# Midterm 2, Feb 17, 2021.

CS289: Great Theory Hits of 21st Century,
Winter 2021. Due Feb 19, 10 AM (Pacific)

Guidelines:

- Write your solutions clearly and when asked to do so, provide complete proofs. You may use results we proved in class without proofs as long as you state what you are using.
- The grading for this midterm will be a bit stricter than last time so please try to include full arguments as required for your solutions.

1. Extend the argument we showed in class to prove that $6n$ vectors suffice to get a 13-spectral sparsifier to show that $O(n)$ vectors suffice to get a 2-spectral sparsifier. It suffices to describe the part of the proof that needs to be redone (you can use notation, and other parts of the discussion from class without redoing them). [6 points]

2. Suppose $v \in \mathbb{R}^d$ is a random vector such that $E[vv^T] = I_d$. In class, I stated without proof that for any matrix $A$, $E_v[p_{A+vv^T}(z)] = (1 - d/dz)p_A(z)$. Prove this claim. [6 points]

   [Hint: Use Matrix determinant Lemma.]

3. An instance $\varphi$ of $3SAT$ on $n$ variables is a collection of clauses $C_1, \ldots, C_m$ where each $C_i$ has at most three literals each (i.e., each $C_i$ is an OR of at most three literals—variables or their negations). Recall the Max-3SAT problem: Given an instance $\varphi$ of 3SAT (i.e., a collection of clauses $C_1, \ldots, C_m$ of three literals each), find the maximum number of clauses, $val(\varphi)$, satisfiable by an assignment. Describe how you can encode Max-3SAT as a linear optimization problem on a polytope $3SAT_n$. Your optimization should have the feature that the polytope $3SAT_n$ *should not* depend on the instance $\varphi$.

   More specifically, recall that when we wrote TSP on a graph as a linear optimization on the TSP polytope, the polytope we were optimizing on did not depend on the graph—only the cost function depended on the graph. Find a similar formalization for Max-3SAT: Find a polytope $3SAT_n$, such that for every instance $\varphi$, there exists a *cost function* $w_\varphi$ such that $val(\varphi) = \max\langle w_\varphi : y\rangle, y \in 3SAT_n$. [6 points]

4. The cut polytope $\mathcal{K}_n$ is the convex hull of all cuts in a graph. Here the number of variables is $\binom{n}{2}$, one variable for each potential edge, and for any set $S \subseteq [n]$, let $x^S$ with $x^S_e = 1$ if edge $e$ has exactly one endpoint in $S$ and 0 otherwise. Then, $\mathcal{K}_n = convex-hull\{x^S : S \subseteq [n]\}$. Prove that the extension complexity of $\mathcal{K}_n$ is $2^{\Omega(n)}$. [6 points]

5. In class, I sketched that for any polytopes $P$ and $P' = \{y : Cy \leq d\}$ with $P \subseteq P'$, the partial-slack matrix $Slack(P, P')$ satisifies $nnr(Slack(P, P')) \leq xc(P)$. Write down the full argument here. [6 points]