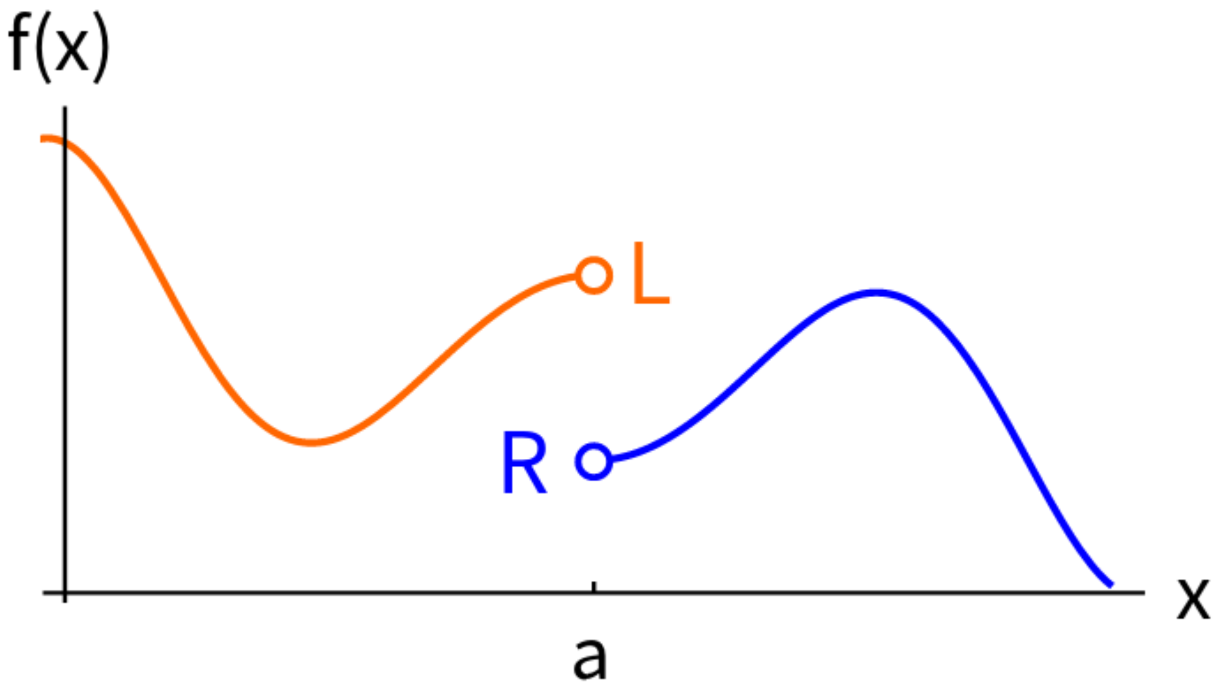
Limits

Definition of One-Sided limit



- Suppose $f(x)$ gets really close to R for values of x that get really close to (but are not equal to) a from the right. Then we say R is the **right-hand limit** of the function $f(x)$ as x approaches a from the right.

We write:

$$f(x) \to R \quad as \quad x \to a^+$$

or

$$\lim_{x \to a^+} f(x) = R$$

- If $f(x)$ gets really close to L for values of x that get really close to (but are not equal to) a from the left, we say that L is the **left-hand limit** of the function $f(x)$ as x approaches a from the left.

We write:

$$f(x) \to L \quad as \quad x \to a^-$$

or

$$\lim_{x \to a^-} f(x) = L$$

Definition of Limit

Limit in words

- If a function $f(x)$ approaches some value L as x approaches a from both the right and the left, then **the limit** or **the overall limit** of $f(x)$ exists and equals L.

Limit in symbols

If

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

then

$$\lim_{x \to a} f(x) = L$$

Alternatively,

$$f(x) \to L \quad as \quad x \to a$$

Remember that x is approaching a but does not equal a

The Limit Laws:

Suppose $\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = M$

- Then we get the following Limit Laws:
  - Limit Law for Addition: $\quad \lim_{x \to a}[f(x) + g(x)] = L + M$

- Limit Law for Subtraction: $\lim_{x \to a}[f(x) - g(x)] = L - M$
- Limit Law for Multiplication: $\lim_{x \to a}[f(x) \cdot g(x)] = L \cdot M$
- Limit Law for Division:
  - If $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$
  - If $M = 0$ but $L \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist
  - If both $M = 0$ and $L = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ might exist, or it might not exist. More work is necessary to determine whether the last type of limit exists, and what it is if it does exist.

Definition of Continuous at a point

- We say that a function f is **continuous at a point** $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

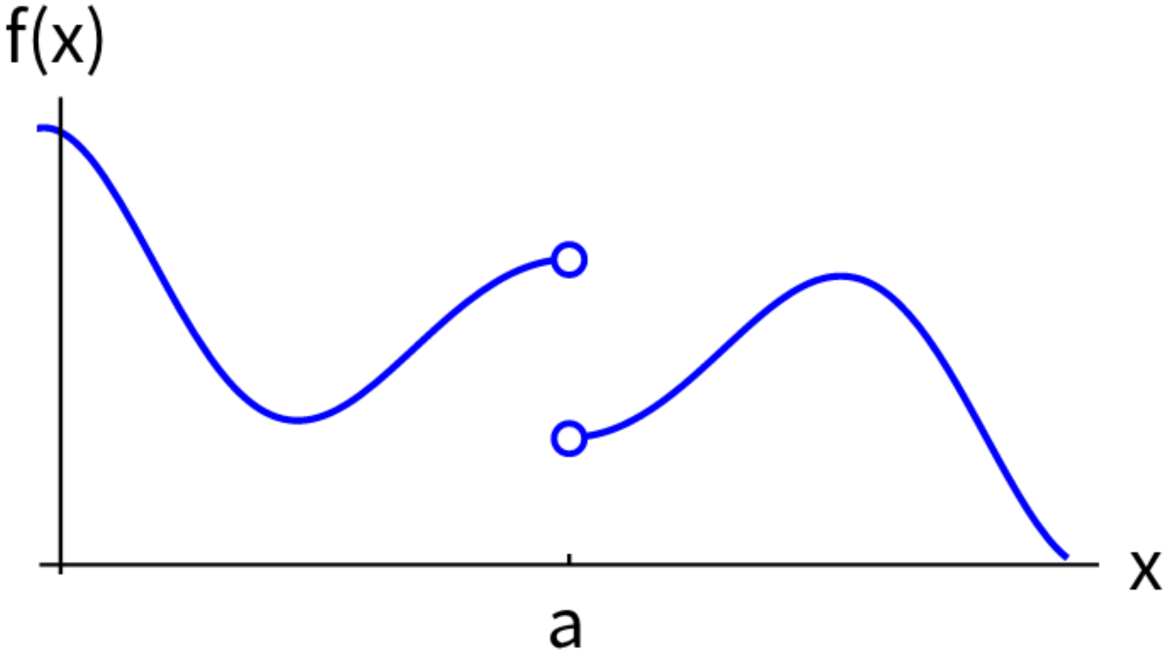
- In particular, if either $f(a)$ or $\lim_{x \to a} f(x)$ fails to exist, then f is discontinuous at a.
- We say that a function f is **right-continuous at a point** $x = a$ if,

$$\lim_{x \to a^+} f(x) = f(a)$$

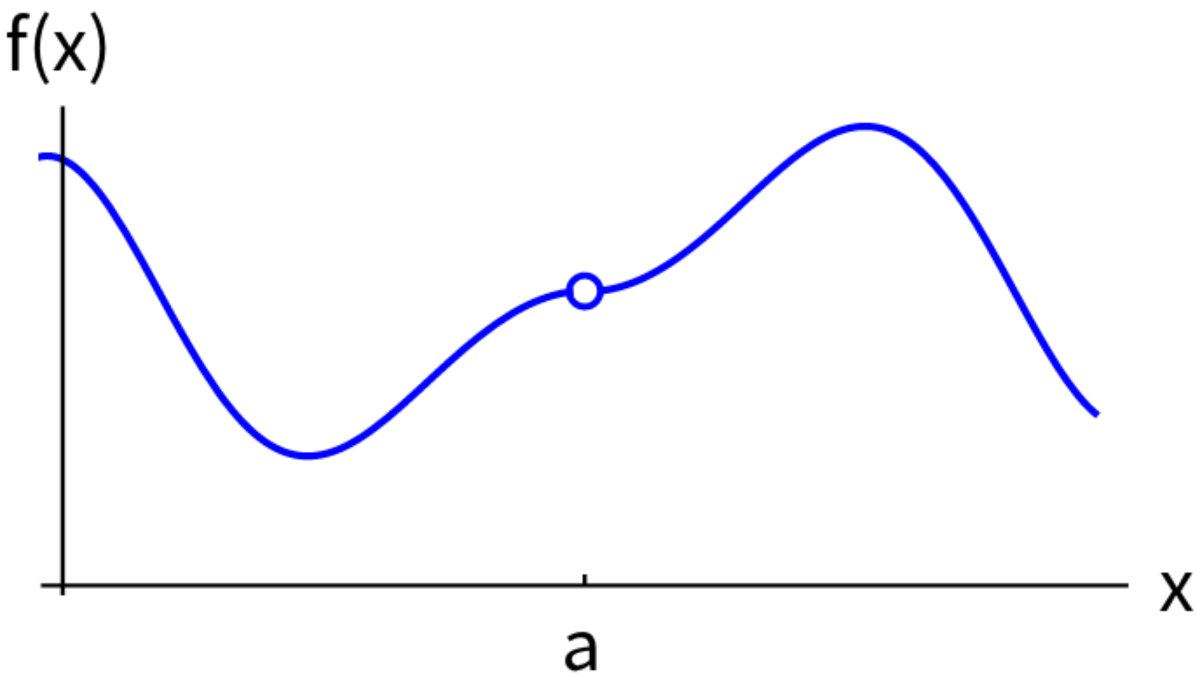
- We say that a function f is **left-continuous at a point** $x = a$ if,

$$\lim_{x \to a^-} f(x) = f(a)$$

- It is sometimes useful to classify certain types of discontinuities.
- If the left-hand limit $\lim_{x \to a^-} f(x)$ and the right-hand limit $\lim_{x \to a^+} f(x)$ both exist at a point $x = a$, but they are not equal, then we say that f has a **jump discontinuity** at $x = a$.



- If the overall limit $\lim_{x \to a} f(x)$ exists, but the overall limit does not equal $f(a)$, then we say that f has a **removable discontinuity** at $x = a$



- The following functions are continuous at all real numbers:

- all polynomials
- $\sqrt[3]{x}$
- $|x|$
- $\cos x$ and $\sin x$
- exponential functions a^x with base $a > 0$

- The following functions are continuous at the specified values of x:
  - $\sqrt{x}$, for $x > 0$
  - $\tan x$, at all x where it is defined
  - logarithmic functions $\log_a x$ with base $a > 0$, for $x > 0$

Intermediate Value Theorem

- If f is a function which is continuous on the interval $[a, b]$, and M lies between the values of $f(a)$ and $f(b)$, then there is at least one point c between a and b such that $f(c) = M$.

(A function f is continuous on a closed interval $[a, b]$, if it is right-continuous at a, left-continuous at b, and continuous at all points between a and b.)