Differentiation

Linear Approximation

- The linear approximation for a function f near a point $x = a$ is given by the following equivalent formulas:

$$\Delta f \approx \left.\frac{df}{dx}\right|_{x=0} \cdot \Delta x \qquad \text{for } \Delta x \text{ near } 0$$

$$f(x) \approx f'(a)(x-a) + f(a) \qquad \text{for } x \text{ near } a$$

The Product Rule

- If $h(x) = f(x)g(x)$, then

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

at all points where the derivatives $f'(x)$ and $g'(x)$ are defined.

The Quotient Rule

- If $h(x) = \frac{f(x)}{g(x)}$ for all x, then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

at all points where f and g are differentiable and $g(x) \neq 0$.

The Chain Rule

- If $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x))g'(x)$$

at all points where the derivatives $f'(g(x))$ and $g'(x)$ are defined.

- Alternatively, if $y = f(u)$, and $u = g(x)$, then

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=g(a)} \cdot \left.\frac{du}{dx}\right|_{x=a}$$

at any point $x = a$ where the derivatives on the right hand side are defined.

Implicit Differentiation

- To implicitly differentiate a function $f(x)g(y) = 1$ with respect to x:

$$\begin{align}
\frac{d}{dx}(f(x)g(y) = 1) \tag{1}\\
\frac{d}{dx}(f(x)g(y)) = \frac{d}{dx}1 \tag{2}\\
\frac{d}{dx}(f(x)g(y)) = 0 \text{ (derivatives of constant functions are 0)} \tag{3}\\
f'(x)g(y) + f(x)\frac{d}{dx}g(y) = 0 \text{ (product rule)} \tag{4}\\
f'(x)g(y) + f(x)g'(y)\frac{dy}{dx} = 0 \text{ (chain rule)} \tag{5}
\end{align}$$

Definition of Inverse Function

- If functions f and g satisfy $g(f(x)) = x$ and $f(g(y)) = y$, then we say g is the inverse of f, and denote it by f^{-1} (Similarly $f = g^{-1}$).

- If a function f has an inverse function f^{-1}, then $f^{-1}(b) = a$ only if $f(a) = b$.

Definition of One-to-One

- A function f is **one-to-one** if $f(a) \neq f(b)$ whenever $a \neq b$. It is one-to-one if and only if its graph satisfies the horizontal line test (no horizontal line intersects its graph at more than one place).

Domain and Range, Interval Notation

- Recall that the **domain** of a function f is the set of allowable input values. For instance, the domain of the function $f(x) = 1/x$ is the set of all non-zero real numbers.
- The **range** of f is the set of all possible output values. For instance, the range of the function $g(x) = x^2$ is the set of all real numbers that are non-negative.
- We often use interval notation to express sets of numbers like domains and ranges. A **closed interval**, denoted $[a, b]$, is the set of numbers x such that $a \leq x \leq b$.
- An **open interval**, denoted (a, b), is the set of numbers x such that $a < x < b$.
- One can have a half-open, half-closed interval. For instance, $[-1, 3)$ is the set of numbers x such that $-1 \leq x < 3$. One can also use $\pm\infty$ as endpoints: $(-\infty, 0)$ is the set of numbers x such that $-\infty < x < 0$ (the set of negative numbers).
- This notation using round parentheses for open intervals is not universal; many mathematicians use reversed square brackets instead. For instance, they would denote the interval $3 < x < 7$ as $]3, 7[$ rather than (3, 7).

The inverse Trig functions

$$\arcsin x = \theta \text{ in } [-\pi/2, \pi/2] \quad \text{such that } \sin\theta = x$$

$$\arccos x = \theta \text{ in } [0, \pi] \quad \text{such that } \cos\theta = x$$

$$\arctan x = \theta \text{ in } [-\pi/2, \pi/2] \quad \text{such that } \tan\theta = x$$

Derivatives of Inverse Functions

- If g is a (full or partial) inverse of a function f, then

  $$g'(x) = \frac{1}{f'(g(x))}$$

  at all x where $f'(g(x))$ exists and is non-zero.

Derivatives of the Inverse Trig Functions

- We now have more basic functions that we can differentiate

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Properties of exponents

- Let a be a positive real number.
- $a^0 = 1$
- $a^1 = a$
- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $a^{m/n} = \sqrt[n]{a^m}$

Properties of exponential functions

- The function $f(x) = a^x$ has base a for a positive real number a.
  - The function a^x is a continuous function.
  - The domain of a^x is all real numbers.
  - The range of a^x is all positive real numbers.

The derivative of an exponential function

- The derivative of the exponential function is

  $$\frac{d}{dx}a^x = M(a)a^x,$$

  where the mystery number $M(a)$ is the slope of the tangent line at zero:

  $$M(a) = \frac{d}{dx}a^x\Big|_{x=0} = \lim_{\Delta x \to 0}\frac{a^{\Delta x}-1}{\Delta x}$$

Definition of e

- The base e is the unique real number so that

$$\frac{d}{dx}e^x = e^x$$

Differentiating exponential functions with other Bases

- We can finally identify our mystery number, and differentiate exponential functions with any base.

  For any positive constant a,

$$\frac{d}{dx}a^x = a^x \ln a$$

Properties of x

- $\log_{10}(x)$ is the inverse function of 10^x.
- The natural log, denoted $\ln(x)$, is the inverse function of e^x
- $\ln e^x = x$
- $e^{\ln x} = x$
- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(a^b) = b \cdot ln(a)$

The derivative of the natural logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x}$$