

# Random Math Problems

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## Integrate by Parts

### Practice Problem 1.1

$$\int \ln(x^2 - x + 2) dx$$

Let  $u = \ln(x^2 - x + 2) \leftrightarrow du = \frac{2x-1}{x^2-x+2}$  and  $dv = dx \leftrightarrow v = x$

$$= x \ln(x^2 - x + 2) - \int \frac{(2x-1)x}{x^2-x+2} dx = x \ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2} dx$$

Long division:  $\frac{2x^2-x}{x^2-x+2} = \frac{x-4}{x^2-x+2} + 2$

$$\int \frac{2x^2 - x}{x^2 - x + 2} dx = \int \left[ \frac{x-4}{x^2-x+2} + 2 \right] dx = 2x + c_1 + \int \frac{x-4}{x^2-x+2} dx = 2x + c_1 + \int \frac{\frac{1}{2}(2x-1) - \frac{7}{2}}{x^2-x+2} dx$$

$$= 2x + c_1 + \frac{1}{2} \int \frac{2x-1}{x^2-x+2} - \int \frac{\frac{7}{2}}{x^2-x+2} dx = 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - \frac{7}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}} dx$$

$$= 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - \frac{7}{2} \int \frac{1}{\frac{7}{4} \left[ \left( \frac{\sqrt{4}}{\sqrt{7}} \left( x - \frac{1}{2} \right) \right)^2 + 1 \right]} dx$$

Let  $u = \sqrt{\frac{4}{7}} \left( x - \frac{1}{2} \right) \leftrightarrow du = \sqrt{\frac{4}{7}} dx$

$$= 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - \frac{7}{2} \frac{4}{7} \sqrt{\frac{7}{4}} \int \frac{1}{u^2 + 1} du = 2x + \frac{1}{2} \ln(x^2 - x + 2) + c_2 - 2 \sqrt{\frac{7}{4}} \arctan \left( \sqrt{\frac{4}{7}} \left( x - \frac{1}{2} \right) \right)$$

So,

$$\int \ln(x^2 - x + 2) dx = x \ln(x^2 - x + 2) - 2x - \frac{1}{2} \ln(x^2 - x + 2) + \sqrt{7} \arctan \left( \sqrt{\frac{4}{7}} \left( x - \frac{1}{2} \right) \right) + C$$

**Practice Problem 1.2**

$$\int x \ln(x) dx$$

Let  $u = \ln(x) \leftrightarrow du = \frac{1}{x} dx$  and  $dv = x dx \leftrightarrow v = \frac{1}{2} x^2$

$$\int x \ln(x) dx = \frac{1}{2} x^2 \ln(x) - \int \frac{1}{2} x dx = \frac{x^2}{2} \ln(x) - \frac{1}{4} x^2 + C = \frac{x^2(2 \ln(x) - 1)}{4} + C$$

**Practice Problem 1.3**

$$\int \ln(x) dx$$

Let  $u = \ln(x) \leftrightarrow du = \frac{1}{x} dx$  and  $dv = 1 dx \leftrightarrow v = x$

$$\int \ln(x) dx = x \ln(x) - \int 1 dx = x \ln(x) - x + C = x(\ln(x) - 1) + C$$

## Exact length of the curve

Given  $f(x)$  and  $a \leq x \leq b$ , the exact length of the curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

### Practice Problem 2.1

Find the exact length of the curve:

$$x = e^y + \frac{1}{4}e^{-y}, \quad 0 \leq y \leq 6$$

Since  $f(y) = e^y + \frac{1}{4}e^{-y}$ , we have  $f'(y) = e^y - \frac{1}{4}e^{-y}$

$$\begin{aligned} L &= \int_0^6 \sqrt{1 + \left(e^y - \frac{1}{4}e^{-y}\right)^2} dy = \int_0^6 \sqrt{1 + \left(e^{2y} + \frac{1}{16}e^{-2y} - \frac{1}{2}\right)} dy = \int_0^6 \sqrt{\frac{16e^{4y} + 8e^{2y} + 1}{16e^{2y}}} dy \\ &= \int_0^6 \sqrt{\frac{(4e^{2y} + 1)^2}{16e^{2y}}} dy = \int_0^6 \frac{4e^{2y} + 1}{4e^y} dy = \int_0^6 e^y + \frac{1}{4}e^{-y} dy = \left[e^y - \frac{1}{4}e^{-y}\right]_0^6 = e^6 - \frac{1}{4e^6} - \frac{3}{4} \end{aligned}$$

## Surface area of the curve when rotated about the axis.

The surface area of the curve when rotated about the x-axis can be found with the following formula:

$$S = 2\pi \int_a^b f(x) \sqrt{(f'(x))^2 + 1} dx$$

### Practice Exercise 3.1

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = x^3, \quad 0 \leq x \leq 3$$

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$S = 2\pi \int_0^3 x^3 \sqrt{1 + [3x^2]^2} dx = 2\pi \int_0^3 x^3 \sqrt{1 + 9x^4} dx$$

Let  $u = 1 + 9x^4$  so  $du = 36x^3 dx$

$$= \frac{2\pi}{36} \int_1^{730} u^{1/2} du = \frac{\pi}{18} \left[ \frac{2}{3} u^{3/2} \right]_1^{730} = \frac{\pi}{27} \left[ u^{3/2} \right]_1^{730} = \frac{\pi}{27} (730\sqrt{730} - 1)$$

### Practice Exercise 3.2

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = \sqrt{6-x}, \quad 4 \leq x \leq 6$$

$$f(x) = (6-x)^{1/2}$$

$$f'(x) = -\frac{1}{2}(6-x)^{-1/2}$$

$$\begin{aligned} S &= 2\pi \int_4^6 \sqrt{6-x} \sqrt{\left(-\frac{1}{2(6-x)^{1/2}}\right)^2 + 1} dx = 2\pi \int_4^6 \sqrt{6-x} \sqrt{\left(\frac{1}{4(6-x)}\right) + 1} dx \\ &= 2\pi \int_4^6 \sqrt{6-x} \sqrt{\left(\frac{1+24-4x}{4(6-x)}\right)} dx = 2\pi \int_4^6 \sqrt{\frac{25-4x}{4}} dx = \pi \int_4^6 (25-4x)^{1/2} dx \end{aligned}$$

let  $u = 25 - 4x$  and  $du = -4dx$

$$= -\frac{\pi}{4} \int_9^1 (u)^{1/2} du = \frac{2\pi}{4 \cdot 3} \left[ u^{3/2} \right]_1^9 = \frac{\pi}{6} [27 - 1] = \frac{13\pi}{3}$$

### Practice Exercise 3.3

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y^2 = x + 1, \quad 0 \leq x \leq 8$$

$$f(x) = \sqrt{x+1}$$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}$$

$$\begin{aligned} S &= 2\pi \int_0^8 \sqrt{x+1} \sqrt{\left(\frac{1}{2}(x+1)^{-1/2}\right)^2 + 1} dx = 2\pi \int_0^8 \sqrt{x+1} \sqrt{\left(\frac{1}{4(x+1)}\right) + 1} dx \\ &= 2\pi \int_0^8 \sqrt{x+1} \sqrt{\frac{4x+5}{4x+4}} dx = \frac{2\pi}{2} \int_0^8 (4x+5)^{1/2} dx \end{aligned}$$

Let  $u = 4x + 5$  and  $du = 4dx$

$$= \frac{\pi}{4} \int_5^{37} u^{1/2} dx = \frac{\pi}{4} \left[ \frac{2}{3} u^{3/2} \right]_5^{37} = \frac{\pi}{6} [37\sqrt{37} - 5\sqrt{5}]$$