Energy, Jacobi Constant, Realms of Possible Motion, Equilibrium Points

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In non-dimensional units, eqs of the CR3BP are:

$$\ddot{x} = 2\dot{y} - \bar{U}_x \ , \quad \ddot{y} = -2\dot{x} - \bar{U}_y \ , \quad \ddot{z} = -\bar{U}_z$$

where $\bar{U}(x,y,z) = \underbrace{-\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}} \underbrace{-\frac{1}{2}(x^2+y^2)} \underbrace{-\frac{1}{2}\mu_1\mu_2}$

$(\dot{\ }) = \frac{d}{dt}$, t = nondim. time, $\mu_1 = 1-\mu$, $\mu_2 = \mu$

$$\boxed{h = \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - L}$$

← this is a constant throughout motion

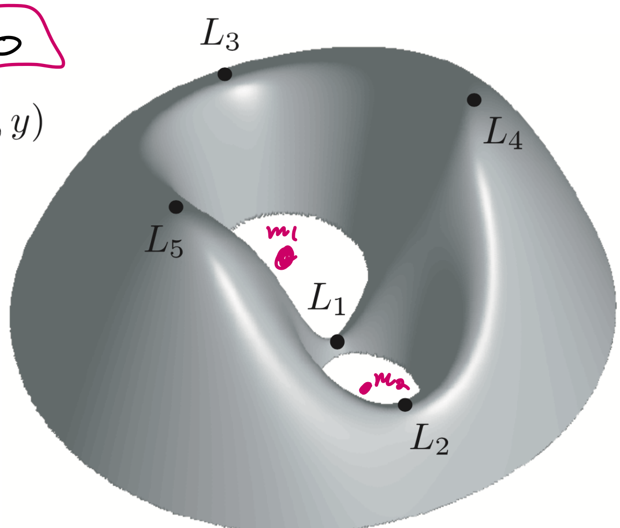
For the CR3BP $\boxed{h = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \bar{U}}$

$\frac{1}{2}v^2 \quad + \quad \bar{U}$

planar

$\boxed{z=0}$

$\bar{U}(x,y)$



← motion of P is motion in this potential plus Coriolis force (vel.-dependent)

Hamiltonian approach to deriving the eqs of motion.

Have a Lagrangian function, $L(\underbrace{q_1,\dots,q_n}, \underbrace{\dot{q}_1,\dots,\dot{q}_n})$

Legendre Transformation

Generalized momenta: $\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$ $(i = 1,\dots,n)$

Hamiltonian function: $H(q_1,\dots,q_n,p_1,\dots,p_n) = \sum_{i=1}^{n} p_i\dot{q}_i - L$

x, y, z write $\dot{q}_i$'s in terms

For the CR3BP $\qquad$ of $\dot{q}_i$'s and q_i's

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y, \quad p_y = \frac{\partial L}{\partial \dot{y}} = x + \dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}$$

$$H = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - L$$

$$\boxed{H = \tfrac{1}{2}\left[(p_x + y)^2 + (p_y - x)^2 + p_z^2\right] + \bar{U}(x, y, z)}$$

In general, Hamilton's eqs are $\quad \dot{q}_i = \dfrac{\partial H}{\partial p_i} \quad (i = 1, \dots, n)$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

For the CR3BP

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y, \quad \dot{y} = \frac{\partial H}{\partial p_y} = p_y - x, \quad \dot{z} = \frac{\partial H}{\partial p_z} = p_z$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - x - \bar{U}_x, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - y - \bar{U}_y$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\bar{U}_z$$

(they are equivalent to earlier eqs)

In general, if $\dfrac{\partial H}{\partial t} = 0 \Rightarrow H =$ constant of motion

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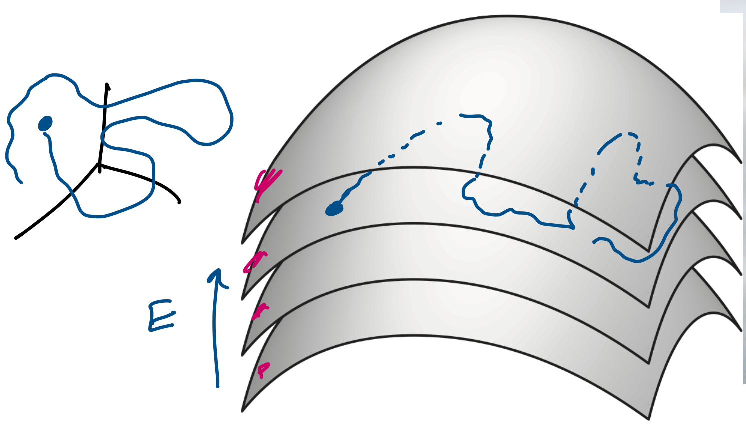
Energy, Jacobi integral

$$h, H, \quad \boxed{E(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \bar{U}(x, y, z)}$$

This is a constant of motion. $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0 \to E$

Geometrically, this means the motion of the particle is limited to a 5-dimensional subspace of the full 6-dimensional phase space.



In astronomical & astrodynamics communities, it's common to use the Jacobi constant scaled as

$$C = -2E = -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - 2\bar{U}$$

$$= -v^2 - 2\bar{U}$$; v is the magnitude of the velocity in the rotating frame

rotating $\boxed{E = \frac{1}{2}v^2 + \bar{U}}$

inertial $\boxed{E = \underbrace{\frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) - \left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right)}_{\text{kinetic \& potential energy}} - \underbrace{\vec{h} \cdot \hat{n}_3}_{\text{z component of angular momentum}}}$

Jacobi constant and realms of possible motion

$$E = \frac{1}{2}v^2 + \bar{U}$$

physically $\frac{1}{2}v^2 \geq 0$

"Realm of possible motion" are those where

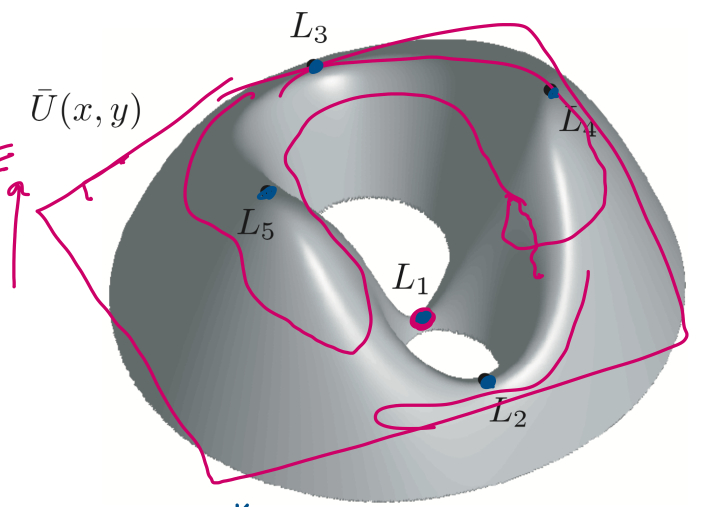
$$\bar{U}(x,y,z) \leq E$$

called "Hill's region"

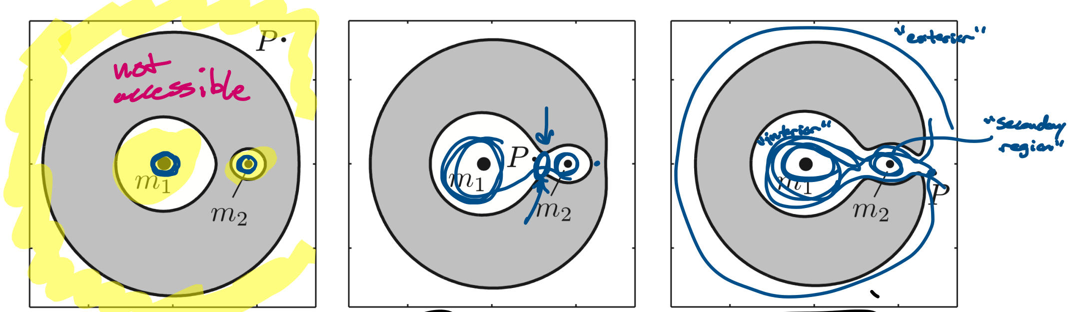
$\boxed{\bar{U}(x,y,z) = E}$ ← boundary of Hill's region

this is where v goes to zero

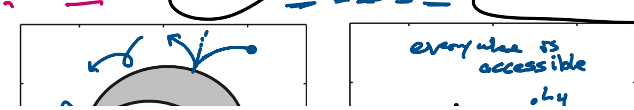
$\bar{U}(x,y) = E$ Hill's surfaces (in 3D) or Hill's curves (in 2D) "zero velocity curves"

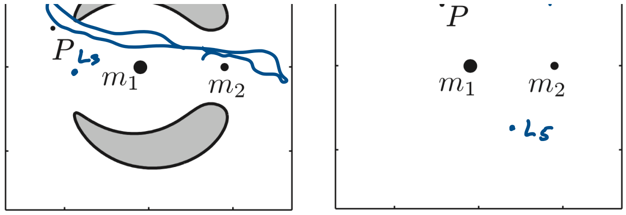


Look at 2D problem $(z = \dot{z} = 0)$



Case 1: $E < E_1$ Case 2: $E_1 \leq E \leq E_2$ Case 3: $E_2 < E < E_3$





Case 4: $E_3 < E < E_4$ Case 5: $E > E_4$

In 3D

2D surfaces of zero velocity in 3D space

Case 3

$\frac{1}{2}v^2 + \bar{U} = E$

$v = \sqrt{2(E - \bar{U})}$