Probability Distributions

Rolf D. Svegstrup

September 29, 2018

1 Bernoulli Distribution

A *Bernoulli random variable*, X, takes the value 1 with probability p and 0 otherwise. Thus,

$$\begin{aligned} E(X) &= p \\ V(X) &= p(1-p). \end{aligned} \tag{1}$$

2 Binomial Distribution

A *Binomial random variable*, X, is a sum of n iid. Bernoulli random variables. We write $X \sim B(n,p)$. Trivially,

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}. \tag{2}$$

From (1) it follows that

$$\begin{aligned} E(X) &= np \\ V(X) &= np(1-p). \end{aligned} \tag{3}$$

3 Poisson Distribution

A *Poisson random variable*, X, is given by ... LOTS OF EXPLANATION HERE.

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}_0 \tag{4}$$

$$\begin{aligned} E(X) &= \lambda \\ V(X) &= \lambda \end{aligned} \tag{5}$$

To prove (5) note that

$$E(X) = e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda.$$

Similarly,

$$E(X^2) = e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} ((k-1)+1) \frac{\lambda^{k-1}}{(k-1)!} = \lambda^2 + \lambda.$$

4 Exponential Distribution

The exponential distribution describes waiting times without memory, e.g., radioactive decay or corporate default (with constant default rate).

A random variable is exponentially distributed, $X \sim \text{Exp}(\lambda)$, if its density function is

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Equivalently, $P(X \geq h) = e^{-\lambda h}$. We note that

$$\begin{aligned} E(X) &= \frac{1}{\lambda} \\ V(X) &= \frac{1}{\lambda^2} \end{aligned} \tag{6}$$

4.1 Memoryless

For $t, \Delta t \geq 0$,

$$P(X \geq t + \Delta t \mid X \geq t) = \frac{e^{-\lambda(t+\Delta t)}}{e^{-\lambda t}} = P(X \geq \Delta t).$$

Equivalently, the exponential hazard function is constant:

$$h(t)dt = P(t < X < t + dt \mid X > t) = \frac{e^{-\lambda t} - e^{-\lambda(t+dt)}}{e^{-\lambda t}} = \lambda dt.$$