Quntum Sheet #1 Solving

Ahmed Khaled to Dr. Norhan on 13 Dec 2025

بسم الله الرحمن الرحيم

Q1 : Possible Qubits States

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

A Valid Quantum State $\leftrightarrow |\alpha|^2 + |\beta|^2 = 1$

, the Measurement In Computational Basis $\{|0\rangle, |1\rangle\}$

$P(0) = |\alpha|^2$

$P(1) = |\beta|^2$

, the Measurement In Hadamard Basis $\{|+\rangle, |-\rangle\}$

$P(+) = |\langle + | \psi\rangle|^2 = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2$

$P(-) = |\langle - | \psi\rangle|^2 = \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2$

Known that,

$\langle + | \psi\rangle$ dot Product; the Projection of the Qubit In $|+\rangle$ Basis, as $\langle - | \psi\rangle$

(a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\frac{1}{2} + \frac{1}{2} = 1$ Valid

$P(0) = \frac{1}{2}, P(1) = \frac{1}{2}$

, In $\pm$ Basis

$P(+) = 1, P(-) = 0$

So, the State Is $|+\rangle$

(b) $\frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2}|0\rangle$

$\frac{3}{4} + \frac{1}{4} = 1$, Valid

$P(0) = \frac{1}{4}, P(1) = \frac{3}{4}$

In $\pm$ Basis

$P(+) = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{2-\sqrt{3}}{4}$

$P(-) = \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 = \frac{2+\sqrt{3}}{4}$

(C) $0.7|0\rangle + 0.3|1\rangle$

$(0.7)^2 + (0.3)^2 \neq 1$

Invalid State !!

(d) $0.8|0\rangle + 0.6|1\rangle$

$(0.8)^2 + (0.6)^2 = 1$, Valid

$P(0) = 0.64, P(1) = 0.36$

In $\pm$ Basis

$P(+) = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{1.96}{2}$, $P(-) = \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 = \frac{0.04}{2}$

(e) $\cos\theta\,|0\rangle + i\sin\theta\,|1\rangle$

$|\cos\theta|^2 + |i\sin\theta|^2 = \cos^2\theta + |\sqrt{-1}^2|\,|\sin^2\theta| = 1$

So, It's Valid.

$P(0) = \cos^2\theta$, $P(1) = \sin^2\theta$

In ± Basis

To be easier, Transform complex number to Euler form

$\cos\theta + i\sin\theta = e^{i\theta}$

~~since~~ $P(+) = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{1}{2}$, $(e^{i\theta})^2 = 1 = |\alpha+\beta|^2$

$P(-) = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{1}{2}$

(F) $\cos^2\theta\,|0\rangle - \sin^2\theta\,|1\rangle$

$|\cos^2\theta|^2 + |-\sin^2\theta|^2 = \cos^4\theta + \sin^4\theta = 1 - 2\sin^2\theta\cos^2\theta \neq 1$

it equals 1 (Valid) only at $\theta = \frac{\pi}{2}$ or $\frac{2\pi}{2}$ or ...

(g) $\left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$

$\left|\frac{1+i}{2}\right|^2 + \left|\frac{1-i}{2}\right|^2 = \frac{2}{4} + \frac{2}{4} = 1$, Valid

$P(0) = \frac{1}{2}$, $P(1) = \frac{1}{2}$

In ± Basis

$P(+) = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \left|\frac{1+i+1-i}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$P(-) = \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 = \left|\frac{1+i-1+i}{\sqrt{2}}\right|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$

Q2 : A Two Qubit System In the Following State

$$\frac{1}{\sqrt{30}}(|00\rangle + 2i\,|01\rangle - 3|10\rangle - 4i|11\rangle)$$

the first Qubit observed to be 1,
What's the State of the System after measurement ?

Collapse and reNormalization Rule

outcome 1 Spans $\{|10\rangle, |11\rangle\}$ Keep this only

$$|\psi\rangle = \frac{1}{\sqrt{30}}(-3\,|10\rangle - 4i\,|11\rangle)$$

$$P(\text{First}=1) = \left|\frac{-3}{\sqrt{30}}\right|^2 + \left|\frac{-4i}{\sqrt{30}}\right|^2 = \frac{5}{6}$$

To get Post measurement State, Re Normalize by divide by $\sqrt{5/6}$

Note $\frac{1}{\sqrt{30}} / \sqrt{5/6} = \sqrt{\frac{30\times5}{6}} = 5$ تخيل!

$$|\psi\rangle_{\text{After measurement}} = \frac{-3}{5}|10\rangle - \frac{4i}{5}|11\rangle$$

What's the Probability the Second Qubit is 1 ?

We Can ReWrite the Collapsed ψ

$$|1\rangle \oplus \left(\frac{-3}{5}|0\rangle + \frac{-4i}{5}|1\rangle\right)$$

We only square $\left|\frac{-4i}{5}\right|^2 = \frac{16}{25} = P(\text{Second}=1 \mid \text{first}=1)$

Q3, First Matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

to get Eignen Values $\quad \det(A - \lambda I) = 0$

$$A - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{pmatrix}$$

multiPly the det by $\sqrt{2}$; does not change the Roots !
to avoid Fractions

$$\begin{vmatrix} 1-\lambda\sqrt{2} & -1 \\ 1 & 1-\lambda\sqrt{2} \end{vmatrix} = (1-\lambda\sqrt{2})^2 + 1 = 0$$

$$(1-\lambda\sqrt{2})^2 = -1 = i^2$$

$$\therefore \quad 1-\lambda\sqrt{2} = \pm i$$

So, $\quad \lambda_1 = \frac{1-i}{\sqrt{2}}, \lambda_2 = \frac{1+i}{\sqrt{2}}$

To get Eigen Vectors

For $\lambda = \frac{1-i}{\sqrt{2}}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{1-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \frac{1-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-ix - y = 0 \quad ①$

$x - iy = 0 \quad ②$

$y = -ix$

Choose $x = 1 \rightarrow y = -i$ $\quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$
is one eigenVector

or Any non Zero Scalar multiPle

$$v_+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For $\lambda = \frac{1+i}{\sqrt{2}}$

Solve $\quad \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$ix - y = 0 \quad ①$

$x + iy = 0 \quad ②$

Choose $x = 1 \rightarrow y = i$

So $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is one eigenVector

Q3, Second Matrix Puali-Y Matrix

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

EigenValues ? Solve $\det|B-\lambda I| = 0$ سهلة!

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0$$

$$\therefore \quad \lambda^2 - (-i^2) = \lambda^2 - 1 = 0$$

So $\lambda_1 = 1$, $\lambda_2 = -1$

EigenVectors ?

For $\lambda_1 = +1$, Solve

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-x - iy = 0$ (1)

$x = -iy$ (2)

Choose $y = 1 \rightarrow x = -i$

So, $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is one EigenVector

For $\lambda = -1$

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x - iy = 0$ (1)

$x = iy$ (2)

So, Choose $y = 1 \rightarrow x = i$

So, $\begin{pmatrix} i \\ 1 \end{pmatrix}$ is one EigenVector

Q4, Uniqueness of Dimension

given $\{ |v_1\rangle, |v_2\rangle, .., |v_n\rangle \}$ a Basis For V

it means

① they're linearly Independent

② they Span V, means every Vector In V is a linear Combination of them

if we Try to Put $n+1$ Arrows In V, one Arrow must be Redundant

Proof

Since $\{ |v_1\rangle .., |v_n\rangle \}$ Spans V, each $|v_k\rangle$ In V Can be Constructed

$$|v_k\rangle = \sum_{j=1}^{n} a_{jk} |v_j\rangle \quad k = 1, 2, ..., n+1$$

Now, consider we Pick one $|v_k\rangle$

$$\sum_{k=1}^{n+1} c_k |v_k\rangle$$

Substitute the expansion of this Particular $|v_k\rangle$ In the Basis

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{n+1} a_{jk} c_k \right) |v_j\rangle$$

the Combination $= 0$ if and only if this Expansions $= 0$

But Since All $|v_j\rangle$'s are linearly Independent,

So each Coefficient must be $= 0$

$$\sum_{k=1}^{n+1} a_{jk} c_k = 0 \qquad \text{For each } j = 1, .., n$$

this is a System of n linear Equations with $n+1$ unknowns $c_1, \cdots, c_{n+1}$

And the System is Homogeneous

then the System must have a non-Trivial Solution (some $c_k \neq 0$)

Contradiction !!

that means $\{ |v_1\rangle, \cdots, |v_n\rangle, |v_{n+1}\rangle \}$ are linearly dependent

Q5, Computational Basis Vectors are $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

the Outer Product Form a Basis For All 2x2 Operators

$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\langle 0|1\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $|1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

So, For Any Operator $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed

$$A = a\,|0\rangle\langle 0| + b\,|0\rangle\langle 1| + c\,|1\rangle\langle 0| + e\,|1\rangle\langle 1|$$

For Operator $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

its Expansion on the Computational Outer Product Basis

$a = \frac{1}{\sqrt{2}}$, $b = \frac{-1}{\sqrt{2}}$, $c = \frac{1}{\sqrt{2}}$, $d = \frac{1}{\sqrt{2}}$

$$A = \frac{1}{\sqrt{2}} \left(|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| \right)$$

For Operator $B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$a = 0$, $b = -i$, $c = i$, $d = 0$

$$B = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$$

Q6, Unitary Matrix $\quad U^{+} U = I$

Conjegate - TransPose

(1) Hadamart Matrix H

Hadamart Is Real, and Symmetric

So, $H = H^{T} = H^{+}$

So $H^{+} H = H^{2}$.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

$\therefore$ H Is Unitary

(2) X-Pauli

X Real, Symmetric

So $X^{+} X = X^{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

(3) Z-Pauli

Z (diagonal, Real)

So $Z = Z^{+}$

$Z^{+} Z = Z^{2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I$

(4) Y-Pauli

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Y^{T} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \; Y^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

So $Y = Y^{+}$

So $Y^{+} Y = Y^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = I$

So, All are Unitary

Q7 - Tensor Product $I \oplus H$, $H \oplus I$ I is Two - dims

Tensor Product ?

How Single-Qubit Operations acts Inside a Two-Qubit System, and Specifying which Qubit is affected.

the Difference between $I \oplus H$ and $H \oplus I$

$I \oplus H$ Do nothing to first Qubit
Apply Hadamard to Second Qubit

$H \otimes I$ Apply Hadamard to first, nothing to the Second.

A, B are Matrices

A are $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

then $A \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}$

$$I \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot H & 0 \cdot H \\ 0 \cdot H & 1 \cdot H \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \cdot I & 1 \cdot I \\ 1 \cdot I & -1 \cdot I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Q8 - Measurement Postulate

let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

is measured using measurement operators

$M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

Show that

$P(M_0) = |\alpha|^2$

$P(M_1) = |\beta|^2$

A measurement is described by a set of operators $\{M_m\}$

each operator m corresponds to on Possible outcome

And Tells us

the Probability of that outcome m

$$P(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

State of $|\psi\rangle$ After the measurement

$$|\psi\rangle = \frac{M_m|\psi\rangle}{\sqrt{P(m)}}$$

$M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$ are Projectors, So

$M_m^\dagger = M_m$, $M_m = M_m^2$, $M_m^\dagger M_m = M_m$

الحياة أسهل

$P(M_0) = \langle\psi| M_0 |\psi\rangle$

First $M_0|\psi\rangle$

$= |0\rangle\langle 0| (\alpha|0\rangle + \beta|1\rangle)$

Since $\langle 0|1\rangle = 0$

$\langle 0|0\rangle = 1$

So, $M_0|\psi\rangle = \alpha|0\rangle$

Second, $\langle\psi| \alpha|0\rangle$

$\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$

$P(M_0) = (\alpha^*\langle 0| + \beta^*\langle 1|)\alpha|0\rangle$

$= \alpha^*\alpha\langle 0|0\rangle = |\alpha|^2$

#

$P(M_1) = \langle\psi| M_1 |\psi\rangle$

First $M_1|\psi\rangle$

$= |1\rangle\langle 1| (\alpha|0\rangle + \beta|1\rangle)$

$= \beta|1\rangle$

Second $\langle\psi| \beta|1\rangle$

$= (\alpha^*\langle 0| + \beta^*\langle 1|)\beta|1\rangle$

$= |\beta|^2$

#

Q10 - For the Puali Matrixes X, Y, and Z

Show $XY = iZ$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$XY = \begin{pmatrix} \boxed{0 \quad 1} \\ \boxed{1 \quad 0} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 0 \\ i \end{matrix}} & \boxed{\begin{matrix} -i \\ 0 \end{matrix}} \end{pmatrix} = \begin{pmatrix} 0+i & 0+0 \\ 0+0 & -i+0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$iZ = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Q9 - Unitary Operations Preserve Norm

If U is unitary, then for any State $|\psi\rangle$

$$\| U |\psi\rangle \| = \|\psi\|$$

Start with LHS

$$\| U |\psi\rangle \|^2 = \langle U\psi | U\psi \rangle$$

Rule $\langle U\psi| = (U|\psi\rangle)^\dagger = \langle\psi| U^\dagger$

then

$$\langle U\psi | U\psi \rangle = \langle\psi| U^\dagger U |\psi\rangle$$

Since U is unitary, $U^\dagger U = I$

So

$$\langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle$$

thus,

$$\| U |\psi \|^2 = \|\psi\|^2$$

unitary Operation does not Stretch Vectors, they only Change Direction
Probabilities, norms Stay Consistent.

Q11 - Apply NOT Gate

Initial Two Qubits State

$$|\psi\rangle = 0.8\,|00\rangle + 0.6\,|11\rangle$$

Apply NOT Gate In Second Qubit
then measure both Qubits In the Computational Basis
Find Probas of All outcomes.

Apply NOT Gate on Second Qubit only
means don't change first one
So, the overall operator

$I \otimes X$

Apply it on $|00\rangle$

$$I \otimes X\,|00\rangle = |0\rangle \otimes X\,|0\rangle = |0\rangle \otimes |1\rangle = |01\rangle$$

on $|11\rangle$

$$I \otimes X\,|11\rangle = |1\rangle \otimes X\,|1\rangle = |1\rangle \otimes |0\rangle = |10\rangle$$

So, the new State

$$|\psi'\rangle = 0.8\,|01\rangle + 0.6\,|10\rangle$$

Probabilities of Possible outcomes ?

$$P(01) = 0.8^2 = 0.64$$
$$P(10) = 0.6^2 = 0.36$$

Consistent :)

Q12 - Entangled State

Product State (Separable State)

one can be written

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

Such that $|\psi_1\rangle, |\psi_2\rangle$ Single-Qubit States

Prove $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is not Seperable

$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$

$$|\psi_1\rangle \oplus |\psi_2\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

Entangled State must have that Form

$$\frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + 0|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$\alpha\gamma = \frac{1}{\sqrt{2}}$ ①

$\alpha\delta = 0$ ②

$\beta\gamma = 0$ ③

$\beta\delta = \frac{1}{\sqrt{2}}$ ④

From ④ β must be non zero, then

From ③ γ must be zero, then

How $\alpha\gamma$ is $\neq$ zero

Contradiction

So, it's entangled #

Q14 - Qualitative (No Algebra)

No-Signaling Principle

Information Cannot be Transmited faster than light.

So, measuring on one Part of entangled System Can Not be used to Send Instantly messages To unique Observer.

No-Cloning theorem

No Machine Can take Any unknown Quantum State $|\psi\rangle$ and Produce $|\psi\rangle \otimes |\psi\rangle$

$$|\psi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

Impossible :)

If Perfect Cloning Were Possible, an Observer Could Clone his Part of entangled State and Perform measurements In multiple Bases to Infer which measurement Was Performed In a Distant System.

This Allow Instantaneous Communication Violating No-Signaling Princeples

Q13 - Distinguishability

If Two States are identical, You Can not distinguish at All

If Two States are Orthogonal, $\langle \psi | \phi \rangle = 0$, You can Perfectly

What control Distinguishability is the overlapping $\langle \psi | \phi \rangle$

Helstorm - Holevo Bound

$$P^*_{success} = \frac{1}{2}\left(1 + \sqrt{1 - |\langle \psi_0 | \psi_1 \rangle|^2}\right)$$

Smaller overlap $\langle \psi_0 | \psi_1 \rangle \Rightarrow$ larger $\sqrt{1 - \text{overlap}} \Rightarrow$ Higher Success

Case1:

$$|\psi_0\rangle = |0\rangle$$

$$|\psi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

$\langle 0 | \psi_1 \rangle = \frac{\sqrt{3}}{2}$, $|\langle 0 | \psi \rangle|^2 = \frac{3}{4}$

$P^*_{succ} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{3}{4}}\right) = \frac{3}{4}$ (Best Possible)

Case 2:

$|\psi_0\rangle = |0\rangle$, $|\psi_1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

$\langle 0 | \psi_1 \rangle = \frac{1}{2}$, squared is $\frac{1}{4}$

overlap

$P^*_{succ} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{4}}\right) = 0.933$ (Best Possible)

the Second Case has Smaller Overlap with $|0\rangle$ than the First State,

So we can distinguish it with higher Probability