HK 1 for PDE-I Fall 2024, 100 Marks Total, Due Sept 23

1. (10 marks) Let $A = (a_{ij})$ and $B = (b_{ij})$ be n by n symmetric matrices. Suppose that A is positive semi-definite and B is negative semi-definite. Prove that $\sum_{i,j=1}^{n} a_{ij}b_{ij} \le 0$.

   Hint: Diagonalize A by an orthogonal matrix P.

2. (10 marks) Assume that Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Let ν be the outward pointing unit normal vector field. Prove that the following Neumann boundary problem has no solution in $C^2(\bar{\Omega})$.

$$\begin{cases} \Delta u = 2024 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

3. (10 marks) Prove that if "$Lu \ge 0$" is replaced by "$Lu = 0$" in the weak maximum principle with $c \le 0$, then

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{2}$$

4. ($10 + 5 = 15$ marks) Assume that Ω is a bounded domain in $\mathbb{R}^n$.

   (i) Prove that the following Dirichlet boundary value problem has at most one solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

   where $f \in C^1(\mathbb{R})$ satisfies $f'(u) \le 0 \ \forall\ u \in \mathbb{R}$.
   (**Hint:** Consider the equation of $w = u_1 - u_2$ and apply the WMP with $c \le 0$, where u_1 and u_2 were any two solutions of (3).)
   (ii) Give an example to show that Part (i) may not hold without the assumption "$f'(u) \le 0$".

5. ($5 + 15 = 20$ marks) Consider the partial differential operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u, \quad x \in \Omega, \tag{4}$$

   where Ω is a bounded domain in $\mathbb{R}^n$, L is strictly elliptic on Ω, and a_{ij} and b_i are bounded on Ω.
   (i) For $n = 1$, give an example of L and u such that

$$\begin{cases} Lu \ge 0 & \text{in } \Omega, \\ u \le 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

yet, "$u \le 0$ on Ω" is not true. (This means weak maximum principle does not hold without sign condition on $c(x)$.)
(ii) Yet, we can replace the sign condition on $c(x)$ by another condition to have "(5) implies $u \le 0$ on Ω": Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (5); assume $\exists$ $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

$$\begin{cases} Lv \le 0 \quad \text{in } \Omega, \quad v > 0 \ \text{ on } \bar{\Omega}, \\ \text{all } v_{x_i} \text{ are bounded on } \Omega. \end{cases} \tag{6}$$

Prove that $u \le 0$ on Ω.
(**Hint:** Let $w = \frac{u}{v}$, which implies $L(vw) \ge 0$ in Ω. Then, find another linear operator $\tilde{L}$ s.t. $\tilde{L}w \ge 0$ in Ω, where "c" in $\tilde{L}$ has the "good sign".)

6. (15 marks) (Weak Maximum principle in narrow domain) This is an application of Problem 5.
Suppose that
(i) L is strictly elliptic on Ω;
(ii) $\exists$ constant M s.t.
$$|a_{ij}|, \quad |b_i|, \quad c^+ \le M \quad \text{on } \Omega; \tag{7}$$
(iii) Ω is a bounded domain, contained in the slab $\{(x_1, x_2, \cdots, x_n) \mid 0 < x_1 < d\}$;
(iv) $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (5).
Prove that $\exists$ small $d_0 > 0$ s.t. if $0 < d < d_0$, then $u \le 0$ on Ω.
(**Hint:** Consider $v(x; d) = \cos(\frac{\pi x_1}{4d})$.)

7. (10 + 10 = 20 marks) Suppose $u \in C^2(\Omega)$ and u is radial, i.e., $u(x) = u(r)$, $\forall\, x \in \Omega \subset \mathbb{R}^n$, where $r = \sqrt{x_1^2 + \cdots + x_n^2} = |x|$.
(i) Prove that
$$\Delta u = u_{rr} + \frac{(n-1)}{r} u_r, \quad \forall\, x \in \Omega,\ x \ne 0.$$
(ii) Let
$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln r, \quad n = 2, \\ \frac{1}{(n-2)\omega_n r^{n-2}}, \quad n \ge 3, \text{ where } \omega_n \text{ is the surface area of the unit ball in } \mathbb{R}^n. \end{cases} \tag{8}$$
Γ is called the fundamental solution of $\Delta u = 0$. Prove that $\Delta\Gamma = 0$ if $x \ne 0$.