Applications

Definition of Critical Points

- The critical points of a function $f(x)$ to be all points x in the domain of $f(x)$ such that
    - $f'(x) = 0$, or
    - $f'(x)$ does not exist

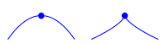
The First Derivative Test

Finding Local Maxima and Minima

- Suppose the function $f(x)$ is continuous a $x = a$ and has a critical point at $x = a$.

f has a local minimum at $x = a$ if $f'(x) < 0$ just to the left of a and $f'(x) > 0$ just to the right of a.

f has a local maximum at $x = a$ if $f'(x) > 0$ just to the left of a and $f'(x) < 0$ just to the right of a.

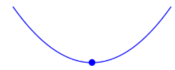


The point $x = a$ is neither a local minimum nor a local maximum of f if $f'(x)$ has the same sign just to the left of a and just to the right of a.

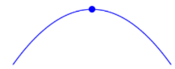
The Second Derivative Test

- Suppose that $x = a$ is a critical point of f, with $f'(a) = 0$.

If $f''(a) > 0$, then f has a local minimum at $x = a$.



If $f''(a) < 0$, then f has a local maximum at $x = a$.



If $f''(a) = 0$, or does not exist, then the test is inconclusive — there might be a local maximum, or a local minimum, or neither.

Definition of Inflection Point

- An inflection point is a point where the concavity of the function changes. That is the second derivative $f''(x)$ changes sign – $f''(x) > 0$ just to the left of x and $f''(x) < 0$ just to the right of x (or vice versa).

General strategy for sketching functions

1. Plot
    - Discontinuities (especially infinite ones)
    - End points (or $x \to \pm\infty$)
    - Easy points ($x = 0$, or $y = 0$)(This is optional.)
2. Plot critical points and values. (Solve $f'(x) = 0$ or undefined)
3. Decide whether $f' < 0$ or $f' > 0$ on each interval between endpoints, critical points, and discontinuities, (Valuable double check)
4. Identify where $f'' < 0$ and $f'' > 0$ (concave down and concave up).
    - Identify inflection points. (Makes graph look nice. Can be used to double check)
5. Combine into graph

Indeterminate Forms

- We call $\frac{0}{0}$ and $\frac{\infty}{\infty}$ **indeterminate forms**, because when we run into them in a limit, they require further analysis to determine whether the numerator or denominator wins the race to 0 or ∞ respectively, or whether they balance out and reach some other finite limit.

L'Hospital's Rule Version 1: Indeterminate from $0/0$

- If

$$\begin{matrix} f(x) \to 0 \\ g(x) \to 0 \end{matrix} \qquad \text{as } x \to a,$$

and the functions f and g are differentiable near the point $x = a$, then limit

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that and the right hand limit exists or is $\pm\infty$.

If

$$\begin{matrix} f(x) \to \pm\infty \\ g(x) \to \pm\infty \end{matrix} \qquad \text{as } x \to a,$$

and the functions f and g are differentiable near the point $x = a$, then limit

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

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Note that:

- We can replace a with a^+ or a^- and the results (versions 1 and 2) still hold.

- We can replace a with $\pm\infty$, and the results (versions 1 and 2) still hold.

Other Indeterminate Forms

- Other indeterminate forms $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0 should be rearranged to be the form

$0/0$ or ∞/∞ in order to apply the L'Hopital's rule.

The Extreme Value Theorem

- If f is continuous on a closed interval $[a, b]$, then there are points at which f attains its maximum and its minimum on $[a, b]$.

Maxima and Minima

- The maxima and minima will be attained at either a critical point or an end point

Related Rates strategy

- To solve related rates problems, it is useful to follow this strategy:

    i. Start with a good picture!

    ii. Identify the relevant variables and rates

    iii. Find an equation relating the relevant variables that always holds.

    iv. Differentiate implicitly.

    v. Plug in and solve!