Options Intro

r is the the continuously compounding interest rate.
Put-Call Parity for European Options (**NO DIVIDENDS**):

$$C_E - P_E = S(0) - Xe^{-rT}$$

We can adapt the put-call parity argument to the case where the underlying asset pays discrete dividends with present value Div_0:

$$C_E - P_E = S(0) - \text{Div}_0 - Xe^{-rT}$$

Or continuously with dividend yield r_{div}:

$$C_E - P_E = S(0)e^{-r_{\text{Div}}T} - Xe^{-rT}$$

To prove using A.O., assume $S_0 + P_E - C_E \neq Xe^{-rT}$.

European Options Bounds

Bounds with **NO DIVIDENDS**:

$$\text{Call: } \max(S(0) - Xe^{-rT}, 0) \leq C_E < S(0) \quad (1)$$
$$\text{Put: } \max(Xe^{-rT} - S(0), 0) \leq P_E < Xe^{-rT} \quad (2)$$

WITH DIVIDENDS:

$$\text{Call: } \max(S(0) - \text{Div}_0 - Xe^{-rT}, 0) \leq C_E < S(0) - \text{Div}_0 \quad (3)$$
$$\text{Put: } \max(Xe^{-rT} - (S(0) - \text{Div}_0), 0) \leq P_E < Xe^{-rT} \quad (4)$$

The lower bounds of the inequality can be demonstrated using **Put-Call Parity** as $C_E, P_E \geq 0$. The upper bounds of the inequality can be proven by showing $C_E < S(0) \iff P_E < Xe^{-rT}$ by first assuming $C_E \geq S(0)$.

Replicating Lemma: If portfolio V_1 is worth the **SAME** as portfolio V_2 at time T, i.e., $V_1(T) = V_2(T)$, then $V_1(0) = V_2(0)$, otherwise an arbitrage opportunity would arise.

Comparisoin Lemma: If portfolio V_1 is worth **NO MORE** than portfolio V_2 at time T, i.e., $V_1(T) \leq V_2(T)$, then $V_1(0) \leq V_2(0)$, otherwise an arbitrage opportunity would arise.

European Options Price Dependence on X

If $X_1 < X_2$, then

$$C_E(X_1) \geq C_E(X_2) \text{ i.e. } X \to C_E(X) \text{ is decreasing} \quad (5)$$
$$P_E(X_1) \leq P_E(X_2) \text{ i.e. } X \to P_E(X) \text{ is increasing} \quad (6)$$

And

$$C_E(X_1) - C_E(X_2) \leq e^{-rT}(X_2 - X_1) \quad (7)$$
$$P_E(X_2) - P_E(X_1) \leq e^{-rT}(X_2 - X_1) \quad (8)$$

Here, the functions $X \to C_E(X)$ and $X \to P_E(X)$ are **Lipschitz with associated constant** $e^{-rT} < 1$.

Convexity: The functions $X \to C_E(X)$ and $X \to P_E(X)$ are **convex**, i.e. for any $X_1, X_2 > 0$ and any $\alpha \in [0, 1]$,

$$C_E(\alpha X_1 + (1-\alpha)X_2) \leq \alpha C_E(X_1) + (1-\alpha)C_E(X_2)$$

$$P_E(\alpha X_1 + (1-\alpha)X_2) \leq \alpha P_E(X_1) + (1-\alpha)P_E(X_2)$$

Let $S \in R_+$ be fixed. The functions $R_+ \ni X \to (S - X)^+ \in R_+$ and $R_+ \ni X \to (X - S)^+ \in R_+$ are **convex**.

European Options Price Dependence on S

If $S_1 < S_2$, then

$$C_E(S_1) \leq C_E(S_2) \text{ i.e. } S \to C_E(S) \text{ is increasing} \quad (9)$$
$$P_E(S_1) \geq P_E(S_2) \text{ i.e. } S \to P_E(S) \text{ is increasing} \quad (10)$$
$$C_E(S_2) - C_E(S_1) \leq S_2 - S_1 \quad (11)$$
$$P_E(S_1) - P_E(S_2) \leq S_2 - S_1 \quad (12)$$

The functions $R_+ \ni S \to C_E(S)$ and $S \to P_E(S)$ are **Lipschitz with associated constant** 1.

Convexity: The functions $S \to C_E(S)$ and $S \to P_E(S)$ are convex, i.e. for any $X_1, X_2 > 0$ and any $\alpha \in [0, 1]$,

$$C_E(\alpha S_1 + (1-\alpha)S_2) \leq \alpha C_E(S_1) + (1-\alpha)C_E(S_2)$$

$$P_E(\alpha S_1 + (1-\alpha)S_2) \leq \alpha P_E(S_1) + (1-\alpha)P_E(S_2)$$

American Options Bounds and Characteristics

Relationship with European Options:

$$C_A \geq C_E(\geq 0), P_A \geq P_E(\geq 0)$$

To prove using A.O., assume $C_A < C_E$.

American Options **WITHOUT dividends**:

$$C_E = C_A$$

To prove this, we only need to consider $C_A > C_E$ leads to an A.O. since $C_A \geq C_E$. There is **NO** corresponding result like $P_E = P_A$.

We can show this by assuming (1) $C_A - P_A < S(0) - X$, (2) $C_A - P_A > S(0) - Xe^{-rT}$. **Put-Call Parity Estimates for American Options**:

$$S(0) - X \leq C_A - P_A \leq S(0) - Xe^{-rT}$$

Put-Call Parity WITH DIVIDENDS:

$$S(0) - \text{Div}_0 - X \leq C_A - P_A \leq S(0) - Xe^{-rT}$$

Bounds with **NO DIVIDENDS**:

$$\text{Call: } \max(S(0) - Xe^{-rT}, 0) \leq C_A < S(0) \quad (13)$$
$$\text{Put: } \max(X - S(0), 0) \leq P_A < X \quad (14)$$

WITH DIVIDENDS:

$$\text{C:} \max(0, S(0) - X, (S(0) - \text{Div}_0) - Xe^{-rT}) \leq C_A < S(0) \quad (15)$$
$$\text{P:} \max(0, X - S(0), Xe^{-rT} - (S(0) - \text{Div}_0)) \leq P_A < X \quad (16)$$

The functions $R_+ \ni X \to C_A(X) \in R_+$ is decreasing, convex, and Lipschitz with associated constant 1.
The functions $R_+ \ni X \to P_A(X) \in R_+$ is increasing, convex, and Lipschitz with associated constant 1.
The functions $R_+ \ni S \to C_A(S) \in R_+$ is increasing, convex, and Lipschitz with associated constant 1.
The functions $R_+ \ni S \to P_A(S) \in R_+$ is decreasing, convex, and Lipschitz with associated constant 1.

If $T_1 < T_2$,

$$C_A(T_1) \leq C_A(T_2) \quad (17)$$
$$P_A(T_1) \leq P_A(T_2) \quad (18)$$

i.e. $T \to C_A(T)$ and $T \to P_A(T)$ are increasing. The one with longer life is worth more.

Options Valuation, BT

The time value of a European call option can **NEVER be negative**:

$$(S(t) - X)^+ \leq \left(S(t) - Xe^{-r(T-t)}\right)^+ \leq C_E(t)$$

However, when a European put option is **deep in the money**, the time value may be negative.

For any European/American call/put option with strike price X, its **time value** attains its **maximum** at $S = X$.

In a binomial tree model, we assume that one-step returns $K(n), n = 1, 2, \ldots$ are i.i.d. such that for each n,

$$K(n) = \begin{cases} u \text{ with probability } p \\ d \text{ with probability } (1-p) \end{cases}$$

where $-1 < d < u$ and $p \in (0, 1)$. Further, we assume that $d < r < u$

Risk-neutral Probability:

$$r = p^\star u + (1 - p^\star)d = E^\star(K(1)) \iff p^\star = \frac{r-d}{u-d}$$

Martingale: The discounted stock prices $\tilde{S}_i(n) = S_i(n)/A(n)$ are martingales for $i = 1, \cdots, k$:

$$E^\star[\tilde{S}_i(n+1)|S(n)] = \tilde{S}_i(n)$$

European Derivative Valuation in N-step BT:

$$D(0,0) = E^\star\left[\frac{f(S(N))}{(1+r)^N}|S(0)\right] = E^\star\left[\frac{f(S(N))}{(1+r)^N}\right] \quad (19)$$

$$= \binom{N}{i}\frac{1}{(1+r)^N}\sum_{i=0}^{N}(p^\star)^{N-i}(1-p^\star)^i f(S(0)(1+u)^{N-i}(1+d)^i) \quad (20)$$

Options Valuation, BT with DIVIDENDS

Ex-dividend stock price:

$$S(n) = S(n-1)(1+K(n)) - \text{Div}(n)$$

We have that

$$S(0) = E^\star\left[\frac{\bar{S}(1)}{1+r}|S(0)\right] = E^\star\left[\frac{S(1)+\text{Div}}{1+r}|S(0)\right]$$

Here, $p^\star$ is calculated using the ex-dividend price $S(0)$ at time 0 and the cum-dividend price $\bar{S}(1)$ at time 1.

We still first calculate its value at the expiry date $D(N) = f(S(N))$. Then we go back to calculate $D(n,j)$ at each node (n,j) using the **risk-neutral probability** $p^\star$. Again, $p^\star_{(n,j)}$ is calculated based on the ex-dividend price $S(n,j)$ and the cum-dividend price $S(n+1)$. This is the **ONLY DIFFERENCE**.

Hedging

Delta Hedging:

$$\Delta_{C_E} = \Phi(d_+) \in [0,1]$$

$$\Delta_{P_E} = \Delta_{C_E} - 1 = \Phi(d_+) - 1 = -\Phi(-d_+)$$

Options Greeks for European Calls:
Delta:

$$\frac{\partial D}{\partial S(t)} = \Delta_{C_E}(t) = \Phi(d_+(t))$$

Gamma:

$$\frac{\partial^2 D}{\partial S(t)^2} = \Gamma_{C_E}(t) = \frac{1}{S(t)\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}d_+^2(t)\right)$$

Theta:

$$\Theta_{C_E}(t) = -\frac{S(t)\sigma}{2\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}d_+^2(t)\right) - rXe^{-r(T-t)}\Phi(d_-(t))$$

Vega:

$$\frac{\partial D}{\partial \sigma} = \nu_{C_E}(t) = \frac{S(t)\sqrt{T-t}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d_+^2(t)\right)$$

Rho:

$$\frac{\partial D}{\partial r} = \rho_{C_E}(t) = (T-t)Xe^{-r(T-t)}\Phi(d_-(t))$$

Options Valuation, B-S Intro

We can extend a BT to a continuous time model: For $t = \frac{m}{T} = m\tau_N$:

$$S_N(t) = S_N(0)\prod_{i=1}^{N}(1+K_N(j)) \tag{21}$$

$$= \exp\left[\ln\left(\prod_{i=1}^{N}(1+K_N(j))\right)\right] \tag{22}$$

$$= S_N(0)\exp\left[\sum_{j=1}^{n} k_N(j)\right] \tag{23}$$

where $k_N(j) = \ln(1+K_N(j)) = \ln\left(\frac{S_N(j\tau_N)}{S_N((j-1)\tau_N)}\right)$ is the **one step log return** of the stock at time j.

Properties of log return: $E(k) = \mu$, $\text{Var}(k) = \sigma^2$, $\ln(1+u_N) = \mu\tau_N + \sigma\sqrt{\tau_N}$, $\ln(1+d_N) = \mu\tau_N - \sigma\sqrt{\tau_N}$.

Scaled Random Walk:

$$W(n) = w_N(n\tau_N) := \frac{1}{\sqrt{N}}w(n) = \frac{1}{\sqrt{N}}\sum_{j=1}^{n}\xi(j) = \sqrt{\tau_N}\sum_{j=1}^{n}\xi(j)$$

The scaled random walk jumps at every τ_N-time unit, $\tau_N = \frac{1}{N}$ by $\pm\frac{1}{\sqrt{N}}$.

Properties of Random Walk: $E[w_N(n\tau_N)] = 0$, $\text{Var}[w_N(n\tau_N)] = \frac{n}{N}$, $W_0 = 0$, $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for $0 \le s \le t$.

Options Valuation, B-S Model

Applying Random Walk to stock prices:

$$k_N(j) = \begin{cases}\ln(1+u_N) & p=\frac{1}{2}\\ \ln(1+d_N) & p=\frac{1}{2}\end{cases} = \begin{cases}\mu\tau_N + \sigma\sqrt{\tau_N}\\ \mu\tau_N - \sigma\sqrt{\tau_N}\end{cases} \tag{24}$$

$$= \mu\tau_N + \sigma\xi(j)\sqrt{\tau_N} \tag{25}$$

$$S_N(t) = S_N(0)\exp\left[\sum_{j=1}^{m}(\mu\tau_N + \sigma\sqrt{\tau_N}\xi(j))\right] \tag{26}$$

$$= S_N(0)\exp\left[m\mu\tau_N + \sigma\sqrt{\tau_N}\sum_{j=1}^{m}(\xi(j))\right] \tag{27}$$

$$= S_N(0)\exp\left[\mu t + \sigma w_N(t)\right] \tag{28}$$

Observe that for every $t \ge 0$,

$$\ln\left(\frac{S(t)}{S(0)}\right) = \mu t + \sigma W(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$$

Black-Scholes SDE (Limit Model): As $N \to \infty$ or equivalently as $\tau_N \to 0$, the CLM gives

$$dS(t) = S(t)\left(\mu + \frac{1}{2}\sigma^2\right)dt + \sigma S(t)dW(t)$$

It has a unique solution

$$S(t) = S(0)\exp(\mu t + \sigma W(t))$$

Risk-neutral Probability under B-S SDE:

$$\mu^\star = r - \frac{1}{2}\sigma^2$$

Ito's Lemma: Assume the process X has the following dynamics: $dX_t = \mu_t dt + \sigma_t dW_t, X_0 = x_0$. Then,

$$df(t,X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t^2 \tag{29}$$

$$= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t dt + \sigma_t dW_t)^2 \tag{30}$$

$$= \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}dW_t \tag{31}$$

since as $dt \to 0, dt^2$ and $dtdW_t$ will tend to zero faster than dt and dW_t^2 and $(dW_t)^2 = dt$.

Martingality of Stochastic Integral: Assume that the function $g(t,x)$ does not grow faster than the exponential function. Then,

$$E\left[\int_0^t g(s,W_s)dW_s\right] = \int_0^0 g(s,W_s)dW_s = 0, t \in R^+$$

Black-Scholes Equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2 = rf$$

European Options Valuation, Pricing

Under the $P^\star$ measure (and by means of Girsanov's theorem) we can rewrite the B-S Model as

$$dS(t) = rS(t)dt + \sigma S(t)dW_t^\star \text{ under } P^\star$$

$\tilde{S}(t) = \tilde{S}(0) + \int_0^t \sigma g(s, W_s^\star)dW_s^\star = \tilde{S}(0)$ is an F^W-martingale under the measure $P^\star$.

Girsanov's Theorem:

$$W_t^\star = W_t + \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}t$$

B-S Formula for European Call/Put:

$$C_E(t) = S(t)\Phi(d_+(t)) - Xe^{-r(T-t)}\Phi(d_-(t))$$

and

$$P_E(t) = Xe^{-r(T-t)}\Phi(-d_-(t)) - S(t)\Phi(-d_+(t))$$

where

$$d_\pm(t) := \frac{\ln\left(\frac{S(t)}{X}\right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$