Midterm 1, Jan 25, 2021.

CS289: Great Theory Hits of 21st Century,
Winter 2021. Due Jan 27, 10 AM (Pacific)

Guidelines:

- Write your solutions clearly and when asked to do so, provide complete proofs. You may use results we proved in class without proofs as long as you state what you are using.

1. Prove that for any D-regular graph G on N vertices, $\lambda_2(G) = \max_{x \perp 1_N} x^T M_G x / \|x\|_2^2$. Here, $x \perp 1_N$ means that x is perpendicular to the all 1's vector. [2.5 points]

2. Let $G = (V, E)$ be a D-regular graph on N vertices with $\lambda(G) \leq \lambda$. Show the following property of G: for any two sets $S, T \subseteq [V]$,

$$\left| \frac{E(S,T)}{D} - \frac{|S| \cdot |T|}{N} \right| \leq \lambda \sqrt{|S| \cdot |T|},$$

   where $E(S,T)$ denotes the number of edges with one endpoint in S and the other in T. [5 points]

3. Prove that the independence number of any (N, D, λ)-graph G is at most $N\lambda/(1+\lambda)$. Note that as a corollary you get that the chrmatic number of an (N, D, λ)-graph is at least $(1+\lambda)/\lambda$. [2.5 points]

   (Recall that independence number is the cardinality of the largest subset of vertices with no edges among them.)

4. Consider the following 'product' on graphs. For G a (N, D)-graph and H a (D, D_1)-graph their new product $F = G \circ H$ is the $(ND, 2D_1)$-graph obtained as follows. Let $Rot_G : [N] \times [D] \to [N] \times [D]$ be a consistent labelling of the edges of G that satisfies the additional constraint that $Rot_G(v, i) = (w, i)$ for some w (that is the edges are labeled with the same number from both sides). Similarly, let $Rot_H : [D] \times [D_1] \to [D] \times [D_1]$ be a consistent labeling for the edges of H. Then, $Rot_F : ([N] \times [D]) \times [2D_1] \to ([N] \times [D]) \times [2D_1]$ is defined as follows: Consider $Rot_F((v, a), i)$

   - If $i \in \{1, \ldots, D_1\}$, then compute $(b, i) = Rot_H(a, i)$, and set $Rot_F((v, a), i) = ((v, b), i)$. That is, you replace each vertex of G with a copy of H and use the edges of H within the cloud for numbers 1 through D_1.
   - If $D_1 + 1 \leq i \leq 2D_1$, then compute $(w, a) = Rot_G(v, a)$, and set $Rot_F((v, a), i) = ((w, a), i)$. That is you connect each vertex in the copy of H to the corresponding vertex in another cloud with D_1 parallel edges.

Prove that if G is a (N, D, λ_G)-graph and H is a (N, D, λ_H)-graph, then $G \circ H$ is a $(N, 2D_1, (7/8 + 1/8(1-(1-\lambda_H)^2(1-\lambda_G)))^{1/3})$-graph. [10 points]

(Hint: First write the normalized adjacency matrix of F using ingredients that we used in the analysis of zig-zag. Then relate the adjacency matrix of F^3 to that of zig-zag.)

5. Let K_n be the complete graph on n vertices. For a parameter $p \in (0,1)$, let G be a random graph obtained by independently keeping each edge of K_n in G with probability p. Prove that for some $C > 0$ if $p \geq C(\log n)/n$, then $(1/p)G$ is cut-sparsifier of K_n with high probability. [5 points]

   (Hint: You can use the Chernoff bound to bound the number of edges crossing a cut and then use union bound appropriately.)

6. Let G be a d-regular graph with eigenvalues of L_G being $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. How big can λ_n be a a function of d? [2.5 points]

7. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be vectors such that $\sum_i v_i v_i^T = I_n$. Prove that $\sum_{i=1}^m \|v_i\|^2 = n$ and that $\|v_i\| \leq 1$ for all i. [2.5 points]