Instructions:

- This assignment is meant to help you grok certain concepts we will use in the course. Please don't copy solutions from any sources.
- Avoid verbosity.
- Questions marked with * are relatively difficult. Don't be discouraged if you cannot solve them right away!
- The assignment needs to be written in latex using the attached tex file. The solution for each question should be written in the solution block in space already provided in the tex file. **Handwritten assignments will not be accepted.**

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1. Suppose, a transformation matrix A, transforms the standard basis vectors of R^3 as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}6\\4\\2\end{bmatrix};\begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}1\\2\\8\end{bmatrix};\begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}1\\5\\7\end{bmatrix}$$

   (a) If the volume of a hypothetical parallelepiped in the un-transformed space is $100 units^3$ what will be volume of this parallelepiped in the transformed space?

Solution:
Let a matrix A_{m*n} is given. The matrix transformation associated with A is :

$$T : R^n -> R^m$$
$$T(x) = Ax$$

This transforms a vector x in R^n to a vector Ax in R^m. R^m is the transformed space and R^n is the original (untransformed space) space.

$$vol(T(P)) = |det(A)|vol(P)$$

And now the volume of a parallelepiped in transformed space is the volume of parallelepiped inoriginal space times the determinant of A

Here, $m = n = 3(given)$

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & 2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 6(14 - 40) - 1(28 - 10) + 1(32 - 4) \\ &= -146 \\ |det(A)| &= 146 \end{aligned}$$

Therefore,
$$vol(T(P)) = 146 * 100 = 146,100\ units^3$$

(b) What will be the volume if the transformation of the basis vectors is as follows :

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} => \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} ; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} => \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} ; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} => \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Solution: The given matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Here using this $R_3 = 2R_2 - R_1$ we find that third row is a linear combination of the first and second row, this extend to $det(A) = 0$.

Therefore, the parallelepiped is flat i.e, the volume of parallelepiped in the transformed space is zero $i.e.$.

(c) Comment on the uniqueness of the second transformation.

Solution: The uniqueness of the second transformation is the parallelepiped to be flat because of the property of any transformation matrix that is $det(A) = 0$.

2. If R^3 is represented by following basis vectors : $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$

   (a) Find the representation of the vector $\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$ (as represented in standard basis) in the above basis.

Solution: Any vector in a vector space V is a linear combination of the basis vectors of the vector space shown as follow:

$$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = a \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$$

$$4a - 3b = 2$$

$$2a - b + 2c = -1$$

$$a + 5b + 7c = 6$$

After solving these3 linear equations with 3 unknowns, we get :

$$a = 2.844$$

$$b = 3.125$$

$$c = -1.781$$

(b) We know that, orthonormal basis simplifies this to a great extent. What would be the representation of vector $\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$ in the orthogonal basis represented by : $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$

Solution: Again similarly, any vector in a vector space V is a linear combination of the basis vectors of the vector space shown as follow:

$$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = a \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = a \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = a \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0 + 0$$

$$a = 8/9$$

Similarly,

$$b = 7/9$$

$$c = 16/9$$

(c) Comment on the advantages of having orthonormal basis.

Solution:
As we increase the dimension of the vectors, calculating the dot product of the basis vectors became computationaly complex therefore if we have the orthogonal basis,it can be computationally less intensive.

3. Consider a square matrix A such that the sum of the entries of every column of A is the same number c. Prove that c is an eigenvalue of transpose of A.

Solution: Let us consider a square matrix A_{m*m} as follow:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33}\ldots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3}\ldots & a_{nn} \end{bmatrix}$$

. where the sum of the entries of every column of A is the same number c.

$$a_{11} + a_{21} + a_{31} + \ldots + a_{n1} = c$$

$$a_{12} + a_{22} + a_{32} + \ldots + a_{n2} = c$$

$$a_{13} + a_{23} + a_{33} + \ldots + a_{n3} = c$$

$$a_{1n} + a_{2n} + a_{3n} + \ldots + a_{nn} = c$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31}\ldots & a_{n1} \\ a_{12} & a_{22} & a_{32}\ldots & a_{n2} \\ a_{13} & a_{23} & a_{33}\ldots & a_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{3n}\ldots & a_{nn} \end{bmatrix}$$

$$A^T X = \lambda X$$

where X is a vector.
Therefore, $(A^T - \lambda I)X = 0$ and $det(A^T - \lambda I) = 0$

$$det(A^T - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31}\ldots & a_{n1} \\ a_{12} & a_{22}-\lambda & a_{32}\ldots & a_{n2} \\ a_{13} & a_{23} & a_{33}-\lambda\ldots & a_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{3n}\ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$det(A^T - \lambda I) = \begin{vmatrix} a_{11}+a_{21}+a_{31}+\ldots+a_{n1}-\lambda & a_{21} & a_{31}\ldots & a_{n1} \\ a_{11}+a_{21}+a_{31}+\ldots+a_{n1}-\lambda & a_{22}-\lambda & a_{32}\ldots & a_{n2} \\ a_{11}+a_{21}+a_{31}+\ldots+a_{n1}-\lambda & a_{23} & a_{33}-\lambda\ldots & a_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ a_{11}+a_{21}+a_{31}+\ldots+a_{n1}-\lambda & a_{2n} & a_{3n}\ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

Thus,

$$det(A^T - \lambda I) = (c-\lambda) \begin{vmatrix} 1 & a_{21} & a_{31}\ldots & a_{n1} \\ 1 & a_{22}-\lambda & a_{32}\ldots & a_{n2} \\ 1 & a_{23} & a_{33}-\lambda\ldots & a_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_{2n} & a_{3n}\ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

$c = \lambda$ Hence, it is proved that c is an eigen value of A^T matrix if the sum of the entries of every column of A^{m*m} is c.

4. Let C be a 2×2 matrix. If the trace of matrix C is 0, then what can you say about matrix C^n where n is a positive integer?

Solution: The sum of eigen values of a matrix is the sum of diagonal elements of the matrix (i.e., trace of C). So, C_{2*2} has two eigen values with same magnitude but opposite signs if matrix C is nonzero matrix. This matrix is also called indefinite matrix.

If the eigen values of C_{2*2} are $-\lambda$ and $+\lambda$ then the eigen values of C^n_{2*2} are $-\lambda^n$ and $+\lambda^n$.

$$Trace(C^n) = 0$$

5. *If c and d are two real numbers then the exponential of $c + d$ is the product of the exponential of c with the exponential of d i.e

$$e^{c+d} = e^c e^d$$

   (a) If we replace them with square matrices C and D, does the equality still holds ? Prove it, if yes, else provide a counterexample.

Solution:
Let us take an example

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$$

So,

$$C + D = \begin{bmatrix} 5 & 8 \\ 8 & 11 \end{bmatrix}$$

$$e^{C+D} = \begin{bmatrix} e^5 & e^8 \\ e^8 & e^{11} \end{bmatrix}$$

$$e^C = \begin{bmatrix} e^1 & e^2 \\ e^3 & e^4 \end{bmatrix}$$

$$e^D = \begin{bmatrix} e^4 & e^6 \\ e^5 & e^7 \end{bmatrix}$$

$$e^C e^D = \begin{bmatrix} e^5 + e^7 & e^7 + e^9 \\ e^7 + e^9 & e^9 + e^{11} \end{bmatrix}$$

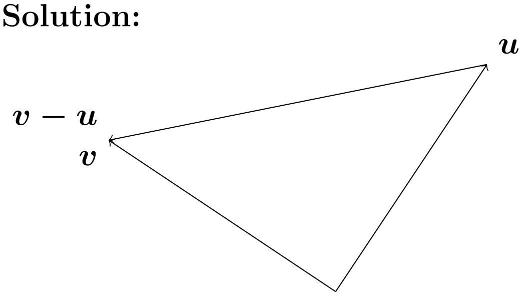
Hence it is proved that, $e^{C+D} \neq e^C e^D$

(b) Are there any special conditions on C and D under which it will always hold ?

Solution: This $e^{C+D} \neq e^C e^D$ holds iff C and D are commutative matrices i.e., $CD = DC$.

6. Prove that if u, v are nonzero vectors in R^2, then $< u, v >=\| u \|\| v \| \, cos\theta$ where θ is the angle between u and v (thinking of u and v as arrows with initial point at the origin).

Solution:



Using the consine's law:

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \, cos\theta$$

where,

$$\|v-u\|^2 = (v-u)(v-u) = \|v\|^2 - 2uv + \|u\|^2$$

$$\|v\|^2 - 2uv + \|u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \, cos\theta$$

$$cos\theta = \frac{u.v}{\|u\|\|v\|}$$

7. **Linear Dependence Lemma**
If the list of vectors $(u_1, \ldots, u_n)$ is linearly dependent in some vector space V, and $u_1 \neq \mathbf{0}$, then prove that there exists an index $i \in \{2, \ldots, n\}$ such that $u_i \in span(u_1, \ldots, u_{i-1})$, and if u_i is removed, the span of the list remains unchanged.

Solution: The list of vectors $(u_1, u_2, \ldots, u_n)$ is linearly dependent where $u_1 \neq \mathbf{0}$ is given. Since the set of vectors is not linearly independent therefore $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$ where $a_1, a_2, \ldots, a_n \in F$ and all a_i are not zero.
Since $u_1 \neq \mathbf{0}$ that means $a_2, a_3, \ldots, a_n \in F$ are not zero. Now there exists $a_j \in F$ such that till jth index $a_j \neq \mathbf{0}$ but $a_{j+1} = 0, a_{j+2} = 0, \ldots, a_m = 0$ onwards.

$$a_1u_1 + a_2u_2 + \ldots + a_{j-1}u_{j-1} + a_ju_j + a_{j+1}u_{j+1} + \ldots + a_nu_n = 0$$

$$a_ju_j = -a_1u_1 - a_2u_2 - \ldots - a_{j-1}u_{j-1} - a_{j+1}u_{j+1} - \ldots - a_nu_n$$
$$u_j = \frac{-a_1u_1 - a_2u_2 - \ldots - a_{j-1}u_{j-1} - a_{j+1}u_{j+1} - \ldots - a_nu_n}{a_j}$$

Since, $a_{j+1} = 0, a_{j+2} = 0, \ldots, a_m = 0$ onwards.

$$a_ju_j = -a_1u_1 - a_2u_2 - \ldots - a_{j-1}u_{j-1}$$

$$u_j = -\frac{a_1}{a_j}u_1 - \frac{a_2}{a_j}u_2 - \ldots - \frac{a_{j-1}}{a_j}u_{j-1}$$

Therefore,
$$u_j \in span(u_1, u_2, \ldots, u_{j-1})$$

Now let us consider a vector u is a linear combination of $u_1, u_2, \ldots, u_n$ such that $u_j \in span(u_1, u_2, \ldots, u_n)$ which can be written as:

$$u = b_1u_1 + b_2u_2 + \ldots + b_{j-1}u_{j-1} + b_ju_j + b_{j+1}u_{j+1} + \ldots + b_nu_n$$

Now substituting u_j in the above equation

$$u = b_1u_1 + b_2u_2 + \ldots + b_{j-1}u_{j-1} + b_j\{-\frac{a_1}{a_j}u_1 - \frac{a_2}{a_j}u_2 - \ldots - \frac{a_{j-1}}{a_j}u_{j-1}\} + b_{j+1}u_{j+1} + \ldots + b_nu_n$$

$$u = (b_1 - \frac{a_1}{a_j}b_j)u_1 + \ldots + (b_{j-1} - \frac{a_{j-1}}{a_j}b_j)u_{j-1} + (b_{j+1} - \frac{a_{j+1}}{a_j}b_j)u_{j+1} + \ldots + (b_n - \frac{a_n}{a_j}b_j)u_n$$

So from this we can say that u is a linear combination of $(u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m)$. Therefore, $span(u_1, u_2, \ldots, u_n) = span(u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m)$ Span of the list remains unchanged.

8. * **Independent Lists Cannot Be Arbitrarily Long**
   Prove that for any finite dimensional vector space, the length of any independent list of vectors is always smaller than or equal to the length of any spanning list of vectors. (Hint: you may want to use the Linear Dependence Lemma)

Solution: Suppose $u_1, u_2, ..., u_m$ is a list of vectors which are linearly independent in V and $w_1, w_2, ..., w_n$ span V.
To prove: $m \leq n$.
We are using a mutli-step process by adding a u and removing a w in every step.
Step 1
Let A be the list of $w_1, w_2, ..., w_n$ which spans V. Since any vector $v \in V$ can be represented as a linear combination of A, A became linearly dependent if we add any vector v to A . The list $u_1, w_1, w_2, ..., w_n$ is linearly dependent. Using Linear Dependence Lemma, The new list A(of length n) consists of u_1 and the remaining w's span V after removing one of the w's .
Step i
$V = span(A)$ is a list. As we saw that the list became linearly independent if we added any vector to this. That means particularly, we obtained a list of $(n + 1)$ vectors is to linearly independent by adding u_j. According to the Linear Dependence Lemma one of the vectors in the list is in the span of the previous ones, and because $u_1, u_2, ..., u_j$ are linearly independent, this vector is one of the w's. Now removing w from A extends the new list A which contains $u_1, u_2, ..., u_j$ and the remaining w's span V
After the furthermore steps, all u's has been added and the process stopped.
So finally at every step if we add a u to A, the Linear Dependence Lemma says that there is some w to be removed. Thus it implies that, $m \leq n$

9. **Cyclic Differences**
Consider the two lists of vectors $A = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)$, and $B = (u_1 - u_2, u_2 - u_3, u_3 - u_4, \ldots, u_{n-1} - u_n, u_n)$ (the last element is the same as A) in some vector space V. Prove or disprove the following statements:

    1. If A is linearly independent, then so is B.
    2. If A is spanning list, then so is B.

Solution: Linear combination of the vectors of A :

$$m_1u_1 + m_2u_2 + ... + m_nu_n$$

Linear combination ofthe vectors of B :

$$l_1(u_1 - u_2) + l_2(u_2 - u_3) + ... + l_nu_n$$

The above equation can be written as:

$$l_1u_1 + (l_2 - l_1)u_2 + (l_3 - l_2)u_3 + ... + (l_n - l_{n-1})u_n$$

1. If A is linearly independent, then so is B
A is linearly independent, If

$$m_1u_1 + m_2u_2 + ... + m_nu_n = 0$$

holds only for $m_1 = m_2 = ... = m_n = 0$.
In the same way, for B,

$$l_1u_1 + (l_2 - l_1)u_2 + (l_3 - l_2)u_3 + ... + (l_n - l_{n-1})u_n = 0$$

The above equality holds when,

$$\begin{aligned} l_1 &= 0 \\ l_2 - l_1 &= 0 \\ &. \\ l_n - l_{n-1} &= 0 \end{aligned}$$

So after solving the above equations, we get $l_1 = l_2 = ... = l_n = 0$. Therefore, B is also linearly independent.

2. If A is a spanning list, then so is B
Any vector can be represented as a linear combination of A if A is a spanning list.

$$v = m_1u_1 + m_2u_2 + ... + m_nu_n$$

We can say that any vector can also be represented as a linear combination of B, if

$$v = l_1u_1 + (l_2 - l_1)u_2 + (l_3 - l_2)u_3 + ... + (l_n - l_{n-1})u_n$$

where

$$\begin{aligned} l_1 &= m_1 \\ l_2 - l_1 &= m_2 \\ &\vdots \\ l_n - l_{n-1} &= m_n \end{aligned}$$

The values of l_i's which can satisfy the above equations. Hence, B is also a spanning list.

10. Compute and compare the L1 norm and Frobenius norm of the matrices given below.

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix}$$

Solution: L1 norm: $\|x\| = max_j \sum_{i=1}^{m} |x_i|$

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix} = max\{7, 9, 4\} = 9$$

$$\begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix} = max\{21, 3, -1\} = 21$$

Frobenius norm: $\|x\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} x_i j^2}$

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix} = \sqrt(16 + 4 + 1 + 4 + 9 + 36 + 1 + 64 + 81) = \sqrt216 = 14.69$$

$$\begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix} = \sqrt(16 + 4 + 1 + 4 + 9 + 36 + 1 + 64 + 81) = \sqrt216 = 14.69$$

So frobenius norm for both of the matrices is same but L-1 norm is different. Therefore L-1 norm depands on the replacement of the certain elements in the matrix.

11. * **Induced Matrix Norms**

In case you didn't already know, a norm $\|.\|$ is any function with the following properties:

1. $\|x\| \geq 0$ for all vectors x.
2. $\|x\| = 0 \iff x = \mathbf{0}$.
3. $\|\alpha x\| = |\alpha|\|x\|$ for all vectors x, and real numbers α.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors x, y.

Now, suppose we're given some vector norm $\|.\|$ (this could be L2 or L1 norm, for example). We would like to use this norm to measure the size of a matrix A. One way is to use the corresponding induced matrix norm, which is defined as $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$.

E.g.: $\|A\|_2 = \sup_x\{\|Ax\|_2 : \|x\|_2 = 1\}$, where $\|.\|_2$ is the standard L2 norm for vectors, defined by $\|x\|_2 = \sqrt{x^T x}$.

Prove the following properties for an arbitrary induced matrix norm:

(a) $\|A\| \geq 0$.

Solution: From the definition of $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$, it is already proved that $\|A\| \geq 0$.

(b) $\|\alpha A\| = |\alpha|\|A\|$ for any real number α.

Solution: α is a real number as given,

$$\|\alpha A\| = sup_x\{\|\alpha Ax\|\} = sup_x\{|\alpha|\|Ax\|\} = |\alpha| sup_x\{\|Ax\|\} = |\alpha|\|A\|$$

(c) $\|A + B\| \leq \|A\| + \|B\|$.

Solution: Using the above equation given in the question,

$$\|A + B\| = sup_x\{\left\|(A + B)x\right\|\} = sup_x\{\|Ax + Bx\|\}$$

$$\|A + B\| <= sup_x\{\|Ax\| + \|Bx\|\} <= sup_x\{\|Ax\|\} + sup_x\{\|Bx\|\}$$
$$\|A + B\| <= \|A\| + \|B\|$$

(d) $\|A\| = 0 \iff A = 0$.

Solution: As $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$ is given and for $A = 0$,

$$\|0\| = su_x\{\|0x\|\} = sup_x\{\|0\|\} = 0$$

(e) $\|AB\| \leq \|A\|\|B\|$.

Solution:

$$\|AB\| = sup_x\{\|ABx\|\} <= sup_x\{\|A\|\|Bx\|\} = \|A\| sup_x\{\|Bx\|\} = \|A\|\|B\|$$

$$\|AB\| <= \|A\|\|B\|$$

(f) $\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value.

Solution: Assume $B = A * A$
As we know that,

$$\|x\|_2 = \sqrt{x^T.x}$$

Similarlly we can write:

$$\|Ax\|_2 = \sqrt{(Ax)^T.(Ax)} = \sqrt{(x^T A^T Ax)} = (x^T \phi^T \wedge \phi x)^{1/2} = (y^T \wedge y)^{1/2}$$

$$\|Ax\|_2 = (\sum_{i=1}^{n} y_i^2 \lambda_i)^{1/2}$$

$$\|Ax\|_2 \leq \max_{1 \leq j \leq n} \sqrt{|\lambda_j| * (\|y\|)}$$

Therefore, as given in the equation of $\|A\| = max\{\|A\| : \|x\| = 1\}$ which implies that $\|A\| \leq \max_{1 \leq j \leq n} \sqrt{|\lambda_j|}$.
$A * A = \phi * \wedge \phi$ Using eigen value decomposition of $A * A$.
where, $\wedge = diag(\lambda_1 \ldots, \lambda_n)$ and $\phi = |\phi_1, \ldots, \phi_n|$ are the matrices of eigen values and eigen vectors of $A * A$ such that: $A * A\phi = \lambda_i \phi_i$ where $i = 1, \ldots, N$.
Suppose all of the eigen values are sorted in descending order shown as follows:

$$\lambda_1 = \lambda_{max} \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 0$$

$$\|A\|_2 = sup_{\|x\|=1}\{\|Ax\|_2\} = \sqrt{\lambda_{max}} = \sigma_{max}(A)$$

Therefore, $\|A\|_2 = \sigma_{max}(A)$ has been proved.

12. Prove that the eigen vectors of a real symmetric(S_{n*n}) matrix are linearly independent and forms a orthogonal basis for R^n.

Solution: Let us consider a matrix S_{n*n} which is real symmetric matrix ($S^T = S$) and have distinct eigen vectors for the corresponding eigen values. Now assume that u and v are two different eigen values corresponding to the eigen vectors λ_1 and λ_2 respectively.Therefore we write as:

$$Su = \lambda_1 u$$

and

$$Sv = \lambda_2 v$$

To prove the orthogonality of these two vectors u and v, we have to prove inner product of these vectors is zero $u.v = 0$.

$$\lambda_1(u.v) = (\lambda_1 u).v = Su.v = (Su)^T.v = u^T S^T.v = u^T S.v = u^T \lambda_2 v = \lambda_2(u.v)$$

$$(\lambda_1 - \lambda_2)(u.v) = 0$$

Since eigen values are distinct so here,

$$\lambda_1 - \lambda_2 \neq \mathbf{0}$$

Therefore,

$$u.v = 0$$

This implies that eigen vectors of a real symmetric matrix are orthogonal and form orthogonal basis and therefore these are linearly independent also.

13. If A_{n*n} is a square symmetric matrix. Prove that solution to the equation $\max_x \{x^T Ax \mid \|x\| = 1\}$ is given by the largest eigen value of A, when x is the eigen vector corresponding to largest eigen value.

Solution:

14. Prove that a full rank square matrix A_{n*n} is always similar to some diagonal matrix D_{n*n}.

Solution: Full rank matrix has pivot element in each column of its echelon matrix (Full matrix has n linearly independent eigen vectors), thus there exists a matrix X which is non-singular matrix and this implies that the matrix A_{n*n} is diagonalizable such that:

$$D_{n*n} = X^{-1} A_{n*n} X$$

where D_{n*n} is diagonal matrix.

15. Consider two vectors x and y separated by angle θ. Suppose an orthonormal transformation represented by matrix A_{n*n} is applied to vectors x and y. Find the relation between θ and the angle between the newly transformed vectors Ax and Ay.

Solution: θ is the angle between two vectors x and y such that:

$$cos\theta = \frac{x.y}{\|x\|\|y\|}$$

$$\theta = cos^{-1}\frac{x.y}{\|x\|\|y\|}$$

An orthonormal transformation represented by matrix A_{n*n} is applied to vectors x and y that means The matrix A_{n*n} is orhtogonal matrix($A^T = A^{-1}$). Suppose the angle between the transformed vectors Ax and Ay is β:

$$cos\beta = \frac{Ax.Ay}{\|Ax\|\|Ay\|}$$

Now,

$$Ax.Ay = (Ax)^T.Ay = x^T A^T Ay = x^T Iy = x.y$$

and

$$\|Ax\|^2 = Ax.Ax = (Ax)^T.Ax = x^T A^T Ax = x^T Ix = x.x = \|x\|^2$$

Therefore, we can write here:

$$cos\beta = \frac{Ax.Ay}{\|Ax\|\|Ay\|} = \frac{x.y}{\|x\|\|y\|}$$

$$\beta = cos^{-1}\frac{x.y}{\|x\|\|y\|}$$

Thus we can say that orthonormal transformation of two vectors preserves length and angle between those vectors.

16. Let $u_1, u_2, ..., u_n$ be a set of n orthonormal vectors. Similarly let $v_1, v_2, ..., v_n$ be another set of n orthonormal vectors.

    (a) Show that $u_1 v_1^T$ is a rank-1 matrix.

Solution:

$$(u_1 v_1^T) v_1 = u_1$$

$u_1 v_1^T v_i = 0$ where $i \in \{2, \ldots, n\}$
$\therefore$ Nullity of the matrix $(u_1 v_1^T) v_1$ (n-1) and rank is $(n - (n-1)) = 1$.

    (b) Show that $u_1 v_1^T + u_2 v_2^T$ is a rank-2 matrix.

Solution:

$$(u_1 v_1^T + u_2^T v_2) v_1 = u_1$$

$$(u_1 v_1^T + u_2^T v_2) v_2 = u_2$$

$(u_1 v_1^T + u_2^T v_2) v_i = 0$ where $i \in \{3, \ldots, n\}$
$\therefore$ Nullity of the matrix $(u_1 v_1^T + u_2^T v_2)$ (n-2) and rank is $(n - (n-2)) = 2$.

    (c) Show that $\sum_{i=1}^n u_i v_i^T$ is a rank-n matrix.

Solution: $(\sum_{i=1}^n u_i v_i) v_j = u_j$ where $j \in \{1, \ldots, n\}$
$\therefore$ Nullity of the matrix $\sum_{i=1}^n u_i v_i$ is 0 and rank is $(n - 0) = n$.