E0 265 : Convex Optimization and Applications
Poissonian image restoration using ADMM

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Outline

Introduction

- In the area of image denoising, a large fraction of work has been devoted to developing regularizers to deal with the presence of noise, and to solve the resulting optimization problem efficiently.

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- The algorithms developed for Gaussian noise models cannot be directly applied to other noise models such as the Poisson case.

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- The resulting optimization problems are convex and usually non-smooth, of very high dimensionality.
- The algorithms developed for Gaussian noise models cannot be directly applied to other noise models such as the Poisson case.
- Poissonian image models are well studies in fields like astronomical, biomedical and, photographic imaging.

Problem Definition

- Given an observed image, possibly corrupted with Poisson noise, restore the original image using various regularizers using ADMM(Alternating direction method of multipliers).

The Poisson Observation Model

- Let $y = (y_1, ..., y_m) \in \mathbb{N}^m$ be the vector of observed image pixels, assumed to be a sample of a random vector $Y = (Y_1, ..., Y_m)$ of m independent Poisson variables, with probability distribution

$$P[Y = y|\lambda] = \prod_{i=1}^{m} \frac{\lambda_i^{y_i} e^{-\lambda_i}}{y_i!} \tag{1}$$

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- $\lambda = (\lambda_1, ..., \lambda_m)$ is the underlying mean intensity vector, assumed to be a linear observation of the image to be estimated x , i.e.,

$$\lambda = Kx \tag{2}$$

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- Log-likelihood

$$\begin{aligned} -logP[Y = y|x] &= \sum_{i=1}^{m} (Kx)_i - y_i log((Kx)_i) + log(y_i!) \\ &= \mathcal{L}(Kx) \\ &= (\mathcal{L} \circ K)(x) \end{aligned} \tag{3}$$

The Poisson Observation Model

- For the case $K = I$ we have,

$$\begin{aligned} \mathcal{L}(z) &= \sum_{i=1}^{m} -y_i log(z_i) + z_i + log(y_i!) \\ &= C + \sum_{i=1}^{m} \zeta(z_i, y_i) \end{aligned} \tag{4}$$

where $C = \sum_{i=1}^{m} log(y_i!)$. ζ is defined as

$$\zeta(z, y) = z + \mathbb{I}_{\mathbb{R}_+}(z) - ylog(z_+) \tag{5}$$

where $\mathbb{I}$ is the indicator function.

Regularization Criteria

- **Total Variation Regularization :**

$$TV(x) = \sum_{i=1}^{n} \sqrt{(\Delta_s^h x)^2 + (\Delta_s^v x)^2} \tag{6}$$

where $(\Delta_s^h xx)$ and $(\Delta_s^v x)$ denote the horizontal and vertical first order differences respectively.

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- The resulting optimization problem is

$$minL^{TV}(x) = \mathcal{L}(Kx) + \tau TV(x) + \mathbb{I}_{\mathbb{R}_+}(x) \qquad (7)$$

where $\tau \in \mathbb{R}_+$ is the regularization parameter.

Regularization Criteria

- **Frame Analysis Regularization :** Penalizes the norm of the representation coefficients of x on some wavelet basis or tight frame, given by Px, where P is the analysis operator associated with the frame.

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$$minL^{FA}(x) = \mathcal{L}(Kx) + \tau||Px||_1 + \mathbb{I}_{\mathbb{R}_+}(x) \tag{8}$$

where $\tau \in \mathbb{R}_+$ is the regularization parameter as in TV regularizer.

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- **Frame Synthesis Regularization :** The unknown image is represented on a frame and then the coefficients of this representation are estimated from the observed data under some regularizer.
- Let $W \in \mathcal{R}^{n\times d}$ be the synthesis matrix, the image is $x = Ws$, where s is the vector of representation coefficients, the resulting optimization problem is

$$minL^{FS}(s) = \mathcal{L}(KWs) + \tau||s||_1 + \mathbb{I}_{\mathbb{R}_+}(Ws) \tag{9}$$

where $\tau \in \mathbb{R}_+$ is the regularization parameter.

The Alternating Direction Method of Multipliers(ADMM)

- Unconstrained problem of the following form

$$min_{z \in \mathcal{R}^d} f_1(z) + f_2(Gz). \tag{10}$$

where $f_1 : \mathcal{R}^d - > \mathcal{R}$ and $f_2 : \mathcal{R}^p - > \mathcal{R}$ and $G \in \mathcal{R}^{p \times d}$.

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where $f_1 : \mathcal{R}^d -> \mathcal{R}$ and $f_2 : \mathcal{R}^p -> \mathcal{R}$ and $G \in \mathcal{R}^{p \times d}$.

- The ADMM for the above problem is

Algorithm *ADMM*

1. Set $k = 0$, choose $\mu > 0$, $\mathbf{u}_0$, and $\mathbf{d}_0$.
2. **repeat**
3. $\quad \mathbf{z}_{k+1} \in \arg\min_{\mathbf{z}} f_1(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z} - \mathbf{u}_k - \mathbf{d}_k\|_2^2$
4. $\quad \mathbf{u}_{k+1} \in \arg\min_{\mathbf{u}} f_2(\mathbf{u}) + \frac{\mu}{2}\|\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u} - \mathbf{d}_k\|_2^2$
5. $\quad \mathbf{d}_{k+1} \leftarrow \mathbf{d}_k - (\mathbf{G}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1})$
6. $\quad k \leftarrow k + 1$
7. **until** stopping criterion is satisfied.

Variant of ADMM

- Generalization of the formulation

$$min_{z \in \mathcal{R}^d} \sum_{j=1}^{J} g_j(H^{(j)} z) \quad (11)$$

where $g_j : \mathcal{R}^{p_j} -> \mathcal{R}$ are closed, proper, convex functions and $H^{(j)} \in \mathcal{R}^{p_j \times d}$ are arbitrary matrices.

- The minimization in (11) can be written using the following correspondences : $f_1 = 0$,

$$G = \begin{bmatrix} H^{(1)} \\ . \\ . \\ . \\ H^{(J)} \end{bmatrix} \in \mathcal{R}^{p \times d}$$

where $p = p_1 + ... + p_j$, and $f_2 : \mathcal{R}^p - > \mathcal{R}$ given by

$$f_2(u) = \sum_{j=1}^{J} g_j(u^{(j)}) \tag{13}$$

- The resulting ADMM has exactly the same structure with

$$d_k = \begin{bmatrix} d_k^{(1)} \\ . \\ . \\ . \\ d_k^{(J)} \end{bmatrix}$$

$$u_k = \begin{bmatrix} u_k^{(1)} \\ . \\ . \\ . \\ u_k^{(J)} \end{bmatrix}$$

Poissonian image reconstruction with TV-Based Regularization

- The objective function with $J = 3$,

$$g_1 = \mathcal{L}, \quad g_2 = \tau TV, \quad g_3 = \mathbb{I}_{\mathbb{R}_+}. \tag{16}$$

and

$$H^{(1)} = \mathcal{K}, \quad H^{(2)} = I, \quad H^{(3)} = I. \tag{17}$$

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- In one experiment the original image was corrupted with Poisson noise and in the other experiment the original image was first blurred with a Gaussian Filter of size 3×3 of unit variance and then corrupted with Poisson noise.

Algorithm

Algorithm *Poisson Image Deconvolution by AL (PIDAL-TV)*

Choose $\mathbf{u}_0^{(1)}, \mathbf{u}_0^{(2)}, \mathbf{u}_0^{(3)}, \mathbf{d}_0^{(1)}, \mathbf{d}_0^{(2)}, \mathbf{d}_0^{(3)}$, μ, and τ. Set $k \leftarrow 0$.

repeat

$\quad \boldsymbol{\zeta}_k^{(1)} \leftarrow \mathbf{u}_k^{(1)} + \mathbf{d}_k^{(1)}$

$\quad \boldsymbol{\zeta}_k^{(2)} \leftarrow \mathbf{u}_k^{(2)} + \mathbf{d}_k^{(2)}$

$\quad \boldsymbol{\zeta}_k^{(3)} \leftarrow \mathbf{u}_k^{(3)} + \mathbf{d}_k^{(3)}$

$\quad \gamma_k \leftarrow \mathbf{K}^T \boldsymbol{\zeta}_k^{(1)} + \boldsymbol{\zeta}_k^{(2)} + \boldsymbol{\zeta}_k^{(3)}$

$\quad \mathbf{z}_{k+1} \leftarrow \left(\mathbf{K}^T\mathbf{K} + 2\,\mathbf{I}\right)^{-1} \gamma_k$

$\quad \boldsymbol{\nu}_k^{(1)} \leftarrow \mathbf{K}\,\mathbf{z}_{k+1} - \mathbf{d}_k^{(1)}$

$\quad \mathbf{u}_{k+1}^{(1)} \leftarrow \arg\min_{\mathbf{v}} \frac{\mu}{2}\|\mathbf{v} - \boldsymbol{\nu}_k^{(1)}\|_2^2 + \sum_{i=1}^{m} \xi(v_i, y_i)$

$\quad \boldsymbol{\nu}_k^{(2)} \leftarrow \mathbf{z}_{k+1} - \mathbf{d}_k^{(2)}$

$\quad \mathbf{u}_{k+1}^{(2)} \leftarrow \arg\min_{\mathbf{v}} \frac{\mu}{2}\|\mathbf{v} - \boldsymbol{\nu}_k^{(2)}\|^2 + \tau\,\mathrm{TV}(\mathbf{v})$.

$\quad \boldsymbol{\nu}_k^{(3)} \leftarrow \mathbf{z}_{k+1} - \mathbf{d}_k^{(3)}$

$\quad \mathbf{u}_{k+1}^{(3)} \leftarrow \arg\min_{\mathbf{v}} \frac{\mu}{2}\|\mathbf{v} - \boldsymbol{\nu}_k^{(3)}\|^2 + \iota_{\mathbb{R}_+^n}(\mathbf{v})$.

$\quad \mathbf{d}_{k+1}^{(1)} \leftarrow \mathbf{d}_k^{(1)} - (\mathbf{K}\,\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(1)})$

$\quad \mathbf{d}_{k+1}^{(2)} \leftarrow \mathbf{d}_k^{(2)} - (\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(2)})$

$\quad \mathbf{d}_{k+1}^{(3)} \leftarrow \mathbf{d}_k^{(3)} - (\mathbf{z}_{k+1} - \mathbf{u}_{k+1}^{(3)})$

$\quad k \leftarrow k + 1$

until some stopping criterion is satisfied.

Poissonian image reconstruction with TV-Based Regularization

Original image(left), Noisy image(middle) and Estimate from PIDAL-TV(right)

Poissonian image reconstruction with TV-Based Regularization

Original image(left), Noisy and blurred image(middle) and Estimate from PIDAL-TV(right)

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- The objective function with $J = 3$,

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- In one experiment the original image was corrupted with Poisson noise and in the other experiment the original image was first blurred with a Gaussian Filter of size 3×3 of unit variance and then corrupted with Poisson noise. The matrix P was initialized with a Haar Wavelet Transformation matrix.

Algorithm

Algorithm *Poisson Image Deconvolution by AL (PIDAL-FA)*

Choose $\mathbf{u}_0^{(1)}, \mathbf{u}_0^{(2)}, \mathbf{u}_0^{(3)}, \mathbf{d}_0^{(1)}, \mathbf{d}_0^{(2)}, \mathbf{d}_0^{(3)}, \mu$, and τ. Set $k \leftarrow 0$.

repeat

$$\boldsymbol{\varsigma}_k^{(1)} \leftarrow \mathbf{u}_k^{(1)} + \mathbf{d}_k^{(1)}$$

$$\boldsymbol{\varsigma}_k^{(2)} \leftarrow \mathbf{u}_k^{(2)} + \mathbf{d}_k^{(2)}$$

$$\boldsymbol{\varsigma}_k^{(3)} \leftarrow \mathbf{u}_k^{(3)} + \mathbf{d}_k^{(3)}$$

$$\gamma_k \leftarrow \mathbf{K}^T \boldsymbol{\varsigma}_k^{(1)} + \mathbf{P}^T \boldsymbol{\varsigma}_k^{(2)} + \boldsymbol{\varsigma}_k^{(3)}$$

$$\mathbf{z}_{k+1} \leftarrow \left(\mathbf{K}^T\mathbf{K} + 2\,\mathbf{I}\right)^{-1} \gamma_k$$

$$\boldsymbol{\nu}_k^{(1)} \leftarrow \mathbf{K}\,\mathbf{z}_{k+1} - \mathbf{d}_k^{(1)}$$

$$\mathbf{u}_{k+1}^{(1)} \leftarrow \frac{1}{2}\left(\nu_{i,k}^{(1)} - \frac{1}{\mu} + \sqrt{\left(\nu_{i,k}^{(1)} - \frac{1}{\mu}\right)^2 + 4\,y_i}\right)$$

$$\boldsymbol{\nu}_k^{(2)} \leftarrow \mathbf{P}\mathbf{z}_{k+1} - \mathbf{d}_k^{(2)}$$

$$\mathbf{u}_{k+1}^{(2)} \leftarrow \arg\min_{\mathbf{v}} \frac{\mu}{2}\|\mathbf{v} - \boldsymbol{\nu}_k^{(2)}\|^2 + \tau\,\|\mathbf{v}\|_1.$$

$$\boldsymbol{\nu}_k^{(3)} \leftarrow \mathbf{z}_{k+1} - \mathbf{d}_k^{(3)}$$

$$\mathbf{u}_{k+1}^{(3)} \leftarrow \max\left\{\boldsymbol{\nu}_k^{(3)}, 0\right\}.$$

$$\mathbf{d}_{k+1}^{(1)} \leftarrow \mathbf{d}_k^{(1)} - \mathbf{K}\,\mathbf{z}_{k+1} + \mathbf{u}_{k+1}^{(1)}$$

$$\mathbf{d}_{k+1}^{(2)} \leftarrow \mathbf{d}_k^{(2)} - \mathbf{P}\mathbf{z}_{k+1} + \mathbf{u}_{k+1}^{(2)}$$

$$\mathbf{d}_{k+1}^{(3)} \leftarrow \mathbf{d}_k^{(3)} - \mathbf{z}_{k+1} + \mathbf{u}_{k+1}^{(3)}$$

$$k := k + 1$$

until some stopping criterion is satisfied.

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Summary

- Studied ADMM based algorithm to handle the optimization problems resulting from regularization approaches to the restoration of Poissonian images.

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- Studied ADMM based algorithm to handle the optimization problems resulting from regularization approaches to the restoration of Poissonian images.
- Experiments show that the proposed algorithm runs fast and is able to restore images.

For Further Reading I

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