Random Math Problems

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Integrate by Parts

Practice Problem 1.1

$$\int \ln(x^2 - x + 2)dx$$

Let $u = \ln(x^2 - x + 2) \leftrightarrow du = \frac{2x-1}{x^2-x+2}$ and $dv = dx \leftrightarrow v = x$

$$= x\ln(x^2 - x + 2) - \int \frac{(2x-1)x}{x^2 - x + 2}dx = x\ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2}dx$$

Long division: $\frac{2x^2-x}{x^2-x+2} = \frac{x-4}{x^2-x+2} + 2$

$$\int \frac{2x^2 - x}{x^2 - x + 2}dx = \int \left[\frac{x-4}{x^2 - x + 2} + 2\right]dx = 2x + c_1 + \int \frac{x-4}{x^2 - x + 2}dx = 2x + c_1 + \int \frac{\frac{1}{2}(2x-1) - \frac{7}{2}}{x^2 - x + 2}dx$$

$$= 2x + c_1 + \frac{1}{2}\int \frac{2x-1}{x^2 - x + 2} - \int \frac{\frac{7}{2}}{x^2 - x + 2}dx = 2x + \frac{1}{2}\ln(x^2 - x + 2) + c_2 - \frac{7}{2}\int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}}dx$$

$$= 2x + \frac{1}{2}\ln(x^2 - x + 2) + c_2 - \frac{7}{2}\int \frac{1}{\frac{7}{4}\left[\left(\frac{\sqrt{4}}{\sqrt{7}}\left(x - \frac{1}{2}\right)\right)^2 + 1\right]}dx$$

Let $u = \sqrt{\frac{4}{7}}\left(x - \frac{1}{2}\right) \leftrightarrow du = \sqrt{\frac{4}{7}}dx$

$$= 2x + \frac{1}{2}\ln(x^2 - x + 2) + c_2 - \frac{7}{2}\frac{4}{7}\sqrt{\frac{7}{4}}\int \frac{1}{u^2 + 1}du = 2x + \frac{1}{2}\ln(x^2 - x + 2) + c_2 - 2\sqrt{\frac{7}{4}}\arctan\left(\sqrt{\frac{4}{7}}\left(x - \frac{1}{2}\right)\right)$$

So,

$$\int \ln(x^2 - x + 2)dx = x\ln(x^2 - x + 2) - 2x - \frac{1}{2}\ln(x^2 - x + 2) + \sqrt{7}\arctan\left(\sqrt{\frac{4}{7}}\left(x - \frac{1}{2}\right)\right) + C$$

Exact length of the curve

Given $f(x)$ and $a \leq x \leq b$, the exact length of the curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Practice Problem 2.1

Find the exact length of the curve:

$x = e^y + \frac{1}{4}e^{-y}, \quad 0 \leq y \leq 6$

Since $f(y) = e^y + \frac{1}{4}e^{-y}$, we have $f'(y) = e^y - \frac{1}{4}e^{-y}$

$$L = \int_0^6 \sqrt{1 + \left(e^y - \frac{1}{4}e^{-y}\right)^2} dy = \int_0^6 \sqrt{1 + \left(e^{2y} + \frac{1}{16}e^{-2y} - \frac{1}{2}\right)} dy = \int_0^6 \sqrt{\frac{16e^{4y} + 8e^{2y} + 1}{16e^{2y}}} dy$$

$$= \int_0^6 \sqrt{\frac{(4e^{2y} + 1)^2}{16e^{2y}}} dy = \int_0^6 \frac{4e^{2y} + 1}{4e^y} dy = \int_0^6 e^y + \frac{1}{4}e^{-y} dy = \left[e^y - \frac{1}{4}e^{-y}\right]_0^6 = e^6 - \frac{1}{4e^6} - \frac{3}{4}$$

Surface area of the curve when rotated about the axis.

The surface area of the curve when rotated about the x-axis can be found with the following formula:

$$S = 2\pi \int_a^b f(x)\sqrt{(f'(x))^2 + 1} dx$$

Practice Exercise 3.1

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = x^3, \quad 0 \leq x \leq 3$$
$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$S = 2\pi \int_0^3 x^3 \sqrt{1 + [3x^2]^2} dx = 2\pi \int_0^3 x^3 \sqrt{1 + 9x^4} dx$$

Let $u = 1 + 9x^4$ so $du = 36x^3 dx$

$$= \frac{2\pi}{36} \int_1^{730} u^{1/2} du = \frac{\pi}{18} \left[\frac{2}{3}u^{3/2}\right]_1^{730} = \frac{\pi}{27} \left[u^{3/2}\right]_1^{730} = \frac{\pi}{27} \left(730\sqrt{730} - 1\right)$$

Practice Exercise 3.2

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = \sqrt{6-x}, \qquad 4 \leq x \leq 6$$

$$f(x) = (6-x)^{1/2}$$
$$f'(x) = -\frac{1}{2}(6-x)^{-1/2}$$
$$S = 2\pi \int_4^6 \sqrt{6-x}\sqrt{\left(-\frac{1}{2(6-x)^{1/2}}\right)^2 + 1dx} = 2\pi \int_4^6 \sqrt{6-x}\sqrt{\left(\frac{1}{4(6-x)}\right) + 1dx}$$
$$= 2\pi \int_4^6 \sqrt{6-x}\sqrt{\left(\frac{1+24-4x}{4(6-x)}\right)}dx = 2\pi \int_4^6 \sqrt{\frac{25-4x}{4}}dx = \pi \int_4^6 (25-4x)^{1/2}dx$$

let $u = 25 - 4x$ and $du = -4dx$

$$= -\frac{\pi}{4}\int_9^1 (u)^{1/2}du = \frac{2\pi}{4*3}\left[u^{3/2}\right]_1^9 = \frac{\pi}{6}[27-1] = \frac{13\pi}{3}$$

Practice Exercise 3.3

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y^2 = x+1, \qquad 0 \leq x \leq 8$$

$$f(x) = \sqrt{x+1}$$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}$$
$$S = 2\pi \int_0^8 \sqrt{x+1}\sqrt{\left(\frac{1}{2}(x+1)^{-1/2}\right)^2 + 1dx} = 2\pi \int_0^8 \sqrt{x+1}\sqrt{\left(\frac{1}{4(x+1)}\right) + 1dx}$$
$$= 2\pi \int_0^8 \sqrt{x+1}\sqrt{\frac{4x+5}{4x+4}} = \frac{2\pi}{2}\int_0^8 (4x+5)^{1/2}dx$$

Let $u = 4x + 5$ and $du = 4dx$

$$= \frac{\pi}{4}\int_5^{37} u^{1/2}dx = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{37} = \frac{\pi}{6}\left[37\sqrt{37} - 5\sqrt{5}\right]$$