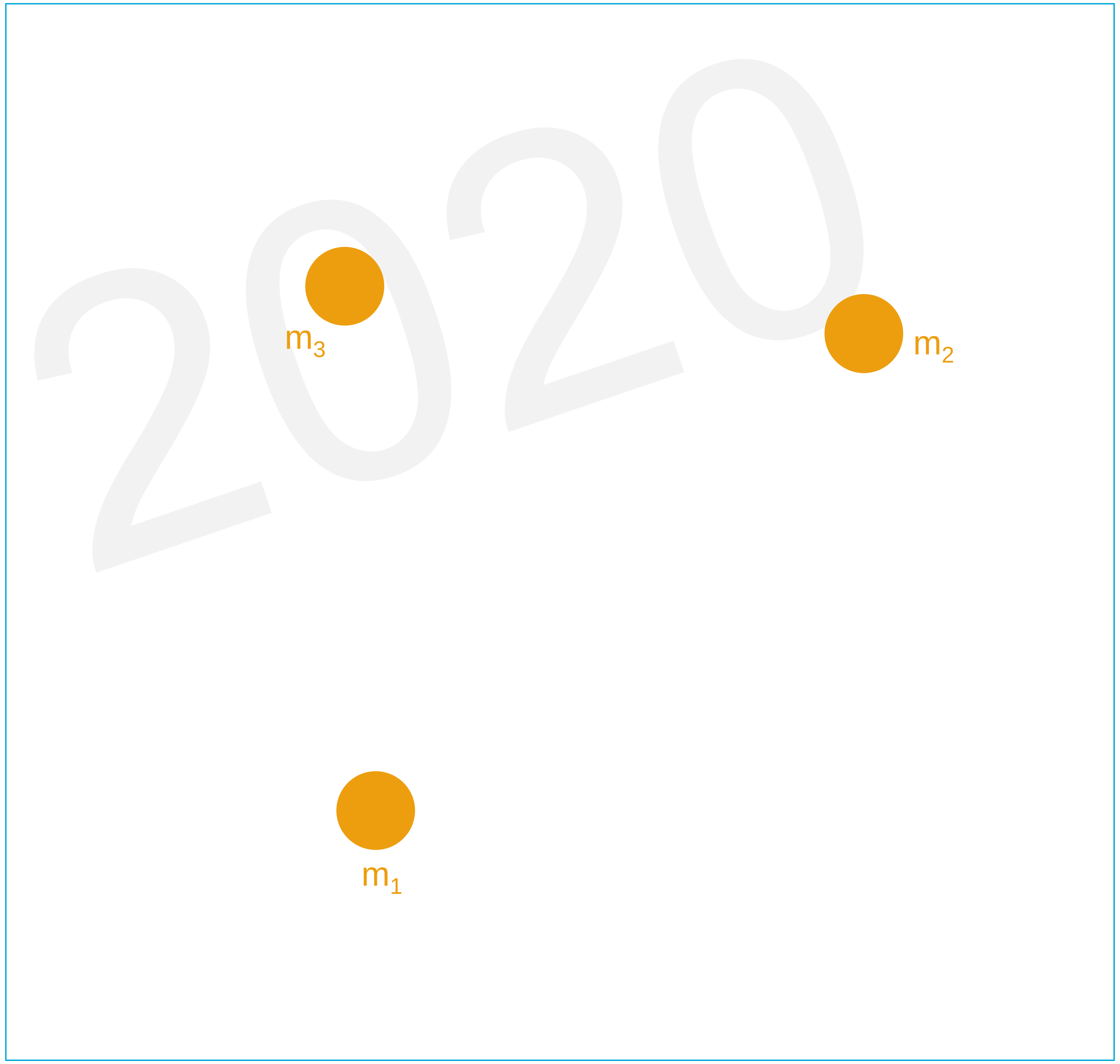
Recap of the cr3bp

Wakker, K.F., *Fundamentals of Astrodynamics*, Lecture Notes, Delft University of Technology, 2015, Sections 3.1-3.3 (3.4,3.6)

cr3bp – the assumptions

- The **three-body** problem
  - Three bodies



cr3bp – the assumptions

- The three-body problem
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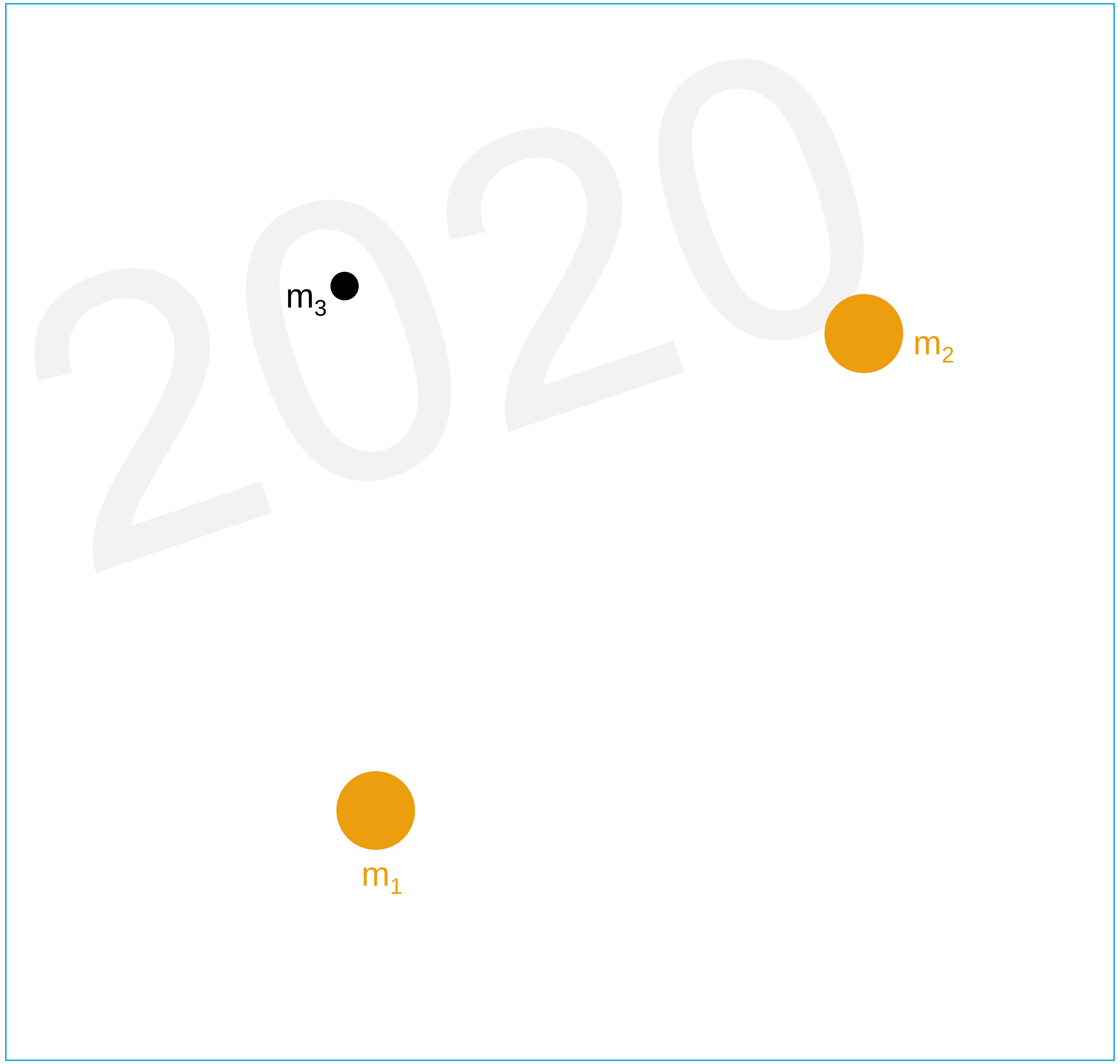
- The **restricted** three-body problem
  - $m_3 << m_1$
  - $m_3 << m_2$
  - *Eg.*, Sun + Earth + spacecraft



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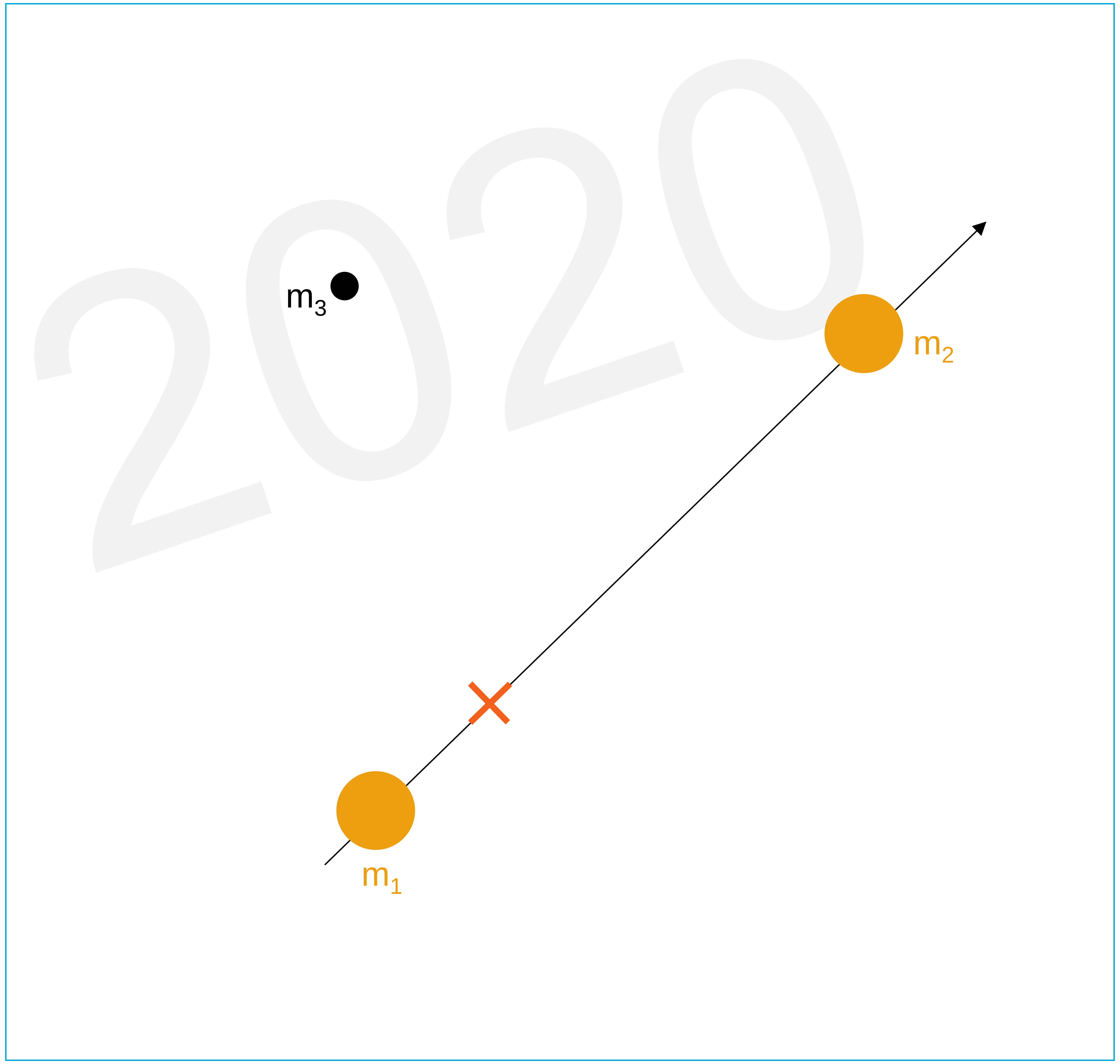
  - **System center of mass** along the line m_1 - m_2



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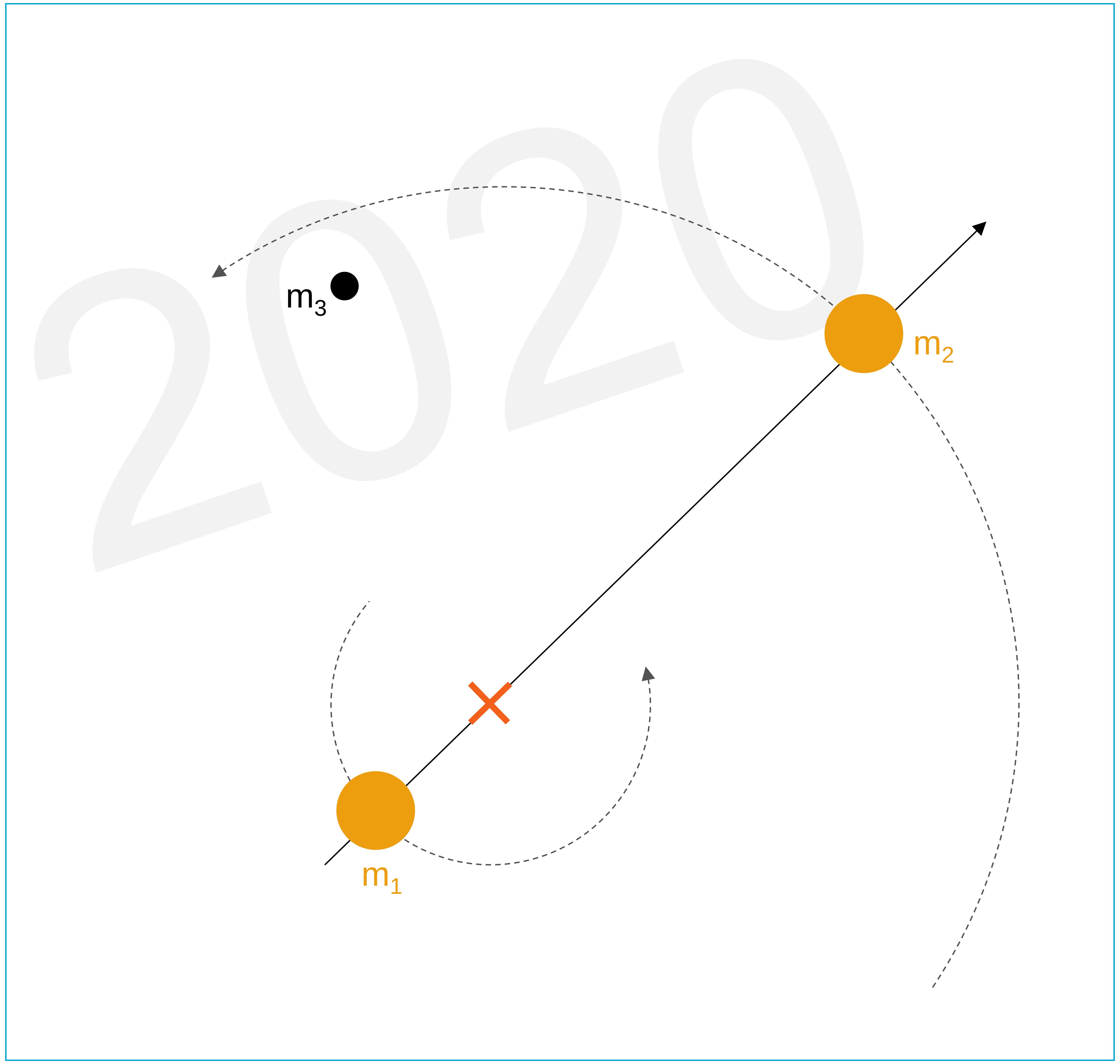
- **Circular** restricted three-body problem
  - m_1 and m_2 orbit in circular orbits around barycenter
  - Location of m_1 and m_2 along connecting line is fixed



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- **Validity of assumptions?**

System	Eccentricity
Sun – Venus	0.007
Sun – Neptune	0.009
Sun – Earth	0.017
Sun – Uranus	0.046
Sun - Jupiter	0.048
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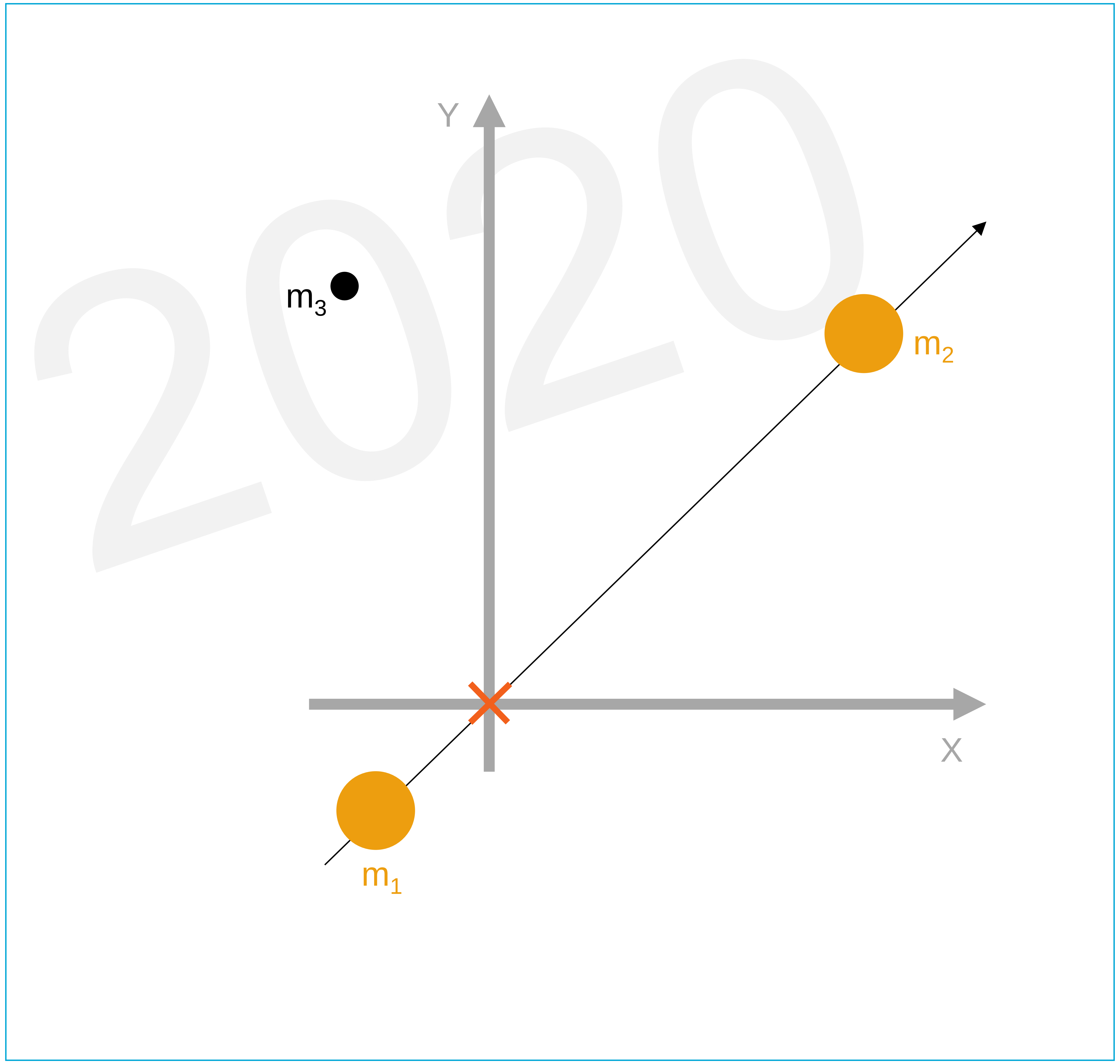
Where to draw the line?

cr3bp – the equations of motion

- Define an **inertial reference frame** I(X,Y,Z)
  - Centered at barycenter
  - (X,Y)-plane in orbital plane of m_1 and m_2



cr3bp – the equations of motion

- Define an inertial reference frame I(X,Y,Z)
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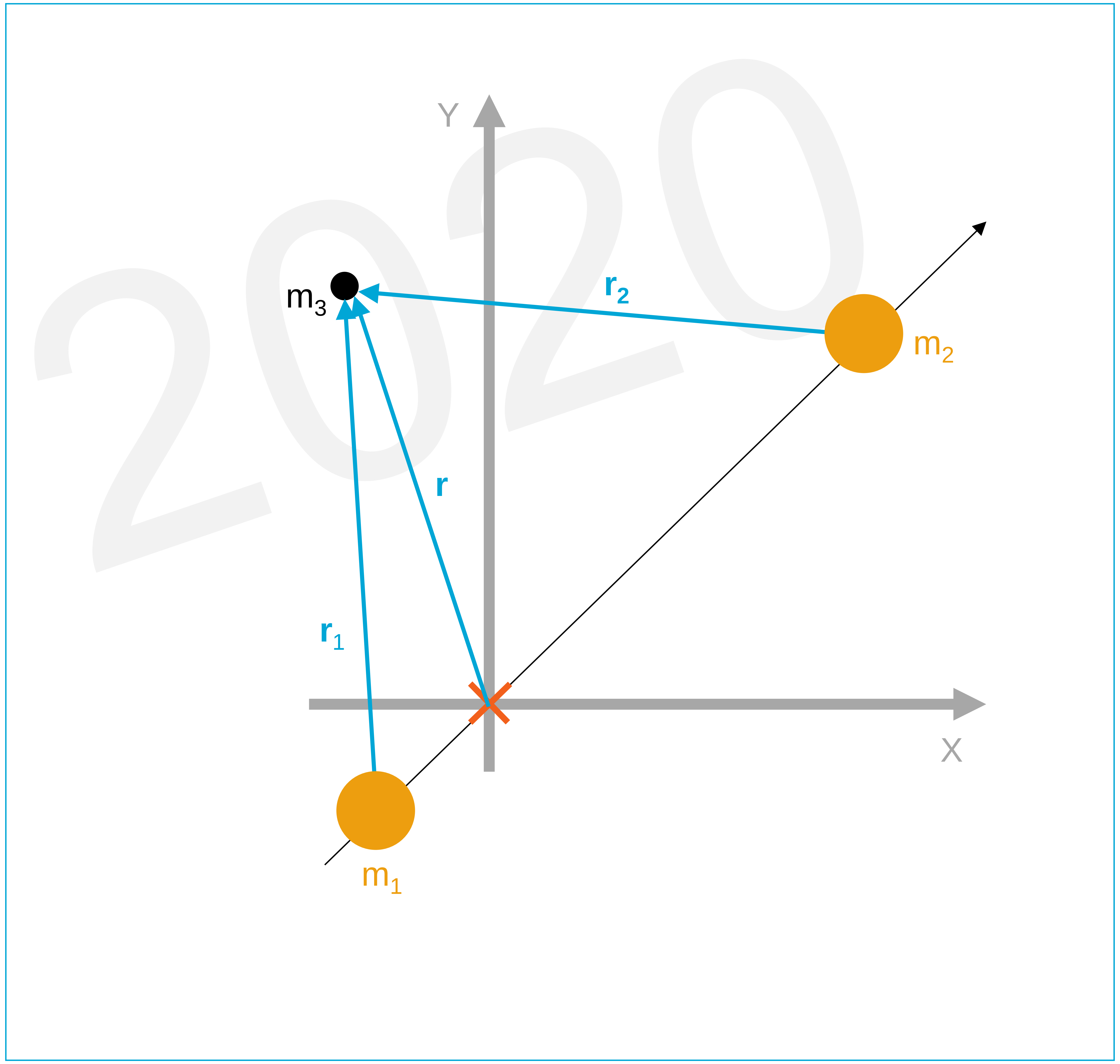
- Define the **position vectors** of m_3 with respect to
  - Barycenter $\rightarrow$ $\mathbf{r}$
  - $m_1 \rightarrow \mathbf{r}_1$
  - $m_2 \rightarrow \mathbf{r}_2$



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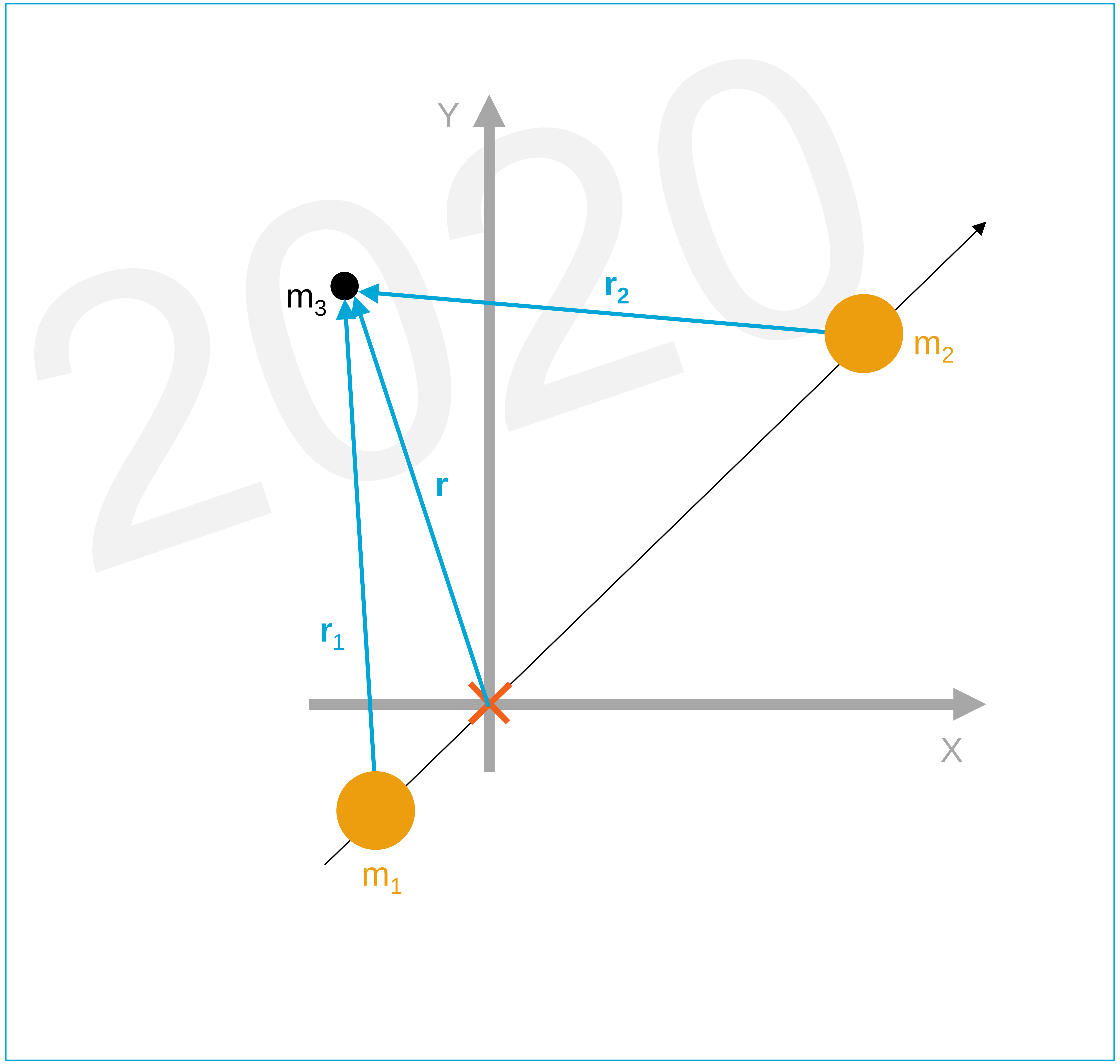
- **Equations of motion** in inertial frame

$$\ddot{\mathbf{r}}\Big|_I = -G\left(\frac{m_1}{r_1^3}\mathbf{r}_1 + \frac{m_2}{r_2^3}\mathbf{r}_2\right)$$



cr3bp – the equations of motion

- Define a **rotating reference frame** R(x,y,z)
  - Centered at barycenter
  - x-axis along the line $m_1 - m_2$
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- Equations of motion in inertial frame

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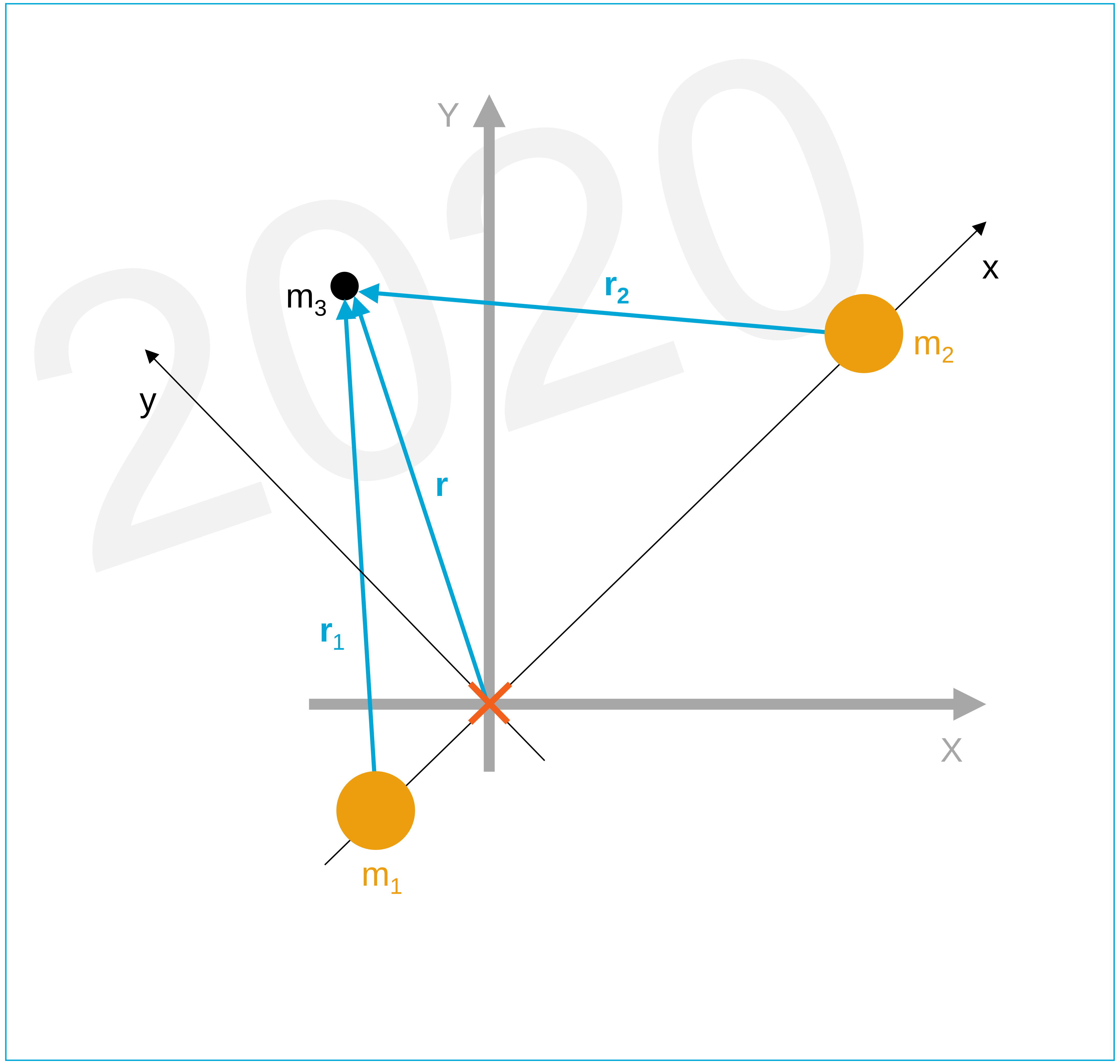

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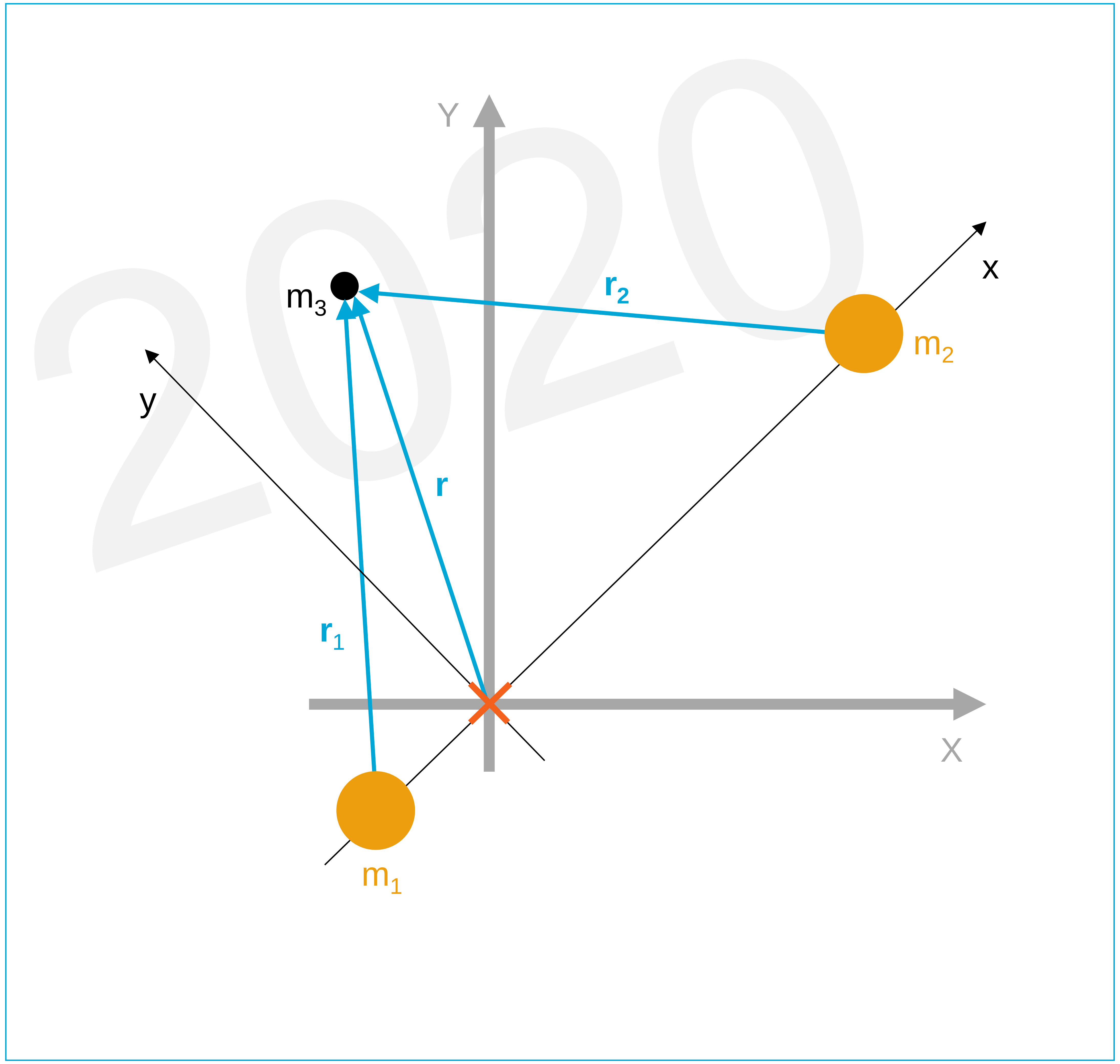

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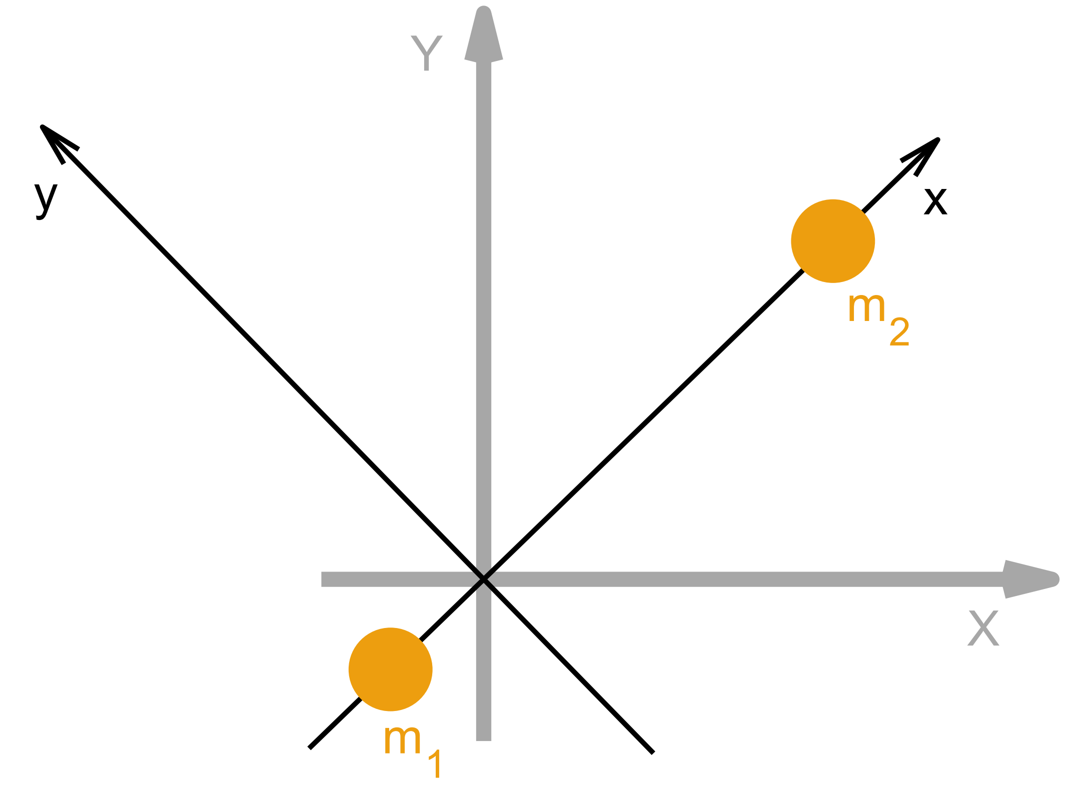

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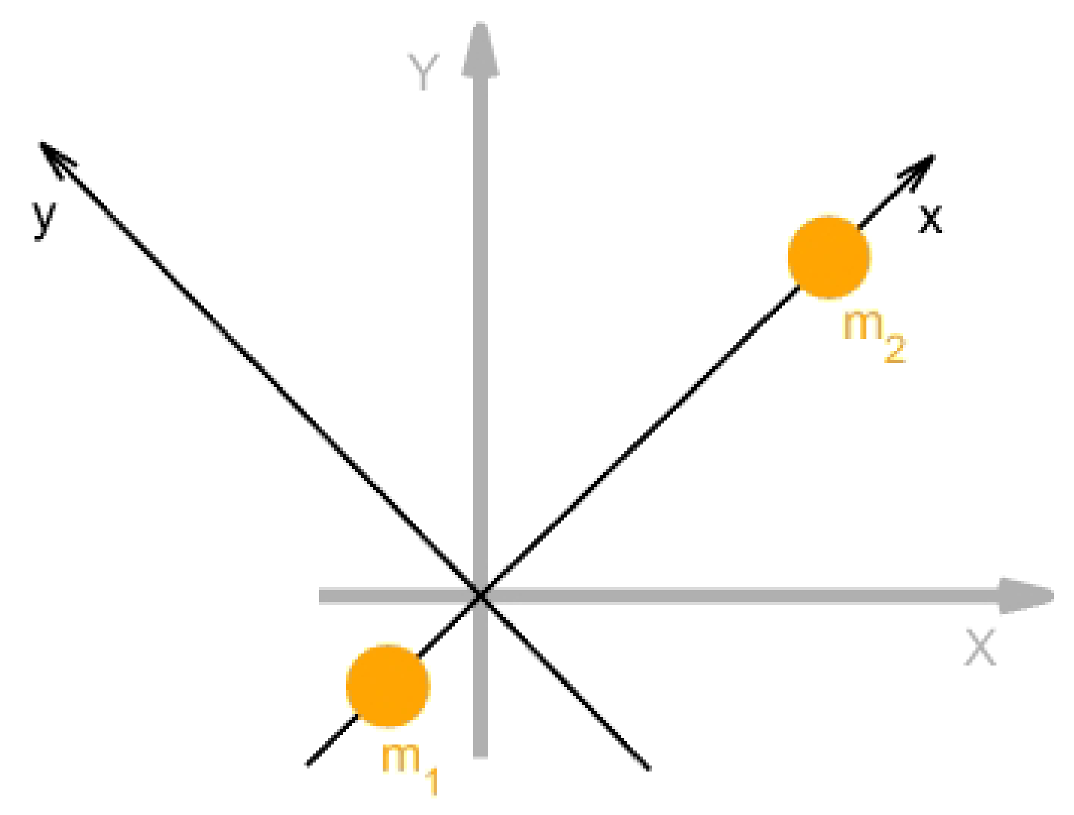

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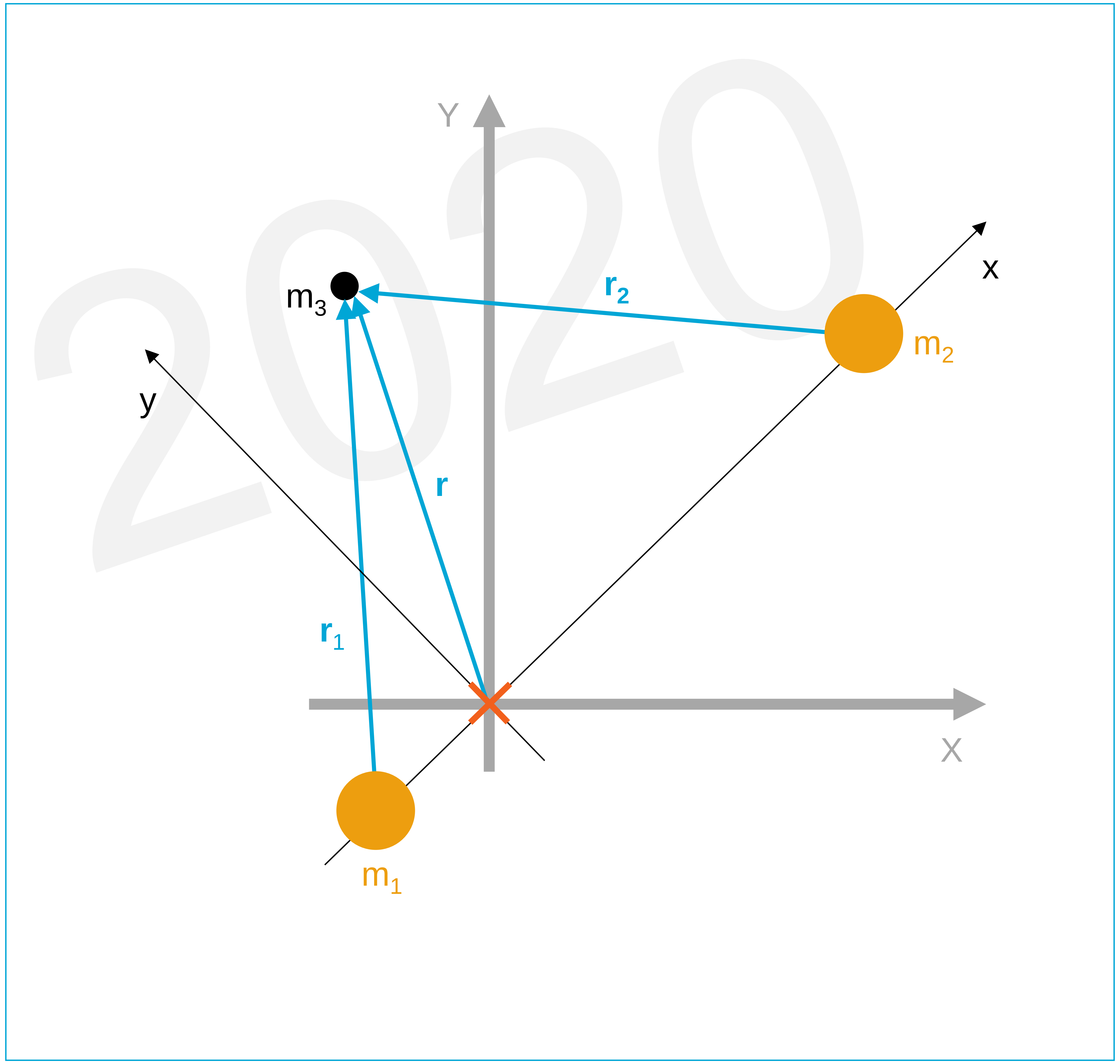
$$\ddot{\mathbf{r}}\big|_R = \ddot{\mathbf{r}}\big|_I + \text{non-inertial terms}$$



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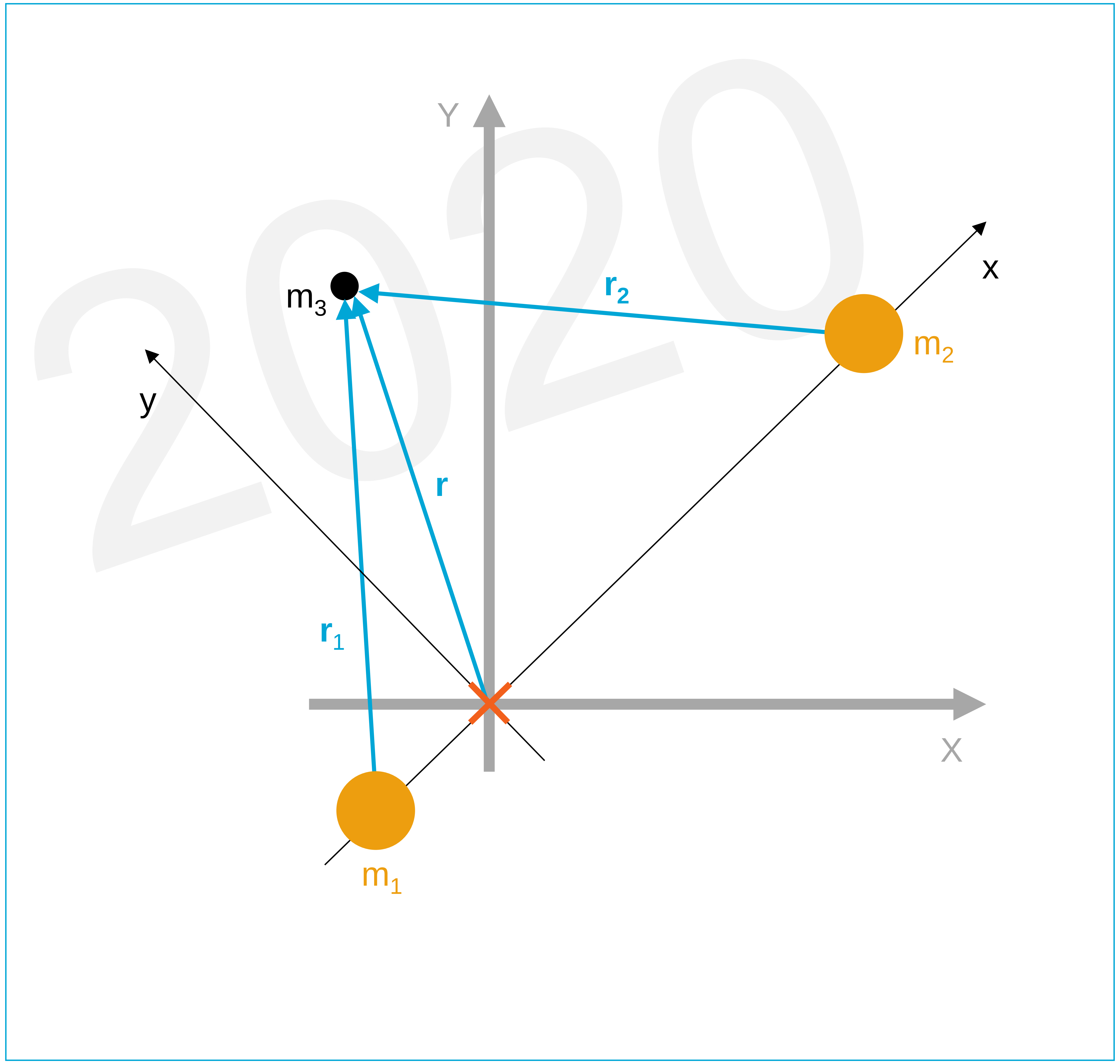

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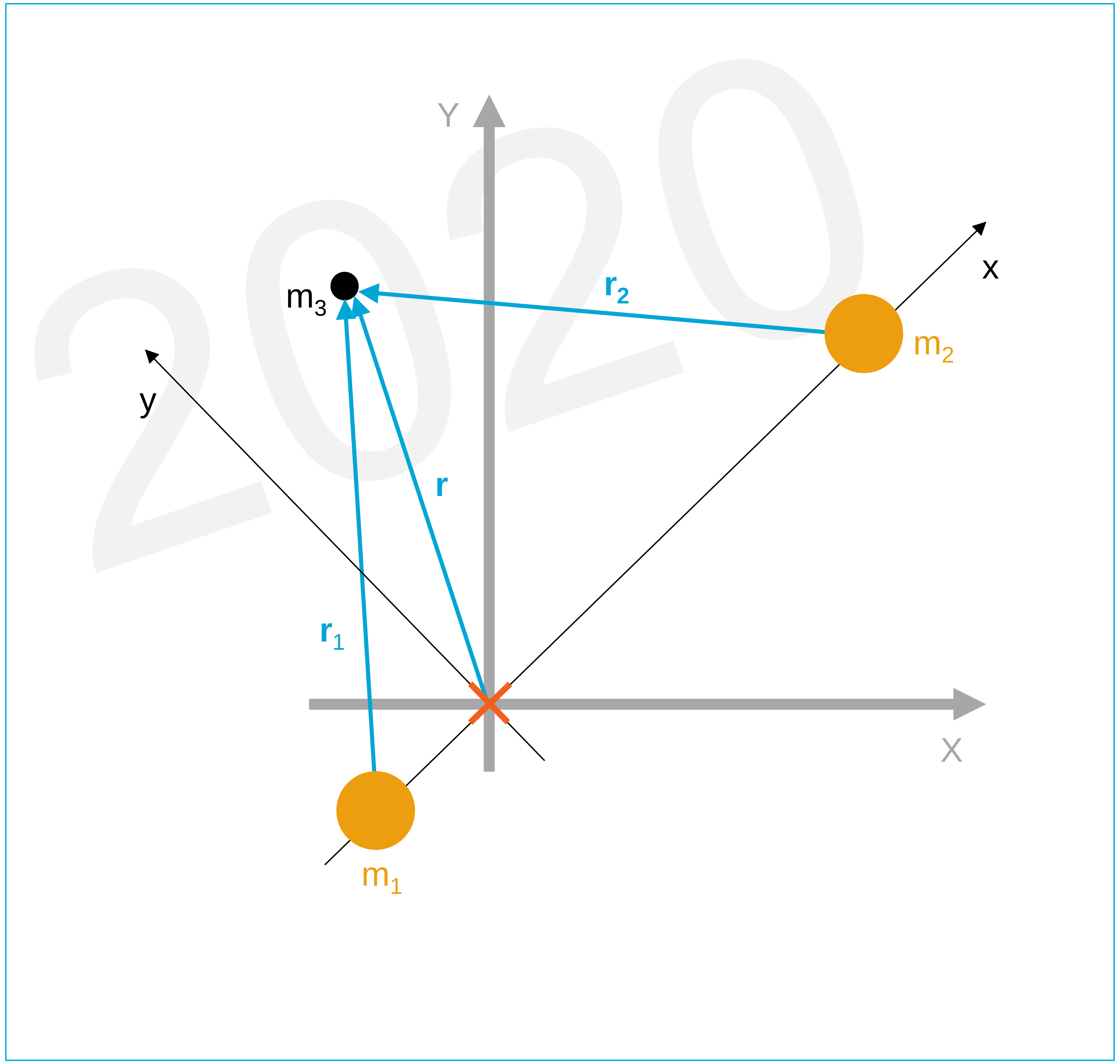

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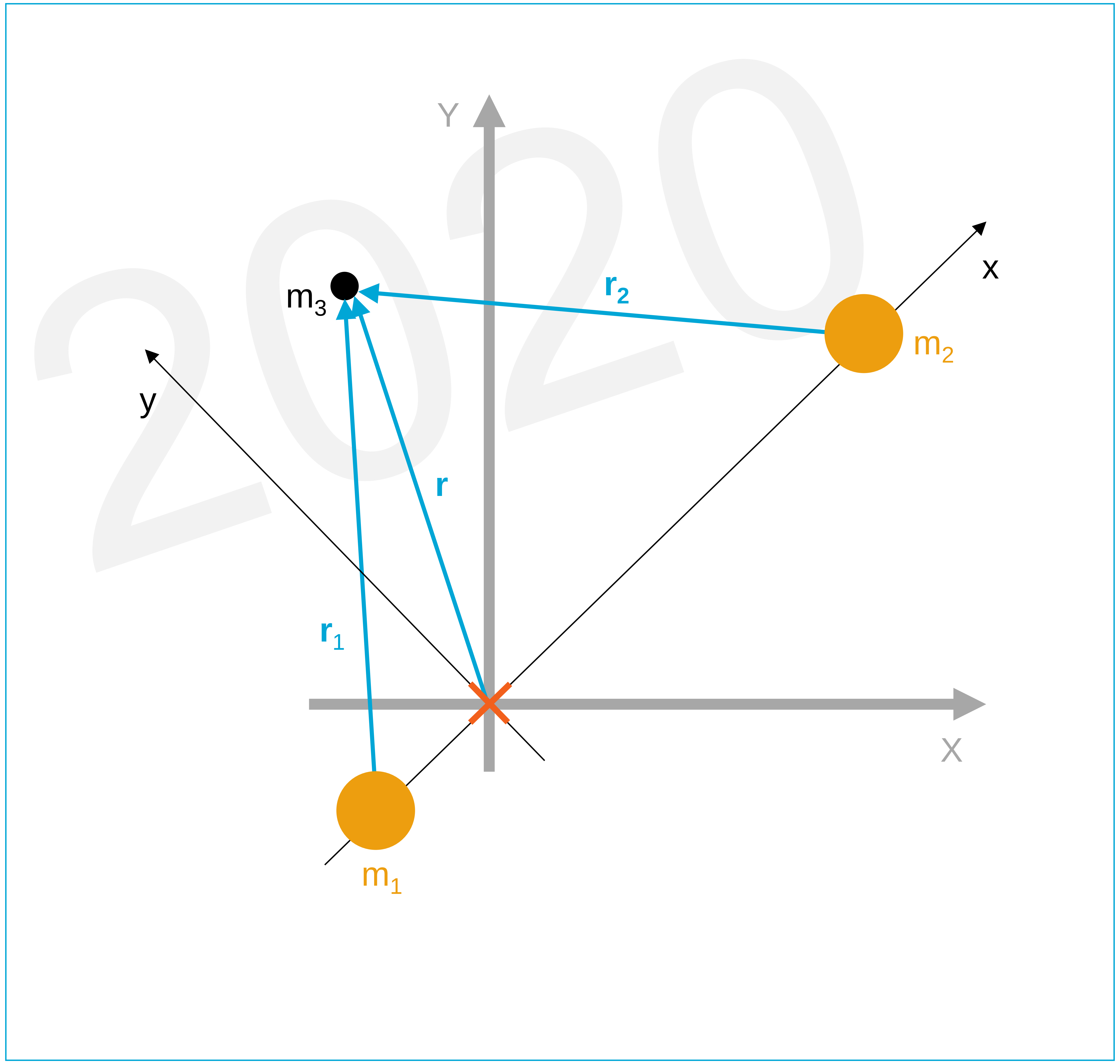
$$\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T$$



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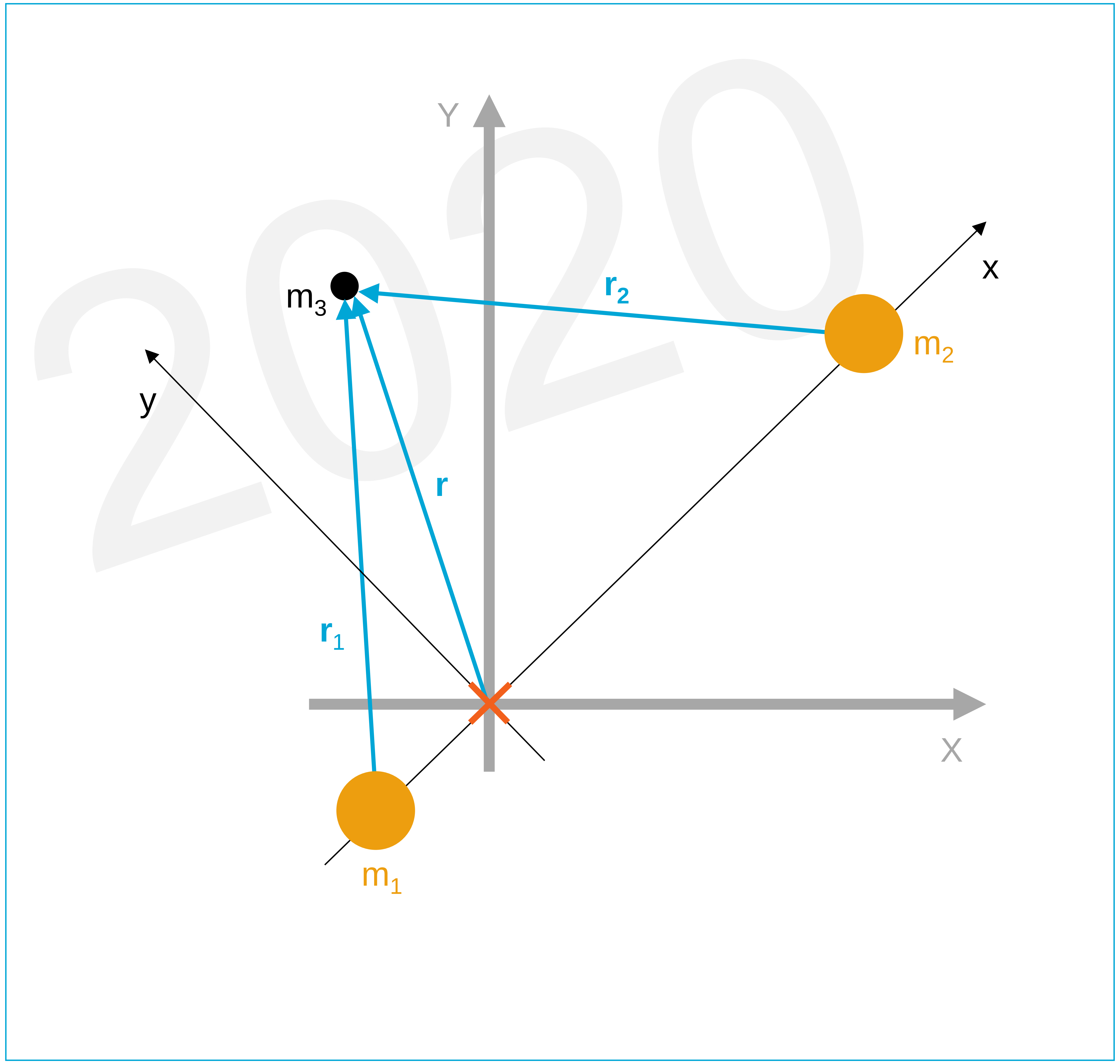
Frame angular velocity vector



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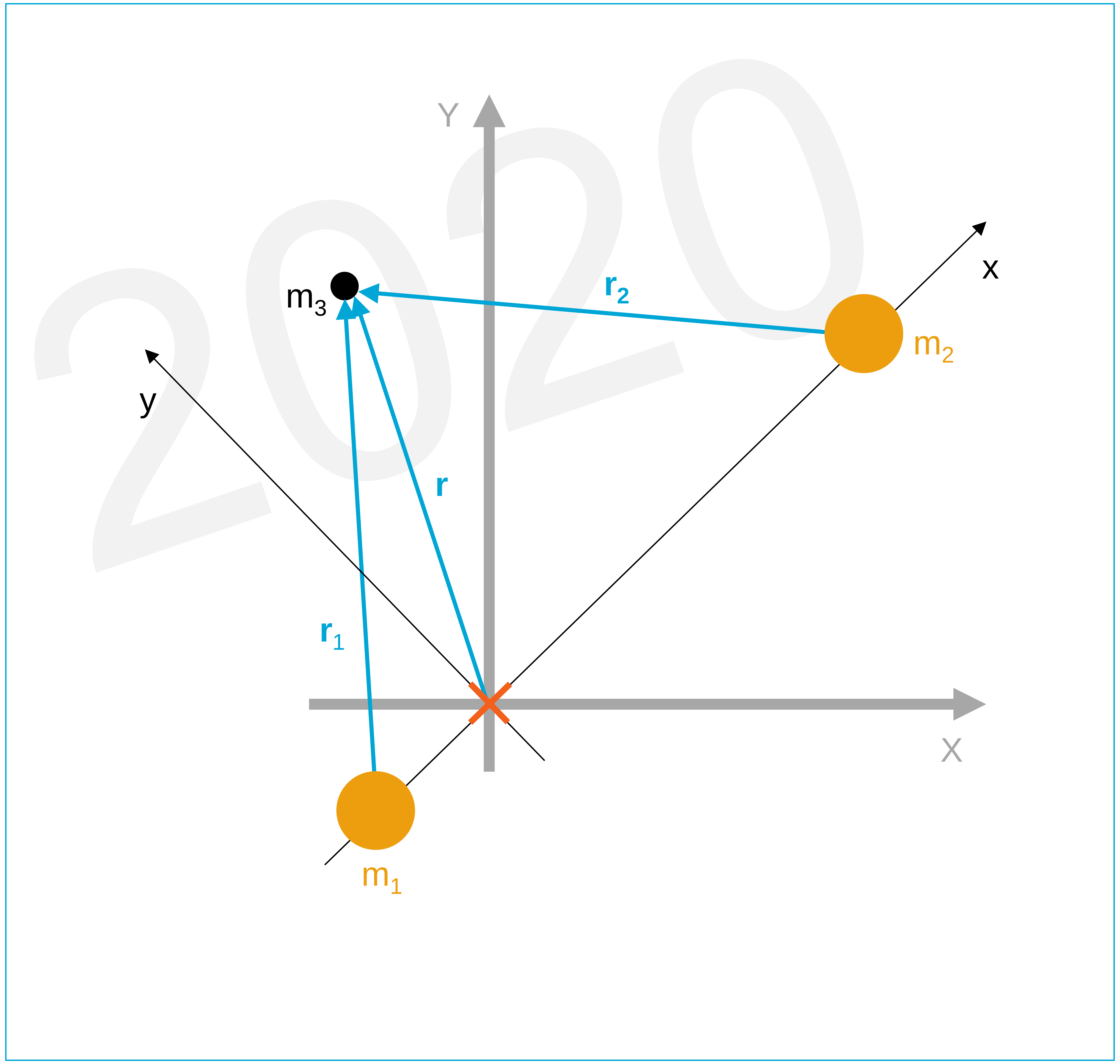

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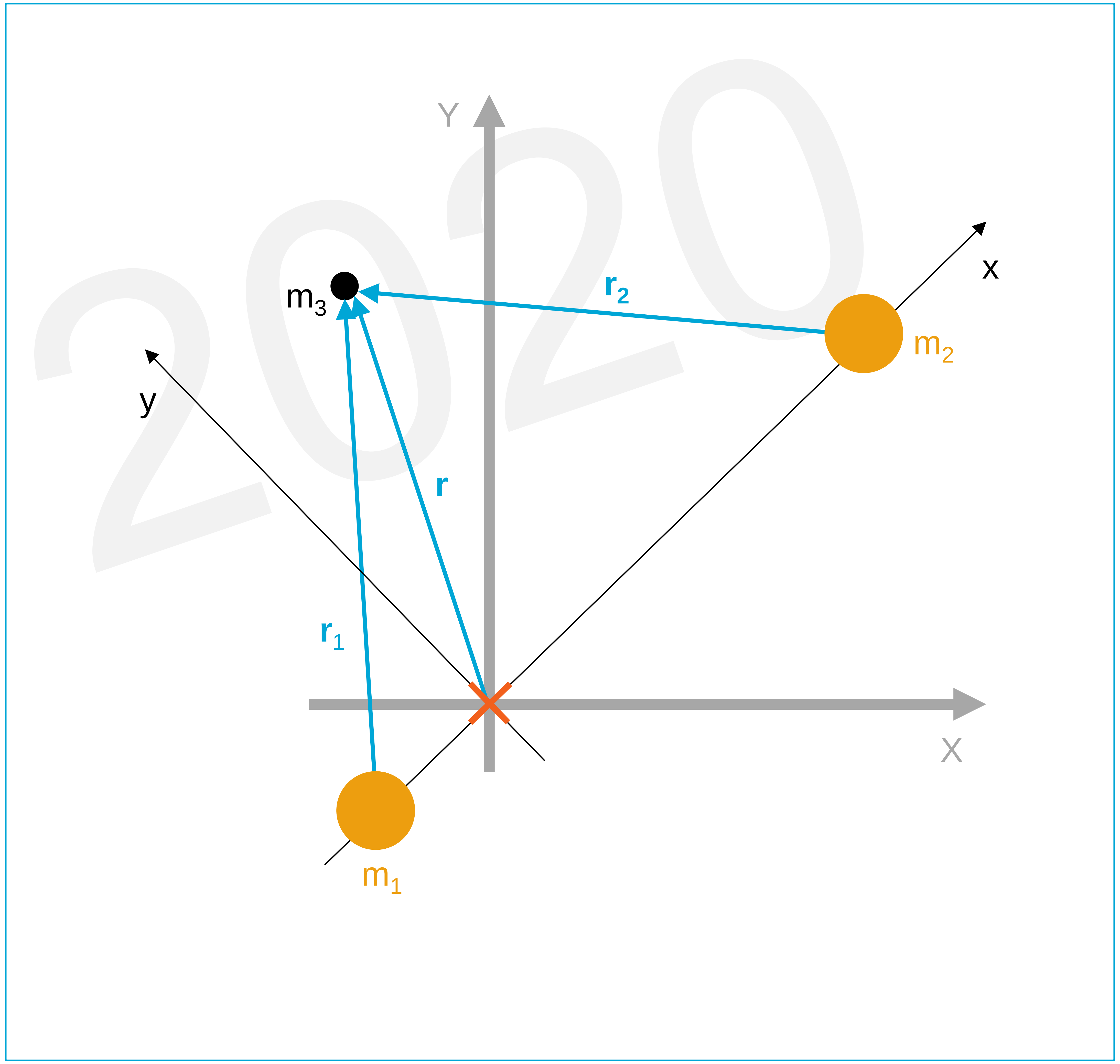
Coriolis acceleration



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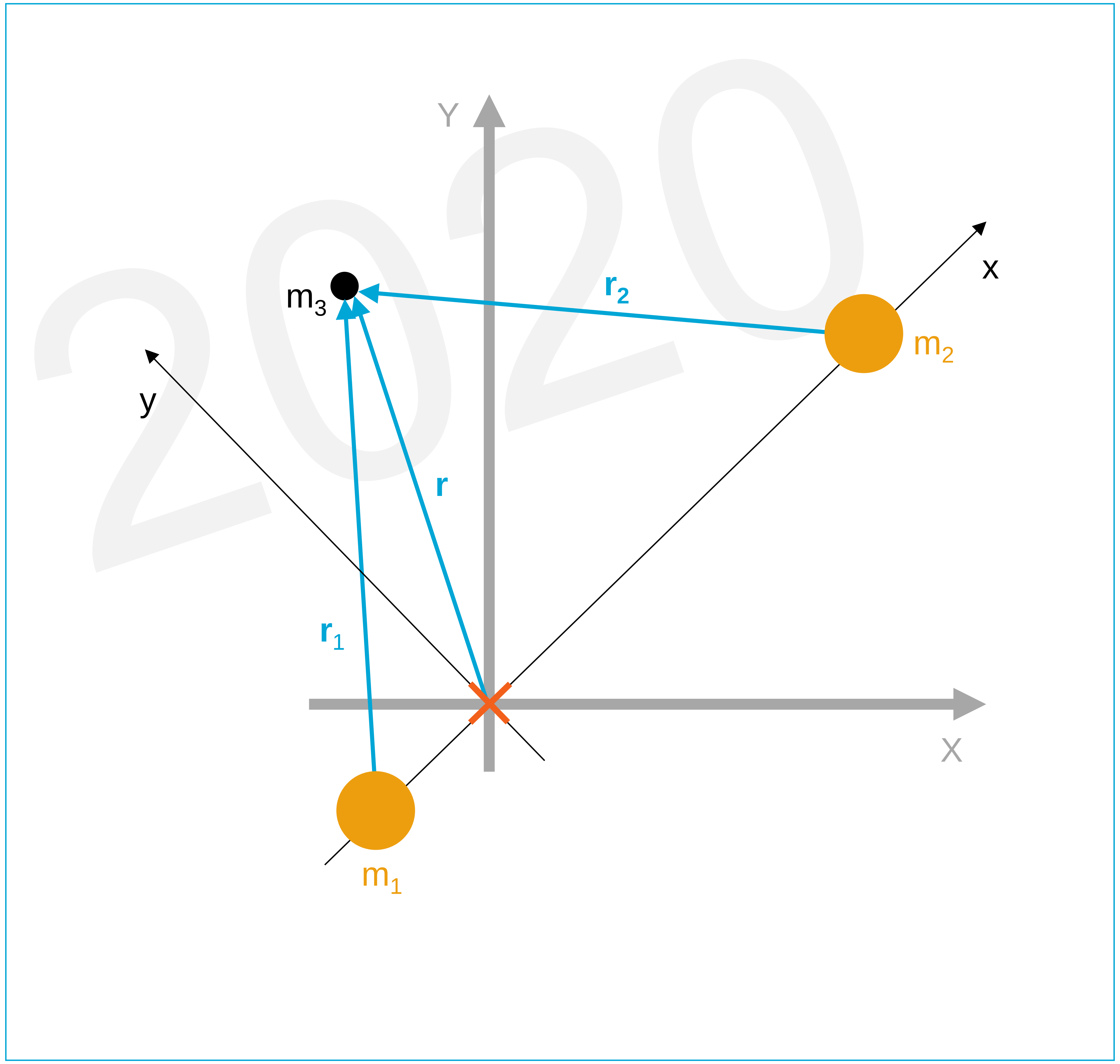
Centripetal acceleration



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Centripetal acceleration

Matlab GUI – "STA_cr3bp"

cr3bp – the equations of motion **in dimensionless form**

- Simplify eqs. of motion by introducing new units for
  - Mass [kg] $\rightarrow$ sum of masses of m_1 and m_2
  - Length [km]
  - Time [s]

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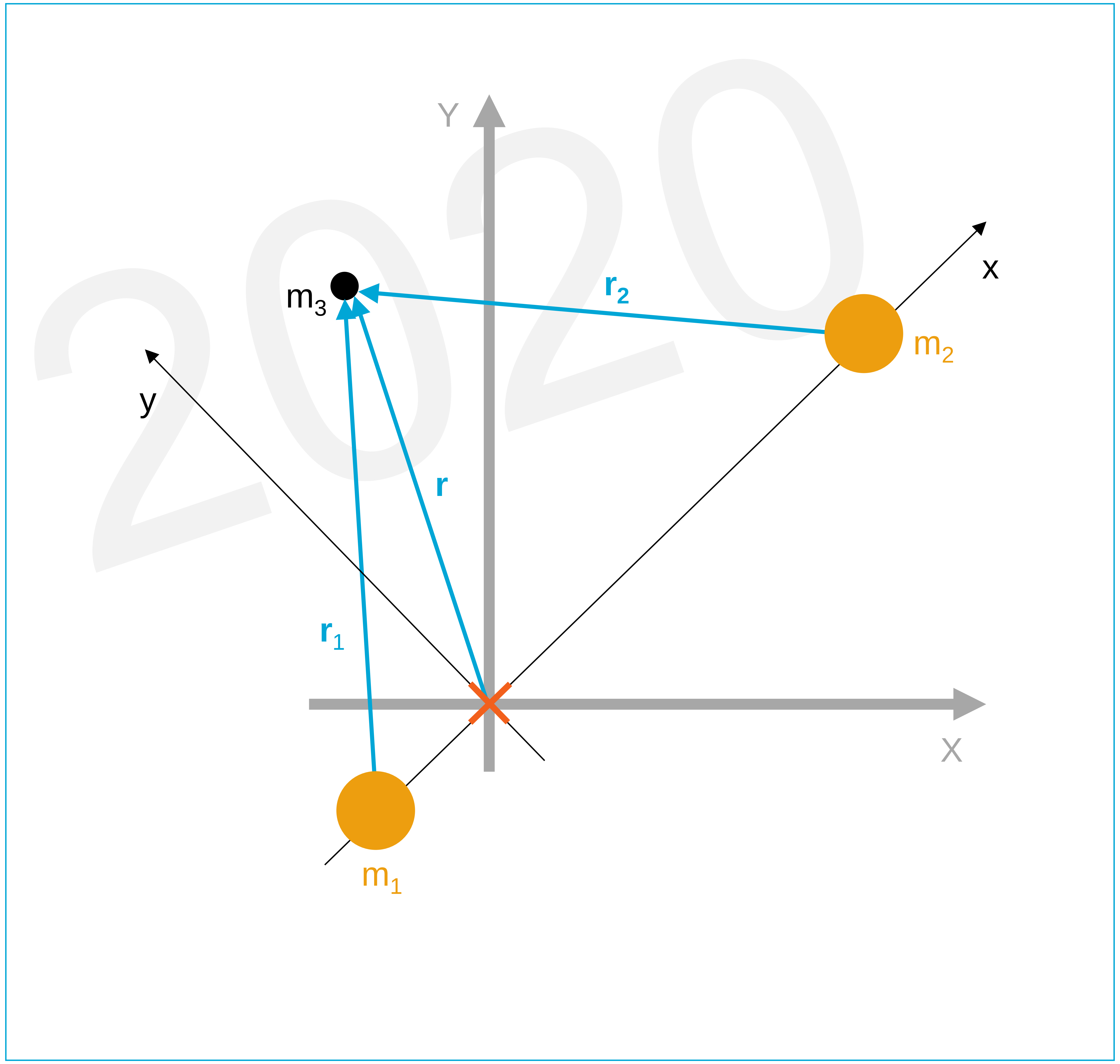

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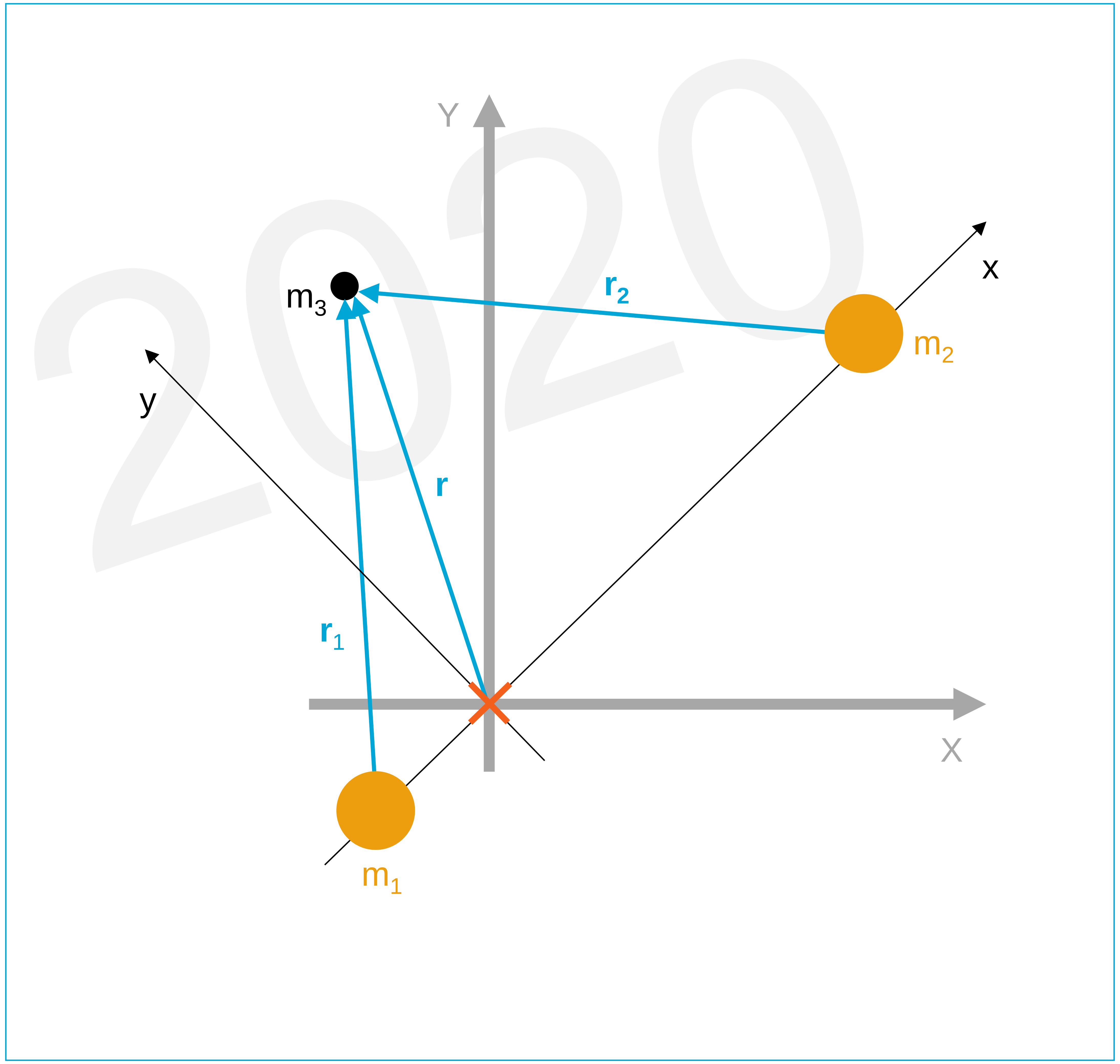

- Equations of motion in rotating frame

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cr3bp – the equations of motion **in dimensionless form**

- Simplify eqs. of motion by introducing new units for
  - Mass [kg] → sum of masses of m_1 and m_2
  - Length [km] → distance between m_1 and m_2
  - Time [s] → inverse of mean motion of m_1 / m_2

- Equations of motion in rotating frame

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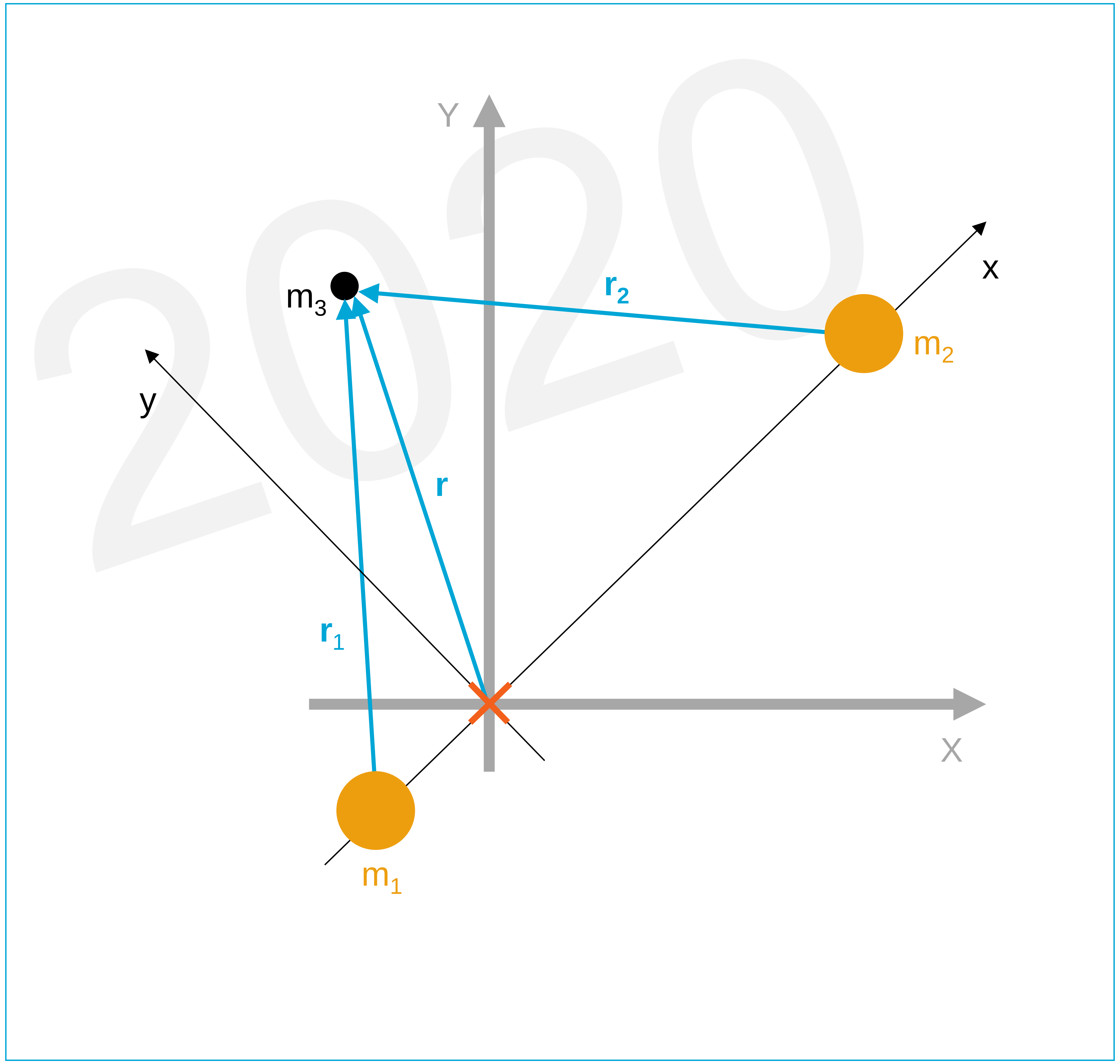

cr3bp – the equations of motion in dimensionless form

	Dimensionless form			Dimensional form		
System	Mass	Length	Time	Unit mass	Unit length	Unit time
Sun – Mercury	1	1	1	1.989×10^{30} kg	0.387 au	1,209,606 s
Sun – Venus	1	1	1	1.989×10^{30} kg	0.723 au	3,089,688 s
Sun – Earth	1	1	1	1.989×10^{30} kg	1 au	5,022,415 s
Sun – Mars	1	1	1	1.989×10^{30} kg	1.524 au	9,446,301 s
Earth – Moon	1	1	1	6.047×10^{24} kg	384,401 km	377,492 s
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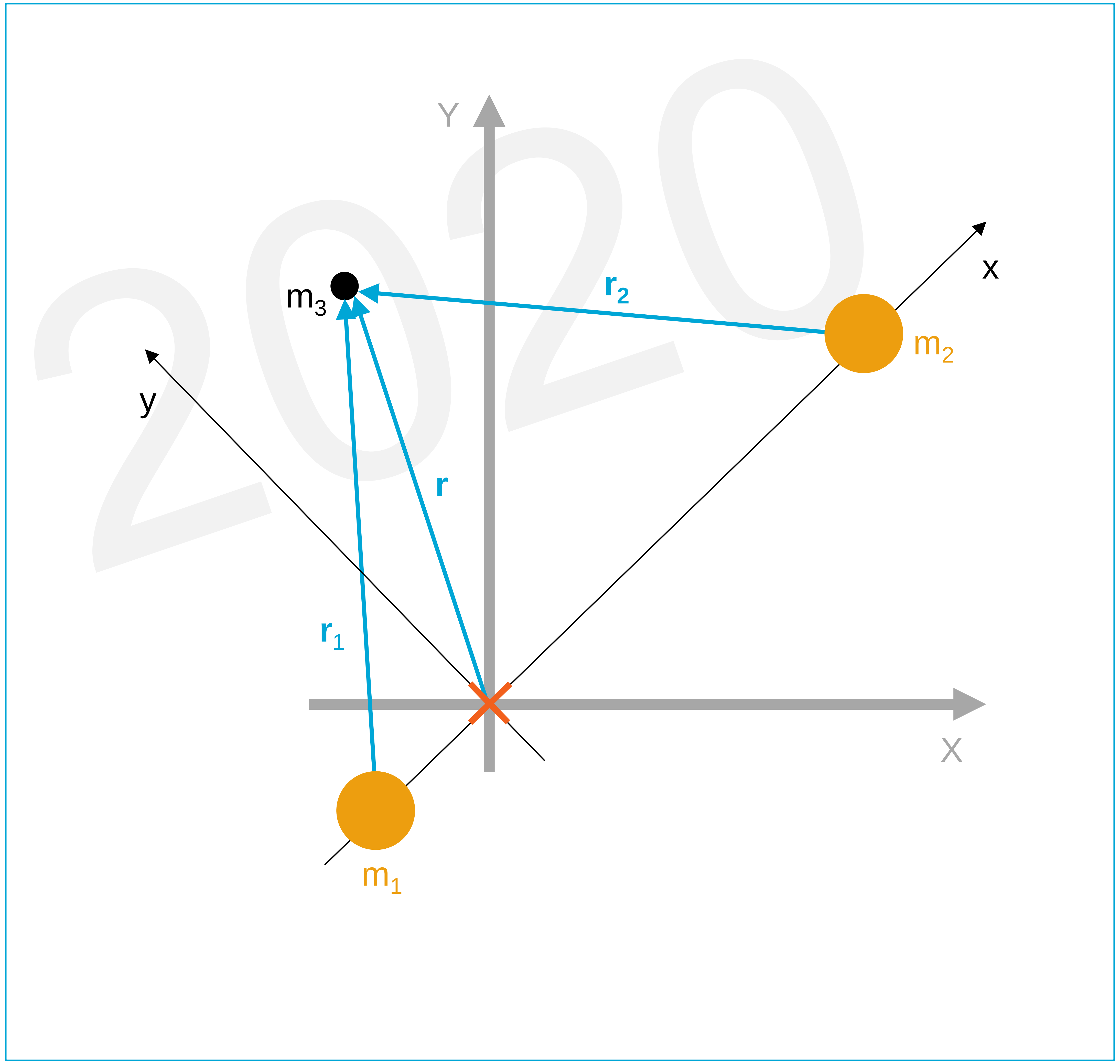
	Dimensionless form				Dimensional form			
System	Mass	Length	Time	Period	Unit mass	Unit length	Unit time	Period
Sun – Mercury	1	1	1	2π	1.989×10^{30} kg	0.387 au	1,209,606 s	0.24 yr
Sun – Venus	1	1	1	2π	1.989×10^{30} kg	0.723 au	3,089,688 s	0.61 yr
Sun – Earth	1	1	1	2π	1.989×10^{30} kg	1 au	5,022,415 s	1 yr
Sun – Mars	1	1	1	2π	1.989×10^{30} kg	1.524 au	9,446,301 s	1.88 yr
Earth – Moon	1	1	1	2π	6.047×10^{24} kg	384,401 km	377,492 s	27.4 days
…								

cr3bp – the equations of motion **in dimensionless form**

- Simplify eqs. of motion by introducing new units for
  - Mass [kg] $\rightarrow$ sum of masses of m_1 and m_2
  - Length [km] $\rightarrow$ distance between m_1 and m_2
  - Time [s] $\rightarrow$ inverse of mean motion of m_1 / m_2
- Some results of the new units
  - $G = 1$
  - $\boldsymbol{\omega} = \begin{bmatrix} 0 & 0 & \omega \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

- Equations of motion in rotating frame

$$\ddot{\mathbf{r}}\big|_R = -G\left(\frac{m_1}{r_1^3}\mathbf{r}_1 + \frac{m_2}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$



cr3bp – the equations of motion **in dimensionless form**

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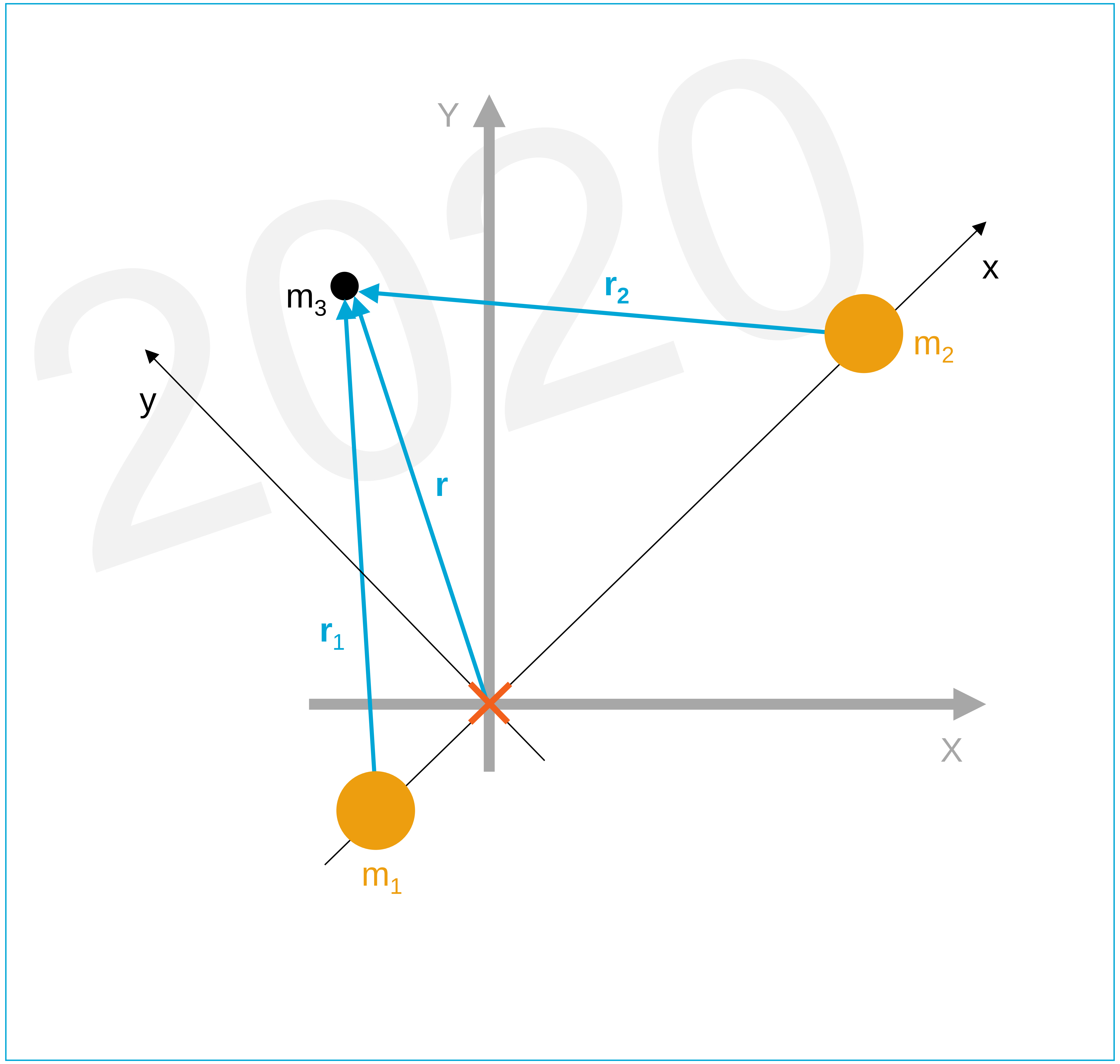
- Introduce **mass parameter**

$$\mu = \frac{m_1}{m_1 + m_2}$$

Erratum: this should be $\mu = \frac{m_2}{m_1 + m_2}$

- Equations of motion in rotating frame

$$\ddot{\mathbf{r}}\big|_R = -G\left(\frac{m_1}{r_1^3}\mathbf{r}_1 + \frac{m_2}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$



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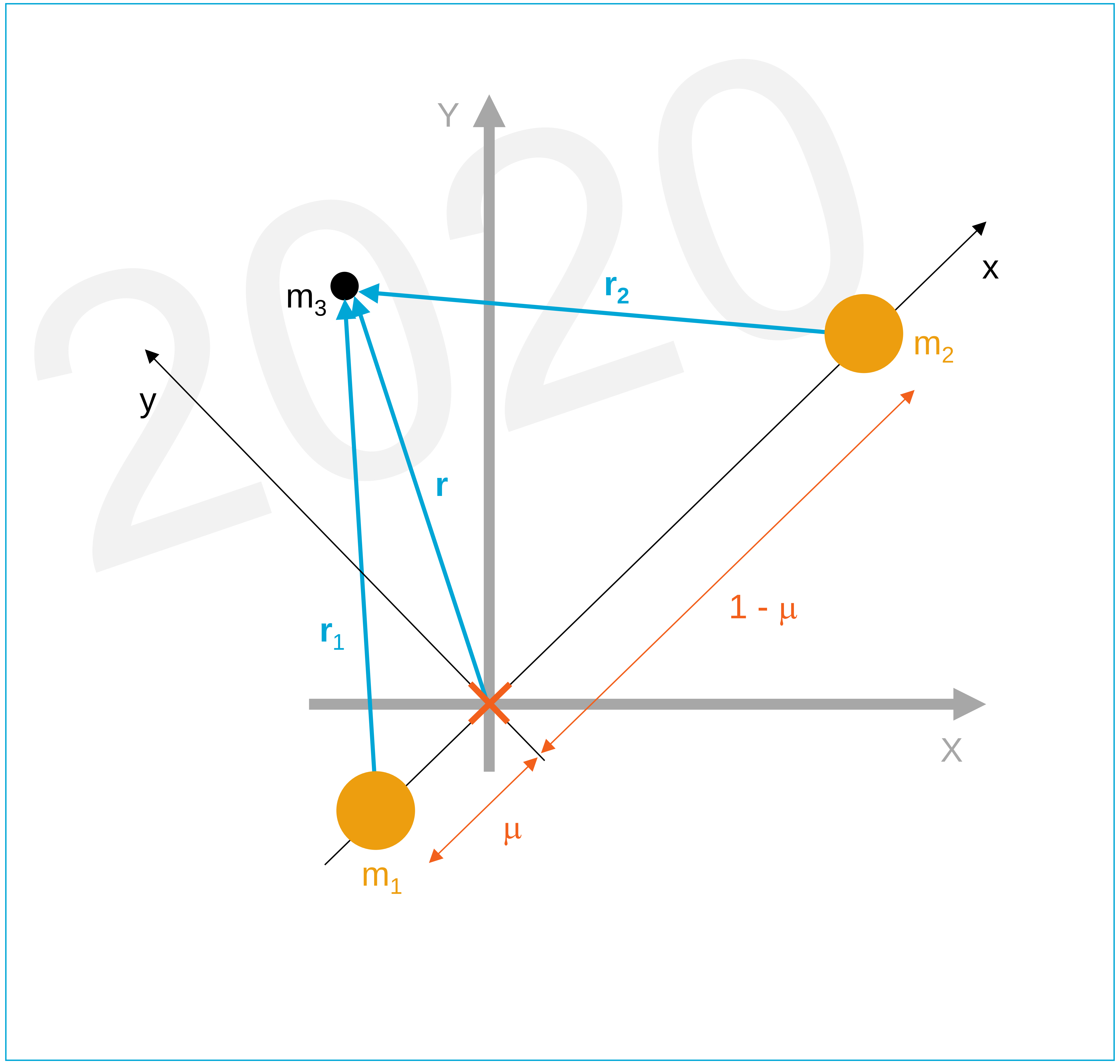

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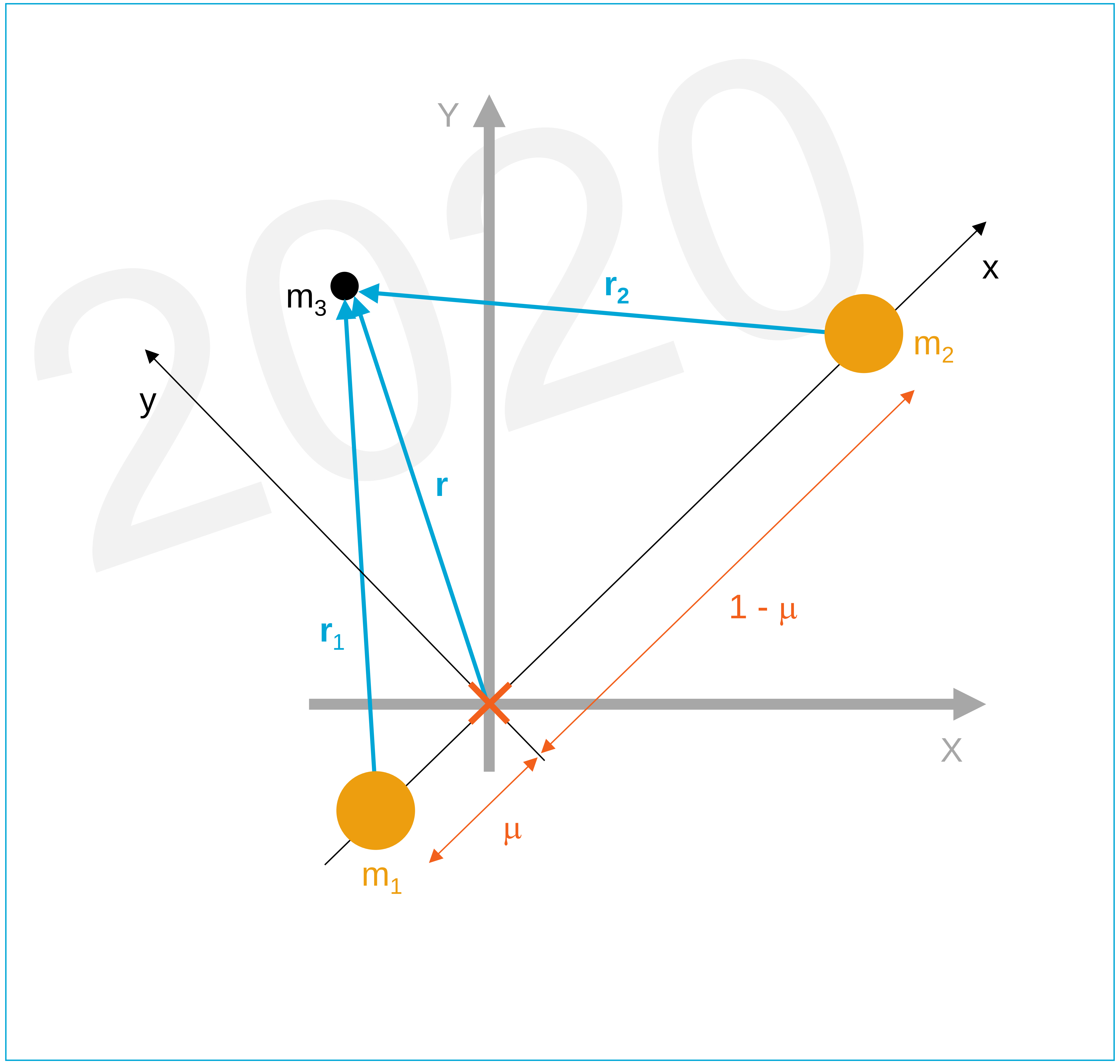
- Equations of motion in rotating frame **– dimensional**

$$\ddot{\mathbf{r}}\Big|_R = -G\left(\frac{m_1}{r_1^3}\mathbf{r}_1 + \frac{m_2}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$



cr3bp – the equations of motion **in dimensionless form**

- Simplify eqs. of motion by introducing new units for
  - Mass [kg] $\rightarrow$ sum of masses of m_1 and m_2
  - Length [km] $\rightarrow$ distance between m_1 and m_2
  - Time [s] $\rightarrow$ inverse of mean motion of m_1 / m_2
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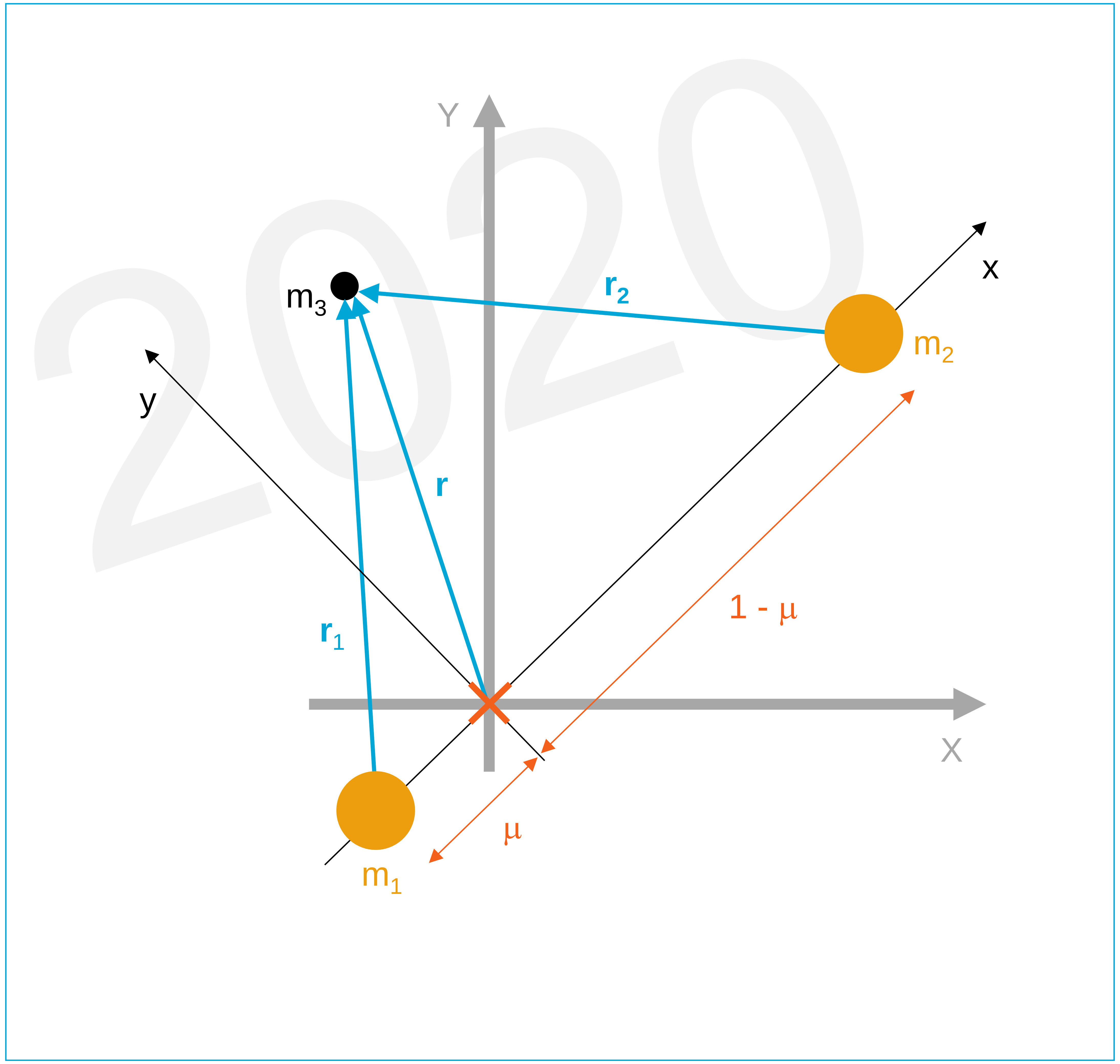
- Equations of motion in rotating frame – **dimensionless**

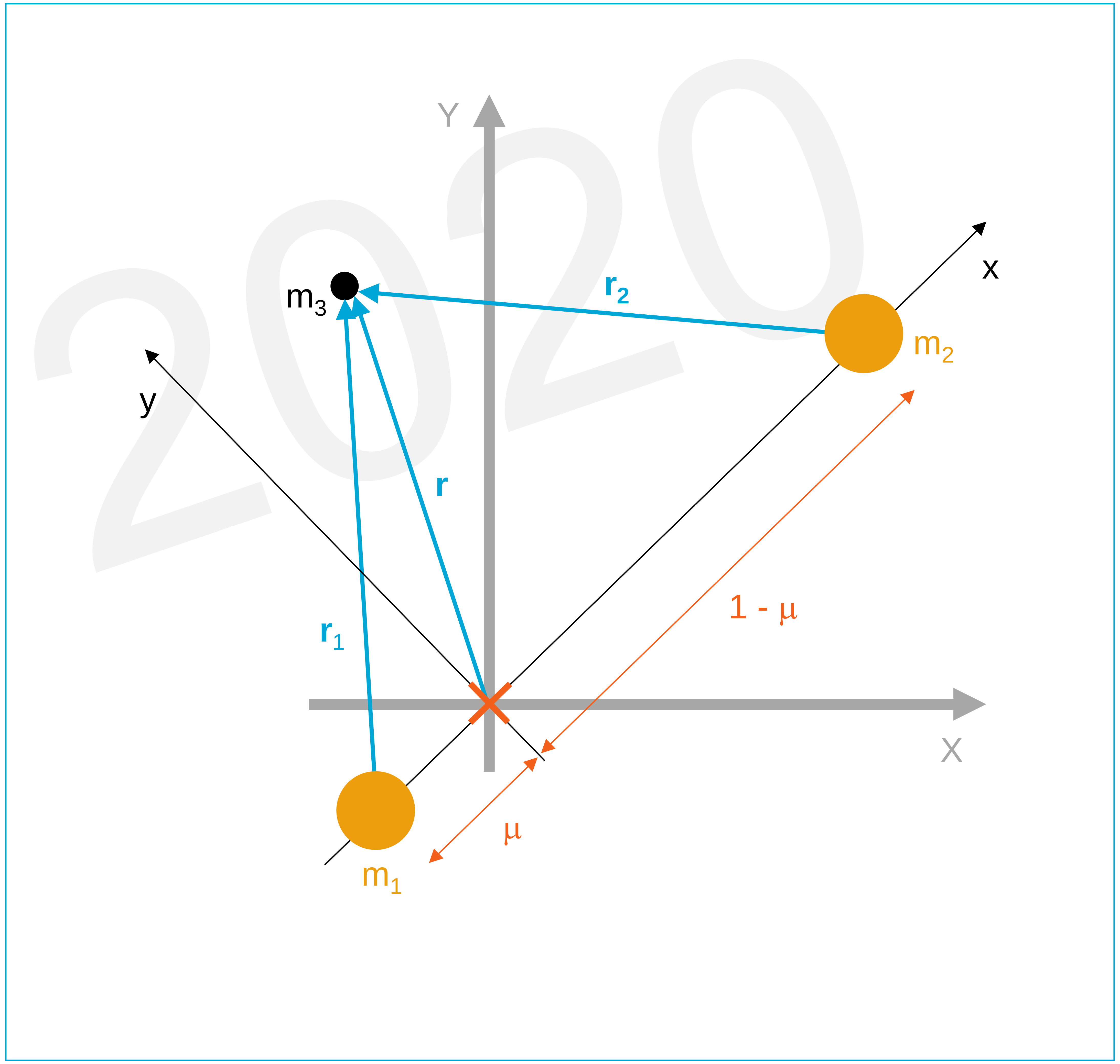
$$\ddot{\mathbf{r}}\big|_R = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$



cr3bp – the equations of motion **in dimensionless form**

- Equations of motion in rotating frame – **dimensionless**

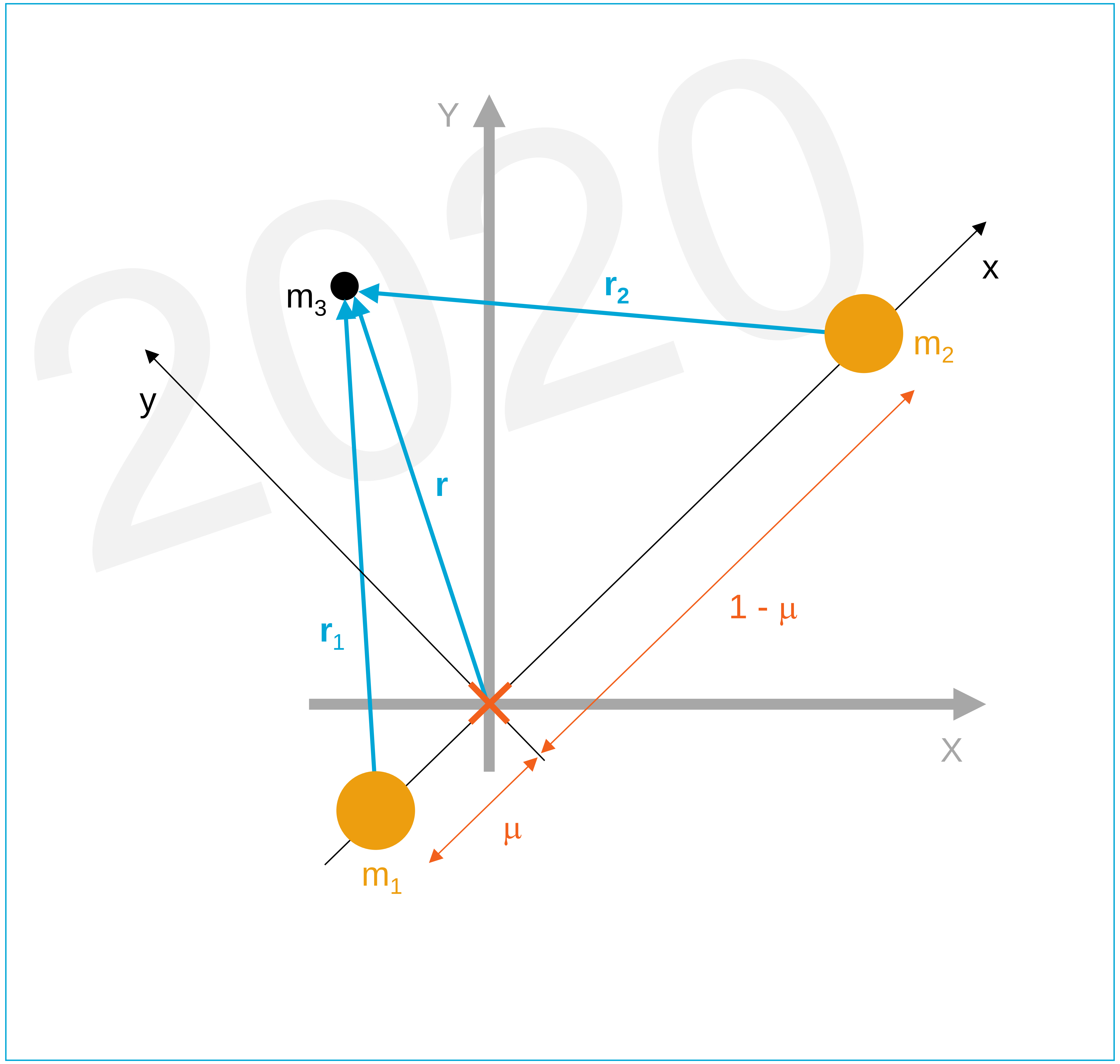
$$\ddot{\mathbf{r}}\big|_R = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$



cr3bp – the equations of motion **in dimensionless form**

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}\right)$$

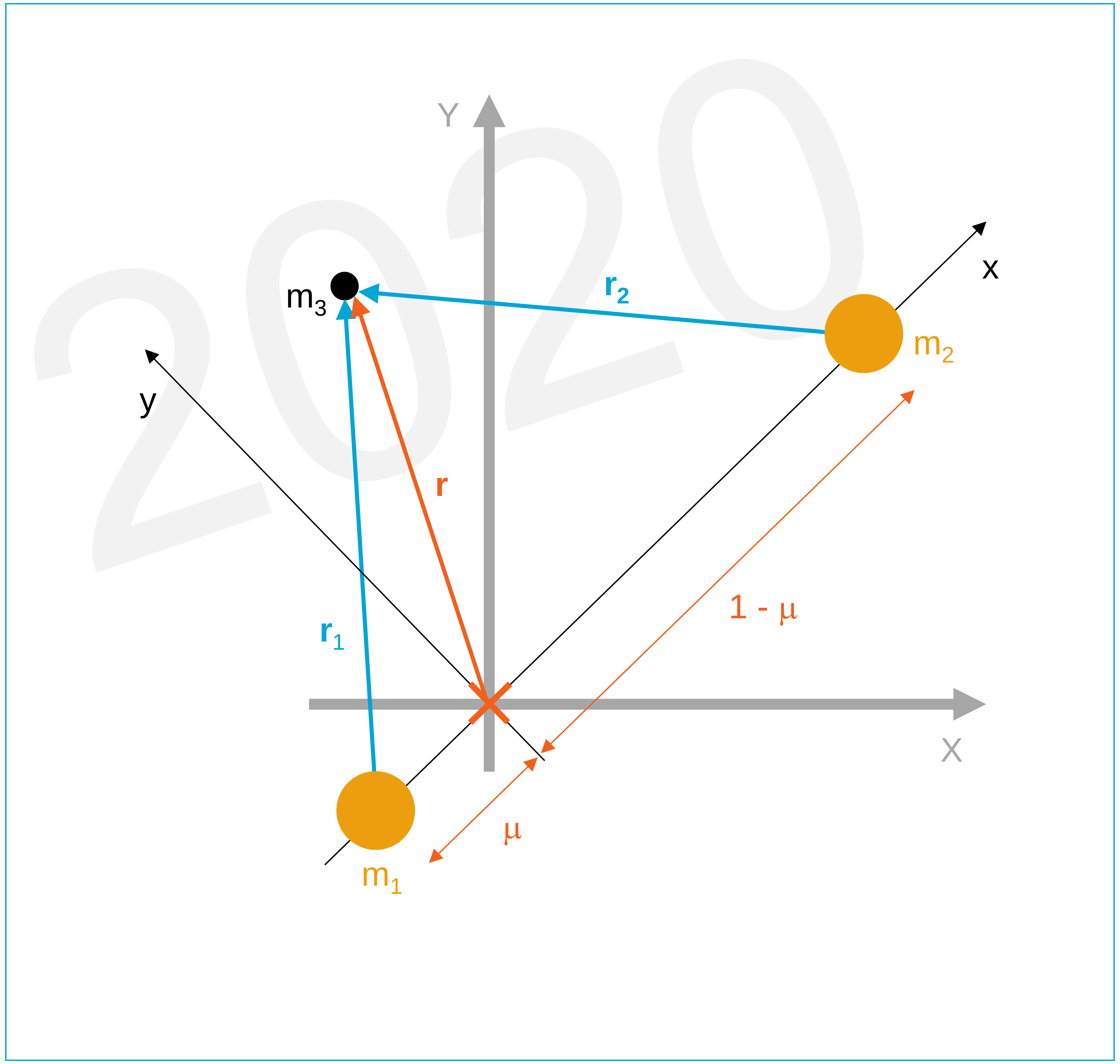


cr3bp – the equations of motion **in dimensionless form**

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}\right)$$

  - $\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T, \dot{\mathbf{r}} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T, \ddot{\mathbf{r}} = \begin{bmatrix} \ddot{x} & \ddot{y} & \ddot{z} \end{bmatrix}^T$



cr3bp – the equations of motion in dimensionless form

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

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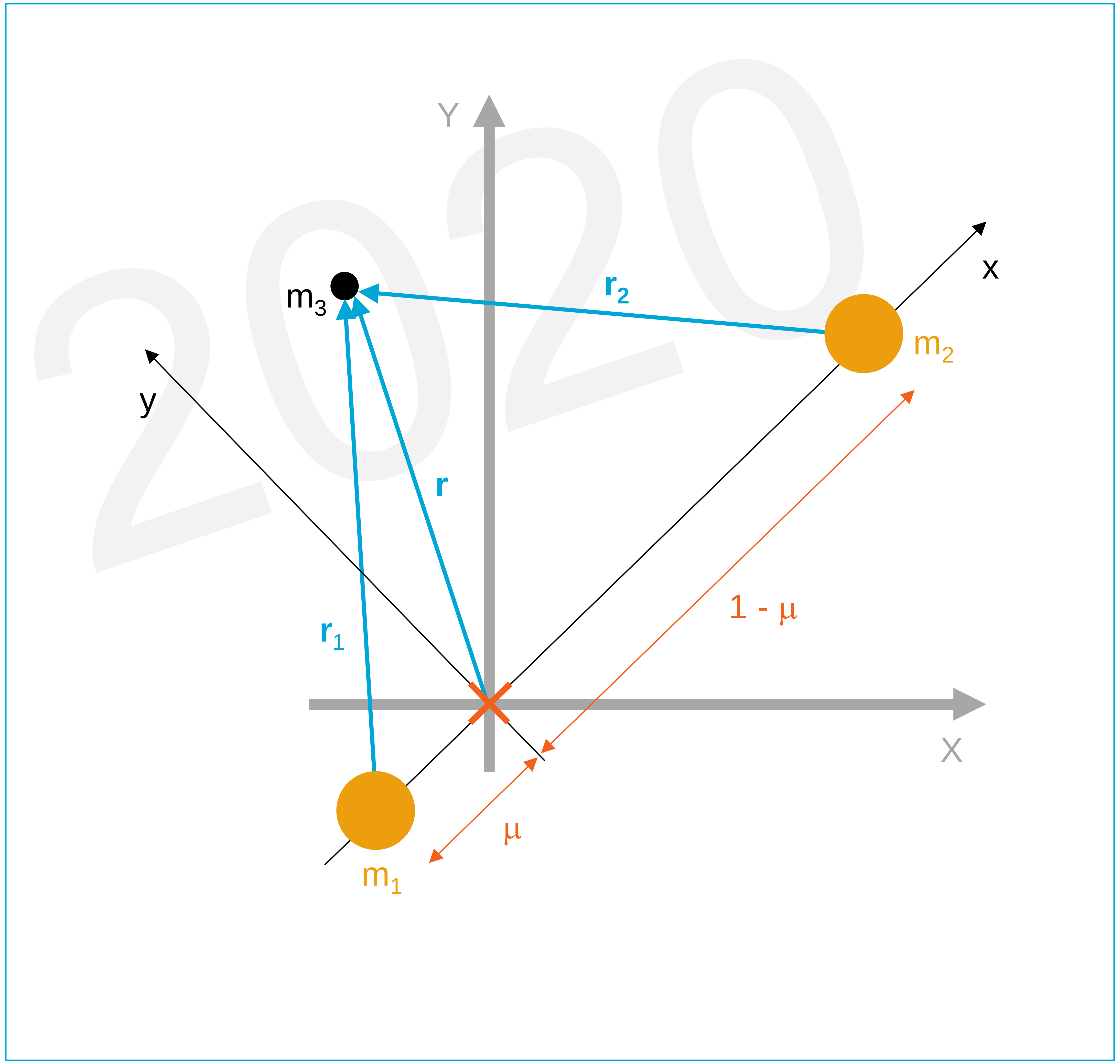

cr3bp – the equations of motion in dimensionless form

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

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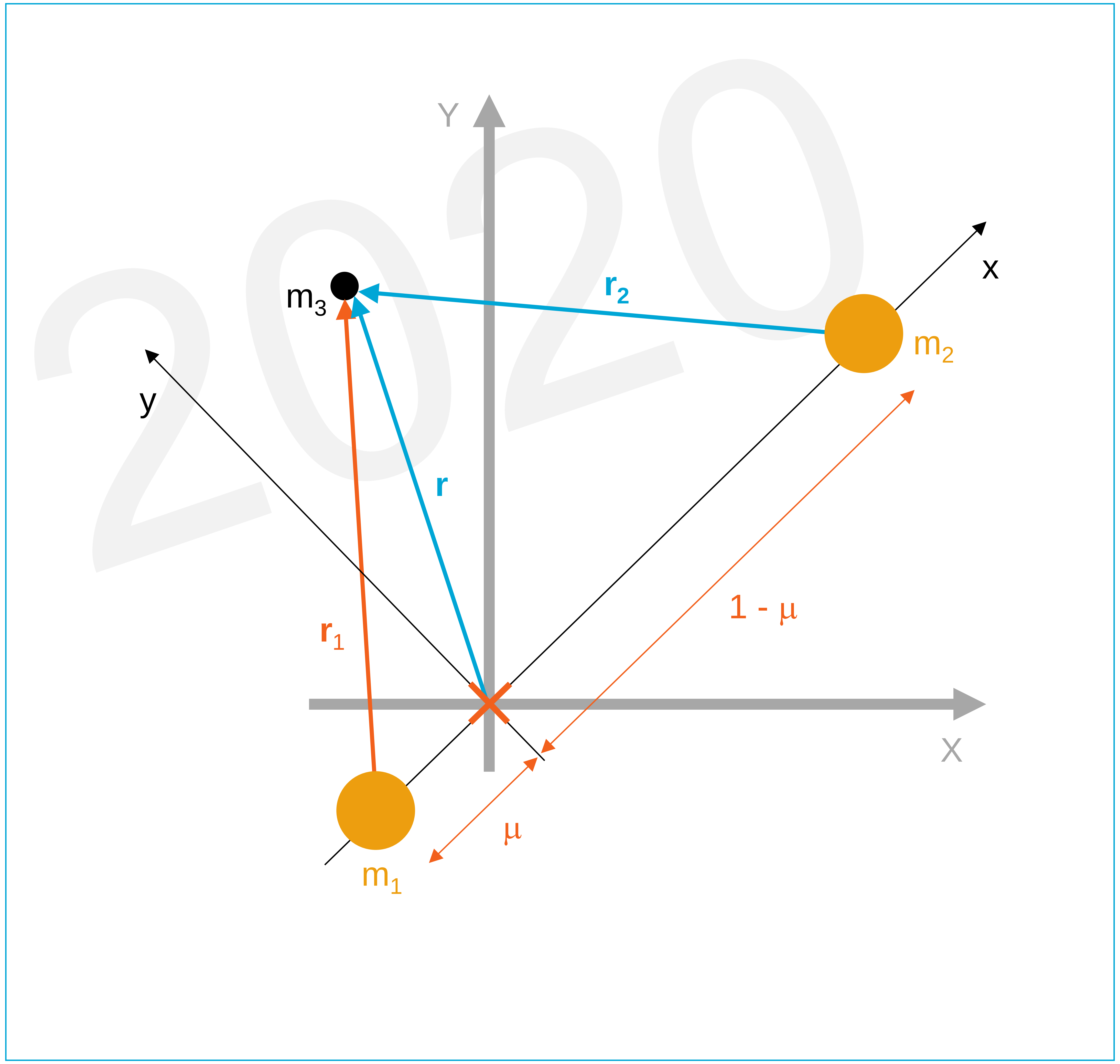
  - $\mathbf{r}_1 = [x + \mu \quad y \quad z]^T$



cr3bp – the equations of motion in dimensionless form

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

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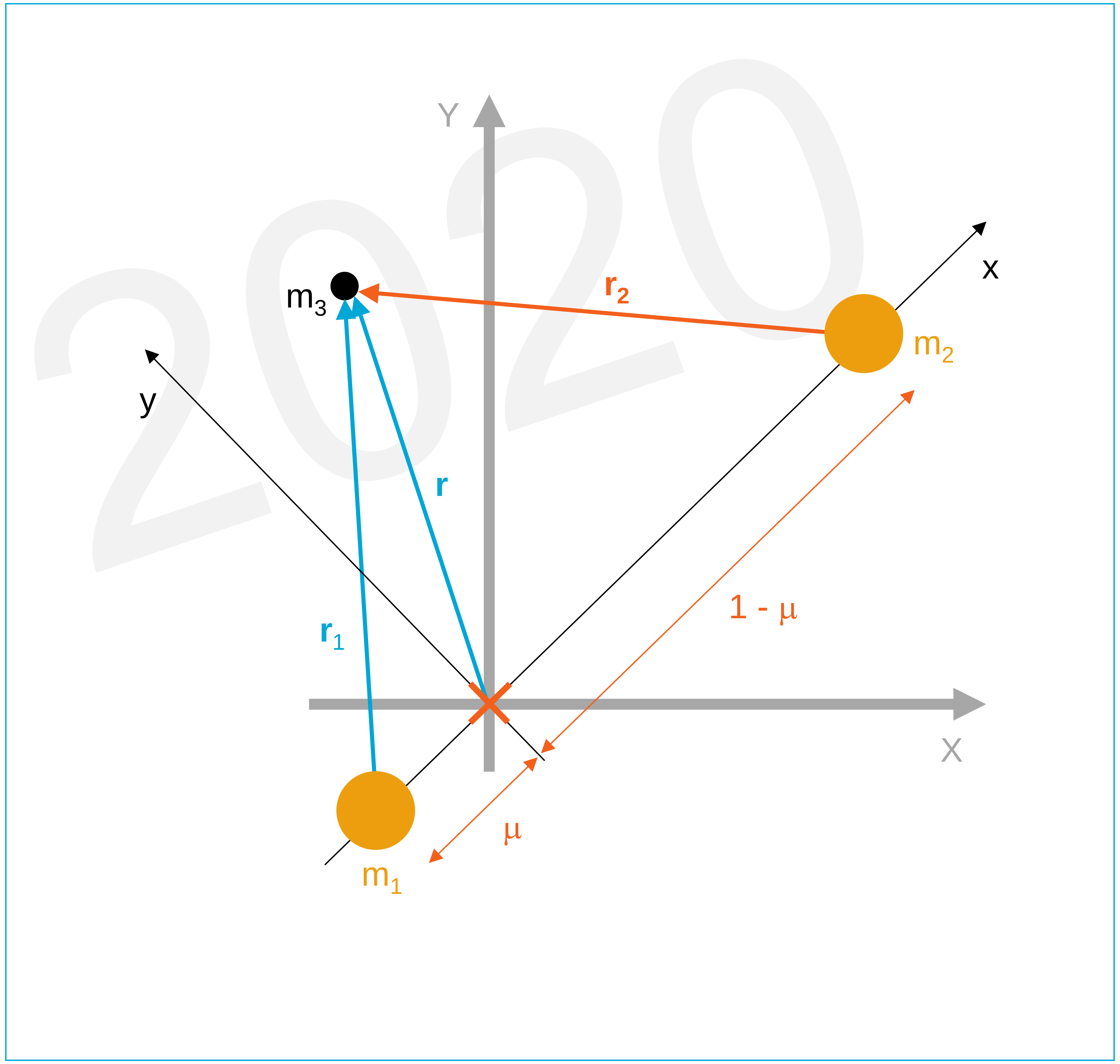
  - $\mathbf{r}_2 = [x-(1-\mu) \quad y \quad z]^T$



cr3bp – the equations of motion in dimensionless form

- Equations of motion in rotating frame – **dimensionless**

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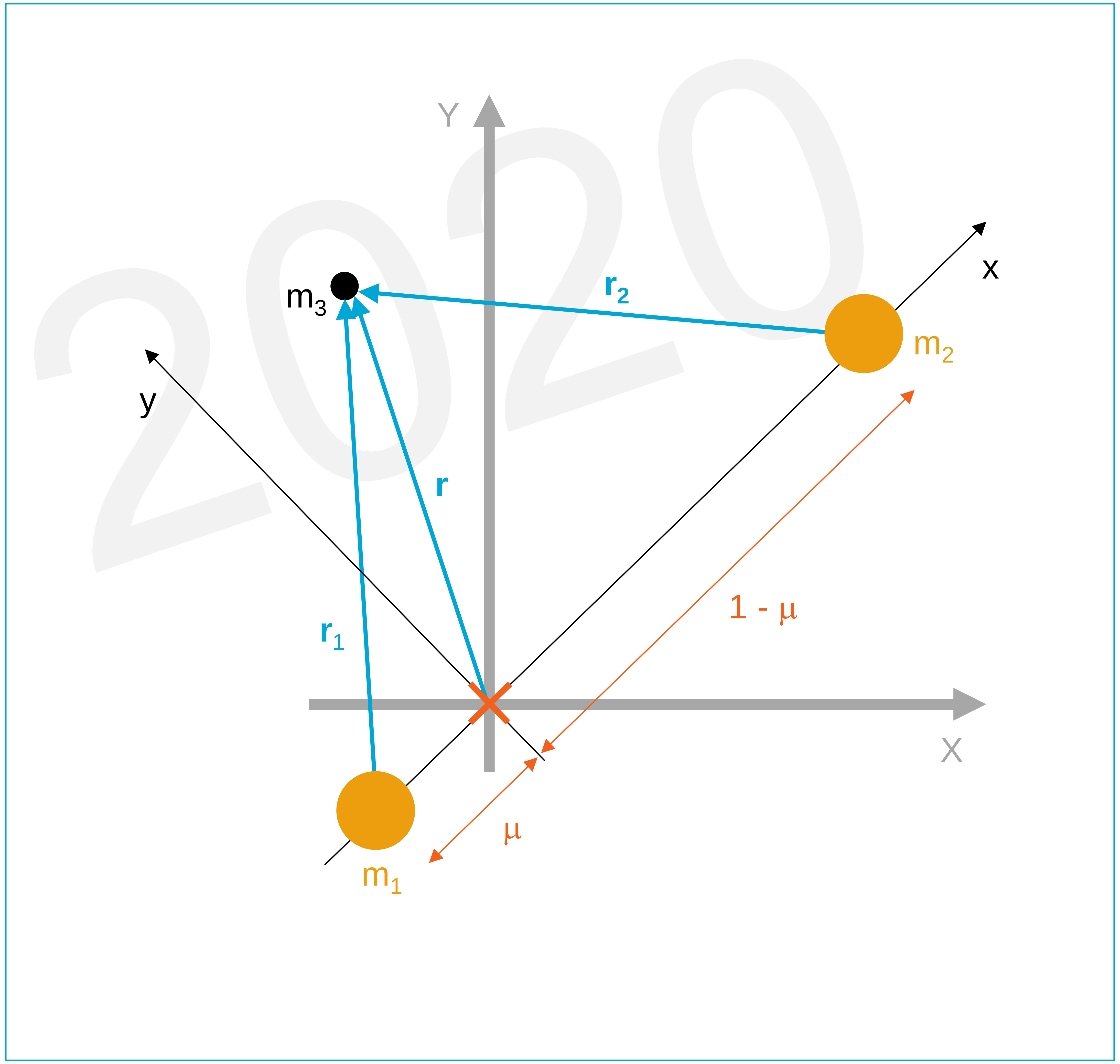
  - $\boldsymbol{\omega} = [0 \quad 0 \quad \omega]^T = [0 \quad 0 \quad 1]^T$



cr3bp – the equations of motion **in dimensionless form**

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

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Matlab GUI – “STA_cr3bp”

cr3bp – the equations of motion **in dimensionless form**

- Equations of motion in rotating frame – **dimensionless**

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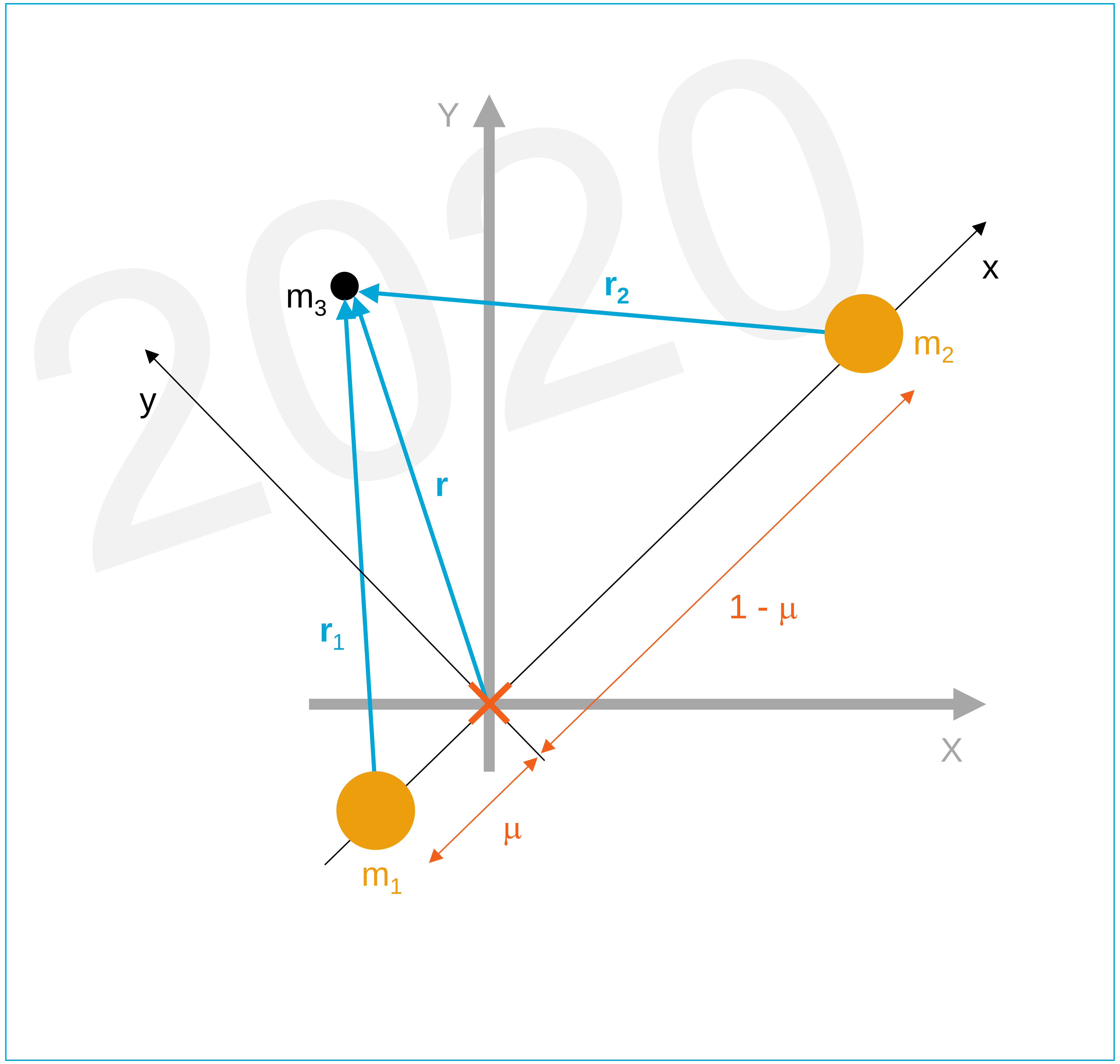
- Equations of motion are fully parameterized by μ



cr3bp – the equations of motion **in dimensionless form**

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

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  - $\mathbf{r}_2 = [x - (1-\mu) \quad y \quad z]^T$
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- Equations of motion are fully parameterized by μ

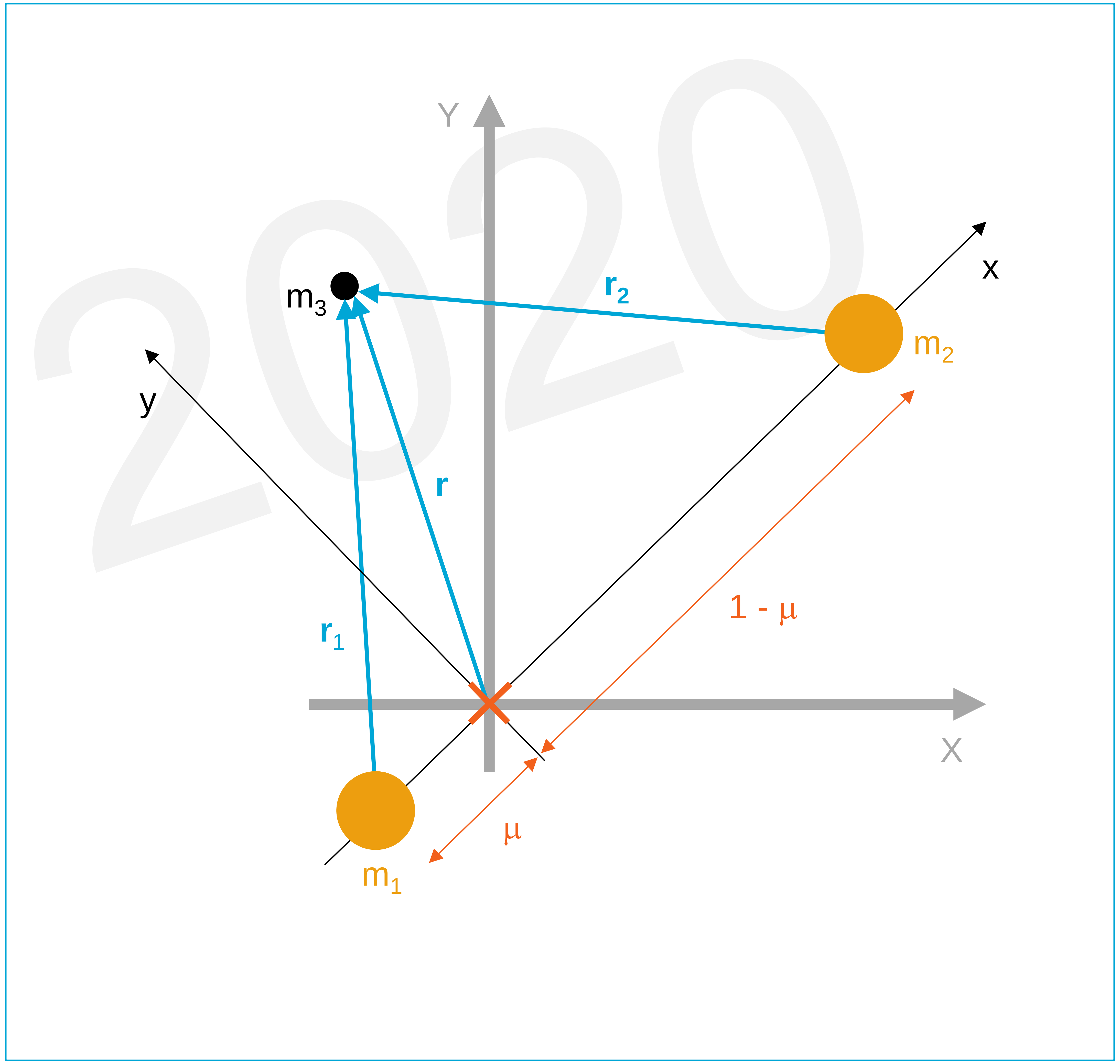
System	μ
Sun – Mercury	1.6600×10^{-7}
Sun – Venus	2.4476×10^{-6}
Sun – Earth	3.0032×10^{-6}
Sun – Mars	3.2268×10^{-7}
Earth – Moon	0.01215

cr3bp – the equations of motion in dimensionless form **+ potential formulation**

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\underbrace{\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right)} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}$$

These terms can be written in potential form



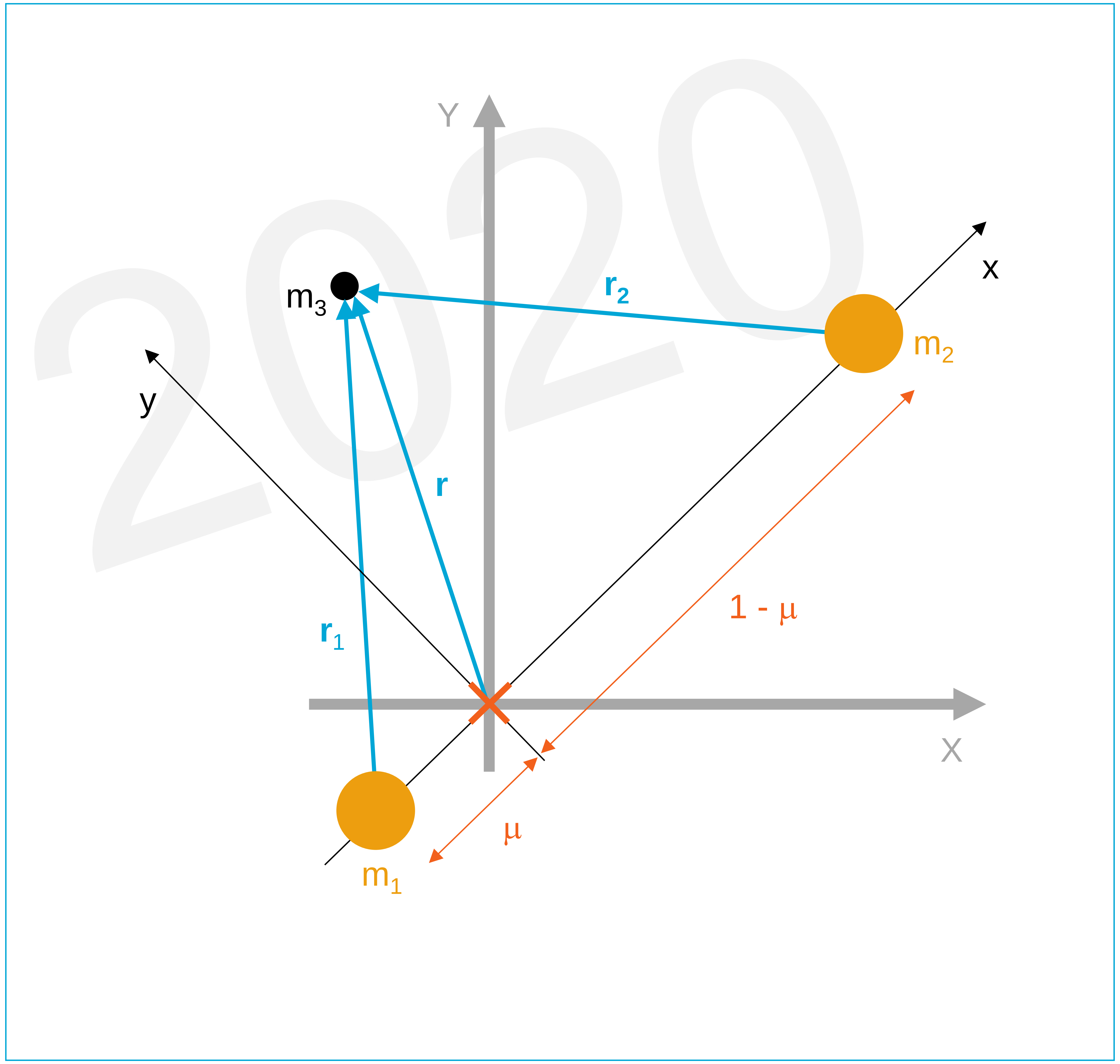
cr3bp – the equations of motion in dimensionless form **+ potential formulation**

- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = -\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}\right)$$

These terms can be written in potential form

$$U = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) - \frac{1}{2}\left(x^2 + y^2\right)$$ **Effective potential**



cr3bp – the equations of motion in dimensionless form + potential formulation

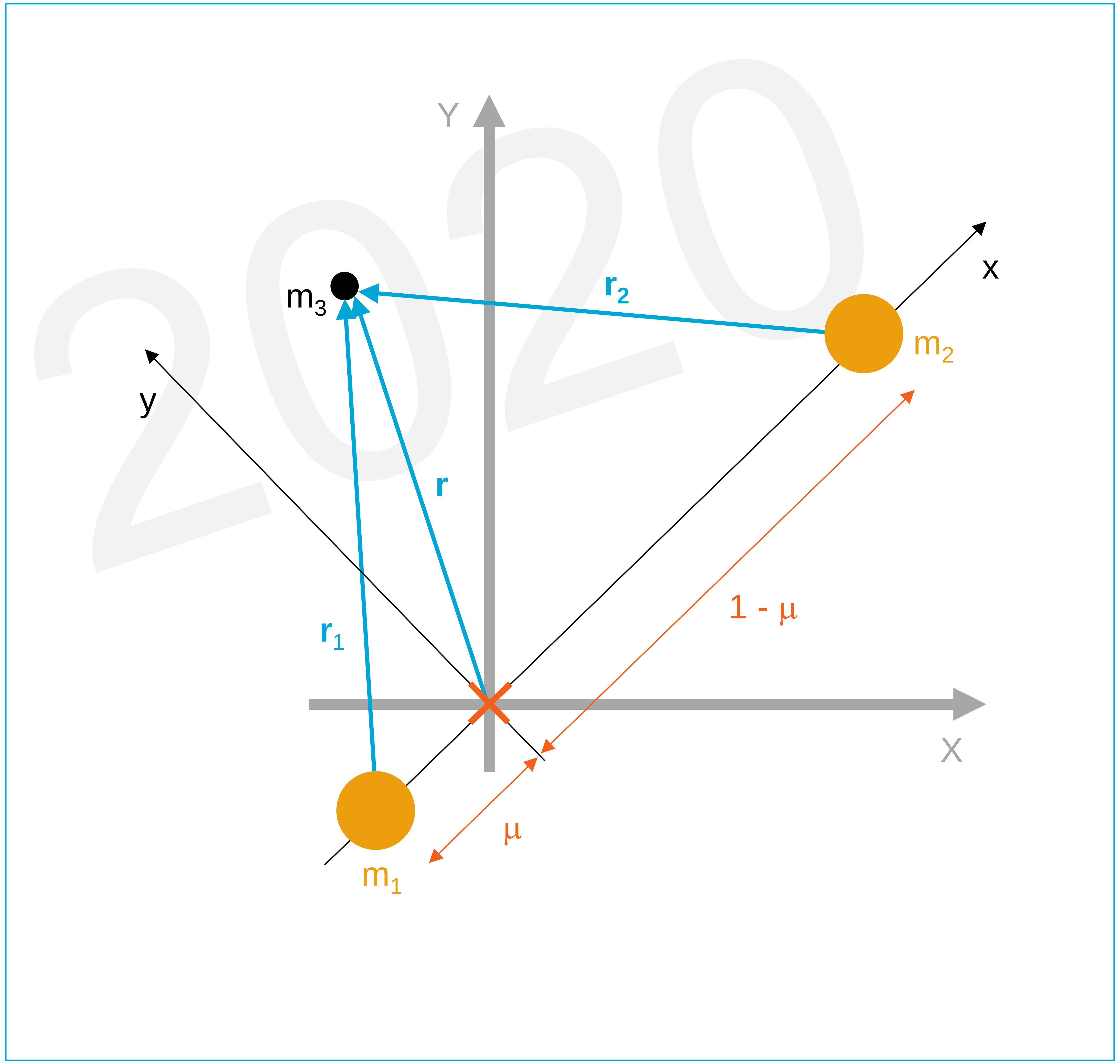
- Equations of motion in rotating frame – **dimensionless**

$$\ddot{\mathbf{r}} = \underbrace{-\left(\frac{1-\mu}{r_1^3}\mathbf{r}_1 + \frac{\mu}{r_2^3}\mathbf{r}_2\right)} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} \underbrace{- \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}$$

These terms can be written in potential form

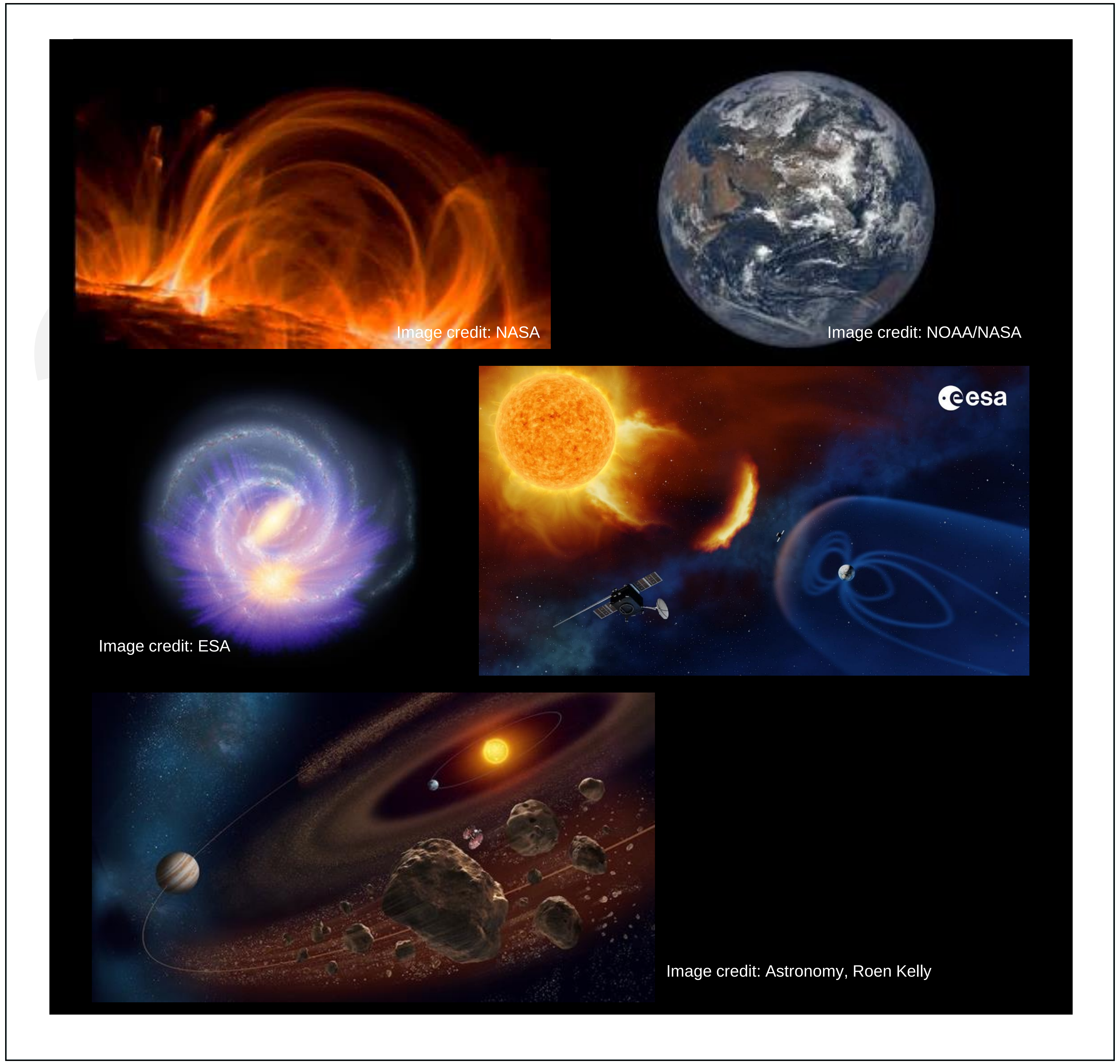
$$U = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) - \frac{1}{2}\left(x^2+y^2\right)$$ Effective potential

$$\ddot{\mathbf{r}} = -2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \nabla U$$



cr3bp – why of interest?

- Interesting dynamical features
  - Equilibria
  - Invariant manifolds
  - Periodic orbits
  - Quasi-periodic orbits
- Exploitable for
  - Space mission applications
  - Understanding the formation and evolution of the Solar System



End of video