

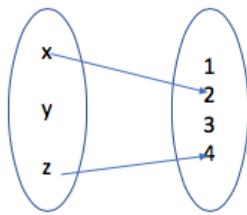
# Discrete Mathematics

Tutorial sheet

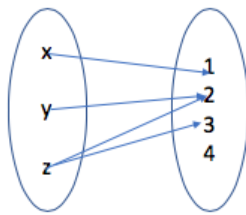
Funtions

## Question 1.

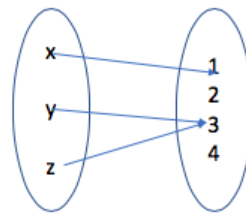
Let  $A$  and  $B$  be two sets with  $A = \{x, y, z\}$  and  $B = \{1, 2, 3, 4\}$ . Which of the following arrow diagrams define functions from  $A$  to  $B$ ?



(i)



(ii)



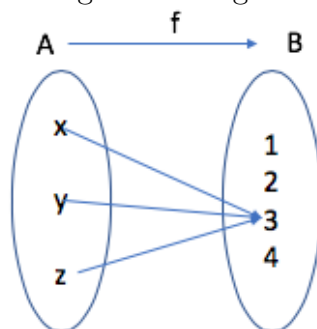
(iii)

Solution:

In (i) there is an element  $x$  of  $A$  that is not mapped to any element of  $B$  as there is no arrow coming from  $x$ . Hence, it is not a function as not every element  $A$  has an image in  $B$ . In (ii) there is an element  $z$  of  $A$  that is mapped to two elements, 2 and 3, of  $B$ . Hence, it is not a function as in a function every element of the domain needs to be mapped to unique element in the co-domain. However, the arrow diagram in (iii) defines a function as every element of  $A$  is mapped to unique image in  $B$ .

## Question 2.

Let  $A$  and  $B$  be two sets with  $A = \{x, y, z\}$  and  $B = \{1, 2, 3, 4\}$ . Let  $f$  from  $A$  to  $B$  defined by the following arrow diagram:



1. Write the domain, the co-domain and the range of  $f$ .
2. Find  $f(x)$  and  $f(y)$ .
3. Write down the set of pre-images of 3 and the set of pre-images of 1.
4. represent  $f$  as a set of ordered pairs.

Solution:

1.  $D_f = A$ ,  $co - D_f = B$  and  $R_f = \{3\}$ .
2.  $f(x) = 3$  and  $f(y) = 3$ .
3. pre-images of  $3 = \{x, y, z\}$  and pre-images of  $1 = \emptyset$ .
4.  $f$  as a set of ordered pairs is  $\{(x, 3), (y, 3), (z, 3)\}$ .

### Question 3.

The Hamming distance function is very important in coding theory. It gives a measure of the difference between two strings of 0's and 1's that have the same length. Let  $S_n$  be the set of all strings of 0's and 1's of length  $n$ . The Hamming function  $H$  is defined as follows:

$$H : S_n \times S_n \rightarrow \mathbb{N} \cup \{0\}$$

$$(s, t) \rightarrow H(s, t) = \text{The number of positions in which } s \text{ and } t \text{ have different values.}$$

For  $n = 5$ , Find  $H(11111, 00000)$ ,  $H(11000, 00000)$ ,  $H(00101, 01110)$  and  $H(10001, 01111)$ .

Solution:

$$H(11111, 00000) = 5, H(11000, 00000) = 2, H(00101, 01110) = 3 \text{ and } H(10001, 01111) = 4.$$

### Question 4.

Digital messages consist of a finite sequence of 0's and 1's. When they are communicated across a transmission channel, they are frequently coded in special ways to reduce the chance that they will be garbled by interfering noise in the transmission lines. A simple way to encode a message of 0's and 1's is to write each bit three times, for example: the message 0010111 would be encoded as 000 000 111 000 111 111 111.

Let  $A$  be the set of all strings of 0's and 1's and let  $E$  and  $D$  be the encoding and the decoding function on the set  $A$  defined for each string,  $s$ , in  $A$  as follows:

$E(s)$  = The string obtained from  $s$  by replacing each bit of  $s$  with the same bit written three times.

$D(s)$  = The string obtained from  $s$  by replacing each consecutive triple of three identical bits of  $s$  by a single copy of that bit.

Find  $E(0110)$ ,  $E(0101)$ ,  $D(000111000111000111111)$  and

$D(111111000111000111000000)$

Solution:

$$E(0110) = 000111111000, E(0101) = 000111000111, D(000111000111000111111) = 0101011$$

and

$$D(111111000111000111000000) = 11010100$$

### Question 5.

Let  $A = \{1, 2, 3, 4, 5, 6\}$ ,  $B = \{a, b, c, d\}$  and  $C = \{w, x, y, z\}$  be three sets.

Let  $f$  and  $g$  be two functions defined as follows:

$f : A \rightarrow B$  is defined by the following table.

$x$	1	2	3	4	5	6
$f(x)$	$a$	$b$	$a$	$c$	$d$	$d$

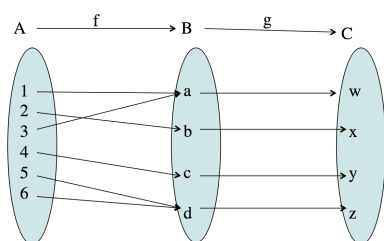
$g : B \rightarrow C$  is defined by the following table.

$x$	$a$	$b$	$c$	$d$
$g(x)$	$w$	$x$	$y$	$z$

1. Draw arrow diagrams to represent the function  $f$  and  $g$ .
2. List the domain; the co-domain and the range of  $f$  and  $g$ .
3. Find  $f(1)$ , the ancestor (pre-image) of  $d$ . and  $(g \circ f)(3)$
4. Show that  $f$  is not a one to one function.
5. Show that  $f$  is an onto function.
6. Show that  $g$  is both one to one and onto.

Solution:

#### 1. Arrow diagram



2.  $D_f = A = \{1, 2, 3, 4, 5, 6\}$   $Co - D_f = B = \{a, b, c, d\}$   $R_f = \{a, b, c, d\}$   
 $D_g = B = \{a, b, c, d\}$   $Co - D_g = C = \{w, x, y, z\}$   $R_g = \{w, x, y, z\}$
3.  $f(1) = a$ , (pre-image) of  $d = \{5, 6\}$ .  $(g \circ f)(3) = g(f(3)) = g(a) = w$
4.  $f$  is not a one to one function as  $f(5) = f(6) = d$ .

5. The arrow diagram shows that every element in the co-domain has at least one pre-image, hence, the function  $f$  is an onto function.
6. It is clear from the arrow diagram that every element of the range of  $g$  has a unique pre-image, hence,  $g$  is a one to one function.  $R_g = Co - D_g$ , hence,  $g$  is an onto function.

**Question 6.**

Suppose you read that a function  $f : \mathbb{Z} \times \mathbb{Z}^+ \rightarrow \mathbb{Q}$  is defined by the formula  $f(m, n) = \frac{m}{n}$  for all  $(m, n) \in \mathbb{Z} \times \mathbb{Z}^+$ .

1. Is  $f$  a one to one function?
2. Is  $f$  an onto function?

Solution:

1.  $f(1, 1) = f(2, 2) = 1$  hence, more than one input can lead to the same output. Hence,  $f$  is not a one to one function.
2. Every rational number can be written with a positive denominator, hence,  $f$  is an onto function.

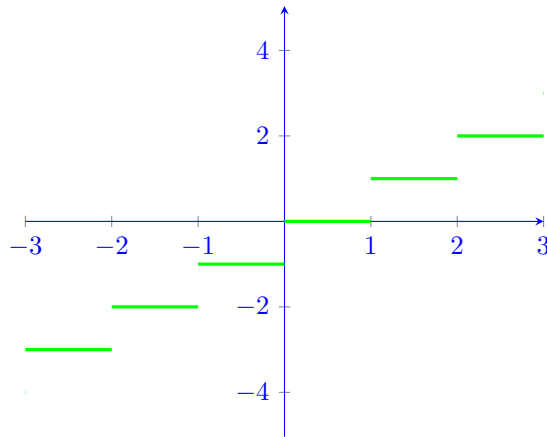
**Question 7.**

Given a function  $f$  defined by  $f(x) = \lfloor x \rfloor$ . where  $f : \mathbb{R} \rightarrow \mathbb{Z}$ ,

1. Plot the graph of the function  $f(x)$  for  $x \in [-3, 3]$ .
2. Use this graph to find  $\lfloor \pi \rfloor$ ,  $\lfloor -2.5 \rfloor$ ,  $\lfloor -1 \rfloor$ .
3. Use the graph in (1) to show that  $f$  is not a one to one (not injective) function.
4. Is  $f$  onto (surjective)? Justify your answer.

Solution:

1. Graph:



2.  $3 < \pi = 3.14 < 4 < 4 \implies [\pi] = 3, [-2.5] = -3$  and  $[-1] = -1$
3. The graph shows that each element of the range has more than one pre-image, i.e.  $[2.5] = [2.] = 2$ . Therefore, the floor function is not a one to one function.
4. for all  $n$  in  $\mathbb{Z}$  there exists at least one pre-images  $x=n$  in  $\mathbb{R}$  such that  $[x] = n$ . Therefore every element of co-domain has a pre-image, hence, the the floor function is an onto function.

### Question 8.

Let  $S$  denote the set of all 3 bit binary strings and  $B = (0, 1, 2, 3)$ . The function  $f : S \rightarrow B$  is defined by the rule

$$f(x) = \text{the number of zeros in } x \text{ for each } x \in S.$$

Find the following.

1. The domain of  $f$ .
2.  $f(001)$  and  $f(101)$ .
3. The set of ancestors of 2.
4. The range of  $f$ .
5. Say whether or not  $f$  is one to one, giving a reason for your answer.
6. Say whether or not  $f$  is onto, giving a reason for your answer.

Solution:

1.  $D_f = \{000, 001, 010, 011, 100, 101, 110, 111\}$
2.  $f(001) = 2$  and  $f(101) = 1$ .
3. The set of ancestors of 2 =  $\{001, 010, 100\}$ .

4. The range of  $f = \{0, 1, 2, 3\}$ .
5.  $f$  is not one to one as 2 has more than one ancestor.
6.  $f$  is onto as the Range of  $f$   $R_f = Co - D_f = \{0, 1, 2, 3\}$ .

**Question 9.**

Let  $f(x) = x \bmod 3$ , where  $f(x)$  is the remainder when  $x$  is divided by 3, and  $f : \mathbb{Z}^+ \rightarrow \{0, 1, 2\}$ .

1. Find  $f(7)$  and  $f(12)$ .
2. Find the ancestors of 2.
3. Say whether or not  $f(x)$  is one to one, justifying your answer.
4. Say whether or not  $f(x)$  is onto, justifying your answer.

Solution:

1.  $f(7) = 7 \bmod 3 = 1$  and  $f(12) = 12 \bmod 3 = 0$ .
2. ancestors of 2 =  $\{2, 5, 8, 11, \dots\}$ .
3.  $f(x)$  is not one to one as  $f(2) = f(5)$  and  $2 \neq 5$
4.  $f(x)$  is onto as each element in  $\{0, 1, 2\}$  has at least one pre-image.

**Question 10.**

Given the following function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x) = 4x - 1$ , for any real number  $x$ .

1. Is  $f$  a one to one function? Prove or give a counter-example.
2. Is  $f$  an onto function? Prove or give a counter-example.
3. Is  $f$  invertible? and why? if the answer yes define  $f^{-1}$ .

Solution:

1. One-to-one: To prove that  $f$  is a one to one function, we need to show that given two real number  $a$  and  $b$  if  $f(a) = f(b)$  then  $a = b$ .  $f(a) = f(b) \implies 4a - 1 = 4b - 1 \implies 4a = 4b \implies a = b$ . Thus,  $f$  is one to one function.
2. Onto: To prove that  $f$  is onto, you must prove that for all  $y \in \mathbb{R}$  there exist  $x \in \mathbb{R}$  such that  $f(x) = y$ . Given a real number  $y$ , we need to show that there exists a real number  $x$  such that  $y = 4x - 1$ .  
if such real number  $x$  exists, then  $4x - 1 = y \implies 4x = y + 1 \implies x = \frac{y+1}{4} \in \mathbb{R}$   
Hence,  $\forall y \in \mathbb{R}, \exists x = \frac{y+1}{4} \in \mathbb{R}$  such that  $f(x) = y$ . Therefore,  $f$  is an onto function.

3.  $f$  is both a one to one and an onto function. Hence,  $f$  is invertible and the inverse function is defined as follow:

$$f^{-1} : \mathbb{R} \rightarrow \mathbb{R} \text{ with } f^{-1}(x) = \frac{x+1}{4}, \forall x \in \mathbb{R}$$

### Question 11.

Given the following function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  with  $g(x) = 4x - 1$ , for any real number  $x$ .

1. Is  $g$  a one to one function? Prove or give a counterexample.
2. Is  $g$  an onto function? Prove or give a counterexample.
3. Is  $g$  invertible? and why? if the answer yes define  $g^{-1}$ .

Solution:

1. One-to-one: To prove that  $f$  is a one to one function, we need to show that given two integers  $a$  and  $b$  if  $g(a) = g(b)$  then  $a = b$ .  $g(a) = g(b) \implies 4a - 1 = 4b - 1 \implies 4a = 4b \implies a = b$ . Thus,  $g$  is one to one function.
2. Onto: To prove that  $g$  is onto, you must prove that for all  $m \in \mathbb{Z}$  there exist  $n \in \mathbb{Z}$  such that  $g(x) = y$ . Given an integer  $m$ , we need to show that there exists an integer  $n$  such that  $m = 4n - 1$ .  
If such integer  $n$  exists, then  $4n - 1 = m \implies 4n = m + 1 \implies n = \frac{m+1}{4}$   
Form  $m = 0, n = \frac{0+1}{4} = \frac{1}{4}$  which is not an integer. hence, 0 has no pre-image, thus  $g$  is not an onto function.
3.  $g$  is a one to one but not an onto function. Thus,  $g$  is not invertible and hence,  $g^{-1}$  doesn't exist.

### Question 12.

Given the following function  $h : \mathbb{R} \rightarrow \mathbb{R}$  with  $h(x) = x^2 - 1$ , for any real number  $x$ .

1. What is co-domain and the range of  $h$
2. Is  $h$  a one to one function? Prove or give a counterexample.
3. Is  $h$  an onto function? Prove or give a counterexample.
4. Is  $h$  invertible? and why? if the answer yes define  $h^{-1}$ .

Solution:

1.  $co - D_h = \mathbb{R}$  and  $R_h = [-1, +\infty[$

2. One-to-one:  $h(2) = 2^2 - 1 = 3$  and  $h(-2) = (-2)^2 - 1 = 3$ , hence,  $h(-2) = h(2)$  but  $-2 \neq 2$ . Thus  $h$  is not a one to one function.
3. Onto: To prove that  $h$  is onto, you must prove that for all  $x \in \mathbb{R}$  there exist  $x \in \mathbb{R}$  such that  $h(x) = y$ . However, all negative real numbers less than -1 have no pre-images. for example -2 has no pre-images. Thus,  $h$  is not an onto function.
4.  $h$  is neither a one to one nor an onto function. Thus,  $h$  is not invertible and hence,  $h^{-1}$  doesn't exist.

### Question 13.

Given the following function  $h : [0, +\infty[ \rightarrow [-1, +\infty[$  with  $h(x) = x^2 - 1$ , for any real number  $x$ .

1. What is co-domain and the range of  $h$
2. Is  $h$  a one to one function? Prove or give a counterexample.
3. Is  $h$  an onto function? Prove or give a counterexample.
4. Is  $h$  invertible? and why? if the answer yes define  $h^{-1}$ .
5. On the same graph, plot the curve of  $h$  and that of  $h^{-1}$  if it exists.

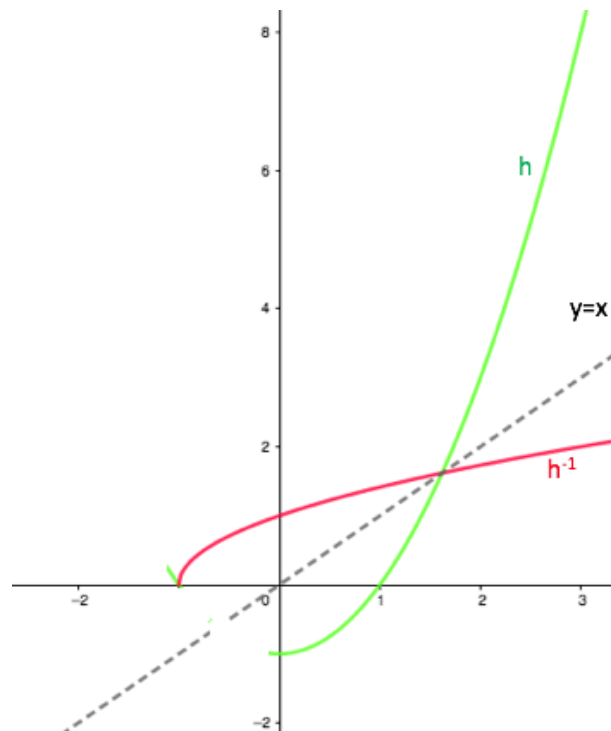
Solution:

1.  $co - D_h = [-1, +\infty[$  and  $R_h = [-1, +\infty[$
2. One-to-one: To prove that  $h$  is a one to one function, we need to show that given two real non negative number  $a$  and  $b$  if  $h(a) = h(b)$  then  $a = b$ .  $h(a) = h(b) \implies a^2 - 1 = b^2 - 1 \implies a^2 = b^2 \implies a = b$ , as  $a, b$  are non-negative real numbers. Thus,  $h$  is one to one function.
3. Onto: To prove that  $h$  is onto, you must prove that for all  $y \in [-1, +\infty[$  there exist  $x \in [0, +\infty[$  such that  $h(x) = y$ . Given a real number  $y \geq -1$ , we need to show that there exists a real number  $x \geq 0$  such that  $y = x^2 - 1$ .  
if such real number  $x$  exists, then  $x^2 - 1 = y \implies x^2 = y + 1 \implies x = \sqrt{y + 1}$   
which is in  $D_h = [0, +\infty[$  as  $y \geq -1$ .  
Hence,  $\forall y \in [-1, +\infty[, \exists x = \sqrt{y + 1} \in [0, +\infty[$  such that  $h(x) = y$ . Therefore,  $h$  is an onto function.
4.  $h$  is both a one to one and an onto function. Hence,  $h$  is invertible and the inverse function is defined as follow:

$$h^{-1} : [-1, +\infty[ \rightarrow [0, +\infty[ \text{ with } h^{-1}(x) = \sqrt{x + 1}, \forall x \in [-1, +\infty[$$

5. The diagram below shows the curves of  $h$  in green and  $h^{-1}$  in red. it also shows these curves symmetric with respect to the line  $y = x$





#### Question 14.

Consider the following function  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  with  $f(x) = 2^{x+3}$ .

1. Show that  $f$  is a bijective function.
2. Find the inverse function  $f^{-1}$ .
3. Plot the both curves of  $f$  and of  $f^{-1}$  on the same graph.

Solution:

1. To show that  $f$  is a bijective function, we need to show that  $f$  is both a one-to-one and an onto function.

One-to-one: Given two real number  $a$  and  $b$ , we need to show that if  $f(a) = f(b)$  then  $a = b$ .

$$f(a) = f(b) \implies 2^{a+3} = 2^{b+3} \implies \log_2(2^{a+3}) = \log_2(2^{b+3}) \implies a+3 = b+3 \implies a = b$$

hence,  $f$  is a one-to-one function.

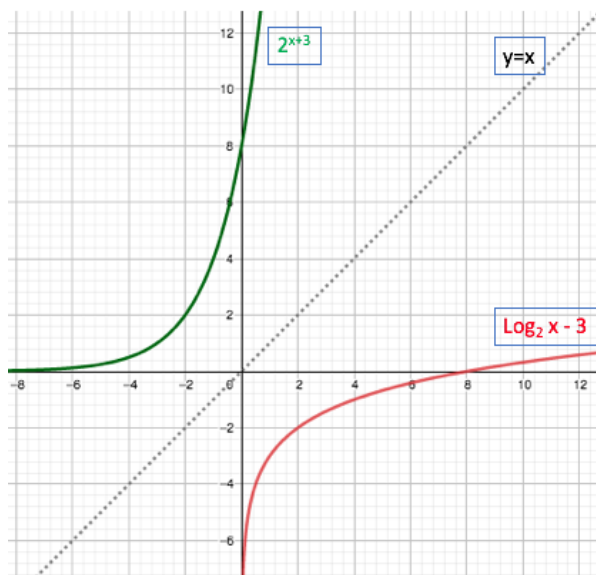
Onto: To prove that  $f$  is onto, you must prove that for all  $y \in \mathbb{R}^+$  there exist  $x \in \mathbb{R}$  such that  $f(x) = y$ . Given a real number positive  $y$ , we need to show that there exists a real number  $x$  such that  $y = 2^{x+3}$ .

if such real number  $x$  exists, then  $2^{x+3} = y \implies \log_2(2^{x+3}) = \log_2 y \implies x+3 = \log_2 y \implies x = \log_2 y - 3$  which is in  $\mathbb{R}$

Hence,  $\forall y \in \mathbb{R}^+, \exists x = \log_2 y - 3 \in \mathbb{R}$  such that  $f(x) = y$ . Therefore,  $f$  is an onto function.

Thus,  $f$  is a bijection.

2.  $f^{-1} : \mathbb{R}^+ \rightarrow \mathbb{R}$ . with  $f^{-1}(x) = \log_2 x - 3$
3. The diagram below shows the curves of  $f$  in green and  $h^{-1}$  in red. it also shows these curves symmetric with respect to the line  $y = x$



### Question 15.

Consider the following function  $f : \mathbb{R} - \{-1\} \rightarrow \mathbb{R}$  with  $f(x) = \frac{2x}{x+1}$ .

1. Show that  $f$  is a one to one function.
2. Show that  $f$  is not an onto function.

Solution:

1. To prove that  $f$  is a one to one function, we need to show that given  $a, b \in \mathbb{R} - \{-1\}$ , if  $f(a) = f(b)$  then  $a = b$ .  $f(a) = f(b) \implies \frac{2a}{a+1} = \frac{2b}{b+1} \implies 2a(b+1) = 2b(a+1) \implies 2ab + 2a = 2ba + 2b \implies 2a = 2b \implies a = b$ . Thus,  $f$  is a one to one function.
2. To prove that  $f$  is onto, you must prove that for all  $y \in \mathbb{R}$  there exist  $x \in \mathbb{R} - \{-1\}$  such that  $f(x) = y$ .  
if such real number  $x$  exists, then  $\frac{2x}{x+1} = y \implies 2x = yx + y \implies 2x - y \implies x = \frac{y}{2-y}$ , this doesn't exist if  $y = 2$ . Hence, 2 has no pre-image. Thus  $f$  is not an onto function

### Question 16.

Find the inverse of the following functions:

$$1. f(x) = e^{x^2-5}$$

$$2. g(x) = e^x + 5$$

Solution:

1. To find the inverse we write  $y = e^{x^2-5}$  and try to find  $x$  in terms of  $y$ .

$$y = e^{x^2-5} \implies \ln y = x^2 - 5 \implies x^2 = \ln y + 5 \implies x = \sqrt{\ln y + 5}$$

$$\text{Thus, } f^{-1} = \sqrt{\ln x + 5}$$

2. To find the inverse we write  $y = e^x + 5$  and try to find  $x$  in terms of  $y$ .

$$y = e^x + 5 \implies y - 5 = e^x \implies x = \ln(y - 5)$$

$$\text{Thus, } g^{-1}(x) = \ln(x - 5)$$

### Question 17.

Find the inverse of the following functions:

$$1. f(x) = \ln(x + 2) + 2$$

$$2. g(x) = \log_2(x - 5) + 3$$

Solution:

1. To find the inverse we write  $y = \ln(x + 2) + 2$  and try to find  $x$  in terms of  $y$ .

$$y = \ln(x + 2) + 2 \implies y - 2 = \ln(x + 2) \implies e^{y-2} = x + 2 \implies x = e^{y-2} - 2$$

$$\text{Thus, } f^{-1} = e^{x-2} - 2$$

2. To find the inverse we write  $y = \log_2(x - 5) + 3$  and try to find  $x$  in terms of  $y$ .

$$y = \log_2(x - 5) + 3 \implies y - 3 = \log_2(x - 5) \implies 2^{y-3} = x - 5 \implies x = 2^{y-3} + 5$$

$$\text{Thus, } g^{-1}(x) = 2^{x-3} + 5$$

### Question 18.

Let  $A, B$  and  $C$  be three sets/and  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions. Prove that if  $g \circ f$  is an onto function then  $g$  must be onto.

Solution:

Proof:  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . if  $g \circ f$  is onto hence for all  $z$  in  $C$  there exists  $x \in A$  such that  $z = (g \circ f)(x) = g(f(x))$ . hence, there exists  $y = f(x) \in B$  such that  $z = g(y)$ . Thus  $g$  is an onto function.,

### Question 19.

Let  $A, B$  and  $C$  be three sets and  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions. Prove that if  $g \circ f$  is a one to one function then  $f$  must be one to one.

Solution:

Proof:  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . Given that  $gof$  is a one to one function, we must show that for all  $a, b \in A$  if  $a \neq b$  then  $f(a) \neq f(b)$ .

$gof$  is a one to one function, hence, Given  $a, b \in A$  with  $a \neq b$  then  $(gof)(a) \neq (gof)(b)$ , this implies that  $g(f(a)) \neq g(f(b))$ . Thus  $f(a) \neq f(b)$  and this implies that  $f$  is a one to one function.

hence, there exists  $y = f(a) \in B$  such that  $z = g(y)$ . Thus  $g$  is an onto function.,

**Question 20.**

Let  $f : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$  with  $f(x, y) = x + \sqrt{2}y$  for all  $x, y \in \mathbb{Q}$

Is  $f$  a one to one function? Prove or give a counter-example.

Solution:

Given  $(a, b)$  and  $(c, d)$  in  $\mathbb{Q} \times \mathbb{Q}$  with  $f(a, b) = f(c, d)$  we need to show that  $a = c$  and  $b = d$ .

$$f(a, b) = f(c, d) \implies a + \sqrt{2}b = c + \sqrt{2}d,$$

Case 1: if  $a = c$  then  $b = d$

Case 2: if  $b = d$  then  $a = c$

Case 3: if  $a \neq c$  and  $b \neq d$ , then  $\sqrt{2} = \frac{a-c}{d-b}$  which is a rational. This is a contradiction as  $\sqrt{2}$  is irrational and can't be written as fraction. Hence,  $a = c$  and  $b = d$ . Thus  $f$  is a one to one function.

End of questions