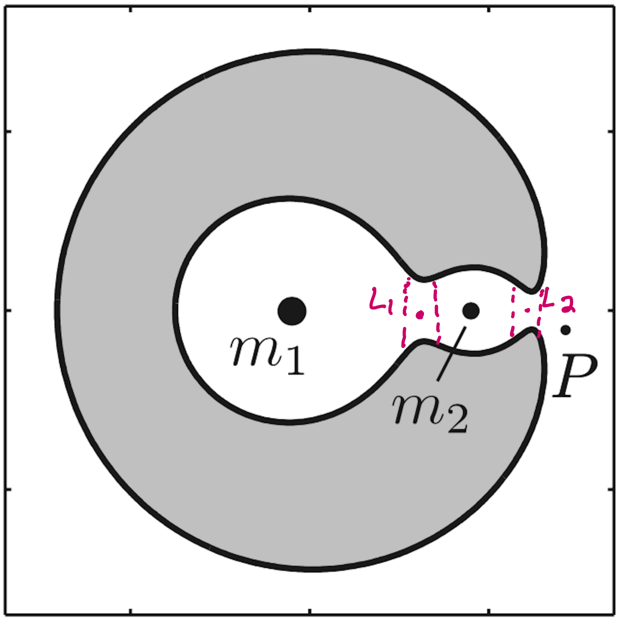
# Motion near L1 and L2: Linearized Equations of Motion

Wednesday, June 22, 2022 7:01 PM




In rotating frame for case 3

$E_2 < E < E_3$

Look at motion near the equil. points $L_1$ and $L_2$
"Lagrange points"
"libration points"

"syzygy"

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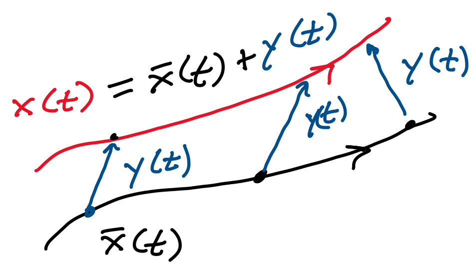
## Linearized Dynamics near a Reference Trajectory

$\underbrace{\dot{x} = f(x)}$, $x \in \mathbb{R}^n$

Given a reference trajectory $\bar{x}(t)$, that solves $\dot{\bar{x}} = f(x)$

Look at nearby trajectories




$\underbrace{x(t) = \bar{x}(t) + y(t)}$, where $y(t)$ is a "displacement" from the reference trajectory

$\dot{x} = \dot{\bar{x}} + \dot{y}$

$\underbrace{|y(t)| \text{ it is "small"}}$

$\dot{x} = f(x) = f(\bar{x}(t) + y(t))$ Taylor series expansion about $\bar{x}$

$\cancel{\dot{\bar{x}}} + \dot{y} = \cancel{f(\bar{x}(t))} + Df(\bar{x}(t))\,y(t) + \mathcal{O}(|y|^2)$

$\dot{\bar{x}}(t) = f(\bar{x}(t))$

$$\boxed{\dot{y} = Df(\bar{x}(t))\,y(t)} + \cancel{\mathcal{O}(|y|^2)} \qquad (1)$$

This equation describes the leading order behavior of the displacement from the reference traj.

$y = 0$ corresponds to the reference traj. and is an equil. point of (1)

$Df$ is the Jacobian matrix of $f$ ($n \times n$ matrix)

$A(t) = Df(\bar{x}(t))$ $\boxed{\dot{y} = A(t)\,y}$

Suppose $\bar{x}(t) = \bar{x}$, a constant, an equil. point of the nonlinear ODE $\dot{x} = f(x)$, $f(\bar{x}) = 0$

In this case, $A$ is a constant matrix and the evolution of displacements is easier to analyze

$$\boxed{\dot{y} = Ay} \Rightarrow y(t) = e^{At} y_0$$

We'll look at the eigenvalues of $A$

$p(\lambda) = \det(A - \lambda \mathbb{I}) = 0$ ← characteristic polynomial

If any eigenvalue has $\mathrm{Re}(\lambda) > 0$ then the equil. point $y = 0$ $(\bar{x})$ is <u>unstable</u>

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## Apply to $L_1$ and $L_2$ in CR3BP

"$\dot{x} = f(x)$" is

$$\underbrace{\begin{aligned} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{aligned}}_{\text{"}\dot{x}\text{"}} \begin{aligned} &= v_x \\ &= v_y \\ &= 2v_y - \bar{U}_x \\ &= \underbrace{-2v_x - \bar{U}_y}_{\text{"}f(x)\text{"}} \end{aligned}$$

where $\bar{U}_x = \frac{\partial \bar{U}}{\partial x}$, $\bar{U}_y = \frac{\partial \bar{U}}{\partial y}$

The Jacobian $Df$ is

$$A = Df(\bar{x}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\bar{U}_{xx} & -\bar{U}_{xy} & 0 & 2 \\ -\bar{U}_{xy} & -\bar{U}_{yy} & -2 & 0 \end{bmatrix}$$

The characteristic eqn will be

$$\det(A - \lambda \mathbb{I}) = 0 \qquad \mathbb{I} = 4\times4 \text{ identity}$$

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \cdots & & & \\ -\bar{U}_{xx} & -\bar{U}_{xy} & -\lambda & 2 \\ -\bar{U}_{xy} & -\bar{U}_{yy} & -2 & -\lambda \end{vmatrix}$$

This reduces to the following polynomial

$$p(\lambda) = \lambda^4 - (4 + \bar{U}_{xx} + \bar{U}_{yy})\lambda^2 + \bar{U}_{xx}\bar{U}_{yy} - \bar{U}_{xy}^2 = 0$$

Note this is "biquadratic", a quadratic in $\lambda^2$

For $L_1$ or $L_2$, $p(\lambda)$ is

$$p(\lambda) = \lambda^4 + (2 - \bar{\mu})\lambda^2 + (1 + \bar{\mu} - 2\bar{\mu}^2)$$

where $\bar{\mu} = \dfrac{\mu}{|x_L - 1 + \mu|^3} + \dfrac{1-\mu}{|x_L + \mu|^3}$ ; $x_L$ is the $x$ location of Lagrange point $L$ (either $L_1$ or $L_2$)

Let $\alpha = \lambda^2$, then we have a quadratic eqn

$$\alpha^2 + (2 - \bar{\mu})\alpha + (1 + \bar{\mu} - 2\bar{\mu}^2) = 0$$

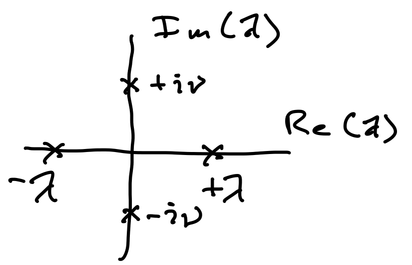
The two roots are

$$\alpha_1 = \tfrac{1}{2}\left(\bar{\mu} - 2 + \sqrt{9\bar{\mu}^2 - 8\bar{\mu}}\right) > 0$$

$$\alpha_2 = \tfrac{1}{2}\left(\bar{\mu} - 2 - \sqrt{9\bar{\mu}^2 - 8\bar{\mu}}\right) < 0$$

Therefore, the eigenvalues are of the form

$\pm\lambda$ and $\pm i\nu$, where $\lambda = \sqrt{\alpha_1}$, $\nu = \sqrt{-\alpha_2}$




so $L_1$ and $L_2$ are unstable

Can also analytically find the corresponding eigenvectors

$$+\lambda \to u_1 \in \mathbb{R}^4, \quad +i\nu \to w_1 \in \mathbb{C}^4$$

$$-\lambda \to u_2 \in \mathbb{R}^4, \quad -i\nu \to w_2 = \overline{w_1}$$

Transform from the usual rotating frame directions $(x, y, v_x, v_y)$ to an eigenbasis $(u_1, u_2, w_1, w_2)$
$(\xi, \eta, \zeta_1, \zeta_2)$

In the original basis w.r.t. equil. point ¦ In eigenbasis,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 2 \\ 0 & -b & -2 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}$$

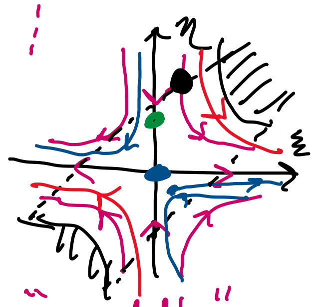
where $a = 2\bar{\mu}+1 > 0$

$b = \bar{\mu} - 1 > 0$

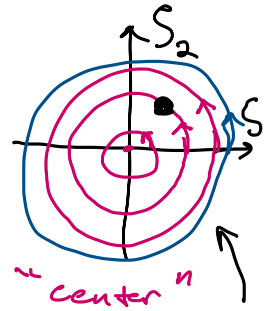
$L_1$ and $L_2$ are called saddle - center equil. pts.

$$\underbrace{\dot{\xi} = \lambda \xi}, \ \underbrace{\dot{\eta} = -\lambda \eta}$$

$$\boxed{\dot{S}_1 = \nu S_2, \ \dot{S}_2 = -\nu S_1}$$

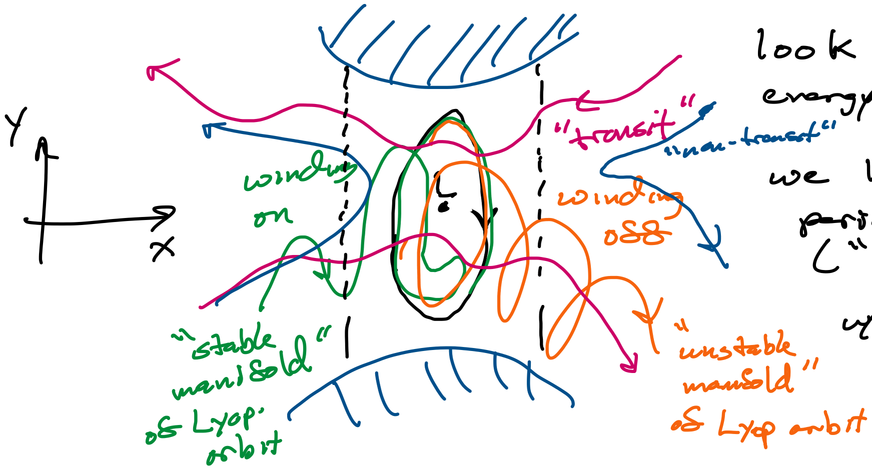



"saddle" projection




"center" projection

center projection shows us that there are periodic orbits (oscillatory motion) around $L_1$ ($L_2$)




look at a single energy (Jacobi constant)

we have a single periodic periodic orbit ("Lyapunov orbit")

w/ period $T = \frac{2\pi}{\nu}$