

# Week 04: Analysis of Algorithms

## Analysis of Algorithms

### Running Time

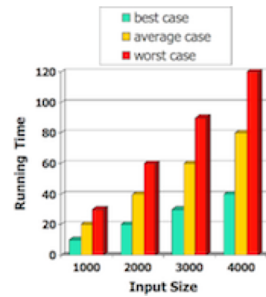
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An **algorithm** is a step-by-step procedure

- for solving a problem
- in a finite amount of time

Most algorithms map input to output

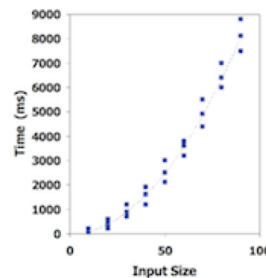
- running time typically grows with input size
- *average time* often difficult to determine
- Focus on *worst case* running time
  - easier to analyse
  - crucial to many applications: finance, robotics, games, ...



### Empirical Analysis

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1. Write program that implements an algorithm
2. Run program with inputs of varying size and composition
3. Measure the actual running time
4. Plot the results



### ... Empirical Analysis

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Limitations:

- requires to implement the algorithm, which may be difficult
- results may not be indicative of running time on other inputs
- same hardware and operating system must be used in order to compare two algorithms

### Theoretical Analysis

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- Uses high-level description of the algorithm instead of implementation ("pseudocode")
- Characterises running time as a function of the input size,  $n$
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

### Pseudocode

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- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

### ... Pseudocode

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Example: Find maximal element in an array

```
arrayMax(A):  
    Input  array A of n integers  
    Output maximum element of A  
  
    currentMax=A[0]  
    for all i=1..n-1 do  
        if A[i]>currentMax then  
            currentMax=A[i]  
        end if  
    end for  
    return currentMax
```

### ... Pseudocode

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Control flow

- if ... then ... [else] ... end if
- while .. do ... end while
- repeat ... until
- for [all][each] .. do ... end for

Function declaration

- `f(arguments):`  
    **Input ...**  
    **Output ...**  
    ...

Expressions

- `=` assignment
- `=` equality testing
- $n^2$  superscripts and other mathematical formatting allowed
- `swap A[i] and A[j]` verbal descriptions of *simple* operations allowed

Exercise #1: Pseudocode

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Formulate the following verbal description in pseudocode:

*In the first phase, we iteratively pop all the elements from stack S and enqueue them in queue Q, then dequeue the element from Q and push them back onto S.*

*As a result, all the elements are now in reversed order on S.*

*In the second phase, we again pop all the elements from S, but this time we also look for the element x.*

*By again passing the elements through Q and back onto S, we reverse the reversal, thereby restoring the original order of the elements on S.*

Sample solution:

```
while ¬empty(S) do
    pop e from S, enqueue e into Q
end while
while ¬empty(Q) do
    dequeue e from Q, push e onto S
end while
found=false
while ¬empty(S) do
    pop e from S, enqueue e into Q
    if e=x then
        found=true
    end if
end while
while ¬empty(Q) do
    dequeue e from Q, push e onto S
end while
```

Exercise #2: Pseudocode

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Implement the following pseudocode instructions in C

1. A is an array of ints

```
...
swap A[i] and A[j]
...
```

2. head points to beginning of linked list

```
...
swap head and head->next
...
```

3. S is a stack

```
...
swap the top two elements on S
...
```

1. `int temp = A[i];`  
   `A[i] = A[j];`  
   `A[j] = temp;`
2. `NodeT *succ = head->next;`  
   `head->next = succ->next;`  
   `succ->next = head;`  
   `head = succ;`
3. `x = StackPop(S);`  
   `y = StackPop(S);`  
   `StackPush(S, x);`  
   `StackPush(S, y);`

The following pseudocode instruction is problematic. Why?

```
...
swap the two elements at the front of queue Q
...
```

The Abstract RAM Model

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RAM = Random Access Machine

- A CPU (central processing unit)
- A potentially unbounded bank of memory cells
  - each of which can hold an arbitrary number, or character
- Memory cells are numbered, and accessing any one of them takes CPU time

Primitive Operations

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- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent of the programming language
- Exact definition not important (we will shortly see why)
- Assumed to take a constant amount of time in the RAM model

Examples:

- evaluating an expression
- indexing into an array
- calling/returning from a function

## Counting Primitive Operations

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By inspecting the pseudocode ...

- we can determine the maximum number of primitive operations executed by an algorithm
- as a function of the input size

Example:

```
arrayMax(A):
    Input  array A of n integers
    Output maximum element of A

    currentMax=A[0]
    for all i=1..n-1 do
        if A[i]>currentMax then
            currentMax=A[i]
        end if
    end for
    return currentMax
```

1  
n+(n-1)  
2(n-1)  
n-1  
  
1  
  
-----  
Total      5n-2

## Estimating Running Times

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Algorithm `arrayMax` requires  $5n - 2$  primitive operations in the *worst* case

- *best* case requires  $4n - 1$  operations (why?)

Define:

- $a$  ... time taken by the fastest primitive operation
- $b$  ... time taken by the slowest primitive operation

Let  $T(n)$  be worst-case time of `arrayMax`. Then

$$a \cdot (5n - 2) \leq T(n) \leq b \cdot (5n - 2)$$

Hence, the running time  $T(n)$  is bound by two **linear** functions

## ... Estimating Running Times

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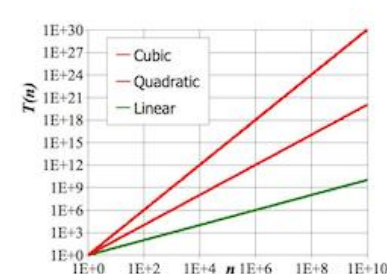
## Seven commonly encountered functions for algorithm analysis

- Constant  $\equiv 1$
- Logarithmic  $\equiv \log n$
- Linear  $\equiv n$
- N-Log-N  $\equiv n \log n$
- Quadratic  $\equiv n^2$
- Cubic  $\equiv n^3$
- Exponential  $\equiv 2^n$

## ... Estimating Running Times

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In a log-log chart, the slope of the line corresponds to the growth rate of the function

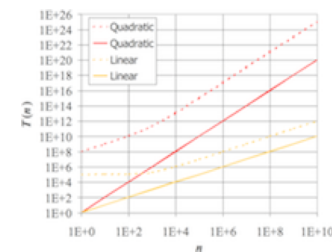


## ... Estimating Running Times

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The growth rate is not affected by constant factors or lower-order terms

- Examples:
  - $10^2n + 10^5$  is a linear function
  - $10^5n^2 + 10^8n$  is a quadratic function



### ... Estimating Running Times

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## Changing the hardware/software environment

- affects  $T(n)$  by a constant factor
- but does not alter the growth rate of  $T(n)$

⇒ Linear growth rate of the running time  $T(n)$  is an intrinsic property of algorithm `arrayMax`

## Exercise #3: Estimating running times

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Determine the number of primitive operations

```
matrixProduct(A,B):
  Input  n×n matrices A, B
  Output n×n matrix A·B

  for all i=1..n do
    for all j=1..n do
      C[i,j]=0
      for all k=1..n do
        C[i,j]=C[i,j]+A[i,k]·B[k,j]
      end for
    end for
  end for
  return C
```

## Exercise #4: Estimating running times

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```
matrixProduct(A,B):
  Input  n×n matrices A, B
  Output n×n matrix A·B

  for all i=1..n do
    for all j=1..n do
      C[i,j]=0
      for all k=1..n do
        C[i,j]=C[i,j]+A[i,k]·B[k,j]
      end for
    end for
  end for
  return C
```

2n+1  
n(2n+1)  
n<sup>2</sup>  
n<sup>2</sup>(2n+1)  
n<sup>3</sup>·5  
  
1  
-----  
Total    7n<sup>3</sup>+4n<sup>2</sup>+3n+2

## Big-Oh

## Big-Oh Notation

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Given functions  $f(n)$  and  $g(n)$ , we say that

$f(n)$  is  $O(g(n))$

if there are positive constants  $c$  and  $n_0$  such that

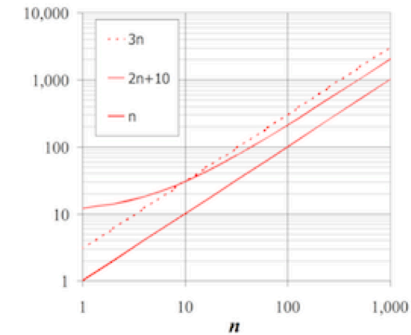
$$f(n) \leq c \cdot g(n) \quad \forall n \geq n_0$$

## ... Big-Oh Notation

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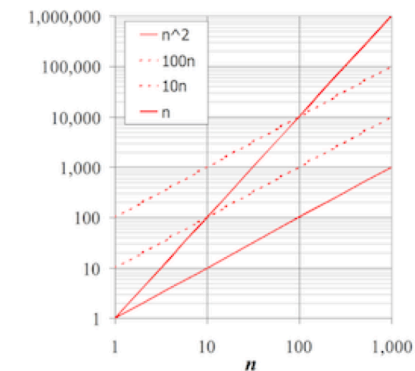
Example: function  $2n + 10$  is  $O(n)$

- $2n+10 \leq c \cdot n$   
⇒  $(c-2)n \geq 10$   
⇒  $n \geq 10/(c-2)$
- pick  $c=3$  and  $n_0=10$



## ... Big-Oh Notation

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Example: function  $n^2$  is not  $O(n)$

- $n^2 \leq c \cdot n$   
⇒  $n \leq c$
- inequality cannot be satisfied since  $c$  must be a constant

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Exercise #5: Big-Oh

Show that

- 1. 7n-2 is O(n)
- 2. 3n<sup>3</sup> + 20n<sup>2</sup> + 5 is O(n<sup>3</sup>)
- 3. 3·log n + 5 is O(log n)

- 1. 7n-2 is O(n)  
need c>0 and n<sub>0</sub>≥1 such that 7n-2 ≤ c·n for n≥n<sub>0</sub>  
⇒ true for c=7 and n<sub>0</sub>=1
- 2. 3n<sup>3</sup> + 20n<sup>2</sup> + 5 is O(n<sup>3</sup>)  
need c>0 and n<sub>0</sub>≥1 such that 3n<sup>3</sup>+20n<sup>2</sup>+5 ≤ c·n<sup>3</sup> for n≥n<sub>0</sub>  
⇒ true for c=4 and n<sub>0</sub>=21
- 3. 3·log n + 5 is O(log n)  
need c>0 and n<sub>0</sub>≥1 such that 3·log n+5 ≤ c·log n for n≥n<sub>0</sub>  
⇒ true for c=8 and n<sub>0</sub>=2

Big-Oh and Rate of Growth

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- Big-Oh notation gives an upper bound on the growth rate of a function
  - "f(n) is O(g(n))" means growth rate of f(n) no more than growth rate of g(n)
- use big-Oh to rank functions according to their rate of growth

	f(n) is O(g(n))	g(n) is O(f(n))
g(n) grows faster	yes	no
f(n) grows faster	no	yes
same order of growth	yes	yes

Big-Oh Rules

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- If f(n) is a polynomial of degree d ⇒ f(n) is O(n<sup>d</sup>)
  - lower-order terms are ignored
  - constant factors are ignored
- Use the smallest possible class of functions
  - say "2n is O(n)" instead of "2n is O(n<sup>2</sup>)"
- Use the simplest expression of the class
  - say "3n + 5 is O(n)" instead of "3n + 5 is O(3n)"

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Exercise #6: Big-Oh

Show that  $\sum_{i=1}^n i$  is O(n<sup>2</sup>)

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

which is O(n<sup>2</sup>)

Asymptotic Analysis of Algorithms

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Asymptotic analysis of algorithms determines running time in big-Oh notation:

- find worst-case number of primitive operations as a function of input size
- express this function using big-Oh notation

Example:

- algorithm `arrayMax` executes at most 5n – 2 primitive operations  
⇒ algorithm `arrayMax` "runs in O(n) time"

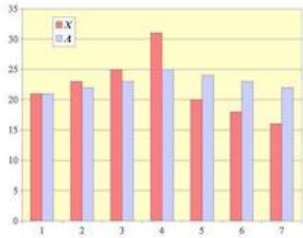
Constant factors and lower-order terms eventually dropped  
⇒ can disregard them when counting primitive operations

Example: Computing Prefix Averages

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- The *i*-th prefix average of an array X is the average of the first i elements:

A[i] = (X[0] + X[1] + ... + X[i]) / (i+1)



NB. computing the array A of prefix averages of another array X has applications in financial analysis

... Example: Computing Prefix Averages

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A *quadratic* algorithm to compute prefix averages:

```
prefixAverages1(X):
  Input  array X of n integers
  Output array A of prefix averages of X

  for all i=0..n-1 do
    s=X[0]
    for all j=1..i do
      s=s+X[j]
    end for
    A[i]=s/(i+1)
  end for
  return A
```

$2 \cdot O(n^2) + 3 \cdot O(n) + O(1) = O(n^2)$

⇒ Time complexity of algorithm prefixAverages1 is  $O(n^2)$

... Example: Computing Prefix Averages

The following algorithm computes prefix averages by keeping a running sum:

```
prefixAverages2(X):
  Input  array X of n integers
  Output array A of prefix averages of X

  s=0
  for all i=0..n-1 do
    s=s+X[i]
    A[i]=s/(i+1)
  end for
  return A
```

Thus, prefixAverages2 is  $O(n)$

Example: Binary Search

The following recursive algorithm searches for a value in a *sorted* array:

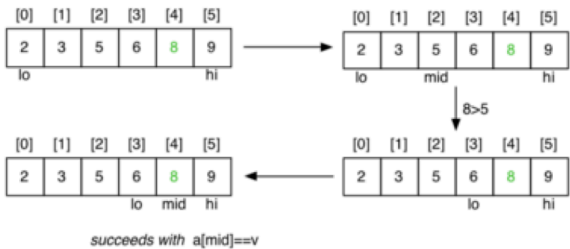
```
search(v,a,lo,hi):
  Input  value v
         array a[lo..hi] of values
  Output true if v in a[lo..hi]
         false otherwise

  mid=(lo+hi)/2
  if lo>hi then return false
  if a[mid]=v then
    return true
  else if a[mid]<v then
```

```
    return search(v,a,mid+1,hi)
  else
    return search(v,a,lo,mid-1)
  end if
```

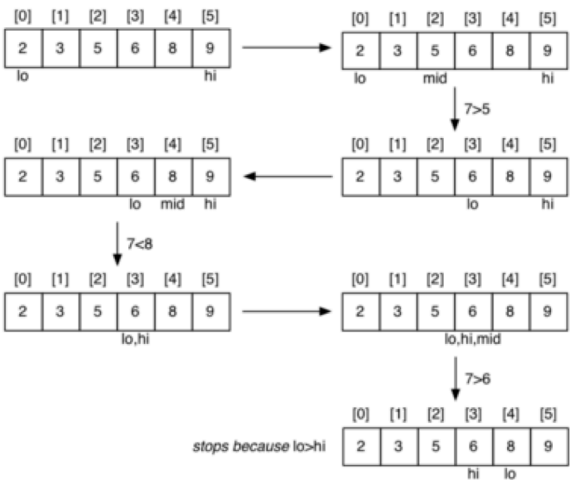
... Example: Binary Search

Successful search for a value of 8:



... Example: Binary Search

Unsuccessful search for a value of 7:



... Example: Binary Search

Cost analysis:

- $C_i = \text{\#calls to search}()$  for array of length  $i$

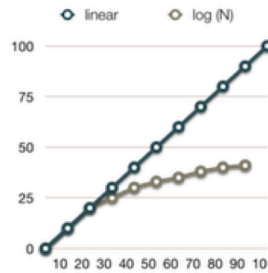
- for best case,  $C_n = 1$
- for  $a[i..j]$ ,  $j < i$  (length=0)
  - $C_0 = 0$
- for  $a[i..j]$ ,  $i \leq j$  (length=n)
  - $C_n = 1 + C_{n/2} \Rightarrow C_n = \log_2 n$

Thus, binary search is  $O(\log_2 n)$  or simply  $O(\log n)$  (why?)

### ... Example: Binary Search

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Why logarithmic complexity is good:



## Math Needed for Complexity Analysis

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- Summations
- Logarithms
  - $\log_b(xy) = \log_b x + \log_b y$
  - $\log_b(x/y) = \log_b x - \log_b y$
  - $\log_b x^a = a \log_b x$
  - $\log_b a = \log_x a / \log_x b$
- Exponentials
  - $a^{(b+c)} = a^b a^c$
  - $a^{bc} = (a^b)^c$
  - $a^b / a^c = a^{(b-c)}$
  - $b = a^{\log_a b}$
  - $b^c = a^{c \cdot \log_a b}$
- Proof techniques
- Summation (addition of sequences of numbers)
- Basic probability (for average case analysis, randomised algorithms)

### Exercise #7: Analysis of Algorithms

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What is the complexity of the following algorithm?

```
splitList(L):
```

```
Input non-empty linked list L
Output L split into two halves
```

```
// use slow and fast pointer to traverse L
slow=head(L), fast=head(L).next
while fast!=NULL ^ fast.next!=NULL do
    slow=slow.next, fast=fast.next.next // advance pointers
end while
cut L between slow and slow.next
```

Answer:  $O(L)$

### Exercise #8: Analysis of Algorithms

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What is the complexity of the following algorithm?

```
binaryConversion(n):
Input positive integer n
Output binary representation of n on a stack

create empty stack S
while n>0 do
    | push (n mod 2) onto S
    | n=[n/2]
end while
return S
```

Assume that creating a stack and pushing an element both are  $O(1)$  operations ("constant")

Answer:  $O(\log n)$

## Relatives of Big-Oh

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*big-Omega*

- $f(n)$  is  $\Omega(g(n))$  if there is a constant  $c > 0$  and an integer constant  $n_0 \geq 1$  such that

$$f(n) \geq c \cdot g(n) \quad \forall n \geq n_0$$

*big-Theta*

- $f(n)$  is  $\Theta(g(n))$  if there are constants  $c', c'' > 0$  and an integer constant  $n_0 \geq 1$  such that

$$c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n) \quad \forall n \geq n_0$$

### ... Relatives of Big-Oh

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- $f(n)$  is  $O(g(n))$  if  $f(n)$  is asymptotically *less than or equal* to  $g(n)$
- $f(n)$  is  $\Omega(g(n))$  if  $f(n)$  is asymptotically *greater than or equal* to  $g(n)$
- $f(n)$  is  $\Theta(g(n))$  if  $f(n)$  is asymptotically *equal* to  $g(n)$

### ... Relatives of Big-Oh

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Examples:

- $\frac{1}{4}n^2$  is  $\Omega(n^2)$ 
  - need  $c > 0$  and  $n_0 \geq 1$  such that  $\frac{1}{4}n^2 \geq c \cdot n^2$  for  $n \geq n_0$
  - let  $c = \frac{1}{4}$  and  $n_0 = 1$
- $\frac{1}{4}n^2$  is  $\Omega(n)$ 
  - need  $c > 0$  and  $n_0 \geq 1$  such that  $\frac{1}{4}n^2 \geq c \cdot n$  for  $n \geq n_0$
  - let  $c = 1$  and  $n_0 = 2$
- $\frac{1}{4}n^2$  is  $\Theta(n^2)$ 
  - since  $\frac{1}{4}n^2$  is in  $\Omega(n^2)$  and  $O(n^2)$

## Complexity Classes

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Problems in Computer Science ...

- some have *polynomial* worst-case performance (e.g.  $n^2$ )
- some have *exponential* worst-case performance (e.g.  $2^n$ )

Classes of problems:

- $P$  = problems for which an algorithm can compute answer in polynomial time
- $NP$  = includes problems for which no  $P$  algorithm is known

Beware: NP stands for "nondeterministic, polynomial time (on a theoretical *Turing Machine*)"

### ... Complexity Classes

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Computer Science jargon for difficulty:

- tractable ... have a polynomial-time algorithm (useful in practice)
- intractable ... no tractable algorithm is known (feasible only for small  $n$ )
- non-computable ... no algorithm can exist

*Computational complexity theory* deals with different degrees of intractability

## Generate and Test Algorithms

## Generate and Test

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In scenarios where

- it is simple to test whether a given state is a solution
- it is easy to generate new states (preferably likely solutions)

then a *generate and test* strategy can be used.

It is necessary that states are generated systematically

- so that we are guaranteed to find a solution, or know that none exists
  - some **randomised** algorithms do not require this, however (more on this later in this course)

### ... Generate and Test

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Simple example: checking whether an integer  $n$  is prime

- generate/test all possible factors of  $n$
- if none of them pass the test  $\Rightarrow n$  is prime

*Generation* is straightforward:

- produce a sequence of all numbers from 2 to  $n-1$

*Testing* is also straightfoward:

- check whether next number divides  $n$  exactly

### ... Generate and Test

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Function for primality checking:

```
isPrime(n):
|   Input   natural number n
|   Output true if n prime, false otherwise
|
|   for all i=2..n-1 do           // generate
|   |   if n mod i = 0 then       // test
|   |       return false         // i is a divisor => n is not prime
|   |   end if
|   end for
|   return true                   // no divisor => n is prime
```

Complexity of `isPrime` is  $O(n)$

Can be optimised: check only numbers between 2 and  $\lfloor \sqrt{n} \rfloor \Rightarrow O(\sqrt{n})$

## Example: Subset Sum

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Problem to solve ...



Is there a subset  $S$  of these numbers with  $sum(S)=1000$ ?

34, 38, 39, 43, 55, 66, 67, 84, 85, 91, 101, 117, 128, 138, 165, 168, 169, 182, 184, 186, 234, 238, 241, 276, 279, 288, 386, 387, 388, 389

General problem:

- given  $n$  integers and a target sum  $k$
- is there a subset that adds up to exactly  $k$ ?

... Example: Subset Sum

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Generate and test approach:

```
subsetsum(A,k):
|   Input  set A of n integers, target sum k
|   Output true if  $\sum_{b \in B} b = k$  for some  $B \subseteq A$ 
|           false otherwise
|
|   for each subset  $S \subseteq A$  do
|   |   if sum(S)=k then
|   |   |   return true
|   |   end if
|   end for
|   return false
```

- How many subsets are there of  $n$  elements?
- How could we generate them?

... Example: Subset Sum

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Given: a set of  $n$  distinct integers in an array  $A$  ...

- produce all subsets of these integers

A method to generate subsets:

- represent sets as  $n$  bits (e.g.  $n=4$ , 0000, 0011, 1111 etc.)
- bit  $i$  represents the  $i^{th}$  input number
- if bit  $i$  is set to 1, then  $A[i]$  is in the subset
- if bit  $i$  is set to 0, then  $A[i]$  is not in the subset
- e.g. if  $A[] = \{1, 2, 3, 5\}$  then 0011 represents  $\{1, 2\}$

... Example: Subset Sum

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Algorithm:

```
subsetsum1(A,k):
|   Input  set A of n integers, target sum k
```

```
   Output true if  $\sum_{b \in B} b = k$  for some  $B \subseteq A$ 
   false otherwise

   for s=0.. $2^n-1$  do
   |   if  $k = \sum_{(i^{th} \text{ bit of } s \text{ is } 1)} A[i]$  then
   |   |   return true
   |   end if
   end for
   return false
```

Obviously, subsetsum1 is  $O(2^n)$

... Example: Subset Sum

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Alternative approach ...

```
subsetsum2(A,n,k)
(return true if any subset of A[0..n-1] sums to k; returns false otherwise)
```

- if the  $n^{th}$  value  $A[n-1]$  is part of a solution ...
  - then the first  $n-1$  values must sum to  $k - A[n-1]$
- if the  $n^{th}$  value is not part of a solution ...
  - then the first  $n-1$  values must sum to  $k$
- base cases:  $k=0$  (solved by  $\{\}$ );  $n=0$  (unsolvable if  $k>0$ )

```
subsetsum2(A,n,k):
|   Input  array A, index n, target sum k
|   Output true if some subset of A[0..n-1] sums up to k
|           false otherwise
|
|   if k=0 then
|   |   return true // empty set solves this
|   else if n=0 then
|   |   return false // no elements => no sums
|   else
|   |   return subsetsum(A,n-1,k-A[n-1])  $\vee$  subsetsum(A,n-1,k)
|   end if
```

... Example: Subset Sum

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Cost analysis:

- $C_i = \# \text{calls to subsetsum2}(\ )$  for array of length  $i$
- for best case,  $C_n = C_{n-1}$  (why?)
- for worst case,  $C_n = 2 \cdot C_{n-1} \Rightarrow C_n = 2^n$

Thus, subsetsum2 also is  $O(2^n)$

Subset Sum is typical member of the class of *NP-complete problems*

- intractable ... only algorithms with exponential performance are known
    - increase input size by 1, double the execution time
    - increase input size by 100, it takes  $2^{100} = 1,267,650,600,228,229,401,496,703,205,376$  times as long to execute
  - but if you can find a polynomial algorithm for Subset Sum, then any other *NP*-complete problem becomes *P*!
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## Summary

- Big-Oh notation
  - Asymptotic analysis of algorithms
  - Examples of algorithms with logarithmic, linear, polynomial, exponential complexity
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- Suggested reading:
    - Sedgewick, Ch.2.1-2.4,2.6
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