and we will show we can reach $n+4$.
Say that the largest number used in summing to n is $k-1$.
Consider adding to the form $k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2 = k^2 - k^2 - 2k - 1 - k^2 - 4k - 4 + k2 + 6k + 9 = 4$. This is in the necessary form, and we have a number 4 larger than before.
Thus, because of the principle of mathematical induction, and since we proved it for 4 consecutive base cases and we can go from n to $n+4$, we proved the existence of at least one way to sum to any given number under this form.

Now we will prove an infinite number exist. Say we have a form n_{k-1}, such that the form sums to n and uses k terms. To this, we can add $k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2 = 4$ and also add $-(k+4)^2 + (k+5)^2 + (k+6)^2 - (k+7)^2 = -4$. We have added 4-4=0, and have changed n_{k-1} to n_{k+7}. Thus, since we can always find another representation using more terms, an infinite number of representations exist and the proof is done.

9. Prove that

$$\sqrt{1 + \sqrt{2 + \cdots + \sqrt{n}}} < 2$$

for each positive integer n.

Solution: We attempt to rephrase the problem without square roots, because that would make the induction step a bit nicer. Denote the expression on the left-hand side by a_n. Then the sequence a_n follows this pattern:

$$a_1^2 = 1$$
$$(a_1^2 - 1)^2 = 2$$
$$((a_2^2 - 1)^2 - 2)^2 = 3$$
$$\vdots$$
$$(\cdots(a_n^2 - 1)^2 - \cdots - n)^2 = n + 1.$$

We wish to show that $a_i < 2$ for all a_i. It is clear that all of the polynomials on the left-hand side are increasing around the vicinity of 2, so $a_i < 2$ if and only if

$(\cdots(2^2-1)^2-\cdots-i)^2 > i+1$. This gives us a new induction problem to solve that is easier to state: If $b_1 = 2^2$ and $b_{i+1} = (b_i - i)^2$, show that $b_i > i$ for all positive integers i.

We go about this using induction to prove a stronger statement: $b_i > 2i$ for all positive integers i. Due to the nature of the induction step, we use two base cases $n = 1$ and $n = 2$, even though b_{i+1} only depends on b_i. The base cases are clear, as $b_1 = 4 > 2$ and $b_2 = 9 > 3$. For the induction step, assume that $b_n > 2n$ for some positive integer n at least 2. Then $b_{n+1} = (b_n - n)^2 > (2n-n)^2 = n^2 > (n+1)$. Note that $n^2 > n+1$ for all positive integers $n \geq 2$ (which is why we needed the base case $n = 2$). This completes the inductive step of the problem, which completes the proof of the original claim.

10. Let $x_1 = 1$, $x_{2n} = 1 + x_n$, and $x_{2n+1} = 1/x_{2n}$. Prove that for every positive rational number r, there exists a unique n such that $x_n = r$.

Solution: Note that every positive rational number r can be expressed as a/b, where a and b are relatively prime positive integers. So we proceed by induction, but on the sum $a+b$. We start with the base case: $a+b = 2$, so $a/b = 1$. We must show that x_1 is the only term in the sequence that equals 1. Clearly the sequence consists of only positive rational numbers, so $x_{2n} > 1$ for all $n \geq 1$. Therefore $x_{2n+1} = 1/x_{2n} < 1$ for all $n \geq 1$. This shows that $x_i \neq 1$ if $i \neq 1$, so for $r = 1$ there exists a unique $n = 1$ such that $x_n = 1$. This completes the base case.

Now we move on to the inductive step: assume that, for some positive integer $k \geq 2$, if a and b are positive integers with $a + b \leq k$, then a/b has a unique n such that $x_n = a/b$. We now consider all pairs of positive integers (a, b) such that $a+b = k+1$. We must show that for each a/b we can find exactly one n with $x_n = a/b$. If $a = b$ then the result is clear, so we can do casework on whether $a/b > 1$ or $a/b < 1$.

If $a/b < 1$, then a/b is in the sequence if and only if b/a is in the sequence (the if direction is clear; the only if direction hinges on demonstrating that if a/b is in the sequence at x_k then k must be odd). In addition, a/b can only be attained through the methods used to attain b/a. And note that $b/a > 1$, so this is in the sequence if and only if $(b-a)/a$ is in the sequence, and we can only get b/a through the method used to get $(b-a)/a$. However, $(b-a)+a = b \leq k$, which means from the induction hypothesis that there is a unique n with $x_n = (b-a)/a$. Therefore there must be a unique m with $x_m = b/a$, and a unique p such that $x_p = a/b$. This conclusion completes this case of the inductive step.

Now we assume that $a/b > 1$. Then a/b is in the sequence if and only if $(b-a)/a$ is in the sequence, but again we can use the induction hypothesis to show that $(b-a)/a$ is uniquely in the sequence, thus a/b is also uniquely in the sequence. This concludes the second case of the inductive step.

As we have covered all possible cases, our inductive step is complete, which completes the proof of the original claim.

11. Let $a_1, a_2, \ldots, a_n$ be pairwise distinct positive integers and M a set of $n-1$ positive integers not containing the number $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is going to jump along the real axis. It starts at the point 0 and makes n jumps to the right of lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the grasshopper can organize its jumps in such a way that it never falls in any point of M.

Solution: We prove this statement using strong induction on n. The $n = 1$ case is clear, and the $n = 2$ case is easy: if M is of size 1, then the grasshopper can either go to a_1 and then to $a_1 + a_2$, or to a_2 and then to $a_1 + a_2$. We now proceed with the induction step: for some $k \leq 2$, assume that such trips can always be arranged for each $n \leq k$. Consider any $a_1, a_2, \ldots, a_{k+1}$, with a corresponding set M of size k. Assume that the a_i are ordered: in particular, $a_1 < a_2 < \cdots < a_{k+1}$. Let $S = a_1 + \cdots + a_{k+1}$. To approach this problem, we do casework on how many of the a_i are in M.

Case 1: $a_{k+1} \in M$, but $a_1, a_2, \ldots, a_k \notin M$.

In this case, there are at most $k-1$ elements of M above a_{k+1}, so if the grasshopper started at a_{k+1} (for the sake of argument we pretend that this is the case), we would be able to find a desired path from a_{k+1} to S. Say that this path starts with a jump of length a_i. Then $a_i + a_{k+1} \notin M$. Construct a path from the origin to S by taking the path we just constructed, and switch the first and second jumps: i.e. the grasshopper takes a jump of a_i, and then a jump of a_{k+1}. This path will not pass over any elements of M, so this is such a desired path. This completes the induction step for this case.

Case 2: $a_{k+1} \notin M$, but we can find at least one $a_1, a_2, \ldots, a_k$ that is in M.

In this case, there are at most $k-1$ elements of M above a_{k+1}, so once again, we can form a desired path from a_{k+1} to S without stepping on any elements of M. But this time, for constructing a path from the origin to S, we can just hop to a_{k+1} first, and then go along this desired path. This case is done.

Case 3: $a_{k+1} \in M$, and at least one $a_1, a_2, \ldots, a_k$ is in M.

Consider the ordered pairs $(a_1, a_1 + a_{k+1}), (a_2, a_2 + a_{k+1}), \ldots, (a_k, a_k + a_{k+1})$. There are k ordered pairs here. There are k elements of M, but only $k-1$ could possibly be in these ordered pairs, since a_{k+1} isn't in any of these ordered pairs. From the Pigeonhole Principle, we must then have that at least one of these pairs $(a_i, a_i + a_{k+1})$ has neither of its entries in M. Then the grasshopper can hop from the origin, to a_i, to $a_i + a_{k+1}$. Then there are at most $k-2$ entries of M above where the grasshopper is at this point, and the grasshopper has $k-1$ jumps to go. From the inductive hypothesis, our grasshopper can finish the desired path in some way. This completes the third case.

Case 4: $a_1, a_2, \ldots, a_{k+1} \notin M$.

This is the last and hardest case. The idea here is to assume for the sake of contradiction that every path from 0 to S must hit some element of M. We then take some $m_a \in M$, and some a_j: construct a path from a_j to S without hitting any of the other elements of M, and from the contradiction assumption, this path must hit m_a. We then go about using the induction hypothesis and the contradiction assumption together to prove that some different $m_b \in M$ must be on this path.

Order the set M as $m_1 < m_2 < \cdots < m_k$. Construct a path from a_{k+1} to S that hits only m_1, and no other elements of M (this exists from the induction hypothesis). We can make a path from 0 to S by jumping from 0 to a_{k+1} first, and then using the path we just constructed to get to S. Take the jump directly after the grasshopper gets to m_1: say that it has length a_x. We now switch a_x with a_{k+1} in our path. So we still get to $m_1 + a_x$, but we use a different path to get there. The path doesn't change at all after m_1. However, the path before $m_1 + a_x$ will not hit m_1, because $a_x < a_{k+1}$. Therefore this path must not hit m_1. From the contradiction hypothesis, this path must hit some different m_b. However, this implies that the original path from a_{k+1} to S also contains $m_b > m_1$, which contradicts our construction of the path only hitting m_1. This gives us a contradiction of our assumption that there are no desired paths: this proves that in this case we are still able to get from the origin to S without hitting any elements of M. We've completed this last case.

In conclusion, in all four cases, we are able to prove that we can construct a desired path for $n + 1$ numbers. This completes the induction step, which shows that we can construct a desired path of any number of jumps, for any set M.