HK 5 for PDE I Fall 2024, 150 marks total, due Dec. 18 in class

1 (10 marks) Suppose that $\Omega \subseteq \mathbb{R}^n$ is bounded. Prove that if $u \in W_0^{1,1}(\Omega)$ and $\nabla u \equiv 0$ in Ω, then $u = 0$ a.e. in Ω.
Hint: For $n \geq 2$, the proof is one sentence long using a theorem that we proved in class. For $n = 1$, use the proof of the following Problem 2.

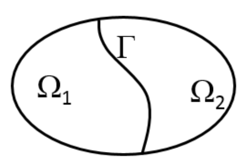
2 (20 marks) Let Ω be a finite interval (a, b). Suppose $u \in W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$. Show that
(i) u is absolutely continuous on $[a, b]$ and $u(a) = 0 = u(b)$.
Hint: By definition, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$; when we say u is absolutely continuous, we mean $\exists\ v$ which is absolutely continuous on $[a, b]$ s.t. $u = v$ a.e. on Ω.
(ii) $u \in C^\alpha(\overline{\Omega})$ with $\alpha = 1 - \frac{1}{p}$ provided $p > 1$.
Remark: This is the 1D case of the Sobolev imbedding:

$$W_0^{1,p}(\Omega) \hookrightarrow C^{1-\frac{n}{p}}(\overline{\Omega}) \quad \text{if } \Omega \subseteq \mathbb{R}^n \text{ bounded }, p > n.$$

Hint: From the proof of (i), you have

$$u(x) = \int_a^x u'(t)\ dt, \quad u(y) = \int_a^y u'(t)\ dt \Rightarrow u(x) - u(y) = \int_y^x u'(t)\ dt.$$

3 (10 marks) Let Ω be a bounded domain in $\mathbb{R}^n$ as shown,



where $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, Γ is the common boundary of the domains Ω_1 and Ω_2, $\Gamma \in C^1$. Suppose $u_1 = u_2$ on Γ, $u_1 \in C^1(\overline{\Omega}_1)$, $u_2 \in C^1(\overline{\Omega}_2)$. Define

$$u = \begin{cases} u_1(x), & x \in \overline{\Omega}_1, \\ u_2(x), & x \in \overline{\Omega}_2. \end{cases}$$

Prove that $u \in W^{1,p}(\Omega)\ \forall\ 1 \leq p \leq \infty$.
Hint: What is your bet on weak $\frac{\partial u}{\partial x_i}$?

4 (10 marks) Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\Omega \subset \mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$. Prove the integration by parts formula:

$$\int_\Omega \frac{\partial u}{\partial x_i} v \, dx = -\int_\Omega u \frac{\partial v}{\partial x_i} \, dx, \quad i = 1, \ldots, n.$$

Hint: $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$; use definition of weak derivative $\frac{\partial v}{\partial x_i}$.

5 (10 marks) Suppose Ω is a bounded domain in $\mathbb{R}^n$, $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and $v \in C^1(\overline{\Omega})$. Show that $uv \in W^{1,p}(\Omega)$ and the weak

$$\nabla(uv) = v\nabla u + u\nabla v.$$

Hint: $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.

6 (10 marks) Suppose $u \in W^{1,p}(\Omega)$ $(1 \leq p < \infty)$ and $\operatorname{supp} u \subset \Omega' \subset\subset \Omega$ (i.e., $\overline{\Omega'}$ is compact and $\overline{\Omega'} \subset \Omega$). Prove that $u \in W_0^{1,p}(\Omega)$.
Hint: Think about the mollification u_ϵ of u.

7 (10 marks) Suppose $1 \leq p < n$. Show that if an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

holds for all functions $u \in C_0^\infty(\mathbb{R}^n)$ and certain constants $C > 0$, $1 \leq q < \infty$ which are independent of u, then necessarily $q = \frac{np}{n-p}$.
Hint: Choose a function $u \in C_0^\infty(\mathbb{R}^n)$, $u \not\equiv 0$, and consider $u_\lambda(x) := u(\lambda x)$ for $\lambda > 0$.

8 (10 marks) Suppose Ω is a bounded domain in $\mathbb{R}^n$. Show that for $f \in L^\infty(\Omega)$, $\lim\limits_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$.

9 (10 marks) (i) Show that for every $\delta > 1$, the series $\displaystyle\sum_{k=1}^{\infty} k\delta^{-k}$ is finite.

(ii) Show that $\delta^{\sum_{k=1}^{\infty} k\delta^{-k}}$ diverges to $+\infty$ as $\delta \to 1+$.

10 (10 marks) Suppose $B_1(0)$ is the unit open ball in $\mathbb{R}^n$ and $n \geq 2$. Prove that $\log\log(1 + \frac{1}{|x|}) \in W^{1,n}(B_1(0))$ but $\notin L^\infty(B_1(0))$.

11 (10 marks) Show that if $u \in W^{1,p}(\mathbb{R}^n)$ with $n < p < \infty$, then $\lim_{|x|\to\infty} u(x) = 0$.

Hint: Use the following Sobolev Inequality: Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^1 boundary. Then for $u \in W^{1,p}(\Omega)$ with $n < p < \infty$, it holds that

$$\|u\|_{L^\infty(\Omega)} \le C(p, n, |\Omega|)\|u\|_{W^{1,p}(\Omega)}.$$

12 (10 marks) Let Ω be a bounded domain in $\mathbb{R}^n$ and let $1 \le p < \infty$. Prove that if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $u|_{\partial\Omega} = 0$, then $u \in W_0^{1,p}(\Omega)$.

Moral of story: This theorem is user-friendly. For example, it helps us to see $\sin x \in W_0^{1,p}((0,\pi))$.

Hint: Just need to show that $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ are in $W_0^{1,p}(\Omega)$. To see $u^+ \in W_0^{1,p}(\Omega)$, consider $v_\varepsilon = (u-\varepsilon)^+$ for $\varepsilon > 0$. Argue that (i) $v_\varepsilon \in W^{1,p}(\Omega)$; (ii) v_ε has a compact support in Ω and hence by Hw 5, # 6, $v_\varepsilon \in W_0^{1,p}(\Omega)$; (iii) $v_\varepsilon \to u^+$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$ by LDCT. Finally, use the fact that $W_0^{1,p}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$.

13 (10 marks) Let $1 \le p < \infty$. Prove $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

Hint: (i) Prove $\exists\, \eta \in C_0^\infty(\mathbb{R}^n)$ s.t.

$$\eta \equiv 1 \text{ on } B_1(0), \quad \eta \equiv 0 \text{ on } (B_2(0))^c, \quad 0 \le \eta \le 1. \text{ (Mollify } \chi_{B_{1.5}(0)}(x).)$$

(ii) $\forall\, R > 0$, define $\eta_R(x) = \eta(\frac{x}{R})$. ($\eta_R$ is called a cut-off function.)
(iii) $\forall\, u \in W^{1,p}(\mathbb{R}^n)$, define $u_R(x) = \eta_R(x)u(x)$.
(iv) Is $u_R \in W_0^{1,p}(\mathbb{R}^n)$?
(v) Prove $u_R \to u$ in $W^{1,p}(\mathbb{R}^n)$.

14 (10 marks) Let Ω be a domain in $\mathbb{R}^n$ and $1 \le p < \infty$. $\forall\, u \in W_0^{1,p}(\Omega)$, define $v(x) = u(x)$ for $x \in \Omega$ and $v(x) = 0$ for $x \notin \Omega$. Prove that $v \in W^{1,p}(\mathbb{R}^n)$.