Singular Value Decomposition

March 5, 2019

Let $A \in \mathbb{R}^{m\times n}$.

Remark 1. *If x is an eigenvector of A^tA with non-zero eigenvalue λ, then Ax is an eigenvector of AA^t with eigenvalue λ. That $(AA^t)(Ax) = \lambda Ax$ is obvious. That $Ax \neq 0$ follows from $\|Ax\|^2 = x^tA^tAx = \lambda x^tx$.*

If x is an eigenvector of A^tA with eigenvalue 0, then $Ax = 0$.

Remark 2. *Similarly if y is an eigenvalue of AA^t with non-zero eigenvalue λ then A^ty is an eigenvalue of A^tA with eigenvalue λ. If λ is zero then $A^ty = 0$.*

Remark 3. *It follows from the above that A^tA and AA^t have the same number of strictly positive eigenvalues.*

Lemma 4. *Assume A^tA has precisely k strictly positive eigenvalues $\lambda_1, \ldots, \lambda_k$. Let $x_1, \ldots, x_k$, be corresponding mutually orthogonal unit eigenvectors for A^tA. Define unit vectors $y_i = Ax_i/\|Ax_i\| = Ax_i/\sqrt{\lambda_i}$.*

Define $\bar{V} \in \mathbb{R}^{n\times k}$ and $\bar{U} \in \mathbb{R}^{m\times k}$ be the matrices whose columns are (x_i) and (y_i), respectively. Let $\bar{\Sigma} = \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_k}) \in \mathbb{R}^{k\times k}$. Then

$$\bar{U}^tA\bar{V} = \bar{\Sigma}. \tag{1}$$

Proof. Let $e_i \in \mathbb{R}^k$ be the ith standard basis vector. Then

$$\bar{U}^tA\bar{V}e_i = \bar{U}^tAx_i = \bar{U}^t\sqrt{\lambda_i}y_i = \sqrt{\lambda_i}e_i. \tag{2}$$

□

Lemma 5. *Complete the columns of $\bar{U}$ and $\bar{V}$ to $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ by adding orthonormal vectors x_i and y_i, $i > k$, with $A^tAx_i = 0$ and $AA^ty_i = 0$, respectively. Extend $\bar{\Sigma}$ to $\Sigma \in \mathbb{R}^{n\times m}$ by adding zeroes. I.e.,*

$$U = \begin{pmatrix}\bar{U} & U_0\end{pmatrix}, V = \begin{pmatrix}\bar{V} & V_0\end{pmatrix}, \Sigma = \begin{pmatrix}\bar{\Sigma} & 0 \\ 0 & 0\end{pmatrix}. \tag{3}$$

Then

$$A = U\Sigma V^t. \tag{4}$$

Proof. Note that U and V are orthogonal matrices. Also,

$$U^t AV = \begin{pmatrix} \bar{U}^t \\ U_0 \end{pmatrix} A \begin{pmatrix} \bar{V} & V_0 \end{pmatrix} = \begin{pmatrix} \bar{U}^t A\bar{V} & \bar{U}^t AV_0 \\ U_0^t A\bar{V} & U_0^t AV_0 \end{pmatrix}. \tag{5}$$

By construction $AV_0 = 0$ and $U_0^t A = 0$. The lemma follows. ☐

Summing up:

Theorem 6. *For any $A \in \mathbb{R}^{m\times n}$, there exists orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ such that $A = U\Sigma V^t$.*

The diagonal values of Σ are the strictly positive eigenvalues of $A^t A$ (or equivalently AA^t) and zeroes.

Remark 7. *To remember the construction of U note that*

$$A^t A = V\Sigma U^t U\Sigma V = V^t \Sigma^t \Sigma V^t. \tag{6}$$

Thus if V is the matrix whose columns of eigenvectors of $A^t A$ and $\Sigma^t\Sigma$ is the diagonal matrix of eigenvalues, then the equation is satisfied.

A similar mnemonic holds made for V.