Computational Physics
Homework 4

Shalma Wegsman

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Problem 1: Simpson's Rule Error

We want to derive the the approximation error for the Simpson's Rule, given by:

$$\epsilon = \frac{1}{90}h^4[f'''(a) - f'''(b)] \tag{1}$$

Let's start by Taylor expanding around x_{i-1} and x_{i+1} and taking the average, like we did for the Trapezoid error in class:

$$A_{SR} \approx \int_{x_{i-1}}^{x_{i+1}} dx \sum_{n=0}^{\infty} \frac{1}{2}\left[\frac{f^{(n)}(x_{i-1})(x-x_{i-1})^n}{n!} + \frac{f^{(n)}(x_{i+1})(x-x_{i+1})^n}{n!}\right]$$

$$\approx \frac{1}{2}(2h)[f(x_{i-1}) + f(x_{i+1})] + \frac{1}{4}(2h)^2[f'(x_{i-1}) + f'(x_{i+1})] + \frac{1}{12}(2h)^3[f''(x_{i-1}) + f''(x_{i+1})]$$

$$+\frac{1}{48}(2h)^4[f'''(x_{i-1}) + f'''(x_{i+1})] + \mathcal{O}((2h)^4)$$

$$\approx h[f(x_{i-1}) + f(x_{i+1})] + (h)^2[f'(x_{i-1}) + f'(x_{i+1})] + \frac{2}{3}h^3[f''(x_{i-1}) + f''(x_{i+1})]$$

$$+\frac{1}{3}h^4[f'''(x_{i-1}) + f'''(x_{i+1})]$$

(Honestly, the instructions/followup email about this question were pretty confusing, so this is as far as I got).

Problem 3: Vector Potential

We want to use the Gauss-Chebyshev quadrature with $W(x) = 1/\sqrt{1-x^2}$ to compute K(k) and E(k). To do that, we need to perform a change of variables:

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

Let $x = \sin\phi$, so the $\frac{dx}{d\phi} = \cos\phi \Rightarrow d\phi = dx/\cos\phi$. Since $\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-x^2}$, we can re-write $K(k)$ as:

$$K(x) = \int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}} = \int_0^1 \frac{dx}{\sqrt{1-(k^2+1)x^2+k^2x^4}}$$

Since we need the integration limits to be $(-1, 1)$, we can utilize the symmetry of the expression to write:

$$K(x) = \frac{1}{2}\int_{-1}^1 \frac{dx}{\sqrt{1-(k^2+1)x^2+k^2x^4}} \tag{2}$$

We do something similar for $E(k)$:

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\,d\phi \tag{3}$$

Setting $x = \sin\phi$:

$$E(x) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx$$