Hw 6 for PDE I Fall 2024 (85 Marks)

1. (15 marks) Consider elliptic equation in the whole $\mathbb{R}^n$:

$$-\Delta u + u = f(x), \quad x \in \mathbb{R}^n, \quad \text{where } f \in L^2(\mathbb{R}^n). \tag{1}$$

   (i) Define weak solutions of (1), which are in $H^1(\mathbb{R}^n)$.
   (ii) Prove the existence and uniqueness of (1).

   Hint: The Riesz Representation Theorem.

2. (15 marks) Let Ω be a bounded domain in $\mathbb{R}^n$ and

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u, \quad x \in \Omega,$$

   where a_{ij}, b_i, and $c \in L^\infty(\Omega)$ and L is strictly elliptic on Ω. Consider

$$\text{(DBVP)} \quad \begin{cases} Lu = f(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi|_{\partial\Omega}. \end{cases}$$

   (i) (5 marks) Suppose that we have the classical scenario, i.e., everything (a_{ij}, b_i, c, u, f, φ) is as smooth as we want (φ is defined on $\bar{\Omega}$). The first step to "solve" (DBVP) is to make the B.C. homogeneous by changing the dependent variable: $v = u - \varphi$. Show that u is a solution of (DBVP) iff v is a solution of

$$\begin{cases} Lv = f - L\varphi & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

   (ii) (10 marks) Suppose we are still in the scenario of (i) and v is a (classical) solution of (2). Show that $\forall\, w \in C_0^\infty(\Omega)$,

$$\int_\Omega [\sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}w_{x_j} + \sum_{i=1}^{n} b_i(x)v_{x_i}w + c(x)vw]dx \tag{3}$$

$$= \quad \int_\Omega f(x)w\ dx - \int_\Omega [\sum_{i,j=1}^{n} a_{ij}(x)\varphi_{x_i}w_{x_j} + \sum_{i=1}^{n} b_i(x)\varphi_{x_i}w + c(x)\varphi w]dx.$$

   Also prove this for $w \in H_0^1(\Omega)$.

Moral of story: u is said to be a weak solution of (DBVP) with $f \in H^{-1}(\Omega)$ and $\varphi \in H^1(\Omega)$, if
(a) $u - \varphi \in H_0^1(\Omega)$ and
(b) $L(u-\varphi) = f - L\varphi$ in H^{-1}–sense.

3. (20 marks) Let Ω and L be as given in Problem 2 with $b_i \equiv 0$ on Ω and $c(x) \geq 0$ a.e. on Ω.
(i) (10 marks) Prove that the homogeneous DBVP

$$\begin{cases} Lu = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{4}$$

has no weak solutions other than $u = 0$ a.e. on Ω.
Hint: Use the definition of weak solution and in it, choose a "smart" v. Warning: you cannot use the classical maximum principle because we are not in the classical scenario!

(ii) (10 marks) Prove that

$$\text{(DBVP)} \quad \begin{cases} Lu = \text{your birthday} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5}$$

has one and only one weak solution.

Comment: You may prove (i) and (ii) simultaneously using the Riesz Representation Theorem or the Lax-Milgram Theorem.

4. (35 marks) **(Neumann BVP)** Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1-smooth $\partial\Omega$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be the unit outer normal of $\partial\Omega$. Let L be given as in Problem 2. Consider

$$\text{(NBVP)} \quad \begin{cases} Lu = f(x) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_A} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $A(x) = (a_{ij}(x))_{n\times n}$ and on $\partial\Omega$, $\nu_A = A\nu$, $\frac{\partial u}{\partial \nu_A} = \nabla u \cdot A\nu = A\nabla u \cdot \nu$ (A is symmetric). (ν_A is called "co-normal" of $\partial\Omega$ with respect to the diffusion matrix A. Physically, "$\frac{\partial u}{\partial \nu_A} = 0$ on $\partial\Omega$" means that the $\partial\Omega$ is well-insulated.)
(i) (10 marks) Suppose everything is smooth. Show that if $u \in C^2(\bar{\Omega})$ is a solution of (NBVP), then $\forall v \in C^1(\bar{\Omega})$,

$$\int_\Omega [\sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x) uv] dx = \int_\Omega f(x) v \, dx. \tag{6}$$

Also prove this for $v \in H^1(\Omega)$.

(ii) (10 marks) Suppose everything is smooth. Suppose $u \in C^2(\bar{\Omega})$ satisfies (6). Show that u is a solution of (NBVP).

Hint: Step 1. In (6), take $v \in C_0^\infty(\Omega)$ to prove that $Lu = f$ holds pointwise in Ω. Step 2. Use the result obtained in Step 1 and take general $v \in C^1(\bar{\Omega})$ in (6) to show the boundary condition.

Moral of story: (NBVP) $\Rightarrow$ (6) at least in the classical sense. This motivates the following definition of weak solutions of (NBVP).

Definition. Let $f \in (H^1(\Omega))^*$. We say u is a weak solution of (NBVP) if (a) $u \in H^1(\Omega)$, (b) (6) holds for every $v \in H^1(\Omega)$ ($\int_\Omega fv\ dx$ replaced by $< f, v >_{(H^1(\Omega))^*, H^1(\Omega)}$).

Remark. This is exactly the corresponding definition for the Dirichlet case, except with $H_0^1(\Omega)$ being replaced by $H^1(\Omega)$!

(iii) (15 marks) Define $L : H^1(\Omega) \to (H^1(\Omega))^*$ by

$$< Lu, v >_{(H^1(\Omega))^*, H^1(\Omega)} = \text{LHS of (6)}, \ \forall\ u, v \in H^1(\Omega).$$

Prove that

(a) $\forall\ u \in H^1(\Omega)$, Lu defined above is indeed in $(H^1(\Omega))^*$.

(b) $L : H^1(\Omega) \to (H^1(\Omega))^*$ is linear and bounded.

(c) u is a weak solution of (NBVP) $\Leftrightarrow$ $u \in H^1(\Omega)$, $Lu = f$ in the $(H^1(\Omega))^*$-sense.