HW 2 for PDE-I Fall 2024, 140 Marks Total, Due Oct 9

1. (30 marks) (i) Let $n \geq 3$. Show that there does not exist a **positive** harmonic function u in the exterior of the unit open ball in $\mathbb{R}^n$, that decays faster than $\frac{1}{r^{n-2}}$ at ∞, i.e.
$$\lim_{r\to\infty} \frac{u(x)}{\frac{1}{r^{n-2}}} = 0.$$
(**Hint:** Define $v(x) = u(x) - \frac{\varepsilon}{r^{n-2}}$, ε being so tiny that $v|_{|x|=1} > 0$. You might need to use the week maximum principle in unbounded domain.)
(ii) Prove that there does not exist a **positive** harmonic function on the exterior of the open unit disc that decays at ∞.
(**Hint:** Let $v_\varepsilon(x) = u(x) + \varepsilon \ln |x|$. What can you say about $\inf_{|x|\geq 1} v_\varepsilon(x)$? You might need to use the week maximum principle in unbounded domain.)
(iii) Prove that there does not exist a **positive**, bounded, and nonconstant harmonic function on the interval $(1, \infty)$.

Note: In problems 2-3, B_1 denotes the closed unit ball in $\mathbb{R}^n$ centered at 0.

2. (10 marks) In class, we proved that if $\Delta u = 0$ in $\overline{B_1^c}$ with $n \geq 3$, and $\lim_{x\to\infty} u(x) = 0$, then $u(x) = O(\frac{1}{|x|^{n-2}})$ as $x \to \infty$. For $n = 2$, prove that it is possible that u decays at ∞ at any algebraic decay rate $\frac{1}{|x|^k}$, $k \geq 1$ integer.
(**Hint:** Consider complex analytic functions such as $\frac{1}{z^k}$.)

3. (20 marks) Suppose that $u \in C^2(B_1 \setminus \{0\})$ is harmonic on $B_1 \setminus \{0\}$ (u may or may not be singular at 0). Prove that if
$$u(x) = \begin{cases} o(\ln r) & \text{as } r = |x| \to 0^+, \quad n = 2, \\ o(\frac{1}{r^{n-2}}) & \text{as } r = |x| \to 0^+, \quad n \geq 3, \end{cases} \tag{1}$$
then
$$\sup_{B_1\setminus\{0\}} u = \sup_{\partial B_1} u, \qquad \inf_{B_1\setminus\{0\}} u = \inf_{\partial B_1} u.$$
(**Hint:** For $n = 2$, consider $u(x) \pm \epsilon \ln r$ for the sup and inf identities, respectively. For $n \geq 3$, consider $u(x) \mp \frac{\epsilon}{r^{n-2}}$ for the sup and inf identities, respectively. Note that the inf identities can also be obtained from the sup identities by considering $-u$.)

4. (10 marks) Let Ω be a bounded domain in $\mathbb{R}^n$ that satisfies the interior sphere condition at every point on $\partial\Omega$. Prove that every solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of
$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$
where ν is an outer pointing vector field on $\partial\Omega$, must be a constant function on Ω.

5. (10 marks) Let $\Omega = (0,1)$. Suppose u, $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $Lu \geq Lv$ in Ω. Here

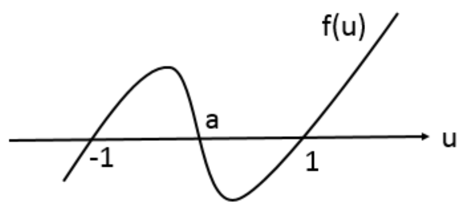
$$Lu = a(x)u'' + b(x)u' + c(x)u,$$

where $\lambda_0 \leq a(x) \leq M$, $|b(x)| \leq M$, and $-M \leq c(x) \leq 0$ for every $x \in \Omega$ for some constants λ_0, $M > 0$. Prove that given $u'(0) \geq v'(0)$ and $u(1) < v(1)$, it holds that $u < v$ in $[0,1]$.

6. ($5 \times 6 = 30$ marks) Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of the "Allen-Cahn/bistable equation" (from material science and math biology):

$$\begin{cases} \Delta u - f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\mathbf{n}$ is the outer normal vector field on $\partial\Omega$; f is an N-shaped, C^1-smooth function as shown:



$f(u) > 0$ for $-1 < u < a$ and $u > 1$; $f(u) < 0$ for $u < -1$ and $a < u < 1$.

Let $M := \max_{\bar{\Omega}} u$, $m := \min_{\bar{\Omega}} u$.
Show that
(i) $M \leq 1$;
Hint: Suppose $M > 1$. Consider two cases: (a) M is attained at some $x_0 \in \Omega$. Then what is the sign of $\Delta u(x_0)$? (b) M is not achieved in Ω (so it is achieved only at a point $x_0 \in \partial\Omega$). Then what can you say about the sign of Δu in a small neighborhood of x_0 in Ω?
(ii) $m \geq -1$.
(iii) $M < 1$ unless $u \equiv 1$ in $\bar{\Omega}$.
(iv) $m > -1$ unless $u \equiv -1$ in $\bar{\Omega}$.
(v) $M > a$ unless $u \equiv -1$ or $u \equiv a$ in $\bar{\Omega}$.
Hint: Argue by contradiction: Suppose $u \not\equiv -1, a$, and $a \geq M \geq m > -1$. Integrating the PDE, we have $\int_\Omega \Delta u \, dx = \int_\Omega f(u(x)) \, dx$.
(vi) $m < a$ unless $u \equiv 1$ or $u \equiv a$ in $\bar{\Omega}$.

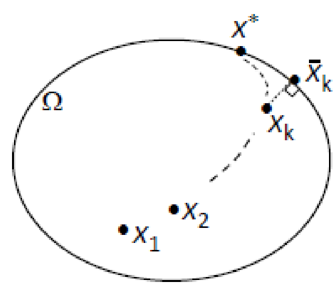
7. (15 marks) Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1-smooth $\partial\Omega$ (so at each $x \in \partial\Omega$, $\exists$ unit outer normal vector $\mathbf{n}(x)$ of $\partial\Omega$). Suppose that $u, v \in C^1(\bar{\Omega})$ satisfy

$$u|_{\partial\Omega} \geq 0, \quad v|_{\partial\Omega} \geq 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u(x)}{\partial \mathbf{n}} < 0 \text{ whenever } x \in \partial\Omega \text{ and } u(x) = 0.$$

Prove that $\exists\, \varepsilon > 0$ s.t. $u + \delta v > 0$ in Ω, $\forall\, \delta \in (0, \varepsilon)$.
Hint: If such ε does not exist, then $\forall\, k \geq 1$, $\exists\, x_k \in \Omega$ s.t.

$$(u + \frac{1}{k} v)(x_k) \leq 0. \tag{4}$$

Since $\bar{\Omega}$ is compact, after passing to a subsequence, $x_k \to$ some $x^* \in \bar{\Omega}$.
The hard case is when $x^* \in \partial\Omega$. Project x_k (for large k) onto $\partial\Omega$ to obtain $\bar{x}_k \in \partial\Omega$. Let $\nu_k = \frac{\bar{x}_k - x_k}{|\bar{x}_k - x_k|}$. Then $\nu_k \to \mathbf{n}(x^*)$ as $k \to \infty$. Let $w_k = u + \frac{1}{k} v$. By (4), we have $w_k(\bar{x}_k) - w_k(x_k) \geq 0$. By the Mean Value Theorem, $\nabla w_k(y_k) \cdot (\bar{x}_k - x_k) \geq 0$, where y_k is between x_k and $\bar{x}_k$.



8. (15 marks) Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2-smooth $\partial\Omega$. Consider operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u, \quad x \in \Omega, \tag{5}$$

where L is strictly elliptic on Ω and $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are uniformly bounded on Ω. Suppose $\exists\, u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ that satisfies

$$Lu \leq 0 \text{ and } u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{6}$$

Prove that if $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$Lv \leq 0 \quad \text{in } \Omega, \quad v \geq 0 \quad \text{on } \partial\Omega, \tag{7}$$

then either $v \geq 0$ in Ω or $\exists$ constant $c_0 > 0$ s.t. $u = -c_0 v$, and in the latter case $Lu = 0 = Lv$ in Ω, $v|_{\partial\Omega} = 0$.
Comments: This is a surprising result with elementary proof. Note we impose no conditions on the sign of $c(x)$.
Hint:

- If $v \geq 0$ in Ω, then nothing needs to be proved.
- If $\exists\, x_0 \in \Omega$ s.t. $v(x_0) < 0$, then define

$$c_0 = \sup\{\varepsilon > 0 \mid u + \delta v > 0 \text{ in } \Omega,\ \delta \in (0, \varepsilon)\}.$$

- Use problem 7 to see that ε exists.

- Is c_0 finite?
- Let $w = u + c_0 v \Rightarrow w \geq 0$ in Ω, $Lw \leq 0$ in Ω.
- Use the strong maximum principle and the Hopf boundary point Lemma on w to show $w \equiv 0$.