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1) $\|a+b\|^2 + \|a-b\|^2$

$= [a+b][a+b] + [a-b][a-b] \qquad [\because \|w\|^2 = w \cdot w]$

$= a \cdot a + a \cdot b + b \cdot a + b \cdot b + a \cdot a + a \cdot (-b) + (-b) \cdot a$
$+ (-b) \cdot (-b)$.

$= \|a\|^2 + \|b\|^2 + \cancel{a \cdot b} + \cancel{b \cdot a} + \|a\|^2 - \cancel{a \cdot b} - \cancel{b \cdot a}$
$+ (-1)^2 \|b\|^2$

$= 2\|a\|^2 + 2\|b\|^2$

$= 2(\|a\|^2 + \|b\|^2)$ ~~{Proved}~~

Hence, proved.

2) $(a+b)^T (a-b) = (a+b) \cdot (a-b)$

$= a \cdot a + a \cdot (-b) + b \cdot a + b \cdot (-b)$

$= \|a\|^2 - \|b\|^2 - a \cdot b + b \cdot a$.

$= \|a\|^2 - \|b\|^2 \qquad [\because b \cdot a = a \cdot b]$

$b \cdot a = \|a\| \, \|b\| \cos \theta$.

$a \cdot b = \|a\| \, \|b\| \cos(-\theta) = \|a\| \, \|b\| \cos \theta$

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1.2 B is a square matrix

Let $M = B + B^T$

Now, $M^T = (B + B^T)^T$

$= B^T + (B^T)^T$ $[\because (X+Y)^T = X^T + Y^T]$

$= B + B^T = M$ $[\because (X^T)^T = X]$

If A is invertible $\therefore$, $AA^{-1} = A^{-1}A = I$.

Now, $A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$.

& $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.

$\therefore$ A^T's inverse is $(A^{-1})^T$.

Clearly, $(A^{-1})^T = (A^T)^{-1}$

1.3 $\bar{x} \in \mathbb{R}^d$, $d \in \mathbb{N}$: L_1 & L_2 norms of $\bar{x}$ are equivalent.

$\therefore \exists\ c_1, c_2 \in \mathbb{R}$ s.t. $0 < c_1 \leq c_2$ and

$$c_1 \|x\|_2 \leq \|x\|_1 \leq c_2 \|x\|_2$$

$$\forall\ x \in \mathbb{R}^d.$$

Now, $\|x\|_1 = \sum_{i=1}^{d} |x_i| = \sum_{i=1}^{d} \cancel{|x_i|} |x_i| \cdot 1$

$$\leq \sqrt{\sum_{i=1}^{d} x_i^2} \sqrt{\sum_{i=1}^{d} 1^2} = \|x\|_2 \sqrt{d}$$

$\Rightarrow \|x\|_1 \leq \sqrt{d}\, \|x\|_2$

Also, $\|x\|_1^2 = \left(\sum_{i=1}^{d} |x_i| \right)^2$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} |x_i| \, |x_j| \geq \sum_{i=1}^{d} |x_i|^2 = \|x\|_2^2$$

$\Rightarrow \|x\|_2^2 \leq \|x\|_1^2 \quad \Rightarrow \|x\|_2 \leq \|x\|_1$

As L_1 & L_2 norms are both positive

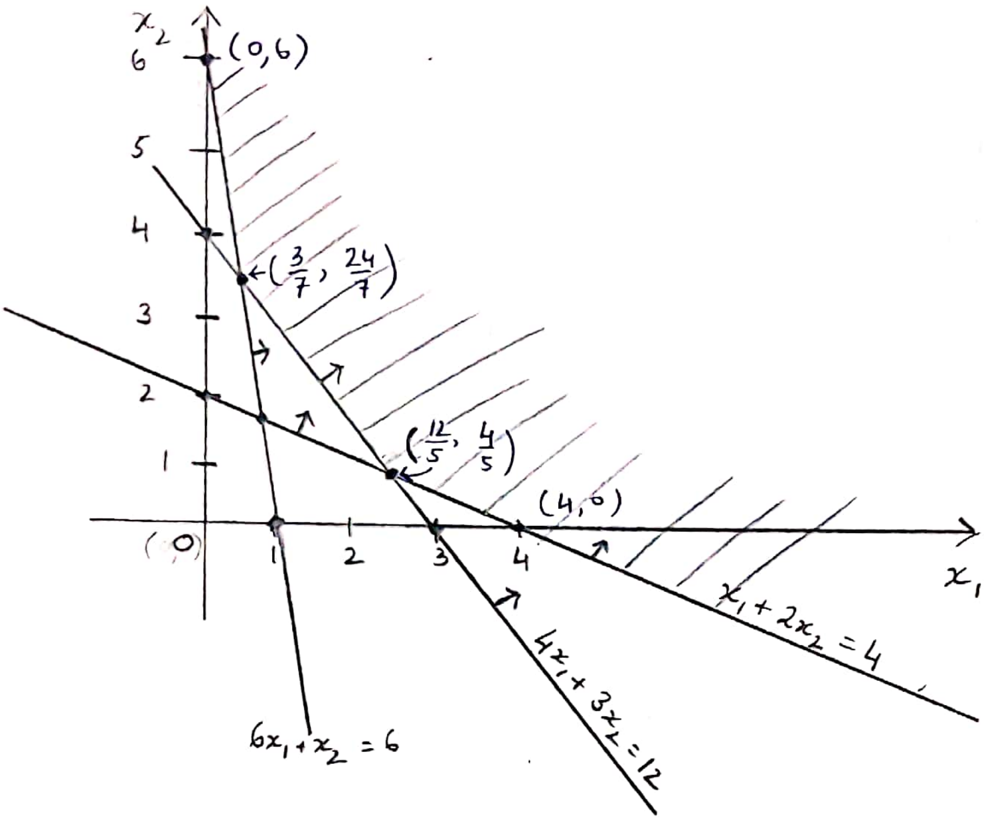
$\therefore \|x\|_2 \leq \|x\|_1 \leq \sqrt{d}\, \|x\|_2$.

As $0 < 1 \leq \sqrt{d} \quad \forall d \in \mathbb{N}$, $c_1 = 1$, $c_2 = \sqrt{d}$

- L_1 & L_2 norms are equivalent.

Hence, proved.

2.1 Minimize $5x_1 + 2x_2$



$$6x_1 + x_2 = 6 \quad -① $$
$$4x_1 + 3x_2 = 12 \quad -② $$

$$18x_1 + \cancel{3x_2} = 18$$
$$\overset{(-)}{+} 4x_1 \overset{(-)}{+} \cancel{3x_2} = \overset{(-)}{+} 12$$

$$14x_1 = 6$$

$$x_1 = \frac{6}{14} = \frac{3}{7}$$

$$6 \cdot \frac{3}{7} + x_2 = 6$$

$$\Rightarrow x_2 = 6 - \frac{18}{7} = \frac{42-18}{7} = \frac{24}{7}$$

$x_1 + 2x_2 = 4$ — (3)

$4x_1 + 3x_2 = 12$ — (4).

$$\begin{aligned} \cancel{4x_1} + 8x_2 &= 16 \\ \overset{(-)}{\cancel{4x_1}} + 3x_2 &= 12. \\ \hline 5x_2 &= 4. \end{aligned}$$

$$x_2 = \frac{4}{5}.$$

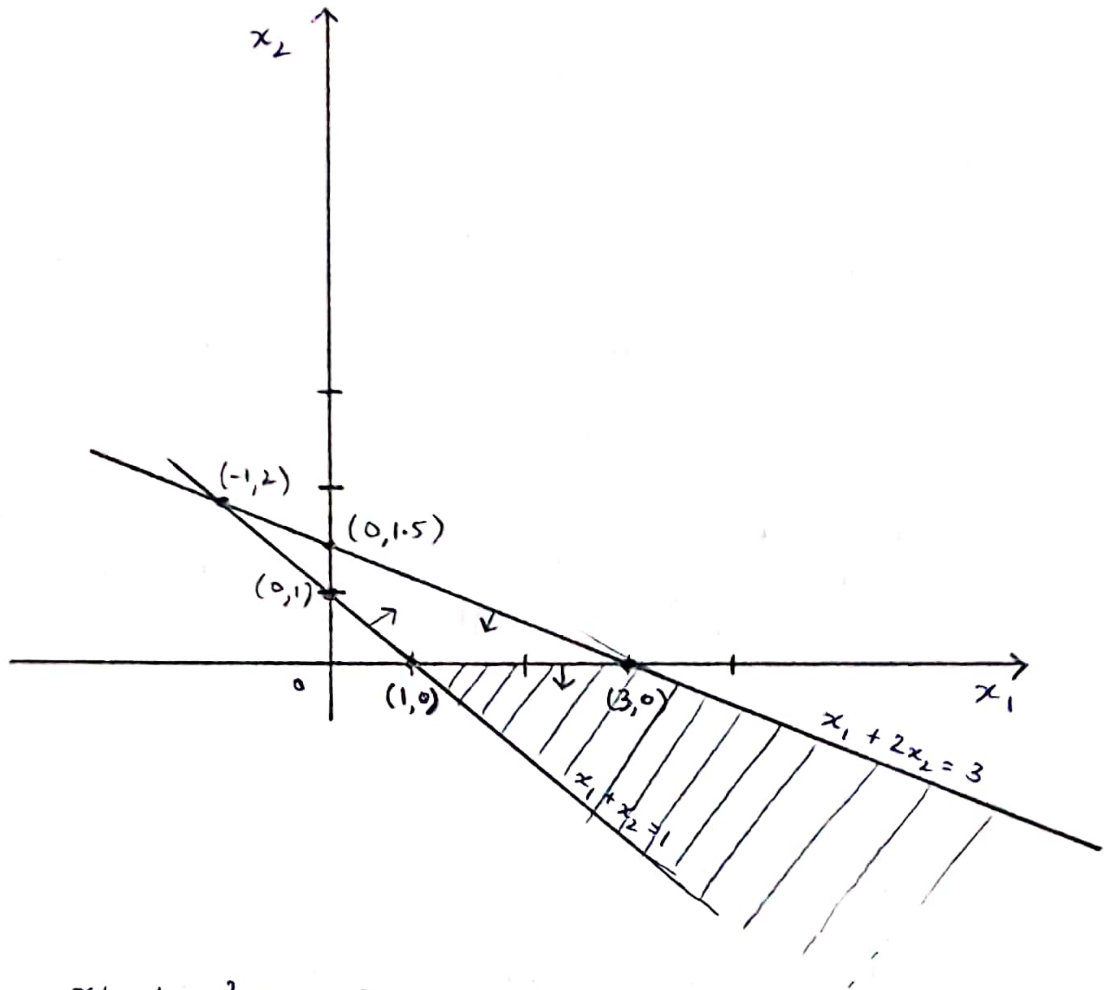
$$x_1 = 4 - \frac{2(4)}{5}.$$

$$= 4 - \frac{8}{5} = \frac{20-8}{5} = \frac{12}{5}.$$

Points	Objective.
$(0, 6)$	$5(0) + 2(6) = 12.$
$\boxed{\left(\frac{3}{7}, \frac{24}{7}\right)}$	$\boxed{5\left(\frac{3}{7}\right) + 2\left(\frac{24}{7}\right) = \frac{63}{7} = 9}.$
$\left(\frac{12}{5}, \frac{4}{5}\right)$	$5\left(\frac{12}{8}\right) + 2\left(\frac{4}{5}\right) = 13.6$
$(4, 0)$	$5(4) + 2(0) = 20.$

∴ Objective function is minimized at $x_1 = \cancel{\frac{12}{8}}\ \frac{3}{7}$ and $x_2 = \cancel{\frac{4}{8}}\ \frac{24}{7}$ when the optimal value is ~~13.6~~ 9

2.2



$$\begin{aligned} \cancel{x_1} + 2x_2 &= 3 \\ \overset{(-)}{+}\cancel{x_1} \overset{(-)}{+} x_2 &= 1 \\ \hline x_2 &= 2 \end{aligned}$$

$x_1 + 2(2) = 3$

$x_1 = -1$

i) Minimize $-x_1 + x_3 = z$

Clearly, $x_3 = -1$ will minimize the objective function z.

In the feasible region lowest value x_1 can take is 1 and the highest value x_1 can take is ∞.

$\therefore$ z is minimized when $x_1 \to \infty$ and $x_3 = -1$ and x_2 can be any value from feasible region. At such an instant the optimal value of z is $-\infty$. $z^* = -\infty$ $x^* = (\infty, -\infty, -1)$

ii) Minimize $x_2 = z$

$\because$ Feasible region is unbounded in negative direction of x_2, x_2 can take $-\infty$.

$\therefore$ z is minimized when $x_2 \to -\infty$ and x_1, x_3 can take any values from feasible regions. At such an instant, the optimal value of z is $-\infty$. $z^* = -\infty$, $x^* = (\infty, -\infty, -1)$

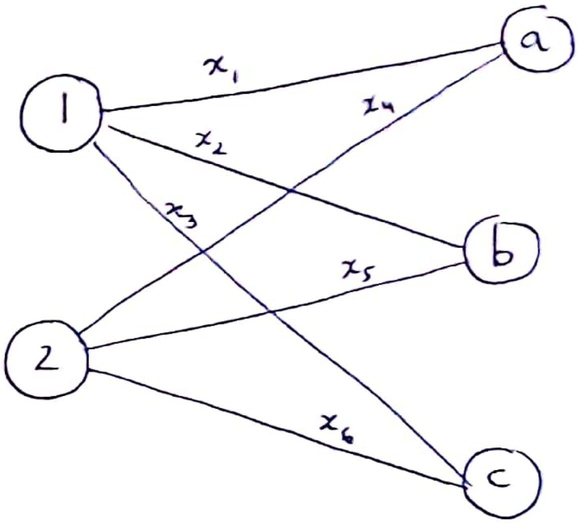
iii) Minimize $-x_3 = z$

$\because$ $-1 \le x_3 \le 1$, smallest value x_3 can take is -1 and largest value x_3 can take is $+1$.

$\therefore$ z is minimized at $x_3 = 1$ when optimal value is -1 and x_2, x_1 can take any value from feasible region.

$x^* = (\infty, -\infty, +1)$ $z^* = -1$

2.3



$x_1 + x_4 \geq 200$.

$x_2 + x_5 \geq 200$.

$x_3 + x_6 \geq 200$.

$x_1 + x_2 + x_3 \leq 250$. $\quad X_1 + X_2 + X_3 \geq 250$.

$x_4 + x_5 + x_6 \leq 450$. $\quad X_4 + X_5 + X_6 \geq 450$.

Minimize $\quad 3.4x_1 + 2.2x_2 + 2.9x_3 + 3.4x_4 + 2.4x_5 + 2.5x_6$

3.1 Adjacency Matrix for an undirected complete graph with 7 vertices ~~is~~

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

No. of length 5 walks from vertex 4 to vertex 7 = $(A)^5$

We use a ~~p~~ Python Script to compute $(A)^5$. ~~a~~ From this matrix we read the value at 4 th row 7 th column.

Required answer = 1111.

3.2 $f(x)$ & $g(x)$: unbounded monotonically increasing on $\mathbb{R}$.

i.e $f(x+h) \geq f(x)$

$g(x+h) \geq g(x) \quad \forall x \in \mathbb{R},\ n \in \mathbb{R},\ h \geq 0$

Now, $f(\cdot) \in o(g(\cdot)) \Rightarrow f(n) \leq c \cdot g(n)$.

for $\forall\ n \geq n_0$ & for some constant c.

Without loss of generality, assume $c \geq 1$,

$\therefore \log(f(n)) \leq \log(c \cdot g(n))$

$\Rightarrow \log(f(n)) \leq \log(c) + \log(g(n))$

[Since log is an increasing function]

$\Rightarrow \log(f(n)) \leq \log(g(n)) + \log(c) \cdot \log(g(n))$

[As $\log(\cdot) \geq 1$ for a sufficiently large $(\cdot)$

$\log_b x \geq 1 \rightarrow x \geq b$ is satisfied

as $x = g(x)$ is monotonically increasing

& $\exists\, n_1$ s.t. $g(n_1) = x \geq b$]

$\Rightarrow \log(f(x)) \leq \{\log(c) + 1\} \log(g(x))$

$= \lambda \log(g(x))$.

for constant λ and $\forall\ n \geq n_1$.

$\Rightarrow \log(f(\cdot)) \in o(\log(\lambda g(\cdot)))$

3.3 $\log(n!) \in \Theta(n \log n)$ is a valid identity.

$$\log(n!) = \log(1 \cdot 2 \cdot 3 \cdots n)$$
$$= \log(1) + \log(2) + \cdots \log(n)$$
$$\leq \log(n) + \log(n) + \cdots \log(n)$$
$$\log(n!) \leq n \log(n)$$

$$\Rightarrow \log(n!) \in O(n \log n) \quad — (1)$$

$$\log(n!) = \log(1 \cdot 2 \cdot 3 \cdots n)$$
$$= \log(1) + \log(2) + \cdots \log(n)$$
$$\geq \log\left(\left\lceil \frac{n}{2} \right\rceil\right) + \log\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + \cdots \log(n)$$
$$\geq \log\left(\left\lceil \frac{n}{2} \right\rceil\right) + \log\left(\left\lceil \frac{n}{2} \right\rceil\right) + \cdots \log\left(\left\lceil \frac{n}{2} \right\rceil\right)$$
$$\log(n!) \geq \left\lceil \frac{n}{2} \right\rceil \log\left(\left\lceil \frac{n}{2} \right\rceil\right)$$

$$\Rightarrow \log(n!) \in \Omega\left(\frac{n}{2} \log\left(\frac{n}{2}\right)\right) \cancel{= \Omega(n \log n)}$$

$$\Rightarrow \log(n!) \in \Omega(n \log n) \quad — (2)$$

From (1) and (2), $\log(n!) \in \Theta(n \log n)$