Solutions to Problems in Morton Hamermesh's "Group Theory and its Application to Physical Problems", 2nd Ed. (Dover, 1989)

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Something that bothers me about the undergraduate physics curriculum at Waterloo is that despite the wide array of quantum theory courses you can (and indeed, must) take, students are not required to take a single course in abstract algebra. Even those in mathematical physics needn't so much as look at the pure math calendar to get their degree.

Modern quantum mechanical theory relies very heavily on the language of groups and algebras –terms that only increase in their importance the higher up the chain you go. You hear things like "Poisson bracket" and "Lie group" and perhaps even "Lie Algebra". You hear about Noether's theorem, but only her work on variational calculus. If you take particle physics, you may hear about special unitary groups and that sort of thing.

It hardly suffices to take these terms for granted if one wants to dive into the weeds of modern-day mathematical physics, and there are few places more worthy of building foundational knowledge than in group theory. So, hopefully you try the problems out before looking at my solutions here. That's the only way to really learn, after all. But this will be here should you get truly stuck, and no mathematician is available to consult.

Hamermesh's text does not number the problems, but the sections are broken up nicely enough that they're not hard to find, provided I index those sections here. I will also write out the questions themselves so that searching for them will be even more simple.

CHAPTER 1: ELEMENTS OF GROUP THEORY

1-1 Correspondences and Transformations.

Problem 1-1.1: The cross ratio of four points on a line is defined as

$$\frac{(x_1-x_2)/(x_3-x_2)}{(x_1-x_4)/(x_3-x_4)} = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)},$$

where x_1, x_2, x_3, x_4 are the coordinates of the four points. Show that the ratio is *invariant* under projective transformation, i.e. that the cross ratio obtained from the image point has the same form as that for the object points.

Solution: All that needs to be done is verify that the image of x under a projective transformation results in the same equation as that stated in the question. The projection of these points gives

$$x_i' = \frac{ax_i+b}{cx_i+d}, \text{ where } ad-bc \neq 0.$$

for $i \in 1,2,3,4$. By direct calculation, we can see that

$$\frac{\left(\frac{ax_1+b}{cx_1+d}-\frac{ax_2+b}{cx_2+d}\right)\left(\frac{ax_3+b}{cx_3+d}-\frac{ax_4+b}{cx_4+d}\right)}{\left(\frac{ax_1+b}{cx_1+d}-\frac{ax_4+b}{cx_4+d}\right)\left(\frac{ax_3+b}{cx_3+d}-\frac{ax_2+b}{cx_2+d}\right)}$$

$$= \frac{((cx_2+d)(ax_1+b)-(cx_1+d)(ax_2+b))((cx_4+d)(ax_3+b)-(cx_3+d)(ax_2+b))}{((cx_4+d)(ax_1+b)-(cx_1+d)(ax_4+b))((cx_2+d)(ax_3+b)-(cx_3+d)(ax_2+b))}$$

after some tedious algebra. After a little more computing, you can simplify this statement to

$$\frac{((ad-bc)(x_1-x_2))((ad-bc)(x_3-x_4))}{((ad-bc)(x_1-x_4))((ad-bc)(x_3-x_2))}$$

$$= \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}$$

$$= \frac{(x_1-x_2)/(x_3-x_2)}{(x_1-x_4)/(x_3-x_4)}$$

which is the same as the cross ratio between object points. Thus, the cross ratio of a line is an invariant under projective transformations of that line.

1-2 Groups. Definition and Examples.

<u>Problem 1-2.1:</u> Show that the cyclic group of order 4 and the Klein 4-group are the only possible structures for groups of order 4.

<u>Solution:</u> This problem would be a very quick exercise if at this point the reader was familiar with Lagrange's theorem, but that hasn't been covered in the text at this point so I will assume the reader doesn't know this one. It's still solvable, but it takes more effort (and paper).

Groups need to fulfill certain conditions, which may be reflected in a Cayley table. In particular, each element in a column (and likewise, a row) may only appear once per column and per row. Further, each element of the group must appear in each column and each row. These greatly reduce the number of binary structures that could represent groups from 4^{16} down to 4. We can write those here:

A	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

,

B	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

C	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

,

D	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

.

Now, it turns out that the the tables A, B, C are identical up to an isomorphism. They are

$$\phi_1(A) = \begin{cases} \phi(e) & = e \\ \phi(a) & = b \\ \phi(b) & = a \\ \phi(c) & = c \end{cases} \qquad \phi_2(A) = \begin{cases} \phi(e) & = e \\ \phi(a) & = c \\ \phi(b) & = b \\ \phi(c) & = a \end{cases}$$

which take $A \to B$ and $A \to C$, respectively. We recognize these three as one group, which is the cyclic 4-group. Existence and uniqueness of this group for any positive integer n is clear from modular arithmetic over the naturals. D is the Klein 4-Group.

Problem 1-2.2: Show that the group of order 4 is Abelian.

Solution: After those two, this one is very quick. We simply observe that the Cayley tables for the order 4 groups are all symmetrical. Thus, the groups they represent are Abelian.

Problem 1-2.3: Give a realization of each group of order 4.

Solution: An example of the cyclic 4-group is $\langle \mathbb{Z}_4, +_4 \rangle$. An example of the Klein 4-group is $\langle \mathbb{Z}_2 \times \mathbb{Z}_2, +_2 \rangle$.

1-3 Subgroups. Cayley's Theorem.

Problem 1-3.1: Give the elements of the regular subgroup of S_6 which is isomorphic with the cyclic group of order 6.

Solution: To see the regular subgroup isomorphic to C_6, we need only write one of the cayley tables for that group and read off the results:

$$\begin{array}{c|cccccc} & e & a & b & c & d & f \\ \hline e & e & a & b & c & d & f \\ a & a & b & c & d & f & e \\ b & b & c & d & f & e & a \\ c & c & d & f & e & a & b \\ d & d & f & e & a & b & c \\ f & f & e & a & b & c & d \end{array} \to \begin{cases} \binom{123456}{123456} & = e \\ \binom{123456}{234561} & = (123456) \\ \binom{123456}{345612} & = (135)(246) \\ \binom{123456}{456123} & = (14)(25)(36) \\ \binom{123456}{561234} & = (153)(264) \\ \binom{123456}{612345} & = (165432) \end{cases}.$$

We can see that this group is indeed regular. These are all the elements, up to some isomorphisms.

Problem 1-3.2: Use Cayley's theorem to find the possible structures of groups of order 6.

Solution: Cayley's theorem states that every group G of order n is isomorphic with a subgroup of the symmetric group S_n. In our case, $n = 6$. From our work above, we have already done much of our task. We can further break down our result: we presume there are no allowed 6-cycles. Then, because the cycles need be of the same length, we can have two 3-cycles and three 2-cycles, plus e. That gives us

$$\{(123), (132), (12), (13), (23), e\},$$

which happens to be the symmetric 3-group, S_3.

Is that it? It is! How do we know there aren't more? Well, suppose we disallow 3-cycles. Then G has only 2-cycles and the identity. The group has to have 6 elements in total, so we need to have 5 2-cycles as well as the identity. But by the theorem, that puts us at trying to find a group isomorphic to a subset of S_{10}, not S_6. It's just too big.

So there are only two ways to structure a group of order 6: the Abelian group C_6 and the non-Abelian group S_3.