Summer Workshop in Mathematics (SWIM)
Kannur University, May, 2025
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Tutorial problems

May 5, 2025

1. Show the following.
   (a) $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 0$.
   (b) $\sum_{i=0}^{n} 4^i = \frac{1}{3}(4^{n+1} - 1)$ for all $n \geq 0$.
   (c) $\sum_{i=0}^{n} 1/3^i = \frac{3}{2}(1 - \frac{1}{3^{n+1}})$ for all $n \geq 0$.
   (d) $2^n \leq n!$ for all $n \geq 4$.

2. Let $f : X \to Y$ be a function between two sets X, Y. Let $A, B \subset X$ and $C, D \subset Y$ be arbitrary subsets. Determine if the following statements are true or false.
   (a) $f(A \cap B) = f(A) \cap f(B)$.
   (b) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
   (c) $f(A \cup B) = f(A) \cup f(B)$.
   (d) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

3. Let A, B, C be subsets of a set X. Show the following.
   (a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
   (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
   (c) $(A \cup B)^c = A^c \cap B^c$.
   (d) $(A \cap B)^c = A^c \cup B^c$.

4. Show the set $\mathbb{Z}$ of integers is countable by using the function $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

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5. Let $f : A \to B, g : B \to C$ be functions. Determine if the following statements are true or false.
   (a) If f, g are injective, then so is $g \circ f$.

(b) If $g \circ f$ is injective, then so is f.

(c) If $g \circ f$ is injective, then so is g.

(d) If f, g are surjective, then so is $g \circ f$.

(e) If $g \circ f$ is surjective, then so is f.

(f) If $g \circ f$ is surjective, then so is g.

6. If $A_1, \ldots, A_k$ are countable, then show that the Cartesian product $A_1 \times \ldots \times A_k$ is also countable.

7. Show that a countable union of countable sets is countable.

8. Which of the following are equivalence relations on the given sets A.

   (a) $A = \mathbb{N} : a \sim b$ if $a - b$ is odd.

   (b) $A = \mathbb{N} : a \sim b$ if $a - b$ is even.

   (c) $A = \mathbb{N} : a \sim b$ if $-4 \leq a - b \leq 4$.

   (d) $A = \mathbb{R} : a \sim b$ if $|a| = |b|$.

   (e) A is the set of all 2×2 real matrices: $A \sim B$ if there exists an invertible matrix $P \in A$ such that $A = PBP^{-1}$.

9. Fix a positive integer n. Consider the equivalence relation on $\mathbb{Z}$ defined by $a \sim b$ if $a - b$ is divisible by n. Determine the equivalence classes for this relation.

10. Negate the following statements. P denotes the set of prime natural numbers.

   (a) $\exists r \in \mathbb{Q}$ such that $r^2 = 2$.

   (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y^3 = x$.

   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y^2 = x$ or $x \leq 0$.

   (d) $\forall p \in P$, if $p > 2$ then p is odd.

   (e) $\forall n \in \mathbb{N}$ such that $n > 0$ $\exists p_1, \ldots, p_r \in P$ such such that $n = p_1 \cdot \ldots \cdot p_r$.

   (f) $\forall$ continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $\forall a, b \in \mathbb{R}$ such that $f(a) < f(b)$, if $f(a) < c < f(b)$, $\exists x \in \mathbb{R}$ such that $f(x) = c$.

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11. Which of the following are subspaces of $\mathbb{R}^n$? When W is a subspace, find a spanning set of W.

(a) $W = \{(a,b) \in \mathbb{R}^2 \mid ab = 0\} \subset \mathbb{R}^2$.

(b) $W = \{(a,b) \in \mathbb{R}^2 \mid a+b = 0\} \subset \mathbb{R}^2$.

(c) $W = \{(a,b) \in \mathbb{R}^2 \mid a+b = 1\} \subset \mathbb{R}^2$.

(d) $W = \{(a,b,c) \in \mathbb{R}^3 \mid a+b = c\} \subset \mathbb{R}^3$.

(e) $W = \{(a,b,c) \in \mathbb{R}^3 \mid ab = c\} \subset \mathbb{R}^3$.

(f) $W = \{(a,b,c) \in \mathbb{R}^3 \mid a+b+c = 0 \text{ and } b-c = 0\} \subset \mathbb{R}^3$.

12. Let $W_1, W_2 \subset \mathbb{R}^n$ be subspaces.

(a) Show that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

(b) Give an example to show that $W_1 \cup W_2$ need not be subspace of $\mathbb{R}^n$.

(c) Define $W_1 + W_2 = \{v_1 + v_2 \mid v_1 \in W_1, v_2 \in W_2\}$. Show that $W_1 + W_2$ is a subspace of $\mathbb{R}^n$.

13. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$. Let $W \subset \mathbb{R}^3$ be the set of vectors $(a,b,c) \in \mathbb{R}^3$ such that

$$A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(a) Find 4 vectors in W. How many vectors are in W?

(b) Show that W is a subspace of $\mathbb{R}^3$.

(c) Find a spanning set of W.

14. Let W_1, W_2 be *distinct* and *proper* subspaces of $\mathbb{R}^2$. Show that $W_1 \cap W_2 = \{0\}$. Is this statement true if W_1, W_2 are distinct and proper subspaces of $\mathbb{R}^3$?

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$M_n(\mathbb{R})$:=The set of real $n \times n$ matrices.

15. Prove that $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$.

16. Show that $\{a + b\sqrt[3]{2} \mid a, b \in \mathbb{Q}\}$ is *not* a subfield of $\mathbb{R}$.

17. Which of the following are vector spaces over $\mathbb{R}$?

(a) $\{p(x) : p(x) = \sum_{i=0}^{n} a_i x^i, n \in \mathbb{N}, a_i \in \mathbb{R}\}$

(b) $\{p(x) : p(x) = \sum_{i=0}^{n} a_i x^i, n \in \mathbb{N}, a_i \in \mathbb{Q}\}$

(c) $\{p(x) : p(x) = \sum_{i=0}^{n} a_i x^i, n \in \mathbb{N}, a_i \in \mathbb{N}\}$

(d) $GL_2(\mathbb{R})$:= The set of all invertible 2×2 matrices.

(e) $\mathbb{R}^2 \setminus \{(x, x) \mid x \in \mathbb{R}^*\}$.

18. Show that the set $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ is a vector space over $\mathbb{R}$.

19. Show that the set of trace zero 2×2 real matrices form a vector space over $\mathbb{R}$. (For a matrix $A \in M_2(\mathbb{R})$, trace(A)=sum of all diagonal entries).

20. A matrix $A \in M_n(\mathbb{R})$ is said to be symmetric if $A^t = A$. Prove that the set of all $n \times n$ real symmetric matrices is a vector space over $\mathbb{R}$.

21. Find bases and dimension of the following vector spaces over $\mathbb{R}$?

    (a) $\{p(x) : p(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{R}\}$

    (b) $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$.

    (c) The set of trace zero 2×2 real matrices.(For a matrix $A \in M_2(\mathbb{R})$, trace(A)=sum of all diagonal entries).

    (d) The set of all 2×2 real symmetric matrices.

    (e) The set of all 3×3 real skew-symmetric matrices.