The Greeks: Derivatives of Option Prices

An Undergraduate Introduction to Financial Mathematics

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- In finance these partial derivatives are referred to as “the Greeks”.
- Unless otherwise specified, all results discussed are valid only for non-dividend paying securities.

Black-Scholes Option Pricing Formulas

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
C(S, t) &= S\Phi\left(w\right) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \\
P(S, t) &= Ke^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) - S\Phi\left(-w\right)
\end{aligned}
$$

Cumulative Distribution Function

The function $\Phi(w)$ is the cumulative distribution function

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which by the Fundamental Theorem of Calculus has derivative

$$\Phi'(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}.$$

Partial Derivatives of w

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\frac{\partial w}{\partial r} &= \frac{\sqrt{T-t}}{\sigma}
\end{aligned}$$

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\frac{\partial w}{\partial r} &= \frac{\sqrt{T - t}}{\sigma} \\
\frac{\partial w}{\partial \sigma} &= \sqrt{T - t} - \frac{w}{\sigma}
\end{aligned}
$$

Important Identity

Claim:

$$S\phi(w) - Ke^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right) = 0$$

Theta Θ (1 of 3)

Theta Θ is the partial derivative with respect to time t.

Time is the only independent variable we are certain will change before expiry. It is also the only deterministic independent variable.

Theta Θ (2 of 3)

$$\begin{aligned}
C &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
\frac{\partial C}{\partial t} &= S\Phi'(w)\frac{\partial w}{\partial t} - rKe^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - Ke^{-r(T-t)}\Phi'\left(w - \sigma\sqrt{T-t}\right)\left[\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right]
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&= \frac{Se^{-w^2/2} - Ke^{-r(T-t)-(w-\sigma\sqrt{T-t})^2/2}}{2\sigma\sqrt{2\pi(T-t)}}\left(\frac{\ln(S/K)}{T-t} - r - \sigma^2/2\right) \\
&\quad - Ke^{-r(T-t)}\left(r\Phi\left(w - \sigma\sqrt{T-t}\right) + \frac{\sigma e^{-(w-\sigma\sqrt{T-t})^2/2}}{2\sqrt{2\pi(T-t)}}\right)
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&\quad - Ke^{-r(T-t)}\Phi'\left(w - \sigma\sqrt{T-t}\right)\left[\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right] \\
&= \frac{Se^{-w^2/2} - Ke^{-r(T-t)-(w-\sigma\sqrt{T-t})^2/2}}{2\sigma\sqrt{2\pi(T-t)}}\left(\frac{\ln(S/K)}{T-t} - r - \sigma^2/2\right) \\
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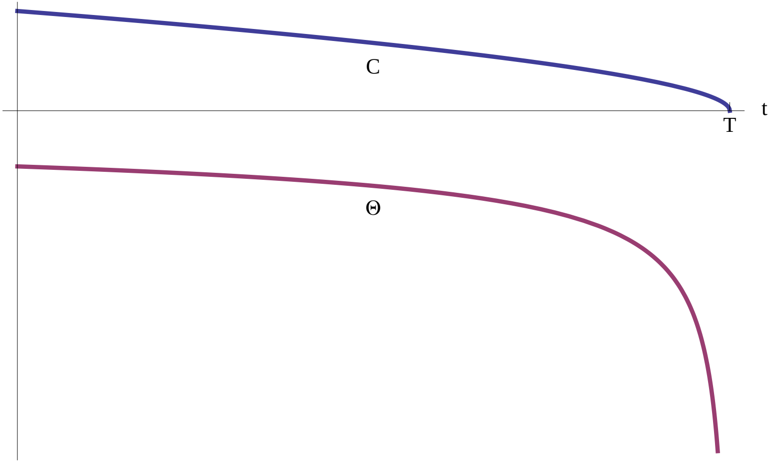
&= -\frac{\sigma S}{2\sqrt{T-t}}\phi\left(w\right) - r\,K\,e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right)
\end{aligned}
$$

Illustration

The value of a European Call decreases as expiry approaches (all other variables and parameters being constant).



For a European Put:

$$\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{Se^{-w^2/2}}{2\sigma\sqrt{2\pi(T-t)}}\left(\frac{\ln(S/K)}{T-t} - r - \sigma^2/2\right) \\
&\quad + Kre^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right) \\
&\quad - \frac{Ke^{-r(T-t)-(w-\sigma\sqrt{T-t})^2/2}}{2\sigma\sqrt{2\pi(T-t)}}\left(\frac{\ln(S/K)}{T-t} - r + \sigma^2/2\right)
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&= -\frac{\sigma S}{2\sqrt{T-t}}\phi(w) + r\,K\,e^{-r(T-t)}\Phi\left(-w + \sigma\sqrt{T-t}\right)
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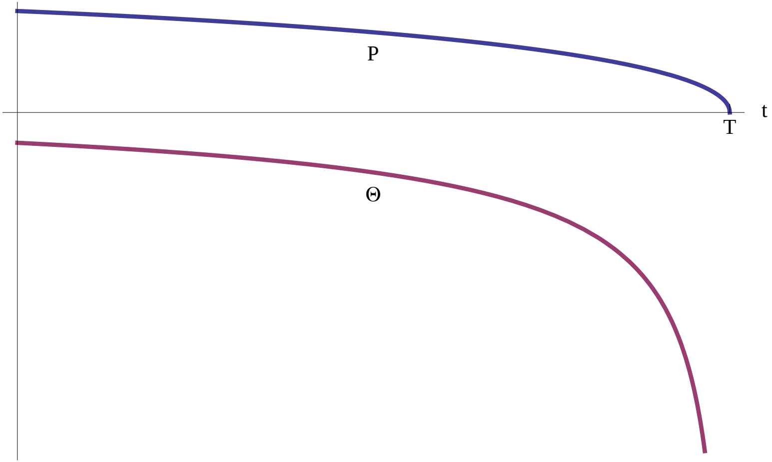

Δ was involved in the derivation of the Black-Scholes PDE and is defined to be the partial derivative with respect to the price of the security.

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\begin{aligned}
C &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
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Delta Δ (1 of 2)

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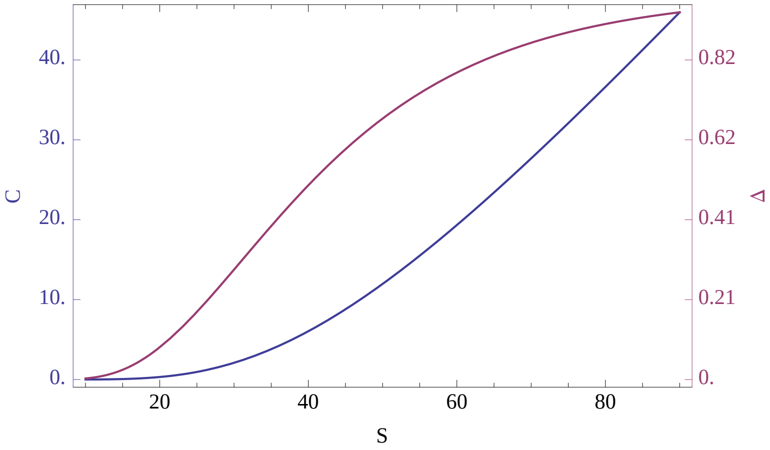
Question: what is the range of Delta for a European call option?

Consider a European Call with $K = 50$, $T = 1$, $r = 0.10$, and $\sigma = 0.50$.



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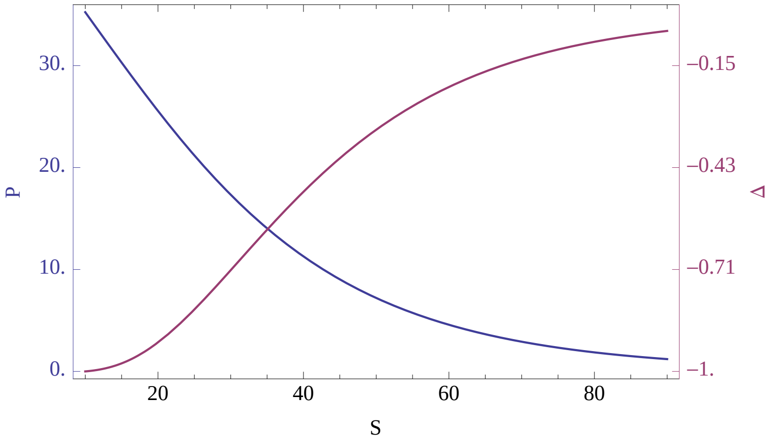
Question: what is the range of Delta for a European put option?

Consider a European Put with $K = 50$, $T = 1$, $r = 0.10$, and $\sigma = 0.50$.



Example (1 of 2)

The current price of a stock is \$77 and its volatility is 35% per year. The risk-free interest rate is 3.25% per year. A portfolio is constructed consisting of one six-month European call option with a strike price of \$80 and the cash obtained from shorting Δ shares of the stock. The portfolio's value is non-random. What is Δ?

Example (2 of 2)

The assumption the portfolio's value is non-random is the assumption

$$(\Delta)S - C = \left(\frac{\partial C}{\partial S}\right) S - C = 0$$

made in deriving the Black-Scholes equation.

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$$\begin{aligned} S &= 77 \quad \sigma = 0.35 \quad T = \frac{6}{12} \\ r &= 0.0325 \quad K = 80 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\ \frac{\partial C}{\partial S} &= \Delta = \Phi\,(w) \end{aligned}$$

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Example (1 of 2)

Suppose a portfolio consists of a share of stock worth \$75 and a European Put option on that stock with a strike price of \$73 and expiry in 3 months. Assume the risk-free interest rate is 10% and the volatility of the stock price is 30%.

Find the Delta of the portfolio consisting of the stock and the option.

Example (2 of 2)

The Delta of the portfolio is

$$\frac{\partial}{\partial S}[S + P] = 1 + \Phi(w) - 1 = \Phi(w)$$

calculated using the variables and parameters below.

$$\begin{aligned} S &= 75 \qquad \sigma = 0.30 \qquad T = 3/12 \\ r &= 0.10 \qquad K = 73 \qquad t = 0 \end{aligned}$$

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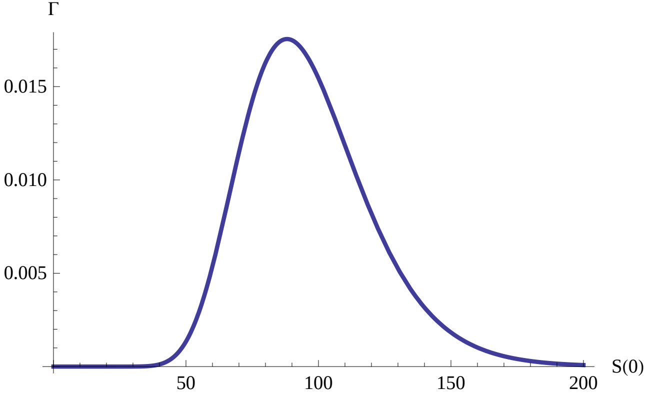
$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ &\approx 0.421858 \\ \Phi(w) &\approx 0.663436 \end{aligned}$$

Gamma Γ

Gamma is the second partial derivative with respect to S, thus

$$\begin{aligned}
\Gamma &= \frac{\partial}{\partial S}\Phi(w) \\
&= \Phi'(w)\frac{\partial w}{\partial S} \\
\frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 C}{\partial S^2} &= \frac{e^{-w^2/2}}{\sigma S\sqrt{2\pi(T-t)}}.
\end{aligned}$$

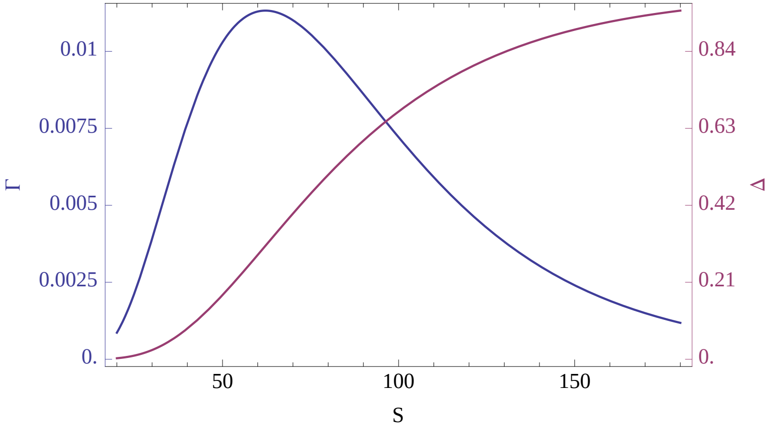
Gamma *vs.* $S(0)$



$$K = 100, \quad \sigma = 0.25 \quad T = 1 \quad r = 0.0325$$

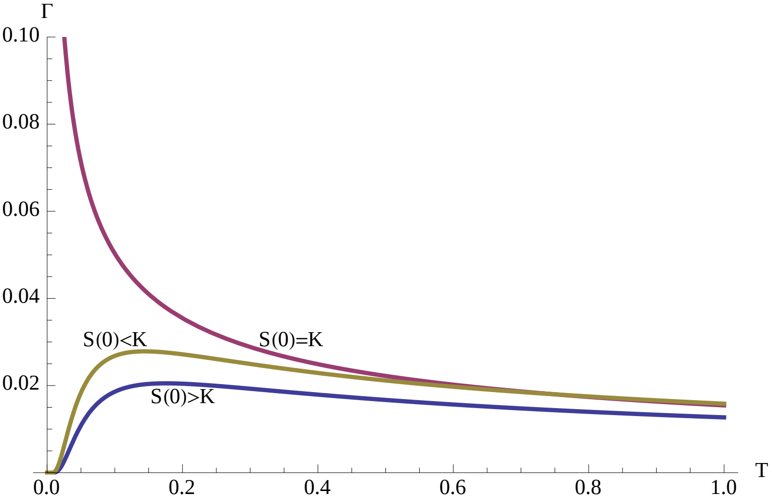
Gamma and Delta

For options far in-the-money or out-of-the-money, there is little change in Δ and thus Γ is nearly zero.



Gamma and At-the-Money Options

Consider an at-the-money option ($S = K$), how does Gamma behave as expiry approaches?

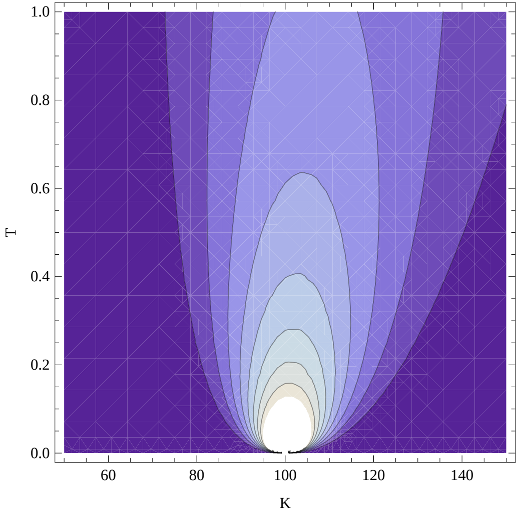


Solution

When $S = K$,

$$\begin{aligned}
\lim_{t \to T^-} \Gamma &= \lim_{t \to T^-} \frac{e^{-((r+\sigma^2/2)(T-t))^2/(2\sigma^2(T-t))}}{\sigma K \sqrt{2\pi(T-t)}} \\
&= \lim_{t \to T^-} \frac{e^{-(r+\sigma^2/2)^2(T-t)/(2\sigma^2)}}{\sigma K \sqrt{2\pi(T-t)}} \\
&= \infty.
\end{aligned}$$

Gamma *vs.* K and T



$$S(0) = 100, \quad \sigma = 0.25 \quad r = 0.0325$$

Relationships Between Δ, Θ, and Γ

Remember the Black-Scholes PDE:

$$rF = F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS}$$

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$$rF = F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS}$$

Since $F_t = \Theta$, $\Delta = F_S$, and $F_{SS} = \Gamma$ then the Black-Scholes equation can be thought of as

$$rF = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma.$$

Changes in Option Values

Using differentials we can approximate changes in the prices of options as underlying variables and parameters change.

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Let $F(S, t)$ be the value of an option at time t when the value of the underlying security is S, then we have the following approximations.

$$
\begin{aligned}
F(S, t + \delta t) &\approx F(S, t) + (\Theta)\delta t \\
F(S + \delta S, t) &\approx F(S, t) + (\Delta)\delta S \\
F(S + \delta S, t) &\approx F(S, t) + (\Delta)\delta S + (\Gamma)(\delta S)^2 \\
F(S + \delta S, t + \delta t) &\approx F(S, t) + (\Theta)\delta t + (\Delta)\delta S + (\Gamma)(\delta S)^2
\end{aligned}
$$

Example (1 of 3)

A six-month call option with a strike price of \$100 on a stock currently valued at \$99 and having a volatility of $\sigma = 0.40$ costs \$12.4911. The risk-free interest rate is $r = 0.08$. Estimate the value of the option at five month to expiry.

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$$\begin{aligned}
C(S, t + \delta t) &\approx C(S, t) + (\Theta)\delta t \\
C(S, \delta t) &\approx C(S, 0) + (\Theta)\delta t \\
C(99, 1/12) &\approx C(99, 0) + (\Theta)(1/12) \\
&= 12.4911 + \frac{-14.5686}{12} \\
&= 11.2771
\end{aligned}$$

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For comparison, the exact value is $C(99, 1/12) = \$11.2322$.

A six-month call option with a strike price of \$100 on a stock currently valued at \$99 and having a volatility of $\sigma = 0.40$ costs \$12.4911. The risk-free interest rate is $r = 0.08$. Estimate the value of the option using Delta if the value of the stock increases to \$101.

Example (2 of 3)

A six-month call option with a strike price of \$100 on a stock currently valued at \$99 and having a volatility of $\sigma = 0.40$ costs \$12.4911. The risk-free interest rate is $r = 0.08$. Estimate the value of the option using Delta if the value of the stock increases to \$101.

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\begin{aligned}
C(S + \delta S, t) &\approx C(S, t) + (\Delta)\delta S \\
C(101, 0) &\approx C(99, 0) + (\Delta)(2) \\
&= 12.4911 + (0.597666)(2) \\
&= 13.6865
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For comparison, the exact value is $C(101, 0) = \$13.7137$.

Example (3 of 3)

A six-month call option with a strike price of \$100 on a stock currently valued at \$99 and having a volatility of $\sigma = 0.40$ costs \$12.4911. The risk-free interest rate is $r = 0.08$. Estimate the value of the option using Delta and Gamma if the value of the stock increases to \$101.

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C(S+\delta S, t) &\approx C(S,t) + (\Delta)\delta S + (\Gamma)(\delta S)^2 \\
C(101, 0) &\approx C(99,0) + (\Delta)(2) + (\Gamma)(4) \\
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\end{aligned}
$$

For comparison, the exact value is $C(101, 0) = \$13.7137$.

Vega $\mathcal{V}$ (1 of 2)

Vega is the partial derivative with respect to volatility σ.

$$
\begin{aligned}
C &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
\frac{\partial C}{\partial \sigma} &= S\Phi'(w)\frac{\partial w}{\partial \sigma} - Ke^{-r(T-t)}\Phi'\left(w - \sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma} - \sqrt{T-t}\right)
\end{aligned}
$$

Vega is the partial derivative with respect to volatility σ.

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C &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
\frac{\partial C}{\partial \sigma} &= S\Phi'(w)\frac{\partial w}{\partial \sigma} - Ke^{-r(T-t)}\Phi'\left(w - \sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma} - \sqrt{T-t}\right) \\
&= \frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-w^2/2}
\end{aligned}$$

According to the Put-Call Parity formula:

$$P + S = C + Ke^{-r(T-t)}$$

Vega $\mathcal{V}$ (2 of 2)

According to the Put-Call Parity formula:

$$\begin{aligned} P + S &= C + Ke^{-r(T-t)} \\ \frac{\partial}{\partial \sigma}(P + S) &= \frac{\partial}{\partial \sigma}(C + Ke^{-r(T-t)}) \end{aligned}$$

According to the Put-Call Parity formula:

$$\begin{aligned} P + S &= C + Ke^{-r(T-t)} \\ \frac{\partial}{\partial \sigma}(P + S) &= \frac{\partial}{\partial \sigma}(C + Ke^{-r(T-t)}) \\ \frac{\partial P}{\partial \sigma} &= \frac{\partial C}{\partial \sigma} \\ \frac{\partial P}{\partial \sigma} &= \frac{S\sqrt{T-t}}{\sqrt{2\pi}} e^{-w^2/2} \end{aligned}$$

Remark: vega is identical for puts and calls.

Example (1 of 2)

Consider a three-month European put option on a stock whose current value is \$200 and whose volatility is 30%. The option has a strike price of \$195 and the risk-free interest rate is 6.25%.

1. Find the vega of the option.
2. If the volatility of the stock increases to 31%, approximate the change in the value of the put.

Example (2 of 2)

$$
\begin{aligned}
S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\
K &= 195 \quad r = 0.0625 \quad t = 0
\end{aligned}
$$

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ \mathcal{V} &= \frac{S\sqrt{T-t}}{\sqrt{2\pi}} e^{-w^2/2} \end{aligned}$$

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx 0.347952 \\ \mathcal{V} &= \frac{S\sqrt{T-t}}{\sqrt{2\pi}} e^{-w^2/2} \end{aligned}$$

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.347952 \\ \mathcal{V} &= \frac{S\sqrt{T - t}}{\sqrt{2\pi}} e^{-w^2/2} \approx 37.5509 \end{aligned}$$

Example (2 of 2)

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.347952 \\ \mathcal{V} &= \frac{S\sqrt{T - t}}{\sqrt{2\pi}} e^{-w^2/2} \approx 37.5509 \end{aligned}$$

Using the linear approximation,

$$dP = \mathcal{V} d\sigma = (37.5509)(0.01) = 0.375509.$$

Rho ρ (1 of 2)

Rho is the partial derivative with respect to the risk-free interest rate r.

$$
\begin{aligned}
C &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
\frac{\partial C}{\partial r} &= S\Phi'(w)\frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - Ke^{-r(T-t)}\Phi'\left(w - \sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r}
\end{aligned}
$$

Rho ρ (1 of 2)

Rho is the partial derivative with respect to the risk-free interest rate r.

$$
\begin{aligned}
C &= S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
\frac{\partial C}{\partial r} &= S\Phi'(w)\frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - Ke^{-r(T-t)}\Phi'\left(w - \sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r} \\
&= K(T-t)e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right)
\end{aligned}
$$

Starting with the Put-Call Parity formula:

$$P + S = C + Ke^{-r(T-t)}$$

Rho ρ (2 of 2)

Starting with the Put-Call Parity formula:

$$\begin{aligned} P + S &= C + Ke^{-r(T-t)} \\ \frac{\partial}{\partial r}(P + S) &= \frac{\partial}{\partial r}(C + Ke^{-r(T-t)}) \end{aligned}$$

Starting with the Put-Call Parity formula:

$$
\begin{aligned}
P + S &= C + Ke^{-r(T-t)} \\
\frac{\partial}{\partial r}(P + S) &= \frac{\partial}{\partial r}(C + Ke^{-r(T-t)}) \\
\frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} - K(T-t)e^{-r(T-t)} \\
\frac{\partial P}{\partial r} &= -K(T-t)e^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right)
\end{aligned}
$$

Example (1 of 2)

Consider a three-month European put option on a stock whose current value is \$200 and whose volatility is 30%. The option has a strike price of \$195 and the risk-free interest rate is 6.25%.

1. Find the rho of the option.
2. If the interest rate increases to 7.00%, approximate the change in the value of the put.

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

$$
\begin{aligned}
S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\
K &= 195 \quad r = 0.0625 \quad t = 0
\end{aligned}
$$

Using these values

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
\rho &= -K(T - t)e^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right)
\end{aligned}
$$

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx 0.347952 \\ \rho &= -K(T-t)e^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right) \end{aligned}$$

Example (2 of 2)

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.347952 \\ \rho &= -K(T - t)e^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) \approx -20.2315 \end{aligned}$$

Example (2 of 2)

$$\begin{aligned} S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\ K &= 195 \quad r = 0.0625 \quad t = 0 \end{aligned}$$

Using these values

$$\begin{aligned} w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.347952 \\ \rho &= -K(T - t)e^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) \approx -20.2315 \end{aligned}$$

Using the linear approximation,

$$dP = \rho dr = (-20.2315)(0.0075) = -0.151737.$$

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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