Week 08: Graph Algorithms 2

Weighted Graphs

Weighted Graphs

Graphs so far have considered

- edge = an association between two vertices/nodes
- may be a precedence in the association (directed)

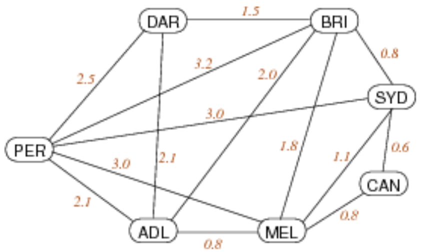
Some applications require us to consider

- a *cost* or *weight* of an association
- modelled by assigning values to edges (e.g. positive reals)

Weights can be used in both directed and undirected graphs.

... Weighted Graphs

Example: major airline flight routes in Australia



Representation: edge = direct flight; weight = approx flying time (hours)

... Weighted Graphs

Weights lead to minimisation-type questions, e.g.

1. Cheapest way to connect all vertices?

- a.k.a. *minimum spanning tree* problem
- assumes: edges are weighted and undirected

2. Cheapest way to get from *A* to *B*?

- a.k.a *shortest path* problem
- assumes: edge weights positive, directed or undirected

Exercise #1: Implementing a Route Finder

If we represent a street map as a graph

- what are the vertices?
- what are the edges?
- are edges directional?
- what are the weights?
- are the weights fixed?

What kind of algorithm would ...

- help us find the "quickest" way to get from A to B?

Weighted Graph Representation

Weights can easily be added to:

- adjacency matrix representation (0/1 → int or float)
- adjacency lists representation (add int/float to list node)

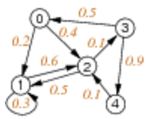
An alternative representation useful in this context:

- edge list representation (list of *(s,t,w)* triples)

All representations work whether edges are directed or not.

... Weighted Graph Representation

Adjacency matrix representation with weights:



Weighted Digraph

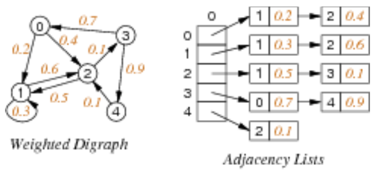
	0	1	2	3	4
0	•	0.2	0.4	•	•
1	•	0.3	0.6	•	•
2	•	0.5	•	0.1	•
3	0.5	•	•	•	0.9
4	•	•	0.1	•	•

Adjacency Matrix

Note: need distinguished value to indicate "no edge".

... Weighted Graph Representation

Adjacency lists representation with weights:

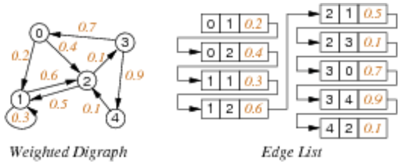


Note: if undirected, each edge appears twice with same weight

... Weighted Graph Representation

9/40

Edge array / edge list representation with weights:



Note: not very efficient for use in processing algorithms, but does
give a possible representation for min spanning trees or shortest paths

... Weighted Graph Representation

10/40

Sample adjacency matrix implementation in C requires minimal changes to previous Graph ADT:

WGraph.h

```
// edges are pairs of vertices (end-points) plus positive weight
typedef struct Edge {
   Vertex v;
   Vertex w;
   int    weight;
} Edge;

// returns weight, or 0 if vertices not adjacent
int adjacent(Graph, Vertex, Vertex);
```

... Weighted Graph Representation

11/40

WGraph.c

```
typedef struct GraphRep {
   int **edges;  // adjacency matrix storing positive weights
                 // 0 if nodes not adjacent
   int   nV;     // #vertices
   int   nE;     // #edges
} GraphRep;
```

```
void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));
   if (g->edges[e.v][e.w] == 0) {  // edge e not in graph
      g->edges[e.v][e.w] = e.weight;
      g->edges[e.w][e.v] = e.weight;
      g->nE++;
   }
}

int adjacent(Graph g, Vertex v, Vertex w) {
   assert(g != NULL && validV(g,v) && validV(g,w));
   return g->edges[v][w];
}
```

Minimum Spanning Trees

Minimum Spanning Trees

13/40

Reminder: *Spanning tree ST* of graph *G(V,E)*

- *spanning* = all vertices, *tree* = no cycles
- *ST* is a subgraph of *G*   (*G'=(V,E')* where *E'* ⊆ *E*)
- *ST* is *connected* and *acyclic*

Minimum spanning tree MST of graph *G*

- *MST* is a spanning tree of *G*
- sum of edge weights is no larger than any other ST

Applications: Computer networks, Electrical grids, Transportation networks …

Problem: how to (efficiently) find MST for graph *G*?

NB: MST may not be unique   (e.g. all edges have same weight ⇒ every ST is MST)

... Minimum Spanning Trees

14/40

Brute force solution:

```
findMST(G):
|  Input  graph G
|  Output a minimum spanning tree of G
|
|  bestCost=∞
|  for all spanning trees t of G do
|  |  if cost(t)<bestCost then
|  |     bestTree=t
|  |     bestCost=cost(t)
|  |  end if
```

```
|  end for
|  return bestTree
```

Example of *generate-and-test* algorithm.

Not useful because #spanning trees is potentially large (e.g. n^{n-2} for a complete graph with n vertices)

... Minimum Spanning Trees

15/40

Simplifying assumption:

- edges in *G* are not directed (MST for digraphs is harder)

Kruskal's Algorithm

16/40

One approach to computing MST for graph *G* with *V* nodes:

1. start with empty MST
2. consider edges in increasing weight order
    - add edge if it does not form a cycle in MST
3. repeat until *V-1* edges are added

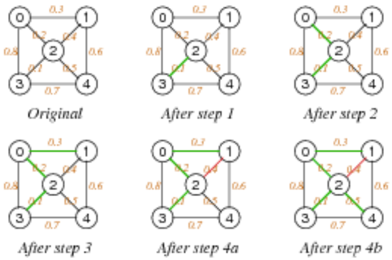
Critical operations:

- iterating over edges in weight order
- checking for cycles in a graph

... Kruskal's Algorithm

17/40

Execution trace of Kruskal's algorithm:

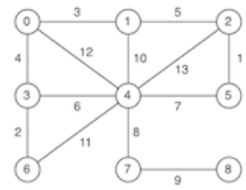


Original, *After step 1*, *After step 2*, *After step 3*, *After step 4a*, *After step 4b*

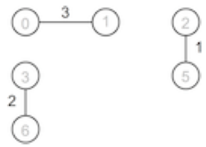
Exercise #2: Kruskal's Algorithm

18/40

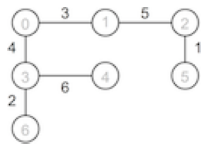
Show how Kruskal's algorithm produces an MST on:



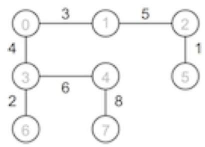
After 3rd iteration:



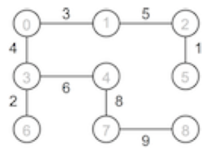
After 6th iteration:



After 7th iteration:



After 8th iteration (*V*-1=8 edges added):



... Kruskal's Algorithm

20/40

Pseudocode:

```
KruskalMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
```

```
sort edges(G) by weight
for each e∈sortedEdgeList do
   MST = MST ∪ {e}
   if MST has a cyle then
      MST = MST \ {e}
   end if
   if MST has n-1 edges then
      return MST
   end if
end for
```

... Kruskal's Algorithm

21/40

Rough time complexity analysis …

- sorting edge list is *O(E·log E)*
- at least *V* iterations over sorted edges
- on each iteration …
  - getting next lowest cost edge is *O(1)*
  - checking whether adding it forms a cycle: cost = ??

Possibilities for cycle checking:

- use DFS … too expensive?
- could use *Union-Find data structure* (see Sedgewick Ch.1)

Prim's Algorithm

22/40

Another approach to computing MST for graph *G=(V,E)*:

1. start from any vertex *s* and empty MST
2. choose edge not already in MST to add to MST
   - must be incident on a vertex already connected to *s* in MST
   - must have minimal weight of all such edges
3. repeat until MST covers all vertices

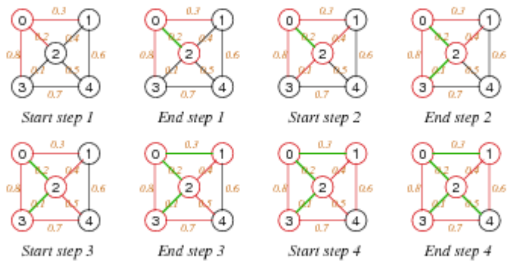
Critical operations:

- checking for vertex being connected in a graph
- finding min weight edge in a set of edges

... Prim's Algorithm

23/40

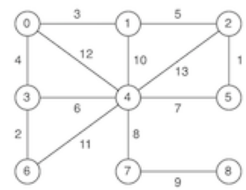
Execution trace of Prim's algorithm (starting at *s*=0):



Exercise #3: Prim's Algorithm

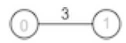
24/40

Show how Prim's algorithm produces an MST on:

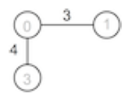


Start from vertex 0

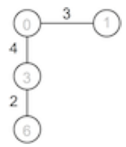
After 1st iteration:



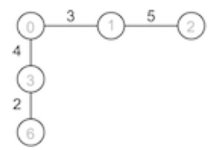
After 2nd iteration:



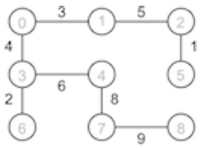
After 3rd iteration:



After 4th iteration:



After 8^{th} iteration (all vertices covered):



... Prim's Algorithm

Pseudocode:

```
PrimMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  usedV={0}
|  unusedE=edges(g)
|  while |usedV|<n do
|  |  find e=(s,t,w)∈unusedE such that {
|  |     s∈usedV ∧ t∉usedV ∧ w is min weight of all such edges
|  |  }
|  |  MST = MST ∪ {e}
|  |  usedV = usedV ∪ {t}
|  |  unusedE = unusedE \ {e}
|  end while
|  return MST
```

Critical operation: finding best edge

... Prim's Algorithm

Rough time complexity analysis …

- *V* iterations of outer loop
- in each iteration …
    - find min edge with set of edges is $O(E) \Rightarrow O(V \cdot E)$ overall
    - find min edge with *priority queue* is $O(\log E) \Rightarrow O(V \cdot \log E)$ overall

Note:

- Using a *priority queue* gives a variation of DFS (stack) and BFS (queue) graph traversal

---

Sidetrack: Priority Queues

Some applications of queues require

- items processed in order of "key"
- rather than in order of entry (FIFO — first in, first out)

Priority Queues (*PQueues*) provide this via:

- `join`: insert item into PQueue (replacing `enqueue`)
- `leave`: remove item with largest key (replacing `dequeue`)

... Sidetrack: Priority Queues

Comparison of different Priority Queue representations:

	sorted array	unsorted array	sorted list	unsorted list
space usage	*MaxN*	*MaxN*	$O(N)$	$O(N)$
join	$O(N)$	$O(1)$	$O(N)$	$O(1)$
leave	$O(N)$	$O(N)$	$O(1)$	$O(N)$
is empty?	$O(1)$	$O(1)$	$O(1)$	$O(1)$

for a PQueue containing *N* items

Other MST Algorithms

Boruvka's algorithm ... complexity $O(E \cdot \log V)$

- the oldest MST algorithm
- start with V separate components
- join components using min cost links
- continue until only a single component

Karger, Klein, and Tarjan ... complexity $O(E)$

- based on Boruvka, but non-deterministic
- randomly selects subset of edges to consider
- for the keen, here's the paper describing the algorithm

Shortest Path

Shortest Path

Path = sequence of edges in graph G $\quad p = (v_0,v_1), (v_1,v_2), \ldots, (v_{m-1},v_m)$

cost(path) = sum of edge weights along path

Shortest path between vertices *s* and *t*

- a simple path $p(s,t)$ where $s = first(p)$, $t = last(p)$

- no other simple path *q(s,t)* has *cost(q)* < *cost(p)*

Assumptions: weighted digraph, no negative weights.

Finding shortest path between two given nodes known as *source-target* SPP problem

Variations: *single-source*, *all-pairs*

Applications: robot navigation, routing in data networks, …

33/40

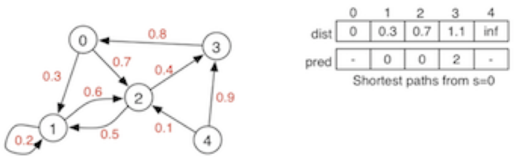
Single-source Shortest Path (SSSP)

Given: weighted digraph *G*, source vertex *s*

Result: shortest paths from *s* to all other vertices

- `dist[]` *V*-indexed array of cost of shortest path from *s*
- `pred[]` *V*-indexed array of predecessor in shortest path from *s*

Example:



34/40

Edge Relaxation

Assume: `dist[]` and `pred[]` as above (but containing data for shortest paths *discovered so far*)

`dist[v]` is length of shortest known path from *s* to *v*
`dist[w]` is length of shortest known path from *s* to *w*

Relaxation updates data for *w* if we find a shorter path from *s* to *w*:

`dist[v]=8, dist[w]=12` `pred[v]=?, pred[w]=?` `dist[v]=8, dist[w]=11` `pred[v]=?, pred[w]=v`

Relaxation along edge *e*=`(v,w,weight)`:

- if **`dist[v]+weight < dist[w]`** then
  update `dist[w]:=dist[v]+weight` and `pred[w]:=w`

35/40

Dijkstra's Algorithm

One approach to solving single-source shortest path …

Data: *G*, *s*, `dist[]`, `pred[]` and

- *vSet*: set of vertices whose shortest path from *s* is known

Algorithm:

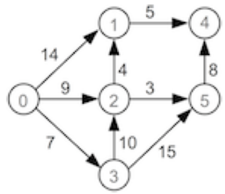
```
dist[]  // array of cost of shortest path from s
pred[]  // array of predecessor in shortest path from s

dijkstraSSSP(G,source):
|  Input graph G, source node
|
|  initialise dist[] to all ∞, except dist[source]=0
|  initialise pred[] to all -1
|  vSet=all vertices of G
|  while vSet≠∅ do
|  |  find s∈vSet with minimum dist[s]
|  |  for each (s,t,w)∈edges(G) do
|  |     relax along (s,t,w)
|  |  end for
|  |  vSet=vSet\{s}
|  end while
```

36/40

Exercise #4: Dijkstra's Algorithm

Show how Dijkstra's algorithm runs on (source node = 0):



	[0]	[1]	[2]	[3]	[4]	[5]
dist	0	∞	∞	∞	∞	∞
pred	–	–	–	–	–	–

dist	0	14	9	7	∞	∞
pred	–	0	0	0	–	–

dist	0	14	9	7	∞	22
pred	–	0	0	0	–	3

dist	0	13	9	7	∞	12
pred	–	2	0	0	–	2

dist	0	13	9	7	20	12
pred	–	2	0	0	5	2

dist	0	13	9	7	18	12
pred	–	2	0	0	1	2

... Dijkstra's Algorithm

Why Dijkstra's algorithm is correct:

Hypothesis.
(a) For visited *s* … *dist[s]* is shortest distance from source
(b) For unvisited *t* … *dist[t]* is shortest distance from source *via visited nodes*

Proof.
Base case: no visited nodes, *dist[source]=0*, *dist[s]=∞* for all other nodes

Induction step:

1. If *s* is unvisited node with minimum *dist[s]*, then *dist[s]* is shortest distance from source to *s*:
   - if ∃ shorter path via only visited nodes, then *dist[s]* would have been updated when processing the predecessor of *s* on this path
   - if ∃ shorter path via an unvisited node *u*, then *dist[u]<dist[s]*, which is impossible if *s* has min distance of all unvisited nodes
2. This implies that (a) holds for *s* after processing *s*
3. (b) still holds for all unvisited nodes *t* after processing *s*:
   - if ∃ shorter path via *s* we would have just updated *dist[t]*
   - if ∃ shorter path without *s* we would have found it previously

---

... Dijkstra's Algorithm

Time complexity analysis …

Each edge needs to be considered once ⇒ *O(E)*.

Outer loop has *O(V)* iterations.

Implementing "`find s∈vSet with minimum dist[s]`"

1. try all `s∈vSet` ⇒ cost = $O(V)$ ⇒ overall cost = $O(E + V^2) = O(V^2)$
2. using a PQueue to implement extracting minimum
   - can improve overall cost to $O(E + V \cdot log\ V)$ (for best-known implementation)

Summary

- Weighted graph representations
- Minimum Spanning Tree (MST)
  - Kruskal, Prim
- Single source shortest path problem
  - Dijkstra

- Suggested reading:
  - Sedgewick, Ch.19.3,20-20.4,21-21.3

Produced: 11 Sep 2017