Binomial Model

Under this framework, if $S(0) = A(0)$, then

$$S^d < A(1) < S^u$$

, otherwise an arbitrage opportunity would arise.
In addition, under the One-step Binomial model, we also have that

$$\frac{S^d}{S(0)} < \frac{A(1)}{A(0)} < \frac{S^u}{S(0)}$$

In general,

$$S(n) = S(0)(1+u)^i(1+d)^{n-1}$$

for $i = 0, 1, ..., n$, i.e. $S(n)$ can take $n+1$ different values - the obtained tree is called the recombining tree. The probability of $S(n)$ such that $S(n) = S(0)(1+u)^i(1+d)^{n-1}$ is

$$P(S(n)) = C(n,i)p^i(1-p)^{n-1}$$

In other words, $S \sim B(n,p)$.

Interest Rates

Simple Interest: Principal is fixed - no interest earned will be added to it.

$$V(t) = P(1+rt)$$

where P is the principal. If we start at some point $s \geq 0$, then $V(t) = P(1 + r(t-s))$

1. Here, $1 + r(t-s)$ is known as the growth factor
2. Reversely, $(1 + r(t-s))^{-1}$ is the discount factor

Periodic Compounding Interest: We reinvest the interest earned.

$$V(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

Here, m is the compounding frequency, and we assume $m = 365$ if interest rate compounded daily.

1. $(1 + \frac{r}{m})^{(t-s)m}$ is the growth factor
2. $(1 + \frac{r}{m})^{-(t-s)m}$ is the discount factor

The sum of the payment is $C \cdot A(r,n)$, where

$$A(r,n) = \sum_{i=1}^{n} = (1+r)^{-i} = \frac{1-(1+r)^{-n}}{r}$$

Hence, PV of the annuity is

$$C\frac{1-(1+r)^{-n}}{r}$$

An annuity is a perpetuity if C is continued to payout forever, i.e. $n \to \infty$:

$$PV = \frac{C}{r}$$

Continuously Compounding Interest: If we calculate interest more frequently such that we let $m \to \infty$, we have that

$$V_m(t) = P\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt}$$

Since we know that $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$, thus

$$V(t) := \lim_{m\to\infty} V_m(t) = Pe^{rt}$$

This is known as continuous compounding, and gives higher future value than periodic compounding.

1. The growth factor is $e^{r(t-s)}$
2. The discount factor is $e^{-r(t-s)}$

The return is

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = e^{r(t-s)} - 1$$

The logarithmic return is

$$k(s,t) = \ln\frac{V(t)}{S(t)} = r(t-s)$$

This is additive, s.t. $k(s,t) + k(t,u) = k(s,u)$.
Effective Rate: For a given compounding method with interest rate r, the effective rate r_e is one that gives the same growth factor over a one year period under annual compounding.

1. For periodic compounding: $(1 + \frac{r}{m})^m$
2. For continous compounding: e^r

Bonds

Zero-coupon Bonds: We define $F \cdot B(t,T)$ to be the value at time t of a zero coupon bond with face value F and maturing at T. Note that $B(T,T) = 1$ and $B(0,T)$ is the current value. By nature we have $V(0) = P = F \cdot B(0,T)$. Hence,

$$V(t) = F \cdot B(t,T) = F\frac{B(t,T)}{B(0,T)}$$

The Present Value (PV) is $V(0) = \frac{F}{1+r}$, if r is annual compounding rate. In reality, we use the market price of bonds to imply the annual compounding. The implied periodic compounding rate with frequency m is determined by

$$B(t,T) \cdot \left(1 + \frac{r_m}{m}\right)^{m(T-t)} = 1 \implies r_m = m(B(t,T)^{\frac{-1}{m(T-t)}} - 1)$$

The implied continuous compounding rate is

$$B(t,T) \cdot e^{r(T-t)} = 1 \implies r = -\frac{\ln B(t,T)}{T-t}$$

Coupon Bonds:

$$V(0) = PV = \sum_{i=1}^{T} Ce^{-ri} + Fe^{-rT}$$

We calculate the value of a coupon bond at time t as the sum of the PV of all future payments to be received in $(t,T]$, IMMEDIATELY after the coupon payment C at time t.
Money Market Account: Then the value of the MMA at time t is

$$A(t) = A(0)\frac{B(t,T)}{B(t,0)} = A(0)e^{rt}, t \leq T$$

Stocks

Returns (NO DIVIDENDS): For two integers $n < m$, we substitute $K(n,m)$ for $K(n\tau, m\tau; \omega)$ as the return over the time interval $[n\tau, m\tau]$ when the scenario ω is realized:

$$K(n,m) := K(n\tau, m\tau, \omega) = \frac{S(m\tau,\omega) - S(n\tau,\omega)}{S(n\tau,\omega)} = \frac{S(m) - S(n)}{S(n)}$$

The return is a random variable as well; rearranging this equality gives

$$S(m) = S(n)(1 + K(n,m))$$

Here we see that the growth factor is $1 + K(n,m)$. Analogously, we can write the log return over the same time interval:

$$k(n,m) := \ln\frac{S(m)}{S(n)}$$

Rearranging this gives

$$S(m) = S(n)e^{k(n,m)}$$

Returns (DIVIDENDS): Since the dividend is decided prior to paying date, the drop of stock price is already reflected in $S(n)$. Thus, $S(n) + \text{div}(n)$ reflects the stock price at time n if dividend were NOT paid. Hence, the one-step return and log return are

$$K(n) := K(n-1,n) = \frac{S(n) + \text{div}(n) - S(n-1)}{S(n-1)}$$

And

$$k(n) := k(n-1,n) = \ln\frac{S(n) + \text{div}(n)}{S(n-1)}$$

If no dividends are paid, then we have the following for one-step returns and log returns:

$$1 + K(n,m) = (1 + K(n+1))(1 + K(n+2))\cdots(1 + K(m))$$

$$k(n,m) = k(n+1) + k(n+2) + \cdots + k(m)$$

$$E(k(n,m)) = E(k(n+1)) + E(k(n+2)) + \cdots + E(k(m))$$

Moreover, if one-step returns $K(n+1), ..., K(m)$ are independent, then

$$1+E(K(n,m)) = (1+E(K(n+1)))(1+E(K(n+2)))\cdots(1+E(K(m)))$$

Portfolio Theory

Correlation of returns between 2 assets is covariance divided by the product of their standard deviations.

$$\rho_{AB} = \mathrm{Corr}(A, B) = \frac{\mathrm{Cov}(A, B)}{\sigma_A \sigma_B}$$

If short sales are not allowed, then

$$\sigma_V \le \max(\sigma_1, \sigma_2)$$

Equivalent to minimizing σ_V. Note that $\rho_{12} \in [1, 1]$. First consider the risk reduction in two extreme cases.

1. If $\rho_{12} = 1$, then $\sigma_V = |w_1\sigma_1 + w_2\sigma_2|$. In particular, when $\sigma_1 \neq \sigma_2$:

$$\sigma_{V=0} \iff w_1 = \frac{-\sigma_2}{\sigma_1 - \sigma_2} \text{ and } w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}$$

Here either w_1 or w_2 is negative, implying that short sales must happen.

2. If $\rho_{12} = -1$, then $\sigma_V = |w_1\sigma_1 - w_2\sigma_2|$. In particular

$$\sigma_V = 0 \iff w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \text{ and } w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

For $-1 \le \rho_{12} \le 1$, the minimal risk is attained when

$$w_1^{\min} := \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

If short sales are not allowed, the smallest risk (may not be 0) is attained at

- $(w_1, w_2) = (0, 1)$ if $w_1^{\min} < 0$
- $(w_1, w_2) = (w_1^{\min}, w_2^{\min} < 0)$ if $0 \le w_1^{\min} \le 1$
- $(w_1, w_2) = (1, 0)$ if $w_1^{\min} > 0$

$$\mu_V = E(K_V) = mw^T$$

$$\mathrm{Var}(K_V) = wCw^T$$

MVP: The weights of the minimum variance portfolio are given by

$$w_{MVP} = \frac{uC^{-1}}{uC^{-1}u^T}$$

We want to solve the problem

$$\min_w wCw^T \text{ subject to } uw^T = 1$$

Weights of portfolios on the MVL ix a level of expected return μ_V. The portfolio with the smallest risk among all portfolios with expected return μ_V has weights

$$w_{\mu_V}^{\min} = \frac{\begin{vmatrix} 1 & c \\ \mu_V & d \end{vmatrix} uC^{-1} + \begin{vmatrix} a & 1 \\ b & \mu_V \end{vmatrix} mC^{-1}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} \quad (1)$$

$$= \frac{(d - c\mu_V)uC^{-1} + (a\mu_V - b)mC^{-1}}{ad - bc} \quad (2)$$

where $a = uC^{-1}u^T, b = mC^{-1}u^T, c = uC^{-1}m^T, d = mC^{-1}m^T$ are all real numbers.Note that a, b, c, d are all constants INDEPENDENT of weight w. We want to solve the problem

$$\min_w wCw^T \text{ subject to } uw^T = 1 \text{ and } mw^T = \mu_V$$

Note that $w_{\mu_V}^{\min}$ is linear in μ_V:

$$w_{\mu_V}^{\min} = A\mu_V + B$$

A portfolio V lies on the MVL if and only if its weight vector w can be written as an affine combination of w_1 and w_2, i.e.

$$w = \alpha w_1 + (1 - \alpha)w_2$$

for some $\alpha \in R$.
The efficient frontier is the set of points whose weights w_{EF} are given by

$$w_{EF} = A\mu + B$$

for $\mu > \mu_{MVP}$ and $A = \frac{(am-cu)C^{-1}}{ad-bc}$ and $B = \frac{(du-bm)C^{-1}}{ad-bc}$ **Market Portfolio**: Under the assumption that $R < \mu_{MVP}$ and if C is invertible, then the market portfolio M exists and its weights are given by

$$w = \frac{(m - Ru)C^{-1}}{(m - Ru)C^{-1}u^T}$$

We again have a maximization problem:

$$\max_w \frac{mw^T - R}{\sqrt{wCw^T}} \text{ subject to } uw^T = 1$$

CAPM

Capital Market Line: The half-line that starts at the risk-free asset and runs through the market portfolio M will be said the Capital Market Line (CML) and it satisfies the equation:

$$\mu = R + \frac{\mu_M - R}{\sigma_M}\sigma$$

where $\frac{\mu_M - R}{\sigma_M}$ is the risk premium. **Beta**: The beta factor of a portfolio/security V is defined by

$$\beta_V = \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2}, \alpha = \mu_V - \beta_V\mu_M$$

Covariance of Portfolios: Let two portfolios V_1,V_2 have associated weights $w_1 := [w_{1,1}, \ldots, w_{1,n}]$ and $w_2 := [w_{2,1}, \ldots, w_{2,n}]$. Then,

$$\mathrm{Cov}(K_{V1}, K_{V2}) = w_1Cw_2^T$$

for $K := [K_1, \ldots, K_n]$ is a row vector of random returns of risky assets.
CAPM/Security Market Line: The expected return μ_V for a feasible portfolio V is given by the CAPM

$$\mu_V = R + \beta_V(\mu_M - R)$$

VaR: The probability that our loss in 1 year, i.e. $(S(0)e^r - S(1))$ is less than some Value at Risk (VaR) amount can be expressed as

$$P(S(0)e^r - S(1) < \mathrm{VaR}) = \alpha$$

Forwards

Forward Price: The no-arbitrage forward price is

$$F(0, T) = S(0)\frac{A(T)}{A(0)} = S(0)e^{rT}$$

where r is the continuous compounding interest rate. If contract starts at time $t \in (0, T)$, then

$$F(t, T) = S(t)\frac{A(T)}{A(t)} = S(t)e^{r(T-t)}$$

Forward Price with DIVIDENDS: The forward price of a stock paying dividend Div at time $t \in (0, T)$ is

$$F(0, T) = [S(0) - e^{-rt}\mathrm{Div}]e^{rT} = \frac{S(0) - B(0, T)\mathrm{Div}}{B(0, T)}$$

More generally, the formula can easily be generalized to the case when dividends are paid more than once within $(0, T)$:

$$F(0, T) = [S(0) - \mathrm{Div}_0]e^{rT} = S(0)e^{rT} - \mathrm{Div}_T$$

where Div_0 (resp. Div_T) stands for present (resp. future) values of all dividends:

$$\mathrm{Div}_0 = \sum_{i=1}^n d_i e^{-rt_i}$$

$$\mathrm{Div}_T = \sum_{i=1}^n d_i e^{r(T-t_i)}$$

Forward Price with REINVESTED DIVIDENDS:

$$F(t, T) = S(t)e^{(r - r_{div})(T-t)} = S(t)e^{-r_{div}(T-t)}e^{r(T-t)}$$

Forward Contracts Value: For any $t \in [0, T]$, the time t value $V(t)$ for a LONG forward contract with forward price $F(0, T)$ is given by

$$V(t) = [F(t, T) - F(0, T)]e^{-r(T-t)}$$

where, as we mentioned before, $F(t, T)$ is the forward price of another forward contract made at time t. The value for a SHORT position is given by

$$V(t) = [F(0, T) - F(t, T)]e^{-r(T-t)}$$

If a stock pays NO dividends, then we have

$$V(t) = [F(t, T) - F(0, T)]e^{-r(T-t)} = [S(t)e^{r(T-t)} - S(0)e^{rT}]e^{-r(T-t)} \quad (3)$$

$$= S(t) - S(0)e^{rt} \quad (4)$$

Future Contracts Value: If interest rate is constant, then $f(0, T) = F(0, T)$. If the underlying stock pays **NO dividends** and the continuous compounding interest rate r is constant, then

$$f(t, T) = S(t)e^{r(T-t)} = F(t, T)$$

$$f(T, T) = F(T, T) = S(T)$$