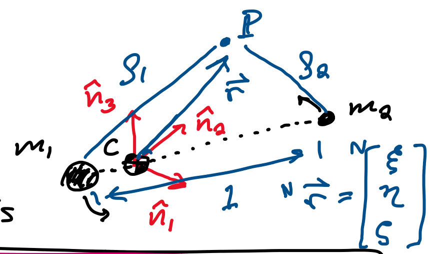
# Equations of Motion in Rotating Frame

Monday, June 6, 2022 5:52 PM




From last time ...

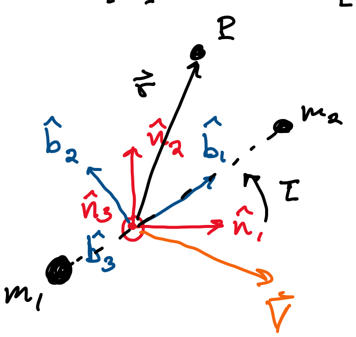
Our non-dimensionalized ODEs

$$\boxed{\begin{aligned} \xi'' &= -\frac{(1-\mu)(\xi+\mu\cos\tau)}{\rho_1^3} - \frac{\mu(\xi-(1-\mu)\cos\tau)}{\rho_2^3} \\ \eta'' &= -\frac{(1-\mu)(\eta+\mu\sin\tau)}{\rho_1^3} - \frac{\mu(\eta-(1-\mu)\sin\tau)}{\rho_2^3} \\ \zeta'' &= -\frac{(1-\mu)\zeta}{\rho_1^3} - \frac{\mu\zeta}{\rho_2^3} \end{aligned}}$$

unit of length, $L$, is distance between primaries
unit of time, $T/2\pi$, where $T$ is period of primaries

$$^N\vec{r} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \quad ^N\vec{r}'' = \begin{bmatrix} \xi'' \\ \eta'' \\ \zeta'' \end{bmatrix}$$

$$\boxed{^N\vec{r} = \underbrace{[NB]}_{[BN]^T}{}^B\vec{r}}$$




$$[BN] = \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

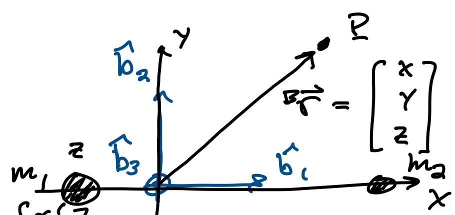
$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = [BN] \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$

$$^B\vec{r} = [BN]{}^N\vec{r} \longrightarrow {}^N\vec{r} = \underbrace{[NB]}{}^B\vec{r}$$

$$A_\tau = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^N\vec{r}' = (A_\tau {}^B\vec{r})'$$

$$= A_\tau' {}^B\vec{r} + A_\tau {}^B\vec{r}'$$




$$= -A_\tau \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + A_\tau \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = A_\tau \begin{bmatrix} x' - y \\ y' + x \\ z' \end{bmatrix}$$

LHS

$$\begin{bmatrix} \xi'' \\ \eta'' \\ \zeta'' \end{bmatrix} = A_\tau \begin{bmatrix} x'' - 2y' - x \\ y'' + 2x' - y \\ z'' \end{bmatrix}$$

Now work on RHS

$$\xi'' = -\frac{(1-\mu)(\xi+\mu\cos\tau)}{\rho_1^3} - \frac{\mu(\xi-(1-\mu)\cos\tau)}{\rho_2^3}$$

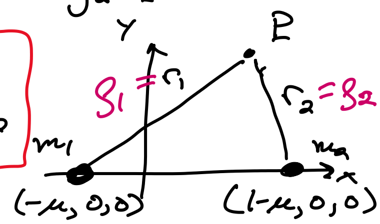
$$= \frac{-(1-\mu)(x\cos\tau - y\sin\tau + \mu\cos\tau)}{r_1^3}$$

$$- \frac{\mu(x\cos\tau - y\sin\tau + \mu\cos\tau - \cos\tau)}{r_2^3}$$

$$= \cos\tau\left[\frac{-(1-\mu)(x+\mu)}{r_1^3}\right] + (-\sin\tau)\left[\frac{-(1-\mu)y}{\rho_1^3}\right]$$

$$+ \cos\tau\left[-\frac{\mu(x-(1-\mu))}{r_2^3}\right] + (-\sin\tau)\left[\frac{-\mu y}{\rho_2^3}\right]$$

$$\boxed{\begin{aligned} r_1^2 &= (x+\mu)^2 + y^2 + z^2 \\ r_2^2 &= (x-(1-\mu))^2 + y^2 + z^2 \end{aligned}}$$




$$\eta'' = \cdots$$

$$\zeta'' = \cdots$$

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-(1-\mu))}{r_2^3} \\ \ddot{y} + 2\dot{x} - y &= -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\ \ddot{z} &= -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3} \end{aligned}}$$

CR3BP eqs of motion in rotating frame

* nondimensionalized
* ODEs do not explicitly depend on time (so many techniques can be used)

Put in first order form,

Introduce $v_x \equiv \dot{x}$, $v_y \equiv \dot{y}$, $v_z \equiv \dot{z}$

$$\begin{aligned} \dot{x} &= v_x & \dot{v}_x &= 2v_y + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-(1-\mu))}{r_2^3} \\ \dot{y} &= v_y & \dot{v}_y &= -2v_x + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\ \dot{z} &= v_z & \dot{v}_z &= -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3} \end{aligned}$$

Give an initial condition $(x(0), y(0), z(0), v_x(0), v_y(0), v_z(0))$ and numerically integrate

The potential energy due to gravity

$$U(x,y,z) = -\frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\mu(1-\mu)$$

The kinetic energy of the particle

$$\underline{K}(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right)$$
$$= \frac{1}{2}\left((\dot{x}-y)^2 + (\dot{y}+x)^2 + \dot{z}^2\right)$$

Lagrangian approach: $\left.\begin{array}{l} q_1,\dots,q_n \\ \dot{q}_1,\dots,\dot{q}_n \end{array}\right\} L = K - \mathcal{U}$

eqs of motion are Lagrange's eqs: $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

$i = 1,\dots,n$

For the CR3BP, $x, y, z, \dot{x}, \dot{y}, \dot{z}$

$L = K - \mathcal{U}$ from up above

x eqn: $\frac{d}{d\tau}(\dot{x} - y) = \dot{y} + x - \frac{\partial \mathcal{U}}{\partial x}$

$$\ddot{x} - 2\dot{y} - x = -\frac{\partial \mathcal{U}}{\partial x}$$

y eqn. $\frac{d}{d\tau}(\dot{y} + x) = -(\dot{x} - y) - \frac{\partial \mathcal{U}}{\partial y}$

$$\ddot{y} + 2\dot{x} - y = -\frac{\partial \mathcal{U}}{\partial y}$$

z eqn: $\ddot{z} = -\frac{\partial \mathcal{U}}{\partial z}$

The RHSs here from $-\frac{\partial \mathcal{U}}{\partial q_i}$ are the same as one gets from the Newtonian approach

For Lagrangian systems where $\frac{\partial L}{\partial \tau} = 0$, there's a constant of motion called "Jacobi integral" or "Jacobi constant"

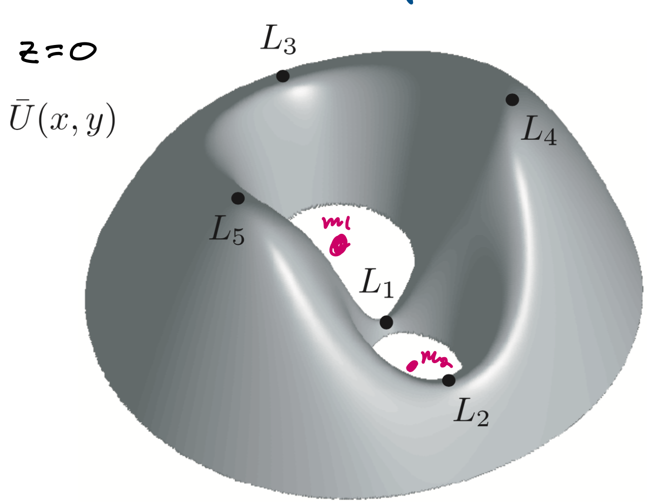
$$h = \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - L$$ ← this is a constant throughout motion

For the CR3BP $h = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \bar{\mathcal{U}}$

where the "effective potential" is $\bar{\mathcal{U}} = \underbrace{-\frac{1}{2}(x^2+y^2)}_{\text{centrifugal force}} + \mathcal{U}$

↑ gravity

z=0




← motion of P is motion in this potential plus Coriolis force (vel.-dependent)