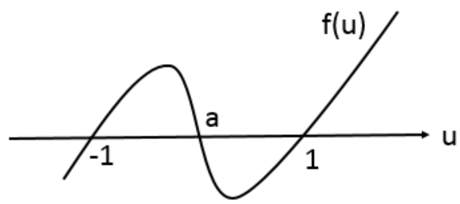
HW 3 for PDE-I Fall 2024, 85 Marks Total, Due Oct 30

1. (30 marks) Consider the initial-boundary value problem for Allen-Cahn/bistable equation

$$\begin{cases} u_t - \Delta u + f(u) = 0, & x \in \Omega, \ \ t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary $\partial\Omega$ (hence $\partial\Omega \times (0,\infty)$ is C^2-smooth); ν is the unit outer normal vector field on $\partial\Omega$; f is an N-shaped, C^1-smooth function as shown:



$f(u) > 0$ for $-1 < u < a$ and $u > 1$; $f(u) < 0$ for $u < -1$ and $a < u < 1$.

Denote by $v(t; v_0)$ the solution of the initial value problem (ODE)

$$\begin{cases} \frac{dv}{dt} = -f(v), \\ v(0) = v_0, \end{cases} \tag{0.2}$$

where v_0 is a constant.

(i) (3 marks) Show that v satisfies the PDE and the boundary condition in (0.1).

(ii) (3 marks) By performing phase line analysis, show that $v(t) \to$ either 1 or -1 as $t \to \infty$, as long as $v_0 \neq a$. (You'll see that 1 and -1 are stable equilibrium points of (0.2), hence the name "bistable equation".)

Let $u(x,t)$ be a $C^{2,1}(\bar{\Omega} \times [0,\infty))$-smooth solution of (0.1).

(iii) (7 marks) Show that

$$v(t; m) \leq u(x,t) \leq v(t; M), \quad \forall \ (x,t) \in \bar{\Omega} \times [0,\infty), \tag{0.3}$$

where $m = \min_{\bar{\Omega}} \varphi$, $M = \max_{\bar{\Omega}} \varphi$.

(iv) (7 marks) Show that if $\varphi > a$, $\forall \ x \in \bar{\Omega}$, then

$$\lim_{t \to \infty} u(x,t) = 1 \tag{0.4}$$

uniformly for $x \in \bar{\Omega}$.

(v) (7 marks) Show that if $\varphi(x) \geq a$ and $\varphi(x) \not\equiv a$ on $\bar{\Omega}$, then

$$\lim_{t \to \infty} u(x,t) = 1 \tag{0.5}$$

uniformly for $x \in \bar{\Omega}$.
Hint: Define $w(x,t) = u(x, t+1)$. Then w satisfies (0.1) with $w(x, 0) = u(x, 1)$. Use comparison to show that $w(x, 0) > a$ on $\bar{\Omega}$.
(vi) (3 marks) Formulate, without proof, the analog of (v) in case $\varphi(x) \le a$ and $\varphi(x) \not\equiv a$ on $\bar{\Omega}$.

2. (18 marks) **(Finite time blow-up)** Consider

$$\text{(IBVP)} \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u + e^u, \ u = u(x,t), \ x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = 0, \ (x,t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = \phi(x), \ x \in \Omega, \end{cases} \tag{0.6}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with C^3 smooth $\partial\Omega$, ν is the unit outer normal of $\partial\Omega$, $\phi \in C^\infty(\bar{\Omega})$ with $\frac{\partial \phi}{\partial \nu}|_{\partial\Omega} = 0$ (called "matching condition"). By advanced existence theory, $\exists \ T_\phi > 0$ (may be ∞) such that (IBVP) has a classical solution u satisfying

(a) $u \in C^{2,1}(\bar{\Omega} \times [0, T_\phi))$;
(b) $\forall T' \in (0, T_\phi)$, u is bounded on $\bar{\Omega} \times [0, T']$;
(c) if $T_\phi < \infty$, then $\limsup\limits_{t \to (T_\phi)-} ||u(\cdot, t)||_{L^\infty(\Omega)} = \infty$.

If (c) happens, we say u **blows up** in finite time. T_ϕ is called the **life-span** of u.
Problems:
(i) (8 marks) Let $v(t)$ solve

$$\begin{cases} \frac{dv}{dt} = e^v, \\ v(0) = 0. \end{cases} \tag{0.7}$$

Show that the life-span T_0 of v is finite.
(ii) (8 marks) Show that $u(x,t) \ge v(t)$ on $\bar{\Omega} \times [0, \min(T_0, T_\phi))$ if $\phi \ge 0$ on $\bar{\Omega}$.
(iii) (2 marks) Explain that $T_\phi \le T_0$ if $\phi \ge 0$ on $\bar{\Omega}$.

3. (27 marks) **(Monotonicity in t)** Consider the initial boundary value problem (nonlinear heat/diffusion equation)

$$\text{(IBVP)} \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(u), & (x,t) \in \Omega \times (0, \infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = \phi(x), & x \in \Omega, \end{cases} \tag{0.8}$$

where $f \in C^1(\mathbb{R})$, Ω is a bounded domain in $\mathbb{R}^n$. Let $u(x, t; \phi)$ be the $C^{2,1}(D) \cap C^0(\bar{D})$ solution of (IBVP) and let $D = \Omega \times (0, \infty)$.
(i) (8 marks) We say $\bar{u} = \bar{u}(x)$ (a function of x only) is a super/upper steady state of (IBVP) if $\bar{u} \in C^2(\Omega) \cap C^0(\bar{\Omega})$, and

$$\begin{cases} -\Delta \bar{u} \ge f(\bar{u}), & x \in \Omega, \\ \bar{u}|_{\partial\Omega} \ge 0. \end{cases} \tag{0.9}$$

Prove that if $\phi \leq \bar{u}$ on $\bar{\Omega}$, then $u(x,t;\phi) \leq \bar{u}(x)$ on $\bar{\Omega} \times [0,\infty)$; moreover, if $\phi \leq \bar{u}, \not\equiv \bar{u}$ on $\bar{\Omega}$, then $u(x,t;\phi) < \bar{u}(x)$, $\forall x \in \Omega,\ t > 0$.

(ii) (8 marks) Prove that $u(x,t;\bar{u})$ is non-increasing in t.

Hint: $\forall\, 0 \leq t_1 < t_2$, let $\delta = t_2 - t_1$. Define $v(x,t) = u(x, t+\delta)$. Show that v satisfies the PDE and BC. Apply C.P. to u and v.

(iii) (8 marks) Prove that if $\bar{u}$ is not a steady state (but still a super/upper one), i.e., if $\bar{u}$ does not satisfy

$$\begin{cases} -\Delta \bar{u} = f(\bar{u}) \ \text{ in } \ \Omega, \\ \bar{u}|_{\partial\Omega} = 0 \end{cases} \tag{0.10}$$

at the same time, then $u(x,t;\bar{u})$ is strictly decreasing in t.

(iv) (3 marks) Formulate the analog for sub/lower steady states (without proof).

4. (10 marks) **(Radial symmetry)** Assume the same conditions as in Problem 3, where now, $\Omega = B_R(0)$, ϕ is radially symmetric, i.e., $\phi = \phi(r)$, $r = |x|$. Prove that u is also radially symmetric in x.

Hint: $\forall$ orthogonal matrix $A_{n\times n}$, define $v(x,t) = u(Ax,t)$. Apply C.P. on u and v.