CS180 Discussion 1B
Week 3: Algorithm Analysis

Ling Ding

Email: lingding@cs.ucla.edu

Outline

- Review of Asymptotic notation
- Survey of common running times
- Exercises

Big O notation (O) vs. Little o notation (o)

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\} .$$

$$o(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\} .$$

- Eg: 2n $= o(n^2)$, but $2n^2 \neq o(n^2)$
- $f(n) = o(g(n))$ implies: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Big Omega notation (Ω) vs. Little Omega notation (ω)

$$\Omega(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$

$$\omega(g(n)) = \{f(n) : \text{ for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\} .$$

- Eg: $n^2/2 = \omega(n)$, but $n^2/2 \neq \omega(n^2)$
- $f(n) = \omega(g(n))$ implies: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

Big Theta notation (Θ)

$$\Theta(g(n)) = \{f(n) : \text{ there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}\,.$$ [1]

- No “little” Theta!!

Asymptotic bounds and limits

Proposition. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some constant $0 < c < \infty$ then $f(n)$ is $\Theta(g(n))$.

Pf.

- By definition of the limit, for any $\varepsilon > 0$, there exists n_0 such that

$$c - \epsilon \leq \frac{f(n)}{g(n)} \leq c + \epsilon$$

  for all $n \geq n_0$.
- Choose $\varepsilon = \frac{1}{2} c > 0$.
- Multiplying by $g(n)$ yields $1/2\, c \cdot g(n) \leq f(n) \leq 3/2\, c \cdot g(n)$ for all $n \geq n_0$.
- Thus, $f(n)$ is $\Theta(g(n))$ by definition, with $c_1 = 1/2\, c$ and $c_2 = 3/2\, c$. ▪

$f(n) = o(g(n))$

Proposition. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$, then $f(n)$ is $O(g(n))$ but not $\Omega(g(n))$.

$f(n) = \omega(g(n))$

Proposition. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$, then $f(n)$ is $\Omega(g(n))$ but not $O(g(n))$.

Properties

Transitivity:

$$
\begin{aligned}
f(n) &= \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) &&\text{imply} && f(n) = \Theta(h(n)) ,\\
f(n) &= O(g(n)) \text{ and } g(n) = O(h(n)) &&\text{imply} && f(n) = O(h(n)) ,\\
f(n) &= \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) &&\text{imply} && f(n) = \Omega(h(n)) ,\\
f(n) &= o(g(n)) \text{ and } g(n) = o(h(n)) &&\text{imply} && f(n) = o(h(n)) ,\\
f(n) &= \omega(g(n)) \text{ and } g(n) = \omega(h(n)) &&\text{imply} && f(n) = \omega(h(n)) .
\end{aligned}
$$

Properties

Reflexivity:

$$
\begin{aligned}
f(n) &= \Theta(f(n)) \,, \\
f(n) &= O(f(n)) \,, \\
f(n) &= \Omega(f(n)) \,.
\end{aligned}
$$

Symmetry:

$$f(n) = \Theta(g(n)) \text{ if and only if } g(n) = \Theta(f(n)) \,.$$

Transpose symmetry:

$$
\begin{aligned}
f(n) &= O(g(n)) \quad \text{if and only if} \quad g(n) = \Omega(f(n)) \,, \\
f(n) &= o(g(n)) \quad \text{if and only if} \quad g(n) = \omega(f(n)) \,.
\end{aligned}
$$

Analogy between asymptotic comparison and the comparison of two real numbers

$$
\begin{array}{lll}
f(n) = O(g(n)) & \text{is like} & a \leq b \,, \\
f(n) = \Omega(g(n)) & \text{is like} & a \geq b \,, \\
f(n) = \Theta(g(n)) & \text{is like} & a = b \,, \\
f(n) = o(g(n)) & \text{is like} & a < b \,, \\
f(n) = \omega(g(n)) & \text{is like} & a > b \,.
\end{array}
$$

Analogy between asymptotic comparison and the comparison of two real numbers

- One property of real numbers, however, **DOES NOT** carry over to asymptotic notation:

Trichotomy: For any two real numbers a and b, exactly one of the following must hold: $a < b$, $a = b$, or $a > b$.

- E.g: we cannot compare the function n and $n^{1+\sin n}$ using asymptotic notation, since the value of the exponent in $n^{1+\sin n}$ oscillates between 0 and 2, taking on all values in between.

Survey of common running times

(See separate slides)

Exercises

13

Thank you!