# An Axiomatic Check-up of Price Indices

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*Abstract*: Axiomatic index theory has developed a variety of tests assessing the suitability of a price index formula. This study surveys and amends these tests and applies them to a large number of price index formulas. Similar past research is being consolidated and the remaining gaps are filled. The comprehensiveness of the results allows for a solid comparative analysis: It is demonstrated that only few index formulas exist which, from an axiomatic point of view, represent serious alternatives to the Laspeyres (or Paasche) index. Contrary to general wisdom, the Fisher index does not appear to be the best alternative

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# 1 Prelude

In many areas of economic analysis, individual price changes have to be aggregated into some "overall price change". This overall price change, in turn, is an essential decision parameter in important fields of economic policy. For instance,

- central banks use the economy's overall price change as a key indicator for determining their *monetary policy*;
- changes in the overall price level of an economy's exports relative to the overall price level of its imports constitutes a change in the economy's terms of trade. Changes in the terms of trade, in turn, are a major concern of an economy's *trade policy*;
- the overall price change can be used to deflate the change in the economy's nominal output. The resulting change in real output is a key indicator for the *economy's performance* and can also be used for computing changes in the economy's productivity, another important performance measure;
- changes in the overall price level imply changes in the purchasing power of wages. These, in turn, are a key input in the process of *wage bargaining* between unions and employers.

*Price index formulas* are a suitable tool for aggregating individual price changes into some overall price change. However, a multitude of price index formulas exists and the numbers these various formulas generate may significantly deviate from each other. Some index formulas may generate rather plausible numbers, whereas other index formulas may produce rather dubious ones. Bias in the measurement of the overall price change, however, leads to ill–defined economic policies, and thus, to significant welfare losses. Furthermore, such a bias may invalidate some of the empirial work testing economic models which feature some sort of overall price change.

*Axiomatic index theory* helps us to discriminate between "reliable" and "unreliable" index formulas. For this purpose, axiomatic index theory develops screening devices which are usually called axioms or *tests*. In these tests, simple postulates are formalized which an index formula must reasonably satisfy. If the index formula violates one or more of these tests, this casts serious doubts on the plausibility of the numbers generated by this index formula.

The systematic development of tests dates back to Walsh (1901) and to Fisher's (1911, 1922) pioneering work. The subject was revived by a monograph of Eichhorn and Voeller (1976). In this monograph, the authors are primarily occupied with exploring the logical relationships between the various tests. Based on these relationships, specific tests are grouped together, such that this set of tests exhibits some interesting property. Three different properties stand out: The set of tests is such that

1. to each of the tests included in the set, at least one price index formula exists which violates this test but satisfies all other tests of the set (*independence between the tests*),
2. no price index formula exists which satisfies all tests included in the set (*consistency between the tests*),
3. a *single* price index formula exists which satisfies all tests included in the set (*characterization of price index formulas*).

Since the publication of Eichhorn and Voeller's (1976) monograph, many other index theorists have contributed to this branch of research. A good survey is provided by Balk (1995).

As a by–product of this research, new tests have been developed. Most of these tests are motivated by the endeavour to establish new sets of tests with one of the above properties. Therefore, these tests sometimes lack economic appeal which may explain why the *economic users* of price indices are less fascinated by this branch of axiomatic index theory. The users of price indices need support for identifying sensible index formulas, that is, they are primarily interested in economically appealing tests and they want to know those index formulas satisfying these tests.

The present paper is primarily intended to serve the specific needs of the *economic users* of price indices. The paper's focus is on those tests which, from the users' perspective, may appear appealing. For a wide range of price index formulas, it is examined whether they satisfy this set of economically motivated tests.

Research of this type is not novel. Not only the classic contributions of Fisher (1911, 1922) but also the work of Eichhorn and Voeller (1976) provide many valuable insights in this direction. In more recent years some appealing new tests have been developed (e.g. "consistency in aggregation") and a number of helpful surveys has been published which also include some assessment as to which index formulas satisfy which test (see, for instance, the surveys of Balk (1995), Diewert (1992), Olt (1996) and of Vogt and Barta (1997)). However, in none of these surveys the primary objective is the systematic examination of price index formulas with respect to *economically motivated* tests. As a consequence, some significant blanks have remained. It is one purpose of the present study to fill in these blanks.

A second purpose of this study is to provide a solid *comparative* analysis of price index formulas. Comparative analysis of price index formulas is not novel, either. Traditionally, these comparisons apply a simple counting criterium: One first defines a set of relevant tests, and then, for each index formula, one investigates how many of these tests the index formula satisfies. The index formula with the highest number of satisfied tests is the winner. Clearly, this is a rather crude way of comparing index formulas.

In the present study, I introduce a more sophisticated approach which is based on *dominance relations* between the various index formulas. Some price index formula $P_A$ dominates (is better than) some alternative price index formula $P_B$, if it satisfies not only all those tests $P_B$ satisfies but at least one additional test. Applying this dominance criterium leads to surprisingly far reaching results: As an alternative to the Laspeyres or Paasche index only four price index formulas remain. Among these index formulas, the Fisher index does *not* appear to be the most suitable one.

This paper is organized as follows: In section 2, I introduce the set of tests considered in this study. Between some of these tests important relationships exist. These are highlighted in section 3. Section 4 lists the price index formulas to be examined. The major contribution of this paper is presented in section 5. In a simple table, it is indicated which index formula satisfies which test. Applying the dominance criterium to this checklist, I can identify some particularly suitable index formulas. Section 6 provides some concluding remarks.

# 2 Statistical Tests

Consider a set $A$ which consists of the integers $i = 1, 2, ..., N$, where each integer $i$ represents a commodity. Let $p_i^t > 0$ denote the unit price and $x_i^t > 0$ the quantity of commodity $i \in A$ in period $t$. Correspondingly, $v_i^t = p_i^t x_i^t$ denotes the monetary value of quantity $x_i^t$ and $V^t = \sum_{i \in A} v_i^t$ is the commodity bundle's monetary value in period $t$. Similarly, $v_i^{st} = p_i^s x_i^t$ denotes the monetary value of $x_i^t$ as measured by $p_i^s$, its price in period $s$. The corresponding aggregate value is $V^{st} = \sum_{i \in A} v_i^{st}$. Only time periods 0 (*base period*) and 1 (*comparison period*) are considered. Variable $r_i$ indicates the price ratio $p_i^1/p_i^0$.

A price index $P$ is a positive function which maps all prices and quantities of the base and comparison period into a single positive price index number. Formally,

$$P : \mathbb{R}_{++}^{4N} \longmapsto \mathbb{R}_{++} \ , \quad (\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \longmapsto P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ ,$$

with $\mathbf{p^t} = (p_1^t, ..., p_N^t)$ and $\mathbf{x^t} = (x_1^t, ..., x_N^t)$.

In order to assess the properties of a price index formula, numerous statistical tests have been developed. In these tests, the price index formula is applied to particular price–quantity scenarios $(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$ and it is examined whether the resulting price index number is in line with the test designer's intuition. In the following, I list a number of important tests. The origin of these tests are given at the end of this section.

**T 1** *The **strict mean value test** postulates that*

$$\min_i \left(p_i^1/p_i^0\right) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max_i \left(p_i^1/p_i^0\right) \ ,$$

*and that, for* $\mathbf{p^1} = \lambda \mathbf{p^0}$*, "$<$" is replaced by "$=$".*

**T 2** *The **proportionality test** postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^0}, \mathbf{x^1}) = \lambda \ , \qquad \textit{for all } \lambda > 0 \ .$$

**T 3** *The **identity test** postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) = 1 \ .$$

Test T1 states that the price index number should lie between the smallest and largest price ratio over all commodities. Test T2 is concerned with a scenario in which comparison period prices result from multiplying all base period prices by the positive number $\lambda$. In this scenario, the price index number should be $\lambda$, too, irrespective of the changes in quantities. Test T2 postulates that without any price change, the price index number should be unity, again irrespective of changes in the quantities.

**T 4** *Suppose* $\widetilde{\mathbf{p}}^\mathbf{0}$*,* $\widetilde{\mathbf{x}}^\mathbf{0}$*,* $\widetilde{\mathbf{p}}^\mathbf{1}$*, and* $\widetilde{\mathbf{x}}^\mathbf{1}$ *result from uniform permutations of* $\mathbf{p^0}$*,* $\mathbf{x^0}$*,* $\mathbf{p^1}$*, and* $\mathbf{x^1}$*. The **symmetry test** (or commodity reversal test) postulates that*

$$P\big(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\big) = P\big(\widetilde{\mathbf{p}}^\mathbf{0}, \widetilde{\mathbf{x}}^\mathbf{0}, \widetilde{\mathbf{p}}^\mathbf{1}, \widetilde{\mathbf{x}}^\mathbf{1}\big) \ .$$

**T 5** *Suppose* $\widetilde{\mathbf{p}}^\mathbf{0}$ *and* $\widetilde{\mathbf{x}}^\mathbf{0}$ *result from uniform permutations of* $\mathbf{p^0}$ *and* $\mathbf{x^0}$*. Then the **permutation test** postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \widetilde{\mathbf{p}}^\mathbf{0}, \widetilde{\mathbf{x}}^\mathbf{0}) = 1 \ .$$

**T 6** *Suppose* $\widetilde{\mathbf{p}}^\mathbf{0}$ *and* $\widetilde{\mathbf{x}}^\mathbf{0}$ *result from uniform permutations of* $\mathbf{p^0}$ *and* $\mathbf{x^0}$*, such that* $p_j^0 = \widetilde{p}_k^0$*,* $p_k^0 = \widetilde{p}_j^0$*,* $x_j^0 = \widetilde{x}_k^0$*,* $x_k^0 = \widetilde{x}_j^0$*, and for all commodities* $i \neq j, k$*,* $p_i^0 = \widetilde{p}_i^0$ *and* $x_i^0 = \widetilde{x}_i^0$*. Then, the **inversion test** postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \widetilde{\mathbf{p}}^\mathbf{0}, \widetilde{\mathbf{x}}^\mathbf{0}) = 1 \ .$$

According to test T4, reordering the commodities should not affect the price index number. Test T5 considers a scenario where the prices and quantities of the comparison period represent a permutation of the prices and quantities of the base period. In such a scenario, the price index number should be unity. Test T6 is similar, but restricts the range of possible permutations to swaps of two commodities, keeping the quantities and prices of all other commodities constant over time. In such scenarios, the price index number should be unity as well.

**T 7** *The* ***commensurability test*** *postulates that*

$$P(\mathbf{p^0}\mathbf{\Lambda}, \mathbf{x^0}\mathbf{\Lambda}^{-1}, \mathbf{p^1}\mathbf{\Lambda}, \mathbf{x^1}\mathbf{\Lambda}^{-1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) ,$$

*where* $\mathbf{\Lambda}$ *is some arbitrary* $N \times N$ *diagonal matrix with positive elements* $\lambda^i$.

**T 8** *The* ***price dimensionality test*** *postulates that*

$$P(\lambda\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) , \qquad \textit{for all } \lambda > 0 .$$

**T 9** *The* ***quantity dimensionality test*** *postulates that*

$$P(\mathbf{p^0}, \lambda\mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) , \qquad \textit{for all } \lambda > 0 .$$

Test T7 postulates that if we change the units in which the quantities are measured (e.g. from gallon to liter), the price index number should remain constant. Similarly, test T8 says that changing the currency in which prices are measured should not affect the price index number. According to test T9, multiplying the quantities of the base and comparison period by some positive constant $\lambda$, should not affect the price index number.

**T 10** *The* ***strict quantity proportionality test*** *postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) = P(\mathbf{p^0}, \delta\mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) , \textit{ for all } \lambda, \delta > 0 .$$

**T 11** *The* ***weak quantity proportionality test*** *postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^0}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^0}) , \qquad \textit{for all } \lambda > 0 .$$

**T 12** *Consider two different scenarios for the comparison period ($t = 1$ and $t = 1^*$), with $\mathbf{p^{1*}} = \lambda\mathbf{p^1}$, and two different scenarios for the base period ($t = 0$ and $t = 0^*$), with $\mathbf{p^{0*}} = (1/\lambda)\mathbf{p^0}$. The* ***linear homogeneity test*** *postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) = \lambda P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P((1/\lambda)\,\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}), \textit{ for all } \lambda > 0.$$

Test T10 says that if all quantities of the comparison period or the base period are multiplied by some constant $\lambda$, this should not affect the price index number. Test T11 postulates that if the relative quantities do not change over time, then the value of the price index should depend only on the prices and the relative quantities. According to test T12, multiplying all prices of the comparison period by some positive number $\lambda$ or multiplying all base period prices by some positive number $1/\lambda$, should give a new price index number which is $\lambda$ times the original price index number.

**T 13** *Consider two different scenarios for the comparison period ($t = 1$ and $t = 1^*$) and the base period ($t = 0$ and $t = 0^*$). If for all commodities $p_i^{1^*} \geq p_i^1$ and for at least one $i$ strict, then the* ***strict monotonicity test*** *postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^*}}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}),$$

*and if for all commodities $p_i^{0^*} \geq p_i^0$ and for at least one $i$ strict, then the strict monotonicity test postulates that*

$$P(\mathbf{p^{0^*}}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ .$$

**T 14** *If for all commodities $p_i^1 \geq p_i^0$ and for at least one $i$ strict, then the* ***weak monotonicity test*** *postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) \ .$$

Test T3 says that increasing the price of at least one comparison period price without lowering any of the other comparison period prices, should increase the price index number. Conversely, increasing at least one base period price without lowering any of the other base period prices should generate a fall in the price index number. If for at least one commodity, its comparison period price is higher than its base period price and for no commodity its comparison period price is smaller than its base period price, then test T4 postulates that the price index number should lie above the price index number resulting from constant prices (if the identity test is satisfied, the latter is unity).

**T 15** *The* ***circularity test*** *postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^2}, \mathbf{x^2}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^2}, \mathbf{x^2}) \ .$$

**T 16** *The* ***time reversal test*** *postulates that*

$$P\big(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\big) = 1 \big/ P\big(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}\big) \ .$$

**T 17** *The* ***constant quantities test*** *postulates that*

$$P\left(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^0}\right) = V^1/V^0 \ .$$

Test 15 is concerned with three periods ($t = 0,1,2$). It says that the price index number which relates periods 0 and 2, should coincide with the product of the price index number relating periods 0 and 1 and the price index number relating periods 1 and 2. According to test T16, interchanging the role of the base and the comparison period should generate a new price index number which is the reciprocal of the original price index number. Test T17 says that if all quantities are constant over time, the price index number should coincide with the ratio of aggregated values.

So far, the analysis was restricted to price indices. It is straightforward to apply the same ideas to quantity indices. A quantity index $Q$ is defined by

$$Q : \mathbb{R}^{4N}_{++} \longmapsto \mathbb{R}_{++} \ , \quad (\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1}) \longmapsto Q(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1}) \ .$$

To all price index formulas $P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1})$, the *corresponding quantity index* $\overleftrightarrow{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1})$ can be defined: Simply swap $\mathbf{p^0}$ and $\mathbf{x^0}$ as well as $\mathbf{p^1}$ and $\mathbf{x^1}$. Furthermore, to all price index formulas $P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1})$, the *complementary quantity index* $\overline{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1})$ can be defined by[2]

$$\overline{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1}) = \left(V^1/V^0\right)/P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) \ .$$

Two important tests of price index formulas are based on postulates with respect to the complementary quantity index:

**T 18** *The* ***factor reversal test*** *postulates that*

$$P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) \cdot \overleftrightarrow{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1}) = V^1/V^0 \ .$$

**T 19** *The* ***product test*** *postulates that*

$$P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) \cdot \overline{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1}) = V^1/V^0 \ ,$$

*where the complementary quantity index* $\overline{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1})$ *must satisfy the following tests:*

- *Proportionality:* $\overline{Q}(\mathbf{x^0},\mathbf{p^0},\lambda\mathbf{x^0},\mathbf{p^1}) = \lambda$, *for all* $\lambda > 0$.
- *Symmetry:* $\overline{Q}(\mathbf{x^0},\mathbf{p^0},\mathbf{x^1},\mathbf{p^1}) = \overline{Q}(\widetilde{\mathbf{x}}^{\mathbf{0}},\widetilde{\mathbf{p}}^{\mathbf{0}},\widetilde{\mathbf{x}}^{\mathbf{1}},\widetilde{\mathbf{p}}^{\mathbf{1}})$, *where "$\sim$" indicates a uniform permutation of the four vectors* $\mathbf{x^0}$, $\mathbf{p^0}$, $\mathbf{x^1}$, *and* $\mathbf{p^1}$.

[2] Fisher (1911) uses the term correlative quantity index.

- *Commensurability:* $\overline{Q}(\mathbf{x^0\Lambda}, \mathbf{p^0\Lambda^{-1}}, \mathbf{x^1\Lambda}, \mathbf{p^1\Lambda^{-1}}) = \overline{Q}(\mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}, \mathbf{p^1})$, *where* $\mathbf{\Lambda}$ *is some arbitrary* $N \times N$ *diagonal matrix with positive elements* $\lambda^i$.

- *Price Dimensionality:* $\overline{Q}(\mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}, \mathbf{p^1}) = \overline{Q}(\mathbf{x^0}, \lambda\mathbf{p^0}, \mathbf{x^1}, \lambda\mathbf{p^1})$, *for all* $\lambda > 0$.

- *Strict Monotonicity: Consider two different scenarios for the comparison period ($t = 1$ and $t = 1^*$) and the base period ($t = 0$ and $t = 0^*$). If for all commodities $x_i^1 \geq x_i^{1*}$ and for at least one $i$ strict, then*

$$\overline{Q}(\mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}, \mathbf{p^1}) > \overline{Q}(\mathbf{x^0}, \mathbf{p^0}, \mathbf{x^{1*}}, \mathbf{p^1}) \ .$$

*If for all commodities $x_i^0 \geq x_i^{0*}$ and for at least one $i$ strict, then*

$$\overline{Q}(\mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}, \mathbf{p^1}) < \overline{Q}(\mathbf{x^{0*}}, \mathbf{p^0}, \mathbf{x^1}, \mathbf{p^1}) \ .$$

Test T18 postulates that the corresponding quantity index is identical to the complementary quantity index. According to test T19 this is not necessary, but the complementary index must satisfy some basic tests related to quantity indices.

In applied work, one often wishes to aggregate over prices not only in a single stage procedure but also in a two or more stage procedure. In a two stage procedure, for instance, the set of commodities is partitioned into disjoint subsets and for each subset a corresponding price index number is computed. In the second step, one calculates from these price index numbers a price index number for the complete set. In a one stage procedure the index number for the complete set is computed directly from the elementary commodities. It is desirable that the two procedures generate identical results.

Formalizing this requirement turns out to be rather tedious. Here I sketch out the definition proposed in von Auer (2000a). I define $\mathbf{r} = (r_1, ..., r_N)$, $\mathbf{v^0} = (v_1^0, ..., v_N^0)$, $\mathbf{v^1} = (v_1^1, ..., v_N^1)$, $\mathbf{v^{10}} = (v_1^{10}, ..., v_N^{10})$ and $\mathbf{v^{01}} = (v_1^{01}, ..., v_N^{01})$. If a single stage price index formula satisfies the commensurability test, then it is true that

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b}) \ ,$$

where $(a, b)$ can take any of the three value–combinations $(0, 01)$, $(1, 10)$, and $(0, 1)$.

In order to do a two stage computation of the price index number, the set of commodities $A$ can be arbitrarily partitioned into $K$ disjoint subsets $A_k$:

$$A = \cup_{k=1}^K A_k, \quad A_k \cap A_l = \emptyset, \ \forall \, k \neq l \ .$$

For each subset $A_k$ one can calculate the *first stage price index* number $P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})$, where $\mathbf{r_k}$ denotes the subvector of $\mathbf{r}$ which consists of all price ratios $r_j$ belonging to subset $A_k$. Analogously, $\mathbf{v_k^a}$ and $\mathbf{v_k^b}$ denote the corresponding subvectors of $\mathbf{v^a}$ and $\mathbf{v^b}$. The functional form of $P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})$ should coincide with the functional form of the single stage index $P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})$. Besides the first stage index numbers $P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})$, one can also calculate the values $V_k^a = \sum_{i \in A_k} v_i^a$ and $V_k^b = \sum_{i \in A_k} v_i^b$. This concludes the first stage of the two stage procedure.

In the second stage, one can compute the corresponding *second stage price index* number $P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})$, with $\mathbf{P^I} = (P_1^I, ..., P_K^I)$, $\mathbf{V^a} = (V_1^a, ..., V_K^a)$ and $\mathbf{V^b} = (V_1^b, ..., V_K^b)$. The functional form of $P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})$ should coincide with the functional form of the single stage index $P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})$. Taken together, the two stages compute an index number for the overall price change. The procedure is denoted as a *two stage price index.*

Based on these concepts, one may formulate the following test:

**T 20** *A (single stage) price index $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$ is **weakly consistent in aggregation**, if and only if it satisfies the commensurability test and for at least one of the three possible $(a, b)$–combinations $(0, 01)$, $(1, 10)$ and $(0, 1)$, for all partitions of $A$*

$$P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b}) = P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b}) \ ,$$

*where each first stage index $P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})$ in vector $\mathbf{P^I}$ has the same functional form as the second stage index $P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})$ and the single stage index $P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})$, except possibly for the number of variables.*[3]

In other words, test T20 postulates that a price index formula should generate the same index number regardless of the chosen partition and the number of stages.

Some of the listed tests have a long history, others are significantly younger: Tests T1 and T4 have been proposed by Olt (1996), tests T2, T12, and T17 by Walsh (1901), test T2 by Laspeyres (1871), tests T4 and T18 by Fisher (1922), tests T5 and T6 by Krtscha (1979), tests T7 and T16 by Pierson (1896), and tests T8, T9, and T3 by Eichhorn and Voeller (1976). Test T10 amalgamates proposals by Vogt (1980) and Diewert (1992), test T11 does not seem to have been proposed before. Test T15 has been proposed by Westergaard (1890) and test T19 combines proposals by Frisch (1930) and Eichhorn and Voeller (1990). Test T20 has been suggested by Auer (2000a), though the basic idea is due to Vartia (1976a) and Balk (1995, 1996).

[3] A stricter variant of Test 20 exists. It restricts the admissable $(a, b)$–combinations to $a = 0$ and $b = 1$. This stricter variant can be traced back to Vartia (1976a) and has been formalized by Balk (1995, 1996).

# 3 Important Relationships Between the Tests

Starting with Eichhorn and Voeller's (1976) work, axiomatic index theory has been primarily concerned with the exploration of logical relationships between the various tests. As pointed out in the introduction to this article, the findings of this research are useful for identifying sets of tests with interesting properties (e.g. independence of all tests included in the set).

Knowing the relationships between the various tests will also be valuable in the context of section 5 of this paper, where I examine which index formulas satisfy which test. Drawing on the logical relationships between the tests, significantly cuts down the number of necessary proofs.

In this respect, various useful relationships exist. The following three relationships are particularly trivial:

**R1)** For $\lambda = 1$, the *proportionality* test (T2) degenerates to the *identity* test (T2).

**R2)** For $\mathbf{p^1}= \lambda\mathbf{p^0}$, the *strict monotonicity* test (T3) degenerates to the *weak monotonicity* test (T4), and the *strict mean value* test (T1) degenerates to the *proportionality* test (T2).

**R3)** For $\mathbf{x^1}= \mathbf{x^0}$, the *strict quantity proportionality* test (T10) degenerates to the *weak quantity proportionality* test (T11).

As demonstrated in Appendix A, also the following relationships can be established:

**R4)** *permutation* (T5) $\Longrightarrow$ *inversion* (T6).

**R5)** *proportionality* (T2) $\wedge$ *strict monotonicity* (T3) $\Longrightarrow$ *strict mean value* (T1).

**R6)** *strict mean value* (T1) $\Longrightarrow$ *weak monotonicity* (T4).

**R7)** Consider the following triple of tests: *commensurability* (T7) *price dimensionality* (T8), and *quantity dimensionality* (T9). No price index exists which violates exactly one out of these three test.

**R8)** *linear homogeneity* (T12) $\wedge$ *identity* (T2) $\Longrightarrow$ *proportionality* (T2).

**R9)** *circularity* (T15) $\wedge$ *identity* (T2) $\Longrightarrow$ $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{p^1})$.

**R10)** *factor reversal* (T18) $\wedge$ *identity* (T2) $\Longrightarrow$ *constant quantities* (T17).

**R11)** *product* (T19) $\Longrightarrow$ *constant quantities* (T17).

**R12)** If a price index satisfies *proportionality* (T2), *symmetry* (T4), *commensurability* (T7), *price dimensionality* (T8), and *strict monotonicity* (T3), then *factor reversal* (T18) $\Longrightarrow$ *product* (T19).

**R13)** *product* (T19) $\Longrightarrow$ *weak quantity proportionality* (T11).

# 4 A List of Established Price Index Formulas

Besides the two most customary indices,

$$\text{Laspeyres}: \; P_L = \frac{\sum_{i \in A} p_i^1 x_i^0}{\sum_{i \in A} p_i^0 x_i^0} = \frac{V^{10}}{V^0}$$

$$\text{Paasche}: \; P_P = \frac{\sum_{i \in A} p_i^1 x_i^1}{\sum_{i \in A} p_i^0 x_i^1} = \frac{V^1}{V^{01}} \, ,$$

also the following index formulas are considered:

$$\text{Fisher}: \; P_F = \sqrt{P_L P_P} = \sqrt{\frac{V^{10} V^1}{V^0 V^{01}}}$$

$$\text{Drobisch}: \; P_{Dr} = \frac{1}{2} \left( P_L + P_P \right)$$

$$\text{Marshall–Edgeworth}: \; P_{ME} = \frac{\sum_{i \in A} p_i^1 (x_i^0 + x_i^1)}{\sum_{i \in A} p_i^0 (x_i^0 + x_i^1)} = \frac{V^{10} + V^1}{V^0 + V^{01}}$$

$$\text{Banerjee}: \; P_B = P_P \frac{P_L + 1}{P_P + 1} = \frac{V^{10}/V^0 + 1}{1 + V^{01}/V^1}$$

$$\text{Walsh(I)}: \; P_{W(I)} = \frac{\sum_{i \in A} p_i^1 \sqrt{x_i^0 x_i^1}}{\sum_{i \in A} p_i^0 \sqrt{x_i^0 x_i^1}} = \frac{\sum_{i \in A} \sqrt{v_i^{10} v_i^1}}{\sum_{i \in A} \sqrt{v_i^0 v_i^{01}}}$$

$$\text{Stuvel}: P_S = \frac{Z}{2} + \sqrt{\left( \frac{Z}{2} \right)^2 + \frac{V^1}{V^0}}$$

$$\text{with } Z = \left( V^{10} - V^{01} \right) / V^0 \, .$$

In addition, I examine the most prominent log–change index numbers:

$$\text{Walsh(II)}: \; \ln P_{W(II)} = \sum_{i \in A} \frac{\sqrt{v_i^0 v_i^1}}{\sum_{j \in A} \sqrt{v_j^0 v_j^1}} \ln r_i$$

$$\text{Walsh–Vartia}: \; \ln P_{WV} = \sum_{i \in A} \frac{\sqrt{v_i^0 v_i^1}}{\sqrt{V^0 V^1}} \ln r_i$$

$$\text{Törnqvist}: \; \ln P_T = \sum_{i \in A} \frac{(v_i^0 / V^0) + (v_i^1 / V^1)}{2} \ln r_i$$

$$\text{Theil}: \; \ln P_{Th} = \sum_{i \in A} \left[ \frac{\sqrt[3]{\frac{1}{2} \left( v_i^0 + v_i^1 \right) v_i^0 v_i^1}}{\sum_{j \in A} \sqrt[3]{\frac{1}{2} \left( v_j^0 + v_j^1 \right) v_j^0 v_j^1}} \right] \ln r_i$$

$$\text{Vartia(I)}: \; \ln P_{V(I)} = \frac{\ln V^1 - \ln V^0}{V^1 - V^0} \sum_{i \in A} \frac{v_i^1 - v_i^0}{\ln v_i^1 - \ln v_i^0} \ln r_i \, ,$$

where for $v_i^1 = v_i^0$ the term $(v_i^1 - v_i^0) / (\ln v_i^1 - \ln v_i^0)$ is set equal to $v_i^0$ and for $V^1 = V^0$ the term $(\ln V^1 - \ln V^0) / (V^1 - V^0)$ is set equal to $1/V^0$;

$$\text{Vartia(II)}: \quad \ln P_{V(II)} = \sum_{i \in A} \frac{\dfrac{[(v_i^1/V^1) - (v_i^0/V^0)]}{[\ln (v_i^1/V^1) - \ln (v_i^0/V^0)]}}{\displaystyle\sum_{j \in A} \dfrac{\left[\left(v_j^1/V^1\right) - \left(v_j^0/V^0\right)\right]}{\left[\ln \left(v_j^1/V^1\right) - \ln \left(v_j^0/V^0\right)\right]}} \ln r_i \ ,$$

where for $v_i^1/V^1 = v_i^0/V^0$,

$$\left[\left(v_i^1/V^1\right) - \left(v_i^0/V^0\right)\right] / \left[\ln \left(v_i^1/V^1\right) - \ln \left(v_i^0/V^0\right)\right] = v_i^0/V^0 \ .$$

In the following section, for all these price index formulas it is examined whether they satisfy the statistical tests introduced in section 2.

# 5 Examination of Price Index Formulas

In section 2, I introduced 20 different tests, and in section 4, I listed 14 different price index formulas. In the present section, it is examined which price index formulas satisfy which tests. In total, $20 \times 14 = 280$ findings are derived. Table 1 presents these findings in transparent form. Filled triangles $\blacktriangle$ indicate that the price index satisfies the respective test and empty triangles $\triangledown$ indicate a violation. For some of the findings, the proof is trivial. For other results the proof is more extensive. A complete documentation of the proofs is provided in Appendix B.

In most countries, national accounting is based on the Laspeyres index $P_L$ (or some slightly modified version of it). Part of its popularity this index owes to its good axiomatic record as documented in Table 1. The major part of its popularity, however, can be credited to its high practicability. For instance, its value can be computed without knowing the quantities of the comparison period $\mathbf{x^1}$. This simplifies and speeds up computations.

Nevertheless, there have been repeated demands to replace the Laspeyres index $P_L$ by some other price index formula, preferably the Fisher index $P_F$. In the present study, I will not comment on these demands. The purpose of the comparative part of this study is more modest: *Given that we look for an alternative to* $P_L$ *(and* $P_P$*)*, which price index formula is the best candidate *when judged by its axiomatic record*? This focus also ensures that only those price index formulas are compared which process the same amount of information, namely $\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}$, and $\mathbf{x^1}$. Note that the index formula $P_L$ discards the inforamtion entailed in $\mathbf{x^1}$ and the index formula $P_P$ discards the information entailed in $\mathbf{x^0}$.

| | $P_L$ | $P_P$ | $P_F$ | $P_{Dr}$ | $P_{ME}$ | $P_B$ | $P_S$ | $P_{W(I)}$ | $P_{W(II)}$ | $P_{WV}$ | $P_T$ | $P_{Th}$ | $P_{V(I)}$ | $P_{V(II)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T1 Str. M. Val. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▲ | ▲ | ▽ | ▲ |
| T2 Proportion. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▲ | ▲ | ▽ | ▲ |
| T2 Identity | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T4 Symmetry | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T5 Permutation | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ |
| T6 Inversion | ▽ | ▽ | ▲ | ▽ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T7 Commens. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T8 Price Dim. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T9 Qua. Dim. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T10 Str. Qua. P. | ▲ | ▲ | ▲ | ▲ | ▽ | ▲ | ▽ | ▲ | ▲ | ▲ | ▲ | ▽ | ▽ | ▲ |
| T11 W. Qua. P. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▽ | ▲ |
| T12 Lin. Hom. | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▽ | ▲ | ▲ | ▽ | ▲ | ▽ | ▽ | ▲ |
| T3 Str. Mon. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ |
| T4 W. Mon. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T15 Circularity | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ |
| T16 Time Rev. | ▽ | ▽ | ▲ | ▽ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ |
| T17 Const. Qua. | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▽ | ▽ | ▽ | ▲ | ▲ |
| T18 Fact. Rev. | ▽ | ▽ | ▲ | ▽ | ▽ | ▽ | ▲ | ▽ | ▽ | ▽ | ▽ | ▽ | ▲ | ▲ |
| T19 Product | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▲ | ▽ | ▽ | ▽ | ▽ | ▽ | ▽ |
| T20 W. Cons. i. A. | ▲ | ▲ | ▽ | ▽ | ▲ | ▽ | ▲ | ▽ | ▽ | ▲ | ▽ | ▽ | ▲ | ▽ |

Table 1: Synopsis of all results. In this table, filled triangles indicate "*satisfied*" and empty triangles indicate "*violated*". A complete documentation of the associated proofs is provided in Appendix B.

For answering this question, Table 1 is the relevant source of information. The usual approach is to rely on a simple *counting principle*: The price index with the smallest number of “fails” is the best candidate. That is, one simply counts the number of unfilled triangles in the respective column. In the present study, as well as in several earlier studies, this procedure would lead to the Fisher index $P_F$. However, inserting or omitting specific tests may manipulate the result generated by the counting principle. Furthermore, this method completely ignores the differences in the importance of the various tests. Combining the counting principle with some weighting scheme might mitigate the last problem, but it creates new problems: It is virtually impossible to come up with a weighting scheme which finds universal approval and there is additional scope for manipulating the final outcome.

For these reasons, I apply a completely different principle: I use the dominance principle instead of the counting principle. The *dominance principle* eliminates all price index formulas which are dominated by at least one other index formula. For instance, $P_F$ satisfies not only all those tests which $P_{W(I)}$ satisfies, but also one additional test (T18). Hence, $P_F$ dominates $P_{W(I)}$ and the latter index can be eliminated from the set of candidates. Applying this approach to Table 1 reduces the number of acceptable replacements of $P_L$ or $P_P$ to four candidates: The Fisher index $P_F$, the Marshall–Edgeworth index $P_{ME}$, the Stuvel–index $P_S$, and the Walsh–Vartia index $P_{WV}$.

To be sure, even the dominance principle allows for manipulation. For instance, embazzling the weak consistency in aggregation test would induce a dominance of $P_F$ over the other three remaining contenders $P_{ME}$, $P_S$, and $P_{WV}$. The larger the list of tests, the smaller the scope for manipulation – and, unfortunately, the fewer dominance relations can be established.

Having reduced the set of candidates to the four formulas $P_F$, $P_{ME}$, $P_S$, and $P_{WV}$, the dominance principle has reached its limits. For further progress the axiomatic deficiencies of these four formulas have to be weighed against each other. Looking at Table 1, it appears justified to eliminate $P_{WV}$. This formula violates a number of rather basic tests including the proportionality test and the strict monotonicity test.

It is more controversial to choose among the remaining three candidates. Each of these candidates has some weakness: $P_F$ violates the weak consistency in aggregation test, $P_{ME}$ violates the factor reversal test and the strict quantity proportionality test, and $P_S$ violates the linear homogeneity test and the strict quantity proportionality test. Among practitioners working with national accounts there is an unanimous position: Weak consistency in aggregation is *indispensible.* This suggests that, from an axiomatic point of view, the Fisher index $P_F$ appears inferior to the other two candidates. It also helps to explain, why in official statistical agencies all over the world, there is strong resistance to the introduction of the Fisher index.

Whether the Marshall–Edgeworth index $P_{ME}$ or the Stuvel index $P_S$ is the best candidate is again a matter of subjective judgement. In my personal view, the factor reversal test appears less relevant than the linear homogeneity test. This would favour the Marshall–Edgeworth index $P_{ME}$. In addition, the formula of $P_{ME}$ is extremely simple whereas the formula of the Stuvel index $P_S$ is not self–explanatory – which is not an axiomatic argument, though. In sum, my personal champion is the Marshall–Edgeworth index $P_{ME}$, a price index formula which has been thoroughly neglected in past debates on the "best" price index formula.

# 6 Concluding Remarks

In this study, 14 different price index formulas have been screened. 20 different statistical tests served as screening devices. The Laspeyres and Paasche index violates some of these tests and one may ask whether other index formulas exist which represent a sensible alternative to the Laspeyres and Paasche index. Using the dominance criterium, the set of alternatives has been narrowed down to four index formulas, one of which (the Walsh–Vartia index) can be safely discarded because of its poor overall performance. Among the remaining three candidates, only the Fisher index violates the weak consistency in aggreation test. Since practitioners regard consistency in aggreation as indispensible, the other two candidates – the Stuvel index and the Marshall–Edgeworth index – appear more suitable than the Fisher index. This is especially true for applied work based on national accounts.

# Appendix A

**R4:** All permutations admissable in the inversion test are also admissable in the permutation test.

**R5:** (adopted from Eichhorn and Voeller (1990)) Proportionality and strict monotonicity imply that

$$\begin{array}{rl} & \min\{r_i\} = P(\mathbf{p^0}, \mathbf{x^0}, \min\{r_i\}\cdot\mathbf{p^0}, \mathbf{x^1}) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \\ & \max\{r_i\} = P(\mathbf{p^0}, \mathbf{x^0}, \max\{r_i\}\cdot\mathbf{p^0}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \\ \Longrightarrow & \min\{r_i\} < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max\{r_i\} \ . \end{array}$$

**R6:** Suppose that $\mathbf{p^0} \leq \mathbf{p^1}$ and for at least one commodity $i$ strict. Then

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) \underbrace{=}_{ident.} 1 < \min(r_i) \underbrace{<}_{str.\ m.\ v.} P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ .$$

**R7:**

$$P(\mathbf{p^0}, \lambda\mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) \underbrace{=}_{com.} P(\lambda\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) \underbrace{=}_{price\ d.} P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ .$$

**R8:** (Olt, 1996)

$$P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^0}, \mathbf{x^1}) \underbrace{=}_{lin.\ hom.} \lambda \cdot P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) \underbrace{=}_{ident.} \lambda \ .$$

**R9:** (Vogt, 1979)

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P\underbrace{(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^{0*}})}_{=1\ (ident.)} \cdot P(\mathbf{p^0}, \mathbf{x^{0*}}, \mathbf{p^1}, \mathbf{x^{1*}}) \cdot P\underbrace{(\mathbf{p^1}, \mathbf{x^{1*}}, \mathbf{p^1}, \mathbf{x^1})}_{=1\ (ident.)}$$

**R10:** (similar in Vogt (1979)) If the price index $P$ satisfies the identity test, then also the corresponding quantity index $\overleftrightarrow{Q}$ satisfies the identity test (for quantity indices). Hence, with the factor reversal test being satisfied ($\overleftrightarrow{Q} = \overline{Q}$), constant quantities generate $P = (V^1/V^0)(1/\overline{Q}) = V^1/V^0$.

**R11:** With the product test being satisfied, we know that the complementary quantity index $\overline{Q}$ satisfies the identity test, and hence, constant quantities generate $P = (V^1/V^0)(1/\overline{Q}) = V^1/V^0$.

**R12**: Given $P$ satisfies the factor reversal test ($\overleftrightarrow{Q} = \overline{Q}$), it satisfies the product test, if and only if it satisfies the proportionality test, the symmetry test, the commensurability test, the price dimensionality test, and the strict monotonicity test.

**R13**: Given the product test is satisfied, the complementary quantity index satisfies the proportionality test. For $\mathbf{x^1} = \lambda\mathbf{x^0}$, this implies that $\overline{Q}(\mathbf{x^0}, \mathbf{p^0}, \lambda\mathbf{x^0}, \mathbf{p^1}) = \lambda$. Hence,

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^0}) = \left(V^1/V^0\right)/\lambda = \lambda\left(V^{10}/V^0\right)/\lambda = V^{10}/V^0,$$

which implies that $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^0})$ is invariant with respect to $\lambda$.

# Appendix B

It is trivial to derive the results:

**T2**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_{W(I)}$]; **T4**[all indices]; **T8**[all indices]; **T10**[$P_L$, $P_P$, $P_{W(I)}$]; **T12**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_{W(I)}$]; **T3**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_{W(I)}$].

From the relationships R1 to R13 presented in section 3 and proved in Appendix A, one can infer the following results:

**T1**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_B$, $P_S$, $P_{W(I)}$] by R5; **T1**[$P_{WV}$, $P_{V(I)}$] by R2
**T2**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_{W(I)}$] by R8; **T2**[$P_{W(II)}$, $P_T$, $P_{Th}$, $P_{V(II)}$] by R2
**T2**[$P_B$, $P_S$] by R1;
**T5**[$P_L$, $P_P$, $P_{Dr}$] by R4;
**T9**[all indices] by R7;
**T10**[$P_{Th}$, $P_{V(I)}$] by R3;
**T11**[all indices except for $P_{Th}$, $P_{V(I)}$, $P_{ME}$, $P_S$] by R3; **T11**[$P_{ME}$, $P_S$] by R13;
**T12**[$P_{WV}$, $P_{V(I)}$] by R8;
**T4**[all indices except for $P_{WV}$, $P_{V(I)}$] by R6;
**T17**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_B$, $P_S$, $P_{W(I)}$] by R11; **T17**[$P_{V(I)}$, $P_{V(II)}$] by R10;
**T18**[$P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$] by R10;
**T19**[$P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$] by R11; **T19**[$P_F$, $P_S$] by R12.

For the remaining results, the following proofs are given:

**T1**[$P_{W(II)}$, $P_T$, $P_{Th}$, $P_{V(II)}$]: $P_{W(II)}$, $P_T$ , $P_{Th}$ and $P_{V(II)}$ can be expressed as $\ln P = \sum w_i \ln r_i$, where $\sum w_i = 1$. Hence, $\ln(\min r_i) < \ln P < \ln(\max r_i)$ and for proportional price changes we get equality signs.

**T2**[$P_B$, $P_S$, $P_{WV}$, $P_{V(I)}$]: For $\mathbf{p^1} = \lambda \mathbf{p^0}$, we get $P_L = P_P = \lambda$, and hence, $P_B = \lambda\,[(\lambda+1)\,/\,(\lambda+1)] = \lambda$. For $P_S$, we get

$$Z = (\lambda V^0 - V^{01})/V^0 = \lambda - V^{01}/V^0 \ ,$$

and hence,

$$\begin{aligned} P_S &= \frac{\lambda - V^{01}/V^0}{2} + \sqrt{\left(\frac{\lambda - V^{01}/V^0}{2}\right)^2 + \lambda V^{01}/V^0} \\ &= \frac{\lambda - V^{01}/V^0}{2} + \sqrt{\left(\frac{\lambda + V^{01}/V^0}{2}\right)^2} = \lambda \,. \end{aligned}$$

For $P_{WV}$, we get

$$\ln P_{WV} = \sum \frac{\sqrt{\lambda v_i^{01} v_i^0}}{\sqrt{\lambda \sum v_i^{01} \sum v_i^0}} \ln \lambda = \ln \lambda \sum \frac{\sqrt{v_i^{01} v_i^0}}{\sqrt{V^{01} V^0}} \ .$$

Since $\sum \left( \sqrt{v_i^{01} v_i^0} / \sqrt{V^{01} V^0} \right) \neq 1$ (exception: $\mathbf{x^1} = \mu \mathbf{x^0}$), we get $\ln P_{WV} \neq \ln \lambda$. In $\ln P_{V(I)}$, the sum of the weights is always smaller than 1 (exception: $v_i^0 / V^0 = v_i^1 / V^1$ for $i = 1, ..., N$, see Vartia (1976a)). Thus, $\ln P_{V(I)} < \ln \lambda$.

**T2**[$P_{W(II)}$, $P_{WV}$, $P_T$,$P_{Th}$, $P_{V(I)}$, $P_{V(II)}$]: All these indices can be expressed as $\ln P = \sum w_i \ln r_i$, where $\ln r_i = 0$ for all $i = 1, ..., N$.

**T5**[all indices except for $P_L$, $P_P$, $P_{Dr}$]: (adopted from von Auer (2000b)) For scenarios covered by the permutation test, we get $V^0 = V^1$ but not necessarily $V^{01} = V^{10}$. This directly leads to $P_F \neq 1$, $P_{ME} \neq 1$ and$P_B = (V^{10} + V^1) / (V^1 + V^{01}) \neq 1$. A permutation scenario with three commodities where $p_3^1 = p_2^0 = 4p_1^0 = 4p_3^0$ and $x_2^1 = x_1^0 = 4x_2^0 = 4x_3^0$ generates $P_{W(I)} = 6/7 \neq 1$. Furthermore, $V^{01} \neq V^{10}$ leads to $Z \neq 0$, and hence, $P_S \neq 1$. The log change index formulas $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$, and $P_{V(II)}$ can be expressed in the general form $\ln P = \sum_{i \in A} w_i \ln r_i$. Furthermore, for all these index formulas, the three–commodity permutation scenario specified above, gives $w_2 \neq w_1 = w_3$ and $\ln r_1 r_3 \neq 0$. Since $\sum_{i=1}^3 \ln r_i = 0$, one gets

$$\begin{aligned} \ln P &= w_1 \ln r_1 + w_2 \left( - \ln r_1 - \ln r_3 \right) + w_1 \ln r_3 \\ &= (w_1 - w_2) \ln r_1 r_3 \neq 0 \ . \end{aligned}$$

**T6**[all indices]: (adopted from von Auer (2000b)) For all scenarios covered by the inversion test, one gets $V^0 = V^1$ and $V^{01} = V^{10}$. Hence, $P_F = 1$, $P_{ME} = 1$, and $P_B = 1$. Furthermore, $V^{01} = V^{10}$ leads to $Z = 0$, and hence, $P_S = 1$. Since $v_j^{01} = v_k^{10}$ and $v_j^0 = v_k^1$ and for all $i \neq j, k$, $v_i^0 = v_i^1 = v_i^{01} = v_i^{10}$, one gets

$$P_{W(I)} = \frac{\sum_{i \neq j,k} v_i^1 + \sqrt{v_j^{10} v_j^1} + \sqrt{v_k^{10} v_k^1}}{\sum_{i \neq j,k} v_i^1 + \sqrt{v_k^{10} v_k^1} + \sqrt{v_j^{10} v_j^1}} = 1 \ .$$

The log–change index formulas $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$, and $P_{V(II)}$ can be expressed in the form $\ln P = \sum_{i \in A} w_i \ln r_i$. Since $r_k = 1/r_j$ and $r_i = 1$ for all $i \neq j, k$, one gets

$$\ln P = w_j \ln r_j + w_k \ln 1/r_j \ . \tag{1}$$

Thus, $\ln P = 0$ requires that $w_j = w_k$. Since $v_j^0 = v_k^1$ and $V^0 = V^1$, the weights $w_i$ of $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$, and $P_{V(II)}$ satisfy this requirement.

$V^0 = V^1$ and $V^{01} = V^{10}$ leads to

$$P_{Dr} = \frac{1}{2}\frac{\left(V^{10}\right)^2 + \left(V^0\right)^2}{V^0 V^{10}} = \frac{1}{2}\frac{\left(V^{10} - V^0\right)^2 + 2V^0 V^{10}}{V^0 V^{10}} \; .$$

Thus, $P_{Dr} = 1$ requires that $\left(V^{10} - V^0\right)^2 = 0$. However, $V^{10} \neq V^0$, unless $p_j^0 = p_j^1 (= p_k^0 = p_k^1)$. Due to $V^{10} \neq V^0$, we get $P_L \neq 1$. Analogous reasoning applies to $P_P$.

**T7**[all indices]: The values of the vectors $\mathbf{v^{st}}$ ($s = 0, 1$ and $t = 0, 1$) are not affected by the re-scaling considered in the commensurability test. Since all of the listed price indices can be expressed exclusively in terms of the vectors $\mathbf{v^{st}}$, the values of the price indices remain unaffected, too.

**T10**[$P_F$, $P_{Dr}$, $P_{ME}$, $P_B$, $P_S$, $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{V(II)}$]: Since $P_L$ and $P_P$ satisfy the strict quantity proportionality test, also $P_F$, $P_{Dr}$, and $P_B$ satisfy this test.

Consider the two commodity case with some inversion scenario, that is, $V^0 = V^1$, $V^{10} = V^{01}$, and $V^0 \neq V^{01}$. We know that in this case $P_{ME} = P_S = 1$. Also Multiplying $\mathbf{x^1}$ by some positive constant $\lambda \neq 1$ should lead to $P_{ME} = P_S = 1$. However, for $P_{ME}$ we get

$$\begin{aligned} P_{ME} = \frac{V^{01} + V^0\lambda}{V^0 + V^{01}\lambda} &\equiv 1 \\ \Rightarrow \qquad V^{01} - V^0 &= \left(V^{01} - V^0\right)\lambda \\ \Rightarrow \qquad V^{01} &= V^0 \; , \end{aligned}$$

which is true only in special cases, and also for $P_S$ we get

$$\begin{aligned} P_S = & \frac{V^{01}(1-\lambda)}{2V^0} + \sqrt{\left(\frac{V^{01}(1-\lambda)}{2V^0}\right)^2 + \lambda} \equiv 1 \\ \Rightarrow \qquad & \left(\frac{V^{01}(1-\lambda)}{2V^0}\right)^2 + \lambda = \left(1 - \frac{V^{01}(1-\lambda)}{2V^0}\right)^2 \\ \Rightarrow \qquad & \left(\frac{V^{01}(1-\lambda)}{2V^0}\right)^2 + \lambda = 1 - 2\frac{V^{01}(1-\lambda)}{2V^0} + \left(\frac{V^{01}(1-\lambda)}{2V^0}\right)^2 \\ \Rightarrow \qquad & \lambda = 1 - \frac{V^{01}(1-\lambda)}{V^0} \\ \Rightarrow \qquad & V^{01}(1-\lambda) = V^0(1-\lambda) \\ \Rightarrow \qquad & V^{01} = V^0 \; . \end{aligned}$$

The indices $P_{W(II)}$, $P_{WV}$, $P_T$, and $P_{V(II)}$ can be expressed as $\ln P = \sum w_i \ln r_i$. After multiplying $\mathbf{x^1}$ or $\mathbf{x^0}$ by any positive constant, the new weights $w_i^*$ are identical to the old weights $w_i$, $i = 1, ..., N$.

**T11**[$P_{Th}$, $P_{V(I)}$]: We know that $P_{V(I)}$ satisfies the constant quantities test (by R10) and the factor reversal test (see proof of T18). Hence,

$$P_{V(I)}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^0}) = V^{10}/V^0$$

and

$$P_{V(I)}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\lambda\mathbf{x^0}) \cdot \overleftrightarrow{Q}_{V(I)}(\mathbf{x^0},\mathbf{p^0},\lambda\mathbf{x^0},\mathbf{p^1}) = \lambda\left(V^{10}/V^0\right)$$

which implies that

$$P_{V(I)}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^0}) = P_{V(I)}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\lambda\mathbf{x^0}) \cdot \overleftrightarrow{Q}_{V(I)}(\mathbf{x^0},\mathbf{p^0},\lambda\mathbf{x^0},\mathbf{p^1}) \Big/ \lambda\ .$$

Hence, for weak proportionality to be satisfied, it must be true that $\overleftrightarrow{Q}_{V(I)}(\mathbf{x^0},\mathbf{p^0},\lambda\mathbf{x^0},\mathbf{p^1}) = \lambda$. However, since $P_{V(I)}$ violates the proportionality test, this is also true for the corresponding quantity index: $\overleftrightarrow{Q}_{V(I)}(\mathbf{x^0},\mathbf{p^0},\lambda\mathbf{x^0},\mathbf{p^1}) \neq \lambda$.

Consider a scenario $(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\lambda\mathbf{x^0})$, such that $r_i = 1$ and $v_i^0 = 1$ for all $i = 2,3,..,N$. For this case, we get

$$\ln P_{Th}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\lambda\mathbf{x^0}) = \frac{\sqrt[3]{\frac{1}{2}\left(v_1^0+v_1^1\right)v_1^0 v_1^1}}{\sqrt[3]{\frac{1}{2}\left(v_1^0+v_1^{10}\lambda\right)v_1^0 v_1^{10}\lambda} + (N-1)\sqrt[3]{\frac{1}{2}\left(1+\lambda\right)\lambda}} \ln r_1\ .$$

Since the value of the denominator depends on $\lambda$, also the value of $P_{Th}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\lambda\mathbf{x^0})$ depends on $\lambda$.

**T12**[$P_B$, $P_S$, $P_{W(II)}$, $P_T$, $P_{Th}$, $P_{V(II)}$]: Since $P_L$ and $P_P$ are linear homogeneous, we get for positive $\lambda \neq 1$:

$$P_B(\mathbf{p^0},\mathbf{x^0},\lambda\mathbf{p^1},\mathbf{x^1}) = \lambda P_P \frac{\lambda P_L + 1}{\lambda P_P + 1} \neq \lambda P_P \frac{P_L + 1}{P_P + 1} = \lambda P_B(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1})\ .$$

Consider an inversion scenario $(\mathbf{p^0},\mathbf{x^0},\widetilde{\mathbf{p}}\mathbf{^0},\widetilde{\mathbf{x}}\mathbf{^0})$, such that $V^0 = V^1$, $V^{10} = V^{01}$, and $V^{10}/V^0 = 2$. Then, for $P_S$ we get: $Z/2 = \left(V^{10} - V^{01}\right)/V^0 = 0$, and hence, $P_S(\mathbf{p^0},\mathbf{x^0},\widetilde{\mathbf{p}}\mathbf{^0},\widetilde{\mathbf{x}}\mathbf{^0}) = 1$. For the scenario $(\mathbf{p^0},\mathbf{x^0},\lambda\widetilde{\mathbf{p}}\mathbf{^0},\widetilde{\mathbf{x}}\mathbf{^0})$, with $\lambda > 0$, we get

$$Z^*/2 = \left(\lambda V^{10} - V^{01}\right)/\left(2V^0\right) = \lambda - 1\ ,$$

and therefore,

$$P_S(\mathbf{p^0},\mathbf{x^0},\lambda\widetilde{\mathbf{p}}\mathbf{^0},\widetilde{\mathbf{x}}\mathbf{^0}) = (\lambda-1) + \sqrt{(\lambda-1)^2 + \lambda} \neq \lambda \cdot 1 = \lambda P_S(\mathbf{p^0},\mathbf{x^0},\widetilde{\mathbf{p}}\mathbf{^0},\widetilde{\mathbf{x}}\mathbf{^0})\ ,$$

unless $\lambda = 1$.

The indices $P_{W(II)}$, $P_T$, $P_{Th}$, and $P_{V(II)}$ can be expressed as $\ln P = \sum w_i \ln r_i$, where $\sum w_i = 1$. After multiplying the prices $p_i^1$ with $\lambda$ ($> 0$), in the price index formulas $P_{W(II)}$, $P_T$, and $P_{V(II)}$, the new weights $w_i^*$ are identical to the old weights $w_i$. Thus,

$$\begin{aligned}
\ln P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) &= \sum w_i^* \ln(\lambda r_i) \\
&= \sum w_i^* \ln \lambda + \sum w_i^* \ln r_i \\
&= \ln \lambda + \ln P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}).
\end{aligned}$$

In contrast, in $P_{Th}$ the weights depend on $\lambda$, and hence, $\sum w_i^* \ln r_i$ usually deviates from $\ln P_{Th}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$.

Since $P_B$, $P_S$, $P_{W(II)}$, $P_T$, $P_{Th}$, and $P_{V(II)}$ satisfy the price dimensionality test and the first part of the linear homogeneity test, we get

$$P(\lambda\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, (1/\lambda)\, \mathbf{p^1}, \mathbf{x^1}) = (1/\lambda)\, P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \, ,$$

which covers the second part of the linear homogeneity test.

**T3**[$P_B$, $P_S$, $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$, $P_{V(II)}$]: $P_B$ can be expressed as $P_B = (P_L + 1)/(1 + 1/P_P)$. For $P_S$ the proof is adopted from Olt (1996). $P_S$ can be rearranged such that

$$\begin{aligned}
\left(P_S - \frac{V^{10} - V^{01}}{2V^0}\right)^2 &= \left(\sqrt{\left(\frac{V^{10} - V^{01}}{2V^0}\right)^2 + \frac{V^1}{V^0}}\right)^2 \\
\Rightarrow \quad P_S^2 - P_S\left(V^{10} - V^{01}\right)/V^0 &= V^1/V^0 \\
\Rightarrow \; V^0 P_S - V^{10} + V^{01} - V^1/P_S &= 0 \, .
\end{aligned}$$

Suppose $p_i^1$ increases. The price $p_i^1$ appears in $V^1$, $V^{10}$, and $P_S$. In order to offset the effect on $V^1$ and $V^{10}$, $P_S$ must increase, too. Analogous reasoning applies to a change in $p_i^0$.

The price indices $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$ and $P_{V(II)}$ can be expressed as $\ln P = \sum w_i \ln r_i$. Suppose that $N = 2$, $p_1^0 = e^6$, $p_1^1 = 1$, $p_1^{1*} = e$, $x_1^0 = 1/e^5$, $x_1^1 = x_1^{1*} = e$, $p_2^0 = p_2^1 = p_2^{1*} = \sqrt{e}$ and $x_2^0 = x_2^1 = x_2^{1*} = \sqrt{e}$. This gives $\ln r_1 = -6$, $\ln r_1^* = \ln(p_1^{1*}/p_1^0) = -5$, $\ln r_2 = 0$, and therefore $\ln P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = -w_1 6$ and $\ln P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1*}}, \mathbf{x^1}) = -w_1^* 5$, where $w_1^*$ is the weight associated with the comparison period $1^*$. Strict monotonicity is violated, if $-w_1^* 5 < -w_1 6$, that is, if $w_1^*/w_1 > 6/5$. For $P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$ and $P_{V(II)}$ we get $w_1^* > 0.6$ and $w_1 = 0.5$.

**T4**[$P_{WV}$, $P_{V(I)}$]: The price indices $P_{WV}$ and $P_{V(I)}$ can be expressed as $\ln P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \sum w_i \ln r_i$. For all commodities $i$ with $p_i^1 = p_i^0$ we get $\ln r_i = 0$ and for all other commodities $\ln r_i > 0$. Since $w_i > 0$ ($i = 1, ..., N$), we get $\ln P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) > 0$. Since $P_{WV}$ and $P_{V(I)}$ satisfy the identity test, we get $\ln P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) = 0$.

**T16**[all indices]:

$$P_L : P_L(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \cdot P_L(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \left(V^{10}/V^0\right)\left(V^1/V^{01}\right) \neq 1$$

$$P_P : P_P(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \cdot P_P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \left(V^1/V^{01}\right)\left(V^0/V^{10}\right) \neq 1$$

$$\begin{aligned} P_{Dr} &: P_{Dr}(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \cdot P_{Dr}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \\ &= (1/2)\left[\left(V^{10}V^{01}\right)/\left(V^0V^1\right) + \left(V^1V^0\right)/\left(V^{01}V^{10}\right)\right] \neq 1 \end{aligned}$$

$$\begin{aligned} P_F &: P_F(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \cdot P_F(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \\ &= \sqrt{(V^{10}/V^0)\,(V^1/V^{01})} \cdot \sqrt{(V^{01}/V^1)\,(V^0/V^{10})} = 1 \\ P_{ME} &: P_{ME}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P_{ME}(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \\ &= \frac{V^{10}+V^1}{V^0+V^{01}} \cdot \frac{V^{01}+V^0}{V^1+V^{10}} = 1 \\ P_B &: P_B(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P_B(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \\ &= \frac{V^0/V^{10}+1}{1+V^{01}/V^1} \cdot \frac{V^{01}/V^1+1}{1+V^0/V^{10}} = 1 \\ P_{W(I)} &: P_{W(I)}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P_{W(I)}(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \\ &= \frac{\sum_{i\in A}\sqrt{v_i^{10}v_i^1}}{\sum_{i\in A}\sqrt{v_i^0 v_i^{01}}} \cdot \frac{\sum_{i\in A}\sqrt{v_i^{01}v_i^0}}{\sum_{i\in A}\sqrt{v_i^1 v_i^{10}}} = 1\ . \end{aligned}$$

We define $Z = \left(V^{10} - V^{01}\right)/V^0$ and $\widetilde{Z} = \left(V^{01} - V^{10}\right)/V^1$. This gives $\widetilde{Z} = -Z(V^0/V^1)$ and

$$\begin{aligned} &P_S(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P_S(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0}) \\ &= \left[\frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}}\right]\left[-\frac{V^0}{V^1}\frac{Z}{2} + \sqrt{\left(\frac{V^0}{V^1}\frac{Z}{2}\right)^2 + \frac{V^0}{V^1}}\right] \\ &= \left[\frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}}\right]\left[-\frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}}\right]\frac{V^0}{V^1} \\ &= \left[-\left(\frac{Z}{2}\right)^2 + \left(\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}\right]\frac{V^0}{V^1} = 1\ . \end{aligned}$$

$P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$, $P_{V(I)}$, and $P_{V(II)}$ can be expressed as $\ln P = \sum w_i \ln r_i$. Swapping the periods does not affect the weights $w_i$. Therefore,

$$\begin{aligned} &\ln\left[P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0})\right] \\ &= \sum w_i\left[\ln r_i + \ln\left(1/r_i\right)\right] = \sum w_i \ln 1 = 0\ . \end{aligned}$$

**T17**[$P_{W(II)}$, $P_{WV}$, $P_T$, $P_{Th}$]: Suppose that $N = 2$, $x_1^0 = x_2^0 = x_1^1 = x_2^1 = \sqrt{e}$, $p_1^0 = p_2^0 = p_2^1 = \sqrt{e}$ and $p_1^1 = 2\sqrt{e}$. In this case, $\ln P_{W(II)}$, $\ln P_{WV}$, $\ln P_T$ and $\ln P_{Th}$ generate values different from $\ln V^1/V^0 = \ln(3/2)$:

$$\ln P_{W(II)} = \left[\sqrt{2}\Big/\left(\sqrt{2}+1\right)\right]\ln 2\,, \qquad \ln P_{WV} = \sqrt{1/3}\ln 2\,,$$
$$\ln P_T = (7/12)\ln 2\,, \qquad \ln P_{Th} = \left[\sqrt[3]{3}\Big/\left(\sqrt[3]{3}+1\right)\right]\ln 2\,.$$

**T18**[$P_L$, $P_P$, $P_F$, $P_{Dr}$, $P_{ME}$, $P_B$, $P_S$, $P_{W(I)}$, $P_{V(I)}$, $P_{V(II)}$]:

$$\begin{aligned}
P_F \cdot Q_F &= \sqrt{P_L \cdot P_P \cdot Q_L \cdot Q_P} = \sqrt{\frac{V^{10}V^1V^{01}V^1}{V^0V^{01}V^0V^{10}}} = V^1/V^0 \\
P_S \cdot Q_S &= \left[\frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}}\right]\left[-\frac{Z}{2} + \sqrt{\left(-\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}}\right] \\
&= -(Z/2)^2 + (Z/2)^2 + V^1/V^0 = V^1/V^0\,.
\end{aligned}$$

The corresponding quantity indices of $P_L$, $P_P$, $P_{Dr}$, $P_{ME}$, and $P_B$ deviate from the respective complementary quantity indices:

$$\begin{aligned}
\overline{Q}_L &= \frac{V^1}{V^0}\frac{V^0}{V^{10}} = Q_P \\
\overline{Q}_P &= \frac{V^1}{V^0}\frac{V^{01}}{V^1} = Q_L \\
\overline{Q}_{Dr} &= \frac{V^1}{V^0}\Big/\left[\frac{1}{2}\left(\frac{V^{10}}{V^0} + \frac{V^1}{V^{01}}\right)\right] = \frac{2}{V^{10}/V^1 + V^0/V^{01}} = \frac{2}{1/Q_P + 1/Q_L} \\
\overline{Q}_{ME} &= \frac{V^1}{V^0}\cdot\frac{V^0+V^{01}}{V^{10}+V^1} = \frac{V^1}{V^{10}}\cdot\frac{1+V^{01}/V^0}{1+V^1/V^{10}} = Q_P\frac{1+Q_L}{1+Q_P} = Q_B \\
\overline{Q}_B &= Q_{ME}\,.
\end{aligned}$$

Defining $s_i = x_i^1/x_i^0$, $P_{W(I)}$ gives

$$\begin{aligned}
\overline{Q}_{W(I)} &= \frac{V^1}{V^0}\frac{\sum\sqrt{v_i^0 v_i^{01}}}{\sum\sqrt{v_i^{10}v_i^1}} = \frac{V^1}{V^0}\frac{\sum v_i^0\sqrt{s_i}}{\sum v_i^1\sqrt{1/s_i}} = \frac{\sum\left(v_i^0/V^0\right)\sqrt{s_i}}{\sum\left(v_i^1/V^1\right)\sqrt{1/s_i}} \\
&\neq \frac{\sum\sqrt{v_i^0v_i^1}\sqrt{s_i}}{\sum\sqrt{v_i^0v_i^1}\sqrt{1/s_i}} = \frac{\sum\sqrt{v_i^{01}v_i^{10}}\sqrt{s_i}}{\sum\sqrt{v_i^{01}v_i^{10}}\sqrt{1/s_i}} = \frac{\sum\sqrt{v_i^{01}v_i^1}}{\sum\sqrt{v_i^0v_i^{10}}} = Q_{W(I)}\,. \qquad (2)
\end{aligned}$$

Swapping prices and quantities, does not alter the weights $w_i$ in $P_{V(I)}$

and $P_{V(II)}$. Therefore, we get

$$\begin{aligned}
\ln\left(P_{V(I)}\cdot Q_{V(I)}\right) &= \sum w_i \ln\left(r_i s_i\right) \\
&= \frac{\ln V^1 - \ln V^0}{V^1 - V^0}\sum \frac{v_i^1 - v_i^0}{\ln v_i^1 - \ln v_i^0}\ln\frac{v_i^1}{v_i^0} \\
&= \frac{\ln V^1 - \ln V^0}{V^1 - V^0}\left(V^1 - V^0\right) = \ln\left(V^1/V^0\right) \ .
\end{aligned}$$

For $P_{V(II)}$ we know that $\sum w_i = 1$, and hence,

$$\begin{aligned}
&\ln\left(P_{V(II)}\cdot Q_{V(II)}\right) \\
&= \sum w_i \ln\left(r_i s_i\right) \\
&= \sum w_i \ln\left(\frac{v_i^1/V^1}{v_i^0/V^0}\cdot\frac{V^1}{V^0}\right) \\
&= \sum w_i \left[\ln\left(v_i^1/V^1\right) - \ln\left(v_i^0/V^0\right)\right] + \sum w_i \ln\left(V^1/V^0\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum \frac{v_i^1/V^1 - v_i^0/V^0}{\sum \frac{v_j^1/V^1 - v_j^0/V^0}{\ln\left(v_j^1/V^1\right) - \ln\left(v_j^0/V^0\right)}} + \ln\left(V^1/V^0\right) \\
&= \frac{1}{\sum \frac{v_j^1/V^1 - v_j^0/V^0}{\ln\left(v_j^1/V^1\right) - \ln\left(v_j^0/V^0\right)}} \underbrace{\left(\sum v_i^1/V^1 - \sum v_i^0/V^0\right)}_{=0} + \ln\left(V^1/V^0\right) \\
&= \ln\left(V^1/V^0\right) \ .
\end{aligned}$$

**T19**[$P_L$, $P_P$, $P_{Dr}$, $P_{ME}$, $P_B$, $P_{W(I)}$, $P_{V(I)}$, $P_{V(II)}$]: We know that $P_P$, $P_L$, $P_{ME}$, and $P_B$ – and using the proof of T18, also $\overline{Q}_L\left(= Q_P\right)$, $\overline{Q}_P\left(= Q_L\right)$, $\overline{Q}_{Dr}\left(= 2/\left(1/Q_P + 1/Q_L\right)\right)$, $\overline{Q}_{ME}\left(= Q_B\right)$, $\overline{Q}_B\left(= Q_{ME}\right)$, and $\overline{Q}_{W(I)}$ – satisfy all tests required by the product test (proportionality, symmetry, commensurability, price dimensionality, strict monotonicity).

The formulas $P_{V(I)}$ and $P_{V(II)}$ satisfy the factor reversal test, but violate strict monotonicity. Hence, $\overline{Q}_{V(I)}$ and $\overline{Q}_{V(II)}$ violate strict monotonicity.

**T20**[all indices]: (Auer, 2003).

**Lemma 1** *Consider a price index formula $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$ that satisfies the commensurability test and can be expressed in the form of equation (**??**). This price index formula passes the weak consistency test if and only if its function $w(\cdot)$ is such that*

$$w\left(v_i^a, v_i^b, \mathbf{v_{-i}^a}, \mathbf{v_{-i}^b}\right) = w\left(v_i^a, v_i^b, \mathbf{v_{k,-i}^a}, \mathbf{v_{k,-i}^b}\right)\cdot w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right) \ . \quad (3)$$

*Proof:* By Definition, $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$ passes the weak consistency test, if and only if

$$f\left(P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})\right) = f\left(P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})\right) \ , \tag{4}$$

where all elements $P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})$ in $\mathbf{P^I}$ represent the same function as $P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})$ and $P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})$. Commensurability ensures that $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})$. Since $f\left(P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})\right)$ can be expressed as in (**??**), $f\left(P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})\right)$ and $f\left(P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})\right)$ can be expressed by

$$f\left(P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})\right) = \sum_{i \in A_k} w\left(v_i^a, v_i^b, \mathbf{v_{k,-i}^a}, \mathbf{v_{k,-i}^b}\right) f\left(r_i\right) \qquad \text{for } k = 1, 2, ..., K \tag{5}$$

$$f\left(P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})\right) = \sum_{k=1}^{K} w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right) f\left(P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})\right) \ . \tag{6}$$

*Sufficiency Part:* For all price index formulas of the form (**??**), it is true that

$$\begin{aligned}
f\left(P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})\right) &= \sum_{k=1}^{K} \sum_{i \in A_k} w\left(v_i^a, v_i^b, \mathbf{v_{-i}^a}, \mathbf{v_{-i}^b}\right) f\left(r_i\right) && (7)\\
&= \sum_{k=1}^{K} w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right) \sum_{i \in A_k} \frac{w\left(v_i^a, v_i^b, \mathbf{v_{-i}^a}, \mathbf{v_{-i}^b}\right)}{w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right)} f\left(r_i\right) \\
\text{[from (3)]} \qquad &= \sum_{k=1}^{K} w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right) \sum_{i \in A_k} w\left(v_i^a, v_i^b, \mathbf{v_{k,-i}^a}, \mathbf{v_{k,-i}^b}\right) f\left(r_i\right) \\
\text{[from (5)]} \qquad &= \sum_{k=1}^{K} w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right) f\left(P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})\right) \\
\text{[from (6)]} \qquad &= f\left(P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})\right) \ .
\end{aligned}$$

Thus, relationship (4) is satisfied. In the derivation of this result, equation (5) was used which ensures that the applied formula $P_k^I(\mathbf{r_k}, \mathbf{v_k^a}, \mathbf{v_k^b})$ is the same as the formulas $P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})$ and $P(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b})$.

*Necessity Part:* Inserting (5) in (6) gives

$$f\left(P^{II}(\mathbf{P^I}, \mathbf{V^a}, \mathbf{V^b})\right) = \sum_{k=1}^{K} \sum_{i \in A_k} w\left(v_i^a, v_i^b, \mathbf{v_{k,-i}^a}, \mathbf{v_{k,-i}^b}\right) w\left(V_k^a, V_k^b, \mathbf{V_{-k}^a}, \mathbf{V_{-k}^b}\right) f\left(r_i\right) \ .$$

In view of (7), for relationship (4) to be true for all $\left(\mathbf{r}, \mathbf{v^a}, \mathbf{v^b}\right)$, it is necessary that (3) is satisfied. q.e.d.

Equipped with this lemma, it is easy to check the consistency properties of the listed price index formulas.

**Theorem 1** *The Laspeyres index $P_L$ and the Paasche index $P_P$ pass the super-consistency test, and hence, also the weak and the strict consistency test.*

*Proof:* Lemma 1 applies. Furthermore, the price index formulas $P_L(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$, $P_L(\mathbf{r},\mathbf{v^0},\mathbf{v^{01}})$, $P_L(\mathbf{r},\mathbf{v^1},\mathbf{v^{10}})$, $P_P(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$, $P_P(\mathbf{r},\mathbf{v^0},\mathbf{v^{01}})$, and $P_P(\mathbf{r},\mathbf{v^1},\mathbf{v^{10}})$ can be stated in the general form (**??**), where their respective functions $w(\cdot)$ and $f(r_i)$ are listed in Table 2. From the table it is easy to see that for all $(a,b)$–variants the function $w(\cdot)$ satisfies condition (3). Hence, by Lemma 2, $P_L$ and $P_P$ pass the super–consistency test. q.e.d.

Table 2: The functions $w(.)$ and $f(r_i)$ associated with the three variants of $P_L$ and $P_P$.

| | $P_L(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$ $P_L(\mathbf{r},\mathbf{v^0},\mathbf{v^{01}})$ | $P_L(\mathbf{r},\mathbf{v^1},\mathbf{v^{10}})$ | $P_P(\mathbf{r},\mathbf{v^0},\mathbf{v^{01}})$ | $P_P(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$ $P_P(\mathbf{r},\mathbf{v^1},\mathbf{v^{10}})$ |
|---|---|---|---|---|
| $w\left(v_i^a, v_i^b, \mathbf{v^a_{-i}}, \mathbf{v^b_{-i}}\right)$ | $v_i^0/V^0$ | $v_i^{10}/V^{10}$ | $v_i^{01}/V^{01}$ | $v_i^1/V^1$ |
| $w\left(v_i^a, v_i^b, \mathbf{v^a_{k,-i}}, \mathbf{v^b_{k,-i}}\right)$ | $v_i^0/V_k^0$ | $v_i^{10}/V_k^{10}$ | $v_i^{01}/V_k^{01}$ | $v_i^1/V_k^1$ |
| $w\left(V_k^a, V_k^b, \mathbf{V^a_{-k}}, \mathbf{V^b_{-k}}\right)$ | $V_k^0/V^0$ | $V_k^{10}/V^{10}$ | $V_k^{01}/V^{01}$ | $V_k^1/V^1$ |
| $f(r_i)$ | $r_i$ | $1/r_i$ | $r_i$ | $1/r_i$ |

Theorem 1 reveals that for multi stage index computations based on the Laspeyres or Paasche index formulas, any of the possible $(a,b)$-combinations can be used. As shown in the following theorem, some price index formulas exist which require that variant $(a,b)=(0,1)$ is applied.

**Theorem 2** *The Walsh-Vartia index $P_{WV}$, the Vartia(I) index $P_{V(I)}$, and the Stuvel index $P_S$ pass the strict, and hence, also the weak consistency test.*

*Proof:* Lemma 1 applies. The price index formulas $P_{WV}(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$ and $P_{V(I)}(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$ can be stated in the general form (**??**), where their respective functions $w(\cdot)$ and $f(r_i)$ are listed in Table 3. From the table it is easy to see that for $(a,b)=(0,1)$ the function $w(\cdot)$ satisfies condition (3). Hence, by Lemma 2 and Definition **??**, $P_{WV}$ and $P_{V(I)}$ pass the strict consistency test.

Table 3: The functions $w(.)$ and $f(r_i)$ associated with $P_{WV}$ and $P_{V(I)}$.

| | $P_{WV}\left(\mathbf{r},\mathbf{v^0},\mathbf{v^1}\right)$ | $P_{V(I)}\left(\mathbf{r},\mathbf{v^0},\mathbf{v^1}\right)$ |
|---|---|---|
| $w\left(v_i^0,v_i^1,\mathbf{v^0_{-i}},\mathbf{v^1_{-i}}\right)$ | $\dfrac{\sqrt{v_i^0 v_i^1}}{\sqrt{V^0 V^1}}$ | $\dfrac{\left(v_i^1\text{-}v_i^0\right)/\left(\ln v_i^1\text{-}\ln v_i^0\right)}{\left(V^1\text{-}V^0\right)/\left(\ln V^1\text{-}\ln V^0\right)}$ |
| $w\left(v_i^a,v_i^b,\mathbf{v^a_{k,-i}},\mathbf{v^b_{k,-i}}\right)$ | $\dfrac{\sqrt{v_i^0 v_i^1}}{\sqrt{V_k^0 V_k^1}}$ | $\dfrac{\left(v_i^1\text{-}v_i^0\right)/\left(\ln v_i^1\text{-}\ln v_i^0\right)}{\left(V_k^1\text{-}V_k^0\right)/\left(\ln V_k^1\text{-}\ln V_k^0\right)}$ |
| $w\left(V_k^a,V_k^b,\mathbf{V^a_{-k}},\mathbf{V^b_{-k}}\right)$ | $\dfrac{\sqrt{V_k^0 V_k^1}}{\sqrt{V^0 V^1}}$ | $\dfrac{\left(V_k^1\text{-}V_k^0\right)/\left(\ln V_k^1\text{-}\ln V_k^0\right)}{\left(V^1\text{-}V^0\right)/\left(\ln V^1\text{-}\ln V^0\right)}$ |
| $f(r_i)$ | $\ln r_i$ | $\ln r_i$ |

$\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$ can be rearranged such that

$$\left(\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})-\frac{V^{10}-V^{01}}{2V^0}\right)^2=\left(\sqrt{\left(\frac{V^{10}-V^{01}}{2V^0}\right)^2+\frac{V^1}{V^0}}\right)^2$$

$$\Rightarrow\quad\left[\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})\right]^2-\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})\left(V^{10}-V^{01}\right)/V^0=V^1/V^0$$

$$\Rightarrow\quad V^0\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})-V^1/\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})-V^{10}+V^{01}=0$$

$$\Rightarrow\quad V^0\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})-V^1/\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})-\sum_{i\in A}\left(v_i^0 r_i-v_i^1/r_i\right)=0\,. \qquad (8)$$

Equation (8) implicitly defines the Stuvel index formula $\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$. Analogously, the corresponding first stage index formula $\overline{P}^I_{S,k}(\mathbf{r_k},\mathbf{v^0_k},\mathbf{v^1_k})$ is implicitly defined by

$$V_k^0\overline{P}^I_{S,k}(\mathbf{r_k},\mathbf{v^0_k},\mathbf{v^1_k})-V_k^1/\overline{P}^I_{S,k}(\mathbf{r_k},\mathbf{v^0_k},\mathbf{v^1_k})-\sum_{i\in A_k}\left(v_i^0 r_i-v_i^1/r_i\right)=0,\quad\text{for all } k\,, \qquad (9)$$

and the corresponding second stage index formula $\overline{P}^{II}_S(\mathbf{\overline{P}^I_{S,k}},\mathbf{V^0},\mathbf{V^1})$, with $\mathbf{\overline{P}^I_{S,k}}=(\overline{P}^I_{S,1},\overline{P}^I_{S,2},...,\overline{P}^I_{S,K})$, is implicitly defined by

$$V^0\bar{P}^{II}_S(\mathbf{\bar{P}^I_{S,k}},\mathbf{V^0},\mathbf{V^1})-V^1/\bar{P}^{II}_S(\mathbf{\bar{P}^I_{S,k}},\mathbf{V^0},\mathbf{V^1})-\sum_{k=1}^{K}\left(V_k^0\bar{P}^I_{S,k}(\mathbf{r_k},\mathbf{v^0_k},\mathbf{v^1_k})-V_k^1/\bar{P}^I_{S,k}(\mathbf{r_k},\mathbf{v^0_k},\mathbf{v^1_k})\right)=0\,. \qquad (10)$$

Since the implicit formulas (8) to (10) are identical, also their explicit counterparts $\overline{P}_S(\mathbf{r},\mathbf{v^0},\mathbf{v^1})$, $\overline{P}^I_{S,k}(\mathbf{r_k},\mathbf{v^0_k},\mathbf{v^1_k})$, and $\overline{P}^{II}_S(\mathbf{\overline{P}^I_{S,k}},\mathbf{V^0},\mathbf{V^1})$ are identical.

It remains to be shown that $\overline{P}_S(\mathbf{r}, \mathbf{v^0}, \mathbf{v^1}) = \overline{P}_S^{II}(\overline{\mathbf{P}}^{\mathbf{I}}_{\mathbf{S,k}}, \mathbf{V^0}, \mathbf{V^1})$. Equation (9) gives

$$\sum_{k=1}^{K} \left( V_k^0 \overline{P}_{S,k}^I(\mathbf{r_k}, \mathbf{v_k^0}, \mathbf{v_k^1}) - V_k^1 / \overline{P}_{S,k}^I(\mathbf{r_k}, \mathbf{v_k^0}, \mathbf{v_k^1}) \right) = \sum_{i \in A} \left( v_i^0 r_i - v_i^1 / r_i \right) \ . \quad (11)$$

Inserting expression (11) in (8) gives

$$V^0 \bar{P}_S(\mathbf{r}, \mathbf{v^0}, \mathbf{v^1}) - V^1 / \bar{P}_S(\mathbf{r}, \mathbf{v^0}, \mathbf{v^1}) - \sum_{k=1}^{K} \left( V_k^0 \bar{P}_{S,k}^I(\mathbf{r_k}, \mathbf{v_k^0}, \mathbf{v_k^1}) - V_k^1 / \bar{P}_{S,k}^I(\mathbf{r_k}, \mathbf{v_k^0}, \mathbf{v_k^1}) \right) = 0. \quad (12)$$

The simultaneous satisfaction of equations (10) and (12) implies that $\overline{P}_S(\mathbf{r}, \mathbf{v^0}, \mathbf{v^1}) = \overline{P}_S^{II}(\overline{\mathbf{P}}^{\mathbf{I}}_{\mathbf{S,k}}, \mathbf{V^0}, \mathbf{V^1})$. Thus, $P_S$ passes the strict consistency test. q.e.d.

For the Marshall–Edgeworth index, the variant $(a, b) = (0, 1)$ would generate inconsistent single stage and two stage index numbers. However, as is shown in the following finding, for the variants $(a, b) = (0, 01)$ and $(a, b) = (1, 10)$, this index formula generates consistent results.

**Theorem 3** *The Marshall–Edgeworth index $P_{ME}$ passes the weak consistency test.*

*Proof:* Lemma 1 applies. The price index formulas $P_{ME}(\mathbf{r}, \mathbf{v^0}, \mathbf{v^{01}})$ and $P_{ME}(\mathbf{r}, \mathbf{v^1}, \mathbf{v^{10}})$ can be restated in the general form of equation (**??**), where their respective functions $w(\cdot)$ and $f(r_i)$ are listed in Table 4 which reveals that condition (3) is satisfied. Hence, by Lemma 2, $P_{ME}$ passes the weak consistency test. q.e.d.

Table 4: The functions $w(.)$ and $f(r_i)$ associated with two variants of $P_{ME}$.

| | $P_{ME}(\mathbf{r}, \mathbf{v^0}, \mathbf{v^{01}})$ | $P_{ME}(\mathbf{r}, \mathbf{v^1}, \mathbf{v^{10}})$ |
|---|---|---|
| $w\left(v_i^0, v_i^1, \mathbf{v^0_{-i}}, \mathbf{v^1_{-i}}\right)$ | $(v_i^0 + v_i^{01}) / (V^0 + V^{01})$ | $(v_i^{10} + v_i^1) / (V^{10} + V^1)$ |
| $w\left(v_i^a, v_i^b, \mathbf{v^a_{k,-i}}, \mathbf{v^b_{k,-i}}\right)$ | $(v_i^0 + v_i^{01}) / (V_k^0 + V_k^{01})$ | $(v_i^{10} + v_i^1) / (V_k^{10} + V_k^1)$ |
| $w\left(V_k^a, V_k^b, \mathbf{V^a_{-k}}, \mathbf{V^b_{-k}}\right)$ | $(V_k^0 + V_k^{01}) / (V^0 + V^{01})$ | $(V_k^{10} + V_k^1) / (V^{10} + V^1)$ |
| $f(r_i)$ | $r_i$ | $1/r_i$ |

It remains to be examined whether Theorems 1 to 3 are exhaustive in the sense that for none of the price index formulas considered in this paper, a weak, strict, or super-consistency property can be established that was not stated in Theorems 1 to 3.

**Theorem 4** *For the listed price index formulas, the consistency properties listed in Theorems 1, 2, and 3 are exhaustive.*

*Proof:* It is easy to check that the functions $w(\cdot)$ associated with the price index formulas $P_{W(I)}$, $P_{W(II)}$, $P_T$, $P_{Th}$, and $P_{V(II)}$ violate condition (3). Hence, by Lemma 2, these price index formulas fail the weak consistency test. Also the price index formulas $P_F$, $P_{Dr}$, and $P_B$ violate the weak consistency test. The simplest way of proving this, is by presenting a numerical counter-example. This is provided in Table 2 of Illustration **??**.

The function $w(\cdot)$ associated with $P_{ME}(\mathbf{r}, \mathbf{v^0}, \mathbf{v^1})$ violates condition (3). Therefore, by Lemma 2 and Definition **??**, the price index formula $P_{ME}$ fails the strict consistency test.

Analogous reasoning applies to the price index formulas $P_{WV}(\mathbf{r}, \mathbf{v^0}, \mathbf{v^{01}})$ and $P_{V(I)}(\mathbf{r}, \mathbf{v^0}, \mathbf{v^{01}})$. Hence, $P_{WV}$ and $P_{V(I)}$ fail the super–consistency test. The counter–example presented in Table 2 shows that also $P_S$ fails this test. q.e.d.

## References

**Auer, L. von** (2000a), *Consistency in Aggregation*, University of Magdeburg, FEMM Working Paper Series, 1/2000, Magdeburg.

**Auer, L. von** (2000b), *Inversion Tests for Price and Quantitiy Indices*, University of Magdeburg, FEMM Working Paper Series, 2/2000, Magdeburg.

**Balk, B. M.** (1995), "Axiomatic Price Index Theory: A Survey," *International Statistical Review*, 63(1), 69-93.

**Balk, B. M.** (1996), "Consistency-in-Aggregation and Stuvel Indices," *Review of Income and Wealth*, 42(3), 353-63.

**Diewert, W. E.** (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited," *Journal of Productivity Analysis*, 3, 211-48.

**Eichhorn, W., J. Voeller** (1976), *Theory of the Price Index*, Lecture Notes in Economics and Mathematical Systems, 140, Berlin: Springer.

**Eichhorn, W., J. Voeller** (1990), "Axiomatic Foundation of Price Indexes and Purchasing Power Parities," in W. E. Diewert (Hrsg.), *Price Level Measurement*, Amsterdam: North-Holland, 321-58.

**Fisher, I.** (1911), *The Purchasing Power of Money*, London: Macmillan, 2nd ed. revised 1913, reprinted 1997 by Pickering and Chatto, London.

**Fisher, I.** (1922), *The Making of Index Numbers*, Boston: Houghton Mifflin, 3rd ed. revised 1927, reprinted 1997 by Pickering and Chatto, London.

**Frisch, R.** (1930), "Necessary and Sufficient Conditions Regarding the Form of an Index Number which Shall Meet Certain of Fisher's Tests," *Journal of the American Statistical Association*, 25, 397-406.

**Krtscha, M.** (1979), *Über Preisindices und deren axiomatische Charakterisierung*, Dissertation, Universität Karlsruhe.

**Laspeyres, E.** (1871), "Die Berechnung einer mittleren Warenpreissteigerung," *Jahrbücher für Nationalökonomie und Statistik*, 16, 296-314.

**Olt, B.** (1996), *Axiom und Struktur in der statistischen Preisindextheorie*, Frankfurt: Peter Lang.

**Pierson, N. G.** (1896), "Further Considerations on Index Numbers," *Economic Journal*, 6, 127-131.

**Vartia, Y. O.** (1976a), *Relative Changes and Index Numbers*, Research Institute of the Finnish Economy, Helsinki.

**Vartia, Y. O.** (1976b), "Ideal Log-Change Index Numbers," *Scandinavian Journal of Statistics*, 3, 121-26.

**Vogt, A.** (1979), *Das Statistische Index Problem im Zwei-Situationen-Fall*, Dissertation No. 6448, ETH Zürich.

**Vogt, A.** (1980), "Der Zeit– und der Faktorumkehrtest als 'Finders of Tests'," *Statistische Hefte*, 21, 66-71.

**Vogt, A., Barta J.** (1997), *The Making of Tests for Index Numbers*, Heidelberg: Physica Verlag.

**Walsh, C. M.** (1901), *The Measurement of General Exchange–Value*, New York: Macmillan.

**Westergaard, H.** (1890), *Die Grundzüge der Theorie der Statistik*, Jena: Fischer.