Derivatives

The Definition of Average Rate of Change

- The **average rate of change** of a function $f(x)$ over an interval $a \leq x \leq b$ is defined to be

$$\frac{f(b) - f(a)}{b - a}$$

Geometrically

- Geometrically, the average rate of change is the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

The Definition of the Derivative

- The derivative of a function $f(x)$ at a point $x = a$ is defined to

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Geometrically

Geometrically, the derivative $f'(a)$ is the slope of the tangent line to the function f through the point $(a, f(a))$.

Properties of derivatives

- The derivative of a function is itself a function, and satisfies the following linearity properties.

Derivatives of Constant Multiples

- If $g(x) = kf(x)$ for some k, then

$$g'(x) = kf'(x)$$

at all points where f is differentiable.

Derivatives of Sums

- If $h(x) = f(x) + g(x)$, then

$$h'(x) = f'(x) + g'(x)$$

at all points where f and g are differentiable.

Derivatives of Differences

- Similarly, if $j(x) = f(x) - g(x)$, then

$$j'(x) = f'(x) - g'(x)$$

at all points where f and g are differentiable.

The Power Rule

- If n is any fixed real number, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof by First Principle for The Power Rule

- **The Trick**:

$$a^n - b^n = (a-b)\underbrace{(a^{n-1} + a^{n-2}b + a^{n-3}b^2 \text{ .... } b^{n-1})}_{\text{n number of terms}}$$

- **The Proof**:

$$\begin{align} f(x) &= x^n \\ f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ f'(x) &= \lim_{h\to 0} \frac{(x+h-x)((x+h)^{n-1} + (x+h)^{n-2}x \text{ ... } x^{n-1})}{h} \\ f'(x) &= nx^{n-1} \end{align}$$

Properties of Leibniz notation

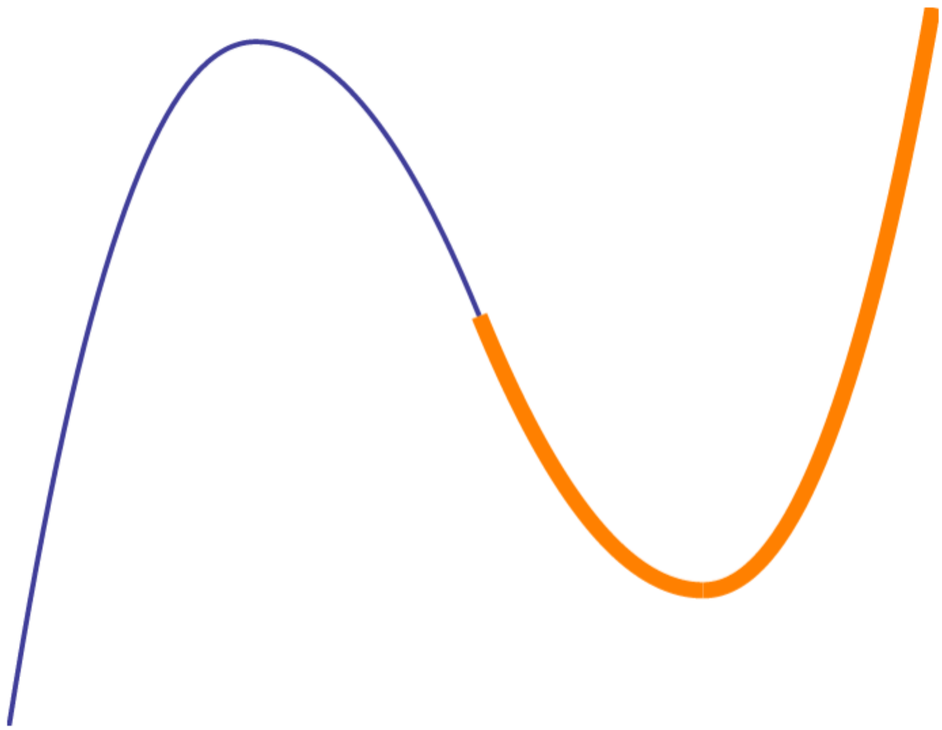
- **Units:** If P has units of pressure, and t has units of time, then $\frac{dP}{dt}$ has units of pressure per time.
- **Evaluating at points:** If we want to take the derivative at a particular point $x = 3$, then we use the notation $\left.\frac{df}{dx}\right|_{x=3}$. The bar is read as "evaluated at".
- **Derivatives act on functions:**
  - We can write $\frac{d(x^2)}{dx}$ for the derivative of x^2.
  - If a formula is long, we can write $\frac{d}{dy}(y^3 + 2y^2)$.

Second Derivative

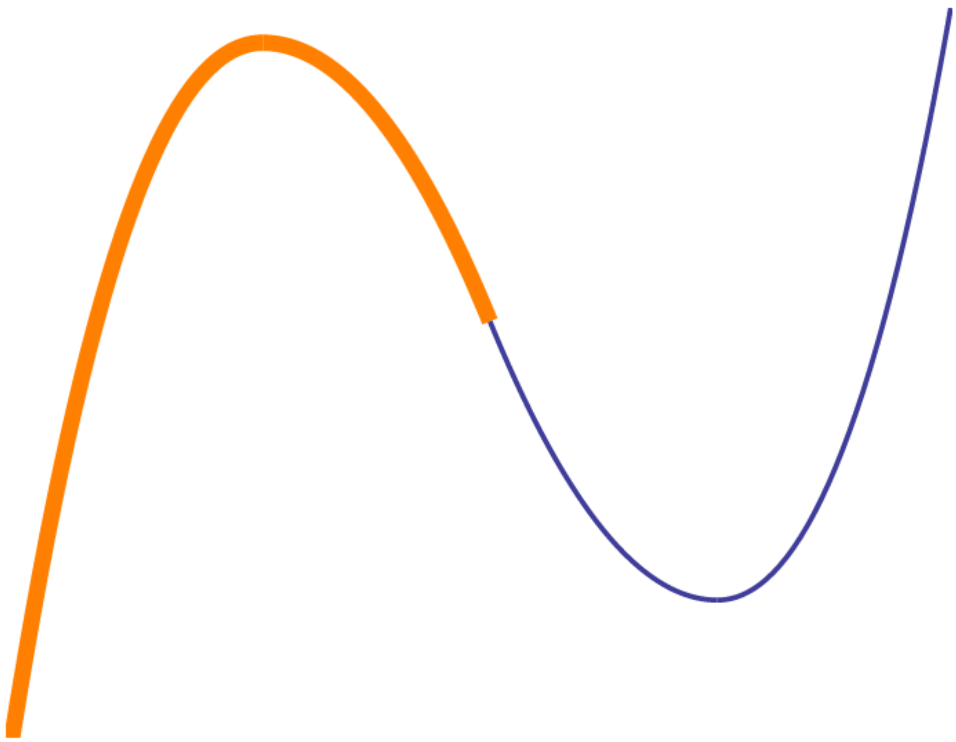
- The second derivative of a function $f(x)$ is the first derivative of $f'(x)$, and is denoted by $f''(x)$ or $\frac{d^2 f}{dx^2}$.

Second Derivative and Concavity

- On intervals where $f'' > 0$, the function f is concave up.



- On intervals where $f'' < 0$, the function f is concave down.



Derivative of sine and cosine

- The first and second derivatives of sine and cosine:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d^2}{dx^2}\sin(x) = -\sin(x)$$

$$\frac{d^2}{dx^2}\cos(x) = -\cos(x)$$

Definition of Significant figure

- The number of *significant figures* is the count of those digits that carry meaning with regards to precision.

- Examples:
  - All non-zero digits are significant - 1235 has 4 significant digits
  - Zeros appearing between non-zero digits are significant - 101 has 3 significant digits
  - Trailing zeros in a number containing a decimal are significant, 32.000 has 5 significant figures.
- Non Examples:
  - Trailing zeros in a number with no decimal are **not** significant - 5400 has 2 significant figures.
  - Leading zeros in a decimal number are **not** significant - 0.0003 has 1 significant figure.
  - Extraneous digits introduced in a computation to greater precision than measured data are **not** significant - if .25 and .50 are measurements accurate to $\pm .01$, then in the product (.25)(.50) = 0.125 the last 5 is **not** significant.