Project Report

QF605 – Fixed Income Securities

April 2023

Stephen Kusrianto

Fernando Oktavianes

Cassidy Gan

Tian Borui

Stanley Wong

Lim Seng Hui

Contents

Introduction

Internal-rate-of-return (IRR) settled swaptions are the main interest rate volatility instruments in the European interest rate markets. Industry practice is to use an approximation formula to price IRR swaptions based on Black model, which is not arbitrage-free. We formulate a unified market model to incorporate both swaptions and constant maturity swaps (CMS) pricing under a single, self-consistent framework. We demonstrate that the model can calibrate market quotes well and is also able to efficiently price IRR-settled swaptions, along with CMS products. We use the model to illustrate the difference in implied volatilities for IRR-settled payer.

The report is organized as follows: Part I presents the bootstrapping of OIS discount factor and LIBOR discount factor to calculate forward swap rates. In Part II, we calibrate the displaced-diffusion model and SABR model to swaption market data to price payer and receiver swaptions. We present the valuation of constant maturity swap (CMS) using the calibrated SABR model in Part III, where we compare the forward swap rates with the CMS rates. Finally, the valuations for decompounded options using static replications are shown in part IV.

Part I (Bootstrapping Swap Curves)

Question 1

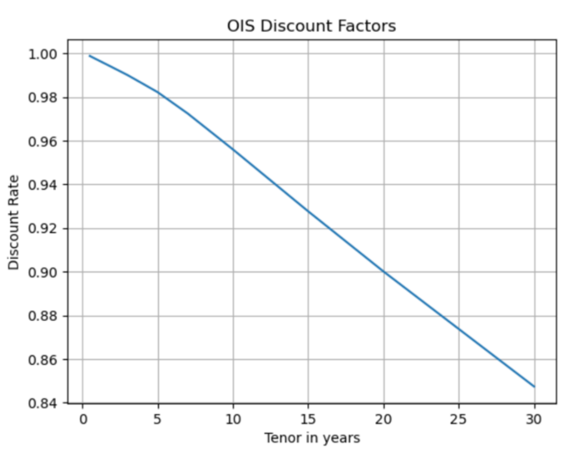
OIS discount factor

To bootstrap the OIS discount rates, we followed the formula provided in the project notes and calculated the six month & first year fed fund rates f0 & f1. From second year onwards, we observe the following relationship:

$$\text{OIS} \sum_{t=i}^{T} (D_t \cdots D_T) = \sum_{t=i}^{T} D_0(T_i)\left[\left(1+\frac{f_0}{360}\right)^{(T-t)}-1\right] \qquad D_0(0,1y) = \left(\frac{1}{1+\frac{F_0(t)}{360}}\right)^{360}$$

The above implies that we can represent the fed fund rates in terms of OIS discount factors, and reduces the OIS Par swap rate calculation to substitute and simplify into one unknown parameter of either f or the discount rate itself. We chose the discount rate since we can perform linear interpolation on the unknowns in between two known discount factors, and simultaneously solve the fed fund rates retroactively. We also took note that it is generally not a feasible practice to interpolate the market observable Swap Rates itself. Below is the OIS discount curve next to LIBOR for comparison:

Tenor	Discount	Tenor	Discount	Tenor	Discount	Tenor	Discount	Tenor	Discount	Tenor	Discount
0.5	0.99875	5.5	0.97974	10.5	0.95314	15.5	0.92486	20.5	0.89744	25.5	0.87111
1.0	0.99701	6.0	0.97729	11.0	0.95030	16.0	0.92210	21.0	0.89481	26.0	0.86847
1.5	0.99527	6.5	0.97485	11.5	0.94747	16.5	0.91935	21.5	0.89217	26.5	0.86584
2.0	0.99353	7.0	0.97240	12.0	0.94463	17.0	0.91660	22.0	0.88954	27.0	0.86321
2.5	0.99177	7.5	0.96967	12.5	0.94179	17.5	0.91384	22.5	0.88691	27.5	0.86057
3.0	0.99002	8.0	0.96693	13.0	0.93896	18.0	0.91109	23.0	0.88427	28.0	0.85794
3.5	0.98807	8.5	0.96419	13.5	0.93612	18.5	0.90834	23.5	0.88164	28.5	0.85531
4.0	0.98612	9.0	0.96145	14.0	0.93328	19.0	0.90558	24.0	0.87901	29.0	0.85267
4.5	0.98415	9.5	0.95871	14.5	0.93045	19.5	0.90283	24.5	0.87637	29.5	0.85004
5.0	0.98218	10.0	0.95598	15.0	0.92761	20.0	0.90007	25.0	0.87374	30.0	0.84740



Question 2

LIBOR discount factor

For LIBOR in a collateralized swap market, we observe the following relationship:

$$\frac{1}{\Delta_i} \cdot \text{IRS} \sum_{t=i}^{T} (D_t \cdots D_T) = \sum_{t=i}^{T} D_0(T_i)\tilde{L}(T_{i-1}, T_i)$$

$$L(T_{i-1}, T_i) = \frac{1}{\Delta_i}\left[\frac{\tilde{D}(0,T_{i-1}) - \tilde{D}(0,T_i)}{\tilde{D}(0,T_i)}\right]$$

As is the case with OIS discount factors, we can perform substitution and refactor the equation into a single unknown discount factor D(0,Ti) tilde, and have an analytical solution even for discount factors in between tenors that are not observable from the IRS market (e.g. the 3 year gap from IRS 7y to IRS 10y), by implementing interpolation ratios between the 'right' discount D(0,Ti) tilde and the second last known, 'left' discount rate D(0,Ti-n) tilde. However, in the interest of time we built a function to append said interpolation formula and perform the calculation numerically via brentq as we go down on each discretized time span. Below is the LIBOR discount curve:

Tenor	Discount	Tenor	Discount	Tenor	Discount	Tenor	Discount	Tenor	Discount	Tenor	Discount
0.5	0.987654	5.5	0.832796	10.5	0.678547	15.5	0.536794	20.5	0.398990	25.5	0.306712
1.0	0.972577	6.0	0.816603	11.0	0.664384	16.0	0.522508	21.0	0.389762	26.0	0.297484
1.5	0.957378	6.5	0.800409	11.5	0.650221	16.5	0.508222	21.5	0.380534	26.5	0.288257
2.0	0.942179	7.0	0.784216	12.0	0.636058	17.0	0.493935	22.0	0.371307	27.0	0.279029
2.5	0.926330	7.5	0.768965	12.5	0.621896	17.5	0.479649	22.5	0.362079	27.5	0.269801
3.0	0.910482	8.0	0.753714	13.0	0.607733	18.0	0.465363	23.0	0.352851	28.0	0.260573
3.5	0.894731	8.5	0.738463	13.5	0.593570	18.5	0.451077	23.5	0.343623	28.5	0.251346
4.0	0.878981	9.0	0.723212	14.0	0.579407	19.0	0.436790	24.0	0.334396	29.0	0.242118
4.5	0.863985	9.5	0.707961	14.5	0.565244	19.5	0.422504	24.5	0.325168	29.5	0.232890
5.0	0.848989	10.0	0.692710	15.0	0.551081	20.0	0.408218	25.0	0.315940	30.0	0.223662

Question 3

Forward Swap Rate

Once we have computed the OIS & LIBOR discount factors, we can apply the standard Forward Swap Rate formula and obtain the following results:

Forward Swap Rate					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

Part II (Swaption Calibration)

We utilized the Forward Swap Rates from Part I to calculate the range of Strikes, which are then fitted to perform least squares regression against the market volatility and achieve optimum parameter for both the Displaced-Diffusion & SABR models.

Question 1

Displaced-Diffusion Model Calibration

To obtain DD model's volatility, we performed a regression on ATM market implied volatility, under the assumption that all models including the market converges at the money. Below is the result:

Table 1: Calibrated Displaced-Diffusion Model Parameters

Sigma					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.226286	0.286685	0.297716	0.259171	0.245596
5Y	0.269714	0.294808	0.295745	0.264635	0.244107
10Y	0.274894	0.281445	0.282755	0.259171	0.239422
Beta					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.096686	0.050000	0.013383	0.050000	0.050001
5Y	0.054541	0.011335	0.012226	0.078820	0.050007
10Y	0.049999	0.049999	0.049999	0.050000	0.060962

Given its shifted lognormal properties, we recognize that the DD volatility is equivalent to Black Scholes' constant vol if the beta parameter is equal to 0. To calibrate effectively to the market, we need to expand the width of our regression to include the observed volatility at the 'wings' (i.e., incorporating ITM & OTM strike ranges and experimenting with the span of said range).

Doing so yields a series of DD volatilities that are no longer constant, following the market profile as the value of beta increases, leading the model to exhibit a behavior that goes closer towards the Bachelier model. This implies that as the beta goes closer to 1, the ITM side of the smile exhibits lower volatilities than Black-Scholes'.

Question 2

SABR Model Calibration

Similar to the steps with DD model, we calibrated the SABR model by running the SABR function provided in QF620 lecture against the market data to obtain the Alpha, Rho and Nu parameters. The only difference is that we want the SABR to mirror the market implied volatility very closely, and thus the regression ought to be performed across all range of strikes.

Table 2: Calibrated SABR Model Parameters

Alpha					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.145647	0.190246	0.204236	0.185893	0.179654
5Y	0.163511	0.198984	0.211334	0.192892	0.177126
10Y	0.172753	0.189718	0.200110	0.193512	0.176453
Nu					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	1.932118	1.624398	1.377737	0.984724	0.665812
5Y	1.303637	1.050600	0.926461	0.659610	0.497210
10Y	0.985005	0.906595	0.849657	0.707408	0.576775
Rho					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	-0.608593	-0.525092	-0.496099	-0.457473	-0.357602
5Y	-0.563848	-0.539905	-0.548774	-0.517958	-0.439314
10Y	-0.526523	-0.525793	-0.529680	-0.534532	-0.484079

With the above parameters, we can run the SABR function again with beta anchored at 0.9 and generate the SABR volatility across the range of strikes. Like the DD model, we have plotted the SABR calibrated volatilities across all dimensions as well. We observe that the SABR model is much more accurate than the DD model, with some cases having nearly perfect calibration towards the market observed volatility.

Question 3

Pricing Payer 2 X 10 & Receiver 8 X 10 swaptions with calibrated models

We priced the following swaptions based on insights gained from Question 1 & 2:

Table 3: Calibrated Displaced-Diffusion & SABR Swaption Prices

Swaptions	1%	2%	3%	4%	5%	6%	7%	8%
DD Payer 2 X 10	0.277455	0.187444	0.107806	0.049296	0.016823	0.004087	0.000686	0.000078
SABR Payer 2 X 10	0.278448	0.190194	0.110692	0.050699	0.019785	0.008639	0.004589	0.002830
DD Rec 8 X 10	0.012145	0.023040	0.040180	0.064933	0.098028	0.139375	0.188116	0.242879
SABR Rec 8 X 10	0.018918	0.036426	0.056666	0.082840	0.119592	0.169357	0.227963	0.290548

Linear Interpolation was done between 1 year, 5 year and 10-year expiry swaptions on the parameters as both 2 & 8 year expiries swaptions are not observable from market data. We built a function to interpolate the parameters so we may evaluate other non-observable swaptions dynamically as well.

Please refer to the appendix (last page) for a complete plotting of the DD volatility smile across all expiries, tenors, and strikes.

Part III (Convexity Correction)

Question 1

The CMS leg is a collection of CMS rates paid over a period. The PV is calculated is given by the formula shown below:

$$\sum_{i=0.5}^{N} D_o\,(0, T_i) \cdot Delta\ \cdot CMS\left(S_{n,N}(T_i)\right)$$

We calculated $CMS\left(S_{n,N}(T_i)\right)$,known as the CMS Rate using the formula shown below:

$$E^T\left[S_{n,N}(T)\right] \;=\; g(F) \;+\; \frac{1}{D(0,T)}\left[\int_0^F h''(K)V^{Rec}(K)dK \;+\; \int_F^\infty h''(K)V^{Pay}(K)dK\right]$$

We also calculated the IRR-Option pricer, $V(K)$ using the formula given below:

$$V(K) \;=\; D(0,T) \cdot IRR\left(S_{n,N}(0)\right).Black76\left(S_{n,N}(0), K, \sigma_{SABR}, T\right)$$

The calibrated SABR model obtained from question 2 to derive volatility (σ_{SABR}) to be fitted into Black 76 function. The SABR perimeters ($\alpha, \beta, \rho\ , v$) are estimated using the least squares estimation from the Scipy optimization package in Python.

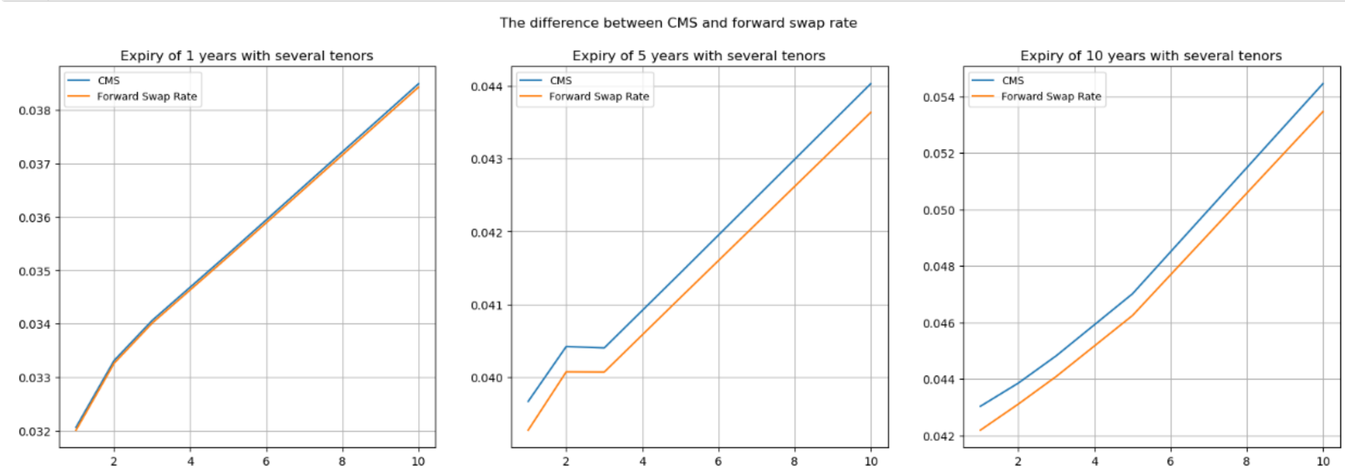
From the findings we have obtained:

- The PV of a leg receiving CMS10y semi-annually over the next 5 years is 0.20207.
- The PV of a leg receiving CMS2y quarterly over the next 10 years is 0.37917.

Question 2

We have compared the forward swap rates with the CMS rate, the results are as shown below:

	Expiry	Tenor	CMS	Forward Swap Rate	difference
0	1	1	0.032065	0.032007	0.000058
1	1	2	0.033312	0.033259	0.000052
2	1	3	0.034063	0.034011	0.000052
3	1	5	0.035309	0.035255	0.000054
4	1	10	0.038494	0.038428	0.000066
5	5	1	0.039695	0.039274	0.000421
6	5	2	0.040444	0.040075	0.00037
7	5	3	0.040425	0.040072	0.000352
8	5	5	0.041454	0.041093	0.00036
9	5	10	0.044051	0.043634	0.000417
10	10	1	0.043130	0.042189	0.000941
11	10	2	0.043941	0.043116	0.000825
12	10	3	0.044907	0.044097	0.00081
13	10	5	0.047097	0.046249	0.000848
14	10	10	0.054535	0.053458	0.001078



Based on the results obtained, the CMS rate is always higher than the forward swap rate. The difference between the two rates can be explained using convexity adjustments. As CMS rates are quoted for longer durations while forward swap rates are quoted for shorter durations, at longer maturities the adjustment for convexity is greater. It can also be inferred that Tenor does not have a significant effect on the convexity adjustment.

Part IV (Decompounded Options)

Question 1

Given a decompounded option pays the following at time $T = 5y$:

$$\text{CMS } 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}$$

For static replication of any constant maturity swap (CMS) payoff $g(F)$, where F is the swap rate, we use the following formula:

$$V_0 = D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)]$$
$$+ \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

Let $F = \text{CMS } 10y$ and $g(K)$ be the payoff function. Differentiating, we get:

$$g(K) = F^{\frac{1}{4}} - 0.2$$
$$g'(K) = \frac{1}{4}F^{-\frac{3}{4}}$$
$$g''(K) = -F^{-\frac{7}{4}}\frac{3}{16}F^{-\frac{7}{4}}$$

Let $\text{IRR}(K)$ be the IRR annuity. Differentiating, we get:

$$\text{IRR}(K) = \sum_{i=1}^{N\times m} \frac{1}{\left(1+\frac{K}{m}\right)^i} = \frac{1}{K}\left[1 - \frac{1}{\left(1+\frac{K}{m}\right)^{N\times m}}\right]$$
$$\text{IRR}'(K) = -\frac{1}{K}\text{IRR}(K) + \frac{1}{m\times K}\frac{N\times m}{\left(1+\frac{K}{m}\right)^{N\times m+1}}$$
$$\text{IRR}''(K) = -\frac{2}{K}\text{IRR}'(K) - \frac{1}{m^2\times K}\frac{N\times m\cdot(N\times m+1)}{\left(1+\frac{K}{m}\right)^{N\times m+2}}$$

For CMS rate payoff, the payoff function is $g(F)$, and the static replication formula is:

$$D(0,T)g(F) + \underbrace{\int_0^F h''(K)V^{rec}(K)dK}_{\text{Put Integral}} + \underbrace{\int_F^\infty h''(K)V^{pay}(K)dK}_{\text{Call Integral}}$$

The present value of this payoff is 0.250042.

Question 2

Suppose the payoff is now

$$\left(CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}\right)^+$$

Let $F = \text{CMS } 10y$ and $g(K)$ be the payoff function. Differentiating, we get:

$$\begin{aligned} g(K) \Rightarrow F^{\frac{1}{4}} - 0.2 &> 0 \\ F &> 0.2^4 \\ F &> 0.0016 \end{aligned}$$

Let L = 0.0016, which is the CMS caplet strike, and the static replication formula is:

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

The present value of this payoff is 0.030343.

We noted that the CMS caplet payoff and IRR-settled payer swaption payoff are functions of the same swap rate distribution. Beyond the strike rate of L, CMS caplet payoff is linear and payer swaption payoff is concave of the swap rate.

Since swaptions are vanilla derivatives and more liquid, we can replicate the CMS caplet payoff using a basket of IRR-settled payer swaptions with increasing strikes starting with the CMS caplet strike L.

Conclusion

In this report, we have derived a model to price decompounded options that pay the CMS rate and CMS caplet in a consistent manner. This has been achieved by formulating a framework which price CMS products of either payoff types under a single market model. The model can capture the convex payoff profiles of IRR-settled swaptions under the static-replication method to obtain a model-independent convexity correction.

References

Tee, Chyng Wen and Kerkhof, Jeroen, A Unified Market Model for Swaptions and Constant Maturity Swaps (August 20, 2019). International Journal of Theoretical and Applied Finance, Vol. 24, No. 4, 2021, https://doi.org/10.1142/S0219024921500266, Available at SSRN: https://ssrn.com/abstract=3441544 or http://dx.doi.org/10.2139/ssrn.3441544

Choi, J., Kwak, M., Tee, C. W., & Wang, Y. (2022). A Black–Scholes user's guide to the Bachelier model. Journal of Futures Markets, 42(5), 959-980

Appendix – Displaced-Diffusion & SABR Model Calibration