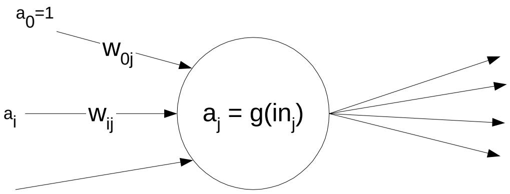
Artificial neural networks

A neural network is a learning system consisting of nodes or units that represent single neurons and directed links connected two nodes. Neural networks are used to solve learning problems involving classification into two or more classes. A mathematical model of a single neuron is shown on Figure to the left.



Each link connecting two neurons i and j has a weight associated with it w_{ij}that represents the strength and a sign of the connection via which the activation of neuron i propagates to neuron j. Each neuron i has a dummy input $a_0 = 1$ with weight w_{0i}. A neuron j receives signals from other neurons via connections and calculates a weighted sum of its inputs:

$$in_j = \sum_{i=0...n} a_i w_{ij}$$

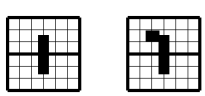
Then it derives the output by calculating an ***activation function g***

$$a_j = g(in_j) = g\left(\sum_{i=0...n} a_i w_{ij}\right)$$

An activation function is usually a threshold or a logistic function:

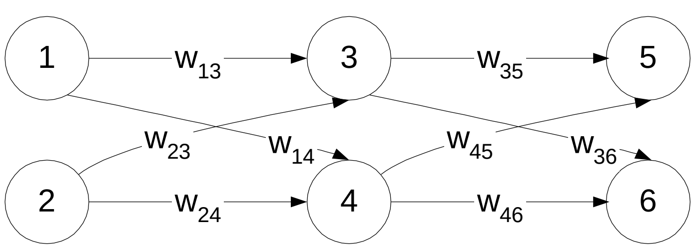
$$g(in_j) = \frac{1}{1 + exp^{-in_j}}$$

We'll give an example of using a neural network to recognize digits. Our network recognizes only two digits 1 and 7. Each example consists of a digit written inside of a 6x6 pixels square. The input vector $\boldsymbol{x} = [x_1, x_2]$ has two values: the first value is the proportion of pixels used by the digit in the upper half of the square and the second value is the proportion of pixels taken by the digit in the lower half of the square.



For example, on Figure to the left, digit 1 uses 2 pixels in the upper half and 2 pixels in the lower half; thus, $\boldsymbol{x_1} = \left[\frac{2}{18}, \frac{2}{18}\right] = \left[\frac{1}{9}, \frac{1}{9}\right] = [0.11, 0.11]$. For digit 7, 4 pixels are taken by the digit in the upper half and 2 in the lower half; thus, $\boldsymbol{x_2} = \left[\frac{4}{18}, \frac{2}{18}\right] = [0.22, 0.11]$.

A neural network for classifying digits into 1 and 7 are shown on Figure below. It has two input nodes 1 and 2, two ***hidden*** nodes 3 and 4, and two output nodes 5 and 6. The output vector $\boldsymbol{y} = [a_5, a_6]$ equals to [0.9, 0.1] if a digit is 1 and [0.1, 0.9] if a digit is 7.



The algorithm iteratively learns the weights of the neural network using the ***back-propagation*** algorithm.

Randomly initialize all weights to small values -0.5...0.5.
Repeat until convergence:
For each example <x, y>

Step 1. For each node in the input layer do

$a_j = x_j$

Step 2. For each layer $l = 2 \dots L$ do

Step 3. For each node j do

$in_j = \sum_i a_i w_{ij}$

$a_j = g(in_j)$

Step 4. For each node in the output layer calculate error:

$\Delta_j = a_j(1 - a_j)(y_j - a_j)$

Step 5. For $l = L - 1 \dots 1$ do

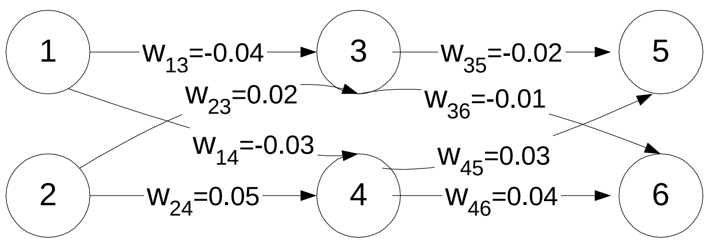
Step 6. For each node in layer l do

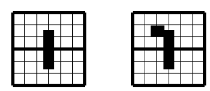
$\Delta_i = a_i(1 - a_i)\sum_j \Delta_j w_{ij}$

Step 7. For each weight w_{ij} in network do

$w_{ij}^{(t)} = w_{ij}^{(t-1)} + \alpha a_i \Delta_j$

Example Run the back-propagation algorithm for two given examples and given the neural network:





$< x_1, \boldsymbol{y}_1 > = < [0.11, 0.11], [0.9, 0.1] >$

$< x_2, \boldsymbol{y}_2 > = < [0.22, 0.11], [0.1, 0.9] >$

In addition to the weights shown on Figure above, each node i has a dummy weight $w_{0i} = 0.01$.

Iteration one

Example $< x_1, \boldsymbol{y}_1 >$:

Step 1: $a_1 = x_{11} = 0.11$, $a_2 = x_{12} = 0.11$

Step 2:

Layer $l = 2$

Step 3:

$$in_3 = \sum_i a_i w_{ij} = w_{03} + a_1 * w_{13} + a_2 * w_{23} = 0.01 + 0.11 * (-0.04) + 0.11 * 0.02 = 0.0078$$

$$a_3 = g(in_3) = \frac{1}{1 + exp^{-0.0078}} = 0.50195$$

$$in_4 = \sum_i a_i w_{ij} = w_{04} + a_1 * w_{14} + a_2 * w_{24} = 0.01 + 0.11 * (-0.03) + 0.11 * 0.05 = 0.0122$$

$$a_4 = g(in_4) = \frac{1}{1 + exp^{-0.0122}} = 0.50305$$

Layer $l = 3$

Step 3:

$$in_5 = \sum_i a_i w_{ij} = w_{05} + a_3 * w_{35} + a_4 * w_{45} = 0.01 + 0.50195 * (-0.02) + 0.50305 * 0.03 = 0.015053$$

$$a_5 = g(in_5) = \frac{1}{1 + exp^{-0.015053}} = 0.503763$$

$$in_6 = \sum_i a_i w_{ij} = w_{06} + a_3 * w_{36} + a_4 * w_{46} = 0.01 + 0.50195 * (-0.01) + 0.50305 * 0.04 = 0.025103$$

$$a_6 = g(in_6) = \frac{1}{1 + exp^{-0.025103}} = 0.506275$$

Step 4:

$\Delta_5 = a_5(1-a_5)(y_5-a_5) = 0.503763(1-0.503763)(0.9-0.503763) = 0.099054$

$\Delta_6 = a_6(1-a_6)(y_6-a_6) = 0.506275(1-0.506275)(0.1-0.506275) = -0.10155$

Step 5:

Layer $l = 2$

Step 6:

$\Delta_3 = a_3(1-a_3)\sum_j \Delta_j w_{3j} = 0.50195(1-0.50195)\big(0.099054 * (-0.02) + (-0.10155) * (-0.01)\big)$
$= -0.00024$

$\Delta_4 = a_4(1-a_4)\sum_j \Delta_j w_{4j} = 0.50305(1-0.50305)(0.099054 * 0.03 + (-0.10155) * 0.04) = -0.00027$

Layer $l = 1$

Step 6:

$\Delta_1 = a_1(1-a_1)\sum_j \Delta_j w_{1j} = 0.11 * (1-0.11) * \big((-0.00024) * (-0.04) + (-0.00027) * (-0.03)\big)$
$= 0.00000173283$

$\Delta_2 = a_2(1-a_2)\sum_j \Delta_j w_{2j} = 0.11 * (1-0.11) * ((-0.00024) * (0.02) + (-0.00027) * (0.05)$
$= -0.00000179157$

Step 7:

$w_{01}^{(t)} = w_{01}^{(t-1)} + \alpha a_0 \Delta_1 = 0.01 + 0.01 * 1 * 0.00000173283 = 0.0100000173283$

$w_{02}^{(t)} = w_{02}^{(t-1)} + \alpha a_0 \Delta_2 = 0.01 + 0.01 * 1 * (-0.00000179157) = 0.0099999820843$

$w_{03}^{(t)} = w_{03}^{(t-1)} + \alpha a_0 \Delta_3 = 0.01 + 0.01 * 1 * (-0.00024) = 0.0099976$

$w_{04}^{(t)} = w_{04}^{(t-1)} + \alpha a_0 \Delta_4 = 0.01 + 0.01 * 1 * (-0.00027) = 0.0099973$

$w_{05}^{(t)} = w_{05}^{(t-1)} + \alpha a_0 \Delta_5 = 0.01 + 0.01 * 1 * 0.099054 = 0.01099054$

$w_{13}^{(t)} = w_{13}^{(t-1)} + \alpha a_1 \Delta_3 = -0.04 + 0.01 * 0.11 * (-0.00024) = -0.040000264$

$w_{14}^{(t)} = w_{14}^{(t-1)} + \alpha a_1 \Delta_4 = -0.03 + 0.01 * 0.11 * (-0.00027) = -0.030000297$

$w_{23}^{(t)} = w_{23}^{(t-1)} + \alpha a_2 \Delta_3 = 0.02 + 0.01 * 0.11 * (-0.00024) = 0.019999736$

$w_{24}^{(t)} = w_{24}^{(t-1)} + \alpha a_2 \Delta_4 = 0.05 + 0.01 * 0.11 * (-0.00027) = 0.049999703$

$w_{35}^{(t)} = w_{35}^{(t-1)} + \alpha a_3 \Delta_5 = -0.02 + 0.01 * 0.50195 * 0.099054 = -0.019502798447$

$w_{36}^{(t)} = w_{36}^{(t-1)} + \alpha a_3 \Delta_6 = -0.01 + 0.01 * 0.50195 * (-0.10155) = -0.010509730225$

$w_{45}^{(t)} = w_{45}^{(t-1)} + \alpha a_4 \Delta_5 = 0.03 + 0.01 * 0.50305 * 0.099054 = 0.030498291147$

$w_{46}^{(t)} = w_{46}^{(t-1)} + \alpha a_4 \Delta_6 = 0.04 + 0.01 * 0.50305 * (-0.10155) = 0.039489152725$