System Specification

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This document is created directly from the definitions in the file `Qualitative.ott`, with minor modifications as listed below.

This document is intended to specify, in a readable form, the subject of the proofs of the paper as well as explain the slight differences between this rendering, the paper, and the generated Coq files `Qualitative_ott.v` and `Qualitative_inf.v`.

The reason for these slight differences is partly due to the restrictions of the Ott locally nameless backend and the LNgen theory generation tool.

1. All parts of the syntax must be defined concretely in the Ott source file.
2. All bound variables need to be explicitly determined.
3. All syntactic forms must bind at most one variable at a time.

The first limitation is simply to accommodate through minor manual edits of the outputs of Ott and LNgen. These edits allow us, for example, to parameterize the development on an arbitrary semiring (see `grade_sig.v`) instead of working with a specific, concrete semiring.

The second limitation affects our generation of the typing rules for pattern matching elimination forms, i.e. T-Case and T-LetPair In these rules, we need to substitute in for the scrutinee y the result type B.

The third limitation causes difficulty for the formalization of the elimination rule for products. The usual pattern matching elimination syntactic form binds two variables, one for each component of the tuple. This is the form that is used in the submission. To accommodate Ott, in the mechanization we replace the pattern matching elimination form for Σ types with a slightly more general, but less familiar, form.

1 System Specification

1.1 Grammar

This language is parameterized over a lattics of grades, written ℓ, and a set of sorts, s, which at must be nonempty.

$tm,\ a,\ b,\ c,\ A,\ B,\ C,\ u,\ v,\ t$	$::=$		terms and types
	$\vert$	$\mathbf{Unit}$	unit type
	$\vert$	$\mathbf{unit}$	unit term
	$\vert$	$\Pi x :^{\ell} A.B$	dependent function type
	$\vert$	$\lambda^{\ell} x : A.a$	function
	$\vert$	$a\ b^{\ell}$	function application
	$\vert$	s	sort
	$\vert$	x	variable
	$\vert$	$A_1 + A_2$	sum type
	$\vert$	$\mathbf{inj}_1\ a$	injection into sum type
	$\vert$	$\mathbf{inj}_2\ a$	injection into sum type
	$\vert$	$\mathbf{case}_{\ell}\ a\ \mathbf{of}\ b_1 ; b_2$	case elimination of sum type
	$\vert$	$\Sigma x{:}^{\ell} A.B$	dependent tuple type
	$\vert$	(a^{ℓ}, b)	tuple creation
	$\vert$	$\mathbf{let}\ (x^{\ell},) = a\ \mathbf{in}\ b$	tuple elimination
	$\vert$	$(x{:}^{\ell} A)\ \&\ B$	
	$\vert$	$(a^{\ell}\ \&\ b)$	
	$\vert$	$\pi_1^{\ell}\ a$	
	$\vert$	$\pi_2^{\ell} a$	
$context,\ \Omega$	$::=$		contexts
	$\vert$	$\varnothing$	
	$\vert$	$x :^{\ell} A$	

1.2 Operational semantics

$\square$ *(Values that are types)*

$$\frac{}{\mathsf{ValueType}\ s}\ \textsc{ValueType-Type} \qquad \frac{}{\mathsf{ValueType}\ \Pi x :^{\ell} A.B}\ \textsc{ValueType-Pi} \qquad \frac{}{\mathsf{ValueType}\ \Sigma x{:}^{\ell} A.B}\ \textsc{ValueType-WSigma}$$

$$\frac{}{\mathsf{ValueType}\ A + B}\ \textsc{ValueType-Sum} \qquad \frac{}{\mathsf{ValueType}\ \mathbf{Unit}}\ \textsc{ValueType-Unit}$$

$\square$ *(Values)*

$$\frac{\mathsf{ValueType}\ a}{\mathsf{ValueType}\ a}\ \textsc{V-ValueType} \qquad \frac{}{\mathsf{ValueType}\ \mathbf{unit}}\ \textsc{V-TmUnit} \qquad \frac{}{\mathsf{ValueType}\ (a^{\ell}, b)}\ \textsc{V-WPair}$$

$$\frac{}{\mathsf{ValueType}\ \mathbf{inj}_1\ a}\ \textsc{V-Inj1} \qquad \frac{}{\mathsf{ValueType}\ \mathbf{inj}_2\ a}\ \textsc{V-Inj2}$$

$\boxed{a \leadsto a'}$ *(Small-step operational semantics)*

$$\textsc{S-AppCong}\quad \frac{a \leadsto a'}{a\ b^{\ell} \leadsto a'\ b^{\ell}}$$

$$\textsc{S-Beta}\quad \frac{}{(\lambda^{\ell} x\!:\!A.a)\ b^{\ell} \leadsto a\{b/x\}}$$

$$\textsc{S-CaseCong}\quad \frac{a \leadsto a'}{\mathbf{case}_{\ell}\ a\ \mathbf{of}\ b_1; b_2 \leadsto \mathbf{case}_{\ell}\ a'\ \mathbf{of}\ b_1; b_2}$$

$$\textsc{S-Case1Beta}\quad \frac{}{\mathbf{case}_{\ell}\ (\mathbf{inj}_1\ a)\ \mathbf{of}\ b_1; b_2 \leadsto b_1\ a^{\ell}}$$

$$\textsc{S-Case2Beta}\quad \frac{}{\mathbf{case}_{\ell}\ (\mathbf{inj}_2\ a)\ \mathbf{of}\ b_1; b_2 \leadsto b_2\ a^{\ell}}$$

$$\textsc{S-LetPairCong}\quad \frac{a \leadsto a'}{\mathbf{let}\ (x^{\ell},) = a\ \mathbf{in}\ b \leadsto \mathbf{let}\ (x^{\ell},) = a'\ \mathbf{in}\ b}$$

$$\textsc{S-LetPairBeta}\quad \frac{}{\mathbf{let}\ (x^{\ell},) = ({a_1}^{\ell}, a_2)\ \mathbf{in}\ b \leadsto b\{a_1/x\}\ {a_2}^{\perp}}$$

1.3 Definitional equality

$\boxed{\Phi \vdash a \equiv_{\ell} b}$ *(Conditional Definitional Equality)*

$$\textsc{CDefEq-Leq}\quad \frac{k_0 \leq k \qquad \Phi \vdash a \equiv_k b}{\Phi \vdash a \equiv_k b}$$

$$\textsc{CDefEq-Nleq}\quad \frac{\neg(k_0 \leq k)}{\Phi \vdash a \equiv_k b}$$

$\boxed{\Phi \vdash a \equiv_{\ell} b}$ *(Definitional Equality)*

$$\textsc{Eq-Refl}\quad \frac{\Phi \vdash_{\ell} a}{\Phi \vdash a \equiv_{\ell} a}$$

$$\textsc{Eq-Sym}\quad \frac{\Phi \vdash a \equiv_{\ell} b}{\Phi \vdash b \equiv_{\ell} a}$$

$$\textsc{Eq-Trans}\quad \frac{\begin{array}{c}\Phi \vdash a \equiv_{\ell} b \\ \Phi \vdash b \equiv_{\ell} c\end{array}}{\Phi \vdash a \equiv_{\ell} c}$$

$$\textsc{Eq-Subst}\quad \frac{\begin{array}{c}\Phi, x\!:\!\ell \vdash b_1 \equiv_k b_2 \\ \Phi \vdash a_1 \equiv_k a_2\end{array}}{\Phi \vdash b_1\{a_1/x\} \equiv_k b_2\{a_2/x\}}$$

$$\textsc{Eq-Beta}\quad \frac{\begin{array}{c}\Phi \vdash_{\ell} a \\ a \leadsto b \qquad \Phi \vdash_{\ell} b\end{array}}{\Phi \vdash a \equiv_{\ell} b}$$

$$\textsc{Eq-Pi}\quad \frac{\begin{array}{c}\Phi \vdash A_1 \equiv_{\ell} A_2 \\ \Phi, x\!:\!\ell \vdash B_1 \equiv_{\ell} B_2\end{array}}{\Phi \vdash \Pi x\!:^{\ell_0} A_1.B_1 \equiv_{\ell} \Pi x\!:^{\ell_0} A_2.B_2}$$

$$\textsc{Eq-Abs}\quad \frac{\begin{array}{c}\Phi \vdash A_1 \equiv_{\ell} A_2 \\ \Phi, x\!:\!\ell_0 \vdash b_1 \equiv_{\ell} b_2\end{array}}{\Phi \vdash \lambda^{\ell_0} x\!:\!A_1.b_1 \equiv_{\ell} \lambda^{\ell_0} x\!:\!A_2.b_2}$$

$$\textsc{Eq-App}\quad \frac{\begin{array}{c}\Phi \vdash b_1 \equiv_{\ell} b_2 \\ \Phi \vdash a_1 \equiv_{\ell} a_2\end{array}}{\Phi \vdash b_1\ {a_1}^{\ell_0} \equiv_{\ell} b_2\ {a_2}^{\ell_0}}$$

Eq-PiFst
$$\frac{\Phi \vdash \Pi x :^{\ell_0} A_1.B_1 \equiv_\ell \Pi x :^{\ell_0} A_2.B_2}{\Phi \vdash A_1 \equiv_\ell A_2}$$

Eq-PiSnd
$$\frac{\Phi \vdash \Pi x :^{\ell_0} A_1.B_1 \equiv_\ell \Pi x :^{\ell_0} A_2.B_2 \quad \Phi \vdash_\ell a}{\Phi \vdash B_1\{a/x\} \equiv_\ell B_2\{a/x\}}$$

Eq-WSigma
$$\frac{\Phi \vdash A_1 \equiv_\ell A_2 \quad \Phi, x : \ell \vdash B_1 \equiv_\ell B_2}{\Phi \vdash \Sigma x{:}^{\ell_0} A_1.B_1 \equiv_\ell \Sigma x{:}^{\ell_0} A_2.B_2}$$

Eq-WSigmaFst
$$\frac{\Phi \vdash \Sigma x{:}^{\ell_0} A_1.B_1 \equiv_\ell \Sigma x{:}^{\ell_0} A_2.B_2}{\Phi \vdash A_1 \equiv_\ell A_2}$$

Eq-WSigmaSnd
$$\frac{\Phi \vdash \Sigma x{:}^{\ell_0} A_1.B_1 \equiv_\ell \Sigma x{:}^{\ell_0} A_2.B_2 \quad \Phi \vdash_\ell a}{\Phi \vdash B_1\{a/x\} \equiv_\ell B_2\{a/x\}}$$

Eq-WPair
$$\frac{\Phi \vdash a_1 \equiv_\ell a_2 \quad \Phi \vdash b_1 \equiv_\ell b_2}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \equiv_\ell ({a_2}^{\ell_0}, b_2)}$$

Eq-LetPair
$$\frac{\Phi \vdash a_1 \equiv_\ell a_2 \quad \Phi, x : \ell_0 \vdash b_1 \equiv_\ell b_2}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1 \,\mathbf{in}\, b_1 \equiv_\ell \mathbf{let}\,(x^{\ell_0},) = a_2 \,\mathbf{in}\, b_2}$$

Eq-Sum
$$\frac{\Phi \vdash A_1 \equiv_\ell A_1' \quad \Phi \vdash A_2 \equiv_\ell A_2'}{\Phi \vdash A_1 + A_2 \equiv_\ell A_1' + A_2'}$$

Eq-SumFst
$$\frac{\Phi \vdash A_1 + A_2 \equiv_\ell A_1' + A_2'}{\Phi \vdash A_1 \equiv_\ell A_1'}$$

Eq-SumSnd
$$\frac{\Phi \vdash A_1 + A_2 \equiv_\ell A_1' + A_2'}{\Phi \vdash A_2 \equiv_\ell A_2'}$$

Eq-Inj1
$$\frac{\Phi \vdash a_1 \equiv_\ell a_1'}{\Phi \vdash \mathbf{inj}_1\, a_1 \equiv_\ell \mathbf{inj}_1\, a_1'}$$

Eq-Inj2
$$\frac{\Phi \vdash a_2 \equiv_\ell a_2'}{\Phi \vdash \mathbf{inj}_2\, a_2 \equiv_\ell \mathbf{inj}_2\, a_2'}$$

Eq-Case
$$\frac{\Phi \vdash a \equiv_\ell a' \quad \Phi \vdash b_1 \equiv_\ell b_1' \quad \Phi \vdash b_2 \equiv_\ell b_2' \quad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\, a \,\mathbf{of}\, b_1; b_2 \equiv_\ell \mathbf{case}_{\ell_0}\, a' \,\mathbf{of}\, b_1'; b_2'}$$

Eq-TyUnit
$$\frac{}{\Phi \vdash \mathbf{Unit} \equiv_\ell \mathbf{Unit}}$$

Eq-TmUnit
$$\frac{}{\Phi \vdash \mathbf{unit} \equiv_\ell \mathbf{unit}}$$

$\boxed{\Phi \vdash^\ell_k a}$ *(Conditional Grading)*

CG-Leq
$$\frac{k_0 \leq k \qquad \Phi \vdash_k a}{\Phi \vdash^{k_0}_k a}$$

CG-Nleq
$$\frac{\neg(k_0 \leq k)}{\Phi \vdash^{k_0}_k a}$$

$\boxed{\Phi \vdash_k a}$ *(Grading)*

$$\frac{}{\Phi \vdash_\ell s}\ \text{G-Type} \qquad \frac{\ell_0 \le \ell \qquad x : \ell_0\ \mathbf{in}\ \Phi}{\Phi \vdash_\ell x}\ \text{G-Var} \qquad \frac{\Phi \vdash_\ell A \qquad \Phi, x : \ell \vdash_\ell B}{\Phi \vdash_\ell \Pi x :^{\ell_0} A.B}\ \text{G-Pi} \qquad \frac{\Phi, x : \ell_0 \vdash_\ell b \qquad \Phi \vdash_\ell^\top A}{\Phi \vdash_\ell \lambda^{\ell_0} x : A.b}\ \text{G-Abs}$$

$$\frac{\Phi \vdash_\ell b \qquad \Phi \vdash_\ell^{\ell_0} a}{\Phi \vdash_\ell b\ a^{\ell_0}}\ \text{G-App} \qquad \frac{\Phi \vdash_\ell A \qquad \Phi, x : \ell \vdash_\ell B}{\Phi \vdash_\ell \Sigma x{:}^{\ell_0} A.B}\ \text{G-WSigma} \qquad \frac{\Phi \vdash_\ell^{\ell_0} a \qquad \Phi \vdash_\ell b}{\Phi \vdash_\ell (a^{\ell_0}, b)}\ \text{G-WPair}$$

$$\frac{\Phi \vdash_\ell a \qquad (\Phi, x : \ell_0) \vdash_\ell c}{\Phi \vdash_\ell \mathbf{let}\,(x^{\ell_0},) = a\ \mathbf{in}\ c}\ \text{G-LetPair} \qquad \frac{\Phi \vdash_\ell A \qquad \Phi \vdash_\ell B}{\Phi \vdash_\ell A + B}\ \text{G-Sum} \qquad \frac{\Phi \vdash_\ell a_1}{\Phi \vdash_\ell \mathbf{inj}_1\ a_1}\ \text{G-Inj1}$$

$$\frac{\Phi \vdash_\ell a_2}{\Phi \vdash_\ell \mathbf{inj}_2\ a_2}\ \text{G-Inj2} \qquad \frac{\Phi \vdash_\ell a \qquad \Phi \vdash_\ell b_1 \qquad \Phi \vdash_\ell b_2 \qquad \ell_0 \le \ell}{\Phi \vdash_\ell \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2}\ \text{G-Case} \qquad \frac{}{\Phi \vdash_\ell \mathbf{Unit}}\ \text{G-TyUnit} \qquad \frac{}{\Phi \vdash_\ell \mathbf{unit}}\ \text{G-TmUnit}$$

1.4 Type System

As in pure type systems, this type system is parameterized by a set of axioms ($\mathbf{axiom}\ s_1\ s_2$) and rules ($\mathbf{rule}\ s_1\ s_2\ s_3$) that govern the treatment of sorts.

$\boxed{\Omega \Vdash a :^\ell A}$ *(Conditional Typing)*

$$\frac{\Omega \vdash a :^\ell A \qquad \ell \le C}{\Omega \Vdash a :^\ell A}\ \text{CT-Leq} \qquad \frac{C \wedge \Omega \vdash a :^C A \qquad C < \ell}{\Omega \Vdash a :^\ell A}\ \text{CT-Top}$$

$\boxed{\Omega \vdash a :^\ell A}$ *(Typing)*

$$\frac{\ell \le C \qquad \mathbf{axiom}\ s_1\ s_2}{\Omega \vdash s_1 :^\ell s_2}\ \text{T-Type} \qquad \frac{\Omega \vdash a :^\ell A \qquad |C \wedge \Omega| \vdash A \equiv_C B \qquad C \wedge \Omega \vdash B :^C s}{\Omega \vdash a :^\ell B}\ \text{T-Conv} \qquad \frac{\ell_0 \le \ell \qquad x :^{\ell_0} A \in \Omega \qquad \ell \le C}{\Omega \vdash x :^\ell A}\ \text{T-Var}$$

$$\frac{\Omega \vdash A :^\ell s_1 \qquad \Omega, x :^\ell A \vdash B :^\ell s_2 \qquad \mathbf{rule}\ s_1\ s_2\ s_3}{\Omega \vdash \Pi x :^{\ell_0} A.B :^\ell s_3}\ \text{T-Pi} \qquad \frac{\Omega, x :^{\ell_0 \vee \ell} A \vdash b :^\ell B \qquad C \wedge \Omega \vdash (\Pi x :^{\ell_0} A.B) :^C s}{\Omega \vdash \lambda^{\ell_0} x : A.b :^\ell \Pi x :^{\ell_0} A.B}\ \text{T-Abs} \qquad \frac{\Omega \vdash b :^\ell \Pi x :^{\ell_0} A.B \qquad \Omega \vdash a :^{\ell_0 \vee \ell} A \qquad \ell_0 \le C}{\Omega \vdash b\ a^{\ell_0} :^\ell B\{a/x\}}\ \text{T-App}$$

$$\textsc{T-AppIrrel}\quad \frac{\begin{array}{c} \Omega \vdash b :^{\ell} \Pi x :^{\ell_0} A.B \\ C \wedge \Omega \vdash a :^{C} A \\ C < \ell_0 \end{array}}{\Omega \vdash b\ a^{\ell_0} :^{\ell} B\{a/x\}}$$

$$\textsc{T-WSigma}\quad \frac{\begin{array}{c} \Omega \vdash A :^{\ell} s_1 \\ \Omega, x :^{\ell} A \vdash B :^{\ell} s_2 \\ \mathbf{rule}\, s_1\, s_2\, s_3 \end{array}}{\Omega \vdash \Sigma x{:}^{\ell_0} A.B :^{\ell} s_3}$$

$$\textsc{T-WPair}\quad \frac{\begin{array}{c} C \wedge \Omega \vdash (\Sigma x{:}^{\ell_0} A.B) :^{C} s \\ \Omega \vdash a :^{\ell_0 \vee \ell} A \\ \Omega \vdash b :^{\ell} B\{a/x\} \\ \ell_0 \leq C \end{array}}{\Omega \vdash (a^{\ell_0}, b) :^{\ell} \Sigma x{:}^{\ell_0} A.B}$$

$$\textsc{T-WPairIrrel}\quad \frac{\begin{array}{c} C \wedge \Omega \vdash (\Sigma x{:}^{\ell_0} A.B) :^{C} s \\ C \wedge \Omega \vdash a :^{C} A \\ C < \ell_0 \\ \Omega \vdash b :^{\ell} B\{a/x\} \end{array}}{\Omega \vdash (a^{\ell_0}, b) :^{\ell} \Sigma x{:}^{\ell_0} A.B}$$

$$\textsc{T-LetPairC}\quad \frac{\begin{array}{c} \Omega \vdash a :^{\ell} \Sigma x{:}^{\ell_0} A.B \\ \Omega, x :^{\ell_0 \vee \ell} A \vdash c :^{\ell} \Pi y :^{k} B.C\{(x^{\ell_0}, y)/z\} \\ \Omega, z :^{\top} (\Sigma x{:}^{\ell_0} A.B) \Vdash C :^{\top} s \end{array}}{\Omega \vdash \mathbf{let}\, (x^{\ell_0},) = a\, \mathbf{in}\, c :^{\ell} C\{a/z\}}$$

$$\textsc{T-Sum}\quad \frac{\begin{array}{c} \Omega \vdash A :^{\ell} s \\ \Omega \vdash B :^{\ell} s \end{array}}{\Omega \vdash A + B :^{\ell} s}$$

$$\textsc{T-Inj1}\quad \frac{\begin{array}{c} \Omega \vdash a_1 :^{\ell} A_1 \\ C \wedge \Omega \vdash A_1 + A_2 :^{C} s \end{array}}{\Omega \vdash \mathbf{inj}_1\, a_1 :^{\ell} A_1 + A_2}$$

$$\textsc{T-Inj2}\quad \frac{\begin{array}{c} \Omega \vdash a_2 :^{\ell} A_2 \\ C \wedge \Omega \vdash A_1 + A_2 :^{C} s \end{array}}{\Omega \vdash \mathbf{inj}_2\, a_2 :^{\ell} A_1 + A_2}$$

$$\textsc{T-CaseC}\quad \frac{\begin{array}{c} \Omega \vdash a :^{\ell} A_1 + A_2 \\ \Omega \vdash b_1 :^{\ell} \Pi x :^{\ell_0} A_1.B\{\mathbf{inj}_1\, x/z\} \\ \Omega \vdash b_2 :^{\ell} \Pi y :^{\ell_0} A_2.B\{\mathbf{inj}_2\, y/z\} \\ \ell_0 \leq \ell \\ ((C \wedge \Omega), z :^{C} A_1 + A_2) \vdash B :^{C} s \end{array}}{\Omega \vdash \mathbf{case}_{\ell_0}\, a\, \mathbf{of}\, b_1; b_2 :^{\ell} B\{a/z\}}$$

$$\textsc{T-TmUnit}\quad \frac{\ell \leq C}{\Omega \vdash \mathbf{Unit} :^{\ell} s} \qquad \textsc{T-TyUnit}\quad \frac{\ell \leq C}{\Omega \vdash \mathbf{unit} :^{\ell} \mathbf{Unit}}$$

1.5 Auxiliary Judgements

$\boxed{\Phi \vdash^{\ell_0}_{\ell} a \sim b}$ *(Conditional Graded Equality)*

$$\textsc{CEq-Leq}\quad \frac{\begin{array}{c} \ell_0 \leq \ell \\ \Phi \vdash a_1 \sim_{\ell} a_2 \end{array}}{\Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2} \qquad \textsc{CEq-Nleq}\quad \frac{\neg(\ell_0 \leq \ell)}{\Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2}$$

$\boxed{\Phi \vdash a \sim_{\ell} b}$ *(Graded Syntactic Equality)*

$$\textsc{GEq-Type}\quad \frac{}{\Phi \vdash s \sim_{\ell} s} \qquad \textsc{GEq-Var}\quad \frac{x : \ell_0\, \mathbf{in}\, \Phi \qquad \ell_0 \leq \ell}{\Phi \vdash x \sim_{\ell} x}$$

$$\textsc{GEq-Pi}\quad \frac{\begin{array}{c} \Phi \vdash A_1 \sim_{\ell} A_2 \\ \Phi, x : \ell \vdash B_1 \sim_{\ell} B_2 \end{array}}{\Phi \vdash \Pi x :^{\ell_0} A_1.B_1 \sim_{\ell} \Pi x :^{\ell_0} A_2.B_2}$$

$$\frac{\Phi, x:\ell_0 \vdash b_1 \sim_\ell b_2 \qquad \Phi \vdash^{\top}_{\ell} A_1 \sim A_2}{\Phi \vdash \lambda^{\ell_0} x:A_1.b_1 \sim_\ell \lambda^{\ell_0} x:A_2.b_2}\ \textsc{GEq-Abs}$$

$$\frac{\Phi \vdash b_1 \sim_\ell b_2 \qquad \Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2}{\Phi \vdash b_1\ {a_1}^{\ell_0} \sim_\ell b_2\ {a_2}^{\ell_0}}\ \textsc{GEq-App}$$

$$\frac{\Phi \vdash A_1 \sim_\ell A_2 \qquad \Phi, x:\ell \vdash B_1 \sim_\ell B_2}{\Phi \vdash \Sigma x{:}^{\ell_0} A_1.B_1 \sim_\ell \Sigma x{:}^{\ell_0} A_2.B_2}\ \textsc{GEq-WSigma}$$

$$\frac{\Phi \vdash^{\ell_0}_{\ell} a_1 \sim a_2 \qquad \Phi \vdash b_1 \sim_\ell b_2}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \sim_\ell ({a_2}^{\ell_0}, b_2)}\ \textsc{GEq-WPair}$$

$$\frac{\Phi \vdash a_1 \sim_\ell a_2 \qquad \Phi, x:\ell_0 \vdash b_1 \sim_\ell b_2}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1\ \mathbf{in}\ b_1 \sim_\ell \mathbf{let}\,(x^{\ell_0},) = a_2\ \mathbf{in}\ b_2}\ \textsc{GEq-LetPair}$$

$$\frac{\Phi \vdash A_1 \sim_\ell A_1' \qquad \Phi \vdash A_2 \sim_\ell A_2'}{\Phi \vdash A_1 + A_2 \sim_\ell A_1' + A_2'}\ \textsc{GEq-Sum}$$

$$\frac{\Phi \vdash a_1 \sim_\ell a_1'}{\Phi \vdash \mathbf{inj}_1\ a_1 \sim_\ell \mathbf{inj}_1\ a_1'}\ \textsc{GEq-Inj1}$$

$$\frac{\Phi \vdash a_2 \sim_\ell a_2'}{\Phi \vdash \mathbf{inj}_2\ a_2 \sim_\ell \mathbf{inj}_2\ a_2'}\ \textsc{GEq-Inj2}$$

$$\frac{\Phi \vdash a \sim_\ell a' \qquad \Phi \vdash b_1 \sim_\ell b_1' \qquad \Phi \vdash b_2 \sim_\ell b_2' \qquad \ell_0 \leq \ell}{\Phi \vdash \mathbf{case}_{\ell_0}\ a\ \mathbf{of}\ b_1; b_2 \sim_\ell \mathbf{case}_{\ell_0}\ a'\ \mathbf{of}\ b_1'; b_2'}\ \textsc{GEq-Case}$$

$$\frac{}{\Phi \vdash \mathbf{Unit} \sim_\ell \mathbf{Unit}}\ \textsc{GEq-TyUnit}$$

$$\frac{}{\Phi \vdash \mathbf{unit} \sim_\ell \mathbf{unit}}\ \textsc{GEq-TmUnit}$$

$\boxed{\Phi \vdash^{\ell_0}_{\ell} a \Rightarrow b}$ *(Conditional Parallel Reduction)*

$$\frac{\ell_0 \leq \ell \qquad \Phi \vdash a_1 \Rightarrow_\ell a_2}{\Phi \vdash^{\ell_0}_{\ell} a_1 \Rightarrow a_2}\ \textsc{CPar-Leq}$$

$$\frac{\neg(\ell_0 \leq \ell)}{\Phi \vdash^{\ell_0}_{\ell} a_1 \Rightarrow a_2}\ \textsc{CPar-Nleq}$$

$\boxed{\Phi \vdash a \Rightarrow_\ell b}$ *(Parallel reduction)*

$$\frac{\Phi \vdash_\ell a}{\Phi \vdash a \Rightarrow_\ell a}\ \textsc{Par-Refl}$$

$$\frac{\Phi \vdash A_1 \Rightarrow_\ell A_2 \qquad \Phi, x:\ell \vdash B_1 \Rightarrow_\ell B_2}{\Phi \vdash \Pi x{:}^{\ell_1} A_1.B_1 \Rightarrow_\ell \Pi x{:}^{\ell_1} A_2.B_2}\ \textsc{Par-Pi}$$

$$\textsc{Par-AppBeta}\quad \frac{\begin{array}{c}\Phi \vdash a \Rightarrow_{\ell} (\lambda^{\ell_0} x\!:\!A.a') \\ \Phi \vdash^{\ell_0}_{\ell} b \Rightarrow b'\end{array}}{\Phi \vdash a\ b^{\ell_0} \Rightarrow_{\ell} a'\{b'/x\}}$$

$$\textsc{Par-App}\quad \frac{\begin{array}{c}\Phi \vdash a \Rightarrow_{\ell} a' \\ \Phi \vdash^{\ell_0}_{\ell} b \Rightarrow b'\end{array}}{\Phi \vdash a\ b^{\ell_0} \Rightarrow_{\ell} a'\ {b'}^{\ell_0}}$$

$$\textsc{Par-Abs}\quad \frac{\begin{array}{c}\Phi, x\!:\ell_0 \vdash b_1 \Rightarrow_{\ell} b_2 \\ \Phi \vdash^{\top}_{\ell} A_1 \Rightarrow A_2\end{array}}{\Phi \vdash \lambda^{\ell_0} x\!:\!A_1.b_1 \Rightarrow_{\ell} \lambda^{\ell_0} x\!:\!A_2.b_2}$$

$$\textsc{Par-WSigma}\quad \frac{\begin{array}{c}\Phi \vdash A_1 \Rightarrow_{\ell} A_2 \\ \Phi, x\!:\ell \vdash B_1 \Rightarrow_{\ell} B_2\end{array}}{\Phi \vdash \Sigma x{:}^{\ell_1} A_1.B_1 \Rightarrow_{\ell} \Sigma x{:}^{\ell_1} A_2.B_2}$$

$$\textsc{Par-WPair}\quad \frac{\begin{array}{c}\Phi \vdash^{\ell_0}_{\ell} a_1 \Rightarrow a_2 \\ \Phi \vdash b_1 \Rightarrow_{\ell} b_2\end{array}}{\Phi \vdash ({a_1}^{\ell_0}, b_1) \Rightarrow_{\ell} ({a_2}^{\ell_0}, b_2)}$$

$$\textsc{Par-WPairBeta}\quad \frac{\begin{array}{c}\Phi \vdash a_1 \Rightarrow_{\ell} ({a_1'}^{\ell_0}, a_2') \\ \Phi, x\!:\ell_0 \vdash b_1 \Rightarrow_{\ell} b_2\{x/x\}\end{array}}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1\,\mathbf{in}\,b_1 \Rightarrow_{\ell} b_2\{a_1'/x\}\ {a_2'}^{\perp}}$$

$$\textsc{Par-LetPair}\quad \frac{\begin{array}{c}\Phi \vdash a_1 \Rightarrow_{\ell} a_2 \\ \Phi, x\!:\ell_0 \vdash b_1 \Rightarrow_{\ell} b_2\end{array}}{\Phi \vdash \mathbf{let}\,(x^{\ell_0},) = a_1\,\mathbf{in}\,b_1 \Rightarrow_{\ell} \mathbf{let}\,(x^{\ell_0},) = a_2\,\mathbf{in}\,b_2}$$

$$\textsc{Par-Sum}\quad \frac{\begin{array}{c}\Phi \vdash A_1 \Rightarrow_{\ell} A_1' \\ \Phi \vdash A_2 \Rightarrow_{\ell} A_2'\end{array}}{\Phi \vdash A_1 + A_2 \Rightarrow_{\ell} A_1' + A_2'}$$

$$\textsc{Par-Inj1}\quad \frac{\Phi \vdash a_1 \Rightarrow_{\ell} a_1'}{\Phi \vdash \mathbf{inj}_1\, a_1 \Rightarrow_{\ell} \mathbf{inj}_1\, a_1'}$$

$$\textsc{Par-Inj2}\quad \frac{\Phi \vdash a_2 \Rightarrow_{\ell} a_2'}{\Phi \vdash \mathbf{inj}_2\, a_2 \Rightarrow_{\ell} \mathbf{inj}_2\, a_2'}$$

$$\textsc{Par-CaseBeta1}\quad \frac{\begin{array}{c}\Phi \vdash a \Rightarrow_{\ell} \mathbf{inj}_1\, a' \\ \Phi \vdash b_1 \Rightarrow_{\ell} b_1' \\ \Phi \vdash b_2 \Rightarrow_{\ell} b_2' \\ \ell_0 \leq \ell\end{array}}{\Phi \vdash \mathbf{case}_{\ell_0}\, a\,\mathbf{of}\, b_1; b_2 \Rightarrow_{\ell} b_1'\ {a'}^{\ell_0}}$$

$$\textsc{Par-CaseBeta2}\quad \frac{\begin{array}{c}\Phi \vdash a \Rightarrow_{\ell} \mathbf{inj}_2\, a' \\ \Phi \vdash b_1 \Rightarrow_{\ell} b_1' \\ \Phi \vdash b_2 \Rightarrow_{\ell} b_2' \\ \ell_0 \leq \ell\end{array}}{\Phi \vdash \mathbf{case}_{\ell_0}\, a\,\mathbf{of}\, b_1; b_2 \Rightarrow_{\ell} b_2'\ {a'}^{\ell_0}}$$

$$\textsc{Par-Case}\quad \frac{\begin{array}{c}\Phi \vdash a \Rightarrow_{\ell} a' \\ \Phi \vdash b_1 \Rightarrow_{\ell} b_1' \\ \Phi \vdash b_2 \Rightarrow_{\ell} b_2' \\ \ell_0 \leq \ell\end{array}}{\Phi \vdash \mathbf{case}_{\ell_0}\, a\,\mathbf{of}\, b_1; b_2 \Rightarrow_{\ell} \mathbf{case}_{\ell_0}\, a'\,\mathbf{of}\, b_1'; b_2'}$$

$\boxed{\Phi \vdash a \Rightarrow_{\ell}^{*} b}$ *(Parallel reduction, reflexive transitive closure)*

$$\text{MP-Refl}\quad \frac{\Phi \vdash_{\ell} a}{\Phi \vdash a \Rightarrow_{\ell}^{*} a} \qquad \text{MP-Step}\quad \frac{\Phi \vdash a \Rightarrow_{\ell} b \qquad \Phi \vdash b \Rightarrow_{\ell}^{*} a'}{\Phi \vdash a \Rightarrow_{\ell}^{*} a'}$$

$\boxed{\Phi \vdash a \Leftrightarrow_{\ell} b}$ *(Joinability)*

$$\text{Join}\quad \frac{\Phi \vdash a_1 \Rightarrow_{\ell}^{*} b_1 \qquad \Phi \vdash a_2 \Rightarrow_{\ell}^{*} b_2 \qquad \Phi \vdash b_1 \sim_{\ell} b_2}{\Phi \vdash a_1 \Leftrightarrow_{\ell} a_2}$$