Ross book, section 2.5 **Murray & Dermott section 3.6**

Lagrange Points: Equilibrium Points in Restricted 3-Body Problem

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In the rotating frame, we seek equilibrium points (points where there is no motion)

In general, $\dot{x} = f(x)$, $x \in \mathbb{R}^n$

equil. points are those points $\bar{x}$ where $\dot{\bar{x}} = \vec{0}$

means we look for $\bar{x}$ s.t. $f(\bar{x}) = \vec{0}$

For the CR3BP, the eqs in first order form:

$$\dot{x} = v_x \qquad \dot{v}_x = 2\cancel{v_y}^{0} - \bar{U}_x$$
$$\dot{y} = v_y \qquad \dot{v}_y = -2\cancel{v_x}^{0} - \bar{U}_y$$
$$\cancel{\dot{z} = v_z} \qquad \cancel{\dot{v}_z = -\bar{U}_z}$$

notation $\bar{U}_{q_i} \equiv \frac{\partial \bar{U}}{\partial q_i}$

Equil. will have $\dot{x} = \dot{y} = \dot{z} = \dot{v}_x = \dot{v}_y = \dot{v}_z = 0$

$\Rightarrow v_x = v_y = v_z = 0$ $\quad$ $\bar{U}_x = \bar{U}_y = \bar{U}_z = 0$

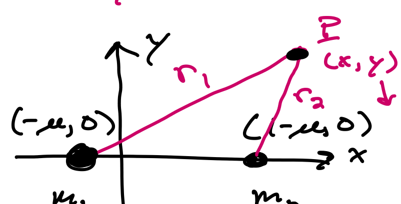
In other words, a "critical point" of $\bar{U}(x,y,z)$

when $z \neq 0$, there are no critical points

Consider $z=0$, $\bar{U}_x = \bar{U}_y = 0$ $\quad$ in the plane of m_1 & m_2

Restrict for now to $y \neq 0$

We'll use r_1, r_2



$$\bar{U}(x,y) = -\frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}(x^2+y^2) - \frac{1}{2}\mu(1-\mu)$$

need in terms of r_1 and r_2

$\bar{U}(r_1, r_2)$

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x-(1-\mu))^2 + y^2$$

Claim: $x^2+y^2 = (1-\mu)r_1^2 + \mu r_2^2 - \mu(1-\mu)$

$$(1-\mu)r_1^2 + \mu r_2^2$$
$$= (1-\mu)\{(x+\mu)^2 + y^2\} + \mu\{(x-(1-\mu))^2 + y^2\}$$

$$(x+\mu)^2+y^2 - \mu(\cancel{x^2}+2\mu x+\mu^2 \cancel{+y^2})$$

$$+\mu(\cancel{x^2}-2(1-\mu)x+(1-\mu)^2 \cancel{+y^2})$$

$$= x^2 \cancel{+2\mu x} + \mu^2 + y^2 \cancel{-2\mu^2 x} - \mu^3 \cancel{-2\mu x} \cancel{+2\mu^2 x}$$

$$+\mu(1-\mu)^2$$

$$= x^2+y^2+\mu^2+\mu(1-2\mu+\mu^2)-\mu^3$$

$$= x^2+y^2+\mu^2+\mu-2\mu^2\cancel{+\mu^3}\cancel{-\mu^3}$$

$$= x^2+y^2+\mu-\mu^2 = x^2+y^2+\mu(1-\mu)$$

$E=\frac{1}{2}v^2+\bar{U}$

$$\boxed{\bar{U}(r_1,r_2) = -\frac{(1-\mu)}{r_1}-\frac{\mu}{r_2}-\frac{1}{2}(1-\mu)r_1^2-\frac{1}{2}\mu r_2^2} = -1-\frac{1}{2} = -\frac{3}{2}$$

$$\bar{U}_x = \bar{U}_{r_1}\frac{\partial r_1}{\partial x} + \bar{U}_{r_2}\frac{\partial r_2}{\partial x} = \bar{U}_{r_1}\left(\frac{x+\mu}{r_1}\right)+\bar{U}_{r_2}\left(\frac{x-(1-\mu)}{r_2}\right)=0$$

$$\bar{U}_y = \bar{U}_{r_1}\frac{\partial r_1}{\partial y} + \bar{U}_{r_2}\frac{\partial r_2}{\partial y} = \bar{U}_{r_1}\left(\frac{y}{r_1}\right)+\bar{U}_{r_2}\left(\frac{y}{r_2}\right)=0$$

$\bar{U}_x = \bar{U}_y = 0$ is equiv. to $\bar{U}_{r_1} = \bar{U}_{r_2} = 0$

So, we want to solve the following sys. of eqs

$$0=\bar{U}_{r_1} = -\cancel{(1-\mu)}r_1 + \frac{\cancel{(1-\mu)}}{r_1^2}$$

$$0=\bar{U}_{r_2} = -\mu r_2 + \frac{\mu}{r_2^2}$$

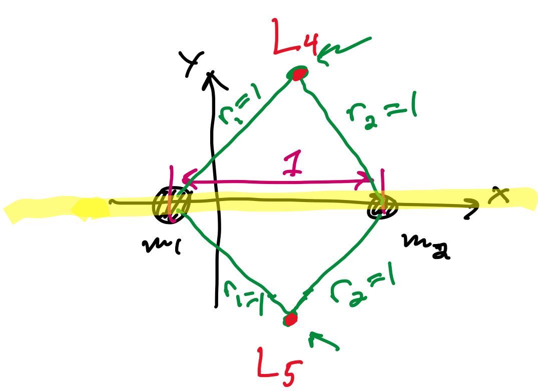
$0=-r_1+\frac{1}{r_1^2}$

$\Rightarrow r_1=1$

similarly

$r_2=1$

So there is a critical point at $r_1=r_2=1$



There are 2 locations satisfying $r_1=1$ and $r_2=1$ in the xy plane

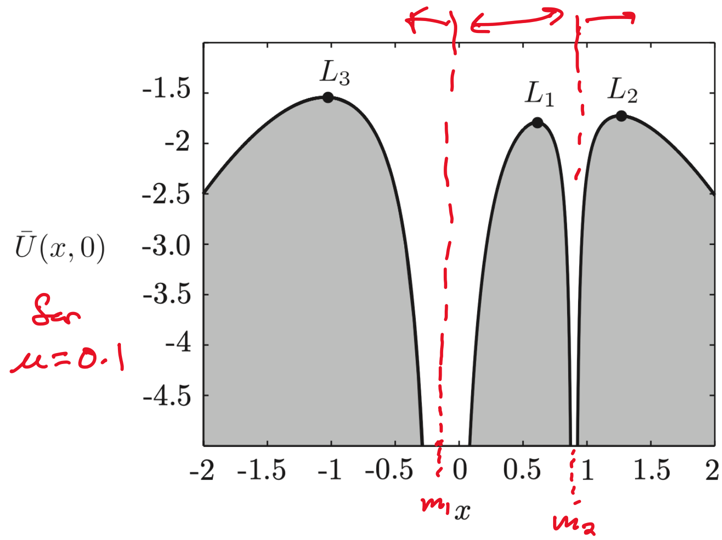
These are the "equilateral points" They form an equilateral triangle (a.k.a. "triangular points")

These are the L_4 and L_5 Lagrange points

<u>Collinear points</u>. Now consider $\boxed{y=0}$ (the x-axis)

Look at critical points of $\bar{U}$ along the x-axis

$$\bar{U}(x,0) = -\frac{1}{2}x^2 - \frac{1-\mu}{|x+\mu|} - \frac{\mu}{|x-(1-\mu)|} - \frac{1}{2}\mu(1-\mu)$$

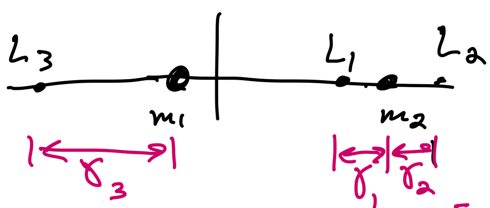


To find the equilibria,

$$\frac{\partial \bar{U}(x,0)}{\partial x} = 0$$

$\Rightarrow$ this gives a quintic (5^{th} order) polynomial for the location

consider the equil. between m_1 and m_2 (L_1)



The distance of L_1 from m_2 is γ_1 which is the real root of the following polynomial

$$\gamma_1^5 - (3-\mu)\gamma_1^4 + (3-2\mu)\gamma_1^3 - \mu\gamma_1^2 + 2\mu\gamma_1 - \mu = 0$$

for $\boxed{\mu = 0.1}$, $\boxed{\gamma_1 = 0.29096}$

An approximation is $\gamma_1 = r_h\left(1 - \frac{1}{3}r_h - \frac{1}{9}r_h^2 + \cdots\right)$

$r_h = \left(\frac{\mu}{3}\right)^{1/3}$ $\quad \gamma_2 = r_h\left(1 + \frac{1}{3}r_h - \frac{1}{9}r_h^2 + \cdots\right)$

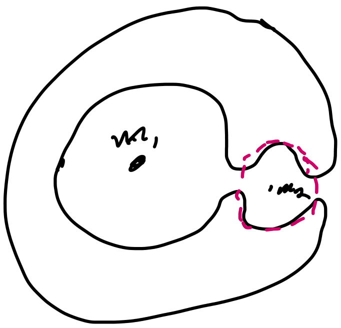
"Hill radius" ; for $\mu=0.1$, $\boxed{r_h = 0.322}$

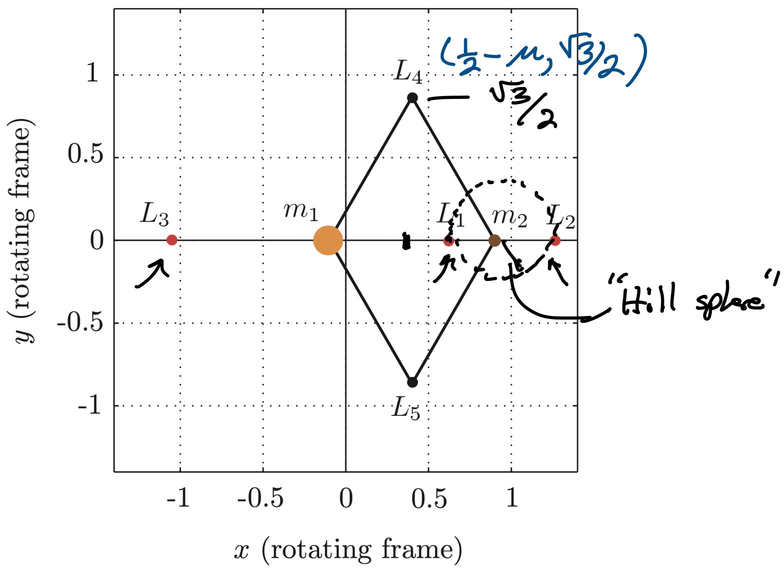
$\boxed{\gamma_2 = 0.3597}$

for $\mu = 0.1$

$x_{L_1} = (1-\mu) - \gamma_1$

$x_{L_2} = (1-\mu) + \gamma_2$

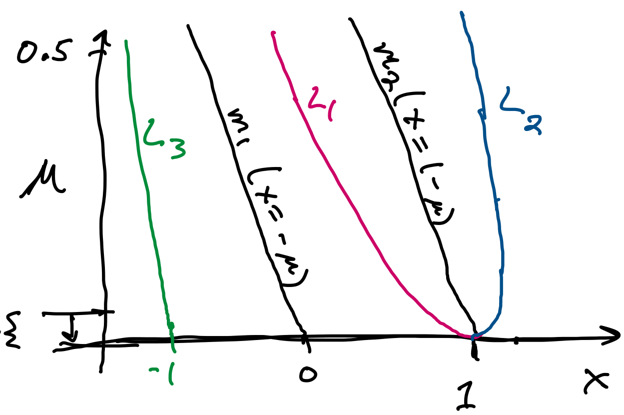


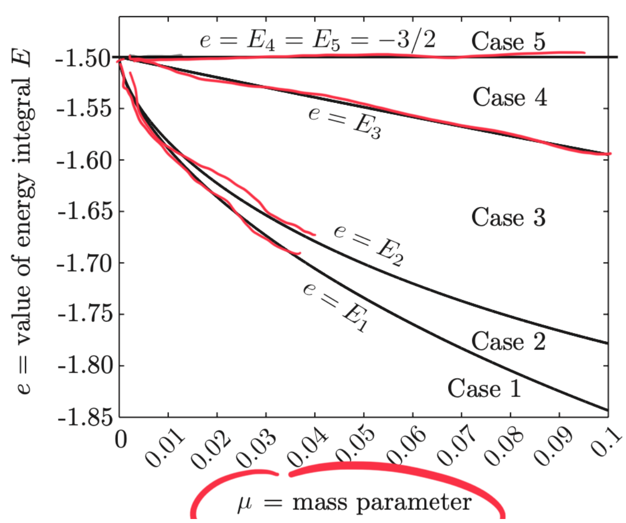


$r_{SOI} \propto \mu^{2/5}$, $r_h \propto \mu^{1/3}$, related "sphere of influence"

Showing equilibria as a "bifurcation" diagram

all of solar system {





Energy values

$E = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \bar{U}(x,y,z)$

of Lagrange points