System Specification

July 9, 2021

This document is created directly from the definitions in the file `Qualitative.ott`, with minor modifications as listed below.

This document is intended to specify, in a readable form, the subject of the proofs of the submitted paper as well as explain the slight differences between this rendering, the submission, and the generated Coq files `Qualitative_ott.v` and `Qualitative_inf.v`.

The reason for these slight differences is partly due to the restrictions of the Ott locally nameless backend and the LNgen theory generation tool.

1. All parts of the syntax must be defined concretely in the Ott source file.
2. All bound variables need to be explicitly determined
3. All syntactic forms must bind at most one variable at a time.

The first limitation is simply to accommodate through minor manual edits of the outputs of Ott and LNgen. These edits allow us to parameterize the development on an arbitrary semiring (see `grade_sig.v`) instead of working with a specific, concrete semiring.

The second limitation affects our generation of the typing rules for pattern matching elimination forms, i.e. T-Case and T-LetPair In these rules, we need to substitute in for the scrutinee y the result type B.

The third limitation causes difficulty for the formalization of the elimination rule for products. The usual pattern matching elimination syntactic form binds two variables, one for each component of the tuple. This is the form that is used in the submission. To accommodate Ott, in the mechanization we replace the pattern matching elimination form for Σ types with a slightly more general, but less familiar, form.

Other differences between the paper and this specification are due to the avoidance of mutually recursive inductive definitions. While guarded equality and extended equality are mutually defined, as in the paper, others, such as the conditional typing judgement, has been inlined into the typing rules. As a result, rules such as T-AppC in the paper appear as two rules (rules T-AppRel and T-AppIrrel).

1 System Specification

1.1 Grammar

$tm,\ a,\ b,\ c,\ A,\ B,\ C,\ u,\ v,\ s,\ t$	::=		terms and types
	|	$\mathbf{Unit}$	
	|	$\mathbf{unit}$	
	|	$\Pi x:^{\ell} A.B$	
	|	$\lambda^{\ell} x.a$	
	|	$a\ b^{\ell}$	
	|	$\mathbf{Type}$	
	|	x	
	|	$A_1 + A_2$	
	|	$\mathbf{inj}_1\ a$	
	|	$\mathbf{inj}_2\ a$	
	|	$\mathbf{case}_{\ell}\ a\ \mathbf{of}\ b_1; b_2$	
	|	$\mathbf{case}\ a\ \mathbf{of}\ b_1; b_2$	
	|	$\Sigma x:^{\ell} A.B$	
	|	(a^{ℓ}, b)	
	|	$\mathbf{let}\ (x^{\ell},) = a\ \mathbf{in}\ b$	
	|	$(x:^{\ell} A)\ \&\ B$	
	|	$(a^{\ell}\ \&\ b)$	
	|	$\pi_1^{\ell}\ a$	
	|	$\pi_2^{\ell} a$	
$context,\ \Omega$	::=		contexts
	|	$\varnothing$	
	|	$x:^{\ell} A$	

1.2 Operational semantics

$\square$ *(Values that are types)*

$$\frac{}{\textsf{ValueType}\ \mathbf{Type}}\ \textsc{ValueType-Type} \qquad \frac{}{\textsf{ValueType}\ \Pi x:^{\ell} A.B}\ \textsc{ValueType-Pi} \qquad \frac{}{\textsf{ValueType}\ \Sigma x:^{\ell} A.B}\ \textsc{ValueType-WSigma}$$

$$\frac{}{\textsf{ValueType}\ A + B}\ \textsc{ValueType-Sum} \qquad \frac{}{\textsf{ValueType}\ \mathbf{Unit}}\ \textsc{ValueType-Unit}$$

$\square$ *(Values)*

$$\frac{\textsf{ValueType}\ a}{\textsf{ValueType}\ a}\ \textsc{V-ValueType} \qquad \frac{}{\textsf{ValueType}\ \mathbf{unit}}\ \textsc{V-TmUnit} \qquad \frac{}{\textsf{ValueType}\ (a^{\ell}, b)}\ \textsc{V-WPair}$$

$$\frac{}{\mathsf{ValueType}\ \mathbf{inj}_1\ a}\ \textsc{V-Inj1} \qquad \frac{}{\mathsf{ValueType}\ \mathbf{inj}_2\ a}\ \textsc{V-Inj2}$$

$\boxed{a \leadsto a'}$ *(Small-step operational semantics)*

$$\frac{a \leadsto a'}{a\ b^{\ell} \leadsto a'\ b^{\ell}}\ \textsc{S-AppCong} \qquad \frac{}{(\lambda^{\ell} x.a)\ b^{\ell} \leadsto a\{b/x\}}\ \textsc{S-Beta}$$

$$\frac{a \leadsto a'}{\mathbf{case}_{\ell}\ a\ \mathbf{of}\ b_1; b_2 \leadsto \mathbf{case}_{\ell}\ a'\ \mathbf{of}\ b_1; b_2}\ \textsc{S-CaseCong} \qquad \frac{}{\mathbf{case}_{\ell}\ (\mathbf{inj}_1\ a)\ \mathbf{of}\ b_1; b_2 \leadsto b_1\ a^{\ell}}\ \textsc{S-Case1Beta}$$

$$\frac{}{\mathbf{case}_{\ell}\ (\mathbf{inj}_2\ a)\ \mathbf{of}\ b_1; b_2 \leadsto b_2\ a^{\ell}}\ \textsc{S-Case2Beta} \qquad \frac{a \leadsto a'}{\mathbf{let}\ (x^{\ell},) = a\ \mathbf{in}\ b \leadsto \mathbf{let}\ (x^{\ell},) = a'\ \mathbf{in}\ b}\ \textsc{S-LetPairCong}$$

$$\frac{}{\mathbf{let}\ (x^{\ell},) = ({a_1}^{\ell}, a_2)\ \mathbf{in}\ b \leadsto b\{a_1/x\}\ {a_2}^{\perp}}\ \textsc{S-LetPairBeta}$$

1.3 Definitional equality

$\boxed{\Phi \vdash a \equiv_{\ell} b}$ *(Definitional Equality)*

$$\frac{\Phi \vdash_{\ell} a}{\Phi \vdash a \equiv_{\ell} a}\ \textsc{Eq-Refl} \qquad \frac{\Phi \vdash a \equiv_{\ell} b}{\Phi \vdash b \equiv_{\ell} a}\ \textsc{Eq-Sym} \qquad \frac{\Phi \vdash a \equiv_{\ell} b \qquad \Phi \vdash b \equiv_{\ell} c}{\Phi \vdash a \equiv_{\ell} c}\ \textsc{Eq-Trans}$$

$$\frac{\Phi, x\!: \ell \vdash b_1 \equiv_k b_2 \qquad \neg(\ell \leq k)}{\Phi \vdash b_1\{a_1/x\} \equiv_k b_2\{a_2/x\}}\ \textsc{Eq-SubstIrrel}$$

$$\frac{\Phi \vdash_{\ell} a \qquad a \leadsto b \qquad \Phi \vdash_{\ell} b}{\Phi \vdash a \equiv_{\ell} b}\ \textsc{Eq-Beta} \qquad \frac{\Phi \vdash A_1 \equiv_{\ell} A_2 \qquad \Phi, x\!: \ell \vdash B_1 \equiv_{\ell} B_2}{\Phi \vdash \Pi x\!:^{\ell_0} A_1.B_1 \equiv_{\ell} \Pi x\!:^{\ell_0} A_2.B_2}\ \textsc{Eq-Pi}$$

$$\frac{\Phi, x\!: \ell_0 \vdash b_1 \equiv_{\ell} b_2}{\Phi \vdash \lambda^{\ell_0} x.b_1 \equiv_{\ell} \lambda^{\ell_0} x.b_2}\ \textsc{Eq-Abs} \qquad \frac{\Phi \vdash b_1 \equiv_{\ell} b_2 \qquad \Phi \vdash a_1 \equiv_{\ell} a_2 \qquad \ell_0 \leq \ell}{\Phi \vdash b_1\ {a_1}^{\ell_0} \equiv_{\ell} b_2\ {a_2}^{\ell_0}}\ \textsc{Eq-AppRel} \qquad \frac{\Phi \vdash b_1 \equiv_{\ell} b_2 \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash b_1\ {a_1}^{\ell_0} \equiv_{\ell} b_2\ {a_2}^{\ell_0}}\ \textsc{Eq-AppIrrel}$$

$$\frac{\Phi \vdash \Pi x\!:^{\ell_0} A_1.B_1 \equiv_{\ell} \Pi x\!:^{\ell_0} A_2.B_2}{\Phi \vdash A_1 \equiv_{\ell} A_2}\ \textsc{Eq-PiFst} \qquad \frac{\Phi \vdash \Pi x\!:^{\ell_0} A_1.B_1 \equiv_{\ell} \Pi x\!:^{\ell_0} A_2.B_2 \qquad \Phi \vdash a_1 \equiv_{\ell} a_2}{\Phi \vdash B_1\{a_1/x\} \equiv_{\ell} B_2\{a_2/x\}}\ \textsc{Eq-PiSnd}$$

$$\frac{\Phi \vdash A_1 \equiv_\ell A_2 \qquad \Phi, x : \ell \vdash B_1 \equiv_\ell B_2}{\Phi \vdash \Sigma x{:}^{\ell_0} A_1 . B_1 \equiv_\ell \Sigma x{:}^{\ell_0} A_2 . B_2}\ \textsc{Eq-WSigma}$$

$$\frac{\Phi \vdash \Sigma x{:}^{\ell_0} A_1 . B_1 \equiv_\ell \Sigma x{:}^{\ell_0} A_2 . B_2}{\Phi \vdash A_1 \equiv_\ell A_2}\ \textsc{Eq-WSigmaFst}$$

$$\frac{\Phi \vdash \Sigma x{:}^{\ell_0} A_1 . B_1 \equiv_\ell \Sigma x{:}^{\ell_0} A_2 . B_2 \qquad \Phi \vdash_\ell a}{\Phi \vdash B_1\{a/x\} \equiv_\ell B_2\{a/x\}}\ \textsc{Eq-WSigmaSnd}$$

$$\frac{\Phi \vdash a_1 \equiv_\ell a_2 \qquad \Phi \vdash b_1 \equiv_\ell b_2 \qquad \ell_0 \leq \ell}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \equiv_\ell ({a_2}^{\ell_0}, b_2)}\ \textsc{Eq-WPairRel}$$

$$\frac{\Phi \vdash b_1 \equiv_\ell b_2 \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \equiv_\ell ({a_2}^{\ell_0}, b_2)}\ \textsc{Eq-WPairIrrel}$$

$$\frac{\Phi \vdash a_1 \equiv_\ell a_2 \qquad \Phi, x : \ell_0 \vdash b_1 \equiv_\ell b_2}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1 \,\mathbf{in}\, b_1 \equiv_\ell \mathbf{let}\,(x^{\ell_0},) = a_2 \,\mathbf{in}\, b_2}\ \textsc{Eq-LetPair}$$

$$\frac{\Phi \vdash A_1 \equiv_\ell A_1' \qquad \Phi \vdash A_2 \equiv_\ell A_2'}{\Phi \vdash A_1 + A_2 \equiv_\ell A_1' + A_2'}\ \textsc{Eq-Sum}$$

$$\frac{\Phi \vdash A_1 + A_2 \equiv_\ell A_1' + A_2'}{\Phi \vdash A_1 \equiv_\ell A_1'}\ \textsc{Eq-SumFst}$$

$$\frac{\Phi \vdash A_1 + A_2 \equiv_\ell A_1' + A_2'}{\Phi \vdash A_2 \equiv_\ell A_2'}\ \textsc{Eq-SumSnd}$$

$$\frac{\Phi \vdash a_1 \equiv_\ell a_1'}{\Phi \vdash \mathbf{inj}_1\, a_1 \equiv_\ell \mathbf{inj}_1\, a_1'}\ \textsc{Eq-Inj1}$$

$$\frac{\Phi \vdash a_2 \equiv_\ell a_2'}{\Phi \vdash \mathbf{inj}_2\, a_2 \equiv_\ell \mathbf{inj}_2\, a_2'}\ \textsc{Eq-Inj2}$$

$$\frac{\Phi \vdash a \equiv_\ell a' \qquad \Phi \vdash b_1 \equiv_\ell b_1' \qquad \Phi \vdash b_2 \equiv_\ell b_2' \qquad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\, a \,\mathbf{of}\, b_1; b_2 \equiv_\ell \mathbf{case}_{\ell_0}\, a' \,\mathbf{of}\, b_1'; b_2'}\ \textsc{Eq-Case}$$

$$\frac{}{\Phi \vdash \mathbf{Unit} \equiv_\ell \mathbf{Unit}}\ \textsc{Eq-TyUnit}$$

$$\frac{}{\Phi \vdash \mathbf{unit} \equiv_\ell \mathbf{unit}}\ \textsc{Eq-TmUnit}$$

$\boxed{\Phi \vdash_k a}$ *(Grading)*

$$\textsc{G-Type}\quad \frac{}{\Phi \vdash_\ell \mathbf{Type}}$$

$$\textsc{G-Var}\quad \frac{\ell_0 \leq \ell \qquad x : \ell_0\ \mathbf{in}\ \Phi}{\Phi \vdash_\ell x}$$

$$\textsc{G-Pi}\quad \frac{\Phi \vdash_\ell A \qquad \Phi, x : \ell \vdash_\ell B}{\Phi \vdash_\ell \Pi x :^{\ell_0} A.B}$$

$$\textsc{G-Abs}\quad \frac{\Phi, x : \ell_0 \vdash_\ell b}{\Phi \vdash_\ell \lambda^{\ell_0} x.b}$$

$$\textsc{G-AppRel}\quad \frac{\Phi \vdash_\ell b \qquad \Phi \vdash_\ell a \qquad \ell_0 \leq \ell}{\Phi \vdash_\ell b\ a^{\ell_0}}$$

$$\textsc{G-AppIrrel}\quad \frac{\Phi \vdash_\ell b \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash_\ell b\ a^{\ell_0}}$$

$$\textsc{G-WSigma}\quad \frac{\Phi \vdash_\ell A \qquad \Phi, x : \ell \vdash_\ell B}{\Phi \vdash_\ell \Sigma x{:}^{\ell_0} A.B}$$

$$\textsc{G-WPairRel}\quad \frac{\Phi \vdash_\ell a \qquad \Phi \vdash_\ell b \qquad \ell_0 \leq \ell}{\Phi \vdash_\ell (a^{\ell_0}, b)}$$

$$\textsc{G-WPairIrrel}\quad \frac{\Phi \vdash_\ell b \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash_\ell (a^{\ell_0}, b)}$$

$$\textsc{G-LetPair}\quad \frac{\Phi \vdash_\ell a \qquad (\Phi, x : \ell_0) \vdash_\ell c}{\Phi \vdash_\ell \mathbf{let}\ (x^{\ell_0},) = a\ \mathbf{in}\ c}$$

$$\textsc{G-Sum}\quad \frac{\Phi \vdash_\ell A \qquad \Phi \vdash_\ell B}{\Phi \vdash_\ell A + B}$$

$$\textsc{G-Inj1}\quad \frac{\Phi \vdash_\ell a_1}{\Phi \vdash_\ell \mathbf{inj}_1\ a_1}$$

$$\textsc{G-Inj2}\quad \frac{\Phi \vdash_\ell a_2}{\Phi \vdash_\ell \mathbf{inj}_2\ a_2}$$

$$\textsc{G-Case}\quad \frac{\Phi \vdash_\ell a \qquad \Phi \vdash_\ell b_1 \qquad \Phi \vdash_\ell b_2 \qquad \ell_0 \leq \ell}{\Phi \vdash_\ell \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2}$$

$$\textsc{G-TyUnit}\quad \frac{}{\Phi \vdash_\ell \mathbf{Unit}}$$

$$\textsc{G-TmUnit}\quad \frac{}{\Phi \vdash_\ell \mathbf{unit}}$$

1.4 Type Systems

$\boxed{\Omega \vdash a :^\ell A}$ *(Typing)*

$$\textsc{T-Type}\quad \frac{\ell \leq C}{\Omega \vdash \mathbf{Type} :^\ell \mathbf{Type}}$$

$$\textsc{T-Conv}\quad \frac{\Omega \vdash a :^\ell A \qquad \lfloor C \wedge \Omega \rfloor \vdash A \equiv_C B \qquad C \wedge \Omega \vdash B :^C \mathbf{Type}}{\Omega \vdash a :^\ell B}$$

$$\textsc{T-Var}\quad \frac{\ell_0 \leq \ell \qquad x :^{\ell_0} A \in \Omega \qquad \ell \leq C}{\Omega \vdash x :^\ell A}$$

$$\textsc{T-Pi}\quad \frac{\Omega \vdash A :^\ell \mathbf{Type} \qquad \Omega, x :^\ell A \vdash B :^\ell \mathbf{Type}}{\Omega \vdash \Pi x :^{\ell_0} A.B :^\ell \mathbf{Type}}$$

$$\textsc{T-Abs}\quad \frac{\Omega, x :^{\ell_0 \vee \ell} A \vdash b :^\ell B \qquad C \wedge \Omega \vdash (\Pi x :^{\ell_0} A.B) :^C \mathbf{Type}}{\Omega \vdash \lambda^{\ell_0} x.b :^\ell \Pi x :^{\ell_0} A.B}$$

$$\textsc{T-App}\quad \frac{\Omega \vdash b :^\ell \Pi x :^{\ell_0} A.B \qquad \Omega \vdash a :^{\ell_0 \vee \ell} A \qquad \ell_0 \leq C}{\Omega \vdash b\ a^{\ell_0} :^\ell B\{a/x\}}$$

$$\textsc{T-AppIrrel}\quad \frac{\Omega \vdash b :^\ell \Pi x :^{\ell_0} A.B \qquad C \wedge \Omega \vdash a :^C A \qquad C < \ell_0}{\Omega \vdash b\ a^{\ell_0} :^\ell B\{a/x\}}$$

$$\textsc{T-WSigma}\quad \frac{\Omega \vdash A :^\ell \mathbf{Type} \qquad \Omega, x :^\ell A \vdash B :^\ell \mathbf{Type}}{\Omega \vdash \Sigma x{:}^{\ell_0} A.B :^\ell \mathbf{Type}}$$

$$\frac{C \wedge \Omega \vdash (\Sigma x{:}^{\ell_0} A.B) :^C \textbf{Type} \quad \Omega \vdash a :^{\ell_0 \vee \ell} A \quad \Omega \vdash b :^{\ell} B\{a/x\} \quad \ell_0 \leq C}{\Omega \vdash (a^{\ell_0}, b) :^{\ell} \Sigma x{:}^{\ell_0} A.B} \ \textsc{T-WPair}$$

$$\frac{C \wedge \Omega \vdash (\Sigma x{:}^{\ell_0} A.B) :^C \textbf{Type} \quad C \wedge \Omega \vdash a :^C A \quad C < \ell_0 \quad \Omega \vdash b :^{\ell} B\{a/x\}}{\Omega \vdash (a^{\ell_0}, b) :^{\ell} \Sigma x{:}^{\ell_0} A.B} \ \textsc{T-WPairIrrel}$$

$$\frac{C \wedge \Omega \vdash (\Pi y{:}^{\ell} B.C) :^C \textbf{Type} \quad \Omega \vdash a :^{\ell} (\Sigma x{:}^{\ell_0} A.B) \quad \Omega, x{:}^{\ell_0 \vee \ell} A \vdash c :^{\ell} (\Pi y{:}^{k} B.C)}{\Omega \vdash \textbf{let}\,(x^{\ell_0},) = a\,\textbf{in}\,c :^{\ell} C} \ \textsc{T-LetPair}$$

$$\frac{\Omega \vdash A :^{\ell} \textbf{Type} \quad \Omega \vdash B :^{\ell} \textbf{Type}}{\Omega \vdash A + B :^{\ell} \textbf{Type}} \ \textsc{T-Sum}$$

$$\frac{\Omega \vdash a_1 :^{\ell} A_1 \quad C \wedge \Omega \vdash A_2 :^C \textbf{Type}}{\Omega \vdash \textbf{inj}_1\, a_1 :^{\ell} A_1 + A_2} \ \textsc{T-Inj1}$$

$$\frac{\Omega \vdash a_2 :^{\ell} A_2 \quad C \wedge \Omega \vdash A_1 :^C \textbf{Type}}{\Omega \vdash \textbf{inj}_2\, a_2 :^{\ell} A_1 + A_2} \ \textsc{T-Inj2}$$

(T-CaseC?)

$$\frac{\ell \leq C}{\Omega \vdash \textbf{Unit} :^{\ell} \textbf{Type}} \ \textsc{T-TmUnit} \qquad \frac{\ell \leq C}{\Omega \vdash \textbf{unit} :^{\ell} \textbf{Unit}} \ \textsc{T-TyUnit}$$

1.5 Auxiliary Judgements

$\boxed{\Phi \vdash^{\ell_0}_{\ell} a \sim b}$ *(Conditional Syntactic Equality)*

$$\frac{\ell_0 \leq \ell \quad \Phi \vdash a_1 \sim_{\ell} a_2}{\Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2} \ \textsc{CEq-Leq} \qquad \frac{\neg(\ell_0 \leq \ell)}{\Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2} \ \textsc{CEq-Nleq}$$

$\boxed{\Phi \vdash a \sim_{\ell} b}$ *(Guarded Syntactic Equality)*

$$\frac{}{\Phi \vdash \textbf{Type} \sim_{\ell} \textbf{Type}} \ \textsc{GEq-Type} \qquad \frac{x : \ell_0\, \textbf{in}\, \Phi \qquad \ell_0 \leq \ell}{\Phi \vdash x \sim_{\ell} x} \ \textsc{GEq-Var}$$

$$\frac{\Phi \vdash A_1 \sim_{\ell} A_2 \quad \Phi, x : \ell \vdash B_1 \sim_{\ell} B_2}{\Phi \vdash \Pi x{:}^{\ell_0} A_1.B_1 \sim_{\ell} \Pi x{:}^{\ell_0} A_2.B_2} \ \textsc{GEq-Pi} \qquad \frac{\Phi, x : \ell_0 \vdash b_1 \sim_{\ell} b_2}{\Phi \vdash \lambda^{\ell_0} x.b_1 \sim_{\ell} \lambda^{\ell_0} x.b_2} \ \textsc{GEq-Abs}$$

GEq-App
$$\frac{\Phi \vdash b_1 \sim_\ell b_2 \qquad \Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2}{\Phi \vdash b_1\ {a_1}^{\ell_0} \sim_\ell b_2\ {a_2}^{\ell_0}}$$

GEq-WSigma
$$\frac{\Phi \vdash A_1 \sim_\ell A_2 \qquad \Phi, x\!:\ell \vdash B_1 \sim_\ell B_2}{\Phi \vdash \Sigma x{:}^{\ell_0} A_1.B_1 \sim_\ell \Sigma x{:}^{\ell_0} A_2.B_2}$$

GEq-WPair
$$\frac{\Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2 \qquad \Phi \vdash b_1 \sim_\ell b_2}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \sim_\ell ({a_2}^{\ell_0}, b_2)}$$

GEq-LetPair
$$\frac{\Phi \vdash a_1 \sim_\ell a_2 \qquad \Phi, x\!:\ell_0 \vdash b_1 \sim_\ell b_2}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1\ \mathbf{in}\ b_1 \sim_\ell \mathbf{let}\,(x^{\ell_0},) = a_2\ \mathbf{in}\ b_2}$$

GEq-Sum
$$\frac{\Phi \vdash A_1 \sim_\ell A'_1 \qquad \Phi \vdash A_2 \sim_\ell A'_2}{\Phi \vdash A_1 + A_2 \sim_\ell A'_1 + A'_2}$$

GEq-Inj1
$$\frac{\Phi \vdash a_1 \sim_\ell a'_1}{\Phi \vdash \mathbf{inj}_1\ a_1 \sim_\ell \mathbf{inj}_1\ a'_1}$$

GEq-Inj2
$$\frac{\Phi \vdash a_2 \sim_\ell a'_2}{\Phi \vdash \mathbf{inj}_2\ a_2 \sim_\ell \mathbf{inj}_2\ a'_2}$$

GEq-Case
$$\frac{\Phi \vdash a \sim_\ell a' \qquad \Phi \vdash b_1 \sim_\ell b'_1 \qquad \Phi \vdash b_2 \sim_\ell b'_2 \qquad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2 \sim_\ell \mathbf{case}_{\ell_0}\ a'\ \mathbf{of}\ b'_1; b'_2}$$

GEq-TyUnit
$$\frac{}{\Phi \vdash \mathbf{Unit} \sim_\ell \mathbf{Unit}}$$

GEq-TmUnit
$$\frac{}{\Phi \vdash \mathbf{unit} \sim_\ell \mathbf{unit}}$$

$\boxed{\Phi \vdash a \Rightarrow_\ell b}$ *(Parallel reduction)*

Par-Refl
$$\frac{\Phi \vdash_\ell a}{\Phi \vdash a \Rightarrow_\ell a}$$

Par-Pi
$$\frac{\Phi \vdash A_1 \Rightarrow_\ell A_2 \qquad \Phi, x\!:\ell \vdash B_1 \Rightarrow_\ell B_2}{\Phi \vdash \Pi x\!:^{\ell_1} A_1.B_1 \Rightarrow_\ell \Pi x\!:^{\ell_1} A_2.B_2}$$

Par-AppBetaRel
$$\frac{\Phi \vdash a \Rightarrow_\ell (\lambda^{\ell_0} x.a') \qquad \Phi \vdash b \Rightarrow_\ell b' \qquad \ell_0 \leq \ell}{\Phi \vdash a\ b^{\ell_0} \Rightarrow_\ell a'\{b'/x\}}$$

Par-AppBetaIrrel
$$\frac{\Phi \vdash a \Rightarrow_\ell (\lambda^{\ell_0} x.a') \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash a\ b^{\ell_0} \Rightarrow_\ell a'\{b'/x\}}$$

$$\frac{\Phi \vdash a \Rightarrow_\ell a' \qquad \Phi \vdash b \Rightarrow_\ell b' \qquad \ell_0 \leq \ell}{\Phi \vdash a\ b^{\ell_0} \Rightarrow_\ell a'\ b'^{\ell_0}}\ \textsc{Par-AppRel}$$

$$\frac{\Phi \vdash a \Rightarrow_\ell a' \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash a\ b^{\ell_0} \Rightarrow_\ell a'\ b'^{\ell_0}}\ \textsc{Par-AppIrrel}$$

$$\frac{\Phi, x : \ell_0 \vdash b_1 \Rightarrow_\ell b_2}{\Phi \vdash \lambda^{\ell_0} x.b_1 \Rightarrow_\ell \lambda^{\ell_0} x.b_2}\ \textsc{Par-Abs}$$

$$\frac{\Phi \vdash A_1 \Rightarrow_\ell A_2 \qquad \Phi, x : \ell \vdash B_1 \Rightarrow_\ell B_2}{\Phi \vdash \Sigma x{:}^{\ell_1} A_1.B_1 \Rightarrow_\ell \Sigma x{:}^{\ell_1} A_2.B_2}\ \textsc{Par-WSigma}$$

$$\frac{\Phi \vdash a_1 \Rightarrow_\ell a_2 \qquad \Phi \vdash b_1 \Rightarrow_\ell b_2 \qquad \ell_0 \leq \ell}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \Rightarrow_\ell ({a_2}^{\ell_0}, b_2)}\ \textsc{Par-WPairRel}$$

$$\frac{\Phi \vdash b_1 \Rightarrow_\ell b_2 \qquad \neg(\ell_0 \leq \ell)}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \Rightarrow_\ell ({a_2}^{\ell_0}, b_2)}\ \textsc{Par-WPairIrrel}$$

$$\frac{\Phi \vdash a_1 \Rightarrow_\ell ({a_1'}^{\ell_0}, a_2') \qquad \Phi, x : \ell_0 \vdash b_1 \Rightarrow_\ell b_2\{x/x\}}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1\ \mathbf{in}\ b_1 \Rightarrow_\ell b_2\{a_1'/x\}\ {a_2'}^{\perp}}\ \textsc{Par-WPairBeta}$$

$$\frac{\Phi \vdash a_1 \Rightarrow_\ell a_2 \qquad \Phi, x : \ell_0 \vdash b_1 \Rightarrow_\ell b_2}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1\ \mathbf{in}\ b_1 \Rightarrow_\ell \mathbf{let}\,(x^{\ell_0},) = a_2\ \mathbf{in}\ b_2}\ \textsc{Par-LetPair}$$

$$\frac{\Phi \vdash A_1 \Rightarrow_\ell A_1' \qquad \Phi \vdash A_2 \Rightarrow_\ell A_2'}{\Phi \vdash A_1 + A_2 \Rightarrow_\ell A_1' + A_2'}\ \textsc{Par-Sum}$$

$$\frac{\Phi \vdash a_1 \Rightarrow_\ell a_1'}{\Phi \vdash \mathbf{inj}_1\ a_1 \Rightarrow_\ell \mathbf{inj}_1\ a_1'}\ \textsc{Par-Inj1}$$

$$\frac{\Phi \vdash a_2 \Rightarrow_\ell a_2'}{\Phi \vdash \mathbf{inj}_2\ a_2 \Rightarrow_\ell \mathbf{inj}_2\ a_2'}\ \textsc{Par-Inj2}$$

$$\frac{\Phi \vdash a \Rightarrow_\ell \mathbf{inj}_1\ a' \qquad \Phi \vdash b_1 \Rightarrow_\ell b_1' \qquad \Phi \vdash b_2 \Rightarrow_\ell b_2' \qquad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2 \Rightarrow_\ell b_1'\ a'^{\ell_0}}\ \textsc{Par-CaseBeta1}$$

$$\frac{\Phi \vdash a \Rightarrow_\ell \mathbf{inj}_2\ a' \qquad \Phi \vdash b_1 \Rightarrow_\ell b_1' \qquad \Phi \vdash b_2 \Rightarrow_\ell b_2' \qquad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2 \Rightarrow_\ell b_2'\ a'^{\ell_0}}\ \textsc{Par-CaseBeta2}$$

$$\textsc{Par-Case}\quad \frac{\Phi \vdash a \Rightarrow_\ell a' \qquad \Phi \vdash b_1 \Rightarrow_\ell b_1' \qquad \Phi \vdash b_2 \Rightarrow_\ell b_2' \qquad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2 \Rightarrow_\ell \mathbf{case}_{\ell_0}\ a'\ \mathbf{of}\ b_1'; b_2'}$$

$\boxed{\Phi \vdash a \Rightarrow_\ell^* b}$ *(Parallel reduction, reflexive transitive closure)*

$$\textsc{MP-Refl}\quad \frac{\Phi \vdash_\ell a}{\Phi \vdash a \Rightarrow_\ell^* a} \qquad \textsc{MP-Step}\quad \frac{\Phi \vdash a \Rightarrow_\ell b \qquad \Phi \vdash b \Rightarrow_\ell^* a'}{\Phi \vdash a \Rightarrow_\ell^* a'}$$

$\boxed{\Phi \vdash a \Leftrightarrow_\ell b}$ *(Joinability)*

$$\textsc{join}\quad \frac{\Phi \vdash a_1 \Rightarrow_\ell^* b_1 \qquad \Phi \vdash a_2 \Rightarrow_\ell^* b_2 \qquad \Phi \vdash b_1 \sim_\ell b_2}{\Phi \vdash a_1 \Leftrightarrow_\ell a_2}$$