HK 4 for PDE I Fall 2024, Due Wed. Nov. 27, in Class, 60 Marks Total

1. (10 marks) Suppose that $f \in C^1((-\infty, c]) \cap C^1([c, \infty)),\quad -\infty < c < \infty$ (f may not be continuous at c). Let $g \in C_0(\mathbb{R})$ (continuous and compact supported). Compute
$$\frac{d}{dx}\int_{-\infty}^{\infty} f(x-y)g(y)dy \tag{0.1}$$
in terms of f', g, $f(c+)$, and $f(c-)$.

   Hint:

   - First, guess a plausible answer.
   - Then, justify the guessed answer.

   In doing so, observe
$$\int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{x-c} f(x-y)g(y)dy + \int_{x-c}^{\infty} f(x-y)g(y)dy = I(x) + II(x) \tag{0.2}$$
   Let
$$f_+(x) = \begin{cases} f(x) & \text{if } x > c, \\ f(c+) + f'(c+)(x-c) & \text{if } x \le c. \end{cases} \tag{0.3}$$
   (How smooth is f_+ in $\mathbb{R}$?)

   Then show
$$I(x) = \int_{-\infty}^{x-c} f_+(x-y)g(y)dy. \tag{0.4}$$
   Now use Fundamental Theorem of Calculus to do $I'(x)$.

2. (10 marks) Let $u \in \mathcal{D}'((a,b))$ s.t. distributional derivative $u' = 0$. Show that $u = \text{const. } C$ in the sense of distribution.

   Hint: Choose a $\phi_0 \in \mathcal{D}(a,b)$ with $\int_a^b \phi_0(x) = 1$. $\forall \phi \in \mathcal{D}((a,b))$, define $\psi(x) = \phi(x) - \phi_0(x)\int_a^b \phi(x)dx$. Show that $\exists\, \theta \in \mathcal{D}((a,b))$ s.t. $\theta' = \psi$, which implies that
$$\phi(x) = \theta'(x) + \phi_0(x)\int_a^b \phi(x)dx. \tag{0.5}$$

3. (40 marks) Let $\Gamma(x)$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^n$, $n \ge 3$. Let $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Define Newtonian potential
$$u(x) = (\Gamma * f)(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy \tag{0.6}$$
   (i) (5 marks) Show that u is well defined in $\mathbb{R}^n$.

Hint: $\int_{\mathbb{R}^n} = \int_{|y-x|\leq 1} + \int_{|y-x|\geq 1}$.

(ii) (5 marks) Show that $||u||_{L^\infty(\mathbb{R}^n)} \leq C(n)(||f||_{L^1(\mathbb{R}^n)} + ||f||_{L^\infty(\mathbb{R}^n)})$, where $C(n)$ is a constant depending only on n.

(iii) (10 marks) Show that u is a distributional solution of the Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n. \tag{0.7}$$

Hint: Fubini

(iv) (10 marks) Prove $u \in C^1(\mathbb{R}^n)$ (regularity of u).

Hint: Let $\mathbf{e_i} = (0, \cdots, 0, 1, 0, \cdots, 0)$. Of course, you start with $\lim_{h\to 0} \frac{u(x+h\mathbf{e_i}-u(x))}{h}$ (x fixed). When trying to use the Lebesgue dominanted convergence theorem (LDCT), the difficulty is that you cannot find a dominanting function in the whole $\mathbb{R}^n$. Well, you can, outside a small ball $B_\delta(x)$ (δ fixed). Then you have to show that the part inside $B_\delta(x)$ is tiny. Do not forget to show $\frac{\partial u}{\partial x_i}$ is continuous in $\mathbb{R}^n$.

(v) (10 marks) If f also has compact support, show that

$$\lim_{|x|\to\infty} |x|^{n-2}u(x) = \int_{\mathbb{R}^n} f(y)dy \frac{1}{n(n-2)|B_1(0)|}. \tag{0.8}$$

(i.e., $u(x) \approx \Gamma(x)\int_{\mathbb{R}^n} f(y)dy$ for $x \approx \infty$.) Give an electrostatic interpretation of this result when $n = 3$.

Hint:

- LDCT; do show me your dominanting function.
- Viewed afar, the support of f is tiny.