Stability of equilibria

Stability – what is stability?

- Lagrange points are locations where, if
  - The negligible mass, m_3, has zero velocity and
  - No acceleration acts on m_3
  - It will stay stationary *w.r.t.* the rotating frame
- Let's investigate what happens

Matlab GUI – "STA_Lpoints_and_Manifolds"

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- Equilibrium is considered **stable** when
  - m_3 is positioned at the equilibrium with zero velocity
  - A small displacement is imposed on the body
  - And the motion stays bounded to the vicinity of the equilibrium

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- With this definition, which equilibria appear stable?

Matlab GUI – "STA_Lpoints_and_Manifolds"

Stability – obtaining a metric

- Investigate the motion in close vicinity of equilibrium
- Through linearisation of the equations of motion

$$\ddot{\mathbf{r}} = -2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \nabla U$$

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- Assume a small linear perturbation

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$$\mathbf{r} \rightarrow \mathbf{r}_0 + \boldsymbol{\delta}$$

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The “0” indicates conditions at the equilibrium

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- Investigate the motion in close vicinity of equilibrium
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- Assume a small linear perturbation and substitute

$$\ddot{\mathbf{r}} = -2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \nabla U$$

$$\mathbf{r} \rightarrow \mathbf{r}_0 + \boldsymbol{\delta}$$

$$\left(\ddot{\mathbf{r}}_0 + \ddot{\boldsymbol{\delta}}\right) = -2\boldsymbol{\omega} \times \left(\dot{\mathbf{r}}_0 + \dot{\boldsymbol{\delta}}\right) - \nabla U\left(\mathbf{r}_0 + \boldsymbol{\delta}\right)$$

Stability – obtaining a metric

- Investigate the motion in close vicinity of equilibrium
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- Expand the effective potential in Taylor series to first order around equilibrium point

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$$\nabla U\left(\mathbf{r}_0 + \boldsymbol{\delta}\right) = \nabla U\left(\mathbf{r}_0\right) + \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

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- Rewrite

$$\ddot{\mathbf{r}}_0 + \ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega} \times \dot{\mathbf{r}}_0 - 2\boldsymbol{\omega} \times \dot{\boldsymbol{\delta}} - \nabla U\left(\mathbf{r}_0\right) - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

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$$\underbrace{\boxed{\ddot{\mathbf{r}}_0} + \ddot{\boldsymbol{\delta}} = \boxed{-2\boldsymbol{\omega} \times \dot{\mathbf{r}}_0} - 2\boldsymbol{\omega} \times \dot{\boldsymbol{\delta}} \boxed{-\nabla U\left(\mathbf{r}_0\right)}}_{\text{= 0 at equilibrium!}} - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

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$$\ddot{\mathbf{r}} = -2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \nabla U$$

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$$\mathbf{r} \to \mathbf{r}_0 + \boldsymbol{\delta}$$

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- Rewrite

$$\ddot{\mathbf{r}}_0 + \ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega}\times\dot{\mathbf{r}}_0 - 2\boldsymbol{\omega}\times\dot{\boldsymbol{\delta}} - \nabla U\left(\mathbf{r}_0\right) - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

- **Linearised equations of motion**

$$\ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega}\times\dot{\boldsymbol{\delta}} - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

Stability – obtaining a metric

- Linearised equations of motion

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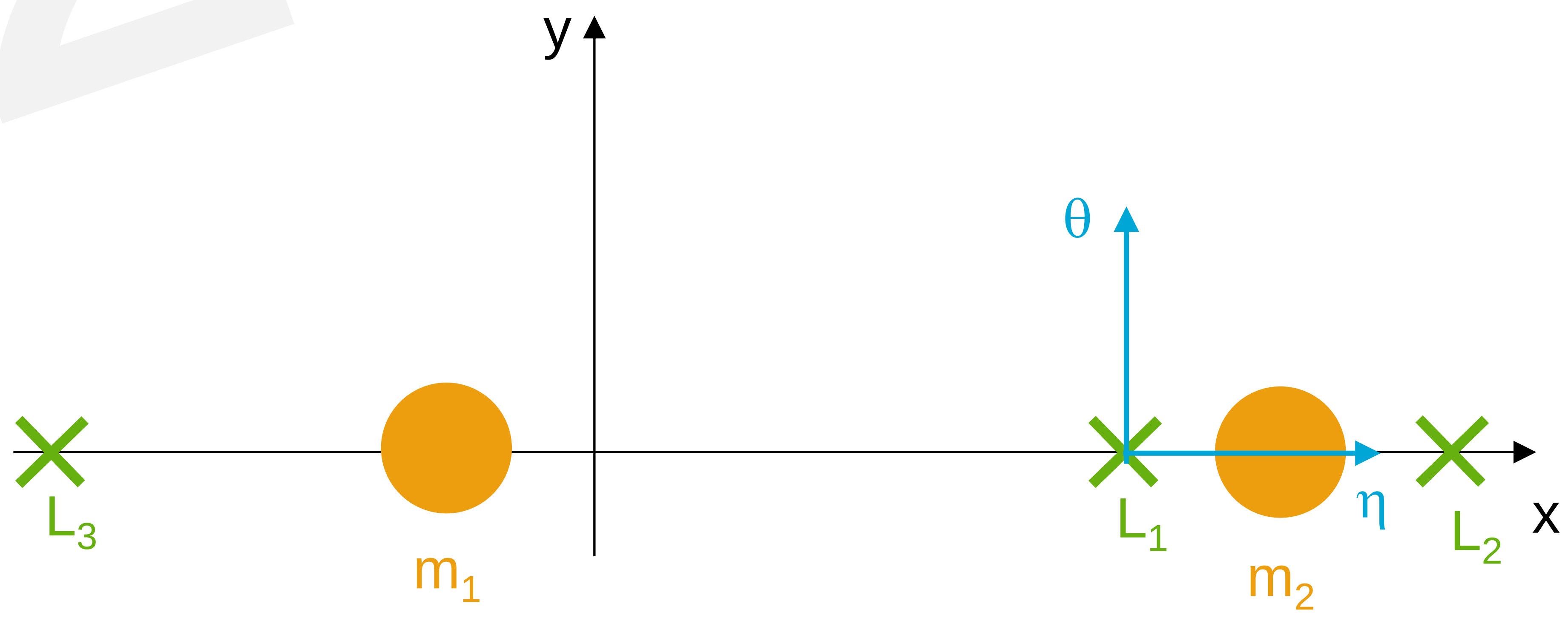
Stability – obtaining a metric

- Linearised equations of motion

- Remember, δ is the perturbation in position *w.r.t.* the equilibrium

$$\ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega} \times \dot{\boldsymbol{\delta}} - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

$$\boldsymbol{\delta} = \begin{bmatrix} \eta & \theta & \xi \end{bmatrix}^T$$



Stability – obtaining a metric

- Linearised equations of motion
- Remember, δ is the perturbation in position *w.r.t.* the equilibrium
- Write in scalar form

$$\ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega}\times\dot{\boldsymbol{\delta}} - \left.\frac{\delta\nabla U}{\delta\mathbf{r}}\right|_0 \boldsymbol{\delta}$$

$$\boldsymbol{\delta} = \begin{bmatrix}\eta & \theta & \xi\end{bmatrix}^T$$

$$\begin{aligned}\ddot{\eta} &= 2\dot{\theta} - \eta U_{xx} - \theta U_{xy} - \xi U_{xz}\\ \ddot{\theta} &= -2\dot{\eta} - \eta U_{yx} - \theta U_{yy} - \xi U_{yz}\\ \ddot{\xi} &= -\eta U_{zx} - \theta U_{zy} - \xi U_{zz}\end{aligned}$$

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- Remember, δ is the perturbation in position *w.r.t.* the equilibrium
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$$\ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega} \times \dot{\boldsymbol{\delta}} - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

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$$\begin{aligned}
\ddot{\eta} &= \quad 2\dot{\theta} - \eta U_{xx} - \theta U_{xy} - \xi U_{xz} \\
\ddot{\theta} &= -2\dot{\eta} - \eta U_{yx} - \theta U_{yy} - \xi U_{yz} \\
\ddot{\xi} &= \qquad\;\; - \eta U_{zx} - \theta U_{zy} - \xi U_{zz}
\end{aligned}$$

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$$\boldsymbol{\delta} = \begin{bmatrix} \eta & \theta & \xi \end{bmatrix}^T$$

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- When you do the math...

$$U_{xy} = U_{yx} \quad , \quad U_{xz} = U_{zx} = U_{yz} = U_{zy} = 0$$

Stability – obtaining a metric

- Linearised equations of motion

$$\ddot{\boldsymbol{\delta}} = -2\boldsymbol{\omega} \times \dot{\boldsymbol{\delta}} - \left.\frac{\delta \nabla U}{\delta \mathbf{r}}\right|_0 \boldsymbol{\delta}$$

- Remember, δ is the perturbation in position *w.r.t.* the equilibrium

$$\boldsymbol{\delta} = \begin{bmatrix} \eta & \theta & \xi \end{bmatrix}^T$$

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$$\begin{aligned} \ddot{\eta} &= 2\dot{\theta} - \eta U_{xx} - \theta U_{xy} - \xi U_{xz} \\ \ddot{\theta} &= -2\dot{\eta} - \eta U_{yx} - \theta U_{yy} - \xi U_{yz} \\ \ddot{\xi} &= \phantom{-2\dot{\eta}} - \eta U_{zx} - \theta U_{zy} - \xi U_{zz} \end{aligned}$$

- When you do the math...

$$U_{xy} = U_{yx} \quad , \quad U_{xz} = U_{zx} = U_{yz} = U_{zy} = 0$$

- What you are left with

$$\begin{aligned} \ddot{\eta} &= 2\dot{\theta} - \eta U_{xx} - \theta U_{xy} \\ \ddot{\theta} &= -2\dot{\eta} - \eta U_{yx} - \theta U_{yy} \\ \ddot{\xi} &= -\xi U_{zz} \end{aligned}$$

Stability – obtaining a metric

- Linearised equations of motion

$$
\begin{aligned}
\ddot{\eta} &= 2\dot{\theta} - \eta U_{xx} - \theta U_{xy} \\
\ddot{\theta} &= -2\dot{\eta} - \eta U_{yx} - \theta U_{yy} \\
\ddot{\xi} &= -\xi U_{zz}
\end{aligned}
$$

Stability – obtaining a metric

- Linearised equations of motion
- **Motion in ξ-direction (out-of-plane direction)** is
  - **Uncoupled** from motion in (η,θ)-plane (in-plane)
  - Represents an **undamped harmonic oscillator**
  - Motion is **purely periodic** and thus **stable**!

$$
\begin{aligned}
\ddot{\eta} &= 2\dot{\theta} - \eta U_{xx} - \theta U_{xy} \\
\ddot{\theta} &= -2\dot{\eta} - \eta U_{yx} - \theta U_{yy} \\
\ddot{\xi} &= -\xi U_{zz}
\end{aligned}
$$

Stability – obtaining a metric

- Linearised equations of motion – **in-plane**

$$\ddot{\eta} = 2\dot{\theta} - \eta U_{xx} - \theta U_{xy}$$
$$\ddot{\theta} = -2\dot{\eta} - \eta U_{yx} - \theta U_{yy}$$

Stability – obtaining a metric

- Linearised equations of motion – in-plane

$$\ddot{\eta} = 2\dot{\theta} - \eta U_{xx} - \theta U_{xy}$$
$$\ddot{\theta} = -2\dot{\eta} - \eta U_{yx} - \theta U_{yy}$$

- Rewrite into vector form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

Stability – obtaining a metric

- Linearised equations of motion – in-plane

$$\ddot{\eta} = 2\dot{\theta} - \eta U_{xx} - \theta U_{xy}$$
$$\ddot{\theta} = -2\dot{\eta} - \eta U_{yx} - \theta U_{yy}$$

- Rewrite into vector form, using
  - The **state vector**, **x**

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} \eta & \theta & \dot{\eta} & \dot{\theta} \end{bmatrix}^T$$

Stability – obtaining a metric

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$$\ddot{\eta} = 2\dot{\theta} - \eta U_{xx} - \theta U_{xy}$$
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- Rewrite into vector form, using
  - The state vector, **x**
  - The **Jacobian**, **A**

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} \eta & \theta & \dot{\eta} & \dot{\theta} \end{bmatrix}^T \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix}$$

Stability – obtaining a metric

- Linearised equations of motion – in-plane
- Rewrite into vector form, using
  - The state vector, **x**
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- The **stability** of a linearised system is
  - given by the **eigenvalues of the Jacobian**
  - that come in conjugate pairs and

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$$\lambda_i = \pm\alpha \pm \beta i$$

Stability – obtaining a metric

- Linearised equations of motion – in-plane
- Rewrite into vector form, using
  - The state vector, **x**
  - The Jacobian, **A**
- The stability of a linearised system is
  - given by the eigenvalues of the Jacobian
  - that come in conjugate pairs and
  - have associated **eigenvectors** (important later on!)

$$\ddot{\eta} = 2\dot{\theta} - \eta U_{xx} - \theta U_{xy}$$
$$\ddot{\theta} = -2\dot{\eta} - \eta U_{yx} - \theta U_{yy}$$

$$\dot{\mathbf{x}} = A\mathbf{x}$$

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$$\lambda_i = \pm\alpha \pm \beta i \qquad \zeta_i$$

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Stability – obtaining a metric

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  - given by the **eigenvalues** of the Jacobian

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$$\lambda_i = \pm\alpha \pm \beta i$$

Stability – obtaining a metric

- The stability of a linearised system is
  - given by the eigenvalues of the Jacobian
- System is **unstable** if at least 1 eigenvalue has **$\alpha > 0$** (positive real part)

$$\dot{\mathbf{x}} = A\mathbf{x}$$

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Stability – obtaining a metric

- The stability of a linearised system is
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- System is unstable if at least 1 eigenvalue has $\alpha > 0$ (positive real part)
- System **is marginally/asymptotically stable** if all eigenvalues have $\alpha \leq 0$ (zero or negative real part)

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  - Purely imaginary
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- Why? Look at the solution of the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix} \qquad \lambda_i = \pm\alpha \pm \beta i$$

$$\eta = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + A_4 e^{\lambda_3 t}$$
$$\theta = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} + B_4 e^{\lambda_3 t}$$

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  - Purely imaginary
  - Or negative real part
- Most cr3bp systems show the same behavior

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix} \qquad \lambda_i = \pm\alpha \pm \beta i$$

Colinear Lagrange points

- $\lambda_{1,2} = \pm\alpha$
- $\lambda_{3,4} = \pm\beta i$

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Colinear Lagrange points

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- $\lambda_{3,4} = \pm\beta i$

Stable or not?

Stability – obtaining a metric

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Colinear Lagrange points

$$\left.\begin{array}{l} \bullet\ \lambda_{1,2} = \pm\alpha \\ \bullet\ \lambda_{3,4} = \pm\beta i \end{array}\right\} \textbf{Unstable} \rightarrow \alpha > 0 \text{ for } \lambda_{1,2}$$

Stability – obtaining a metric

- The stability of a linearised system is
  - given by the eigenvalues of the Jacobian
- System is unstable if at least 1 eigenvalue has α > 0 (positive real part)
- System is marginally/asymptotically stable if all eigenvalues have $\alpha \le 0$ (zero or negative real part)
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Colinear Lagrange points

$$\left.\begin{array}{l} \bullet\ \lambda_{1,2} = \pm\alpha \\ \bullet\ \lambda_{3,4} = \pm\beta i \end{array}\right\}$$ **Unstable** → α > 0 for $\lambda_{1,2}$

Triangular Lagrange points

$$\left.\begin{array}{l} \bullet\ \lambda_{1,2} = \pm\beta_1 i \\ \bullet\ \lambda_{3,4} = \pm\beta_2 i \end{array}\right\}$$ Stable or not?

Stability – obtaining a metric

- The stability of a linearised system is
  - given by the eigenvalues of the Jacobian
- System is unstable if at least 1 eigenvalue has $\alpha > 0$ (positive real part)
- System is marginally/asymptotically stable if all eigenvalues have $\alpha \leq 0$ (zero or negative real part)
  - Purely imaginary
  - Or negative real part
- Most cr3bp systems show the same behavior

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix} \qquad \lambda_i = \pm\alpha \pm \beta i$$

Colinear Lagrange points

$$\left.\begin{array}{l} \bullet\ \lambda_{1,2} = \pm\alpha \\ \bullet\ \lambda_{3,4} = \pm\beta i \end{array}\right\} \textbf{Unstable} \rightarrow \alpha > 0 \text{ for } \lambda_{1,2}$$

Triangular Lagrange points

$$\left.\begin{array}{l} \bullet\ \lambda_{1,2} = \pm\beta_1 i \\ \bullet\ \lambda_{3,4} = \pm\beta_2 i \end{array}\right\} \textbf{Stable} \rightarrow \alpha = 0 \text{ for all } \lambda$$

Stability – obtaining a metric

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Colinear Lagrange points

$$\left.\begin{aligned} \lambda_{1,2} &= \pm\alpha \\ \lambda_{3,4} &= \pm\beta i \end{aligned}\right\}$$

Unstable → $\alpha > 0$ for $\lambda_{1,2}$

Triangular Lagrange points

$$\left.\begin{aligned} \lambda_{1,2} &= \pm\beta_1 i \\ \lambda_{3,4} &= \pm\beta_2 i \end{aligned}\right\}$$

Stable → $\alpha = 0$ for all λ

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- **Assignment**
  - Compute stability of your Lagrange points

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{bmatrix} \qquad \lambda_i = \pm\alpha \pm \beta i$$

Colinear Lagrange points

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Triangular Lagrange points

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End of video