GENERALIZED ADDITIVE MODELS

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GENERAL STRUCTURE

Generalized Additive Models (GAMs)

 Provides general framework for adding non-linear functions to standard linear model.

$$y = \beta_0 + f_1(x_1) + f_2(x_2) + \dots + f_p(x_p) + \varepsilon$$

Can be used for regression or classification problems.

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Adding **potentially** complex, individual relationships together.

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Can be used for regression or classification problems.

Many potential complex relationships to try and model with.

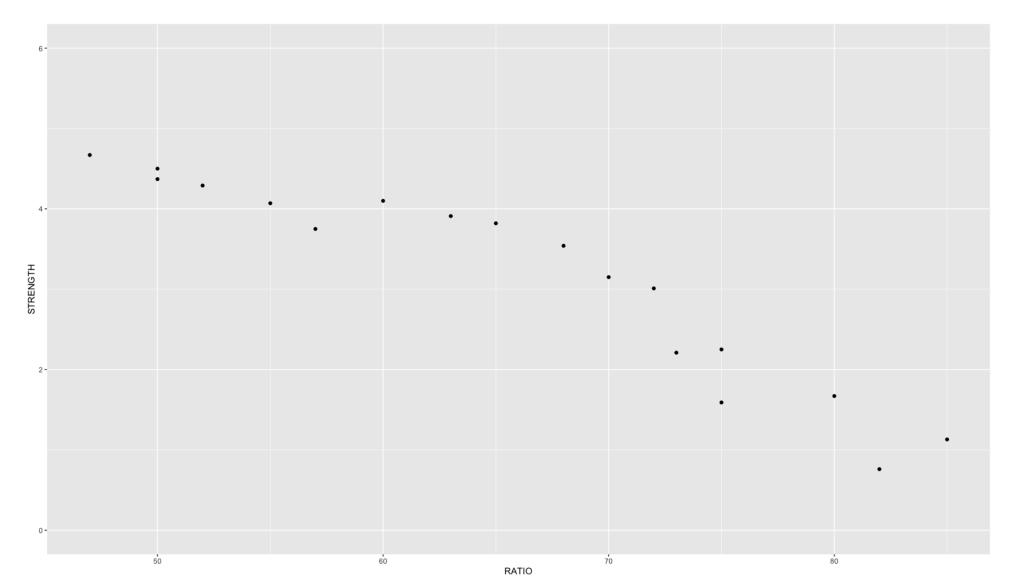


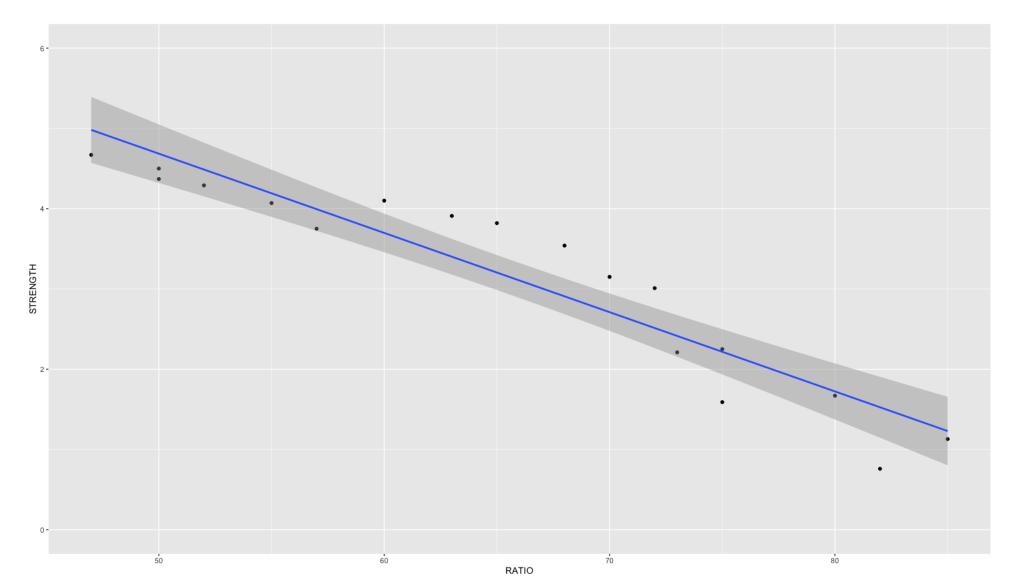
PIECEWISE LINEAR REGRESSION

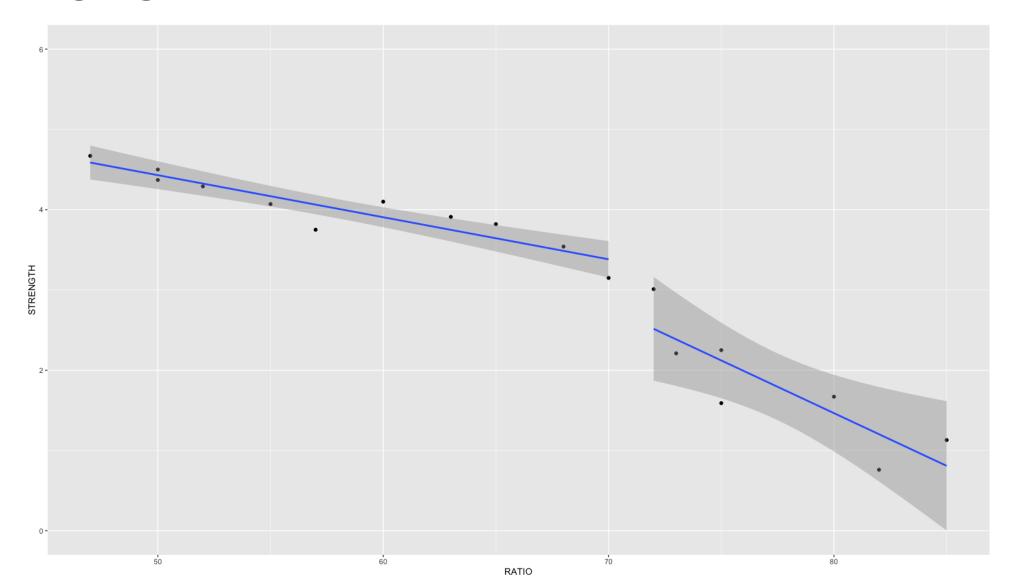
- The slope of the linear relationship between a predictor variable and a response variable can change over different values of the predictor variable.
- Typical straight-line model $\hat{y} = \beta_0 + \beta_1 x_1$ will not be a good fit for this type of data.

Cement Data

- The comprehensive strength of concrete depends on the proportion of water mixed with cement.
- The comprehensive strength decreases at a much faster rate for batches with a greater than 70% water/cement ratio.







- A model where different straight-line relationships for different intervals in the predictor variable is called the **piecewise linear regression model**.
- The model is the following for two slopes:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 (x_1 - k) x_2 + \varepsilon$$

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- The model is the following for two slopes:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 (x_1 + k) x_2 + \varepsilon$$

Knot value for x_1 .

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- The model is the following for two slopes:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 (x_1 - k) x_2 + \varepsilon$$

$$x_2 = \begin{cases} 1, & x_1 > k \\ 0, & x_1 \le k \end{cases}$$

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$$x_2 = 0$$

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

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- The model is the following for two slopes:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 (x_1 - k) x_2 + \varepsilon$$

$$x_2 = 0$$

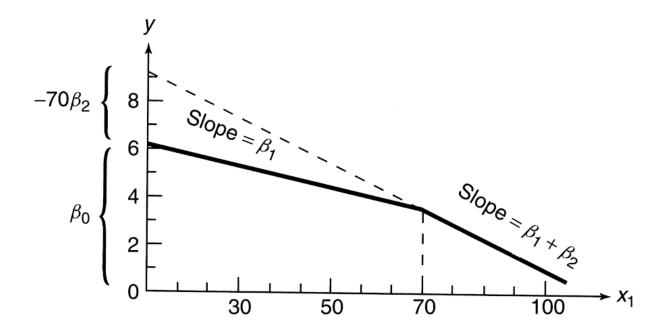
$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

$$x_2 = 1$$

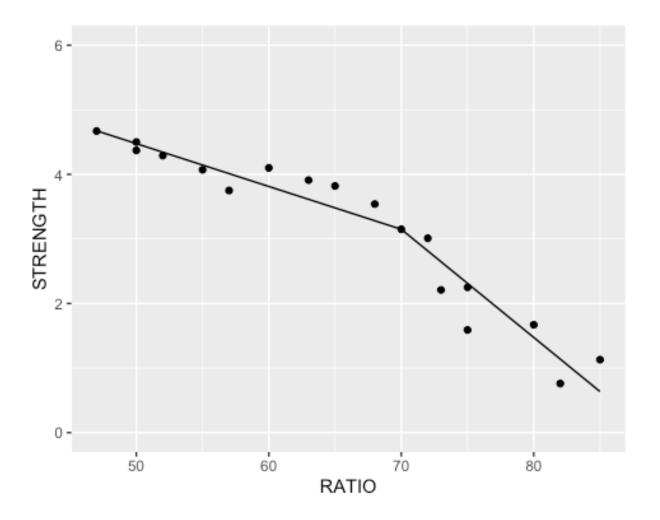
$$y = (\beta_0 - k\beta_2) + (\beta_1 + \beta_2) x_1 + \varepsilon$$

Cement Data

- The comprehensive strength of concrete depends on the proportion of water mixed with cement.
- The comprehensive strength decreases at a much faster rate for batches with a greater than 70% water/cement ratio.



```
cement.lm <- lm(STRENGTH ~ RATIO + X2STAR, data = cement)</pre>
summary(cement.lm)
                                                       (x_1 - k)x_2
## Call:
## lm(formula = STRENGTH ~ RATIO + X2STAR, data = cement)
##
## Residuals:
       Min
##
                 10 Median
                                  30
                                          Max
## -0.72124 -0.09753 -0.00163 0.24297 0.49393
##
## Coefficients:
              Estimate Std. Error t value Pr(>|t|)
##
## (Intercept) 7.79198 0.67696 11.510 7.62e-09 ***
## RATIO -0.06633 0.01123 -5.904 2.89e-05 ***
## X2STAR -0.10119 0.02812 -3.598 0.00264 **
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3286 on 15 degrees of freedom
## Multiple R-squared: 0.9385, Adjusted R-squared: 0.9303
## F-statistic: 114.4 on 2 and 15 DF, p-value: 8.257e-10
```



Extensions – Discontinuous

- The previous approach dealt with piecewise functions that are continuous.
- The following is the discontinuous set-up for two straight lines:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 (x_1 - k) x_2 + \beta_3 x_2 + \varepsilon$$

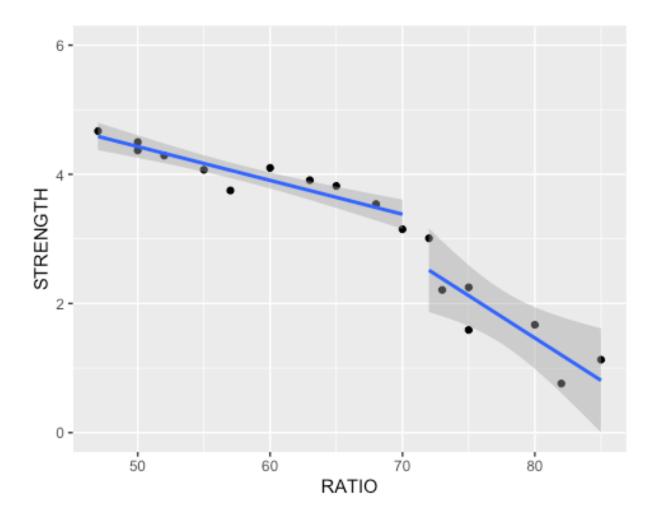
$$x_2 = \begin{cases} 1, & x_1 > k \\ 0, & x_1 \le k \end{cases}$$

Extensions – Discontinuous

```
cement.lm <- lm(STRENGTH ~ RATIO + X2STAR + X2, data = cement)</pre>
summary(cement.lm)
                                                (x_1-k)x_2
##
## Call:
## lm(formula = STRENGTH ~ RATIO + X2STAR + X2, data = cement)
##
## Residuals:
               10 Median
      Min
                               3Q
                                      Max
## -0.53167 -0.15513 0.06171 0.17239 0.49451
##
## Coefficients:
##
             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 7.04975
                      0.68558 10.283 6.6e-08 ***
         -0.05240 0.01174 -4.463 0.000536 ***
## RATIO
         ## X2STAR
## X2
            ## ---
## Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2916 on 14 degrees of freedom
## Multiple R-squared: 0.9548, Adjusted R-squared: 0.9451
## F-statistic: 98.57 on 3 and 14 DF, p-value: 1.188e-09
```

Extensions – Discontinuous

```
qplot(RATIO, STRENGTH, group = X2,
    geom = c('point', 'smooth'),
    method = 'lm', data = cement,
    ylim = c(0,6))
```



Extensions

- The same modeling approach can be applied to any piecewise regression.
- The following is the set-up for three straight lines:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 (x_1 - k_1) x_2 + \beta_3 (x_1 - k_2) x_3 + \varepsilon$$

$$x_2 = \begin{cases} 1, & x_1 > k_1 \\ 0, & \text{if not} \end{cases} \quad x_3 = \begin{cases} 1, & x_1 > k_2 \\ 0, & \text{if not} \end{cases}$$

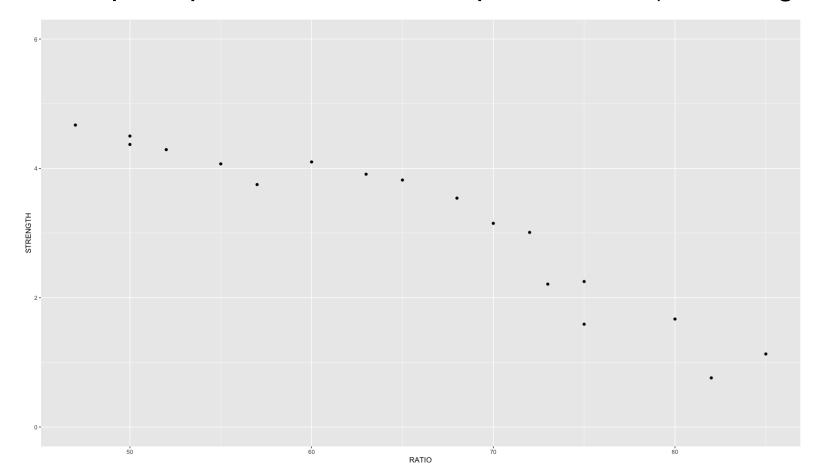


MARS (AND EARTH)

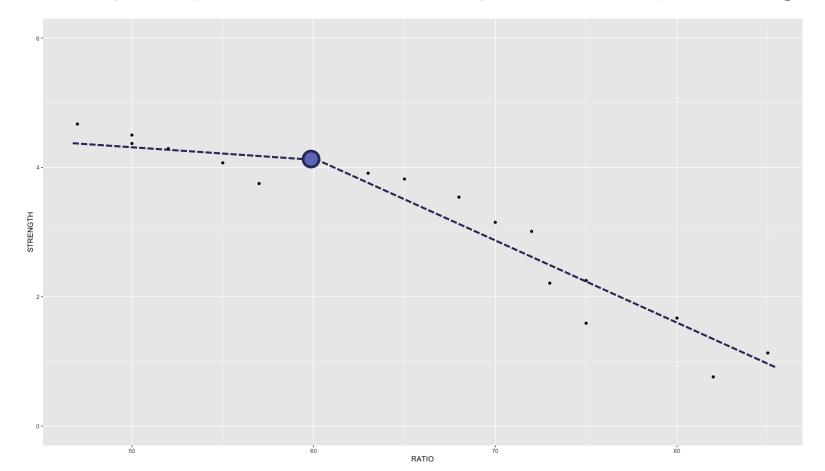
Multivariate Adaptive Regression Splines (MARS)

- Multivariate adaptive regression splines (MARS) is a non-parametric technique that is still has a linear form to the model (additive) but has nonlinearities and interaction between variables.
- Essentially, uses **piecewise** regression approach to split into pieces then potentially uses either linear or nonlinear patterns for each piece.

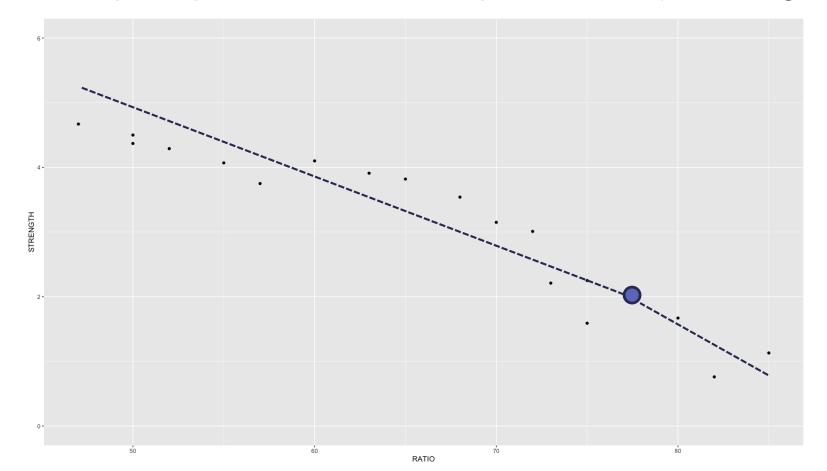
 MARS first looks for the point in the range of x where two linear functions on either side of the point provides the least squared error (linear regression).



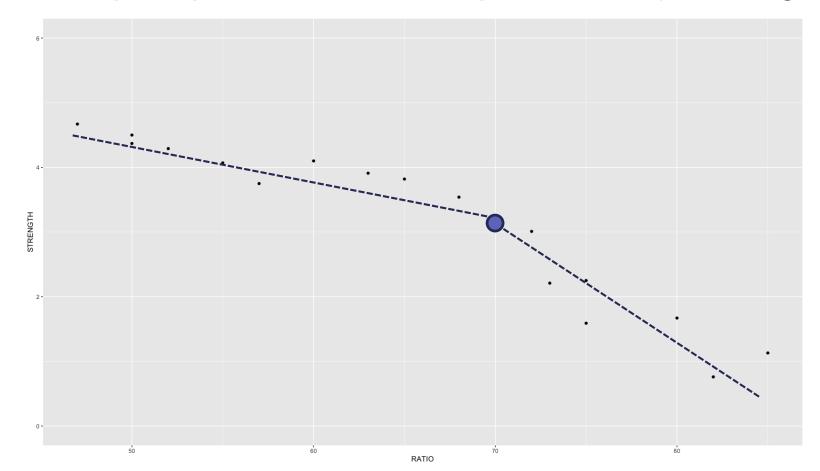
• MARS first looks for the point in the range of x where two linear functions on either side of the point provides the least squared error (linear regression).



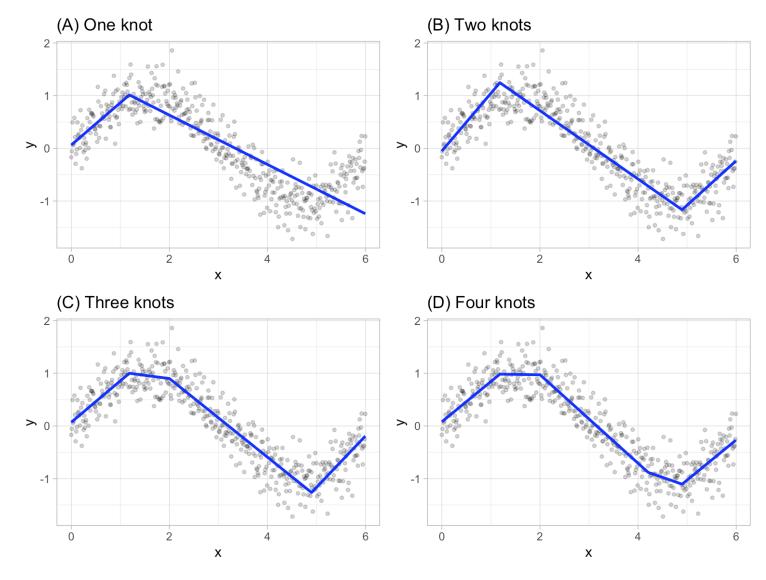
• MARS first looks for the point in the range of x where two linear functions on either side of the point provides the least squared error (linear regression).



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Many Knots



How Many Knots?

- Algorithm continues on each piece of the piecewise function until many knots are found (WILL OVERFIT YOUR DATA).
- Then works backwards ("prunes") to remove knots that do not contribute significantly to out of sample accuracy.
 - Calculation is performed by the generalized cross-validation (GCV)
 procedure computational shortcut for leave-one-out cross-validation.
- Does this for all variables!

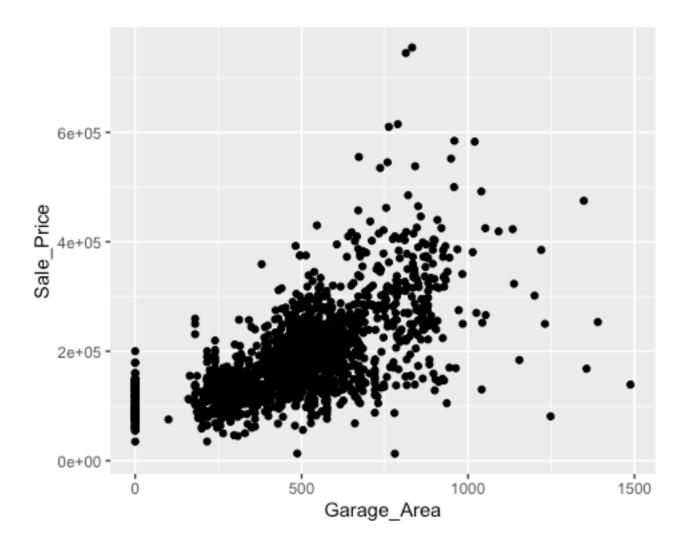
$$y = \beta_0 + f_1(x_1) + f_2(x_2) + \dots + f_p(x_p) + \varepsilon$$

EARTH

- Enhanced Adaptive Regression Through Hinges (EARTH) is the implementation of the MARS algorithm in most software.
- MARS is trademarked by Salford Systems, so instead we use EARTH.

EARTH (and MARS)

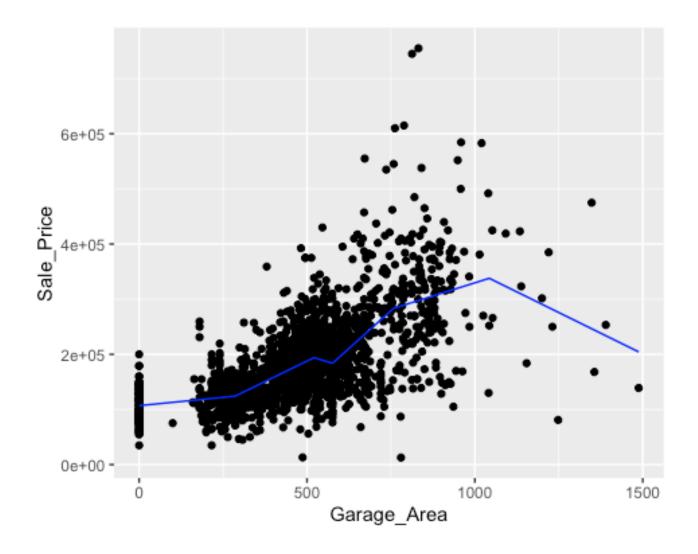
EARTH (and MARS)



EARTH (and MARS)

```
mars1 <- earth(Sale Price ~ Garage Area, data = training)</pre>
summary(mars1)
## Call: earth(formula=Sale Price~Garage Area, data=training)
##
##
                      coefficients
## (Intercept)
                        124159.039
## h(286-Garage Area)
                           -60.257
                     297.277
## h(Garage Area-286)
## h(Garage Area-521)
                     -483.642
## h(Garage Area-576)
                     733.859
## h(Garage_Area-758) -356.460
## h(Garage Area-1043)
                          -490.873
##
## Selected 7 of 7 terms, and 1 of 1 predictors
## Termination condition: RSq changed by less than 0.001 at 7 terms
## Importance: Garage_Area
## Number of terms at each degree of interaction: 1 6 (additive model)
## GCV 3427475346 RSS 6.94092e+12
                                      GRSq 0.4492014
                                                        RSq 0.4556309
```

EARTH (and MARS)



EARTH (and MARS)

```
mars1 <- earth(Sale Price ~ ., data = training)</pre>
summary(mars1)
## Call: earth(formula=Sale Price~., data=training)
##
##
                       coefficients
## (Intercept)
                          319493.46
## Central AirY
                           20289.49
## h(4-Bedroom AbvGr)
                            9214.66
## h(Bedroom AbvGr-4)
                     -23009.05
## h(Half Bath-1)
                  -45378.31
## h(2-Fireplaces) -14408.56
## h(Fireplaces-2) -26072.58
## h(Garage Area-539)
                             101.97
## h(Garage Area-1043)
                            -294.30
## h(Gr Liv Area-2049)
                            65.21
## h(Gr Liv Area-3194)
                            -159.79
##
## Selected 21 of 24 terms, and 10 of 14 predictors
## Termination condition: Reached nk 29
## Importance: First Flr SF, Second Flr SF, Year Built, Garage Area, ...
## Number of terms at each degree of interaction: 1 20 (additive model)
## GCV 1033819964
                  RSS 2.036439e+12
                                      GRSq 0.8338641
                                                        RSq 0.8402842
```

Variable Importance

```
mars1 <- earth(Sale Price ~ ., data = training)</pre>
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## Call: earth(formula=Sale Price~., data=training)
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                   RSS 2.036439e+12
                                       GRSq 0.8338641
                                                         RSq 0.8402842
```

Variable Importance

- There is one "subset" for each model size (1 term, 2 terms, etc.) the best model of that size.
- Ranks variables by how many of the subsets that variable appears in.
 - More subsets of models it appears in (in the best 1 variable model, in the best 2 variable model, etc.), then the better the variable.
- RSS (residual sum of squares or sum of squares error) is scaled version of decrease in residual sum of squares relative to the previous subset.
- GCV is approximation of RSS on leave-one-out-cross validation.

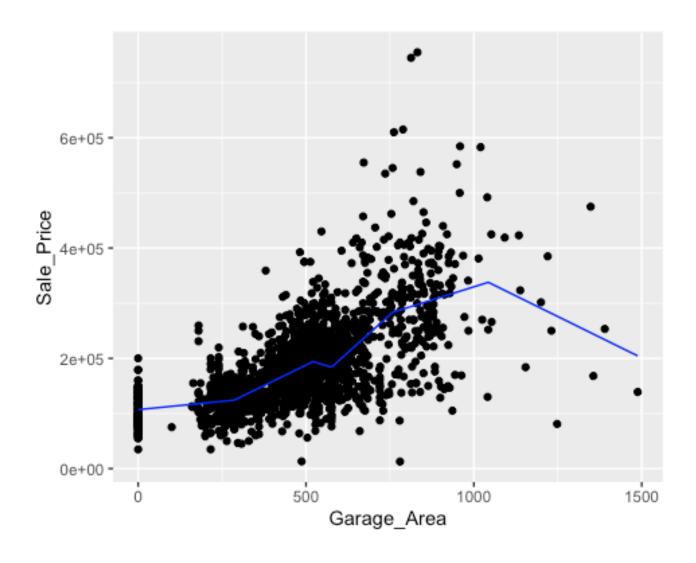
Variable Importance

evimp(mars1)

##		nsubsets	gcv	rss
##	First_Flr_SF	20	100.0	100.0
##	Second_Flr_SF	19	71.7	71.9
##	Year_Built	18	50.9	51.3
##	Garage_Area	17	34.3	35.0
##	Fireplaces	16	31.0	31.7
##	Gr_Liv_Area	15	27.6	28.4
##	Central_AirY	12	20.0	20.9
##	Bedroom_AbvGr	11	18.1	19.0
##	Lot_Area	10	16.2	17.2
##	Half_Bath	4	7.4	8.2

Interpretability

 Can view the "relationship" between predictors and target variable.





SMOOTHING

Smoothing

- GAMs can be made up of any non-parametric function of the predictor variables.
- Another popular technique is to use smoothing functions so the piecewise linear regressions are not so jagged.
- Many different types of smoothing functions:
 - LOESS (localized regression)
 - Smoothing splines
 - Regression splines

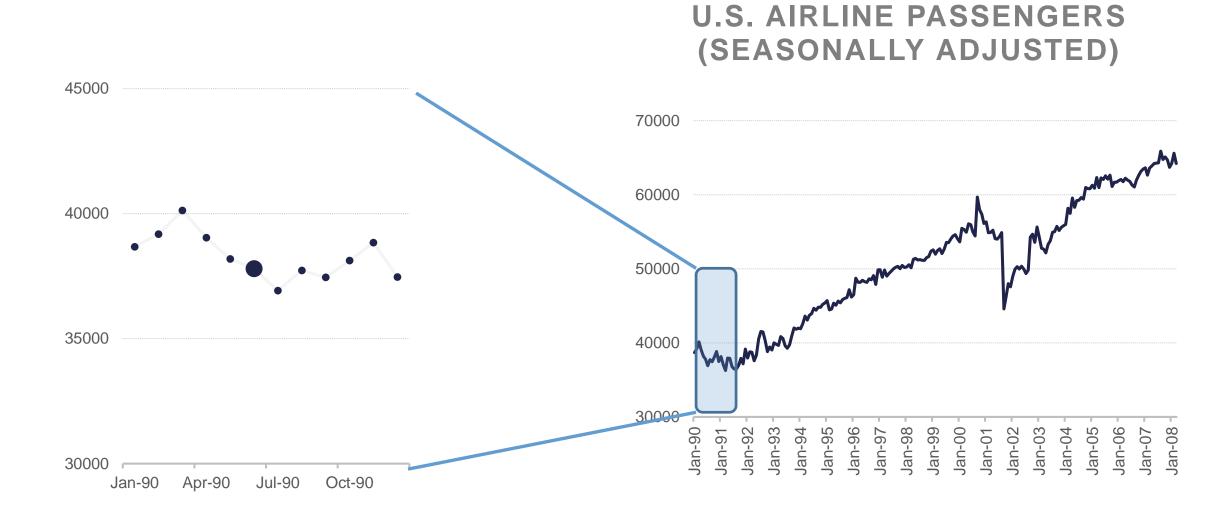
Smoothing

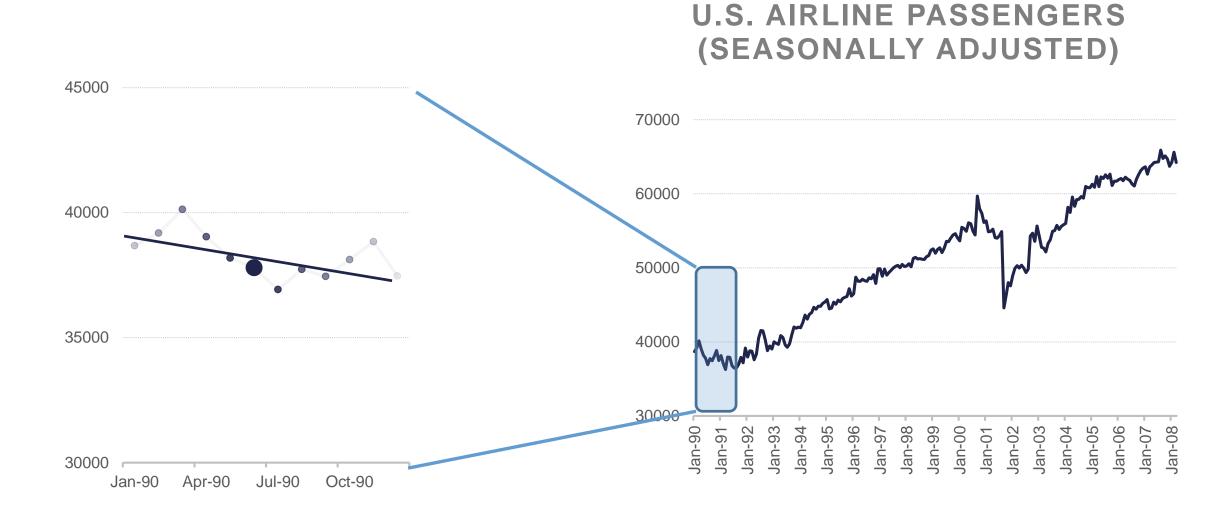
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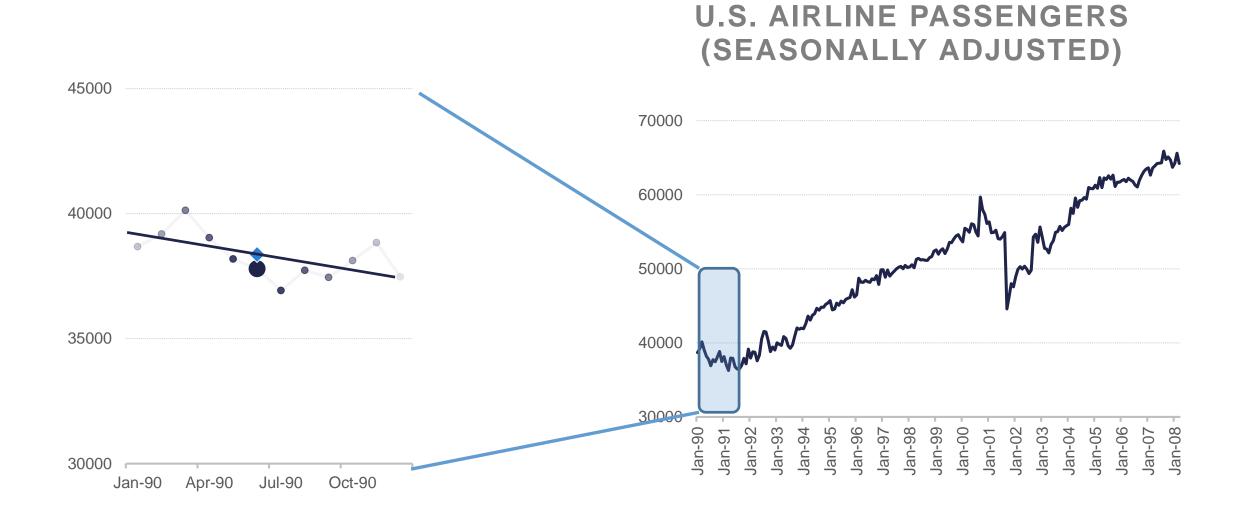
- Locally estimated scatterplot smoothing (LOESS) is a popular smoothing technique.
- Same technique used in STL decomposition in time series

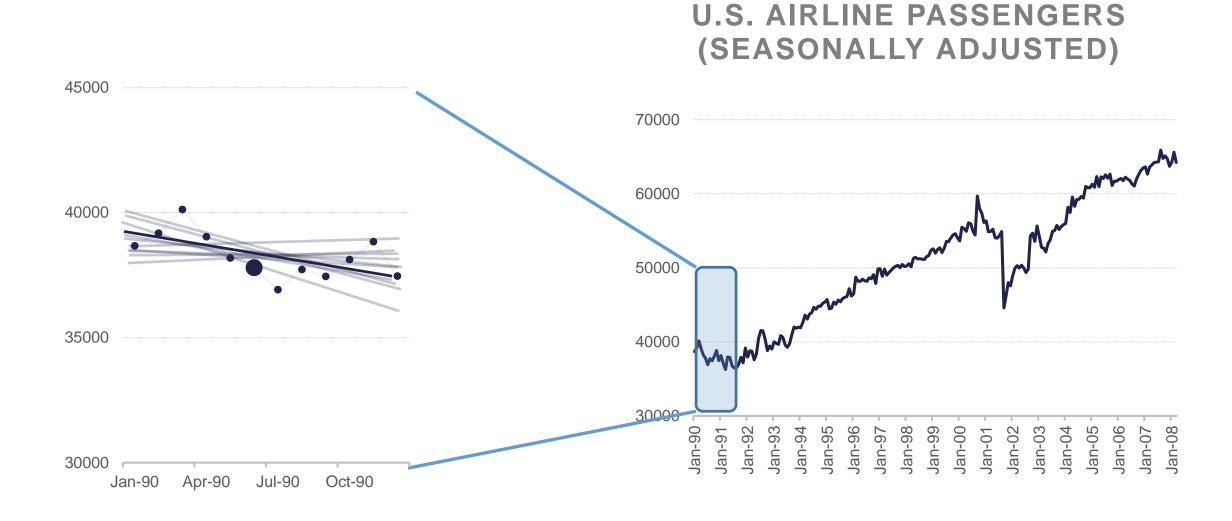
U.S. AIRLINE PASSENGERS (SEASONALLY ADJUSTED)

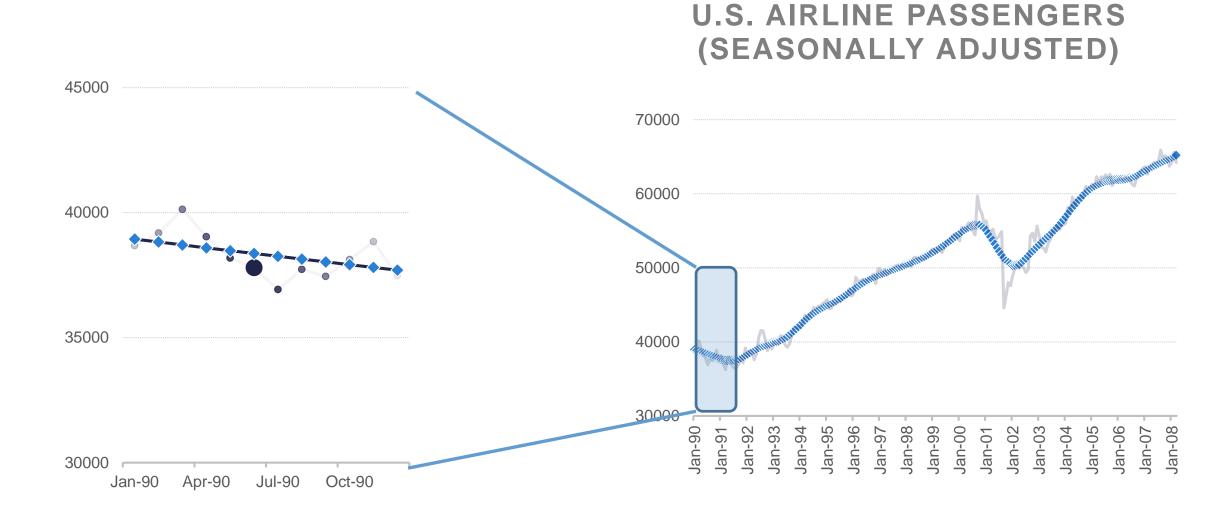








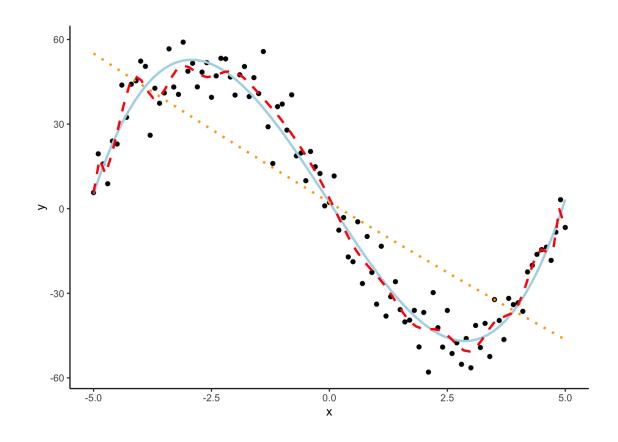




Smoothing

- GAMs can be made up of any non-parametric function of the predictor variables.
- Another popular technique is to use smoothing functions so the piecewise linear regressions are not so jagged.
- Many different types of smoothing functions:
 - LOESS (localized regression)
 - Smoothing splines
 - Regression splines

- Smoothing splines take a different approach as compared to LOESS.
- Smoothing splines have a knot at every single observation for piecewise regression – OVERFITTING!
- Use penalty parameter to counterbalance the "wiggle" of the spline.



• Smoothing splines try to find the function $s(x_i)$ that optimally fits x to the target variable y through this equation:

$$\min \sum_{i=1}^{n} (y_i - s(x_i))^2 + \lambda \int s''(t_i)^2 dt$$

• Smoothing splines try to find the function $s(x_i)$ that optimally fits x to the target variable y through this equation:

$$\min \left[\sum_{i=1}^{n} \left(y_i - s(x_i) \right)^2 \right] + \lambda \int s''(t_i)^2 dt$$

Sum of squared error!

• Smoothing splines try to find the function $s(x_i)$ that optimally fits x to the target variable y through this equation:

$$\min \sum_{i=1}^{n} (y_i - s(x_i))^2 + \lambda \int s''(t_i)^2 dt$$

Penatly (λ) applied to integral of second derivative of smoothing function

• Smoothing splines try to find the function $s(x_i)$ that optimally fits x to the target variable y through this equation:

$$\min \sum_{i=1}^{n} (y_i - s(x_i))^2 + \lambda \int s''(t_i)^2 dt$$

Penatly (λ) applied to integral of "slope of slopes" which is large when lots of "wiggle"

• Smoothing splines try to find the function $s(x_i)$ that optimally fits x to the target variable y through this equation:

$$\min \sum_{i=1}^{n} (y_i - s(x_i))^2 + \lambda \int s''(t_i)^2 dt$$

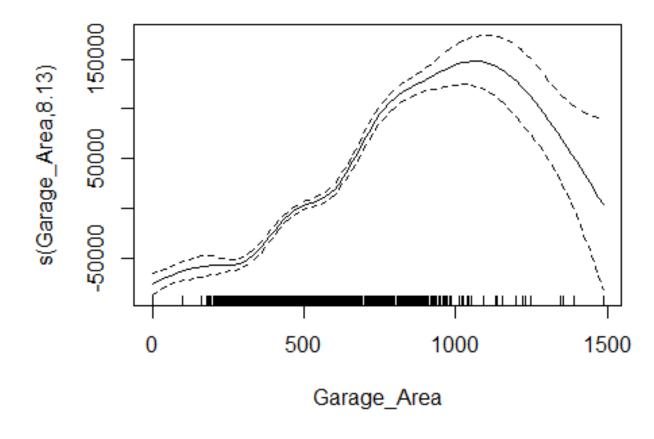
Penatly (λ) estimated with another approximation of leave one out cross validation

Smoothing

- GAMs can be made up of any non-parametric function of the predictor variables.
- Another popular technique is to use smoothing functions so the piecewise linear regressions are not so jagged.
- Many different types of smoothing functions:
 - LOESS (localized regression)
 - Smoothing splines
 - Regression splines computationally nicer version of smoothing splines

```
gam1 <- mgcv::gam(Sale Price ~ s(Garage Area), data = training)</pre>
summary(gam1)
## Family: gaussian
## Link function: identity
##
## Formula:
## Sale Price ~ s(Garage Area)
##
## Parametric coefficients:
##
              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 180897
                             1290 140.2 <2e-16 ***
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Approximate significance of smooth terms:
                   edf Ref.df F p-value
##
## s(Garage_Area) 8.134 8.769 192.5 <2e-16 ***
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.05 '.' 0.1 ' ' 1
##
## R-sq.(adj) = 0.451 Deviance explained = 45.3\%
## GCV = 3.4301e+09 Scale est. = 3.4148e+09 n = 2051
```

plot(gam1)



```
gam2 <- mgcv::gam(Sale Price ~ s(Bedroom AbvGr, k = 5) +</pre>
                                s(Year_Built) +
                                s(Mo Sold) +
                                s(Lot_Area) +
                                s(First_Flr_SF) +
                                s(Second_Flr_SF) +
                                s(Garage_Area) +
                                s(Gr_Liv_Area) +
                                s(TotRms AbvGrd) +
                                Street +
                               Central Air +
                                factor(Fireplaces) +
                               factor(Full Bath) +
                               factor(Half_Bath)
                  , method = 'REML', data = training)
summary(gam2)
```

```
## Parametric coefficients:
##
                       Estimate Std. Error t value Pr(>|t|)
                                      19681
                                              6.917 6.19e-12 ***
## (Intercept)
                         136139
                                      12710
                                              2.178
                                                      0.0295 *
## StreetPave
                          27689
## Central AirY
                          18012
                                       3168
                                              5.685 1.50e-08 ***
## factor(Fireplaces)1
                          14070
                                       1666
                                              8.443
                                                     < 2e-16 ***
## factor(Fireplaces)2
                          27137
                                       3146
                                              8.626
                                                     < 2e-16 ***
## factor(Fireplaces)3
                          15704
                                      10552
                                              1.488
                                                      0.1368
## factor(Fireplaces)4
                         -79595
                                      31469
                                             -2.529
                                                      0.0115 *
## factor(Full Bath)1
                          -5341
                                      14528
                                                      0.7132
                                             -0.368
## factor(Full Bath)2
                                      14827
                                             -0.747
                                                      0.4552
                         -11074
## factor(Full Bath)3
                           1226
                                      15787
                                              0.078
                                                      0.9381
## factor(Full Bath)4
                                      24326
                          -16271
                                             -0.669
                                                      0.5037
## factor(Half_Bath)1
                                              0.953
                            2102
                                       2206
                                                      0.3408
## factor(Half Bath)2
                                             -4.226 2.48e-05 ***
                          -38507
                                       9111
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
. . .
```

```
## Approximate significance of smooth terms:
                    edf Ref.df
##
                                    F p-value
## s(Bedroom_AbvGr) 2.653 3.165 18.789 <2e-16 ***
## s(Year_Built) 6.445 7.543 101.758 <2e-16 ***
## s(Mo Sold) 1.516 1.868
                               0.993 0.4507
## s(Lot Area) 7.186 8.193 11.726 <2e-16 ***
## s(First Flr SF) 8.063 8.765 15.548 <2e-16 ***
## s(Second Flr SF) 8.212 8.818
                               7.806 <2e-16 ***
## s(Garage Area) 7.426 8.328
                               21.654 <2e-16 ***
## s(Gr Liv Area) 8.545 8.882 14.834 <2e-16 ***
## s(TotRms AbvGrd) 3.805 4.738
                               1.921 0.0783 .
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## R-sq.(adj) = 0.851 Deviance explained = 85.6\%
## -REML = 23957 Scale est. = 9.2392e+08 n = 2051
```

GAM's with Selected Splines

```
sel.gam2 <- mgcv::gam(Sale Price ~ s(Bedroom AbvGr, k = 5) +</pre>
                                    s(Year Built) +
                                    s(Mo Sold) +
                                    s(Lot Area) +
                                    s(First Flr SF) +
                                    s(Second Flr SF) +
                                    s(Garage Area) +
                                    s(Gr Liv Area) +
                                    s(TotRms AbvGrd) +
                                    Street +
                                    Central Air +
                                   factor(Fireplaces) +
                                   factor(Full Bath) +
                                   factor(Half Bath)
                      , method = 'REML',
                        select = TRUE, data = training)
summary(sel.gam2)
```

```
## Approximate significance of smooth terms:
                       edf Ref.df
                                      F p-value
## s(Bedroom AbvGr) 2.333381
                               4 14.833 < 2e-16 ***
## s(Year Built)
                  6.945994
                               9 83.962 < 2e-16 ***
## s(Mo Sold)
                  0.007522
                               9 0.001 0.33142
                  7.561273
                               9 11.790 < 2e-16 ***
## s(Lot Area)
                               9 32.390 < 2e-16 ***
## s(First Flr SF) 8.602695
## s(Second Flr SF) 0.940878
                               9 1.716 3.89e-06 ***
## s(Garage Area)
                  6.688676
                               9 19.626 < 2e-16 ***
## s(Gr Liv Area)
                  4.839941
                               9 7.634 < 2e-16 ***
## s(TotRms AbvGrd) 3.740058
                                9 1.280 0.00845 **
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.05 '.' 0.1 ' ' 1
## R-sq.(adi) = 0.845 Deviance explained = 84.9\%
## -REML = 24081 Scale est. = 9.6657e+08 n = 2051
```



SUMMARY

Generalized Additive Models (GAMs)

 Provides general framework for adding non-linear functions to standard linear model.

$$y = \beta_0 + f_1(x_1) + f_2(x_2) + \dots + f_p(x_p) + \varepsilon$$

Can be used for regression or classification problems.

Generalized Additive Models (GAMs)

Advantages

- Allows a nonlinear relationship without trying out many transformations manually
- Improved predictions
- Still has some "interpretation"
- Computationally fast

Disadvantages

- Can incorporate interactions but can take time
- Not good for large numbers of variables – prescreening needed!
- Multicollinearity still a problem

