General Instructions:

This document contains two case studies each with equal score. You are expected to attempt both case studies and submit the solution by the deadline.

Each of these case studies introduce the basics of a problem and pose a few questions on them. A question might have multiple sub-parts, all of which need to be answered. The scores for the sub-parts have been indicated against the question. The solution format for each question has been specified alongside the question and the final submission format is described at the end of this document.

You are free to use online resources to improve your understanding of the problem statement.

If you have any doubts, please email: jpmqrmentorship.mumbai@jpmorgan.com

With the subject: Quant Mentorship 2022 Doubt {Your First Name}_{Your Last Name}

Please note that responses will be slow over the weekend.

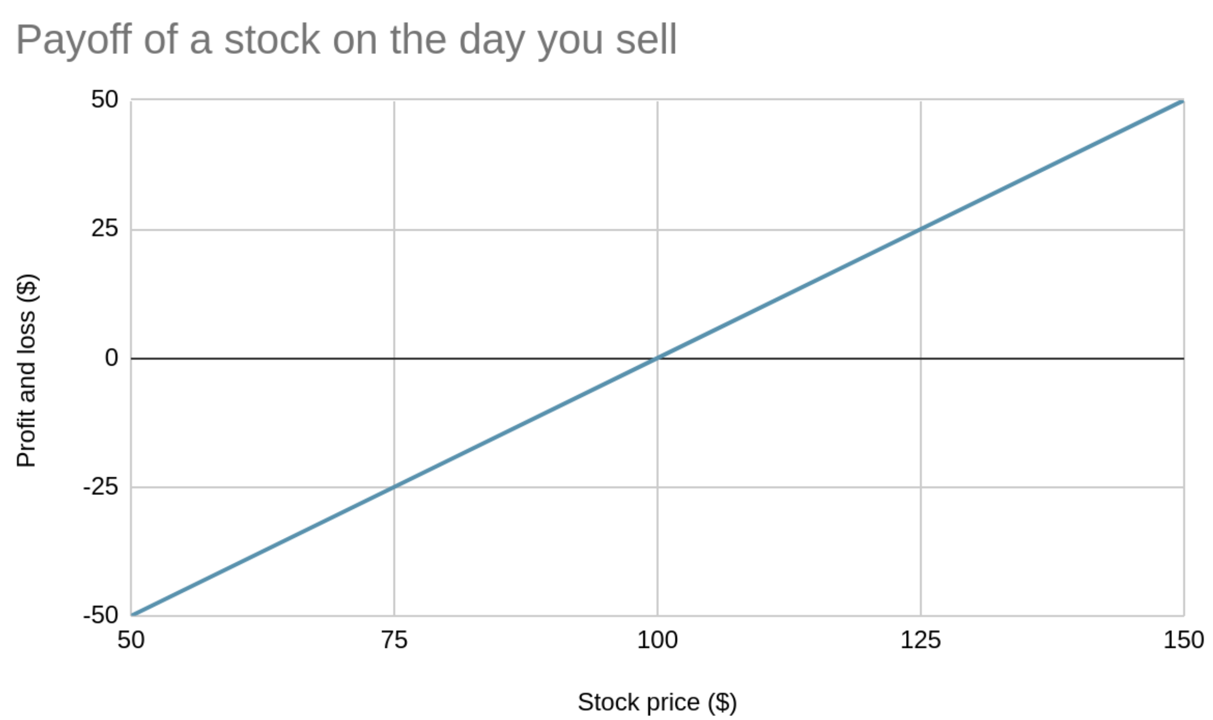
Plagiarism of any kind will not be tolerated.

Case Study – A: Derivatives

Theory:

Stocks:

A stock (or a share) is a small ownership of a company that can be freely traded (allows you to buy and sell from other participants like you). Like any other asset, the price of the stock can fluctuate based on the forces of supply and demand. Higher the demand, greater the price; and higher the supply, lower the price. If you buy a stock at, for example, $100 today, and after 3 months, its value rises to $120, you would have made a profit of $20. Likewise, if it went down to $80, you would have made a loss of $20.



Are stocks the only way you can participate in the market? Read further sections to find out!

Derivatives:

In the previous example, you have already purchased the stock, and plan to book your profit or loss after 3 months. Obviously, you have purchased it in the hope that the price rises so that you can sell at a higher price than what you bought for, but nobody can guarantee on what happens 3-months from now, right?!

Theoretically speaking, the stock price can go to 0 at the worst case and you will lose all your money. Now what if there was some way which you could use to floor your losses? This is where derivatives come in. '*Options, Futures and Other Derivatives'*, an extremely popular book on derivatives by John C. Hull, defines a derivative to be a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. Very often the variables underlying derivatives are the prices of traded assets. A stock option, for example, is a derivative whose value is dependent on the price of a stock. However, derivatives can be dependent on almost any variable, from the price of hogs to the amount of snow falling at a certain ski resort! Let's look at some famous derivatives -

Call Option:

Going back to our earlier example, your goal is to place a bet that the price of the stock goes up after 3 months, and at the same time, you want to floor your losses. A call option is a contract that gives you the right, but not obligation, to buy the stock at a pre-specified price (called strike price), on a certain date in the future (3 months, in our example), irrespective of the prevailing market price on the expiry date. You can *buy* a call option by paying a small premium to the seller of the option.

Imagine you bought a call option after paying a small premium. As the holder of the call option, you now have the right to buy the stock at $100, 3 months from now, even if the market price is greater than $100. If the stock price ends up <= $100, you can let your option expire without any action, and the maximum amount you can lose is the initial premium that you paid to buy the option (so the loss is floored)!

In the above example, you bought a call option in the hope that the prices of the underlying stock will go up. What if you wanted to place a bet on the stock price going down? In that case, you can *sell* a call option! You will get paid a small premium by the buyer of the call option, and if your estimate is correct and the stock price does go down, the buyer won't exercise the option and you will get to keep the initial premium paid to you!

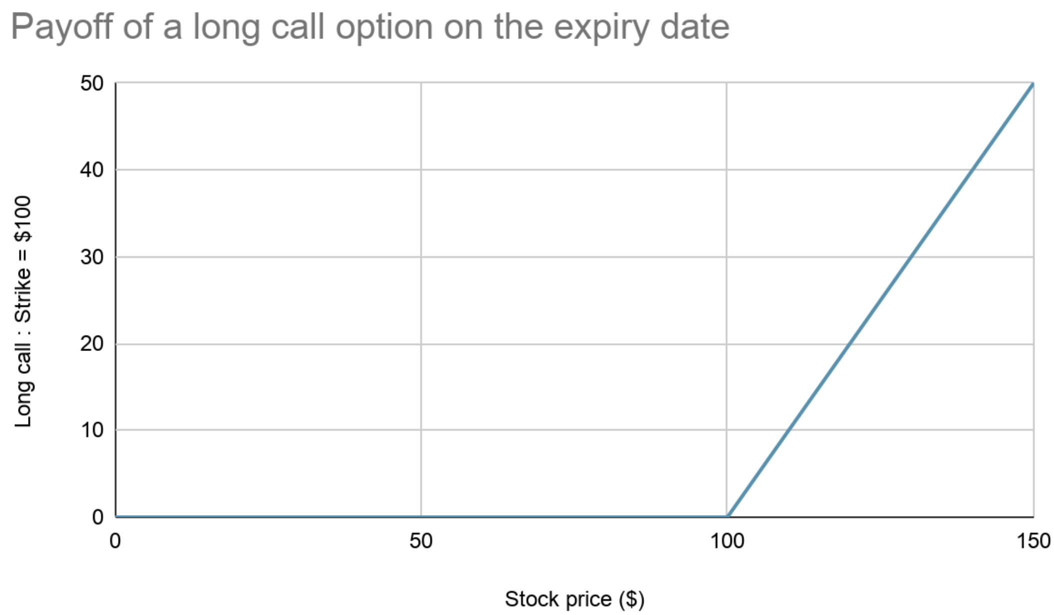
One fascinating point to note here is that you don't actually need to have any stock with you while selling a call option. When you sell a call option, your only obligation is to give the buyer the underlying stock at expiry at the predetermined strike price, that too only if the buyer chooses to exercise the call option. And so you have the liberty to not have the underlying stock with you until the expiry date arrives!

Since you don't need to have the stock with you when you are buying or selling the option, your initial investment is very minimal unlike the initial case where you were buying the stock at the current market price today, and hoping that it would increase in price 3 months later.

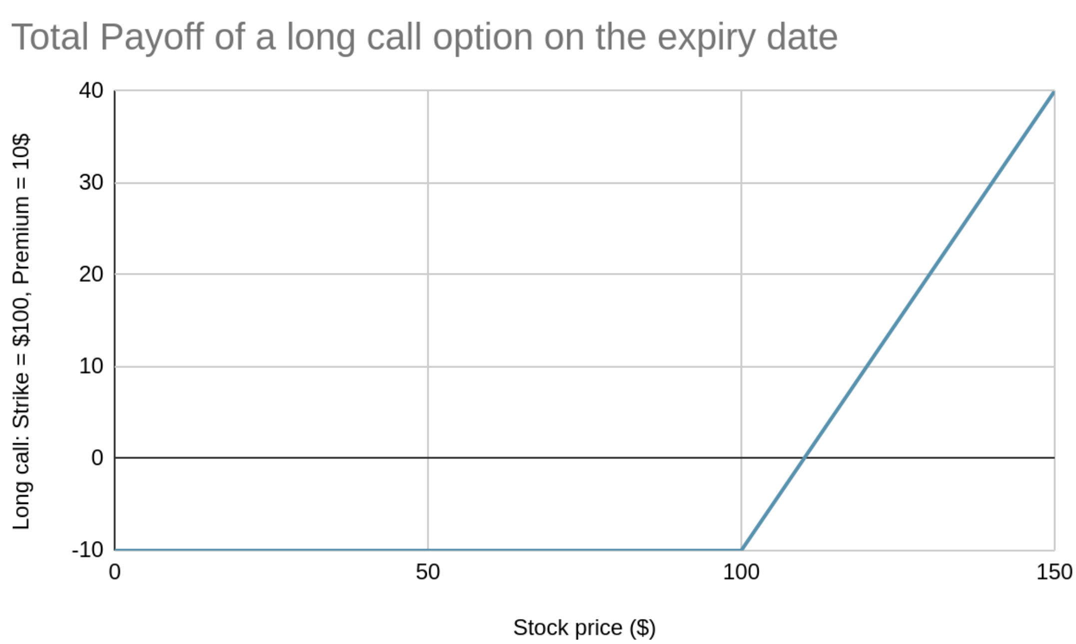
Note: 'Buying' of an asset is termed as a 'long position' and 'Selling' of an asset is termed as a 'short position'.

See the below diagrams to see how the payoff looks like on the date of expiry for the call option:

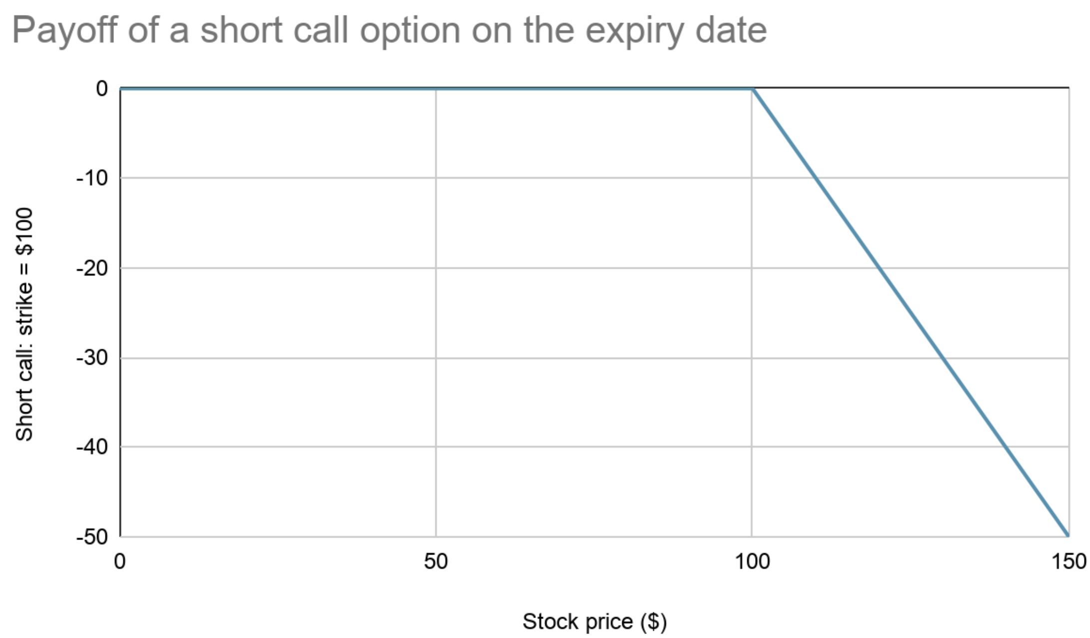
1. A long call with Strike = $100 has the payoff as C = max (stock price on expiry date - strike, 0).



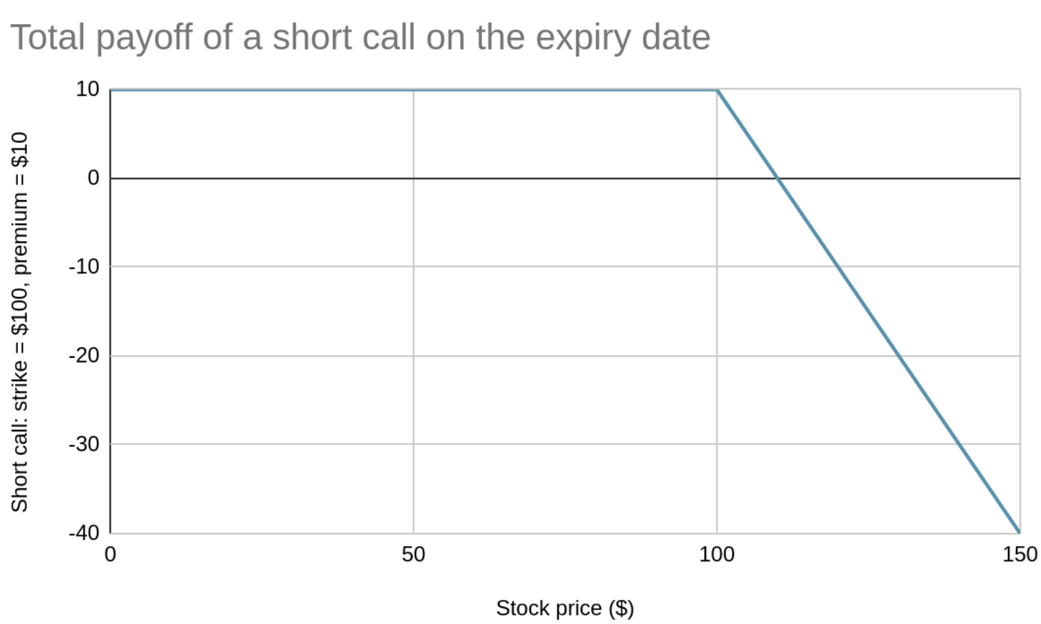
2. As you paid an initial premium (assume $10) to buy the call option, the above payoff graph actually gets shifted down by the premium amount.



3. As you guessed, short call (i.e. selling a call option) payoff with Strike = $100 has the payoff of long call reflected on the x-axis.



4. As this time you received the initial premium (assume $10) from the buyer of the option, the payoff graph will get shifted up by the premium amount.



Put option:

Just as the call option gave the buyer of the option the right to buy the stock at a predetermined price (strike) on the expiry date, a put option will give the buyer of the option the right to sell the stock at a predetermined price (strike) on the expiry date, irrespective of the prevailing market price on the expiry date. You can buy a put option by paying a small premium to the seller of the option.

Imagine you bought a put option after paying a small premium. As the holder of the put option, you now have the right to sell the stock at $100, 3 months from now, even if the market price is < $100. If the stock price ends up >= $100, you can let your option expire without any action, and the maximum amount you can lose is the initial premium that you paid to buy the option.

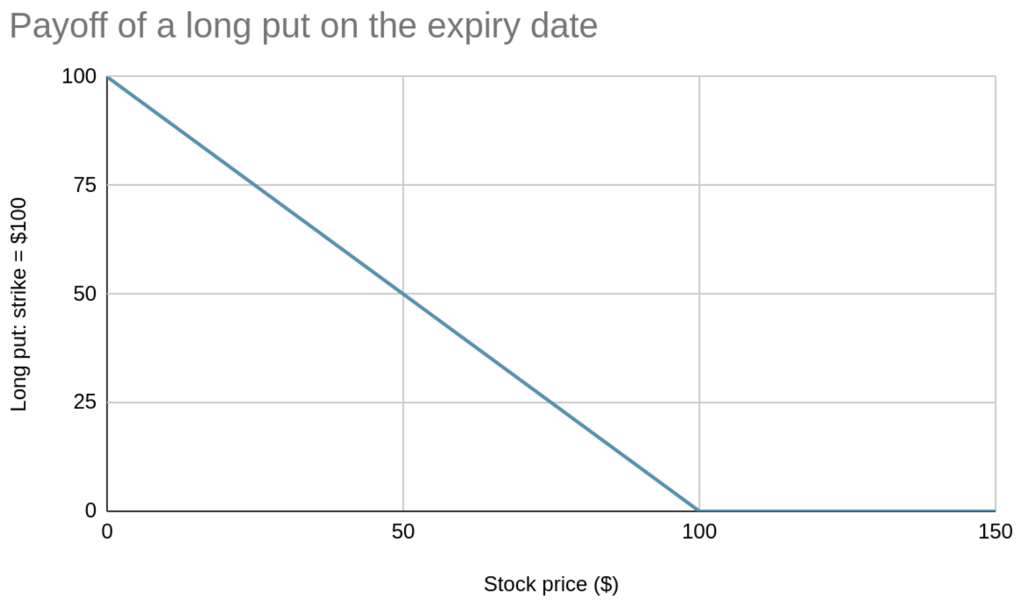
Q. What is the difference between buying a put option and selling a call option?

Ans.

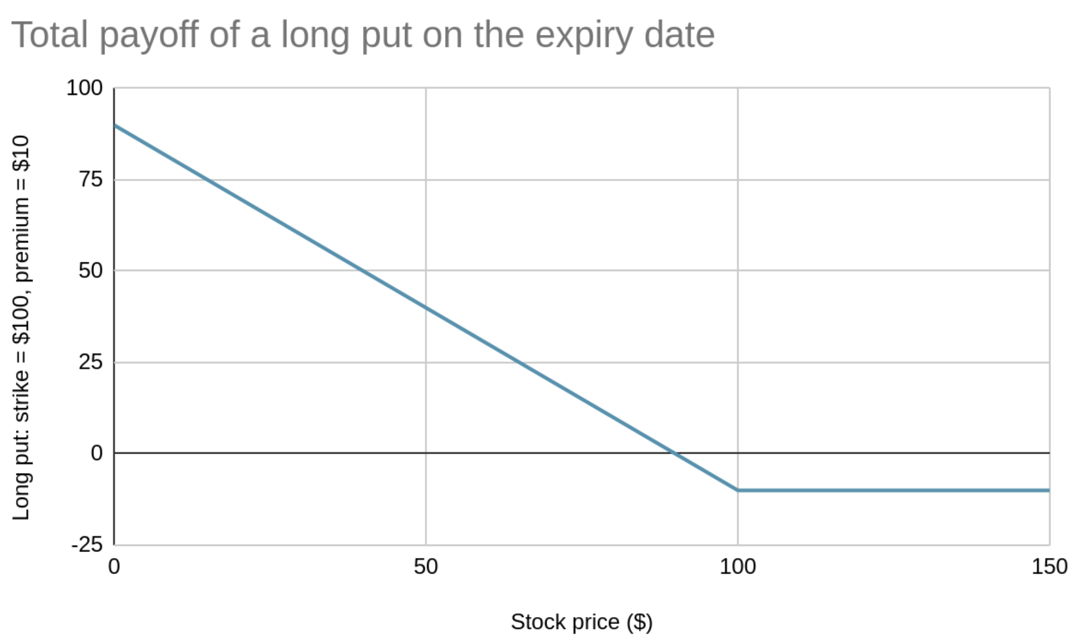
a. Different names, duh.
b. Buying a put option will need you to pay a premium, while selling a call option will earn you a premium.
c. A call option gets exercised when the stock price at expiry is greater than the strike price. As a seller of a call option, the most you can possibly get from the deal is the initial premium you were initially paid. A put option gets exercised when the stock price at expiry is smaller than the strike price. As a buyer of a put option, your upside gets maxed out at [strike – premium] (if the stock price goes down to $0 at expiry, you can buy the stock for free from the market on the expiration date and your put option will give you the ability to sell the stock at the pre-agreed strike price to the seller of that put option).

See the below diagrams to see how the payoff looks like on the date of expiry for the put option:

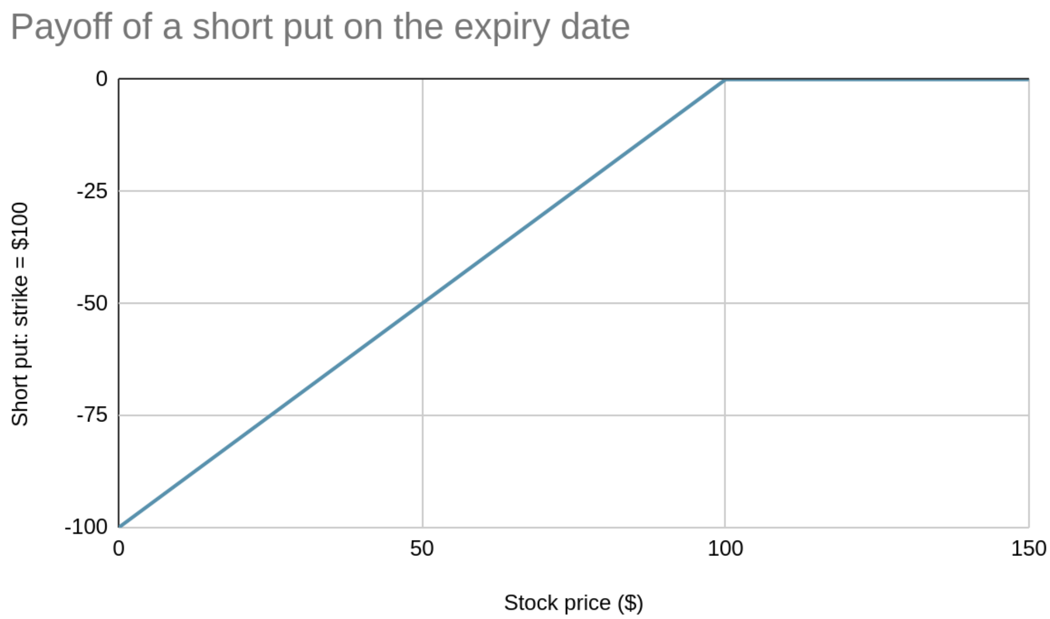
1. Buying a put option with Strike = $100 has the payoff as P = max (strike - stock price on the expiry date, 0).



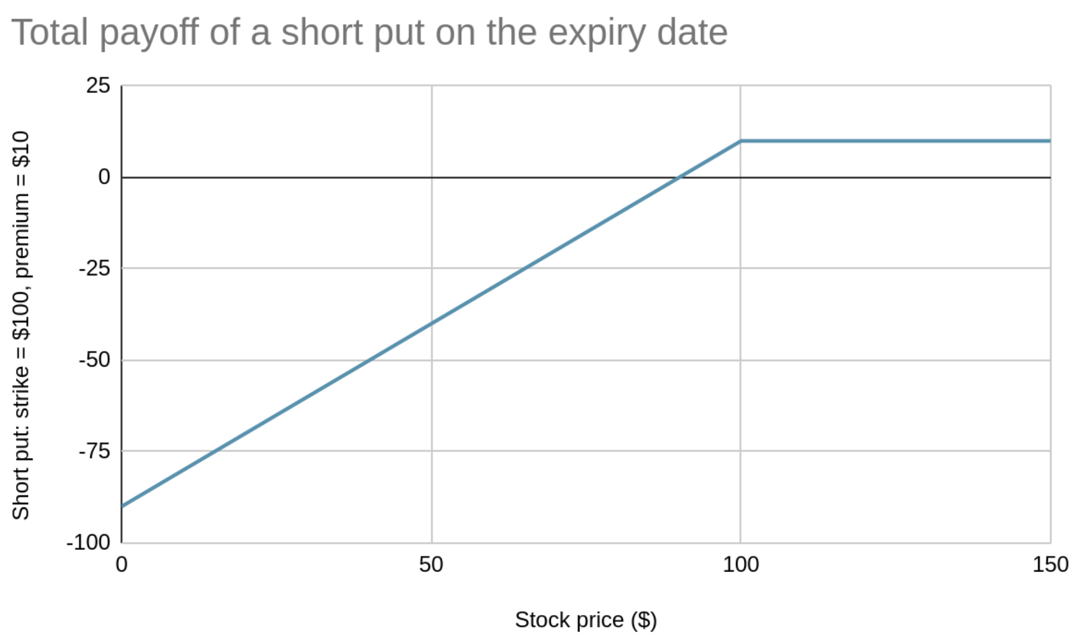
2. As you paid an initial premium (assume $10) to buy the put option, the payoff graph gets shifted down by the premium amount.



3. Selling a put payoff with Strike = $100 has the payoff of buying a put reflected on the x-axis.



4. As you received the initial premium (assume $10) from the buyer of the put option, the payoff graph gets shifted up by the premium amount.



Problem Statements:

Total Points: 100

There are 5 questions in this section. The weightage for each question has been mentioned individually, along with the format of the solution that is expected for each question. A common instruction for all questions is to **show your work**; your journey is almost as important as your destination!

The aim of this case study is to give you a glimpse into how the payoffs/premiums mentioned above are calculated and used. Don't fret, let's start at the very beginning of this saga, continuous compounding!

Q1. Continuous Compounding **[5 Points]**

a. What's the amount you get if you invest $10,000, on 15th Jan 2022, at an interest rate of 5% per annum, compounded semi-annually after 10 years?
b. What would be the final amount if the invested amount was compounded weekly?
c. If compounded daily?
d. As you might have noticed, the amount is increasing with each reduction in the compounding time period. Can you calculate the formulation if the compounding is done continuously instead of discrete time intervals like "daily" or "weekly"? First try and calculate in terms of the following parameters, and then plug in the values from part (a) to compare with discrete intervals! Parameters - $(P) principal, (r) annual interest rate and (T) years.

   Hint:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Q2. Call/Put Option Pricing **[10 Points]**

a. Money today is more valuable than money tomorrow. Why? Simply because you can invest the money today and earn a risk-free (i.e. FD-like) interest 'r', continuously compounded.

   So, if you were in the market looking for a call option at strike $K, expiry in T years from now, for a stock which is currently trading at $S, how much would you be willing to pay today?

   Remember, the strike $K is in the future, i.e. at time T, so it is less valuable right now than it will be at time T. It's helpful to think of part (d) in the first question and use that result here (If your final amount is $K and time to maturity/expiry is T years, what's the principal amount…). Assume r is the annual "risk-free" interest rate.

b. Repeat the analysis from part (a), this time for a put option with the same parameters – Strike $K, Time to expiry T years, Annual interest rate r.

Q3. BSM pricing **[20 Points]**

A slight bummer - the prices you just derived above are not actually what you'll see in the real world. You see, real-life markets react to events (news) in and around them, which contribute to the price of stocks going up and down. We call this stock price movement uncertainty "volatility" (often represented by σ, usually a small positive fraction).

Black-Scholes-Merton (BSM) is a model which factors in the volatility of the stock's price into its pricing of call/put options. The derivation of these formulae is out of scope for this case-study, so take the following equations at face value :)

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t)$$
$$= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$$

where,

- $C(S_t, t)$ - BSM price of a call option at time t and stock price S_t (in \$)
- $P(S_t, t)$ - BSM price of a put option at time t and stock price S_t (in \$)
- S_t - Stock price at time t (in \$)
- σ - Stock volatility (absolute, not percent)
- K - Option strike price (in \$)
- T - Time to maturity (in years)
- N - CDF of the standard normal distribution
- r - annual risk free rate of interest (absolute, not percent)

a. Stock volatility (σ) is meant to account for uncertainty is stock price movement between t and T. If we are very close to expiry, it stands to reason that σ won't be able to impact the call/put option prices by much. For $t -> T$, find the BSM price for both call and put options. Do these agree with the formulae you derived in the previous ques. without inculcating σ?

b. For 3 given points in time ($t = 0, 0.5\ and\ 1\ years\ respectively$), plot the BSM price of a call option C, with stock price S_t as the x-axis; consider the range of S_t to be from \$1 to \$100, K = \$50, r = 12 % p.a., σ = 0.30, T = 1 year.

c. Repeat the above analysis for put options, by plotting the BSM price of a put option P vs. S_t at the same time points and parameters as mentioned above.

To avoid any ambiguity – you are expected to create a total of 6 two-dimensional plots in parts (b) and (c), each of which will have a $ option price/premium on the y-axis and $ stock price on the x-axis.

Q4. Delta calculation **[15 Points]**

For the scope of this case study, the above analysis is sufficient in terms of how an option is priced. Let's now move onto some interesting applications!

Delta is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price. Suppose that the delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price increases by about 60% of that amount.

a. With the BSM pricing formulae given in the previous question, analytically calculate the delta of call and put options in terms of t, S_t, T, σ and K. Calculate these deltas in the six scenarios mentioned in the previous question for $S_t = \$125$.
b. Now try to numerically calculate these deltas as the slope of the plots you made in the previous question at $S_t = \$125$. How close did you get?

Q5. Delta Hedging **[50 Points]**

As explained in the previous question, the delta (Δ) of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. It is the number of units of the stock we should hold for each option shorted in order to create what is called a "riskless portfolio". In the example where the delta of a call option is 0.6, say you are long in n units of the call option. If you now short 0.6*n units of the underlying stock directly, your total exposure to the market (called a portfolio) will have [n units of a call option, -0.6*n units of a stock]. This portfolio's overall value doesn't really depend on where the stock price goes! We can now sit-back and not worry about what happens to the stock.

The construction of a riskless portfolio (by being long/short in the underlying stock by a certain amount – long if delta is –ve, short if delta is +ve) is referred to as delta hedging. Note that the delta of a call option is positive (option price increases with increase in stock price), whereas the delta of a put option is negative (option price decreases with increase in stock price).

a. In the previous question, you learnt how to calculate delta for call/put options. From the theory above you can now build a riskless portfolio if you are given a bunch of calls and puts pretty easily. Since creating riskless portfolios for calls and puts is so easy (calculating deltas from the formulae you derived in the last question and buying/selling the corresponding number of underlying stocks), its usual practice to break down complex derivative payoffs into a linear combination of calls and puts, and hedge the weighted sum of their deltas.

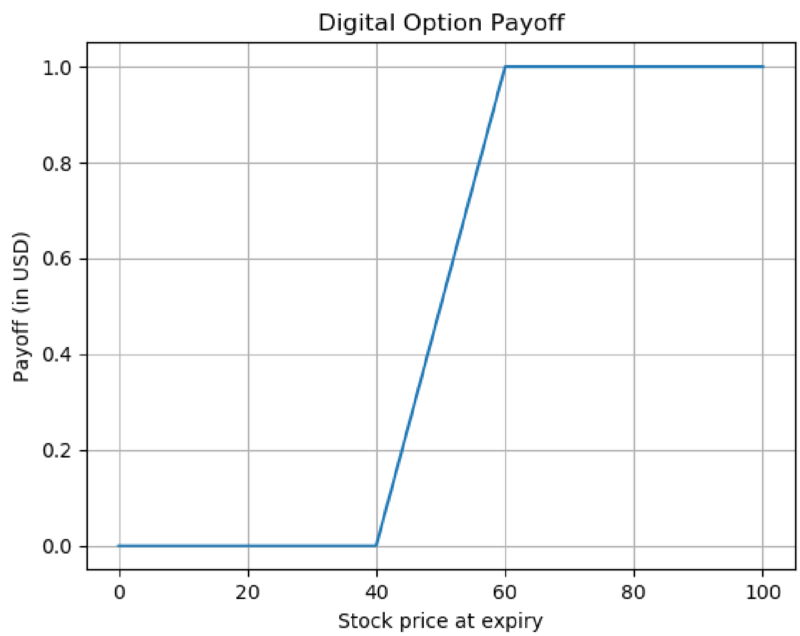
   One such derivative with the below payoff is called a digital option. Show that this payoff can be broken down into an equivalent linear combination of call and put options. Once you do, show how delta hedging can be done as a linear combination of the delta 'contributions' from said call and put options. What will be the final riskless portfolio in this case if you start with n units of the below digital option. **[15 Points]**

Note: the below payoff has been plotted against stock price at expiry, i.e. S_T, rather than S_t. Take r = 12 % p.a., σ = 0.3 and T = 1 year, and create riskless portfolios for 3 scenarios at t = 0:

(i) $S_t = \$30$,

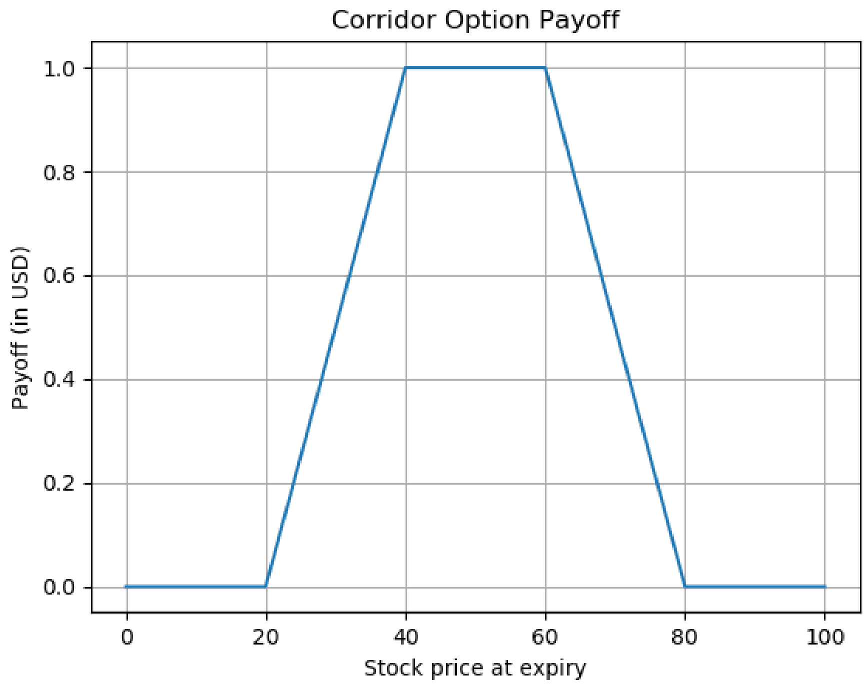
(ii) $S_t = \$50$, and

(iii) $S_t = \$70$.



b. The strategy used in part (a) can theoretically be applied to any derivative with a piecewise linear payoff vs S_T. In that spirit, consider the following derivative given in the below figure.

[15 Points]



Take r = 12 % p.a., σ = 0.3 and T = 1 year, and create riskless portfolios for 5 scenarios at t = 0 (assuming that we start with n units of this derivative):

(i) $S_t = \$10$,

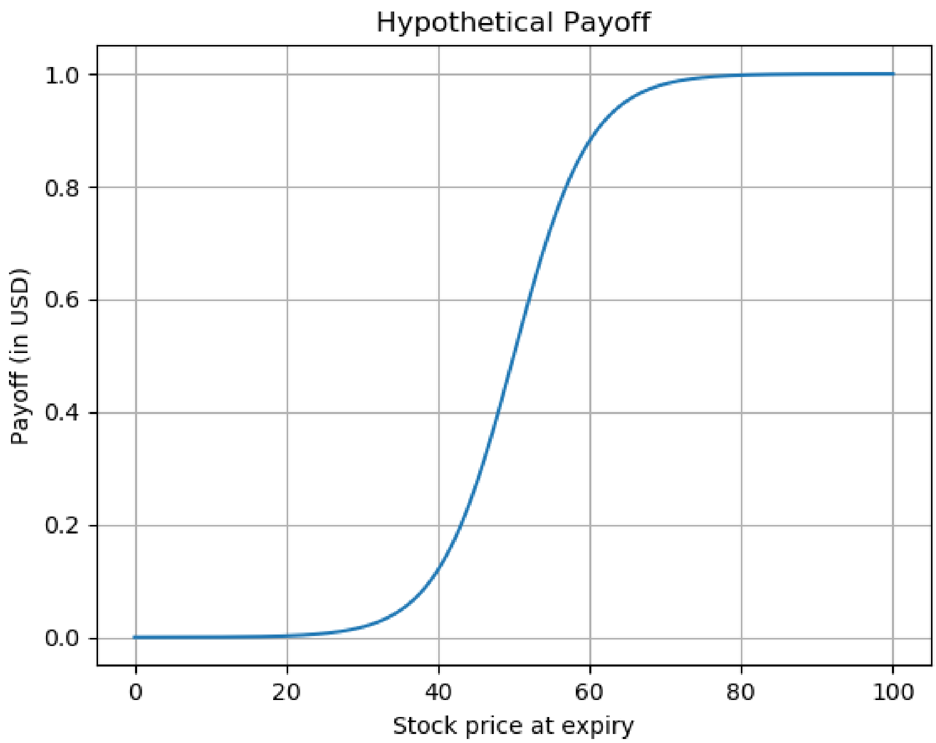
(ii) $S_t = \$30$,

(iii) $S_t = \$50$,

(iv) $S_t = \$70$, and

(v) $S_t = \$90$.

c. An interesting extension of this concept is to consider a case where payoff vs S_T is not piecewise linear, like the payoff given below.



Describe a general purpose strategy/technique/hypothesis to create riskless portfolios for such a payoff for any given S_0.

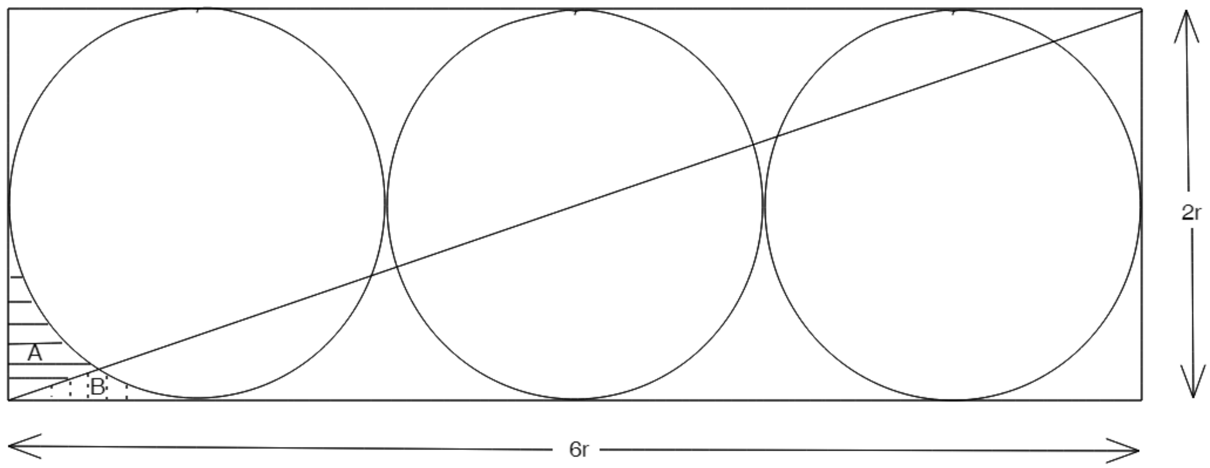
To test your strategy/technique/hypothesis, create riskless portfolios for the 5 scenarios and parameters given in part (b). **[20 Points]**

Case Study – B

Question-1 Circles?

Consider the below figure. There are three circles of equal radius ‘r’ arranged side by side inside a rectangle of length ‘6r’ and breadth ‘2r’. A diagonal is drawn in the rectangle. Area ‘A’ is the bounded region to the left of the first circle, above the intersecting diagonal shaded with solid horizontal lines.

Area ‘B’ is the area between the first circle and the diagonal shaded with dotted lines.



Let AreaRatio be defined as:

AreaRatio = area of ‘A’ / (area of ‘A’ + area of ‘B’)

Q1.1. What is the AreaRatio in the set up described above? **[5 Points]**

Expected Solution format:

a. A single ratio or percentage.
b. Answer can be arrived at programmatically or manually.
c. If solution is determined manually, please include a detailed explanation of the solution.
d. Alternately, attach the code if the solution was arrived at programmatically.

If there are ‘n’ circles arranged inside a rectangle of dimensions ‘2nr’ by ‘2r’ in a similar fashion,

Q1.2. What is the range of AreaRatio? **[5 Points]**

Expected Solution format:

a. A range clearly indicating inclusion or exclusion of endpoints. E.g. (10,100]
b. Explanation: Mathematical or with code.

Q1.3. What is the minimum value of 'n' for which the AreaRatio is greater than or equal to:

a. 50% **[2.5 Points]**
b. 70% **[2.5 Points]**
c. 90% **[5 Points]**
d. 99.9% **[5 Points]**
e. 99.99% **[5 Points]**
f. 100% **[5 Points]**

Expected Solution format:

a. A single value of 'n' for each case.
b. Answer can be arrived at manually or programmatically. Note that non-programmatic answer for this problem will be awarded a lower score.

Question-2 Keep Calm and Carry on …

Humans in general tend to switch between languages when communicating. For example the phrase: "Cojelo con take it easy" comprises of Spanish tokens "Cojelo con" and English tokens "take it easy". Let us define a model around language switching: at every token position 't', the variable 'y_i' denotes the language of the speaker. 'y_i' can be one of two states S or E. The word is then produced using a unigram language model for that language. Assume that we know the language model parameters and the final sentence, we need you to derive an algorithm to guess the most likely language transition states.

For Example: the possible language transition strings for "Cojelo con take it easy" can be "S S E E E" or "S E E E E" or "S E S E E" etc.

Language model parameters:

vocabulary = ("cojelo", "con", "take", "it", "easy")

(Hint: The vocabulary is exhaustive and no other word exists in the universe in question)

languages = ("E", "S")

Sentence start probability :

Ps['E'] = 0.6

Ps['S'] = 0.4

Transition probability:

Pt['S'|'E'] = 0.7

Pt['E'|'E'] = 0.3

Pt['S'|'S'] = 0.4

Pt['E'|'S'] = 0.6

Emission probability :

Pe[‘cojelo’|’E’] = 0.1

Pe[‘con’|’E’] = 0.2

Pe[‘take’|’E’] = 0.3

Pe[‘it’|’E’] = 0.2

Pe[‘easy’|’E’] = 0.2

Pe[‘cojelo’|’S’] = 0.3

Pe[‘con’|’S’] = 0.3

Pe[‘take’|’S’] = 0.15

Pe[‘it’|’S’] = 0.15

Pe[‘easy’|’S’] = 0.1

Q2.1. Given the language model parameters above, can you guess the most likely language transition string and corresponding probability for:

a. Cojelo Con Take It Easy **[5 points]**
b. Con Take It Easy **[5 points]**
c. Easy Con **[5 points]**
d. Cojelo Easy Take It **[5 points]**

Expected Solution format:

a. Expected answer is in the form of a string followed by a float (correct to 5 decimal points) indicative of probability of that language String

Example solution to (A): “E E E E E”: 0.23400 (Note that this example is not the correct answer to (A))

b. This problem can be solved programmatically or manually. Both carry equal score.
c. Please explain your solution.
d.

Q2.2. Write a program that takes a string (eg: “Cojelo Con”) as input and prints the most probable language string and the probability of the language string (correct to 5 decimal points) separated by (“ : “). **[45 points]**

Expected Solution format:

a. Program that computes the above.

Or

b. Pseudo code that can compute the output as described.

Note:

i. Pseudo code has a lower max score than actual code for this problem.
ii. There are multiple correct approaches to the problem. However, the most optimal solution will carry the highest score.

Sample Output: “EE” : 0.12340

Final Submission Format:

Create a well formatted word/pdf document describing your solution to each question, clearly enumerated and in the same order as the questions. Include your personal details such as name, institute name, branch and year in the first page.

For each question, if the expected solution is a written answer/ plot / table or pseudo code, include it in line in this document. Name this document in the format

{FirstName}_{LastName}_{CollegeName(short)}_JPMQuantMentorshipCaseStudy

If the expected solution is a program, state the file name of the program.

Instructions for program submissions:

Any programming language you are comfortable with is permitted. You are permitted to use standard libraries in your chosen language.

a. The program submitted must compulsorily have a main function that must call all the test cases described in the question when run.

b. The program must be commented and intelligible.

c. The program's file name must be of the format:

{CANDIDATE_NAME}_{QUESTION_NO_SUB_PART}_MAIN.

d. If there are additional modules that are imported in the script with the main function, name those files as {CANDIDATE_NAME}_Module_{ SCRIPT_NAME}.

e. You must include a short description of your program in the solution document.

f. Finally, zip the solution document and code (if any) together and name it

{FirstName}_{LastName}_{CollegeName(short)}_JPMQuantMentorshipCaseStudy

Mail this compressed file to the jpmqrmentorship.mumbai@jpmorgan.com.