

Random Math Problems

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Integrate by Parts

Practice Problem 1

$$\int \ln(x^2 - x + 2) dx$$

Let $u = \ln(x^2 - x + 2) \leftrightarrow du = \frac{2x-1}{x^2-x+2}$ and $dv = dx \leftrightarrow v = x$

$$= x \ln(x^2 - x + 2) - \int \frac{(2x-1)x}{x^2-x+2} dx = x \ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2} dx$$

Long division: $\frac{2x^2-x}{x^2-x+2} = \frac{x-4}{x^2-x+2} + 2$

$$\int \frac{2x^2 - x}{x^2 - x + 2} dx = \int \left[\frac{x-4}{x^2-x+2} + 2 \right] dx = 2x + c_1 + \int \frac{x-4}{x^2-x+2} dx = 2x + c_1 + \int \frac{\frac{1}{2}(2x-1) - \frac{7}{2}}{x^2-x+2} dx$$

$$= 2x + c_1 + \frac{1}{2} \int \frac{2x-1}{x^2-x+2} - \int \frac{\frac{7}{2}}{x^2-x+2} dx = 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - \frac{7}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}} dx$$

$$= 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - \frac{7}{2} \int \frac{1}{\frac{7}{4} \left[\left(\frac{\sqrt{4}}{\sqrt{7}} \left(x - \frac{1}{2} \right) \right)^2 + 1 \right]} dx$$

Let $u = \sqrt{\frac{4}{7}} \left(x - \frac{1}{2} \right) \leftrightarrow du = \sqrt{\frac{4}{7}} dx$

$$= 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - \frac{7}{2} \frac{4}{7} \sqrt{\frac{7}{4}} \int \frac{1}{u^2 + 1} du = 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - 2 \sqrt{\frac{7}{4}} \arctan \left(\sqrt{\frac{4}{7}} \left(x - \frac{1}{2} \right) \right)$$

So,

$$\int \ln(x^2 - x + 2) dx = x \ln(x^2 - x + 2) - 2x - \frac{1}{2} \ln(x^2 - x + 2) + \sqrt{7} \arctan \left(\sqrt{\frac{4}{7}} \left(x - \frac{1}{2} \right) \right) + C$$

Surface area of the curve when rotated about the axis.

The surface area of the curve when rotated about the x-axis can be found with the following formula:

$$S = 2\pi \int_a^b f(x) \sqrt{(f'(x))^2 + 1} dx$$

Practice Exercise 1

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = x^3, \quad 0 \leq x \leq 3$$

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$S = 2\pi \int_0^3 x^3 \sqrt{1 + [3x^2]^2} dx = 2\pi \int_0^3 x^3 \sqrt{1 + 9x^4} dx$$

Let $u = 1 + 9x^4$ so $du = 36x^3 dx$

$$= \frac{2\pi}{36} \int_1^{730} u^{1/2} du = \frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_1^{730} = \frac{\pi}{27} \left[u^{3/2} \right]_1^{730} = \frac{\pi}{27} (730\sqrt{730} - 1)$$

Practice Exercise 2

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = \sqrt{6-x}, \quad 4 \leq x \leq 6$$

$$f(x) = (6-x)^{1/2}$$

$$f'(x) = -\frac{1}{2} (6-x)^{-1/2}$$

$$\begin{aligned} S &= 2\pi \int_4^6 \sqrt{6-x} \sqrt{\left(-\frac{1}{2(6-x)^{1/2}}\right)^2 + 1} dx = 2\pi \int_4^6 \sqrt{6-x} \sqrt{\left(\frac{1}{4(6-x)}\right) + 1} dx \\ &= 2\pi \int_4^6 \sqrt{6-x} \sqrt{\left(\frac{1+24-4x}{4(6-x)}\right)} dx = 2\pi \int_4^6 \sqrt{\frac{25-4x}{4}} dx = \pi \int_4^6 (25-4x)^{1/2} dx \end{aligned}$$

let $u = 25 - 4x$ and $du = -4dx$

$$= -\frac{\pi}{4} \int_9^1 (u)^{1/2} du = \frac{2\pi}{4 \cdot 3} \left[u^{3/2} \right]_1^9 = \frac{\pi}{6} [27 - 1] = \frac{13\pi}{3}$$

Practice Exercise 3

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y^2 = x + 1, \quad 0 \leq x \leq 8$$

$$f(x) = \sqrt{x+1}$$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}$$

$$\begin{aligned} S &= 2\pi \int_0^8 \sqrt{x+1} \sqrt{\left(\frac{1}{2}(x+1)^{-1/2}\right)^2 + 1} dx = 2\pi \int_0^8 \sqrt{x+1} \sqrt{\left(\frac{1}{4(x+1)}\right) + 1} dx \\ &= 2\pi \int_0^8 \sqrt{x+1} \sqrt{\frac{4x+5}{4x+4}} dx = \frac{2\pi}{2} \int_0^8 (4x+5)^{1/2} dx \end{aligned}$$

Let $u = 4x + 5$ and $du = 4dx$

$$= \frac{\pi}{4} \int_5^{37} u^{1/2} dx = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_5^{37} = \frac{\pi}{6} [37\sqrt{37} - 5\sqrt{5}]$$