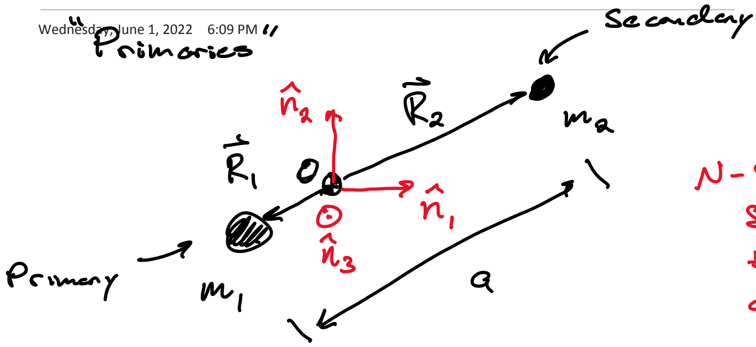
Equations of Motion of Circular Restricted 3-Body Problem

Wednesday, June 1, 2022 6:09 PM

"Primaries"



$m_1 > m_2$

N-frame is an inertial frame w/ origin at the center of mass of $m_1 - m_2$ system

m_1 and m_2 are assumed to be on a circular orbit about their common center of mass

Kepler's law states: $n^2 = \frac{G(m_1+m_2)}{a^3}$

n is orbital rate (rad/s), $T = \frac{2\pi}{n}$

$m_1 \vec{R}_1 + m_2 \vec{R}_2 = \vec{0}$ ← center of mass corollary

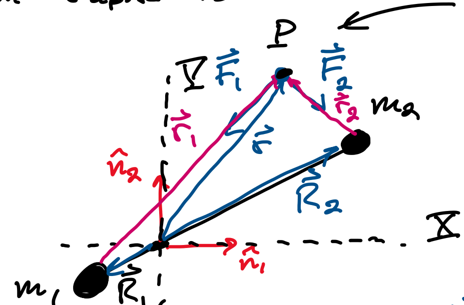
$\vec{R}_2 = -\frac{m_1}{m_2}\vec{R}_1$ | $R_1 + R_2 = a$

$R_1\left(\frac{m_1+m_2}{m_2}\right) = a \Rightarrow R_1 = \underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{\mu\, =\, \text{mass parameter}} a$

$\mu_{Sun-Earth} \approx 3\times10^{-6}$

$\mu_{Earth-Moon} \approx 10^{-2}$ $\qquad 0 < \mu \leq 0.5$

$\mu_{Sun-Jupiter} \approx 10^{-3}$



$\vec{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \vec{R}_1 = \begin{bmatrix} X_1 \\ Y_1 \\ 0 \end{bmatrix}$

$\vec{R}_2 = \begin{bmatrix} X_2 \\ Y_2 \\ 0 \end{bmatrix}$

$\vec{r}_1 = \vec{r} - \vec{R}_1 = \begin{bmatrix} X - X_1 \\ Y - Y_1 \\ Z \end{bmatrix}$

$r_1^2 = (X-X_1)^2 + (Y-Y_1)^2 + Z^2$ | $\vec{r}_2 = \vec{r} - \vec{R}_2 = \begin{bmatrix} X - X_2 \\ Y - Y_2 \\ Z \end{bmatrix}$

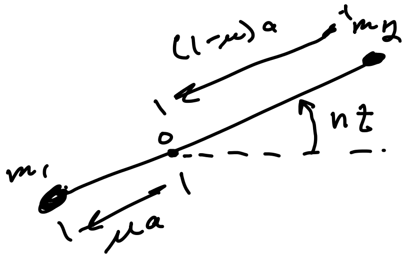
$r_2^2 = (X-X_2)^2 + (Y-Y_2)^2 + Z^2$ $\qquad \hat{r}_i = \frac{\vec{r}_i}{r_i}$

Newton's 2nd Law

$$\boxed{\cancel{m}\ddot{\vec{r}} = -\frac{G m_1 \cancel{m}}{r_1^2}\hat{r}_1 - \frac{G m_2 \cancel{m}}{r_2^2}\hat{r}_2}$$

$$\boxed{\ddot{X} = -\frac{G m_1}{r_1^3}(X - X_1) - \frac{G m_2}{r_2^3}(X - X_2)}$$

etc. for $\ddot{Y}$, $\ddot{Z}$



$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} -\mu a \cos(nt) \\ -\mu a \sin(nt) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} (1-\mu) a \cos(nt) \\ (1-\mu) a \sin(nt) \\ 0 \end{bmatrix}$$

Nondimensionalizing the governing equation:

How many variables? $[X]=L$ $[Y]=L$ $[Z]=L$ $[t]=T$ $[a]=L$

$K=9$ $[G]=\frac{L^3}{MT^2}$ $[m_1]=M$ $[m_2]=M$ $[n]=\frac{1}{T}$

How many dimensions required to describe them, L (length), M (mass), T (time)

$r=3$

According to the Buckingham Pi theorem, $k-r=6$ nondimensional numbers are needed to describe the systems

Choose unit of length to be a

$\xi = \frac{X}{a}$, $\eta = \frac{Y}{a}$, $\zeta = \frac{Z}{a}$ | $m_1 = (1-\mu)(m_1+m_2)$, $m_2 = \mu(m_1+m_2)$

$$\ddot{X} = G(m_1+m_2)\left[-\frac{(1-\mu)(\xi a + \mu a\cos(nt))}{a^3 \rho_1^3} - \frac{\mu(\xi a - (1-\mu) a \cos(nt))}{a^3 \rho_2^3}\right]$$

$$r_1 = a\rho_1, \quad \rho_1^2 = (\xi + \mu\cos(nt))^2 + (\eta + \mu\sin(nt))^2 + \zeta^2$$

similarly, $r_2 = a\rho_2$

$$\xi'' = \underbrace{\left(\frac{G(m_1+m_2)}{n^2 a^3}\right)}_{\gamma}\left[-\frac{(1-\mu)(\xi + \mu\cos(\tau))}{\rho_1^3} - \frac{\mu(\xi - (1-\mu)\cos(\tau))}{\rho_2^3}\right]$$

$\ddot{\xi} = \frac{d^2\xi}{dt^2}$, use $\tau = nt$ as the nondimensional unit of time

$$\frac{d\xi}{dt} = \frac{d\xi}{d\tau}\cancel{\frac{d\tau}{dt}}^{\,n} \qquad \frac{d\tau}{dt} = n$$

$$\ddot{\xi} = \frac{d^2\xi}{dt^2} = n^2\frac{d^2\xi}{d\tau^2} = n^2\xi''$$

$$\gamma = \frac{G(m_1+m_2)}{n^2 a^3} = 1$$

Kepler's law

$$n^2 = \frac{G(m_1+m_2)}{a^3}$$

Our non-dimensionalized ODEs

$$\boxed{\begin{aligned}
\xi'' &= -\frac{(1-\mu)(\xi+\mu\cos\tau)}{\rho_1^3} - \frac{\mu(\xi-(1-\mu)\cos\tau)}{\rho_2^3}\\
\eta'' &= -\frac{(1-\mu)(\eta+\mu\sin\tau)}{\rho_1^3} - \frac{\mu(\eta-(1-\mu)\sin\tau)}{\rho_2^3}\\
\zeta'' &= -\frac{(1-\mu)\zeta}{\rho_1^3} - \frac{\mu\zeta}{\rho_2^3}
\end{aligned}}$$

$\xi \quad \eta \quad \zeta \quad \tau \quad \cancel{\boxed{\gamma=1}} \quad \mu$ ← 6 nondim. variables

↑ analogous to Reynold's number