MODERN ROBOTICS

Ch. 8 - Dynamics of open chains

Equation of motion: $\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$

$\theta \triangleq$ Joint variables
$\tau \triangleq$ Joint forces and torques
$M \triangleq$ Mass matrix
$h \triangleq$ forces combining centripedal, coriolis, gravity and friction terms

⚠ equation gets complicated quickly

Forward dynamics

$$\ddot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta}))$$

8.1 Lagrangian formulation

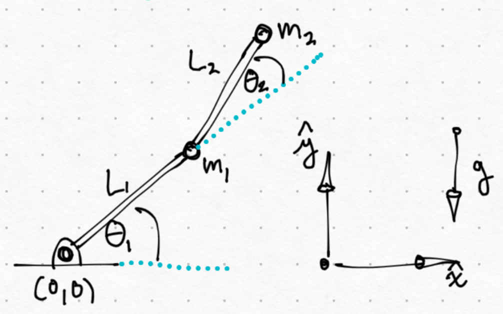
Choose independent generalized coordinates q that describe the systems configuration. These define generalized forces f. Then:

$$\mathcal{L}(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

is the Lagrange equation and the generalized forces are

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$$

Lagrange formulation for 2R open chain

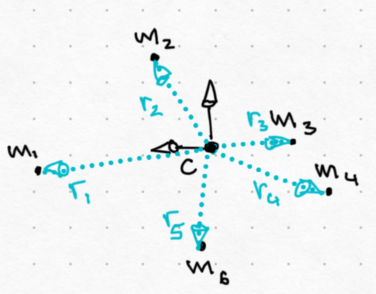


$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = K(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) - P(\theta_1, \theta_2)$$
$$= \tfrac{1}{2} m_1 (L_1\dot{\theta}_1)^2 + \tfrac{1}{2} m_2((L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\dot{\theta}_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2)$$
$$- m_1 g L_1 \sin\theta_1 - m_2 g(L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2))$$

Centripedal terms: $\dot{\theta}_i^2$, Coriolis terms: $\dot{\theta}_i\dot{\theta}_j$, $i \neq j$

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) + \underbrace{J^T(\theta) F_{tip}}_{\text{term to create wrench at the tip}}$$

8.2 Dynamics of a single rigid body



Rigid body consisting of multiple point masses with positions relative to the center of mass C. This body is moving with a twist $\mathcal{V}_b = (\omega_b, v_b)$. Renaming the position of the masses $p_i(t)$ with initial positions r_i we obtain:

$$\dot{p}_i = v_b + \omega_b \times p_i$$
$$\ddot{p}_i = \dot{v}_b + \dot{\omega}_b \times p_i + \omega_b \times (v_b + \omega_b \times p_i)$$

Substituting r_i for p_i and using different notation:

$$\ddot{p}_i = \dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i$$
$$\Rightarrow f_i = m_i \ddot{p}_i = m_i(\dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i)$$

We can define a moment $m_i = [r_i] f_i$. Then

$$\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \sum_i m_i \\ \sum_i f_i \end{bmatrix}$$

Simplified expressions:

$$f_b = m(\dot{v}_b + [\omega_b] v_b)$$
$$m_b = \underbrace{\mathcal{I}_b \dot{\omega}_b + [\omega_b]\mathcal{I}_b \omega_b}_{\text{Eulers Equation}} \quad \text{with} \quad \mathcal{I}_b = -\sum_i m_i [r_i]^2 \text{ the rotational inertia matrix}$$

Rotational kinetic Energy: $K = \frac{1}{2}\omega_b^T \mathcal{I}_b \omega_b$

The principal axis of inertia are given by the Eigenvectors of $\mathcal{I}_b$, the principal moments of inertia are given by the Eigenvalues. If {b} basis are the Eigenvectors the moments become easier to compute:

$$m_b = \begin{bmatrix} \mathcal{I}_{xx}\dot{\omega}_x + (\mathcal{I}_{zz} - \mathcal{I}_{yy})\omega_y\omega_z \\ \mathcal{I}_{yy}\dot{\omega}_y + (\mathcal{I}_{xx} - \mathcal{I}_{zz})\omega_z\omega_x \\ \mathcal{I}_{zz}\dot{\omega}_z + (\mathcal{I}_{yy} - \mathcal{I}_{xx})\omega_x\omega_y \end{bmatrix} \quad \text{⚠ if \{b\}'s basis are EV of } \mathcal{I}_b$$

For easier computation this basis should be chosen.

Steiner's Theorem: The inertia matrix $\mathcal{I}_q$ about a frame aligned with {b} but at a point q in {b} is related to the inertia matrix $\mathcal{I}_b$ calculated at the centers of mass by:

$$\mathcal{I}_q = \mathcal{I}_b + m(q^T q \cdot Id - q q^T)$$

Spatial inertia matrix: $\mathcal{G}_b \in \mathbb{R}^{6\times 6}$, $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & m \cdot Id \end{bmatrix}$, This allows for an expression for the kinetic Energy:

$$E_{kin} = \tfrac{1}{2}\omega_b^T \mathcal{I}_b \omega_b + \tfrac{1}{2} m v_b^T v_b = \tfrac{1}{2}\mathcal{V}_b^T \mathcal{G}_b \mathcal{V}_b$$

Spatial momentum: $\mathcal{P}_b \in \mathbb{R}^6$, $\mathcal{P}_b = \mathcal{G}_b \mathcal{V}_b$

Lie Bracket: $[ad_{\mathcal{V}_1}]\mathcal{V}_2 = ad_{\mathcal{V}_1}(\mathcal{V}_2) = \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix}\begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix}$

Dynamics Equations for a single rigid body:

$$\mathcal{F}_b = \mathcal{G}_b \dot{\mathcal{V}}_b - ad^T_{\mathcal{V}_b}(\mathcal{P}_b) = \mathcal{G}_b \dot{\mathcal{V}}_b - [ad_{\mathcal{V}_b}]^T \mathcal{G}_b \mathcal{V}_b$$
