Invariant manifolds

Invariant manifolds

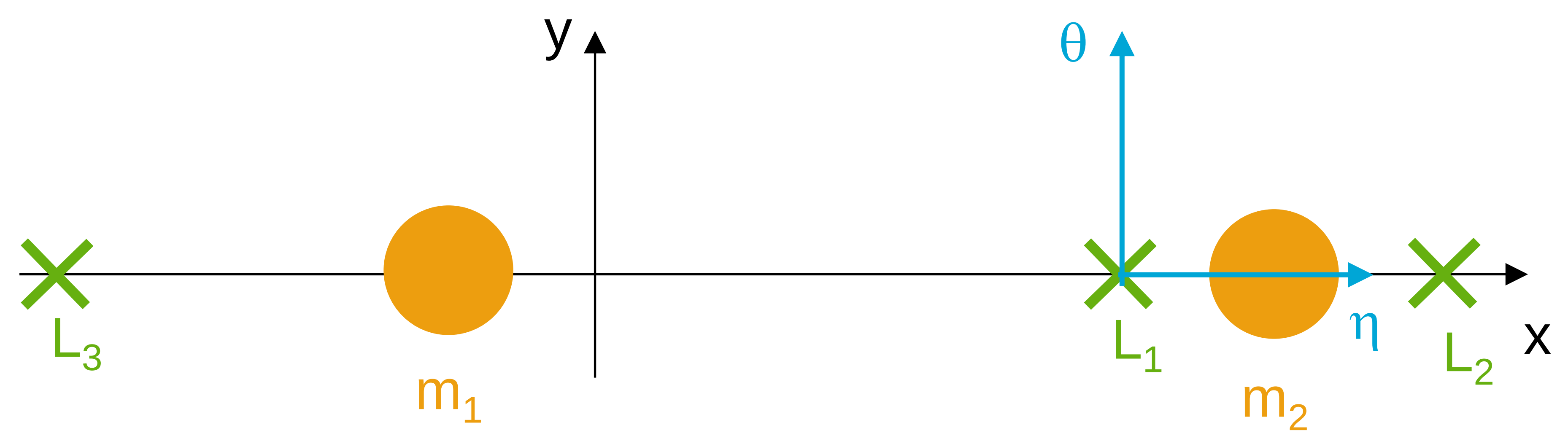
- Linearised equations of motion around the equilibrium

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} \eta & \theta & \dot{\eta} & \dot{\theta} \end{bmatrix}^T$$

$$\lambda_i = \pm\alpha \pm \beta i$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix}$$



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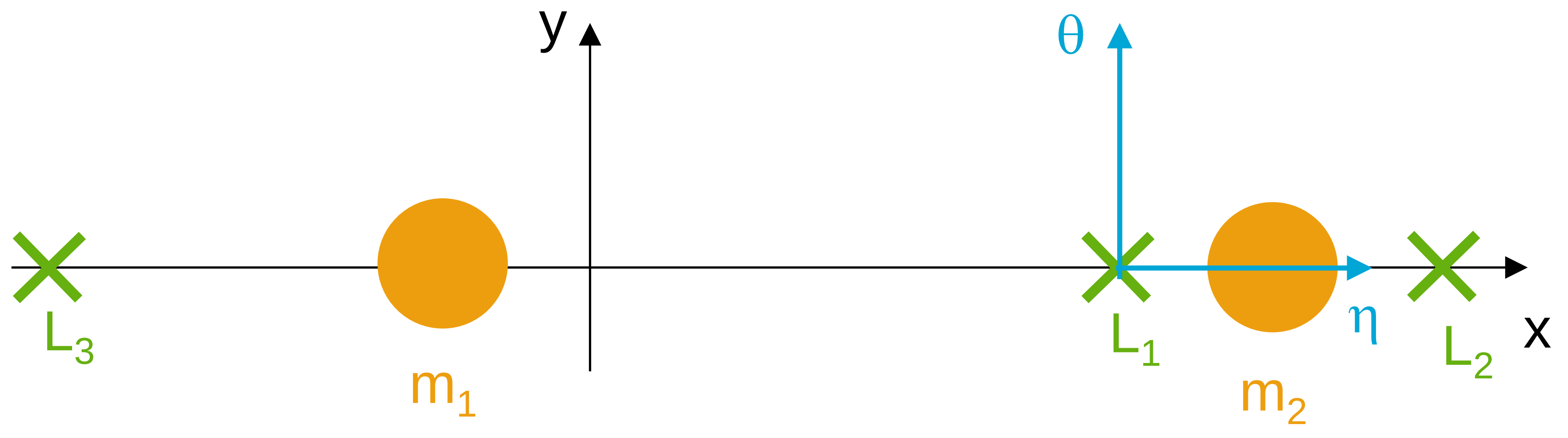
Colinear Lagrange points

- $\lambda_{1,2} = \pm\alpha$
- $\lambda_{3,4} = \pm\beta i$



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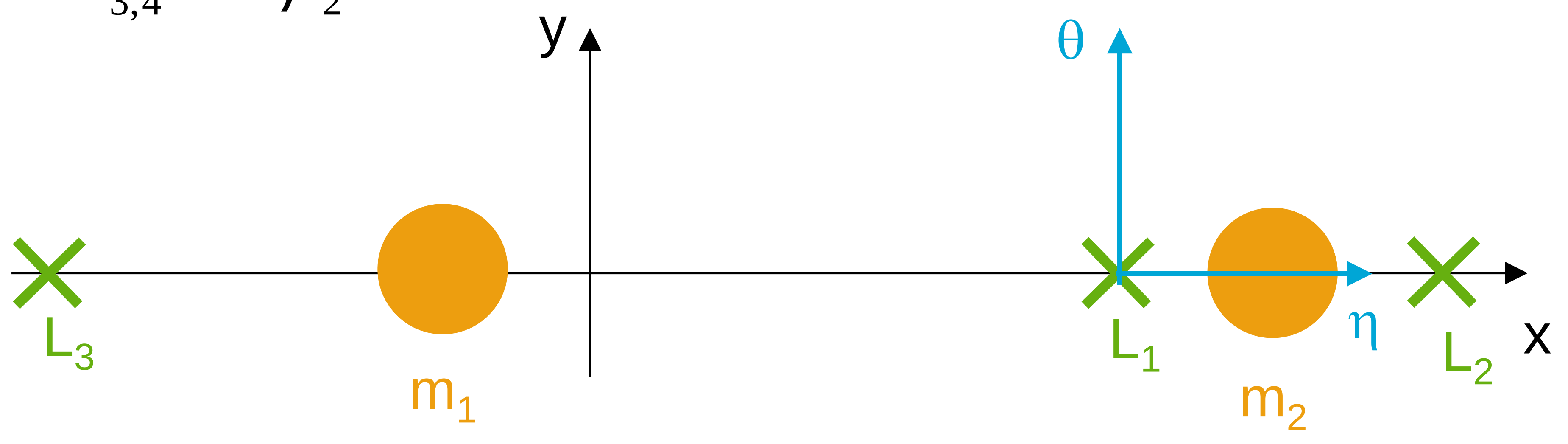
$$\lambda_i = \pm\alpha \pm \beta i$$

Colinear Lagrange points

- $\lambda_{1,2} = \pm\alpha$
- $\lambda_{3,4} = \pm\beta i$

Triangular Lagrange points

- $\lambda_{1,2} = \pm\beta_1 i$
- $\lambda_{3,4} = \pm\beta_2 i$



Invariant manifolds

- Linearised equations of motion around the equilibrium
- If you excite the motion associated with **imaginary eigenvalues** → Stable motion → Periodic motion around the Lagrange points (workshop 7)

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$$\lambda_i = \pm\alpha \pm \beta i$$

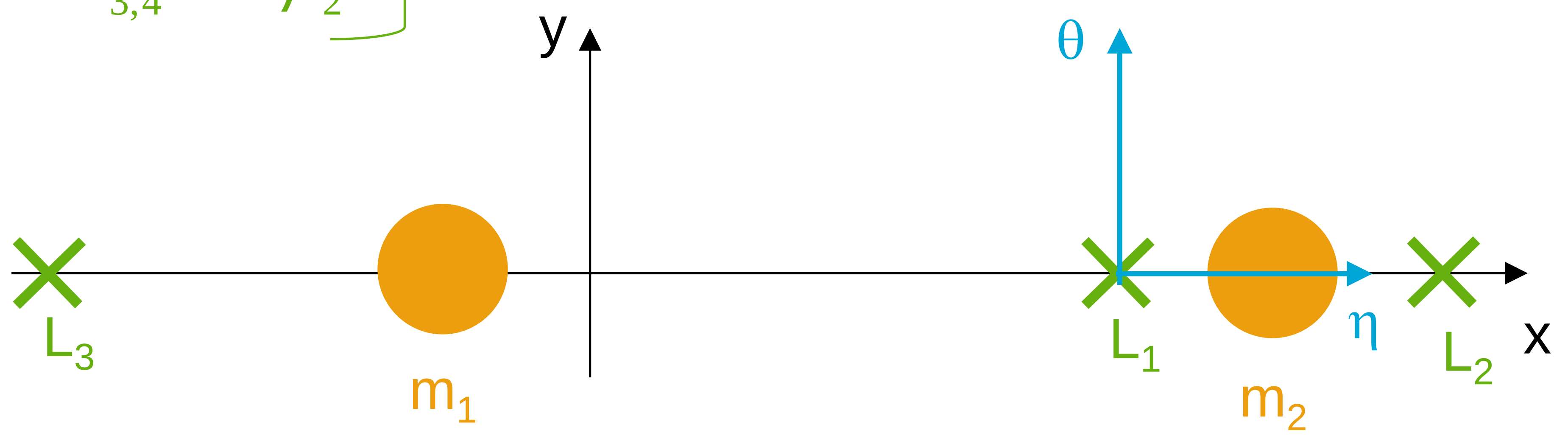
Colinear Lagrange points

- $\lambda_{1,2} = \pm\alpha$
- $\lambda_{3,4} = \pm\beta i$ — **Periodic motion**

Triangular Lagrange points

- $\lambda_{1,2} = \pm\beta_1 i$
- $\lambda_{3,4} = \pm\beta_2 i$

Periodic motion



Invariant manifolds

- Linearised equations of motion around the equilibrium
- If you excite the motion associated with **imaginary eigenvalues** → Stable motion → Periodic motion around the Lagrange points (workshop 7)
- If you excite the motion associated with the **real eigenvalues** → (Un)stable motion → Invariant manifolds towards/away from the Lagrange points

$$\dot{\mathbf{x}} = A\mathbf{x}$$

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$$\lambda_i = \pm\alpha \pm \beta i$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix}$$

Colinear Lagrange points

- $\lambda_{1,2} = \pm\alpha$ **Motion away/towards**
- $\lambda_{3,4} = \pm\beta i$

Triangular Lagrange points

- $\lambda_{1,2} = \pm\beta_1 i$
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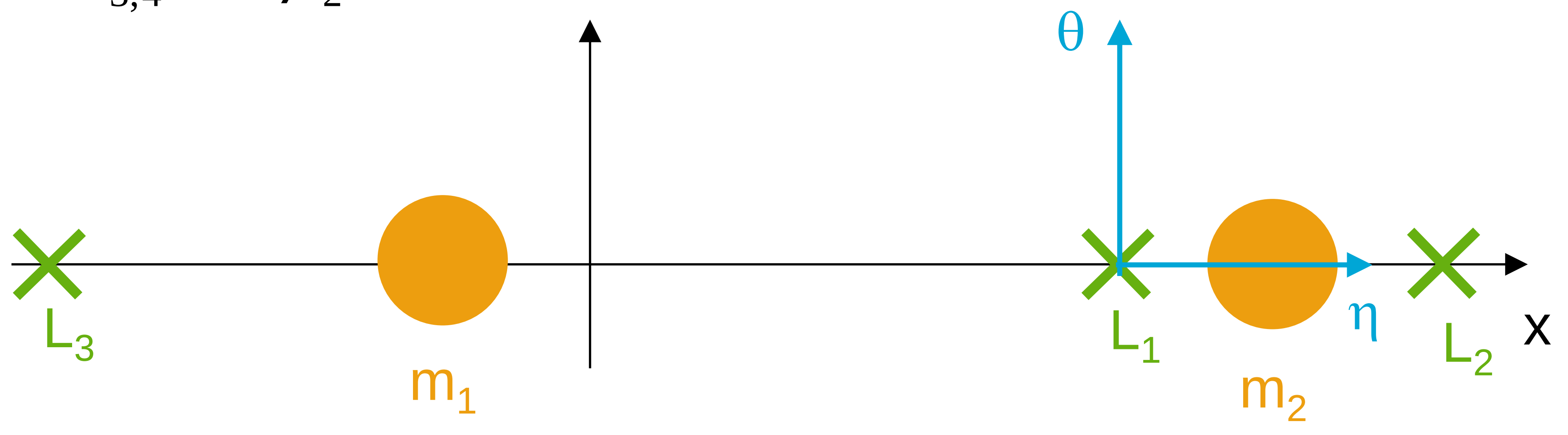

Invariant manifolds

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- If you excite the motion associated with the **real eigenvalues** → (Un)stable motion → Invariant manifolds towards/away from the Lagrange points
- Focus on the unstable eigenvalues of the colinear Langrange points
  - $\lambda_1 = +\alpha$ → motion **away** from equilibrium
  - $\lambda_2 = -\alpha$ → motion **towards** equilibrium

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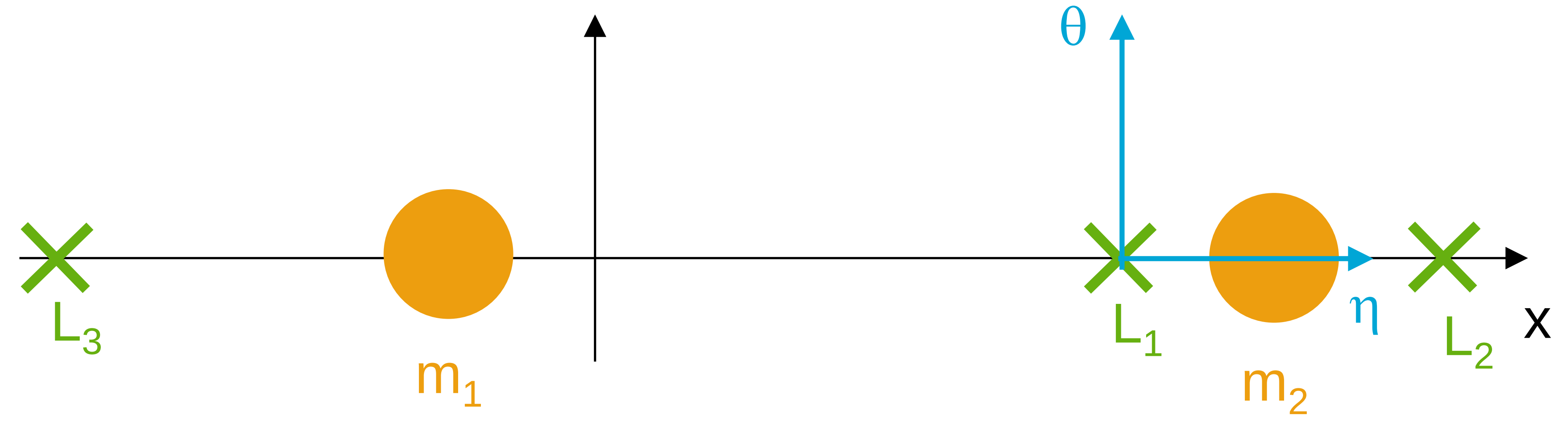
Colinear Lagrange points

- $\lambda_1 = +\alpha$ — Motion **away**
- $\lambda_2 = -\alpha$ — Motion **towards**



Invariant manifolds

- Linearised equations of motion around the equilibrium
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  - $\lambda_1 = +\alpha$ → motion **away** from equilibrium
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- How to "excite" the motion? Through the eigenvectors!

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$$\lambda_i = \pm\alpha \pm \beta i$$

Colinear Lagrange points

- $\lambda_1 = +\alpha$ — Motion **away** → ζ_1
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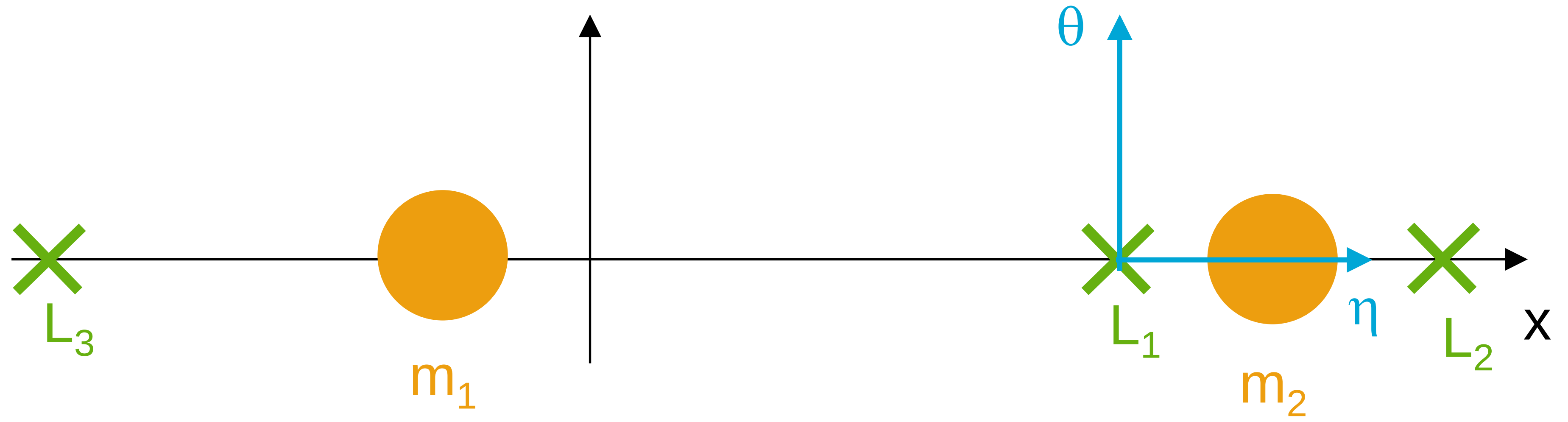
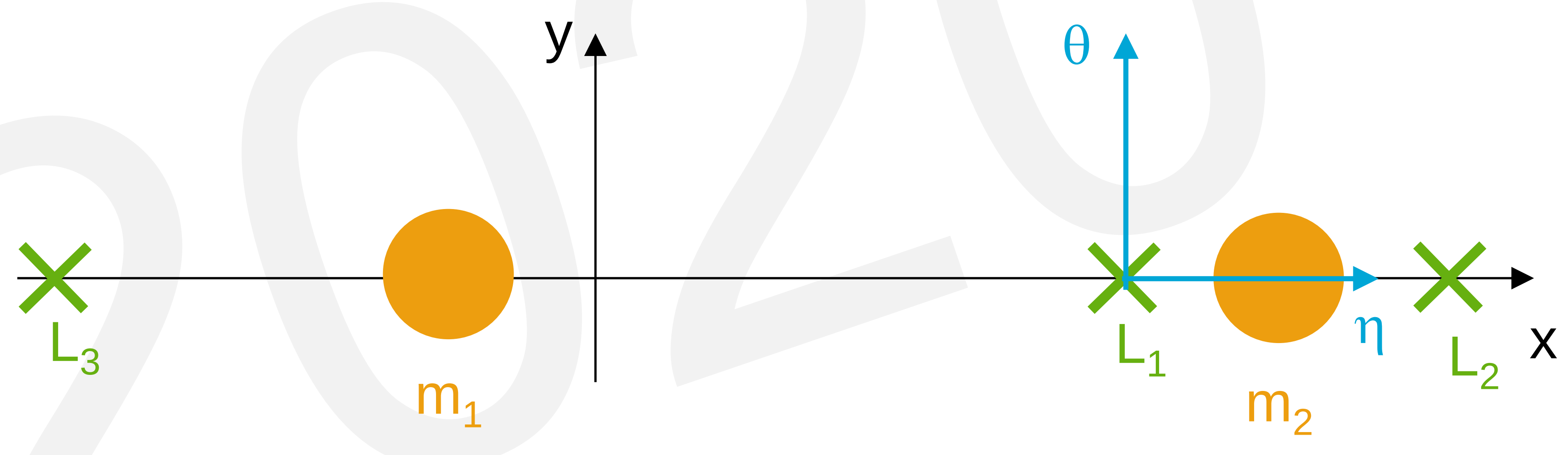


Invariant manifolds

- How to obtain the manifolds?
  - Define the state of m_3 at the Lagrange point

$$\mathbf{X}_0 = \begin{bmatrix} \mathbf{r}_0 & \dot{\mathbf{r}}_0 \end{bmatrix}^T = \begin{bmatrix} x_0 & y_0 & \dot{x}_0 & \dot{y}_0 \end{bmatrix}^T$$



Invariant manifolds

- How to obtain the manifolds?
  - Define the state of m_3 at the Lagrange point
  - Perturb this state in the direction of the real eigenvectors

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$$\mathbf{X}_i = \mathbf{X}_0 \pm \varepsilon \hat{\boldsymbol{\zeta}}_i$$

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One of the two real eigenvectors

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A small perturbation ~10^{-5} – 10^{-6}

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Perturb in either direction

Invariant manifolds

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Initial state of one of the manifolds

Matlab GUI – "STA_Lpoints_and_Manifolds"

Invariant manifolds

- How to obtain the manifolds?
  - Define the state of m_3 at the Lagrange point
  - Perturb this state in the direction of the real eigenvectors
  - To obtain the **unstable** manifold (motion away)
    - **Forward** integrate

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- Once inserted into the stable manifold, you **freely** move towards the Lagrange point

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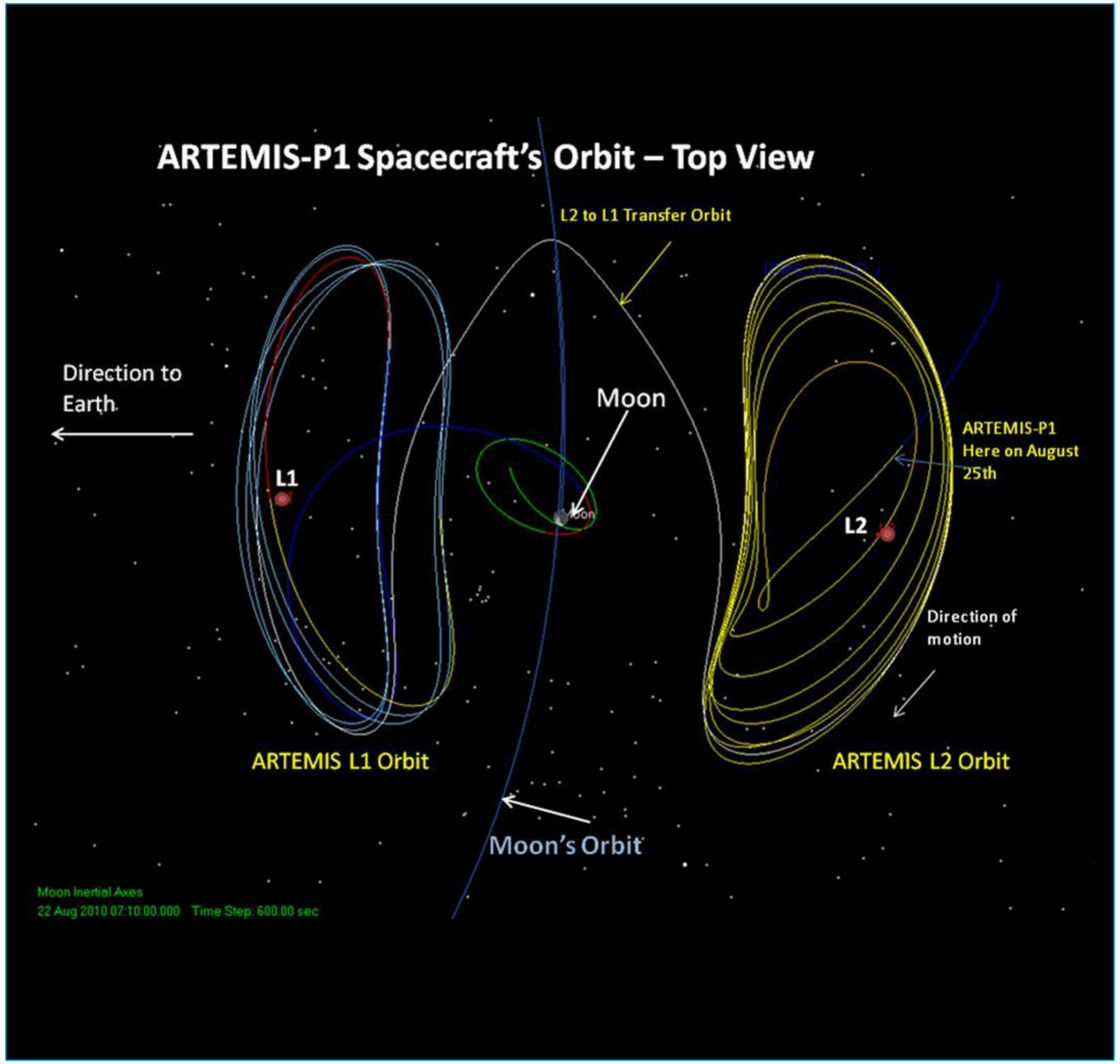
- **Mission design applications → heteroclinic connections!**



End of video