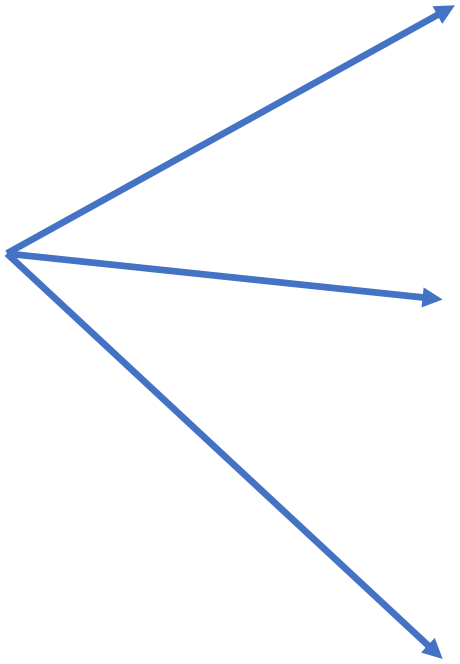
More Complex ANOVA & Regression

Institute for Advanced Analytics
MSA Class of 2021

Two-Way ANOVA

n-Way ANOVA

Continuous Target Variable



One-Way ANOVA
- 1 variable
- k categories

Two-Way ANOVA
- 2 variables
- k_1 and k_2 categories

⋮

***n*-Way ANOVA**
- n variables
- $k_1, k_2, \ldots, k_n$ categories

Additional Linear Models Terminology

- **Model** – a mathematical relationship between explanatory variables and response variables
- **Effect** – the expected change in the response that occurs with a change in the value of an explanatory variable
  - **Main Effect** – the effect of a single explanatory variable (for example, x_1, x_2, x_3)

Exploring the Data

- Similar to One-Way ANOVA, we need to explore variables to add to our generalized ANOVA model.
- Previously looked at Heating Quality.
- Want to also look at Central Air availability.

Exploring the Data

```
train %>%
  group_by(Heating_QC, Central_Air) %>%
  summarise(mean = mean(Sale_Price),
            sd = sd(Sale_Price),
            max = max(Sale_Price),
            min = min(Sale_Price))
```

Exploring the Data

```
## # A tibble: 10 x 6
## # Groups:   Heating_QC [5]
##    Heating_QC Central_Air    mean     sd    max    min
##    <ord>      <fct>         <dbl>  <dbl>  <int>  <int>
##  1 Poor       N             50050  52255.  87000  13100
##  2 Poor       Y            107000     NA  107000 107000
##  3 Fair       N             84748. 28267. 158000  37900
##  4 Fair       Y            145165. 38624. 230000  50000
##  5 Typical    N            103469. 34663. 209500  12789
##  6 Typical    Y            142003. 39657. 375000  60000
##  7 Good       N            110811. 38455. 214500  59000
##  8 Good       Y            160113. 54158. 415000  52000
##  9 Excellent  N            115062. 33271. 184900  64000
## 10 Excellent  Y            216401. 88518. 745000  58500
```

Exploring the Data

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## # A tibble: 10 x 6
## # Groups:   Heating_QC [5]
##    Heating_QC Central_Air    mean      sd    max    min
##    <ord>      <fct>          <dbl>  <dbl>  <int>  <int>
##  1 Poor       N              50050  52255.  87000  13100
##  2 Poor       Y             107000     NA  107000 107000
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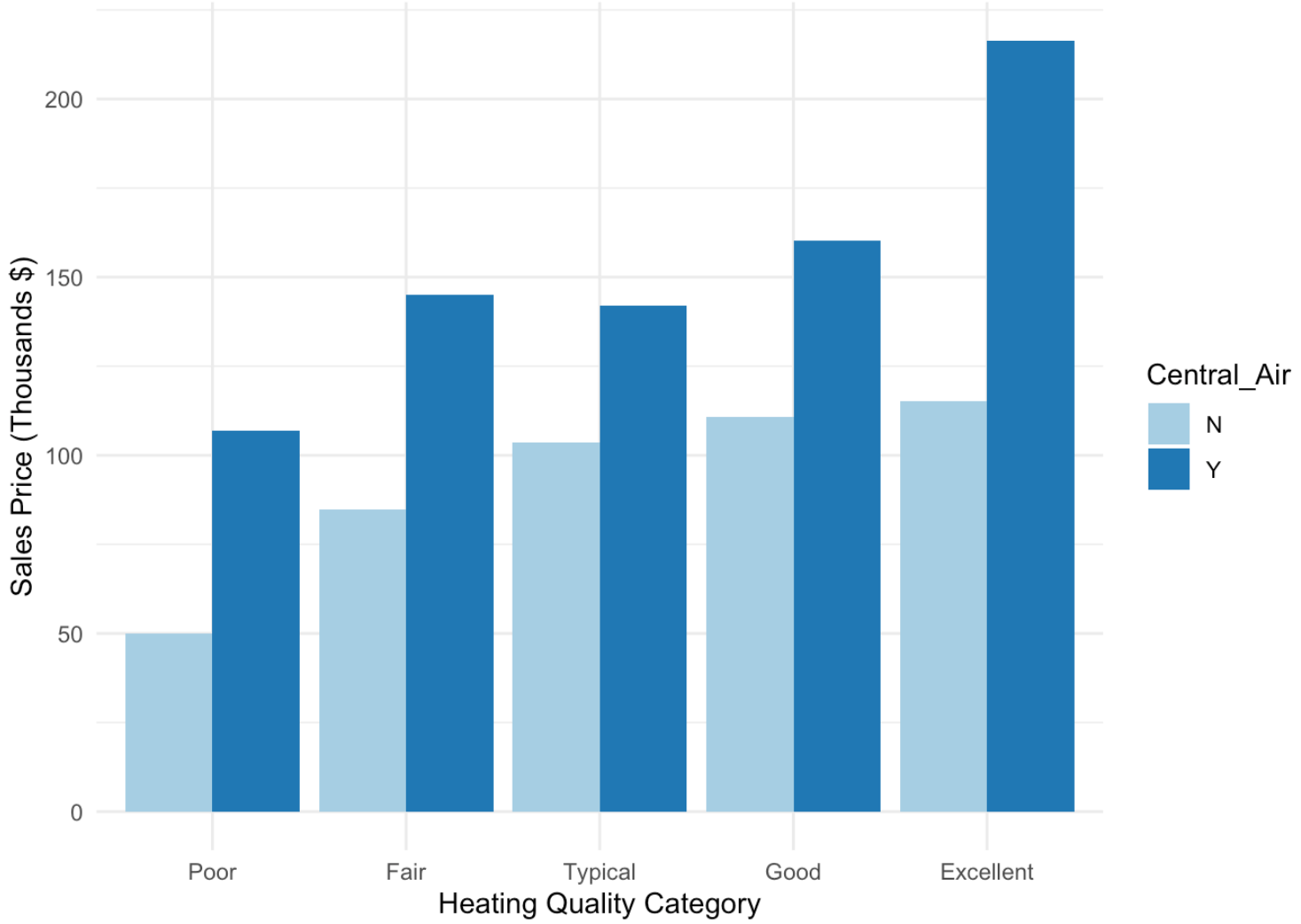
```

Exploring the Data

- Appears to have differences between heating quality levels as well as presence of central air.
- Need statistical proof.



Two-Way ANOVA

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$$

Sales Price	Base Level	Heating Quality	Central Air	Unexplained Error

Two-Way ANOVA

```r
ames_aov2 <- aov(Sale_Price ~ Heating_QC + Central_Air, data = train)
summary(ames_aov2)
```

```
##                Df    Sum Sq   Mean Sq F value   Pr(>F)
## Heating_QC      4 2.891e+12 7.228e+11  147.60  < 2e-16 ***
## Central_Air     1 2.903e+11 2.903e+11   59.28 2.11e-14 ***
## Residuals    2045 1.002e+13 4.897e+09
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Two-Way ANOVA

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```

Each variable has F test

Two-Way ANOVA

```
ames_aov2 <- aov(Sale_Price ~ Heating_QC + Central_Air, data = train)
summary(ames_aov2)
```

5-1 = 4 for Heating Quality

2-1 = 1 for Central Air

```
##                Df    Sum Sq   Mean Sq F value   Pr(>F)
## Heating_QC      4 2.891e+12 7.228e+11  147.60  < 2e-16 ***
## Central_Air     1 2.903e+11 2.903e+11   59.28 2.11e-14 ***
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```

Post-Hoc Testing

- Similar to One-Way ANOVA, if we have statistical differences among the categories, we want to know where these statistical differences exist.
- Use same approaches as before.

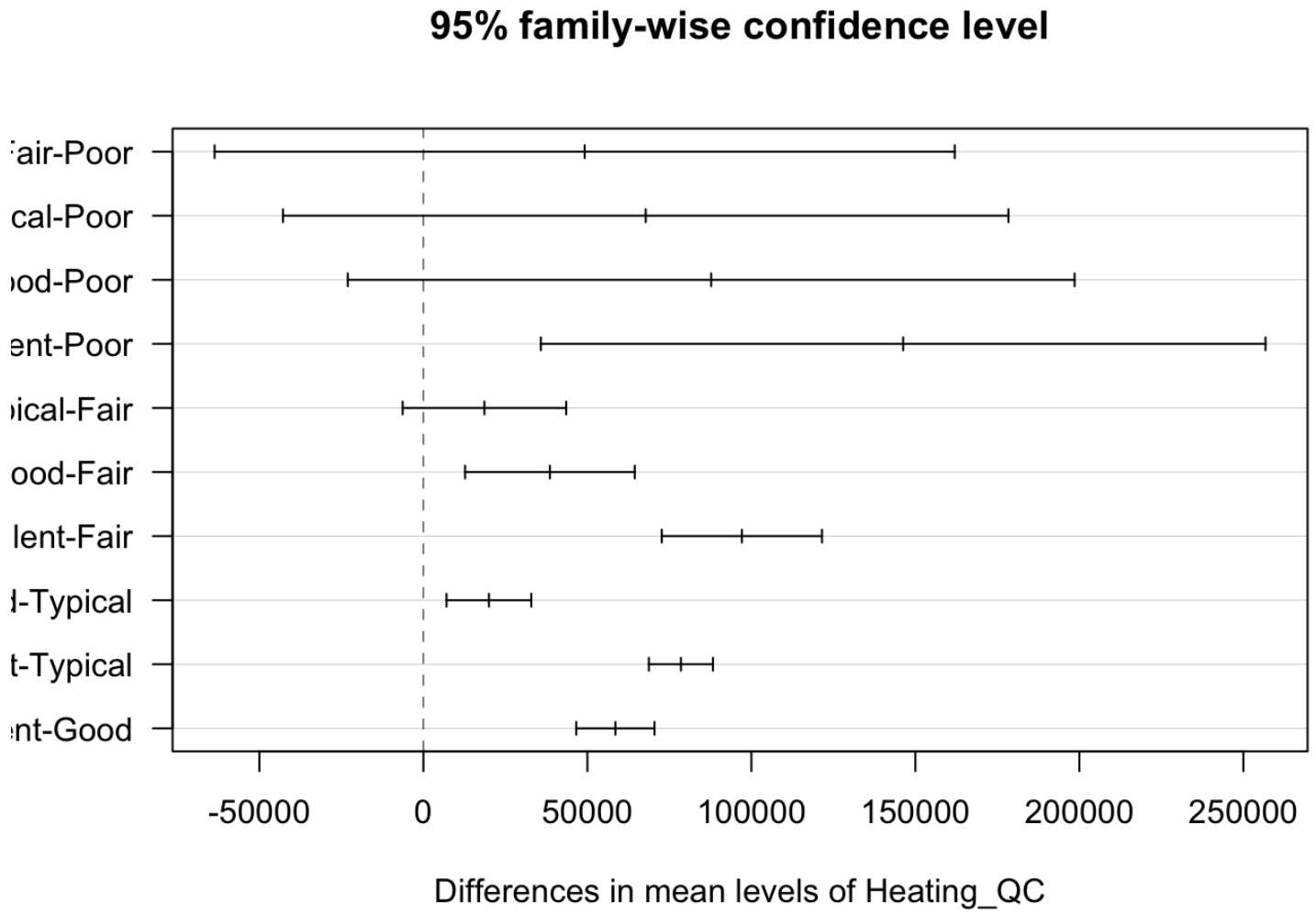
Post-Hoc Testing

```
tukey.ames2 <- TukeyHSD(ames_aov2)
print(tukey.ames2)
plot(tukey.ames2, las = 1)
```

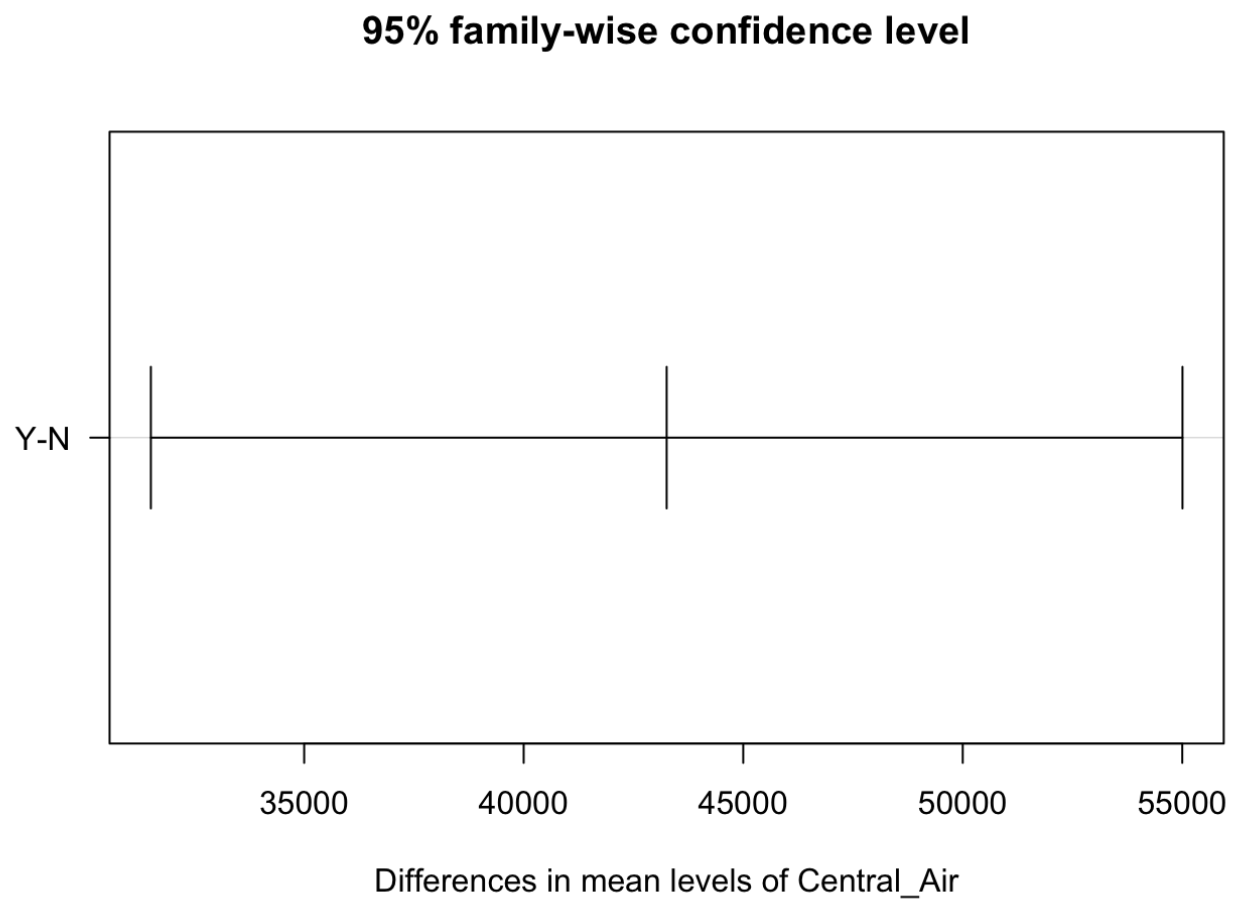
Post-Hoc Testing

```
## $Heating_QC
##                        diff         lwr       upr     p adj
## Fair-Poor           49176.42 -63650.448 162003.29 0.7571980
## Typical-Poor        67781.01 -42800.320 178362.35 0.4506761
## Good-Poor           87753.89 -23040.253 198548.03 0.1945181
## Excellent-Poor     146288.89  35818.859 256758.92 0.0028361
## Typical-Fair        18604.59  -6326.425  43535.61 0.2484556
## Good-Fair           38577.47  12718.894  64436.04 0.0004622
## Excellent-Fair      97112.47  72679.867 121545.07 0.0000000
## Good-Typical        19972.87   7050.230  32895.52 0.0002470
## Excellent-Typical   78507.88  68746.678  88269.07 0.0000000
## Excellent-Good      58535.00  46602.229  70467.78 0.0000000
##
## $Central_Air
##         diff      lwr      upr p adj
## Y-N 43256.57 31508.27 55004.87     0
```

Post-Hoc Testing



Post-Hoc Testing



Two-Way ANOVA with Interactions

Additional Linear Models Terminology

- **Model** – a mathematical relationship between explanatory variables and response variables
- **Effect** – the expected change in the response that occurs with a change in the value of an explanatory variable
  - **Main Effect** – the effect of a single explanatory variable (for example, x_1, x_2, x_3)
  - **Interaction Effect** – the effect of one variable changes as levels of another variable changes (for example, $x_1 \times x_2, x_1 \times x_2 \times x_3$)

Two-Way ANOVA with Interactions

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

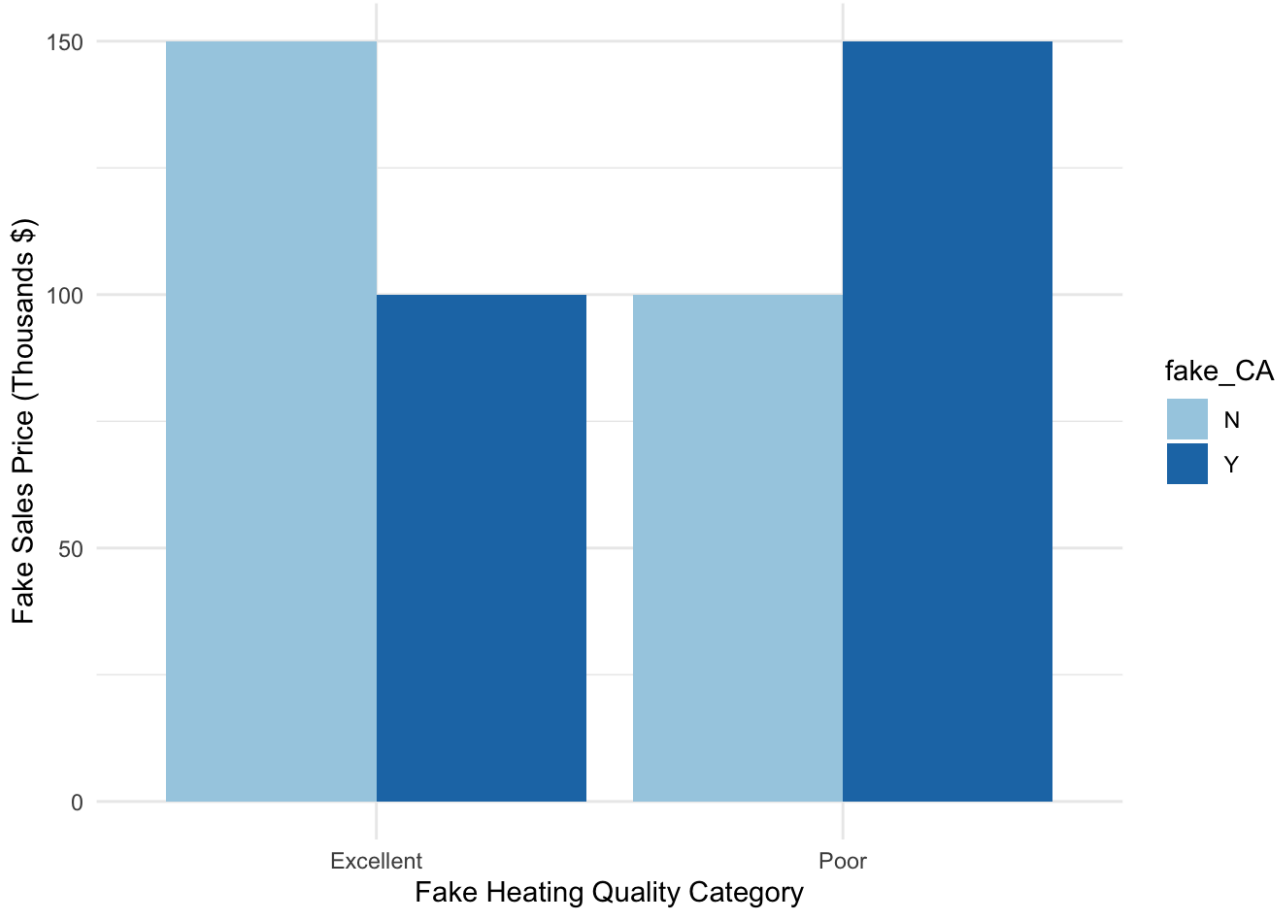
Y_{ijk}	μ	α_i	β_j	$(\alpha\beta)_{ij}$	ε_{ijk}
Sales Price	Base Level	Heating Quality	Central Air	Heating Quality & Central Air	Unexplained Error

Interactions

- For our model, an interaction between Central Air and Heating Quality would imply both of the following:
  - The impact of Heating Quality on Sale Price differs across levels of Central Air (ex. Difference in price between Excellent and Poor HQ changes if Central Air is or is **not** present)
  - The impact of Central Air on Sale Price differs across levels of Heating Quality (ex. Difference in price between having and not having Central Air changes across levels of HQ)

Interactions – Caution

- Interactions can potentially **mask** the effects of the variables.



Interactions

```
ames_aov_int <- aov(Sale_Price ~ Heating_QC*Central_Air, data = train)
summary(ames_aov_int)
```

Interactions

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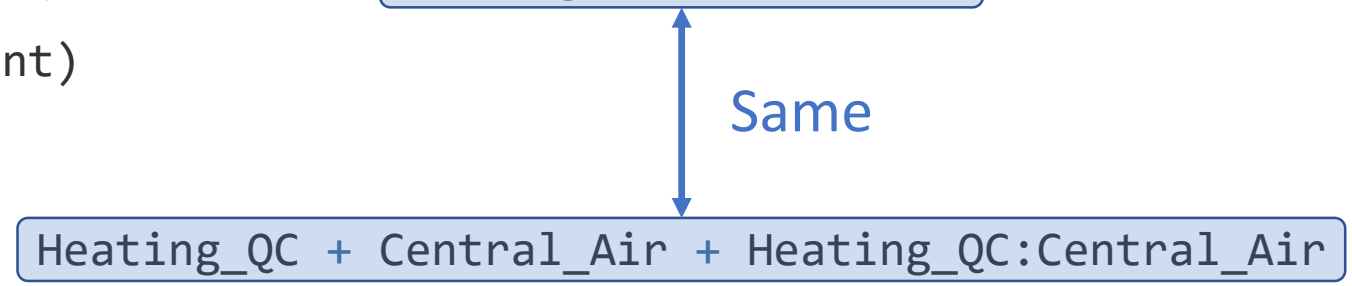

Interactions

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```

```
##                          Df    Sum Sq   Mean Sq F value   Pr(>F)
## Heating_QC                4 2.891e+12 7.228e+11 147.897  < 2e-16 ***
## Central_Air               1 2.903e+11 2.903e+11  59.403 1.99e-14 ***
## Heating_QC:Central_Air    4 3.972e+10 9.930e+09   2.032   0.0875 .
## Residuals              2041 9.975e+12 4.887e+09
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interactions

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summary(ames_aov_int)
```

```
##                          Df    Sum Sq   Mean Sq F value   Pr(>F)    
## Heating_QC                4 2.891e+12 7.228e+11 147.897  < 2e-16 ***
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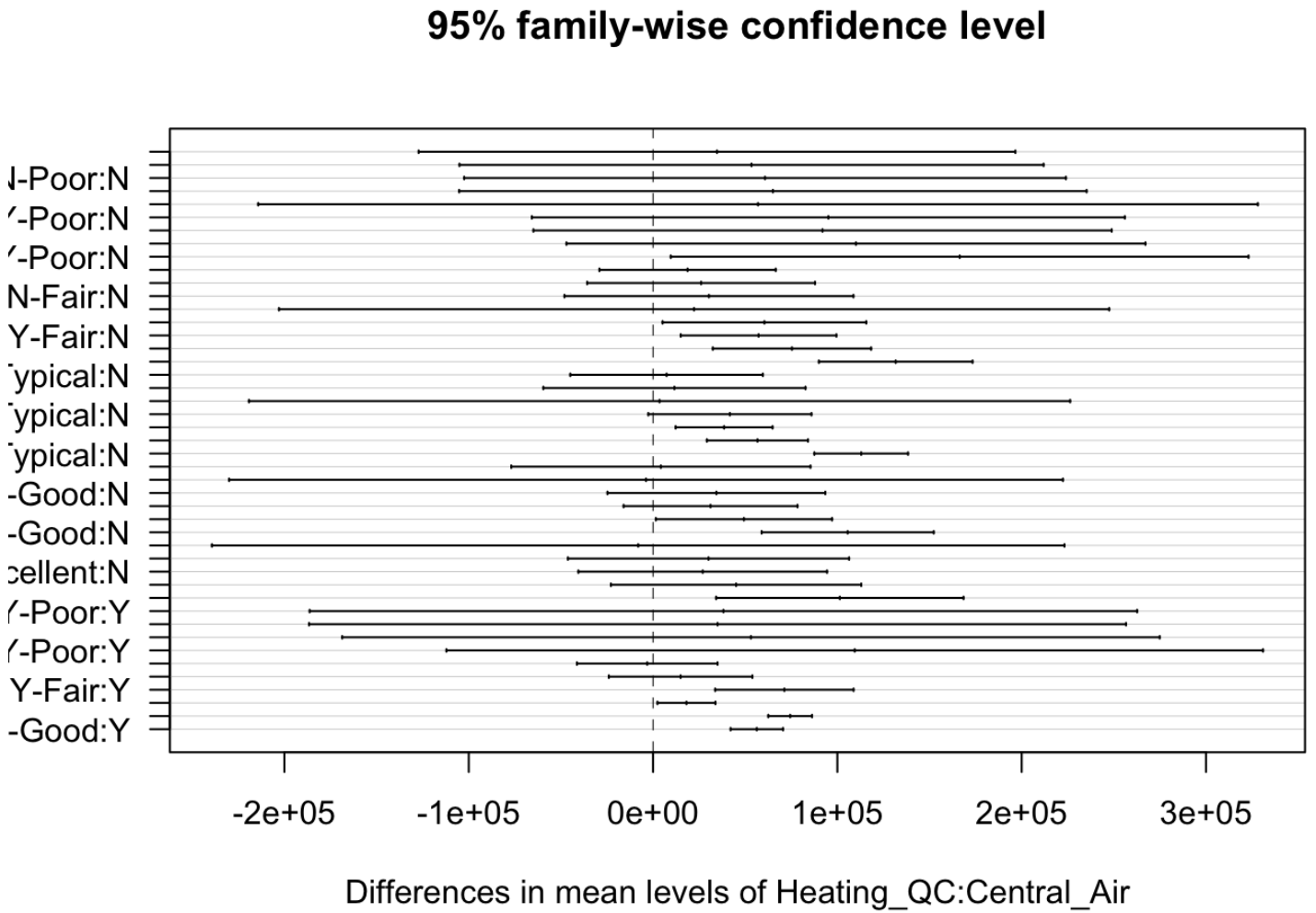
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Post-hoc Testing

```
tukey.ames_int <- TukeyHSD(ames_aov_int)
plot(tukey.ames_int, las = 1)
```

Post-hoc Testing



Within Effects Testing (Slicing)

- If an interaction exists, we are probably curious to see which of the levels of one variable are different within the level of another variable.
- Testing every possible combination might be overwhelming.
- **Slicing** performs an F-test for means for one variable within the level of another variable.

Sliced ANOVA

```
CA_aov <- train %>%
            group_by(Central_Air) %>%
            nest() %>%
            mutate(aov = map(data, ~summary(aov(Sale_Price ~ Heating_QC, data = .x))))
print(CA_aov$aov)
```

Sliced ANOVA

```
## [[1]]
##                Df    Sum Sq   Mean Sq F value Pr(>F)
## Heating_QC      4 2.242e+12 5.606e+11   108.5 <2e-16 ***
## Residuals    1899 9.809e+12 5.165e+09
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## [[2]]
##                Df    Sum Sq   Mean Sq F value  Pr(>F)
## Heating_QC      4 1.774e+10 4.435e+09   3.793 0.00582 **
## Residuals     142 1.660e+11 1.169e+09
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Central Air – YES

Heating Quality levels different

Central Air – NO

Heating Quality levels different

Assumptions

- The assumptions of *n*-Way ANOVA are the same as with One-Way ANOVA.
  - Independence of observations
  - Equality of variance
  - Normality of categories

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- The assumptions of *n*-Way ANOVA are the same as with One-Way ANOVA.
  - Independence of observations
  - Equality of variance – Levene Test only available for interactions
  - Normality of categories

Assumptions

- The assumptions of *n*-Way ANOVA are the same as with One-Way ANOVA.
  - Independence of observations
  - Equality of variance (of errors from model)
  - Normality of categories (or errors from model)

Since ANOVA is essentially a linear regression, can use diagnostic approaches of linear regression to assess.

Discussed in later section of course!

Randomized Block Design with ANOVA

OPTIONAL SELF STUDY

Observational or Retrospective Studies

- Groups can be naturally occurring.
  - Ex: Gender and ethnicity
- Random assignment might be unethical or untenable.
  - Ex: Smoking or credit risk groups
- Often you look at what already happened (retrospective) instead of following through to the future (prospective).
- You have little control over other factors contributing to the outcome measure.

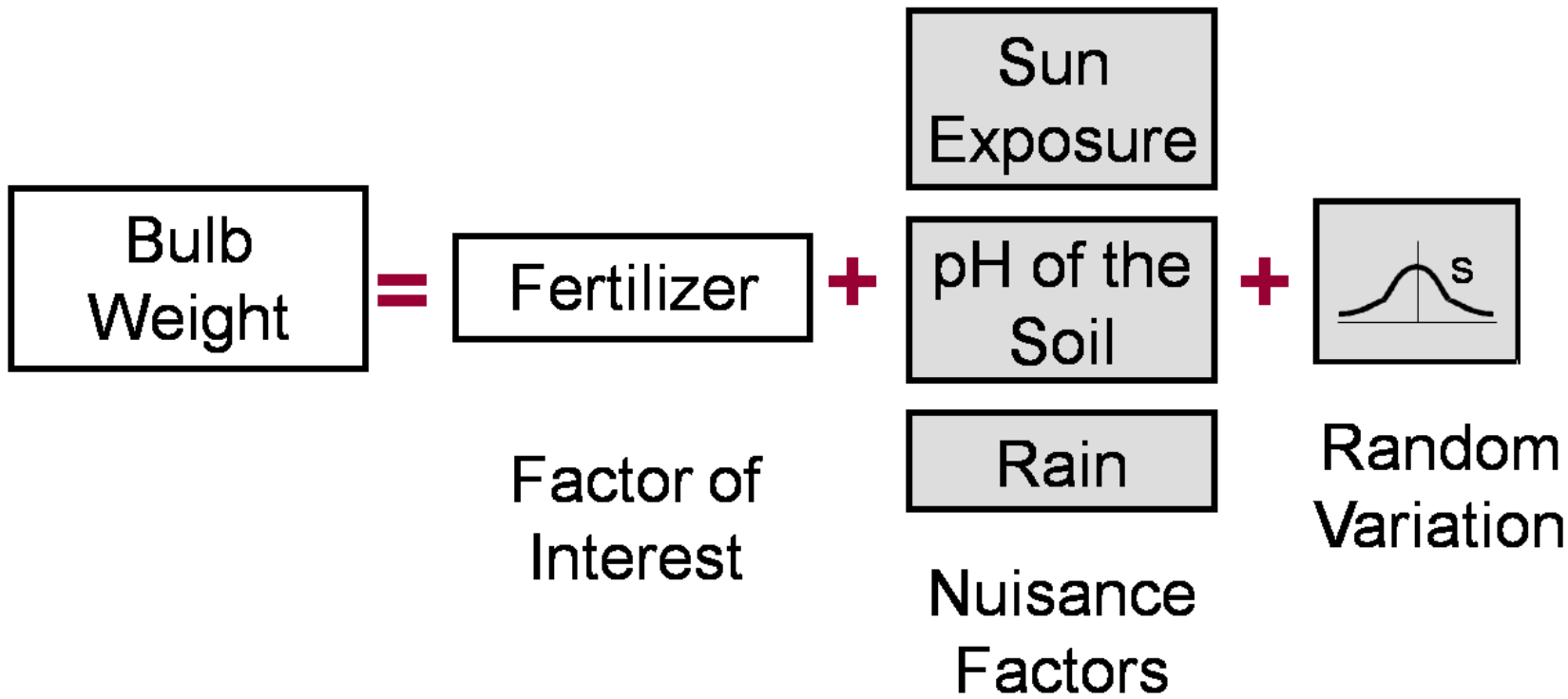
Controlled Experiments

- Random assignment might be desirable to eliminate selection bias.
- You often want to look at the outcome measure prospectively.
- You can manipulate the factors of interest and can more reasonably claim causation.
- You can design your experiment to control for other **nuisance factors** contributing to the outcome measure.

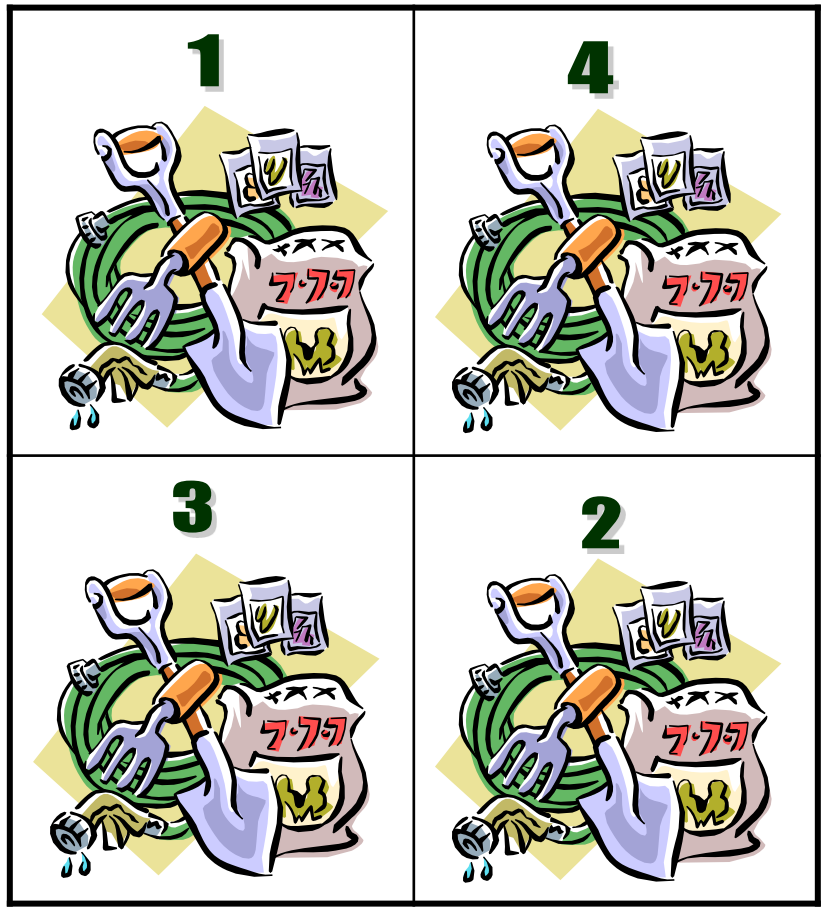
Garlic Bulb Weight Data

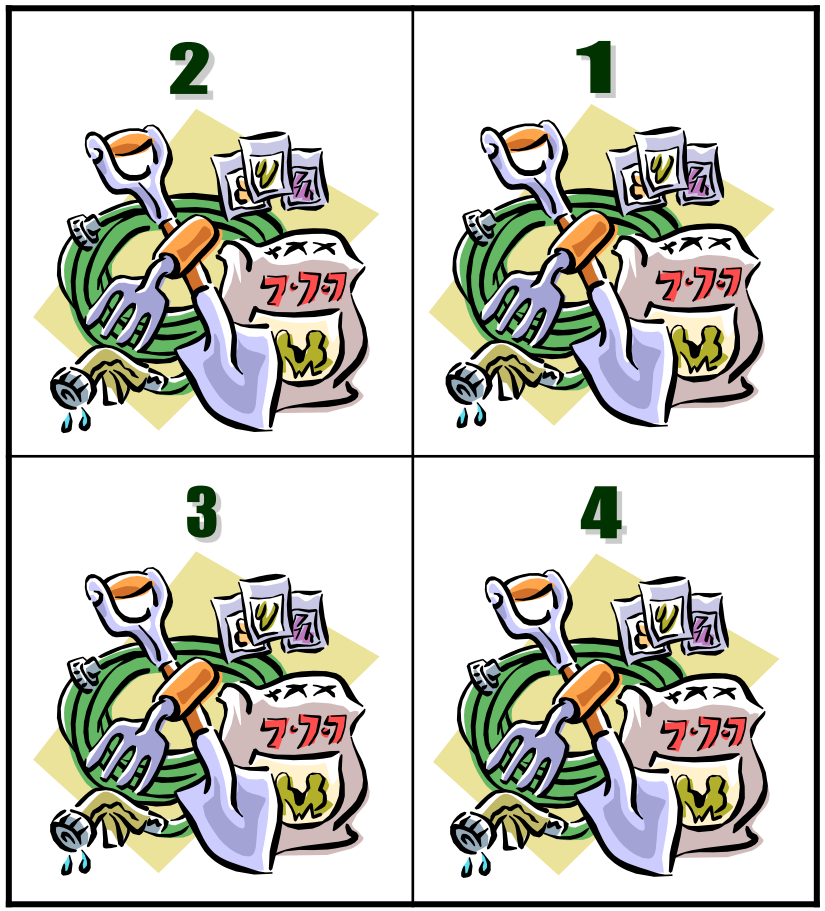
- This dataset contains the average garlic bulb weight from different plots of land.
- Compare the effects of fertilizer on average bulb weight.
- Potential nuisance factors:
  - Sun exposure
  - pH for the soil
  - Rain amounts
- **Blocking** to account for these nuisance factors.

Nuisance Factors



Assigning Treatments within Blocks





Exploring Garlic Data

```
## # A tibble: 32 x 6
##    Sector Position Fertilizer BulbWt Cloves BedId
##     <dbl>    <dbl>      <dbl>  <dbl>  <dbl> <dbl>
##  1      1        1          3  0.259   11.6 22961
##  2      1        2          4  0.207   12.6 23884
##  3      1        3          1  0.275   12.1 19642
##  4      1        4          2  0.245   12.1 20384
##  5      2        1          3  0.215   11.6 20303
##  6      2        2          4  0.170   12.7 21004
##  7      2        3          1  0.225   12.0 16117
##  8      2        4          2  0.168   11.9 19686
##  9      3        1          4  0.217   12.4 26527
## 10      3        2          3  0.226   11.7 23574
## # … with 22 more rows
```

Include Blocking Variable in Model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$$

Bulb Weight

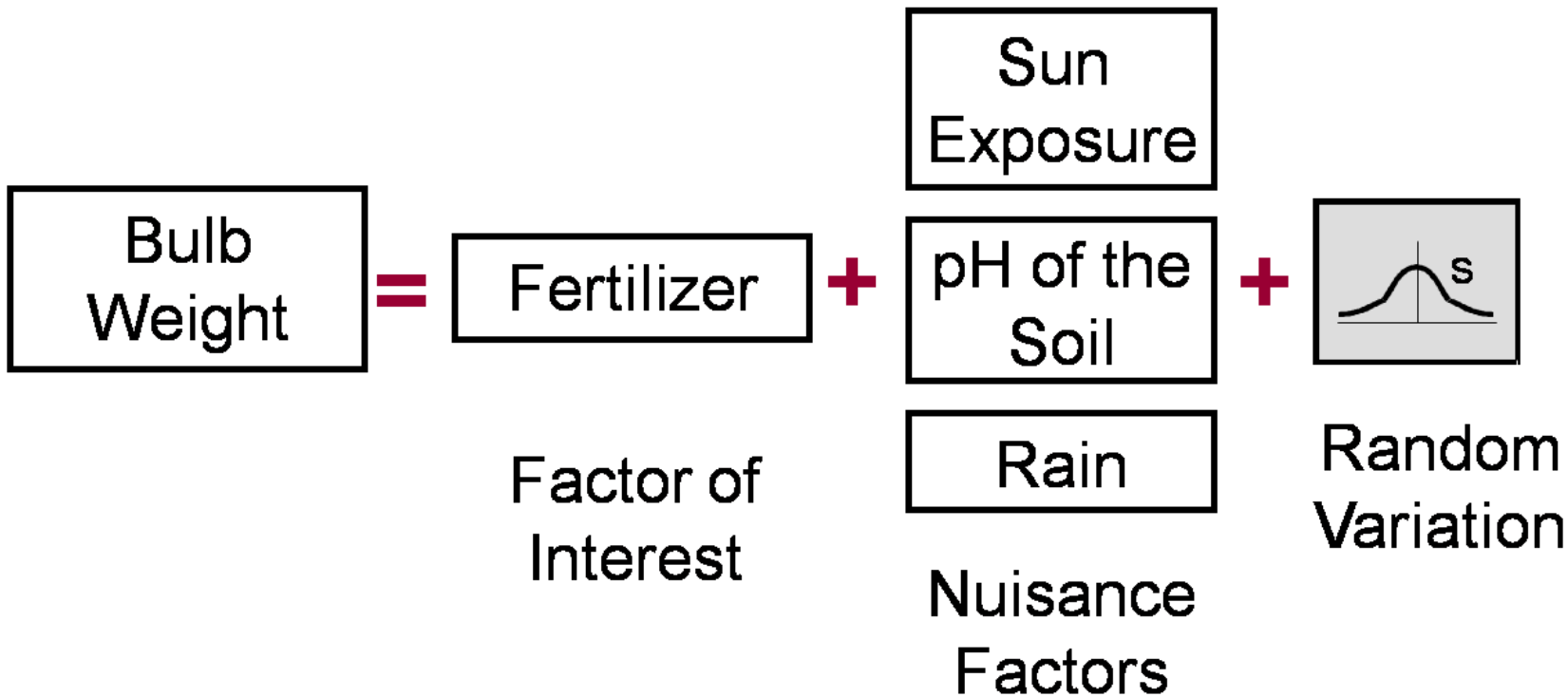
Base Level

Fertilizer

Sector (Block)

Unexplained Error

Nuisance Factors



ANOVA with Random Block Design

```r
block_aov <- aov(BulbWt ~ factor(Fertilizer) + factor(Sector), data = block)
summary(block_aov)
```

```
##                     Df   Sum Sq   Mean Sq F value   Pr(>F)
## factor(Fertilizer)  3 0.005086 0.0016954   4.307 0.016222 *
## factor(Sector)      7 0.017986 0.0025695   6.527 0.000364 ***
## Residuals          21 0.008267 0.0003937
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

ANOVA with Random Block Design

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## ---
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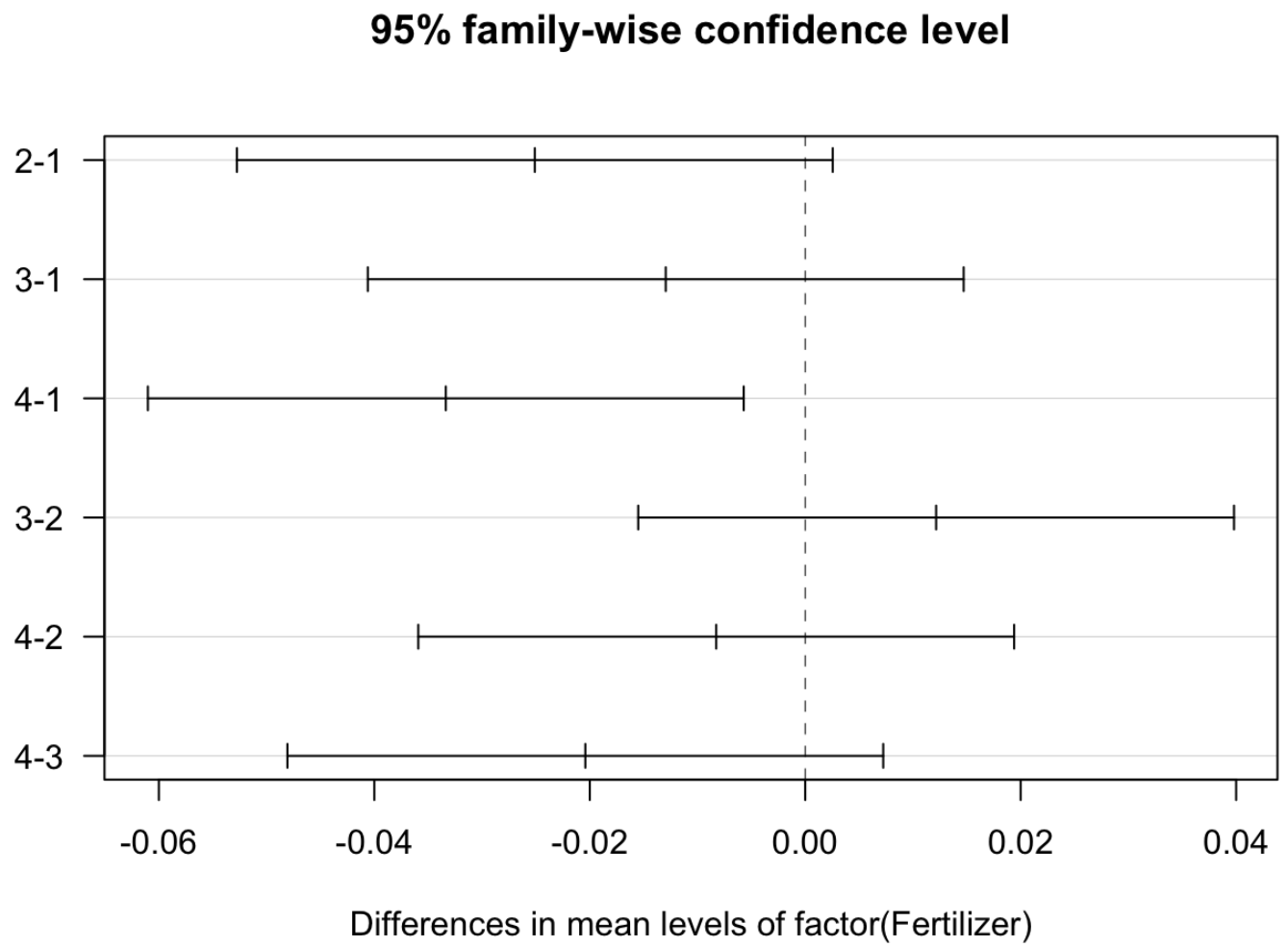
```

Post-hoc Testing

```
tukey.block <- TukeyHSD(block_aov)
plot(tukey.block, las = 1)
```

Post-hoc Testing



Including a Blocking Variable in the Model

- Additional assumptions are as follows:
  - Treatments are randomly assigned within each block.
  - The effects of the treatment factor are constant across the levels of the blocking variable.

- In the garlic example, the design is balanced, which means that there is the same number of garlic samples for every **Fertilizer**/**Sector** combination.

Multiple Linear Regression

CONCEPTS

Regression Modeling

- Most practical applications of regression modeling involve using more complicated models than the simple linear regression model.
- Typically it is better to have more than one variable in a regression model.
- Models with more than one predictor variable are called **multiple regression models**.

Multiple Linear Regression

- Models with more than one predictor variable are called **multiple regression models**.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \varepsilon$$

Multiple Linear Regression

- Models with more than one predictor variable are called **multiple regression models**.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \varepsilon$$

- With multiple variables in the model, the interpretation of $\hat{\beta}_j$ changes slightly.
- The estimate $\hat{\beta}_j$ is the predicted (or expected or average) change in y with a one unit increase in x_j **given all other variables are held constant.**

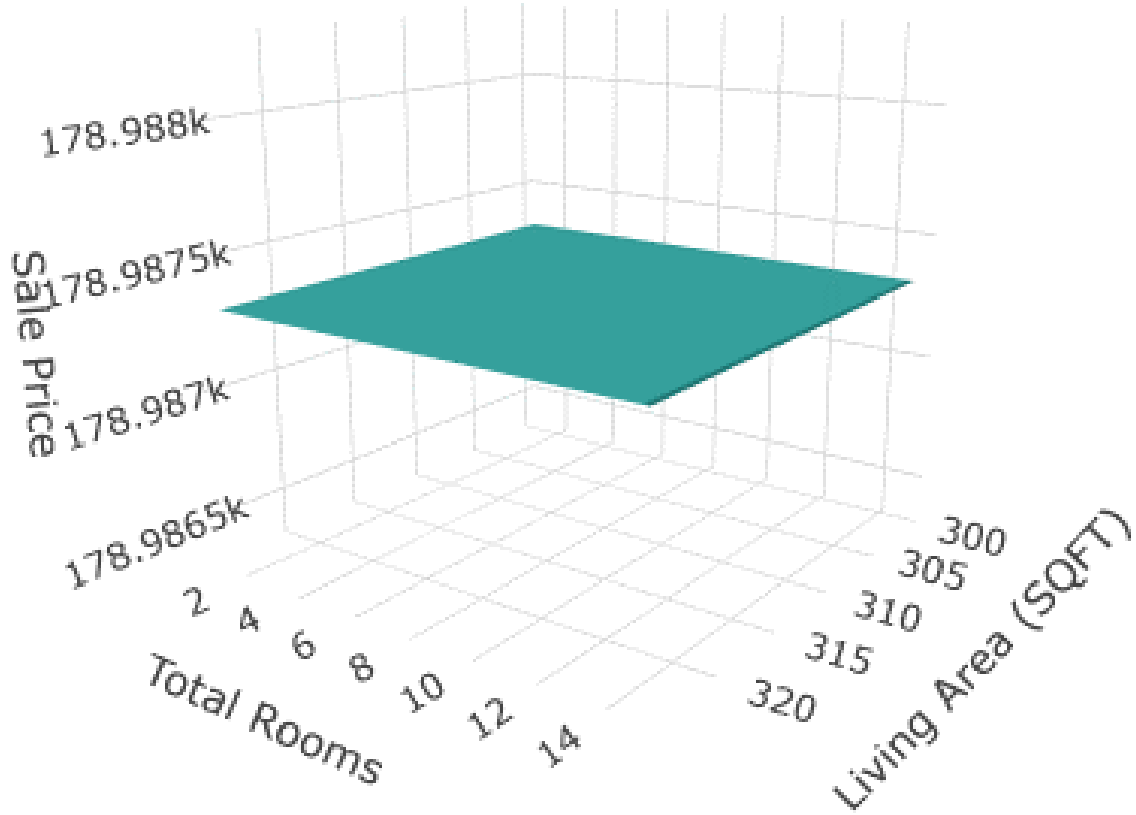
Fitting the Model

- The method for finding the line of best fit for multiple linear regression is the exact same for simple linear regression – the least squares method.
- The only thing that has changed is the predicted value of the response, $\hat{y}_i$.

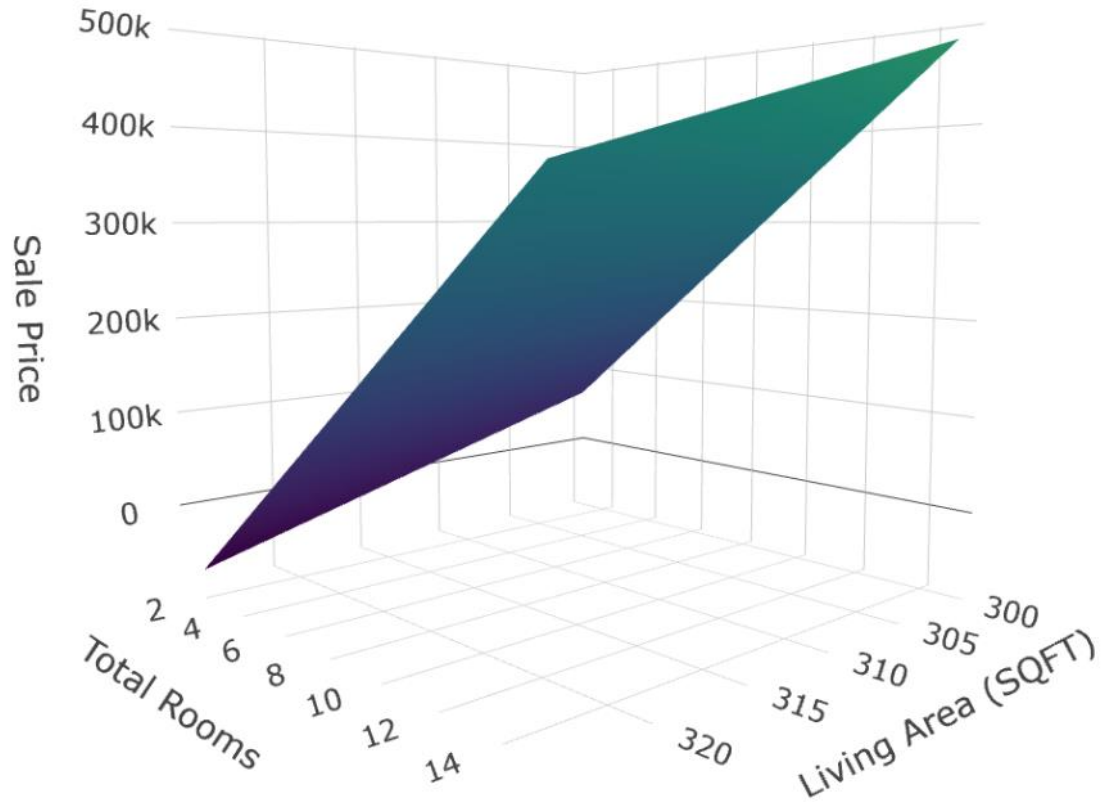
$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1,i} + \cdots + \hat{\beta}_k x_{k,i}$$

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Picturing the Model: No Relationship



Picturing the Model: A Relationship



Multiple ***Linear*** Regression

- The *linear* in multiple linear regression has nothing to do with the visualization of the fitted plane (or line in 2-dimensions).
- *Linear* refers to the linear combination of variables in the model.

- For example, mathematically, z is a linear combination of x and y (a and b are just constants/numbers):

$$z = ax + by$$

Multiple ***Linear*** Regression

- The *linear* in multiple linear regression has nothing to do with the visualization of the fitted plane (or line in 2-dimensions).
- *Linear* refers to the linear combination of variables in the model.

- Mathematically, y is a linear combination of the x's and error term:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \varepsilon$$

Multiple ***Linear*** Regression

- Linear regression with 4 variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \varepsilon$$

Multiple ***Linear*** Regression

- Linear regression with 4 variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \varepsilon$$

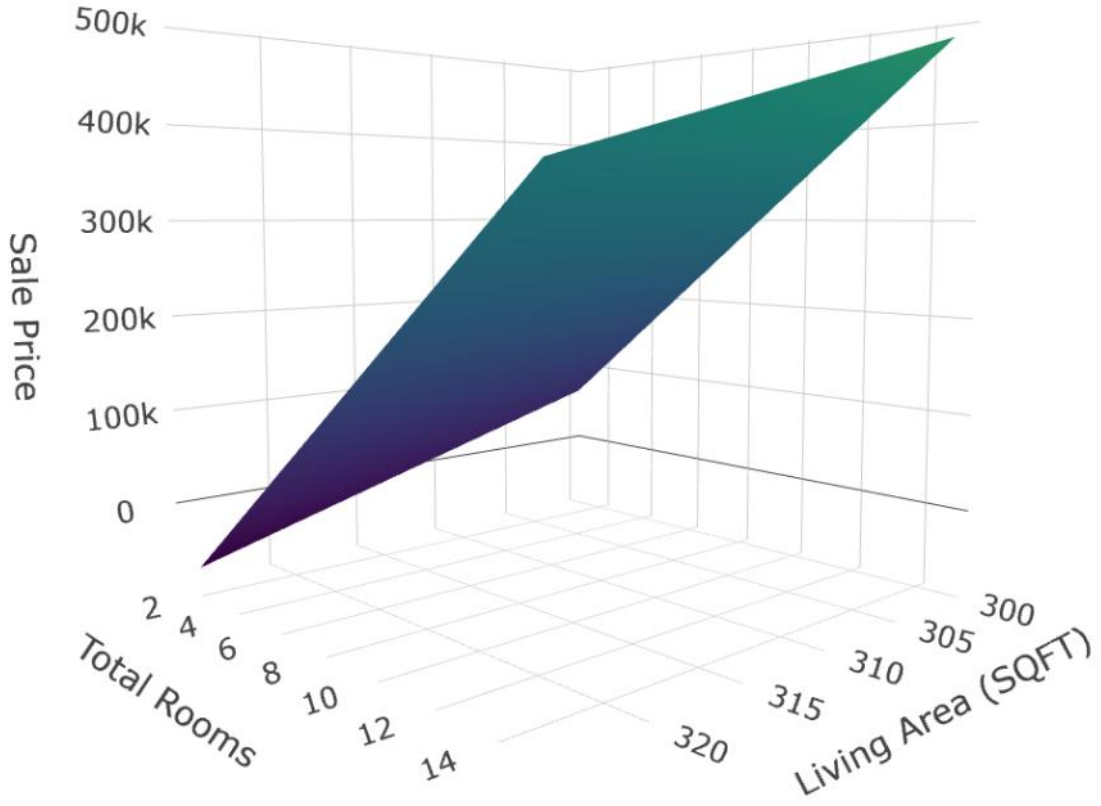
- Let $x_3 = x_1^2$ and $x_4 = x_2^2$:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_2^2 + \varepsilon$$

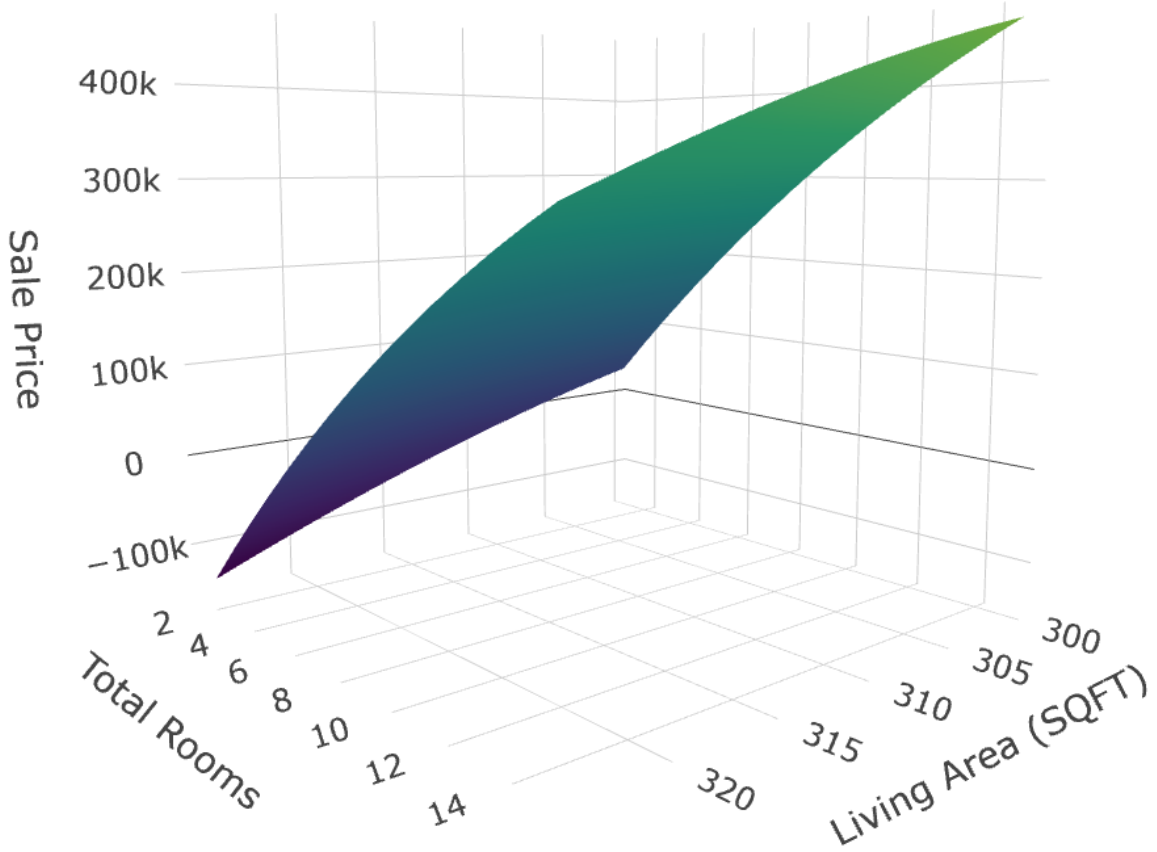
- Still linear regression!

Both Linear Regressions

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$



$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_2^2 + \varepsilon$$



Multiple Linear Regression

GLOBAL & LOCAL INFERENCE

Global vs. Local Test

- In simple linear regression we could just look at the t-test for our slope parameter estimate to determine the utility of our model.
- With multiple parameter estimates comes multiple t-tests.
- Ideally there should be a way of determining whether the model is adequate for predicting y overall, instead of looking at every individual parameter estimate.

Global Hypothesis Test

- **Null Hypothesis:**
  - None of the variables are useful in predicting the target variable.
  - $\beta_1 = \beta_2 = \ldots = \beta_k = 0$

- **Alternative Hypothesis:**
  - At least one variable is useful in predicting the target variable.
  - Not all β_is equal zero.

Global F-test

- The test statistic for this hypothesis test follows an *F*-distribution and is calculated as follows:

$$F = \frac{\left(\frac{SSR}{k}\right)}{\left(\frac{SSE}{n-(k+1)}\right)}$$

Global F-test

- The test statistic for this hypothesis test follows an *F*-distribution and is calculated as follows:

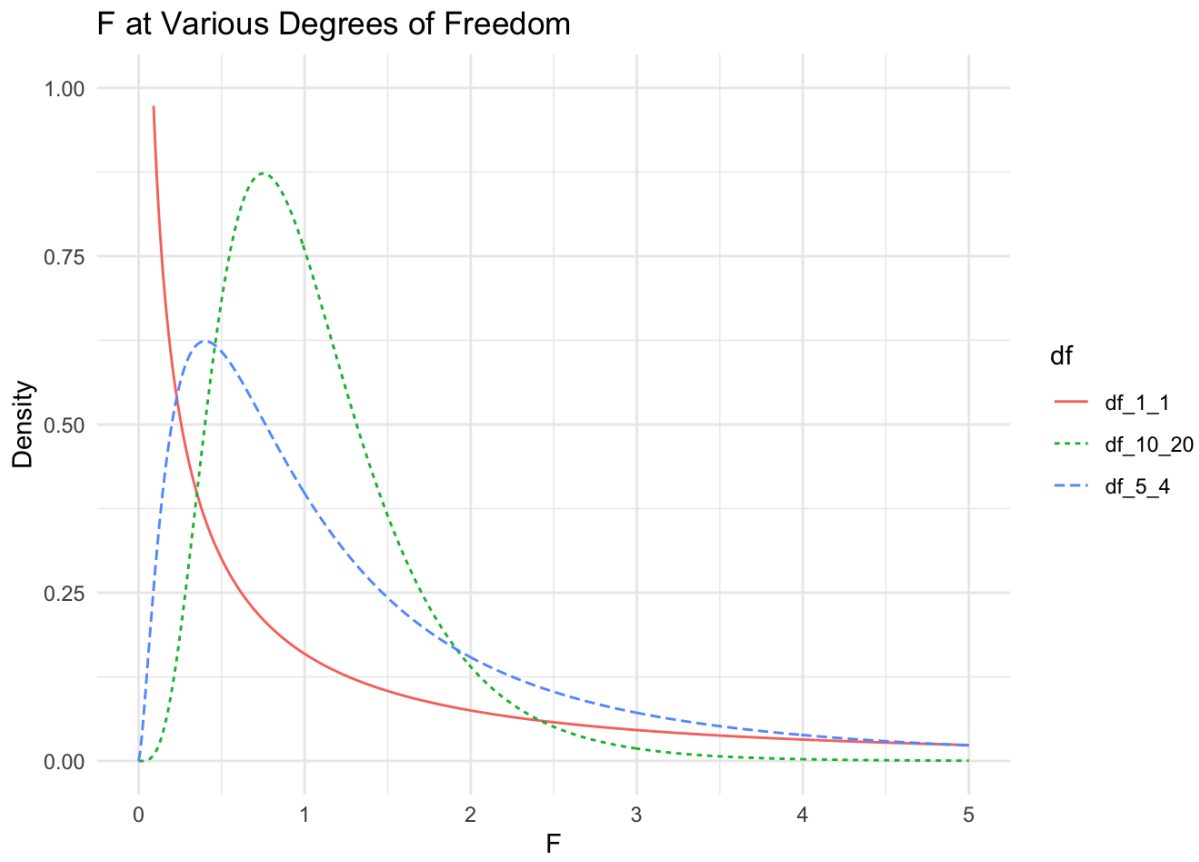
$$F = \frac{\left(\frac{SSR}{k}\right)}{\left(\frac{SSE}{n-(k+1)}\right)}$$

Average amount of variation each variable explains (i.e. Mean Square Regression)

"Average" amount of variation left per data point (i.e. Mean Square Error)

F-Distribution

- The F-test comes from the **F-distribution**.
- Characteristics of the F-distribution:
  1. Bounded Below By Zero
  2. Right Skewed
  3. Numerator **and** Denominator Degrees of Freedom



Global vs. Local Test

- If the global F-test is significant (at least one variable is useful), then we would dive down into the individual t-tests to find which variables are useful.
- Need to test if the values of each of these coefficients are zero to determine if a relationship exists between the response variable y and that **specific** explanatory variable x.

$$H_0: \beta_j = 0, \qquad \text{for } j = 1, \ldots, k$$

$$H_a: \beta_j \neq 0, \qquad \text{for } j = 1, \ldots, k$$

Multiple Linear Regression

```
ames_lm2 <- lm(Sale_Price ~ Gr_Liv_Area + TotRms_AbvGrd, data = train)
summary(ames_lm2)
```

Multiple Linear Regression

```
## Call:
## lm(formula = Sale_Price ~ Gr_Liv_Area + TotRms_AbvGrd, data = train)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -528656  -30077   -1230   21427  361465
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    42562.657   5365.721   7.932 3.51e-15 ***
## Gr_Liv_Area      136.982      4.207  32.558  < 2e-16 ***
## TotRms_AbvGrd -10563.324   1370.007  -7.710 1.94e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 56630 on 2048 degrees of freedom
## Multiple R-squared:  0.5024, Adjusted R-squared:  0.5019
## F-statistic:  1034 on 2 and 2048 DF,  p-value: < 2.2e-16
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```

Multiple Linear Regression

EVALUATING A MODEL

Assumptions for Linear Regression

- The mean of the Ys is accurately modeled by a **linear** function of the Xs.
- The random error term, ε, is assumed to have a **normal** distribution with a mean of zero.
- The random error term, ε, is assumed to have a **constant variance**, σ^2.
- The errors are **independent**.

- **No perfect collinearity**

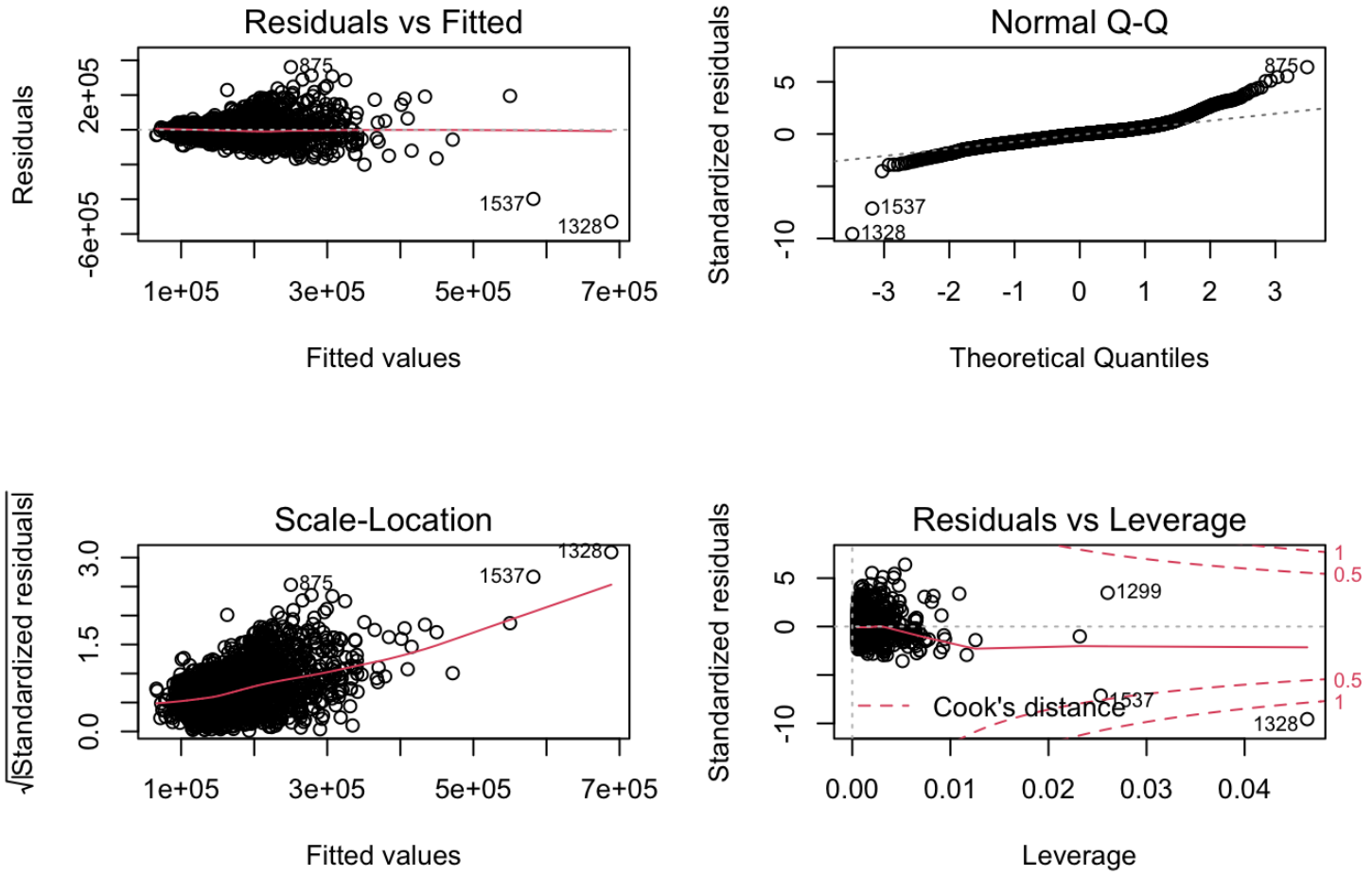
Multicollinearity

- **Multicollinearity –** predictor variables are correlated with each other.

- No **perfect** collinearity (multicollinearity) – predictor variables are a perfect linear combination of each other.
  - Only care when collinearity has **drastic** impact.
  - Linear regression only breaks when collinearity is perfect.

Assumptions through Residuals

```
plot(ames_lm2)
```



Multiple Linear vs. Simple Linear Regression

- **Main Advantage**
  - Multiple linear regression enables you to investigate the relationship among Y and several independent variables simultaneously.

- **Main Disadvantages**
  - Increased complexity makes it more difficult to do the following:
    - ascertain which model is “best”
    - interpret the models

Common Applications

- Multiple linear regression is a powerful tool for the following tasks:
  - **Predict** – to develop a model to predict future values of a response variable (Y) based on its relationships with other predictor variables (Xs)
  - **Explain** – to develop an understanding of the relationships between the response variable and predictor variables

Predict

- The terms in the model, the values of their coefficients, and their statistical significance are of secondary importance.
- The focus is on producing a model that is the best at predicting future values of Y as a function of the Xs. The predicted value of Y is given by this formula:

$$\underline{\hat{Y}} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \ldots + \hat{\beta}_k X_k$$

Explain

- The focus is on understanding the relationship between the dependent variable and the independent variables.
- Consequently, the statistical significance of the coefficients is important as well as the magnitudes and signs of the coefficients.

$$\hat{Y} = \underline{\hat{\beta}_0} + \underline{\hat{\beta}_1} X_1 + \ldots + \underline{\hat{\beta}_k} X_k$$

Problem with R^2

- The problem with the calculation of R^2 in a multiple linear regression is that the addition of any variable (good or bad) will make the R^2 value increase if even slightly.

$$R^2 = 1 - \frac{SSE}{TSS}$$

Will never increase with the addition of a variable.

Example with Randomness

```
##                                Estimate Std. Error t value  Pr(>|t|)
## (Intercept)                   42562.657   5365.721   7.932  3.51e-15 ***
## Gr_Liv_Area                     136.982      4.207  32.558   < 2e-16 ***
## TotRms_AbvGrd                -10563.324   1370.007  -7.710  1.94e-14 ***
## ---
## Multiple R-squared: 0.5024, Adjusted R-squared: 0.502
```

```
##                                Estimate Std. Error t value  Pr(>|t|)
## (Intercept)                   42589.091   5364.877   7.939  3.34e-15 ***
## Gr_Liv_Area                     136.927      4.207  32.548   < 2e-16 ***
## TotRms_AbvGrd                -10552.425   1369.808  -7.704  2.05e-14 ***
## rnorm(length(Sale_Price), 0, 1)   1629.854   1259.478   1.294     0.196
## ---
## Multiple R-squared: 0.5028, Adjusted R-squared: 0.502
```

Adjusted Coefficient of Determination

- To account for this problem, most people use the **adjusted coefficient of determination**, R_a^2.
- The calculations are as follows:

$$R_a^2 = 1 - \left[\left(\frac{n-1}{n-(k+1)}\right)\left(\frac{SSE}{TSS}\right)\right]$$

OR

$$R_a^2 = 1 - \left[\left(1 - R^2\right)\left(\frac{n-1}{n-(k+1)}\right)\right]$$

Adjusted Coefficient of Determination

- The R_a^2 penalizes a model for adding a variable that does not provide any useful information.

$$R_a^2 \leq R^2$$

- The adjusted coefficient of determination loses its interpretation (because it could be negative!), but is better at determining utility of a model.

Example with Randomness

```
##                                Estimate Std. Error t value  Pr(>|t|)
## (Intercept)                   42562.657   5365.721   7.932  3.51e-15 ***
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## ---
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## ---
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```

Multiple Linear Regression

CATEGORICAL PREDICTORS

Dummy Variables

- Categorical variables need to be coded differently because they are not numerical in nature.
- **Dummy variables** are a common way to code categorical variables.
- Central Air Example (Y, N):

$$x_1 = \begin{cases} 1 & \text{if Y} \\ 0 & \text{if N} \end{cases}$$

Dummy Variable Interpretation

- Categorical variables need to be coded differently because they are not numerical in nature.
- **Dummy variables** are a common way to code categorical variables.
- Central Air Example (Y, N):

$$x_1 = \begin{cases} 1 & \text{if Y} \\ 0 & \text{if N} \end{cases}$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

Average difference between category Y and N.

Dummy Variable Interpretation

- Central Air Example (Y, N):

$$x_1 = \begin{cases} 1 & \text{if Y} \\ 0 & \text{if N} \end{cases}$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

Average difference between category Y and N. BUT WHY?

Dummy Variable Interpretation

- Central Air Example (Y, N):

$$x_1 = \begin{cases} 1 & \text{if Y} \\ 0 & \text{if N} \end{cases}$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

Average difference between category Y and N. BUT WHY?

$$\hat{y}_Y = \hat{\beta}_0 + \hat{\beta}_1 \cdot 1 = \hat{\beta}_0 + \hat{\beta}_1$$

$$\hat{y}_N = \hat{\beta}_0 + \hat{\beta}_1 \cdot 0 = \hat{\beta}_0$$

$$\hat{y}_{Y-N} = (\hat{\beta}_0 + \hat{\beta}_1) - \hat{\beta}_0 = \hat{\beta}_1$$

Effects Coding Interpretation

- Categorical variables need to be coded differently because they are not numerical in nature.
- **Effects coding** is another common way to code categorical variables.
- Central Air Example (Y, N):

$$x_1 = \begin{cases} 1 & \text{if Y} \\ -1 & \text{if N} \end{cases}$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

Average difference between category Y and the overall average of categories Y & N.

Effects Coding Interpretation

- Central Air Example (Y, N):

$$x_1 = \begin{cases} 1 & \text{if Y} \\ -1 & \text{if N} \end{cases}$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

Average difference between category Y and the overall average of categories Y & N. BUT WHY?

$$\hat{y}_Y = \hat{\beta}_0 + \hat{\beta}_1 \cdot 1 = \hat{\beta}_0 + \hat{\beta}_1$$

$$\hat{y}_N = \hat{\beta}_0 + \hat{\beta}_1 \cdot (-1) = \hat{\beta}_0 - \hat{\beta}_1$$

$$\hat{y}_{Avg.} = \frac{\left((\hat{\beta}_0 + \hat{\beta}_1) + (\hat{\beta}_0 - \hat{\beta}_1)\right)}{2} = \hat{\beta}_0$$

$$\hat{y}_{Y-Avg} = (\hat{\beta}_0 + \hat{\beta}_1) - \hat{\beta}_0 = \hat{\beta}_1$$

Multiple Linear Regression

```
ames_lm2 <- lm(Sale_Price ~ Gr_Liv_Area + TotRms_AbvGrd + Central_Air, data = train)
summary(ames_lm2)
```

Multiple Linear Regression

```
## Residuals:
##     Min      1Q  Median      3Q     Max
## -510745  -28984   -2317   20273  356742
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -7169.259   6778.879  -1.058     0.29
## Gr_Liv_Area      129.594      4.131  31.374  < 2e-16 ***
## TotRms_AbvGrd  -8980.938   1335.669  -6.724 2.29e-11 ***
## Central_AirY   54513.082   4762.926  11.445  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 54910 on 2047 degrees of freedom
## Multiple R-squared:  0.5323, Adjusted R-squared:  0.5316
## F-statistic: 776.6 on 3 and 2047 DF,  p-value: < 2.2e-16
```