OPEN QUANTUM SYSTEMS WITH TIME-DELAYED INTERACTIONS

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ABSTRACT

The subject of this thesis is the modelling of non-Markovian open quantum systems, in particular those that display memory effects due to time-delayed interactions. I study these systems in the context of a model of an open quantum system that interacts with a bosonic environment by way of particle exchanges. While the model is flexible enough to describe quite general memory effects, this work focuses to a large extent on a single prototypical system. This system emits into a coherent feedback loop with a time delay, and it captures the essential phenomena we aim to model.

Within the context of this general model, I develop a general approach to spontaneous emission from singly-excited systems and show how to calculate the corresponding emission spectra. I also evaluate two commonly-used perturbative techniques in the context of spontaneous emission from singly-excited systems, coming to the conclusion that perturbation theory is of limited use in modelling open quantum systems with delayed feedback.

I then turn attention to exact equations of motion for more general non-Markovian open quantum systems. I derive Heisenberg–Langevin equations that describe a large class of such systems, and solve these equations analytically for a series of simple bosonic systems. I also use the theory of cascaded open quantum systems to simulate systems with delayed coherent feedback in the limit of long time delay, and compare the results of these simulations with the analytic solutions obtained in the Heisenberg picture. Finally, I show that once we have a solution to a given Heisenberg–Langevin equation we can derive an alternative, time-local equation of motion that describes equivalent dynamics, and that these time-local Heisenberg–Langevin equations can in turn be used to derive corresponding time-local master equations. These master equations provide an exact description of the non-Markovian dynamics, but this treatment is limited to linear, bosonic systems—those systems for which the Heisenberg–Langevin equations can be solved in closed form. I discuss the prospect of generalising these master equations to take into account nonlinear and fermionic systems.

Dedicated to the loving memory of Trudi Arndt Whalen

1955 – 2008

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CONTENTS

1

INTRODUCTION

This thesis grew out of an attempt to generalise the theory of cascaded open quantum systems. This theory was developed independently by Carmichael,[1] Gardiner,[2] and Kolobov and Sokolov,[3] and describes the situation sketched in figure 1.1a, where the retarded output from one open quantum system is used to drive a second system. In the standard treatment of cascaded systems there is no backscatter from the second system, so the coupling between the systems is one-way. In this situation the propagation delay between the systems is an arbitrary parameter whose value has no influence on the predictions of the model: the delay can be removed – set to zero, effectively – by way of a simple transformation of the time variable. In effect, the two systems are replaced by a bipartite system with an irreversible, but equal-time, coupling between subsystems, as depicted in figure 1.1b. The cascaded systems formalism is easily generalised to describe irreversible coupling between the systems in both directions, so long as the propagation delay associated with the coupling is sufficiently small that it may be neglected.[4] We cannot, however, transform away a non-negligible delay in both directions. The original aim of the work presented in this thesis was to generalise the theory of cascaded open quantum systems to take into account non-zero round-trip time delays.

It is quite easy to see, however, that cascaded systems per se are not the origin of the problem. Figure 1.1c depicts the problematic case: coupling between the systems in both directions, with a non-negligible propagation delay. Here we can make the same transformation of the time variable to eliminate the propagation delay from the first system to the second. This time, however, the result is the system shown in figure 1.1d: a bipartite system (with irreversible equal-time coupling from the first subsystem into the second) in which the retarded output from the second subsystem is fed back into the first—essentially, the transformation combines the two delays into one. The coupling from the first system into the second is well-understood; it is the coherent feedback with a time delay that cannot be described in existing open systems approaches. Systems like the one depicted in figure 1.1e, in which there is delayed feedback but not necessarily a cascade of irreversible couplings, therefore present us with the same challenges as the two-way cascade in figure 1.1c. It is this kind of system that will, for the most part, concern us in this thesis.

These problems are not unique to feedback with a discrete time delay. In the systems illustrated in figures 1.1c–1.1e, the environment 'remembers' the state of the system and feeds this information back coherently – that is as a quantum field – with a time delay. It is these memory effects that are the origin of the difficulties we face in modelling such systems. While systems displaying delayed feedback, like the one sketched in figure 1.1e, will be the main focus of the examples presented in this thesis, any time-delayed interaction in an

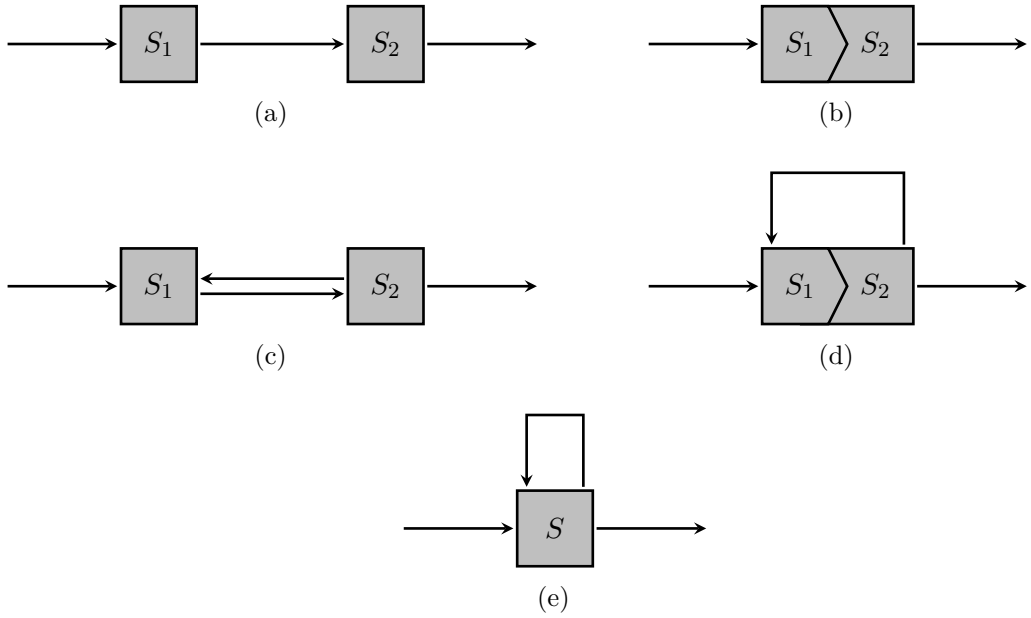


Figure 1.1: Schematic depictions of cascaded systems and systems with feedback. The systems depicted in (a) and (b) are equivalent, as are those depicted in (c) and (d).

open quantum system potentially carries with it significant challenges. With this in mind, I aim to develop tools for modelling systems with quite arbitrary memories. In general, systems displaying memory effects fall into the category of non-Markovian open quantum systems.[5] While the theory of Markovian (loosely, memoryless) open quantum systems is well-understood,[6,7] deriving equations of motion for non-Markovian open quantum systems is a notoriously difficult problem, owing to the absence of any systematic tool that can be used irrespective of the specifics of the system–environment interaction.[8] Depending on the definition of non-Markovian dynamics one chooses to adopt, certain subtleties arise in the connection between memory and non-Markovianity. This topic will be discussed briefly in chapter 2, although the classification of dynamics as Markovian or otherwise is not a focus of this thesis. Rather, this thesis focuses on the technical problem of deriving and solving the equations of motion that arise in the presence of environmental memory.

1.1 OUTLINE

I will now briefly outline the content of this thesis. In broad strokes, chapters 2 and 3 comprise introductory material, while chapters 4–7 discuss various methods for modelling non-Markovian open quantum systems, with delayed coherent feedback appearing in several places as an example. The main results of this thesis appear in chapters 4, 6, and 7.

Chapter 2 is an introduction to the theory of open quantum systems. Although the models considered in this thesis are mostly non-Markovian, in the first three sections of chapter 2 I survey, by way of background, the standard theory of Markovian open quantum systems. I then go on, in section 2.4, to briefly introduce some of the concepts

related to non-Markovian open quantum systems, including the matter of the definition of non-Markovian dynamics. The subtleties of this definition will not concern us, however, as the subject of this thesis is solving the equations of motion that stem from specific models of open quantum systems with memory 'built in' to the dynamics. In chapter 3 I introduce a general model of an open quantum system that admits memory effects, to be used throughout the following chapters. I use this model in section 3.3 to describe a system that emits into a coherent feedback loop—this system captures the essential phenomena we aim to model. The remainder of chapter 3 comprises further preliminary material and the fixing of notation.

In chapter 4, I consider the case where there is only a single excitation in the system. I develop a general approach to spontaneous emission from singly-excited systems in section 4.1, and apply this formalism to the simple feedback loop mentioned above in section 4.2, and to a simple network of systems in section 4.4. I also derive a master equation for singly-excited systems in section 4.3. In sections 4.5–4.7, I discuss how to go about calculating spontaneous emission spectra for these singly-excited systems, and compute spectra for a series of simple examples. In chapter 5 I turn attention to the use of perturbation theory in the study of open quantum systems. I review two commonly-used perturbative techniques and demonstrate the use of these techniques in the context of spontaneous emission from singly-excited systems, making use of some of the results obtained in chapter 4. I investigate whether these techniques are likely to be useful in modelling delayed coherent feedback, and come to the conclusion that perturbation theory is of limited use when it comes to non-Markovian open quantum systems, at least those with time-delayed feedback.

The equations of motion derived in chapter 4 describing spontaneous emission are exact, but limited to singly-excited systems. Beginning in chapter 6 I turn to exact equations of motion for more general non-Markovian open quantum systems. Section 6.1 focuses on using Heisenberg–Langevin equations to describe a large class of such systems. In section 6.2 I review the theory of cascaded open quantum systems, before using this formalism to simulate systems with delayed coherent feedback in section 6.3. I then derive analytic solutions to non-Markovian Heisenberg–Langevin equations for a series of simple bosonic systems, and compare these exact solutions with the results of the aforementioned simulations. In chapter 7, I show that once we have a solution to a given Heisenberg–Langevin equation we can derive an alternative, time-local equation of motion (in the Heisenberg picture) that describes equivalent dynamics. Finally, I use these time-local Heisenberg–Langevin equations to derive the corresponding time-local master equations for non-Markovian open quantum systems. This master equation treatment is limited to linear, bosonic systems—those systems for which the Heisenberg–Langevin equations can be solved in closed form. I discuss the prospect of adapting these master equations to take into account nonlinear and fermionic systems.

1.2 NOTATION AND CONVENTIONS

As a practical matter, I will briefly mention some of the notation and conventions I have used in this thesis. Firstly, I have used a system of natural units throughout, in which $\hbar = c = k_B = 1$. Secondly, I have not used any special notation (such as 'hats') to distinguish operators from classical variables, but the nature of any given quantity should be clear from the context. Thirdly, most of the parameters that appear in the models in this thesis have dimensions of either time or inverse time, and I have used arbitrary units for these quantities in all the examples presented. In practice this means that I have arbitrarily chosen parameters that illustrate some interesting – in my opinion, of course – feature of the system being modelled, without any particular experimental context in mind.

2

OPEN QUANTUM SYSTEMS

The standard Markovian theory of open quantum systems is textbook material,[6] and I assume familiarity with this theory – along with the 'culture' of quantum optics – throughout this thesis. Nevertheless, I have elected to very briefly review this material here, so that I might later discuss the approximations made in deriving the Lindblad master equation from physical models.

This chapter does not comprise original content. Sections 2.1–2.3 follow the treatment presented by Breuer and Petruccione,[7] as does the discussion of classical stochastic processes in section 2.4, and much of the chapter is substantially based on the corresponding content in my own master's thesis.[9] Much of the content of this chapter is quite elementary, but I have elected to include it nonetheless in order to fix notation.

2.1 CLOSED AND OPEN QUANTUM SYSTEMS

The Schrödinger equation for the time evolution of the state $|\psi(t)\rangle$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -iH(t)|\psi(t)\rangle \ . \tag{2.1}$$

The solution of the Schrödinger equation with initial condition $|\psi(0)\rangle$ can be written

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \ , \tag{2.2}$$

where $U(t)$ is the unitary time-evolution operator. The time-evolution operator also satisfies the Schrödinger equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -iH(t)U(t) \, , \tag{2.3}$$

with initial condition

$$U(0) = I \, . \tag{2.4}$$

The solution of (2.3) with initial condition (2.4) can be written as

$$U(t) = \mathrm{T}_{\leftarrow} \exp\left(-i\int_0^t \mathrm{d}t_1 \, H(t_1)\right) \ , \tag{2.5}$$

where the time-ordering operator $\mathrm{T}_{\leftarrow}$ orders products of time-dependent operators such that their time arguments increase from right to left. It may be the case that the system under

consideration is driven by external forces. If, however, the dynamics of the system can still be formulated solely in terms of a Hamiltonian $H(t)$ (which, as the notation suggests, may in general be time-dependent) the system is still generally referred to as *closed*, even though strictly speaking such a system is interacting with an external environment. A system whose Hamiltonian is independent of time is called an *isolated* system.[7] Of course, when the Hamiltonian H is time-independent the above solution reduces to $U(t) = \exp(-iHt)$.

If the system under consideration is in a mixed state, we characterise it by way of the density matrix $\rho(t)$. The equation of motion for the density matrix is of course the von Neumann equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i\,[H(t), \rho(t)]\ . \tag{2.6}$$

The von Neumann equation can also be written in a form analogous to the well-known Liouville equation in classical mechanics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathcal{L}(t)\rho(t)\,, \tag{2.7}$$

where the *Liouville super-operator* (or simply *Liouvillian*) is defined by

$$\mathcal{L}(t)O = -i\,[H(t), O]\ , \tag{2.8}$$

where O is any operator in the system Hilbert space. A super-operator is an operator that acts on the vector space of operators. If we suppose that at the initial time $t = 0$ the state of the system is characterised by the density matrix $\rho(0)$ we obtain, much as in (2.5), the formal solution

$$\rho(t) = \mathrm{T}_{\leftarrow} \exp\left(\int_0^t \mathrm{d}t_1\, \mathcal{L}(t_1)\right) \rho(0)\,. \tag{2.9}$$

If the Hamiltonian is time-independent, then so is the Liouvillian $\mathcal{L}$, and we get $\rho(t) = \exp(\mathcal{L}t)\rho(0)$.

We now turn to composite quantum systems. It is a foundational postulate of quantum mechanics that the Hilbert space of a composite system is the Hilbert space tensor product of the state spaces associated with the component systems. To be specific: consider two quantum systems S_A and S_B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The state space of the composite system $S = S_A + S_B$ is given by

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B\,. \tag{2.10}$$

If we take a fixed orthonormal basis $\{|\psi_{A,j}\rangle\}$ for $\mathcal{H}_A$ and $\{|\psi_{B,j}\rangle\}$ for $\mathcal{H}_B$, a general state in $\mathcal{H}$ can be written $|\psi\rangle = \sum_{jk} a_{jk}\,|\psi_{A,j}\rangle \otimes |\psi_{B,k}\rangle$. Thus, the elements $|\psi_{A,j}\rangle \otimes |\psi_{B,k}\rangle$ form a basis for $\mathcal{H}$.

If O_A and O_B are operators acting in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, their tensor product is

defined by

$$(O_A \otimes O_B)\left(|\psi_{A,j}\rangle \otimes |\psi_{B,k}\rangle\right) = \left(O_A |\psi_{A,j}\rangle\right) \otimes \left(O_B |\psi_{B,k}\rangle\right) . \tag{2.11}$$

The action of an operator on an arbitrary state is defined by a linear extension of the above formula. Any operator O on $\mathcal{H}$ can be written $O = \sum_j O_{A,j} \otimes O_{B,j}$; specifically, the observables of system S_A take the form $O_A \otimes I_B$, and the observables of S_B take the form $I_A \otimes O_B$. If one is only interested in observables of the subsystem S_A, say, one can make use of the reduced density matrix defined by

$$\rho_A = \mathrm{tr}_B\, \rho\,, \tag{2.12}$$

where tr_B denotes the partial trace over $\mathcal{H}_B$. The reduced density matrix ρ_A completely describes the statistical properties of all observables belonging to S_A—that is, observables of the form $O_A \otimes I_B$. The expectation value of such an observable can be determined using the formula $\langle O \rangle = \mathrm{tr}_A(O_A \rho_A)$.

An *open quantum system* is a quantum system S (which we will frequently refer to as simply the *system*) which is coupled to another quantum system E, called the *environment*. The system $S + E$, which we will refer to as the *combined system*, is usually assumed to be closed, with dynamics described by the von Neumann equation. The dynamics of S, however, cannot in general be described in terms of unitary time-evolution as the dynamics of a closed system can. This is because the state of S will evolve as a consequence of both its internal dynamics and of its interaction with the environment, the interaction giving rise to correlations between the system and the environment.

If we denote the Hilbert spaces of S and E respectively by $\mathcal{H}_S$ and $\mathcal{H}_E$, then the Hilbert space of the combined system, $S + E$, is given by $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$. The Hamiltonian of the combined system will take on the form

$$H(t) = H_S \otimes I_E + I_S \otimes H_E + H_{SE}(t)\,, \tag{2.13}$$

where H_S and H_E are the free Hamiltonians for the system and environment respectively, and $H_{SE}(t)$ describes the interaction between the two. If the environment E has infinitely many degrees of freedom, it is referred to as a *reservoir*. A reservoir in thermal equilibrium is a *bath*.

Sometimes a complete treatment of the dynamics of the combined system would be too complex to be practical. For example, a reservoir consisting of infinitely many degrees of freedom is governed by an infinite-dimensional system of differential equations, and we might not be able to find a way to deal with such equations directly. Provided that all measurements of interest will pertain to the system S rather than its environment, we may attempt to develop a simpler description in the reduced state space $\mathcal{H}_S$. As explained above, all observables of S take the form $A_S \otimes I_E$, where A_S is an operator acting on $\mathcal{H}_S$. If the state of the combined system $S + E$ is described by ρ, then the expectation values of

all observables acting on $\mathcal{H}_S$ are determined using

$$\langle A_S \rangle = \mathrm{tr}_S(A_S \rho_S)\,, \tag{2.14}$$

where

$$\rho_S = \mathrm{tr}_E\, \rho \tag{2.15}$$

is referred to as the reduced density matrix of the open quantum system S; tr_S and tr_E denote partial traces over the system and environment respectively. Since the evolution of $\rho(t)$ is determined by the von Neumann equation (2.6), we can take the partial trace on both sides to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\,\mathrm{tr}_E\left[H(t), \rho(t)\right]\,. \tag{2.16}$$

Similarly, since $\rho(t)$ evolves unitarily, we have

$$\rho_S(t) = \mathrm{tr}_E\left(U(t)\rho(0)U^\dagger(t)\right)\,. \tag{2.17}$$

The purpose of the theory of open quantum systems is to enable us to evaluate the system expectation values (2.14) without directly considering the degrees of freedom of the environment.

Before we move on, we will briefly review the transformation into the interaction picture for a closed quantum system. First, we define $U_0(t) = \exp(-iH_0 t)$, where H_0 is some chosen time-independent Hamiltonian – any Hamiltonian defines a valid interaction picture – along with the interaction picture unitary evolution operator

$$U_I(t) = U_0^\dagger(t)U(t)\,. \tag{2.18}$$

In the interaction picture, the evolution of the state is generated by the Hamiltonian in the interaction picture

$$H_I(t) = U_0^\dagger(t)\left(H(t) - H_0\right)U_0(t)\,, \tag{2.19}$$

giving

$$\rho_I(t) = U_I(t)\rho(0)U_I^\dagger(t)\,. \tag{2.20}$$

The evolution of operators is generated by H_0: if O is some Schrödinger picture operator, we have $O_I(t) = U_0^\dagger(t)OU_0(t)$. If we differentiate (2.20),

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_I(t) = -i\left[H_I(t), \rho_I(t)\right]\,. \tag{2.21}$$

Using these definitions, we can calculate the expectation value of an operator O as a

function of time:

$$\langle O(t)\rangle = \mathrm{tr}\left(O_I(t)\rho_I(t)\right) . \tag{2.22}$$

We can also define the interaction picture reduced density matrix,

$$\rho_{SI}(t) = \mathrm{tr}_E\, \rho_I(t)\,, \tag{2.23}$$

so that we can obtain the expectation values of system operators using

$$\langle A_S(t)\rangle = \mathrm{tr}_S\left(A_{SI}(t)\rho_{SI}(t)\right) . \tag{2.24}$$

2.2 QUANTUM DYNAMICAL SEMIGROUPS

Here we will briefly outline the formal mathematics behind the Markovian quantum master equation known as the *Lindblad master equation*, or simply the *Lindblad equation*.

Suppose that the state of the combined system $S + E$ at the initial time $t = 0$ can be expressed as an uncorrelated product state $\rho(0) = \rho_S(0) \otimes \rho_E$, where ρ_E is some reference state of the environment. From (2.17), the transformation describing the evolution of the reduced system from $t = 0$ to some $t > 0$ can be written

$$\rho_S(0) \mapsto \rho_S(t) = \Phi(t)\rho_S(0) = \mathrm{tr}_E\left(U(t)(\rho_S(0)\otimes\rho_E)U^\dagger(t)\right) . \tag{2.25}$$

If we take ρ_E and the final time $t > 0$ to be fixed, the above relation defines a map from the space of density matrices of the reduced system, $\mathcal{S}(\mathcal{H}_S)$, to itself:

$$\Phi(t)\colon \mathcal{S}(\mathcal{H}_S) \to \mathcal{S}(\mathcal{H}_S)\,. \tag{2.26}$$

The map $\Phi(t)$ is known as a *dynamical map*, and is required to be completely positive. For $\Phi(t)$ to be positive, $\Phi(t)A$ must be positive for any positive operator A acting on $\mathcal{H}_S$—this condition is clearly required for $\Phi(t)\rho$ to be a valid density matrix for any input density matrix ρ. Now consider a second Hilbert space $\mathcal{H}'$ of arbitrary dimension d, and a combined operation $\Phi(t) \otimes I_d$ that acts on $\mathcal{H}_S \otimes \mathcal{H}'$, where I_d is the identity map on $d \times d$ matrices. If $\Phi(t) \otimes I_d$ is positive for all d, then $\Phi(t)$ is said to be completely positive, which means that $\Phi(t)$ can be applied to part of a larger system and still map density matrices to density matrices. It can easily be shown that the dynamical map $\Phi(t)$ can always be written entirely in terms of operators in $\mathcal{H}_S$.

If we allow t to vary, we get a one-parameter family of dynamical maps,

$$\{\Phi(t) \mid t \geq 0\}\,, \tag{2.27}$$

with $\Phi(0)$ the identity. This family of maps completely characterises the future time-evolution of the open quantum system.[7] The dynamical map corresponding to a time-

homogeneous Markov process satisfies the semigroup property,

$$\Phi(t_1)\Phi(t_2) = \Phi(t_1 + t_2)\,, \quad t_1, t_2 \geq 0\,. \tag{2.28}$$

A *quantum dynamical semigroup* is a continuous, one-parameter family of dynamical maps satisfying the semigroup property (2.28). The physical conditions under which the dynamics of the system can be assumed to Markovian will be examined later.

Given a quantum dynamical semigroup there exists, subject to certain mathematical conditions,[7] a linear map $\mathcal{L}$ that is the infinitesimal generator of the semigroup. Thus, we can represent the dynamical map in exponential form,

$$\Phi(t) = \exp\left(\mathcal{L}t\right)\,, \tag{2.29}$$

which yields the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \mathcal{L}\rho_S(t)\,, \tag{2.30}$$

known as the *Markovian quantum master equation*. The generator $\mathcal{L}$ is a super-operator and a generalisation of the Liouvillian introduced above. When confusion is unlikely, we will refer to the generator $\mathcal{L}$ as the Liouvillian as well.

If $\dim \mathcal{H}_S = N$ the space of operators on $\mathcal{H}_S$ has dimension N^2, and we can define a basis for the this space comprising orthonormal operators F_j, $j = 1, 2, \ldots, N^2$. The most general form of the generator is given by

$$\mathcal{L}\rho_S = -i\left[H, \rho_S\right] + \sum_{j=1}^{N^2-1} \gamma_j \left(A_j \rho_S A_j^\dagger - \frac{1}{2} A_j^\dagger A_j \rho_S - \frac{1}{2}\rho_S A_j^\dagger A_j\right)\,, \tag{2.31}$$

where the so-called *Lindblad operators* A_j are linear combinations of the basis functions F_j. The master equation (2.30) corresponding to this generator is known as the *master equation in Lindblad form*, or simply the *Lindblad equation*. The proof that (2.31) is the most general form for the generator of a quantum dynamical semigroup, in the case of a finite-dimensional Hilbert space, was given by Gorini et al.,[10] building on pioneering work by Kossakowski.[11] A related theorem was proved by Lindblad, for whom the equation is named.[12]

2.3 A MICROSCOPIC DERIVATION OF THE LINDBLAD MASTER EQUATION

Consider a quantum mechanical system S weakly coupled to a reservoir E. The combined system's Hamiltonian is of the form

$$H = H_S + H_E + H_{SE}\,. \tag{2.32}$$

If we now transform into an interaction picture generated by $H_0 = H_S + H_E$ and formally integrate the interaction picture von Neumann equation (2.21), we obtain

$$\rho_I(t) = \rho(0) - i\int_0^t \mathrm{d}t_1 \left[H_{SEI}(t_1), \rho_I(t_1)\right] . \tag{2.33}$$

Substituting this formal solution back into (2.19) and tracing over the reservoir degrees of freedom gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = -\int_0^t \mathrm{d}t_1 \,\mathrm{tr}_E \left[H_{SEI}(t), \left[H_{SEI}(t_1), \rho_I(t_1)\right]\right] , \tag{2.34}$$

where we have assumed that

$$\mathrm{tr}_E \left[H_{SEI}(t), \rho(0)\right] = 0 . \tag{2.35}$$

This condition is not really an assumption, because we can always absorb terms into H_0 to ensure that it holds. In fact, this is how we get a coherent driving field – which is really the result of the environment being in a coherent state – entering as an external force in the system Hamiltonian.[4] In this thesis we will always take (2.35) to be true, and introduce phenomenological driving terms into our model Hamiltonians as necessary.

In order to eliminate the density matrix $\rho_I(t)$ of the combined system from the right-hand side of (2.34), we make an approximation known as the *Born approximation*. Provided the coupling between the system and reservoir is weak, the influence of the system on the reservoir will be small, and we can assume that the effect of the interaction on the reduced density matrix ρ_E of the reservoir is negligible. Thus, we can approximate the state of the combined system by

$$\rho_I(t) \approx \rho_{SI}(t) \otimes \rho_E . \tag{2.36}$$

Substituting (2.36) into (2.34), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = -\int_0^t \mathrm{d}t_1 \,\mathrm{tr}_E \left[H_{SEI}(t), \left[H_{SEI}(t_1), \rho_{SI}(t_1) \otimes \rho_E\right]\right] . \tag{2.37}$$

Provided that the time-scales over which the correlation functions of the environment decay are much smaller than the time-scale of the system's evolution, we can neglect memory effects in the dynamics of the reduced system, and the evolution of the reduced system will be Markovian. In (2.37), however, the time-evolution of the system state depends on all the past states of the system, so to obtain Markovian dynamics from (2.37) we need to make a series of approximations. The first of these is known (a little misleadingly) as the *Markov approximation*: the approximation that the time-evolution of the state of the system at time t depends only on the present state $\rho_{SI}(t)$, and not on the past states. The Markov approximation consists of two steps: the first step is to replace $\rho_{SI}(t_1)$ in the

integrand above with $\rho_{SI}(t)$. The resulting equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = -\int_0^t \mathrm{d}t_1 \,\mathrm{tr}_E\left[H_{SEI}(t), [H_{SEI}(t-t_1), \rho_{SI}(t)\otimes\rho_E]\right] , \tag{2.38}$$

is known as the *Redfield equation*.[13] Note that we have substituted $t - t_1$ for t_1 above. The Redfield equation is local in time with respect to $\rho_{SI}(t)$. However, it is not yet a Markovian master equation since the evolution of the reduced density matrix still implicitly depends on the initial preparation at $t = 0$. To remove this dependence from the Redfield equation, we need to make a series of additional approximations. The first of these is the second step in the Markov approximation: letting the upper limit of the integral go to infinity. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = -\int_0^\infty \mathrm{d}t_1 \,\mathrm{tr}_E\left[H_{SEI}(t), [H_{SEI}(t-t_1), \rho_{SI}(t)\otimes\rho_E]\right] . \tag{2.39}$$

This approximation is justified provided that the integrand disappears sufficiently fast for $t_1 \gg \tau_E$, where τ_E is the reservoir correlation time (the time-scale on which reservoir correlation functions decay). The relaxation time of the system τ_R is the typical time taken for the system to return equilibrium after having been perturbed, and defines the characteristic time-scale over which the system state varies appreciably. Thus, τ_R must be large compared with τ_E for the Markov approximation to apply.

Returning briefly to the Born approximation (2.36), it is worth noting that this approximation does not mean that the open quantum system causes no excitations in the reservoir: rather, the underlying assumption is that excitations created in the reservoir by the system decay over times that are not resolved. The Markov approximation provides a description of the dynamics on a coarse-grained time-scale larger than the time-scale of the reservoir correlations, so the Born approximation is justified in this context. Indeed, any dynamical behaviour on time-scales of the order of magnitude of the reservoir correlation time τ_E is not resolved by a Markovian quantum master equation.

In many quantum optics applications, the physical assumption underlying the Markov approximation amounts to the assumption that any excitations created by the system in the environment propagate away – never to return – in a time very short compared to the characteristic time-scale of the system evolution. The subject of this thesis is the modelling of systems in which this assumption cannot be made, because the excitations that the system creates in the reservoir are fed back into the system. In short, such systems are non-Markovian, and as such are much more difficult to model than the Markovian systems considered in this section. We will return to the subject of non-Markovian open quantum systems in section 2.4.

Before we continue with our derivation of the Lindblad master equation, we will make an interesting digression. To this end we characterise the strength of the interaction between

system and environment by a coupling strength ϵ, writing

$$H_{SE} \to \epsilon H_{SE}\,. \tag{2.40}$$

The Markov approximation simplifies the calculations substantially: equation (2.37), where only the Born approximation has been made, is a rather complicated integro-differential equation, and the Markov approximation removes much of the complexity. Despite this fact, to second order in the coupling strength ϵ, omitting the Markov approximation (and making only the Born approximation) does not, in general, improve the accuracy of the calculation.[14] It is interesting to note that both the Born and Markov approximations are only valid to second order in the coupling strength. We will consider higher orders of perturbation theory in chapter 5.

The approximations made above in equations (2.36), (2.38), and (2.39) are known collectively as the *Born-Markov approximation*. However, in general they do not guarantee that the equation of motion (2.39) defines the generator of a quantum dynamical semigroup.[15,16] A further approximation is necessary: a secular approximation in which one averages over those terms in the master equation that oscillate rapidly. This approximation is known as the *rotating-wave approximation*. Before this approximation can be made, though, some additional steps are required.

In the Schrödinger picture, the interaction Hamiltonian can always be expanded in the form

$$H_{SE} = \sum_{j,\omega} A_j(\omega) \otimes B_j\,, \tag{2.41}$$

where B_j are reservoir operators, and $A_j(\omega)$ are degenerate eigenoperators of the system Hamiltonian H_S, such that

$$[H_S, A_j(\omega)] = -\omega A_j(\omega)\,, \tag{2.42}$$

$$\left[H_S, A_j^\dagger(\omega)\right] = \omega A_j^\dagger(\omega)\,, \tag{2.43}$$

where $A_j(-\omega) = A_j^\dagger(\omega)$. Both $\sum_\omega A_j(\omega)$ and B_j are Hermitian operators for all j. From these relations we find the corresponding interaction picture operators:

$$e^{iH_S t} A_j(\omega) e^{-iH_S t} = e^{-i\omega t} A_j(\omega)\,, \tag{2.44}$$

$$e^{iH_S t} A_j^\dagger(\omega) e^{-iH_S t} = e^{i\omega t} A_j^\dagger(\omega)\,. \tag{2.45}$$

The interaction Hamiltonian – in the interaction picture – can now be written in the form

$$\begin{aligned} H_{SEI}(t) &= \sum_{j,\omega} e^{-i\omega t} A_j(\omega) \otimes B_j(t) \\ &= \sum_{j,\omega} e^{i\omega t} A_j^\dagger(\omega) \otimes B_j^\dagger(t)\,, \end{aligned} \tag{2.46}$$

where

$$B_j(t) = e^{iH_E t} B_j e^{-iH_E t} \,. \tag{2.47}$$

Note that the assumption (2.35) now becomes

$$\mathrm{tr}\,(B_j(t)\rho_E) = \langle B_j(t)\rangle = 0\,, \tag{2.48}$$

that is, we have in fact assumed that the expectation values of $B_j(t)$ in the reservoir vanish.

Substituting the form (2.46) of the interaction Hamiltonian into (2.39), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = \sum_{j,\omega'}\sum_{k,\omega} e^{i(\omega'-\omega)t}\Gamma_{jk}(\omega)\left[A_k(\omega)\rho_{SI}(t), A_j^\dagger(\omega')\right] + \text{h.c.}, \tag{2.49}$$

where

$$\Gamma_{jk}(\omega,t) = \int_0^\infty \mathrm{d}t_1\, e^{i\omega t_1}\left\langle B_j^\dagger(t)B_k(t-t_1)\right\rangle \,. \tag{2.50}$$

Supposing that ρ_E is a stationary state of the reservoir Hamiltonian, the reservoir correlation functions are homogeneous in time,

$$\left\langle B_j^\dagger(t)B_k(t-t_1)\right\rangle = \left\langle B_j^\dagger(t_1)B_k(0)\right\rangle \,, \tag{2.51}$$

which means that the quantities (2.50) no longer depend on t, and we can replace them with $\Gamma_{jk}(\omega) = \Gamma_{jk}(\omega,0)$.

In order for the Markov approximation to be justified, the time τ_E over which the reservoir correlation functions decay must, as we have already remarked, be much smaller than the system relaxation time τ_R. If we were dealing with a general environment consisting of a collection of harmonic oscillator modes with a discrete frequency spectrum, the reservoir correlation functions (2.51) would be quasi-periodic in t_1—this is a consequence of the quantum recurrence theorem.[17–19] Rapid decay of the reservoir correlations requires a continuum of frequencies, which is in fact the case when we consider a reservoir, which has infinitely many degrees of freedom. In such a case, Poincaré recurrence times become infinite and irreversible dynamics can emerge.[7]

We are now in a position to make the rotating-wave approximation we mentioned earlier. The typical time-scale τ_S on which the system S evolves is defined by a typical value of $|\omega'-\omega|^{-1}$, where $\omega' \neq \omega$. If $\tau_S \gg \tau_R$, then the terms for which $\omega' \neq \omega$ oscillate very rapidly over the typical time-scale on which the system state varies, and can be neglected. Assuming that this condition is satisfied, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = \sum_{\omega}\sum_{jk}\Gamma_{jk}(\omega)\left[A_k(\omega)\rho_{SI}(t), A_j^\dagger(\omega)\right] + \text{h.c.} \tag{2.52}$$

We now decompose $\Gamma_{jk}(\omega)$ into Hermitian and anti-Hermitian parts:

$$\Gamma_{jk}(\omega) = \frac{1}{2}\gamma_{jk}(\omega) + iS_{jk}(\omega)\,. \tag{2.53}$$

For fixed ω, it is clear that the matrix formed by

$$S_{jk}(\omega) = \frac{1}{2i}\left(\Gamma_{jk}(\omega) - \Gamma^*_{kj}(\omega)\right) \tag{2.54}$$

is Hermitian; and it can be shown that the matrix formed by

$$\gamma_{jk}(\omega) = \Gamma_{jk}(\omega) + \Gamma^*_{kj}(\omega) \tag{2.55}$$

is positive—this follows from the positive-definiteness of the reservoir correlation functions $\langle B_j^\dagger(t_1)B_k(0)\rangle$ combined with Bochner's theorem.[7]

Substituting (2.53) into (2.52), and noting that the indices j and k run over the same set of operators, finally leads to the interaction picture master equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_{SI}(t) = &-i\left[H_{LS}, \rho_{SI}(t)\right] \\ &+ \sum_{\omega}\sum_{jk}\gamma_{jk}(\omega)\left(A_k(\omega)\rho_{SI}(t)A_j^\dagger(\omega) - \frac{1}{2}\left\{A_j^\dagger(\omega)A_k(\omega), \rho_{SI}(t)\right\}\right)\,,\end{aligned} \tag{2.56}$$

where

$$H_{LS} = \sum_{\omega}\sum_{jk} S_{jk}(\omega)A_j^\dagger A_k(\omega) \tag{2.57}$$

is known as the *Lamb shift* Hamiltonian, because it results in a Lamb shift–type renormalisation of the unperturbed energy levels, induced by the system–environment coupling. From equations (2.44) and (2.45) it is easy to show that the Lamb shift Hamiltonian commutes with the system Hamiltonian, $[H_S, H_{LS}] = 0$. The master equation (2.56) can be brought into Lindblad form (2.31) by diagonalising the matrices formed by $\gamma_{jk}(\omega)$.

To transform the master equation back into the Schrödinger picture, we note that $\rho_S(t) = \exp(-iH_0t)\rho_{SI}(t)\exp(iH_0t)$. If we differentiate and use the properties (2.42) and (2.43), we find that the Schrödinger picture master equation is obtained from (2.56) by simply adding H_S to H_{LS}.

2.4 TIME-LOCAL AND NON-MARKOVIAN QUANTUM MASTER EQUATIONS

If the Born–Markov and secular approximations outlined in section 2.3 are not made – or cannot be made – the master equation for the reduced system evolution will not, in general, take the Lindblad form (2.30). Nonetheless, master equations can often be obtained, at least in a formal sense, for quite general systems. As is shown in appendix A, subject to the assumption of a factorising initial condition $\rho(t_0) = \rho_S(t_0)\otimes\rho_E$ the master equation

for a general open quantum system can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\left[H_S(t), \rho_S(t)\right] + \int_{t_0}^{t} \mathrm{d}t_1\, \mathcal{K}(t, t_1)\rho_S(t_1)\,, \tag{2.58}$$

where the Hamiltonian $H_S(t)$ describes coherent evolution, and the integration kernel $\mathcal{K}(t, t_1)$ may include memory effects. The time evolution generated by this master equation can be represented using a generalisation of the dynamical map (2.26):

$$\rho_S(t) = \Phi(t, t_0)\rho_S(t_0)\,. \tag{2.59}$$

Assuming the map $\Phi(t, t_0)$ is invertible, the derivative of (2.59) may be expressed in a time-local form defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \mathcal{L}(t)\rho_S(t) = \dot{\Phi}(t, t_0)\Phi^{-1}(t, t_0)\rho_S(t)\,. \tag{2.60}$$

Hall et al. have shown that any such time-local master equation for a quantum system with an N-dimensional Hilbert space can be written in a canonical, Lindblad-like form,[20]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) &= \mathcal{L}(t)\rho_S(t) \\ &= -i\left[H(t), \rho_S(t)\right] + \sum_{j=1}^{N^2-1} \gamma_j(t)\left(A_j(t)\rho_S(t)A_j^\dagger(t) - \frac{1}{2}\left\{A_j^\dagger(t)A_j(t), \rho_S(t)\right\}\right)\,,\end{aligned} \tag{2.61}$$

where the operators $A_j(t)$ are traceless and form an orthonormal basis set – that is, $\mathrm{tr}\, A_j(t) = 0$ and $\mathrm{tr}(A_j^\dagger(t)A_k(t)) = \delta_{jk}$ – and where the decoherence rates $\gamma_j(t)$ can be either positive or negative (depending on the underlying unitary dynamics). By integrating (2.60) we can see that the dynamical map can be written in terms of the generator $\mathcal{L}(t)$:

$$\Phi(t, t_0) = \mathrm{T}_{\leftarrow} \exp\left(\int_{t_0}^{t} \mathrm{d}t_1\, \mathcal{L}(t_1)\right)\,. \tag{2.62}$$

So, we know that a master equation of the form (2.61) exists, subject to the map $\Phi(t, t_0)$ being invertible for all t—the master equation does not, strictly speaking, exist at any t for which the map is not invertible. Finding the master equation corresponding to a specific underlying dynamics is, of course, another matter.

We now turn our attention to what it means for a master equation like (2.61) to be Markovian or, alternatively, non-Markovian. According to one point of view, which might be called the 'traditional' view, the Lindblad generator (2.31) *defines* a Markovian open quantum system, and any dynamics which deviate from this are deemed non-Markovian. Because this definition classifies some trivial phenomena as non-Markovian – a time-dependent driving field, for example, or damping which 'switches on' at some specific time – there have been in recent years attempts to refine the definition of non-Markovianity. I

will now briefly review some results from this field. The discussion below is based on a much more comprehensive review article by Rivas et al., to which the reader is referred for more information.[5]

Before we consider quantum non-Markovianity, we need to consider some related ideas from the theory of classical stochastic processes. For any stochastic process $X(t)$, we can define the conditional probability density that the process takes on the value x_2 at time t_2, given that the process took the value x_1 at some previous time t_1. This conditional probability is known as the *propagator*, and is denoted by $T(x_2, t_2|x_1, t_1)$. The probability $p(x, t)$ that the process takes on the value x at time t is related to the probability density at the initial time t_0 by

$$p(x,t) = \sum_{x'} T(x,t|x',t_0)p(x',t_0)\,. \tag{2.63}$$

By definition, the propagator must be normalised:

$$\int \mathrm{d}x_2\, T(x_2,t_2|x_1,t_1) = 1\,. \tag{2.64}$$

If the propagator is non-negative,

$$T(x_2,t_2|x_1,t_1) \geq 0\,, \tag{2.65}$$

and additionally satisfies the *Chapman–Kolmogorov equation*,

$$T(x_3,t_3|x_1,t_1) = \int \mathrm{d}x_2\, T(x_3,t_3|x_2,t_2)T(x_2,t_2|x_1,t_1)\,, \tag{2.66}$$

for any $t_3 \geq t_2 \geq t_1$, then the process $X(t)$ is said to be *divisible*. All Markov processes are divisible, but not all divisible processes are Markovian.[5,7] In the case of a non-Markovian process, $T(x_2, t_2|x_1, t_1)$ might not be well-defined as a conditional probability for t_1 other than the initial time t_0. If, however $T(x_1, t_1|x_0, t_0)$ is invertible for every t_1, then we can write down an 'effective' propagator

$$T(x_2,t_2|x_1,t_1) = \sum_{x_0} T(x_2,t_2|x_0,t_0)T^{-1}(x_1,t_1|x_0,t_0)\,, \tag{2.67}$$

which satisfies (2.63) even though it cannot be interpreted as a conditional probability density. If this propagator also satisfies equations (2.64)–(2.66), then the process in question is divisible, despite being non-Markovian. A non-Markovian process that is also divisible cannot be distinguished from a classical "time-inhomogeneous Markov" (that is, time-inhomogeneous but memoryless) process at the level of one-point probabilities. The complete hierarchy of time-conditional probabilities can, however, be used to distinguish divisible and non-Markovian processes.

We now return to quantum processes. Consider a density matrix that can be decomposed

as

$$\rho(t_0) = \sum_{x_0} p(x_0, t_0) \left|x_0\right\rangle \left\langle x_0\right| , \tag{2.68}$$

where $p(x_0, t_0)$ is a classical probability distribution, and $|x_0\rangle$ is a state labelled by x_0. If the process x_0 is divisible, then the density matrix at a future time t_1 can be written as

$$\rho(t_1) = \Phi(t_1, t_0)\rho(t_0) = \sum_{x_1} \sum_{x_0} T(x_1, t_1|x_0, t_0) p(x_0, t_0) \left|x_1\right\rangle \left\langle x_1\right| , \tag{2.69}$$

where the transition matrix satisfies equations (2.64)–(2.66). It is easy enough to see from the properties of a divisible process that $\Phi(t_1, t_0)$ is positive and trace-preserving, and that it obeys the composition law

$$\Phi(t_3, t_1) = \Phi(t_3, t_2)\Phi(t_2, t_1) , \qquad t_3 \geq t_2 \geq t_1 . \tag{2.70}$$

We thus make the following definition: a quantum system subject to a time evolution given by some family of trace-preserving linear maps $\Phi(t_2, t_1)$, $t_2 \geq t_1 \geq t_0$ is divisible if, for every t_2 and t_1, $\Phi(t_2, t_1)$ is a completely positive map that obeys the composition law (2.70). The Lindblad–Gorini–Kossakowski–Sudarshan theorem states that an operator $\mathcal{L}(t)$ is the generator of a divisible quantum process if and only if it can be written in the form (2.61) with $\gamma_j(t) \geq 0$ for all j and t.[5,21,22] Note that a dynamical map whose generator is given by (2.61) with arbitrary time-dependent decoherence rates may or may not be completely positive. Any time-local master equation with $\gamma_j(t) < 0$ (for some j and t) that *does* generate a completely positive map will therefore describe a non-divisible quantum process with a corresponding dynamical map that does not satisfy the composition law (2.70).

Much as in the classical case, a divisible quantum evolution cannot be distinguished from a time-inhomogeneous Lindblad evolution based on one-point probabilities. Rivas et al. argue that in quantum mechanics, measurements disturb the system, and so we cannot sample a quantum process without affecting its posterior statistics: we are limited to one-point functions. For this reason, these authors hold that divisibility and Markovianity are indistinguishable in the quantum case.[5]

Finally, I will briefly outline another measure of non-Markovianity, proposed by Breuer et al.[23] As explained by Haikka et al.,[24] these authors define non-Markovian dynamics as a temporary information backflow from the environment into the system, which manifests as an increase in the distinguishability of pairs of evolving quantum states (in Markovian dynamics, according to this point of view, pairs of states become less distinguishable as time goes on). As such, according to this definition a dynamical map Φ is non-Markovian if there exists a pair of initial states, $\rho_1(0)$ and $\rho_2(0)$ such that for some time t, $\sigma(t, \rho_1, \rho_2) = \dot{D}(\rho_1(t), \rho_2(t)) > 0$, with the distinguishability given by $D(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2||$, where $||A|| = \mathrm{tr}\sqrt{A^\dagger A}$ is the trace distance and, naturally, $\rho_{1,2}(t) = \Phi(t, 0)\rho_{1,2}(0)$. If a map is Markovian according to the divisibility view, it is also Markovian according to the definition

advanced by Breuer et al., but the converse is not true in general. This is because negativity of one of the decay rates, although sufficient to break the divisibility, does not guarantee that $\sigma(t) > 0$.[24] There are further definitions of non-Markovianity still, many of which are detailed in the review article by Rivas et al.[5]

In this thesis, I use the term 'non-Markovian' loosely, to refer to non-trivial memory effects exhibited by the systems I aim to model, and I will not attempt to quantify the degree of non-Markovianity displayed by these systems according to any of the extant definitions. This thesis is concerned with the challenges associated with the particular memory effects we are studying, and as such I will introduce additional concepts from the theory of non-Markovian open quantum systems only as necessary in the following chapters.

3

PRELIMINARIES

3.1 A MODEL OF AN OPEN QUANTUM SYSTEM

Consider a model of an open quantum system S coupled to an environment E. The Hamiltonian that defines the model takes the form

$$H(t) = H_S(t) + H_{SE} + H_E\,, \tag{3.1}$$

where $H_S(t)$ and H_E are the free Hamiltonians of the system and environment respectively, and H_{SE} describes the interaction between the two. The time-dependence of $H_S(t)$ allows for external driving of the system. The system has a collection of canonical operators a_j and $a_j^\dagger$, where the index j labels different subsystems of S—that is to say, a_j and $a_j^\dagger$ are lowering and raising operators for either fermionic or bosonic systems. Similarly, the environment has canonical operators $b_{\alpha l_\alpha}$ and $b_{\alpha l_\alpha}^\dagger$, where α runs over the various independent subenvironments and l_α runs over the modes of the α^{th} subenvironment. We will suppose that system and environment operators commute at equal times, but at this stage it isn't necessary to specify the algebra of either the system or the environment operators. It will turn out to be convenient to partition $H(t)$ in the alternative form

$$H(t) = H_0 + H_{S1}(t) + H_{SE}\,, \tag{3.2}$$

where

$$H_0 = \sum_j \omega_j a_j^\dagger a_j + H_E = \sum_j \omega_j a_j^\dagger a_j + \sum_\alpha \sum_{l_\alpha} \omega_{l_\alpha} b_{\alpha l_\alpha}^\dagger b_{\alpha l_\alpha} \tag{3.3}$$

and

$$H_{S1}(t) = H_S(t) - \sum_j \omega_j a_j^\dagger a_j\,, \tag{3.4}$$

so that $H_{S1}(t)$ accounts for detunings from the central frequencies ω_j as well as the internal dynamics of the system and any external driving. The ω_j should not be confused with physical resonant frequencies—we are free to choose these parameters according to our needs. The system and environment interact by way of particle exchanges, as described by the interaction part of the Hamiltonian,

$$H_{SE} = \sum_j \sum_\alpha \sum_{l_\alpha} \left(\kappa_{j\alpha l_\alpha} a_j^\dagger b_{\alpha l_\alpha} + \kappa_{j\alpha l_\alpha}^* b_{\alpha l_\alpha}^\dagger a_j \right)\,, \tag{3.5}$$

where $\kappa_{j\alpha l_\alpha}$ are coupling constants. In quantum optics, an interaction term of this form is usually the result of making the rotating-wave approximation, but this isn't always the case: a similar interaction term arises from a so-called 'modes-of-the-universe' approach to modelling the coupling between a cavity with a partially-transmitting mirror and external field modes, in the limit where the transmission is weak.[25–27]

We denote by ρ the Schrödinger picture initial state, which is of course the state at all times in the Heisenberg picture. We assume that this state is separable:

$$\rho = \rho_S \otimes \rho_E \,, \tag{3.6}$$

where ρ_S is the state of the system and ρ_E is the state of the environment. We choose an environment in a thermal state, so we have

$$\rho_E = \frac{\exp\left(-\sum_\alpha T_\alpha^{-1} \sum_{l_\alpha} (\omega_{l_\alpha} - \mu_\alpha) b^\dagger_{\alpha l_\alpha} b_{\alpha l_\alpha}\right)}{\operatorname{tr}\left(\exp\left(-\sum_\alpha T_\alpha^{-1} \sum_{l_\alpha} (\omega_{l_\alpha} - \mu_\alpha) b^\dagger_{\alpha l_\alpha} b_{\alpha l_\alpha}\right)\right)} \,. \tag{3.7}$$

In the case where the environment is a reservoir – with infinite degrees of freedom – the thermal equilibrium state (3.7) defines a bath (see section 2.1).

3.2 THE ROTATING FRAME

In quantum optics, the free evolution of the system usually occurs on a much shorter time-scale than the behaviour of interest, so it is desirable to neglect the part of the evolution generated by H_0. This is achieved through a transformation to a *rotating frame*, so named because this transformation is analogous to choosing a rotating reference frame in classical mechanics. While it often isn't stated explicitly, transforming into a rotating frame is equivalent to transforming into and calculating expectation values in an appropriate interaction picture—in this way, we remove the free oscillation from even those expectation values where it would otherwise be present.

To perform this transformation, we define a 'slowly-moving' expectation value

$$\langle O(t) \rangle_{\text{slow}} = \operatorname{tr}\left(O U_I(t) \rho(0) U_I^\dagger(t)\right) = \operatorname{tr}\left(O \rho_I(t)\right) = \operatorname{tr}\left(O_I(t) \rho(0)\right) \,, \tag{3.8}$$

where $U_I(t)$ is given by (2.18). Note that only one of either the state or the operator can be interpreted as time-dependent in (3.8), not both as was the case in the interaction picture expectation value (2.22): this is what distinguishes the rotating frame from the interaction picture, and what makes the expectation value (3.8) 'slow'. To transform into a rotating frame appropriate for our model, we transform $H(t)$, given by (3.2), using the free Hamiltonian (3.3). This yields an interaction picture Hamiltonian

$$H_I(t) = H_{SI}(t) + H_{SEI}(t) \,. \tag{3.9}$$

The system part of this Hamiltonian is related to the quantities in (3.2) by

$$H_{SI}(t) = U_0^\dagger(t) H_{S1}(t) U_0(t)\,, \tag{3.10}$$

while the interaction part of (3.9), derived from H_{SE}, is given by

$$H_{SEI}(t) = \sum_j \left(a_j^\dagger B_j(t) + B_j^\dagger(t) a_j \right)\,, \tag{3.11}$$

with

$$B_j(t) = \sum_\alpha \sum_{l_\alpha} \kappa_{j\alpha l_\alpha} \exp\left(i(\omega_j - \omega_{l_\alpha})t\right) b_{\alpha l_\alpha}\,. \tag{3.12}$$

It is clear that transforming into the rotating frame leads to a derivative model in which $H_I(t)$ plays the same role as $H(t)$ did in the original model. From now on, all the expectation values we calculate will be slowly-moving expectation values, so for simplicity's sake we will drop the 'slow' label, as well as the subscripts I which denote interaction picture quantities—except where the transformation into the rotating frame is explicitly noted. We can calculate the expectation value (3.8) using either the Schrödinger picture or the Heisenberg picture, with the interaction picture Hamiltonian (3.9) generating the time evolution.

3.3 SPECTRAL DENSITIES AND DISCRETE DELAYS

Using the environment operators (3.12), we define two functions that describe the behaviour of the environment:

$$\begin{aligned} F_{jk}(t_1, t_2) &= \left[B_j(t_1), B_k^\dagger(t_2) \right] \\ &= e^{i(\omega_j t_1 - \omega_k t_2)} \sum_\alpha \int_{-\infty}^{\infty} \mathrm{d}\omega\, J_{\alpha jk}(\omega) e^{-i\omega(t_1 - t_2)}\,, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} G_{jk}(t_1, t_2) &= \left\langle B_k^\dagger(t_2) B_j(t_1) \right\rangle \\ &= e^{i(\omega_j t_1 - \omega_k t_2)} \sum_\alpha \int_{-\infty}^{\infty} \mathrm{d}\omega\, J_{\alpha jk}(\omega) n_\alpha(\omega) e^{-i\omega(t_1 - t_2)}\,. \end{aligned} \tag{3.14}$$

Here we have defined the *spectral densities*

$$J_{\alpha jk}(\omega) = \sum_{l_\alpha} \kappa_{j\alpha l_\alpha} \kappa^*_{k\alpha l_\alpha} \delta(\omega - \omega_{l_\alpha})\,, \tag{3.15}$$

and

$$n_\alpha(\omega) = \frac{1}{e^{(\omega - \mu_\alpha)/T_\alpha} - 1} \tag{3.16}$$

is the Bose-Einstein distribution corresponding to the α^{th} subenvironment. This quantity is related to the mean photon number in that subenvironment – at the initial time – in the sense that $\langle b^\dagger_{\alpha l_\alpha} b_{\alpha l_\alpha}\rangle = \mathrm{tr}(b^\dagger_{\alpha l_\alpha} b_{\alpha l_\alpha}\rho_E) = n_\alpha(\omega_{l_\alpha})$, where ρ_E is given by (3.7). The chemical potential μ_α is, of course, zero for a photon bath. The quantities $F_{jk}(t_1,t_2)$ and $G_{jk}(t_1,t_2)$ are known as the dissipation and noise kernels, respectively.

We can always choose $\omega_j = \omega_0$ for all j, and account for any detunings from the central frequency ω_0 in the system part of the interaction picture Hamiltonian $H_S(t)$. If we do this, $F_{jk}(t_1,t_2)$ and $G_{jk}(t_1,t_2)$ both become time-homogeneous, and can be written as

$$F_{jk}(t_1,t_2) = \sum_\alpha \int_{-\infty}^{\infty} \mathrm{d}\omega\, J'_{\alpha jk}(\omega) e^{-i\omega(t_1-t_2)}\,, \tag{3.17}$$

and

$$G_{jk}(t_1,t_2) = \sum_\alpha \int_{-\infty}^{\infty} \mathrm{d}\omega\, J'_{\alpha jk}(\omega) n_\alpha(\omega+\omega_0) e^{-i\omega(t_1-t_2)}\,, \tag{3.18}$$

where we will refer to

$$J'_{\alpha jk}(\omega) = J_{\alpha jk}(\omega+\omega_0) \tag{3.19}$$

as the *frequency-shifted spectral densities*. For convenience, we define the alternative notation

$$F_{jk}(t_1-t_2) = F_{jk}(t_1-t_2,0)\,, \tag{3.20}$$

$$G_{jk}(t_1-t_2) = G_{jk}(t_1-t_2,0)\,. \tag{3.21}$$

Therefore, whenever we choose ω_j such that the dissipation and noise kernels take a time-homogeneous form, we can make the replacements $F_{jk}(t_1,t_2) \to F_{jk}(t_1-t_2)$ and $G_{jk}(t_1,t_2) \to G_{jk}(t_1-t_2)$. It should be understood that when we use the time-homogeneous notation (3.20) and (3.21) we are implicitly referring to a rotating frame in which $\omega_j = \omega_0$ for all j. Also, whenever the interaction between system and environment can be described in terms of a single dissipation kernel and a single noise kernel – such as in the case of a single subsystem – we will denote the kernels (and, indeed, many other dynamical variables) by lowercase letters: $f(t-t_1)$ and $g(t-t_1)$. Of course, in the case of a single subsystem the dissipation and noise kernels will always be homogeneous.

The spectral density is an important quantity: it specifies to a large extent the behaviour of the environment. It is therefore pertinent, at this point, to illustrate what kinds of spectral densities we might be interested in. In general, we're interested in environments that lead to coherent feedback with discrete time delays. The term *quantum feedback* usually refers to measurement-based quantum control schemes, in which the result of some measurement is fed back into the system by way of an external drive.[28–31] Coherent quantum feedback is different in that it is not mediated by a measurement: it is a result of the dynamics of the reservoir and of the nature of the system–reservoir coupling.[32–34] Grimsmo et al. have proposed a control scheme – designed to reduce the time a damped,

driven system takes to reach a steady state – based on delayed coherent feedback.[35] In this thesis, I will not be considering applications of coherent feedback to control systems. Rather, I will focus on modelling systems involving delayed coherent feedback with the greatest possible generality and flexibility.

As an illustration of the kind of system we aim to model, consider the system depicted in figure 3.1: a cavity coupled to an environment at two different spatial locations to form a coherent feedback loop. In the context of our model, we have a single system degree of freedom – a standing-wave mode of the cavity, for example – coupled to a one-dimensional unipartite environment—in principle this could be, for example, an optical fibre, a waveguide, or a superconducting transmission line. For the purposes of this illustration, we will model the field in this environment as a massless real scalar field $\varphi(x,t)$, and we will assume a quantisation interval $-M/2 \leq x \leq M/2$. The classical Lagrangian for this field is given by

$$L = \int_{-M/2}^{M/2} \mathrm{d}x \left(\frac{1}{2} \Big(\frac{\partial}{\partial t} \varphi(x,t) \Big)^2 - \frac{1}{2} \Big(\frac{\partial}{\partial x} \varphi(x,t) \Big)^2 \right) . \tag{3.22}$$

The corresponding Euler–Lagrange equation is a wave equation $\partial_{tt}\varphi = \partial_{xx}\varphi$. Implicit in the Lagrangian (3.22) is the assumption that the propagation speed of waves in this environment is c; recall that we are using natural units, so $c = 1$ by definition. We assume periodic boundary conditions, $\varphi(-M/2,t) = \varphi(M/2,t)$, and thus the general solution for $\varphi(x,t)$ can be written in the form

$$\varphi(x,t) = \frac{1}{\sqrt{M}} \sum_{l=-\infty}^{\infty} \varphi_l(t) e^{i\omega_l x} , \qquad \omega_l = \frac{2\pi l}{M} . \tag{3.23}$$

At this point we quantise the field by promoting $\varphi_l(t)$ to operators, and imposing the canonical commutation relations

$$\left[\varphi_l, \pi_{l'}^\dagger \right] = i\delta_{ll'} , \tag{3.24}$$

where $\pi_l = \partial_t \varphi_l$. However, to be consistent with the notation used everywhere else in this thesis, we won't use any special notation to distinguish operators from classical variables. It should be understood that from this point on, all fields are operator-valued.

Because the field $\varphi(x,t)$ is defined to be real, we can infer that $\varphi_{-l} = \varphi_l^\dagger$ and $\pi_{-l} = \pi_l^\dagger$. The normal mode expansion then allows us to perform the spatial integration in the Lagrangian (3.22), which gives

$$L = \sum_{l=-\infty}^{\infty} \left(\frac{1}{2} \pi_l^\dagger \pi_l - \frac{{\omega_l}^2}{2} \varphi_l^\dagger \varphi_l \right) . \tag{3.25}$$

The Hamiltonian obtained from the Lagrangian (3.25) is in the typical form of a set of

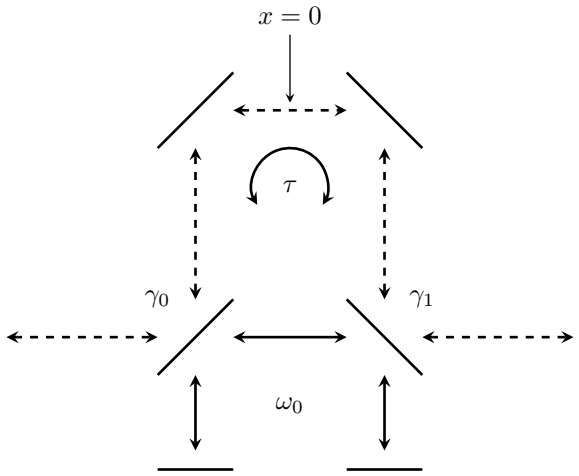


Figure 3.1: Schematic depiction of a system – drawn here as a standing-wave cavity – coupled to a one-dimensional feedback reservoir with time delay τ. The dashed lines represent the reservoir field.

harmonic oscillators, and as such we define boson annihilation and creation operators,

$$b_l = \sqrt{\frac{|\omega_l|}{2}} \left(\varphi_l + \frac{i}{|\omega_l|} \pi_l \right) , \tag{3.26}$$

$$b_l^\dagger = \sqrt{\frac{|\omega_l|}{2}} \left(\varphi_l^\dagger - \frac{i}{|\omega_l|} \pi_l^\dagger \right) , \tag{3.27}$$

which satisfy the commutation relations

$$\left[b_l, b_{l'}^\dagger \right] = \delta_{ll'} \, . \tag{3.28}$$

The field (3.23) may now be rewritten in the form

$$\varphi(x,t) = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{2|\omega_l| M}} \left(b_l(t) e^{i\omega_l x} + b_l^\dagger(t) e^{-i\omega_l x} \right) , \tag{3.29}$$

and by using the canonical operators (3.26) and (3.27) we can show that the Hamiltonian corresponding to (3.25) is given by

$$H_E = \sum_{l=-\infty}^{\infty} |\omega_l| \left(b_l^\dagger b_l + \frac{1}{2} \right) . \tag{3.30}$$

If we add a term describing the system (one mode of the cavity) we obtain the free Hamiltonian for the system and environment,

$$H_0 = \omega_0 a^\dagger a + \sum_{l=-\infty}^{\infty} |\omega_l| b_{pl}^\dagger b_{pl} \, , \tag{3.31}$$

where we have neglected the zero-point energy, which of course has no physical importance

in this context.

In order to effect the feedback loop with discrete time delay, we assume frequency-independent coupling between the system and the modes of the environment at two spatial locations, $x = \pm\tau/2$. The interaction part of the Hamiltonian is given by

$$H_{SE} = (a^\dagger + a) \sum_{l=-\infty}^{\infty} \left(\sqrt{\frac{\gamma_0}{2M}} \left(b_l e^{-i\,\mathrm{sgn}(l)(|\omega_l|\tau/2-\phi_0)} + b_l^\dagger e^{i\,\mathrm{sgn}(l)(|\omega_l|\tau/2-\phi_0)} \right) \right.$$
$$\left. + \sqrt{\frac{\gamma_1}{2M}} \left(b_l e^{i\,\mathrm{sgn}(l)(|\omega_l|\tau/2+\phi_1)} + b_l^\dagger e^{-i\,\mathrm{sgn}(l)(|\omega_l|\tau/2+\phi_1)} \right) \right) , \tag{3.32}$$

with $\gamma_0, \gamma_1 \geq 0$. The phases ϕ_0 and ϕ_1 are phenomenological factors, introduced in order to account for phase shifts such as those at the mirrors. We now make the rotating-wave approximation by neglecting the so-called *counter-rotating terms*, that is, all terms involving the products $a^\dagger b_l^\dagger$ and $a b_l$. This brings the interaction Hamiltonian into the same form as (3.5):

$$H_{SE} = \sum_{l=-\infty}^{\infty} \left(\kappa_l a^\dagger b_l + \kappa_l^* b_l^\dagger a \right) , \tag{3.33}$$

with

$$\kappa_l = \sqrt{\frac{\gamma_0}{2M}} e^{-i\,\mathrm{sgn}(l)(|\omega_l|\tau/2-\phi_0)} + \sqrt{\frac{\gamma_1}{2M}} e^{i\,\mathrm{sgn}(l)(|\omega_l|\tau/2+\phi_1)} . \tag{3.34}$$

If we transform into the rotating frame, as described in section 3.2, we find

$$H_{SEI}(t) = a^\dagger B(t) + B^\dagger(t) a , \tag{3.35}$$

with

$$B(t) = \sum_{l=-\infty}^{\infty} \kappa_l e^{i(\omega_0 - |\omega_l|)t} b_l . \tag{3.36}$$

The corresponding spectral density is

$$\begin{aligned} J(\omega) &= \sum_{l=-\infty}^{\infty} |\kappa_l|^2 \delta(\omega - |\omega_l|) \\ &= \sum_{l=-\infty}^{\infty} \left(\frac{\gamma_0 + \gamma_1}{2M} + \frac{\sqrt{\gamma_0\gamma_1}}{M} \cos(|\omega_l|\tau + \phi_1 - \phi_0) \right) \delta(\omega - |\omega_l|) . \end{aligned} \tag{3.37}$$

We now take the continuum limit, in which $M \to \infty$. This can be entirely accounted for by modifying the spectral density, which becomes

$$J(\omega) = \theta(\omega) \left(\frac{\gamma_0 + \gamma_1}{2\pi} + \frac{\sqrt{\gamma_0\gamma_1}}{\pi} \cos(\omega\tau + \phi_1 - \phi_0) \right) , \tag{3.38}$$

where $\theta(\omega)$ is the Heaviside step function, used here in the distributional sense: the spectral density only has support for $\omega > 0$. In this continuum limit, it is appropriate to use the

frequency-shifted spectral density

$$J'(\omega) = \theta(\omega + \omega_0) \left(\frac{\gamma_0 + \gamma_1}{2\pi} + \frac{\sqrt{\gamma_0 \gamma_1}}{\pi} \cos(\omega\tau + \phi) \right) , \tag{3.39}$$

where we have assumed that we are able to hold the phase $\omega_0\tau$ constant, and we have combined the various phase factors into an overall phase $\phi = \omega_0\tau + \phi_1 - \phi_0$.

In the limit of very large ω_0, $\omega_0 \to \infty$, it is clear that $\theta(\omega + \omega_0) \to 1$ for all ω. In this limit, therefore, equation (3.39) becomes

$$J'(\omega) = \frac{\gamma_0 + \gamma_1}{2\pi} + \frac{\sqrt{\gamma_0 \gamma_1}}{\pi} \cos(\omega\tau + \phi) . \tag{3.40}$$

Substituting (3.40) into (3.17) yields the dissipation kernel for the simple system depicted in figure 3.1:

$$f(t) = \sqrt{\gamma_0 \gamma_1} e^{i\phi} \delta(t - \tau) + (\gamma_0 + \gamma_1)\delta(t) + \sqrt{\gamma_0 \gamma_1} e^{-i\phi} \delta(t + \tau) . \tag{3.41}$$

All the work in this thesis is informed by the context of quantum optics, where typical systems have a central frequency $\omega_0 \gg 0$ that defines by far the smallest time-scale.[36] Therefore, it is reasonable for us to use the form (3.40) of the frequency-shifted spectral density. This limit is often referred to somewhat impressionistically as 'extending the lower limit of the frequency integration to $-\infty$' or something similar, which sometimes leads to misunderstandings. For example, Dolce et al. went so far as to say that this approximation "has no physical justification".[37] But we are not extending the limits of any integral: we are letting $\omega_0 \to \infty$ which, while it represents an idealisation, is justified to a very good degree of approximation when the free oscillation of the system occurs on a much smaller time-scale than the other time-scales in the system.

The model outlined above, with the particular frequency-shifted spectral density (3.40), represents a prototype model for the kinds of system–environment interaction we are interested in. All the systems considered in this thesis will be based on this model to a greater or lesser extent.

3.4 THE TWO ROTATING-WAVE APPROXIMATIONS

So far we have mentioned the rotating-wave approximation (RWA) in two different contexts: in section 2.3 we used it in our derivation of the Lindblad master equation, while in this chapter we have defined a model which includes an RWA–type coupling from the outset. Even though they bear the same name, these are actually two different, though related, approximations. Fleming et al., who have provided a comprehensive review and criticism of both approximations, refer to them respectively as the *post-trace* and *pre-trace* rotating-wave approximations.[27] Both approximations consist of neglecting terms that oscillate rapidly in the interaction picture, with the reasoning that these terms will

make a negligible contribution on coarse-grained time-scales. The post-trace RWA neglects these rapidly-oscillating terms in the master equation, while the pre-trace RWA drops the counter-rotating terms in the Hamiltonian.

Several authors have raised concerns about the use of the RWA in both its forms. Because the form of the interaction Hamiltonian (3.5) is the same as the form obtained by making the pre-trace RWA, some of these concerns are relevant to our model. As Fleming et al. discuss, it is well-known that models derived using the pre-trace RWA are unsuitable for calculating environmentally-induced frequency shifts. In this thesis, we are looking to examine only the effects of time delays within the context of this model, so we will not concern ourselves with frequency shifts and associated issues. A more serious issue, raised by Ford and O'Connell, is that in general the combined Hamiltonian obtained by making the pre-trace RWA is not bounded below.[36] Here we will recapitulate their argument and discuss its implications for our model. Once again, for the purpose of illustration we will consider the simplified model with Hamiltonian given by (3.31) and (3.33), namely

$$H = \omega_0 a^\dagger a + \sum_{l=-\infty}^{\infty} |\omega_l| b_l^\dagger b_l + \sum_{l=-\infty}^{\infty} \left(\kappa_l a^\dagger b_l + \kappa_l^* b_l^\dagger a \right) . \tag{3.42}$$

We define a direct product of coherent states of the non-interacting oscillators,

$$|\psi\rangle = |\zeta\rangle \prod_{l=-\infty}^{\infty} |\eta_l\rangle = \exp\left(\zeta a^\dagger - \zeta^* a\right) \prod_{l=-\infty}^{\infty} \exp\left(\eta_l b_l^\dagger - \eta_l^* b_l\right) |0\rangle \ , \tag{3.43}$$

where $|0\rangle$ denotes the vacuum state of the combined system. It follows that

$$\langle\psi|H|\psi\rangle = \omega_0 |\zeta|^2 + \sum_{l=-\infty}^{\infty} |\omega_l| |\eta_l|^2 + \sum_{l=-\infty}^{\infty} \left(\kappa_l \zeta^* \eta_l + \kappa_l^* \eta_l^* \zeta\right) . \tag{3.44}$$

To obtain an upper bound for the ground state energy, we minimise $\langle\psi|H|\psi\rangle$ while holding ζ constant. This yields $\eta_l = -\zeta\kappa_l^*/|\omega_l|$, which means that the minimum value of the expectation value of the Hamiltonian is

$$\langle\psi|H|\psi\rangle = \left(\omega_0 - \sum_{l=-\infty}^{\infty} \frac{|\kappa_l|^2}{|\omega_l|}\right) |\zeta|^2 . \tag{3.45}$$

If, therefore,

$$\omega_0 < \sum_{l=-\infty}^{\infty} \frac{|\kappa_l|^2}{|\omega_l|} \tag{3.46}$$

we will always be able to choose $|\zeta|^2$ such that the expectation value (3.45) becomes arbitrarily large and negative. Since the expectation value of any self-adjoint operator is within the spectrum of the operator, this implies that when (3.46) holds the spectrum of H has no lower bound. This would mean that the Hamiltonian has no ground state—a

completely unphysical situation.

Ford and O'Connell show in their paper that the condition (3.46) holds for a large class of physical models. The situation is different with the model we are studying, because the couplings κ_l are chosen phenomenologically to effect certain phenomena of interest, namely feedback loops. Nonetheless, the right-hand side of (3.46) is divergent even in the very simple case where κ_l is a constant, so the issue remains relevant to the models studied in this thesis. However, Ford and O'Connell do acknowledge that the problem can be averted (albeit not uniquely) by adding a term to the Hamiltonian that renormalises the system frequency, and thereby cancels the divergent contribution on the right-hand side of (3.46). This correction amounts to making the replacement

$$\omega_0 \to \omega_0 + \sum_{l=-\infty}^{\infty} \frac{|\kappa_l|^2}{|\omega_l|} \tag{3.47}$$

in the Hamiltonian (3.42). Given that we have already assumed in section 3.3 that $\omega_0 \to \infty$, we are free to absorb this correction into the central frequency, and thus avert the issue of the boundedness of the Hamiltonian. These arguments are easily generalised to multipartite systems. I recognise that this defence of the models used in this thesis may not be completely satisfactory; however, I believe that these models are interesting enough to nonetheless warrant study and investigation.

3.5 THE LAPLACE TRANSFORM

The Laplace transform is usually defined as

$$\mathscr{L}_t(x)(s) \stackrel{?}{=} \int_{0^-}^{\infty} \mathrm{d}t\, e^{-st} x(t)\,,$$

where the lower limit of 0^- is chosen so that any mass point located at $t = 0$, such as a contribution from a Dirac delta function, is entirely captured by the transform. Some authors replace the lower limit with 0^+, but neither of these conventions is appropriate for our purposes. The Dirac delta functions appearing in (3.41) have a physical origin that requires that we replace the lower limit with 0 exactly, as we will explain below.

If we derive the frequency-shifted spectral density from (3.37) without first taking the continuum limit, we get

$$J'(\omega) = \sum_{l=1}^{\infty} \Big(\frac{\gamma_0 + \gamma_1}{2\pi} + \frac{\gamma_0\gamma_1}{\pi} \cos(\omega_l \tau + \phi_1 - \phi_0) \Big)\, \delta(\omega + \omega_0 - \omega_l)\,. \tag{3.48}$$

According to section 3.3, the time-homogeneous dissipation kernel is given by the Fourier transform of the frequency-shifted spectral density, so equation (3.48) yields

$$F(t) = \gamma_0\gamma_1 e^{i\phi} D(t-\tau) + (\gamma_0 + \gamma_1) D(t) + \gamma_0\gamma_1 e^{-i\phi} D(t+\tau)\,, \tag{3.49}$$

where

$$D(t) = \frac{1}{2\pi}\sum_{l=1}^{\infty} e^{-i(\omega_l-\omega_0)t}\,, \qquad \omega_l = \frac{2\pi l}{L}\,. \tag{3·50}$$

If l_0 is the greatest l for which $\omega_0 \geq \omega_l$, then we can rewrite (3·50) as

$$D(t) = \frac{1}{2\pi}e^{i\Delta_{l_0}t}\sum_{l=-l_0+1}^{\infty} e^{-i\omega_l t}\,, \qquad \omega_l = \frac{2\pi l}{L}\,, \tag{3·51}$$

where $\Delta_{l_0} = \omega_0 - \omega_{l_0} \geq 0$. In the limit $\omega_0 \to \infty$, we find

$$D(t) = e^{i\Delta_{l_0}t}\lim_{N\to\infty} D_N(t)\,, \tag{3·52}$$

where

$$D_N(t) = \frac{1}{2\pi}\sum_{l=-N}^{N} e^{-i\omega_l t} = \frac{1}{2\pi}\frac{\sin((2N+1)\pi t/L)}{\sin(\pi t/L)} \tag{3·53}$$

is a Dirichlet kernel. It is well-known that $\lim_{N\to\infty} D_N(t) = \delta(t)$ on the interval $-L/2 \leq t \leq L/2$. In the limit where $L \to \infty$ and $\Delta_{l_0} \to 0$, we find that $D(t) \to \delta(t)$ on the whole real line. In this way we recover (3·41). The first several Dirichlet kernels are shown in figure 3.2.

Any Dirac delta functions arising from the models we consider in this thesis will be derived in the same way: from (3·52) in the $L \to \infty$ limit. It follows from the symmetry of the Dirichlet kernels $D_N(t)$ that the convention we should adopt is that for any test function $x(t)$,

$$\int_0^\infty \mathrm{d}t\,\delta(t)x(t) = \frac{x(0)}{2}\,. \tag{3·54}$$

Thus, we need to choose the convention for the Laplace transform which yields

$$\mathscr{L}_t(\delta(t-\tau))(s) = \begin{cases} \frac{1}{2} & \tau = 0 \\ e^{-\tau s} & \tau > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3·55}$$

So, we define

$$\tilde{x}(t) = \mathscr{L}_t(x)(s) = \int_0^\infty \mathrm{d}t\, e^{-st}x(t)\,, \tag{3·56}$$

where the lower limit is to be interpreted as being exactly zero.

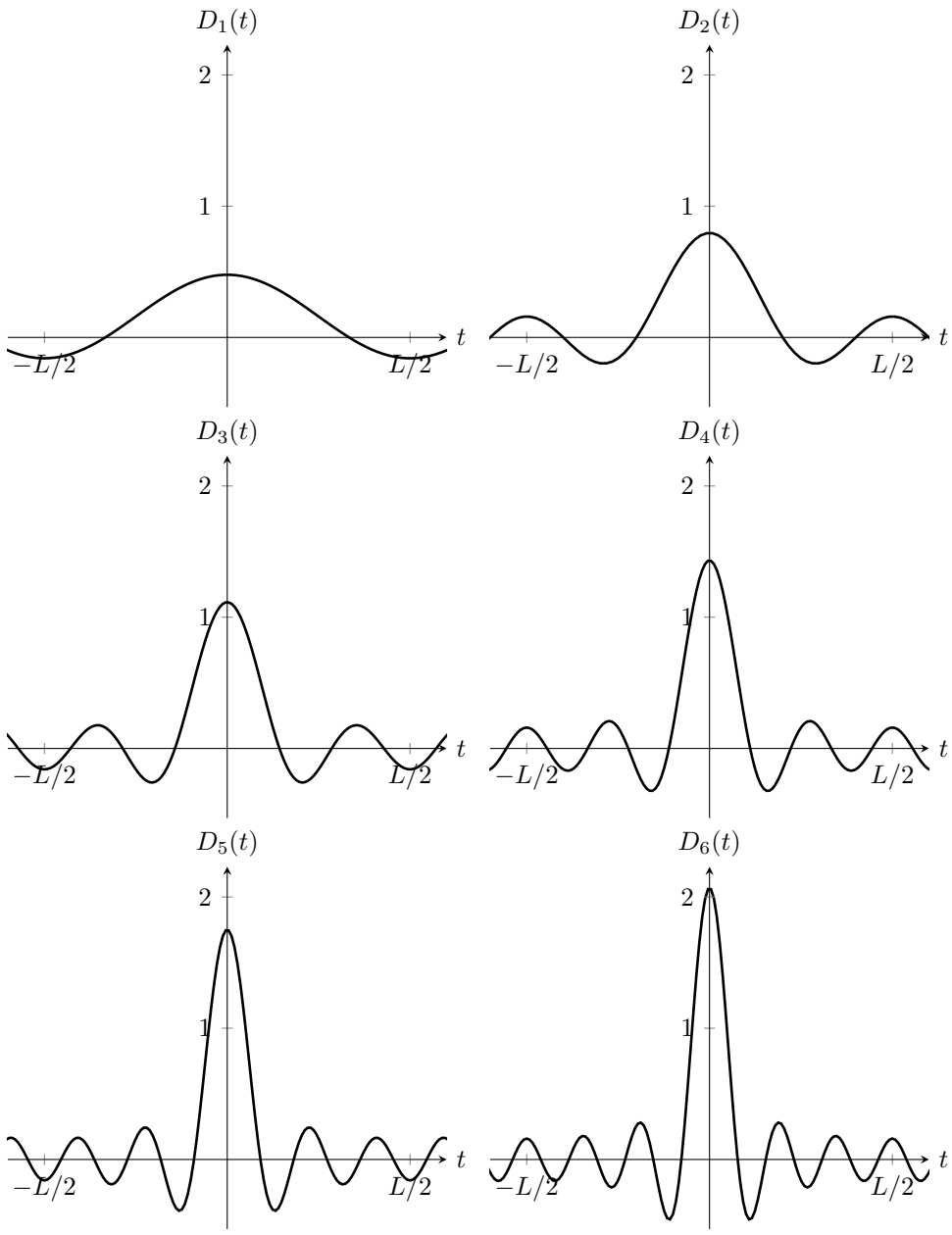


Figure 3.2: The first six Dirichlet kernels, as defined by (3.53).

4

SPONTANEOUS EMISSION FROM SINGLY-EXCITED SYSTEMS

If the Hamiltonian (3.2) of the model described in section 3.1 conserves the number of excitations in the combined system, and the system initially contains only a single excitation, the resulting differential equations are often simple enough that they admit an exact solution, or at least that they are easily solved numerically. In this chapter we will choose a Hamiltonian that has this property and perform a number of example calculations.

Sections 4.1 and 4.3 represent a straightforward generalisation of a textbook derivation by Breuer and Pettrucione.[7]

4.1 DECAY OF A SINGLY-EXCITED SYSTEM

The model we are going to investigate is a modification of the model in section 3.1. For the purposes of this calculation, we don't need to specify the algebra of the operators—both the system and reservoir degrees of freedom can be either bosonic or fermionic. The only properties we require are

$$a_j a_{j'}^\dagger \left|0\right\rangle = \delta_{jj'} \left|0\right\rangle \quad \text{and} \quad b_{\alpha l} b_{\alpha' l'}^\dagger \left|0\right\rangle = \delta_{\alpha\alpha'}\delta_{ll'} \left|0\right\rangle \,, \tag{4.1}$$

where the vacuum $\left|0\right\rangle$ is defined by

$$a_j \left|0\right\rangle = b_{\alpha l_\alpha} \left|0\right\rangle = 0 \quad \text{for all} \quad j,\, \alpha,\, l_\alpha \,. \tag{4.2}$$

As I mentioned above, we want to look at the case where the Hamiltonian (3.2) conserves total excitation number. The total excitation number operator is given by

$$N = \sum_j a_j^\dagger a_j + \sum_\alpha \sum_{l_\alpha} b_{\alpha l_\alpha}^\dagger b_{\alpha l_\alpha} \,, \tag{4.3}$$

and it's easy to verify that, given the definitions above, so long as we have

$$[H_{S1}(t), N] = \left[H_{S1}(t), \sum_j a_j^\dagger a_j \right] = 0 \,, \tag{4.4}$$

we also have

$$[H(t), N] = 0 \,, \tag{4.5}$$

and that N is therefore a conserved quantity. Thus, we have a certain amount of freedom in choosing $H_{S1}(t)$: so long as the condition (4.4) holds, the analysis below will still apply.

The initial state we will consider here is that of a singly-excited combined system: the excitation is localised within the system and the reservoir is in a vacuum state. Such a state can be expanded in the form

$$\rho(0) = p_0 \left|0\right\rangle \left\langle 0\right| + p_1 \left|\psi(0)\right\rangle \left\langle \psi(0)\right| , \tag{4.6}$$

where $\left|\psi(0)\right\rangle$ can be written as

$$\left|\psi(0)\right\rangle = c_0 \left|0\right\rangle + \sum_j c_j(0) a_j^\dagger \left|0\right\rangle . \tag{4.7}$$

Naturally, we have $p_0 + p_1 = 1$. Conservation of total excitation number means that an initial state of this form evolves, after a time t, into a state with the expansion

$$\left|\psi(t)\right\rangle = c_0 \left|0\right\rangle + \sum_j c_j(t) a_j^\dagger \left|0\right\rangle + \sum_\alpha \sum_{l_\alpha} c_{\alpha l_\alpha}(t) b_{\alpha l_\alpha}^\dagger \left|0\right\rangle , \tag{4.8}$$

where c_0, $c_j(t)$, and $c_{\alpha l_\alpha}(t)$ are, in general, complex numbers. We have $c_{\alpha l_\alpha}(0) = 0$ by definition. We require, naturally, that the state $\left|\psi\right\rangle$ be normalised:

$$|c_0|^2 + \sum_j |c_j(t)|^2 + \sum_\alpha \sum_{l_\alpha} |c_{\alpha l_\alpha}(t)|^2 = 1 . \tag{4.9}$$

The first term in the state (4.8) is of course the ground state; the latter two terms are referred to as the *single-excitation manifold*. Substituting (4.9) into the Schrödinger equation yields differential equations for the amplitudes

$$\frac{\mathrm{d}}{\mathrm{d}t} c_j(t) = -i \sum_k \Delta_{jk}(t) c_k(t) - i \sum_\alpha \sum_{l_\alpha} \kappa_{j\alpha l_\alpha} \exp\left(i(\omega_j - \omega_{l_\alpha})t\right) c_{\alpha l_\alpha}(t) , \tag{4.10}$$

where

$$\Delta_{jk}(t) = \left\langle 0 \,\middle|\, a_j H_{S1}(t) a_k^\dagger \,\middle|\, 0 \right\rangle , \tag{4.11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} c_{\alpha l_\alpha}(t) = -i \sum_k \kappa_{k\alpha l_\alpha}^* \exp\left(-i(\omega_k - \omega_{l_\alpha})t\right) c_k(t) . \tag{4.12}$$

Formal integration of (4.12) gives

$$c_{\alpha l_\alpha}(t) = -i \sum_k \int_0^t \mathrm{d}t_1 \, \kappa_{k\alpha l_\alpha}^* \exp\left(-i(\omega_k - \omega_{l_\alpha})t_1\right) c_k(t_1) , \tag{4.13}$$

and substituting this into (4.10) yields the Volterra integro-differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} c_j(t) = -i \sum_k \Delta_{jk}(t) c_k(t) - \sum_k \int_0^t \mathrm{d}t_1 \, F_{jk}(t, t_1) c_k(t_1) , \tag{4.14}$$

where $F_{jk}(t, t_1)$ is given by (3.13). The integro-differential equations (4.14) can be written as a single differential equation for a vector $\vec{c}(t)$, whose j^{th} element is $c_j(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{c}(t) = -i\Delta(t)\vec{c}(t) - \int_0^t \mathrm{d}t_1\, F(t, t_1)\vec{c}(t_1)\,, \tag{4.15}$$

where $\Delta(t)$ and $F(t, t_1)$ are matrix-valued functions whose elements are $\Delta_j(t)$ and $F_{jk}(t, t_1)$ respectively. Given that the requirement (4.4) rules out any kind of driving we would, under many circumstances, expect $\Delta(t)$ to be constant. Nonetheless, time-dependent detunings are not a physical impossibility, so we will leave the time-dependence in place for the time being for the sake of generality.

4.2 A SIMPLE FEEDBACK LOOP

Consider a simplification of the spectral density (3.40) – which corresponds to the system depicted in figure 3.1 – where we define $\gamma = (\gamma_0 + \gamma_1)/2$ and $\eta = \sqrt{\gamma_0\gamma_1}$:

$$J'(\omega) = \frac{\gamma}{\pi} + \frac{\eta}{\pi}\cos(\omega\tau + \phi)\,, \tag{4.16}$$

with $\tau \geq 0$. We can readily show that $\gamma \geq \eta$. If we suppose $H_{S1}(t) = 0$, leading to $\Delta(t) = 0$, the differential equation (4.15) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = -\int_0^t \mathrm{d}t_1\, f(t - t_1)c(t_1)\,, \tag{4.17}$$

where the dissipation kernel is given by

$$\begin{aligned} f(t - t_1) &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, J'(\omega)e^{i\omega(t-t_1)} \\ &= \eta e^{i\phi}\delta(t - \tau - t_1) + 2\gamma\delta(t - t_1) + \eta e^{-i\phi}\delta(t + \tau - t_1)\,. \end{aligned} \tag{4.18}$$

Substituting (4.18) into (4.17) yields the delay-differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = -\gamma c(t) - \theta(t - \tau)\eta e^{i\phi}c(t - \tau)\,, \tag{4.19}$$

where $\theta(t)$ is the Heaviside step function. Equation (4.19), despite being very simple, has some interesting properties.

For $t \leq \tau$, equation (4.19) can be solved immediately, giving

$$c(t) = c(0)e^{-\gamma t} \quad \text{for} \quad 0 < t \leq \tau\,. \tag{4.20}$$

For $t > \tau$, it has a characteristic equation

$$\lambda + \gamma + \eta e^{i\phi - \lambda\tau} = 0\,. \tag{4.21}$$

When $\tau = 0$, we have $\lambda = -\gamma - \eta e^{i\phi}$. Otherwise, there are an infinite number of solutions to this equation for complex λ:

$$\lambda_k = -\gamma + \frac{1}{\tau} W_k\left(-\eta\tau e^{i\phi+\gamma\tau}\right), \quad k \in \mathbb{Z}, \tag{4.22}$$

where $W_k(z)$ is the k^{th} branch of the Lambert W function. The Lambert W function is defined as the inverse function of $w\exp(w)$.[38] In other words, the defining equation for $W(z)$ is $z = W(z)\exp(W(z))$, for any complex number z. Thus, a general solution to (4.19) is given by $c(t) = \sum_{k\in\mathbb{Z}} g_k e^{-\lambda_k t}$, where the coefficients g_k must be chosen such that (4.20) is satisfied. Clearly the sum over k must be truncated in order to actually evaluate $c(t)$, so in practice we will need to choose g_k such that $c(t) \approx c(0)e^{-\gamma t}$ for $0 < t < \tau$, using an estimation method such as least squares. This does work, but it isn't a particularly practical way of finding solutions for specific parameters. Fortunately there's an alternative analytic solution.

The solution to equation (4.19) can be written as a piecewise function:

$$c(t) = c(0)e^{-\gamma t} \sum_{m=0}^{n} \frac{1}{m!}\left(-\eta e^{i\phi+\gamma\tau}(t - m\tau)\right)^m \quad \text{when} \quad n\tau < t \leq (n+1)\tau\,. \tag{4.23}$$

This solution can be obtained in various ways – I found it using repeated integration, in the manner of the 'method of steps' – and is easily verified by way of substitution into (4.19). To the best of my knowledge, the solution (4.23) was first developed by Milonni and Knight in 1974.[39] It reappears in more recent work on "atomic" emission in front of a mirror.[40–42] The Laplace transform of (4.19) leads to

$$\tilde{c}(s) = \frac{c(0)}{s + \gamma + \eta e^{i\phi - \tau s}}\,, \tag{4.24}$$

for $t > \tau$. Provided $\lim_{t\to\infty} c(t)$ is finite, the Laplace final value theorem gives:

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} s\tilde{c}(s) = \begin{cases} \dfrac{c(0)}{1+\gamma\tau} & \gamma + \eta e^{i\phi} = 0 \\ 0 & \gamma + \eta e^{i\phi} \neq 0\,, \end{cases} \tag{4.25}$$

where the case $\gamma + \eta e^{i\phi} = 0$ is evaluated using L'Hôpital's rule. Figure 4.1 shows several examples of solutions of (4.19).

The condition $\gamma + \eta e^{i\phi} = 0$ in equation (4.25) corresponds to destructive interference between the field in the feedback loop and the field at the outputs from the system. We can see how this comes about by looking at the delay-differential equation (4.19) for the excited state amplitude $c(t)$. If the condition $\eta e^{i\phi} = -\gamma$ is met, whenever the dynamics give us $c(t-\tau) = c(t)$ we have $\dot{c}(t) = 0$: the damping is suppressed, or to put it another way, the system and feedback loop 'dynamically decouple' from the rest of the environment. Physically, this can be pictured as two fields, namely the field in the feedback loop and the output field from the system, adding together at the system output and cancelling out.

Clearly, if the system reaches a nontrivial steady state in which $c(t-\tau) = c(t)$ the decoupling will persist, and excitations can become trapped in the system–loop subsystem. This is precisely what happens in the example depicted in figure 4.1b. The destructive interference displayed in this example corresponds to negative feedback, which acts to stabilise the system in a persistent state of partial excitation. The other possibility is constructive interference, as depicted in figure 4.1a, which brings about positive feedback that initially increases the decay rate and in the long run destabilises the system. Figures 4.1c and 4.1d depict examples of attenuated positive and negative feedback, respectively.

Finally, it is worth noting that the example at hand is in fact an instance of specifically *quantum* interference, in the sense that it is probability amplitudes that interfere with one another. In particular, because there is only a single photon in the combined system, it is the probability amplitudes for escaping the system via two pathways – directly or via the feedback loop – that interfere.

4.3 THE MASTER EQUATION

Suppose we have a solution $\vec{c}(t)$ to equation (4.15) (or equivalently (4.14)). Because of the linearity of the differential equations, we will always be able to write such a solution as

$$\vec{c}(t) = V(t)\vec{c}(0)\,, \tag{4.26}$$

where $V(t)$ is a matrix-valued Green's function. This function satisfies a matrix version of (4.15),

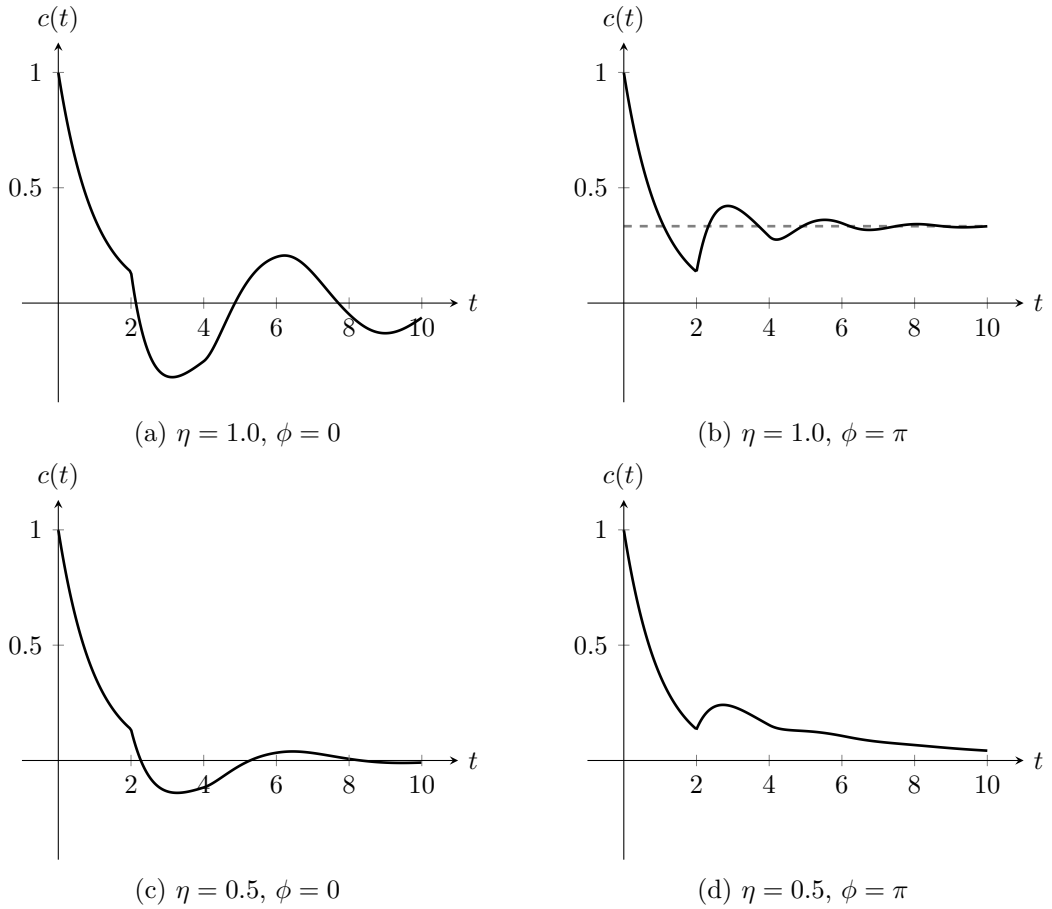
$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -i\Delta(t)V(t) - \int_0^t \mathrm{d}t_1\, F(t,t_1)V(t_1)\,, \qquad V(0) = I\,. \tag{4.27}$$

It may be the case that $V(t)$ can be calculated from (4.27) using the Laplace transform, but whether or not this is practical depends on the nature of $\Delta(t)$ and $F(t,t_1)$. In particular, the calculation will be much easier in the case where $\Delta(t)$ is a constant function of time, and where we have – as discussed in section 3.3 – $\omega_j = \omega_0$ for all j, such that the integral in (4.27) becomes a convolution. We will return to the Laplace transform approach in section 4.5. Regardless of whether or not we have an analytic expression for $V(t)$, so long as a solution exists we can differentiate (4.26) and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{c}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right)\vec{c}(0)\,. \tag{4.28}$$

Assuming $V(t)$ is also an invertible matrix, we can then invert (4.26) to obtain $\vec{c}(0) = V^{-1}(t)\vec{c}(t)$, and substitute this into (4.28). This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{c}(t) = -\Gamma(t)\vec{c}(t)\,, \tag{4.29}$$



(a) $\eta = 1.0$, $\phi = 0$

(b) $\eta = 1.0$, $\phi = \pi$

(c) $\eta = 0.5$, $\phi = 0$

(d) $\eta = 0.5$, $\phi = \pi$

Figure 4.1: Examples of solutions to the delay-differential equation (4.19), with $\gamma = 1$ and $\tau = 2$ (in arbitrary units), for several values of η and ϕ. The dashed line in (b) indicates the non-zero final value for that case; the other three cases shown eventually decay completely.

where

$$\Gamma(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right)V^{-1}(t) = i\Delta(t) + \int_0^t \mathrm{d}t_1\, F(t,t_1)V(t_1)V^{-1}(t)\,. \tag{4.30}$$

Equation (4.29) is a time-local equation that describes the same dynamics as equation (4.15).

Using (4.29) and (4.30), we now aim to construct a master equation for the reduced system dynamics. The reduced density matrix will be given by

$$\rho_S(t) = p_0 \,\mathrm{tr}_E \left|0\right\rangle \left\langle 0\right| + p_1 \,\mathrm{tr}_E \left|\psi(t)\right\rangle \left\langle \psi(t)\right| \,, \tag{4.31}$$

where $|\psi(t)\rangle$ is given by (4.8). We now define the reduced system ground state $|0_S\rangle$, which is annihilated by a_j for all j, as well as the reservoir ground state $|0_E\rangle$, which is annihilated by all b_{al_α}. The combined system ground state is thus given by $|0\rangle = |0_S\rangle \otimes |0_E\rangle$. The states $b^\dagger_{\alpha l_\alpha}|0_E\rangle$, along with $|0_E\rangle$, constitute a basis over which the partial trace can be

taken. Evaluating the trace in equation (4.31) therefore gives

$$\rho_S(t) = p_1 \sum_j c_0 c_j^*(t) \left|0_S\right\rangle \left\langle 0_S\right| a_j + p_1 \sum_j c_j(t) c_0^* a_j^\dagger \left|0_S\right\rangle \left\langle 0_S\right| + p_1 \sum_{jk} c_j(t) c_k^*(t) a_j^\dagger \left|0_S\right\rangle \left\langle 0_S\right| a_k + \left(1 - p_1 \sum_j |c_j(t)|^2\right) \left|0_S\right\rangle \left\langle 0_S\right| . \tag{4.32}$$

The latter term in (4.32), derived from the trace over the environmental degrees of freedom, follows from (4.9) and ensures that $\mathrm{tr}\,\rho_S(t) = 1$ at all times.

The derivative of (4.32) can be written compactly in the form of a master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \sum_{jk} \left(\Gamma_{jk}(t) \left[a_k \rho_S(t), a_j^\dagger\right] + \Gamma_{jk}^*(t) \left[a_j, \rho_S(t) a_k^\dagger\right]\right) . \tag{4.33}$$

Except for the time-dependent coefficients, this equation is in Lindblad form. The real part of $\Gamma(t)$ induces damping of the system at a time-dependent rate, which may be negative for some times; the imaginary part contributes a set of time-dependent frequency shifts. The master equation (4.33), despite the appearance of generality, only applies (or can only be said to apply, at this stage at least) to the singly-excited systems considered above. Put another way, equation (4.33) contains exactly the same information as equation (4.15). We don't actually gain anything from writing down a master equation description, because we've already solved the model using another method.

The treatment of the decay of singly-excited systems in section 4.1 and the corresponding master equation (4.33) are quite well-known: as I stated at the beginning of this chapter, the basic ideas are textbook material. In 2000, for example, Anastopoulos and Hu derived a master equation equivalent to (4.33) in the case of a single system degree of freedom.[43] As has been pointed out by Cresser,[44,45] the main results leading to the master equation (4.33) were first obtained by Garraway in 1997.[46,47]

As an example, we will now derive the master equation for the simple feedback loop we considered in section 4.2. In this case, we already have the exact solution (4.23), which can be written as $c(t) = v(t)c(0)$, where $v(t)$ in this instance is a complex scalar function given by

$$v(t) = e^{-\gamma t} \sum_{m=0}^{n} \frac{1}{m!} \left(-\eta e^{i\phi + \gamma\tau}(t - m\tau)\right)^m \quad \text{when} \quad n\tau < t \leq (n+1)\tau . \tag{4.34}$$

From this solution we can immediately write down a master equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \gamma(t) \left[a\rho_S(t), a^\dagger\right] + \gamma^*(t) \left[a, \rho_S(t) a^\dagger\right] , \tag{4.35}$$

with

$$\gamma(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t} v(t)\right) \frac{1}{v(t)} = \gamma + \theta(t - \tau)\eta e^{i\phi} \frac{v(t-\tau)}{v(t)} , \tag{4.36}$$

which follows from (4.19). Obviously, (4.36) is discontinuous at $t = \tau$, but this is a consequence of the unphysical, separable initial condition (3.6). More worryingly, it is clear from figures 4.1a and 4.1c that $v(t)$ crosses zero at several points: $\gamma(t)$ will be singular at these points. If we want to integrate the master equation (4.35) numerically, therefore, we have to be careful to use integration routines that can handle such singularities. Indeed, the presence of these singularities may in some cases prevent us from integrating this master equation altogether. Because I have not made extensive use of master equations such as equation (4.35) in this thesis, I have not undertaken a systematic investigation of these numerical issues. Andersson et al. have discussed some of the difficulties of dealing with singular damping.[48]

We can, however, make one important observation about $\gamma(t)$. We can tell by inspection of figures 4.1b and 4.1d that we must have $\gamma(t) < 0$ for some t. As mentioned in section 2.4, negative decay rates have been identified by Rivas et al., among other authors, as an indicator of non-Markovian behaviour—meaning that this system is non-Markovian according to the typical definition used in the literature.[5] Examination of figures 4.1a and 4.1c shows that the damping $\gamma(t)$ can be positive for some $t > \tau$, when feedback would already have taken effect. However, it's important to recall that negative damping at *some* time is sufficient to break the divisibility for *all* times and thereby establish the dynamics as non-Markovian according to the criterion espoused by Rivas et al.

4.4 A SIMPLE NETWORK OF SUBSYSTEMS WITH TIME DELAY

The solutions developed in the previous section can be applied in certain cases to other systems. As an example, consider the combined system depicted in figure 4.2, which consists of N subsystems that interact only via separate one-dimensional reservoirs. While this system does not involve feedback, it does involve time delays, and as such the solutions developed in section 4.2 will be of use. One possible physical set-up corresponding to this picture is a sequence of superconducting qubits interacting via several transmission-line resonators (or 'quantum wires').

The subsystems have canonical operators a_j and $a_j^\dagger$, $j = 1, 2, \ldots, N$. They are identical, with energy level spacings ω_0. We will assume that there is at most one excitation in the combined system, so as before it doesn't matter whether the system canonical operators have bosonic or fermionic commutation relations. We have $N + 1$ identical one-dimensional reservoirs, with canonical operators $b_{\alpha l}$ and $b_{\alpha l}^\dagger$, $\alpha = 1, 2, \ldots, N + 1$. Each reservoir α supports counter-propagating fields, and therefore permits coupling in both directions. Following section 3.3, the field operators of the reservoirs are given by

$$\varphi_\alpha(x_\alpha) = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{2|\omega_l|M}} \left(b_{\alpha l} e^{i\omega_l x_\alpha} + b_{\alpha l}^\dagger e^{-i\omega_l x_\alpha} \right), \qquad \omega_l = \frac{2\pi l}{M}, \tag{4.37}$$

where each reservoir is given its own coordinate x_α and origin $x_\alpha = 0$. In this model each

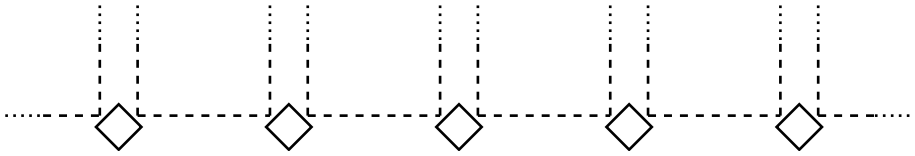


Figure 4.2: Schematic depiction of a network of subsystems: the diamonds denote the subsystems, and the dashed lines represent the separate reservoir fields via which the systems couple to one another.

subsystem j couples to two reservoirs: it couples to reservoir j at $x_j = \tau/2$ and to reservoir $j+1$ at $x_{j+1} = -\tau/2$, creating a time-delayed coupling in both directions between adjacent subsystems, with a propagation delay τ. Once again, we assume frequency-independent coupling between the system and the modes of the environment, and we make the rotating-wave approximation. The resulting system–environment interaction Hamiltonian is given by

$$H_{SE} = \sqrt{\frac{\gamma}{2M}} \sum_{j=1}^{N} \sum_{l=-\infty}^{\infty} \left(a_j^\dagger \left(b_{jl} e^{i\omega_l \tau/2} + b_{(j+1)l} e^{-i\omega_l \tau/2} \right) + \left(b_{jl}^\dagger e^{-i\omega_l \tau/2} + b_{(j+1)l}^\dagger e^{i\omega_l \tau/2} \right) a_j \right). \tag{4.38}$$

Note that all of the system–reservoir interactions have the same coupling strength, $\sqrt{\gamma/2}$. Moving into the interaction picture, we find that (4.38) can be written in the same form as (3.11), with

$$B_j(t) = \sum_{\alpha=1}^{N} \sum_{l=-\infty}^{\infty} \kappa_{j\alpha l} e^{i(\omega_0 - |\omega_l|)t} b_{\alpha l}\,, \tag{4.39}$$

and

$$\kappa_{j\alpha l} = \sqrt{\frac{\gamma}{2M}} \left(e^{i\omega_l \tau/2} \delta_{\alpha,j} + e^{-i\omega_l \tau/2} \delta_{\alpha,j+1} \right). \tag{4.40}$$

Note that in (4.39) we have chosen the same central frequency, ω_0, for each system. The spectral density is given by

$$J_{\alpha jk}(\omega) = \frac{\gamma}{2M} \sum_{l=-\infty}^{\infty} \left(e^{i\omega_l \tau/2} \delta_{\alpha,j} + e^{-i\omega_l \tau/2} \delta_{\alpha,j+1} \right) \times \left(e^{-i\omega_l \tau/2} \delta_{\alpha,k} + e^{i\omega_l \tau/2} \delta_{\alpha,k+1} \right) \delta(\omega - |\omega_l|)\,. \tag{4.41}$$

As in section 3.3, we take the continuum limit $M \to \infty$, and then the limit $\omega_0 \to \infty$, which gives the frequency-shifted spectral density

$$J'_{\alpha jk}(\omega) = \frac{\gamma}{4\pi} \sum_{p=\pm 1} \left(e^{ip(\omega\tau+\phi)/2} \delta_{\alpha,j} + e^{-ip(\omega\tau+\phi)/2} \delta_{\alpha,j+1} \right) \times \left(e^{-ip(\omega\tau+\phi)/2} \delta_{\alpha,k} + e^{ip(\omega\tau+\phi)/2} \delta_{\alpha,k+1} \right), \tag{4.42}$$

where, as before, we have assumed that we can hold the phase $\phi = \omega_0 \tau$ constant. If we sum over the various reservoirs, we obtain

$$\sum_{\alpha=1}^{N} J'_{\alpha jk}(\omega) = \begin{cases} (\pi)^{-1}\gamma & j = k \\ (2\pi)^{-1}\gamma \cos(\omega\tau + \phi) & j \pm 1 = k\,. \end{cases} \tag{4.43}$$

In this case,

$$F_{jk}(t - t_1) = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{\alpha=1}^{N} J'_{\alpha jk}(\omega) e^{-i\omega(t-t_1)}\,, \tag{4.44}$$

so we find

$$F_{jk}(t - t_1) = \begin{cases} 2\gamma\delta(t - t_1) & j = k \\ \dfrac{1}{2}\gamma\left(e^{i\phi}\delta(t - \tau - t_1) + e^{-i\phi}\delta(t + \tau - t_1)\right) & j \pm 1 = k\,. \end{cases} \tag{4.45}$$

All the subsystems have identical, constant energy level spacings ω_0, and we chose ω_0 to be the central frequency when we moved into the interaction picture. It follows from (4.4) and from the fact that the systems do not interact other than via the environment that the detunings (4.11) are given by $\Delta_{jk}(t) = 0$. Substituting these results into (4.14) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}c_j(t) = -\gamma c_j(t) - \frac{1}{2}\gamma e^{i\phi}\left(c_{j-1}(t - \tau) + c_{j+1}(t - \tau)\right)\,. \tag{4.46}$$

We now aim to find the normal modes of the coupled delay differential equations (4.46). To do so, we take Laplace transforms, which gives

$$s\tilde{c}_j(s) - c_j(0) = -\gamma\tilde{c}_j(s) - \frac{1}{2}\gamma e^{i\phi - \tau s}\left(\tilde{c}_{j-1}(s) + \tilde{c}_{j+1}(s)\right)\,, \tag{4.47}$$

which can be written in the form

$$M\tilde{\vec{c}}(s) = \vec{c}(0) \quad \text{with} \quad M = \mathrm{tri}\left(\frac{1}{2}\gamma e^{i\phi-\tau s}, s + \gamma, \frac{1}{2}\gamma e^{i\phi-\tau s}\right)\,. \tag{4.48}$$

We now need to diagonalise the $N \times N$ tridiagonal matrix M. It turns out that $M\vec{x}_n = \lambda_n \vec{x}_n$, with

$$\lambda_n = s + \gamma + \gamma e^{i\phi - \tau s}\cos\left(\frac{n\pi}{N+1}\right)\,, \tag{4.49}$$

and

$$(\vec{x}_n)_m = \sqrt{\frac{2}{N+1}}\sin\left(\frac{mn\pi}{N+1}\right)\,. \tag{4.50}$$

The indices n and m both run over the set $\{1, \ldots, N\}$. Note that the eigenvectors are orthonormal, $\vec{x}_n^T\vec{x}_{n'} = \delta_{nn'}$. Diagonalising gives $M = PDP^T$, with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $P = (\vec{x}_1 \;\; \vec{x}_2 \;\; \cdots \;\; \vec{x}_N)$. Of course, we also have $\tilde{\vec{c}}(s) = M^{-1}\vec{c}(0)$. Suppose the initial

condition is one of the normal modes, $\vec{c}(0) = \vec{x}_n$. Then we have $\vec{c}(t) = \vec{x}_n(t)$, where

$$\tilde{\vec{x}}_n(s) = \tilde{v}_n(s)\vec{x}_n \quad \text{with} \quad \tilde{v}_n(s) = \Big(s + \gamma + \gamma e^{i\phi - \tau s} \cos\Big(\frac{n\pi}{N+1}\Big)\Big)^{-1}, \tag{4.51}$$

which corresponds to the delay-differential equation for $v_n(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} v_n(t) = -\gamma v_n(t) - \theta(t-\tau)\gamma \cos\Big(\frac{n\pi}{N+1}\Big) e^{i\phi} v_n(t-\tau)\,, \qquad v_n(0) = 1\,, \tag{4.52}$$

which gives the time dependence of the mode. The poles of $\tilde{v}_n(s)$ occur when

$$s + \gamma = -\gamma e^{i\phi - \tau s} \cos\Big(\frac{n\pi}{N+1}\Big)\,. \tag{4.53}$$

Here γ and τ are both real and positive. Taking the real part gives

$$\Re(s) + \gamma = -\gamma \cos(\phi) e^{-\tau \Re(s)} \cos\Big(\frac{n\pi}{N+1}\Big)\,. \tag{4.54}$$

Using $-1 < \cos(n\pi/(N+1)) < 1$, we find that $-2\gamma < \Re(s) < 0$, so all the poles of $\tilde{v}_n(s)$ are in the left half-plane. We can therefore apply the Laplace final value theorem, and we obtain $v_n(\infty) = 0$ for all n and for all allowed parameters. Therefore, all initial states eventually decay to the vacuum—just as we would expect physically, as there is no backscatter at the ends of the network of subsystems.

The delay-differential equations (4.52) and (4.19) are identical, once one substitutes $\eta = \gamma \cos(n\pi/(N+1))$ into the latter. Thus, the solution to (4.52) is

$$v_n(t) = e^{-\gamma t} \sum_{l=0}^{m} \frac{1}{l!} \Big(-\gamma \cos\Big(\frac{n\pi}{N+1}\Big) e^{i\phi + \gamma\tau}(t - l\tau)\Big)^l \quad \text{when} \quad m\tau < t \le (m+1)\tau\,. \tag{4.55}$$

Examples of $|v_n(t)|^2$ are plotted in figures 4.3 and 4.4, for $N = 5$ and $N = 6$ respectively. We can make a simple classification of the different curves depicted in these plots. For convenience, let us define

$$\eta_n = e^{i\phi} \cos\Big(\frac{n\pi}{N+1}\Big)\,. \tag{4.56}$$

It is instructive to note that when $\tau = 0$, (4.52) takes on a particularly simple form:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_n(t) = -\gamma(1 + \eta_n) v_n(t)\,, \qquad v_n(0) = 1\,. \tag{4.57}$$

Even though it is difficult to exhaustively classify the dynamics of our network of subsystems, we can use (4.55)–(4.57) to make a few statements about its behaviour:

1. When $\eta_n = 0$ we have 'normal' exponential decay: $|v_n(t)|^2 = \exp(-2\gamma t)$.
2. When $\eta_n \in \mathbb{R}$ and $\eta_n < 0$, we have $|v_n(t)|^2 \ge \exp(-2\gamma t)$ for all t, with $|v_n(t)|^2 > \exp(-2\gamma t)$ holding for at least some t. Thus, we have partial suppression of the

spontaneous emission in these cases, which we might loosely refer to as 'subradiance'. When $\tau = 0$ this becomes a strict inequality, $|v_n(t)|^2 > \exp(-2\gamma t)$, for all t.

3. When $\eta_n \in \mathbb{R}$, $\eta_n > 0$, and $\tau = 0$, we have $|v_n(t)|^2 < \exp(-2\gamma t)$, representing cooperative spontaneous emission or, loosely, 'superradiance'.

4. When $\eta_n \in \mathbb{R}$, $\eta_n > 0$, and $\tau > 0$, we find that $|v_n(t)|^2 < \exp(-2\gamma t)$ for $\tau < t < 2\tau$; $|v_n(t)|^2$ then proceeds to oscillate about $\exp(-2\gamma t)$ for $t > 2\tau$ before finally decaying completely. This case is difficult to describe, because we see a sort of cooperative spontaneous emission – at least initially – but there is also a partial suppression of the decay in some time intervals.

All in all, the complex interplay between enhanced and suppressed spontaneous emission that we see in this system does not have a direct analogue in the usual vocabulary of quantum optics. Of course, this isn't really a concern because for a complete specification of the dynamics one can simply refer to the exact solution (4.55).

4.5 SPONTANEOUS EMISSION SPECTRA

In terms of the general model considered in sections 3.1–4.1, the *spontaneous emission spectrum* of the α^{th} reservoir, denoted $S_\alpha(\omega)$, is defined as the conditional probability density for finding a photon of frequency ω in the reservoir as $t \to \infty$, given that there was a photon in the system initially. Under ordinary circumstances we would calculate this using

$$\begin{aligned} S_\alpha(\omega) &= \frac{1}{\Delta\omega_{l_\alpha} \langle N(0)\rangle} \sum_{l_\alpha} \lim_{t\to\infty} \left\langle \psi(t) \right| b^\dagger_{\alpha l_\alpha} b_{\alpha l_\alpha} \left| \psi(t) \right\rangle \delta(\omega - \omega_{l_\alpha}) \\ &= \frac{1}{\Delta\omega_{l_\alpha} \sum_j |c_j(0)|^2} \sum_{l_\alpha} \lim_{t\to\infty} |c_{\alpha l_\alpha}(t)|^2 \delta(\omega - \omega_{l_\alpha}) \,, \end{aligned} \tag{4.58}$$

where $\Delta\omega_{l_\alpha}$ is the spacing of the environment modes, and where $|\psi(t)\rangle$ is given by (4.8). We should recall that $c_{\alpha l}(0) = 0$ by definition (see section 4.1). In our model, however, we have feedback – potentially of a very general kind – and this needs to be taken into account. If any photons find themselves oscillating back and forth, trapped in a feedback loop, then one might reasonably be worried that this limit might not exist. So what we really need to calculate is the probability of finding a photon of frequency ω outside the part of the reservoir where there are interactions with the system. Surprisingly, however, under certain circumstances – including the ones we are interested in – it turns out formula (4.58) does not require any corrections to account for the feedback loop. This is shown in appendix B.

Using the definition of the spectrum (4.58) along with equation (4.13), and taking the continuum limit, we get

$$S_\alpha(\omega) = \frac{1}{\sum_j |c_j(0)|^2} \sum_{jk} J_{\alpha jk}(\omega) \int_0^\infty \mathrm{d}t_1 \, e^{i(\omega_j - \omega)t_1} c_j^*(t_1) \int_0^\infty \mathrm{d}t_2 \, e^{-i(\omega_k - \omega)t_2} c_k(t_2) \,. \tag{4.59}$$

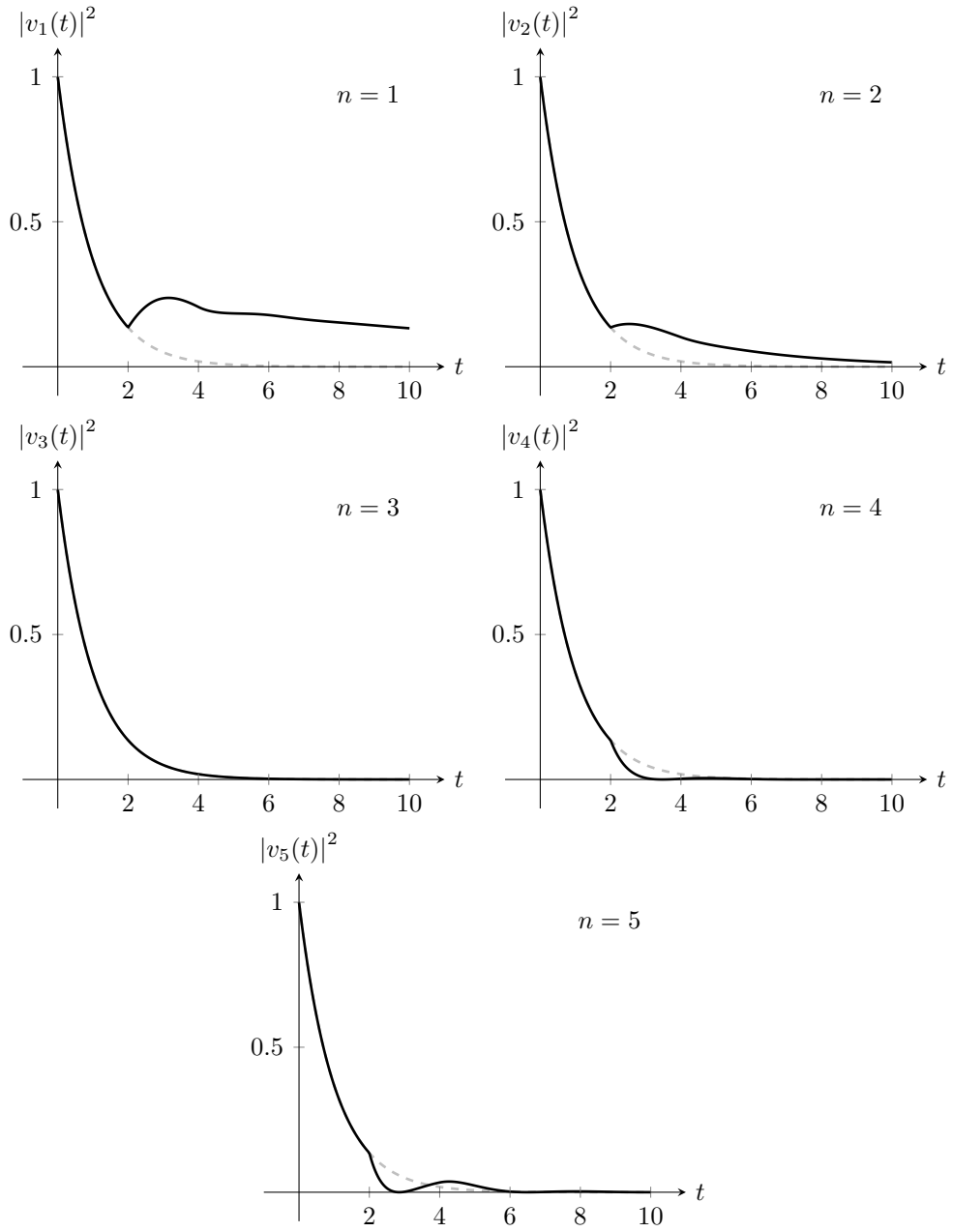


Figure 4.3: Plots of $|v_n(t)|^2$ for $n = 1, \ldots, N$. In this case $N = 5$. Parameters: $\gamma = 1/2$, $\tau = 2$, and $\phi = \pi$. The exponential decay curve $\exp(-2\gamma t)$ is plotted as a dashed grey line.

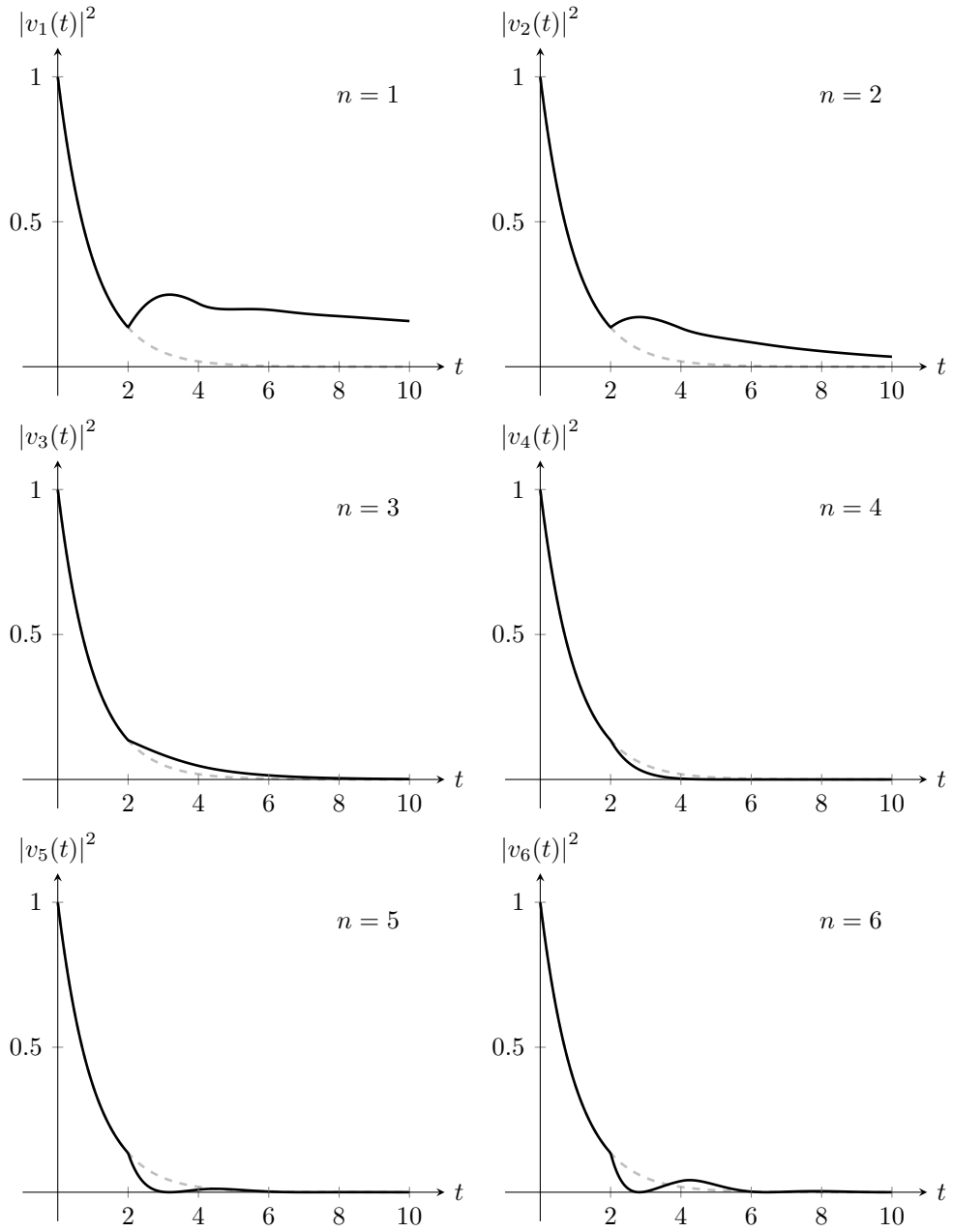


Figure 4.4: Plots of $|v_n(t)|^2$ for $n = 1, \ldots, N$. In this case $N = 6$. Parameters: $\gamma = 1/2$, $\tau = 2$, and $\phi = \pi$. The exponential decay curve $\exp(-2\gamma t)$ is plotted as a dashed grey line.

With the help of the Laplace transform, we can write this as

$$S_\alpha(\omega) = \frac{1}{\sum_j |c_j(0)|^2} \sum_{jk} J_{\alpha jk}(\omega) \tilde{c}_j^*(i\omega - i\omega_j) \tilde{c}_k(-i\omega + i\omega_k) \,. \tag{4.60}$$

Note that we can't make use of the Wiener–Khinchin theorem here, because the functions $c_j(t)$ are not wide-sense stationary processes for all times. Formulas for calculating spectra in non-stationary cases like spontaneous emission do already exist in quantum optics, but these formulas aren't directly applicable to the case at hand.[4,49]

Once again a simplification is possible if $\omega_j = \omega_0$ for all j. In this case we find that the emission spectrum is given by

$$S_\alpha(\omega + \omega_0) = \frac{1}{\sum_j |c_j(0)|^2} \sum_{jk} J'_{\alpha jk}(\omega) \tilde{c}_j^*(i\omega) \tilde{c}_k(-i\omega) \,. \tag{4.61}$$

This form of the emission spectrum, where ω represents the detuning from the resonant frequency ω_0, is in fact the form we are usually interested in.

In section 3.3 we defined the dissipation kernels, $F_{jk}(t, t_1)$, given by equation (3.13), which then appeared – in matrix form – as a memory kernel in (4.15). The Laplace transform of (4.15) is given by

$$s\tilde{\vec{c}}(s) - \vec{c}(0) = -i\,\mathscr{L}_t\left(\Delta(t)\vec{c}(t)\right)(s) - \mathscr{L}_t\left(\int_0^t \mathrm{d}t_1\, F(t, t_1)\vec{c}(t_1)\right)(s) \,. \tag{4.62}$$

The Laplace transforms in (4.62) are not, in general, straightforward to evaluate. But there are two simplifications available to us. Firstly, as mentioned in section 4.1, $\Delta(t)$ is not expected – based on the physics – to be time-dependent. Secondly, as mentioned in section 3.3, we can always choose our rotating frame such that the dissipation kernels can be written in the time-homogeneous form (3.17), in which case we have $F(t, t_1) = F(t - t_1)$. With these modifications, equation (4.62) becomes

$$s\tilde{\vec{c}}(s) - \vec{c}(0) = -i\Delta\tilde{\vec{c}}(s) - \tilde{F}(s)\tilde{\vec{c}}(s) \,. \tag{4.63}$$

Equation (4.63) is easily solved for $\tilde{\vec{c}}(s)$. Substituting $-i\omega$ for s in the solution, we find

$$\left(-i\omega I + i\Delta + \tilde{F}(-i\omega)\right)\tilde{\vec{c}}(-i\omega) = \vec{c}(0) \,. \tag{4.64}$$

This system of linear equations can be solved either analytically or numerically (as appropriate) for every ω of interest. Combined with the frequency-shifted spectral density (3.19), also evaluated for each ω, the spectrum (4.61) can be calculated. This constitutes a general algorithm for calculating spontaneous emission spectra in the systems we're looking at.

4.6 SIMPLE EXAMPLES OF SPONTANEOUS EMISSION SPECTRA

Let's return to the simple feedback loop treated in section 4.2. In this case the frequency-shifted spectral density is given by equation (4.16); the differential equation for $c(t)$ is (4.19) and its Laplace transform is (4.24). The frequency-shifted emission spectrum is thus given by

$$\begin{aligned} S(\omega+\omega_0) &= \frac{1}{|c(0)|^2}\left(\frac{\gamma}{\pi}+\frac{\eta}{\pi}\cos(\omega\tau+\phi)\right)\left|\frac{c(0)}{-i\omega+\gamma+\eta e^{i(\omega\tau+\phi)}}\right|^2 \\ &= \frac{1}{\pi}\frac{\gamma+\eta\cos(\omega\tau+\phi)}{\omega^2+\gamma^2+\eta^2+2\gamma\eta\cos(\omega\tau+\phi)-2\omega\eta\sin(\omega\tau+\phi)}\,. \end{aligned} \tag{4.65}$$

Examples of this spectrum are shown in figure 4.5. It follows from $\gamma \geq \eta$ that the spectrum is always positive, and of course if $\eta = 0$ the spectrum becomes a standard Lorentzian. The formula (4.65) for the spectrum has a removable singularity at $\omega = 0$ when there is destructive interference between the input and output, for example for $\gamma = \eta$ and $\phi = \pi$.

A slightly more complicated example is a cavity QED system – much along the lines of the damped Jaynes–Cummings model[50,51] – similar to the simple feedback loop except for the addition of a two-state "atom" coupled to the cavity. The atomic energy levels are separated by a frequency $\Delta + \omega_0$, where ω_0 is once again the resonant frequency of the cavity. We now have two decay channels – two reservoirs – corresponding respectively to the spontaneous emission from the atom, and to the decay of the cavity. Only the cavity emits into a feedback reservoir: the atom exhibits standard Markovian damping. As per usual, we assume a system–environment coupling of the form (3.5), where the frequency-shifted spectral density for the cavity's interaction with the feedback reservoir is

$$J_c'(\omega) = \frac{\gamma_c}{\pi}\left(1+\cos(\omega\tau+\phi)\right)\,, \tag{4.66}$$

whereas the corresponding quantity for the atom is simply

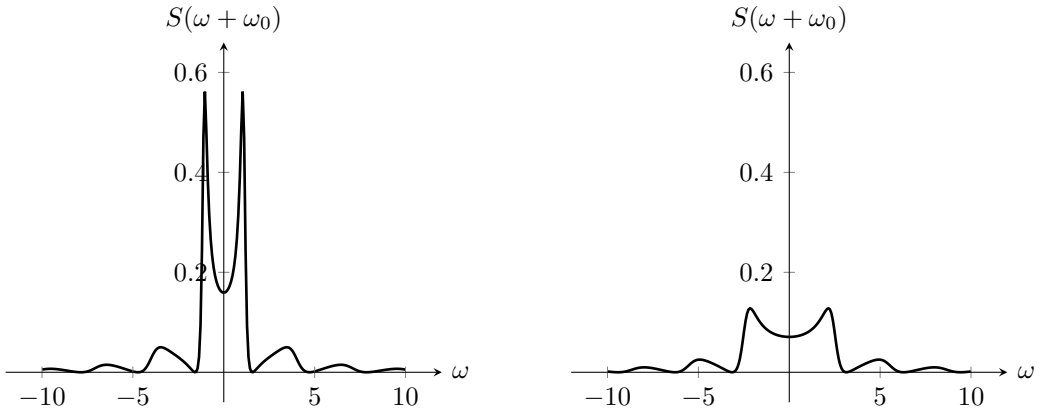
$$J_a'(\omega) = \frac{\gamma_a}{\pi}\,. \tag{4.67}$$

In the single-excitation manifold, this model is described by the coupled differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}c_c(t) = -igc_a(t) - \gamma_c\left(c_c(t)+\theta(t-\tau)e^{i\phi}c(t-\tau)\right)\,, \tag{4.68}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c_a(t) = -igc_c(t) - i\Delta c_a(t) - \gamma_a c_a(t)\,, \tag{4.69}$$

where $c_c(t)$ and $c_a(t)$ are the amplitudes of the singly-excited cavity and atom states respectively, and g parametrises the strength of the coupling between the cavity and the



(a) $\phi = 0$ (b) $\phi = \pi$

Figure 4.5: Spontaneous emission spectra for the simple feedback loop (figure 3.1), with $\gamma = 1$, $\eta = 1$, and $\tau = 2$ (in arbitrary units), for two different values of ϕ. In 4.5b, there is a removable singularity at $\omega = 0$ (not shown).

atom. Taking the Laplace transforms of equations (4.68) and (4.69) gives

$$s\tilde{c}_c(s) - c_c(0) = -ig\tilde{c}_a(s) - \gamma_c\left(1 + e^{-\tau s + i\phi}\right)\tilde{c}_c(s)\,, \tag{4.70}$$

$$s\tilde{c}_a(s) - c_a(0) = -ig\tilde{c}_c(s) - i\Delta\tilde{c}_a(s) - \gamma_a\tilde{c}_a(s)\,. \tag{4.71}$$

Assuming $c_a(t) = 0$, we can now evaluate the spectrum of the light emitted from the feedback reservoir coupled to the cavity. Solving (4.71) gives

$$\tilde{c}_a(s) = \frac{-ig}{s + i\Delta + \gamma_a}\tilde{c}_c(s)\,. \tag{4.72}$$

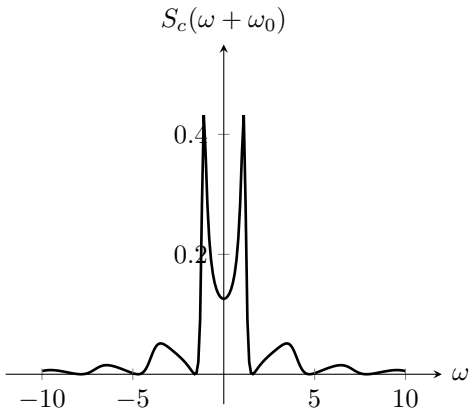
Substituting this into (4.70) leads to

$$\tilde{c}_c(s) = c_c(0)\left(s + \frac{g^2}{s + i\Delta + \gamma_a} + \gamma_c + \gamma_c e^{-\tau s + i\phi}\right)^{-1}. \tag{4.73}$$

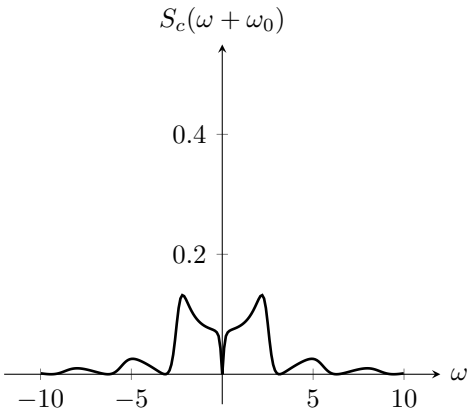
Finally, we get:

$$S_c(\omega+\omega_0) = \frac{1}{\pi}\left(\gamma_c + \gamma_c\cos(\omega\tau + \phi)\right)\left|\left(-i\omega + \frac{g^2}{-i\omega + i\Delta + \gamma_a} + \gamma_c + \gamma_c e^{i(\omega\tau+\phi)}\right)^{-1}\right|^2. \tag{4.74}$$

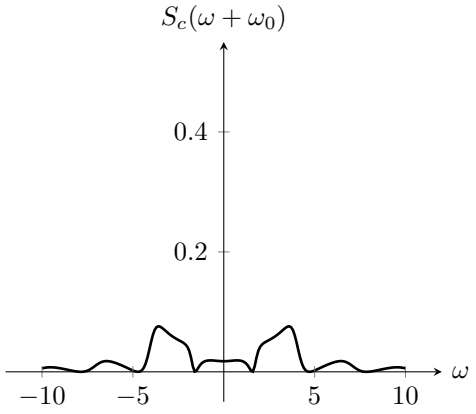
Some examples are shown in figure 4.6. The spontaneous emission spectrum for a coupled atom and cavity without delay has been calculated previously by Carmichael.[4]



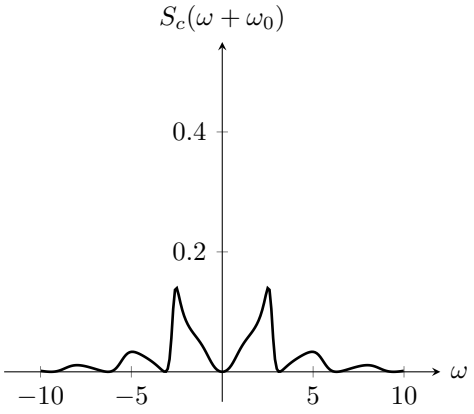
(a) $\tau = 2$, $g = 1/2$, $\phi = 0$



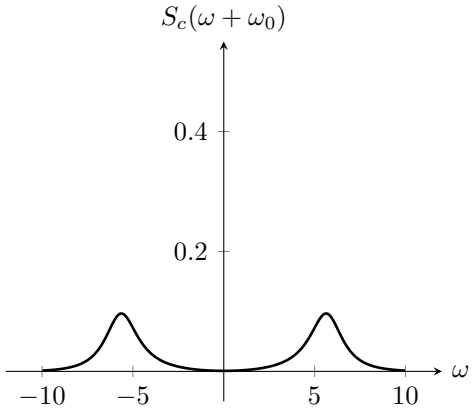
(b) $\tau = 2$, $g = 1/2$, $\phi = \pi$



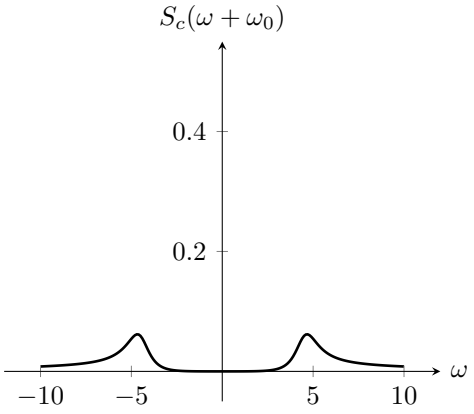
(c) $\tau = 2$, $g = 2$, $\phi = 0$



(d) $\tau = 2$, $g = 2$, $\phi = \pi$



(e) $\tau = 1/4$, $g = 5$, $\phi = 0$



(f) $\tau = 1/4$, $g = 5$, $\phi = \pi$

Figure 4.6: Spectra of the light emitted from the cavity in a simple cavity QED model, with $\gamma_c = \gamma_a = 1$ and $\Delta = 0$ (in arbitrary units), for various values of τ, g, and ϕ.

4.7 A CAVITY QED SYSTEM

In order to demonstrate what a more complicated calculation using these techniques looks like, we will now consider a cavity QED system comprising two identical subsystems; it would be straightforward to generalise this to many non-identical subsystems. Each subsystem consists of a two-state atom coupled on resonance to a cavity that supports two counter-propagating modes. The system, which is depicted schematically in figure 4.7, is inspired by the work of Mirza et al.[52]—our calculation generalises theirs in the sense that it does not neglect propagation delays between the subsystems. We can define two internal interaction Hamiltonians:

$$H_{S1} = g\left(\left(a_{1L}^\dagger + a_{1R}^\dagger\right)a_{1A} + a_{1A}^\dagger\left(a_{1L} + a_{1R}\right)\right), \tag{4.75}$$

$$H_{S2} = g\left(\left(a_{2L}^\dagger + a_{2R}^\dagger\right)a_{2A} + a_{2A}^\dagger\left(a_{2L} + a_{2R}\right)\right), \tag{4.76}$$

where a_{nL}, a_{nR}, and a_{nA} refer to the left- and right-propagating modes, and the atom of system n. As well as the usual Markovian damping of each atom and cavity mode, the two subsystems couple by way of a reservoir with a propagation delay, of exactly the type we considered in section 4.4. We assume an overall interaction of the form (3.5), and simply define the frequency-shifted spectral density for each system–reservoir interaction. Firstly, we have the Markovian damping, denoted by a subscript M:

$$J'_{MnA,nA,nA}(\omega) = \frac{\gamma_A}{\pi}, \qquad J'_{MnL,nL,nL}(\omega) = J'_{MnR,nR,nR}(\omega) = \frac{\gamma_C}{\pi}. \tag{4.77}$$

Secondly, the feedback reservoir, denoted by a subscript F, that couples the two subsystems:

$$\begin{aligned} J'_{FL,nL,nL}(\omega) &= J'_{FR,nR,nR}(\omega) = \frac{\gamma_F}{\pi}, \\ J'_{FL,1L,2L}(\omega) &= J'_{FR,2R,1R}(\omega) = \frac{\gamma_F}{\pi}e^{i(\omega\tau+\phi)}, \\ J'_{FL,2L,1L}(\omega) &= J'_{FR,1R,2R}(\omega) = \frac{\gamma_F}{\pi}e^{-i(\omega\tau+\phi)}, \end{aligned} \tag{4.78}$$

with $\tau \geq 0$. Note that the second two lines in (4.78) are complex conjugates—this follows from the definition of the spectral densities. In (4.77) and (4.78), as before, the subscripts L, R, and A refer to the left- and right-propagating fibre modes and the atoms, while $n = 1, 2$ labels the two subsystems. Thus, for example, the frequency-shifted spectral density $J'_{FL,1L,2L}(\omega)$ describes the coupling from the left-propagating mode of the second subsystem into the corresponding mode of the first subsystem, via the left-propagating modes of the feedback reservoir (that is, the fibre).

The various definitions (4.75)–(4.78) lead to a system of six coupled delay-differential equations, which can be written in vector form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{c}(t) = -i\Delta\vec{c}(t) - \Gamma_0\vec{c}(t) - \Gamma_\tau\theta(t-\tau)\vec{c}(t-\tau), \tag{4.79}$$

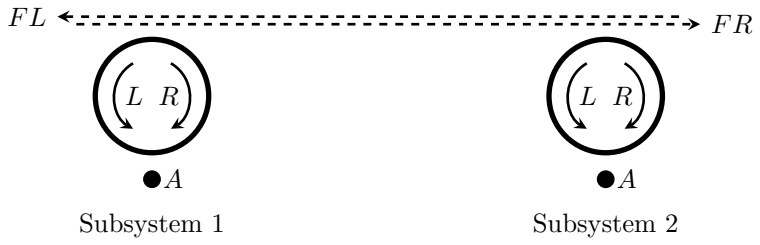


Figure 4.7: Schematic depiction of two identical cavity QED systems, each comprising a microtoroid resonator with two counter-propagating modes, coupled on resonance to a two-state atom (black dots labelled A). The systems interact via a fibre (denoted by the dashed lines) with a time delay. The left- and right-propagating modes of the each resonator couple, respectively, to the left- and right-propagating modes of the fibre.

where

$$\vec{c}(t) = \left(c_{1L}(t), c_{1R}(t), c_{1A}(t), c_{2L}(t), c_{2R}(t), c_{2A}(t)\right)^T , \tag{4.80}$$

and where we have defined three matrices:

$$\Delta_{jk} = g\left(e_{13} + e_{23} + e_{31} + e_{32} + e_{46} + e_{56} + e_{64} + e_{65}\right) , \tag{4.81}$$

$$\Gamma_{0jk} = \left(\gamma_C + \gamma_F\right)\left(e_{11} + e_{22} + e_{44} + e_{55}\right) + \gamma_A\left(e_{33} + e_{66}\right) , \tag{4.82}$$

$$\Gamma_{\tau jk} = \theta(\tau) 2\gamma_F e^{i\phi}\left(e_{14} + e_{52}\right) . \tag{4.83}$$

Here e_{jk} are basis matrices, defined by $\left(e_{jk}\right)_{lm} = \delta_{jl}\delta_{km}$.

It would be simple at this point to integrate (4.79) numerically, but simply inspecting the dynamics is not a very interesting enterprise. On the other hand, we are now also in a position to calculate spontaneous emission spectra for the cavity QED system. The spectra of interest here are those of the reservoirs FL and FR, which together form the feedback loop. We can easily obtain expressions for these spectra, using the formula (4.61) and the spectral densities (4.78):

$$S_{FL}(\omega + \omega_0) = \frac{1}{\langle N(0)\rangle}\frac{\gamma_F}{\pi}\left|\tilde{c}_{1L}(-i\omega) + e^{i(\omega\tau+\phi)}\tilde{c}_{2L}(-i\omega)\right|^2 , \tag{4.84}$$

$$S_{FR}(\omega + \omega_0) = \frac{1}{\langle N(0)\rangle}\frac{\gamma_F}{\pi}\left|e^{i(\omega\tau+\phi)}\tilde{c}_{1R}(-i\omega) + \tilde{c}_{2R}(-i\omega)\right|^2 . \tag{4.85}$$

The Laplace transform of the system of equations (4.79) satisfies equation (4.64) with $\tilde{F}(-i\omega) = \Gamma_0 + \Gamma_\tau e^{i\omega\tau}$. Solving (4.64) numerically for each ω yields the vector of Laplace-transformed amplitudes $\tilde{\vec{c}}(-i\omega)$, and substituting these amplitudes into (4.84) and (4.85) yields the spectra—examples are plotted in figure 4.8. To gain a full understanding of the physical content of these spectra, as well as those shown earlier in figures 4.5 and 4.6, requires further research.

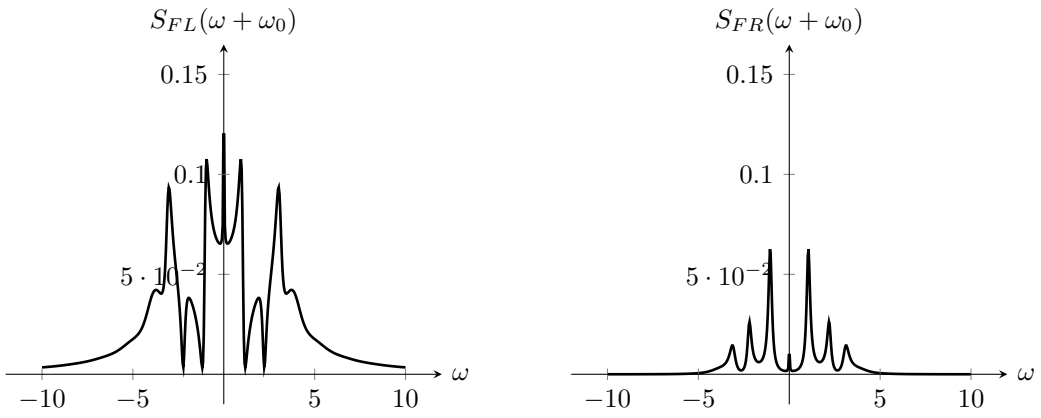


Figure 4.8: Spectra of the light coming out the ends of the fibre in figure 4.7, with $g = 2$, $\gamma_A = 1/2$, $\gamma_C = 0$, $\gamma_F = 1$, $\tau = 2$, and $\phi = 0$ (in arbitrary units). The excitation is initially localised in the $1L$ mode.

4.8 HOW DO WE DEAL WITH MULTIPLE EXCITATIONS?

To go beyond the treatment of the model presented in this chapter, we have to consider what happens when there is more than one excitation in the combined system. Unfortunately, the preceding analysis cannot be extended to consider multiple excitations. To see why, we consider a system with two excitations. To avoid having to deal with a profusion of confusing indices, I will use the simple Hamiltonian (3.35), which describes a single system degree of freedom and a unipartite environment.

In the case where the system contains at most N excitations, the density operator (4.6) generalises to

$$\rho(t) = \sum_{n=0}^{N} p_n \left|\psi_n(t)\right\rangle \left\langle\psi_n(t)\right| , \qquad \sum_{n=0}^{N} p_n = 1 , \tag{4.86}$$

where the pure state $|\psi_n(t)\rangle$ is a normalised superposition of all the states with at most n excitations in the combined system. We have $|\psi_0(t)\rangle = |0\rangle$, of course; and $|\psi_1(t)\rangle$ corresponds to (4.8), although it is convenient to use slightly different notation:

$$|\psi_1(t)\rangle = c_{1,0}\,|0\rangle + c_{1,1}(t)a^\dagger\,|0\rangle + \sum_{l_1} c_{1,0;l_1}(t)b_{l_1}^\dagger\,|0\rangle \ . \tag{4.87}$$

The pure state with at most two excitations follows the same pattern:

$$\begin{aligned}|\psi_2(t)\rangle = {} & c_{2,0}\,|0\rangle + c_{2,1}(t)a^\dagger\,|0\rangle + \sum_{l_1} c_{2,0;l_1}(t)b_{l_1}^\dagger\,|0\rangle + c_{2,2}(t)a^\dagger a^\dagger\,|0\rangle \\ & + \sum_{l_1} c_{2,1;l_1}(t)b_{l_1}^\dagger a^\dagger\,|0\rangle + \sum_{l_1}\sum_{l_2\le l_1} c_{2,0;l_1,l_2}(t)b_{l_1}^\dagger b_{l_2}^\dagger\,|0\rangle \ .\end{aligned} \tag{4.88}$$

This expansion is valid regardless of whether the algebra of the operators is bosonic or fermionic. It's easy to see how this treatment might be further extended to take into

account three or more excitations. Consider the state with label n that has m excitations in total, of which m_S are in the system and m_E are in the environment. This state will have a corresponding amplitude $c_{n,m_S;l_1,l_2,\ldots,l_{m_E}}(t)$, where $l_1 \geq l_2 \geq \cdots \geq l_{m_E}$. Just as we did for the single-excitation manifold in section 4.1, we can derive a system of coupled differential equations for the amplitudes by substituting the corresponding pure state $|\psi_n(t)\rangle$ into the Schrödinger equation generated by the Hamiltonian (3.35). Because the Hamiltonian conserves total excitation number, only amplitudes with equal $m = m_S + m_E$ will couple to one another.

The problem, of course, is that the equations of motion for the amplitudes are utterly intractable for $n > 1$. Indeed, this very intractability is the reason that the theory of open quantum systems exists in the first place! In chapters 5–7, we will examine a collection of ideas regarding how we might move our investigation beyond singly-excited systems.

5

THE LIMITATIONS OF PERTURBATION THEORY

Consider the model of spontaneous emission we studied in chapter 4. In particular, consider a single, undriven, singly-excited system degree of freedom emitting into a bosonic vacuum reservoir. As we established in chapter 3, this system can be described by the interaction picture Hamiltonian

$$H_{SE}(t) = \sigma_+ B(t) + B^\dagger(t)\sigma_- \,, \tag{5.1}$$

where, because we are limiting ourselves to the case of a single excitation, we have used the usual notation for a two-state system. Note that the subscript I has been dropped from the Hamiltonian (5.1), as per the discussion at the end of section 3.2. We have, already derived an exact, time-local master equation for this system in section 4.3, namely (4.33). In the present case, this master equation reduces to the simpler form given by (4.35):

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \mathcal{L}(t)\rho_S(t) = \gamma(t)\left[\sigma_-\rho_S(t), \sigma_+\right] + \gamma^*(t)\left[\sigma_-, \rho_S(t)\sigma_+\right] . \tag{5.2}$$

The time-dependent decay rate is, as before, given by

$$\gamma(t) = -\frac{\dot{v}(t)}{v(t)} \,, \tag{5.3}$$

with the Green's function $v(t)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\int_0^t \mathrm{d}t_1\, f(t-t_1)v(t_1) \,, \qquad v(0) = 1 \,. \tag{5.4}$$

The dissipation kernel is given by

$$f(t-t_1) = \left[B(t), B^\dagger(t)\right] = \int_{-\infty}^{\infty} \mathrm{d}\omega\, J'(\omega)e^{-i\omega(t-t_1)} \,. \tag{5.5}$$

It turns out that the simplicity of this system allows us to carry out a perturbation expansion to much higher order than would otherwise be practical. As such, we can use this system to evaluate the usefulness of perturbative techniques when it comes to the delayed coherent feedback we aim to model.

Suppose, therefore, that the frequency-shifted spectral density is given by equation (4.16), which describes a coherent feedback loop with a propagation delay $\tau \geq 0$. This spectral density leads to the memory kernel (4.18), which we repeat here for convenience:

$$f(t-t_1) = \eta e^{i\phi}\delta(t-\tau-t_1) + 2\gamma\delta(t-t_1) + \eta e^{-i\phi}\delta(t+\tau-t_1) \,, \tag{5.6}$$

where $\eta \leq \gamma$. The exact solution for $v(t)$ with this dissipation kernel is given by (4.34). In section 2.3, we noted that the validity of the Born and Markov approximations relies on any excitations created by the system in the environment decaying in a time very short compared to the relaxation time of the system. In the case at hand, the characteristic time-scale on which the system evolves is given, roughly, by γ^{-1}; the corresponding quantity for the reservoir is τ. Thus, the Born and Markov approximations are valid in this system only when $\tau \ll \gamma^{-1}$, or $\gamma\tau \ll 1$. When this condition does not hold, we would expect the Born approximation to fail spectacularly. Moreover, we would expect perturbative corrections at higher orders in the system–environment coupling strength (recall that the Born and Markov approximations are valid to second order) to ameliorate the problem only slightly. In this chapter I will present two examples that I think vividly illustrate this issue.

5.1 A NAÏVE APPROACH: THE BORN APPROXIMATION

The Born approximation amounts to the assumption that the system does not cause any excitations in the reservoir on timescales that we are able to resolve. As I outlined above, this is manifestly untrue for systems that display coherent feedback, because the excitations that the system creates in the reservoir are fed back into the system! Nonetheless, in this section I will present a specific illustration of failure of the Born approximation, and compare the results of this flawed derivation with the the exact results we obtained in chapter 4.

Consider the master equation in the Born approximation (2.37). If we substitute the Hamiltonian (3.9) into this master equation, assuming for simplicity's sake that the internal Hamiltonian of the system is given by $H_{SI}(t) = 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\sum_{jk}\int_0^t \mathrm{d}t_1 \,\mathrm{tr}_E\left[a_j^\dagger B_j(t) + B_j^\dagger(t)a_j, \left[a_k^\dagger B_k(t_1) + B_k^\dagger(t_1)a_k, \rho_S(t_1)\otimes\rho_E\right]\right], \tag{5.7}$$

where ρ_E is given by (3.7) and $B_j(t)$ is given by (3.12). If we expand the commutators in (5.7) and evaluate the trace over the environment degrees of freedom, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \sum_{jk}\int_0^t \mathrm{d}t_1 \Big(&\left(F_{jk}(t,t_1) + G_{jk}(t,t_1)\right)\left[a_k\rho_S(t_1), a_j^\dagger\right] \\ &+ \left(F_{jk}^*(t,t_1) + G_{jk}^*(t,t_1)\right)\left[a_j, \rho_S(t_1)a_k^\dagger\right] \\ &+ G_{jk}(t,t_1)\left[a_j^\dagger, \rho_S(t_1)a_k\right] + G_{jk}^*(t,t_1)\left[a_k^\dagger\rho_S(t_1), a_j\right]\Big),\end{aligned} \tag{5.8}$$

where $F_{jk}(t,t_1)$ and $G_{jk}(t,t_1)$ are, respectively, the dissipation and noise kernels given by equations (3.17) and (3.18). In a 2013 paper, Zhang et al. use the Born approximation

to derive an equation equivalent to (5.8).[53] They note that it is unknown what criteria the kernels $F_{kl}(t,t_1)$ and $G_{jk}(t,t_1)$ must satisfy to guarantee complete positivity of the evolution. There are, however, some simple cases for which complete positivity has been proven, such as the case of an environment with a Lorentzian spectral density—this is the case studied by Zhang et al. In a great many cases – the Markovian case, in particular – the Born approximation is a good approximation, but when it comes to the systems we're interested in the master equation (5.8) is wrong in dramatic fashion. The purpose of this section is to demonstrate that the Born approximation is not an appropriate assumption when dealing with delayed coherent feedback.

To simplify our illustration, we will consider the model of single system degree of freedom given at the start of this chapter. We have assumed a vacuum reservoir, so the noise kernel is given by $g(t-t_1)=0$. Under these conditions, the master equation (5.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \int_0^t \mathrm{d}t_1 \left(f(t-t_1)\left[\sigma_-\rho_S(t_1),\sigma_+\right] + f^*(t-t_1)\left[\sigma_-,\rho_S(t_1)\sigma_+\right]\right) . \tag{5.9}$$

In general a singly-excited system can, at time t, be described by a reduced density matrix

$$\rho_S(t) = y(t)\left|1\right\rangle\left\langle 1\right| + x(t)\left|1\right\rangle\left\langle 0\right| + x^*(t)\left|0\right\rangle\left\langle 1\right| + (1-y(t))\left|0\right\rangle\left\langle 0\right| . \tag{5.10}$$

The exact time-evolution for this system follows immediately from (4.26), and can be described in terms of the function $v(t)$, which satisfies (5.4). In terms of this function, the exact solution – the solution we *should* get – is given by $x(t)=v(t)x(0)$ and $y(t)=|v(t)|^2y(0)$. However, the master equation (5.9) gives a rather different result. If we substitute (5.10) into this master equation, we obtain two differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\int_0^t \mathrm{d}t_1\, f(t-t_1)x(t_1)\,, \tag{5.11}$$

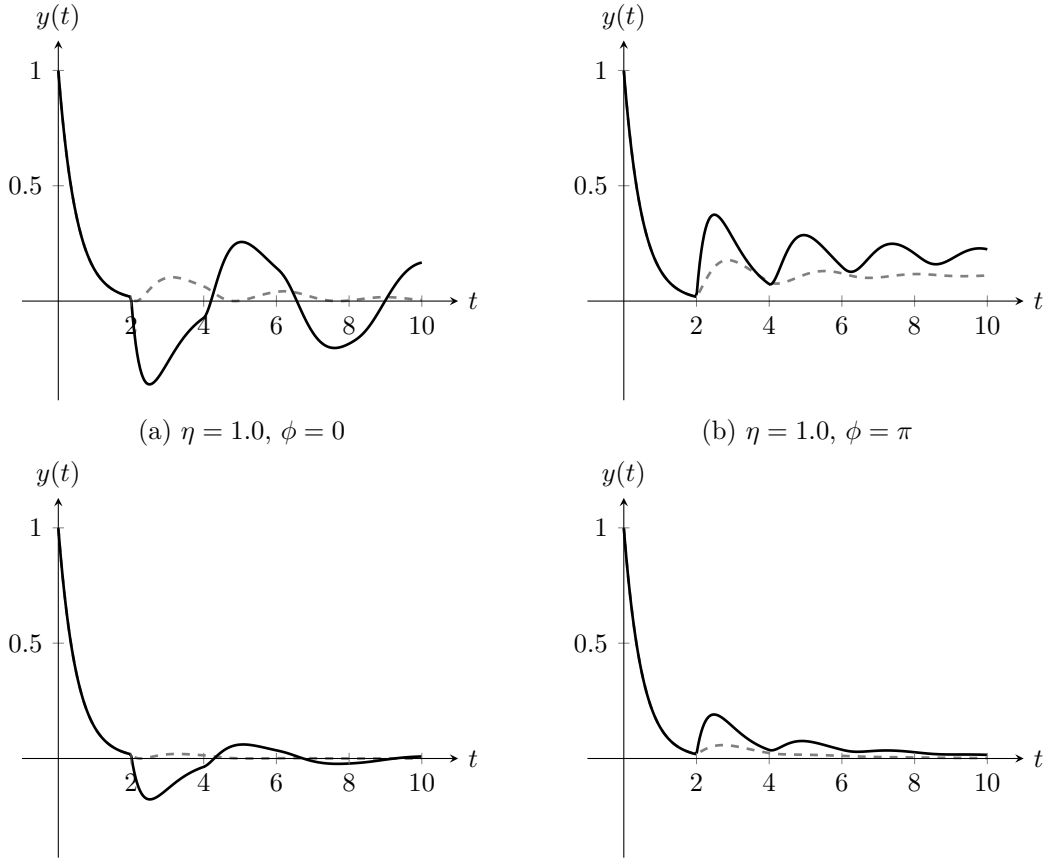
$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -\int_0^t \mathrm{d}t_1\, \left(f(t-t_1)+f^*(t-t_1)\right)y(t_1)\,. \tag{5.12}$$

While (5.11) does in fact agree with the exact solution, (5.12) in general does not. Of course, (5.12) reproduces the exact solution in the Markovian case.

As an example, consider the feedback loop we discussed above. The dissipation kernel for this system is given by equation (5.5), and substituting this kernel into (5.12) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -2\gamma y(t) - 2\eta\cos(\phi)y(t-\tau)\,. \tag{5.13}$$

Several examples of the function $y(t)$, the mean number of excitations in the system, are plotted in figure 5.1. The difference between the exact results and the results derived from the Born approximation is clear to see. In particular, when $\phi=0$ the Born approximation yields a solution that is not even physically consistent, with $y(t)<0$ for some values of t.



(a) $\eta = 1.0$, $\phi = 0$ (b) $\eta = 1.0$, $\phi = \pi$

(c) $\eta = 0.5$, $\phi = 0$ (d) $\eta = 0.5$, $\phi = \pi$

Figure 5.1: Examples of $y(t)$, with $\gamma = 1$ and $\tau = 2$ (in arbitrary units), and with $y(0) = 1$, for several values of η and ϕ. The solid lines represent the solution derived from the Born approximation (5.13), while the dashed lines represent the exact solution.

5.2 A TIME-LOCAL PERTURBATION EXPANSION

It is not my purpose here to provide an exhaustive review of the perturbative techniques used in the field of open quantum systems, or even of the techniques specifically used to model non-Markovian open quantum systems. On the contrary, what I will present below is a treatment of spontaneous emission using one particular method, namely a time-local perturbation expansion. The method I will use has a lot in common with the time-convolutionless projection operator technique, which is a method of obtaining a time-local perturbation expansion of the exact master equation that is based on the Nakajima–Zwanzig projection operator technique. Breuer and Petruccione provide a good exposition of this method in their textbook.[7] Here I'm providing what I think is a simpler derivation, which does not involve projection operators. The formalism I present in section 5.2.1 is very similar to that developed by Fleming and Hu.[54]

5.2.1 *Derivation of the perturbation expansion*

We define the time-dependent Liouvillian super-operator $L(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \epsilon L(t)\rho(t) = -i\left[H(t), \rho(t)\right] , \tag{5.14}$$

where the Hamiltonian is of the form (3.9), $H(t) = H_S(t) + H_{SE}(t)$, and where ϵ is an expansion parameter. We can write the solution to (5.14) in the form of (2.9), and it's useful in this case to expand the Dyson series explicitly:

$$\begin{aligned}\rho(t) &= \mathrm{T}_{\leftarrow} \exp\left(\epsilon \int_{t_0}^{t} \mathrm{d}t_1\, L(t_1)\right) \rho(t_0) \\ &= \left(1 + \epsilon \int_{t_0}^{t} \mathrm{d}t_1\, L(t_1) + \epsilon^2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\, L(t_1)L(t_2) \right. \\ &\qquad \left. + \epsilon^3 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3\, L(t_1)L(t_2)L(t_3) + \cdots \right) \rho(t_0) \,. \end{aligned} \tag{5.15}$$

We assume that the initial state of the system is in the separable form (3.6), so that we can write $\rho(t_0) = \rho_S(t_0) \otimes \rho_E$. Therefore, the reduced density matrix can be derived from (5.15):

$$\rho_S(t) = \Phi(t, t_0)\rho_S(t_0) \,, \tag{5.16}$$

where the reduced system propagator is given by

$$\Phi(t, t_0) = \left\langle \mathrm{T}_{\leftarrow} \exp\left(\epsilon \int_{t_0}^{t} \mathrm{d}t_1\, L(t_1)\right) \right\rangle_E \,. \tag{5.17}$$

Supposing that there exists an inverse of the reduced propagator, we can express it using a Neumann series

$$\Phi^{-1}(t, t_0) = \sum_{n=0}^{\infty} \left(1 - \Phi(t, t_0)\right)^n \,, \tag{5.18}$$

and write the master equation for $\rho_S(t)$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \mathcal{L}(t)\rho_S(t) \,, \tag{5.19}$$

where

$$\mathcal{L}(t) = \dot{\Phi}(t, t_0)\Phi^{-1}(t, t_0) \,. \tag{5.20}$$

We can now perform a perturbation expansion of the generator $\mathcal{L}(t)$, and thereby the master equation, by expanding in powers of ϵ:

$$\mathcal{L}(t) = \sum_{n=0}^{\infty} \epsilon^n \mathcal{L}_n(t)\,. \tag{5.21}$$

We can also make the expansions

$$\Phi(t,t_0) = \sum_{n=0}^{\infty} \epsilon^n \Phi_n(t,t_0)\,, \tag{5.22}$$

and

$$\dot{\Phi}(t,t_0) = \sum_{n=1}^{\infty} \epsilon^n \dot{\Phi}_n(t,t_0)\,, \tag{5.23}$$

where

$$\Phi_n(t,t_0) = \int_{t_0}^{t} \mathrm{d}t_1 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n \left\langle L(t_1)\cdots L(t_n)\right\rangle_E\,, \tag{5.24}$$

and thus

$$\dot{\Phi}_n(t,t_0) = \int_{t_0}^{t} \mathrm{d}t_1 \cdots \int_{t_0}^{t_{n-2}} \mathrm{d}t_{n-1} \left\langle L(t)L(t_1)\cdots L(t_{n-1})\right\rangle_E\,. \tag{5.25}$$

Assuming the odd moments of L in the environment vanish – as they do in the thermal reservoir (3.7), and indeed in any Gaussian environment – we find that for n odd $\Phi_n(t,t_0) = \dot{\Phi}_n(t,t_0) = 0$, and equation (5.22) becomes

$$\Phi(t,t_0) = 1 + \epsilon^2\Phi_2(t,t_0) + \epsilon^4\Phi_4(t,t_0) + \dots\,. \tag{5.26}$$

Simply expanding the inverse propagator to second order, $\Phi^{-1}(t,t_0) = 1 - \epsilon^2\Phi_2(t,t_0) + \dots$, allows us to write down the second- and fourth-order contributions. The tedious process of obtaining the higher-order contributions can be automated with the aid of a computer algebra system: in this case I used SymPy.[55] As such, we obtain

$$\begin{aligned}
\mathcal{L}_2(t) &= \dot{\Phi}_2(t,t_0) \\
&= \int_{t_0}^{t} \mathrm{d}t_1 \left\langle L(t)L(t_1)\right\rangle_E\,, &(5.27)\\
\mathcal{L}_4(t) &= -\dot{\Phi}_2(t,t_0)\Phi_2(t,t_0) + \dot{\Phi}_4(t,t_0) \\
&= -\left(\int_{t_0}^{t} \mathrm{d}t_1 \left\langle L(t)L(t_1)\right\rangle_E\right)\left(\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \left\langle L(t_1)L(t_2)\right\rangle_E\right) \\
&\quad + \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3 \left\langle L(t)L(t_1)L(t_2)L(t_3)\right\rangle_E\,, &(5.28)\\
\mathcal{L}_6(t) &= \dot{\Phi}_2(t,t_0)\Phi_2(t,t_0)^2 - \dot{\Phi}_2(t,t_0)\Phi_4(t,t_0) - \dot{\Phi}_4(t,t_0)\Phi_2(t,t_0) + \dot{\Phi}_6(t,t_0)\,, &(5.29)
\end{aligned}$$

$$\begin{aligned}\mathcal{L}_8(t) = {} & \dot{\Phi}_2(t,t_0)\Phi_2(t,t_0)\Phi_4(t,t_0) - \dot{\Phi}_2(t,t_0)\Phi_2(t,t_0)^3 + \dot{\Phi}_2(t,t_0)\Phi_4(t,t_0)\Phi_2(t,t_0) \\ & - \dot{\Phi}_2(t,t_0)\Phi_6(t,t_0) + \dot{\Phi}_4(t,t_0)\Phi_2(t,t_0)^2 - \dot{\Phi}_4(t,t_0)\Phi_4(t,t_0) \\ & - \dot{\Phi}_6(t,t_0)\Phi_2(t,t_0) + \dot{\Phi}_8(t,t_0)\,, \end{aligned} \tag{5.30}$$

and so on, up to the highest order the computer in question can handle. The second-order contribution to the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \epsilon^2\mathcal{L}_2(t)\rho_S(t)\,, \tag{5.31}$$

with $\mathcal{L}_2(t)$ given by (5.27), is in fact the Redfield equation (2.38).

5.2.2 *The perturbation expansion for spontaneous emission*

Our aim here is to compare the master equation (5.2) with the results of the perturbation expansion developed above. It is often very difficult to evaluate the higher-order contributions to a perturbation expansion, but in the case of spontaneous emission from a singly-excited system it turns out that we can use our knowledge of the exact solution to evaluate the expansion much more easily than might otherwise be the case.

We can easily see from (5.2) that σ_+ and σ_- are eigenoperators of the generator $\mathcal{L}(t)$, so that we have

$$\mathcal{L}(t)\sigma_+ = -\gamma(t)\sigma_+ \qquad \text{and} \qquad \mathcal{L}(t)\sigma_- = -\gamma^*(t)\sigma_-\,. \tag{5.32}$$

We now want to know what happens when we act on a reduced density matrix with the even moments of L and then trace out the environment—this will lead to the perturbation expansion that we will compare to (5.32). It's straightforward to show the eigenoperator relations

$$L(t_1)L(t_2)\sigma_+ \otimes |0\rangle\langle 0| = -f(t_1-t_2)\sigma_+ \otimes |0\rangle\langle 0|\,, \tag{5.33}$$

$$L(t_1)L(t_2)\sigma_- \otimes |0\rangle\langle 0| = -f^*(t_1-t_2)\sigma_- \otimes |0\rangle\langle 0|\,. \tag{5.34}$$

From these relations we immediately find

$$\langle L(t_1)L(t_2)\cdots L(t_{2k-1})L(t_{2k})\rangle_E\,\sigma_+ = (-1)^k f(t_1-t_2)\cdots f(t_{2k-1}-t_{2k})\sigma_+\,, \tag{5.35}$$

$$\langle L(t_1)L(t_2)\cdots L(t_{2k-1})L(t_{2k})\rangle_E\,\sigma_- = (-1)^k f^*(t_1-t_2)\cdots f^*(t_{2k-1}-t_{2k})\sigma_-\,. \tag{5.36}$$

We therefore find that

$$\mathcal{L}_{2k}(t)\sigma_+ = -\gamma_{2k}(t)\sigma_+ \qquad \text{and} \qquad \mathcal{L}_{2k}(t)\sigma_- = -\gamma_{2k}^*(t)\sigma_-\,, \tag{5.37}$$

where $\gamma_{2k}(t)$ is obtained from the expansion of $\mathcal{L}_{2k}(t)$ by replacing each pair $L(t_j)L(t_k)$ with the correlation function $f(t_j-t_k)$. Comparing (5.32) and (5.37), we can see that the

time-dependent coefficient $\gamma(t)$ can be expanded perturbatively as

$$\gamma(t) = \sum_{k=0}^{\infty} \epsilon^{2k} \gamma_{2k}(t) \,. \tag{5.38}$$

We can now use equations (5.35) and (5.36) along with (5.38) to evaluate the contributions to $\gamma(t)$ at various orders in perturbation theory.

We define two functions $q_{2k}(t)$ and $p_{2k}(t)$ such that

$$\dot{\Phi}_{2k}(t,0)\sigma_+ = p_{2k}(t)\sigma_+ \qquad \text{and} \qquad \Phi_{2k}(t,0)\sigma_+ = q_{2k}(t)\sigma_+ \,. \tag{5.39}$$

It follows that

$$p_{2k}(t) = (-1)^k \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{2k-2}} \mathrm{d}t_{2k-1}\, f(t-t_1) \cdots f(t_{2k-2} - t_{2k-1}) \,, \tag{5.40}$$

$$\begin{aligned} q_{2k}(t) &= (-1)^k \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{2k-1}} \mathrm{d}t_{2k}\, f(t_1-t_2) \cdots f(t_{2k-1} - t_{2k}) \\ &= \int_0^t \mathrm{d}t_1\, p_{2k}(t_1) \,. \end{aligned} \tag{5.41}$$

Furthermore, we obtain the recursion relation

$$p_{2k}(t) = -\int_0^t \mathrm{d}t_1\, f(t-t_1) q_{2k-2}(t) \,. \tag{5.42}$$

Using the mutual recursion relations (5.41) and (5.42) along with the base case $q_0(t) = 1$, we can iteratively obtain the full series. Once we have $p_{2k}(t)$ and $q_{2k}(t)$, we can use the relation $\Phi^{-1}(t,0)\Phi(t,0)\sigma_+ = \gamma(t)\sigma_+$, along with the Neumann series (5.18), to obtain the time-dependent coefficient

$$\gamma(t) = -\left(\sum_{k=0}^{\infty} \epsilon^{2k} p_{2k}(t)\right) \sum_{n=0}^{\infty} \left(1 - \sum_{l=0}^{\infty} \epsilon^{2l} q_{2l}(t)\right)^n . \tag{5.43}$$

This result also follows directly from an iterative solution of equation (5.4) for $v(t)$, using $\gamma(t) = -\dot{v}(t)/v(t)$ as given in (5.3). To evaluate the perturbative contributions $\gamma_{2k}(t)$ we can use a computer algebra system to expand (5.43) in powers of the coupling strength, just as we did for the formal propagators to obtain equations (5.27)–(5.30).

The foregoing analysis describes spontaneous emission from a singly-excited system in general, but as it happens we are interested in feedback reservoirs of the kind described in section 4.2. As we have already established, such a reservoir, with a discrete time delay $\tau \geq 0$, is described by the frequency-shifted spectral density (4.16). Here, we use a modification of this spectral density:

$$J'(\omega) = \frac{1}{\epsilon^2}\left(\frac{\gamma_0}{\pi} + \frac{\eta}{\pi}\cos(\omega\tau + \phi)\right) , \tag{5.44}$$

where we have included a factor of ϵ^2 in the denominator to cancel the expansion parameters

in the final result. This leads to a dissipation kernel

$$f(t-t_1) = \frac{1}{\epsilon^2}\left(\eta e^{i\phi}\delta(t-\tau-t_1) + 2\gamma_0\delta(t-t_1) + \eta e^{-i\phi}\delta(t+\tau-t_1)\right) . \tag{5.45}$$

Substituting (5.45) into (5.42) yields

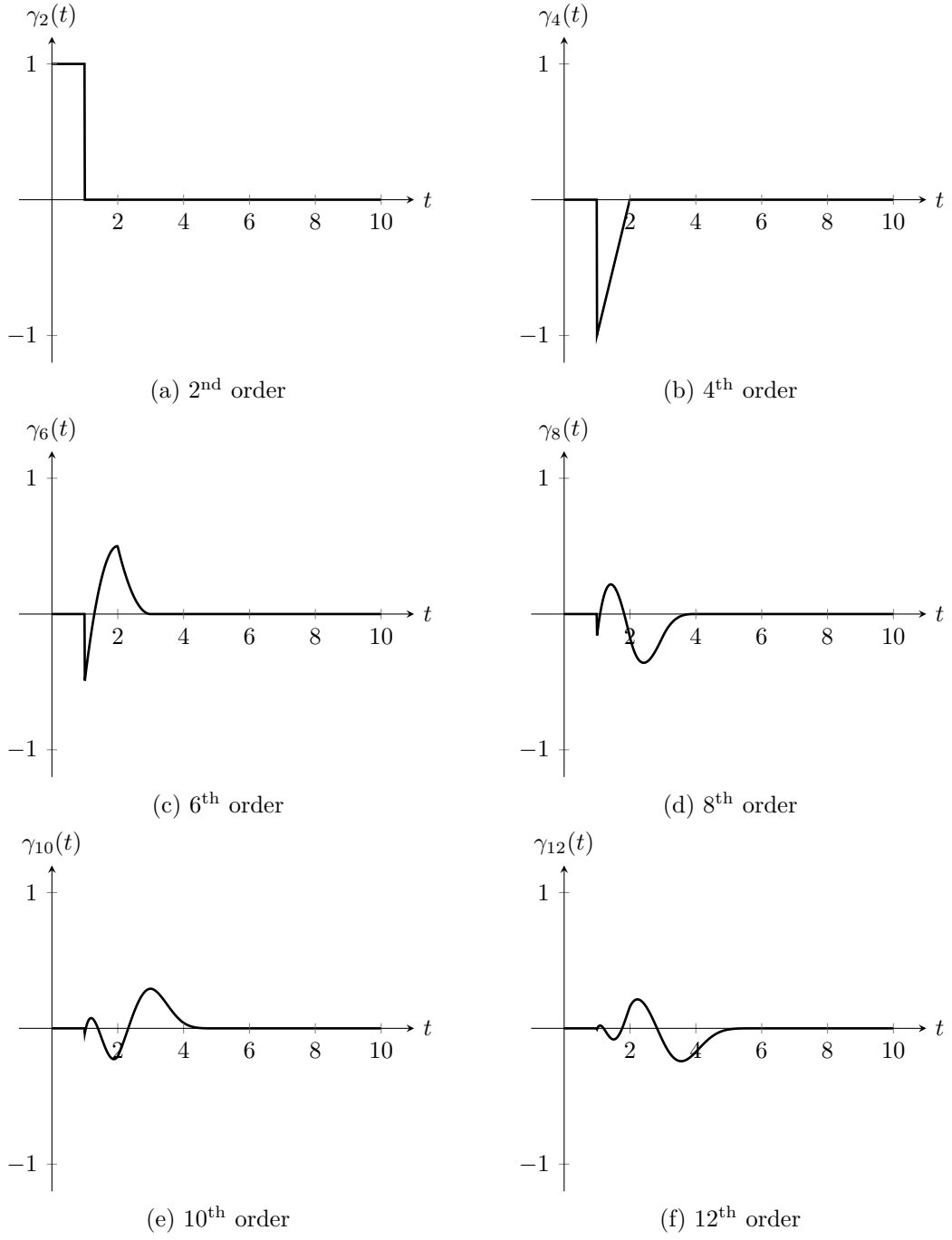
$$p_{2k}(t) = -\frac{1}{\epsilon^2}\left(\gamma_0 q_{2k-2}(t) + \theta(t-\tau)\eta e^{i\phi} q_{2k-2}(t-\tau)\right) , \tag{5.46}$$

which is readily evaluated numerically. As such, we can numerically evaluate the perturbation expansion for this system and compare it with the analytic solutions (4.34) and (4.36) we obtained previously.

Various results derived from this perturbation expansion are depicted in figures 5.2–5.5. Figure 5.2 shows the contributions to $\gamma(t)$ up to 12^{th} order in the coupling strength, while figure 5.3 shows approximations of $\gamma(t)$ up to and including the corresponding orders, and figure 5.4 shows the resulting values of $v(t)$ derived from these approximate decay rates. The complete results up to both 30^{th} and 40^{th} order are depicted in figure 5.5.

All the results shown in these figures correspond to the case of negative feedback in which $\phi = \pi$. As we noted in section 4.3, the time-dependent coefficient $\gamma(t)$ is singular for every value of t for which $v(t) = 0$. Any such singularities restrict the radius of convergence of the perturbation expansion: the expansion only converges between $t = 0$ and the value of t at which the first zero-crossing occurs. In the case of positive feedback, where $\phi = 0$, $v(t)$ displays zero-crossings for a wide range of parameters, and as such the perturbation expansion yields largely spurious results.

We can see from figure 5.5 that even when the perturbation expansion does converge, the result displays clear differences from the exact solution up to a very high order in the coupling strength. This example of spontaneous emission does not prove that the time-local perturbation expansion is of no use at all when modelling open quantum systems with delayed coherent feedback. What it does show is that there exist parameter regimes in which this technique does not give a satisfactory answer, and others in which we cannot even complete the calculation. As such, in chapters 6 and 7 we will turn our attention to exact, non-perturbative equations of motion.



(a) 2^{nd} order

(b) 4^{th} order

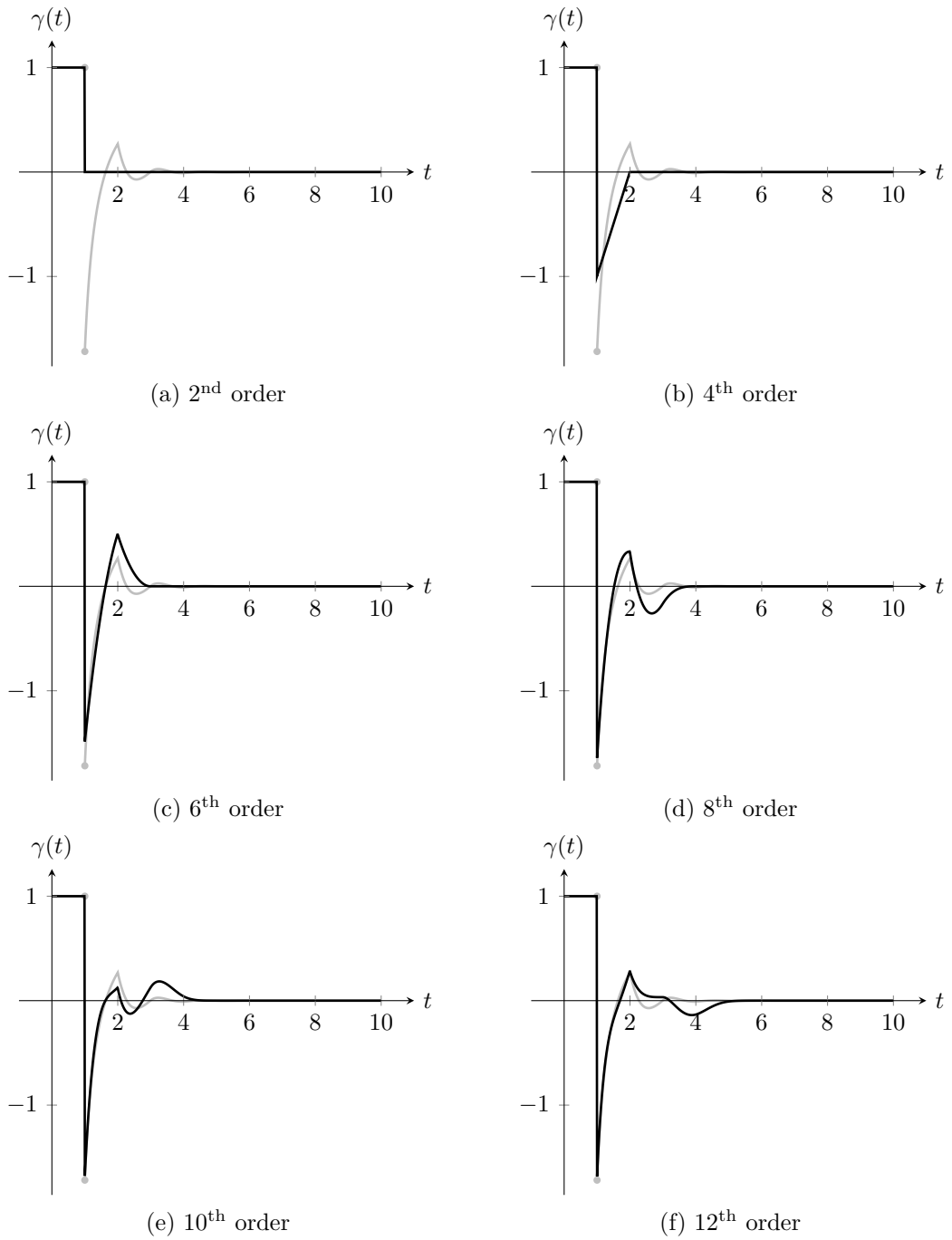
(c) 6^{th} order

(d) 8^{th} order

(e) 10^{th} order

(f) 12^{th} order

Figure 5.2: Contributions of various orders in perturbation theory. Parameters: $\gamma_0 = \eta = 1$, $\tau = 1$, $\phi = \pi$.



(a) 2^{nd} order

(b) 4^{th} order

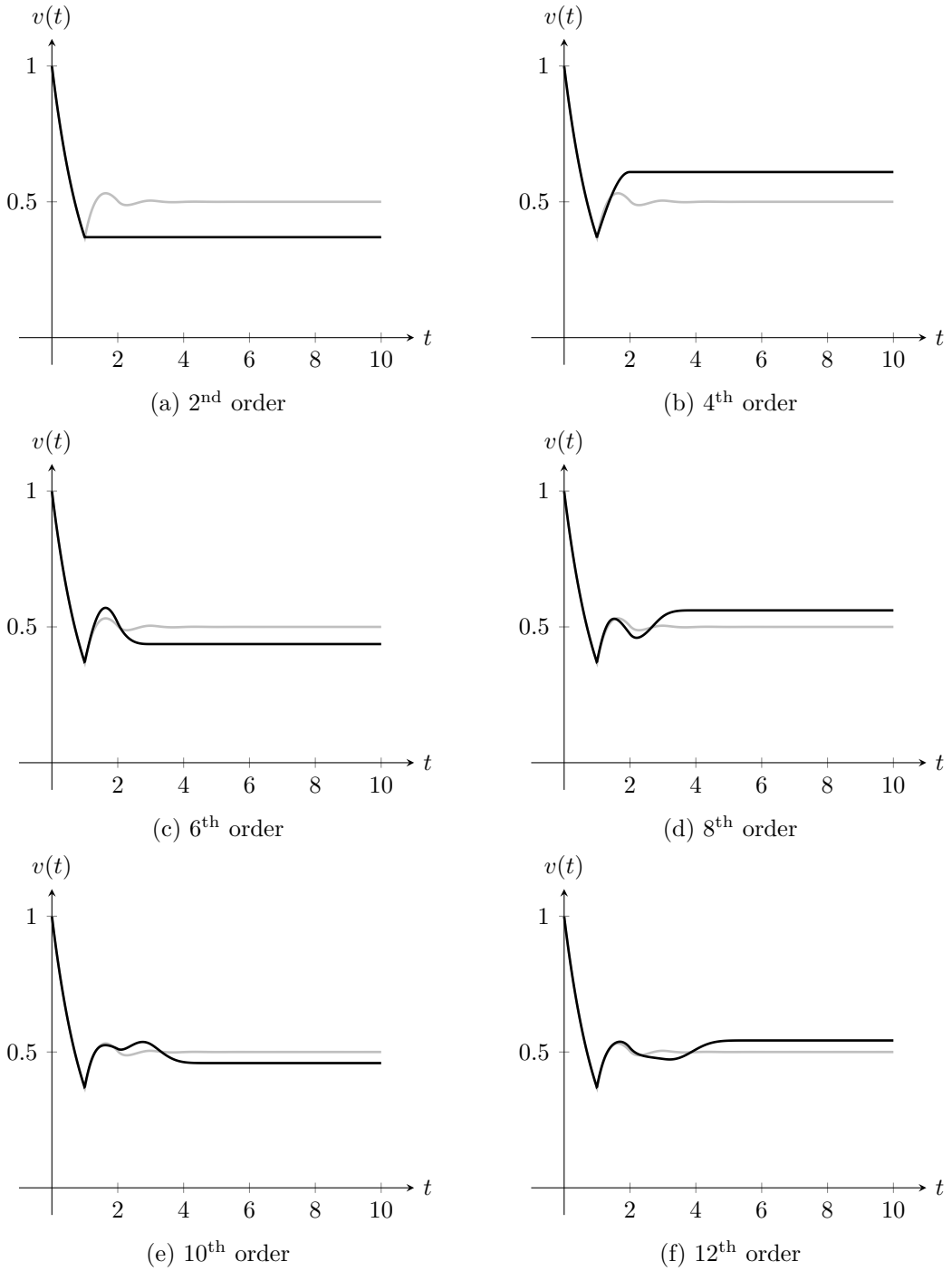
(c) 6^{th} order

(d) 8^{th} order

(e) 10^{th} order

(f) 12^{th} order

Figure 5.3: $\gamma(t)$ approximated to various orders in perturbation theory. The grey lines denote the exact results. Parameters: $\gamma_0 = \eta = 1$, $\tau = 1$, $\phi = \pi$.



(a) 2nd order

(b) 4th order

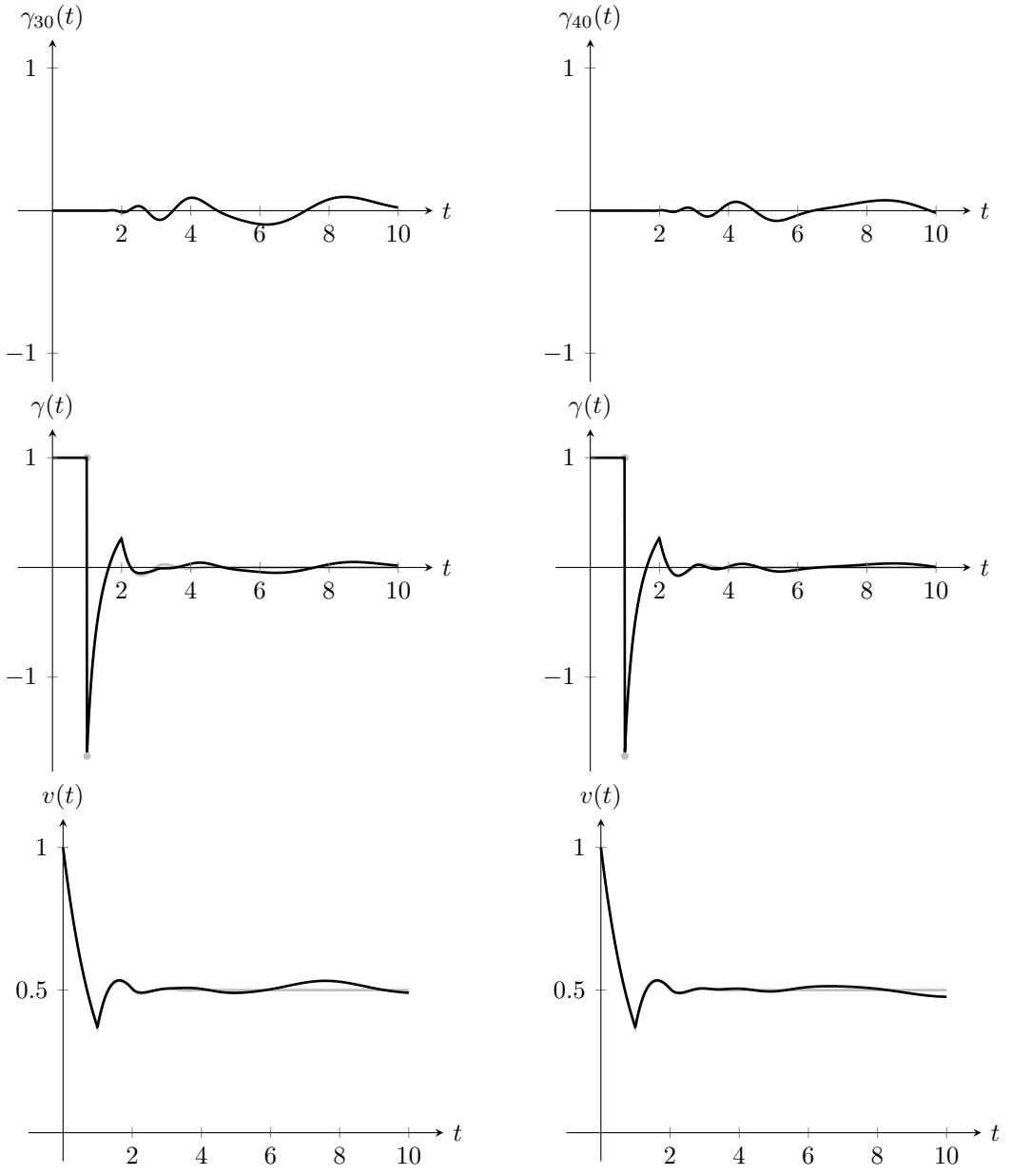
(c) 6th order

(d) 8th order

(e) 10th order

(f) 12th order

Figure 5.4: $v(t)$ approximated to various orders in perturbation theory. The grey lines denote the exact results. Parameters: $\gamma_0 = \eta = 1$, $\tau = 1$, $\phi = \pi$.



(a) 30^{th} order (b) 40^{th} order

Figure 5.5: Higher-order contributions and results. The grey lines denote the exact results. Parameters: $\gamma_0 = \eta = 1$, $\tau = 1$, $\phi = \pi$.

6

HEISENBERG–LANGEVIN EQUATIONS

While we were able to develop an exact treatment of spontaneous emission from singly-excited systems in chapter 4, we saw in section 4.8 that there is no obvious way to extend such a treatment to systems with multiple excitations. Similarly, it turned out that the approximate methods discussed in chapter 5 were of limited use when it came to modelling delayed coherent feedback.

In section 6.1 we will derive exact Heisenberg–Langevin equations describing a large class of non-Markovian open quantum systems. In section 6.2 we will review the theory of cascaded open quantum systems, and in section 6.3 we will use this formalism to simulate systems displaying delayed feedback. Finally, in section 6.4, we will develop analytic solutions to non-Markovian Heisenberg–Langevin equations for a series of simple systems.

We will use the model defined in section 3.1, but for convenience we will reiterate some of the relevant quantities here. We will perform all calculations in the rotating frame (see section 3.2) in which the motion – whether it's the motion of the operators in the Heisenberg picture, or of the states in the Schrödinger picture – is generated by the interaction Hamiltonian $H(t) = H_S(t) + H_{SE}(t)$, where $H_S(t)$ involves only system operators and describes the internal dynamics of the system; and $H_{SE}(t)$, which describes the system–environment interaction, is given by

$$H_{SE}(t) = \sum_j \left(a_j^\dagger B_j(t) + B_j^\dagger(t) a_j \right) , \tag{6.1}$$

where

$$B_j(t) = \sum_\alpha \sum_{l_\alpha} \kappa_{j\alpha l_\alpha} \exp\left(i(\omega_j - \omega_{l_\alpha})t\right) b_{\alpha l_\alpha} . \tag{6.2}$$

Because the system and environment canonical operators commute at equal times, and because the time-dependence of $B_j(t)$ is explicit rather than being generated by Heisenberg picture evolution, we find that $[a_j^\dagger, B_j(t)] = [a_j, B_j(t)] = 0$. We don't want to specify the algebra of the system canonical operators a_j and $a_j^\dagger$ just yet. The environment, however, will be assumed to be an assemblage of harmonic oscillators, with raising and lowering operators that satisfy the bosonic commutation relations

$$\left[b_{\alpha l}, b_{\alpha' l'}^\dagger \right] = \delta_{\alpha\alpha'} \delta_{ll'} . \tag{6.3}$$

6.1 DERIVATION OF THE HEISENBERG–LANGEVIN EQUATIONS

The Heisenberg equation in the rotating frame is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}O(t) = \frac{\partial}{\partial t}O(t) - i\left[O(t), H(t)\right] . \tag{6.4}$$

Thus, the Heisenberg equation for a system operator $A(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{\partial}{\partial t}A(t) - i\left[A(t), H_S(t)\right] - i\left[A(t), H_{SE}(t)\right] . \tag{6.5}$$

Substituting (6.1) into this equation, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{\partial}{\partial t}A(t) - i\left[A(t), H_S(t)\right] + i\sum_j \left(\left[a_j^\dagger(t), A(t)\right] E_j(t) - E_j^\dagger(t)\left[A(t), a_j(t)\right] \right) , \tag{6.6}$$

where

$$E_j(t) = U^\dagger(t) B_j(t) U(t) = \sum_\alpha \sum_{l_\alpha} \kappa_{j\alpha l_\alpha} \exp\left(i(\omega_j - \omega_{l_\alpha})t\right) b_{\alpha l_\alpha}(t) , \tag{6.7}$$

Here the unitary operator $U(t)$ is defined by $\dot{U}(t) = -iH(t)U(t)$, $U(0) = I$. Because the reservoir operators obey bosonic commutation relations, we can rewrite $E_j(t)$ in a convenient form that explicitly shows the dependence on system operators. We first need to derive Heisenberg equations for the environment operators $b_{\alpha l_\alpha}(t)$. These follow straightforwardly from (6.4):

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{\alpha l_\alpha}(t) = -i \sum_k \kappa^*_{k\alpha l_\alpha} \exp\left(-i(\omega_k - \omega_{l_\alpha})t\right) a_k(t) . \tag{6.8}$$

Formally integrating this equation gives

$$b_{\alpha l_\alpha}(t) = b_{\alpha l_\alpha} - i \sum_k \int_0^t \mathrm{d}t_1 \, \kappa^*_{k\alpha l_\alpha} \exp\left(-i(\omega_k - \omega_{l_\alpha})t_1\right) a_k(t_1) . \tag{6.9}$$

From this we find

$$E_j(t) = B_j(t) - i \sum_k \int_0^t \mathrm{d}t_1 \, F_{jk}(t, t_1) a_k(t_1) , \tag{6.10}$$

along with the conjugate equation for $E_j^\dagger(t)$, where the dissipation kernel $F_{jk}(t, t_1)$ is given by (3.13) with the spectral densities (3.15). As explained in section 3.3, if we choose $\omega_j = \omega_0$ for all j, the dissipation kernel is given by (3.17) with the frequency-shifted spectral densities (3.19). We will suppose that this choice has been made, and as such we will make the replacement $F_{jk}(t, t_1) \to F_{jk}(t - t_1)$ in accordance with (3.20).

Using (6.10), the Heisenberg equation (6.6) can now be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{\partial}{\partial t}A(t) - i\left[A(t), H_S(t)\right] \\ + \sum_{jk}\int_0^t \mathrm{d}t_1 \left(F_{jk}(t,t_1)\left[a_j^\dagger(t), A(t)\right]a_k(t_1) + F_{jk}^*(t,t_1)a_k^\dagger(t_1)\left[A(t), a_j(t)\right]\right) \\ + i\sum_j \left(\left[a_j^\dagger(t), A(t)\right]B_j(t) - B_j^\dagger(t)\left[A(t), a_j(t)\right]\right) . \end{aligned} \tag{6.11}$$

This equation is often referred to as a Heisenberg–Langevin equation, because the operators $B_j(t)$ can be thought of as noise inputs, analogous to those in the classical Langevin equation.[56,57] Because we made no approximations or assumptions in deriving it, equation (6.11) is exact within the context of the model described above. As noted by Gardiner and Collett,[25] the Heisenberg–Langevin equation has alternative forms—a consequence of the commutation of $E_j(t)$ and $E_j^\dagger(t)$ with all system operators at time t. However, for our purposes (6.11) is the appropriate form of the equation: the normal-ordering of the environmental noise terms simplifies the calculations when averages over vacuum reservoir states are taken. Recent work on equations similar to (6.11) includes that by Diósi,[58] and by Zhang et al.[53]

Additionally, we can calculate some useful commutators. Because $A(t)$ is a system operator, it commutes with all reservoir operators at equal times. Thus, we can use (6.2) and (6.9) to find:

$$\left[B_j(t), A(t_1)\right] = i\sum_k \int_0^{t_1} \mathrm{d}t_2\, F_{jk}(t,t_2)\left[a_k(t_2), A(t_1)\right] , \tag{6.12}$$

$$\left[A(t_1), B_j^\dagger(t)\right] = -i\sum_k \int_0^{t_1} \mathrm{d}t_2\, F_{jk}^*(t,t_2)\left[A(t_1), a_k^\dagger(t_2)\right] . \tag{6.13}$$

These commutation relations represent a generalisation of similar relations derived by Gardiner et al.,[25,57,59] and a special case of the relations (6.12) and (6.13) was derived by Dorner and Zoller.[40]

Using the Heisenberg equation (6.11) together with the commutation relations (6.12) and (6.13) we can, in principle, calculate any system quantity of interest. The most direct way to do this is to develop the so-called *moment hierarchy*: the 'tree' of moment equations needed to evaluate a given expectation value $\langle A(t)\rangle$. The moment hierarchy is a system of coupled partial differential equations for expectation values and correlation functions involving system operators. For non-trivial systems, integrating the moment hierarchy is a demanding exercise that we would rather avoid if possible. I have presented a proposal for a method of integrating the moment hierarchy in appendix C, but I will not pursue it because of the intrinsic difficulty associated with this approach.

Instead, we will investigate alternative methods of simulating the dynamics described by the Heisenberg–Langevin equations. In section 6.2, I will introduce the theory of cascaded open quantum systems, and in section 6.3 I will investigate whether or not it is possible to

simulate delayed coherent feedback using this formalism. In section 6.4, we will develop analytic solutions for the Heisenberg–Langevin equations in some simple cases, and in chapter 7 we will look at the connection between these solutions and exact, time-local master equations.

6.2 CASCADED OPEN QUANTUM SYSTEMS

The theory of cascaded open quantum systems describes the situation in which the retarded output from one open quantum system is used to drive a second system, and was developed independently by Carmichael,[1] Gardiner,[2] and Kolobov and Sokolov.[3] In this section, we will develop the formalism of cascaded open quantum systems, following Gardiner's approach. In section 6.3 we will adapt this formalism – insofar as we are able – to simulate delayed feedback.

We can derive the master equation for cascaded open quantum systems quite directly within the formalism we have already developed. Consider the system depicted in figure 6.1, consisting of two subsystems interacting with a common reservoir that couples the subsystems in one direction only—here we consider only a single reservoir, for simplicity's sake. We will further simplify the derivation by considering only vacuum reservoirs in this section: we take the state of the combined system to be described by $\rho = \rho_S \otimes |0\rangle\langle 0|$. The first system couples to the reservoir at $x = -\tau/2$, while the second couples to the reservoir at $x = \tau/2$, where we take the field propagation to be in the positive x direction. The first system has canonical operators a_1 and $a_1^\dagger$, and the second a_2 and $a_2^\dagger$—operators from different subsystems commute at equal times. We will take the system–reservoir interaction Hamiltonian to have the form (3.11):

$$H_{SE}(t) = \sum_{j=1,2} \left(a_j^\dagger B_j(t) + B_j^\dagger(t) a_j \right) , \tag{6.14}$$

where

$$B_j(t) = \sum_l \kappa_{j,l} \exp\left(i(\omega_0 - \omega_l)t\right) b_l \, . \tag{6.15}$$

The coupling constants are given by

$$\kappa_{1,l} = \frac{1}{2\pi}\sqrt{\gamma_1} e^{-i\omega_l \tau/2} \qquad \text{and} \qquad \kappa_{2,l} = \frac{1}{2\pi}\sqrt{\gamma_2} e^{i\omega_l \tau/2} \, . \tag{6.16}$$

It will be convenient to write

$$B_j(t) = \sqrt{\gamma_j} B(t - x_j - \tau/2) \, , \tag{6.17}$$

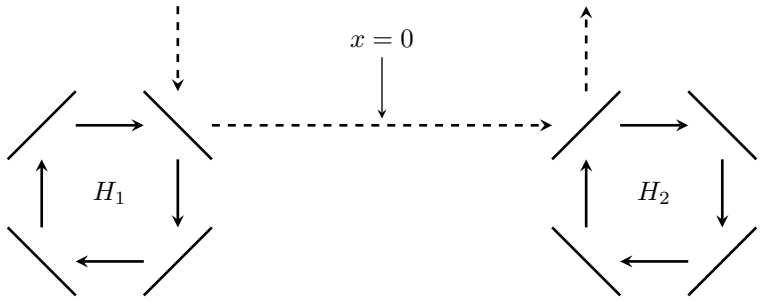


Figure 6.1: Cascaded open quantum systems, depicted schematically as ring cavities. The dashed lines represent the reservoir field that couples the systems.

where

$$B(t) = \frac{1}{2\pi} \exp(i\omega_0 t) \sum_l \exp\left(-i\omega_l (t + \tau/2)\right) b_l \,, \tag{6.18}$$

and where x_j is the location at which the j^{th} subsystem couples to the reservoir, as described above.

From the definitions above, we can see that the interaction between the two subsystems and the reservoir can be fully described with the aid of four frequency-shifted spectral densities. In the continuum limit – in which we obtain a travelling-wave reservoir – these are given by:

$$J'_{11}(\omega) = \frac{\gamma_1}{2\pi} \,, \tag{6.19}$$

$$J'_{22}(\omega) = \frac{\gamma_2}{2\pi} \,, \tag{6.20}$$

$$J'_{12}(\omega) = \frac{\sqrt{\gamma_1 \gamma_2}}{2\pi} e^{-i(\omega\tau + \phi)} \,, \tag{6.21}$$

$$J'_{21}(\omega) = \frac{\sqrt{\gamma_1 \gamma_2}}{2\pi} e^{i(\omega\tau + \phi)} \,. \tag{6.22}$$

The corresponding dissipation kernels are given by

$$F_{11}(t - t_1) = \gamma_1 \delta(t - t_1) \,, \tag{6.23}$$

$$F_{22}(t - t_1) = \gamma_2 \delta(t - t_1) \,, \tag{6.24}$$

$$F_{12}(t - t_1) = \sqrt{\gamma_1 \gamma_2} e^{-i\phi} \delta(t + \tau - t_1) \,, \tag{6.25}$$

$$F_{21}(t - t_1) = \sqrt{\gamma_1 \gamma_2} e^{i\phi} \delta(t - \tau - t_1) \,. \tag{6.26}$$

If we substitute these kernels into the Heisenberg–Langevin equation (6.11), we find for a

system operator $A(t)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{\partial}{\partial t}A(t) - i\left[A(t), H_1 + H_2\right] \\ + \gamma_1\left(\left[a_1^\dagger(t), A(t)\right]a_1(t) + a_1^\dagger(t)\left[A(t), a_1(t)\right]\right) \\ + \gamma_2\left(\left[a_2^\dagger(t), A(t)\right]a_2(t) + a_2^\dagger(t)\left[A(t), a_2(t)\right]\right) \\ + \theta(t-\tau)\sqrt{\gamma_1\gamma_2}\left(e^{i\phi}\left[a_2^\dagger(t), A(t)\right]a_1(t-\tau) + e^{-i\phi}a_1^\dagger(t-\tau)\left[A(t), a_2(t)\right]\right) \\ + i\sqrt{\gamma_1}\left(\left[a_1^\dagger(t), A(t)\right]B(t) - B^\dagger(t)\left[A(t), a_1(t)\right]\right) \\ + i\sqrt{\gamma_2}\left(\left[a_2^\dagger(t), A(t)\right]B(t-\tau) - B^\dagger(t-\tau)\left[A(t), a_2(t)\right]\right). \qquad (6.27)\end{aligned}$$

If $A_1(t)$ is an operator of the first subsystem, and $A_2(t)$ of the second, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_1(t) = \frac{\partial}{\partial t}A_1(t) - i\left[A_1(t), H_1\right] + \gamma_1\left(\left[a_1^\dagger(t), A_1(t)\right]a_1(t) + a_1^\dagger(t)\left[A_1(t), a_1(t)\right]\right) \\ + i\sqrt{\gamma_1}\left(\left[a_1^\dagger(t), A_1(t)\right]B(t) - B^\dagger(t)\left[A_1(t), a_1(t)\right]\right), \qquad (6.28) \\ \frac{\mathrm{d}}{\mathrm{d}t}A_2(t) = \frac{\partial}{\partial t}A_2(t) - i\left[A_2(t), H_2\right] + \gamma_2\left(\left[a_2^\dagger(t), A_2(t)\right]a_2(t) + a_2^\dagger(t)\left[A_2(t), a_2(t)\right]\right) \\ + \theta(t-\tau)\sqrt{\gamma_1\gamma_2}\left(e^{i\phi}\left[a_2^\dagger(t), A_2(t)\right]a_1(t-\tau) + e^{-i\phi}a_1^\dagger(t-\tau)\left[A_2(t), a_2(t)\right]\right) \\ + i\sqrt{\gamma_2}\left(\left[a_2^\dagger(t), A_2(t)\right]B(t-\tau) - B^\dagger(t-\tau)\left[A_2(t), a_2(t)\right]\right). \qquad (6.29)\end{aligned}$$

Note that the equation of motion (6.29) for a generic operator of the second system $A_2(t)$ depend on operators of the first system via the appearance of $a_1(t-\tau)$, whereas the equation (6.28) for an operator of the first system $A_1(t)$ has no dependence on operators of the second system: this demonstrates the one-way coupling.

The Heisenberg–Langevin equation (6.27) is a non-Markovian equation. However, the memory effect in this system is trivial: the delay parameter τ has no effect on the dynamics because the phase associated with the free propagation of the field from one subsystem to another, ϕ, is already accounted for. Therefore, the delay can be removed using a simple transformation, namely letting $\tau \to 0^+$. Doing this gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{\partial}{\partial t}A(t) - i\left[A(t), H_1 + H_2\right] \\ + \gamma_1\left(\left[a_1^\dagger(t), A(t)\right]a_1(t) + a_1^\dagger(t)\left[A(t), a_1(t)\right]\right) \\ + \gamma_2\left(\left[a_2^\dagger(t), A(t)\right]a_2(t) + a_2^\dagger(t)\left[A(t), a_2(t)\right]\right) \\ + \sqrt{\gamma_1\gamma_2}\left(e^{i\phi}\left[a_2^\dagger(t), A(t)\right]a_1(t) + e^{-i\phi}a_1^\dagger(t)\left[A(t), a_2(t)\right]\right) \\ + i\sqrt{\gamma_1}\left(\left[a_1^\dagger(t), A(t)\right]B(t) - B^\dagger(t)\left[A(t), a_1(t)\right]\right) \\ + i\sqrt{\gamma_2}\left(\left[a_2^\dagger(t), A(t)\right]B(t) - B^\dagger(t)\left[A(t), a_2(t)\right]\right). \qquad (6.30)\end{aligned}$$

Taking the expectation value of this Heisenberg–Langevin equation in the vacuum reservoir

yields the moment equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle = \frac{\partial}{\partial t}\langle A(t)\rangle - i\langle[A(t), H_1 + H_2]\rangle \\ + \gamma_1\left\langle\left[a_1^\dagger(t), A(t)\right]a_1(t) + a_1^\dagger(t)\left[A(t), a_1(t)\right]\right\rangle \\ + \gamma_2\left\langle\left[a_2^\dagger(t), A(t)\right]a_2(t) + a_2^\dagger(t)\left[A(t), a_2(t)\right]\right\rangle \\ + \sqrt{\gamma_1\gamma_2}\left\langle e^{i\phi}\left[a_2^\dagger(t), A(t)\right]a_1(t) + e^{-i\phi}a_1^\dagger(t)\left[A(t), a_2(t)\right]\right\rangle . \end{aligned} \tag{6.31}$$

Defining a reduced system density operator $\rho_S(t)$ that satisfies $\langle A(t)\rangle = \mathrm{tr}(A\rho_S(t))$, we can simply read off the master equation for this operator from (6.31):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\left[H_1 + H_2, \rho_S(t)\right] + \gamma_1\left(\left[a_1\rho_S(t), a_1^\dagger\right] + \left[a_1, \rho_S(t)a_1^\dagger\right]\right) \\ + \gamma_2\left(\left[a_2\rho_S(t), a_2^\dagger\right] + \left[a_2, \rho_S(t)a_2^\dagger\right]\right) \\ + \sqrt{\gamma_1\gamma_2}\left(e^{i\phi}\left[a_1\rho_S(t), a_2^\dagger\right] + e^{-i\phi}\left[a_2, \rho_S(t)a_1^\dagger\right]\right) . \end{aligned} \tag{6.32}$$

Equation (6.32) is the master equation for cascaded open quantum systems—or, more accurately, two such systems. The extension to multiple systems is straightforward, as are extensions to take into account additional reservoirs or reservoirs at non-zero temperature.

6.3 SIMULATING FEEDBACK WITH CASCADED SYSTEMS

In a 2010 paper, Menicucci et al. presented an experimental design for producing an arbitrarily large continuous-variable cluster state, for the purposes of quantum information processing and continuous variable cluster state quantum computation.[60] One element of their design is a linear continuous-variable cluster state comprising a single squeezer and a single controlled-Z gate, where the output from the gate is fed back into the system with a time delay τ. This apparatus is sketched in figure 6.2a. The source of squeezed light is coupled to the gate with a periodic coupling strength, implementing a pulsed input (the period of the input pulses is matched to the time delay). The idea behind this proposal is a postulated equivalence between this system and the system sketched in figure 6.2b. The latter system comprises a chain of subsystems prepared and driven identically to the single-gate system, except that the output from a given subsystem, rather than being fed back into the same gate, is used to drive the next subsystem in the sequence.

It is not my intention here to undertake a detailed examination of this experimental design—the particulars of the proposal, and those of cluster-state quantum computation, are available in the literature.[60,61] Instead, we will investigate the possibility of using the theory of cascaded systems to simulate delayed coherent feedback, in an analogue of the well-known method of solving a classical delay-differential equation by re-casting it as a multivariate Markov process.[62]

In section 6.1 we developed the general Heisenberg–Langevin equation (6.11), which

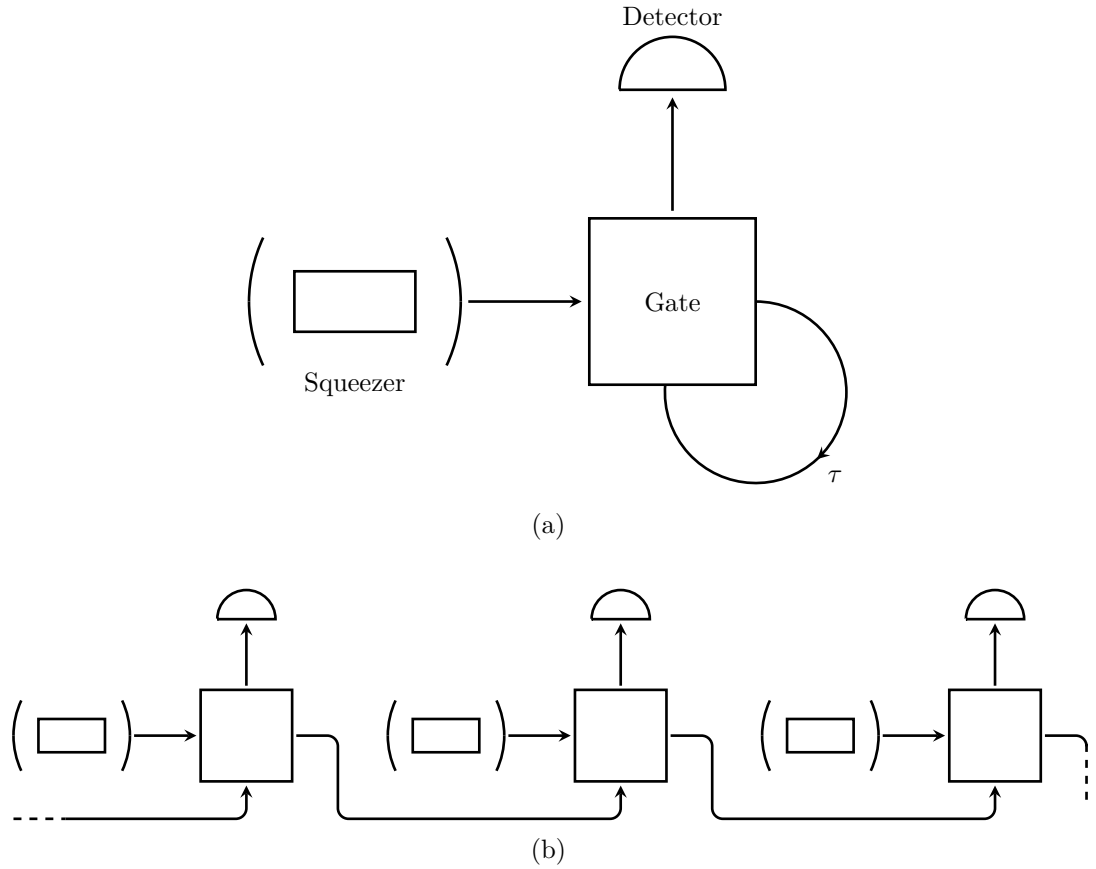


Figure 6.2: The quantum computation scheme proposed by Menicucci et al.[60] The system with delayed coherent feedback (a) is thought to be equivalent, under certain conditions, to the chain of identical subsystems (b).

describes non-Markovian processes including coherent feedback. Taking the expectation value of equation (6.11) in the vacuum reservoir gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle = \frac{\partial}{\partial t}\langle A(t)\rangle - i\langle [A(t), H_S(t)]\rangle \\ + \sum_{jk}\int_0^t \mathrm{d}t_1 \Big(F_{jk}(t,t_1)\left\langle \left[a_j^\dagger(t), A(t)\right] a_k(t_1)\right\rangle \\ + F_{jk}^*(t,t_1)\left\langle a_k^\dagger(t_1)\left[A(t), a_j(t)\right]\right\rangle\Big)\,. \end{aligned} \tag{6.33}$$

Substituting the dissipation kernel (4.18) into (6.33) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle = \frac{\partial}{\partial t}\langle A(t)\rangle - i\langle [A(t), H_S(t)]\rangle + \gamma\left(\left\langle \left[a^\dagger(t), A(t)\right] a(t)\right\rangle + \left\langle a^\dagger(t)\left[A(t), a(t)\right]\right\rangle\right) \\ + \theta(t-\tau)\eta\left(e^{i\phi}\left\langle \left[a^\dagger(t), A(t)\right] a(t-\tau)\right\rangle + e^{-i\phi}\left\langle a^\dagger(t-\tau)\left[A(t), a(t)\right]\right\rangle\right)\,. \end{aligned} \tag{6.34}$$

The similarity between equation (6.34) and the moment equation for cascaded systems (6.31) is indeed striking—one might infer from the similarity of these two equations that a system with delayed feedback is, in some sense, equivalent to a chain of cascaded systems.

The master equation for such a chain will form the basis of our candidate method of simulating delayed coherent feedback. We make the ansatz that the state of the system over N intervals of length τ can be represented as a density matrix describing a fictitious chain of N subsystems: $\rho_{S,N}(t) \in \bigotimes_{n=0}^{N-1} \mathcal{H}_S$, where $\mathcal{H}_S$ is the Hilbert space of the reduced system. Following Menicucci et al., we will take each subsystem in the chain to be prepared identically to the real system. Thus, the density matrix of the fictitious system, at the initial time $t = 0$, will be given by

$$\rho_{S,N}(0) = \underbrace{\rho_S(0) \otimes \cdots \otimes \rho_S(0)}_{N \text{ times}}. \tag{6.35}$$

The density matrix obeys the iterative equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{S,N}(t) = \mathcal{L}_m \rho_{S,N}(t), \qquad m\tau < t \leq (m+1)\tau, \tag{6.36}$$

where

$$\begin{aligned}\mathcal{L}_m \rho_{S,N}(t) = \sum_{n=0}^{m} \left(-i\left[H_{S,n}, \rho_{S,N}(t)\right] + \gamma\left(\left[a_n \rho_{S,N}(t), a_n^\dagger\right] + \left[a_n, \rho_{S,N}(t) a_n^\dagger\right]\right)\right) \\ + \eta \sum_{n=1}^{m} \left(e^{i\phi}\left[a_n \rho_{S,N}(t), a_{n-1}^\dagger\right] + e^{-i\phi}\left[a_{n-1}, \rho_{S,N}(t) a_n^\dagger\right]\right),\end{aligned} \tag{6.37}$$

and where

$$O_n = \underbrace{I \otimes \cdots \otimes I}_{n \text{ times}} \otimes O \otimes \underbrace{I \otimes \cdots \otimes I}_{(N-n-1) \text{ times}}, \tag{6.38}$$

for any operator O acting on a single subsystem, such as H_S or a. Thus, we are identifying a_n in (6.37) with $a(t - n\tau)$ in the moment equation (6.34). Expectation values of the zeroth subsystem correspond to the 'real' expectation values we aim to calculate.

The way this algorithm works is that all N subsystems comprising the chain are accounted for in the initial state (6.35), but the n^{th} subsystem only begins evolving when $t = n\tau$. Physically, this is equivalent to adding an additional subsystem to the chain at the beginning of each time interval. While it is less computationally efficient to have all the subsystems present over all intervals – even those intervals where they are not needed – this is not a major concern because the principal bottleneck, computationally speaking, is the size of the state required to describe the system in the last interval. Furthermore, the present form of the algorithm is easier to implement than a version in which the state grows with each time interval. The dynamics described by the master equation (6.36) can be simulated using well-known numerical techniques. The results shown in this section were obtained using QuTiP.[63,64]

Before we can perform any simulations of the master equation (6.36), however, we need to specify the algebra of the operators a_n and $a_n^\dagger$. It is important to realise that even if the

real system we are trying to simulate contains at most a single excitation, the distinction between bosonic and fermionic commutation relations still needs to be made because the corresponding fictitious system, with state (6.35), contains multiple excitations. Depending on whether we take the algebra of the fictitious system operators to be bosonic or fermionic, the results of our simulation may or may not agree with the exact solution. To contrast the results obtained from different choices of operator algebra, we will consider the first two intervals – where each interval is of length τ – and compare the dynamics generated by the exact Heisenberg–Langevin equations with those generated by the cascade ansatz (6.36), in the case where the real system contains at most one excitation.

Consider the time interval $0 \leq t \leq \tau$. In this first interval, before the feedback has any effect, equation (6.36) reduces to a simple Lindblad master equation for a single system, given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{S,2}(t) = \gamma\left(\left[a_0\rho_{S,2}(t), a_0^\dagger\right] + \left[a_0, \rho_{S,2}(t)a_0^\dagger\right]\right). \tag{6.39}$$

The moment equation (6.34) reduces to an equivalent equation on this time interval.

In the second interval $\tau < t \leq 2\tau$, the feedback takes effect and it is no longer obvious what a Schrödinger picture description of the dynamics should look like. According to the cascade algorithm given above, to describe the dynamics in this interval we need two subsystems, with annihilation operators $a_0 = a \otimes I$ and $a_1 = I \otimes a$ respectively. The state of these two subsystems will be described by a density matrix $\rho_{S,2}(t)$ that obeys the master equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_{S,2}(t) = {} & \gamma\left(\left[a_0\rho_{S,2}(t), a_0^\dagger\right] + \left[a_0, \rho_{S,2}(t)a_0^\dagger\right]\right) + \gamma\left(\left[a_1\rho_{S,2}(t), a_1^\dagger\right] + \left[a_1, \rho_{S,2}(t)a_1^\dagger\right]\right) \\ & + \eta\left(e^{i\phi}\left[a_1\rho_{S,2}(t), a_0^\dagger\right] + e^{-i\phi}\left[a_0, \rho_{S,2}(t)a_1^\dagger\right]\right).\end{aligned} \tag{6.40}$$

This master equation can be used to obtain a moment hierarchy for the excitation number $\langle a_0^\dagger a_0\rangle$, which begins with the two equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a_0^\dagger a_0\right\rangle(t) = -2\gamma\left\langle a_0^\dagger a_0\right\rangle(t) - \eta\left(e^{i\phi}\left\langle a_0^\dagger a_1\right\rangle(t) + e^{-i\phi}\left\langle a_1^\dagger a_0\right\rangle(t)\right), \tag{6.41}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a_0^\dagger a_1\right\rangle(t) = -2\gamma\left\langle a_0^\dagger a_1\right\rangle(t) + \eta e^{-i\phi}\left\langle a_1^\dagger\left[a_0^\dagger, a_0\right]a_1\right\rangle(t). \tag{6.42}$$

When it comes to the differential equation for the second-order correlation function $\langle a_1^\dagger\left[a_0^\dagger, a_0\right]a_1\rangle(t)$, we obtain a different equation depending on the algebra we choose for the system operators. In the bosonic case, $[a_0, a_0^\dagger] = 1$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a_1^\dagger\left[a_0^\dagger, a_0\right]a_1\right\rangle(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a_1^\dagger a_1\right\rangle(t) = 2\gamma\left\langle a_1^\dagger a_1\right\rangle(t). \tag{6.43}$$

On the other hand, in the fermionic case, $\{a_0, a_0^\dagger\} = 1$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a_1^\dagger \left[a_0^\dagger, a_0\right] a_1\right\rangle(t) = 2\gamma\left\langle a_1^\dagger a_1\right\rangle(t) - 8\gamma\left\langle a_1^\dagger a_0^\dagger a_0 a_1\right\rangle(t)\,. \tag{6.44}$$

Equations (6.41)–(6.44) describe the fictitious system. We now need to derive the corresponding equations for the real system. If we substitute the dissipation kernel (4.18) into (6.11) and use $H_S(t) = 0$, we obtain a simple Heisenberg–Langevin equation for $a(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = \left[a^\dagger(t), a(t)\right]\left(\gamma a(t) + \theta(t-\tau)\eta e^{i\phi}a(t-\tau)\right)\,, \tag{6.45}$$

This equation leads to a moment hierarchy for $\langle a^\dagger a\rangle$, which on the interval $\tau < t \leq 2\tau$ begins with the two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a^\dagger(t)a(t)\right\rangle = -2\gamma\left\langle a^\dagger(t)a(t)\right\rangle - \eta\left(e^{i\phi}\left\langle a^\dagger(t)a(t-\tau)\right\rangle + e^{-i\phi}\left\langle a^\dagger(t-\tau)a(t)\right\rangle\right)\,, \tag{6.46}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a^\dagger(t)a(t-\tau)\right\rangle = -2\gamma\left\langle a^\dagger(t)a(t-\tau)\right\rangle + \eta e^{-i\phi}\left\langle a^\dagger(t-\tau)\left[a^\dagger(t), a(t)\right]a(t-\tau)\right\rangle\,. \tag{6.47}$$

As in the case of the fictitious system, the next moment equation in the hierarchy differs depending on the algebra of the system operators. Using the truncation scheme outlined in appendix C – and recalling that in the present case we have assumed that the real system contains at most one excitation – we can show that regardless of whether the operators $a(t)$ and $a^\dagger(t)$ obey bosonic or fermionic commutation relations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a^\dagger(t-\tau)\left[a^\dagger(t), a(t)\right]a(t-\tau)\right\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a^\dagger(t-\tau)a(t-\tau)\right\rangle = 2\gamma\left\langle a^\dagger(t-\tau)a(t-\tau)\right\rangle\,. \tag{6.48}$$

According to the proposed correspondence between real and fictitious systems, (6.43) agrees with (6.48), but (6.44) does not unless $\langle a_1^\dagger a_0^\dagger a_0 a_1\rangle(t) = 0$. Otherwise, the correspondence cannot be valid unless the fictitious system is taken to have bosonic commutation relations, $[a_n, a_n^\dagger] = 1$ for all n. This argument is easily generalised to τ-intervals beyond the first and second.

One case in which we can apply the cascade algorithm to either a bosonic or fermionic system is the case of an undriven system in the limit of large τ, as compared with $1/\gamma$. In this limit the system will always be, to a very good degree of approximation, in its ground state at the time the output from the previous interval is fed back in. This means that all relevant correlations between time intervals, such as $\langle a_1^\dagger a_0^\dagger a_0 a_1\rangle$, will decay completely before the feedback from the previous interval reaches the system, and the bosonic and fermionic moment hierarchies for the fictitious cascade system will both agree with the corresponding hierarchies for the real system.

As an example, we will consider the case of a single two-level system. The system is

undriven, so we have $H_{S,n}(t) = H_S(t) = 0$. We will take the initial state to be the excited state $\rho_S(0) = |1\rangle\langle 1|$, where $|0\rangle$ and $|1\rangle$ are the ground and excited states of the system respectively. Substituting $\rho_S(0)$ into (6.35) gives the initial condition for the fictitious chain of cascaded systems, and simulating the relaxation of this initial state using the master equation (6.36) yields the results shown in figure 6.3. In this simulation, the system operators have been assumed to obey bosonic commutation relations, albeit with the Fock space of each subsystem truncated at the two-excitation level.

As we would expect from the discussion above, the results of our simulation approximately agree with the analytic solutions we developed in section 4.2 for large τ. However, even though we have taken all operators to have bosonic commutation relations, the simulations disagree with the analytic solutions for small τ. This is easily explained: this method neglects the correlations between time intervals, and only gives the correct result in the large-τ limit where the correlations decay completely before the feedback from the previous interval reaches the system.

6.4 ANALYTIC SOLUTIONS OF HEISENBERG–LANGEVIN EQUATIONS

Our aim in this section is to find analytic solutions for selected Heisenberg–Langevin equations. This can be a rather involved process, so we will consider a series of equations of increasing complexity, solving each in turn. This will allow me to introduce, one by one, certain ideas that are helpful when it comes to solving these equations.

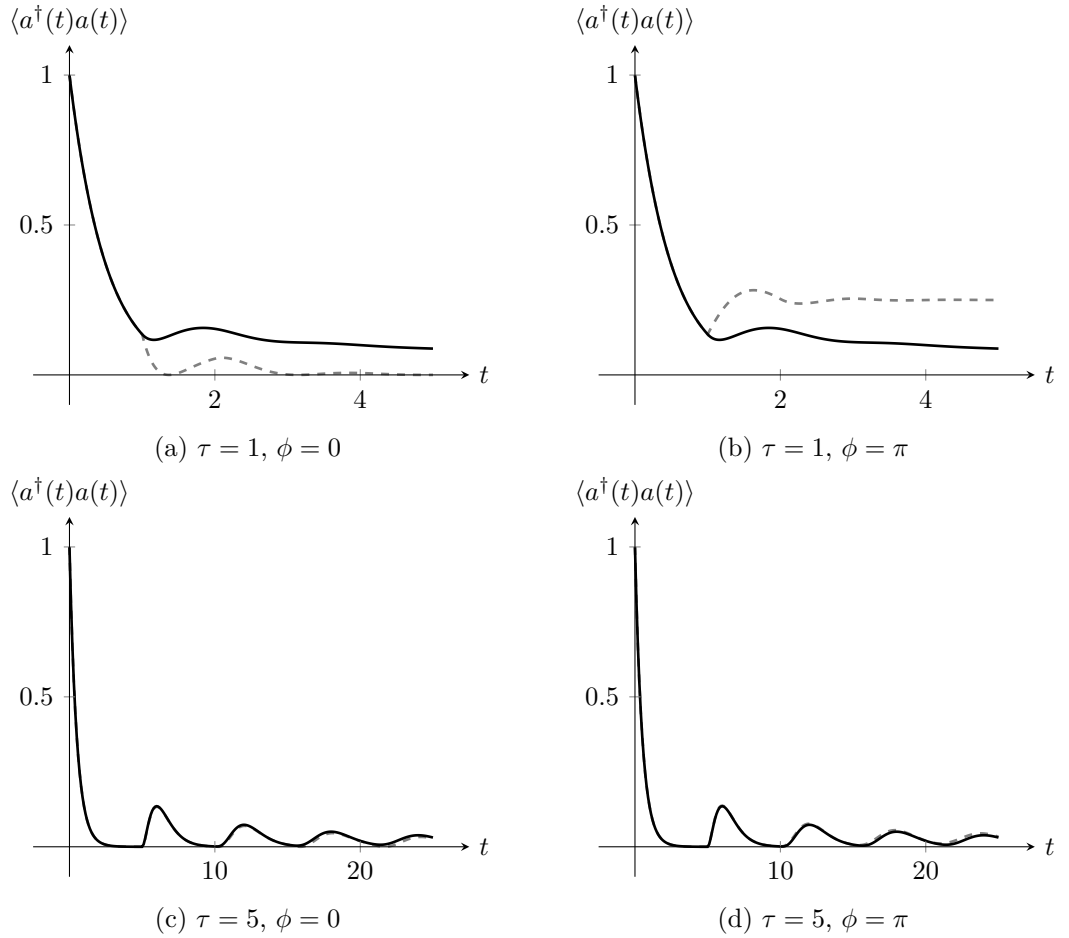
All the systems we consider in sections 6.4.1–6.4.4 will be of the kind specified in chapter 3, and as such obey the Heisenberg–Langevin equation (6.11). Furthermore, these systems will all be driven by an external field, which for the sake of generality we allow to have an arbitrary time-dependence.

6.4.1 *A single bosonic subsystem*

To begin with we'll consider a very simple form of our model: a single system degree of freedom coupled to a single environment. In this case, the Heisenberg–Langevin equation (6.11) reduces to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = &-i\left[A(t), H_S(t)\right] \\ &+ \int_0^t \mathrm{d}t_1 \left(f(t-t_1)\left[a^\dagger(t), A(t)\right]a(t_1) + f^*(t-t_1)a^\dagger(t_1)\left[A(t), a(t)\right]\right) \\ &+ i\left(\left[a^\dagger(t), A(t)\right]B(t) - B^\dagger(t)\left[A(t), a(t)\right]\right), \end{aligned} \tag{6.49}$$

where we have assumed that the operator $A(t)$ has no explicit time dependence. We choose a system Hamiltonian representing an external driving field detuned from the resonant frequency of the system, where in general we allow the driving field amplitude to vary with



(a) $\tau = 1$, $\phi = 0$ (b) $\tau = 1$, $\phi = \pi$

(c) $\tau = 5$, $\phi = 0$ (d) $\tau = 5$, $\phi = \pi$

Figure 6.3: Results of the cascaded systems simulation of delayed feedback (solid line) compared with the analytic solution (dashed line), with $\gamma = \eta = 1$ (in arbitrary units), for various values of τ and ϕ. The simulation was performed with at most two excitations in each subsystem.

time:

$$H_S(t) = \left(\Omega(t)a^\dagger + \Omega^*(t)a\right) + \Delta a^\dagger a\,. \tag{6.50}$$

Thus, the Heisenberg–Langevin equation (6.49) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = &-i\left[A(t), \left(\Omega(t)a^\dagger(t) + \Omega^*(t)a(t)\right) + \Delta a^\dagger(t)a(t)\right] \\ &+ \int_0^t \mathrm{d}t_1 \left(f(t-t_1)\left[a^\dagger(t), A(t)\right]a(t_1) + f^*(t-t_1)a^\dagger(t_1)\left[A(t), a(t)\right]\right) \\ &+ i\left(\left[a^\dagger(t), A(t)\right]B(t) - B^\dagger(t)\left[A(t), a(t)\right]\right)\,.\end{aligned} \tag{6.51}$$

We have not yet defined the algebra of the system operators a and $a^\dagger$. Here we will assume that they have bosonic commutation relations, $[a, a^\dagger] = 1$. In this case, we find from (6.51)

that the Heisenberg–Langevin equation for $a(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = -i\Delta a(t) - \int_0^t \mathrm{d}t_1\, f(t-t_1)a(t_1) - i\left(\Omega(t) + B(t)\right)\,. \tag{6.52}$$

Equation (6.52) admits a simple analytic solution, which we can find using the Laplace transform. Taking the Laplace transform of (6.52) gives

$$s\tilde{a}(s) - a = -i\Delta\tilde{a}(s) - \tilde{f}(s)\tilde{a}(s) - i\left(\tilde{\Omega}(s) + \tilde{B}(s)\right)\,. \tag{6.53}$$

Note that the driving field appears here as a displacement of the environmental noise operator $B(t)$; this is of course to be expected, because a coherent excitation of the environment is often the physical origin of the driving field in systems such as this one. Rearranging (6.53) gives

$$\tilde{a}(s) = \tilde{v}(s)\left(a - i\left(\tilde{\Omega}(s) + \tilde{B}(s)\right)\right)\,, \tag{6.54}$$

where

$$\tilde{v}(s) = \frac{1}{s + i\Delta + \tilde{f}(s)}\,. \tag{6.55}$$

We can immediately see that $v(t)$ is the Green's function we have met several times before, in this case satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -i\Delta v(t) - \int_0^t \mathrm{d}t_1\, f(t-t_1)v(t_1)\,, \qquad v(0) = 1\,. \tag{6.56}$$

Inverting the transforms in equation (6.54) gives the solution

$$a(t) = v(t)a - i(v * \Omega)(t) - i(v * B)(t)\,, \tag{6.57}$$

where we have used the following notation for the Laplace convolution:

$$(f * g)(t) = \int_0^t \mathrm{d}t_1\, f(t-t_1)g(t_1) = \int_0^t \mathrm{d}t_1\, f(t_1)g(t-t_1)\,. \tag{6.58}$$

It will turn out to be convenient to define the shorthand

$$C(t) = (v * B)(t) = \int_0^t \mathrm{d}t_1\, v(t-t_1)B(t_1)\,, \tag{6.59}$$

such that (6.57) can also be written as

$$a(t) = v(t)a - i(v * \Omega)(t) - iC(t)\,. \tag{6.60}$$

The solution we have obtained here can easily be verified by trial substitution. Using the differentiation under the integral sign theorem, one can easily show that for any (potentially

operator-valued) $y(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(v * y)(t) = v(0)y(t) + \int_0^t \mathrm{d}t_1\, \dot{v}(t - t_1)y(t_1)\,. \tag{6.61}$$

Using (6.56), and after a simple change of integration variable, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(v * y)(t) = y(t) - i\Delta(v * y)(t) - \int_0^t \mathrm{d}t_1\, f(t - t_1)(v * y)(t_1)\,. \tag{6.62}$$

This identity can be used to show that the derivative of (6.60) is equivalent to (6.52).

Using these results, we can perform a simple example calculation. Suppose we want to calculate the moment $\langle a^\dagger(t)a(t)\rangle$ in a vacuum reservoir, where $a(t)$ is given by (6.60). We immediately obtain:

$$\langle a^\dagger(t)a(t)\rangle = |v(t)|^2 \langle a^\dagger a\rangle - iv^*(t) \langle a^\dagger\rangle (v*\Omega)(t) + iv(t) \langle a\rangle (v^**\Omega^*)(t) + |(v*\Omega)(t)|^2\,. \tag{6.63}$$

If we assume that the system is initially in its ground state, meaning $\langle a\rangle = \langle a^\dagger\rangle = \langle a^\dagger a\rangle = 0$, we get

$$\langle a^\dagger(t)a(t)\rangle = |(v * \Omega)(t)|^2 = \left|\int_0^t \mathrm{d}t_1\, v(t - t_1)\Omega(t_1)\right|^2. \tag{6.64}$$

If, furthermore, we choose (phenomenologically) a constant coherent driving field, $\Omega(t) = \Omega$, equation (6.64) takes the simple form

$$\langle a^\dagger(t)a(t)\rangle = |\Omega|^2\left|\int_0^t \mathrm{d}t_1\, v(t_1)\right|^2. \tag{6.65}$$

More generally, if the system is initially in its ground state, it is driven into a coherent state with amplitude given by the convolution $-i(v * \Omega)(t)$. To see this, we evaluate the normal-ordered product

$$\left\langle (a^\dagger(t))^p (a(t))^q \right\rangle = \Big(i(v^* * \Omega^*)(t)\Big)^p \Big(-i(v * \Omega)(t)\Big)^q, \tag{6.66}$$

where we have used the fact that all the initial operator averages are zero.

Consider the feedback loop first introduced in section 4.2. As we have already established, this model admits an analytic solution for the Green's function $v(t)$, given by (4.34). Given this solution for $v(t)$, the integral in (6.65) is easily performed by numerical quadrature, leading to the results shown in figures 6.4 and 6.5. The driven harmonic oscillator displays the sort of interference effects discussed in section 4.2, where the feedback loop field interferes with the system output. In this case, however, the system state is a coherent state at all times (provided the initial condition is also a coherent state) so the interference can be thought of in the same way as interference between classical wave amplitudes, unlike in the single-photon regime investigated in chapter 4 where the interference is quantum-mechanical in nature. Recall that when $\gamma + \eta e^{i\phi} = 0$, there is the possibility

of destructive interference between the field in the loop and the system output field: in the case of a singly-excited system we found that the excitation became trapped in the system–loop subsystem. It is much the same in the present case, and the trapping cases for our driven system are shown in figures 6.5a and 6.5b. We can see from these figures that the combination of trapping and driving results in exponential growth of the cavity occupation number. This unbounded growth is the result of an idealised model, of course: indefinite trapping of excitations is not something we would see in, for example, any real optical system.

As an additional example, consider the time-varying driving field amplitude

$$\Omega(t) = \begin{cases} \Omega & t \bmod \tau < T \\ 0 & \text{otherwise.} \end{cases} \tag{6.67}$$

This function describes a pulsed coherent drive, where the pulses turn on at the beginning of every τ-interval and have duration T. The resulting mean excitation number is easily obtained, for a given $v(t)$, using equation (6.64). We choose, once again, the Green's function (4.34). Figure 6.6 compares exact results derived from (6.64) with results from a set of simulations using the cascade algorithm presented in section 6.3. Because the parameters used to obtain figure 6.6 place the system in the large-τ regime, the simulations are in close agreement with the exact results.

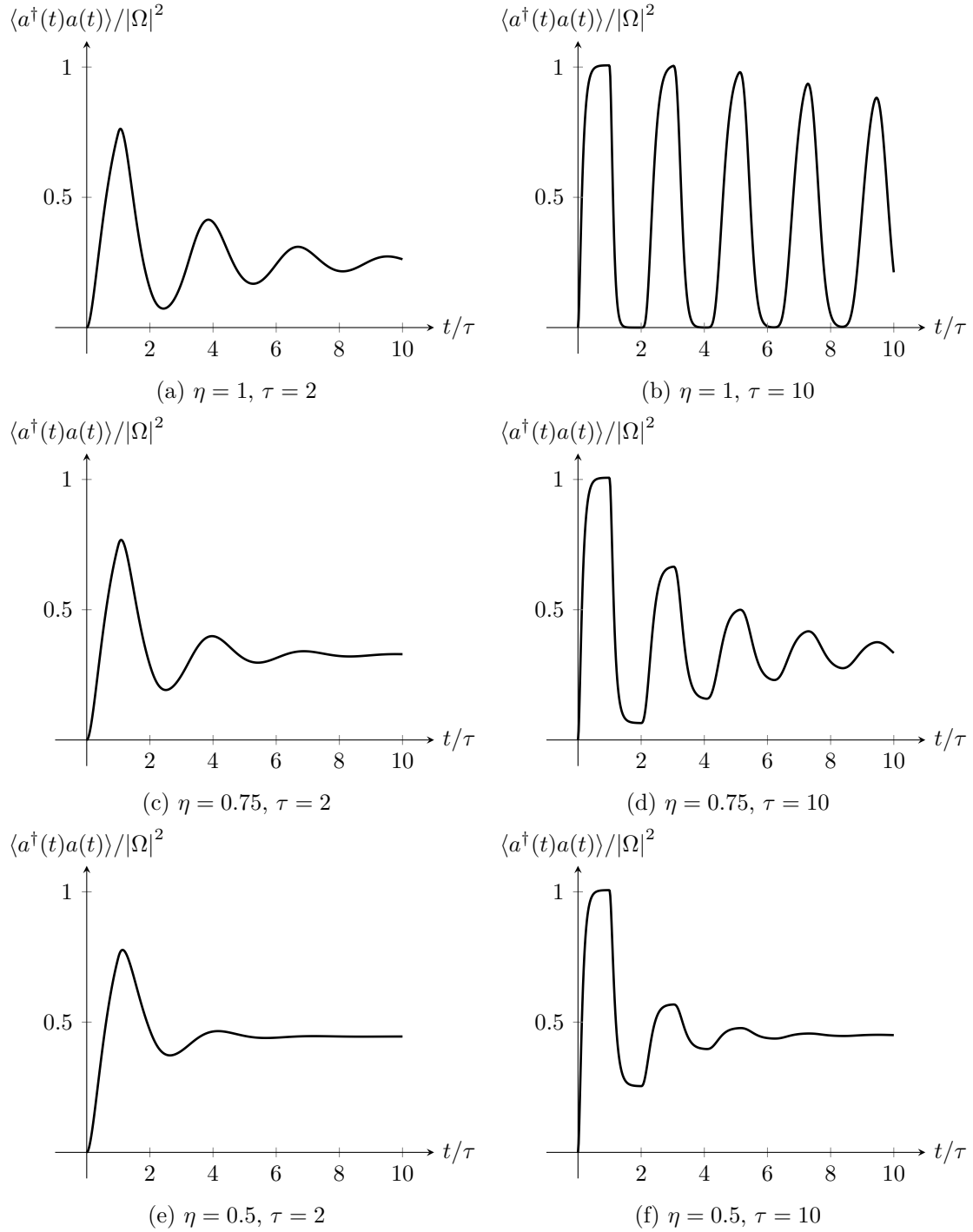
As noted in section 6.3, our cascade simulation algorithm yields correct results in the limit of large τ regardless of whether the system operators obey bosonic or fermionic commutation relations. Figure 6.7 depicts the results of several such simulations for a fermionic system, driven by a field whose amplitude varies according to (6.67). As it stands, the exact solutions of Heisenberg–Langevin equations we have derived only apply to bosonic systems, and as such cannot be used to replicate the results of these latter simulations.

6.4.2 *Multiple bosonic subsystems*

We can generalise the analysis in the section 6.4.1 to take into account multiple subsystems, N in total, which we will label $j = 1, \ldots, N$. We choose a system Hamiltonian

$$H_S(t) = \sum_j \left(\Omega_j(t) a_j^\dagger + \Omega_j^*(t) a_j \right) + \sum_{jk} \Delta_{jk} a_j^\dagger a_k \,, \tag{6.68}$$

where $\Delta_{jk}^* = \Delta_{kj}$. Additionally, we choose $\omega_j = \omega_0$ for all j so that we can use the time-homogeneous form (3.20) of the dissipation kernel. Substituting this Hamiltonian



(a) $\eta = 1$, $\tau = 2$

(b) $\eta = 1$, $\tau = 10$

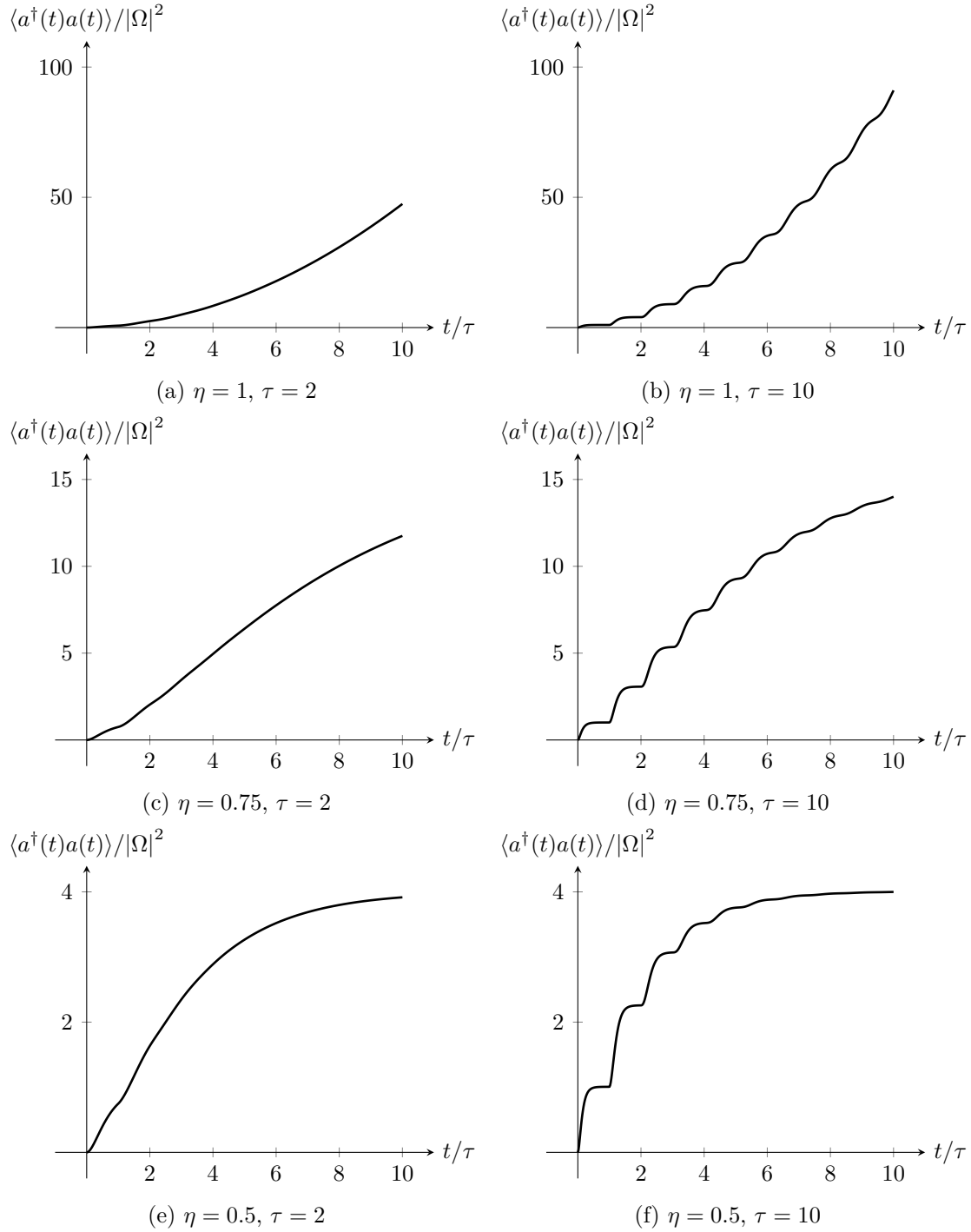
(c) $\eta = 0.75$, $\tau = 2$

(d) $\eta = 0.75$, $\tau = 10$

(e) $\eta = 0.5$, $\tau = 2$

(f) $\eta = 0.5$, $\tau = 10$

Figure 6.4: Excitation number in a coherently driven bosonic system emitting into a simple feedback loop, as given by equation (6.65), with $\gamma = 1$ (in arbitrary units) and $\phi = 0$, for several values of η and τ.



(a) $\eta = 1$, $\tau = 2$

(b) $\eta = 1$, $\tau = 10$

(c) $\eta = 0.75$, $\tau = 2$

(d) $\eta = 0.75$, $\tau = 10$

(e) $\eta = 0.5$, $\tau = 2$

(f) $\eta = 0.5$, $\tau = 10$

Figure 6.5: Excitation number in a coherently driven bosonic system emitting into a simple feedback loop, as given by equation (6.65), with $\gamma = 1$ (in arbitrary units) and $\phi = \pi$, for several values of η and τ. Note that the scale of the ordinate differs between pairs of plots.

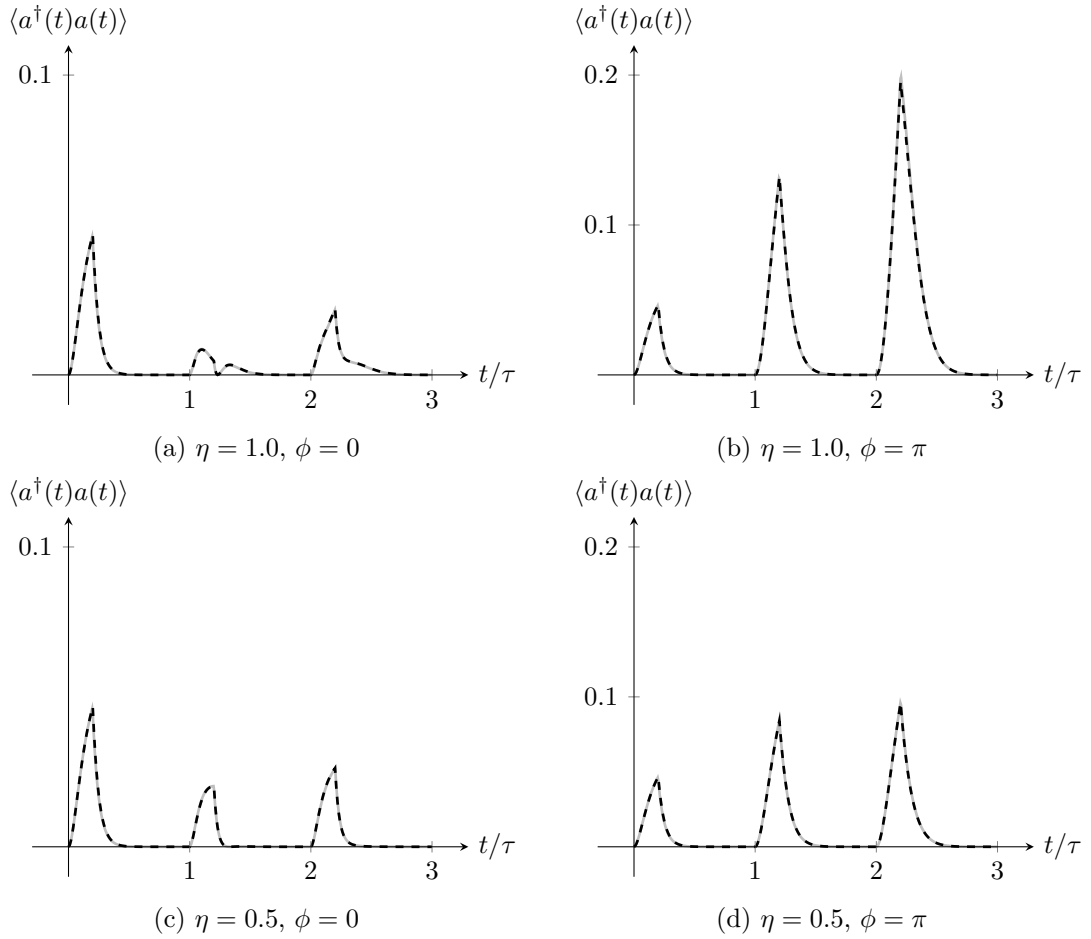


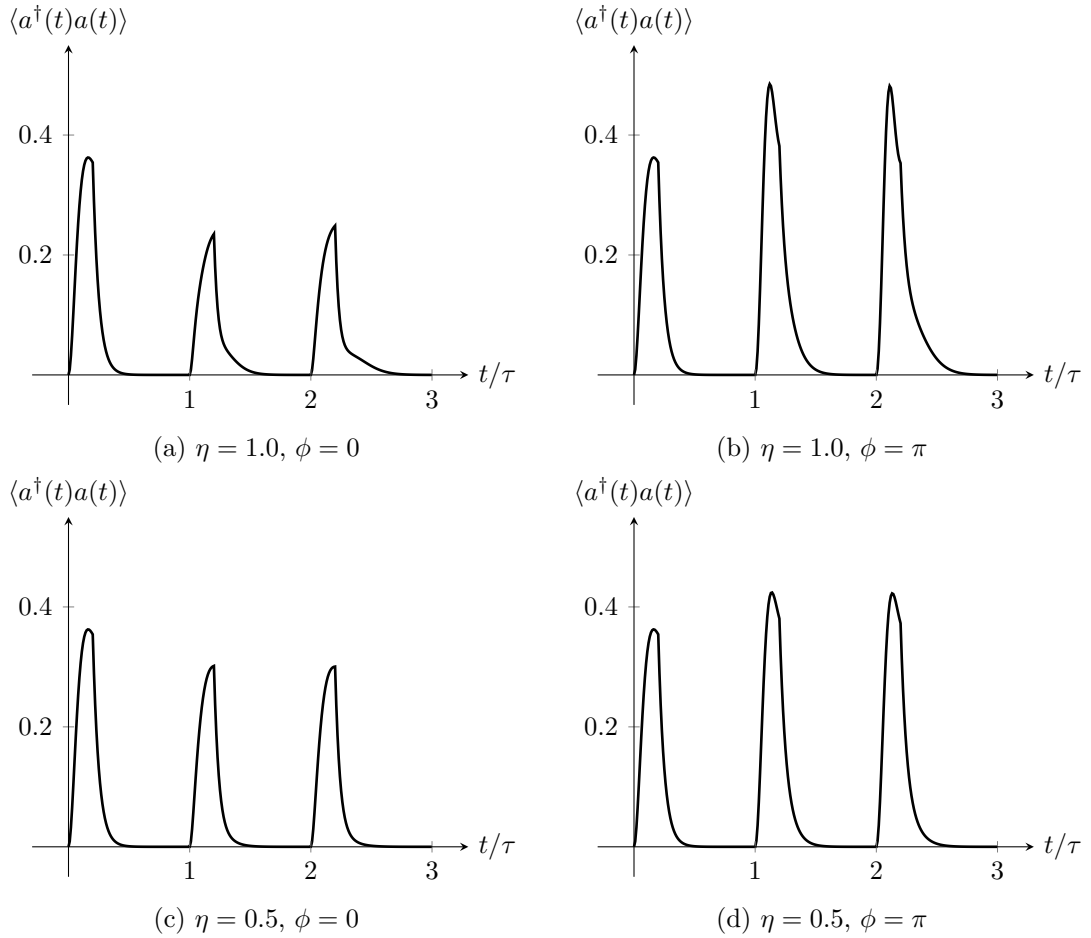
Figure 6.6: Excitation number in a driven bosonic system emitting into a simple feedback loop. The time-varying amplitude of the driving is described by the function (6.67). Parameters: $\gamma = 1$, $\tau = 10$, $\Omega = 1/4$, and $T = 2$ (in arbitrary units), for several values of η and ϕ. The grey lines denote the exact results derived from (6.64). The dashed black lines denote the results of a simulation based on the theory of cascaded systems, with at most two excitations in each subsystem.

into the Heisenberg–Langevin equation (6.11) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\left[A(t), \sum_j \left(\Omega_j(t)a_j^\dagger(t) + \Omega_j^*(t)a_j(t)\right) - i\sum_{jk}\Delta_{jk}a_j^\dagger(t)a_k(t)\right] \\ + \sum_{jk}\int_0^t \mathrm{d}t_1 \left(F_{jk}(t-t_1)\left[a_j^\dagger(t), A(t)\right]a_k(t_1) + F_{jk}^*(t-t_1)a_k^\dagger(t_1)\left[A(t), a_j(t)\right]\right) \\ + i\sum_j\left(\left[a_j^\dagger(t), A(t)\right]B_j(t) - B_j^\dagger(t)\left[A(t), a_j(t)\right]\right), \end{aligned} \tag{6.69}$$

for an operator $A(t)$ without any explicit time-dependence. Once again, we will assume that the system operators obey bosonic commutation relations,

$$\left[a_j, a_k^\dagger\right] = \delta_{jk}\,. \tag{6.70}$$



(a) $\eta = 1.0$, $\phi = 0$ (b) $\eta = 1.0$, $\phi = \pi$

(c) $\eta = 0.5$, $\phi = 0$ (d) $\eta = 0.5$, $\phi = \pi$

Figure 6.7: Excitation number in a driven two-state system emitting into a simple feedback loop, derived from a simulation based on the theory of cascaded systems. The time-varying amplitude of the driving is described by the function (6.67). Parameters: $\gamma = 1$, $\tau = 10$, $\Omega = 1$, and $T = 2$ (in arbitrary units), for several values of η and ϕ.

In this case we can use (6.69) to find

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j(t) = -i\sum_k \Delta_{jk}a_k(t) - \sum_k \int_0^t \mathrm{d}t_1\, F_{jk}(t-t_1)a_k(t_1) - i\left(\Omega_j(t) + B_j(t)\right)\,. \quad (6.71)$$

We have N such equations, one for each $a_j(t)$, and it's convenient to write these equations in vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{a}(t) = -i\Delta\vec{a}(t) - \int_0^t \mathrm{d}t_1\, F(t-t_1)\vec{a}(t_1) - i\left(\vec{\Omega}(t) + \vec{B}(t)\right)\,, \quad (6.72)$$

where Δ and $F(t-t_1)$ are $N \times N$ matrices.

Just as we did in the case of a single system degree of freedom, we can take the Laplace transform of (6.72), and solve to obtain

$$\tilde{\vec{a}}(s) = \tilde{V}(s)\left(\vec{a} - i\left(\tilde{\vec{\Omega}}(t) + \tilde{\vec{B}}(s)\right)\right)\,, \quad (6.73)$$

where

$$\tilde{V}(s) = \left(sI_N + i\Delta + \tilde{F}(s)\right)^{-1}. \tag{6.74}$$

The Green's function $V(t)$ therefore satisfies a form of (4.27), specialised to the present case:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -i\Delta V(t) - \int_0^t \mathrm{d}t_1\, F(t-t_1)V(t_1)\,, \qquad V(0) = I_N\,. \tag{6.75}$$

Inverting the transforms in (6.73) gives

$$\vec{a}(t) = V(t)\vec{a} - i(V * \vec{\Omega})(t) - i\vec{C}(t)\,, \tag{6.76}$$

where

$$\vec{C}(t) = (V * \vec{B})(t)\,. \tag{6.77}$$

Naturally, equation (6.76) can be decomposed into its components to give

$$a_j(t) = \sum_k V_{jk}(t)a_k - i\sum_k (V_{jk} * \Omega_k)(t) - iC_j(t)\,, \tag{6.78}$$

where

$$C_j(t) = \sum_k (V_{jk} * B_k)(t)\,. \tag{6.79}$$

Once again, it is clear that if the system is initially in a vacuum state, each subsystem is driven into a coherent state: after a time t, the j^{th} subsystem is in a coherent state with amplitude $-i\sum_k (V_{jk} * \Omega_k)(t)$.

6.4.3 *A single bosonic subsystem with parametric down-conversion*

Returning to the case of a single system degree of freedom we considered in section 6.4.1, we now augment the system Hamiltonian (6.50) with an additional, nonlinear term. This gives us a new Hamiltonian:

$$H_S(t) = \left(\Omega(t)a^\dagger + \Omega^*(t)a\right) + \Delta a^\dagger a + \frac{1}{2}\left(\chi a^\dagger a^\dagger + \chi^* aa\right)\,. \tag{6.80}$$

As in section 6.4.1 we have the commutation relation $[a, a^\dagger] = 1$. The new term, $(\chi a^\dagger a^\dagger + \chi^* aa)/2$, corresponds to parametric down-conversion—the generator of squeezed light.[65,66]

Substituting (6.80) into the Heisenberg–Langevin equation (6.49) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\left[A(t), \left(\Omega(t)a^\dagger(t) + \Omega^*(t)a(t)\right) + \Delta a^\dagger(t)a(t) + \frac{1}{2}\left(\chi a^\dagger(t)a^\dagger(t) + \chi^* a(t)a(t)\right)\right] \\ + \int_0^t \mathrm{d}t_1 \left(f(t-t_1)\left[a^\dagger(t), A(t)\right]a(t_1) + f^*(t-t_1)a^\dagger(t_1)\left[A(t), a(t)\right]\right) \\ + i\left(\left[a^\dagger(t), A(t)\right]B(t) - B^\dagger(t)\left[A(t), a(t)\right]\right) . \end{aligned} \tag{6.81}$$

This equation leads to a pair of coupled Heisenberg–Langevin equations for $a(t)$ and $a^\dagger(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = -i\Delta a(t) - i\chi a^\dagger(t) - \int_0^t \mathrm{d}t_1\, f(t-t_1)a(t_1) - i\left(\Omega(t) + B(t)\right) , \tag{6.82}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a^\dagger(t) = i\Delta a^\dagger(t) + i\chi^* a(t) - \int_0^t \mathrm{d}t_1\, f^*(t-t_1)a^\dagger(t_1) + i\left(\Omega^*(t) + B^\dagger(t)\right) . \tag{6.83}$$

So long as the Heisenberg–Langevin equations remain linear, as they do in this case, a simple analytic solution like the one developed in section 6.4.1 is still possible. In order to solve these equations, we first write them in vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{a}'(t) = -i\Delta'\vec{a}'(t) - \int_0^t \mathrm{d}t_1\, F'(t-t_1)\vec{a}'(t) - i\left(\vec{\Omega}'(t) + \vec{B}'(t)\right) , \tag{6.84}$$

where we have defined the vector quantities

$$\vec{a}'(t) = \begin{pmatrix} a(t) \\ a^\dagger(t) \end{pmatrix} , \qquad \vec{\Omega}' = \begin{pmatrix} \Omega(t) \\ -\Omega^*(t) \end{pmatrix} , \qquad \vec{B}'(t) = \begin{pmatrix} B(t) \\ -B^\dagger(t) \end{pmatrix} , \tag{6.85}$$

and the matrices

$$\Delta' = \begin{pmatrix} \Delta & \chi \\ -\chi^* & -\Delta \end{pmatrix} , \qquad F'(t-t_1) = \begin{pmatrix} f(t-t_1) & 0 \\ 0 & f^*(t-t_1) \end{pmatrix} . \tag{6.86}$$

Equation (6.84) is solved in the same way as equation (6.73). Taking the Laplace transform of (6.84) and solving yields

$$\tilde{\vec{a}}'(s) = \tilde{V}'(s)\left(\vec{a} - i\left(\tilde{\vec{\Omega}}'(t) + \tilde{\vec{B}}'(s)\right)\right) , \tag{6.87}$$

where

$$\tilde{V}'(s) = \left(sI_2 + i\Delta' + \tilde{F}'(s)\right)^{-1} . \tag{6.88}$$

This Green's function therefore satisfies a modified form of (4.27):

$$\frac{\mathrm{d}}{\mathrm{d}t}V'(t) = -i\Delta'V'(t) - \int_0^t \mathrm{d}t_1\, F'(t-t_1)V'(t_1) , \qquad V'(0) = I_2 . \tag{6.89}$$

Inverting the transforms in (6.87) gives

$$\vec{a}'(t) = V'(t)\vec{a} - i(V' * \vec{\Omega}')(t) - i\vec{C}'(t)\,, \tag{6.90}$$

where

$$\vec{C}'(t) = (V' * \vec{B}')(t)\,. \tag{6.91}$$

To get an explicit solution for $a(t)$, and thereby $a^\dagger(t)$, we can decompose $V'(t)$ as

$$V'(t) = \begin{pmatrix} v_0(t) & v_1(t) \\ v_1^*(t) & v_0^*(t) \end{pmatrix}\,, \tag{6.92}$$

where (6.89) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(t) = -i\Delta v_0(t) - i\chi v_1^*(t) - \int_0^t \mathrm{d}t_1\, f(t-t_1)v_0(t_1)\,, \qquad v_0(0) = 1\,, \tag{6.93}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v_1(t) = -i\Delta v_1(t) - i\chi v_0^*(t) - \int_0^t \mathrm{d}t_1\, f(t-t_1)v_1(t_1)\,, \qquad v_1(0) = 0\,. \tag{6.94}$$

Substituting (6.92) into (6.90) yields

$$a(t) = v_0(t)a + v_1(t)a^\dagger - i\left((v_0 * \Omega)(t) - (v_1 * \Omega^*)(t)\right) - iC(t)\,, \tag{6.95}$$

where in this case we have

$$C(t) = (v_0 * B)(t) - (v_1 * B^\dagger)(t)\,, \tag{6.96}$$

as distinct from the examples we studied in sections 6.4.1 and 6.4.2.

6.4.4 *Multiple bosonic subsystems with parametric down-conversion*

Just as section 6.4.2 generalised section 6.4.1 to take into account multiple subsystems, here we will generalise section 6.4.3. Once again, we have N subsystems, labelled $j = 1, \ldots, N$, with operators satisfying the commutation relations (6.70). We choose the Hamiltonian:

$$H_S(t) = \sum_j \left(\Omega_j(t)a_j^\dagger + \Omega_j^*(t)a_j\right) + \sum_{jk}\Delta_{jk}a_j^\dagger a_k + \frac{1}{2}\sum_{jk}\left(\chi_{jk}a_j^\dagger a_k^\dagger + \chi_{jk}^* a_k a_j\right)\,, \tag{6.97}$$

where $\Delta_{jk}^* = \Delta_{kj}$ (as before) and $\chi_{jk} = \chi_{kj}$. As we did in section 6.4.2, we choose $\omega_j = \omega_0$ for all j so that we can use the time-homogeneous form (3.20) of the dissipation kernel.

Substituting this Hamiltonian into the Heisenberg–Langevin equation (6.11) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\Bigg[A(t), \sum_j \left(\Omega_j(t)a_j^\dagger(t) + \Omega_j^*(t)a_j(t)\right) - i\sum_{jk}\Delta_{jk}a_j^\dagger(t)a_k(t) \\ + \frac{1}{2}\sum_{jk}\left(\chi_{jk}a_j^\dagger(t)a_k^\dagger(t) + \chi_{jk}^*a_k(t)a_j(t)\right)\Bigg] \\ + \sum_{jk}\int_0^t \mathrm{d}t_1 \left(F_{jk}(t-t_1)\left[a_j^\dagger(t), A(t)\right]a_k(t_1) + F_{jk}^*(t-t_1)a_k^\dagger(t_1)\left[A(t), a_j(t)\right]\right) \\ + i\sum_j\left(\left[a_j^\dagger(t), A(t)\right]B_j(t) - B_j^\dagger(t)\left[A(t), a_j(t)\right]\right), \end{aligned} \tag{6.98}$$

for an operator $A(t)$ without any explicit time-dependence. Equation (6.98) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j(t) = -i\sum_k \Delta_{jk}a_k(t) - i\sum_k \chi_{jk}a_k^\dagger(t) - \sum_k\int_0^t \mathrm{d}t_1\, F_{jk}(t-t_1)a_k(t_1) - i\left(\Omega_j(t) + B_j(t)\right), \tag{6.99}$$

along with the conjugate equations for $a_j^\dagger(t)$. If we combine the operators $a_j(t)$ into a vector $\vec{a}(t)$ and the operators $a_j^\dagger(t)$ into $\vec{a^\dagger}(t)$, we can define a new vector containing all these operators:

$$\vec{a}'(t) = \begin{pmatrix}\vec{a}(t) \\ \vec{a^\dagger}(t)\end{pmatrix}. \tag{6.100}$$

We also define additional vector quantities

$$\vec{\Omega}'(t) = \begin{pmatrix}\vec{\Omega}(t) \\ -\vec{\Omega}^*(t)\end{pmatrix}, \qquad \vec{B}'(t) = \begin{pmatrix}\vec{B}(t) \\ -\vec{B^\dagger}(t)\end{pmatrix}, \tag{6.101}$$

along with the block matrices

$$\Delta' = \begin{pmatrix}\Delta & \chi \\ -\chi^* & -\Delta\end{pmatrix}, \qquad F'(t-t_1) = \begin{pmatrix}F(t-t_1) & 0 \\ 0 & F^*(t-t_1)\end{pmatrix}, \tag{6.102}$$

where I remind the reader that here Δ, χ, and $F(t-t_1)$ are all $N \times N$ matrices. Thus, both Δ' and $F'(t-t_1)$ are $2N \times 2N$ matrices. With these definitions, the differential equations (6.99) for the operators $a_j(t)$, as well as the conjugate equations for the operators $a_j^\dagger(t)$, can be combined into a single vector differential equation with a form identical to that of equation (6.84), but with the various quantities in the equation given by (6.100)–(6.102). The solution for $\vec{a}'(t)$ therefore takes on the form given by (6.90), where $V'(t)$ in this case satisfies the differential equation (6.89), once again with the corresponding parameters replaced with those given by (6.100)–(6.102), and with the additional modification that the initial condition is now $V'(0) = I_{2N}$.

To get an explicit solution for $\vec{a}(t)$ we use the same procedure as in section 6.4.3, decomposing $V'(t)$ as

$$V'(t) = \begin{pmatrix} V_0(t) & V_1(t) \\ V_1^*(t) & V_0^*(t) \end{pmatrix}, \tag{6.103}$$

where (6.89) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0(t) = -i\Delta V_0(t) - i\chi V_1^*(t) - \int_0^t \mathrm{d}t_1\, F(t-t_1)V_0(t_1)\,, \qquad V_0(0) = I_N\,, \tag{6.104}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) = -i\Delta V_1(t) - i\chi V_0^*(t) - \int_0^t \mathrm{d}t_1\, F(t-t_1)V_1(t_1)\,, \qquad V_1(0) = 0\,. \tag{6.105}$$

Substituting (6.103) into (6.90) yields

$$\vec{a}(t) = V_0(t)\vec{a} + V_1(t)\vec{a^\dagger} - i\left((V_0 * \vec{\Omega})(t) - (V_1 * \vec{\Omega}^*)(t)\right) - i\vec{C}(t)\,, \tag{6.106}$$

where in this case we have

$$C(t) = (V_0 * \vec{B})(t) - (V_1 * \vec{B^\dagger})(t)\,. \tag{6.107}$$

Equation (6.106) can now be decomposed into its components to give

$$a_j(t) = \sum_k \left(V_{0,jk}(t)a_k + V_{1,jk}(t)a_k^\dagger\right) - i\sum_k \left((V_{0,jk} * \Omega_k)(t) - (V_{1,jk} * \Omega_k^*)(t)\right) - iC_j(t)\,, \tag{6.108}$$

where

$$C_j(t) = \sum_k \left((V_{0,jk} * B_k)(t) - (V_{1,jk} * B_k^\dagger)(t)\right)\,. \tag{6.109}$$

Equations (6.106) and (6.107) – or equivalently (6.108) and (6.109) – represent the solutions of the most general Heisenberg–Langevin equations we have yet written down, namely those for the assemblage of interacting harmonic oscillators with system Hamiltonian (6.97). These solutions can be used to evaluate all desired system expectation values and correlation functions. In the Schrödinger picture, on the other hand, calculating multi-time correlation functions is somewhat unnatural—after all, these expressions are defined by way of the Heisenberg picture. In the presence of damping, the evolution of the reduced system – as described by a master equation – is non-unitary, and this prevents us from calculating general multi-time correlation functions. In the Markovian case, we can calculate certain correlation functions using the quantum regression formula,[49,67–69] but while there do exist perturbative formulas for calculating two-time correlation functions in the non-Markovian case,[70,71] there is nothing analogous to the quantum regression formula for general non-Markovian systems. This makes the Heisenberg picture the more natural

choice when correlation functions are required.

Despite these attractive features of the Heisenberg–Langevin equation approach, master equations are very frequently employed in the study of open quantum systems. As such, in the next chapter we will show that some of the analytic solutions we have obtained can be used to derive a Schrödinger picture description of the dynamics, in the form of time-local master equations.

7

TIME-LOCAL MASTER EQUATIONS

In chapter 6 we derived Heisenberg–Langevin equations for system operators, which were exact within our model. As useful as these equations are, one often wants a Schrödinger picture description of the dynamics: a master equation. In this chapter we turn our attention to master equations, in particular those we can derive from the Heisenberg–Langevin equations we already know about.

Suppose we have a differential equation for the expectation value of an arbitrary system operator, $\langle A(t)\rangle$. A master equation for a reduced system density operator $\rho_S(t)$ must gives the same results as this moment equation, in the sense that we should have

$$\mathrm{tr}\left(A_S(t)\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle \ , \tag{7.1}$$

where $A_S(t)$ is the (in general time-dependent) Schrödinger picture system operator corresponding to the Heisenberg picture operator $A(t)$. If we simply take the expectation value of the general Heisenberg–Langevin equation (6.11), however, we are immediately presented with a problem: the resulting moment equation will involve two-time, and in general multi-time, correlation functions. As mentioned at the end of the previous chapter, there is no quantum regression formula for calculating correlation functions in general non-Markovian systems. In the absence of any systematic way of evaluating multi-time correlation functions in the Schrödinger picture, we cannot write down a master equation corresponding to the Heisenberg dynamics. We can, however, try to rewrite the Heisenberg equation (6.11) in a time-local form that permits a transformation back into the Schrödinger picture.

In section 7.1 we derive time-local master equations corresponding to the solutions of Heisenberg–Langevin equations we obtained in sections 6.4.1 and 6.4.2. In section 7.2 we discuss whether the master equation can be generalised to allow for fermionic subsystems.

7.1 DERIVATION OF THE TIME-LOCAL MASTER EQUATIONS

Consider equations (6.6) and (6.10). For an operator $A(t)$ without any explicit time-dependence, these equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\left[A(t), H_S(t)\right] + i\sum_j \left(\left[a_j^\dagger(t), A(t)\right]E_j(t) - E_j^\dagger(t)\left[A(t), a_j(t)\right]\right) \ , \tag{7.2}$$

and

$$E_j(t) = B_j(t) - i\sum_k \int_0^t \mathrm{d}t_1\, F_{jk}(t-t_1)a_k(t_1)\,, \tag{7.3}$$

where we have additionally assumed – as we always can – that $H_S(t)$ is defined in such a way that we can use the homogeneous form of the dissipation kernel, which is given by (3.17). In the case where the system operators have bosonic commutation relations, $[a_j, a_k^\dagger] = \delta_{jk}$, substituting $a_j(t)$ into (7.2) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j(t) = -i\left[a_j(t), H_S(t)\right] - iE_j(t)\,. \tag{7.4}$$

We similarly obtain the conjugate equation for $a_j^\dagger(t)$. Thus, equation (7.2) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\left[A(t), H_S(t)\right] - \sum_j \Bigg(&\left[a_j^\dagger(t), A(t)\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}a_j(t) + i\left[a_j(t), H_S(t)\right]\right)\\ &+\left(\frac{\mathrm{d}}{\mathrm{d}t}a_j^\dagger(t) + i\left[a_j^\dagger(t), H_S(t)\right]\right)\left[A(t), a_j(t)\right]\Bigg)\,.\end{aligned} \tag{7.5}$$

If we substitute the linear system Hamiltonian (6.68) into this equation, several terms cancel and we obtain the simpler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -\sum_j \left(\left[a_j^\dagger(t), A(t)\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}a_j(t)\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}a_j^\dagger(t)\right)\left[A(t), a_j(t)\right]\right)\,. \tag{7.6}$$

In section 6.4, we obtained a series of analytic solutions of Heisenberg–Langevin equations. Equation (7.6) will allow us to transform the solutions we obtained in section 6.4.1 and section 6.4.2 – the linear cases – into time-local Heisenberg–Langevin equations, from which we will be able to derive time-local master equations for these systems.

7.1.1 *A single bosonic subsystem*

Let us begin by considering the simplest case: the single system degree of freedom described by the system Hamiltonian (6.50). In this case, equation (7.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -\left[a^\dagger(t), A(t)\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}a(t)\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}a^\dagger(t)\right)\left[A(t), a(t)\right]\,, \tag{7.7}$$

where $a(t)$ satisfies the differential equation (6.52). This result can also be obtained directly from equation (6.51). Using the solution of (6.52) we have already obtained, namely (6.57), we can rewrite equation (6.52) in a time-local form and thereby derive a time-local form of the Heisenberg–Langevin equation for this system. Differentiating (6.57) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right)a - i\frac{\mathrm{d}}{\mathrm{d}t}(v * \Omega)(t) - i\frac{\mathrm{d}}{\mathrm{d}t}C(t)\,. \tag{7.8}$$

Using equation (6.57) again, we can rewrite (7.8) in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right)\frac{1}{v(t)}\left(a(t)+i(v*\Omega)(t)+iC(t)\right)-i\frac{\mathrm{d}}{\mathrm{d}t}(v*\Omega)(t)-i\frac{\mathrm{d}}{\mathrm{d}t}C(t)\\ &= -\gamma(t)a(t)-i\left(\xi(t)+D(t)\right)\,, \end{aligned} \tag{7.9}$$

where

$$\gamma(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right)\frac{1}{v(t)}\,, \tag{7.10}$$

$$\xi(t) = \left(\gamma(t)+\frac{\mathrm{d}}{\mathrm{d}t}\right)(v*\Omega)(t)\,, \tag{7.11}$$

$$D(t) = \left(\gamma(t)+\frac{\mathrm{d}}{\mathrm{d}t}\right)C(t)\,. \tag{7.12}$$

Substituting (7.9) into (7.7) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) &= -i\left[A(t),\xi(t)a^\dagger(t)+\xi^*(t)a(t)\right]+\gamma(t)\left[a^\dagger(t),A(t)\right]a(t)+\gamma^*(t)a^\dagger(t)\left[A(t),a(t)\right]\\ &\quad+i\left(\left[a^\dagger(t),A(t)\right]D(t)-D^\dagger(t)\left[A(t),a(t)\right]\right)\,. \end{aligned} \tag{7.13}$$

Equation (7.13) is a time-local form of the Heisenberg–Langevin equation (6.51).

If we take expectation values on both sides of equation (7.13), we obtain the moment equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle A(t)\right\rangle &= -i\left\langle\left[A(t),\xi(t)a^\dagger(t)+\xi^*(t)a(t)\right]\right\rangle\\ &\quad+\gamma(t)\left\langle\left[a^\dagger(t),A(t)\right]a(t)\right\rangle+\gamma^*(t)\left\langle a^\dagger(t)\left[A(t),a(t)\right]\right\rangle\\ &\quad+i\left\langle\left[a^\dagger(t),A(t)\right]D(t)-D^\dagger(t)\left[A(t),a(t)\right]\right\rangle\,. \end{aligned} \tag{7.14}$$

To evaluate the last term in this equation, we will need to calculate expectation values of the form $\langle(a^\dagger(t))^m(a(t))^{m'-1}D(t)\rangle$. We will use Wick's theorem, as presented in appendix D, to factor out the second-order moment $\langle a^\dagger(t)D(t)\rangle$ from this expectation value. If we take the conjugate of (6.57), we get

$$a^\dagger(t) = v^*(t)a^\dagger+i(v^**\Omega^*)(t)+iC^\dagger(t)\,. \tag{7.15}$$

It is clear, therefore, that each factor of $a^\dagger(t)$ in the expectation value contributes a factor of $iC^\dagger(t)$, which makes a non-zero contribution when contracted (paired off) with $D(t)$. The other factors $v^*(t)a^\dagger$ and $i(v^**\Omega^*)(t)$ make no contribution to the second-order moment, because the average of $D(t)$ in the thermal environment (3.7) is zero. We therefore find $\langle a^\dagger(t)D(t)\rangle = i\langle C^\dagger(t)D(t)\rangle$. We can use the commutator $[a(t),\cdot]$ to count off the factors of $a^\dagger(t)$ and thereby enumerate the contractions. Thus, equation (D.7) immediately gives the

relation

$$\left\langle (a^\dagger(t))^m (a(t))^{m'-1} D(t) \right\rangle = i \left\langle C^\dagger(t) D(t) \right\rangle \left\langle \left[a(t), (a^\dagger(t))^m (a(t))^{m'-1} \right] \right\rangle . \tag{7.16}$$

To extend this to a general system operator $A(t)$, we note that any system operator can be written as a sum of normal-ordered products of $a^\dagger(t)$ and $a(t)$, potentially with (most generally) an operator-valued pre-factor which by construction does not involve any environment operators. Therefore, because of the linearity of the commutator, equation (7.16) leads to the identity

$$\langle A(t) D(t) \rangle = i \left\langle C^\dagger(t) D(t) \right\rangle \langle [a(t), A(t)] \rangle . \tag{7.17}$$

Using equation (D.8), we can follow an analogous process to obtain another identity

$$\left\langle D^\dagger(t) A(t) \right\rangle = -i \left\langle D^\dagger(t) C(t) \right\rangle \left\langle [A(t), a^\dagger(t)] \right\rangle . \tag{7.18}$$

Using the identities (7.17) and (7.18), we can readily show that the last term in (7.14) is given by

$$i \left\langle [a^\dagger(t), A(t)]\, D(t) - D^\dagger(t)\, [A(t), a(t)] \right\rangle = \lambda(t) \left\langle [[a^\dagger(t), A(t)], a(t)] \right\rangle , \tag{7.19}$$

where we have defined

$$\lambda(t) = \left\langle C^\dagger(t) D(t) \right\rangle + \left\langle D^\dagger(t) C(t) \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t} w(t) + \gamma(t) w(t) + w(t) \gamma^*(t) , \tag{7.20}$$

with

$$w(t) = \left\langle C^\dagger(t) C(t) \right\rangle = \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, v(t - t_1) g(t - t_1) v^*(t - t_2) . \tag{7.21}$$

Recall that the notation $g(t - t_1)$ was introduced in section 3.3 to be used in the case of a single noise kernel. Equation (7.14) now becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle A(t) \rangle = & -i \left\langle [A(t), \xi(t) a^\dagger(t) + \xi^*(t) a(t)] \right\rangle \\ & + \gamma(t) \left\langle [a^\dagger(t), A(t)]\, a(t) \right\rangle + \gamma^*(t) \left\langle a^\dagger(t)\, [A(t), a(t)] \right\rangle \\ & + \lambda(t) \left\langle [[a^\dagger(t), A(t)], a(t)] \right\rangle . \end{aligned} \tag{7.22}$$

This moment equation involves only system operators—the noise operators have been formally eliminated.

Our aim here is to find a master equation for a reduced system density operator $\rho_S(t)$ that gives the same results as (7.22), in the sense that the condition (7.1) should hold for any Schrödinger picture system operator $A(t)$. Because (7.22) is in a time-local form

involving only system operators, we can simply read off the master equation by inspection:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\left[\xi(t)a^\dagger + \xi^*(t)a, \rho_S(t)\right] + \gamma(t)\left[a\rho_S(t), a^\dagger\right] + \gamma^*(t)\left[a, \rho_S(t)a^\dagger\right] + \lambda(t)\left[[a, \rho_S(t)], a^\dagger\right] . \tag{7.23}$$

This equation is equivalent to the bosonic form of the master equation derived by Lei and Zhang, in the case of a single system degree of freedom.[72] The zero-temperature version of this master equation was derived much earlier by Cresser.[44]

Equation (7.23) can also be written in the canonical form (2.61). To do this, we first use (7.20) to express (7.23) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = &-i\left[\xi(t)a^\dagger + \xi^*(t)a, \rho_S(t)\right] \\ &+ \left(\gamma(t) + \langle C^\dagger(t)D(t)\rangle\right)\left[a\rho_S(t), a^\dagger\right] + \left(\gamma^*(t) + \langle D^\dagger(t)C(t)\rangle\right)\left[a, \rho_S(t)a^\dagger\right] \\ &+ \langle C^\dagger(t)D(t)\rangle\left[a^\dagger, \rho_S(t)a\right] + \langle D^\dagger(t)C(t)\rangle\left[a^\dagger\rho_S(t), a\right] .\end{aligned} \tag{7.24}$$

If we now define $\gamma_1(t)/2 + i\Delta_1(t) = \gamma(t) + \langle C^\dagger(t)D(t)\rangle$ and $\gamma_2(t)/2 + i\Delta_2(t) = \langle D^\dagger(t)C(t)\rangle$, equation (7.24) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = &-i\left[\xi(t)a^\dagger + \xi^*(t)a + \Delta_1(t)a^\dagger a + \Delta_2(t)aa^\dagger, \rho_S(t)\right] \\ &+ \gamma_1(t)\left(a\rho_S(t)a^\dagger - \frac{1}{2}\left\{a^\dagger a, \rho_S(t)\right\}\right) + \gamma_2(t)\left(a^\dagger\rho_S(t)a - \frac{1}{2}\left\{aa^\dagger, \rho_S(t)\right\}\right) ,\end{aligned} \tag{7.25}$$

which is indeed in the form (2.61), with $H(t) = \xi(t)a^\dagger + \xi^*(t)a + \Delta_1(t)a^\dagger a + \Delta_2(t)aa^\dagger$, $A_1 = a$, and $A_2 = a^\dagger$.

When the frequency-shifted spectral density is constant we recover – subject to a small approximation – a Markovian master equation, as we will show below. If we choose the constant spectral density $J'(\omega) = \gamma/\pi$, the dissipation kernel becomes $f(t-t_1) = 2\gamma\delta(t-t_1)$, giving $\gamma(t) = \gamma + i\Delta$, $\xi(t) = \Omega$, and $D(t) = B(t)$. In this case we find that $v(t) = \exp(-\gamma t)$ and therefore that

$$\langle C^\dagger(t)D(t)\rangle = \int_0^t \mathrm{d}t_1\, e^{-\gamma(t-t_1)}g(t-t_1) , \tag{7.26}$$

with $\langle D^\dagger(t)C(t)\rangle = \langle C^\dagger(t)D(t)\rangle^*$. The noise kernel, given by (3.14), is difficult to evaluate, but we can get around this by making this is the approximation mentioned above: we neglect the frequency dependence of the thermal distribution of photon numbers in the environment, and therefore let $n(\omega + \omega_0) \approx \bar{n}$. This yields

$$g(t-t_1) = \frac{\gamma\bar{n}}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\, e^{-i\omega(t-t_1)} = \bar{n}f(t-t_1) , \tag{7.27}$$

which gives

$$\langle C^\dagger(t)D(t)\rangle = \langle D^\dagger(t)C(t)\rangle = \gamma\bar{n}\,, \tag{7.28}$$

which in turn means that (7.24) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = &-i\left[\Omega\left(a^\dagger + a\right) + \Delta a^\dagger a, \rho_S(t)\right] \\ &+ \gamma(1+\bar{n})\left(2a\rho_S(t)a^\dagger - a^\dagger a\rho_S(t) - \rho_S(t)a^\dagger a\right) \\ &+ \gamma\bar{n}\left(2a^\dagger\rho_S(t)a - aa^\dagger\rho_S(t) - \rho_S(t)aa^\dagger\right)\,,\end{aligned} \tag{7.29}$$

which is of course the standard Lindblad master equation.

7.1.2 *Multiple bosonic subsystems*

We now consider the case of multiple subsystems. As mentioned above, substituting the Hamiltonian (6.68) for multiple subsystems into equation (7.5) yields the Heisenberg–Langevin equation (7.6). In this case $a_j(t)$ satisfies the differential equation (6.71). Using the same procedure as in section 7.1.1, we can easily show that the derivative of (6.76) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{a}(t) = -\Gamma(t)\vec{a}(t) - i\left(\vec{\xi}(t) + \vec{D}(t)\right)\,, \tag{7.30}$$

where

$$\Gamma(t) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right)V^{-1}(t)\,, \tag{7.31}$$

$$\vec{\xi}(t) = \left(\Gamma(t) + \frac{\mathrm{d}}{\mathrm{d}t}\right)(V * \vec{\Omega})(t)\,, \tag{7.32}$$

$$\vec{D}(t) = \left(\Gamma(t) + \frac{\mathrm{d}}{\mathrm{d}t}\right)\vec{C}(t)\,. \tag{7.33}$$

Equation (7.30) is easily decomposed to yield the time-local differential equations for the individual $a_j(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j(t) = -\sum_k \Gamma_{jk}(t)a_k(t) - i\left(\xi_j(t) + D_j(t)\right)\,. \tag{7.34}$$

Equation (7.6) therefore becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = &-i\sum_j \left[A(t), \xi_j(t)a_j^\dagger(t) + \xi_j^*(t)a_j(t)\right] \\ &+ \sum_{jk}\left(\Gamma_{jk}(t)\left[a_j^\dagger(t), A(t)\right]a_k(t) + \Gamma_{jk}^*(t)a_k^\dagger(t)\left[A(t), a_j(t)\right]\right) \\ &+ i\sum_j\left(\left[a_j^\dagger(t), A(t)\right]D_j(t) - D_j^\dagger(t)\left[A(t), a_j(t)\right]\right). \end{aligned} \tag{7.35}$$

This is a time-local form of the Heisenberg–Langevin equation (6.69).

Taking expectation values on both sides of (7.35) yields the moment equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle = &-i\sum_j \left\langle\left[A(t), \xi_j(t)a_j^\dagger(t) + \xi_j^*(t)a_j(t)\right]\right\rangle \\ &+ \sum_{jk}\left(\Gamma_{jk}(t)\left\langle\left[a_j^\dagger(t), A(t)\right]\right\rangle a_k(t) + \Gamma_{jk}^*(t)\left\langle a_k^\dagger(t)\left[A(t), a_j(t)\right]\right\rangle\right) \\ &+ i\sum_j\left\langle\left[a_j^\dagger(t), A(t)\right]D_j(t) - D_j^\dagger(t)\left[A(t), a_j(t)\right]\right\rangle. \end{aligned} \tag{7.36}$$

We now need to factorise the last term in (7.36), as we did in section 7.1.1. The argument – involving Wick's theorem – that we used previously generalises straightforwardly to the multivariate case. First, we will consider the second-order moment $\langle a_k^\dagger(t)D_j(t)\rangle$. Taking the conjugate of (6.78) yields

$$a_k^\dagger(t) = \sum_l V_{kl}^*(t)a_l^\dagger + i\sum_l (V_{kl}^* * \Omega_l^*)(t) + iC_k^\dagger(t)\,. \tag{7.37}$$

Each factor of $a_k^\dagger(t)$ in the expectation value therefore contributes a factor of $iC_k^\dagger(t)$, which makes a non-zero contribution when contracted with $D_j(t)$. The other terms in $a_k^\dagger(t)$ make no contribution to the second-order moment, so we find that $\langle a_k^\dagger(t)D_j(t)\rangle = i\,\langle C_k^\dagger(t)D_j(t)\rangle$. Now consider the expectation value $\langle\prod_k \left(a_k^\dagger(t)\right)^{m_k}(a_k(t))^{m_k'-1}D_j(t)\rangle$, and recall that the environment is assumed to be in the thermal state (3.7). Wick's theorem, in the form (D.7), can be applied separately to each of the normal-ordered operator products in this expectation value, and just as we did in section 7.1.1, we can use the commutators $[a_k(t), \cdot]$ to enumerate the contractions. This process yields the result

$$\begin{aligned}&\left\langle\prod_k\left(a_k^\dagger(t)\right)^{m_k}(a_k(t))^{m_k'-1}D_j(t)\right\rangle \\ &\qquad = i\sum_k\left\langle C_k^\dagger(t)D_j(t)\right\rangle\left\langle\left[a_k(t), \prod_l\left(a_l^\dagger(t)\right)^{m_l}(a_l(t))^{m_l'-1}\right]\right\rangle. \end{aligned} \tag{7.38}$$

Any product $A(t)$ can be written as a sum of normal-ordered products like those in the

equation above. Thus, this equation immediately generalises to

$$\langle A(t)D_j(t)\rangle = i\sum_k \left\langle C_k^\dagger(t)D_j(t)\right\rangle \langle [a_k(t), A(t)]\rangle \ . \tag{7.39}$$

An analogous argument leads to the conjugate equation

$$\left\langle D_j^\dagger(t)A(t)\right\rangle = -i\sum_k \left\langle D_j^\dagger(t)C_k(t)\right\rangle \left\langle \left[A(t), a_k^\dagger(t)\right]\right\rangle \ . \tag{7.40}$$

Using the identities (7.39) and (7.40), we can readily show that the last term in (7.36) can be written as

$$i\sum_j \left\langle \left[a_j^\dagger(t), A(t)\right] D_j(t) - D_j^\dagger(t)\left[A(t), a_j(t)\right]\right\rangle = \sum_{jk}\Lambda_{jk}(t)\left\langle \left[\left[a_j^\dagger(t), A(t)\right], a_k(t)\right]\right\rangle \ , \tag{7.41}$$

where we define corresponding generalisations of the quantities introduced in section 7.1.1:

$$\Lambda_{jk}(t) = \left\langle C_k^\dagger(t)D_j(t)\right\rangle + \left\langle D_k^\dagger(t)C_j(t)\right\rangle \ , \tag{7.42}$$

and

$$W_{jk}(t) = \left\langle C_k^\dagger(t)C_j(t)\right\rangle \ , \tag{7.43}$$

leading to

$$\Lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t}W(t) + \Gamma(t)W(t) + W(t)\Gamma^\dagger(t)\,, \tag{7.44}$$

and

$$W(t) = \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_1\, V(t-t_1)G(t_1,t_2)V^\dagger(t-t_2)\,, \tag{7.45}$$

where $G(t_1, t_2)$ is the matrix formed by the noise kernels (3.14).

Substituting (7.41) into the moment equation (7.36) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle = &-i\sum_j \left\langle \left[A(t), \xi_j(t)a_j^\dagger(t) + \xi_j^*(t)a_j(t)\right]\right\rangle \\ &+ \sum_{jk}\left(\Gamma_{jk}(t)\left\langle \left[a_j^\dagger(t), A(t)\right]\right\rangle a_k(t) + \Gamma_{jk}^*(t)\left\langle a_k^\dagger(t)\left[A(t), a_j(t)\right]\right\rangle\right) \\ &+ \sum_{jk}\Lambda_{jk}(t)\left\langle \left[\left[a_j^\dagger(t), A(t)\right], a_k(t)\right]\right\rangle \ .\end{aligned} \tag{7.46}$$

Once again, because the moment equation is now in a time-local form involving only system

operators, we can read off the master equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = &-i\sum_j \left[\xi_j(t)a_j^\dagger + \xi_j^*(t)a_j, \rho_S(t)\right] \\ &+ \sum_{jk}\left(\Gamma_{jk}(t)\left[a_k\rho_S(t), a_j^\dagger\right] + \Gamma_{jk}^*(t)\left[a_j, \rho_S(t)a_k^\dagger\right]\right) \\ &+ \sum_{jk}\Lambda_{jk}(t)\left[[a_k, \rho_S(t)], a_j^\dagger\right] . \end{aligned} \tag{7.47}$$

This equation is equivalent to the bosonic form of the master equation derived by Lei and Zhang.[72] There also exists a corresponding master equation for fermionic systems in a fermionic reservoir, which was derived by Jin et al.[73] Both the bosonic and fermionic versions of this master equation were solved in the general case by Zhang et al.[74] Equation (7.47) generalises the master equation (4.33) we derived in section 4.3 to take into account coherent driving, and a reservoir at non-zero temperature—but only in the bosonic case.

As an example of a derivation of a time-local master equation, we will consider a system comprising a pair of interacting bosonic subsystems. We assume that the first subsystem interacts with a feedback reservoir, and that the frequency-shifted spectral density describing this interaction is given by

$$J_1'(\omega) = \frac{\gamma}{\pi}\left(1 + \cos(\omega\tau + \phi)\right) . \tag{7.48}$$

For the sake of simplicity, we will assume that the second subsystem is undamped, and that the two subsystems are on resonance. The interaction between the two subsystem is described by the Hamiltonian

$$H_S = g\left(a_1^\dagger a_2 + a_2^\dagger a_1\right) . \tag{7.49}$$

Following the analysis presented in section 6.4, we can easily derive the following Heisenberg–Langevin equations for the annihilation operators of the two subsystems:

$$\frac{\mathrm{d}}{\mathrm{d}t}a_1(t) = -iga_2(t) - \gamma\left(a_1(t) + \theta(t-\tau)e^{i\phi}a_1(t-\tau)\right) - iB_1(t) , \tag{7.50}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a_2(t) = -iga_1(t) - iB_2(t) . \tag{7.51}$$

The solutions to these equations are given by

$$\vec{a}(t) = V(t)\vec{a} - i\vec{C}(t) , \tag{7.52}$$

where, following section 6.4.2, we have $\vec{a} = (a_1, a_2)^T$, and $\vec{C}(t) = (V * \vec{B})(t)$ with $\vec{B}(t) = (B_1(t), B_2(t))^T$. The Green's function $V(t)$ is a 2×2 matrix satisfying the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -AV(t) - \theta(t-\tau)Be^{i\phi}V(t-\tau) , \qquad V(0) = I_2 , \tag{7.53}$$

where

$$A = \begin{pmatrix} \gamma & ig \\ ig & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} . \tag{7.54}$$

Equation (7.53) can easily be solved by numerical integration. The solution (7.52) can of course also be written as a pair of equations:

$$a_1(t) = V_{11}(t)a_1 + V_{12}(t)a_2 - iC_1(t) , \tag{7.55}$$
$$a_2(t) = V_{21}(t)a_1 + V_{22}(t)a_2 - iC_2(t) . \tag{7.56}$$

Therefore, if we assume that the environment is in a vacuum state, the mean number of excitations in the each subsystem at time t is given by

$$\begin{aligned} \left\langle a_1^\dagger(t)a_1(t) \right\rangle &= |V_{11}(t)|^2 \left\langle a_1^\dagger a_1 \right\rangle + V_{11}^*(t)V_{12}(t) \left\langle a_1^\dagger a_2 \right\rangle \\ &\quad + V_{12}^*(t)V_{11}(t) \left\langle a_2^\dagger a_1 \right\rangle + |V_{12}(t)|^2 \left\langle a_2^\dagger a_2 \right\rangle , \end{aligned} \tag{7.57}$$
$$\begin{aligned} \left\langle a_2^\dagger(t)a_2(t) \right\rangle &= |V_{21}(t)|^2 \left\langle a_1^\dagger a_1 \right\rangle + V_{21}^*(t)V_{22}(t) \left\langle a_1^\dagger a_2 \right\rangle \\ &\quad + V_{22}^*(t)V_{21}(t) \left\langle a_2^\dagger a_1 \right\rangle + |V_{22}(t)|^2 \left\langle a_2^\dagger a_2 \right\rangle . \end{aligned} \tag{7.58}$$

If we additionally assume that the second subsystem is initially in its ground state, these expectation values reduce to

$$\left\langle a_1^\dagger(t)a_1(t) \right\rangle = |V_{11}(t)|^2 \left\langle a_1^\dagger a_1 \right\rangle , \tag{7.59}$$
$$\left\langle a_2^\dagger(t)a_2(t) \right\rangle = |V_{21}(t)|^2 \left\langle a_1^\dagger a_1 \right\rangle . \tag{7.60}$$

In a vacuum reservoir, and because this system is undriven, the master equation (7.47) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \sum_{jk} \left(\Gamma_{jk}(t) \left[a_k \rho_S(t), a_j^\dagger \right] + \Gamma_{jk}^*(t) \left[a_j, \rho_S(t) a_k^\dagger \right] \right) , \tag{7.61}$$

where $\Gamma(t)$ is given by (7.31). This master equation has exactly the same form as equation (4.33), but applies to bosonic systems containing multiple excitations. Figure 7.1 compares the analytic solutions (7.59) and (7.60) with the results obtained by integrating the master equation (7.61), showing that these two approaches agree closely.

Before we move on, I want to comment briefly on the models considered in sections 6.4.3 and 6.4.4, in which we augmented the Hamiltonian to include a nonlinear term describing parametric down-conversion. Even in the presence of this non-linearity, the Heisenberg–Langevin equations for the system operators $a_j(t)$ and $a_j^\dagger(t)$ – such as equation (6.99) – can be written in a time-local form using much the same procedure as we used above. The time-local equations of motion for the creation and annihilation operators may then be decoupled by way of a Bogoliubov transformation. Alternatively, this transformation may be

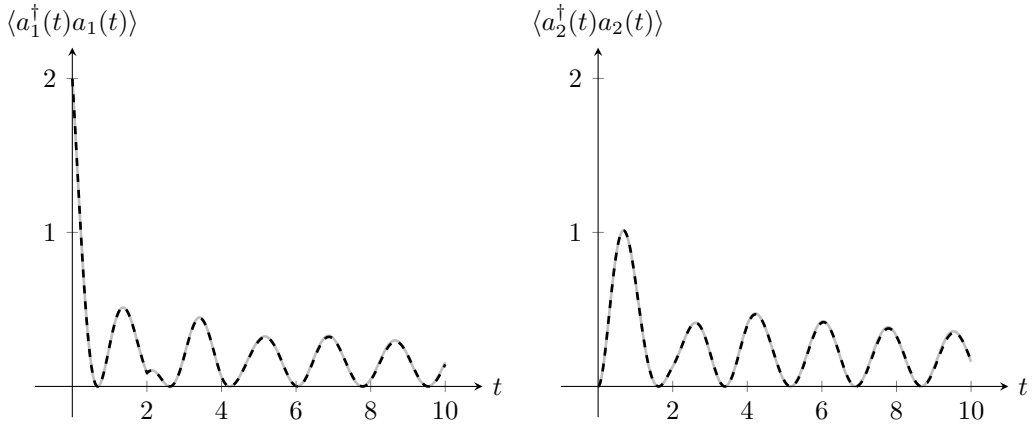


Figure 7.1: Excitation numbers in a pair of coupled, bosonic systems. The first system is coupled to a feedback reservoir. The grey lines depict the results obtained by numerically integrating the differential equations for the Green's functions describing the system. The dashed black lines depict the results of a simulation of the master equation, with at most three excitations in each of the two subsystems. Parameters: $g = 2$, $\gamma = 1$, $\tau = 2$, and $\phi = 0$ (in arbitrary units).

performed in the solution (6.90)—this brings the Green's function $V'(t)$ into block-diagonal form and yields a solution of the form (6.76), from which time-local Heisenberg–Langevin equations are easily obtained. Either way, once we have obtained a time-local form of the Heisenberg–Langevin equations in which $a_j(t)$ and $a_j^\dagger(t)$ are no longer coupled, the master equation can be obtained using the procedure described above in this section.

One alternative approach is that pursued by Chang and Law,[75] who have derived an exact time-local master equation for a single bosonic subsystem with parametric down-conversion. They use a method very similar to mine, deriving and solving a Heisenberg–Langevin equation and using the result to obtain a time-local master equation. Their model, however, preserves the counter-rotating terms in the system–environment interaction Hamiltonian.

7.2 FERMIONIC SUBSYSTEMS

We will now examine whether the master equations developed in section 7.1 can be generalised to allow for fermionic, or two-level, subsystems. Consider a single two-level subsystem obeying the fermionic commutation relation $[\sigma_+, \sigma_-] = \sigma_z$. The analogous equations to (7.2) and (7.3) for this system are

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\,[A(t), H_S(t)] + i\left([\sigma_+(t), A(t)]\,E(t) - E^\dagger(t)\,[A(t), \sigma_-(t)]\right)\,, \tag{7.62}$$

where we have assumed that the operator $A(t)$ does not have any explicit time-dependence, and

$$E(t) = B(t) - i\int_0^t \mathrm{d}t_1\, f(t-t_1)\sigma_-(t_1)\,. \tag{7.63}$$

Substituting $\sigma_-(t)$ into (7.62) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t) = -i\left[\sigma_-(t), H_S(t)\right] + i\sigma_z(t)E(t)\,. \tag{7.64}$$

Left-multiplying equation (7.64) by $\sigma_z(t)$ yields $\sigma_z(t)(\dot{\sigma}_-(t) + i\left[\sigma_-(t), H_S(t)\right]) = iE(t)$, so the Heisenberg equation for a fermionic subsystem (7.62) can be written in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -i\left[A(t), H_S(t)\right] + \left[\sigma_+(t), A(t)\right]\sigma_z(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t) + i\left[\sigma_-(t), H_S(t)\right]\right) \\ + \left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_+(t) + i\left[\sigma_+(t), H_S(t)\right]\right)\sigma_z(t)\left[A(t), \sigma_-(t)\right]\,.\end{aligned} \tag{7.65}$$

As in the bosonic case, we can simplify this equation somewhat by choosing an appropriate system Hamiltonian. We choose a linear Hamiltonian, similar to (6.50) but adapted to the case of a fermionic subsystem:

$$H_S(t) = (\Omega(t)\sigma_+ + \Omega^*(t)\sigma_-) + \Delta\sigma_+\sigma_-\,. \tag{7.66}$$

Substituting this Hamiltonian into equation (7.65) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \left[\sigma_+(t), A(t)\right]\sigma_z(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t)\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_+(t)\right)\sigma_z(t)\left[A(t), \sigma_-(t)\right]\,. \tag{7.67}$$

Using equations (7.63) and (7.64), and the system Hamiltonian (7.66), we can derive a Heisenberg–Langevin equation for $\sigma_-(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t) = \sigma_z(t)\left(i\Delta\sigma_-(t) + \int_0^t \mathrm{d}t_1\, f(t-t_1)\sigma_-(t_1) + i\left(\Omega(t) + B(t)\right)\right)\,. \tag{7.68}$$

If, therefore, we were able to write the derivative of $\sigma_-(t)$, and thereby $\sigma_+(t)$, in a time-local form we would be able to use (7.67) to derive a time-local master equation for a two-level subsystem, using the same procedure as in section 7.1. Unfortunately, the Heisenberg–Langevin equation (7.68) and the corresponding equations for $\sigma_+(t)$ and $\sigma_z(t)$ are not as easily solved as are the analogous equations in the bosonic case. In fact, to the best of my knowledge no explicit, closed-form solution to (7.68) exists—even in the Markovian case. In the absence of analytic solutions to the Heisenberg–Langevin equations for $\sigma_-(t)$ and $\sigma_+(t)$, we cannot directly transform equation (7.65) into a time-local form. However, a time-local master equation describing a non-Markovian, driven two-level system has been reported in the literature, and in the following we will consider how this equation connects to equations we have already derived, as well as whether or not it is correct.

Suppose the system is very weakly excited, that is to say it stays near its ground state at all times. In this case, $\sigma_z(t) \approx -1$ for all t, and the Heisenberg–Langevin equation (7.68) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t) = -i\Delta\sigma_-(t) - \int_0^t \mathrm{d}t_1\, f(t-t_1)\sigma_-(t_1) - i\left(\Omega(t) + B(t)\right)\,, \tag{7.69}$$

while (7.67) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -\left[\sigma_+(t), A(t)\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t)\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_+(t)\right)\left[A(t), \sigma_-(t)\right] . \tag{7.70}$$

While we were unable to solve (7.68), we *can* solve (7.69). In fact, equations (7.69) and (7.70) have exactly the same form as the analogous equations – (6.52) and (7.7) – for a single bosonic subsystem, so the results from that case carry over directly. Equation (7.69) has solution

$$\sigma_-(t) = v(t)\sigma_- - i(v * \Omega)(t) - i(v * B)(t)\,, \tag{7.71}$$

where $v(t)$ obeys (6.56). Differentiating this solution yields a time-local form of (7.69),

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_-(t) = -\gamma(t)\sigma_-(t) - i\left(\xi(t) + D(t)\right) , \tag{7.72}$$

where $\gamma(t)$, $\xi(t)$, and $D(t)$ are given by equations (7.10)–(7.12), and substituting (7.72) into (7.70) yields a general time-local Heisenberg–Langevin equation for a weakly excited two-level system:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) = &-i\left[A(t), \xi(t)\sigma_+(t) + \xi^*(t)\sigma_-(t)\right] \\ &+ \gamma(t)\left[\sigma_+(t), A(t)\right]\sigma_-(t) + \gamma^*(t)\sigma_+(t)\left[A(t), \sigma_-(t)\right] \\ &+ i\left(\left[\sigma_+(t), A(t)\right]D(t) - D^\dagger(t)\left[A(t), \sigma_-(t)\right]\right) .\end{aligned} \tag{7.73}$$

Taking the expectation value of both sides of this equation in a vacuum reservoir yields a moment equation from which we can immediately read off a master equation for a damped, driven two-level system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\left[\xi(t)\sigma_+ + \xi^*(t)\sigma_-, \rho_S(t)\right] + \gamma(t)\left[\sigma_-\rho_S(t), \sigma_+\right] + \gamma^*(t)\left[\sigma_-, \rho_S(t)\sigma_+\right] . \tag{7.74}$$

Because this master equation was derived under the assumption of weak excitation, it only applies in the limit of very weak driving. Equation (7.74) can also be 'derived' from the master equation (7.23) for a single, linear, bosonic subsystem. In a very weakly excited harmonic oscillator, energy levels above the first excited state have negligible occupation number, so the oscillator behaves just like a two-level system and we can simply replace the bosonic creation and annihilation operators in (7.23) with the corresponding fermionic operators. The fact that the reservoir is in its vacuum state gives $\lambda(t) = 0$, and the result is exactly (7.74).

Equation (7.74) is equivalent to the master equation reported by Shen et al.[76] The same equation was derived earlier by Anastopoulos and Hu,[77] but never published due to "possible ambiguities in the Grassmannian variable approach for finite temperature and coherent-state fields".[78] Shresta et al. later developed an alternative approach – which does

not involve master equations – to the problem of a two-level system in a thermal bath, but without a coherent drive.[79,80]

Shen et al. present their master equation as an exact equation which applies in all parameter regimes, but a straightforward calculation shows that this cannot possibly be true. Examples of the 'effective' decoherence rate $\gamma(t)$ and driving field strength $\xi(t)$, which are given – as they are in the bosonic case considered in section 7.1.1 – by equations (7.10) and (7.11) are shown in figure 7.2. Figure 7.3 illustrates the results obtained by numerically integrating the master equation (7.74) using the same parameters, for several different values of the driving field strength Ω, and with the system initially in its excited state. We can see from figure 7.3c that when $\Omega = 3$ (in the arbitrary units used in this simulation) the mean atomic excitation, which should of course be constrained by $0 \leq \langle \sigma_+(t)\sigma_-(t) \rangle \leq 1$, reaches unphysical values less than zero and greater than one.[81] This immediately tells us that at least in certain parameter regimes, this master equation does not describe a positive evolution. While we can be confident that equation (7.74) is applicable in the weak driving limit, finding a time-local master equation for the driven two-state system that applies in arbitrary parameter regimes remains an open problem.

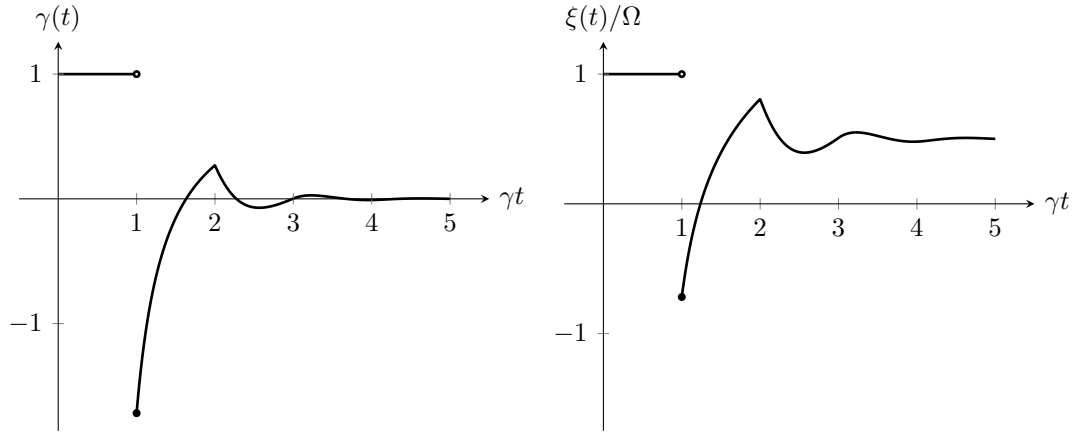


Figure 7.2: Time-dependent coefficients of the master equation (7.74), with parameters $\gamma = 1$ and $\tau = 1$ (in arbitrary units), and $\phi = \pi$.

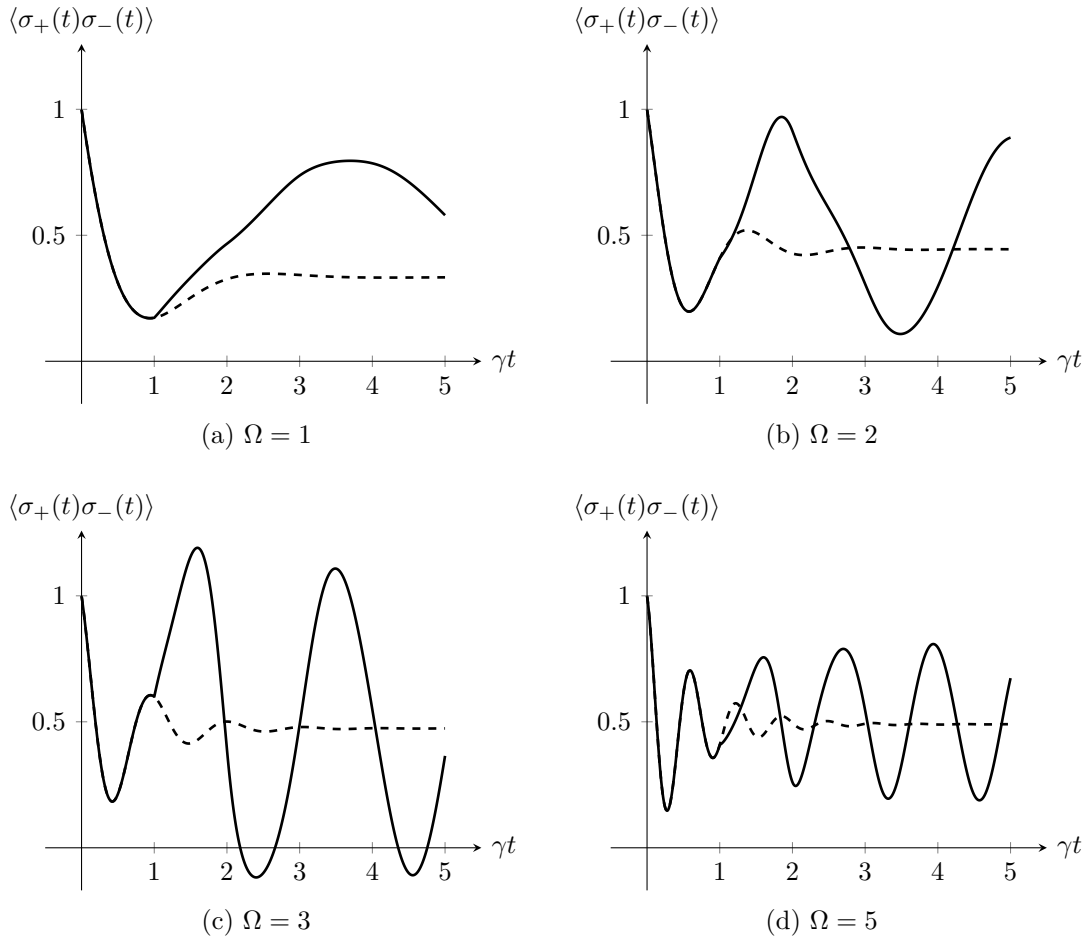


Figure 7.3: Mean atomic excitation number obtained by numerically integrating of the master equation (7.74), with parameters $\gamma = 1$, $\tau = 1$, and several values of Ω (in arbitrary units), and $\phi = \pi$. In these simulations, the two-level system is initially in its excited state. The dashed lines depict the results of analogous simulations without any feedback.

8

CONCLUSION

To conclude, I will briefly summarise the progress we have made on the problem of modelling open quantum systems with time-delayed interactions. I will also point out some possible directions for future research.

8.1 SUMMARY

After reviewing the theory of open quantum systems – both Markovian and non-Markovian – in chapter 2, I introduced in chapter 3 a general model of an open quantum system that interacts with the environment by way of particle exchanges. This model admits quite arbitrary memory effects, and in section 3.3 I used this model to describe a system exhibiting the kind of non-Markovian dynamics we set out to model: the simple feedback loop depicted in figure 3.1. This model represents a prototype for the kinds of system–environment interaction we are interested in, and all the systems considered in this thesis are based on it to a greater or lesser extent.

In chapter 4, I looked at what happens when there is only a single excitation in the system. Because the system–environment interaction outlined in chapter 3 conserves excitation number, I was able to develop a general approach to singly-excited systems in section 4.1. I applied this formalism to a simple feedback loop in section 4.2, and to a simple network of systems in section 4.4. I also derived a master equation for singly-excited systems – both bosonic and fermionic – in section 4.3. In sections 4.5–4.7, I showed how to calculate spontaneous emission spectra for these singly-excited systems, and illustrated this procedure using a series of simple examples.

In chapter 5 I reviewed a pair of commonly-used perturbative techniques – the Born approximation in section 5.1, and a variant of the time-convolutionless projection operator technique in section 5.2 – and evaluated the usefulness of these techniques in modelling systems with delayed feedback. I compared results from these techniques to the exact results obtained in chapter 4. As it turned out, a naïve application of the Born approximation gave dramatically incorrect results in the parameter regime of interest. The time-convolutionless perturbation expansion – a more systematic approach – converged too slowly to be of much use in further modelling.

In chapters 6 and 7 I turned attention to exact equations of motion for non-Markovian open quantum systems, beginning in section 6.1 by deriving Heisenberg–Langevin equations describing a large class of such systems. In section 6.2 I reviewed the theory of cascaded open quantum systems, and in section 6.3 used this formalism to simulate systems with

delayed coherent feedback, finding that there is a close correspondence between cascaded systems and systems with delayed feedback—but only in the limit of long delay.

I obtained analytic solutions to non-Markovian Heisenberg–Langevin equations for a series of simple bosonic systems in section 6.4. In chapter 7, I showed that given a solution to a Heisenberg–Langevin equation, a time-local equation of motion can be derived that describes equivalent dynamics. I also found that time-local Heisenberg–Langevin equations may be used to derive corresponding time-local master equations for a non-Markovian open quantum systems, thereby making contact with several Schrödinger picture approaches that have been pursued in the literature.[72–74] In particular, I was able to derive exact time-local master equations for linear, bosonic systems.

8.2 DIRECTIONS FOR FUTURE RESEARCH

In section 4.5, I developed a method (described in more detail in appendix B) of calculating spontaneous emission spectra for singly-excited systems, even in the presence of feedback loops. However, this method lacks a satisfactory physical interpretation. Developing a more intuitive illustration of the method we have used is a direction for future investigation. Moreover, as noted in section 4.7, gaining a full understanding of the physical content of the spectra shown in figures 4.5, 4.6 and 4.8 also requires further research.

I showed in section 6.4 how to solve the Heisenberg–Langevin equations for linear, bosonic systems. I also showed how to recast the equations of motion in a time-local form and eventually obtain time-local master equations. Nonlinear systems, including two-level systems, have presented us with greater difficulty—in this case we cannot even solve the integro-differential Heisenberg–Langevin equations, let alone derive time-local equations of motion. In section 7.2, I showed how to derive a time-local master equation for two-level systems in the limit of very weak excitation, but a generally-applicable equation remains elusive. Establishing whether such a master equation can be derived will be the subject of ongoing research.

A

PROJECTION OPERATOR DERIVATION OF THE EXACT NON-MARKOVIAN MASTER EQUATION

Here I will briefly outline how the general, exact master equation (2.58) can be derived using the Nakajima–Zwanzig projection operator technique. The argument presented in this appendix is adapted from section 9.1 of the book by Breuer and Petruccione.[7]

We begin with a generic model of an open quantum system defined by the interaction picture Hamiltonian

$$H(t) = H_S(t) + H_{SE}(t)\,, \tag{A.1}$$

where $H_S(t)$ accounts for interactions internal to the system and $H_{SE}(t)$ describes the system–environment interaction. We define a time-dependent Liouvillian super-operator $L(t)$ such that the interaction picture equation of motion for the density matrix $\rho(t)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = L(t)\rho(t) = -i\,[H(t), \rho(t)]\ . \tag{A.2}$$

We now define the projection super-operator $\mathcal{P}$ according to

$$\mathcal{P}\rho = \mathrm{tr}_E\,\rho \otimes \rho_E = \rho_S \otimes \rho_E\,, \tag{A.3}$$

where ρ_E, known in this context as the *reference state*, can be any chosen state of the environment—here we choose ρ_E to be time-independent. The density matrix $\mathcal{P}\rho(t)$ is known as the *relevant* part of the density matrix of the combined system. We also define a second super-operator $\mathcal{Q}$ that satisfies

$$\mathcal{P} + \mathcal{Q} = I\,. \tag{A.4}$$

This definition immediately gives $\mathcal{Q}\rho = \rho - \mathcal{P}\rho$, where $\mathcal{Q}\rho$ is referred to as the *irrelevant* part of ρ. The two projection operators clearly satisfy the additional properties $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q}^2 = \mathcal{Q}$, and $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$.

We now apply each of the projection operators in turn to the von Neumann equation (A.2). Using the time-independence of the reference state and the resolution of the identity (A.4), we obtain the coupled differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\rho(t) = \mathcal{P}L(t)\mathcal{P}\rho(t) + \mathcal{P}L(t)\mathcal{Q}\rho(t)\,, \tag{A.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}\rho(t) = \mathcal{Q}L(t)\mathcal{P}\rho(t) + \mathcal{Q}L(t)\mathcal{Q}\rho(t)\,. \tag{A.6}$$

We now formally solve equation (A.6) for an initial condition $\rho(t_0)$ to get

$$\mathcal{Q}\rho(t) = \mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0) + \int_{t_0}^{t} \mathrm{d}t_1\, \mathcal{G}(t,t_1)\mathcal{Q}L(t_1)\mathcal{P}\rho(s)\,, \tag{A.7}$$

where we have defined the propagator

$$\mathcal{G}(t,t_1) = \mathrm{T}_{\leftarrow} \exp\left(\int_{t_1}^{t} \mathrm{d}t_1'\, \mathcal{Q}L(t_1')\right)\,, \tag{A.8}$$

and substitute the solution (A.7) into equation (A.5) to obtain the *Nakajima–Zwanzig equation*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\rho(t) = \mathcal{P}L(t)\mathcal{P}\rho(t) + \mathcal{P}L(t)\mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0) + \int_{t_0}^{t} \mathrm{d}t_1\, \mathcal{P}L(t)\mathcal{G}(t,t_1)\mathcal{Q}L(t_1)\mathcal{P}\rho(t_1)\,. \tag{A.9}$$

The only assumption we shall make, beyond the specification of the model itself, is that of a factorising initial condition,

$$\rho(t_0) = \rho_S(t_0)\otimes\rho_E\,. \tag{A.10}$$

This gives $\mathcal{P}\rho(t_0) = \rho(t_0)$ and $\mathcal{Q}\rho(t_0) = 0$, which means that the second term in (A.9) vanishes. We now turn our attention to the first term in (A.9). If we write this term out explicitly, we get

$$\mathcal{P}L(t)\mathcal{P}\rho(t) = -i\left[H_S(t),\rho_S(t)\right]\otimes\rho_E - i\,\mathrm{tr}_E\left[H_{SE}(t),\rho_S(t)\otimes\rho_E\right]\otimes\rho_E\,. \tag{A.11}$$

We now suppose that

$$\mathrm{tr}_E\left[H_{SE}(t),\rho_S(t)\otimes\rho_E\right] = 0\,. \tag{A.12}$$

This condition amounts to the same thing as the assumption (2.35) made in section 2.3 and as in that case, (A.12) is not really an assumption because terms can always be absorbed into $H_S(t)$ to ensure that it holds. Using (A.11) and (A.12), and defining the convolution kernel

$$\mathcal{K}(t,t_1) = \mathcal{P}L(t)\mathcal{G}(t,t_1)\mathcal{Q}L(t_1)\mathcal{P}\,, \tag{A.13}$$

the Nakajima–Zwanzig equation (A.9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\left[H_S(t),\rho_S(t)\right] + \int_{t_0}^{t} \mathrm{d}t_1\, \mathcal{K}(t,t_1)\rho_S(t_1)\,. \tag{A.14}$$

B

CORRECTIONS TO SPONTANEOUS EMISSION SPECTRA

In section 4.5, we presented the formula (4.58) for the *spontaneous emission spectrum* of the α^{th} reservoir, denoted $S_\alpha(\omega)$. We will repeat this formula here for convenience:

$$\begin{aligned} S_\alpha(\omega) &= \frac{1}{\Delta\omega_{l_\alpha}\langle N(0)\rangle}\sum_{l_\alpha}\lim_{t\to\infty}\left\langle\psi(t)\right|b^\dagger_{\alpha l_\alpha}b_{\alpha l_\alpha}\left|\psi(t)\right\rangle\delta(\omega-\omega_{l_\alpha}) \\ &= \frac{1}{\Delta\omega_{l_\alpha}\sum_j|c_j(0)|^2}\sum_{l_\alpha}\lim_{t\to\infty}|c_{\alpha l_\alpha}(t)|^2\delta(\omega-\omega_{l_\alpha})\,, \end{aligned} \tag{B.1}$$

where $|\psi(t)\rangle$ is given by (4.8). Our aim here is to derive corrections to spontaneous emission spectra to account for the presence of feedback loops, and thereby show that under certain circumstances, the formula (B.1) does not require any such corrections.

What we really need to calculate, rather than (B.1), is the probability of finding a photon of frequency ω outside the part of the reservoir where there are interactions with the system—a photon that has irreversibly decayed from the system. To do this, we have to assume that ignoring a sufficiently large chunk of the reservoir will exclude the problematic regions. We can certainly imagine pathological systems in which this won't be the case (such as an environment comprising a single, finite loop), but we most likely wouldn't be asking about emission from such systems anyway. As an illustration, suppose the field in reservoir α is given by a slight generalisation of (4.37),

$$\varphi_\alpha(x_\alpha) = \sum_{l_\alpha=-\infty}^{\infty}\frac{1}{\sqrt{2|\omega_{l_\alpha}|M}}\left(b_{\alpha l_\alpha}e^{i\omega_{l_\alpha}x_\alpha}+b^\dagger_{\alpha l_\alpha}e^{-i\omega_{l_\alpha}x_\alpha}\right), \qquad \omega_{l_\alpha}=\frac{2\pi l_\alpha}{M}\,. \tag{B.2}$$

We can now define the fields inside and outside the potentially problematic interaction region,

$$\varphi_\alpha(x_\alpha) = \varphi_{\alpha,\text{in}}(x_\alpha,t)+\varphi_{\alpha,\text{out}}(x_\alpha,t)\,, \tag{B.3}$$

where

$$\varphi_{\alpha,\text{in}}(x_\alpha,t) = \Pi_M((x_\alpha+x_{\alpha,0}(t))/T)\varphi_\alpha(x_\alpha)\,. \tag{B.4}$$

Here $\Pi_M(x)$ is the rectangular pulse wave of period M and pulse duration 1, T is a constant defining a sufficiently large section of the reservoir to ignore (naturally, $0\le T\le M$), and $x_{\alpha,0}(t)$ is the centre of the interaction region we want to ignore. If we let $M\to\infty$ then $\Pi_M(x)\to\Pi(x)$, the rectangular function. In the Schrödinger picture, the interaction region would typically be centred on $x_{\alpha,0}(t)=0$. However, in the rotating frame the free

propagation of the reservoir is absent, and so the interaction region counter-intuitively moves with time and is centred on $x_{\alpha,0}(t) = t$. The Fourier series expansion for the rectangular pulse wave is

$$\Pi_M(x/T) = \frac{|T|}{M} \sum_{l=-\infty}^{\infty} e^{i\omega_l x} \operatorname{sinc}\left(\frac{T\omega_l}{2\pi}\right), \qquad \omega_l = \frac{2\pi l}{M}, \tag{B.5}$$

where the normalised sinc function is given by $\operatorname{sinc}(x) = (\pi x)^{-1}\sin(\pi x)$. Thus, we can write (4.37) in the form

$$\varphi_\alpha(x_\alpha) = \sum_{l_\alpha=-\infty}^{\infty} \frac{1}{\sqrt{2|\omega_{l_\alpha}|M}} \left((b_{\alpha l_\alpha,\mathrm{in}} + b_{\alpha l_\alpha,\mathrm{out}})\, e^{i\omega_{l_\alpha} x_\alpha} + \left(b^\dagger_{\alpha l_\alpha,\mathrm{in}} + b^\dagger_{\alpha l_\alpha,\mathrm{out}}\right) e^{-i\omega_{l_\alpha} x_\alpha} \right), \tag{B.6}$$

where $b_{\alpha l_\alpha} = b_{\alpha l_\alpha,\mathrm{in}} + b_{\alpha l_\alpha,\mathrm{out}}$. It follows from the convolution theorem for Fourier series that

$$b_{\alpha l_\alpha,\mathrm{in}}(t) = \frac{|T|}{M} \sum_{l'=-\infty}^{\infty} e^{i\omega_{l'} x_{\alpha,0}(t)} \operatorname{sinc}\left(\frac{T\omega_{l'}}{2\pi}\right) b_{\alpha(l_\alpha - l')}\,. \tag{B.7}$$

Thus, we can modify (B.1) to calculate the probability of finding a photon of frequency ω in the reservoir but outside the interaction region as $t \to \infty$:

$$\begin{aligned} S_\alpha(\omega) &= \frac{M}{2\pi \langle N(0)\rangle} \sum_{l_\alpha} \lim_{t\to\infty} \left\langle \psi(t) \,\middle|\, b^\dagger_{\alpha l_\alpha,\mathrm{out}}(t) b_{\alpha l_\alpha,\mathrm{out}}(t) \,\middle|\, \psi(t) \right\rangle \delta(\omega - |\omega_{l_\alpha}|) \\ &= \frac{M}{2\pi \sum_j |c_j(0)|^2} \sum_{l_\alpha} \lim_{t\to\infty} \left| c_{\alpha l_\alpha}(t) - \frac{|T|}{M} \sum_{l'} e^{i\omega_{l'} x_{\alpha,0}(t)} \operatorname{sinc}\left(\frac{T\omega_{l'}}{2\pi}\right) c_{\alpha(l_\alpha - l')}(t) \right|^2 \\ &\qquad \times \delta(\omega - |\omega_{l_\alpha}|)\,, \end{aligned} \tag{B.8}$$

where we have used the fact that $\Delta\omega_{l_\alpha} = 2\pi/M$ in this case. It is necessary to use the corrected formula (B.8) instead of (B.1) in any case where problematic feedback is present and where M is finite. However, in the interaction picture, where $x_{\alpha,0}(t) = t$, and in the continuum limit discussed in section 3.3, in which $M \to \infty$, equation (B.7) becomes a Fourier transform:

$$b_{\alpha,\mathrm{in}}(\omega, t) = \frac{|T|}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega_1\, e^{i\omega_1 t} \operatorname{sinc}\left(\frac{T\omega_1}{2\pi}\right) b_\alpha(\omega - \omega_1)\,. \tag{B.9}$$

We can now apply the Riemann–Lebesgue lemma, which gives the limits

$$\lim_{t\to\infty} b_{\alpha,\mathrm{in}}(\omega, t) = 0\,, \qquad \text{and} \qquad \lim_{t\to\infty} b_{\alpha,\mathrm{out}}(\omega, t) = b_\alpha(\omega)\,. \tag{B.10}$$

Thus, in the continuum limit we get the highly counter-intuitive result that no correction is required in equation (B.1).

To demonstrate the use of the corrected formula (B.8), I will now consider a simulation with finite M. We want to construct this simulation so that in the limit $M \to \infty$ it corresponds to the simple feedback loop we studied in section 4.2. Suppose that the system's evolution is generated by the effective Hamiltonian

$$H_{\text{eff}} = \sum_{l=-N}^{N} \Delta_l b_l^\dagger b_l + a^\dagger \left(\sum_{l=-N}^{N} \kappa_l b_l \right) + \left(\sum_{l=-N}^{N} \kappa_l^* b_l^\dagger \right) a \,, \qquad \Delta_l = \frac{2\pi l}{M} \,. \tag{B.11}$$

We can study the dynamics of this system using an analogous procedure to that used in section 4.1. We assume that the system is singly-excited, with a state of the form

$$|\psi(t)\rangle = c_0 |0\rangle + c(t) a^\dagger |0\rangle + \sum_{l=-N}^{N} c_l(t) b_l^\dagger |0\rangle \,, \tag{B.12}$$

with $c_l(0) = 0$. Regardless of whether the operators obey bosonic or fermionic commutation relations, it is easy to show that the effective Hamiltonian (B.11) leads to the following differential equation for the Schrödinger picture amplitude $c(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} c(t) = -\int_0^t \mathrm{d}t_1 \, f(t - t_1) c(t_1) \,, \tag{B.13}$$

where we have defined the dissipation kernel

$$f(t - t_1) = \int_{-\infty}^{\infty} \mathrm{d}\Delta \, J_{\text{eff}}(\Delta) e^{-i\Delta(t - t_1)} \,, \tag{B.14}$$

and the effective spectral density

$$J_{\text{eff}}(\Delta) = \sum_{l=-N}^{N} |\kappa_l|^2 \delta(\Delta - \Delta_l) \,. \tag{B.15}$$

If we choose the coupling constants

$$\kappa_l = \sqrt{\frac{4\gamma}{M}} \cos\left(\frac{\Delta_l \tau + \phi}{2} \right) \,, \tag{B.16}$$

then the effective spectral density becomes

$$J_{\text{eff}}(\Delta) = \frac{2\gamma}{M} \sum_{l=-N}^{N} \left(1 + \cos(\Delta_l \tau + \phi)\right) \delta(\Delta - \Delta_l) \,. \tag{B.17}$$

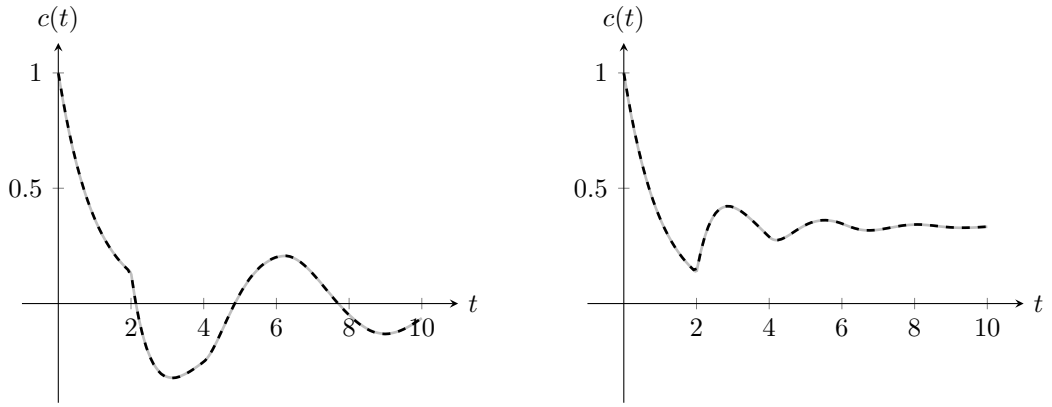
In the continuum limit where both $M \to \infty$ and $N \to \infty$, this becomes

$$J_{\text{eff}}(\Delta) = \frac{\gamma}{\pi} \left(1 + \cos(\Delta\tau + \phi)\right) \,, \tag{B.18}$$

which is equivalent to the frequency-shifted spectral density (4.16) that describes spontaneous emission from a single system degree of freedom, with $\eta = \gamma$. Thus, the Hamilto-

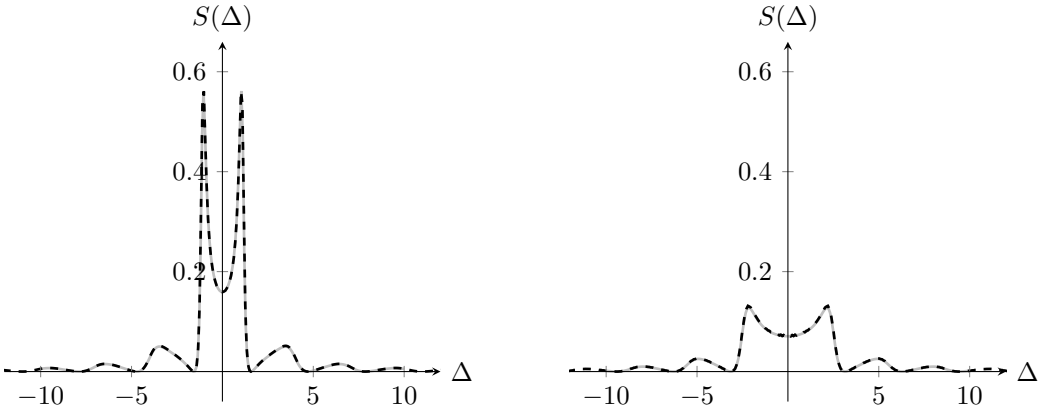
nian (B.11) with the couplings (B.16) can be said to approximate the model defined by (4.16), which of course we have already solved.

The dynamics generated by the Hamiltonian (B.11), for some finite N, can be simulated using well-known numerical techniques. In this case, I obtained the results shown in figures B.1 and B.2 using QuTiP.[63,64] The spectra shown in figure B.2 were computed using the corrected formula (B.8), but because there is no trapping when $\phi = 0$, figure B.2a could have been computed directly using the unmodified formula (B.1). On the other hand, when $\phi = \pi$, as it does in figure B.2b, one obtains spurious results unless the correction is made.



(a) $\phi = 0$ (b) $\phi = \pi$

Figure B.1: The amplitude $c(t)$ for a simple feedback loop, with $\gamma = 1$ and $\tau = 2$ (in arbitrary units), for two different values of ϕ. The grey lines depict the exact results, while the dashed lines depict the results of simulations (with 1001 environment modes, and $M = 10\tau$).



(a) $\phi = 0$ (b) $\phi = \pi$

Figure B.2: Spontaneous emission spectra for a simple feedback loop, with $\gamma = 1$ and $\tau = 2$ (in arbitrary units), for two different values of ϕ. The grey lines depict the exact results, while the dashed lines depict the results of simulations (with 1001 environment modes, and $M = 40\tau$; the spectrum was evaluated at $t = 20\tau$).

C

INTEGRATING THE MOMENT HIERARCHY: A PROPOSAL

In order to evaluate the moment equations we derived in section 6.1, such as (6.33), we need to be able to calculate moments of the form $\langle A(t+t_1)a_j(t_1)\rangle$ and $\langle a_j^\dagger(t_1)A(t+t_1)\rangle$. Assuming that we know $\langle A(t_1)a_j(t_1)\rangle$ and $\langle a_j^\dagger(t_1)A(t_1)\rangle$, we can easily use the Heisenberg–Langevin equation (6.11), along with the commutation relations (6.12) and (6.13), to derive partial differential equations to propagate these correlation functions forward. Assuming that the environment is in a vacuum state, we have $\rho = \rho_S \otimes |0\rangle\langle 0|$, and as such we find:

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle A(t+t_1)a_j(t_1)\rangle \\
&\quad = -i\langle [A(t+t_1), H_S(t+t_1)]\, a_j(t_1)\rangle \\
&\qquad + \sum_{jk}\int_0^{t+t_1} \mathrm{d}t_2 \Big(F_{jk}(t+t_1,t_2)\Big\langle \Big[a_j^\dagger(t+t_1), A(t+t_1)\Big]\, a_k(t_2)a_j(t_1)\Big\rangle \\
&\qquad\qquad + F_{jk}^*(t+t_1,t_2)\Big\langle a_k^\dagger(t_2)\,[A(t+t_1), a_j(t+t_1)]\, a_j(t_1)\Big\rangle\Big) \\
&\qquad + \sum_{jk}\int_0^{t_1} \mathrm{d}t_2\, F_{jk}(t+t_1,t_2)\Big\langle \Big[a_j^\dagger(t+t_1), A(t+t_1)\Big]\,[a_k(t_2), a_j(t_1)]\Big\rangle\,,
\end{aligned} \tag{C.1}$$

along with a very similar conjugate equation for $\langle a_j^\dagger(t_1)A(t+t_1)\rangle$. For simplicity's sake, we have assumed that $A(t)$ has no explicit time-dependence. By repeating this process for all the higher-order correlation functions that appear, we develop the moment hierarchy mentioned in section 6.1. In a vacuum reservoir, the commutation relations (6.12) and (6.13) can be used to completely eliminate the dependence on the reservoir operators, as we have done in equation (C.1).

Even after going through this procedure, it still isn't possible to perform practical calculations, because the moment hierarchy is, in general, infinite. A similar problem presents itself when we simulate Markovian dynamics in the Schrödinger picture, where we have an exact differential equation – the Lindblad equation – but nevertheless usually cannot solve the dynamics without some kind of additional approximation. One way of getting around this problem in the Schrödinger picture is to truncate the Fock space of the system so that the system operators and the density matrix can be represented as finite arrays in computer memory. To integrate the moment hierarchy, we need an analogous procedure in the Heisenberg picture—I propose such a scheme below.

Various system operators with different time arguments appear in the damping terms of the moment hierarchy equations. These operators can be thought of as referring to excitations that are localised in the subsystem (of the combined system) comprising the system and the feedback loop—excitations that have not yet escaped irreversibly into the

reservoir. In the spirit of a Fock space truncation, we assume that there are at most N excitations in this subsystem at any given time. As such, we should neglect any expectation values with contain more than N consecutive factors of any given a_j or $a_j^\dagger$.

This truncation scheme is, of course, exact whenever we can make a definite statement about the maximum number of excitations in the system. Suppose, for instance, that at the initial time there are at most N excitations in the combined system. The form of the system–environment interaction Hamiltonian (6.1) conserves total excitation number, so whenever the system Hamiltonian $H_S(t)$ also conserves total excitation number there will be at most N excitations in the system at all times. Under these conditions, truncating the moment hierarchy at the N-excitation level will lead to the exact result. Otherwise, the truncation scheme outlined above will constitute an approximation, and its validity will depend on the parameters of the system in question.

As a simple example, we will consider the same model of spontaneous emission as we did in section 4.2: the spectral density will be given by (4.16), giving the dissipation kernel (4.18), and the system Hamiltonian will be given by $H_S(t) = 0$. Given the restriction in this case to a single excitation, the truncation scheme outlined above instructs us to neglect any expectation values that contain multiple consecutive factors of either a or $a^\dagger$. Putting all this together, equation (C.1) leads to:

$$\frac{\partial}{\partial t}\left\langle A(t+t_1)a(t_1)\right\rangle = \int_0^{t+t_1} \mathrm{d}t_2\, f^*(t+t_1-t_2)\left\langle a^\dagger(t_2)\left[A(t+t_1), a(t+t_1)\right]a(t_1)\right\rangle , \tag{C.2}$$

$$\frac{\partial}{\partial t}\left\langle a^\dagger(t_1)A(t+t_1)\right\rangle = \int_0^{t+t_1} \mathrm{d}t_2\, f(t+t_1-t_2)\left\langle a^\dagger(t_1)\left[a^\dagger(t+t_1), A(t+t_1)\right]a(t_2)\right\rangle . \tag{C.3}$$

From these equations, as well as (6.33), we can immediately obtain the moment hierarchy for $\langle a^\dagger a\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle a^\dagger(t)a(t)\right\rangle = -\int_0^t \mathrm{d}t_1 \left(f(t-t_1)\left\langle a^\dagger(t)a(t_1)\right\rangle + f^*(t-t_1)\left\langle a^\dagger(t_1)a(t)\right\rangle\right) , \tag{C.4}$$

$$\frac{\partial}{\partial t}\left\langle a^\dagger(t+t_1)a(t_1)\right\rangle = -\int_0^{t+t_1} \mathrm{d}t_2\, f^*(t+t_1-t_2)\left\langle a^\dagger(t_2)a(t_1)\right\rangle , \tag{C.5}$$

$$\frac{\partial}{\partial t}\left\langle a^\dagger(t_1)a(t+t_1)\right\rangle = -\int_0^{t+t_1} \mathrm{d}t_2\, f(t+t_1-t_2)\left\langle a^\dagger(t_1)a(t_2)\right\rangle . \tag{C.6}$$

Substituting the dissipation kernel (4.18) into equations (C.5) and (C.6) yields

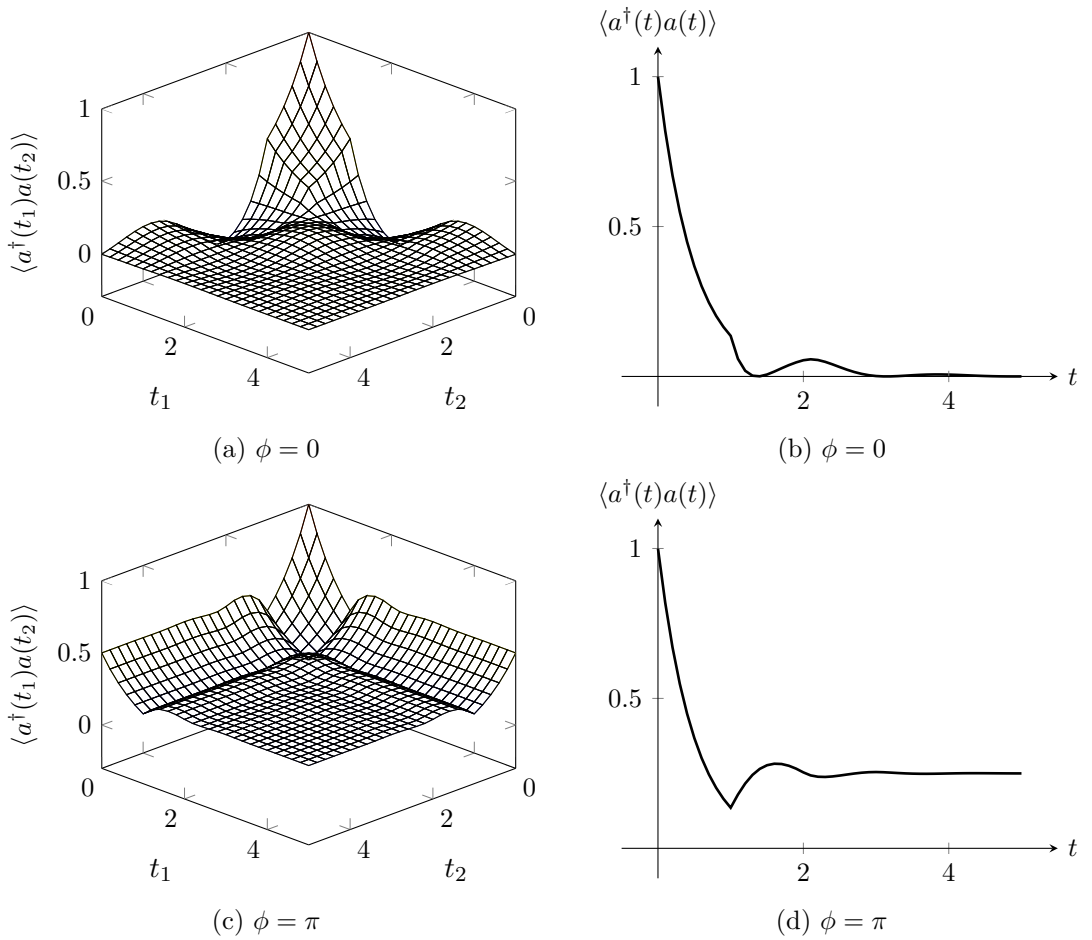
$$\frac{\partial}{\partial t_1}\left\langle a^\dagger(t_1)a(t_2)\right\rangle = -\gamma\left\langle a^\dagger(t_1)a(t_2)\right\rangle - \eta e^{-i\phi}\left\langle a^\dagger(t_1-\tau)a(t_2)\right\rangle , \tag{C.7}$$

$$\frac{\partial}{\partial t_2}\left\langle a^\dagger(t_1)a(t_2)\right\rangle = -\gamma\left\langle a^\dagger(t_1)a(t_2)\right\rangle - \eta e^{i\phi}\left\langle a^\dagger(t_1)a(t_2-\tau)\right\rangle . \tag{C.8}$$

Now, as it happens, we can easily infer the solution to these equations: $\langle a^\dagger(t_1)a(t_2)\rangle = v^*(t_1)v(t_2)\langle a^\dagger a\rangle$, where $v(t)$ is given by (4.34). But it's interesting to ask how we might go

about solving equations (C.7) and (C.8) naïvely, without any foreknowledge of the solution. Partial differential equations such as these can be integrated numerically on a grid. For example, by integrating (C.7) we can, given an initial condition $\langle a^\dagger(0)a(0)\rangle$, easily find $\langle a^\dagger(t_1)a(0)\rangle$ for various values of t_1 with a suitably-chosen spacing. These expectation values can then be used as initial conditions for (C.8). Figure C.1 shows two examples, which are in agreement with the analytic solutions presented in figure 4.1.

Equations (C.4)–(C.6) could have been obtained much more easily by deriving the Heisenberg–Langevin equations for $a(t)$ and $a^\dagger(t)$, as we did in section 6.3. My intention in this section has been to present a proof of concept for my proposed method of integrating the moment hierarchy. I expect that when it comes to nontrivial examples, generating the moment hierarchy and implementing this truncation scheme will require the help of a computer algebra system.



(a) $\phi = 0$ (b) $\phi = 0$

(c) $\phi = \pi$ (d) $\phi = \pi$

Figure C.1: First-order correlation functions $\langle a^\dagger(t_1)a(t_2)\rangle$ and excitation numbers $\langle a^\dagger(t)a(t)\rangle$, calculated numerically with $\gamma = \eta = 1$ and $\tau = 1$, for two different values of ϕ.

D

WICK'S THEOREM AND THE ENVIRONMENT CORRELATION IDENTITIES

Consider a state

$$\rho = \frac{\exp\left(-T^{-1}(\omega-\mu)b^\dagger b\right)}{\operatorname{tr}\left(\exp\left(-T^{-1}\sum(\omega-\mu)b^\dagger b\right)\right)} = \frac{1}{Z}\exp\left(-T^{-1}(\omega-\mu)b^\dagger b\right) . \tag{D.1}$$

We aim to calculate the expectation value, for $m \geq 1$:

$$\begin{aligned}
\left\langle \left(b^\dagger\right)^m b^m \right\rangle &= \frac{1}{Z}\operatorname{tr}\left(\left(b^\dagger\right)^m b^m e^{-T^{-1}(\omega-\mu)b^\dagger b}\right) \\
&= \frac{1}{Z}\operatorname{tr}\left(\left(b^\dagger\right)^m b^{m-1} e^{-T^{-1}(\omega-\mu)b^\dagger b} e^{T^{-1}(\omega-\mu)b^\dagger b} b e^{-T^{-1}(\omega-\mu)b^\dagger b}\right) \\
&= e^{-(\omega-\mu)/T}\left\langle b\left(b^\dagger\right)^m b^{m-1}\right\rangle \\
&= e^{-(\omega-\mu)/T}\left\langle \left(\left(b^\dagger\right)^m b + \left[b, \left(b^\dagger\right)^m\right]\right) b^{m-1}\right\rangle \\
&= e^{-(\omega-\mu)/T}\left(\left\langle \left(b^\dagger\right)^m b^m\right\rangle + m\left\langle \left(b^\dagger\right)^{m-1} b^{m-1}\right\rangle\right) .
\end{aligned} \tag{D.2}$$

Equation (D.2) is a recursion relation, and can also be written as

$$\left\langle \left(b^\dagger\right)^m b^m \right\rangle = \frac{m}{e^{(\omega-\mu)/T}-1}\left\langle \left(b^\dagger\right)^{m-1} b^{m-1}\right\rangle . \tag{D.3}$$

With the trivial base case $\langle (b^\dagger)^0 b^0\rangle = 1$, we find that

$$\left\langle b^\dagger b\right\rangle = n(\omega) , \tag{D.4}$$

where $n(\omega) = \left(e^{(\omega-\mu)/T}-1\right)^{-1}$ is the Bose–Einstein distribution. For general $m \geq 1$,

$$\left\langle \left(b^\dagger\right)^m b^{m'}\right\rangle = \delta_{mm'} m!\left\langle b^\dagger b\right\rangle^m = m\left\langle b^\dagger b\right\rangle\left\langle \left(b^\dagger\right)^{m-1} b^{m'-1}\right\rangle . \tag{D.5}$$

This is Wick's theorem for the harmonic oscillator.[82,83] This result is also known, in probability theory, as Isserlis' theorem or the Gaussian moment theorem—the thermal state (D.1) is a Gaussian state.[84–86] Equation (D.5) can also be obtained by evaluating the m^{th} factorial moment of the thermal distribution,[87] or by expanding ρ in a Fock state basis.

Now suppose $C = \sum_k c_k b_k$ and $D = \sum_k d_k b_k$ are two arbitrary linear combinations of a set of operators satisfying $[b_k, b_{k'}] = [b_k^\dagger, b_{k'}^\dagger] = 0$ and $[b_k, b_{k'}^\dagger] = \delta_{kk'}$. For our present

purposes, we want to find an expression for an expectation value of the form

$$\left\langle (C^\dagger)^m C^{m'-1} D \right\rangle = \mathrm{tr}\left((C^\dagger)^m C^{m'-1} D\rho \right) . \tag{D.6}$$

Applying (D.5) separately to each term in the expansion of (D.6) yields the factorisation

$$\left\langle (C^\dagger)^m C^{m'-1} D \right\rangle = m \left\langle C^\dagger D \right\rangle \left\langle (C^\dagger)^{m-1} C^{m'-1} \right\rangle . \tag{D.7}$$

There are many alternative factorisations, but this is the particular one we are looking for. We can see that there are m terms in the right-hand side of (D.7), one for each factor of $C^\dagger$ on the left-hand side. Thus, we can say that we have chosen to 'pair' each one of these factors, in turn, with D. In the context of quantum field theory, where Wick's theorem is most often used, these 'pairings' are known as *contractions*. In an exactly analogous way we can show that

$$\left\langle D^\dagger (C^\dagger)^{m'-1} C^m \right\rangle = m' \left\langle D^\dagger C \right\rangle \left\langle (C^\dagger)^{m'-1} C^{m-1} \right\rangle . \tag{D.8}$$

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   Cascaded systems with symmetric irreversible coupling are considered in section 19.2.6. Details of the origin of coherent driving fields as excitations of the environment can be found in note 9.9 on page 25. Section 19.3.1 discusses the calculation of an optical spectrum, for a particular non-stationary case, from the perspective of quantum trajectory theory. The spontaneous emission spectrum for a coupled atom and cavity, without delay, is calculated in section 13.3.1.
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   The theory of classical stochastic processes is covered in chapter 1.

   The theory of quantum master equations appears in chapter 3: in particular, section 3.2 is concerned with quantum Markov processes and quantum dynamical semigroups, and section 3.3 contains a microscopic derivation of the Lindblad master equation. Completely positive maps are discussed on page 86, and the mathematical conditions required for the existence of an infinitesimal generator of a semigroup are discussed on pages 118–119.

   The theory of non-Markovian quantum processes is covered in chapters 9 and 10. The Nakajima–Zwanzig and time-convolutionless projection operator techniques are developed in chapter 9. Spontaneous emission is discussed in section 10.1.
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